Low Regularity Solutions for Gravity Water Waves

By

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Abstract

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The gravity water waves equations are a system of partial differential equations which govern the evolution of the interface between a vacuum and an incompressible, irrotational fluid in the presence of gravity. In the case of two dimensions, these equations model non-breaking waves at the surface of a body of water, such as a lake or ocean, while in one dimension, they model non-breaking waves propagating in a channel.

We are concerned with the well-posedness of the Cauchy problem for the gravity water waves equations: We seek to show that given an initial configuration of the vacuum-fluid interface and an initial fluid velocity field beneath the interface, there is a unique solution to the gravity water waves equations which matches the given initial data. In particular, we are concerned with the situation where the initial data has low regularity, corresponding to surface waves which are not necessarily smooth.

The classical regularity threshold for the well-posedness of the water waves system requires initial velocity field in H^s, with $s > \frac{d}{2} + 1$, and can be obtained by proving standard energy conservation estimates. On the other hand, it has been shown that for dispersive equations (equations describing phenomena which disperse waves of different frequencies), one can lower well-posedness regularity thresholds below that which is attainable by energy conservation alone. This was first realized for the nonlinear wave equation via dispersive estimates known as Strichartz estimates, and was first applied toward the well-posedness of gravity water waves by Alazard-Burq-Zuily.

However, this approach was implemented as a partial result, using Strichartz estimates with loss relative to what one expects based on the corresponding linearized model problem. In this dissertation, we prove well-posedness with initial velocity field in H^s, $s > \frac{d}{2} + 1 - \mu$, where $\mu = \frac{1}{10}$ in the case $d = 1$ and $\mu = \frac{1}{5}$ in the case $d \geq 2$, extending the previous result of Alazard-Burq-Zuily. In the case of one dimension, using a further refined argument, we establish the well-posedness for $s > \frac{1}{2} + 1 - \frac{1}{8}$, corresponding to proving lossless Strichartz estimates. This provides the sharp regularity threshold with respect to the approach of combining Strichartz estimates with energy estimates.

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Contents

Chapter 1

Introduction

1. The Gravity Water Waves Equations

The gravity water waves equations are a system of partial differential equations which govern the evolution of the interface between a vacuum and an incompressible, irrotational fluid in the presence of gravity. We are concerned with the threshold for the well-posedness of these equations with respect to the regularity of the initial data.

Let Ω denote a time dependent fluid domain contained in a fixed domain $\mathcal{O}$, located between a free surface and a fixed bottom:

$$\Omega = \{(t, x, y) \in [0, 1] \times \mathcal{O} \ ; \ y < \eta(t, x)\}$$

where $\mathcal{O} \subseteq \mathbb{R}^d \times \mathbb{R}$ is a given connected open set, with $x \in \mathbb{R}^d$ representing the horizontal spatial coordinate and $y \in \mathbb{R}$ representing the vertical spatial coordinate. We also assume the free surface

$$\Sigma = \{(t, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R} : y = \eta(t, x)\}$$

is separated from the fixed bottom $\Gamma = \partial\Omega \backslash \Sigma$ by a curved strip of depth $h > 0$:

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \eta(t, x) - h < y < \eta(t, x)\} \subseteq \mathcal{O}. \tag{1.1}$$

We consider an incompressible, irrotational fluid flow. In this setting the fluid velocity field v may be given by $\nabla_{x,y}\phi$ where $\phi : \Omega \to \mathbb{R}$ is a harmonic velocity potential,

$$\Delta_{x,y}\phi = 0.$$

We consider the situation of a constant downward gravitational force, but no surface tension. The water waves system is then given by

$$\begin{cases} \partial_t \phi + \dfrac{1}{2}|\nabla_{x,y}\phi|^2 + P + gy = 0 & \text{in } \Omega, \\ \partial_t \eta = \partial_y \phi - \nabla_x \eta \cdot \nabla_x \phi & \text{on } \Sigma, \\ P = 0 & \text{on } \Sigma, \\ \partial_\nu \phi = 0 & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $g > 0$ is acceleration due to gravity, ν is the normal to Γ, and P is the pressure, recoverable from the other unknowns by solving an elliptic equation. Here the first equation is the Euler equation in the presence of gravity, the second is the kinematic condition ensuring fluid particles at the interface remain at the interface, the third indicates no surface tension, and the fourth indicates a solid bottom.

A substantial literature regarding the well-posedness of the system (1.2) has been produced. We refer the reader to [ABZ14a], [ABZ14b], [Lan13] for a more complete history and references. In our direction, well-posedness of (1.2) in Sobolev spaces was first established by Wu [Wu97], [Wu99]. This well-posedness result was improved by Alazard-Burq-Zuily to a lower regularity at the threshold of a Lipschitz velocity field using only energy estimates [ABZ14a]. This was further sharpened to velocity fields with only a BMO derivative by Hunter-Ifrim-Tataru [HIT16].

It has been known that by taking advantage of an equation's dispersive properties, one can lower its well-posedness regularity threshold below that which is attainable by energy estimates alone. This was first realized in the context of low regularity Strichartz estimates for the nonlinear wave equation, in the works of Bahouri-Chemin [BC99b], [BC99a], Tataru [Tat00], [Tat01], [Tat02], Klainerman-Rodnianski [KR03], and Smith-Tataru [ST05]. Strichartz estimates have been similarly been studied for the water waves equations with surface tension; see [CHS10], [ABZ11b], [dPN15], [dPN16], [Ngu17].

This low regularity Strichartz paradigm was first applied toward gravity water waves by Alazard-Burq-Zuily in [ABZ14b]. The argument proceeds by first establishing a paradifferential formulation of the water waves system (described precisely in Section 2.2)

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)u = f \tag{1.3}$$

where T_V denotes the low-high frequency paraproduct with the vector field V, and T_γ denotes the low-high paradifferential operator with symbol γ. Here, γ is a real symbol of order $\frac{1}{2}$, thus making explicit the dispersive character of the equations. Then by proving a Strichartz estimate for the paradifferential equation (1.3), one can obtain *a priori* estimates for the full nonlinear system. However, to be useful toward the low regularity well-posedness theory, this Strichartz estimate must be proven assuming only a correspondingly low regularity of the coefficient vector field V and symbol γ. This difficulty is addressed in [ABZ14b] by truncating these coefficients to frequencies below λ^δ, where $\delta = 2/3 < 1$ is an optimized constant, essentially regularizing V and γ.

However, this truncation approach comes at the cost of derivative losses in the Strichartz estimates and corresponding losses in the well-posedness threshold. In fact, it is likely that using only the regularity of the coefficients V and γ in (1.3), it is impossible to prove Strichartz estimates without loss. For instance, in the context of the wave equation with Lipschitz metric, counterexamples to sharp Strichartz estimates were provided by Smith-Sogge [SS94] and Smith-Tataru [ST02]. Likewise, the former discusses counterexamples to sharp L^p estimates for eigenfunctions of elliptic operators with Lipschitz coefficients.

Thus, to further improve the well-posedness threshold (at least, through the low regularity Strichartz paradigm), one needs to invoke the additional structure of V and γ as solutions to the water waves system. In Chapter 2, we implement this by observing an integration structure that reveals extra regularity in the change of variables from Eulerian to Lagrangian coordinates. This approach applies to all dimensions with no additional difficulty, but is insufficient for achieving sharp Strichartz estimates. In Chapter 3, we entirely remove the derivative loss for the Strichartz estimates, implying the greatest improvement to the well-posedness threshold that can be attained using the low regularity Strichartz paradigm. Because this argument is quite delicate, we consider only the case $d = 1$ of one surface dimension, where the equations are somewhat simpler. Lastly, in Chapter 4, we outline the proof of well-posedness, given the low regularity Strichartz estimates.

2. Statement of Results

2.1. **Zakharov-Craig-Sulem formulation.** We first reduce the free boundary problem (1.2) to a system of equations on the free surface. Following Zakharov [Zak68] and Craig-Sulem [CS93], we have unknowns (η, ψ) where η is the vertical position of the fluid surface as before, and

$$\psi(t, x) = \phi(t, x, \eta(t, x))$$

is the velocity potential ϕ restricted to the surface. Then (henceforth, $\nabla = \nabla_x$):

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta + \dfrac{1}{2}|\nabla \psi|^2 - \dfrac{1}{2}\dfrac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases} \tag{2.1}$$

Here, $G(\eta)$ is the Dirichlet to Neumann map with boundary η:

$$(G(\eta)\psi)(t, x) = \sqrt{1 + |\nabla \eta|^2}(\partial_n \phi)|_{y=\eta(t,x)}.$$

See [ABZ11a], [ABZ14a] for a precise construction of $G(\eta)$ in a domain with a general bottom. In addition, it was shown in [ABZ13] that if a solution (η, ψ) of (2.1) belongs to $C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d))$ for $T > 0$ and $s > \frac{d}{2} + \frac{1}{2}$, then one can define a velocity potential ϕ and a pressure P satisfying the Eulerian system (1.2). We will consider the case $s > \frac{d}{2} + \frac{1}{2}$ throughout.

2.2. **Paradifferential reduction.** We establish Strichartz estimates in terms of the paradifferential reduction of the water waves system developed in [ABZ14a], [ABZ14b], which expresses the unknowns of the water waves system as the solution to an explicit dispersive equation. We recall the reduction in this subsection.

We denote the horizontal and vertical components of the velocity field restricted to the surface η by

$$V(t, x) = (\nabla \phi)|_{y=\eta(t,x)}, \quad B(t, x) = (\partial_y \phi)|_{y=\eta(t,x)}$$

and the good unknown of Alinhac by

$$U_s = \langle D_x \rangle^s V + T_{\nabla \eta} \langle D_x \rangle^s B.$$

The traces (V, B) can be expressed directly in terms of the unknowns (η, ψ) of the Zakharov-Craig-Sulem formulation,

$$B = \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad \nabla \psi = V + B\nabla \eta. \tag{2.2}$$

Denote the principal symbol of the Dirichlet to Neumann map

$$\Lambda(t, x, \xi) = \sqrt{(1 + |\nabla \eta|^2)|\xi|^2 - (\nabla \eta \cdot \xi)^2},$$

and the Taylor coefficient

$$a(t, x) = -(\partial_y P)|_{y=\eta(t,x)}.$$

We will assume the now-classical Taylor sign condition, $a(t, \cdot) \geq a_{min} > 0$. This expresses the fact that the pressure increases going from the air to the fluid domain. It is satisfied in the case of infinite bottom [Wu99] or small perturbations of flat bottoms [Lan05]. In the case of one surface dimension, also see [HIT16] and [HGIT17] for alternative proofs of this fact with infinite bottom and flat bottom, respectively. The water waves system is known to be ill-posed when the condition is not satisfied [Ebi87].

The reduction to a dispersive equation involves a symmetrization and complexification of the Eulerian system. As a result, it is convenient to define the symmetrized symbol

$$\gamma = \sqrt{a\Lambda}$$

and the complex unknown

$$u = \langle D_x \rangle^{-s}(U_s - iT_{\sqrt{a/\Lambda}} \langle D_x \rangle^s \nabla \eta). \tag{2.3}$$

Lastly, let $\mathcal{F}$ denote a non-decreasing positive function depending on h, g, and a_{min} and define

$$M_s(t) = \|(\eta, \psi)(t)\|_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} + \|(V, B)(t)\|_{H^s(\mathbb{R}^d)}$$
$$Z_r(t) = 1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}(\mathbb{R}^d)} + \|(V, B)(t)\|_{W^{r,\infty}(\mathbb{R}^d)}.$$

Then we have the following paradifferential equation for u (see Appendix E.1 for the definitions and notation of the paradifferential calculus), which holds in general dimension:

Proposition 2.1 ([ABZ14b, Corollary 2.7]). *Let*

$$0 < T \leq 1, \quad s > \frac{d}{2} + \frac{3}{4}, \quad r > 1.$$

Consider a smooth solution $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d))$ to (2.1) satisfying, uniformly on $t \in [0, T]$, (1.1) and the Taylor sign condition $a(t, \cdot) \geq a_{min} > 0$.

Then u given by (2.3) satisfies (1.3),

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f,$$

with

$$\|f(t)\|_{H^s(\mathbb{R}^d)} \leq \mathcal{F}(M_s(t)) Z_r(t).$$

Remark 2.2. The proposition was stated for $s > \frac{d}{2} + \frac{3}{4}$ in [ABZ14b], but using a more careful elliptic analysis, this can be reduced to $s > \frac{d}{2} + \frac{1}{2}$.

2.3. **Strichartz estimates and local well-posedness.** In this section we state our main result, a Strichartz estimate for arbitrary smooth solutions u to the equation (1.3). Write, denoting $I = [0, T]$,

$$\mathcal{M}_s(T) = \|(\eta, \psi)\|_{L^\infty(I; H^{s+\frac{1}{2}}(\mathbb{R}^d))} + \|(V, B)\|_{L^\infty(I; H^s(\mathbb{R}^d))}$$
$$\mathcal{Z}_r(T) = 1 + \|\eta\|_{L^p(I; W^{r+\frac{1}{2},\infty}(\mathbb{R}^d))} + \|(V, B)\|_{L^p(I; W^{r,\infty}(\mathbb{R}^d))}.$$

Theorem 2.3. *Let $d \geq 1$ and*

$$0 < T \leq 1, \quad s > \frac{d}{2} + \frac{3}{4}, \quad r > 1.$$

Consider a smooth solution $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d))$ to (2.1) satisfying, uniformly on $t \in [0, T]$, (1.1) and the Taylor sign condition $a(t, \cdot) \geq a_{min} > 0$.

Further, let

$$\sigma \in \mathbb{R}, \quad \epsilon > 0,$$

$$\begin{cases} \mu = \frac{1}{10}, \ p = 4 & \text{if } d = 1 \\ \mu = \frac{1}{5}, \ p = 2 & \text{if } d \geq 2. \end{cases}$$

Then a smooth solution u on I to (1.3),

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f$$

with coefficients V, γ obtained from (η, ψ), satisfies

$$\|u\|_{L^p(I;W^{\sigma+\mu-\frac{d}{2}-\epsilon,\infty}(\mathbb{R}^d))} \leq \mathcal{F}(\mathcal{M}_s(T) + \mathcal{Z}_r(T)) \left(\|f\|_{L^1(I;H^{\sigma-\frac{1}{10}}(\mathbb{R}^d))} + \|u\|_{L^\infty(I;H^\sigma(\mathbb{R}^d))} \right).$$

In the $d = 1$ case, we further improve the derivative gain μ to a sharp $\frac{1}{8}$ gain:

Theorem 2.4. *Let $d = 1$ and*

$$0 < T \leq 1, \quad s > \frac{d}{2} + \frac{7}{8}, \quad r > 1.$$

Consider a smooth solution $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}))$ to (2.1) satisfying, uniformly on $t \in [0, T]$, (1.1) and the Taylor sign condition $a(t, \cdot) \geq a_{min} > 0$. Further, let

$$\sigma \in \mathbb{R}, \quad \sigma' = \sigma + \frac{1}{8}, \quad p = 2, \quad \epsilon > 0.$$

Then a smooth solution u on I to (1.3),

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f$$

with coefficients V, γ obtained from (η, ψ), satisfies

$$\|u\|_{L^2(I;W^{\sigma'-\frac{d}{2}-\epsilon,\infty}(\mathbb{R}))} \leq \mathcal{F}(\mathcal{M}_s(T) + \mathcal{Z}_r(T))(\|f\|_{L^1(I;H^\sigma(\mathbb{R}))} + \|u\|_{L^\infty(I;H^\sigma(\mathbb{R}))}).$$

Remark 2.5. We make the following remarks regarding the Strichartz gain μ:

- For comparison, solutions to the constant coefficient linearized equation satisfy

  $$\|e^{-it|D_x|^{\frac{1}{2}}} u_0\|_{L^4([0,1];L^\infty(\mathbb{R}))} \lesssim \|u_0\|_{H^{\frac{3}{8}}(\mathbb{R})},$$

  $$\|e^{-it|D_x|^{\frac{1}{2}}} u_0\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|u_0\|_{H^{\frac{3}{4}}(\mathbb{R}^2)}.$$

  Thus, we expect at most a gain of $\mu = \frac{1}{8}$ derivatives over Sobolev embedding in the case $d = 1$, as achieved by Theorem 2.4, and a gain of $\mu = \frac{1}{4}$ in the case $d \geq 2$.
- Note that Theorem 2.4 does not achieve L^4 in time; this remains an open question for now.
- Such estimates were first established in [ABZ14b] with $\mu = \frac{1}{24}$ in the case $d = 1$ and $\mu = \frac{1}{12}$ in the case $d \geq 2$.

Theorems 2.3 and 2.4 combine with Proposition 2.1 to yield *a priori* energy and Hölder estimates. Combining these with contraction and limiting arguments, we will obtain the following local well-posedness with Strichartz estimates:

Theorem 2.6. *Let $d \geq 1$,*

$$\begin{cases} \mu = \frac{1}{10}, \ p = 4 & \text{if } d = 1 \\ \mu = \frac{1}{5}, \ p = 2 & \text{if } d \geq 2, \end{cases}$$

and

$$s > \frac{d}{2} + 1 - \mu, \quad 1 < r < s - \left(\frac{d}{2} - \mu\right).$$

Consider initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ *satisfying*

i) $(V_0, B_0) \in H^s(\mathbb{R}^d)$,

ii) the positive depth condition

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \eta_0(x) - h < y < \eta_0(x)\} \subseteq \mathcal{O},$$

iii) the Taylor sign condition

$$a_0(x) \geq a_{min} > 0.$$

Then there exists $T > 0$ *such that the system (2.1) with initial data* (η_0, ψ_0) *has a unique solution* $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d))$ *such that*

i) $(\eta, \psi) \in L^p([0, T]; W^{r+\frac{1}{2},\infty}(\mathbb{R}^d))$,

ii) $(V, B) \in C([0, T]; H^s(\mathbb{R}^d)) \cap L^p([0, T]; W^{r,\infty}(\mathbb{R}^d))$,

iii) the positive depth condition (1.1) holds on $t \in [0, T]$ *with* $h/2$ *in place of* h,

iv) the Taylor sign condition $a(t, x) \geq a_{min}/2$ *holds on* $t \in [0, T]$.

Further, if $(\eta_{0,n}, \psi_{0,n})$ *uniformly form a sequence of such initial data with*

$$(\eta_{0,n}, \psi_{0,n}) \to (\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d),$$

then the corresponding solutions converge:

$$(\eta_n, \psi_n, V_n, B_n) \to (\eta, \psi, V, B) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d)).$$

Theorem 2.7. *Theorem 2.6 holds with*

$$d = 1, \quad \mu = \frac{1}{8}, \quad p = 2.$$

3. Overview of the Arguments

Here we outline the main ideas used in Chapters 2 and 3 to prove the Strichartz estimates of Theorem 2.3 and 2.4 respectively. The proof of well-posedness, Theorems 2.6 and 2.7, is relatively standard in the context of quasilinear equations, and is presented in Chapter 4.

3.1. **General surface dimensions.** To prove the low regularity Strichartz estimate of Theorem 2.3, we use a parametrix construction for (1.3) based on the FBI transform (see [Tat04], [KT05]). This transform decomposes the initial data into space and frequency localized components which evolve along the Hamilton flow, yielding a parametrix resembling a Fourier integral operator with complex phase. We use this parametrix to establish fixed-time dispersive estimates, so that a classical TT^* argument then yields the Strichartz estimate.

A difficulty one encounters in using this strategy is that the order 1 transport term $T_V \cdot \nabla$ of (1.3) dominates the order 1/2 dispersive term T_γ when considering the regularity of the Hamilton flow associated to (1.3), which is central to the fixed-time dispersive estimates.

This difficulty is addressed in [ABZ14b] by performing a change of variables $x \mapsto X$ to straighten the vector field $\partial_t + V \cdot \nabla$, essentially passing to Lagrangian coordinates,

$$\partial_t v + ip(t, y, D)v = f(X) \tag{3.1}$$

where p is a symbol of order $1/2$. However, this change of variables comes at the cost of a low regularity Jacobian term $\partial_x X$ appearing in the symbol p, forcing losses in the Strichartz estimates.

Here, we use the fact that V arises from a solution to the Euler equations to observe that we may write

$$V \approx (\partial_t + V \cdot \nabla)F$$

where F has improved regularity over V. Using this integration structure, we observe that the change of variables X has a hidden Strichartz regularity. This may be used to observe a corresponding hidden regularity for the symbol p, which in fact matches the Strichartz regularity of the original symbol $\gamma \in L_t^2 C_x^{1/2}$.

However, a heuristic wave packet analysis shows that the order $1/2$ dispersive equation requires symbol regularity $\gamma \in L_t^2 C_x^{2/3}$ to establish Strichartz estimates without loss relative to the constant coefficient case. Thus, as in [ABZ14b], we still need to truncate the coefficients to frequencies below λ^δ, with $\delta = 9/10$ an optimized constant. We remark that as part of this approach, one decomposes the time interval into a number of short intervals on which one proves the Strichartz estimates, before recombining the intervals at a loss depending on the number of short intervals. Here, we are able to obtain an additional gain over the implementation in [ABZ14b] by performing the time interval partition more delicately, balancing the the length of each short interval with the regularity of the symbol on that interval (also see [Tat02]).

3.2. **Sharp estimates in one surface dimension.** The strategy we use to prove Theorem 2.4 relies on a parametrix construction that approximates solutions to (1.3) as a square-summable superposition of discrete wave packets. The Strichartz estimates then follow from a geometric observation capturing the dispersion of the packets, combined with a counting argument. This is in contrast with the Hadamard parametrix used in [ABZ14b], or the parametrix based on the FBI transform that we will use in the case of general dimensions in Chapter 2, both of which use dispersive estimates combined with a classical TT^* argument to obtain the Strichartz estimates. The strategy we follow here more closely resembles the parametrix construction and counting argument used for the wave equation in [ST05]. Also in contrast with both [ABZ14b] and the general dimensional argument in Chapter 2, we incorporate the transport term of (1.3) directly in the parametrix construction, rather than changing variables to Lagrangian coordinates.

To proceed without derivative losses, we can only truncate the coefficients of (1.3) to frequencies below λ, in further contrast with [ABZ14b] and Chapter 2, which truncate the coefficients of (1.3) to frequencies λ^δ, with δ a constant below 1. As a result, our wave packet parametrix construction and subsequent geometric analysis must be performed at lower regularity. This presents difficulties on two main fronts.

First, if we observe only the regularity of the symbol of (1.3), the associated Hamilton characteristics along which wave packets travel do not have sufficient regularity to allow a meaningful geometric analysis of the dispersion on the full time interval. To overcome this, we seek an integration structure for the symbol along the Hamilton flow, to show that the characteristics are in fact bilipschitz. We remark that this integration structure is more complex than that in Chapter 2, as it involves the full symbol $V\xi + \gamma$ rather than V alone.

Second, the natural wave packet scale at frequency λ for a dispersive equation of order 1/2 is

$$\delta x \approx \lambda^{-\frac{3}{4}}, \quad \delta\xi \approx \lambda^{\frac{3}{4}}, \quad \delta t \approx 1.$$

However, the coefficients of (1.3), when truncated only to frequencies below λ, are approximately constant only on the scale $\delta x \approx \lambda^{-1}$, so we cannot expect a generic bump function localized on the wave packet scale to be a useful approximate solution. We address this by using an exact eikonal phase function for our wave packets, at the cost of losing the exact localization in frequency. In turn, this interferes with the square summability of the wave packets. To gain the additional frequency localization needed for the summability, we use the fact that, along wave packets, the coefficients of (1.3) enjoy extra regularity in the form of dispersive local smoothing estimates on the wave packet scale $\delta x \approx \lambda^{-\frac{3}{4}}$.

4. Notation

We record the notation and setting that we will use throughout.

4.1. Regularity indices. We assume

$$s > \frac{d}{2} + \frac{3}{4}, \quad 1 < r < s - \frac{d}{2} + \frac{1}{4}, \quad r \leq \frac{3}{2}.$$

Note that some propositions will evidently hold without the upper bounds on r. Also let $0 < \epsilon < 1 - r$ denote a small number, and for any regularity index $m \in \mathbb{R}$, let

$$m+ = m + \epsilon, \quad m- = m - \epsilon.$$

We will allow $m+, m-$, and ϵ to vary from line to line.

In Chapter 2, we let

$$\begin{cases} \mu = \frac{1}{10}, \ p = 4 & \text{if } d = 1 \\ \mu = \frac{1}{5}, \ p = 2 & \text{if } d \geq 2. \end{cases}$$

In Chapter 3, we set $d = 1$ and $p = 2$. In place of setting $\mu = 1/8$, we use the notation

$$m' = m + \frac{1}{8}$$

for brevity. We will further assume

$$s > \frac{d}{2} + \frac{7}{8}$$

where $d = 1$. Note that we will often preserve the notation of general d to provide context.

4.2. Quantities from the water waves system. We write

$$\nabla = \nabla_x, \quad \Delta = \Delta_x, \quad \langle D \rangle = \langle D_x \rangle, \quad |D| = |D_x|.$$

Unless otherwise specified, let $0 < T \leq 1$, $I = [0, T]$, and $(\eta, \psi) \in C^1(I; H^{s+\frac{1}{2}}(\mathbb{R}^d))$ denote a smooth solution to (2.1) satisfying, uniformly on $t \in I$, the positive depth condition (1.1). From (η, ψ) we can extract the harmonic velocity potential $\phi(t, x, y)$ satisfying

$$\Delta_{x,y}\phi = 0, \qquad \phi|_{y=\eta(t,x)} = \psi,$$

the velocity field $v(t, x, y)$,

$$v = \nabla_{x,y}\phi,$$

the pressure $P(t,x,y)$,

$$-P = \partial_t\phi + \frac{1}{2}|\nabla_{x,y}\phi|^2 + gy,$$

the horizontal and vertical components $(V,B)(t,x)$ of the velocity field restricted to the surface η,

$$V = (\nabla\phi)|_{y=\eta(t,x)}, \quad B = (\partial_y\phi)|_{y=\eta(t,x)},$$

and the Taylor coefficient $a(t,x)$,

$$a = -(\partial_y P)|_{y=\eta(t,x)}.$$

We assume $(V,B) \in L^\infty(I; H^s(\mathbb{R}^d))$, as well as the Taylor sign condition $a \geq a_{min} > 0$ as discussed in the introduction. We remark that throughout, we use the estimates on a provided in Proposition D.8. Lastly, we obtain the principal symbol $\Lambda(t,x,\xi)$ of the Dirichlet to Neumann map associated to $\eta(t,\cdot)$,

$$\Lambda = \sqrt{(1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2},$$

and define the symmetrized symbol

$$\gamma = \sqrt{a\Lambda}.$$

Observe that in the case $d=1$, we have

$$\Lambda = |\xi|, \quad \gamma = \sqrt{a|\xi|}.$$

Lastly, for brevity we let L denote the vector field

$$L = \partial_t + V\cdot\nabla.$$

4.3. **Frequency decomposition and truncation.** Throughout, we denote frequencies by $\lambda, \mu, \kappa \in 2^{\mathbb{N}}$. We will fix λ by the end of Section 5 in Chapter 2 and Section 10 in Chapter 3, and assume μ satisfies $\lambda^{\frac{3}{4}} \ll \mu \leq c\lambda$, where the small absolute constant c is discussed further below. We typically use κ to denote a general frequency parameter.

We recall the standard Littlewood-Paley decomposition. Fix $\varphi(\xi) \in C_0^\infty(\mathbb{R}^d)$ with support in $\{|\xi| \leq 2\}$ such that $\varphi \equiv 1$ on $\{|\xi| \leq 1\}$. Then define for $u = u(x)$,

$$\widehat{S_\kappa u}(\xi) := (\varphi(\xi/\kappa) - \varphi(2\xi/\kappa))\widehat{u}(\xi) =: \psi(\xi/\kappa)\widehat{u} =: \psi_\kappa(\xi)\widehat{u}$$

which has support $\{|\xi| \in [\kappa/2, 2\kappa]\}$. Also allow $S_{<\kappa}$, etc., in the natural way, and denote $S_0 = S_{<1}$. Lastly, let $\tilde{S}_\kappa = \sum_{\kappa/4 \leq \tilde{\kappa} \leq 4\kappa} S_{\tilde{\kappa}}$ denote a frequency projection with widened support so that $\tilde{S}_\kappa S_\kappa = S_\kappa$.

For a small absolute constant $c > 0$, write

$$A \ll B, \quad A \approx B$$

when, respectively,

$$A \leq cB, \quad (1-c)A \leq B \leq A/(1-c).$$

In Chapter 2, we denote, for $0 < \delta < 1$,

$$V_\delta = S_{\leq\lambda^\delta} V, \quad \eta_\delta = S_{\leq\lambda^\delta}\eta, \quad \gamma_\delta = S_{\leq\lambda^\delta}\gamma, \quad L_\delta = \partial_t + V_\delta\cdot\nabla.$$

In Chapter 3, we will set an additional absolute constant

$$0 < c_1 \ll c \ll 1,$$

where $c_1\lambda$ will be the frequency truncation for the paradifferential equation on which we prove Strichartz estimates. We denote

$$a_\kappa = S_{\leq c_1\kappa} a, \quad V_\kappa = S_{\leq c_1\kappa} V, \quad \eta_\kappa = S_{\leq c_1\kappa}\eta, \quad \gamma_\kappa = S_{\leq c_1\kappa}\gamma.$$

4.4. **A priori quantities.** Throughout, $\mathcal{F}$ will denote a non-decreasing positive function which may change from line to line, and which may depend on the depth $h > 0$, the constant $a_{min} > 0$ of the Taylor sign condition, and the gravitational constant $g > 0$. For brevity, we denote

$$\begin{aligned} M(t) = M_s(t) &= \|(\eta, \psi)(t)\|_{H^{s+\frac{1}{2}}} + \|(V, B)(t)\|_{H^s}, \\ Z(t) = Z_r(t) &= 1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|(V, B)(t)\|_{W^{r,\infty}}. \end{aligned}$$

We also write for brevity,

$$\begin{aligned} \mathcal{M}(T) = \mathcal{M}_s(T) &= \|(\eta, \psi)\|_{L^\infty(I; H^{s+\frac{1}{2}})} + \|(V, B)\|_{L^\infty(I; H^s)}, \\ \mathcal{Z}(T) = \mathcal{Z}_r(T) &= \|\eta\|_{L^p(I; W^{r+\frac{1}{2},\infty})} + \|(V, B)\|_{L^p(I; W^{r,\infty})}, \\ \mathcal{F}(T) &= \mathcal{F}(\mathcal{M}_s(T) + \mathcal{Z}_r(T)). \end{aligned}$$

Observe that

$$\|\mathcal{F}(M(t))\|_{L^\infty_t(I)} + \|Z(t)\|_{L^p_t(I)} \leq \mathcal{F}(T).$$

Chapter 2

Strichartz Estimates for General Surface Dimensions

In this chapter, we establish Theorem 2.3, the Strichartz estimates in general dimensions but with a loss of derivatives.

We outline the chapter below. First, in Section 5, we perform a standard reduction of the Strichartz estimate to a dyadic frequency localized form.

In Section 7, we straighten the vector field $\partial_t + V \cdot \nabla$ to remove the non-dispersive principal term in (1.3). This is done via a change of variables obtained by solving the flow $\dot{X} = V(t, X)$, essentially passing to Lagrangian coordinates. This method was already applied in [ABZ14b].

We will observe some structure of the vector field V to discover extra regularity in the flow X, which was a limiting factor in the Strichartz estimates proved in [ABZ14b]. We discuss this structure in Section 6. As part of this analysis, we will be required to estimate our errors, and in particular the paralinearization error of the Dirichlet to Neumann map, in Hölder norm. These estimates are established in Appendices A and D.

In Section 8, we construct a parametrix and prove Strichartz estimates for the equation after the change of variables. Here, we apply a wave packet parametrix, which has the advantage that it only requires control of the Hamilton flow on the λ-frequency wave packet scale associated to our dispersive operator, $\Delta x \approx \lambda^{-3/4}$.

As in [ABZ14b], the low regularity of our symbol will limit our Strichartz estimate, without derivative loss, to short time intervals. In Section 9, we partition the unit time interval into such short time intervals, implying a Strichartz estimate on a unit time interval with a loss depending on the number of intervals. We obtain a gain by performing this partition in a way that balances the the length of the time interval with the regularity of our symbol on that interval (also see [Tat02]).

5. Frequency Localization

In this section we reduce Theorem 2.3 to a frequency localized form.

5.1. **Dyadic decomposition.** First we reduce to the corresponding dyadic frequency estimates:

Proposition 5.1. *Consider a smooth solution u_λ to (1.3) on I where $u = u_\lambda(t, \cdot)$ and $f = f_\lambda$ have frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$. Then*

$$\|u_\lambda\|_{L^p(I;W^{\mu-\frac{d}{2}-,\infty})} \leq \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

Proof of Theorem 2.3. Given u solving (1.3), $u_\lambda = S_\lambda u$ solves (1.3) with inhomogeneity

$$S_\lambda f + [T_V \cdot \nabla, S_\lambda]u + i[T_\gamma, S_\lambda]u.$$

Note that this has frequency support $\{|\xi| \in [\lambda/4, 4\lambda]\}$ (strictly speaking, we should modify Proposition 5.1 to address this wider support, but we neglect this detail throughout for simplicity). By the paradifferential commutator estimate (E.2),

$$\|[T_V \cdot \nabla, S_\lambda]u\|_{L^1(I;H^{\sigma-\frac{1}{10}})} \lesssim \|V\|_{L^1(I;W^{1,\infty})}\|\tilde{S}_\lambda u\|_{L^\infty(I;H^\sigma)}.$$

Similarly, by (E.2) and the estimates on the Taylor coefficient as in Corollary D.10,

$$\|[T_\gamma, S_\lambda]u\|_{H^{\sigma-\frac{1}{10}}} \lesssim M^{\frac{1}{2}}_{\frac{1}{2}}(\gamma)\|\tilde{S}_\lambda u\|_{H^\sigma} \leq \mathcal{F}(M(t))Z(t)\|\tilde{S}_\lambda u\|_{H^\sigma}$$

and hence

$$\|[T_\gamma, S_\lambda]u\|_{L^1(I;H^{\sigma-\frac{1}{10}})} \leq \mathcal{F}(T)\|\tilde{S}_\lambda u\|_{L^\infty(I;H^\sigma)}.$$

We then decompose u into frequency pieces u_λ on which we can apply Proposition 5.1 (multiplied by λ^σ, using the frequency localization):

$$\begin{aligned}
\|u\|_{L^p(I;W^{\sigma+\mu-\frac{d}{2}-,\infty})} &\leq \sum_{\lambda=0} \|u_\lambda\|_{L^p(I;W^{\sigma+\mu-\frac{d}{2}-,\infty})} \\
&\leq \mathcal{F}(T)\sum_{\lambda=0} \lambda^{-\epsilon}(\|S_\lambda f\|_{L^1(I;H^{\sigma-\frac{1}{10}})} + \|\tilde{S}_\lambda u\|_{L^\infty(I;H^\sigma)}) \\
&\leq \mathcal{F}(T)(\|f\|_{L^1(I;H^{\sigma-\frac{1}{10}})} + \|u\|_{L^\infty(I;H^\sigma)})
\end{aligned}$$

as desired. □

5.2. **Symbol truncation.** Next, we reduce Proposition 5.1 to an estimate with frequency truncated symbols.

Proposition 5.2. *Let $\frac{9}{10} \leq \delta < 1$. Consider a smooth solution u_λ to*

$$(\partial_t + T_{V_\delta} \cdot \nabla + iT_{\gamma_\delta})u_\lambda = f_\lambda \tag{5.1}$$

on I where $u_\lambda(t,\cdot)$ and f_λ have frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$. Then

$$\|u_\lambda\|_{L^p(I;W^{\mu-\frac{d}{2}-,\infty})} \leq \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

Proof of Proposition 5.1. This follows from Proposition 5.2 by including $(T_V - T_{V_\delta})\cdot\nabla u_\lambda$ and $(T_\gamma - T_{\gamma_\delta})u_\lambda$ as components of the inhomogeneous term, along with the following estimates, proven below:

$$\begin{aligned}
\|(T_V - T_{V_\delta}) \cdot \nabla u_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} &\leq \mathcal{F}(T)\|u_\lambda\|_{L^\infty(I;L^2)} \\
\|(T_\gamma - T_{\gamma_\delta})u_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} &\leq \mathcal{F}(T)\|u_\lambda\|_{L^\infty(I;L^2)}.
\end{aligned}$$

For the first estimate, observe that

$$(T_V - T_{V_\delta}) \cdot \nabla u_\lambda = (S_{\lambda^\delta < \cdot \leq \lambda/8}V) \cdot \nabla u_\lambda$$

which satisfies

$$\|(S_{\lambda^\delta < \cdot \leq \lambda/8}V) \cdot \nabla u_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} \lesssim \lambda^{-\frac{1}{10}}\lambda^{1-\delta}\|V\|_{L^1(I;W^{1,\infty})}\|u_\lambda\|_{L^\infty(I;L^2)}$$

where $1 - \delta - \frac{1}{10} \leq 0$ as desired.

The second estimate is similar, using (E.1) to obtain

$$\|(T_\gamma - T_{\gamma_\delta})u_\lambda\|_{H^{-\frac{1}{10}}} \lesssim \lambda^{-\frac{1}{10}} M_0^{\frac{1}{2}}(S_{>\lambda^\delta}\gamma)\|u_\lambda\|_{H^{\frac{1}{2}}} \\ \lesssim \lambda^{-\frac{1}{10}}\lambda^{\frac{1}{2}-\frac{1}{2}\delta} M_{\frac{1}{2}}^{\frac{1}{2}}(\gamma)\|u_\lambda\|_{L^2}.$$

Then $\frac{1}{2} - \frac{1}{2}\delta - \frac{1}{10} < 0$ is better than needed. Using the estimates on γ provided in Corollary D.10 and integrating in time yields the desired result. □

5.3. **Pseudodifferential symbol.** To construct a parametrix, it is convenient to replace the paradifferential symbol T_{γ_δ} with the pseudodifferential symbol $\gamma_\delta = \gamma_\delta(t, x, \xi)$, and likewise the paraproduct T_{V_δ} by V_δ. It is harmless to do so since u_λ is frequency localized.

Since Proposition 5.2 reduces to Proposition 5.3 as seen below, the remainder of CHapter 2 will be dedicated to the proof of Proposition 5.3:

Proposition 5.3. *Let $\frac{9}{10} \leq \delta < 1$. Consider a smooth solution u_λ to*

$$(\partial_t + V_\delta \cdot \nabla + i\gamma_\delta(t, x, D))u_\lambda = f_\lambda \tag{5.2}$$

on I where $u_\lambda(t, \cdot)$ and f_λ have frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$. Then

$$\|u_\lambda\|_{L^p(I;W^{\mu-\frac{d}{2}-,\infty})} \leq \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

Proof of Proposition 5.2. We may apply Proposition 5.3 with inhomogeneity

$$f_\lambda + (T_{\gamma_\delta} - \gamma_\delta)u_\lambda + (T_{V_\delta} - V_\delta) \cdot \nabla u_\lambda.$$

Using for instance [Ngu15, Proposition 2.7] and the estimates on γ as in Corollary D.10,

$$\|(T_{\gamma_\delta} - \gamma_\delta)u_\lambda\|_{H^{-\frac{1}{10}}} \lesssim M_{\frac{1}{2}}^{\frac{1}{2}}(\gamma)\|u_\lambda\|_{H^{-\frac{1}{10}}} \leq \mathcal{F}(M(t))Z(t)\|u_\lambda\|_{H^{-\frac{1}{10}}}.$$

Integrating in time, we have

$$\|(T_{\gamma_\delta} - \gamma_\delta)u_\lambda\|_{L^1(I;H^{-\frac{1}{10}})} \leq \mathcal{F}(T)\|u_\lambda\|_{L^\infty(I;H^{-\frac{1}{10}})}$$

which is better than needed.

Similarly,

$$\|(T_{V_\delta} - V_\delta) \cdot \nabla u_\lambda\|_{H^{-\frac{1}{10}}} \lesssim M_1^1(V \cdot \xi)\|u_\lambda\|_{H^{-\frac{1}{10}}} \lesssim \|V\|_{W^{1,\infty}}\|u_\lambda\|_{H^{-\frac{1}{10}}}.$$

Integrating in time yields the desired estimate.

□

6. Flow of the Vector Field

6.1. **Integrating the vector field.** Recall that the traces of the velocity field on the surface (V, B) satisfy the relations [ABZ14a, Proposition 4.3]:

$$L\nabla\eta = \nabla B - \nabla V \cdot \nabla\eta = G(\eta)V + \nabla\eta G(\eta)B + \Gamma_x + \nabla\eta\Gamma_y, \quad G(\eta)B = -\nabla \cdot V - \Gamma_y. \tag{6.1}$$

Here, Γ_y, Γ_x arise only in the case of finite bottom, and are discussed in Appendix D.2.

Observe that the vector field V of (1.3) may be integrated along L by using (6.1), as in the following proposition. This additional structure will imply improved regularity on the flow of the vector field.

Proposition 6.1. *Let $\alpha \geq \frac{1}{2}$. Then*

$$\partial_x V_\delta = L_\delta T_{q^{-1}} \partial_x \nabla \eta_\delta + g \tag{6.2}$$

where $q = \Lambda I - i\nabla\eta \cdot \xi^T$ is a matrix-valued symbol of order 1, and g satisfies

$$\begin{aligned}\|g\|_{W^{\alpha,\infty}(\mathbb{R}^d)} \leq \lambda^{\delta(\alpha-\frac{1}{2})} &\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}\times H^s \times H^s}) \\ &\cdot (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|(V, B)\|_{W^{r,\infty}}).\end{aligned}$$

Remark 6.2. The proposition only uses $s > \frac{d}{2} + \frac{1}{2}$.

Proof. We may assume $\alpha = \frac{1}{2}$; for the general case, observe that all the terms of (6.2) have frequency support $\{|\xi| \lesssim \lambda^\delta\}$.

We have from (6.1)

$$L\nabla\eta = G(\eta)V - (\nabla\eta)\nabla \cdot V + \Gamma_x.$$

Step 1. Paralinearization. Paralinearize the terms

$$\begin{aligned} G(\eta)V &= T_\Lambda V + R(\eta)V, \\ (\nabla\eta)\nabla \cdot V &= T_{\nabla\eta}\nabla \cdot V + T_{\nabla\cdot V}\nabla\eta + R(\nabla\eta, \nabla\cdot V)\end{aligned}$$

of (12.7). Then rearranging,

$$T_q V = T_\Lambda V - T_{\nabla\eta}\nabla\cdot V = L\nabla\eta - R(\eta)V + T_{\nabla\cdot V}\nabla\eta + R(\nabla\eta, \nabla\cdot V) - \Gamma_x.$$

We estimate the error terms on the right hand side. By Proposition D.3, (E.1), (E.6), and Proposition D.7 respectively,

$$\begin{aligned} \|R(\eta)V\|_{W^{\frac{1}{2},\infty}} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|V\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|V\|_{W^{r,\infty}}) \\ \|T_{\nabla\cdot V}\nabla\eta\|_{W^{\frac{1}{2},\infty}} &\lesssim \|V\|_{W^{1,\infty}}\|\eta\|_{W^{\frac{3}{2},\infty}} \\ \|R(\nabla\eta, \nabla\cdot V)\|_{W^{\frac{1}{2},\infty}} &\lesssim \|\eta\|_{W^{\frac{3}{2},\infty}}\|V\|_{W^{1,\infty}} \\ \|\Gamma_x\|_{W^{\frac{1}{2},\infty}} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}).\end{aligned}$$

We conclude

$$T_q V = L\nabla\eta + g_1$$

where

$$\begin{aligned}\|g_1\|_{W^{\frac{1}{2},\infty}} \leq &\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}\times H^s\times H^s}) \\ &\cdot (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|(V, B)\|_{W^{r,\infty}}).\end{aligned}$$

For brevity, denote the right hand side by E.

Step 2. Inversion of q. To invert T_q, note the outer product $\nabla\eta\cdot\xi^T$ has only real eigenvalues, so $\Lambda I - i\nabla\eta\cdot\xi^T = -i(\nabla\eta\cdot\xi^T + i\Lambda I)$ is invertible except when $\Lambda = 0$. Furthermore, since $\Lambda \geq |\xi|$, we see that for fixed $|\xi| \geq \frac{1}{2}$ the inverse is a smooth function of $\nabla\eta$. We write

$$V = T_{q^{-1}}L\nabla\eta + T_{q^{-1}}g_1 + (1 - T_{q^{-1}}T_q)V.$$

We again estimate the error terms on the right hand side. For the first error term $T_{q^{-1}}g_1$, q^{-1} is a symbol of order -1 and is a smooth function of $\nabla\eta$, so that by (E.15) and Sobolev embedding with $s - \frac{1}{2} > \frac{d}{2}$,

$$M_0^{-1}(q^{-1}) \leq \mathcal{F}(\|\nabla\eta\|_{L^\infty}) \leq \mathcal{F}(\|\nabla\eta\|_{H^{s-\frac{1}{2}}}) \tag{6.3}$$

and similarly

$$M^{-1}_{\frac{1}{2}}(q^{-1}) \leq \mathcal{F}(\|\nabla\eta\|_{H^{s-\frac{1}{2}}})\|\nabla\eta\|_{C^{\frac{1}{2}}_*}.$$

Then by (E.1) and the estimate on g_1 in the previous step,

$$\|T_{q^{-1}}g_1\|_{W^{\frac{3}{2},\infty}} \lesssim M^{-1}_0(q^{-1})\|g_1\|_{W^{\frac{1}{2},\infty}} \leq E.$$

Similarly, by (E.3), we control the second error term by

$$\begin{aligned}\|(1-T_{q^{-1}}T_q)V\|_{W^{\frac{3}{2},\infty}} &\lesssim \left(M^{-1}_{1/2}(q^{-1})M^1_0(q) + M^{-1}_0(q^{-1})M^1_{1/2}(q)\right)\|V\|_{W^{1,\infty}} \\ &\lesssim \|\eta\|_{W^{r+\frac{1}{2},\infty}}\|V\|_{W^{1,\infty}}.\end{aligned}$$

We conclude

$$V = T_{q^{-1}}L\nabla\eta + g_2$$

with

$$\|g_2\|_{W^{\frac{3}{2},\infty}} \leq E.$$

Step 3. Frequency localization and differentiation. Applying $\partial_x S_{\leq\lambda^\delta}$ to both sides of our identity, we have

$$\partial_x V_\delta = T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}L\nabla\eta + \partial_x S_{\leq\lambda^\delta}g_2 + [\partial_x S_{\leq\lambda^\delta}, T_{q^{-1}}]L\nabla\eta.$$

For the first error term, we have

$$\|\partial_x S_{\leq\lambda^\delta}g_2\|_{W^{\frac{1}{2},\infty}} \lesssim \|g_2\|_{W^{\frac{3}{2},\infty}}.$$

To estimate the second error term, use the first identity of (6.1), (E.3), and (6.3),

$$\begin{aligned}&\|[\partial_x S_{\leq\lambda^\delta}, T_{q^{-1}}]L\nabla\eta\|_{W^{\frac{1}{2},\infty}} \\ &\qquad\lesssim \left(M^{-1}_{1/2}(q^{-1}) + M^{-1}_0(q^{-1})\right)\|\partial B - \nabla\eta\cdot\partial V\|_{L^\infty} \\ &\qquad\lesssim \|\eta\|_{W^{r+\frac{1}{2},\infty}}(\|B\|_{W^{1,\infty}} + \|\nabla\eta\|_{L^\infty}\|\partial V\|_{L^\infty}) \\ &\qquad\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|(V,B)\|_{W^{1,\infty}}\|\eta\|_{W^{r+\frac{1}{2},\infty}}.\end{aligned}$$

We conclude

$$\partial_x V_\delta = T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}L\nabla\eta + g_3$$

with

$$\|g_3\|_{W^{\frac{1}{2},\infty}} \leq E.$$

Step 4. Vector field paralinearization. We paralinearize the vector field in order to commute past it in the next step. Writing the paraproduct expansion

$$(V\cdot\nabla)\nabla\eta = (T_V\cdot\nabla)\nabla\eta + T_{\nabla(\nabla\eta)}\cdot V + R(V,\nabla(\nabla\eta)),$$

we have

$$\partial_x V_\delta = T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}(\partial_t + T_V\cdot\nabla)\nabla\eta + g_3 + T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}(R(V,\nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)}\cdot V).$$

Then by (E.1), (E.6), and (E.9),

$$
\begin{aligned}
\|T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}(R(V,\nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)}\cdot V)\|_{W^{\frac12,\infty}} \\
&\lesssim M_0^0(q^{-1}\xi)\|S_{\leq\lambda^\delta}(R(V,\nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)}\cdot V)\|_{W^{\frac12,\infty}} \\
&\lesssim \|\nabla\eta\|_{L^\infty}\|V\|_{W^{1,\infty}}\|\nabla^2\eta\|_{W^{-\frac12,\infty}} \\
&\lesssim \|\eta\|_{H^{s+\frac12}}\|V\|_{W^{1,\infty}}\|\eta\|_{W^{\frac32,\infty}}.
\end{aligned}
$$

We may thus replace V with T_V, yielding, for g_4 satisfying the same estimate as g_3,

$$\partial_x V_\delta = T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}(\partial_t + T_V\cdot\nabla)\nabla\eta + g_4.$$

Step 5. Vector field commutator estimate. Applying Proposition D.11 with $m = 0$, $r = \frac12$ and $\epsilon = 1$, we may exchange $T_{q^{-1}}\partial_x S_{\leq\lambda^\delta}(\partial_t + T_V\cdot\nabla)\nabla\eta$ for $LT_{q^{-1}}\partial_x\nabla\eta_\delta$ with an error bounded in $W^{\frac12,\infty}$ by (using that q is a smooth function of $\nabla\eta$ and the first identity of (6.1))

$$
\begin{aligned}
&M_0^0(q^{-1}\xi)\|V\|_{W^{1,\infty}}\|\nabla\eta\|_{B^{\frac12}_{\infty,1}} + M_0^0((\partial_t + V\cdot\nabla)q^{-1}\xi)\|\nabla\eta\|_{W^{\frac12,\infty}} \\
&\lesssim (\|\eta\|_{W^{1,\infty}}\|V\|_{W^{1,\infty}}\|\eta\|_{B^{\frac32}_{\infty,1}} + \|\partial B - \nabla\eta\cdot\partial V\|_{L^\infty}\|\eta\|_{W^{\frac32,\infty}}) \\
&\leq \mathcal{F}(\|\eta\|_{H^{s+\frac12}})\|(V,B)\|_{W^{1,\infty}}\|\eta\|_{W^{r+\frac12,\infty}} \leq E.
\end{aligned}
$$

Step 6. Vector field truncation. Lastly we frequency truncate the vector field, using (E.12) and (E.1):

$$
\begin{aligned}
\|((S_{>\lambda^\delta}V)\cdot\nabla)T_{q^{-1}}\partial_x\nabla\eta_\delta\|_{W^{\frac12,\infty}} \\
&\lesssim \|S_{>\lambda^\delta}V\|_{W^{\frac12,\infty}}M_0^0(q^{-1}\xi)\|\nabla\eta_\delta\|_{W^{1,\infty}} + \|S_{>\lambda^\delta}V\|_{L^\infty}M_0^0(q^{-1}\xi)\|\nabla\eta_\delta\|_{W^{\frac32,\infty}} \\
&\lesssim \|\eta\|_{H^{s+\frac12}}(\lambda^{\frac12\delta}\|S_{>\lambda^\delta}V\|_{W^{\frac12,\infty}}\|\nabla\eta_\delta\|_{W^{\frac12,\infty}} + \lambda^\delta\|S_{>\lambda^\delta}V\|_{L^\infty}\|\nabla\eta_\delta\|_{W^{\frac12,\infty}}) \\
&\lesssim \|\eta\|_{H^{s+\frac12}}\|V\|_{W^{r,\infty}}\|\nabla\eta_\delta\|_{W^{\frac12,\infty}}.
\end{aligned}
$$

□

6.2. **Regularity of the flow.** We straighten the vector field $L_\delta = \partial_t + V_\delta\cdot\nabla$ by considering the system

$$
\begin{cases}
\dot X(s) = V_\delta(s, X(s)) \\
X(s_0) = x.
\end{cases}
\tag{6.4}
$$

Since we assume that $V \in L^\infty([0,T]\times\mathbb{R}^d)$, V_δ is smooth with bounded derivatives so that this system has a unique solution defined on $I = [0,T]$, which we denote $X(s,x)$. For emphasis, we may also write $X = X(s)$ or $X(x)$.

Proposition 6.3 ([ABZ14b, Proposition 2.16])**.** *The map $X(s,\cdot)$ is smooth, with the following estimates:*

$$\|(\partial_x X)(s,\cdot) - Id\|_{L^\infty(\mathbb{R}^d)} \leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})|s-s_0|^{1/2} \tag{6.5}$$

$$\|(\partial_x^\alpha X)(s,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\lambda^{\delta(|\alpha|-1)}|s-s_0|^{1/2}, \quad |\alpha| \geq 1 \tag{6.6}$$

In the case that V arises from a solution to (2.1), we can improve upon the regularity of X by using the integrability of V along the vector field established in the previous section:

Proposition 6.4. *Consider a smooth solution (η, ψ) to the system (2.1). Let $s > \frac{d}{2} + \frac{1}{2}$ and $r > 1$. There exists $s_0 \in I$ such that for $\frac{1}{2} \leq \alpha < 1$,*

$$\|\partial_x X\|_{L^p(I;C^\alpha_*(\mathbb{R}^d))} \leq \lambda^{\delta(\alpha - \frac{1}{2})}\mathcal{F}(\|\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}, \|(\psi, V, B)\|_{L^\infty(I;H^s)}, \|V\|_{L^2(I;W^{1,\infty})}) \cdot (1 + \|\eta\|_{L^2(I;W^{r+\frac{1}{2},\infty})})(1 + \|\eta\|_{L^p(I;W^{r+\frac{1}{2},\infty})} + \|(V, B)\|_{L^2(I;W^{r,\infty})}). \tag{6.7}$$

Proof. In the following let $\partial_i = \partial_{x_i}$, and sum over repeated indices. Also denote $|D|^\alpha S_\mu = |D|^\alpha_\mu$ for brevity. Differentiating the system for X, we have

$$\frac{d}{ds}(|D|^\alpha_\mu \partial_j X)(s) = |D|^\alpha_\mu((\partial_k V_\delta)(s, X(s))\partial_j X^k). \tag{6.8}$$

We decompose the right hand side into paraproducts:

$$|D|^\alpha_\mu(T_{(\partial_k V_\delta)(s,X(s))}\partial_j X^k + T_{\partial_j X^k}((\partial_k V_\delta)(s, X(s))) + R((\partial_k V_\delta)(s, X(s)), \partial_j X^k)) =: I + II + III.$$

We estimate I using (E.10):

$$\||D|^\alpha_\mu T_{(\partial_k V_\delta)(s,X(s))}\partial_j X^k\|_{L^\infty_x} \lesssim \|T_{(\partial_k V_\delta)(s,X(s))}\partial_j X^k\|_{C^\alpha_{*,x}} \lesssim \|V\|_{W^{1,\infty}}\|\partial_j X\|_{C^\alpha_{*,x}}.$$

III satisfies the same estimate but using (E.6) in place of (E.10).

To study II, use Proposition 6.1 and the fact that X is the flow of V_δ to write

$$\begin{aligned}(\partial_k V_\delta)(s, X(s)) &= ((\partial_t + V_\delta \cdot \nabla)T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s)) + g_k(s, X(s)) \\ &= \frac{d}{ds}((T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s))) + g_k(s, X(s))\end{aligned}$$

and hence

$$\begin{aligned}II = \frac{d}{ds}(|D|^\alpha_\mu T_{\partial_j X^k}((T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s)))) &- |D|^\alpha_\mu T_{\frac{d}{ds}\partial_j X^k}((T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s))) \\ &+ |D|^\alpha_\mu T_{\partial_j X^k}(g_k(s, X(s))).\end{aligned}$$

The first term of II will be moved to the left hand side of (6.8). The second term of II may be estimated using the system for X, (E.10), and (6.5):

$$\begin{aligned}&\||D|^\alpha_\mu T_{\frac{d}{ds}\partial_j X^k}((T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s)))\|_{L^\infty_x} \\ &\quad\lesssim \|T_{\partial_\ell V^k_\delta(s,X(s))\partial_j X^\ell}((T_{1/q}\partial_k \nabla \eta_\delta)(s, X(s)))\|_{C^\alpha_{*,x}} \\ &\quad\leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|V\|_{W^{1,\infty}}\|(T_{1/q}\partial_x \nabla \eta_\delta)(s, X(s))\|_{C^\alpha_{*,x}}.\end{aligned}$$

In turn, by the Lipschitz regularity of X from (6.5), Proposition E.11, (E.1), and (6.3),

$$\begin{aligned}\|(T_{1/q}\partial_x \nabla \eta_\delta)(s, X(s))\|_{C^\alpha_{*,x}} &\leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|(T_{1/q}\partial_x \nabla \eta_\delta)(s, x)\|_{C^\alpha_{*,x}} \\ &\leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})M^0_0(q^{-1}\xi)\|\nabla \eta_\delta\|_{W^{\alpha,\infty}_x} \\ &\leq \lambda^{\delta(\alpha - \frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|\eta\|_{H^{s+\frac{1}{2}}}\|\eta\|_{W^{r+\frac{1}{2},\infty}_x}.\end{aligned}$$

We estimate the third term of II similarly, using the Lipschitz regularity of X from (6.5), Proposition E.11, and Proposition 6.1 to see that $g \in W^{\frac{1}{2},\infty}$:

$$\begin{aligned}
\||D|^\alpha_\mu T_{\partial_j X^k}(g_k(s, X(s)))\|_{L^\infty_x} &\lesssim \|T_{\partial_j X^k}(g_k(s, X(s)))\|_{C^\alpha_{*,x}} \\
&\leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|g(s, X(s))\|_{C^\alpha_{*,x}} \\
&\leq \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|g\|_{W_x^{\frac{1}{2},\infty}}.
\end{aligned}$$

Collecting the above estimates for I, II and III, we can write (6.8) as

$$\frac{d}{ds}|D|^\alpha_\mu(\partial_j X - F)(s, x) = G(s, x) + H(s, x)$$

where

$$\begin{aligned}
F &= T_{\partial_j X^k}(T_{1/q}\partial_k\nabla\eta_\delta)(s, X(s)), \\
\|G\|_{L^\infty_x} &\lesssim \|V\|_{W^{1,\infty}_x}\|\partial_j X\|_{C^\alpha_{*,x}},
\end{aligned}$$

and

$$\|H\|_{L^\infty_x} \leq \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}, \|V\|_{L^2(I;W^{1,\infty})})(\|V\|_{W^{1,\infty}_x}\|\eta\|_{W_x^{r+\frac{1}{2},\infty}} + \|g\|_{W_x^{\frac{1}{2},\infty}}).$$

Integrating in s, we may further write (6.8) as

$$|D|^\alpha_\mu(\partial_j X - F)(s, x) = |D|^\alpha_\mu(\partial_j X - F)(s_0, x) + \int_{s_0}^s G(\sigma, x) + H(\sigma, x)\, d\sigma$$

so that

$$\||D|^\alpha_\mu(\partial_j X - F)(s)\|_{L^\infty_x} \lesssim \||D|^\alpha_\mu(\partial_j X - F)(s_0)\|_{L^\infty_x} + \int_{s_0}^s \|V\|_{W^{1,\infty}_x}\|\partial_j X\|_{C^\alpha_{*,x}} + \|H\|_{L^\infty_x}\, d\sigma.$$

Then taking the supremum over μ and using triangle inequality on the right hand side,

$$\begin{aligned}
\|(\partial_j X - F)(s)\|_{C^\alpha_{*,x}} \lesssim \|(\partial_j X - F)(s_0)\|_{C^\alpha_{*,x}} &+ \int_{s_0}^s \|V\|_{W^{1,\infty}_x}\|\partial_j X - F\|_{C^\alpha_{*,x}} \\
&+ \|H\|_{L^\infty_x} + \|V\|_{W^{1,\infty}_x}\|F\|_{C^\alpha_{*,x}}\, d\sigma.
\end{aligned}$$

Then by the Gronwall and Hölder inequalities,

$$\begin{aligned}
\|(\partial_j X - F)(s)\|_{C^\alpha_{*,x}} &\leq \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}) \\
&\quad\cdot (\|(\partial_j X - F)(s_0)\|_{C^\alpha_{*,x}} + \|H\|_{L^1(I;L^\infty)} + \|F\|_{L^2(I;C^\alpha_*)}).
\end{aligned}$$

Finally, integrating in time,

$$\begin{aligned}
\|(\partial_j X - F)(s)\|_{L^p(I;C^\alpha_*)} &\leq T^{\frac{1}{p}}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}) \\
&\quad\cdot (\|(\partial_j X - F)(s_0)\|_{C^\alpha_{*,x}} + \|H\|_{L^1(I;L^\infty)} + \|F\|_{L^2(I;C^\alpha_*)}).
\end{aligned} \tag{6.9}$$

It remains to estimate the terms of the right hand side, and $\|F\|_{L^p(I;C^\alpha_*)}$ on the left, by the right hand side of (6.7). The term with H is already suitably estimated above, using additionally Hölder in time and the estimate on g from Proposition 6.1. It remains to study F.

F may be estimated in the same way as the second term of II:

$$\|F\|_{C^\alpha_{*,x}} = \|T_{\partial_j X^k}(T_{1/q}\partial_k\nabla\eta_\delta)(s, X(s))\|_{C^\alpha_{*,x}} \leq \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|\eta\|_{H^{s+\frac{1}{2}}}\|\eta\|_{W^{r+\frac{1}{2},\infty}}.$$

We conclude

$$\|F\|_{L^p(I;C^\alpha_*)} \leq \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}\|\eta\|_{L^p(I;W^{r+\frac{1}{2},\infty})}$$

as desired, and similarly with $L^2(I;C^\alpha_*)$.

It remains to estimate $\|(\partial_j X - F)(s_0)\|_{C^\alpha_*}$. Note there exists $s_0 \in I$ such that

$$\|\eta(s_0)\|^p_{W^{r+\frac{1}{2},\infty}} \leq T^{-1}\|\eta\|^p_{L^p(I;W^{r+\frac{1}{2},\infty})}.$$

Fixing such an s_0, by the previous estimate on $\|F\|_{C^\alpha_*}$,

$$\begin{aligned}
\|(\partial_j X - F)(s_0)\|_{C^\alpha_*} &\lesssim 1 + \|F(s_0)\|_{C^\alpha_*} \\
&\leq 1 + \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|\eta(s_0)\|_{H^{s+\frac{1}{2}}}\|\eta(s_0)\|_{W^{r+\frac{1}{2},\infty}} \\
&\leq 1 + T^{-\frac{1}{p}}\lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}\|\eta\|_{L^p(I;W^{r+\frac{1}{2},\infty})}.
\end{aligned}$$

Then $T^{\frac{1}{p}}\|(\partial_j X - F)(s_0)\|_{C^\alpha_*}$ from the right hand side of (6.9) is bounded by the right hand side of (6.7). □

It will be convenient to have estimates on the higher derivatives of X, to later see that our operator has a symbol in a classical symbol class:

Proposition 6.5. *Consider a smooth solution (η, ψ) to the system (2.1). Let $s > \frac{d}{2} + \frac{1}{2}$ and $r > 1$. There exists $s_0 \in I$ such that for $|\alpha| \geq 2$,*

$$\begin{aligned}
\|\partial^\alpha_x X\|_{L^p(I;L^\infty(\mathbb{R}^d))} \leq \lambda^{\delta(|\alpha|-\frac{3}{2})}\mathcal{F}(\|\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}, \|(\psi, V, B)\|_{L^\infty(I;H^s)}, \|V\|_{L^2(I;W^{1,\infty})}) \\
\cdot (1 + \|\eta\|_{L^2(I;W^{r+\frac{1}{2},\infty})})(1 + \|\eta\|_{L^p(I;W^{r+\frac{1}{2},\infty})} + \|(V,B)\|_{L^2(I;W^{r,\infty})}).
\end{aligned}$$

Proof. The proof is similar to that of Proposition 6.4, except easier as it does not require the paradifferential calculus, using only the chain and product rules directly. We differentiate both sides of the flow for V_δ,

$$\frac{d}{ds}\partial^\alpha_x X = \partial^\alpha_x(V_\delta(s, X(s))).$$

On the right hand side, the term for which all the derivatives fall on a single copy of X is treated as a Gronwall term. The term on which all the derivatives fall on V should be handled using Proposition 6.1 as in the proof of Proposition 6.4. The remaining terms are estimated either as with the analogous terms in the proof of Proposition 6.4, or inductively. □

7. Change of Variables

On a sufficiently small time interval $[s_0, s_0 + T']$, (6.5) implies that $\partial_x X$ is invertible. It is also straightforward to check that $x \mapsto X(t,x)$ is proper, so we can conclude by Hadamard's theorem that $x \mapsto X(t,x)$ is a smooth diffeomorphism for each $t \in [s_0, s_0 + T']$. The length of the time interval depends only on $\|V\|_{L^2(I;W^{1,\infty})}$, so we may partition $[0,T]$ into a number of time intervals of length T' on which $x \mapsto X(t,x)$ is a diffeomorphism. Without loss of generality, consider the first interval $[0,T']$.

We now return to the setting of Proposition 5.3 to perform the change of variables $x \mapsto X(t,x)$. Consider a smooth solution u_λ to (5.2). Writing

$$v_\lambda(t,y) := u_\lambda(t, X(t,y)),$$

we have by (6.4) that

$$\partial_t v_\lambda(t,y) = (\partial_t u_\lambda)(t, X(t,y)) + V_\delta(t, X(t,y)) \cdot (\nabla u_\lambda)(t, X(t,y))$$

and hence

$$\partial_t v_\lambda(t,y) + i(\gamma_\delta u_\lambda)(t, X(t,y)) = f_\lambda(t, X(t,y)).$$

Next, we write the dispersive term in terms of v_λ. Fix t in the following and omit it for brevity. We have

$$(\gamma_\delta u_\lambda)(X(y)) = \int e^{i(X(y)-x')\eta} \gamma_\delta(X(y),\eta) u_\lambda(x')\ dx' d\eta.$$

By the frequency support of u_λ, we may write

$$(\gamma_\delta u_\lambda)(X(y)) = \int e^{i(X(y)-x')\eta} \gamma_\delta(X(y),\eta)\psi_\lambda(\eta) u_\lambda(x')\ dx' d\eta,$$

though abusing notation by writing ψ in place of the appropriate smooth cutoff with broader support. To make the change of variables $x' = X(y')$, we use the following notation:

$$H(y,y') = \int_0^1 (\partial_x X)(hy + (1-h)y')\ dh, \quad M(y,y') = (H(y,y')^t)^{-1},$$

$$J(y,y') = |\det((\partial_x X)(y'))|\,|\det M(y,y')|.$$

Then

$$(\gamma_\delta u_\lambda)(X(y)) = \int e^{i(X(y)-X(y'))\eta} \gamma_\delta(X(y),\eta)\psi_\lambda(\eta) u_\lambda(X(y'))\, |\det((\partial_x X)(y'))|\ dy' d\eta.$$

Then make a second change of variables $\eta = M(y,y')\xi$, noting the identity $X(y) - X(y') = H(y,y')(y-y')$:

$$(\gamma_\delta u_\lambda)(t, X(y)) = \int e^{i(y-y')\xi} \gamma_\delta(X(y), M(y,y')\xi)\psi_\lambda(M(y,y')\xi) v_\lambda(y') J(y,y')\ dy' d\xi.$$

We thus have

$$\partial_t v_\lambda(t,y) + i(p(t,y,y',D)v_\lambda)(t,y) = f_\lambda(t, X(t,y)) \tag{7.1}$$

where

$$p(t,y,y',\xi) = \gamma_\delta(X(y), M(y,y')\xi)\psi_\lambda(M(y,y')\xi) J(y,y').$$

7.1. **Symbol regularity.** It remains to study the regularity and curvature properties of p needed for Strichartz estimates.

Proposition 7.1. *There exists $T' > 0$ sufficiently small depending on $\|V\|_{L^2(I;W^{1,\infty})}$ such that for $I = [0,T']$ and $\frac{1}{2} \le \alpha < 1$,*

$$\|\partial_\xi^\beta p(t,y,y',\xi)\|_{L^p(I; L^\infty_\xi C^\alpha_{*,y,y'})} \le \lambda^{\frac{1}{2}-|\beta|+\delta(\alpha-\frac{1}{2})}\mathcal{F}(T).$$

Proof. Choose T' sufficiently small so that $x \mapsto X(t,x)$ is a diffeomorphism for each $t \in I$. Let $m_{ij}(y,y')$ denote the entries of the matrix $M(y,y')$. Then $\partial_\xi^\beta p$ is a sum of products of the form, with $\beta_1 + \beta_2 = \beta$,

$$(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi) P_\beta(m_{ij}(y,y')) J(y,y')$$

where P_β is a polynomial of degree $|\beta|$.

By (6.5), for $T' > 0$ sufficiently small, $\|M\|, \|M^{-1}\| \le 1/2$ so that $(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi)$ and hence p have support $\{|\xi| \in [c\lambda, \lambda/c]\}$. Thus, in the following, $L^\infty_\xi = L^\infty_\xi(\{|\xi| \in [c\lambda, \lambda/c]\})$ unless otherwise specified.

By using the product estimate (E.12) and recalling the estimates on γ from Corollary D.10, it suffices to show the following estimates:

$$\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)\|_{L^p(I;L^\infty_\xi C^\alpha_*)} \le \lambda^{\frac{1}{2}-|\beta_1|+\delta(\alpha-\frac{1}{2})}\mathcal{F}(T) \cdot \Big(\sum_{|b|\le|\beta|+1} \sup_{|\xi|=1} \|\partial_\xi^b\gamma\|_{L^\infty_{t,x}} + \|\sup_{|\xi|=1}\|\partial_\xi^b\gamma\|_{W_x^{\frac{1}{2},\infty}}\|_{L^p_t(I)}\Big) \tag{7.2}$$

$$\|(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi)\|_{L^p(I;L^\infty_\xi C^\alpha_*)} \le \lambda^{-|\beta_2|+\delta(\alpha-\frac{1}{2})}\mathcal{F}(T) \tag{7.3}$$

$$\|P_\beta(m_{ij})\|_{L^p(I;C^\alpha_*)} + \|J(y,y')\|_{L^p(I;C^\alpha_*)} \le \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(T) \tag{7.4}$$

$$\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)\|_{L^\infty_{t,y,y',\xi}} \lesssim \lambda^{\frac{1}{2}-|\beta_1|} \sup_{|\xi|=1}\|\partial_\xi^{\beta_1}\gamma\|_{L^\infty_{t,x}} \tag{7.5}$$

$$\|(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi)\|_{L^\infty_{t,y,y',\xi}} \lesssim \lambda^{-|\beta_2|} \tag{7.6}$$

$$\|P_\beta(m_{ij})\|_{L^\infty_{t,y,y'}} + \|J(y,y')\|_{L^\infty_{t,y,y'}} \le \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}). \tag{7.7}$$

First we show (7.5). Using $T' > 0$ sufficiently small so that $\|M\|, \|M^{-1}\| \le 1/2$, by homogeneity we have

$$\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)\|_{L^\infty_{t,y,y',\xi}} \lesssim \lambda^{\frac{1}{2}-|\beta_1|} \sup_{|\xi|=1}\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\xi)\|_{L^\infty_{t,y}}.$$

Then (7.5) is clear. (7.6) is similarly proven.

To see (7.7), note that by (6.5),

$$\|m_{ij}(y,y')\|_{L^\infty_{t,y,y'}} \le \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}). \tag{7.8}$$

Then (7.7) holds, as $P_\beta(m_{ij})$ and J are polynomials in m_{ij}.

Next we prove (7.4). From (6.7) and the definition of M,

$$\|m_{ij}(y,y')\|_{L^p(I;C^\alpha_{*,y,y'})} \le \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(T). \tag{7.9}$$

Then (7.4) holds, using the product estimate (E.12), the fact that $P_\beta(m_{ij})$ and J are polynomials in m_{ij}, and (7.8).

To prove (7.3), write $(F)_\lambda(\cdot) = F((\cdot)/\lambda)$ so that we have

$$(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi) = \lambda^{-|\beta_2|}(\partial_\xi^{\beta_2}\psi)_\lambda(M(y,y')\xi).$$

Then view $(\partial_\xi^{\beta_2}\psi)_\lambda((\cdot)\xi)$ as a smooth function vanishing near 0 to apply the Moser-type estimate (E.15)

$$\|(\partial_\xi^{\beta_2}\psi)_\lambda(M(y,y')\xi)\|_{C^\alpha_{*,y,y'}} \le \mathcal{F}(\|m_{ij}(y,y')\|_{L^\infty_{y,y'}})\|m_{ij}(y,y')\|_{C^\alpha_{*,y,y'}}.$$

Then the desired estimate is obtained by taking $L^p_tL^\infty_\xi$ and using (7.8) and (7.9).

It remains to show (7.2). We are in a position to apply Proposition E.12, by writing $x = (y,y') \in \mathbb{R}^{2d}$, $a(x,\zeta) = (\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\zeta)$, and $f(x) = f(y,y') = M(y,y')\xi$. Since the range of f may be assumed to be $\{|\zeta| \approx \lambda\}$ for T' sufficiently small, we may smoothly cut off $a(x,\zeta)$ to have support $\{|\zeta| \approx \lambda\}$. We obtain

$$\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)\|_{C^\alpha_{*,y,y'}} \lesssim \sup_{|\zeta|\approx\lambda} \|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\zeta)\|_{C^\alpha_{*,y}} + \sup_{|\zeta|\approx\lambda} \|(\nabla_\xi\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\zeta)\|_{L^\infty_y}\|M(y,y')\xi\|_{C^\alpha_{*,y,y'}}.$$

The second term is estimated as before by taking $L^p_t L^\infty_\xi$ and using homogeneity of γ and (7.9) to obtain a bound by

$$\|(\nabla_\xi\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\xi)\|_{L^\infty_{t,y,\xi}}\|M(y,y')\|_{L^p(I;C^\alpha_{*,y,y'})} \leq \lambda^{\frac{1}{2}-(|\beta_1|+1)+\delta(\alpha-\frac{1}{2})}\mathcal{F}(T)\sup_{|\xi|=1}\|\nabla_\xi\partial_\xi^{\beta_1}\gamma_\delta\|_{L^\infty_{t,x}}$$

which is better than desired.

For the first term, we apply Proposition E.11 and the Lipschitz regularity of X from (6.5):

$$\|(\partial_\xi^{\beta_1}\gamma_\delta)(X(y),\zeta)\|_{C^\alpha_{*,y}} \leq \|(\partial_\xi^{\beta_1}\gamma_\delta)(y,\zeta)\|_{C^\alpha_{*,y}}\|\partial_x X\|^\alpha_{L^\infty} \leq \lambda^{\delta(\alpha-\frac{1}{2})}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})})\|(\partial_\xi^{\beta_1}\gamma_\delta)(y,\zeta)\|_{W_y^{\frac{1}{2},\infty}}.$$

Then taking $\sup_{|\zeta|\approx\lambda}$, using homogeneity of γ, and taking L^p_t yields the desired estimate. $\square$

It will be convenient to have the following symbol property for p, for later application of the mapping properties of $p(t,y,y',D)$.

Proposition 7.2. *Fix $k \in \mathbb{N}$. There exists $T' > 0$ sufficiently small, depending on*

$$\|V\|_{L^2(I;W^{1,\infty})},$$

such that for $I = [0,T']$, $|\alpha| + |\alpha'| + |\beta| \leq k$, and $|\alpha| + |\alpha'| \geq 1$,

$$\|\partial_y^\alpha\partial_{y'}^{\alpha'}\partial_\xi^\beta p(t,y,y',\xi)\|_{L^p(I;L^\infty_{y,y',\xi})} \leq \lambda^{\frac{1}{2}-|\beta|+\delta(|\alpha|+|\alpha'|-\frac{1}{2})}\mathcal{F}(T).$$

Proof. We consider the case where $|\alpha| \geq 1$ and $\alpha' = 0$. The general case is similar.

Choose T' sufficiently small so that $x \mapsto X(t,x)$ is a diffeomorphism for each $t \in I$. Let $m_{ij}(y,y')$ denote the entries of the matrix $M(y,y')$. Then $\partial_\xi^\beta p$ is a sum of products of the form, with $\beta_1 + \beta_2 = \beta$,

$$(\partial_\xi^{\beta_1}\gamma_\delta)(X(y), M(y,y')\xi)(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi)P_\beta(m_{ij}(y,y'))J(y,y')$$

where P_β is a polynomial of degree $|\beta|$.

By (6.5), for $T' > 0$ sufficiently small, $\|M\|, \|M^{-1}\| \leq 1/2$ so that $(\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi)$ and hence p have support $\{|\xi| \in [c\lambda, \lambda/c]\}$. Thus, in the following, $L^\infty_\xi = L^\infty_\xi(\{|\xi| \in [c\lambda,\lambda/c]\})$ unless otherwise specified.

By using the product estimate (E.12) and recalling the estimates on γ from Corollary D.10, it suffices to show the following estimates, for $|\nu|\geq 1$:

$$
\begin{aligned}
\|\partial_y^\nu((\partial_\xi^{\beta_1}\gamma_\delta)(X(y),M(y,y')\xi))\|_{L^p(I;L^\infty_{y,y',\xi})} &\leq \lambda^{\frac12-|\beta_1|+\delta(|\nu|-\frac12)}\mathcal{F}(T)\\
&\quad\cdot\Big(\sum_{|b|\leq k}\sup_{|\eta|=1}\|\partial_\xi^b\gamma\|_{L^\infty_{t,x}}+\|\sup_{|\eta|=1}\|\partial_\xi^b\gamma\|_{W_x^{\frac12,\infty}}\|_{L_t^p(I)}\Big)\\
\|\partial_y^\nu((\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi))\|_{L^p(I;L^\infty_{y,y',\xi})} &\leq \lambda^{-|\beta_2|+\delta(|\nu|-\frac12)}\mathcal{F}(T)\\
\|\partial_y^\nu P_\beta(m_{ij})\|_{L^p(I;L^\infty)}+\|\partial_y^\nu J(y,y')\|_{L^p(I;L^\infty_{y,y',\xi})} &\leq \lambda^{\delta(|\nu|-\frac12)}\mathcal{F}(T)\\
\|\partial_y^{\nu'}((\partial_\xi^{\beta_1}\gamma_\delta)(X(y),M(y,y')\xi))\|_{L^\infty_{t,y,y',\xi}} &\lesssim \lambda^{\frac12-|\beta_1|+\delta|\nu'|}\sup_{|\xi|=1}\|\partial_\xi^{\beta_1}\gamma\|_{L^\infty_{t,x}}.\\
\|\partial_y^{\nu'}((\partial_\xi^{\beta_2}\psi_\lambda)(M(y,y')\xi))\|_{L^\infty_{t,y,y',\xi}} &\lesssim \lambda^{-|\beta_1|+\delta|\nu'|}\\
\|\partial_y^{\nu'}P_\beta(m_{ij})\|_{L^\infty_{t,y,y'}}+\|\partial_y^{\nu'}J(y,y')\|_{L^\infty_{t,y,y'}} &\leq \lambda^{\delta|\nu'|}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}).
\end{aligned}
$$

Other than the first, these estimates are straightforward, similar to the proof of Proposition 7.1. The difference is that here we apply Proposition 6.5 in place of Proposition 6.4.

We will only focus on the first estimate. For brevity, write $\beta=\beta_1$. By the chain rule, $\partial_y^\nu(\partial_\xi^\beta\gamma_\delta)(X(y),M(y,y')\xi)$ consists of terms of the form

$$K:=(\partial_y^a\partial_\xi^{\beta+b}\gamma_\delta)(X(y),M(y,y')\xi)\prod_{j=1}^r\left(\partial_y^{\ell_j}X(y)\right)^{p_j}\left(\partial_y^{\ell_j}M(y,y')\xi\right)^{q_j}$$

where

$$\sum p_j=a,\ \sum q_j=b,\ |\ell_j|\geq 1,\ \sum(|p_j|+|q_j|)\ell_j=\nu.$$

We study the frequency contribution of each member of the product K. For the first term, we can remove the $M(y,y')$ as before by choosing T' sufficiently small, and using homogeneity:

$$
\begin{aligned}
\|(\partial_y^a\partial_\xi^{\beta+b}\gamma_\delta)(X(y),M(y,y')\xi)\|_{L^\infty_{y,y',\xi}} &\lesssim \lambda^{\frac12-(|\beta|+b)}\sup_{|\xi|=1}\|(\partial_y^a\partial_\xi^{\beta+b}\gamma_\delta)(X(y),\xi)\|_{L^\infty_{t,y}}\\
&\leq \lambda^{\frac12-|\beta|}\sum_{|b|\leq k}\sup_{|\xi|=1}\|(\partial_y^a\partial_\xi^b\gamma_\delta)(x,\xi)\|_{L^\infty_{t,x}}.
\end{aligned}
$$

Assuming for now that $|a|\geq 1$, then by the frequency localization of γ_δ,

$$\|(\partial_y^a\partial_\xi^{\beta+b}\gamma_\delta)(X(y),M(y,y')\xi)\|_{L^p(I;L^\infty_{y,y',\xi})}\lesssim \lambda^{\frac12-|\beta|+\delta(|a|-\frac12)}\sum_{|b|\leq k}\|\sup_{|\xi|=1}\|\partial_\xi^b\gamma_\delta(x,\xi)\|_{W^{\frac12,\infty}}\|_{L^p(I)}.$$

For the second term, we apply (6.5) and (6.6):

$$\prod_{j=1}^r\|\partial_y^{\ell_j}X(y)\|_{L^\infty_{t,y}}^{p_j}\leq \lambda^{\delta\sum|p_j|(|\ell_j|-1)}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}).\tag{7.10}$$

For the third term, from (6.5), we have

$$\|\partial_y^{\nu'}m_{ij}(y,y')\|_{L^\infty_{t,y,y'}}\leq \lambda^{\delta|\nu'|}\mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}).$$

Applying this,

$$\prod_{j=1}^{r} \|\partial_y^{\ell_j} M(y,y')\xi\|_{L^\infty_{t,y,y',\xi}}^{q_j} \leq \lambda^{\delta \sum |q_j||\ell_j|} \mathcal{F}(\|V\|_{L^2(I;W^{1,\infty})}). \tag{7.11}$$

Then putting the three terms together, the exponent on λ is

$$\delta\left(|a| - \frac{1}{2}\right) + \delta \sum_{j=1}^{r} |p_j|(|\ell_j| - 1) + \delta \sum_{j=1}^{r} |q_j||\ell_j| = \delta\left(|a| - \frac{1}{2}\right) + \delta(|\nu| - |a|) = \delta\left(|\nu| - \frac{1}{2}\right)$$

as desired.

It remains to consider the case $|a| = 0$, in which case we have $|b| \geq 1$, so that without loss of generality, $|q_1| \geq 1$. We apply (7.5) and (7.10) on the first and second terms in K. For the third term, we apply (7.11) except on a single copy of $\partial_y^{\ell_1} M(y,y')\xi$ for which we apply (6.7):

$$\|\partial_y^{\ell_1} M(y,y')\xi\|_{L^p(I;L^\infty_{y,y',\xi})} \leq \lambda^{\delta(|\ell_1| - \frac{1}{2})} \lambda \mathcal{F}(T).$$

Then we would have the desired exponent on λ, once we account for the extra λ term arising from the fact that $\xi \approx \lambda$. Note from the third term of K that this extra factor in fact appears $\sum q_j = b$ times, which is cancelled by the $4\lambda^{-b}$ gain from the first term of K. □

7.2. **Time regularity.** In this section we study the regularity of the symbol p in time.

Proposition 7.3. *Fix $I = [0, T']$ as in Proposition 7.1. Then*

$$\|\partial_t \partial_\xi^2 p(t,y,y',\xi)\|_{L^p(I;L^\infty_{y,y',\xi})} \leq \lambda^{-3/2} \mathcal{F}(T)$$

where $L^\infty_\xi = L^\infty_\xi(\{|\xi| \in [c\lambda, \lambda/c]\})$.

Proof. Recall

$$p(t,y,y',\xi) = \gamma_\delta(X(t,y), M(y,y')\xi)\psi_\lambda(M(y,y')\xi)J(y,y').$$

By (6.5), for $T' > 0$ sufficiently small, $\|M\|, \|M^{-1}\| \leq 1/2$. Thus on $\{|\xi| \in [c\lambda, \lambda/c]\}$ with the appropriate constant, $\psi_\lambda(M(y,y')\xi) \equiv 1$. Restricting our attention to this domain,

$$p(t,y,y',\xi) = \gamma_\delta(X(y), M(y,y')\xi)J(y,y').$$

Then, by homogeneity, it suffices to show

$$\|\partial_t \partial_\xi^2 p(t,y,y',\xi)\|_{L^p(I;L^\infty_{y,y',\xi})} \leq \mathcal{F}(T)$$

on $\{|\xi| = 1\}$.

Let $m_{ij}(y,y')$ be the entries of the matrix $M(y,y')$. Then $\partial_\xi^2 p$ is a product of three terms,

$$(\partial_\xi^2 \gamma_\delta)(X(y), M(y,y')\xi)P_2(m_{ij}(y,y'))J(y,y')$$

where P_2 is a polynomial of degree 2. By the product rule, (6.5), and the L^∞ bound of Corollary D.10, it suffices to study the time derivative on each of $\gamma_{\xi\xi\delta}$, J, and $P_2(m_{ij})$ individually. Throughout the proof, since $\|M\|, \|M^{-1}\| \leq 1/2$, we will use the fact that we may scale to $|M(y,y')\xi| = 1$ with an acceptable loss.

Recall

$$J(y,y') = |\det(\partial_x X)(y')||\det M(y,y')|$$

is a polynomial in $\partial_x X$ and the matrix coefficients m_{ij} of M. Further, the matrix coefficients m_{ij} are smooth functions of $\partial_x X$. Thus using again (6.5), to study the time derivative of J it suffices to study the time derivative of $\partial_x X$, . We have by the flow of V_δ that

$$\frac{d}{dt}\partial_i X = (\partial_k V_\delta)(t, X(t))\partial_i X^k$$

and thus

$$\|\frac{d}{dt}\partial_i X\|_{L^p(I;L^\infty)} \lesssim \|V\|_{L^p(I;W^{r,\infty})}\|\partial_i X^k\|_{L^\infty(I;L^\infty)} \leq \mathcal{F}(\|V\|_{L^p(I;W^{r,\infty})}).$$

The analysis of $P_2(m_{ij})$ is similar.

Next we turn to $\gamma_{\xi\xi\delta}$. Write

$$\frac{d}{dt}\gamma_{\xi\xi\delta} = \partial_t\gamma_{\xi\xi\delta} + \partial_t X \cdot \nabla_x \gamma_{\xi\xi\delta} + \partial_t M \cdot \nabla_\xi \gamma_{\xi\xi\delta}.$$

Similar to before, we estimate $\partial_t M \cdot \nabla_\xi \gamma_{\xi\xi\delta}$ by applying Corollary D.10 and noting that $\partial_t M$ satisfies the same estimates as $\partial_t \partial_x X$, discussed above.

For the first two terms, note by the flow of V_δ that

$$\partial_t\gamma_{\xi\xi\delta} + \partial_t X \cdot \nabla_x \gamma_{\xi\xi\delta} = (\partial_t + V_\delta \cdot \nabla)\gamma_{\xi\xi\delta}.$$

We need to remove the frequency localization to exploit the material derivative, $L = \partial_t + V \cdot \nabla$. First, we can replace V_δ with V via the following estimate:

$$\|(S_{>\lambda^\delta} V) \cdot \nabla \gamma_{\xi\xi\delta}\|_{L^\infty} \lesssim \lambda^\delta \|S_{>\lambda^\delta} V\|_{L^\infty}\|\gamma_{\xi\xi\delta}\|_{L^\infty} \lesssim \|V\|_{W^{1,\infty}}\|\gamma_{\xi\xi}\|_{L^\infty}$$

and thus

$$\|(S_{>\lambda^\delta} V) \cdot \nabla \gamma_{\xi\xi\delta}\|_{L^p(I;L^\infty)} \lesssim \|V\|_{L^p(I;W^{1,\infty})}\|\gamma_{\xi\xi}\|_{L^\infty(I;L^\infty)}$$

as desired (estimating $\gamma_{\xi\xi}$ via Corollary D.10 as above). Second, by the commutator estimate

$$\|[V, \nabla S_{\leq \lambda^\delta}]\gamma_{\xi\xi}\|_{L^\infty} \lesssim \|V\|_{W^{r,\infty}}\|\gamma_{\xi\xi}\|_{L^\infty}$$

it remains to bound

$$S_{\leq\lambda^\delta}(\partial_t \gamma_{\xi\xi} + \nabla \cdot (V\gamma_{\xi\xi})).$$

Third, by the product rule, we are reduced to studying

$$(\partial_t + V \cdot \nabla)\gamma_{\xi\xi}$$

by observing

$$\|(\nabla \cdot V)\gamma_{\xi\xi}\|_{L^\infty} \leq \|V\|_{W^{1,\infty}}\|\gamma_{\xi\xi}\|_{L^\infty}.$$

To estimate $(\partial_t + V \cdot \nabla)\gamma_{\xi\xi}$, it suffices to estimate each of

$$(\partial_t + V \cdot \nabla)a, \quad (\partial_t + V \cdot \nabla)\Lambda.$$

The former is estimated in [ABZ14b, Proposition C.1]:

$$\|(\partial_t + V \cdot \nabla)a\|_{W^{\epsilon,\infty}} \leq \mathcal{F}(\|(\eta,\psi)\|_{H^{s+\frac{1}{2}}}, \|(V,B)\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|(V,B)\|_{W^{r,\infty}}).$$

Then take the L^p integral in time.

For the latter, it suffices to estimate

$$(\partial_t + V \cdot \nabla)\nabla\eta = G(\eta)V + \nabla\eta G(\eta)B + \Gamma_x + \nabla\eta\Gamma_y.$$

We may estimate the terms arising from the bottom using Sobolev embedding and [ABZ14a, Proposition 4.3] (which uses the notation $\gamma := \Gamma_x + \nabla\eta\Gamma_y$):

$$\|\Gamma_x + \nabla\eta\Gamma_y\|_{L^\infty} \lesssim \|\Gamma_x + \nabla\eta\Gamma_y\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}}).$$

Finally, it remains to estimate $G(\eta)V$ (as $\nabla\eta G(\eta)B$ is similar). We have

$$\|G(\eta)V\|_{L^\infty} \leq \|G(\eta) - T_\Lambda V\|_{L^\infty} + \|T_\Lambda V\|_{L^\infty}.$$

and

$$\|T_\Lambda V\|_{L^\infty} \lesssim M_0^1(\Lambda)\|V\|_{C_*^1} \lesssim \|\nabla\eta\|_{L^\infty}\|V\|_{W^{r,\infty}}.$$

For the paralinearization error, we may use [ABZ14b, Theorem 1.4] and Sobolev embedding, to obtain

$$\|G(\eta) - T_\Lambda V\|_{L^\infty} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|V\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|V\|_{W^{r,\infty}}).$$

Taking L^p in time yields the desired estimate.

□

7.3. **Curvature estimates.** We recall the following estimate on the Hessian of γ:

Proposition 7.4 ([ABZ14b, Proposition 2.11])**.** *There exists $c_0 > 0$ and $\lambda_0 > 0$ such that*

$$|\det \partial_\xi^2 \gamma_\delta(t, x, \xi)| \geq c_0$$

for all $\lambda \geq \lambda_0$ and $(t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \{|\xi| \approx 1\}$.

We can obtain the same estimates for p:

Corollary 7.5. *There exists $T' > 0$ (depending on $\|V\|_{L^2(I;W^{1,\infty})}$), $c_0 > 0$, and λ_0 such that*

$$|\det \partial_\xi^2 p(t, y, y', \xi)| \geq c_0 \lambda^{-\frac{3d}{2}}$$

for all $\lambda \geq \lambda_0$ and $(t, y, y', \xi) \in [0, T'] \times \mathbb{R}^{2d} \times \{|\xi| \in [c\lambda, \lambda/c]\}$.

Proof. Recall

$$p(t, y, y', \xi) = \gamma_\delta(X(y), M(y, y')\xi)\psi_\lambda(M(y, y')\xi)J(y, y').$$

By (6.5), for $T > 0$ sufficiently small, $\|M\|, \|M^{-1}\| \leq 1/2$. Thus on $\{|\xi| \in [c\lambda, \lambda/c]\}$ with the appropriate constant, $\psi_\lambda(M(y, y')\xi) \equiv 1$. Restricting our attention to this domain,

$$p(t, y, y', \xi) = \gamma_\delta(X(y), M(y, y')\xi)J(y, y').$$

Thus, we have

$$\partial_{\xi_i\xi_j}^2 p(t, y, y', \xi) = M(y, y')^T(\partial_{\xi_i\xi_j}^2 \gamma_\delta)(X(y), M(y, y')\xi)M(y, y')J(y, y').$$

By (6.5), $M(y, y') \approx Id$ and $J(y, y') \approx 1$ on $[0, T']$ for T' sufficiently small. Thus by Proposition 7.4 and homogeneity,

$$|\det \partial_\xi^2 p(t, y, y', \xi)| \gtrsim \inf_{x\in\mathbb{R}^d} |\det \partial_\xi^2 \gamma_\delta(t, x, \xi)| \gtrsim c_0 \lambda^{-\frac{3d}{2}}.$$

□

8. Strichartz Estimates for Order 1/2 Evolution Equations

8.1. **The parametrix construction.** In this section we consider a general evolution equation of the form

$$\begin{cases} (D_t + a^w(t,x,D))u = f, & \text{in } (0,1)\times\mathbb{R}^d \\ u(0) = u_0, & \text{on } \mathbb{R}^d \end{cases} \tag{8.1}$$

where $a(t,x,\xi)$ is a real symbol continuous in t and smooth with respect to x and ξ. In this setting a^w is self-adjoint and thus generates an isometric evolution $S(t,s)$ on $L^2(\mathbb{R}^d)$. We outline the construction of a phase space representation of the fundamental solution for (8.1), following [KT05], [Tat04], [MMT08].

The FBI transform [Tat04]

$$(Tf)(x,\xi) = 2^{-\frac{d}{2}}\pi^{-\frac{3d}{4}}\int e^{-\frac{1}{2}(x-y)^2}e^{i\xi(x-y)}f(y)\,dy$$

is an isometry from $L^2(\mathbb{R}^d)$ to phase space $L^2(\mathbb{R}^{2d})$ with an inversion formula

$$f(y) = (T^*Tf)(y) = 2^{-\frac{d}{2}}\pi^{-\frac{3d}{4}}\int e^{-\frac{1}{2}(x-y)^2}e^{-i\xi(x-y)}(Tf)(x,\xi)\,dxd\xi.$$

First we would like to describe the phase space localization properties of $S(t,s)$ relative to the Hamilton flow corresponding to (8.1),

$$\begin{cases} \dot{x} = a_\xi(t,x,\xi) \\ \dot{\xi} = -a_x(t,x,\xi). \end{cases} \tag{8.2}$$

More precisely, let

$$(x^t,\xi^t) = (x^t(x,\xi),\xi^t(x,\xi))$$

denote the solution to (10.1), with initial data (x,ξ) at time 0. Let $\chi(t,s)$ denote the family of canonical transformations on phase space $L^2(\mathbb{R}^{2d})$ corresponding to (10.1),

$$\chi(t,s)(x^s,\xi^s) = (x^t,\xi^t).$$

Then we would like an estimate on the kernel $\tilde{K}$ of the phase space operator $TS(t,s)T^*$, of the form

$$|\tilde{K}(t,x,\xi,s,y,\eta)| \lesssim (1+|(x,\xi)-\chi(t,s)(y,\eta)|)^{-N}.$$

Such an estimate has been established in [Tat04] for the class of symbols $a \in S^{0,(k)}_{0,0}$ satisfying

$$|\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)| \le c_{\alpha,\beta}, \qquad |\alpha|+|\beta| \ge k,$$

for $k=2$. This was generalized in [MMT08] to the class of symbols $a \in S^{(k)}L^1_\chi$ satisfying

$$\sup_{x,\xi}\int_0^1 |\partial_x^\alpha\partial_\xi^\beta a(t,\chi(t,0)(x,\xi))|\,dt \le c_{\alpha,\beta}, \qquad |\alpha|+|\beta|\ge k.$$

For our purposes it suffices to consider an intermediate class of symbols $a \in L^1S^{0,(k)}_{0,0}$ satisfying

$$\|\partial_x^\alpha\partial_\xi^\beta a\|_{L^1_t([0,1];L^\infty(\mathbb{R}^{2d}))} \le c_{\alpha,\beta}, \qquad |\alpha|+|\beta|\ge k.$$

Precisely, we have the following corollary of [MMT08] for this smaller class of symbols:

Theorem 8.1. *Let $a(t,x,\xi) \in L^1S^{0,(2)}_{0,0}$. Then*

(1) The Hamilton flow (10.1) is well-defined and bilipschitz.

(2) *The kernel $\tilde{K}(t,s)$ of the phase space operator $TS(t,s)T^*$ decays rapidly away from the graph of the Hamilton flow,*

$$|\tilde{K}(t,x,\xi,s,y,\eta)| \lesssim (1+|(x,\xi)-\chi(t,s)(y,\eta)|)^{-N}.$$

Then we have the following phase space representation for the exact solution to (8.1):

Theorem 8.2. *Let $a(t,x,\xi) \in L^1S^{0,(2)}_{0,0}$. Then the kernel $K(t,s)$ of the evolution operator $S(t,s)$ for $D_t + a^w$ can be represented in the form*

$$K(t,y,s,\tilde{y}) = \int e^{-\frac{1}{2}(\tilde{y}-x^s)^2} e^{-i\xi^s(\tilde{y}-x^s)} e^{i(\psi(t,x,\xi)-\psi(s,x,\xi))} e^{i\xi^t(y-x^t)} G(t,s,x,\xi,y)\, dxd\xi$$

where the function G satisfies

$$|(x^t-y)^\gamma \partial_x^\alpha \partial_\xi^\beta \partial_y^\nu G(t,s,x,\xi,y)| \lesssim c_{\gamma,\alpha,\beta,\nu}.$$

Proof. This is a consequence of Theorem 8.1 and [Tat04, Theorem 4], once we prove that the canonical transformation $\chi(t,s)$ is smooth with uniform bounds,

$$|\partial_x^\alpha \partial_\xi^\beta \chi(x,\xi)| \le c_{\alpha,\beta}, \qquad |\alpha|+|\beta|>0.$$

This can be proven for instance by using the argument in Section 2 of [MMT08] showing that χ is uniformly bilipschitz, along with an induction. □

8.2. **Dispersive estimates.** We combine the representation formula in Theorem 8.2 with a curvature condition to yield a dispersive estimate.

First we define a class of symbols analogous to the class λS^k_λ in [KT05]. Define the class of symbols $a(t,x,\xi) \in L^1S^{m,(k)}_{1,\delta}(\lambda)$ satisfying

$$\|\partial_x^\alpha \partial_\xi^\beta a\|_{L^1_t([0,1];L^\infty(\mathbb{R}^{2d}))} \le c_{\alpha,\beta}\lambda^{m-|\beta|+\delta(|\alpha|-k)}, \qquad |\alpha| \ge k.$$

Note this definition makes sense even for noninteger k.

We will work with symbols which also partially satisfy uniform bounds. Define the class of symbols $a \in S^m_1(\lambda)$ satisfying

$$|\partial_\xi^\beta a(t,x,\xi)| \le c_\beta \lambda^{m-|\beta|}.$$

Proposition 8.3. *Let $a \in L^1S^{\frac{1}{2},(\frac{2}{3})}_{1,\frac{3}{4}}(\lambda) \cap S^{\frac{1}{2}}_1(\lambda)$ such that for each $(t,x,\xi) \in [0,1]\times\mathbb{R}^d\times\{|\xi| \in [c\lambda,\lambda/c]\}$, $\partial_\xi^2 a$ satisfies*

$$|\det \partial_\xi^2 a(t,x,\xi)| \ge c\lambda^{-\frac{3d}{2}}.$$

Also assume that

$$\|\lambda^{3/2}\partial_t\partial_\xi^2 a\|_{L^1_t([0,1];L^\infty_{x,\xi})} \le c_1 \ll c$$

where $L^\infty_\xi = L^\infty_\xi(\{|\xi| \in [c\lambda,\lambda/c]\})$.

Let u_0 have frequency support $\{|\xi| \in [c\lambda,\lambda/c]\}$. Then there exists $0 < T \le 1$ such that for all $|t-s| < T$, we have

$$\|S(t,s)u_0\|_{L^\infty} \lesssim \lambda^{\frac{3d}{4}}|t-s|^{-\frac{d}{2}}\|u_0\|_{L^1}.$$

Proof. Without loss of generality let $s = 0$. Fix T small to be chosen later. We fix $t_0 \in [0, T]$ and prove the estimate when $t = t_0$. To do so, we reduce the problem to an estimate for $t = 1$ by rescaling. Write $u = S(t, s)u_0$ and set

$$v(t, x) = u(t_0 \cdot t, \lambda^{-3/4}\sqrt{t_0}x).$$

Then v solves

$$(D_t + \tilde{a}^w(t, x, D))v = 0, \qquad v(0) = u_0(0, \lambda^{-3/4}\sqrt{t_0}x) =: v_0$$

where

$$\tilde{a}(t, x, \xi) = t_0 a\left(t_0 \cdot t, \lambda^{-3/4}\sqrt{t_0}x, \lambda^{3/4}\frac{\xi}{\sqrt{t_0}}\right).$$

Then it suffices to show

$$\|v(1)\|_{L^\infty} \lesssim \|v_0\|_{L^1}.$$

We first show $\tilde{a} \in L^1 S^{0,(2)}_{0,0}$ in order to apply Theorem 8.2. Indeed,

$$\begin{aligned}\|\partial_\xi^\beta \tilde{a}\|_{L^1_t([0,1];L^\infty)} &\leq \|\partial_\xi^\beta \tilde{a}\|_{L^\infty} = \lambda^{\frac{3}{4}|\beta|}t_0^{-\frac{1}{2}|\beta|}t_0\|\partial_\xi^\beta a\|_{L^\infty} \\ &\leq \lambda^{\frac{3}{4}|\beta|}t_0^{1-\frac{1}{2}|\beta|}c_\beta\lambda^{\frac{1}{2}-|\beta|} = c_\beta(t_0^{-1}\lambda^{-\frac{1}{2}})^{\frac{1}{2}(|\beta|-2)}.\end{aligned} \tag{8.3}$$

Note that when $t_0^{-1}\lambda^{-\frac{1}{2}} \geq 1$, the dispersive estimate is trivial by Sobolev embedding, so we may assume $t_0^{-1}\lambda^{-\frac{1}{2}} \leq 1$. Thus when $|\beta| \geq 2$,

$$\|\partial_\xi^\beta \tilde{a}\|_{L^1_t([0,1];L^\infty)} \leq c_\beta.$$

On the other hand, for $|\alpha| \geq 1$, again using $t_0^{-1}\lambda^{-\frac{1}{2}} \leq 1$,

$$\begin{aligned}\|\partial_x^\alpha\partial_\xi^\beta \tilde{a}\|_{L^1_t([0,1];L^\infty)} &\leq \lambda^{-\frac{3}{4}|\alpha|}t_0^{\frac{1}{2}|\alpha|}\lambda^{\frac{3}{4}|\beta|}t_0^{-\frac{1}{2}|\beta|}t_0\|\partial_x^\alpha\partial_\xi^\beta a(t_0 \cdot t)\|_{L^1_t([0,1];L^\infty)} \\ &\leq c_{\alpha,\beta}\lambda^{-\frac{3}{4}|\alpha|}t_0^{\frac{1}{2}|\alpha|}\lambda^{\frac{3}{4}|\beta|}t_0^{-\frac{1}{2}|\beta|}\lambda^{\frac{1}{2}-|\beta|+\frac{3}{4}(|\alpha|-\frac{2}{3})} \\ &= c_{\alpha,\beta}t_0^{\frac{1}{2}|\alpha|}(t_0^{-1}\lambda^{-\frac{1}{2}})^{\frac{1}{2}|\beta|} \leq c_{\alpha,\beta}t_0^{\frac{1}{2}|\alpha|} \leq c_{\alpha,\beta}.\end{aligned} \tag{8.4}$$

Thus, we may use the representation formula in Theorem 8.2,

$$v(t, y) = \int G(t, x, \xi, y)e^{-\frac{1}{2}(\tilde{y}-x)^2+i\xi^t(y-x^t)-i\xi(y-x)+i\psi(t,x,\xi)}v_0(\tilde{y})\,dxd\xi d\tilde{y}.$$

By the frequency support of v_0 in $B = \{|\xi| \approx \lambda^{\frac{1}{4}}t_0^{\frac{1}{2}}\}$, the contribution of the complement of B to the integral is negligible, so it suffices to study

$$\int\int_B |G(t, x, \xi, y)|\,d\xi\, e^{-\frac{1}{2}(\tilde{y}-x)^2}|v_0(\tilde{y})|\,dxd\tilde{y} \lesssim \|v_0\|_{L^1}\sup_x \int_B |G(t, x, \xi, y)|\,d\xi.$$

It remains to show

$$\int_B |G(1, x, \xi, y)|\,d\xi \lesssim 1.$$

Given the bound for G in Theorem 8.2, this reduces to

$$\int_B (1 + |x^1 - y|)^{-N}\,d\xi \lesssim 1.$$

To show this, we study the dependence of $x^1 = x^1(x, \xi)$ on ξ. Write

$$X = \partial_\xi x^t, \quad \Xi = \partial_\xi \xi^t,$$

which by the Hamilton flow (10.1) for $\tilde{a}$ solve

$$\begin{cases} \dot{X} = \tilde{a}_{\xi x} X + \tilde{a}_{\xi\xi}\Xi, & X(0) = 0 \\ \dot{\Xi} = -\tilde{a}_{xx} X - \tilde{a}_{x\xi}\Xi, & \Xi(0) = I. \end{cases} \tag{8.5}$$

From (8.4), we have

$$\|\tilde{a}_{x\xi}\|_{L^1_t([0,1];L^\infty)} + \|\tilde{a}_{xx}\|_{L^1_t([0,1];L^\infty)} \lesssim \sqrt{t_0}.$$

Similarly, (8.3) implies $\|\partial_{\xi\xi}\tilde{a}\|_{L^\infty} \lesssim 1$. Further, since the Hamilton flow for $\tilde{a}$ is bilipschitz by Theorem 8.1, we have

$$\|X\|_{L^\infty} + \|\Xi\|_{L^\infty} \lesssim 1.$$

Thus, we have

$$\Xi(t) = \Xi(0) + \int_0^t \dot{\Xi}\, ds = I - \int_0^t \tilde{a}_{xx} X + \tilde{a}_{x\xi}\Xi\, ds = I + O(\sqrt{t_0})$$

and hence

$$\tilde{a}_{\xi\xi}\Xi = \tilde{a}_{\xi\xi} + O(\sqrt{t_0})$$

Then

$$X(t) = X(0) + \int_0^t \dot{X}\, ds = \int_0^t \tilde{a}_{\xi x} X + \tilde{a}_{\xi\xi}\Xi\, ds = \int_0^t \tilde{a}_{\xi\xi}\, ds + O(\sqrt{t_0}). \tag{8.6}$$

We would like to replace $\tilde{a}_{\xi\xi} = \tilde{a}_{\xi\xi}(s, x^s, \xi^s)$ in the integral by $\tilde{a}_{\xi\xi}(0, x, \xi)$. Combining (10.1), (8.4), and (8.3), we have

$$\|\tilde{a}_{\xi\xi x}\dot{x}\|_{L^1_t([0,1];L^\infty)} = \|\tilde{a}_{\xi\xi x}\tilde{a}_{\xi}\|_{L^1_t([0,1];L^\infty)} \le c_{\alpha,\beta} t_0^{\frac{1}{2}} (t_0^{-1}\lambda^{-\frac{1}{2}}) \cdot c_\beta (t_0^{-1}\lambda^{-\frac{1}{2}})^{-\frac{1}{2}} \le c_{\alpha,\beta} c_\beta \lambda^{-\frac{1}{4}}$$

$$\|\tilde{a}_{\xi\xi\xi}\dot{\xi}\|_{L^1_t([0,1];L^\infty)} = \|\tilde{a}_{\xi\xi\xi}\tilde{a}_{x}\|_{L^1_t([0,1];L^\infty)} \le c_\beta (t_0^{-1}\lambda^{-\frac{1}{2}})^{\frac{1}{2}} \cdot c_{\alpha,\beta} t_0^{\frac{1}{2}} \le c_{\alpha,\beta} c_\beta \lambda^{-\frac{1}{4}}.$$

In addition, recalling the assumption that $\lambda^{3/2}\partial_t\partial_\xi^2 a \in L^1_t([0,1];L^\infty)$, we have

$$\|\partial_t\partial_\xi^2\tilde{a}\|_{L^1_t([0,1];L^\infty)} = \lambda^{3/2}\|\partial_t\partial_\xi^2 a\|_{L^1_t([0,t_0];L^\infty)} \le c_1.$$

Thus,

$$\begin{aligned} \tilde{a}_{\xi\xi}(s, x^s, \xi^s) &= \tilde{a}_{\xi\xi}(0, x, \xi) + \int_0^s \frac{d}{dt}\tilde{a}_{\xi\xi}\, dr \\ &= \tilde{a}_{\xi\xi}(0, x, \xi) + \int_0^s \partial_t\partial_\xi^2\tilde{a} + \tilde{a}_{\xi\xi x}\dot{x} + \tilde{a}_{\xi\xi\xi}\dot{\xi}\, dr \\ &= \tilde{a}_{\xi\xi}(0, x, \xi) + O(c_1) + O(\lambda^{-\frac{1}{4}}). \end{aligned}$$

We conclude from (8.6) that

$$\partial_\xi x^1 = X(1) = \tilde{a}_{\xi\xi}(0, x, \xi) + O(c_1) + O(\lambda^{-\frac{1}{4}} + \sqrt{t_0}) = \tilde{a}_{\xi\xi}(0, x, \xi) + O(c_1) + O(\sqrt{t_0}).$$

For $t_0 \in [0, T]$, with T chosen sufficiently small,

$$|\det \partial_\xi x^1| = |\det \tilde{a}_{\xi\xi}| + O(c_1) + O(\sqrt{t_0}) \gtrsim 1$$

and hence

$$\int_B (1 + |x^1 - y|)^{-N}\, d\xi \lesssim \int (1 + |x^1 - y|)^{-N}\, dx^1 \lesssim 1$$

as desired.

□

8.3. **Strichartz estimate.** We establish a Strichartz estimate from the previous dispersive estimate. Say (p,q,ρ) is *admissible* if

$$\frac{2}{p}+\frac{d}{q}=\frac{d}{2}, \quad 2\le p,q\le\infty, \quad (p,q)\neq(2,\infty), \quad \rho=\frac{3}{2p}.$$

Theorem 8.4. *Consider a symbol $a(t,x,\xi)$ as in Proposition 8.3. Let u and f_1 have frequency support $\{|\xi|\in[c\lambda,\lambda/c]\}$ and solve*

$$(D_t+a^w)u=f_1+f_2, \qquad u(0)=u_0$$

on $t\in[0,1]=I$. Then for (p,q,ρ) admissible we have

$$\||D|^{-\rho}u\|_{L^p(I;L^q)}\lesssim\||D|^{\rho}f_1\|_{L^{p'}(I;L^{q'})}+\|f_2\|_{L^1(I;L^2)}+\|u_0\|_{L^2}.$$

Proof. It suffices to prove this in a sufficiently small time interval $[0,T]$, as we can iterate it to obtain it in the full interval $[0,1]$.

The proof follows the standard TT^* formalism. By Duhamel's formula, it suffices to show

$$S(t,s):L^2\to\lambda^{\rho}L^pL^q \tag{8.7}$$

$$1_{t>s}S(t,s):\lambda^{-\rho}L^{p'}L^{q'}\to\lambda^{\rho}L^pL^q. \tag{8.8}$$

In turn, by a TT^* argument for (8.7) and the Christ-Kiselev lemma for (8.8), it suffices to show

$$S(t,s):\lambda^{-\rho}L^{p'}L^{q'}\to\lambda^{\rho}L^pL^q.$$

To show this, we interpolate the trivial energy bound

$$\|S(t,s)\|_{L^2\to L^2}\le 1$$

with the dispersive estimate in Proposition 8.3

$$\|S(t,s)\|_{L^1\to L^\infty}\lesssim\lambda^{\frac{3d}{4}}|t-s|^{-\frac{d}{2}}$$

and then apply the Hardy-Littlewood-Sobolev inequality. □

8.4. **Symbol truncation and rescaling.** We will consider symbols

$$a(t,x,y,\xi)\in L^1_t([0,T];L^\infty_\xi\dot W^{\frac{2}{3},\infty}_{x,y})$$

and study the evolution equation (8.1) but with the Kohn-Nirenberg quantization. We can reduce Strichartz estimates in this setting to Theorem 8.4 by frequency truncating in the spatial variables.

In our applications, we will also need to rescale to time intervals of variable length.

Corollary 8.5. *Let $I=[0,T]$ with $T>0$. Consider a symbol $a(t,x,y,\xi)$ with support on $\{|\xi|\in[c\lambda,\lambda/c]\}$ and satisfying*

$$\|\partial_\xi^\beta a\|_{L^\infty_{t,x,y,\xi}}\lesssim\lambda^{\frac12-|\beta|}, \qquad T^{\frac13}\|\partial_\xi^\beta a\|_{L^1_t(I;L^\infty_\xi\dot W^{\frac23,\infty}_{x,y})}\lesssim\lambda^{\frac12-|\beta|}.$$

For each $(t,x,y,\xi)\in[0,1]\times\mathbb{R}^{2d}\times\{|\xi|\in[c\lambda,\lambda/c]\}$, assume $\partial_\xi^2 a$ satisfies

$$|\det\partial_\xi^2 a(t,x,y,\xi)|\ge c\lambda^{-\frac{3d}{2}}.$$

Also assume that

$$\|\lambda^{3/2}\partial_t\partial_\xi^2 a\|_{L^1_t(I;L^\infty_{x,y,\xi})}\le c_1\ll c$$

where $L^\infty_\xi=L^\infty_\xi(\{|\xi|\in[\lambda/2,2\lambda]\})$.

Let u have frequency support $\{|\xi| \in [c\lambda, \lambda/c]\}$ and solve

$$(D_t + a(t,x,y,D))u = f, \qquad u(0) = u_0. \tag{8.9}$$

Then for (p,q,ρ) admissible we have

$$\||D|^{-\rho}u\|_{L^p(I;L^q)} \lesssim \|f\|_{L^1(I;L^2)} + \|u_0\|_{L^\infty(I;L^2)}. \tag{8.10}$$

Proof. By applying the scaling

$$\tilde{a} = a(Tt, T^2x, T^2y, \xi), \quad \tilde{u} = u(Tt, T^2x), \quad \tilde{f} = Tf(Tt, T^2x)$$

and the scaling invariance of the symbol a and the desired estimate (8.10), we may assume $T = 1$.

Next, we frequency truncate the symbol a in the x and y variables. Write, with $\delta = \frac{3}{4}$,

$$a = S_{\leq \lambda^\delta} a + S_{>\lambda^\delta} a =: a_\delta + a_{>\delta}.$$

Then we may assume $a = a_\delta$, since viewing $a_{>\delta}(t,x,y,D)u$ as a component of f on the right hand side of (8.9),

$$\begin{aligned}
\|a_{>\delta}(t,x,y,D)u\|_{L^1(I;L^2)} &= \lambda^{\frac{1}{2}}\|\lambda^{-\frac{1}{2}}a_{>\delta}(t,x,y,D)u\|_{L^1(I;L^2)} \\
&\lesssim \lambda^{\frac{1}{2}}\|\lambda^{-\frac{1}{2}}a_{>\delta}\|_{L^1(I;L^\infty_{x,y,\xi})}\|u\|_{L^\infty(I;L^2)} \\
&\lesssim \|\lambda^{-\frac{1}{2}}a\|_{L^1(I;L^\infty_\xi \dot{W}^{\frac{2}{3},\infty})}\|u\|_{L^\infty(I;L^2)}
\end{aligned}$$

which by assumption is bounded by $\|u\|_{L^\infty(I;L^2)}$ on the right hand side of (8.10).

Next we pass to the Weyl quantization. Since $a = a_\delta$, we have

$$a \in L^1 S^{\frac{1}{2},(\frac{2}{3})}_{1,\delta,\delta}.$$

We can write (see [Tay91, Proposition 0.3.A])

$$a(t,x,y,D) = a(t,x,x,D) + r(t,x,D), \qquad r \in L^1 S^{\frac{1}{2}-1+\delta(1-\frac{1}{2}),(0)}_{1,\delta} = L^1 S^{-\frac{1}{8},(0)}_{1,\delta}$$

so that

$$\|r(t,x,D)u\|_{L^1(I;L^2)} \lesssim \|u\|_{L^\infty(I;L^2)}.$$

Similarly, writing $\tilde{a}(t,x,\xi) = a(t,x,x,\xi)$, we can write

$$\tilde{a}(t,x,D) = \tilde{a}^w(t,x,D) + \tilde{r}(t,x,D), \qquad \tilde{r} \in L^1 S^{-\frac{1}{8},(0)}_{1,\delta}$$

and hence

$$\|\tilde{r}(t,x,D)u\|_{L^1(I;L^2)} \lesssim \|u\|_{L^\infty(I;L^2)}.$$

Viewing $(r+\tilde{r})u$ as an inhomogeneous term bounded by $\|u\|_{L^\infty(I;L^2)}$, we may assume

$$(D_t + \tilde{a}^w)u = f. \tag{8.11}$$

Since

$$\tilde{a} \in L^1 S^{\frac{1}{2},(\frac{2}{3})}_{1,\frac{3}{4}}(\lambda)$$

and the remaining properties required of $\tilde{a}$ in Proposition 8.3 are straightforward to check, we may apply Theorem 8.4 with $f_2 = f$ and $f_1 = 0$ to yield (8.10).

□

9. Strichartz Estimates for Rough Symbols

As our symbol p in (7.1) is only in $L^1_t([0,T]; L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})$, we cannot directly apply Corollary 8.5. We will consider Strichartz estimates on time intervals of variable length, adapted to our lower regularity.

9.1. Strichartz estimates on variable time intervals.

First we establish a counterpart to Corollary 8.5 for lower regularity symbols.

Corollary 9.1. *Corollary 8.5 holds with a symbol $a(t,x,y,\xi)$ satisfying*

$$T^{\frac14}\lambda^{\frac18}\|\partial_\xi^\beta a\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})} \lesssim \lambda^{\frac12-|\beta|}$$

in place of

$$T^{\frac13}\|\partial_\xi^\beta a\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac23,\infty}_{x,y})} \lesssim \lambda^{\frac12-|\beta|}.$$

Proof. We frequency truncate the symbol a in the x and y variables. Set

$$\mu = \|a\|^2_{L^1_t(I;L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})}$$

and write

$$a = S_{\le\mu}a + S_{>\mu}a =: a_\mu + a_{>\mu}.$$

Then we may assume $a = a_\mu$, since viewing $a_{>\mu}(t,x,y,D)u$ as a component of f on the right hand side of (8.9),

$$\begin{aligned}\|a_{>\mu}(t,x,y,D)u\|_{L^1(I;L^2)} &= \lambda^{\frac12}\|\lambda^{-\frac12}a_{>\mu}(t,x,y,D)u\|_{L^1(I;L^2)}\\ &\lesssim \lambda^{\frac12}\|\lambda^{-\frac12}a_{>\mu}\|_{L^1(I;L^\infty_{x,y,\xi})}\|u\|_{L^\infty(I;L^2)}\\ &\lesssim \mu^{-\frac12}\lambda^{\frac12}\|\lambda^{-\frac12}a\|_{L^1(I;L^\infty_\xi \dot W^{\frac12,\infty})}\|u\|_{L^\infty(I;L^2)}\end{aligned}$$

which by the choice of μ is bounded by $\|u\|_{L^\infty(I;L^2)}$ on the right hand side of (8.10).

Then we have

$$T^{\frac13}\|\partial_\xi^\beta a_\mu\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac23,\infty}_{x,y})} \lesssim \mu^{\frac16}T^{\frac13}\|\partial_\xi^\beta a\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})}.$$

By choice of μ and assumption,

$$\mu^{\frac16} = \|a\|^{\frac13}_{L^1_t(I;L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})} \lesssim T^{-\frac1{12}}\lambda^{\frac18}.$$

Again by assumption,

$$\|\partial_\xi^\beta a\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac12,\infty}_{x,y})} \lesssim T^{-\frac14}\lambda^{\frac38-|\beta|}.$$

Collecting these estimates, we conclude

$$T^{\frac13}\|\partial_\xi^\beta a_\mu\|_{L^1_t(I;L^\infty_\xi \dot W^{\frac23,\infty}_{x,y})} \lesssim T^{-\frac1{12}}\lambda^{\frac18}T^{\frac13}T^{-\frac14}\lambda^{\frac38-|\beta|} = \lambda^{\frac12-|\beta|}$$

which permits us to apply Corollary 8.5. $\square$

9.2. **Time interval partition.** We will apply Corollary 9.1 to the elements of a partition of our time interval to obtain a Strichartz estimate with loss:

Proposition 9.2. *Let* $I = [0, T']$ *with* $0 < T' \leq 1$. *Consider a symbol* $a(t, x, y, \xi)$ *with support on* $\{|\xi| \in [c\lambda, \lambda/c]\}$ *and satisfying*

$$\|\partial_\xi^\beta a\|_{L^\infty_{t,x,y,\xi}} \lesssim \lambda^{\frac{1}{2}-|\beta|}, \qquad \|\partial_\xi^\beta a\|_{L^1_t(I; L^\infty_\xi \dot{W}^{\frac{1}{2},\infty}_{x,y})} \lesssim \lambda^{\frac{1}{2}-|\beta|}.$$

For each $(t, x, y, \xi) \in [0, T'] \times \mathbb{R}^{2d} \times \{|\xi| \in [c\lambda, \lambda/c]\}$, *assume* $\partial_\xi^2 a$ *satisfies*

$$|\det \partial_\xi^2 a(t, x, y, \xi)| \geq c\lambda^{-\frac{3d}{2}}.$$

Also assume that

$$\|\lambda^{3/2}\partial_t \partial_\xi^2 a\|_{L^1_t(I; L^\infty_{x,y,\xi})} \leq c_1 \ll c$$

where $L^\infty_\xi = L^\infty_\xi(\{|\xi| \in [c\lambda, \lambda/c]\})$.

Let u *have frequency support* $\{|\xi| \in [c\lambda, \lambda/c]\}$ *and solve*

$$(D_t + a(t, x, y, D))u = f, \qquad u(0) = u_0.$$

Then for (p, q, ρ) *admissible we have*

$$\||D|^{-\rho} u\|_{L^p(I; L^q)} \lesssim \lambda^{-\frac{1}{10p'}} \|f\|_{L^1(I; L^2)} + \lambda^{\frac{1}{10p}} \|u_0\|_{L^\infty(I; L^2)}. \tag{9.1}$$

Proof. Decompose $[0, T']$ into maximal subintervals

$$0 = t_0 < t_1 < \ldots < t_k = T'$$

satisfying both

$$\|f\|_{L^1([t_j, t_{j+1}]; L^2)} \leq \lambda^{-\frac{1}{10}} \|f\|_{L^1(I; L^2)} \tag{9.2}$$

and

$$(t_{j+1} - t_j)^{\frac{1}{4}} \lambda^{\frac{1}{8}} \lambda^{-\frac{1}{2}+|\beta|} \|\partial_\xi^\beta a\|_{L^1_t([t_j, t_{j+1}]; L^\infty_\xi \dot{W}^{\frac{1}{2},\infty}_{x,y})} \leq 1 \tag{9.3}$$

for each $0 \leq |\beta| \leq N$. We claim that the number k of intervals satisfies

$$k \approx \lambda^{\frac{1}{10}}.$$

The lower bound follows from (9.2). For the upper bound, observe that for each j, equality must hold either in (9.2) or (9.3) for some β. The number k_0 of intervals for which equality in (9.2) holds is at most $\lambda^{\frac{1}{10}}$ as desired. On each of the k_β intervals in which equality holds in (9.3) with β, we have, since $c^4 + c^{-1} \gtrsim 1$ for any c,

$$\lambda^{\frac{1}{10}}(t_{j+1} - t_j) + \lambda^{-\frac{1}{40}} \lambda^{\frac{1}{8}} \lambda^{-\frac{1}{2}+|\beta|} \|\partial_\xi^\beta a\|_{L^1_t([t_j, t_{j+1}]; L^\infty_\xi \dot{W}^{\frac{1}{2},\infty}_{x,y})} \gtrsim 1.$$

Then summing over all such intervals, we have

$$\lambda^{\frac{1}{10}} T' + \lambda^{\frac{1}{10}} \gtrsim \lambda^{\frac{1}{10}} \sum_j (t_{j+1} - t_j) + \lambda^{-\frac{1}{40}} \lambda^{\frac{1}{8}} \sum_j \lambda^{-\frac{1}{2}+|\beta|} \|\partial_\xi^\beta a\|_{L^1_t([t_j, t_{j+1}]; L^\infty_\xi \dot{W}^{\frac{1}{2},\infty}_{x,y})} \gtrsim k_\beta.$$

and thus

$$k = k_0 + \sum_\beta k_\beta \lesssim \lambda^{\frac{1}{10}}.$$

as desired.

We conclude from (9.3) that a restricted to $[t_j, t_{j+1}]$ satisfies the conditions of Corollary 9.1 with $T = t_{j+1} - t_j$. We obtain

$$\||D|^\rho u\|_{L^p([t_j,t_{j+1}];L^q)} \lesssim \|f\|_{L^1([t_j,t_{j+1}];L^2)} + \|u\|_{L^\infty([t_j,t_{j+1}];L^2)}.$$

Using (9.2) we have

$$\||D|^\rho u\|_{L^p([t_j,t_{j+1}];L^q)} \lesssim \lambda^{-\frac{1}{10}}\|f\|_{L^1(I;L^2)} + \|u\|_{L^\infty([t_j,t_{j+1}];L^2)}.$$

Raising to the power p and summing over j, we obtain

$$\||D|^\rho u\|_{L^p(I;L^q)} \lesssim \lambda^{\frac{1-p}{10p}}\|f\|_{L^1(I;L^2)} + \lambda^{\frac{1}{10p}}\|u\|_{L^\infty(I;L^2)}.$$

□

9.3. **Proof of Proposition 5.3.** In this section we combine the estimates on the symbol p from Section 7 and the Strichartz estimates from the previous section to prove Proposition 5.3. Recall that if we define $v_\lambda(t,y) = u_\lambda(t, X(t,y))$, then v_λ satisfies (7.1),

$$\partial_t v_\lambda(t,y) + i(p(t,y,y',D)v_\lambda)(t,y) = f_\lambda(t, X(t,y)).$$

Up to a perturbative inhomogeneous error, we will apply Proposition 9.2 with $u = v_\lambda$ and $a = p(t,y,y',\xi)$. In the following, implicit constants will depend on $\mathcal{F}(T)$.

For instance, we may partition $[0,T]$ into a number of time intervals of length T', depending on $\|V\|_{L^2(I;W^{1,\infty})}$ and hence $\mathcal{F}(T)$. Without loss of generality, we consider the first such interval $I = [0,T']$.

Step 1. We recall the properties of p required by Proposition 9.2. For the curvature lower bound on p, we recall Corollary 7.5.

We have $\lambda^{3/2}\partial_t\partial_\xi^2 p \in L_t^p(I; L^\infty_{y,y',\xi})$ by Proposition 7.3. Then on a sufficiently short time interval I, we have by Hölder in time,

$$\|\lambda^{3/2}\partial_t\partial_\xi^2 p\|_{L_t^1(I;L^\infty_{x,\xi})} \le c_1.$$

Proposition 7.1 implies

$$\|\partial_\xi^\beta p\|_{L_t^1(I;L_\xi^\infty \dot{W}^{\frac{1}{2},\infty}_{y,y'})} \lesssim \lambda^{\frac{1}{2}-|\beta|}.$$

It is also easy to see that $\lambda^{-\frac{1}{2}+|\beta|}\partial_\xi^\beta p \in L^\infty_{t,y,y',\xi}$ from the proof of Proposition 7.1, by using only $L^\infty_{t,y,y',\xi}$ in place of $L_t^p(I; L_\xi^\infty C_*^\alpha)$.

Step 2. Before we can apply Proposition 9.2, we need to frequency localize

$$v_\lambda(t,y) = u_\lambda(t, X(t,y)).$$

By the computations for the change of variables in Section 7, but with $\psi_\lambda(\xi)$, the symbol of S_λ, in place of γ_δ, we have

$$v_\lambda(t,y) = u_\lambda(t, X(y)) = (S_\lambda u_\lambda)(t, X(y)) = (\chi(t,y,y',D)v_\lambda)(t,y)$$

where

$$\chi(t,y,y',\xi) = \psi_\lambda(M(y,y')\xi)J(y,y').$$

Further, following the proof of Proposition 7.2 with χ in place of p, we have

$$\chi \in L^1 S^{0,(\frac{1}{2})}_{1,\delta,\delta}$$

and hence, recalling that $0 < \delta < 1$,

$$\chi(t, y, y', D) = \chi(t, y', y', \xi) + r(t, y', D), \qquad r \in L^1 S_{1,\delta}^{-1+\delta(1-\frac{1}{2}),(0)} \subseteq L^1 S_{1,\delta}^{-\frac{1}{2},(0)}. \tag{9.4}$$

Writing $\tilde{\chi}(t, y', \xi) = \chi(t, y', y', \xi)$, we conclude from (7.1) and (9.4) that

$$(\partial_t \tilde{\chi} v_\lambda)(t, y) + i(p\tilde{\chi} v_\lambda)(t, y) = f_\lambda(t, X(t, y)) - (\partial_t r v_\lambda)(t, y) - i(p r v_\lambda)(t, y). \tag{9.5}$$

Step 3. From the estimate of Proposition 9.2, we see that an estimate

$$\|\partial_t r v_\lambda - i p r v_\lambda\|_{L^1(I;L^2)} \lesssim \|f_\lambda(t, X(t, y))\|_{L^1(I;L^2)} + \|v_\lambda\|_{L^\infty(I;L^2)} \tag{9.6}$$

would suffice.

First we estimate prv_λ. Observe that

$$p \in L^\infty S_{1,\delta,\delta}^{\frac{1}{2},(0)},$$

by using the uniform bound (6.6) in place of (6.7) (and similarly choosing the uniform bound in Corollary D.10) in the proof of Proposition 7.2. Thus we have

$$\|prv_\lambda\|_{L^1(I;L^2)} \lesssim \|rv_\lambda\|_{L^1(I;H^{\frac{1}{2}})} \lesssim \|v_\lambda\|_{L^\infty(I;L^2)}$$

as desired.

Next, we estimate $\partial_t r v_\lambda$. Write

$$\partial_t r v_\lambda = \dot{r} v_\lambda + r\partial_t v_\lambda = \dot{r} v_\lambda - irpv_\lambda + rf_\lambda(t, X(t, y)).$$

rpv_λ is estimated in the same way as prv_λ. To estimate rf_λ, view $r = \chi - \tilde{\chi}$ and without loss of generality, estimate χf_λ. Similar to the analysis of p with uniform bounds, we have

$$\chi \in L^\infty S_{1,\delta,\delta}^{0,(0)},$$

from which we obtain

$$\|\chi(f_\lambda(t, X(t, y)))\|_{L^1(I;L^2)} \lesssim \|f_\lambda(t, X(t, y))\|_{L^1(I;L^2)}.$$

Similarly, to estimate $\dot{r} v_\lambda$, view $\dot{r} = \dot{\chi} - \dot{\tilde{\chi}}$ and without loss of generality, estimate $\dot{\chi} v_\lambda$. From the definition of χ and the relation $\dot{X} = V_\delta$, we have (with implicit constant depending on $\|V\|_{L^p(I;W^{1,\infty})}$)

$$\dot{\chi} \in L^1 S_{1,\delta,\delta}^{0,(0)},$$

from which we obtain

$$\|\dot{\chi} v_\lambda\|_{L^1(I;L^2)} \lesssim \|v_\lambda\|_{L^\infty(I;L^2)}$$

as desired.

Step 4. Applying Proposition 9.2 to (9.5) and applying the estimate (9.6), we obtain

$$\begin{aligned} \||D|^{-\rho} \tilde{\chi} v_\lambda\|_{L^p(I;L^q)} &\lesssim \lambda^{-\frac{1}{10p'}} (\|f_\lambda(t, X(t, y))\|_{L^1(I;L^2)} + \|v_\lambda\|_{L^\infty(I;L^2)}) + \lambda^{\frac{1}{10p}} \|\tilde{\chi} v_\lambda\|_{L^\infty(I;L^2)} \\ &\lesssim \lambda^{-\frac{1}{10p'}} \|f_\lambda(t, X(t, y))\|_{L^1(I;L^2)} + \lambda^{\frac{1}{10p}} \|v_\lambda\|_{L^\infty(I;L^2)}. \end{aligned}$$

We would like to replace $\tilde{\chi} v_\lambda$ on the left hand side by v_λ. Recall the difference is rv_λ. We have using Sobolev embedding and $r \in L^1 S_{1,\delta}^{-\frac{1}{2},(0)}$,

$$\||D|^{-\rho} rv_\lambda\|_{L^p(I;L^q)} \lesssim \||D|^{-\rho} rv_\lambda\|_{L^p(I;H^{\frac{2}{p}+\epsilon})} \lesssim \|rv_\lambda\|_{L^p(I;H^{\frac{1}{2p}+\epsilon})} \lesssim \|v_\lambda\|_{L^\infty(I;H^{\frac{1}{2p}+\epsilon-\frac{1}{2}})}$$

which more than suffices as $p \geq 2$.

Lastly, by (6.5), X is bilipschitz uniformly in time, so we may replace v_λ with u_λ and $f_\lambda(X)$ with f_λ:

$$\||D|^{-\rho}u_\lambda\|_{L^p(I;L^q)} \lesssim \lambda^{-\frac{1}{10p'}}\|f_\lambda\|_{L^1(I;L^2)} + \lambda^{\frac{1}{10p}}\|u_\lambda\|_{L^\infty(I;L^2)}. \tag{9.7}$$

Step 5. Rearranging (9.7),

$$\||D|^{-\rho}u_\lambda\|_{L^p(I;L^q)} \lesssim \lambda^{\frac{1}{10p}}(\lambda^{-\frac{1}{10}}\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

In the case $d = 1$ and $p = 4$,

$$\||D|^{-\rho}u_\lambda\|_{L^4(I;L^\infty)} \lesssim \lambda^{\frac{1}{40}}(\lambda^{-\frac{1}{10}}\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}),$$

We conclude by the frequency localization of u_λ, f_λ that

$$\|u_\lambda\|_{L^4(I;W^{s_1-\frac{d}{2}+\frac{1}{10},\infty})} \leq \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;H^{s_1-\frac{1}{10}})} + \|u_\lambda\|_{L^\infty(I;H^{s_1})})$$

as desired.

In the case $d \geq 2$, choose $p = 2 + \epsilon'$. Then by Bernstein's,

$$\||D|^{-\rho}u_\lambda\|_{L^\infty_x} \lesssim \lambda^{\frac{d}{q}}\||D|^{-\rho}u_\lambda\|_{L^q_x} = \lambda^{\frac{d}{2}-\frac{2}{p}}\||D|^{-\rho}u_\lambda\|_{L^q_x}$$

and hence

$$\|u_\lambda\|_{L^p(I;L^\infty)} \lesssim \lambda^{\rho+\frac{d}{2}-\frac{2}{p}}\lambda^{\frac{1}{10p}}(\lambda^{\frac{1}{10}}\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

Compute that, for small $\epsilon_i > 0$ depending on ϵ',

$$\rho + \frac{d}{2} - \frac{2}{p} = \frac{d}{2} - \frac{1}{4} + \epsilon_1$$

and

$$\frac{1}{10p} = \frac{1}{20} - \epsilon_2$$

so that

$$\|u_\lambda\|_{L^p(I;L^\infty)} \lesssim \lambda^{\frac{d}{2}-\frac{1}{4}+\epsilon_1+\frac{1}{20}-\epsilon_2}(\lambda^{\frac{1}{10}}\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

We conclude by the frequency localization of u_λ, f_λ and collecting ϵ_i into ϵ that

$$\|u_\lambda\|_{L^p(I;W^{s_1-\frac{d}{2}+\frac{1}{5}-\epsilon,\infty})} \leq \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;H^{s_1-\frac{1}{10}})} + \|u_\lambda\|_{L^\infty(I;H^{s_1})}).$$

After applying a Hölder estimate in time to replace p on the left hand side by 2, this concludes the proof of Proposition 5.3.

Chapter 3

Sharp Strichartz Estimates for One Surface Dimension

In this chapter, we establish Theorem 2.4, which improves upon Theorem 2.3 in the case $d = 1$ by providing the sharp Strichartz estimate.

We outline the chapter below. In Section 10, we record some additional notation and preliminary frequency localization that we use throughout the chapter.

In Section 11, we prove local smoothing estimates. We also remark that in Appendices A and B, we prove local versions of the elliptic estimates for the Dirichlet problem with rough boundary, and of the paralinearization of the Dirichlet to Neumann map, respectively. In Appendix C, we likewise establish local Sobolev estimates on the Taylor coefficient a.

In Sections 12 and 13, we integrate the symbol of our evolution equation along its Hamilton flow, and use this to establish improved the regularity properties for the flow. In Section 14, we construct the wave packet parametrix. Finally, in Section 15, we prove Strichartz estimates for the parametrix, before establishing the estimate for the exact solution.

10. Notation and Preliminaries

We record the notation and setting that we will use throughout this chapter, in addition to the notation discussed in Section 4.

10.1. **The Hamiltonian and wave packets.** We will use the Hamiltonian $H(t, x, \xi)$ given by

$$H = V_\lambda \xi + \sqrt{a_\lambda |\xi|}.$$

Note we omit notating λ on H for brevity since we will fix λ by the end of this section.

Recall the Hamilton equations associated to H,

$$\begin{cases} \dot{x}(t) = H_\xi(t, x(t), \xi(t)) \\ \dot{\xi}(t) = -H_x(t, x(t), \xi(t)) \\ (x(s), \xi(s)) = (x, \xi). \end{cases} \tag{10.1}$$

We also denote the solution $(x(t), \xi(t))$ to (10.1), at time $t \in I$ with initial data (x, ξ) at time $s \in I$, variously by

$$(x_s^t(x, \xi), \xi_s^t(x, \xi)) = (x^t(x, \xi), \xi^t(x, \xi)) = (x_s^t, \xi_s^t) = (x^t, \xi^t).$$

We consider the discrete set of phase space indices,

$$\mathcal{T} = \{T = (x,\xi) \in \lambda^{-\frac{3}{4}}\mathbb{Z} \times \lambda^{\frac{3}{4}}\mathbb{Z} : |\xi| \approx \lambda\},$$

and in particular often write $T = (x,\xi)$ for brevity.

We let $s_0 \in I$ denote the time chosen as in Proposition 13.3. Once s_0 is introduced, we write

$$(x^t, \xi^t) = (x^t_{s_0}, \xi^t_{s_0})$$

unless otherwise indicated. Further, associated to $T = (x,\xi) \in \mathcal{T}$, we let $s_T \in I$ denote the time chosen as in Lemma 13.8.

10.2. **Local smoothing.** We define weights to exhibit local smoothing estimates. Fix a Schwartz weight $w \in C^\infty(\mathbb{R})$ with frequency support $\{|\xi| \leq 1\}$ and $w \geq 1$ on $\{|x| \leq 1\}$. We denote a scaled translate of w by

$$w_{x_0,\kappa}(x) = w(\kappa^{\frac{3}{4}}(x - x_0))$$

for $x_0 \in \mathbb{R}$. We will use throughout that $w_{x_0,\kappa}$ has frequency support $\{|\xi| \leq \kappa^{\frac{3}{4}}\}$.

It will also be convenient to have compactly supported weights. Fix a smooth bump function $\chi \in C_0^\infty(\mathbb{R})$ supported on $[-1,1]$, with $\chi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Also let $\tilde{\chi}$ denote a widened bump function supported on $[-2,2]$, with $\tilde{\chi} \equiv 1$ on $[-1,1]$. As with w, we also write $\chi_{x_0,\kappa}, \tilde{\chi}_{x_0,\kappa}$ for the scaled translates of $\chi, \tilde{\chi}$.

We define the following local seminorm to measure local smoothing:

$$\|f\|_{LS^\sigma_{x_0,\lambda}} = \sum_{\kappa \leq c\lambda} \|w_{x_0,\kappa} S_\kappa f\|_{H^\sigma}.$$

We will also define a more technical local seminorm which we use to measure local smoothing on products. Let $\xi_0 \in \mathbb{R}$ with $|\xi_0| \in [\lambda/2, 2\lambda]$. First we construct a symmetric λ-frequency projection $S_{\xi_0,\lambda,\mu}$ with a $c\mu$-width gap at ξ_0, as follows: Let

$$p(\xi) = 1_{[0,\infty)}(\xi + \lambda)\psi_\lambda(\xi + \lambda)(1 - \chi(\mu^{-1}\xi/c))$$

and define

$$S_{\xi_0,\lambda,\mu} = p(D - \xi_0) + p(-D + \xi_0).$$

Throughout, to simplify exposition, we will abuse notation and use $S_{\xi_0,\lambda,\mu}$ to denote the same construction with $\psi_{\lambda'}$ in place of ψ_λ, where λ and λ' have bounded ratio.

Then we define the following local seminorm for $\sigma \in \mathbb{R}$, which collects local measurements on low, high, and balanced frequencies, respectively by row:

$$\begin{aligned}\|f\|_{LS^\sigma_{x_0,\xi_0,\lambda,\mu}} = &\sum_{\kappa \in [c\mu, c\lambda]} (\kappa^{\frac{3}{4}}\mu^{-1}\|S_\kappa f\|_{H^\sigma} + \|w_{x_0,\kappa} S_\kappa f\|_{H^\sigma}) \\ &+ \lambda^{\frac{3}{4}}\mu^{-1}\|S_{\geq c\lambda} f\|_{H^\sigma} + \sum_{\kappa \geq \lambda/c} \|w_{x_0,\lambda} S_\kappa f\|_{H^\sigma} \\ &+ \|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} f\|_{H^\sigma}.\end{aligned}$$

For brevity, we combine these by denoting

$$\begin{aligned}\mathcal{L}(t,x_0,\xi_0,\lambda) &= \mathcal{L}_s(t,x_0,\xi_0,\lambda)\\ &= \|(\eta,\psi)(t)\|_{LS^{s'+\frac{1}{2}}_{x_0,\lambda}} + \|(V,B)(t)\|_{LS^{s'}_{x_0,\lambda}}\\ &\quad + \max_{\lambda^{\frac{3}{4}}\ll\mu\le c\lambda}\max_{0\le\sigma\le s}\lambda^{-\sigma}\mu^{s'}(\|(\eta,\psi)(t)\|_{LS^{\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}} + \|(V,B)(t)\|_{LS^{\sigma}_{x_0,\xi_0,\lambda,\mu}}).\end{aligned}$$

10.3. **Frequency localization.** Similar to Section 5 in Chapter 2, we can reduce Theorem 2.4 to the following frequency localized and symbol truncated form. The proof is similar, so we omit it:

Proposition 10.1. *Consider a smooth solution u_λ to*

$$(\partial_t + T_{V_\lambda}\cdot\nabla + iT_{\gamma_\lambda})u_\lambda = f_\lambda \tag{10.2}$$

on I where $u_\lambda(t,\cdot)$ and f_λ have frequency support $\{|\xi|\in[\lambda/2,2\lambda]\}$. Then

$$\|u_\lambda\|_{L^2(I;W^{\frac{1}{8}-\frac{d}{2}-,\infty})} \le \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

As in Section 5, it is convenient to replace the paradifferential symbol T_{γ_λ} by the pseudo-differential symbol $\gamma_\lambda = \gamma_\lambda(t,x,\xi)$, and likewise the paraproduct T_{V_λ} by V_λ. Further, recall that since $d=1$, $\gamma = \sqrt{a|D|}$. For technical reasons, we further replace γ_λ by $\sqrt{a_\lambda|D|}$.

Proposition 10.2. *Consider a smooth solution u_λ to*

$$(\partial_t + V_\lambda\partial_x + i\sqrt{a_\lambda|D|})u_\lambda = f \tag{10.3}$$

on I where $u_\lambda(t,\cdot)$ has frequency support $\{|\xi|\in[\lambda/2,2\lambda]\}$. Then

$$\|u_\lambda\|_{L^2(I;W^{\frac{1}{8}-\frac{d}{2}-,\infty})} \le \mathcal{F}(T)(\|f_\lambda\|_{L^1(I;L^2)} + \|u_\lambda\|_{L^\infty(I;L^2)}).$$

Proof of Proposition 10.1. We may apply Proposition 10.2 with inhomogeneity

$$f_\lambda + (T_{\gamma_\lambda}-\gamma_\lambda)u_\lambda + (T_{V_\lambda}-V_\lambda)\cdot\nabla u_\lambda + (\gamma_\lambda - \sqrt{a_\lambda|D|})u_\lambda.$$

Using for instance [Ngu15, Proposition 2.7] and the estimates on the Taylor coefficient as in Corollary D.10,

$$\|(T_{\gamma_\lambda}-\gamma_\lambda)u_\lambda\|_{L^2} \lesssim M^{\frac{1}{2}}_{\frac{1}{2}}(\gamma)\|u_\lambda\|_{L^2} \le \mathcal{F}(M(t))Z(t)\|u_\lambda\|_{L^2}.$$

Integrating in time, we have

$$\|(T_{\gamma_\lambda}-\gamma_\lambda)u_\lambda\|_{L^1(I;L^2)} \le \mathcal{F}(T)\|u_\lambda\|_{L^\infty(I;L^2)}$$

which is controlled by the right hand side of the estimate in Proposition 5.2.

Similarly,

$$\|(T_{V_\lambda}-V_\lambda)\cdot\nabla u_\lambda\|_{L^2} \lesssim M^1_1(V\cdot\xi)\|u_\lambda\|_{L^2} \lesssim \|V\|_{W^{1,\infty}}\|u_\lambda\|_{L^2}.$$

Integrating in time yields the desired estimate.

It remains to estimate $(\gamma_\lambda - \sqrt{a_\lambda|D|})u_\lambda$. First, we have

$$\begin{aligned}\|(\gamma-\gamma_\lambda)u_\lambda\|_{L^2} &\lesssim \|S_{>c_1\lambda}\sqrt{a}\|_{L^\infty}\|u_\lambda\|_{H^{\frac{1}{2}}}\\ &\le \|\sqrt{a}\|_{W^{\frac{1}{2},\infty}}\|u_\lambda\|_{L^2}.\end{aligned}$$

Using the Taylor coefficient estimates of Proposition D.8 and integrating in time, this is controlled by the right hand side of the estimate in Proposition 5.2. We may thus exchange γ_λ for γ.

Lastly, it is easy to compute

$$\gamma - \sqrt{a_\lambda |D|} = \frac{a_{>c_1\lambda}}{\sqrt{a} + \sqrt{a_\lambda}} \sqrt{|D|}.$$

Thus, using the Taylor sign condition,

$$\begin{aligned}\|(\gamma - \sqrt{a_\lambda |D|}) u_\lambda\|_{L^2} &\lesssim \frac{1}{\sqrt{a_{min}}} \|a_{>c_1\lambda}\|_{L^\infty} \|u_\lambda\|_{H^{\frac{1}{2}}} \\ &\lesssim \frac{1}{\sqrt{a_{min}}} \|a\|_{W^{\frac{1}{2},\infty}} \|u_\lambda\|_{L^2}.\end{aligned}$$

Using the usual Taylor coefficient estimates and integrating in time yields the desired result. □

For the remainder of the chapter, it remains to prove Proposition 10.2. In particular, we fix the frequency λ throughout.

11. Local Smoothing Estimates

In this section we establish local smoothing estimates for the paradifferential dispersive equation (1.3), and then show that these estimates are inherited by (η, ψ, V, B).

Observe that the order 1/2 dispersive term of (1.3) contributes velocity $\mu^{-\frac{1}{2}} \ll 1$ to a frequency μ component of a solution u. On the other hand, the transport term of the equation contributes up to a unit scale velocity to all frequency components of u, dominating the dispersive velocity, $1 \gg \mu^{-\frac{1}{2}}$. As a result, we only expect to see a local smoothing effect by following the transport vector field.

In fact, for our purposes we are interested in local smoothing along λ-frequency Hamilton characteristics, on packets of width $\delta x \approx \lambda^{-\frac{3}{4}}$. Since at frequency λ,

$$\partial_\xi^2 |\xi|^{\frac{1}{2}} = \lambda^{-\frac{3}{2}},$$

we expect to see local smoothing effects for frequencies of u with frequency separation greater than

$$\lambda^{-\frac{3}{4}} / \lambda^{-\frac{3}{2}} = \lambda^{\frac{3}{4}}$$

from the frequency center of a packet.

11.1. The Hamilton flow.

Recall we denote the Hamiltonian for (10.3) by

$$H(t, x, \xi) = V_\lambda \xi + \sqrt{a_\lambda |\xi|}.$$

Thus, the Hamilton equations (10.1) associated to H are given by

$$\begin{cases} \dot{x}^t = V_\lambda + \frac{1}{2}\sqrt{a_\lambda} |\xi^t|^{-\frac{3}{2}} \xi \\ \dot{\xi}^t = -\partial_x V_\lambda \xi^t - \partial_x \sqrt{a_\lambda} |\xi^t|^{\frac{1}{2}}. \end{cases}$$

In the following lemma, we observe that ξ^t preserves data satisfying $|\xi| \in [\lambda/2, 2\lambda]$:

Lemma 11.1. *If* $|\xi| \in [\lambda/2, 2\lambda]$ *and* $|t-s|^{\frac{1}{2}} \mathcal{F}(T) \ll 1$, *then*

$$\|\xi_s^t(x, \xi)\|_{L_x^\infty} \approx |\xi|.$$

Proof. We work with $|\xi^t|^{\frac{1}{2}}$ instead of ξ^t itself, due to its appearance in H. By (10.1),

$$\frac{d}{dt}|\xi^t|^{\frac{1}{2}} = -\frac{1}{2}|\xi^t|^{-\frac{3}{2}}\xi^t H_x,$$

so we have (using at the end the Taylor sign condition, and the frequency localization)

$$\begin{aligned} ||\xi^t|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}}| \leq \frac{1}{2}\int_s^t |\xi^\tau|^{-\frac{1}{2}}|H_x|\,d\tau &\leq \frac{1}{2}\int_s^t |\partial_x V_\lambda|\xi^\tau|^{\frac{1}{2}}| + |\partial_x\sqrt{a_\lambda}|\,d\tau \\ &\leq \frac{1}{2}\int_s^t \|\partial_x V_\lambda\|_{L^\infty_x}|\xi^\tau|^{\frac{1}{2}} + \|\partial_x\sqrt{a_\lambda}\|_{L^\infty_x}\,d\tau \\ &\lesssim \int_s^t \|V\|_{W^{1,\infty}}|\xi^\tau|^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\|a_\lambda\|_{W^{\frac{1}{2},\infty}}\,d\tau. \end{aligned}$$

Using the Taylor coefficient estimates of Proposition D.8,

$$|\xi^t|^{\frac{1}{2}} \leq |\xi|^{\frac{1}{2}} + \int_s^t Z(t)|\xi^\tau|^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\mathcal{F}(M(t))Z(t)\,d\tau. \tag{11.1}$$

Applying Gronwall, Cauchy-Schwarz, and the assumption $|\xi|^{\frac{1}{2}} \lesssim \lambda^{\frac{1}{2}}$,

$$|\xi^t|^{\frac{1}{2}} \lesssim \lambda^{\frac{1}{2}}(1 + |t-s|^{\frac{1}{2}}\mathcal{F}(T))\exp(|t-s|^{\frac{1}{2}}\mathcal{F}(T)) \leq 2\lambda^{\frac{1}{2}}.$$

Use this in (11.1) to obtain

$$||\xi^t|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}}| \leq \int_s^t Z(t)|\xi^\tau|^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\mathcal{F}(M(t))Z(t)\,d\tau \leq \lambda^{\frac{1}{2}}|t-s|^{\frac{1}{2}}\mathcal{F}(T) \ll \lambda^{\frac{1}{2}}.$$

□

Remark 11.2. In the remainder of the chapter, we may assume T is small enough so that the condition

$$|t-s|^{\frac{1}{2}}\mathcal{F}(T) \ll 1$$

of the lemma is satisfied for all $t, s \in I$ (otherwise, we can iterate the argument a number of times depending on $\mathcal{F}(T)$). In particular, ξ^t maintains separation from 0, so that the flow (x^t, ξ^t) remains defined on I.

11.2. **Estimates on the dispersive equation.** In this subsection, we prove local smoothing estimates for (1.3) along Hamilton characteristics. Throughout this section, we consider the case of $\xi_0 > 0$, but the corresponding results for $\xi_0 < 0$ are similar.

Proposition 11.3. *Consider a smooth solution u to*

$$(\partial_t + V_\lambda\partial_x + i\sqrt{a_\lambda|D|})u = f \tag{11.2}$$

on I. Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$. Let $u(t,\cdot)$ have frequency support $[c\lambda/2, \xi^t - c\mu]$. Then

$$\|w_{x^t,\lambda}u\|_{L^2(I;L^2)} \leq \mu^{-\frac{1}{2}}\lambda^{\frac{3}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

The same holds for $u(t,\cdot)$ with frequency supports $[\xi^t + c\mu, 2\lambda/c]$ or $[-2\lambda/c, -c\lambda/2]$.

Proof. We use the positive commutator method. Let $v \in C^\infty(\mathbb{R})$ satisfy $\sqrt{v'} = w$. It is also convenient to write

$$p(D) = |D|^{\frac{1}{2}} \tilde{S}_\lambda$$

so that we may exchange $|D|^{\frac{1}{2}}$ for $p(D)$ in (11.2). Lastly, for brevity, write

$$\tilde{w}(t,x) = w_{x^t,\lambda} = w(\lambda^{\frac{3}{4}}(x - x^t)), \quad \tilde{v}(t,x) = v(\lambda^{\frac{3}{4}}(x - x^t)), \quad \tilde{v}' = \partial_x \tilde{v}.$$

Using (11.2), we have

$$\begin{aligned} \frac{d}{dt}\langle \tilde{v}u, u\rangle &= \langle \tilde{v}\partial_t u - \tilde{v}'\dot{x}^t u, u\rangle + \langle \tilde{v}u, \partial_t u\rangle \\ &= \langle \tilde{v}\partial_t u + \partial_x(V_\lambda \tilde{v}u) + ip(D)\sqrt{a_\lambda}\tilde{v}u, u\rangle - \langle \tilde{v}'\dot{x}^t u, u\rangle \\ &= \langle \tilde{v}f, u\rangle + \langle i[p(D), \sqrt{a_\lambda}]\tilde{v}u, u\rangle + \langle i\sqrt{a_\lambda}[p(D), \tilde{v}]u, u\rangle \\ &\quad + \langle \tilde{v}'(V_\lambda - \dot{x}^t)u, u\rangle + \langle (\partial_x V_\lambda)\tilde{v}u, u\rangle. \end{aligned} \tag{11.3}$$

We successively consider each term on the right hand side.

We will obtain positivity from the third term on the right hand side of (11.3). Write the kernel of $[p(D), \tilde{v}]$ using a Taylor expansion:

$$\begin{aligned} &\int e^{i(x-y)\xi} p(\xi)(\tilde{v}(y) - \tilde{v}(x))\, d\xi \\ &\quad = \int e^{i(x-y)\xi} p(\xi) \left((y-x)\tilde{v}'(x) + (y-x)^2 \int_0^1 \tilde{v}''(h(x-y)+y)h\, dh \right) d\xi \end{aligned}$$

Then integrating by parts, we obtain

$$\begin{aligned} \int e^{i(x-y)\xi} \left(-ip'(\xi)\tilde{v}'(x) - p''(\xi) \int_0^1 \tilde{v}''(h(x-y)+y)h\, dh \right) d\xi \\ =: K_1(x,y) + K_2(x,y). \end{aligned}$$

Observe that K_1 is the kernel of

$$-i\tilde{v}'p'(D) = -i\lambda^{\frac{3}{4}} w^2 p'(D)$$

so that K_1 will contribute the positive operator. We observe that $\tilde{v}$ has been chosen so that the second order component K_2 is integrable, independent of λ. Indeed, observe that we may write, for a unit-scaled bump function q,

$$p(\xi) = \lambda^{\frac{1}{2}} q(\lambda^{-1}\xi)$$

so that

$$\lambda^{\frac{3}{2}} \int e^{i(x-y)\xi} p''(\xi)\, d\xi = \lambda \widehat{q''}(\lambda(y-x))$$

is an integrable kernel. Further, since $\lambda^{-\frac{3}{2}}\tilde{v}'' = \widetilde{v''}$ is bounded, we have uniformly in (x,y),

$$\lambda^{-\frac{3}{2}} \left| \int_0^1 \tilde{v}''(h(y-x)+x)h\, dh \right| \lesssim 1.$$

We conclude

$$\int \sqrt{a_\lambda}(x) K_2(x,y) u(y)\overline{u(x)}\, dy dx \lesssim \|a\|_{L^\infty} \|u\|_{L^2}^2.$$

We can similarly estimate the second and last terms on the right hand side of (11.3) as errors (observe that $\tilde{v}$ has frequency support below $\lambda^{\frac{3}{4}} \ll \lambda$):

$$\begin{aligned}\|[p(D), \sqrt{a_\lambda}]\tilde{v}u\|_{L^2} &\lesssim \|\partial_x \sqrt{a_\lambda}\|_{L^\infty} \|\tilde{v}u\|_{H^{-\frac{1}{2}}} \\ &\leq \lambda^{\frac{1}{2}} \mathcal{F}(M(t)) Z(t) \lambda^{-\frac{1}{2}} \|u\|_{L^2}, \\ \|(\partial_x V_\lambda)\tilde{v}u\|_{L^2} &\leq \mathcal{F}(M(t))Z(t)\|u\|_{L^2}.\end{aligned}$$

It remains to consider the fourth term on the right hand side of (11.3),

$$\langle \tilde{v}'(V_\lambda - \dot{x}^t)u, u\rangle = \langle \tilde{v}'(V_\lambda(x) - V_\lambda(x^t))u, u\rangle - \frac{1}{2}\langle \tilde{v}'\sqrt{a_\lambda}|\xi^t|^{-\frac{3}{2}}\xi^t u, u\rangle.$$

We bound the first of these terms as an error:

$$\begin{aligned}\|\tilde{v}'(V_\lambda(x) - V_\lambda(x^t))u\|_{L^2} &\lesssim \|\lambda^{\frac{3}{4}} v'(\lambda^{\frac{3}{4}}(x - x^t))(x - x^t)\|_{L^\infty} \|V_\lambda\|_{C^1} \|u\|_{L^2} \\ &\lesssim Z(t)\|u\|_{L^2}.\end{aligned}$$

We conclude from (11.3) that, on the frequency support of u, and using the choice of v in terms of w,

$$\begin{aligned}\frac{d}{dt}\langle \tilde{v}u, u\rangle = {} & \frac{1}{2}\lambda^{\frac{3}{4}}\langle \sqrt{a_\lambda}\tilde{w}^2(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})u, u\rangle \\ & + O((\mathcal{F}(M(t))Z(t))\|u\|_{L^2}^2 + \|f\|_{L^2}\|u\|_{L^2}).\end{aligned} \tag{11.4}$$

Next, we symmetrize the first term on the right hand side. We write

$$\tilde{w}^2(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})u = \tilde{w}(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})\tilde{w}u + \tilde{w}[|D|^{-\frac{1}{2}}, \tilde{w}]u$$

where the commutator is bounded in L^2_x by

$$\|\tilde{w}'\|_{L^\infty}\|u\|_{H^{-\frac{3}{2}}} \leq \lambda^{\frac{3}{4}}\lambda^{-\frac{3}{2}}\|u\|_{L^2}.$$

Using additionally the bounds on the Taylor coefficient, this may be absorbed into the error on the right hand side. Similarly, we write, using the frequency localization of u,

$$\begin{aligned}\sqrt{a_\lambda}(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})\tilde{w}u = {} & (|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}\sqrt{a_\lambda}(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}\tilde{w}u \\ & - [(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}, \sqrt{a_\lambda}](|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}\tilde{w}u.\end{aligned}$$

Then, observing that $\tilde{w}$ has frequency support below $\lambda^{\frac{3}{4}} \ll \lambda$, we similarly may estimate the commutator by

$$\|\partial_x\sqrt{a_\lambda}\|_{L^\infty}\|\tilde{w}u\|_{H^{-\frac{3}{2}}} \leq \lambda^{\frac{1}{2}}\mathcal{F}(M(t))Z(t)\lambda^{-\frac{3}{2}}\|u\|_{L^2}$$

which is better than needed with respect to the power of λ to absorb into the right hand side.

After the symmetrization, integrating (11.4) in time, we obtain

$$\lambda^{\frac{3}{4}}\|a_\lambda^{\frac{1}{4}}(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}\tilde{w}u\|_{L^2(I;L^2)}^2 \leq \mathcal{F}(T)\|u\|_{L^\infty(I;L^2)}^2 + \|f\|_{L^1(I;L^2)}^2.$$

Using the lower bound from the Taylor sign condition and taking square roots,

$$\|(|D|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}})^{\frac{1}{2}}\tilde{w}u\|_{L^2(I;L^2)} \leq \lambda^{-\frac{3}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

Lastly, observe that for ξ in the frequency support of u (using Lemma 11.1),

$$|\xi|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}} \geq |\xi^t - c\mu|^{-\frac{1}{2}} - |\xi^t|^{-\frac{1}{2}} \gtrsim \lambda^{-\frac{3}{2}}\mu.$$

Using that the frequency support of $\tilde{w}$ at $\lambda^{\frac{3}{4}} \ll \mu \leq c\lambda$ essentially leaves that of u unchanged, we obtain

$$\lambda^{-\frac{3}{4}}\mu^{\frac{1}{2}}\|\tilde{w}u\|_{L^2(I;L^2)} \leq \lambda^{-\frac{3}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)})$$

as desired. The case of frequency support $[\xi^t + c\mu, 2\lambda/c]$ is similar, and the case of support $[-2\lambda/c, -c\lambda/2]$ is better than needed.

□

Using similar, cruder analyses, we have corresponding estimates for the cases of u with low and high frequencies:

Proposition 11.4. *Consider a smooth solution u to*

$$(\partial_t + V_\kappa \partial_x + i\sqrt{a_\kappa |D|})u = f$$

on I. Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$. Let $u(t,\cdot)$ have frequency support $\{|\xi| \in [\kappa/2, 2\kappa]\}$. If $\kappa \leq c\lambda$, then

$$\|w_{x^t,\kappa}u\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

If $\kappa \geq \lambda/c$, then

$$\|w_{x^t,\lambda}u\|_{L^2(I;L^2)} \leq \lambda^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

We can prove paradifferential counterparts, using essentially the same analysis as in Section 5:

Corollary 11.5. *Consider a smooth solution u to (1.3) on I. Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$.*

i) If $u(t,\cdot)$ has frequency support on one of

$$[c\lambda/2, \xi^t - c\mu], \quad [\xi^t + c\mu, 2\lambda/c], \quad [-2\lambda/c, -c\lambda/2],$$

then

$$\|w_{x^t,\lambda}u\|_{L^2(I;L^2)} \leq \mu^{-\frac{1}{2}}\lambda^{\frac{3}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

ii) If $u(t,\cdot)$ has frequency support $\{|\xi| \in [\kappa/2, 2\kappa]\}$ and $\kappa \leq c\lambda$, then

$$\|w_{x^t,\kappa}u\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

iii) If $u(t,\cdot)$ has frequency support $\{|\xi| \in [\kappa/2, 2\kappa]\}$ and $\kappa \geq \lambda/c$, then

$$\|w_{x^t,\lambda}u\|_{L^2(I;L^2)} \leq \lambda^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;L^2)} + \|f\|_{L^1(I;L^2)}).$$

Lastly, we apply Corollary 11.5 to frequency localized pieces of a smooth solution u to (1.3), with no assumed frequency localization. Recall that we construct a symmetric λ-frequency projection $S_{\xi_0,\lambda,\mu}$ with a $c\mu$-width gap at ξ_0, as in Section 10.2.

Corollary 11.6. *Consider a smooth solution u to (1.3) on I. Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$.*

i) We have

$$\|w_{x^t,\lambda}S_{\xi^t,\lambda,\mu}u\|_{L^2(I;H^s)} \leq \mu^{-\frac{3}{2}}\lambda^{\frac{11}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;H^s)} + \|f\|_{L^1(I;H^s)}).$$

ii) For $\kappa \leq c\lambda$,

$$\|w_{x^t,\kappa}S_\kappa u\|_{L^2(I;H^s)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;H^s)} + \|f\|_{L^1(I;H^s)}).$$

iii) For $\kappa \geq \lambda/c$,

$$\|w_{x^t,\lambda}S_\kappa u\|_{L^2(I;H^s)} \leq \lambda^{-\frac{1}{8}}\mathcal{F}(T)(\|u\|_{L^\infty(I;H^s)} + \|f\|_{L^1(I;H^s)}).$$

Proof. For (ii) and (iii), we may apply Corollary 11.5 on $S_\kappa u$ with inhomogeneity

$$S_\kappa f + [T_V\partial, S_\kappa]u + i[T_{\sqrt{a}}|D|^{\frac{1}{2}}, S_\kappa]u.$$

The commutators are estimated in the same way as the corresponding inhomogeneous commutators in Section 5, though we do not sum the frequency pieces here.

The case of (i) is similar. However, for p as in the definition of the symbol of $S_{\xi^t,\lambda,\mu}$, we observe that p' is not of order -1 uniformly in μ and λ. Rather, we need to consider instead $\mu\lambda^{-1}p'$. Additionally, we need to estimate

$$[\partial_t, p(D-\xi^t)]u = \dot{\xi}^t p'(D-\xi^t)u = (-\partial_x V_\lambda \xi^t - \partial_x\sqrt{a_\lambda}|\xi^t|^{\frac{1}{2}})p'(D-\xi^t)u$$

in $L^1(I;L^2)$ as an inhomogeneous term (and likewise for the similar $p(-D+\xi^t)$ case). Using Lemma 11.1 to see that $\xi^t \approx \xi_0$, we obtain

$$\|(-\partial_x V_\lambda \xi^t - \partial_x\sqrt{a_\lambda}|\xi^t|^{\frac{1}{2}})p'(D-\xi^t)u\|_{L^1(I;L^2)} \lesssim \lambda(\|V\|_{L^1(I;C^1)} + \|a\|_{L^1(I;C^{\frac{1}{2}})})\mu^{-1}\|u\|_{L^\infty(I;L^2)}.$$

Using the Taylor coefficient estimates of Proposition D.8 yields the desired estimate. □

11.3. **Estimates on the surface and velocity field.** In this section we establish local smoothing estimates on the original unknowns (η, ψ, V, B), using Proposition 2.1.

Proposition 11.7. *Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$.*

i) We have

$$\|w_{x^t,\lambda}S_{\xi^t,\lambda,\mu}(\eta,\psi)\|_{L^2(I;H^{s+\frac{1}{2}})} + \|w_{x^t,\lambda}S_{\xi^t,\lambda,\mu}(V,B)\|_{L^2(I;H^s)} \leq \mu^{-\frac{3}{2}}\lambda^{\frac{11}{8}}\mathcal{F}(T).$$

ii) For $\kappa \leq c\lambda$,

$$\|w_{x^t,\kappa}S_\kappa(\eta,\psi)\|_{L^2(I;H^{s+\frac{1}{2}})} + \|w_{x^t,\kappa}S_\kappa(V,B)\|_{L^2(I;H^s)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

iii) For $\kappa \geq \lambda/c$,

$$\|w_{x^t,\lambda}S_\kappa(\eta,\psi)\|_{L^2(I;H^{s+\frac{1}{2}})} + \|w_{x^t,\lambda}S_\kappa(V,B)\|_{L^2(I;H^s)} \leq \lambda^{-\frac{1}{8}}\mathcal{F}(T).$$

Proof. We establish (ii); the other cases are similar.

By Proposition 2.1, we have that

$$u = \langle D\rangle^{-s}(\langle D\rangle^s V + T_{\nabla\eta}\langle D\rangle^s B - iT_{\sqrt{a/|D|}}\langle D\rangle^s\nabla\eta)$$

solves

$$(\partial_t + T_V\partial_x + iT_{\sqrt{a}}\sqrt{|D|})u = f$$

with

$$\|f\|_{L^1(I;H^s)} \leq \mathcal{F}(T).$$

Further, using Sobolev embedding and (E.1), it is easy to see that

$$\|u\|_{L^\infty(I;H^s)} \leq \mathcal{F}(T).$$

We conclude by Corollary 11.6,

$$\|w_{x^t,\kappa}S_\kappa u\|_{L^2(I;H^s)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Using the frequency localization of $w_{x^t,\kappa}$ at $\kappa^{-\frac{3}{4}} \ll \kappa$, we may commute $\langle D\rangle^s$ to obtain

$$\|w_{x^t,\kappa}\langle D\rangle^s S_\kappa u\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Step 1. First we estimate η. Taking the imaginary part of u,

$$\|w_{x^t,\kappa} S_\kappa T_{\sqrt{a}} |D|^{-\frac{1}{2}}\langle D\rangle^s \nabla\eta\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

We may commute S_κ with $T_{\sqrt{a}}$ by absorbing the commutator into the right hand side, using (E.2) with the estimates on the Taylor coefficient a provided in Proposition D.8:

$$\begin{aligned}\|[S_\kappa, T_{\sqrt{a}}]|D|^{-\frac{1}{2}}\langle D\rangle^s\nabla\tilde{S}_\kappa\eta\|_{L^2(I;L^2)} &\leq \mathcal{F}(T)\||D|^{-\frac{1}{2}}\langle D\rangle^s\nabla\tilde{S}_\kappa\eta\|_{L^\infty(I;H^{-\frac{1}{2}})} \\ &\leq \mathcal{F}(T)\|\tilde{S}_\kappa\eta\|_{L^\infty(I;H^s)} \leq \kappa^{-\frac{1}{2}}\mathcal{F}(T)\|\tilde{S}_\kappa\eta\|_{L^\infty(I;H^{s+\frac{1}{2}})}\end{aligned}$$

so that

$$\|w_{x^t,\kappa} T_{\sqrt{a}} |D|^{-\frac{1}{2}}\langle D\rangle^s \nabla S_\kappa\eta\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

We may also exchange $T_{\sqrt{a}}$ with $\sqrt{a}$, absorbing the error into the right hand side using (E.8), (E.7), and Proposition D.8:

$$\begin{aligned}\|T_{|D|^{-\frac{1}{2}}\langle D\rangle^s\nabla S_\kappa\eta}\sqrt{a}\|_{L^2(I;L^2)} &\lesssim \||D|^{-\frac{1}{2}}\langle D\rangle^s\nabla S_\kappa\eta\|_{L^2(I;C_*^{\frac{1}{2}-s})}\|\sqrt{a}\|_{L^\infty(I;H^{s-\frac{1}{2}})} \\ &\leq \kappa^{-\frac{1}{2}}\mathcal{F}(T), \\ \|R(|D|^{-\frac{1}{2}}\langle D\rangle^s\nabla S_\kappa\eta, \sqrt{a})\|_{L^2(I;L^2)} &\lesssim \||D|^{-\frac{1}{2}}\langle D\rangle^s\nabla S_\kappa\eta\|_{L^2(I;C_*^{\frac{1}{2}-s})}\|\sqrt{a}\|_{L^\infty(I;H^{s-\frac{1}{2}})} \\ &\leq \kappa^{-\frac{1}{2}}\mathcal{F}(T),\end{aligned}$$

so that

$$\|\sqrt{a}w_{x^t,\kappa}|D|^{-\frac{1}{2}}\langle D\rangle^s \nabla S_\kappa\eta\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Using the Taylor sign condition, $a \geq a_{min} > 0$,

$$\|w_{x^t,\kappa}|D|^{-\frac{1}{2}}\langle D\rangle^s \nabla S_\kappa\eta\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Lastly, recalling the frequency localization of $w_{x^t,\kappa}$, we obtain the desired result.

Step 2. Next, we estimate B. Taking the real part of u,

$$\|w_{x^t,\kappa}S_\kappa(\langle D\rangle^s V + T_{\nabla\eta}\langle D\rangle^s B)\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Similar to Step 1, we may commute S_κ with $T_{\nabla\eta}$, obtaining

$$\|w_{x^t,\kappa}(\langle D\rangle^s \partial_x^{-1} S_\kappa\partial_x V + T_{\nabla\eta}\langle D\rangle^s S_\kappa B)\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Recalling from Proposition 4.5 of [ABZ14a],

$$\partial_x V = -G(\eta)B - \Gamma_y.$$

We may thus exchange $\partial_x V$ with $-G(\eta)B$, absorbing the Γ_y by using Proposition D.4 to estimate

$$\|\langle D\rangle^s\partial_x^{-1}S_\kappa\Gamma_y\|_{L^2(I;L^2)} \lesssim \kappa^{-\frac{1}{2}}\|\Gamma_y\|_{L^2(I;H^{s-\frac{1}{2}})} \leq \kappa^{-\frac{1}{2}}\mathcal{F}(T).$$

Further, we can exchange $G(\eta)B$ with $|D|B$ by using Proposition B.15 to estimate

$$\|\langle D\rangle^s\partial_x^{-1}S_\kappa(G(\eta)B - |D|B)\|_{L^2(I;L^2)} \lesssim \kappa^{-\frac{1}{2}}\|G(\eta)B - |D|B\|_{L^2(I;H^{s-\frac{1}{2}})} \leq \kappa^{-\frac{1}{2}}\mathcal{F}(T).$$

We conclude

$$\|w_{x^t,\kappa}(-\partial_x^{-1}|D| + T_{\nabla\eta})\langle D\rangle^s S_\kappa B\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Similar to Step 1, we may exchange $T_{\nabla\eta}$ with $\nabla\eta$, and using as usual the frequency localization of $w_{x^t,\kappa}$, commute $w_{x^t,\kappa}$ with $\partial_x^{-1}|D|$:

$$\|(-\partial_x^{-1}|D| + (\nabla\eta))w_{x^t,\kappa}\langle D\rangle^s S_\kappa B\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

We can restore the paralinearization by estimating (using the frequency localization of $w_{x^t,\kappa}$ and $S_\kappa B$)

$$\begin{aligned}\|T_{w_{x^t,\kappa}\langle D\rangle^s S_\kappa B}\nabla\eta\|_{L^2(I;L^2)} &\lesssim \|w_{x^t,\kappa}\langle D\rangle^s S_\kappa B\|_{L^2(I;C_*^{\frac{1}{2}-s})}\|\nabla\eta\|_{L^\infty(I;H^{s-\frac{1}{2}})} \\ &\leq \kappa^{-\frac{1}{2}}\mathcal{F}(T).\end{aligned}$$

and similarly for the balanced-frequency term. We conclude

$$\|T_{i\xi^{-1}|\xi|+\nabla\eta}w_{x^t,\kappa}\langle D\rangle^s S_\kappa B\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Lastly, by (E.1),

$$\|T_{(i\xi^{-1}|\xi|+\nabla\eta)^{-1}}T_{i\xi^{-1}|\xi|+\nabla\eta}w_{x^t,\kappa}\langle D\rangle^s S_\kappa B\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

which we may exchange for the desired estimate by (E.2).

Step 3. We estimate V. Using the estimate on B from Step 2 along with an analysis similar to that in Step 1 to commute $T_{\nabla\eta}$, we have

$$\|w_{x^t,\kappa}S_\kappa T_{\nabla\eta}\langle D\rangle^s B\|_{L^2(I;L^2)} \leq \kappa^{-\frac{1}{8}}\mathcal{F}(T).$$

Recalling the estimate on the real part of u at the beginning of Step 2, we may absorb this into the right hand side. The remaining term is the desired estimate.

Step 4. Lastly, we estimate ψ using the formula

$$\nabla\psi = V + B\nabla\eta.$$

Note that it suffices to show

$$\|w_{x^t,\kappa}\langle D\rangle^{s-\frac{3}{8}}S_\kappa\nabla\psi\|_{L^2(I;L^2)} \leq \mathcal{F}(T).$$

We easily have

$$\|w_{x^t,\kappa}\langle D\rangle^{s-\frac{3}{8}}S_\kappa V\|_{L^2(I;L^2)} \leq \mathcal{F}(T)$$

so it remains to show

$$\|w_{x^t,\kappa}\langle D\rangle^{s-\frac{3}{8}}S_\kappa B\nabla\eta\|_{L^2(I;L^2)} \leq \mathcal{F}(T).$$

We may commute B with $\langle D\rangle^{s-\frac{3}{8}}S_\kappa$ by estimating

$$\|[\langle D\rangle^{s-\frac{3}{8}}S_\kappa, B]\nabla\eta\|_{L^2(I;L^2)} \lesssim \|B\|_{L^2(I;C^1)}\|\eta\|_{L^2(I;H^{s-\frac{3}{8}})} + \|B\|_{L^2(I;H^{s-\frac{3}{8}})}\|\nabla\eta\|_{L^\infty} \leq \mathcal{F}(T).$$

Then it remains to show

$$\|Bw_{x^t,\kappa}\langle D\rangle^{s-\frac{3}{8}}S_\kappa\nabla\eta\|_{L^2(I;L^2)} \leq \mathcal{F}(T),$$

which is immediate from the local estimate on η.

□

As a straightforward consequence of case (ii) in Proposition 11.7, we have

Corollary 11.8. *Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$. Then*

$$\|(\eta, \psi)(t)\|_{L^2(I; LS^{s'+\frac{1}{2}-}_{x^t,\lambda})} + \|(V, B)(t)\|_{L^2(I; LS^{s'-}_{x^t,\lambda})} \leq \mathcal{F}(T).$$

The other cases will be used in considering local Sobolev estimates on products in the following subsection.

11.4. **Local smoothing on products.** Recall that we define the following local seminorm for $\sigma \in \mathbb{R}$ (see Section 10.2):

$$\begin{aligned}\|f\|_{LS^\sigma_{x_0,\xi_0,\lambda,\mu}} = &\sum_{\kappa \in [c\mu, c\lambda]} (\kappa^{\frac{3}{4}} \mu^{-1} \|S_\kappa f\|_{H^\sigma} + \|w_{x_0,\kappa} S_\kappa f\|_{H^\sigma}) \\ &+ \lambda^{\frac{3}{4}} \mu^{-1} \|S_{\geq c\lambda} f\|_{H^\sigma} + \sum_{\kappa \geq \lambda/c} \|w_{x_0,\lambda} S_\kappa f\|_{H^\sigma} \\ &+ \|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} f\|_{H^\sigma}.\end{aligned}$$

This definition is motivated by the following balanced-frequency product estimate:

Proposition 11.9. *Let $\alpha + \beta = \alpha' + \beta' > 0$. Then*

$$\|w_{x_0,\lambda} S_\mu R(f, g)\|_{H^{\alpha+\beta}} \lesssim \|f\|_{LS^\alpha_{x_0,\xi_0,\lambda,\mu}} \|g\|_{C^\beta_*} + \|S_\lambda f\|_{C^{\alpha'}_*} \|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} g\|_{H^{\beta'}}.$$

In the case $\alpha = \beta = 0$, we have

$$\|w_{x_0,\lambda} S_\mu R(f, g)\|_{L^2} \lesssim \|f\|_{LS^0_{x_0,\xi_0,\lambda,\mu}} \|g\|_{C^{0+}_*} + \|S_\lambda f\|_{C^{\alpha'}_*} \|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} g\|_{H^{\beta'}}.$$

Proof. For simplicity, we set $\alpha = \beta = \alpha' = \beta' = 0$ and accordingly use L^∞ in place of C^α_*; the generalization is easily obtained. First write

$$R(f, g) = R(S_{\leq c\lambda} f, S_{\leq c\lambda} g) + R(S_{c\lambda \leq \cdot \leq \lambda/c} f, S_{c\lambda \leq \cdot \leq \lambda/c} g) + R(S_{\geq \lambda/c} f, S_{\geq \lambda/c} g). \tag{11.5}$$

From the first term of (11.5), we have

$$w_{x_0,\lambda} S_\mu R(S_{\leq c\lambda} f, S_{\leq c\lambda} g) = w_{x_0,\lambda} S_\mu R(S_{c\mu \leq \cdot \leq c\lambda} f, S_{c\mu \leq \cdot \leq c\lambda} g).$$

A term of this sum takes the form

$$w_{x_0,\lambda} S_\mu ((S_\kappa f)(S_\kappa g))$$

with $\kappa \in [c\mu, c\lambda]$. Observe that

$$\|w_{x_0,\lambda} S_\mu ((S_\kappa f)(S_\kappa g))\|_{L^2} \lesssim \|w_{x_0,\kappa} S_\mu ((S_\kappa f)(S_\kappa g))\|_{L^2}.$$

We commute

$$\begin{aligned}\|[w_{x_0,\kappa}, S_\mu]((S_\kappa f)(S_\kappa g))\|_{L^2} &= \|[w_{x_0,\kappa}, S_\mu] \tilde{S}_\mu ((S_\kappa f)(S_\kappa g))\|_{L^2} \\ &\lesssim \kappa^{\frac{3}{4}} \mu^{-1} \|\tilde{S}_\mu ((S_\kappa f)(S_\kappa g))\|_{L^2} \\ &\lesssim \kappa^{\frac{3}{4}} \mu^{-1} \|S_\kappa f\|_{L^2} \|g\|_{L^\infty}.\end{aligned}$$

Thus it remains to consider

$$S_\mu w_{x_0,\kappa} ((S_\kappa f)(S_\kappa g))$$

which we estimate

$$\|S_\mu w_{x_0,\kappa} ((S_\kappa f)(S_\kappa g))\|_{L^2} \lesssim \|w_{x_0,\kappa} S_\kappa f\|_{L^2} \|g\|_{L^\infty}.$$

For the latter two terms of (11.5), we commute

$$\begin{aligned}\|[w_{x_0,\lambda}, S_\mu] R(S_{\geq c\lambda} f, S_{\geq c\lambda} g)\|_{L^2} &= \|[w_{x_0,\lambda}, S_\mu]\tilde{S}_\mu R(S_{\geq c\lambda} f, S_{\geq c\lambda} g)\|_{L^2} \\ &\lesssim \lambda^{\frac{3}{4}}\mu^{-1}\|\tilde{S}_\mu R(S_{\geq c\lambda} f, S_{\geq c\lambda} g)\|_{L^2} \\ &\lesssim \lambda^{\frac{3}{4}}\mu^{-1}\|S_{\geq c\lambda} f\|_{L^2}\|S_{\geq c\lambda} g\|_{C_*^{0+}}\end{aligned}$$

so it remains to consider

$$S_\mu w_{x_0,\lambda}(R(S_{c\lambda\leq\cdot\leq\lambda/c} f, S_{c\lambda\leq\cdot\leq\lambda/c} g) + R(S_{\geq\lambda/c} f, S_{\geq\lambda/c} g)). \tag{11.6}$$

Consider a term of the latter sum in (11.6), which takes the form

$$S_\mu w_{x_0,\lambda}((S_\kappa f)(S_\kappa g))$$

with $\kappa \geq \lambda/c$. We easily estimate

$$\|S_\mu w_{x_0,\lambda}((S_\kappa f)(S_\kappa g))\|_{L^2} \lesssim \|w_{x_0,\lambda} S_\kappa f\|_{L^2}\|g\|_{L^\infty}.$$

It remains to consider (an absolute number of) terms of the former sum in (11.6),

$$S_\mu w_{x_0,\lambda}((S_\lambda f)(S_\lambda g)).$$

First recall that $w_{x_0,\lambda}$ has frequency support below $\lambda^{\frac{3}{4}} \ll \mu$ and so essentially does not disturb the frequency pieces of $S_\lambda f$ and $S_\lambda g$ at the $c\mu$-scale. Then observe that the outermost S_μ eliminates $c\mu$-width frequency pieces of $S_\lambda f$ and $S_\lambda g$ that are not separated in absolute value by at least $\mu/8$.

Thus, write

$$\psi_\lambda(\xi) = p(\xi - \xi_0) + p(-\xi + \xi_0) + q(\xi - \xi_0) + q(-\xi + \xi_0)$$

where p was constructed in the definition of $S_{\xi_0,\lambda,\mu}$ and q has support $\{|\xi| \leq c\mu\}$. Using the previous observation, we may write

$$\begin{aligned}S_\mu w_{x_0,\lambda}((S_\lambda f)(S_\lambda g)) = S_\mu w_{x_0,\lambda}(&(S_{\xi_0,\lambda,\mu} f)(S_\lambda g) \\ &+ ((q(D - \xi_0) + q(-D + \xi_0))f)(S_{\xi_0,\lambda,\mu} g)).\end{aligned}$$

Then

$$\|S_\mu w_{x_0,\lambda}(S_\lambda f)(S_\lambda g)\|_{L^2} \lesssim \|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} f\|_{L^2}\|S_\lambda g\|_{L^\infty} + \|S_\lambda f\|_{L^\infty}\|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu} g\|_{L^2}.$$

□

In the special case where at least one of f, g is truncated to low frequencies $\{|\xi| \leq c\lambda\}$, we see from the proof of Proposition 11.9 that we can use instead the simpler local seminorm,

$$\|f\|_{LS^\sigma_{x_0,\lambda}} = \sum_{\kappa\leq c\lambda} \|w_{x_0,\kappa} S_\kappa f\|_{H^\sigma}.$$

Corollary 11.10. *Let $\alpha + \beta \geq 0$, and $f = S_{\leq c\lambda} f$ or $g = S_{\leq c\lambda} g$. Then*

$$\|w_{x_0,\lambda} S_\mu R(f, g)\|_{H^{\alpha+\beta}} \lesssim (\|f\|_{LS^\alpha_{x_0,\lambda}} + \lambda^{\frac{3}{4}}\mu^{-1}\|S_{\leq c\lambda} f\|_{H^\alpha})\|g\|_{C_*^\beta}.$$

Lastly, using Proposition 11.7, we observe that we estimate (η, ψ, V, B) in the full local seminorm:

Corollary 11.11. *Let $0 \le \sigma \le s$. Let (x^t, ξ^t) be a solution to (10.1) with initial data (x_0, ξ_0) satisfying $\xi_0 \in [\lambda/2, 2\lambda]$. Then*

$$\lambda^{-\sigma}\mu^{s'}\|(\eta,\psi)\|_{L^2(I;LS^{\sigma+\frac{1}{2}}_{x^t,\xi^t,\lambda,\mu})} + \lambda^{-\sigma}\mu^{s'}\|(V,B)\|_{L^2(I;LS^{\sigma}_{x^t,\xi^t,\lambda,\mu})} \le \mathcal{F}(T).$$

Proof. We consider V; the other terms are similar.

First consider the terms of $LS^{\sigma}_{x^t,\xi^t,\lambda,\mu}$ that do not have local weights. We have

$$\lambda^{-\sigma}\kappa^{\frac{3}{4}}\mu^{-1}\|S_\kappa V\|_{H^\sigma} \lesssim \kappa^{\frac{3}{4}-s}\mu^{-1}\|S_\kappa V\|_{H^s} \le \kappa^{-\frac{1}{8}}\mu^{-s'}\mathcal{F}(T).$$

Summing geometrically with respect to κ yields the desired estimate. Similarly,

$$\lambda^{-\sigma}\lambda^{\frac{3}{4}}\mu^{-1}\|S_{\ge c\lambda}f\|_{H^\sigma} \lesssim \lambda^{\frac{3}{4}-s}\mu^{-1}\|S_{\ge c\lambda}f\|_{H^s} \le \lambda^{-\frac{1}{8}}\mu^{-s'}\mathcal{F}(T)$$

which is better than needed.

It remains to consider the three terms of $LS^{\sigma}_{x^t,\xi^t,\lambda,\mu}$ with local weights. First consider the low frequency sum, using case (ii) of Proposition 11.7:

$$\lambda^{-\sigma}\sum_{\kappa\in[c\mu,c\lambda]}\|w_{x^t,\kappa}S_\kappa V\|_{L^2(I;H^\sigma)} \le \sum_{\kappa\in[c\mu,c\lambda]}\kappa^{-s-\frac{1}{8}}\mathcal{F}(T) \le \mu^{-s'}\mathcal{F}(T).$$

For the high frequency sum, use case (iii):

$$\lambda^{-\sigma}\sum_{\kappa\ge\lambda/c}\|w_{x^t,\lambda}S_\kappa V\|_{L^2(I;H^\sigma)} \le \sum_{\kappa\ge\lambda/c}\kappa^{-(s-\sigma)}\lambda^{-\sigma-\frac{1}{8}}\mathcal{F}(T) \le \lambda^{-s'}\mathcal{F}(T)$$

which is better than needed.

Lastly, for the λ-frequency term, use case (i):

$$\lambda^{-\sigma}\|w_{x^t,\lambda}S_{\xi^t,\lambda,\mu}V\|_{L^2(I;H^\sigma)} \lesssim \lambda^{-s}\mu^{-\frac{3}{2}}\lambda^{\frac{11}{8}}\mathcal{F}(T) \le \mu^{-s'}\mathcal{F}(T).$$

□

12. Integration Along the Hamilton Flow

To motivate the results of this section, recall the symbol of the operator of our evolution is

$$H(t,x,\xi) = V_\lambda\xi + \sqrt{a_\lambda|\xi|}$$

with associated Hamilton equations

$$\begin{cases}\dot{x} = H_\xi(t,x,\xi) = V_\lambda + \frac{1}{2}\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}\xi\\ \dot{\xi} = -H_x(t,x,\xi) = -\partial_x V_\lambda\xi - \partial_x\sqrt{a_\lambda}|\xi|^{\frac{1}{2}}.\end{cases}$$

To construct a useful wave packet parametrix for the evolution $D_t + H(t,x,D)$, we require the Hamilton flow of H to be bilipschitz. In turn, this relies on the regularity of $\partial_x^2 H$. To circumvent this in our low regularity setting, we will write $\partial_x^2 H$ as a derivative of some $F = F(t,x,\xi)$ along the flow $(x(t),\xi(t))$:

$$\partial_x^2 H \approx (\partial_t + \dot{x}\partial_x + \dot{\xi}\partial_\xi)F \tag{12.1}$$

in a sense to be made precise.

By a straightforward computation,

$$\partial_x^2 H = \partial_x^2 V_\lambda\xi + \frac{1}{2}\left(-\frac{1}{2}\frac{(\partial_x a_\lambda)^2}{\sqrt{a_\lambda^3}} + \frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}}\right)|\xi|^{\frac{1}{2}}.$$

Then (12.1) becomes (omitting "lower order terms" from both sides)

$$\partial_x^2 V_\lambda \xi + \frac{1}{2}\frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}}|\xi|^{\frac{1}{2}} \approx \left(\partial_t + V_\lambda \partial_x + \frac{1}{2}\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}\xi\partial_x\right) F. \tag{12.2}$$

We will see below that in fact, for $F_1 = F_1(t,x)$ to be determined,

$$\partial_x^2 V_\lambda \xi \approx (\partial_t + V_\lambda \partial_x) F_1 \xi, \quad \frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}}|\xi|^{\frac{1}{2}} \approx (\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}\xi\partial_x) F_1 \xi \tag{12.3}$$

in a sense to be made precise.

12.1. **Vector field identities.** First we recall identities involving the vector field $L = \partial_t + V \cdot \nabla$.

Recall that the traces of the velocity field on the surface (V, B) can be expressed directly in terms of η, ψ via the formulas (2.2). Further, as a simple consequence of these formulas and the first equation of the system (2.1),

$$L\eta = B. \tag{12.4}$$

Also recall from [ABZ14a, Propositions 4.3, 4.5],

$$\begin{cases} LB = a - g, \\ LV = -a\nabla\eta, \\ L\nabla\eta = G(\eta)V + \nabla\eta G(\eta)B + \Gamma_x + \nabla\eta\Gamma_y, \end{cases} \tag{12.5}$$

$$G(\eta)B = -\nabla \cdot V - \Gamma_y. \tag{12.6}$$

Here, Γ_x, Γ_y arise only in the case of finite bottom, and is described with estimates in Appendix D.2. In the remainder of the section, we will consider the case of infinite bottom, so that

$$\Gamma_x = \Gamma_y = 0.$$

This assumption is only for convenience; the appropriate local estimates on Γ_x, Γ_y may be obtained by using elliptic arguments similar to those in Appendix C.

12.2. **Integrating the vector field.** Observe that two of the identities from (12.5) and (12.6) imply

$$L\nabla\eta = (G(\eta) - (\nabla\eta)\mathrm{div})V, \tag{12.7}$$

which is the basis for our integration of $\partial_x^2 V_\lambda$ below. Recall we write Λ for the principal symbol of the Dirichlet to Neumann map.

Proposition 12.1. *We have*

$$\partial_x^2 V_\lambda = (\partial_t + V_\lambda \cdot \nabla) T_{q^{-1}} \partial_x^2 \nabla\eta_\lambda + G_V \tag{12.8}$$

where $q = \Lambda - i\nabla\eta\xi$ is a symbol of order 1, *and G_V satisfies*

$$\|G_V\|_{L^\infty} \leq \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t)^2,$$
$$\|w_{x_0,\lambda} S_\mu G_V\|_{H^{\frac{1}{2}}} \leq \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t)(Z(t) + \mathcal{L}(t, x_0, \xi_0, \lambda)).$$

Proof. Starting from (12.7), we incrementally paralinearize, apply the symbolic calculus, frequency truncate, and commute. First, we observe that

$$\|w_{x_0,\lambda} S_\mu f\|_{H^\sigma} \lesssim \|w_{x_0,\mu} S_\mu f\|_{H^\sigma}$$

so we may exchange $w_{x_0,\lambda}$ with $w_{x_0,\mu}$ when necessary.

Step 1. Paralinearization. Paralinearize the terms

$$\begin{aligned} G(\eta)V &= T_\Lambda V + R(\eta)V, \\ (\nabla\eta)\nabla\cdot V &= T_{\nabla\eta}\nabla\cdot V + T_{\nabla\cdot V}\nabla\eta + R(\nabla\eta, \nabla\cdot V) \end{aligned}$$

of (12.7). Then rearranging,

$$T_q V = (T_\Lambda - T_{\nabla\eta}\mathrm{div})V = L\nabla\eta - R(\eta)V + T_{\nabla\cdot V}\nabla\eta + R(\nabla\eta, \nabla\cdot V).$$

We estimate the error terms on the right hand side. By (D.3) (appropriately sharpened for the ϵ gain), (E.1), and (E.6) respectively,

$$\begin{aligned} \|R(\eta)V\|_{W^{\frac12+,\infty}} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac12}}, \|V\|_{H^s})(1 + \|\eta\|_{W^{r+\frac12,\infty}})(1 + \|\eta\|_{W^{r+\frac12,\infty}} + \|V\|_{W^{r,\infty}}) \\ \|T_{\nabla\cdot V}\nabla\eta\|_{W^{\frac12+,\infty}} &\lesssim \|V\|_{W^{r,\infty}}\|\eta\|_{W^{r+\frac12,\infty}} \\ \|R(\nabla\eta, \nabla\cdot V)\|_{W^{\frac12+,\infty}} &\lesssim \|\eta\|_{W^{r+\frac12,\infty}}\|V\|_{W^{r,\infty}}. \end{aligned}$$

For the local Sobolev estimates, by Proposition B.14,

$$\|w_{x_0,\lambda} S_\mu R(\eta)V\|_{H^{s'-\frac12}} \leq \mathcal{F}(M(t))Z(t)(\lambda^{\frac12}\mu^{-\frac12} + Z(t) + \lambda^{\frac12}\mu^{-\frac12}\mathcal{L}(t, x_0, \xi_0, \lambda)).$$

By Proposition B.1,

$$\|w_{x_0,\mu} S_\mu T_{\nabla\cdot V}\nabla\eta\|_{H^{s'-\frac12}} \lesssim \|\nabla\cdot V\|_{L^\infty}(\|w_{x_0,\mu} S_\mu \nabla\eta\|_{H^{s'-\frac12}} + \|\nabla\eta\|_{H^{s-\frac12}}).$$

Lastly, by Proposition 11.9,

$$\begin{aligned} &\mu^{s'-\frac12}\|w_{x_0,\lambda} S_\mu R(\nabla\eta, \nabla\cdot V)\|_{L^2} \\ &\qquad \lesssim \mu^{s'-\frac12}\|\nabla\eta\|_{LS^0_{x_0,\xi_0,\lambda,\mu}}\|\nabla\cdot V\|_{C^{0+}_*} + \|S_\lambda\nabla\eta\|_{C^{\frac12}_*}\mu^{s'-\frac12}\|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}\nabla\cdot V\|_{H^{-\frac12}} \\ &\qquad \lesssim Z(t)\lambda^{\frac12}\mu^{-\frac12}\mathcal{L}(t, x_0, \xi_0, \lambda). \end{aligned}$$

We conclude

$$T_q V = L\nabla\eta + G_1$$

where

$$\begin{aligned} \|G_1\|_{W^{\frac12+,\infty}} &\leq \mathcal{F}(M(t))Z(t)(1 + \|\eta\|_{W^{r+\frac12,\infty}}), \\ \|w_{x_0,\lambda} S_\mu G_1\|_{H^{s'-\frac12}} &\leq \mathcal{F}(M(t))Z(t)(\lambda^{\frac12}\mu^{-\frac12} + Z(t) + \lambda^{\frac12}\mu^{-\frac12}\mathcal{L}(t, x_0, \xi_0, \lambda)). \end{aligned}$$

Following the above proof of the local Sobolev estimate, but using the standard Sobolev counterparts to all of the local Sobolev estimates, one also obtains the (global) Sobolev counterpart,

$$\|G_1\|_{H^{s-\frac12}} \leq \mathcal{F}(M(t))Z(t).$$

Step 2. Inversion of q. We write

$$V = T_{q^{-1}}L\nabla\eta + T_{q^{-1}}G_1 + (1 - T_{q^{-1}}T_q)V.$$

For the first error term $T_{q^{-1}}G_1$, q^{-1} is a symbol of order -1 and is a smooth function of $\nabla\eta$, so that by (E.15) and Sobolev embedding with $s - \frac{1}{2} > \frac{d}{2}$,

$$M_0^{-1}(q^{-1}) \le \mathcal{F}(\|\nabla\eta\|_{L^\infty}) \le \mathcal{F}(\|\nabla\eta\|_{H^{s-\frac{1}{2}}}) \tag{12.9}$$

and similarly

$$M_{\frac{1}{2}}^{-1}(q^{-1}) \le \mathcal{F}(\|\nabla\eta\|_{H^{s-\frac{1}{2}}})\|\nabla\eta\|_{C_*^{\frac{1}{2}}}.$$

Then by (E.1) and the estimate on G_1 in the previous step,

$$\|T_{q^{-1}}G_1\|_{W^{\frac{3}{2}+,\infty}} \lesssim M_0^{-1}(q^{-1})\|G_1\|_{W^{\frac{1}{2}+,\infty}} \le \mathcal{F}(M(t))Z(t)(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}).$$

For the local Sobolev counterpart, use instead Proposition B.3 with Step 1:

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu T_{q^{-1}}G_1\|_{H^{s'+\frac{1}{2}}} &\lesssim M_0^{-1}(q^{-1})(\|w_{x_0,\lambda}S_\mu G_1\|_{H^{s'-\frac{1}{2}}} + \lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|G_1\|_{H^{s'+\frac{1}{2}-1-\frac{1}{8}}})\\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|w_{x_0,\lambda}S_\mu G_1\|_{H^{s'-\frac{1}{2}}} + \lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|G_1\|_{H^{s-\frac{1}{2}}})\\ &\le \mathcal{F}(M(t))Z(t)(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + Z(t) + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\mathcal{L}(t,x_0,\xi_0,\lambda)).\end{aligned}$$

Using (E.3), we control the second error term by

$$\begin{aligned}\|(1 - T_{q^{-1}}T_q)V\|_{W^{\frac{3}{2}+,\infty}} &\lesssim \left(M_{\frac{1}{2}}^{-1}(q^{-1})M_0^1(q) + M_0^{-1}(q^{-1})M_{\frac{1}{2}}^1(q)\right)\|V\|_{W^{r,\infty}}\\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{W^{r+\frac{1}{2},\infty}}\|V\|_{W^{1,\infty}}.\end{aligned}$$

Likewise, using instead Proposition B.3 for the local Sobolev estimate,

$$\begin{aligned}\|w_{x_0,\mu}S_\mu(1 - T_{q^{-1}}T_q)V\|_{H^{s'+\frac{1}{2}}} &\lesssim \left(M_{\frac{1}{2}}^{-1}(q^{-1})M_0^1(q) + M_0^{-1}(q^{-1})M_{\frac{1}{2}}^1(q)\right)\\ &\quad\cdot(\|w_{x_0,\mu}S_\mu V\|_{H^{s'}} + \|\tilde{S}_\mu V\|_{H^s})\\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|V\|_{H^s})\|\eta\|_{W^{r+\frac{1}{2},\infty}}\|w_{x_0,\mu}S_\mu V\|_{H^{s'}}.\end{aligned}$$

We conclude

$$V = T_{q^{-1}}L\nabla\eta + G_2$$

with

$$\begin{aligned}\|G_2\|_{W^{\frac{3}{2}+,\infty}} &\le \mathcal{F}(M(t))Z(t)(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}),\\ \|w_{x_0,\lambda}S_\mu G_2\|_{H^{s'+\frac{1}{2}}} &\le \mathcal{F}(M(t))Z(t)(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + Z(t) + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\mathcal{L}(t,x_0,\xi_0,\lambda)).\end{aligned}$$

As before, it is easy to follow the local Sobolev proof to show additionally

$$\|G_2\|_{H^{s+\frac{1}{2}}} \le \mathcal{F}(M(t))Z(t).$$

Step 3. Frequency truncation and differentiation. Applying $\partial_x^2 S_{\le c_1\lambda}$ to both sides of our identity, we have

$$\partial_x^2 V_\lambda = T_{q^{-1}}\partial_x^2 S_{\le c_1\lambda}L\nabla\eta + \partial_x^2 S_{\le c_1\lambda}G_2 + [\partial_x^2 S_{\le c_1\lambda}, T_{q^{-1}}]L\nabla\eta.$$

For the first error term, we have

$$\begin{aligned}\|\partial_x^2 S_{\le c_1\lambda}G_2\|_{L^\infty} &\lesssim \lambda^{\frac{1}{2}-}\|G_2\|_{W^{\frac{3}{2}+,\infty}} \lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t)^2,\\ \|w_{x_0,\lambda}S_\mu\partial_x^2 G_2\|_{H^{\frac{1}{2}}} &\lesssim \mu^{\frac{1}{2}-}\|w_{x_0,\lambda}S_\mu G_2\|_{H^{s'+\frac{1}{2}}} \lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t)(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)).\end{aligned}$$

To estimate the second error term, first rewrite, using the definition of the paradifferential calculus,

$$[\partial_x^2 S_{\leq c_1\lambda}, T_{q^{-1}}] S_{\leq c\lambda} L \nabla\eta.$$

Using (12.4), write

$$L\nabla\eta = \nabla L\eta - \nabla V \cdot \nabla\eta = \nabla B - \nabla V \cdot \nabla\eta. \tag{12.10}$$

Then by (E.3) and (12.9),

$$\begin{aligned}
\|[\partial_x^2 S_{\leq c_1\lambda}, T_{q^{-1}}] S_{\leq c\lambda} L\nabla\eta\|_{L^\infty} &\lesssim \lambda^{\frac{1}{2}-} \left(M_{\frac{1}{2}}^{-1}(q^{-1}) + M_0^{-1}(q^{-1})\right) \|\nabla B - \nabla V \cdot \nabla\eta\|_{C_*^r} \\
&\lesssim \lambda^{\frac{1}{2}-} \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{W^{r+\frac{1}{2},\infty}} \left(\|B\|_{C_*^r} + \|\nabla\eta\|_{L^\infty} \|\partial V\|_{C_*^r}\right) \\
&\leq \lambda^{\frac{1}{2}-} \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{W^{r+\frac{1}{2},\infty}} \|(V,B)\|_{C_*^r}.
\end{aligned}$$

For the local Sobolev counterpart, use instead Proposition B.3:

$$\begin{aligned}
&\|w_{x_0,\mu} S_\mu [\partial_x^2 S_{\leq c_1\lambda}, T_{q^{-1}}] L\nabla\eta\|_{H^{\frac{1}{2}}} \\
&\qquad \lesssim \lambda^{\frac{1}{2}-} \left(M_{\frac{1}{2}}^{-1}(q^{-1}) + M_0^{-1}(q^{-1})\right) \\
&\qquad\qquad \cdot (\|w_{x_0,\mu} S_\mu(\nabla B - \nabla V \cdot \nabla\eta)\|_{H^{s'-1}} + \|\tilde{S}_\mu(\nabla B - \nabla V \cdot \nabla\eta)\|_{H^{s-1}}) \\
&\qquad \lesssim \lambda^{\frac{1}{2}-} \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{W^{r+\frac{1}{2},\infty}} \\
&\qquad\qquad \cdot (\mathcal{L}(t,x_0,\xi_0,\lambda) + \|w_{x_0,\mu} S_\mu(\nabla V \cdot \nabla\eta)\|_{H^{s'-1}} + M(t)).
\end{aligned}$$

For the remaining middle term, using Corollary B.2,

$$\begin{aligned}
\|w_{x_0,\mu} S_\mu(\nabla V \cdot \nabla\eta)\|_{H^{s'-1}} &\lesssim (\|w_{x_0,\mu} \tilde{S}_\mu \nabla V\|_{H^{s'-1}} + \|\nabla V\|_{H^{s-1}}) \|\nabla\eta\|_{L^\infty} \\
&\quad + \|\nabla V\|_{L^\infty} (\|w_{x_0,\mu} \tilde{S}_\mu \nabla\eta\|_{H^{s'-1}} + \|\nabla\eta\|_{H^{s-1}}) \\
&\quad + \|\nabla\eta\|_{C_*^{\frac{1}{8}}} \|\nabla V\|_{H^{s-1}} \\
&\leq \mathcal{F}(M(t))(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)).
\end{aligned}$$

We conclude

$$\partial_x^2 V_\lambda = T_{q^{-1}} \partial_x^2 S_{\leq c_1\lambda} L\nabla\eta + G_3$$

with

$$\begin{aligned}
\|G_3\|_{L^\infty} &\leq \lambda^{\frac{1}{2}-} \mathcal{F}(M(t)) Z(t)^2, \\
\|w_{x_0,\lambda} S_\mu G_3\|_{H^{\frac{1}{2}}} &\leq \lambda^{\frac{1}{2}-} \mathcal{F}(M(t)) Z(t)(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda))
\end{aligned}$$

Step 4. Vector field paralinearization. We paralinearize the vector field in order to commute past it in the next step. Writing the paraproduct expansion

$$(V \cdot \nabla)\nabla\eta = (T_V \cdot \nabla)\nabla\eta + T_{\nabla(\nabla\eta)} \cdot V + R(V, \nabla(\nabla\eta)),$$

we have

$$\partial_x^2 V_\lambda = T_{q^{-1}} \partial_x^2 S_{\leq c_1\lambda}(\partial_t + T_V \cdot \nabla)\nabla\eta + G_3 + T_{q^{-1}} \partial_x^2 S_{\leq c_1\lambda}(R(V, \nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)} \cdot V).$$

Then by (E.1), (E.6), and (E.9),

$$\begin{aligned}\|T_{q^{-1}}\partial_x^2 S_{\le c_1\lambda}(R(V,\nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)}\cdot V)\|_{L^\infty} \\ &\lesssim M_0^1(q^{-1}\xi^2)\|S_{\le c_1\lambda}(R(V,\nabla(\nabla\eta)) + T_{\nabla(\nabla\eta)}\cdot V)\|_{W^{1,\infty}} \\ &\lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\nabla\eta\|_{L^\infty}\|V\|_{W^{r,\infty}}\|\nabla^2\eta\|_{C_*^{-\frac{1}{2}}} \\ &\lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{H^{s+\frac{1}{2}}}\|V\|_{W^{r,\infty}}\|\eta\|_{W^{\frac{3}{2},\infty}}.\end{aligned}$$

For the local Sobolev estimate, first consider $R(V,\nabla(\nabla\eta))$. By Proposition B.3,

$$\begin{aligned}&\|w_{x_0,\lambda}S_\mu T_{q^{-1}}\partial_x^2 R(V,\nabla(\nabla\eta))\|_{H^{\frac{1}{2}}} \\ &\quad\lesssim M_0^{-1}(q^{-1})(\|w_{x_0,\lambda}S_\mu\partial_x^2 R(V,\nabla(\nabla\eta))\|_{H^{\frac{1}{2}-1}} + \lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|\tilde{S}_\mu\partial_x^2 R(V,\nabla(\nabla\eta))\|_{H^{\frac{1}{2}-1-\frac{1}{8}}}) \\ &\quad\le \mathcal{F}(M(t))(\|w_{x_0,\lambda}S_\mu\partial_x^2 R(V,\nabla(\nabla\eta))\|_{H^{-\frac{1}{2}}} + \lambda^{\frac{3}{8}}\|\tilde{S}_\mu R(V,\nabla(\nabla\eta))\|_{H^1}) \\ &\quad\le \mathcal{F}(M(t))(\|w_{x_0,\lambda}S_\mu\partial_x^2 R(V,\nabla(\nabla\eta))\|_{H^{-\frac{1}{2}}} + \lambda^{\frac{1}{2}-}Z(t)).\end{aligned}$$

For the remaining first term, first note that

$$\begin{aligned}\|(\partial_x w_{x_0,\lambda})S_\mu\partial_x R(V,\nabla(\nabla\eta))\|_{H^{-\frac{1}{2}}} &\lesssim \mu^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\|S_\mu R(V,\nabla(\nabla\eta))\|_{H^1} \\ &\lesssim \lambda^{\frac{3}{8}}\|S_\mu R(V,\nabla(\nabla\eta))\|_{H^1} \\ &\lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t).\end{aligned}$$

We have a similar situation (better than needed) when both ∂_x^2 fall on w, so it remains to consider

$$\|w_{x_0,\lambda}S_\mu R(V,\nabla(\nabla\eta))\|_{H^{\frac{3}{2}}}.$$

Applying Proposition 11.9,

$$\|w_{x_0,\lambda}S_\mu R(V,\nabla(\nabla\eta))\|_{L^2} \lesssim \|V\|_{LS^{\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}\|\nabla^2\eta\|_{C_*^{r-\frac{3}{2}}} + \|V\|_{C_*^r}\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\nabla^2\eta\|_{H^{-1}}$$

and thus

$$\|w_{x_0,\lambda}S_\mu R(V,\nabla(\nabla\eta))\|_{H^{\frac{3}{2}}} \le \mathcal{F}(M(t))Z(t)\lambda^{\frac{1}{2}-}\mathcal{L}(t,x_0,\xi_0,\lambda).$$

For the local Sobolev estimate of $T_{\nabla(\nabla\eta)}\cdot V$, the analysis is similar and easier, using Proposition B.1 in the place of Proposition 11.9.

We conclude that we may replace V with T_V, yielding, for G_4 satisfying the same estimates as G_3,

$$\partial_x^2 V_\lambda = T_{q^{-1}}\partial_x^2 S_{\le c_1\lambda}(\partial_t + T_V\cdot\nabla)\nabla\eta + G_4.$$

Step 5. Vector field commutator estimate. By the definition of the paradifferential calculus, we may freely exchange $T_{q^{-1}}\partial_x^2 S_{\le c_1\lambda}(\partial_t + T_V\cdot\nabla)\nabla\eta$ for

$$T_{q^{-1}}\partial_x^2 S_{\le c_1\lambda}(\partial_t + T_V\cdot\nabla)\nabla S_{\le c\lambda}\eta.$$

In turn, applying Proposition D.11 with $m = 1$, $r = 0$ and $\epsilon = 1$, we may exchange this for

$$LT_{q^{-1}}\partial_x^2\nabla\eta_\lambda$$

with an error bounded in L^∞ by (using that q is a smooth function of $\nabla\eta$, and (12.10))

$$
\begin{aligned}
&M_0^1(q^{-1}\xi^2)\|V\|_{W^{1,\infty}}\|\nabla S_{\leq c\lambda}\eta\|_{B^1_{\infty,1}} + M_0^1(Lq^{-1}\xi^2)\|\nabla S_{\leq c\lambda}\eta\|_{W^{1,\infty}} \\
&\lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|\eta\|_{W^{1,\infty}}\|V\|_{W^{1,\infty}}\|\eta\|_{B^{\frac{3}{2}+}_{\infty,1}} + \|\nabla B - \nabla\eta\cdot\nabla V\|_{L^\infty}\|\eta\|_{W^{\frac{3}{2},\infty}}) \\
&\leq \lambda^{\frac{1}{2}-}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|(V,B)\|_{W^{1,\infty}}\|\eta\|_{W^{r+\frac{1}{2},\infty}} \leq \lambda^{\frac{1}{2}-}\mathcal{F}(M(t))Z(t)(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}).
\end{aligned}
$$

For the local Sobolev estimate, apply instead Proposition B.3 to estimate

$$
\begin{aligned}
\|w_{x_0,\mu}S_\mu[T_{q^{-1}}\partial_x^2 S_{\leq c_1\lambda},\partial_t + T_V\cdot\nabla]\nabla\eta(t)\|_{H^{\frac{1}{2}}} &\\
\lesssim (M_0^1(q^{-1}\xi^2)\|V(t)\|_{W^{r,\infty}} + M_0^1((\partial_t + V\cdot\nabla)q^{-1}\xi^2)) &\\
\cdot(\|w_{x_0,\mu}S_\mu\nabla\eta(t)\|_{H^{\frac{1}{2}+1}} + \|\nabla\eta(t)\|_{H^{\frac{1}{2}+1-\frac{1}{4}}}) &\\
\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})Z(t)(\mu^2\|w_{x_0,\mu}S_\mu\eta(t)\|_{H^{\frac{1}{2}}} + \mu^{\frac{1}{2}-}) &\\
\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})Z(t)\lambda^{\frac{1}{2}-}(\mathcal{L}(t,x_0,\xi_0,\lambda)+1). &
\end{aligned}
$$

Note that we also need to restore V, from T_V. First consider the error arising from the balanced terms. Note that since we have one term truncated to low frequencies, we may use use Corollary 11.10 instead of Proposition 11.9:

$$
\begin{aligned}
\|w_{x_0,\lambda}S_\mu R(V,\nabla T_{q^{-1}}\partial_x^2\nabla\eta_\lambda)\|_{H^{0+}} &\lesssim (\|V\|_{LS^{s'}_{x_0,\lambda}} + \lambda^{\frac{3}{4}}\mu^{-1}\|S_{\leq c\lambda}V\|_{H^{s'}})\|\nabla T_{q^{-1}}\partial_x^2\nabla\eta_\lambda\|_{C_*^{-\frac{3}{2}}} \\
&\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|V\|_{LS^{s'}_{x_0,\lambda}} + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\|V\|_{H^s})\|\eta\|_{C_*^{\frac{3}{2}}} \\
&\leq \mathcal{F}(M(t))Z(t)\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}(\mathcal{L}(t,x_0,\xi_0,\lambda)+1).
\end{aligned}
$$

The high-low terms $T_{\nabla T_{q^{-1}}\partial_x^2\nabla\eta_\lambda}V$ are similarly estimated, using Proposition B.1 in the place of Proposition 11.9.

We conclude, for G_5 satisfying the same estimates as G_4,

$$\partial_x^2 V_\lambda = LT_{q^{-1}}\partial_x^2\nabla\eta_\lambda + G_5.$$

Step 6. Vector field truncation. Lastly, we frequency truncate the vector field, using (E.1):

$$
\begin{aligned}
\|((S_{>c\lambda}V)\cdot\nabla)T_{q^{-1}}\partial_x^2\nabla\eta_\lambda\|_{L^\infty} &\lesssim \|S_{>c\lambda}V\|_{L^\infty}M_0^1(q^{-1}\xi^2)\|\nabla\eta_\lambda\|_{W^{2,\infty}} \\
&\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\lambda^{1-}\|S_{>c\lambda}V\|_{W^{\frac{1}{2},\infty}}\|\nabla\eta_\lambda\|_{W^{r-\frac{1}{2},\infty}} \\
&\lesssim \lambda^{\frac{1}{2}-}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|V\|_{W^{r,\infty}}\|\nabla\eta_\lambda\|_{W^{r-\frac{1}{2},\infty}}.
\end{aligned}
$$

For the local Sobolev counterpart, note that it suffices to consider

$$w_{x_0,\lambda}S_\mu(\tilde{S}_{c_1\lambda}V)\nabla T_{q^{-1}}\partial_x^2\nabla\eta_\lambda,$$

which was essentially estimated at the end of Step 5. □

12.3. **Integrating the dispersive term.** In this subsection we integrate $\partial_x^2 a_\lambda$, the coefficient of the dispersive term in our symbol H.

Proposition 12.2. *We have*

$$\frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}} = \sqrt{a_\lambda}\partial_x T_{q^{-1}}\partial_x^3\eta_\lambda + (\partial_t + V_\lambda\partial_x)F_a + G_a$$

where $q = \Lambda - i\nabla\eta\xi$ is a symbol of order 1, and F_a, G_a satisfy

$$\begin{aligned}
\|F_a\|_{L^2} &\le \lambda^{\frac{5}{8}-}\mathcal{F}(M(t)),\\
\|F_a\|_{H^{\frac{1}{2}}} &\le \lambda^{\frac{5}{8}-}\mathcal{F}(M(t))Z(t),\\
\|w_{x_0,\lambda}S_\mu\sqrt{a_\lambda}\partial_x F_a\|_{H^{\frac{1}{2}}} &\le \lambda^{\frac{3}{2}-}\mathcal{F}(M(t))Z(t)(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)),\\
\|G_a\|_{L^\infty} &\le \lambda^{1-}\mathcal{F}(M(t))Z(t)^2,\\
\|w_{x_0,\lambda}S_\mu G_a\|_{H^{\frac{1}{2}}} &\le \lambda^{1-}\mathcal{F}(M(t))Z(t)(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)).
\end{aligned}$$

Proof. We will focus on the local Sobolev estimates, as the Hölder estimates are easier, using the appropriate Hölder counterparts. Throughout, we use identities from Section 12.1. Beginning with the identity

$$G(\eta)B = -\partial_x V$$

and the paralinearization

$$G(\eta)B = |D|B + R(\eta)B,$$

write

$$B = |D|^{-1}|D|B = |D|^{-1}(G(\eta)B - R(\eta)B) = -|D|^{-1}\partial_x V - |D|^{-1}R(\eta)B$$

Then apply L to both sides, so that using the identities

$$LB = a - g, \quad LV = -a\nabla\eta,$$

we obtain

$$\begin{aligned}
a - g &= -L|D|^{-1}\partial_x V - L|D|^{-1}R(\eta)B\\
&= |D|^{-1}\partial_x(a\nabla\eta) - L|D|^{-1}R(\eta)B - [V, |D|^{-1}\partial_x]\partial_x V.
\end{aligned}$$

Applying $|D|$ to both sides, using the product rule, and applying a paradifferential decomposition,

$$(|D| - T_{\nabla\eta}\partial_x)a = a\partial_x^2\eta + ((\nabla\eta) - T_{\nabla\eta})\partial_x a - |D|L|D|^{-1}R(\eta)B - |D|[V, |D|^{-1}\partial_x]\partial_x V.$$

Observe that on the left hand side, we have $T_q a$. Lastly, we apply $\partial_x^2 S_{\le c_1\lambda}T_{q^{-1}}$ to both sides of the identity, to obtain

$$\partial_x^2 a_\lambda = \partial_x^2 S_{\le c_1\lambda}T_{q^{-1}}a\partial_x^2\eta + E$$

where E denotes error terms,

$$\begin{aligned}
E = \partial_x^2 S_{\le c_1\lambda}T_{q^{-1}}(((\nabla\eta) - T_{\nabla\eta})\partial_x a - |D|L|D|^{-1}R(\eta)B - |D|[V, |D|^{-1}\partial_x]\partial_x V)&\\
+ \partial_x^2 S_{\le c_1\lambda}(T_{q^{-1}}T_q - 1)a.&
\end{aligned} \tag{12.11}$$

Except for some commuting and rearrangement, it remains to estimate the errors E in L^∞ or locally with the weight $w_{x_0,\lambda}S_\mu$ in $H^{\frac{1}{2}}$. As usual, we observe that

$$\|w_{x_0,\lambda}S_\mu f\|_{H^\sigma} \lesssim \|w_{x_0,\mu}S_\mu f\|_{H^\sigma}$$

so we may exchange $w_{x_0,\lambda}$ with $w_{x_0,\mu}$ when necessary.

Step 1. First we consider from (12.11) the paraproduct error with the Taylor coefficient,

$$((\nabla\eta) - T_{\nabla\eta})\partial_x a = T_{\partial_x a}\partial_x\eta + R(\partial_x\eta, \partial_x a).$$

From the first of these terms, we estimate

$$w_{x_0,\lambda}S_\mu\partial_x^2 T_{q^{-1}}T_{\partial_x a}\partial_x\eta$$

using successive application of Propositions B.1 and B.3 to commute the localization under the paraproducts and paradifferential operators. Note that we have a total of 2 derivatives applied to η (using that q^{-1} is of order -1), but we have an additional 1/2 from measuring E in $H^{\frac{1}{2}}$, and an additional $\frac{1}{2}$ from measuring $\partial_x a \in C_*^{-\frac{1}{2}}$. Using that $s' + \frac{1}{2} > 2$ and the estimate

$$\|w_{x_0,\mu} S_\mu \eta\|_{H^3} \lesssim \mu^{1-} \|w_{x_0,\mu} S_\mu \eta\|_{H^{s'+\frac{1}{2}}} \leq \mu^{1-} \mathcal{L}(t, x_0, \xi_0, \lambda),$$

we see that we have a net loss of μ^{1-}, which is better than needed. (Throughout, we will use the fact that we may commute $w_{x_0,\lambda} S_\mu$ under $\partial_x^2 T_{q^{-1}}$ and thus effectively treat it as a single low frequency derivative.)

For the second term, note that the bilinear estimate of Proposition 11.9 is not directly applicable here, because we do not have an estimate on a in $LS^\alpha_{x_0,\xi_0,\lambda,\mu}$. Thus, we instead integrate, writing

$$R(\partial_x \eta, \partial_x a) = R(\partial_x \eta, \partial_x L B) = R(\partial_x \eta, L \partial_x B + (\partial_x V) \partial_x B).$$

The term with $(\partial_x V)\partial_x B$ is balanced and hence easily estimated. Indeed, we have (note we do not use the spatial localization)

$$\begin{aligned}
\|S_\mu \partial_x^2 T_{q^{-1}} R(\partial_x \eta, (\partial_x V)\partial_x B)\|_{H^{\frac{1}{2}}} &\leq \mu^{1-} \mathcal{F}(M(t)) \|S_\mu R(\partial_x \eta, (\partial_x V)\partial_x B)\|_{H^{\frac{1}{2}+}} \\
&\leq \mu^{1-} \mathcal{F}(M(t)) \|\partial_x \eta\|_{H^{s-\frac{1}{2}}} \|\partial_x V\|_{L^\infty} \|\partial_x B\|_{L^\infty}.
\end{aligned}$$

It thus remains to consider

$$R(\partial_x \eta, L\partial_x B) = \sum_\kappa (S_\kappa \partial_x \eta)(S_\kappa L \partial_x B) = L R(\partial_x \eta, \partial_x B) + \sum_\kappa [(S_\kappa \partial_x \eta) S_\kappa, L] \partial_x B.$$

Consider the commutator. It is easy to commute S_κ with L as discussed above, as this balances the derivatives onto the vector field V of L. Thus we consider

$$\sum_\kappa [(S_\kappa \partial_x \eta), L] S_\kappa \partial_x B = -\sum_\kappa (L S_\kappa \partial_x \eta)(S_\kappa \partial_x B).$$

Again, we may commute L with a single derivative, so that using $L\eta = B$, we may exchange this for

$$-\sum_\kappa (S_\kappa \partial_x B)(S_\kappa \partial_x B) = -R(\partial_x B, \partial_x B)$$

which we may estimate using Proposition B.3 to commute with the paradifferential operator and Proposition 11.9 to handle the $R(\cdot,\cdot)$ term:

$$\begin{aligned}
\|w_{x_0,\lambda} S_\mu \partial_x^2 T_{q^{-1}} R(\partial_x B, \partial_x B)\|_{H^{\frac{1}{2}}} &\leq \mathcal{F}(M(t)) (\|w_{x_0,\lambda} S_\mu R(\partial_x B, \partial_x B)\|_{H^{\frac{1}{2}+1}} \\
&\quad + \lambda^{\frac{3}{8}} \mu^{-\frac{3}{8}} \|S_\mu R(\partial_x B, \partial_x B)\|_{H^{\frac{1}{2}+1-\frac{1}{8}}}) \\
&\leq \mathcal{F}(M(t)) (\mu^{\frac{3}{2}} \|w_{x_0,\lambda} S_\mu R(\partial_x B, \partial_x B)\|_{L^2} + \lambda^{\frac{3}{8}} \mu^{\frac{5}{8}-} Z(t)) \\
&\leq \mathcal{F}(M(t)) (\mu^{\frac{3}{2}} \|B\|_{C_*^r} \|B\|_{LS^1_{x_0,\xi_0,\lambda,\mu}} + \lambda^{1-} Z(t)) \\
&\leq \lambda^{1-} \mathcal{F}(M(t)) Z(t) (\mathcal{L}(t, x_0, \xi_0, \lambda) + 1).
\end{aligned}$$

We conclude that the first error term from (12.11) satisfies the appropriate estimates, except for a term

$$LR(\partial_x \eta, \partial_x B),$$

which we consider later as part of F_a.

Step 2. Next, consider the second error term from (12.11),

$$|D|L|D|^{-1}R(\eta)B.$$

As discussed in Step 1, we may easily commute L with a single derivative, so we are left with a term

$$LR(\eta)B$$

which we also consider later as part of F_a.

Step 3. For the third error term from (12.11),

$$|D|[V,|D|^{-1}\partial_x]\partial_x V,$$

we reduce to paraproducts. First, we have by Proposition B.3,

$$\begin{aligned}\|w_{x_0,\mu}S_\mu[T_V,|D|^{-1}\partial_x]\partial_x V\|_{H^{\frac{3}{2}+}} &\lesssim \|V\|_{C^1}(\|w_{x_0,\mu}S_\mu V\|_{H^{\frac{3}{2}+}} + \|V\|_{H^{\frac{3}{2}-\frac{1}{4}+}}) \\ &\lesssim \|V\|_{C^1}(\|V\|_{LS^{s'}_{x_0,\lambda}} + \|V\|_{H^s})\end{aligned}$$

which suffices. The other low-high terms are similarly estimated using Proposition B.3. On the other hand, a typical balanced-frequency term may be estimated by Proposition 11.9,

$$\|w_{x_0,\lambda}S_\mu R(V,|D|^{-1}\partial_x^2 V)\|_{H^{\frac{3}{2}+}} \lesssim \mu^{s'}\|V\|_{C^r}\|w_{x_0,\lambda}S_\mu V\|_{LS^0_{x_0,\xi_0,\lambda,\mu}}.$$

Step 4. Lastly, we estimate the fourth error term from (12.11), using the commutator estimate of Proposition B.3 with the local Taylor coefficient estimate of Corollary C.6,

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu(T_{q^{-1}}T_q - 1)a\|_{H^{\frac{5}{2}}} &\lesssim (M^1_{\frac{1}{2}}(q)M^{-1}_0(q^{-1}) + M^1_0(q)M^{-1}_{\frac{1}{2}}(q^{-1})) \\ &\quad \cdot (\|w_{x_0,\lambda}S_\mu a\|_{H^2} + \lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|\tilde{S}_\mu a\|_{H^{2-\frac{1}{8}}}) \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})Z(t)(\mu^{1-}\|w_{x_0,\lambda}S_\mu a\|_{H^{s'-\frac{1}{2}}} + \lambda^{\frac{3}{8}}\mu^{\frac{5}{8}-}\|\tilde{S}_\mu a\|_{H^{s-\frac{1}{2}}}) \\ &\leq \mathcal{F}(M(t))Z(t)\lambda^{1-}(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)).\end{aligned}$$

We conclude that

$$\partial_x^2 a_\lambda = \partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}a\partial_x^2\eta + \partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}L(R(\partial_x\eta,\partial_x B) + R(\eta)B) + G_1 \tag{12.12}$$

where G_1 satisfies the desired estimate.

Step 5. In the first term on the right hand side of (12.12), we need to commute the Taylor coefficient a to the front. To do so, we check below that we may exchange a for T_a, after which we can commute and apply the commutator estimate of Proposition B.3 as usual. Having done so, we may also commute the paradifferential operators to the desired arrangement,

$$T_a\partial_x T_{q^{-1}}\partial_x^3\eta_\lambda.$$

To estimate

$$\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}(a - T_a)\partial_x^2\eta,$$

we apply the same steps as in Step 1, ending instead with the integrated term

$$LR(B,\partial_x^2\eta).$$

After commuting the paradifferential operators to the desired arrangement above, it remains to estimate

$$(a_\lambda - T_a)\partial_x T_{q^{-1}}\partial_x^3\eta_\lambda = (a_\lambda - T_{a_\lambda})\partial_x T_{q^{-1}}\partial_x^3\eta_\lambda.$$

The low-high term uses Proposition B.1 as usual, while for the balanced term

$$R(a_\lambda, \partial_x T_{q^{-1}}\partial_x^3\eta_\lambda),$$

we apply Corollary 11.10 since both factors are frequency truncated. Finally, multiplying both sides of (12.12), after the above modifications, by $(a_\lambda)^{-\frac{1}{2}}$, we obtain

(12.13)

$$\frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}} = \sqrt{a_\lambda}\partial_x T_{q^{-1}}\partial_x^3\eta_\lambda + (a_\lambda)^{-\frac{1}{2}}\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}L(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B) + G_2$$

Here, observe that we may use

$$(a_\lambda)^{-\frac{1}{2}} \in H^{s-\frac{1}{2}} \subseteq H^{\frac{1}{2}+}, \quad (a_\lambda)^{-\frac{1}{2}} \in W^{\frac{1}{2},\infty}$$

with a paraproduct decomposition and Proposition B.1 to easily estimate $G_2 = (a_\lambda)^{-\frac{1}{2}}G_1$ by the same right hand side as G_1.

Step 6. It remains to commute L to the front of the second term in (12.13),

$$(a_\lambda)^{-\frac{1}{2}}\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}L(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B).$$

To do so, as in Step 5, we exchange L for a paradifferential counterpart, $\partial_t + T_V\partial_x$. We remark that the discussion in Step 1 about commuting L applies here, in that the computations in this step do not require spatial localization and thus are relatively straightforward. We may commute the vector field using [ABZ14a, Lemma 2.15] to obtain

$$(a_\lambda)^{-\frac{1}{2}}(\partial_t + T_V\partial_x)\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B).$$

After restoring L from $\partial_t + T_V\partial_x$, we have

$$(a_\lambda)^{-\frac{1}{2}}L\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B).$$

Lastly, we may commute L to the front by using the Taylor coefficient estimates of Proposition D.8,

$$L(a_\lambda)^{-\frac{1}{2}} \in L^\infty \cap H^{s-1},$$

again not requiring spatial localization. Likewise, we may exchange L for $\partial_t + V_\lambda\partial_x$.

Step 7. Lastly, we check that

$$F_a := (a_\lambda)^{-\frac{1}{2}}\partial_x^2 S_{\leq c_1\lambda}T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)$$

satisfies the desired estimates. Using the algebra property of $H^{\frac{1}{2}+}$, as well as the tame (linear in $Z(t)$) paralinearization estimate of Proposition B.15,

$$\begin{aligned}
&\|(a_\lambda)^{-\frac{1}{2}}\partial_x^2 S_{\leq c_1\lambda} T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)\|_{H^{\frac{1}{2}}} \\
&\qquad \lesssim \|(a_\lambda)^{-\frac{1}{2}}\|_{H^{\frac{1}{2}+}} \|\partial_x^2 S_{\leq c_1\lambda} T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)\|_{H^{\frac{1}{2}+}} \\
&\qquad \lesssim \mathcal{F}(M(t))\|S_{\leq c_1\lambda} T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)\|_{H^{s'+1}} \\
&\qquad \leq \lambda^{\frac{5}{8}-}\mathcal{F}(M(t))\|R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B\|_{H^{s-\frac{1}{2}}} \\
&\qquad \leq \lambda^{\frac{5}{8}-}\mathcal{F}(M(t))Z(t)
\end{aligned}$$

as desired. The L^2 estimate is similar, using instead Proposition B.16.

We also estimate $w_{x_0,\lambda} S_\mu \sqrt{a_\lambda}\partial_x F_a$. When the derivative falls on $(a_\lambda)^{-\frac{1}{2}}$, we have the previous estimate except with an additional loss,

$$\|\partial_x(a_\lambda)^{-\frac{1}{2}}\|_{H^{\frac{1}{2}}} \leq \lambda^{5/8}\mathcal{F}(M(t))$$

which is better than needed. When the derivative falls on the rest of the term, we have

$$\|w_{x_0,\lambda} S_\mu \partial_x^3 T_{q^{-1}}(R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)\|_{H^{\frac{1}{2}}}.$$

By applying Proposition B.3 to commute under the paradifferential operator, it remains to estimate

$$\|w_{x_0,\lambda} S_\mu (R(\partial_x\eta, \partial_x B) + R(B, \partial_x^2\eta) + R(\eta)B)\|_{H^{5/2}}.$$

To the first quadratic term (the second is similar), we apply Proposition 11.9,

$$\begin{aligned}
\|w_{x_0,\lambda} S_\mu R(\partial_x\eta, \partial_x B)\|_{H^{5/2}} &\lesssim \mu^2(\|\partial_x\eta\|_{C_*^{r-\frac{1}{2}}}\|B\|_{LS^1_{x_0,\xi_0,\lambda,\mu}} + \|\partial_x B\|_{C_*^{r-1}}\|\eta\|_{LS^{\frac{3}{2}}_{x_0,\xi_0,\lambda,\mu}}) \\
&\leq \lambda\mu^{\frac{1}{2}-}Z(t)\mathcal{L}(t, x_0, \xi_0, \lambda)
\end{aligned}$$

as desired. Likewise, by Proposition B.14,

$$\|w_{x_0,\lambda} S_\mu R(\eta)B\|_{H^{s'-\frac{1}{2}}} \leq \mathcal{F}(M(t))Z(t)(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + Z(t) + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\mathcal{L}(t, x_0, \xi_0, \lambda))$$

so that

$$\begin{aligned}
\|w_{x_0,\lambda} S_\mu R(\eta)B\|_{H^{5/2}} &\leq \mu^{\frac{3}{2}-}\mathcal{F}(M(t))Z(t)(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + Z(t) + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\mathcal{L}(t, x_0, \xi_0, \lambda)) \\
&\leq \lambda^{\frac{3}{2}-}\mathcal{F}(M(t))Z(t)(Z(t) + \mathcal{L}(t, x_0, \xi_0, \lambda)).
\end{aligned}$$

□

12.4. **Integrating the symbol.** Combining Propositions 12.1 and 12.2, we obtain the integration (12.2) of $\partial_x^2 H$:

Corollary 12.3. *Let*

$$F_1(t, x) = T_{q^{-1}}\partial_x^3\eta_\lambda, \quad F_{\frac{1}{2}}(t, x) = F_a$$

where F_a is given in the proof of Proposition 12.2. Then we may write

$$\partial_x^2 H = (\partial_t + H_\xi\partial_x - H_x\partial_\xi)(F_1\xi + F_{\frac{1}{2}}|\xi|^{\frac{1}{2}}) + G_1\xi + G_{\frac{1}{2}}|\xi|^{\frac{1}{2}} + G_0|\xi|^{-1}\xi \tag{12.14}$$

where

$$\begin{aligned}
\|F_\alpha\|_{L^2} &\leq \lambda^{\frac{9}{8}-\alpha-}\mathcal{F}(M(t)),\\
\|F_\alpha\|_{L^\infty} &\leq \lambda^{\frac{3}{2}-\alpha-}\mathcal{F}(M(t))Z(t),\\
\|w_{x_0,\lambda}S_\mu F_\alpha\|_{H^{\frac{1}{2}}} &\leq \lambda^{\frac{3}{2}-\alpha-}\mathcal{F}(M(t))(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)),\\
\|G_\alpha\|_{L^\infty} &\leq \lambda^{\frac{3}{2}-\alpha-}\mathcal{F}(M(t))Z(t)^2,\\
\|w_{x_0,\lambda}S_\mu G_\alpha\|_{H^{\frac{1}{2}}} &\leq \lambda^{\frac{3}{2}-\alpha-}\mathcal{F}(M(t))Z(t)(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)).
\end{aligned}$$

Proof. It is easy to check that F_1 satisfies the desired estimates using Proposition B.3. Note that $F_{\frac{1}{2}}$ is even easier, using Sobolev embedding and not requiring the spatial localization. It remains to compute the errors G_i of the integration.

First compute

$$\begin{aligned}
H_\xi &= V_\lambda + \frac{1}{2}\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}\xi,\\
H_x &= \partial_x V_\lambda \xi + \partial_x\sqrt{a_\lambda}|\xi|^{\frac{1}{2}},\\
\partial_x^2 H &= \partial_x^2 V_\lambda \xi + \frac{1}{2}\left(-\frac{1}{2}\frac{(\partial_x a_\lambda)^2}{\sqrt{a_\lambda^3}} + \frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}}\right)|\xi|^{\frac{1}{2}}.
\end{aligned}$$

Consider the three terms constituting $\partial_x^2 H$. For the first and last, recall the identities of Propositions 12.1 and 12.2,

$$\begin{aligned}
\partial_x^2 V_\lambda &= (\partial_t + V_\lambda\partial_x)F_1 + G_V\\
\frac{\partial_x^2 a_\lambda}{\sqrt{a_\lambda}} &= \sqrt{a_\lambda}\partial_x F_1 + (\partial_t + V_\lambda\partial_x)F_{\frac{1}{2}} + G_a.
\end{aligned}$$

Substituting into the expression for $\partial_x^2 H$, we see that the two terms containing F_1 combine to form

$$(\partial_t + H_\xi\partial_x)F_1\xi,$$

so that

$$\partial_x^2 H = (\partial_t + H_\xi\partial_x)F_1\xi + G_V\xi + \frac{1}{2}\left(-\frac{1}{2}\frac{(\partial_x a_\lambda)^2}{\sqrt{a_\lambda^3}} + (\partial_t + V_\lambda\partial_x)F_{\frac{1}{2}} + G_a\right)|\xi|^{\frac{1}{2}}.$$

On the right hand side, putting aside for now the term $(\partial_t + V_\lambda\partial_x)F_{\frac{1}{2}}|\xi|^{\frac{1}{2}}$, and adding and subtracting $H_x\partial_\xi F_1\xi$, we have

$$\begin{aligned}
(\partial_t + H_\xi\partial_x - H_x\partial_\xi)F_1\xi + (G_V + (\partial_x V_\lambda)F_1)\xi + \frac{1}{2}\left(-\frac{1}{2}\frac{(\partial_x a_\lambda)^2}{\sqrt{a_\lambda^3}} + G_a + (\partial_x\sqrt{a_\lambda})F_1\right)|\xi|^{\frac{1}{2}}\\
=: (\partial_t + H_\xi\partial_x - H_x\partial_\xi)F_1\xi + G_1\xi + G_{\frac{1}{2}}|\xi|^{\frac{1}{2}}.
\end{aligned}$$

It remains to check the estimates on G_i. Note that G_V, G_a satisfy the desired estimates by Propositions 12.1 and 12.2. We consider the remaining terms, focusing on the local Sobolev estimates since the Hölder estimates are easier, using the appropriate Hölder counterparts. First, for

$$(\partial_x V_\lambda)F_1,$$

we use a paraproduct decomposition,

$$(\partial_x V_\lambda)F_1 = T_{\partial_x V_\lambda} F_1 + T_{F_1}\partial_x V_\lambda + R(\partial_x V_\lambda, F_1).$$

The first and second terms are straightforward, using Proposition B.1 with the estimates on $F_1 \in L^\infty \cap H^{\frac{1}{2}}$ of the proposition. The third term uses Corollary 11.10.

The remaining two terms,

$$-\frac{1}{2}\frac{(\partial_x a_\lambda)^2}{\sqrt{a_\lambda^3}} + (\partial_x\sqrt{a_\lambda})F_1$$

are similarly treated using a paraproduct decomposition (using also the Taylor coefficient bounds of Proposition D.8 and the Taylor sign condition).

We assess the term

$$\frac{1}{2}(\partial_t + V_\lambda\partial_x)F_{\frac{1}{2}}|\xi|^{\frac{1}{2}}$$

similarly, adding errors as necessary to the G_i terms. However, to form

$$\frac{1}{2}(\partial_t + H_\xi\partial_x)F_{\frac{1}{2}}|\xi|^{\frac{1}{2}},$$

we simply add and subtract

$$\frac{1}{4}\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}\xi\partial_x F_{\frac{1}{2}}|\xi|^{\frac{1}{2}} = \frac{1}{4}\sqrt{a_\lambda}\partial_x F_{\frac{1}{2}}|\xi|^{-1}\xi.$$

By the corresponding estimate of Proposition 12.2, this term satisfies the appropriate estimate to be absorbed into G_0. Also note that inserting the term $H_x\partial_\xi F_{\frac{1}{2}}|\xi|^{\frac{1}{2}}$ is easier than before, in particular not using spatial localization. □

13. Regularity of the Hamilton Flow

Recall we have the Hamiltonian

$$H(t,x,\xi) = V_\lambda\xi + \sqrt{a_\lambda|\xi|}.$$

In this section we discuss the regularity properties of the associated Hamilton flow.

13.1. **The linearized equations.** Recall the Hamilton equations (10.1) associated to H,

$$\begin{cases} \dot{x}(t) = H_\xi(t,x(t),\xi(t)) \\ \dot{\xi}(t) = -H_x(t,x(t),\xi(t)) \\ (x(s),\xi(s)) = (x,\xi), \end{cases}$$

and that we denote the solution $(x(t),\xi(t))$ to (10.1), at time $t\in I$ with initial data (x,ξ) at time $s\in I$, variously by

$$(x_s^t(x,\xi),\xi_s^t(x,\xi)) = (x^t(x,\xi),\xi^t(x,\xi)) = (x_s^t,\xi_s^t) = (x^t,\xi^t).$$

Denoting

$$p(t,x,\xi) = \begin{pmatrix} x^t(x,\xi) \\ \xi^t(x,\xi)\end{pmatrix},$$

we have the linearization of the Hamilton equations (10.1):

$$\frac{d}{dt}\partial_x p(t) = \begin{pmatrix} H_{\xi x}(t,x^t,\xi^t) & H_{\xi\xi}(t,x^t,\xi^t) \\ -H_{xx}(t,x^t,\xi^t) & -H_{x\xi}(t,x^t,\xi^t)\end{pmatrix}\partial_x p(t). \tag{13.1}$$

By a heuristic computation using the uncertainty principle and the dispersion relation $|\xi|^{\frac{1}{2}}$, the scale on which wave packets cohere is

$$\delta x \approx \lambda^{-\frac{3}{4}}, \quad \delta\xi \approx \lambda^{\frac{3}{4}}, \quad \delta t \approx 1.$$

As a result, it is natural to define

$$P(t,x,\xi) = \begin{pmatrix} X^t(x,\xi) \\ \Xi^t(x,\xi) \end{pmatrix} = \begin{pmatrix} \lambda^{\frac{3}{4}} x^t(x,\xi) \\ \lambda^{-\frac{3}{4}} \xi^t(x,\xi) \end{pmatrix},$$

and with a slight abuse of notation,

$$\tilde{H}_{\xi\xi} = \lambda^{\frac{3}{2}} H_{\xi\xi}, \quad \tilde{H}_{x\xi} = H_{x\xi}, \quad \tilde{H}_{xx} = \lambda^{-\frac{3}{2}} H_{xx}.$$

Then we have

$$\frac{d}{dt}\partial_x P(t) = \begin{pmatrix} \tilde{H}_{\xi x}(t,x^t,\xi^t) & \tilde{H}_{\xi\xi}(t,x^t,\xi^t) \\ -\tilde{H}_{xx}(t,x^t,\xi^t) & -\tilde{H}_{x\xi}(t,x^t,\xi^t) \end{pmatrix} \partial_x P(t). \tag{13.2}$$

For the remainder of this subsection we estimate the entries of the matrix of the linearized Hamilton flow 13.2.

Lemma 13.1. *On* $\{|\xi| \in [\lambda/2, 2\lambda]\}$,

$$\begin{aligned} \|\tilde{H}_{\xi\xi}(t)\|_{L^\infty_x} &\leq \mathcal{F}(M(t)), \\ \|\tilde{H}_{x\xi}(t)\|_{L^\infty_x} &\leq \mathcal{F}(M(t))Z(t), \\ \|w_{x_0,\lambda}\tilde{H}_{\xi\xi}(t)\|_{H_x^{\frac{1}{2}+}} &\leq \lambda^{0+}\mathcal{F}(M(t)), \\ \|w_{x_0,\lambda}\tilde{H}_{x\xi}(t)\|_{H_x^{\frac{1}{2}+}} &\leq \lambda^{0+}\mathcal{F}(M(t))(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)). \end{aligned}$$

Proof. We have

$$H_{\xi\xi} = -\frac{1}{4}\sqrt{a_\lambda}|\xi|^{-\frac{3}{2}}$$

from which the first estimate is immediate on $\{|\xi| \in [\lambda/2, 2\lambda]\}$, using Proposition D.8. We also have

$$H_{x\xi} = \partial_x V_\lambda + \frac{1}{2}(\partial_x\sqrt{a_\lambda})|\xi|^{-\frac{3}{2}}\xi.$$

Then the second bound is obtained using Proposition D.8, the Taylor sign condition, frequency localization, and the assumption on ξ. The third estimate on $H_{\xi\xi}$ is immediate as there is evidently sufficient regularity on a.

For the fourth estimate on $H_{x\xi}$, consider first the transport term V_λ. We have using Proposition B.3 to commute the localization under ∂_x (or simply by using the product rule),

$$\|w_{x_0,\mu}S_\mu\partial_x V\|_{H^{\frac{1}{2}+}} \lesssim \|w_{x_0,\mu}S_\mu V\|_{H^{\frac{3}{2}+}} + \|V\|_{H^{s-}}.$$

Further, we have

$$\|w_{x_0,\lambda}S_{\lesssim\lambda^{\frac{3}{4}}}\partial_x V\|_{H^{\frac{1}{2}+}} \lesssim \lambda^{0+}\|\partial_x V\|_{L^\infty}.$$

Summing this with logarithmically many values of μ, we conclude

$$\|w_{x_0,\lambda}\partial_x V\|_{H^{\frac{1}{2}+}} \leq \mathcal{F}(M(t))(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)).$$

From the dispersive term, we have

$$\begin{aligned}\|w_{x_0,\lambda}\partial_x\sqrt{a_\lambda}\|_{H^{\frac12+}} &\lesssim \|w_{x_0,\lambda}a_\lambda^{-\frac12}\partial_x a_\lambda\|_{H^{\frac12}} \\ &\lesssim \|a_\lambda^{-\frac12}\|_{H^{\frac12+}}\|w_{x_0,\lambda}\partial_x a_\lambda\|_{L^\infty} + \|a_\lambda^{-\frac12}\|_{L^\infty}\|w_{x_0,\lambda}\partial_x a_\lambda\|_{H^{\frac12+}}.\end{aligned}$$

The first three terms are all easily estimated using Proposition D.8, so it remains to consider the last term in $H^{\frac12+}$.

By Corollary C.6, we have

$$\|w_{x_0,\lambda}S_{\gg\lambda^{\frac34}}a_\lambda\|_{H^{\frac32+}} \lesssim \sum_{\lambda^{\frac34}\ll\mu\leq c_1\lambda}\mu^{-\epsilon}\|w_{x_0,\lambda}S_\mu a\|_{H^{s'}} \leq \lambda^{\frac12}\mathcal{F}(M(t))(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)).$$

Thus, it remains to estimate

$$\|w_{x_0,\lambda}S_{\lesssim\lambda^{\frac34}}\partial_x a\|_{H^{\frac12+}} + \|(\partial_x w_{x_0,\lambda})S_{\gg\lambda^{\frac34}}a_\lambda\|_{H^{\frac12+}}.$$

The first is easily estimated,

$$\|w_{x_0,\lambda}S_{\lesssim\lambda^{\frac34}}\partial_x a\|_{H^{\frac12+}} \lesssim \lambda^{\frac38+}\|S_{\lesssim\lambda^{\frac34}}\partial_x a\|_{L^2} \lesssim \lambda^{\frac12}\|a\|_{H^{s-\frac12}}.$$

The second is estimated using the algebra property of $H^{\frac12+}$,

$$\begin{aligned}\|(\partial_x w_{x_0,\lambda})S_{\gg\lambda^{\frac34}}a_\lambda\|_{H^{\frac12+}} &\lesssim \|\partial_x w_{x_0,\lambda}\|_{H^{\frac12+}}\|S_{\gg\lambda^{\frac34}}a_\lambda\|_{H^{\frac12+}} \\ &\lesssim \lambda^{\frac34+}\|S_{\gg\lambda^{\frac34}}a_\lambda\|_{H^{\frac12+}} \\ &\lesssim \lambda^{\frac12}\|a\|_{H^{s-\frac12}}.\end{aligned}$$

□

A direct estimate on $\tilde{H}_{xx} = \lambda^{-\frac32}H_{xx}$ as in Lemma 13.1 would yield estimates that are too crude for a bilipschitz flow (the estimate would depend on a positive power of λ). However, as discussed in Section 12, we can integrate H_{xx} along (x^t,ξ^t), with the following estimates on the integral:

Corollary 13.2. *We may write*

$$\partial_x^2 H(t,x^t,\xi^t) = \frac{d}{dt}F(t,x^t,\xi^t) + G(t,x^t,\xi^t) \tag{13.3}$$

where on $\{|\xi|\in[\lambda/2,2\lambda]\}$,

$$\begin{aligned}\|F(t,\cdot,\xi)\|_{L^2} &\leq \lambda^{\frac98-}\mathcal{F}(M(t)), \\ \|F(t,\cdot,\xi)\|_{L^\infty} &\leq \lambda^{\frac32-}\mathcal{F}(M(t))Z(t), \\ \|w_{x_0,\lambda}F(t,\cdot,\xi)\|_{H^{\frac12+}} &\leq \lambda^{\frac32-}\mathcal{F}(M(t))(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)), \\ \|G(t,\cdot,\xi)\|_{L^\infty} &\leq \lambda^{\frac32-}\mathcal{F}(M(t))Z(t)^2, \\ \|w_{x_0,\lambda}G(t,\cdot,\xi)\|_{H^{\frac12+}} &\leq \lambda^{\frac32-}\mathcal{F}(M(t))Z(t)(Z(t)+\mathcal{L}(t,x_0,\xi_0,\lambda)).\end{aligned}$$

Proof. By Corollary 12.3,

$$\partial_x^2 H = (\partial_t + H_\xi\partial_x - H_x\partial_\xi)(F_1\xi + F_{\frac12}|\xi|^{\frac12}) + G_1\xi + G_{\frac12}|\xi|^{\frac12} + G_0|\xi|^{-1}\xi.$$

Then set

$$F(t,x,\xi) = F_1\xi + F_{\frac{1}{2}}|\xi|^{\frac{1}{2}}, \quad G(t,x,\xi) = G_1\xi + G_{\frac{1}{2}}|\xi|^{\frac{1}{2}} + G_0|\xi|^{-1}\xi,$$

from which the desired identity is easily verified, along with the desired estimates on $\{|\xi| \in [\lambda/2, 2\lambda]\}$. We make some additional remarks regarding the local Sobolev estimates: By Corollary 12.3, we have

$$\|w_{x_0,\lambda} S_\mu F(t,\cdot,\xi)\|_{H^{\frac{1}{2}}} \leq \lambda^{\frac{3}{2}-}\mathcal{F}(M(t))(Z(t) + \mathcal{L}(t,x_0,\xi_0,\lambda)).$$

Since the first term of F,

$$F_1\xi = T_{q^{-1}}\partial_x^3\eta_\lambda\xi,$$

is localized to frequencies $\leq c\lambda$, we can sum these estimates over $\lambda^{\frac{3}{4}} \ll \mu \leq c\lambda$ with logarithmic loss. The component $\lesssim \lambda^{\frac{3}{4}}$ is estimated directly as in the proof of Lemma 13.1.

The other terms of F and G are not localized to frequencies $\leq c\lambda$, but only due to a coefficient in $C^{\frac{1}{2}}$. Precisely, they may be written as

$$AS_{\lesssim c_1\lambda}B$$

where $A \in C^{\frac{1}{2}}$ (for instance, $A = (a_\lambda)^{-\frac{1}{2}}$). Thus, we may estimate the high frequency component

$$S_{>c_1\lambda}AS_{\lesssim c_1\lambda}B$$

by absorbing half a derivative into A instead of using the local Sobolev estimates of Corollary 12.3.

□

We denote in the following, where F, G are as in Corollary 13.2,

$$\tilde{F} = \lambda^{-\frac{3}{2}}F, \quad \tilde{G} = \lambda^{-\frac{3}{2}}G$$

so that

$$\partial_x^2\tilde{H}(t,x^t,\xi^t) = \frac{d}{dt}\tilde{F}(t,x^t,\xi^t) + \tilde{G}(t,x^t,\xi^t).$$

13.2. **The bilipschitz flow.** We can combine the integration of Corollary 13.2 with Gronwall to obtain estimates on $p(t)$ which exhibit the sense in which (x^t,ξ^t) is bilipschitz.

Proposition 13.3. *There exists $s_0 \in I$ such that for $J \subseteq \mathbb{R}$ with $|J| \lesssim \lambda^{-\frac{3}{4}}$, and solutions (x^t,ξ^t) to (10.1) with initial data (x,ξ) satisfying $|\xi| \in [\lambda/2, 2\lambda]$, at initial time*

i) $s = s_0 \in I$,

$$\|\partial_x X^t - \lambda^{\frac{3}{4}}I\|_{L_t^\infty(I;L_x^\infty)} \ll \lambda^{\frac{3}{4}}.$$

ii) $s \in I$,

$$\|\partial_x X^t - \lambda^{\frac{3}{4}}I\|_{L_x^2(J)} + \|\partial_x\Xi^{s_0}\|_{L_x^2(J)} \ll \lambda^{\frac{3}{8}}.$$

Proof. Recall (13.2),

$$\frac{d}{dt}\partial_x P(t) = \begin{pmatrix} \tilde{H}_{\xi x}(t,x^t,\xi^t) & \tilde{H}_{\xi\xi}(t,x^t,\xi^t) \\ -\tilde{H}_{xx}(t,x^t,\xi^t) & -\tilde{H}_{x\xi}(t,x^t,\xi^t) \end{pmatrix}\partial_x P(t).$$

Step 1. First, we integrate to obtain the right object on which to apply Gronwall. Using Corollary 13.2, write (omitting the parameters (t, x^t, ξ^t) for brevity)

$$\frac{d}{dt}\partial_x P(t) = \begin{pmatrix} 0 & 0 \\ -\dot{\tilde{F}} & 0 \end{pmatrix} \partial_x P(t) + \begin{pmatrix} \tilde{H}_{\xi x} & \tilde{H}_{\xi\xi} \\ -\tilde{G} & -\tilde{H}_{x\xi} \end{pmatrix} \partial_x P(t)$$

$$= -\frac{d}{dt}\left(\begin{pmatrix} 0 & 0 \\ \tilde{F} & 0 \end{pmatrix} \partial_x P(t)\right) + \begin{pmatrix} 0 & 0 \\ \tilde{F} & 0 \end{pmatrix} \frac{d}{dt}\partial_x P(t) + \begin{pmatrix} \tilde{H}_{\xi x} & \tilde{H}_{\xi\xi} \\ -\tilde{G} & -\tilde{H}_{x\xi} \end{pmatrix} \partial_x P(t).$$

Substituting (13.2) into the right hand side and rearranging,

$$\frac{d}{dt}\left(\begin{pmatrix} I & 0 \\ \tilde{F} & I \end{pmatrix} \partial_x P(t)\right) = \begin{pmatrix} \tilde{H}_{\xi x} & \tilde{H}_{\xi\xi} \\ \tilde{F}\tilde{H}_{\xi x} - \tilde{G} & \tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi} \end{pmatrix} \partial_x P(t).$$

For brevity, write this as

$$\frac{d}{dt}(I_F \partial_x P)(t) = (A\partial_x P)(t). \tag{13.4}$$

Integrating and rewriting the right hand side to apply Gronwall,

$$(I_F\partial_x P)(t) = (I_F\partial_x P)(s) + \int_s^t (AI_F^{-1})(r)(I_F\partial_x P)(r)\, dr. \tag{13.5}$$

Step 2. As a preliminary step, we establish fixed-time estimates in L_x^∞. Applying uniform norms and Gronwall,

$$\|(I_F\partial_x P)(t)\|_{L_x^\infty} \lesssim \|(I_F\partial_x P)(s)\|_{L_x^\infty} \exp\left(\int_s^t \|(AI_F^{-1})(r)\|_{L_x^\infty}\, dr\right).$$

Noting that

$$I_F^{-1} = \begin{pmatrix} I & 0 \\ -\tilde{F} & I \end{pmatrix}$$

and

$$(I_F\partial_x P)(s) = I_F(s)\begin{pmatrix} \lambda^{\frac{3}{4}} I \\ 0 \end{pmatrix} = \lambda^{\frac{3}{4}}\begin{pmatrix} I \\ \tilde{F}(s) \end{pmatrix},$$

we have

$$\|\partial_x P(t)\|_{L_x^\infty} \lesssim \lambda^{\frac{3}{4}}(1 + \|\tilde{F}(t)\|_{L_x^\infty})(1 + \|\tilde{F}(s)\|_{L_x^\infty}) \exp\left(\|AI_F^{-1}\|_{L_t^1(I;L_x^\infty)}\right). \tag{13.6}$$

To estimate the right hand side, first consider the exponential term:

$$AI_F^{-1} = \begin{pmatrix} \tilde{H}_{\xi x} & \tilde{H}_{\xi\xi} \\ \tilde{F}\tilde{H}_{\xi x} - \tilde{G} & \tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi} \end{pmatrix}\begin{pmatrix} I & 0 \\ -\tilde{F} & I \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{H}_{\xi x} - \tilde{F}\tilde{H}_{\xi\xi} & \tilde{H}_{\xi\xi} \\ \tilde{F}\tilde{H}_{\xi x} - \tilde{G} - \tilde{F}\tilde{H}_{\xi\xi}\tilde{F} + \tilde{H}_{x\xi}\tilde{F} & \tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi} \end{pmatrix}$$

Then using Corollary 13.2, Lemma 13.1, and Cauchy-Schwarz,

$$\begin{aligned} \|AI_F^{-1}\|_{L_t^1(I;L_x^\infty)} &\lesssim T^{\frac{1}{2}}(\|\tilde{H}_{\xi x}\|_{L_t^2(I;L_x^\infty)} + \|\tilde{H}_{\xi\xi}\|_{L_t^2(I;L_x^\infty)}) + \|\tilde{G}\|_{L_t^1(I;L_x^\infty)} \\ &\quad + \|\tilde{F}\|_{L_t^2(I;L_x^\infty)}\left(\|\tilde{H}_{\xi\xi}\|_{L_t^2(I;L_x^\infty)} + \|\tilde{H}_{x\xi}\|_{L_t^2(I;L_x^\infty)}\right) \\ &\quad + \|\tilde{F}\|^2_{L_t^2(I;L_x^\infty)}\|\tilde{H}_{\xi\xi}\|_{L_t^\infty(I;L_x^\infty)} \\ &\le (T^{\frac{1}{2}} + \lambda^{0-})\mathcal{F}(T) \ll 1. \end{aligned} \tag{13.7}$$

Here we have chosen T sufficiently small, iterating the argument over multiple time intervals as necessary, and λ sufficiently large. For smaller λ, the frequency localized Strichartz estimates can be proven directly with Sobolev embedding.

To estimate $\tilde{F}(s)$, we choose $s_0 \in I$ such that

$$\|\tilde{F}(s_0)\|_{L_x^\infty} \leq T^{-1}\|\tilde{F}\|_{L_t^1(I;L_x^\infty)} \leq T^{-\frac{1}{2}}\|\tilde{F}\|_{L_t^2(I;L_x^\infty)}.$$

Then using Corollary 13.2 and choosing λ sufficiently large as before,

$$\|\tilde{F}(s_0)\|_{L_x^\infty} \leq T^{-\frac{1}{2}}\lambda^{0-}\mathcal{F}(T) \ll T^{-\frac{1}{2}}. \tag{13.8}$$

Thus, by setting $s = s_0$ in (13.6) and using Corollary 13.2 on $\tilde{F}(t)$, we find

$$\|\partial_x P(t)\|_{L_x^\infty} \leq \lambda^{\frac{3}{4}}T^{-\frac{1}{2}}\mathcal{F}(M(t))Z(t). \tag{13.9}$$

Step 3. Next we improve upon Step 2 by showing that $\partial_x X^t \approx \lambda^{\frac{3}{4}} I$ uniformly in t (thus establishing the first estimate in part (i)). From the top row of (13.5),

$$\partial_x X^t = \lambda^{\frac{3}{4}} I + \int_{s_0}^t \tilde{H}_{\xi x}(r)\partial_x X^r + \tilde{H}_{\xi\xi}(r)\partial_x \Xi^r \, dr.$$

Using (13.9), Lemma 13.1, and Cauchy-Schwarz,

$$\|\partial_x X^t - \lambda^{\frac{3}{4}} I\|_{L_x^\infty} \leq \int_{s_0}^t \|\tilde{H}_{\xi x}(r)\partial_x X^r\|_{L_x^\infty} + \|\tilde{H}_{\xi\xi}(r)\partial_x \Xi^r\|_{L_x^\infty} \, dr \leq \lambda^{\frac{3}{4}}T^{-\frac{1}{2}}\mathcal{F}(T) \tag{13.10}$$

and in particular,

$$\|\partial_x X^t\|_{L_t^\infty(I;L_x^\infty)} \leq \lambda^{\frac{3}{4}}T^{-\frac{1}{2}}\mathcal{F}(T).$$

Using this in place of (13.9), revisit (13.10) and perform the estimate again to obtain

$$\begin{aligned}\|\partial_x X^t - \lambda^{\frac{3}{4}} I\|_{L_x^\infty} &\leq T^{\frac{1}{2}}\mathcal{F}(T)(\lambda^{\frac{3}{4}}T^{-\frac{1}{2}}\|\tilde{H}_{\xi x}\|_{L_t^2(I;L_x^\infty)} + \|\partial_x \Xi^t\|_{L_t^2(I;L_x^\infty)}) \\ &\leq \lambda^{\frac{3}{4}}\mathcal{F}(T).\end{aligned}$$

Repeating this process one more time, we obtain

$$\|\partial_x X^t - \lambda^{\frac{3}{4}} I\|_{L_x^\infty} \leq \lambda^{\frac{3}{4}}T^{\frac{1}{2}}\mathcal{F}(T) \ll \lambda^{\frac{3}{4}}. \tag{13.11}$$

Step 4. Lastly, we establish $L_x^2(J)$ estimates in (ii). Recalling that

$$(I_F\partial_x P)(s) = I_F(s)\begin{pmatrix}\lambda^{\frac{3}{4}} I \\ 0\end{pmatrix} = \lambda^{\frac{3}{4}}\begin{pmatrix} I \\ \tilde{F}(s)\end{pmatrix} := \lambda^{\frac{3}{4}} I_x + \lambda^{\frac{3}{4}}\begin{pmatrix} 0 \\ \tilde{F}(s)\end{pmatrix},$$

it is natural to write, in place of (13.5),

$$(I_F\partial_x P - \lambda^{\frac{3}{4}} I_x)(t) = \lambda^{\frac{3}{4}}\begin{pmatrix} 0 \\ \tilde{F}(s)\end{pmatrix} + \int_s^t (AI_F^{-1})(r)(I_F\partial_x P - \lambda^{\frac{3}{4}} I_x)(r) + \lambda^{\frac{3}{4}}(AI_F^{-1})(r)I_x \, dr.$$

Then apply $L_x^2(J)$ and Gronwall:

$$\begin{aligned}&\|(I_F\partial_x P - \lambda^{\frac{3}{4}} I_x)(t)\|_{L_x^2(J)} \\ &\qquad \lesssim \lambda^{\frac{3}{4}}(\|\tilde{F}(s)\|_{L_x^2(J)} + \|AI_F^{-1}\|_{L_t^1(I;L_x^2(J))})\exp\left(\int_s^t \|(AI_F^{-1})(r)\|_{L_x^\infty} \, dr\right).\end{aligned}$$

We estimate the right hand side using the (global) L^2 estimate of Corollary 13.2, (13.7), and Cauchy-Schwarz in space, concluding

$$\|(I_F\partial_x P - \lambda^{\frac{3}{4}} I_x)(t)\|_{L_x^2(J)} \ll \lambda^{\frac{3}{8}}. \tag{13.12}$$

From the top row, we obtain

$$\|\partial_x X^t - \lambda^{\frac{3}{4}} I\|_{L^2_x(J)} \ll \lambda^{\frac{3}{8}} \tag{13.13}$$

as desired.

It remains to establish the estimate on $\partial_x \Xi^{s_0}$. First, recalling,

$$I_F^{-1} = \begin{pmatrix} I & 0 \\ -\tilde{F} & I \end{pmatrix},$$

we have using Corollary 13.2 with (13.12),

$$\|\partial_x P(t)\|_{L^2_x(J)} \lesssim (1 + \|\tilde{F}(t)\|_{L^\infty_x})\|(I_F \partial_x P)(t)\|_{L^2_x(J)} \leq \lambda^{\frac{3}{8}} \mathcal{F}(M(t)) Z(t). \tag{13.14}$$

Using again the top row of (13.5),

$$\partial_x X^t = \lambda^{\frac{3}{4}} I + \int_{s_0}^t \tilde{H}_{\xi x}(r) \partial_x X^r + \tilde{H}_{\xi\xi}(r) \partial_x \Xi^r \, dr,$$

we apply $L^2_x(J)$, and use (13.13, (13.14), Lemma 13.1, and Cauchy-Schwarz to obtain

$$\begin{aligned} \|\partial_x X^t - \lambda^{\frac{3}{4}} I\|_{L^2_x(J)} &\lesssim \int_s^t \|\tilde{H}_{\xi x}(r)\|_{L^\infty} \|\partial_x X^r\|_{L^2_x(J)} + \|\tilde{H}_{\xi\xi}(r)\|_{L^\infty} \|\partial_x \Xi^r\|_{L^2_x(J)} \, dr \\ &\leq \lambda^{\frac{3}{8}} T^{\frac{1}{2}} \mathcal{F}(T). \end{aligned}$$

Combining this with (13.8),

$$\|\tilde{F}(s_0)(\partial_x X^t - \lambda^{\frac{3}{4}} I)\|_{L^2_x(J)} \leq T^{-\frac{1}{2}} \lambda^{0-} \lambda^{\frac{3}{8}} T^{\frac{1}{2}} \mathcal{F}(T) \ll \lambda^{\frac{3}{8}}. \tag{13.15}$$

Returning next to the bottom row of (13.12), we obtain

$$\|\tilde{F}(t) \partial_x X^t + \partial_x \Xi^t\|_{L^2_x(J)} \ll \lambda^{\frac{3}{8}}.$$

Setting $t = s_0$ and applying (13.15),

$$\|\lambda^{\frac{3}{4}} \tilde{F}(s_0) + \partial_x \Xi^{s_0}\|_{L^2_x(J)} \ll \lambda^{\frac{3}{8}}.$$

Lastly, using the (global) L^2 estimate of Corollary 13.2 (here using as well as the bilipschitz property (13.13) since $F(s_0) = F(s_0, x^{s_0}, \xi^{s_0})$), we obtain

$$\|\partial_x \Xi^{s_0}\|_{L^2_x(J)} \ll \lambda^{\frac{3}{8}},$$

completing the two estimates of (ii).

□

We also require estimates on $\partial_\xi P(t)$ to exhibit the spreading of the characteristics, which we later use to show dispersive estimates:

Proposition 13.4. *Let $s_0 \in I$ be as in Proposition 13.3. Then for solutions (x^t, ξ^t) to (10.1) with initial data (x, ξ) satisfying $|\xi| \in [\lambda/2, 2\lambda]$, at initial time $s \in I$,*

$$\|\partial_\xi \Xi^{s_0} - \lambda^{-\frac{3}{4}} I\|_{L^\infty_x} \ll \lambda^{-\frac{3}{4}}.$$

Further, if $t \geq s$,

$$-\partial_\xi X^t \gtrsim \lambda^{-\frac{3}{4}}(t - s),$$

and if $s > t$,

$$\partial_\xi X^t \gtrsim \lambda^{-\frac{3}{4}}(s - t).$$

In either case,

$$\|\partial_\xi X^t\|_{L^\infty_x} \leq \lambda^{-\frac{3}{4}}\mathcal{F}(T)|t-s|.$$

Proof. Similar to (13.2), we have

$$\frac{d}{dt}\partial_\xi P(t) = \begin{pmatrix} \tilde{H}_{\xi x}(t,x^t,\xi^t) & \tilde{H}_{\xi\xi}(t,x^t,\xi^t) \\ -\tilde{H}_{xx}(t,x^t,\xi^t) & -\tilde{H}_{x\xi}(t,x^t,\xi^t) \end{pmatrix} \partial_\xi P(t).$$

Step 1. We first establish L^∞_x estimates, uniform in time. In analogy to (13.5), we can integrate $\tilde{H}_{xx}$ to write

$$(I_F\partial_\xi P)(t) = (I_F\partial_\xi P)(s) + \int_s^t (AI_F^{-1})(r)(I_F\partial_\xi P)(r)\,dr.$$

Observing that

$$(I_F\partial_\xi P)(s) = \begin{pmatrix} 0 \\ \lambda^{-\frac{3}{4}} I \end{pmatrix} := \lambda^{-\frac{3}{4}} I_\xi,$$

we may write

$$(I_F\partial_\xi P - \lambda^{-\frac{3}{4}}I_\xi)(t) = \int_s^t (AI_F^{-1})(r)(I_F\partial_\xi P - \lambda^{-\frac{3}{4}}I_\xi)(r) + \lambda^{-\frac{3}{4}}(AI_F^{-1})(r)I_\xi\,dr \tag{13.16}$$

Applying uniform norms, Gronwall, and a subset of the computations ending in (13.7), we have

$$\begin{aligned}\|(I_F\partial_\xi P - \lambda^{-\frac{3}{4}}I_\xi)(t)\|_{L^\infty_x} &\lesssim \lambda^{-\frac{3}{4}}\|AI_F^{-1}I_\xi\|_{L^1_t(I;L^\infty_x)} \exp\left(\int_s^t \|(AI_F^{-1})(r)\|_{L^\infty_x}\,dr\right) \\ &\leq \lambda^{-\frac{3}{4}}T^{\frac{1}{2}}\mathcal{F}(T).\end{aligned}$$

From the top row, we have

$$\|\partial_\xi X^t\|_{L^\infty_x} \leq \lambda^{-\frac{3}{4}}T^{\frac{1}{2}}\mathcal{F}(T)$$

and from the bottom row,

$$\|\tilde{F}(t)\partial_\xi X^t + \partial_\xi \Xi^t - \lambda^{-\frac{3}{4}}I\|_{L^\infty_x} \leq \lambda^{-\frac{3}{4}}T^{\frac{1}{2}}\mathcal{F}(T) \ll \lambda^{-\frac{3}{4}}. \tag{13.17}$$

In particular, combining these two estimates, setting $t = s_0$, and using (13.8), we find

$$\|\partial_\xi \Xi^{s_0} - \lambda^{-\frac{3}{4}}I\|_{L^\infty_x} \leq \lambda^{-\frac{3}{4}}(T^{-\frac{1}{2}}\lambda^{0-}T^{\frac{1}{2}}\mathcal{F}(T) + c) \ll \lambda^{-\frac{3}{4}}$$

as desired.

Step 2. Next we show that $\partial_\xi X^t$ is linear in the time t. From the top row of (13.16),

$$\partial_\xi X^t = \int_s^t \tilde{H}_{\xi x}(r)\partial_\xi X^r + \tilde{H}_{\xi\xi}(r)\partial_\xi \Xi^r\,dr.$$

Substituting the formula for $\partial_\xi \Xi^t$ from (13.17), we have

$$\partial_\xi X^t = \int_s^t (\tilde{H}_{\xi x} - \tilde{H}_{\xi\xi}\tilde{F})(r)\partial_\xi X^r + \tilde{H}_{\xi\xi}(r)(\lambda^{-\frac{3}{4}}I + K(r))\,dr \tag{13.18}$$

where

$$\|K(t)\|_{L^\infty_x} \ll \lambda^{-\frac{3}{4}}.$$

Using Gronwall followed by Lemma 13.1 and (13.7) (noting that $\tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi}$ is one of the coefficients of AI_F^{-1}), we have

$$\begin{aligned}\|\partial_\xi X^t\|_{L^\infty_x} &\lesssim |t-s|\|\tilde{H}_{\xi\xi}(t)(\lambda^{-\frac{3}{4}}I + K(t))\|_{L^\infty_t(I;L^\infty_x)} \exp\left(\int_s^t \|(AI_F^{-1})(r)\|_{L^\infty_x}\,dr\right)\\ &\leq \lambda^{-\frac{3}{4}}\mathcal{F}(T)|t-s|\end{aligned}$$

as desired.

In turn, using this estimate in again (13.18), we can write, using Corollary 13.2, Lemma 13.1, and Cauchy-Schwarz,

$$\|\partial_\xi X^t - \int_s^t \tilde{H}_{\xi\xi}(r)(\lambda^{-\frac{3}{4}}I + K(r))\,dr\|_{L^\infty_x} \leq T^{\frac{1}{2}}\lambda^{-\frac{3}{4}}\mathcal{F}(T)|t-s| \ll \lambda^{-\frac{3}{4}}|t-s|.$$

Lastly, by the Taylor sign condition and Lemma 11.1,

$$-\tilde{H}_{\xi\xi} = \frac{1}{4}\lambda^{\frac{3}{2}}\sqrt{a_\lambda}|\xi^t|^{-\frac{3}{2}} \gtrsim \sqrt{a_{min}} > 0$$

so that if $t \geq s$,

$$-\int_s^t \tilde{H}_{\xi\xi}(r)\,dr > \sqrt{a_{min}}(t-s)$$

and if $s > t$,

$$\int_s^t \tilde{H}_{\xi\xi}(r)\,dr > \sqrt{a_{min}}(s-t).$$

We conclude by the triangle inequality that if $t > s$,

$$-\partial_\xi X^t \gtrsim \lambda^{-\frac{3}{4}}(t-s),$$

and if $s > t$,

$$\partial_\xi X^t \gtrsim \lambda^{-\frac{3}{4}}(s-t).$$

□

13.3. **Geometry of the characteristics.** In this subsection we discuss the geometry of the characteristics (x^t, ξ^t). Throughout, $s_0 \in I$ denotes the s_0 discussed in Propositions 13.3 and 13.4.

First we observe that if two characteristics intersect at time t with similar frequencies, then they must have had similar initial frequencies.

Proposition 13.5. *For solutions (x^t, ξ^t) to (10.1) with initial data (x_i, ξ_i) at initial time $s_0 \in I$, if*

$$|x^t(x_1,\xi_1) - x^t(x_2,\xi_2)| \leq \lambda^{-\frac{3}{4}},$$

then

$$|\xi_1 - \xi_2| \leq 2|\xi^t(x_1,\xi_1) - \xi^t(x_2,\xi_2)| + \lambda^{\frac{3}{4}}.$$

Proof. Write $\xi^{s_0} = \xi^{s_0}_t$ and

$$\xi_1 - \xi_2 = \xi^{s_0}(x^t(x_1,\xi_1),\xi^t(x_1,\xi_1)) - \xi^{s_0}(x^t(x_2,\xi_2),\xi^t(x_2,\xi_2)). \tag{13.19}$$

Using the fundamental theorem of calculus and (ii) in Proposition 13.3 with inital time $s = t$, we have (assuming without loss of generality that $x^t(x_1,\xi_1) \le x^t(x_2,\xi_2)$)

$$|\xi^{s_0}(x^t(x_2,\xi_2),\xi^t(x_2,\xi_2)) - \xi^{s_0}(x^t(x_1,\xi_1),\xi^t(x_2,\xi_2))| \le \int_{x^t(x_1,\xi_1)}^{x^t(x_2,\xi_2)} |\partial_x \xi^{s_0}(z,\xi^t(x_2,\xi_2))|\,dz \ll \lambda^{\frac{3}{4}}.$$

On the other hand, using Proposition 13.4,

$$|\xi^{s_0}(x^t(x_1,\xi_1),\xi^t(x_1,\xi_1)) - \xi^{s_0}(x^t(x_1,\xi_1),\xi^t(x_2,\xi_2))| \le \int_{\xi^t(x_1,\xi_1)}^{\xi^t(x_2,\xi_2)} |\partial_\xi \xi^{s_0}(x^t(x_1,\xi_1),\eta)|\,d\eta \le 2|\xi^t(x_1,\xi_1) - \xi^t(x_2,\xi_2)|.$$

Using these two estimates with (13.19), we obtain

$$|\xi_1 - \xi_2| \le 2|\xi^t(x_1,\xi_1) - \xi^t(x_2,\xi_2)| + \lambda^{\frac{3}{4}}.$$

□

Next, we observe that two characteristics which intersect at two distant times must have similar initial frequencies:

Proposition 13.6. *For solutions (x^t,ξ^t) to (10.1) with initial data (x_i,ξ_i) at initial time $s_0 \in I$, if*

$$|x^s(x_1,\xi_1) - x^s(x_2,\xi_2)| \le \lambda^{-\frac{3}{4}}, \quad |x^t(x_1,\xi_1) - x^t(x_2,\xi_2)| \le \lambda^{-\frac{3}{4}}$$

for $t, s \in I$, then

$$|\xi_1 - \xi_2| \lesssim \lambda^{\frac{3}{4}}|t-s|^{-1}.$$

Proof. Write $x^r = x^r_{s_0}$ unless otherwise indicated, and

$$x^t(x_1,\xi_1) - x^t(x_2,\xi_2) = x^t_s(x^s(x_1,\xi_1),\xi^s(x_1,\xi_1)) - x^t_s(x^s(x_2,\xi_2),\xi^s(x_2,\xi_2)). \tag{13.20}$$

Using the fundamental theorem of calculus and (ii) in Proposition 13.3,

$$|x^t_s(x^s(x_2,\xi_2),\xi^s(x_2,\xi_2)) - x^t_s(x^s(x_1,\xi_1),\xi^s(x_2,\xi_2))| \le \int_{x^s(x_1,\xi_1)}^{x^s(x_2,\xi_2)} |\partial_x x^t_s(z,\xi^s(x_2,\xi_2))|\,dz \lesssim \lambda^{-\frac{3}{4}}.$$

On the other hand, considering without loss of generality the case $t \ge s$ and $\xi^s(x_2,\xi_2) \ge \xi^s(x_1,\xi_1)$, we may use Proposition 13.4 to estimate

$$x^t_s(x^s(x_1,\xi_1),\xi^s(x_1,\xi_1)) - x^t_s(x^s(x_1,\xi_1),\xi^s(x_2,\xi_2)) = -\int_{\xi^s(x_1,\xi_1)}^{\xi^s(x_2,\xi_2)} \partial_\xi x^t_s(x^s(x_1,\xi_1),\eta)\,d\eta \gtrsim \lambda^{-\frac{3}{2}}(\xi^s(x_2,\xi_2) - \xi^s(x_1,\xi_1))(t-s).$$

Using these two estimates with (13.20), we obtain

$$|x^t(x_1,\xi_1) - x^t(x_2,\xi_2)| + \lambda^{-\frac{3}{4}} \gtrsim \lambda^{-\frac{3}{2}}(\xi^s(x_2,\xi_2) - \xi^s(x_1,\xi_1))(t-s).$$

We have an upper bound on the left hand side by assumption, and a lower bound on the right hand side by Proposition 13.5:

$$2\lambda^{-\frac{3}{4}} \gtrsim \frac{1}{2}\lambda^{-\frac{3}{2}}(|\xi_1 - \xi_2| - \lambda^{\frac{3}{4}})(t-s).$$

Rearranging yields the claim.

□

Conversely, we see that if two packets intersect at a given time, then they intersect at nearby times:

Proposition 13.7. *For solutions (x^t, ξ^t) to (10.1) with initial data (x_i, ξ_i) at initial time $s_0 \in I$,*

$$|x^t(x_1,\xi_1) - x^t(x_2,\xi_2)| \lesssim |x^s(x_1,\xi_1) - x^s(x_2,\xi_2)| + \lambda^{-\frac{1}{2}}\mathcal{F}(T)|t-s|.$$

Proof. Write $x^r = x^r_{s_0}$ unless otherwise indicated, and

$$x^t(x_1,\xi_1) - x^t(x_2,\xi_2) = x^t_s(x^s(x_1,\xi_1), \xi^s(x_1,\xi_1)) - x^t_s(x^s(x_2,\xi_2), \xi^s(x_2,\xi_2)). \tag{13.21}$$

Using the fundamental theorem of calculus and (ii) in Proposition 13.3,

$$\begin{aligned} x^t_s(x^s(x_2,\xi_2), \xi^s(x_2,\xi_2)) - x^t_s(x^s(x_1,\xi_1), \xi^s(x_2,\xi_2)) &= \int_{x^s(x_1,\xi_1)}^{x^s(x_2,\xi_2)} \partial_x x^t_s(z, \xi^s(x_2,\xi_2))\, dz \\ &\approx x^s(x_2,\xi_2) - x^s(x_1,\xi_1). \end{aligned}$$

On the other hand, we may use the last estimate of Proposition 13.4 to obtain

$$\begin{aligned} |x^t_s(x^s(x_1,\xi_1), \xi^s(x_1,\xi_1)) - x^t_s(x^s(x_1,\xi_1), \xi^s(x_2,\xi_2))| &\leq \int_{\xi^s(x_1,\xi_1)}^{\xi^s(x_2,\xi_2)} |\partial_\xi x^t_s(x^s(x_1,\xi_1), \eta)|\, d\eta \\ &\leq \lambda^{-\frac{1}{2}}\mathcal{F}(T)|t-s|. \end{aligned}$$

Using these two estimates with (13.21), we obtain

$$|x^t(x_1,\xi_1) - x^t(x_2,\xi_2)| \lesssim |x^s(x_1,\xi_1) - x^s(x_2,\xi_2)| + \lambda^{-\frac{1}{2}}\mathcal{F}(T)|t-s|.$$

□

13.4. **Characteristic local smoothing.** The analysis so far has addressed the motion of the characteristics (x^t, ξ^t) in physical space, and hence the motion of the wave packets we construct in Section 14. However, for the square summability of the wave packets in our parametrix construction, we will need control of an extra half derivative on the phase of the packets. We obtain this gain via local smoothing estimates on the characteristics.

Before establishing the local smoothing estimate, we establish a bilipschitz property for the Hamilton characteristics, but relative to initial data at a time $s_T \in I$ amenable to local smoothing:

Lemma 13.8. *Let $T = (x_0, \xi_0)$ with $|\xi_0| \in [\lambda/2, 2\lambda]$. There exists $s_T \in I$ such that for solutions (x^t, ξ^t) to (10.1) with initial data (x, ξ) at initial time s_T, and satisfying $|\xi| \in [\lambda/2, 2\lambda]$,*

$$\|w_{x^{s_T},\lambda}(\partial_x X^t - \lambda^{\frac{3}{4}} I)\|_{L^\infty_t(I;L^\infty_x)} \ll \lambda^{\frac{3}{4}}, \quad \|w_{x^{s_T},\lambda}\tilde{F}(s_T)\|_{H^{\frac{1}{2}+}_x} \ll T^{-\frac{1}{2}}$$

where we denote $x^{s_T} = x^{s_T}_{s_0}(x_0, \xi_0)$.

Proof. The proof is similar to the proof of Proposition 13.3, except we multiply by $w_{x^s,\lambda}$ before applying L^∞_x, with s to be chosen as follows: When estimating

$$(I_F \partial_x P)(s) = I_F(s) \begin{pmatrix} \lambda^{\frac{3}{4}} I \\ 0 \end{pmatrix} = \lambda^{\frac{3}{4}} \begin{pmatrix} I \\ \tilde{F}(s) \end{pmatrix},$$

we use that

$$\|w_{x^s,\lambda}\tilde{F}(s)\|_{L^\infty_x} \lesssim \|w_{x^s,\lambda}\tilde{F}(s)\|_{H_x^{\frac{1}{2}+}}.$$

Then we choose $s = s_T$ such that

$$\|w_{x^{s_T},\lambda}\tilde{F}(s_T)\|_{H_x^{\frac{1}{2}+}} \leq T^{-1}\|w_{x^t,\lambda}\tilde{F}\|_{L^1_t(I;H_x^{\frac{1}{2}+})} \leq T^{-\frac{1}{2}}\|w_{x^t,\lambda}\tilde{F}\|_{L^2_t(I;H_x^{\frac{1}{2}+})}.$$

Then using Corollary 13.2, combined with the fact that by Corollaries 11.8 and 11.11,

$$\|\mathcal{L}(t,x^t,\xi^t,\lambda)\|_{L^2_t} \leq \lambda^{0+}\mathcal{F}(T),$$

we have

$$\|w_{x^{s_T},\lambda}\tilde{F}(s_T)\|_{H_x^{\frac{1}{2}+}} \leq T^{-\frac{1}{2}}\lambda^{0-}\mathcal{F}(T) \ll T^{-\frac{1}{2}}.$$

□

Proposition 13.9. *Let $T = (x_0,\xi_0)$ with $|\xi_0| \in [\lambda/2, 2\lambda]$. There exists $s_T \in I$ such that for solutions (x^t,ξ^t) to (10.1) with initial data (x,ξ) at initial time s_T, and satisfying $|\xi| \in [\lambda/2, 2\lambda]$,*

$$\|\chi_{x^{s_T},\lambda}(\partial_x X^t - \lambda^{\frac{3}{4}}I)\|_{H^{\frac{1}{2}+}} \ll \lambda^{\frac{3}{4}},$$

$$\|\chi_{x^{s_T},\lambda}\partial_x \Xi^t\|_{L^2_t(I;H_x^{\frac{1}{2}+})} \ll \lambda^{\frac{3}{4}},$$

$$\|\chi_{x^{s_T},\lambda}\partial_x \Xi^t\|_{L^\infty_t(I;L^2_x)} \ll \lambda^{\frac{3}{8}-}$$

where we denote $x^{s_T} = x^{s_T}_{s_0}(x_0,\xi_0)$.

Proof. We begin by multiplying (13.5) by $\chi_{x^{s_T},\lambda}(x)$, applying $H^{\frac{1}{2}+}$ norms, and using the algebra property,

$$\begin{aligned}\|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(t)\|_{H^{\frac{1}{2}+}} &\lesssim \|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(s_T)\|_{H^{\frac{1}{2}+}} \\ &\quad + \int_{s_T}^t \|\tilde{\chi}_{x^{s_T},\lambda}(AI_F^{-1})(r)\|_{H^{\frac{1}{2}+}}\|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(r)\|_{H^{\frac{1}{2}+}}\, dr.\end{aligned}$$

Then applying Gronwall,

(13.22)

$$\|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(t)\|_{H^{\frac{1}{2}+}} \lesssim \|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(s_T)\|_{H^{\frac{1}{2}+}} \exp\left(\int_{s_T}^t \|\tilde{\chi}_{x^{s_T},\lambda}(AI_F^{-1})(r)\|_{H^{\frac{1}{2}+}}\, dr\right).$$

We consider the exponential term. Consider a typical term in the matrix AI_F^{-1},

$$\|\chi_{x^{s_T},\lambda}(x)\tilde{H}_{x\xi}(r,x^r,\xi^r)\|_{H^{\frac{1}{2}+}}.$$

In turn, $\tilde{H}_{x\xi}$ is a sum of a transport term and a dispersive term. We consider the transport term,

$$\|\chi(\lambda^{\frac{3}{4}}(x-x^{s_T}))(\partial_x V_\lambda)(r,x^r(x,\xi))\|_{H^{\frac{1}{2}+}}.$$

Using the bilipschitz property of Lemma 13.8, this may be rewritten with an inserted weight,

$$\|\chi(\lambda^{\frac{3}{4}}(x-x^{s_T}))\tilde{\chi}(\lambda^{\frac{3}{4}}(x^r(x,\xi)-x^r(x^{s_T},\xi)))(\partial_x V_\lambda)(r,x^r(x,\xi))\|_{H^{\frac{1}{2}+}}.$$

Then using the algebra property of $H^{\frac{1}{2}+}$, it suffices to estimate

$$\|\tilde{\chi}(\lambda^{\frac{3}{4}}(x^r(x,\xi)-x^r(x^{s_T},\xi)))(\partial_x V_\lambda)(r,x^r(x,\xi))\|_{H^{\frac{1}{2}+}}.$$

We remark that measuring the weight in $H^{\frac{1}{2}+}$ nets a loss of λ^{0+}. We may choose $T \leq \lambda^{0-}$ to negate this, iterating the argument as necessary.

Using again the bilipschitz property of Lemma 13.8 with the diffeomorphism estimate Proposition E.13, it suffices to estimate

$$\|\tilde{\chi}(\lambda^{\frac{3}{4}}(x - x^r(x^{s_T}, \xi)))(\partial_x V_\lambda)(r, x)\|_{H^{\frac{1}{2}+}} = \|\tilde{\chi}_{x^r,\lambda}\partial_x V_\lambda\|_{H^{\frac{1}{2}+}} \lesssim \|w_{x^r,\lambda}\partial_x V_\lambda\|_{H^{\frac{1}{2}+}}.$$

Then using the last estimate of Lemma 13.1 (more accurately, its proof), use again the assumption $T \leq \lambda^{0-}$ to conclude that

$$\|\chi_{x^{s_T},\lambda}(x)(\partial_x V_\lambda)(r, x^r)\|_{L^1(I;H^{\frac{1}{2}+})} \leq T^{\frac{1}{2}}\lambda^{0+}\mathcal{F}(T) \ll 1.$$

Similar analyses apply to the other terms of AI_F^{-1}, with repeated use of the algebra property of $H^{\frac{1}{2}+}$. We remark that other terms also have ξ^r factors, which require (after possibly applying a Moser composition estimate)

$$\|\chi(\lambda^{\frac{3}{4}}(x - x^{s_T}))\xi^r(x, \xi)\|_{H^{\frac{1}{2}+}} \lesssim \lambda^{1+}, \tag{13.23}$$

which we establish in Lemma 13.10 below.

We conclude from (13.22), using the estimate on F from Lemma 13.8,

$$\|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(t)\|_{H^{\frac{1}{2}+}} \lesssim \lambda^{\frac{3}{4}}T^{-\frac{1}{2}}.$$

Further, recalling

$$I_F^{-1} = \begin{pmatrix} I & 0 \\ -\tilde{F} & I \end{pmatrix}$$

and using the same analysis as above, combined with (the proof of) Corollary 13.2,

$$\begin{aligned}\|\chi_{x^{s_T},\lambda}(\partial_x P)(t)\|_{H^{\frac{1}{2}+}} &\lesssim (1 + \|\tilde{\chi}_{x^{s_T},\lambda}\tilde{F}(t)\|_{L^\infty_x})\|\chi_{x^{s_T},\lambda}(I_F\partial_x P)(t)\|_{H^{\frac{1}{2}+}} \\ &\leq \lambda^{\frac{3}{4}}T^{-\frac{1}{2}}\mathcal{F}(M(t))(Z(t) + \mathcal{L}(t, x^t, \xi^t, \lambda)).\end{aligned}$$

Using an argument analogous to Step 3 in the proof of Proposition 13.3, we can boost this estimate for $\partial_x X^t$ to

$$\|\chi_{x^{s_T},\lambda}(\partial_x X^t - \lambda^{\frac{3}{4}}I)\|_{H^{\frac{1}{2}+}} \ll \lambda^{\frac{3}{4}}. \tag{13.24}$$

We also adapt a similar argument to obtain improvements for $\partial_x \Xi^t$. From the bottom row of (13.5),

$$-\tilde{F}(t)\partial_x X^t + \partial_x \Xi^t = \lambda^{\frac{3}{4}}\tilde{F}(s_T) + \int_{s_T}^t (\tilde{F}\tilde{H}_{\xi x} - \tilde{G})(r)\partial_x X^r + (\tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi})(r)\partial_x \Xi^r\, dr.$$

Multiplying by $\chi_{x^{s_T},\lambda}(x)$, applying $H^{\frac{1}{2}+}$ norms, using the algebra property, and applying Gronwall,

$$\begin{aligned}\|\chi_{x^{s_T},\lambda}\partial_x\Xi^t\|_{H^{\frac{1}{2}+}} \lesssim\ &(\|\chi_{x^{s_T},\lambda}(\lambda^{\frac{3}{4}}\tilde{F}(s_T) + \tilde{F}(t)\partial_x X^t)\|_{H^{\frac{1}{2}+}} \\ &+ \|\chi_{x^{s_T},\lambda}(\tilde{F}\tilde{H}_{\xi x} - \tilde{G})(r)\partial_x X^r\|_{L^1_r(I;H^{\frac{1}{2}+})}) \\ &\cdot \exp\left(\int_{s_T}^t \|\tilde{\chi}_{x^{s_T},\lambda}(\tilde{F}\tilde{H}_{\xi\xi} - \tilde{H}_{x\xi})(r)\|_{H_x^{\frac{1}{2}+}}\, dr\right).\end{aligned} \tag{13.25}$$

The exponential term is estimated similar to the exponential term before. The first and second terms are also similar, using additionally the uniform estimate on $\partial_x X^t$ from (13.24). We conclude

$$\|\chi_{x^{s_T},\lambda}\partial_x\Xi^t\|_{H^{\frac{1}{2}+}} \leq \lambda^{\frac{3}{4}}(cT^{-\frac{1}{2}} + \lambda^{-\epsilon}\mathcal{F}(M(t))(Z(t) + \mathcal{L}(t, x^t, \xi^t, \lambda))).$$

Integrating in time, we obtain

$$\|\chi_{x^{s_T},\lambda}\partial_x\Xi^t\|_{L^2_t(I;H_x^{\frac{1}{2}+})} \ll \lambda^{\frac{3}{4}}$$

as desired.

The third estimate is similar to the second, but using L^2_x in the place of $H_x^{\frac{1}{2}}$. In particular, in estimating the right hand side of (13.25), we use the (global) L^2 estimate of Corollary 13.2,

$$\|\tilde{F}(t)\|_{L^2} \leq \lambda^{-\frac{3}{8}-}\mathcal{F}(M(t)) \leq \lambda^{-\frac{3}{8}-}\mathcal{F}(T)$$

which provides the relative gain, and further is uniform in time. □

It remains to establish the following local estimate (13.23) on ξ^t:

Lemma 13.10. *Let $T = (x_0, \xi_0)$ with $|\xi_0| \in [\lambda/2, 2\lambda]$. There exists $s_T \in I$ such that for solutions (x^t, ξ^t) to (10.1) with initial data (x, ξ) at initial time s_T, and satisfying $|\xi| \in [\lambda/2, 2\lambda]$,*

$$\|\chi_{x^{s_T},\lambda}\xi^t\|_{H^{\frac{1}{2}+}} \lesssim \lambda^{1+}$$

where we denote $x^{s_T} = x^{s_T}_{s_0}(x_0, \xi_0)$.

Proof. Beginning with

$$\xi^t = \xi_0 - \int_{s_T}^t H_x(r)\,dr = \xi - \int_{s_T}^t (\partial_x V_\lambda)(r)\xi^r + (\partial_x\sqrt{a_\lambda})(r)|\xi^r|^{\frac{1}{2}}\,dr,$$

multiply by $\chi_{x^{s_T},\lambda}(x)$, apply $H^{\frac{1}{2}+}$ norms, and use the algebra property to obtain

$$\|\chi_{x^{s_T},\lambda}\xi^t\|_{H^{\frac{1}{2}+}} \lesssim \xi\|\chi_{x^{s_T},\lambda}\|_{H^{\frac{1}{2}+}} + \int_{s_T}^t (\|\tilde{\chi}_{x^{s_T},\lambda}(\partial_x V_\lambda)(r)\|_{H^{\frac{1}{2}+}} \\ + \|\tilde{\chi}_{x^{s_T},\lambda}(\partial_x\sqrt{a_\lambda})(r)\|_{H^{\frac{1}{2}+}}\lambda^{-\frac{1}{2}})\|\chi_{x^{s_T},\lambda}\xi^r\|_{H^{\frac{1}{2}+}}\,dr.$$

Applying Gronwall, we obtain

$$\|\chi_{x^{s_T},\lambda}\xi^t\|_{H^{\frac{1}{2}+}} \lesssim \lambda^{1+}\exp\left(\int_{s_T}^t (\|\tilde{\chi}_{x^{s_T},\lambda}(\partial_x V_\lambda)(r)\|_{H^{\frac{1}{2}+}} + \|\tilde{\chi}_{x^{s_T},\lambda}(\partial_x\sqrt{a_\lambda})(r)\|_{H^{\frac{1}{2}+}}\lambda^{-\frac{1}{2}})\,dr\right).$$

We estimate the right hand side as in the proof of Proposition 13.9 to obtain the lemma. □

13.5. **The eikonal equation.** In the following, we let y denote the spatial variable, so that in particular we may let $T = (x, \xi)$ without conflict. Further let $s_T \in I$ be chosen as in Lemma 13.8, and

$$(x^{s_T}, \xi^{s_T}) = (x^{s_T}_{s_0}(x, \xi), \xi^{s_T}_{s_0}(x, \xi))$$

where $s_0 \in I$ is chosen as in Proposition 13.3. Consider the eikonal equation,

$$\partial_t \psi_{x,\xi}(t, y) + \tilde{H}(t, y, \partial_y \psi_{x,\xi}(t, y)) = 0, \qquad \psi_{x,\xi}(s_T, y) = \xi^{s_T}(y - x^{s_T}), \tag{13.26}$$

whose characteristics are given by the Hamilton flow. The fact that the Hamilton flow is bilipschitz corresponds to two derivatives on the eikonal solution:

Proposition 13.11. *Let $T = (x, \xi)$ with $|\xi| \in [\lambda/2, 2\lambda]$. Then for a solution $\psi_{x,\xi}$ to (13.26),*

$$\|\chi_{x^t_{s_0}(x,\xi),\lambda} \partial_y^2 \psi_{x,\xi}\|_{L^2_t(I; H_y^{\frac{1}{2}+})} \ll \lambda^{\frac{3}{2}},$$

$$\|\chi_{x^t_{s_0}(x,\xi),\lambda} \partial_y^2 \psi_{x,\xi}\|_{L^\infty_t(I; L^2_y)} \ll \lambda^{\frac{9}{8}-}.$$

Proof. Using the characteristics for $\psi_{x,\xi}$, write

$$\xi^t(z, \xi^{s_T}) = (\partial_y \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))$$

so that

$$\partial_x \xi^t(z, \xi^{s_T}) = (\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))(\partial_x x^t)(z, \xi^{s_T}).$$

By Lemma 13.8, $|(\partial_x x^t)(z, \xi^{s_T})| > \frac{1}{2}$ on the support of $\chi_{x^{s_T},\lambda}(z)$, so we may write

$$\chi_{x^{s_T},\lambda}(z)(\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T})) = \chi_{x^{s_T},\lambda}(z)(\partial_x \xi^t)(z, \xi^{s_T})(\tilde{\chi}_{x^{s_T},\lambda}(z)(\partial_x x^t)(z, \xi^{s_T}))^{-1}.$$

Using the algebra property of $H^{\frac{1}{2}+}$, as well as a Moser estimate on the second term, we obtain by Proposition 13.9,

$$\|\chi_{x^{s_T},\lambda}(z)(\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))\|_{L^2_t(I; H_z^{\frac{1}{2}+})} \ll \lambda^{\frac{3}{2}}.$$

It is convenient to exchange χ with $\tilde{\chi}$, with the straightforward modifications to the proof:

$$\|\tilde{\chi}_{x^{s_T},\lambda}(z)(\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))\|_{L^2_t(I; H_z^{\frac{1}{2}+})} \ll \lambda^{\frac{3}{2}}.$$

Similar to the proof of Proposition 13.9, we may use the bilipschitz property to insert a weight, followed by the algebra property to obtain

$$\begin{aligned}
&\|\chi_{x^t(x^{s_T},\xi^{s_T}),\lambda}(x^t(z, \xi^{s_T}))(\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))\|_{L^2_t(I; H_z^{\frac{1}{2}+})} \\
&\quad = \|\chi_{x^t(x^{s_T},\xi^{s_T}),\lambda}(x^t(z, \xi^{s_T}))\tilde{\chi}_{x^{s_T},\lambda}(z)(\partial_y^2 \psi_{x,\xi})(t, x^t(z, \xi^{s_T}))\|_{L^2_t(I; H_z^{\frac{1}{2}+})} \\
&\quad \ll \lambda^{\frac{3}{2}}.
\end{aligned}$$

Lastly, setting $y = x^t(z, \xi^{s_T})$ and combining the diffeomorphism estimate Lemma E.13 with the bilipschitz property of Lemma 13.8, we obtain the first estimate of the proposition.

The proof of the L^2 estimate is similar, using the corresponding third estimate of Proposition 13.9. ☐

As a consequence, we observe that $\partial_y \psi_{x,\xi}$ and $\xi^t(x, \xi)$ are comparable when following a characteristic on the $\lambda^{-\frac{3}{4}}$ spatial scale:

Corollary 13.12. *Let $T = (x, \xi)$ with $|\xi| \in [\lambda/2, 2\lambda]$. Then for solutions $\psi_{x,\xi}$ to (13.26),*

$$\|\chi_{x^t_{s_0}(x,\xi),\lambda}(y)(\partial_y \psi_{x,\xi}(t, y) - \xi^t_{s_0}(x, \xi))\|_{L^\infty_y} \ll \lambda^{\frac{3}{4}-}.$$

Proof. We write $(x^t, \xi^t) = (x^t_{s_0}, \xi^t_{s_0})$. Using the fundamental theorem of calculus, write

$$\partial_y \psi_{x,\xi}(t, y) - \xi^t(x, \xi) = \partial_y \psi_{x,\xi}(t, y) - \partial_y \psi_{x,\xi}(t, x^t(x, \xi)) = \int_{x^t}^{y} \partial_y^2 \psi_{x,\xi}(t, z)\, dz$$

so that using Cauchy-Schwarz followed by Proposition 13.11,

$$|\partial_y \psi_{x,\xi}(t, y) - \xi^t(x, \xi)| \leq |y - x^t(x, \xi)|^{\frac{1}{2}} \|\partial_y^2 \psi_{x,\xi}\|_{L^2_y([x^t,y])} \ll \lambda^{\frac{9}{8}-}|y - x^t(x, \xi)|^{\frac{1}{2}}.$$

Then restricting to $|y - x^t(x, \xi)| \approx \lambda^{-\frac{3}{4}}$ via the cutoff $\chi_{x^t,\lambda}$, we have the desired estimate. □

14. Wave Packet Parametrix

In this section we construct the wave packet parametrix. Define the index set

$$\mathcal{T} = \{T = (x, \xi) \in \lambda^{-\frac{3}{4}}\mathbb{Z} \times \lambda^{\frac{3}{4}}\mathbb{Z} : |\xi| \in [\lambda/2, 2\lambda]\}.$$

Then a *wave packet* $u_T = u_{x,\xi}$ centered at (x, ξ) is a function of the form

$$u_T(t, y) = \lambda^{\frac{3}{8}} \chi_T(t, y) e^{i\psi_T(t,y)}$$

where

$$\chi_T(t, y) = \chi(\lambda^{\frac{3}{4}}(y - x^t_{s_0}(x, \xi)))$$

and

$$\psi_T(t, y) = \psi_{x,\xi}(t, y)$$

solves the eikonal equation (13.26). Also note that we write

$$\chi'_T = (\chi')_T = (\chi')(\lambda^{\frac{3}{4}}(y - x^t(x, \xi)))$$

so that in particular

$$\lambda^{\frac{3}{8}} \chi'_T(t, y) e^{i\psi_T(t,y)}$$

is a wave packet.

14.1. Approximate solution.

A wave packet is an approximate solution to

$$(D_t + H(t, y, D))u = 0$$

in the follow sense:

Proposition 14.1. *Let u_T be a wave packet. Then we may write $(D_t + H(t, y, D))S_\lambda u_T$ as a sum of terms, each taking one of the following forms:*

i) $A(t, y)p_T(D)v_T(t, y)e^{i\psi_T(t,y)}$ *where* $\|A(t)\|_{L^\infty_y} \leq \mathcal{F}(T)$, $|p_T^{(N)}(\eta)| \lesssim \lambda^{-N}$, $v_T = v_T\tilde{\chi}_T$, *and*

$$\|v_T(t)\|_{L^2_y} \leq \mathcal{F}(M(t))Z(t), \quad \|\partial_y v_T(t)\|_{L^2_y} \leq \lambda^{\frac{3}{4}}\mathcal{F}(M(t))Z(t).$$

ii) $A(t, y)p_T(D)v_T(t, y)e^{i\psi_T(t,y)}$ *as before except* $v_T = v_T\tilde{\chi}_T$ *instead satisfies*

$$\|v_T(t)\|_{L^2_y} \ll 1, \quad \||D|^{\frac{1}{2}} v_T(t)\|_{L^2_t(I;L^2_y)} \ll \lambda^{\frac{3}{8}}.$$

iii) $[S_\lambda, V_\lambda]v_T e^{i\psi_T}$ *with* $v_T = v_T\tilde{\chi}_T$ *satisfying*

$$\|v_T(t)\|_{L^2_y} \lesssim \lambda, \quad \|\partial_y v_T(t)\|_{L^2_y} \lesssim \lambda^{1+\frac{3}{4}}.$$

iv) $[S_\lambda, \sqrt{a_\lambda}]v_T e^{i\psi_T}$ *with* $v_T = v_T\tilde{\chi}_T$ *satisfying*

$$\|v_T(t)\|_{L^2_y} \lesssim \lambda^{\frac{1}{2}}, \quad \|\partial_y v_T(t)\|_{L^2_y} \lesssim \lambda^{\frac{1}{2}+\frac{3}{4}}.$$

Proof. Step 1. First we compute and arrange the error terms. By a direct computation using (10.1) and (13.26),

$$\begin{aligned}(D_t + H(t,y,D))S_\lambda u_T = (&H(t,y,D)S_\lambda - S_\lambda H(t,y,\partial_y\psi_T(t,y)) \\ &- S_\lambda H_\xi(t,x^t,\xi^t)(D_y - \partial_y\psi_T(t,y)))u_T.\end{aligned}$$

To organize the right hand side, first we exchange the coefficient $H_\xi(t,x^t,\xi^t)$ for

$$H_\xi(t,y,\xi^t)$$

with the goal of separating the spatial variable y of H from the Taylor expansion in ξ. Writing

$$v_{T,1} := H_\xi(t,y,\xi^t) - H_\xi(t,x^t,\xi^t),$$

we have

$$\begin{aligned}(D_t + H(t,y,D))S_\lambda u_T = (&H(t,y,D)S_\lambda - S_\lambda H(t,y,\partial_y\psi_T(t,y)) \\ &- S_\lambda H_\xi(t,y,\xi^t)(D_y - \partial_y\psi_T(t,y)))u_T - i\lambda^{\frac{3}{4}}S_\lambda v_{T,1}\lambda^{\frac{3}{8}}\chi_T' e^{i\psi_T}.\end{aligned}$$

The right hand side may be viewed as a symbol expansion of $H(t,y,\eta)$ around $\partial_y\psi_T(t,y)$. However, this center is y-dependent. To remedy this, we can shift this center to ξ^t by defining the error

$$V_{T,2}u_T := (H(t,y,\xi^t) + H_\xi(t,y,\xi^t)(\partial_y\psi_T(t,y) - \xi^t) - H(t,y,\partial_y\psi_T(t,y)))u_T$$

and writing

$$\begin{aligned}(D_t + H(t,y,D))S_\lambda u_T = (&H(t,y,D)S_\lambda - S_\lambda H(t,y,\xi^t) - S_\lambda H_\xi(t,y,\xi^t)(D_y - \xi^t))u_T \\ &- i\lambda^{\frac{3}{4}}S_\lambda v_{T,1}\lambda^{\frac{3}{8}}\chi_T' e^{i\psi_T} + S_\lambda V_{T,2}u_T.\end{aligned}$$

Observe that on the right hand side, we now have a symbol expansion of $H(t,y,\eta)$ in η to second order, modulo commutators with S_λ, and in particular, the transport term $V_\lambda\xi$ of H vanishes. In turn, this allows us to separate variables in H. More precisely, writing the dispersive term with the notation

$$\sqrt{a_\lambda|\eta|} = \sqrt{a_\lambda}p(\eta),$$

we may write, using the Lagrange remainder and observing that $p''(\eta) = |\eta|^{-\frac{3}{2}}$,

$$\begin{aligned}(H(t,y,D)S_\lambda &- S_\lambda H(t,y,\xi^t) - S_\lambda H_\xi(t,y,\xi^t)(D_y - \xi^t))u_T \\ &= \sqrt{a_\lambda(y)}(p(D) - p(\xi^t) - p'(\xi^t)(D_y - \xi^t))S_\lambda u_T \\ &\quad - [S_\lambda, H(t,y,\xi^t) + H_\xi(t,y,\xi^t)(D_y - \xi^t)]u_T \\ &= \sqrt{a_\lambda(y)}S_\lambda|q_T(D)|^{-\frac{3}{2}}(D - \xi^t)^2u_T \\ &\quad - [S_\lambda, H(t,y,\xi^t) + H_\xi(t,y,\xi^t)(D_y - \xi^t)]u_T.\end{aligned}$$

By a routine computation,

$$\begin{aligned}(D - \xi^t)^2u_T = &-\lambda^{\frac{3}{2}}\lambda^{\frac{3}{8}}\chi_T'' e^{i\psi_T} - 2i\lambda^{\frac{3}{4}}(\partial_y\psi_T - \xi^t)\lambda^{\frac{3}{8}}\chi_T' e^{i\psi_T} + (\partial_y\psi_T - \xi^t)^2\lambda^{\frac{3}{8}}\chi_T e^{i\psi_T} \\ &- i\partial_y^2\psi_T\lambda^{\frac{3}{8}}\chi_T e^{i\psi_T}.\end{aligned}$$

Since $V_{T,2}$ also takes the form of a Taylor expansion, an analogous analysis applies. However, it is convenient to instead use the integral form of the remainder:

$$\begin{aligned} V_{T,2} &= \sqrt{a_\lambda(y)}(p(\xi^t) + p'(\xi^t)(\partial_y \psi_T(t,y) - \xi^t) - p(\partial_y \psi_T(t,y))) \\ &= -\sqrt{a_\lambda(y)} \int_{\xi^t}^{\partial_y \psi_T(t,y)} |\eta|^{-\frac{3}{2}} (\partial_y \psi_T(t,y) - \eta)\, d\eta \\ &=: \sqrt{a_\lambda(y)} v_{T,2}. \end{aligned}$$

Lastly, write $p_T(\eta) = \lambda^{\frac{3}{2}} |q_T(\eta)|^{-\frac{3}{2}}$, observing that on the support the symbol of S_λ, we have $|p_T^{(N)}(\eta)| \lesssim \lambda^{-N}$. We conclude

$$\begin{aligned} (14.1)\quad (D_t + H(t,y,D)) S_\lambda u_T = &\sqrt{a_\lambda(y)} S_\lambda p_T(D) (-\lambda^{\frac{3}{8}} \chi_T'' e^{i\psi_T} - 2i\lambda^{-\frac{3}{4}} (\partial_y \psi_T - \xi^t) \lambda^{\frac{3}{8}} \chi_T' e^{i\psi_T} \\ &+ \lambda^{-\frac{3}{2}} (\partial_y \psi_T - \xi^t)^2 \lambda^{\frac{3}{8}} \chi_T e^{i\psi_T} - i\lambda^{-\frac{3}{2}} \partial_y^2 \psi_T \lambda^{\frac{3}{8}} \chi_T e^{i\psi_T}) \\ &- i\lambda^{\frac{3}{4}} S_\lambda v_{T,1} \lambda^{\frac{3}{8}} \chi_T' e^{i\psi_T} + \sqrt{a_\lambda(y)} S_\lambda v_{T,2} u_T \\ &- [S_\lambda, H(t,y,\xi^t) + H_\xi(t,y,\xi^t)(D_y - \xi^t)] u_T \\ &+ [S_\lambda, \sqrt{a_\lambda(y)}] v_{T,2} u_T. \end{aligned}$$

Step 2. Next, we check that the non-commutator terms on the right hand side of (14.1) may be put into the desired form with estimates.

We show the first three terms on the right hand side of (14.1) (from the first and second rows) may be placed in case (i). We bound $\sqrt{a_\lambda(y)}$ using Proposition D.8. Using Corollary 13.12, we have

$$\|(\partial_y \psi_T - \xi^t) \tilde{\chi}_T\|_{L_y^\infty} \ll \lambda^{\frac{3}{4}},$$

and we easily have

$$\|\lambda^{\frac{3}{8}} (\chi_T, \chi_T', \chi_T'')\|_{L_y^2} \lesssim 1.$$

It is easy to check that the coefficients satisfy the L^2 estimates desired of v_T by using the widened cutoff. For instance, we have

$$\|(\partial_y \psi_T - \xi^t) \lambda^{\frac{3}{8}} \chi_T'\|_{L_y^2} \lesssim \|(\partial_y \psi_T - \xi^t) \tilde{\chi}_T\|_{L_y^\infty} \|\lambda^{\frac{3}{8}} \chi_T'\|_{L_y^2} \ll \lambda^{\frac{3}{4}}.$$

To estimate the first derivatives, use also Proposition 13.11,

$$\|(\partial_y^2 \psi_T) \chi_T\|_{L_y^2} \ll \lambda^{9/8},$$

and likewise

$$\|\partial_y \lambda^{\frac{3}{8}} (\chi_T, \chi_T', \chi_T'')\|_{L_y^2} \lesssim \lambda^{\frac{3}{4}}.$$

Thus we also obtain the L^2 estimates desired of $\partial_y v_T$.

Consider the fourth term on the right hand side of (14.1) (from the second row), with $\partial_y^2 \psi_T$. The two estimates of Proposition 13.11 yield both estimates desired of v_T for case (ii).

Next, we place the fifth term of (14.1), with

$$v_{T,1} = H_\xi(t,y,\xi^t) - H_\xi(t,x^t,\xi^t),$$

in case (i). Using the fundamental theorem of calculus,

$$H_\xi(t,y,\xi^t) - H_\xi(t,x^t,\xi^t) = \int_{x^t}^{y} H_{x\xi}(t,z,\xi^t)\, dz.$$

Restricting to the support of χ_T and thus $|y - x^t| \lesssim \lambda^{-\frac{3}{4}}$, and using the uniform bound on $H_{x\xi}$ from Lemma 13.1, we have

$$\|(H_\xi(t, y, \xi^t) - H_\xi(t, x^t, \xi^t))\chi_T'\|_{L_y^\infty} \leq \lambda^{-\frac{3}{4}}\mathcal{F}(M(t))Z(t)$$

and thus, using a widened cutoff as before, the L^2 estimate desired of v_T. The first derivative is easier, using directly Lemma 13.1.

We place the the sixth term of (14.1), with

$$v_{T,2} = \int_{\xi^t}^{\partial_y \psi_T(t,y)} |\eta|^{-\frac{3}{2}}(\partial_y \psi_T(t, y) - \eta)\, d\eta,$$

in case (i). Since

$$\xi^t(z, \xi) = (\partial_y \psi_T)(t, x^t(z, \xi)),$$

by Lemma 11.1, we may restrict to $\eta \approx \lambda$. Also restricting to the support of χ_T and thus $|y - x^t| \lesssim \lambda^{-\frac{3}{4}}$, we have using Corollary 13.12 twice,

$$\|v_{T,2}\chi_T\|_{L_y^\infty} \ll \lambda^{-\frac{3}{2}}\lambda^{\frac{3}{4}}\lambda^{\frac{3}{4}} = 1$$

and thus the desired L_y^2 estimate using a widened cutoff. It remains to estimate the first derivative. By a direct compuation,

$$\partial_y v_{T,2} = \int_{\xi^t}^{\partial_y \psi_T(t,y)} |\eta|^{-\frac{3}{2}}\partial_y^2 \psi_T(t, y)\, d\eta.$$

Then (restricting as usual to $|y - x^t| \lesssim \lambda^{-\frac{3}{4}}$) using Proposition 13.11 to estimate $\partial_y^2 \psi_T$ in L_y^2, and Corollary 13.12 to estimate the width of the limit of integration,

$$\|(\partial_y v_{T,2})\lambda^{\frac{3}{8}}\chi_T\|_{L^2} \ll \lambda^{-\frac{3}{2}}\lambda^{9/8}\lambda^{\frac{3}{4}}\lambda^{\frac{3}{8}} = \lambda^{\frac{3}{4}}$$

as desired.

Step 3. It remains to consider the commutator terms on the right hand side of (14.1). There are a total of five, as the Hamiltonian contains two terms.

First, we have

$$[S_\lambda, V_\lambda \xi^t]u_T = [S_\lambda, V_\lambda]\xi^t \lambda^{\frac{3}{8}}\chi_T e^{i\psi_T}.$$

Then $v_T = \xi^t \lambda^{\frac{3}{8}}\chi_T$ satisfies, by Lemma 11.1,

$$\|v_T\|_{L^2} \lesssim \lambda, \quad \|\partial_y v_T\|_{L^2} \lesssim \lambda\lambda^{\frac{3}{4}}.$$

Second, we have

$$[S_\lambda, \sqrt{a_\lambda}|\xi^t|^{\frac{1}{2}}]u_T = [S_\lambda, \sqrt{a_\lambda}]|\xi^t|^{\frac{1}{2}}\lambda^{\frac{3}{8}}\chi_T e^{i\psi_T}$$

so that $v_T = |\xi^t|^{\frac{1}{2}}\lambda^{\frac{3}{8}}\chi_T$ satisfies

$$\|v_T\|_{L^2} \lesssim \lambda^{\frac{1}{2}}, \quad \|\partial_y v_T\|_{L^2} \lesssim \lambda^{\frac{1}{2}}\lambda^{\frac{3}{4}}.$$

Third,

$$\begin{aligned}[S_\lambda, V_\lambda(D_y - \xi^t)]u_T &= [S_\lambda, V_\lambda](D_y - \xi^t)(\lambda^{\frac{3}{8}}\chi_T e^{i\psi_T}) \\ &= [S_\lambda, V_\lambda](-i\lambda^{\frac{3}{4}}\lambda^{\frac{3}{8}}\chi_T' e^{i\psi_T} + \lambda^{\frac{3}{8}}\chi_T(\partial_y \psi_T - \xi^t)e^{i\psi_T}).\end{aligned}$$

An analysis similar to that in Step 2 shows that these are better than needed for case (iii). The fourth term,

$$[S_\lambda, \sqrt{a_\lambda}|\xi^t|^{-\frac{3}{2}}\xi^t(D_y - \xi^t)]u_T = [S_\lambda, \sqrt{a_\lambda}]|\xi^t|^{-\frac{3}{2}}\xi^t\big(- i\lambda^{\frac{3}{4}}\lambda^{\frac{3}{8}}\chi_T' e^{i\psi_T} + \lambda^{\frac{3}{8}}\chi_T(\partial_y\psi_T - \xi^t)e^{i\psi_T}\big)$$

is similarly better than needed.

Lastly, the fifth term with $v_{T,2}$ is similar to the analysis of the corresponding term in Step 2, and again is better than needed.

□

14.2. **Orthogonality.** In this subsection we observe that a collection of wave packets is orthogonal in an appropriate sense. First, we establish the orthogonality of functions in a form as described in the following proposition:

Proposition 14.2. *Let $\{U_T\}$ be functions of the form*

$$U_T(t,y) = A(t,y)p_T(D)v_T(t,y)e^{iy\xi^t(x,\xi)}$$

where $T = (x,\xi) \in \mathcal{T}$, $\|A(t)\|_{L^\infty_y} \leq \mathcal{F}(T)$, $|p_T^{(N)}(\eta)| \lesssim \lambda^{-N}$, and $v_T = v_T\tilde{\chi}_T$. Then

$$\|\sum_{T\in\mathcal{T}} U_T(t)\|^2_{L^2_y} \lesssim \lambda^{-\frac{3}{4}}(\log\lambda)\mathcal{F}(T)\sum_{T\in\mathcal{T}}\|v_T\|^2_{H^{\frac{1}{2}}_y}.$$

Proof. First note that A is independent of T and thus may be factored out and estimated immediately. As a result, we may assume $A \equiv 1$ below.

Step 1. First we reduce the sum over $T = (x,\xi) \in \mathcal{T}$ to fixed x. Consider $\xi \in \{\lambda^{\frac{3}{4}}\mathbb{Z} : |\xi| \in [\lambda/2, 2\lambda]\}$ and $k \in \lambda^{-\frac{3}{4}}\mathbb{Z}$. Consider two packets $(x_1,\xi), (x_2,\xi) \in \mathcal{T}$ intersecting (t,k). By Proposition 13.3, we have

$$\lambda^{-\frac{3}{4}} \geq |x^t(x_1,\xi) - x^t(x_2,\xi)| = |\partial_x x^t(z,\xi)(x_1 - x_2)| \approx |x_1 - x_2|.$$

Since $x_i \in \lambda^{-\frac{3}{4}}\mathbb{Z}$, we conclude that among packets with frequency ξ, there is at most an absolute number whose supports intersect (t,k). For simplicity, we assume there is at most one such packet. Then we may index the packets $\mathcal{T}$ by (k,ξ).

Write

$$\sum_{T\in\mathcal{T}} U_T(t) =: \sum_k\sum_\xi U_T(t) =: \sum_k U^k(t).$$

Note that the $U^k(t)$ have essentially finite overlap. Thus, using polynomial weights and Cauchy-Schwarz,

$$\|\sum_T U_T(t)\|^2_{L^2_y} \lesssim \sum_k \|U^k(t)\|^2_{L^2_y}$$

so that it suffices to show the orthogonality with fixed k. In other words, we may assume $\mathcal{T}$ consists of packets intersecting (t,k), each with distinct ξ.

Step 2. We abuse notation by equating $\xi = (x, \xi) = T$ as indices, and denote $\xi^t = \xi^t(x, \xi)$. By Plancherel's and Cauchy-Schwarz,

$$\begin{aligned}\|\sum_\xi U_T(t)\|_{L^2_y}^2 = \|\sum_\xi \widehat{U}_T(t)\|_{L^2_\eta}^2 &= \int |\sum_\xi \widehat{U}_T(t)\langle \eta - \xi^t\rangle^{\frac{1}{2}}\langle \eta - \xi^t\rangle^{-\frac{1}{2}}|^2 \, d\eta \\ &\lesssim \int \sum_\xi |\widehat{U}_T(t)|^2 \langle \eta - \xi^t\rangle \sum_\xi \langle \eta - \xi^t\rangle^{-1} \, d\eta \\ &\lesssim \left(\sum_\xi \int |\widehat{U}_T(t)|^2\langle \eta - \xi^t\rangle \, d\eta\right)\left(\sup_\eta \sum_\xi \langle \eta - \xi^t\rangle^{-1}\right).\end{aligned}$$

It remains to estimate the two terms.

Step 3. First consider the supremum over η. Fix any $\eta \in [\lambda/2, 2\lambda]$ and write

$$(k, \eta) = (x^t(z, \zeta), \xi^t(z, \zeta)).$$

Then since

$$|x^t(z, \zeta) - x^t(x, \xi)| = |k - x^t(x, \xi)| \leq \lambda^{-\frac{3}{4}},$$

we may apply Proposition 13.5,

$$|\zeta - \xi| \leq 2|\xi^t(z, \zeta) - \xi^t(x, \xi)| + \lambda^{\frac{3}{4}} = 2|\eta - \xi^t| + \lambda^{\frac{3}{4}}.$$

We conclude, using that there are at most $\lambda^{\frac{1}{4}}$ frequencies ξ in $\{|\xi| \in [\lambda/2, 2\lambda]\}$,

$$\sum_\xi \langle \eta - \xi^t\rangle^{-1} \lesssim \sum_\xi \langle \zeta - \xi\rangle^{-1} \lesssim \lambda^{-\frac{3}{4}} \log \lambda.$$

Then take the supremum over η to obtain

$$\sup_\eta \sum_\xi \langle \eta - \xi^t\rangle^{-1} \lesssim \lambda^{-\frac{3}{4}} \log \lambda.$$

Step 4. For the sum over ξ, we have, using a change of variables and Plancherel's,

$$\begin{aligned}\int |\widehat{U}_T(t)|^2\langle \eta - \xi^t\rangle \, d\eta &= \int |p_T(\eta + \xi^t)\widehat{v_T}(t)|^2 \langle \eta\rangle \, d\eta \\ &= \|\langle D\rangle^{\frac{1}{2}} p_T(D + \xi^t) v_T(t)\|_{L^2_y}^2 \\ &\lesssim \|\langle D\rangle^{\frac{1}{2}} v_T(t)\|_{L^2_y}^2.\end{aligned}$$

Combining the above estimates, we conclude

$$\|\sum_\xi U_T(t)\|_{L^2_y}^2 \leq \lambda^{-\frac{3}{4}}(\log \lambda)\mathcal{F}(T)\sum_\xi \|v_T\|^2_{H_y^{\frac{1}{2}}}$$

as desired. □

We then have the following orthogonality of wave packets:

Corollary 14.3. *Let $\{U_T\}$ be functions of the form*

$$U_T(t, y) = A(t, y)p_T(D)v_T(t, y)e^{i\psi_T(t,y)}$$

where $\|A(t)\|_{L^\infty_y} \leq \mathcal{F}(T)$, $|p_T^{(N)}(\eta)| \lesssim \lambda^{-N}$, *and* $v_T = v_T\tilde{\chi}_T$. *Then*

$$\|\sum_{T\in\mathcal{T}} U_T(t)\|_{L^2_y}^2 \lesssim (\log\lambda)\mathcal{F}(T)\sum_{T\in\mathcal{T}}\left(\|v_T(t)\|_{L^2_y}^2 + \lambda^{-\frac{3}{4}}\||D|^{\frac{1}{2}}v_T(t)\|_{L^2_y}^2\right),$$

and

$$\|\sum_{T\in\mathcal{T}} U_T(t)\|_{L^2_y}^2 \lesssim (\log\lambda)\mathcal{F}(T)\sum_{T\in\mathcal{T}}\left(\|v_T(t)\|_{L^2_y}^2 + \lambda^{-\frac{3}{4}}\|\partial_y v_T(t)\|_{L^2_y}\|v_T(t)\|_{L^2_y}\right).$$

Proof. Apply Proposition 14.2 with, in the place of v_T,

$$v_T e^{i(\psi_T - y\xi^t)}.$$

Then write

$$\begin{aligned}\|v_T e^{i(\psi_T-y\xi^t)}\|_{H_y^{\frac{1}{2}}}^2 &= \int (\langle D\rangle v_T e^{i(\psi_T-y\xi^t)})(\bar{v}_T e^{-i(\psi_T-y\xi^t)})\,dy \\ &= \int ([\langle D\rangle, \tilde{\chi}_T e^{i(\psi_T-y\xi^t)}]v_T)(\bar{v}_T e^{-i(\psi_T-y\xi^t)})\,dy + \int (\langle D\rangle v_T)\bar{v}_T\,dy.\end{aligned}$$

The first integral may be estimated using Cauchy-Schwarz and Corollary 13.12:

$$\begin{aligned}\|[\langle D\rangle, \tilde{\chi}_T e^{i(\psi_T-y\xi^t)}]v_T\|_{L^2_y}\|v_T\|_{L^2_y} &\lesssim \|\tilde{\chi}_T(\partial_y\psi_T - \xi^t)e^{i(\psi_T-y\xi^t)} + \lambda^{\frac{3}{4}}\tilde{\chi}_T' e^{i(\psi_T-y\xi^t)}\|_{L^\infty}\|v_T\|_{L^2_y}^2 \\ &\lesssim \lambda^{\frac{3}{4}}\|v_T\|_{L^2_y}^2.\end{aligned}$$

The second integral may be estimated by

$$\|\langle D\rangle^{\frac{1}{2}}v_T\|_{L^2_y}^2 \lesssim \|v_T\|_{L^2_y}^2 + \||D|^{\frac{1}{2}}v_T\|_{L^2_y}^2.$$

Combining the above estimates, we conclude

$$\|\sum_{\xi} U_T(t)\|_{L^2_y}^2 \leq (\log\lambda)\mathcal{F}(T)\sum_{\xi}\left(\|v_T(t)\|_{L^2_y}^2 + \lambda^{-\frac{3}{4}}\||D|^{\frac{1}{2}}v_T\|_{L^2_y}^2\right)$$

as desired.

For the second estimate of the proposition, we instead estimate the second integral above via

$$\langle\langle D\rangle v_T, v_T\rangle \lesssim \|v_T\|_{H_y^1}\|v_T\|_{L^2} \lesssim \|v_T\|_{L^2}^2 + \|\partial_y v_T\|_{L^2}\|v_T\|_{L^2}.$$

□

Combining Proposition 14.1 with Corollary 14.3, we obtain

Corollary 14.4. *Let* $\{c_T\}_{T\in\mathbb{T}} \in \ell^2(\mathcal{T})$ *and*

$$u = \sum_{T\in\mathcal{T}} c_T u_T$$

where u_T *are wave packets. Then*

$$\|(D_t + H(t,y,D))S_\lambda u\|_{L^2(I;L^2_x)}^2 \leq \lambda^{0+}\mathcal{F}(T)\sum_{T\in\mathcal{T}}|c_T|^2$$

and

$$\|(D_t + H(t,y,D))S_\lambda u\|_{L^1(I;L^2_x)}^2 \ll \sum_{T\in\mathcal{T}}|c_T|^2.$$

Proof. The second estimate is immediate from the first, having chosen $T \leq \lambda^{0-}$. The first is obtained by using the estimates on the four types of terms from Proposition 14.1 with the matching orthogonality result of Corollary 14.3. The last two types of terms also require straightforward commutator estimates, to absorb a factor of λ into a derivative on V and $\lambda^{\frac{1}{2}}$ into half of a derivative on a. □

14.3. **Matching the initial data.** To conclude the parametrix construction, it remains to verify that we may use a linear combination of wave packets to match general initial data. In order to achieve this, it is convenient to further specify our choice of cutoff χ to satisfy

$$\sum_{m\in\mathbb{Z}} \chi(y-m)^2 = 1$$

so that

$$v_T(y) = \lambda^{\frac{3}{8}} \chi_T(y) e^{iy\xi}$$

forms a tight frame, in that for $f \in L^2(\mathbb{R})$ with frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$,

$$f = \sum_{T\in\mathcal{T}} c_T v_T, \quad c_T = \int f(y)\overline{v_T}(y)\, dy.$$

However, as we shall see below, it is convenient to instead define

$$v_T(y) = \lambda^{\frac{3}{8}} \chi_T(y) e^{i(\psi_T(s_0,x)+\xi(y-x))} = \lambda^{\frac{3}{8}} \chi_T(y) e^{i(\psi_T(s_0,x)+(\partial_y\psi_T)(s_0,x)(y-x))}$$

which still forms a tight frame.

Proposition 14.5. *Given $u_0 \in L^2_y$ with frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$, there exists $\{a_T\}_{T\in\mathcal{T}} \in \ell^2(\mathcal{T})$ such that*

$$u_0 = \sum_{T\in\mathcal{T}} a_T u_T(s_0)$$

and

$$\sum_{T\in\mathcal{T}} |a_T|^2 \lesssim \|u_0\|^2_{L^2_y}.$$

Proof. Write using the above frame,

$$u_0 = \sum_{T\in\mathcal{T}} c_T v_T.$$

Using these coefficients, construct the linear combination of wave packets

$$\tilde{u} = \sum_{T\in\mathcal{T}} c_T u_T.$$

We consider the difference $u_0 - \tilde{u}(s_0)$, first observing

$$e^{i(\psi_T(s_0,x)+\xi(y-x))} - e^{i\psi_T(s_0,y)} = e^{i(\psi_T(s_0,x)+\xi(y-x))}(1 - e^{i(\psi_T(s_0,y)-\psi_T(s_0,x)-\xi(y-x))}).$$

Then we apply Proposition 14.2 with, in the place of v_T,

$$e^{i(\psi_T(s_0,x)-x\xi)}\chi_T(1 - e^{i(\psi_T(s_0,y)-\psi_T(s_0,x)-(\partial_y\psi_T)(s_0,x)(y-x))}).$$

This is small in L^2_y by Proposition 13.11 with a Taylor expansion:

$$\|\chi_T(1 - e^{i(\psi_T(s_0,y)-\psi_T(s_0,x)-(\partial_y\psi_T)(s_0,x)(y-x))})\|_{L^2_y} \ll \lambda^{-\frac{3}{8}-}.$$

Similarly, its first derivative is small in L^2_y by using the L^∞_y estimate of Corollary 13.12. We conclude that

$$\|u_0 - \tilde{u}(s_0)\|^2_{L^2_y} \ll \sum_{T\in\mathcal{T}} |c_T|^2 = \|u_0\|^2_{L^2}.$$

We then obtain the claim by iterating.

□

In the next subsection, we will use a Duhamel argument to match the source term. To do so, we will need to be able to match initial data at arbitrary time $s \in I$:

Corollary 14.6. *Given $u_0 \in L^2_y$ with frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$ and $s \in I$, there exists $\{a_T\}_{T\in\mathcal{T}} \in \ell^2(\mathcal{T})$ such that*

$$u_0 = S_\lambda \sum_{T\in\mathcal{T}} a_T u_T(s) =: S_\lambda \tilde{u}(s)$$

and

$$\sum_{T\in\mathcal{T}} |a_T|^2 \lesssim \|u_0\|^2_{L^2_y}.$$

Proof. Consider the exact solution u to

$$(\partial_t + H)u = 0, \qquad u(s) = u_0.$$

Using Proposition 14.5, we may construct

$$\tilde{u} = \sum_{T\in\mathcal{T}} a_T u_T$$

satisfying

$$\tilde{u}(s_0) = u(s_0).$$

Using energy estimates with Corollary 14.4, we have

$$\begin{aligned}
\|S_\lambda(\tilde{u} - u)\|_{L^\infty_t(I;L^2_y)} &\lesssim \|(\partial_t + H(t,y,D))S_\lambda(\tilde{u}-u)\|_{L^1_t(I;L^2_y)} \\
&\quad + \|[\partial_x, V_\lambda] + [|D|^{\frac{1}{2}}, \sqrt{a_\lambda}]S_\lambda(\tilde{u}-u)\|_{L^1_t(I;L^2_y)} \\
&\lesssim T^{\frac{1}{2}}(\|(\partial_t + H(t,y,D))S_\lambda(\tilde{u}-u)\|_{L^2_t(I;L^2_y)} \\
&\quad + \|[\partial_x, V_\lambda] + [|D|^{\frac{1}{2}}, \sqrt{a_\lambda}]S_\lambda(\tilde{u}-u)\|_{L^2_t(I;L^2_y)}) \\
&\lesssim T^{\frac{1}{2}}(\lambda^{0+}\mathcal{F}(T)\|u_0\|_{L^2_y} + \|[H, S_\lambda]u\|_{L^2_t(I;L^2_y)} \\
&\quad + (\|V\|_{L^2_t(I;C^1)} + \|a\|_{L^2_t(I;C^{\frac{1}{2}})})\|S_\lambda(\tilde{u}-u)\|_{L^\infty_t(I;L^2_y)}) \\
&\leq T^{\frac{1}{2}}\lambda^{0+}\mathcal{F}(T)(\|u_0\|_{L^2_y} + \|u\|_{L^\infty_t(I;L^2_y)} + \|S_\lambda(\tilde{u}-u)\|_{L^\infty_t(I;L^2_y)}) \\
&\leq T^{\frac{1}{2}}\lambda^{0+}\mathcal{F}(T)(\|u_0\|_{L^2_y} + \|S_\lambda(\tilde{u}-u)\|_{L^\infty_t(I;L^2_y)}).
\end{aligned}$$

Choosing $T^{\frac{1}{2}} \leq \lambda^{0-}$ so that $T^{\frac{1}{2}}\lambda^{0+}\mathcal{F}(T) \ll 1$, we conclude

$$\|S_\lambda\tilde{u}(s) - u_0\|_{L^2_y} \ll \|u_0\|_{L^2_y}.$$

Iterating, we obtain the claim.

□

14.4. **Matching the source.** We use a Duhamel formula with an iteration argument to match the source term in Proposition 10.2:

Proposition 14.7. *Consider the solution u_λ to*

$$(\partial_t + H(t,y,D))u_\lambda = f, \qquad u_\lambda(s_0) = u_0$$

where $u_\lambda(t,\cdot)$ has frequency support $\{|\xi| \in [\lambda/2, 2\lambda]\}$. We may write

$$u_\lambda = \tilde{u} + \int_{s_0}^t \tilde{u}_s(t,y)\, ds$$

where $\tilde{u}$ is the construction in Proposition 14.5, and

$$\tilde{u}_s = \sum_{T\in\mathcal{T}} a_{T,s} u_T$$

with

$$\sum_{T\in\mathcal{T}} |a_{T,s}|^2 \lesssim \|f(s)\|_{L^2_y}^2 + \|u_0\|_{L^2_y}^2.$$

Proof. Let $f \in L^1_t L^2_y$. We construct the function

$$Tf(t,y) = \int_{s_0}^t \tilde{u}_s(t,y)\, ds$$

where $\tilde{u}_s$ is the function constructed in Corollary 14.6 with $f(s)$ in the place of u_0, thus satisfying

$$S_\lambda \tilde{u}_s(s) = f(s)$$

and

$$\sum_{T\in\mathcal{T}} |a_{T,s}|^2 \lesssim \|f(s)\|_{L^2_y}^2.$$

Then we see

$$(\partial_t + H(t,y,D))S_\lambda Tf = f(t,y) + \int_{s_0}^t (\partial_t + H(t,y,D))S_\lambda \tilde{u}_s(t,y)\, ds$$

so that

$$\|(\partial_t + H(t,y,D))S_\lambda Tf - f\|_{L^1_t(I;L^2_y)} \ll \|f\|_{L^1_t(I;L^2_y)}.$$

Thus, iterating, we see we may write the solution u_λ to

$$(\partial_t + H(t,y,D))u_\lambda = f, \qquad u_\lambda(s_0) = 0$$

in the form

$$u_\lambda = Tf = \int_{s_0}^t \tilde{u}_s(t,y)\, ds.$$

Using Proposition 14.5, we can repeat the above argument to write the solution u_λ to

$$(\partial_t + H(t,y,D))u_\lambda = f, \qquad u_\lambda(s_0) = u_0$$

in the form

$$u_\lambda = \tilde{u} + Tf.$$

□

15. Strichartz Estimates

In this section we establish Strichartz estimates on sums of wave packets, as defined in Section 14.

15.1. **Packet overlap.** We record some estimates on the overlap of the packets.

First, we have a trivial overlap bound:

Proposition 15.1. *Let $t \in I$ and $y \in \mathbb{R}$. There are $\lesssim \lambda^{\frac{1}{4}}$ packets $T = (x, \xi) \in \mathcal{T}$ intersecting (t, y).*

Proof. Consider two packets $(x_1, \xi), (x_2, \xi) \in \mathcal{T}$ intersecting (t, y). By Proposition 13.3, we have

$$\lambda^{-\frac{3}{4}} \geq |x^t(x_1, \xi) - x^t(x_2, \xi)| = |\partial_x x^t(z, \xi)(x_1 - x_2)| \approx |x_1 - x_2|.$$

Since $x_i \in \lambda^{-\frac{3}{4}}\mathbb{Z}$, there are at most an absolute number of such packets. Since there are $\approx \lambda^{\frac{1}{4}}$ frequencies ξ, we obtain the claim. □

Second, we have a bound on the number of packets which intersect at two times:

Corollary 15.2. *Let $t, s \in I$ and $y, z \in \mathbb{R}$. There are $\lesssim |t - s|^{-1}$ packets $T = (x, \xi) \in \mathcal{T}$ intersecting both (t, y) and (s, z).*

Proof. Consider two packets $(x_1, \xi_1), (x_2, \xi_2) \in \mathcal{T}$ intersecting both (t, y) and (s, z), so that we have

$$|x^s(x_1, \xi_1) - x^s(x_2, \xi_2)| \leq \lambda^{-\frac{3}{4}}, \quad |x^t(x_1, \xi_1) - x^t(x_2, \xi_2)| \leq \lambda^{-\frac{3}{4}}.$$

Thus we may apply Proposition 13.6,

$$|\xi_1 - \xi_2| \lesssim \lambda^{\frac{3}{4}}|t - s|^{-1}.$$

Since $\xi_i \in \{\lambda^{\frac{3}{4}}\mathbb{Z} : |\xi| \in [\lambda/2, 2\lambda]\}$, we obtain the claim. □

15.2. **Counting argument.** We have the following Strichartz estimate on sums of wave packets. Note that we do not consider phase cancellation, so the estimate is established by analyzing packet overlap only.

Proposition 15.3. *For each $T = (x, \xi) \in \mathcal{T}$, let the $u_T = u_T(t, y)$ denote the wave packet centered at T. Then*

$$\|\sum_{T \in \mathcal{T}} c_T u_T\|^2_{L^2(I;L^\infty)} \leq \lambda^{\frac{3}{4}} \mathcal{F}(T)(\log \lambda)^4 \sum_{T \in \mathcal{T}} |c_T|^2.$$

Proof. Normalizing, we may assume $\sum |c_T|^2 = 1$.

Step 1. Dyadic pidgeonholing. By Proposition 13.7, two packets overlap for at least time $\lambda^{-\frac{1}{4}}$ (this should depend on $\mathcal{F}(T)$ but to simplify our exposition we omit it). As a result, it suffices to estimate the following Riemann sum in t, over $\lambda^{\frac{1}{4}}$ times $t_j \in I$ separated by $\lambda^{-\frac{1}{4}}$, and $y_j \in \mathbb{R}$ arbitrary:

$$\sum_{j \in J} \lambda^{-\frac{1}{4}} (\sum_{T \in \mathcal{T}} |c_T| \chi_T(t_j, y_j))^2 \lesssim (\log \lambda)^4.$$

By the trivial overlap bound Proposition 15.1,

$$\sum_{T \in \mathcal{T}} |\chi_T(t_j, y_j)| \lesssim \lambda^{\frac{1}{4}},$$

so we may restrict the sum over $\mathcal{T}$ to T such that $|c_T| \geq \lambda^{-\frac{1}{4}}$. Since $|c_T| \leq 1$ trivially, we may choose dyadic values $c \in [\lambda^{-\frac{1}{4}}, 1]$ and partition the sum over $\mathcal{T}$ by grouping the T such that $|c_T| \approx c$. Accepting the logarithmic loss, it suffices to consider the one member of the partition (denoting the subcollection by $\mathcal{T}_c$):

$$\sum_{j \in J} \lambda^{-\frac{1}{4}} (\sum_{T \in \mathcal{T}_c} |c_T| \chi_T(t_j, y_j))^2 \lesssim (\log \lambda)^2.$$

On the other hand, again by Proposition 15.1, we may choose dyadic values $L \in [1, \lambda^{\frac{1}{4}}]$ and partition the sum over J by grouping the j such that $\approx L$ packets in $\mathcal{T}_c$ intersect (t_j, y_j). Accepting the logarithmic loss, it suffices to show (denoting the subcollection by J_L):

$$\sum_{j \in J_L} \lambda^{-\frac{1}{4}} (\sum_{T \in \mathcal{T}_c} |c_T| \chi_T(t_j, y_j))^2 \lesssim \log \lambda.$$

If we denote the number of points in J_L by M,

$$\sum_{j \in J_L} \lambda^{-\frac{1}{4}} (\sum_{T \in \mathcal{T}_c} |c_T| \chi_T(t_j, y_j))^2 \lesssim M \lambda^{-\frac{1}{4}} (cL)^2.$$

It thus suffices to show

$$M(cL)^2 \lesssim \lambda^{\frac{1}{4}} \log \lambda.$$

Further, if we denote the number of packets in $\mathcal{T}_c$ by N,

$$Nc^2 \approx \sum_{T \in \mathcal{T}_c} c_T^2 \leq \sum_{T \in T} c_T^2 = 1$$

so that $N \lesssim c^{-2}$. It thus suffices to show

$$ML^2 \lesssim \lambda^{\frac{1}{4}} N \log \lambda. \tag{15.1}$$

Step 2. Double counting. For each packet $T \in \mathcal{T}_c$, denote by n_T the number of points contained in the support of u_T. Note that

$$ML \approx \sum_{T \in \mathcal{T}_c} n_T = \sum_{n_T = 1} n_T + \sum_{n_T \geq 2} n_T \leq N + \sum_{n_T \geq 2} n_T. \tag{15.2}$$

If

$$\sum_{n_T \geq 2} n_T \leq N,$$

then we conclude $ML \lesssim N$. Using also the trivial overlap bound Proposition 15.1 on one copy of L, we obtain an estimate even better than (15.1).

Thus we may assume

$$\sum_{n_T \geq 2} n_T > N.$$

Then (15.2) with Cauchy-Schwarz gives

$$M^2 L^2 \lesssim (\sum_{n_T \geq 2} n_T)^2 \lesssim N \sum_{n_T \geq 2} n_T^2 =: NK. \tag{15.3}$$

We estimate K by a double counting. We claim that K counts the triples $(i, j, T) \in J_L \times J_L \times \mathcal{T}_c$, $i \neq j$, such that the packet u_T intersects (t_i, y_i) and (t_j, y_j). Indeed, if $n_T = 1$, T doesn't contribute to K, and otherwise, T contributes $\approx n_T^2$ to K.

On the other hand, by Corollary 15.2, each pair of points (t_i, y_i) and (t_j, y_j) is covered by $\lesssim |t_i - t_j|^{-1}$ packets. Thus

$$K \lesssim \sum_{1\leq i\neq j\leq M} |t_i - t_j|^{-1}.$$

The sum is maximzed when the t_j are as close as possible, as consecutive multiples of $\lambda^{-\frac{1}{4}}$:

$$K \lesssim \lambda^{\frac{1}{4}} \sum_{1\leq i\neq j\leq M} |i-j|^{-1} \lesssim \lambda^{\frac{1}{4}} M \log M \lesssim M\lambda^{\frac{1}{4}} \log \lambda.$$

Substituting in (15.3), we obtain (15.1).

□

Combined with Proposition 14.7, we obtain Proposition 10.2.

Chapter 4

Local Well-Posedness

In this chapter, we outline the proof of the well-posedness Theorems 2.6 and 2.7 as a consequence of energy, Strichartz, and contraction estimates. Much of the material discussed here is presented in detail in [ABZ14b, Chapter 3]. However, here we discuss additional details regarding continuity with respect to time and continuous dependence on initial data. We also refer to [Ngu16], which discusses continuous dependence on initial data at the energy threshold, $s > \frac{d}{2} + 1$.

In this chapter, we denote for brevity

$$U = (\eta, \psi, V, B), \quad \mathcal{H}^m = H^{m+\frac{1}{2}} \times H^{m+\frac{1}{2}} \times H^m \times H^m.$$

16. Preliminary Estimates

16.1. **Continuity of the Dirichlet to Neumann map.** In establishing continuous dependence on initial data, as well as passing to the limit in the equations, we will require continuity of the Dirichlet to Neumann map with respect to its nonlinear dependence on the surface η. We record two of these results in this section. Note that these are general results; we do not need to assume that (η, ψ) solve (2.1).

Proposition 16.1. [ABZ14b, Proposition 3.8] *We have*

$$\begin{aligned}\|G(\eta_1)f - G(\eta_2)f\|_{H^{s-\frac{3}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}})(\|f\|_{H^s}\|\eta_1 - \eta_2\|_{W^{r-\frac{1}{2},\infty}} \\ + \|f\|_{H^s \cap W^{r,\infty}}\|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}).\end{aligned}$$

Proposition 16.2. [ABZ14b, Proposition 3.13] *We have*

$$\|G(\eta_1)f - G(\eta_2)f\|_{H^{-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty}})\|f\|_{H^{\frac{1}{2}}}\|\eta_1 - \eta_2\|_{W^{1,\infty}}.$$

As a consequence, we have that (V, B) depend continuously on (η, ψ) in the following sense:

Corollary 16.3. [ABZ14b, Corollary 3.16] *Let $(\eta_n, \psi_n), (\eta, \psi) \in H^{s+\frac{1}{2}}$ uniformly; here it suffices to assume only $s > \frac{d}{2} + \frac{1}{2}$. Assume*

$$\eta_n \to \eta \in W^{1,\infty},$$

$$\psi_n \to \psi \in H^{\frac{1}{2}}.$$

Then

$$G(\eta_n)\psi_n \to G(\eta)\psi \in H^{-\frac{1}{2}},$$
$$(V_n, B_n) \to (V, B) \in H^{-\frac{1}{2}}.$$

16.2. **High frequency energy estimates.** We establish the following energy estimate on the high frequency component of a solution, which will complement contraction estimates used to handle the low frequencies:

Proposition 16.4. *Consider $I = [0, T]$ and a solution $(\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}})$ (not necessarily smooth) to (2.1), satisfying the properties discussed in Section 4. Then denoting $U = (\eta, \psi, V, B)$,*

$$\|S_{>\lambda}U(t)\|_{\mathcal{H}^s} \leq \mathcal{F}(T)\left(\lambda^{0-} + \|S_{>\lambda}U(0)\|_{\mathcal{H}^s} + \int_0^t Z_r(\tau)\|\tilde{S}_{>\lambda}U(\tau)\|_{\mathcal{H}^s}\, d\tau\right),$$

and in particular, by Gronwall's inequality,

$$\|S_{>\lambda}U\|_{L^\infty(I;\mathcal{H}^s)} \leq \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U(0)\|_{\mathcal{H}^s} + \sqrt{T}\|\tilde{S}_\lambda U\|_{L^\infty(I;\mathcal{H}^s)}).$$

Proof. This energy estimate is essentially [Ngu16, (3.81)]; we may follow the argument used there. The primary task is to commute the frequency projection $S_{>\lambda}$ with the paralinear formulation (1.3), from which one obtains the desired frequency projected energy estimates along the lines of Proposition 18.1. In turn, commuting $S_{>\lambda}$ with (1.3) requires a frequency projected paralinearization of the Dirichlet to Neumann map. This is analogous to the content of Appendix B except simpler, since the localization occurs only in frequency and not space.

We remark on the following two differences compared to the result and proof in [Ngu16]. First, we use the cruder pair of frequency projections $S_{>\lambda}$, $\tilde{S}_{>\lambda}$ in place of $S^2_{>\lambda}$, $S_{>\lambda}$, which suffices to handle the technical difficulties and simplifies the presentation.

Second, we work with lower Sobolev index s. As a result, throughout the argument in [Ngu16], we refrain from using Sobolev embedding to estimate Hölder norms. Instead, we allow our estimates to depend not only on $\mathcal{M}_s(T)$, but also on the Hölder norms,

$$\mathcal{F}(T) = \mathcal{F}(\mathcal{M}_s(T) + \mathcal{Z}_r(T)).$$

□

17. Uniqueness and Continuity of the Solution Map

In this section, we record a contraction estimate in a lower regularity Sobolev space, which will contribute to our uniqueness and continuity results.

17.1. **Contraction and uniqueness.** Uniqueness is an immediate consequence of the following contraction estimate on the difference of two solutions:

Proposition 17.1. *Consider $I = [0, T]$ and two solutions $(\eta_1, \psi_1), (\eta_2, \psi_2) \in C(I; H^{s+\frac{1}{2}})$ (not necessarily smooth) to (2.1), satisfying the properties discussed in Section 4. Then denoting $U_i = (\eta_i, \psi_i, V_i, B_i)$, we have*

$$\|U_1 - U_2\|_{L^\infty(I;\mathcal{H}^{s-1})} \leq \mathcal{F}(T)\|U_1(0) - U_2(0)\|_{\mathcal{H}^{s-1}}.$$

Here $\mathcal{F}(T)$, the notation in Section 4, measures both $(\eta_1, \psi_1, V_1, B_1)$ and $(\eta_2, \psi_2, V_2, B_2)$.

Proof. The proof is provided in [ABZ14b, Section 3.2]. The approach is standard, by writing an equation for the difference of two solutions. In particular, we make use of Proposition 16.2.

The main new difficulty involves measuring some product error terms in low or negative Sobolev spaces. This is resolved by observing that these error terms may be expressed in differentiated form by using the structure of the equations. □

17.2. **Continuous dependence on initial data.** Combining the previous contraction estimate with the high frequency energy estimates, we obtain continuous dependence of the solution map on the initial data:

Proposition 17.2. *Consider $I = [0,T]$ and solutions $(\eta_n, \psi_n), (\eta, \psi) \in C(I; H^{s+\frac{1}{2}})$ (not necessarily smooth) to (2.1), uniformly satisfying the properties discussed in Section 4, as well as $\mathcal{Z}_r(T) < \infty$. If*

$$(\eta_n, \psi_n, V_n, B_n)(0) \to (\eta, \psi, V, B)(0) \in \mathcal{H}^s,$$

then

$$(\eta_n, \psi_n, V_n, B_n) \to (\eta, \psi, V, B) \in C(I; \mathcal{H}^s).$$

Proof. Denote

$$U_n = (\eta_n, \psi_n, V_n, B_n), \quad U = (\eta, \psi, V, B).$$

We write

$$\|U_n - U\|_{L^\infty(I;\mathcal{H}^s)} \lesssim \|S_{\leq\lambda}(U_n - U)\|_{L^\infty(I;\mathcal{H}^s)} + \|S_{>\lambda}U_n\|_{L^\infty(I;\mathcal{H}^s)} + \|S_{>\lambda}U\|_{C(I;\mathcal{H}^s)}.$$

For the first term on the right hand side, we write, using the contraction estimate of Proposition 17.1,

$$\|S_{\leq\lambda}(U_n - U)\|_{L^\infty(I;\mathcal{H}^s)} \lesssim \lambda\|S_{\leq\lambda}(U_n - U)\|_{L^\infty(I;\mathcal{H}^{s-1})} \leq \lambda\mathcal{F}(T)\|U_n(0) - U(0)\|_{\mathcal{H}^{s-1}}.$$

For the second term, using the high frequency energy estimate of Proposition 16.4, along with the contraction of Proposition 17.1 again,

$$\begin{aligned}
\|S_{>\lambda}U_n\|_{L^\infty(I;\mathcal{H}^s)} &\leq \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U_n(0)\|_{\mathcal{H}^s} + \|\tilde{S}_\lambda U_n\|_{L^\infty(I;\mathcal{H}^s)}) \\
&\leq \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U(0)\|_{\mathcal{H}^s} + \|\tilde{S}_\lambda U\|_{L^\infty(I;\mathcal{H}^s)} \\
&\quad + \|S_{>\lambda}(U_n(0) - U(0))\|_{\mathcal{H}^s} + \|\tilde{S}_\lambda(U_n - U)\|_{L^\infty(I;\mathcal{H}^s)}) \\
&\leq \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U(0)\|_{\mathcal{H}^s} + \|\tilde{S}_\lambda U\|_{L^\infty(I;\mathcal{H}^s)} \\
&\quad + \|U_n(0) - U(0)\|_{\mathcal{H}^s} + \lambda\|U_n(0) - U(0)\|_{\mathcal{H}^{s-1}}).
\end{aligned}$$

Collecting these estimates, we conclude

$$\begin{aligned}
\|U_n - U\|_{L^\infty(I;\mathcal{H}^s)} \leq \mathcal{F}(T)(&\lambda^{0-} + \|S_{>\lambda}U(0)\|_{\mathcal{H}^s} + \|\tilde{S}_{>\lambda}U\|_{C(I;\mathcal{H}^s)} \\
&+ \|U_n(0) - U(0)\|_{\mathcal{H}^s} + \lambda\|U_n(0) - U(0)\|_{\mathcal{H}^{s-1}}).
\end{aligned}$$

Choosing λ large, the top row is arbitrarily small. Then the bottom row is small for all large n, since $U_n(0) \to U(0) \in \mathcal{H}^s$. □

17.3. **Continuity with respect to time.** Here, we show that a solution with initial data in $\mathcal{H}^s$ is continuous with respect to time t, with values in $\mathcal{H}^s$.

Proposition 17.3. *Consider $I = [0,T]$ and a solution $(\eta,\psi) \in L^\infty(I; H^{s+\frac{1}{2}})$ (not necessarily smooth) to (2.1), satisfying the properties discussed in Section 4, and $\mathcal{F}(T) < \infty$. Also assume $(\eta,\psi,V,B) \in C(I;\mathcal{H}^{s'})$ for some $s' \leq s$. Then $(\eta,\psi,V,B) \in C(I;\mathcal{H}^s)$.*

Proof. Consider a sequence $t_n \to t \in [0,T]$. We write

$$\|U(t_n) - U(t)\|_{\mathcal{H}^s} \lesssim \|S_{\leq\lambda}(U(t_n) - U(t))\|_{\mathcal{H}^s} + \|S_{>\lambda}U(t_n)\|_{\mathcal{H}^s} + \|S_{>\lambda}U(t)\|_{\mathcal{H}^s}.$$

For the first term on the right hand side, we write

$$\|S_{\leq\lambda}(U(t_n) - U(t))\|_{\mathcal{H}^s} \lesssim \lambda^{s-s'}\|U(t_n) - U(t)\|_{\mathcal{H}^{s'}}.$$

For the second term, using the high frequency energy estimate of Proposition 16.4,

$$\begin{aligned}\|S_{>\lambda}U(t_n)\|_{\mathcal{H}^s} &\leq \mathcal{F}(T)\left(\lambda^{0-} + \|S_{>\lambda}U(t)\|_{\mathcal{H}^s} + \int_t^{t_n} Z_r(t)\|\tilde{S}_{>\lambda}U(\tau)\|_{\mathcal{H}^s}\,d\tau\right)\\ &\leq \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U(t)\|_{\mathcal{H}^s} + |t_n - t|^{1/2}).\end{aligned}$$

Collecting these estimates, we conclude

$$\|U(t_n) - U(t)\|_{\mathcal{H}^s} \lesssim \lambda^{s-s'}\|U(t_n) - U(t)\|_{\mathcal{H}^{s'}} + \mathcal{F}(T)(\lambda^{0-} + \|S_{>\lambda}U(t)\|_{\mathcal{H}^s} + |t_n - t|^{1/2}).$$

Choosing λ large, λ^{0-} and $\|S_{>\lambda}U(t)\|_{\mathcal{H}^s}$ are arbitrarily small. Then the remaining terms are small for all large n, using $t_n \to t$ and that U is continuous in time with values in $\mathcal{H}^{s'}$. ☐

18. Existence

18.1. **A priori estimates.** In this subsection we collect the main *a priori* estimates, combining energy estimates with the Strichartz estimates discussed in depth in the previous chapters.

First, we have the following energy estimate for $\mathcal{M}_s(T)$.

Proposition 18.1. [ABZ14b, Theorem D.1] *We have for any $T_0 \in (0,T]$,*

$$\mathcal{M}_s(T_0) \leq \mathcal{F}(\mathcal{F}(M_s(0)) + T_0\mathcal{F}(T_0)).$$

Proof. The proof is provided in [ABZ14b, Appendix D], and essentially uses the paralinear formulation (1.3) to obtain an energy estimate for u. The primary task is to convert the estimate on u into estimates on (η,ψ,V,B). ☐

Next, we have the Strichartz estimate for $\mathcal{Z}_r(T)$.

Proposition 18.2. *Additionally assume*

$$s > \frac{d}{2} + 1 - \mu, \quad r < s - \frac{d}{2} + \mu$$

on the regularity indices s, r. We have for any $T_0 \in (0,T]$,

$$\mathcal{Z}_r(T_0) \leq \mathcal{F}(T_0\mathcal{F}(T_0)).$$

Proof. The corresponding result for the case of Strichartz gains

$$\mu = \frac{1}{24},\ d = 1, \quad \text{and} \quad \mu = \frac{1}{12},\ d \geq 2$$

was stated and proved in [ABZ14b, Section 3.1]. The proof for our result with higher μ values is similar, except starting instead with our sharper Strichartz estimates, Theorems 2.3 and 2.4. As with the energy estimate of Proposition 18.1, the primary task is to convert the Strichartz estimate on u into estimates on (η, ψ, V, B). □

A straightforward consequence of Propositions 18.1 and 18.2 is the following *a priori* estimate:

Corollary 18.3. *Additionally assume*

$$s > \frac{d}{2} + 1 - \mu, \quad r < s - \frac{d}{2} + \mu$$

on the regularity indices s, r. For any $A > 0$, there exists $B, T > 0$ such for any smooth solution (η, ψ) to (2.1) on time interval $[0, T_0] \subseteq [0, T]$, satisfying the properties discussed in Section 4, and with initial data of size $M_s(0) \leq A$,

$$\mathcal{M}_s(T_0) + \mathcal{Z}_r(T_0) \leq B.$$

18.2. **Limit of smooth solutions.** We construct the solution (η, ψ) of Theorems 2.6 and 2.7 as a limit of smooth solutions. Given the previous results of this chapter, it remains to show the following existence result:

Proposition 18.4. *Given initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}$ as in Theorems 2.6 and 2.7, there exists $T > 0$ such that the system (2.1) with initial data (η_0, ψ_0) has a solution $(\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}})$ with $(\eta, \psi, V, B) \in C(I; \mathcal{H}^{s'})$ for any $s' < s$.*

Proof. The construction of (η, ψ) is sketched in Section 3.4 of [ABZ14b]. We recall the approach here, and include some additional details.

Construct a sequence (η_n, ψ_n) of solutions to (2.1) with smooth initial data $(\eta_n(0), \psi_n(0))$ converging to (η_0, ψ_0) in $H^{s+\frac{1}{2}}$. Write $U_n = (\eta_n, \psi_n, V_n, B_n)$ and denote the lifespan of (η_n, ψ_n) by $T_n > 0$.

The energy estimate on high frequencies from Proposition 16.4 shows that as long as $\mathcal{F}(T_n) < \infty$, (η_n, ψ_n) is smooth, and in particular, we may extend it past $[0, T_n]$. Thus, by the *a priori* estimate of Corollary 18.3, we have a uniform lower bound $T > 0$ on the lifespan of our smooth solutions (η_n, ψ_n), as well as uniform estimates on $\mathcal{M}_s^n(T), \mathcal{Z}_r^n(T)$.

Next we observe that the contraction of Proposition 17.1 implies that U_n forms a Cauchy sequence in $L^\infty(I; \mathcal{H}^{s-1})$, and thus

$$U_n \to U \in L^\infty(I; \mathcal{H}^{s-1})$$

for some $U = (\eta, \psi, V, B)$. Since U_n is bounded in $L^\infty(I; \mathcal{H}^s)$, by interpolation

$$U_n \to U \in L^\infty(I; \mathcal{H}^{s-})$$

and thus by Sobolev embedding,

$$\eta_n \to \eta \in L^\infty(I; W^{1,\infty}),$$

$$\psi_n \to \psi \in L^\infty(I; H^{\frac{1}{2}}).$$

By Corollary 16.3, we may pass to the limit in the equations, and thus obtain a solution (η, ψ) such that $U \in C(I; \mathcal{H}^{s'})$ for some $s' < s$. Since $U \in L^\infty(I; \mathcal{H}^s)$, we may interpolate this to $U \in C(I; \mathcal{H}^{s'})$ for any $s' < s$.

$\square$

Chapter 5

Appendices

The goal of this appendix is to recall and establish various estimates for the elliptic Dirichlet problem with rough boundary, and then apply them toward objects from the water waves equations, including the pressure P, Taylor coefficient a, and Dirichlet to Neumann map $G(\eta)$. For the reader's convenience, we also provide notation and estimates from the paradifferential calculus in Appendix E.

Throughout the appendix, we use the notation discussed in Sections 4 and 10, with the following modifications. Here, we assume only

$$s > \frac{d}{2} + \frac{1}{2}.$$

Further, for all those estimates involving a local weight w or S, we consider only the case $d = 1$.

The results of Appendices A and B are time independent, so there we omit the variable t. In particular, we denote $\nabla = \nabla_x$ and $\Delta = \Delta_x$. Let $I = [-1, 0]$ denote a vertical space interval, and define the following spaces for a vertical spatial interval $J \subseteq \mathbb{R}$:

$$X^\sigma(J) = C^0_z(J; H^\sigma) \cap L^2_z(J; H^{\sigma+\frac{1}{2}})$$
$$Y^\sigma(J) = L^1_z(J; H^\sigma) + L^2_z(J; H^{\sigma-\frac{1}{2}})$$
$$U^\sigma(J) = C^0_z(J; C^\sigma_*) \cap L^2_z(J; C^{\sigma+\frac{1}{2}}_*).$$

Appendix A. Elliptic Estimates for the Dirichlet Problem

In this section we recall and establish various estimates for the elliptic Dirichlet problem with rough boundary. This problem was studied in [ABZ14a], which discusses Sobolev estimates, and [ABZ14b], which discuss Hölder estimates. Here, our goal is to establish sharper Hölder estimates, as well as local Sobolev counterparts.

A.1. **Flattening the boundary.** Consider the Dirichlet problem with rough boundary. Denote the strip of constant depth h along the surface by

$$\Omega_1 = \{(x, y) \in \mathbb{R}^{d+1} : \eta(x) - h < y < \eta(x)\},$$

and on Ω_1, let θ satisfy

$$\Delta_{x,y}\theta(x, y) = F, \qquad \theta|_{y=\eta(x)} = f. \tag{A.1}$$

We would like estimates on θ and its derivatives. To address the rough boundary η and corresponding rough domain, we change variables to a problem with flat boundary and domain, following [Lan05], [ABZ14a]. Denote the flat strip by

$$\tilde{\Omega}_1 = \{(x, z) \in \mathbb{R}^{d+1} : z \in (-1, 0)\}.$$

Then define the Lipschitz diffeomorphism $\rho : \tilde{\Omega}_1 \to \Omega_1$ by

$$\rho(x, z) = (1 + z)e^{\delta z\langle D\rangle}\eta(x) - z(e^{-(1+z)\delta\langle D\rangle}\eta(x) - h)$$

where δ is chosen small as in [ABZ14a, Lemma 3.6]. Lastly, for $u : \Omega_1 \to \mathbb{R}$, let $\tilde{u} : \tilde{\Omega}_1 \to \mathbb{R}$ denote

$$\tilde{u}(x, z) = u(x, \rho(x, z)).$$

Then on the flat domain $\tilde{\Omega}_1$, $\tilde{\theta}$ satisfies

$$(\partial_z^2 + \alpha\Delta + \beta \cdot \nabla\partial_z - \gamma\partial_z)\tilde{\theta} = F_0, \qquad \tilde{\theta}|_{z=0} = f \tag{A.2}$$

where

$$\alpha = \frac{(\partial_z\rho)^2}{1 + |\nabla\rho|^2}, \quad \beta = -2\frac{\partial_z\rho\nabla\rho}{1 + |\nabla\rho|^2}, \quad \gamma = \frac{\partial_z^2\rho + \alpha\Delta\rho + \beta \cdot \nabla\partial_z\rho}{\partial_z\rho}$$

and

$$F_0(x, z) = \alpha\tilde{F}.$$

The diffeomorphism ρ has essentially the same regularity as η, with additional smoothing when averaged over z. We recall the precise estimates below:

Proposition A.1. *The diffeomorphism ρ satisfies*

$$\|(\partial_z\rho)^{-1}\|_{C^0(I;C_*^{r-1})} + \|\nabla_{x,z}\rho\|_{C^0(I;C_*^{r-1})} \lesssim \|\eta\|_{H^{s+\frac{1}{2}}} \tag{A.3}$$

$$\|(\partial_z\rho - h, \nabla\rho)\|_{X^{s-\frac{1}{2}}(I)} + \|\partial_z^2\rho\|_{X^{s-\frac{3}{2}}(I)} \lesssim \|\eta\|_{H^{s+\frac{1}{2}}} \tag{A.4}$$

$$\|\nabla_{x,z}\rho\|_{U^{r-\frac{1}{2}}(I)} + \|\partial_z^2\rho\|_{U^{r-\frac{3}{2}}(I)} \lesssim 1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} \tag{A.5}$$

$$\|\nabla_{x,z}\rho\|_{L^1(I;H^{s+\frac{1}{2}})} + \|\partial_z^2\rho\|_{L^1(I;H^{s-\frac{1}{2}})} \leq \|\eta\|_{H^{s+\frac{1}{2}}} \tag{A.6}$$

$$\|\nabla_{x,z}\rho\|_{L^1(I;C_*^{r+\frac{1}{2}})} + \|\partial_z^2\rho\|_{L^1(I;C_*^{r-\frac{1}{2}})} \leq 1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}. \tag{A.7}$$

Proof. The first estimate is a consequence of [ABZ14a, Lemmas 3.6, 3.7] combined with Sobolev embedding. The first terms of the second and third estimates are from [ABZ14a, Lemma 3.7] and [ABZ14b, Lemma B.1] respectively. The corresponding estimates on $\partial_z^2\rho$ are proven similarly from the definition of ρ. Lastly, the proofs of the fourth and fifth estimates are similar to those of the second and third respectively, by using L_z^1 in place of L_z^2. □

Then the above coefficients α, β, γ, expressed in terms of ρ, satisfy similar estimates with proofs straightforward from paraproduct estimates:

Proposition A.2. *For α, β, γ defined as above,*

$$\|(\alpha - h^2, \beta)\|_{X^{s-\frac{1}{2}}(I)} + \|\gamma\|_{X^{s-\frac{3}{2}}(I)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \tag{A.8}$$

$$\|(\alpha, \beta)\|_{U^{r-\frac{1}{2}}(I)} + \|\gamma\|_{L^2(I;C_*^{r-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}) \tag{A.9}$$

$$\|(\alpha, \beta)\|_{L^1(I;C_*^{r+\frac{1}{2}})} + \|\gamma\|_{L^1(I;C_*^{r-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}). \tag{A.10}$$

To establish local elliptic estimates, we would like a local counterpart to (A.2), which we obtain by simply commuting:

Proposition A.3. *Let $0 \le \sigma \le s - \frac{1}{2}$ and $z_0 \in [-1,0]$, $J = [z_0, 0]$. Consider $\tilde{\theta}$ solving (A.2). Denote $wS = w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}$ or $wS = w_{x_0,\kappa} S_\kappa$ with $\kappa \le c\lambda$. Then we can write*

$$(\partial_z^2 + \alpha\Delta + \beta \cdot \nabla\partial_z - \gamma\partial_z) wS\tilde{\theta} = wSF_0 + F_4, \qquad (wS\tilde{\theta})|_{z=0} = wSf \tag{A.11}$$

where

$$\|F_4\|_{Y^\sigma(J)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}) \|\nabla_{x,z}\tilde{\theta}\|_{X^{\sigma-\frac{1}{4}}(J)}.$$

Proof. First observe that by Sobolev embedding,

$$\|\nabla_{x,z}\tilde{\theta}\|_{L^\infty(J; C_*^{\sigma-s+\frac{1}{2}+})} \lesssim \|\nabla_{x,z}\tilde{\theta}\|_{L^\infty(J; H^{\sigma-\frac{1}{4}})},$$
$$\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J; C_*^{\sigma-s+1+})} \lesssim \|\nabla_{x,z}\tilde{\theta}\|_{L^2(J; H^{\sigma+\frac{1}{4}})}$$

so we may freely use the Hölder norms on the right hand side of our error estimate.

We have

$$F_4 = [\alpha\Delta + \beta \cdot \nabla\partial_z - \gamma\partial_z, wS]\tilde{\theta}.$$

We discuss the term $\alpha\Delta$ of the commutator below; the term $\beta \cdot \nabla\partial_z$ is similar. First, we exchange α with T_α as follows. Estimate

$$\|T_{\Delta wS\tilde{\theta}}\alpha\|_{H^{\sigma-\frac{1}{2}}} \lesssim \|\Delta wS\tilde{\theta}\|_{C_*^{\sigma-s}} \|\alpha\|_{H^{s-\frac{1}{2}}} \lesssim \|\nabla\tilde{\theta}\|_{C_*^{\sigma-s+1}} \|\alpha\|_{H^{s-\frac{1}{2}}}$$

followed by integrating in z and using the estimates on α in Corollary A.2. The balanced-frequency term is similar. We likewise have

$$\|wST_{\Delta\tilde{\theta}}\alpha\|_{H^{\sigma-\frac{1}{2}}} \lesssim \|\Delta\tilde{\theta}\|_{C_*^{\sigma-s}} \|\alpha\|_{H^{s-\frac{1}{2}}} \lesssim \|\nabla\tilde{\theta}\|_{C_*^{\sigma-s+1}} \|\alpha\|_{H^{s-\frac{1}{2}}}.$$

Then it remains to estimate

$$[T_\alpha\Delta, wS]\tilde{\theta} = T_\alpha[\Delta, w]S\tilde{\theta} + [T_\alpha, wS]\Delta\tilde{\theta}.$$

For the first commutator, we have, estimating α via Corollary A.2,

$$\|T_\alpha[\Delta, w]S\tilde{\theta}\|_{L^2(J; H^{\sigma-\frac{1}{2}})} \lesssim \|\alpha\|_{L^\infty(J; L^\infty)} \|[\Delta, w]S\tilde{\theta}\|_{L^2(J; H^{\sigma-\frac{1}{2}})}$$
$$\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|[\Delta, w]S\tilde{\theta}\|_{L^2(J; H^{\sigma-\frac{1}{2}})}.$$

Then, using κ to denote the frequency of the projection S,

$$\|[\Delta, w]S\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}} \lesssim \|\nabla w\|_{L^\infty} \|\nabla S\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}} + \|\Delta w\|_{L^\infty} \|S\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}}$$
$$\lesssim \kappa^{\frac{3}{4}} \|\nabla S\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}} + \kappa^{\frac{3}{2}} \|S\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}}$$
$$\lesssim \|\nabla\tilde{\theta}\|_{H^{\sigma+\frac{1}{4}}}.$$

Integrating in z, we obtain the desired estimate.

A typical term in the sum defining the second commutator is

$$[(S_{\le\kappa/8}\alpha)S_\kappa, wS]\Delta\tilde{\theta} = (S_{\le\kappa/8}\alpha)[S_\kappa, w]S\Delta\tilde{\theta} + w[(S_{\le\kappa/8}\alpha), S]S_\kappa\Delta\tilde{\theta}$$

with κ comparable to the frequency of the projection S. The first of these commutators is estimated in the same way as the previous paragraph, putting one derivative on w. For the second commutator, using the frequency localization, we may drop the weight w. In the case $wS = w_{x_0,\kappa} S_\kappa$,

$$\|[(S_{\le\kappa/8}\alpha), S]S_\kappa\Delta\tilde{\theta}\|_{H^{\sigma-\frac{1}{2}}} \lesssim \lambda^{\frac{3}{2}-r} \|\alpha\|_{C_*^{r-\frac{1}{2}}} \|S_k\Delta\tilde{\theta}\|_{H^{\sigma-\frac{3}{2}}} \lesssim (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}) \|\nabla\tilde{\theta}\|_{H^\sigma}.$$

In the case $wS = w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}$, commuting induces a $\mu^{-1}\lambda \le \lambda^{\frac14}$ loss:

$$\|[(S_{\le \lambda/8}\alpha), S] S_\lambda \Delta\tilde\theta\|_{H^{\sigma-\frac12}} \lesssim \lambda^{\frac32 - r + \frac14} \|\alpha\|_{C_*^{r-\frac12}} \|S_\lambda \Delta\tilde\theta\|_{H^{\sigma-\frac32}} \lesssim (1 + \|\eta\|_{W^{r+\frac12,\infty}}) \|\nabla\tilde\theta\|_{H^{\sigma+\frac14}}.$$

It remains to consider

$$[\gamma\partial_z, wS]\tilde\theta.$$

Here we do not use the commutator, estimating the two terms separately. We decompose into paraproducts,

$$\gamma\partial_z\tilde\theta = T_\gamma \partial_z\tilde\theta + T_{\partial_z\tilde\theta}\gamma + R(\gamma, \partial_z\tilde\theta).$$

We have by (E.8), (E.7), and Corollary A.2,

$$\begin{aligned}\|T_\gamma\partial_z\tilde\theta\|_{L^1(J;H^\sigma)} + \|R(\gamma,\partial_z\tilde\theta)\|_{L^1(J;H^\sigma)} &\lesssim \|\gamma\|_{L^2(J;C_*^{r-1})} \|\partial_z\tilde\theta\|_{L^2(J;H^\sigma)} \\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})(1 + \|\eta\|_{W^{r+\frac12,\infty}}) \|\partial_z\tilde\theta\|_{L^2(J;H^\sigma)}.\end{aligned}$$

Similarly,

$$\|T_{\partial_z\tilde\theta}\gamma\|_{L^2(J;H^{\sigma-\frac12})} \lesssim \|\gamma\|_{L^2(J;H^{s-1})} \|\partial_z\tilde\theta\|_{L^\infty(J;C_*^{\sigma-s+\frac12+})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}}) \|\partial_z\tilde\theta\|_{L^\infty(J;C_*^{\sigma-s+\frac12+})}$$

□

A.2. **Factoring the elliptic equation.** To establish elliptic estimates, we factor the equation (A.2) as the product of forward and backward paralinearized parabolic evolutions,

$$(\partial_z - T_a)(\partial_z - T_A)\tilde\theta \approx F_0 \tag{A.12}$$

(in a sense to be made precise) where we define

$$a = \frac12(-i\beta\cdot\xi - \sqrt{4\alpha|\xi|^2 - (\beta\cdot\xi)^2}), \quad A = \frac12(-i\beta\cdot\xi + \sqrt{4\alpha|\xi|^2 - (\beta\cdot\xi)^2}).$$

A pair of parabolic estimates then yields the desired elliptic estimates.

Now we estimate the factoring error in (A.12). In preparation, first we record estimates on the symbols a and A in (A.12). Define

$$\mathcal{M}^m_\rho(a) = \sup_{z\in I} M^m_\rho(a(z)), \qquad \mathcal{M}^{m,2}_\rho(a) = \|M^m_\rho(a(z))\|_{L^2_z(I)}.$$

Proposition A.4. *For a, A defined as above,*

$$\mathcal{M}^1_0(a) + \mathcal{M}^1_0(A) \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})$$
$$\mathcal{M}^1_{\frac12}(a) + \mathcal{M}^1_{\frac12}(A) + \mathcal{M}^1_{-\frac12}(\partial_z A) \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})(1 + \|\eta\|_{W^{r+\frac12,\infty}})$$
$$\mathcal{M}^{1,2}_1(a) + \mathcal{M}^{1,2}_r(A) + \mathcal{M}^{1,2}_0(\partial_z A) \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})(1 + \|\eta\|_{W^{r+\frac12,\infty}})$$

Proof. The first estimate is from [ABZ14a, Lemma 3.22]. The second estimate is from [ABZ14b, (B.45)]. The third estimate is similar, but is proven using the L^2_z part of Proposition A.1, instead of C^0_z. □

Proposition A.5. *Let $0 \le \sigma < r - \frac12$ and $z_0 \in [-1, 0]$, $J = [z_0, 0]$. Consider $\tilde\theta$ solving (A.2). Then we can write*

$$(\partial_z - T_a)(\partial_z - T_A)\tilde\theta = F_0 + F_1 + F_2 + F_3$$

where for $i \ge 1$,

$$\|F_i\|_{L^1(J;C^\sigma_*)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac12}})(1 + \|\eta\|_{W^{r+\frac12,\infty}}) \|\nabla_{x,z}\tilde\theta\|_{U^{\sigma-\frac12}(J)}.$$

Proof. Here we have used F_1, F_2, F_3 to represent the errors arising from, respectively, the first order term on the left hand side of (A.2), the paralinearization errors, and the lower order terms from applying the symbolic calculus. We estimate these one by one.

We begin with F_3. Factor

$$\begin{aligned}(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} &= \partial_z^2\tilde{\theta} + T_a T_A \tilde{\theta} - T_a \partial_z \tilde{\theta} - \partial_z T_A \tilde{\theta} \\ &= \partial_z^2\tilde{\theta} + T_{aA}\tilde{\theta} - (T_a + T_A)\partial_z\tilde{\theta} + (T_a T_A - T_{aA})\tilde{\theta} - (\partial_z T_A - T_A \partial_z)\tilde{\theta} \\ &= \partial_z^2\tilde{\theta} + T_\alpha \Delta\tilde{\theta} + T_\beta \cdot \nabla\partial_z\tilde{\theta} + (T_a T_A - T_{aA})\tilde{\theta} - T_{\partial_z A}\tilde{\theta}.\end{aligned}$$

For the first error term, by (E.3) and Proposition A.4,

$$\begin{aligned}\|(T_a T_A - T_{aA})\tilde{\theta}\|_{L^1(J;C^\sigma_*)} &\lesssim (\mathcal{M}^{1,2}_r(a)\mathcal{M}^1_0(A) + \mathcal{M}^1_0(a)\mathcal{M}^{1,2}_r(A))\|\tilde{\theta}\|_{L^2(J;C^{\sigma+2-r}_*)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\tilde{\theta}\|_{L^2(J;C^{\sigma+1}_*)}.\end{aligned}$$

Note that by the definition of the inhomogeneous paradifferential operator, we may exchange v for $S_{>1/10}\tilde{\theta}$ in the previous inequalities, and hence bound

$$\|S_{>1/10}\tilde{\theta}\|_{L^2(J;C^{\sigma+1}_*)} \lesssim \|\nabla\tilde{\theta}\|_{L^2(J;C^\sigma_*)}.$$

Similarly, by (E.1),

$$\begin{aligned}\|T_{\partial_z A}\tilde{\theta}\|_{L^1(J;C^\sigma_*)} &\lesssim \mathcal{M}^{1,2}_0(\partial_z A)\|S_{>1/10}\tilde{\theta}\|_{L^2(J;C^\sigma_*)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla\tilde{\theta}\|_{L^2(J;C^\sigma_*)}.\end{aligned}$$

We hence have

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + T_\alpha\Delta\tilde{\theta} + T_\beta \cdot \nabla\partial_z\tilde{\theta} + F_3$$

where

$$\|F_3\|_{L^1(J;C^\sigma_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla\tilde{\theta}\|_{L^2(J;C^\sigma_*)}.$$

Next we estimate the error F_2 consisting of errors from paralinearization. Write

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + \alpha\Delta\tilde{\theta} + \beta \cdot \nabla\partial_z\tilde{\theta} + (T_\alpha - \alpha)\Delta\tilde{\theta} + (T_\beta - \beta)\cdot\nabla\partial_z\tilde{\theta}$$

and expand

$$(T_\alpha - \alpha)\Delta\tilde{\theta} + (T_\beta - \beta)\cdot\nabla\partial_z\tilde{\theta} = -(T_{\Delta\tilde{\theta}}\alpha + R(\alpha, \Delta\tilde{\theta}) + T_{\nabla\partial_z\tilde{\theta}} \cdot \beta + R(\beta, \nabla\partial_z\tilde{\theta})).$$

By (E.9) and (E.6),

$$\begin{aligned}\|T_{\Delta\tilde{\theta}}\alpha\|_{L^1(J;C^\sigma_*)} + \|R(\alpha, \Delta\tilde{\theta})\|_{L^1(J;C^\sigma_*)} &\lesssim \|\Delta\tilde{\theta}\|_{L^2(J;C^{\sigma-1}_*)}\|\alpha\|_{L^2(J;C^r_*)} \\ \|T_{\nabla\partial_z\tilde{\theta}} \cdot \beta\|_{L^1(J;C^\sigma_*)} + \|R(\beta, \nabla\partial_z\tilde{\theta})\|_{L^1(J;C^\sigma_*)} &\lesssim \|\nabla\partial_z\tilde{\theta}\|_{L^2(J;C^{\sigma-1}_*)}\|\beta\|_{L^\infty(J;C^r_*)}.\end{aligned}$$

We estimate α and β via (A.9) while

$$\|\Delta\tilde{\theta}\|_{C^{\sigma-1}_*} \lesssim \|\nabla\tilde{\theta}\|_{C^\sigma_*}, \quad \|\nabla\partial_z\tilde{\theta}\|_{C^{\sigma-1}_*} \lesssim \|\partial_z\tilde{\theta}\|_{C^\sigma_*}.$$

We hence have

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + \alpha\Delta\tilde{\theta} + \beta \cdot \nabla\partial_z\tilde{\theta} + F_2 + F_3$$

where

$$\|F_2\|_{L^1(J;C^\sigma_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J;C^\sigma_*)}.$$

Lastly, we estimate the first order term appearing on the left hand side of (A.2):

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = F_0 + \gamma\partial_z\tilde{\theta} + F_2 + F_3.$$

To estimate $F_1 := \gamma\partial_z\tilde{\theta}$, we decompose into paraproducts,

$$\gamma\partial_z\tilde{\theta} = T_\gamma\partial_z\tilde{\theta} + T_{\partial_z\tilde{\theta}}\gamma + R(\gamma, \partial_z\tilde{\theta}).$$

We have by (E.10), (E.6), and (A.9),

$$\begin{aligned}\|T_\gamma\partial_z\tilde{\theta}\|_{L^1(J;C^\sigma_*)} + \|R(\gamma, \partial_z\tilde{\theta})\|_{L^1(J;C^\sigma_*)} &\lesssim \|\gamma\|_{L^2(J;C^{r-1}_*)}\|\partial_z\tilde{\theta}\|_{L^2(J;C^\sigma_*)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\partial_z\tilde{\theta}\|_{L^2(J;C^\sigma_*)}.\end{aligned}$$

Similarly, using (A.10)

$$\begin{aligned}\|T_{\partial_z\tilde{\theta}}\gamma\|_{L^1(J;C^\sigma_*)} &\lesssim \|\gamma\|_{L^1(J;C^{r-\frac{1}{2}}_*)}\|\partial_z\tilde{\theta}\|_{L^\infty(J;C^{\sigma+\frac{1}{2}-r}_*)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\partial_z\tilde{\theta}\|_{L^\infty(J;C^{\sigma-\frac{1}{2}}_*)}.\end{aligned}$$

□

A.3. **Elliptic estimates.** Now that we have estimates on the coefficients and error in the factored equation (A.12), we can apply parabolic estimates, which we now recall:

Proposition A.6 ([ABZ14b, Proposition B.4])**.** *Let $\rho \in (0,1)$, $J = [z_0, z_1] \subseteq \mathbb{R}$, and $p \in \Gamma^1_\rho(J \times \mathbb{R}^d)$ with*

$$Re\ p(z; x, \xi) \geq c|\xi|.$$

Consider a solution w to

$$(\partial_z + T_p)w = F_1 + F_2, \quad w|_{z=z_0} = w_0.$$

Then for any $q \in [1, \infty]$ and $(r_0, r) \in \mathbb{R}^2$ with $r_0 < r$, and $\delta > 0$,

$$\|w\|_{C^0(J;C^r_*)} \lesssim \|w_0\|_{C^r_*} + \|F_1\|_{L^1(J;C^r_*)} + \|F_2\|_{L^q(J;C^{r-1+\frac{1}{q}+\delta}_*)} + \|w\|_{L^\infty(J;C^{r_0}_*)}$$

with a constant depending on $r_0, r, \rho, c, \delta, q$, and $\mathcal{M}^1_\rho(p)$.

By a simple modification of the proof of this result, we also have

Proposition A.7. *Let $\rho \in (0,1)$, $J = [z_0, z_1] \subseteq \mathbb{R}$, and $p \in \Gamma^1_\rho(J \times \mathbb{R}^d)$ with*

$$Re\ p(z; x, \xi) \geq c|\xi|.$$

Consider a solution w to

$$(\partial_z + T_p)w = F, \quad w|_{z=z_0} = w_0.$$

Then for any $q \in [1, \infty]$ and $(r_0, r) \in \mathbb{R}^2$ with $r_0 < r$, and $\delta > 0$,

$$\|w\|_{L^1(J;C^{r+1}_*) \cap L^2(J;C^{r+\frac{1}{2}}_*)} \lesssim \|w_0\|_{C^{r+\delta}_*} + \|F\|_{L^1(J;C^{r+\delta}_*)} + \|w\|_{L^2(J;C^{r_0}_*)}$$

with a constant depending on $r_0, r, \rho, c, \delta, q$, and $\mathcal{M}^1_\rho(p)$.

We now apply these parabolic estimates twice to (A.12) to obtain an "inductive" elliptic estimate:

Proposition A.8. *Let $0 \le \sigma < r - \frac{1}{2}$, $\delta > 0$, and $-1 < z_1 < z_0 < 0$. Denote $J_0 = [z_0, 0], J_1 = [z_1, 0]$. Consider $\tilde{\theta}$ solving (A.2). Then*

$$\begin{aligned}\|\nabla_{x,z}\tilde{\theta}\|_{C^0(J_0;C^\sigma_*)} &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}}(J_1)} \\ &\quad + \|f\|_{C^{\sigma+1}_*} + \|F_0\|_{L^1(J_1;C^{\sigma-\frac{1}{2}}_*)} \\ \|\nabla_{x,z}\tilde{\theta}\|_{L^1(J_0;C^{\sigma+1}_*)\cap L^2(J_0;C^{\sigma+\frac{1}{2}}_*)} &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}+\delta}(J_1)} \\ &\quad + \|f\|_{C^{\sigma+1+\frac{\delta}{2}}_*} + \|F_0\|_{L^1(J_1;C^{\sigma+\delta}_*)}.\end{aligned}$$

Proof. First, we would like to apply the parabolic estimate with symbol $-a$ (satisfying $\mathrm{Re}(-a) \gtrsim |\xi|$) on the equation

$$(\partial_z - T_a)w = (\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = F_0 + F_1 + F_2 + F_3.$$

However, it is convenient to apply the parabolic estimate on w with vanishing initial condition, so instead we set

$$w := \chi(z)(\partial_z - T_A)\tilde{\theta}$$

where χ is a smooth cutoff vanishing on $[-1, z_1]$ and $\chi = 1$ on $J_0 = [z_0, 0] \subseteq (-1, 0]$. We then have

$$(\partial_z - T_a)w = \chi(z)(F_0 + F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)\tilde{\theta} =: F'.$$

We estimate $\chi'(z)(\partial_z - T_A)\tilde{\theta}$ directly. By (E.1),

$$\|T_A\tilde{\theta}\|_{L^1(J_1;C^\sigma_*)} \lesssim \mathcal{M}^1_0(A)\|\tilde{\theta}\|_{L^1(J_1;C^{\sigma+1}_*)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\tilde{\theta}\|_{L^2(J_1;C^{\sigma+1}_*)}.$$

As in estimate of F_3, we may replace v by $S_{>1/10}\tilde{\theta}$ by the inhomogeneous paradifferential calculus and hence estimate

$$\|S_{>1/10}\tilde{\theta}\|_{L^2(J_1;C^{\sigma+1}_*)} \lesssim \|\nabla\tilde{\theta}\|_{L^2(J_1;C^\sigma_*)}.$$

We conclude

$$\|\chi'(z)(\partial_z - T_A)\tilde{\theta}\|_{L^1(J_1;C^\sigma_*)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;C^\sigma_*)}.$$

Combining this estimate with the estimates of Proposition A.5, we have

$$\begin{aligned}\|F'\|_{L^1(J_1;C^\sigma_*)} &\lesssim \|F_1 + F_2 + F_3 + \chi'(z)(\partial_z - T_A)\tilde{\theta}\|_{L^1(J_1;C^\sigma_*)} + \|F_0\|_{L^1(J_1;C^\sigma_*)} \\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}}(J_1)} + \|F_0\|_{L^1(J_1;C^\sigma_*)}.\end{aligned}$$

We can now apply Proposition A.6 with

$$\rho = \frac{1}{2}, \quad J = J_1, \quad p = -a, \quad q = \infty, \quad r = \sigma.$$

Note we may also choose $r_0 = \sigma - 1$, using the above analysis on $\chi'(z)(\partial_z - T_A)\tilde{\theta}$ with $w = \chi(z)(\partial_z - T_A)\tilde{\theta}$, yielding as well the same estimates. Thus

$$\begin{aligned}\|w\|_{C^0(J_1;C^\sigma_*)} &\lesssim \|F'\|_{L^1(J_1;C^\sigma_*)} + \|w\|_{L^\infty(J_1;C^{\sigma-1}_*)} \\ &\le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}}(J_1)} + \|F_0\|_{L^1(J_1;C^\sigma_*)}. \qquad \text{(A.13)}\end{aligned}$$

Similarly, for the second estimate of our proposition, we use Proposition A.7 with

$$\rho = \frac{1}{2}, \quad J = J_1, \quad p = -a, \quad r = \sigma + \frac{\delta}{2}$$

and $\delta/2$ in the place of δ. We again choose $r_0 = \sigma - 1$. Using $\sigma + \delta$ in place of σ in the above estimate for F' (we may assume δ is small enough so that $\sigma + \frac{1}{2} + \delta < r$)

$$\begin{aligned}\|w\|_{L^1(J_1;C_*^{\sigma+1+\frac{\delta}{2}})} &\lesssim \|F'\|_{L^1(J_1;C_*^{\sigma+\delta})} + \|w\|_{L^2(J_1;C_*^{\sigma-1})} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}+\delta}(J_1)} + \|F_0\|_{L^1(J_1;C_*^{\sigma+\delta})}.\end{aligned}$$

Next we apply the second parabolic estimate. On $J_0 = [z_0, 0]$, we have $\chi = 1$ and hence

$$(\partial_z - T_A)\tilde{\theta} = w.$$

Define $\tilde{\theta}^*(x, z) = \tilde{\theta}(x, -z)$ and w^*, etc. in the analogous way, so that $\mathrm{Re}(A) \gtrsim |\xi|$ and

$$(\partial_z + T_A)\tilde{\theta}^* = -w^*, \qquad z \in [0, -z_0] = J_0^*.$$

We again apply Proposition A.6 with

$$\rho = \frac{1}{2}, \quad J = J_0^*, \quad p = A, \quad q = \infty, \quad r = \sigma + 1, \quad r_0 = 0.$$

We obtain

$$\|\tilde{\theta}^*\|_{C^0(J_0^*;C_*^{\sigma+1})} \lesssim \|f\|_{C_*^{\sigma+1}} + \|w^*\|_{L^\infty(J_0^*;C_*^\sigma)} + \|\tilde{\theta}^*\|_{L^\infty(J_0^*;L^\infty)}.$$

For the second estimate of our proposition, we again apply Proposition A.7 with

$$\rho = \frac{1}{2}, \quad J = J_0^*, \quad p = A, \quad r = \sigma + 1, \quad r_0 = 0$$

and $\delta/2$ in the place of δ to obtain

$$\|\tilde{\theta}^*\|_{L^1(J_0^*;C_*^{\sigma+2}) \cap L^2(J_0^*;C_*^{\sigma+\frac{3}{2}})} \lesssim \|f\|_{C_*^{\sigma+1+\frac{\delta}{2}}} + \|w^*\|_{L^1(J_0^*;C_*^{\sigma+1+\frac{\delta}{2}})} + \|\tilde{\theta}^*\|_{L^2(J_0^*;L^\infty)}.$$

Now the rest of the proof for the $L^1(J_0; C_*^{\sigma+2}) \cap L^2(J_0; C_*^{\sigma+\frac{3}{2}})$ estimate mirrors that of the $C^0(J_0; C_*^{\sigma+1})$ estimate, detailed in the following.

We estimate the last term on the right hand side by writing

$$\tilde{\theta}^*(z) = \tilde{\theta}^*(0) + \int_0^z \partial_z\tilde{\theta}^* = f + \int_0^z \partial_z\tilde{\theta}^*$$

and hence

$$\|\tilde{\theta}^*\|_{L^\infty(J_0^*;L^\infty)} \lesssim \|f\|_{L^\infty} + \|\partial_z\tilde{\theta}^*\|_{L^2(J_0^*;L^\infty)} \lesssim \|f\|_{C_*^{\sigma+1}} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}}(J_0)}.$$

Collecting the above estimates, we conclude

$$\begin{aligned}\|\nabla\tilde{\theta}\|_{C^0(J_0;C_*^\sigma)} \lesssim \|\tilde{\theta}^*\|_{C^0(J_0;C_*^{\sigma+1})} &\lesssim \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}}(J_1)} \\ &\quad + \|f\|_{C_*^{\sigma+1}} + \|F_0\|_{L^1(J_1;C_*^\sigma)}.\end{aligned}$$

To attain the same estimate on $\partial_z\tilde{\theta}$, write

$$\partial_z\tilde{\theta} = T_A\tilde{\theta} + w.$$

$T_A\tilde{\theta}$ enjoys the same estimate as $\nabla\tilde{\theta}$ by using Proposition A.4 and that A is of order 1, and w already has the desired estimate above. □

We also recall the corresponding (global) Sobolev elliptic estimate for (A.2):

Proposition A.9. [ABZ14a, Proposition 3.16] *Let* $-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}$ *and* $-1 < z_1 < z_0 < 0$. *Denote* $J_0 = [z_0, 0], J_1 = [z_1, 0]$. *Consider* $\tilde{\theta}$ *solving (A.2). Then*

$$\|\nabla_{x,z}\tilde{\theta}\|_{X^\sigma(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(J_1)} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_1)}).$$

Applying this to (A.11), we have the following local Sobolev counterpart:

Corollary A.10. *Let* $0 \leq \sigma \leq s - \frac{1}{2}$ *and* $-1 < z_1 < z_0 < 0$. *Denote* $J_0 = [z_0, 0], J_1 = [z_1, 0]$. *Consider* $\tilde{\theta}$ *solving (A.2). Denote* $wS = w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}$ *or* $wS = w_{x_0,\kappa} S_\kappa$ *with* $\kappa \leq c\lambda$. *Then*

$$\begin{aligned}\|\nabla_{x,z} wS\tilde{\theta}\|_{X^\sigma(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(&\|wSf\|_{H^{\sigma+1}} + \|wSF_0\|_{Y^\sigma(J_1)} \\ &+ (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(\|f\|_{H^{\sigma+\frac{3}{4}}} + \|F_0\|_{Y^{\sigma-\frac{1}{4}}(J_1)} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_1)})).\end{aligned}$$

Proof. Applying Proposition A.9 to (A.11), we have

$$\|\nabla_{x,z} wS\tilde{\theta}\|_{X^\sigma(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|wSf\|_{H^{\sigma+1}} + \|wSF_0 + F_4\|_{Y^\sigma(J_1)} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_1)}).$$

By Proposition A.3,

$$\|F_4\|_{Y^\sigma(J_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^{\sigma+\frac{1}{4}})}).$$

Then apply Proposition A.9 on

$$\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^{\sigma+\frac{1}{4}})}.$$

□

A.4. **Estimates in the harmonic case.** In the special case of (A.1) where θ satisfies

$$\Delta_{x,y}\theta(x, y) = 0, \qquad \theta|_{y=\eta(x)} = f, \qquad \partial_n\theta|_\Gamma = 0, \tag{A.14}$$

as is the case for instance when defining the Dirichlet to Neumann map, we have the following "base case" estimate:

Proposition A.11. [ABZ14a, Remark 3.15] *Consider* θ *solving (A.14). Then*

$$\|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\frac{1}{2}}}.$$

Combined with Proposition A.9, this yields:

Proposition A.12. *Let* $-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}$ *and* $z_0 \in (-1, 0]$, $J = [z_0, 0]$. *Consider* θ *solving (A.14). Then*

$$\|\nabla_{x,z}\tilde{\theta}\|_{X^\sigma(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\sigma+1}}.$$

We also have a local counterpart, combining with Corollary A.10:

Proposition A.13. *Let* $0 \leq \sigma \leq s - \frac{1}{2}$ *and* $z_0 \in (-1, 0]$, $J = [z_0, 0]$. *Consider* θ *solving (A.14). Denote* $wS = w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}$ *or* $wS = w_{x_0,\kappa} S_\kappa$ *with* $\kappa \leq c\lambda$. *Then*

$$\|\nabla_{x,z} wS\tilde{\theta}\|_{X^\sigma(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|wSf\|_{H^{\sigma+1}} + (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|f\|_{H^{\sigma+\frac{3}{4}}}).$$

Lastly, we state a Hölder counterpart:

Proposition A.14. *Let* $0 \leq \sigma < r - \frac{1}{2}$ *and* $z_0 \in (-1, 0]$, $J = [z_0, 0]$. *Consider* $\tilde{\theta}$ *solving (A.14). Then*

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^\sigma(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|f\|_{H^{\sigma+1+}} + \|f\|_{C_*^{\sigma+1+}}.$$

Proof. First, as a straightforward consequence of Proposition A.8,

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^\sigma(J)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}+}(J_1)} + \|f\|_{C_*^{\sigma+1+}}.$$

Then applying Sobolev embedding and Proposition A.12,

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-\frac{1}{2}+}(J_1)} \lesssim \|\nabla_{x,z}\tilde{\theta}\|_{X^{\sigma+}(J_1)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\sigma+1+}}.$$

□

Appendix B. Local Dirichlet to Neumann Paralinearization

In this section we paralinearize the Dirichlet to Neumann map, with a local Sobolev estimate on the error. Recall the Dirichlet to Neumann map is given by solving (A.14),

$$\Delta_{x,y}\theta = 0, \qquad \theta|_{y=\eta(x)} = f, \qquad \partial_n\theta|_\Gamma = 0,$$

and setting

$$(G(\eta)f)(x) = \sqrt{1 + |\nabla\eta|^2}(\partial_n\theta)|_{y=\eta(x)} = ((\partial_y - \nabla\eta\cdot\nabla)\theta)|_{y=\eta(x)}.$$

In the flattened coordinates discussed in Appendix A.1, we have the homogeneous counterpart to (A.2) (recall that we write $\tilde{u}(x,z) = u(x,\rho(x,z))$ where ρ is the diffeomorphism that flattens the boundary defined by the graph of η),

$$(\partial_z^2 + \alpha\Delta + \beta\cdot\nabla\partial_z - \gamma\partial_z)\tilde{\theta} = 0, \qquad \tilde{\theta}|_{z=0} = f, \tag{B.1}$$

and we may write the Dirichlet to Neumann map as

$$G(\eta)f = \left(\frac{1 + |\nabla\rho|^2}{\partial_z\rho}\partial_z\tilde{\theta} - \nabla\rho\cdot\nabla\tilde{\theta}\right)\bigg|_{z=0}.$$

To paralinearize the latter term $\nabla\rho\cdot\nabla\tilde{\theta}$ with errors in local Sobolev norm, we will require local counterparts to the estimates on the diffeomorphism ρ from Proposition A.1. We establish these in the next two subsections. To paralinearize the former $\partial_z\tilde{\theta}$ term, we use the homogeneous case of the factoring (A.12). A single parabolic estimate then provides the paralinearization $\partial_z\tilde{\theta} \approx T_A\tilde{\theta}$.

B.1. Commutator and product estimates.

Before addressing the diffeomorphism ρ, we observe some general commutator and product estimates regarding the local weights used in Section 11. First, we observe that the various weights of the form wS essentially commute with paraproducts T_a:

Proposition B.1. *Let $m \in \mathbb{R}$ and $\rho > 0$.*

i) We have

$$\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}T_a u\|_{H^{m'}} \lesssim \|a\|_{C_*^{-\rho}}(\|w_{x_0,\lambda}\tilde{S}_{\xi_0,\lambda,\mu}u\|_{H^{m'+\rho}} + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\|\tilde{S}_\mu u\|_{H^{m+\rho}}),$$

$$\|w_{x_0,\lambda}S_\mu T_a u\|_{H^{m'}} \lesssim \|a\|_{C_*^{-\rho}}(\|w_{x_0,\lambda}\tilde{S}_\mu u\|_{H^{m'+\rho}} + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\|\tilde{S}_\mu u\|_{H^{m+\rho}}).$$

ii) For $\kappa \le c\lambda$,

$$\|w_{x_0,\kappa}S_\kappa T_a u\|_{H^{m'}} \lesssim \|a\|_{C_*^{-\rho}}(\|w_{x_0,\kappa}\tilde{S}_\kappa u\|_{H^{m'+\rho}} + \|u\|_{H^{m'+\rho-\frac{1}{4}}}).$$

iii) For $\kappa \ge \lambda/c$,

$$\|w_{x_0,\lambda}S_\kappa T_a u\|_{H^{m'}} \lesssim \|a\|_{C_*^{-\rho}}(\|w_{x_0,\lambda}\tilde{S}_\kappa u\|_{H^{m'+\rho}} + \|u\|_{H^{m'+\rho-\frac{1}{4}}}).$$

In the case $\rho = 0$, the same estimates hold with L^∞ in the place of $C_*^{-\rho}$.

Proof. We first prove case (iii), noting that (ii) may be proven in the same way. Using the frequency support of $w_{x_0,\lambda}$ at $\lambda^{\frac{3}{4}}$, we may write $S_\kappa w_{x_0,\lambda} T_a u$ as a sum of finitely many terms of the form

$$S_\kappa w_{x_0,\lambda}(S_{\leq \tilde{\kappa}/8} a) S_{\tilde{\kappa}} u = S_\kappa (S_{\leq \tilde{\kappa}/8} a) w_{x_0,\lambda} S_{\tilde{\kappa}} u$$

where $\tilde{\kappa} \in [\kappa/8, 8\kappa]$. Then estimate

$$\|S_\kappa (S_{\leq \tilde{\kappa}/8} a) w_{x_0,\lambda} S_{\tilde{\kappa}} u\|_{H^{m'}} \lesssim \kappa^m \|S_{\leq \tilde{\kappa}/8} a\|_{L^\infty} \|w_{x_0,\lambda} S_{\tilde{\kappa}} u\|_{L^2} \lesssim \|a\|_{C_*^{-\rho}} \|w_{x_0,\lambda} S_{\tilde{\kappa}} u\|_{H^{m'+\rho}}.$$

It then remains to commute, using the frequency support of $w_{x_0,\lambda}$:

$$\|[w_{x_0,\lambda}, S_\kappa] T_a u\|_{H^{m'}} \lesssim \kappa^{-\frac{1}{4}} \|T_a \tilde{S}_\kappa u\|_{H^{m'}} \lesssim \|a\|_{C_*^{-\rho}} \|\tilde{S}_\kappa u\|_{H^{m'+\rho-\frac{1}{4}}}.$$

The second estimate of (i) is similar, except that when commuting,

$$\|[w_{x_0,\lambda}, S_\mu] T_a u\|_{H^{m'}} \lesssim \lambda^{\frac{3}{4}} \mu^{-1} \|T_a \tilde{S}_\mu u\|_{H^{m'}} \lesssim \lambda^{\frac{3}{4}} \mu^{-\frac{7}{8}} \|a\|_{C_*^{-\rho}} \|\tilde{S}_\mu u\|_{H^{m+\rho}}.$$

Then observe that

$$\lambda^{\frac{3}{4}} \mu^{-\frac{7}{8}} \leq \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}}.$$

The first estimate of (i) is similar, differing only in the commutator. □

As a simple consequence, we have the following local product estimates:

Corollary B.2. *Let $m \in \mathbb{R}$ and $\rho > 0$.*

i) We have

$$\begin{aligned}\|w_{x_0,\lambda} S_{\xi_0,\lambda,\mu}(uv)\|_{H^{m'}} \lesssim{}& (\|w_{x_0,\lambda} \tilde{S}_{\xi_0,\lambda,\mu} u\|_{H^{m'}} + \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|u\|_{H^m}) \|v\|_{L^\infty} \\ &+ \|u\|_{C_*^{-\rho}} (\|w_{x_0,\lambda} \tilde{S}_{\xi_0,\lambda,\mu} v\|_{H^{m'+\rho}} + \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|v\|_{H^{m+\rho}}) \\ &+ \|v\|_{C_*^{\frac{1}{8}}} \|u\|_{H^m},\end{aligned}$$

$$\begin{aligned}\|w_{x_0,\lambda} S_\mu(uv)\|_{H^{m'}} \lesssim{}& (\|w_{x_0,\lambda} \tilde{S}_\mu u\|_{H^{m'}} + \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|u\|_{H^m}) \|v\|_{L^\infty} \\ &+ \|u\|_{C_*^{-\rho}} (\|w_{x_0,\lambda} \tilde{S}_\mu v\|_{H^{m'+\rho}} + \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|v\|_{H^{m+\rho}}) \\ &+ \|v\|_{C_*^{\frac{1}{8}}} \|u\|_{H^m},\end{aligned}$$

ii) For $\kappa \leq c\lambda$,

$$\begin{aligned}\|w_{x_0,\kappa} S_\kappa(uv)\|_{H^{m'}} \lesssim{}& (\|w_{x_0,\kappa} \tilde{S}_\kappa u\|_{H^{m'}} + \|u\|_{H^m}) \|v\|_{L^\infty} \\ &+ \|u\|_{C_*^{-\rho}} (\|w_{x_0,\kappa} \tilde{S}_\kappa v\|_{H^{m'+\rho}} + \|v\|_{H^{m+\rho}}) \\ &+ \|v\|_{C_*^{\frac{1}{8}}} \|u\|_{H^m},\end{aligned}$$

iii) For $\kappa \geq \lambda/c$,

$$\begin{aligned}\|w_{x_0,\lambda} S_\kappa(uv)\|_{H^{m'}} \lesssim{}& (\|w_{x_0,\lambda} \tilde{S}_\kappa u\|_{H^{m'}} + \|u\|_{H^m}) \|v\|_{L^\infty} \\ &+ \|u\|_{C_*^{-\rho}} (\|w_{x_0,\lambda} \tilde{S}_\kappa v\|_{H^{m'+\rho}} + \|v\|_{H^{m+\rho}}) \\ &+ \|v\|_{C_*^{\frac{1}{8}}} \|u\|_{H^m},\end{aligned}$$

In the case $\rho = 0$, the same estimates hold with L^∞ in the place of $C_*^{-\rho}$.

Proof. Write

$$wS(uv) = wS(T_u v + T_v u + R(u,v)).$$

For the third term, we may use (E.7),

$$\|R(u,v)\|_{H^{m+\frac{1}{8}}} \lesssim \|u\|_{H^m} \|v\|_{C_*^{\frac{1}{8}}}.$$

Then the first and second terms are estimated by Proposition B.1.

□

We will also need a generalization to paradifferential operators:

Proposition B.3. *Let $m, \tilde{m}, k \in \mathbb{R}$, $\rho, \tilde{\rho} \in [0,1]$, $a \in \Gamma^m_{\tilde{\rho}}$, $b \in \Gamma^{\tilde{m}}_{\tilde{\rho}}$, and $\kappa^{\frac{3}{4}} \ll \mu$. Then*

$$\begin{aligned}
\|[w_{x_0,\kappa} S_\mu, T_a]u\|_{H^k} &\lesssim M_0^m(a) \kappa^{\frac{3\rho}{4}} \mu^{-\frac{3\rho}{4}} \|\tilde{S}_\mu u\|_{H^{k+m-\frac{\rho}{4}}}, \\
\|w_{x_0,\kappa} S_\mu T_a u\|_{H^k} &\lesssim M_0^m(a)(\|w_{x_0,\kappa} S_\mu u\|_{H^{k+m}} + \kappa^{\frac{3\rho}{4}} \mu^{-\frac{3\rho}{4}} \|\tilde{S}_\mu u\|_{H^{k+m-\frac{\rho}{4}}}), \\
\|w_{x_0,\kappa} S_\mu (T_a T_b - T_{ab}) u\|_{H^k} &\lesssim (M_{\tilde{\rho}}^m(a) M_0^{\tilde{m}}(b) + M_0^m(a) M_{\tilde{\rho}}^{\tilde{m}}(b)) \\
&\quad \cdot (\|w_{x_0,\kappa} S_\mu u\|_{H^{k+m+\tilde{m}-\tilde{\rho}}} + \kappa^{\frac{3\rho}{4}} \mu^{-\frac{3\rho}{4}} \|\tilde{S}_\mu u\|_{H^{k+m+\tilde{m}-\tilde{\rho}-\frac{\rho}{4}}}).
\end{aligned}$$

If additionally a is homogeneous in ξ, we have

$$\begin{aligned}
\|w_{x_0,\kappa} S_\mu [T_a, \partial_t + T_V \cdot \nabla] u(t)\|_{H^k} &\lesssim (M_0^m(a) \|V(t)\|_{W^{r,\infty}} + M_0^m((\partial_t + V \cdot \nabla) a)) \\
&\quad \cdot (\|w_{x_0,\kappa} S_\mu u(t)\|_{H^{k+m}} + \kappa^{\frac{3\rho}{4}} \mu^{-\frac{3\rho}{4}} \|u(t)\|_{H^{k+m-\frac{\rho}{4}}}).
\end{aligned}$$

Proof. We immediately have by (E.1),

$$\|T_a w_{x_0,\kappa} S_\mu u\|_{H^k} \lesssim M_0^m(a) \|w_{x_0,\kappa} S_\mu u\|_{H^{k+m}}.$$

This implies the second estimate once we prove the first estimate.

First observe that by the frequency localization of $w_{x_0,\kappa}$ at $\kappa^{\frac{3}{4}} \ll \mu$, we may replace $w_{x_0,\kappa}$ with $T_{w_{x_0,\kappa}}$. Then, we may apply (E.2),

$$\begin{aligned}
\|[w_{x_0,\kappa} S_\mu, T_a]u\|_{H^k} &\lesssim (M_\rho^m(a) M_0^0(w_{x_0,\kappa}) + M_0^m(a) M_\rho^0(w_{x_0,\kappa})) \|\tilde{S}_\mu u\|_{H^{k+m-\rho}} \\
&\lesssim (M_\rho^m(a) + M_0^m(a) \kappa^{\frac{3\rho}{4}}) \|\tilde{S}_\mu u\|_{H^{k+m-\rho}}.
\end{aligned}$$

However, also observe that since $w_{x_0,\kappa} S_\mu$ is order 0, a sharper analysis (see for instance [Tay08, Theorem 3.4.A]) shows that in fact

$$\|[w_{x_0,\kappa} S_\mu, T_a]u\|_{H^k} \lesssim M_0^m(a) M_\rho^0(w_{x_0,\kappa}) \|\tilde{S}_\mu u\|_{H^{k+m-\rho}} \lesssim M_0^m(a) \kappa^{\frac{3\rho}{4}} \|\tilde{S}_\mu u\|_{H^{k+m-\rho}}.$$

For the third estimate, use that the paradifferential calculus essentially forms an algebra (see the remarks preceding Corollary 3.4.G in [Tay08]), with

$$T_a T_b - T_{ab} = T_A \in OP\Gamma_0^{m+\tilde{m}-\tilde{\rho}}.$$

Then apply the second estimate, with A in the place of a and $m + \tilde{m} - \tilde{\rho}$ in the place of m:

$$\|w_{x_0,\kappa} S_\mu T_A u\|_{H^k} \lesssim M_0^{m+\tilde{m}-\tilde{\rho}}(A)(\|w_{x_0,\kappa} S_\mu u\|_{H^{k+m+\tilde{m}-\tilde{\rho}}} + \kappa^{\frac{3\rho}{4}} \mu^{-\frac{3\rho}{4}} \|\tilde{S}_\kappa u\|_{H^{k+m+\tilde{m}-\tilde{\rho}-\frac{\rho}{4}}}).$$

Then the third estimate follows from (E.2). One similarly obtains the fourth estimate, using [ABZ14a, Lemma 2.15] in place of (E.2). □

B.2. **Local estimates on the diffeomorphism.** In this subsection we establish local counterparts to Proposition A.1. For brevity, it is convenient to let wS denote any of the localizing operators found in the $LS^\sigma_{x_0,\lambda}$ or $LS^\sigma_{x_0,\xi_0,\lambda,\mu}$ seminorms. When using this notation, let κ denote the frequency of S, so that $S = S\tilde{S}_\kappa$.

Proposition B.4. *Let $J \subseteq [-1, 0]$. The diffeomorphism ρ satisfies*

$$\|wS(\partial_z\rho, \nabla\rho)\|_{X^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|wS\eta\|_{H^{s'+\frac{1}{2}}}).$$

Proof. Consider first $\nabla = \nabla_x$. Recall that on $z \in (-1, 0)$,

$$\rho(x, z) = (1 + z)e^{\delta z\langle D\rangle}\eta(x) - z(e^{-(1+z)\delta\langle D\rangle}\eta(x) - h)$$

so without loss of generality, we consider $e^{\delta z\langle D\rangle}\eta$. Since

$$\nabla e^{\delta z\langle D\rangle} wS\eta$$

may be estimated as in the proof of Proposition A.1 with $wS\eta$ in the place of η, it suffices to estimate the commutator

$$[wS, \nabla e^{\delta z\langle D\rangle}]\eta = [w, \nabla e^{\delta z\langle D\rangle}]S\eta$$

in $X^{s'-\frac{1}{2}}(J)$.

Since $\nabla e^{\delta z\langle D\rangle}$ is an order 1 operator uniformly in $z \in (-1, 0)$,

$$\|[w, \nabla e^{\delta z\langle D\rangle}]S\eta\|_{H^{s'-\frac{1}{2}}} \lesssim \kappa^{-\frac{1}{4}}\|S\eta\|_{H^{s'+\frac{1}{2}}} \lesssim \|\eta\|_{H^{s+\frac{3}{8}}}.$$

This yields the $L^\infty_z(J; H^{s'-\frac{1}{2}})$ estimate.

For the L^2_z estimate, we also have that $|z|^{\frac{1}{2}}\langle D\rangle^{-\frac{1}{2}}\nabla e^{\delta z\langle D\rangle}$ is an order 0 operator uniformly in z, so that

$$\begin{aligned} |z|^{-\frac{1}{2}}\|\langle D\rangle^{\frac{1}{2}}[wS, |z|^{\frac{1}{2}}\langle D\rangle^{-\frac{1}{2}}\nabla e^{\delta z\langle D\rangle}]\eta\|_{H^{s'}} &\lesssim |z|^{-\frac{1}{2}}\|[w, |z|^{\frac{1}{2}}\langle D\rangle^{-\frac{1}{2}}\nabla e^{\delta z\langle D\rangle}]S\eta\|_{H^{s'+\frac{1}{2}}} \\ &\lesssim \kappa^{-\frac{1}{4}}|z|^{-\frac{1}{2}}\|S\eta\|_{H^{s'+\frac{1}{2}}} \lesssim |z|^{-\frac{1}{2}}\|\eta\|_{H^{s'+\frac{1}{4}}}. \end{aligned}$$

Integrating in z yields the desired estimate in $L^2_z(J; H^s)$, provided we also estimate the commutator

$$\begin{aligned} |z|^{-\frac{1}{2}}\|[\langle D\rangle^{\frac{1}{2}}, wS]|z|^{\frac{1}{2}}\langle D\rangle^{-\frac{1}{2}}\nabla e^{\delta z\langle D\rangle}\eta\|_{H^{s'}} &\lesssim \kappa^{-\frac{1}{4}}|z|^{-\frac{1}{2}}\||z|^{\frac{1}{2}}\langle D\rangle^{-\frac{1}{2}}\nabla e^{\delta z\langle D\rangle}S\eta\|_{H^{s'+\frac{1}{2}}} \\ &\lesssim |z|^{-\frac{1}{2}}\|\eta\|_{H^{s'+\frac{1}{4}}} \end{aligned}$$

which we again integrate in z.

For the estimate on $\partial_z\rho$, recall that on $z \in (-1, 0)$,

$$\partial_z\rho = h + e^{\delta z\langle D\rangle}\eta - e^{-(1+z)\delta\langle D\rangle}\eta + (1 + z)\delta\langle D\rangle e^{\delta z\langle D\rangle}\eta + z\delta\langle D\rangle e^{-(1+z)\delta\langle D\rangle}\eta. \tag{B.2}$$

Without loss of generality, we consider $e^{\delta z\langle D\rangle}\eta$ and $\langle D\rangle e^{\delta z\langle D\rangle}\eta$. The former is easy to estimate, as $e^{\delta z\langle D\rangle}$ is uniformly bounded on H^s. For the latter, as before, it suffices to estimate the commutator

$$[wS, \langle D\rangle e^{\delta z\langle D\rangle}]\eta.$$

This is estimated in the same way as in the previous analysis of $\nabla\rho$, with $\langle D\rangle$ in the place of ∇. ∎

We also have the analogous estimates on the second derivatives:

Corollary B.5. *Let $J \subseteq [-1,0]$. The diffeomorphism ρ satisfies*

$$\|wS\nabla^2_{x,z}\rho\|_{X^{s'-\frac{3}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|wS\eta\|_{H^{s'+\frac{1}{2}}}).$$

Proof. First consider $\nabla\nabla_{x,z}\rho$. We easily see by commuting ∇ that

$$\|wS\nabla_x\nabla_{x,z}\rho\|_{X^{s'-\frac{3}{2}}(J)} \lesssim \|wS\nabla_{x,z}\rho\|_{X^{s'-\frac{1}{2}}(J)} + \|\nabla_{x,z}\rho\|_{X^{s'-\frac{3}{4}}(J)}$$

on which we can apply the estimates on ρ from Propositions B.4 and A.1.

The estimate on ∂_z^2 is similarly elementary, after computing $\partial_z^2\rho$ from definition. □

Next, we establish estimates on the coefficients of (A.2). First, we require local estimates on reciprocals. Notably, we use an algebraic argument instead of a Moser estimate:

Proposition B.6. *Let $J \subseteq [-1,0]$. The diffeomorphism ρ satisfies*

$$\begin{aligned}\|wS(\partial_z\rho)^{-1}\|_{X^{s'-\frac{1}{2}}(J)} &+ \|wS(1+|\nabla\rho|^2)^{-1}\|_{X^{s'-\frac{1}{2}}(J)} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|wS\eta\|_{H^{s'+\frac{1}{2}}}).\end{aligned}$$

Proof. We consider the first term on the left hand side, and the $L^\infty_z(J; H^{s'-\frac{1}{2}})$ case. The other cases are obtained with the appropriate modifications.

First, by commuting w and S as in the proof of Proposition B.1, it suffices to consider

$$\|Sw(\partial_z\rho)^{-1}\|_{H^{s'-\frac{1}{2}}}.$$

Then we have, using that $\partial_z\rho \geq \min(h/2, 1)$ by Lemma 3.6 of [ABZ14a],

$$\|Sw(\partial_z\rho)^{-1}\|_{H^{s'-\frac{1}{2}}} \lesssim \|(\partial_z\rho)\langle D\rangle^{s'-\frac{1}{2}}Sw(\partial_z\rho)^{-1}\|_{L^2},$$

so it suffices to estimate

$$\|[(\partial_z\rho), \langle D\rangle^{s'-\frac{1}{2}}S]w(\partial_z\rho)^{-1}\|_{L^2}.$$

As with the commutator and product estimates of the previous subsection, the main burden is to absorb the extra 1/8 derivatives in either a local Sobolev or Hölder norm. For this, we reduce to paraproducts. The balanced-frequency terms are easily estimated using $C_*^{\frac{1}{8}}$ on one term, so we consider only the low-high terms. First, we have the commutator, in the case that S is a typical frequency projection S_κ,

$$\|[T_{\partial_z\rho}, \langle D\rangle^{s'-\frac{1}{2}}S]w(\partial_z\rho)^{-1}\|_{L^2} \lesssim \|\partial_z\rho\|_{C_*^{\frac{1}{2}}}\|(\partial_z\rho)^{-1}\|_{H^{s'-1}}$$

which suffices by using the ρ estimates of Proposition A.1. In the case $S = S_{\xi_0,\lambda,\mu}$, we have a $\mu^{-1}\lambda \leq \lambda^{\frac{1}{4}}$ loss, which is still better than needed. We also have

$$\|T_{\langle D\rangle^{s'-\frac{1}{2}}Sw(\partial_z\rho)^{-1}}\partial_z\rho\|_{L^2} \lesssim \|\langle D\rangle^{s'-\frac{1}{2}}Sw(\partial_z\rho)^{-1}\|_{C_*^{\frac{1}{2}-s}}\|\partial_z\rho\|_{H^{s-\frac{1}{2}}} \lesssim \|(\partial_z\rho)^{-1}\|_{C_*^{\frac{1}{8}}}\|\partial_z\rho\|_{H^{s-\frac{1}{2}}}.$$

It remains to consider the term

$$ST_{w(\partial_z\rho)^{-1}}\partial_z\rho$$

in $H^{s'-\frac{1}{2}}$. First observe that by (E.2),

$$\begin{aligned}\|S(T_{w(\partial_z\rho)^{-1}} &- T_{(\partial_z\rho)^{-1}}T_w)\partial_z\rho\|_{H^{s'-\frac{1}{2}}} \\ &\lesssim (M^0_{\frac{1}{2}}(w)M^0_0((\partial_z\rho)^{-1}) + M^0_0(w)M^0_{\frac{1}{2}}((\partial_z\rho)^{-1}))\|\tilde{S}_\kappa\partial_z\rho\|_{H^{s'-1}} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\kappa^{\frac{3}{8}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\tilde{S}_\kappa\partial_z\rho\|_{H^{s'-1}}\end{aligned}$$

so that we may consider

$$ST_{(\partial_z\rho)^{-1}}T_w\partial_z\rho = ST_{(\partial_z\rho)^{-1}}w\partial_z\rho.$$

Similar to before, we may commute $[S, T_{(\partial_z\rho)^{-1}}]$ to

$$T_{(\partial_z\rho)^{-1}}Sw\partial_z\rho$$

with the additional but acceptable loss in the case $S = S_{\xi_0,\lambda,\mu}$. We commute S and w as in the proof of Proposition B.1 (and as at the beginning of this proof). Lastly, observe that

$$\|T_{(\partial_z\rho)^{-1}}wS\partial_z\rho\|_{H^{s'-\frac{1}{2}}} \lesssim \|(\partial_z\rho)^{-1}\|_{L^\infty}\|wS\partial_z\rho\|_{H^{s'-\frac{1}{2}}}$$

on which we can apply the estimates on ρ from Propositions B.4 and A.1.

□

Corollary B.7. *Let $J \subseteq [-1, 0]$. For α, β, and γ defined as above,*

$$\|wS(\alpha,\beta)\|_{X^{s'-\frac{1}{2}}(J)} + \|wS\gamma\|_{X^{s'-\frac{3}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|wS\eta\|_{H^{s'+\frac{1}{2}}}).$$

Proof. For α and β, apply the product and reciprocal estimates, Propositions B.2 and B.6 respectively, with Proposition A.1. For the product estimates, use $\rho = 0$.

For γ, use the same estimates with $\rho = 1/2$.

□

Lastly, we estimate the coefficients in the local seminorm:

Corollary B.8. *Let $J \subseteq [-1, 0]$ and $0 \leq \sigma \leq s - \frac{1}{2}$. For α, β, and γ defined as above,*

$$\begin{aligned}\mu^{\sigma'}\|(\alpha,\beta)\|_{L^\infty(J;LS^{s-\sigma-\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu})\cap L^2(J;LS^{s-\sigma}_{x_0,\xi_0,\lambda,\mu})} &+ \mu^{\sigma'}\|\gamma\|_{L^\infty(J;LS^{s-\sigma-\frac{3}{2}}_{x_0,\xi_0,\lambda,\mu})\cap L^2(J;LS^{s-\sigma-1}_{x_0,\xi_0,\lambda,\mu})} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}).\end{aligned}$$

Proof. We consider the L^2_z case; the L^∞_z case is similar. By Corollary B.7, we have in all three cases of wS,

$$\|wS(\alpha,\beta)\|_{L^2(J;H^{s-\sigma})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\kappa^{-\sigma'}\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \kappa^{-\sigma'}\|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|wS\eta\|_{H^{s-\sigma+\frac{1}{2}}}).$$

Thus, we have

$$\|(\alpha,\beta)\|_{L^2(J;LS^{s-\sigma}_{x_0,\xi_0,\lambda,\mu})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\mu^{-\sigma'}\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \mu^{-\sigma'}\|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}).$$

The γ estimate is similar using Corollary B.7.

□

B.3. **Factoring the elliptic equation.** In this subsection we estimate the error on the right hand side of (A.12), in the local energy norm.

Proposition B.9. *Let $0 \leq \sigma \leq s - \frac{1}{2}$ and $z_0 \in [-1, 0]$, $J = [z_0, 0]$. Consider θ solving (A.14). Then we can write*

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = F_1 + F_2 + F_3$$

where for $i \geq 1$,

$$\|F_i\|_{Y^\sigma(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J;H^\sigma)} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J)}),$$

$$\|w_{x_0,\lambda}S_\mu F_i\|_{Y^{\sigma'}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J)}$$
$$+ (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(\mu^{\sigma'}\|\nabla_{x,z}w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\theta\|_{L^2(J;L^2)} + \|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'})}$$
$$+ \|\nabla_{x,z}\tilde{\theta}\|_{L^2(J;H^\sigma)})).$$

Proof. Here we have used F_1, F_2, and F_3 to represent the errors arising from, respectively, the first order term on the left hand side of (A.2), the paralinearization errors, and the lower order terms from applying the symbolic calculus.

We remark that in contrast to the Sobolev estimates on F_i in [ABZ14a], and similar to the estimates on F_i in [ABZ14b] and Proposition A.5, we use Hölder estimates whenever appropriate to obtain an absolute gain of 1/2 derivatives (or the equivalent integral gain in L^p_z) on F_i when compared to $\nabla^2\tilde{\theta}$. Also contrast with Proposition A.3, which obtains only a 1/4 derivative gain on F_4.

First, we observe that

$$\|w_{x_0,\lambda}S_\mu F_i\|_{Y^{\sigma'}(J)} \lesssim \|w_{x_0,\mu}S_\mu F_i\|_{Y^{\sigma'}(J)}$$

so we may exchange $w_{x_0,\lambda}$ with $w_{x_0,\mu}$ when necessary.

We begin with F_3. Factor

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + T_\alpha\Delta\tilde{\theta} + T_\beta\cdot\nabla\partial_z\tilde{\theta} + (T_aT_A - T_{aA})\tilde{\theta} - T_{\partial_z A}\tilde{\theta}.$$

For the first error term, by the commutator estimate (E.2) and the estimates on a, A from Proposition A.4,

$$\begin{aligned}\|(T_aT_A - T_{aA})\tilde{\theta}\|_{L^1(J;H^\sigma)} &= \|(T_aT_A - T_{aA})S_{\geq 1/10}\tilde{\theta}\|_{L^1(J;H^\sigma)}\\ &\lesssim (\mathcal{M}^{1,2}_1(a)\mathcal{M}^1_0(A) + \mathcal{M}^1_0(a)\mathcal{M}^{1,2}_1(A))\|S_{\geq 1/10}\tilde{\theta}\|_{L^2(J;H^{\sigma+2-1})}\\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)}.\end{aligned}$$

Further, for the local estimate, we have by the third local commutator estimate of Proposition B.3,

$$\begin{aligned}\|w_{x_0,\mu}S_\mu(T_aT_A - T_{aA})\tilde{\theta}\|_{L^1(J;H^{\sigma'})} &\lesssim (\mathcal{M}^{1,2}_1(a)\mathcal{M}^1_0(A) + \mathcal{M}^1_0(a)\mathcal{M}^{1,2}_1(A))\\ &\quad\cdot(\|w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'+2-1})} + \|\tilde{S}_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma+2-1})})\\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\\ &\quad\cdot(\|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'})} + \|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)}).\end{aligned}$$

Similarly, using instead (E.1),

$$\begin{aligned}\|T_{\partial_z A}\tilde{\theta}\|_{L^1(J;H^\sigma)} &\lesssim \mathcal{M}^{1,2}_0(\partial_z A)\|S_{>1/10}\tilde{\theta}\|_{L^2(J;H^{\sigma+1})}\\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)},\end{aligned}$$

and the second estimate of Proposition B.3 for the local estimate,

$$\begin{aligned}\|w_{x_0,\mu}S_\mu T_{\partial_z A}\tilde{\theta}\|_{L^1(J;H^{\sigma'})} &\lesssim \mathcal{M}^{1,2}_0(\partial_z A)(\|w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'+1})} + \|\tilde{S}_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma+1})})\\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\\ &\quad\cdot(\|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'})} + \|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)}).\end{aligned}$$

We hence have

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + T_\alpha\Delta\tilde{\theta} + T_\beta \cdot \nabla\partial_z\tilde{\theta} + F_3$$

where

$$\begin{aligned}\|F_3\|_{L^1(J;H^\sigma)} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)},\\ \|w_{x_0,\lambda}S_\mu F_3\|_{L^1(J;H^{\sigma'})} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\\ &\quad \cdot (\|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J;H^{\sigma'})} + \|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)}).\end{aligned}$$

Next we estimate the error F_2 consisting of errors from paralinearization. Write

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + \alpha\Delta\tilde{\theta} + \beta \cdot \nabla\partial_z\tilde{\theta} + (T_\alpha - \alpha)\Delta\tilde{\theta} + (T_\beta - \beta) \cdot \nabla\partial_z\tilde{\theta} + F_3$$

and expand

$$(T_\alpha - \alpha)\Delta\tilde{\theta} + (T_\beta - \beta) \cdot \nabla\partial_z\tilde{\theta} = -(T_{\Delta\tilde{\theta}}\alpha + R(\alpha, \Delta\tilde{\theta}) + T_{\nabla\partial_z\tilde{\theta}} \cdot \beta + R(\beta, \nabla\partial_z\tilde{\theta})).$$

By (E.8) and (E.7),

$$\begin{aligned}\|T_{\Delta\tilde{\theta}}\alpha\|_{L^1(J;H^\sigma)} + \|R(\alpha, \Delta\tilde{\theta})\|_{L^1(J;H^\sigma)} &\lesssim \|\Delta\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s})}\|\alpha\|_{L^2(J;H^s)}\\ \|T_{\nabla\partial_z\tilde{\theta}} \cdot \beta\|_{L^1(J;H^\sigma)} + \|R(\beta, \nabla\partial_z\tilde{\theta})\|_{L^1(J;H^\sigma)} &\lesssim \|\nabla\partial_z\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s})}\|\beta\|_{L^2(J;H^s)}.\end{aligned}$$

We estimate α and β via Corollary A.2 while

$$\|\Delta\tilde{\theta}\|_{C_*^{\sigma-s}} \lesssim \|\nabla\tilde{\theta}\|_{C_*^{\sigma-s+1}}, \quad \|\nabla\partial_z\tilde{\theta}\|_{C_*^{\sigma-s}} \lesssim \|\partial_z\tilde{\theta}\|_{C_*^{\sigma-s+1}}.$$

For the local estimates, we consider the α terms; the β terms are similar. By Proposition B.1,

$$\|w_{x_0,\mu}S_\mu T_{\Delta\tilde{\theta}}\alpha\|_{H^{\sigma'}} \lesssim \|\Delta\tilde{\theta}\|_{C_*^{\sigma-s}}(\|w_{x_0,\mu}S_\mu\alpha\|_{H^{s'}} + \|\alpha\|_{H^s}).$$

After integrating in z and using Cauchy-Schwarz, we may use Corollary B.7 and Corollary A.2 respectively to estimate the α terms.

Next we consider $w_{x_0,\lambda}S_\mu R(\alpha, \Delta\tilde{\theta})$. By Proposition 11.9,

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu R(\alpha, \Delta\tilde{\theta})\|_{H^{\sigma'}} &\lesssim \mu^{\sigma'}\|w_{x_0,\lambda}S_\mu R(\alpha, \Delta\tilde{\theta})\|_{L^2}\\ &\lesssim \mu^{\sigma'}(\|\alpha\|_{LS^{s-\sigma}_{x_0,\xi_0,\lambda,\mu}}\|\Delta\tilde{\theta}\|_{C_*^{\sigma-s+}} + \|S_\lambda\alpha\|_{C_*^r}\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\Delta\tilde{\theta}\|_{H^{-1}}).\end{aligned}$$

We use Corollary B.8 and Corollary A.2 to estimate the α terms, concluding

$$\begin{aligned}&\|w_{x_0,\lambda}S_\mu R(\alpha, \Delta\tilde{\theta})\|_{L^1(J;H^{\sigma'})}\\ &\quad\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})\|\nabla\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s+1+})}\\ &\qquad + (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\mu^{\sigma'}(\|\nabla w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^2(J;L^2)} + \|S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^2(J;H^{1-\frac{1}{4}})})).\end{aligned}$$

We hence have

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = \partial_z^2\tilde{\theta} + \alpha\Delta\tilde{\theta} + \beta \cdot \nabla\partial_z\tilde{\theta} + F_2 + F_3 = F_0 + \gamma\partial_z\tilde{\theta} + F_2 + F_3$$

where

$$\|F_2\|_{L^1(J;H^\sigma)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s+1})},$$

$$\begin{aligned}&\|w_{x_0,\mu}S_\mu F_2\|_{L^1(J;H^{\sigma'})}\\&\quad\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}+\|\eta\|_{W^{r+\frac{1}{2},\infty}}+\mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})\|\nabla\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s+1+})}\\&\qquad+(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}})(\mu^{\sigma'}\|\nabla w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^2(J;L^2)}+\|\nabla\tilde{\theta}\|_{L^2(J;H^\sigma)})).\end{aligned}$$

Lastly, we estimate the first order term $F_1=\gamma\partial_z\tilde{\theta}$. We decompose into paraproducts,

$$\gamma\partial_z\tilde{\theta}=T_\gamma\partial_z\tilde{\theta}+T_{\partial_z\tilde{\theta}}\gamma+R(\gamma,\partial_z\tilde{\theta}).$$

We have by (E.8), (E.7), and Corollary A.2,

$$\begin{aligned}\|T_\gamma\partial_z\tilde{\theta}\|_{L^1(J;H^\sigma)}+\|R(\gamma,\partial_z\tilde{\theta})\|_{L^1(J;H^\sigma)}&\lesssim\|\gamma\|_{L^2(J;C_*^{r-1})}\|\partial_z\tilde{\theta}\|_{L^2(J;H^\sigma)}\\&\leq\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\partial_z\tilde{\theta}\|_{L^2(J;H^\sigma)}.\end{aligned}$$

Similarly,

$$\|T_{\partial_z\tilde{\theta}}\gamma\|_{L^2(J;H^{\sigma-\frac{1}{2}})}\lesssim\|\gamma\|_{L^2(J;H^{s-1})}\|\partial_z\tilde{\theta}\|_{L^\infty(J;C_*^{\sigma-s+\frac{1}{2}+})}\leq\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\partial_z\tilde{\theta}\|_{L^\infty(J;C_*^{\sigma-s+\frac{1}{2}+})}.$$

For the local estimates, by Proposition B.1,

$$\begin{aligned}\|w_{x_0,\mu}S_\mu T_\gamma\partial_z\tilde{\theta}\|_{H^{\sigma'}}&\lesssim\|\gamma\|_{C_*^{r-1}}(\|w_{x_0,\mu}S_\mu\partial_z\tilde{\theta}\|_{H^{\sigma'}}+\|\partial_z\tilde{\theta}\|_{H^\sigma}),\\\|w_{x_0,\mu}S_\mu T_{\partial_z\tilde{\theta}}\gamma\|_{H^{\sigma'-\frac{1}{2}}}&\lesssim\|\partial_z\tilde{\theta}\|_{C_*^{\sigma-s+\frac{1}{2}+}}(\|w_{x_0,\mu}S_\mu\gamma\|_{H^{s'-1}}+\|\gamma\|_{H^{s-1}}).\end{aligned}$$

After integrating in z and using Cauchy-Schwarz for the first estimate, and simply integrating in L^2_z for the second, we use Corollary B.7 and Corollary A.2 to estimate the γ terms.

Lastly, by Proposition 11.9,

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu R(\gamma,\partial_z\tilde{\theta})\|_{H^{\sigma'}}&\lesssim\mu^{\sigma'}\|w_{x_0,\lambda}S_\mu R(\gamma,\partial_z\tilde{\theta})\|_{L^2}\\&\lesssim\mu^{\sigma'}(\|\gamma\|_{LS^{s-\sigma-1}_{x_0,\xi_0,\lambda,\mu}}\|\partial_z\tilde{\theta}\|_{C_*^{\sigma-s+1+}}+\|S_\lambda\gamma\|_{C_*^{r-1}}\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\partial_z\tilde{\theta}\|_{L^2}).\end{aligned}$$

We use Corollary B.8 and Corollary A.2 to estimate the γ terms, concluding

$$\begin{aligned}&\|w_{x_0,\lambda}S_\mu R(\gamma,\partial_z\tilde{\theta})\|_{L^1(J;H^{\sigma'})}\\&\quad\leq\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}+\|\eta\|_{W^{r+\frac{1}{2},\infty}}+\mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})\|\partial_z\tilde{\theta}\|_{L^2(J;C_*^{\sigma-s+1+})}\\&\qquad+(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}})\mu^{\sigma'}\|\partial_z w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^2(J;L^2)}).\end{aligned}$$

□

B.4. **Paralinearization of ∂_z.** Now that we have estimates on the inhomogeneous error for the factored equation (A.12), we can apply parabolic estimates, which we now recall:

Proposition B.10. [ABZ14a, Proposition 2.18] *Let $r\in\mathbb{R}$, $\rho\in(0,1)$, $J=[z_0,z_1]\subseteq\mathbb{R}$, and let $p\in\Gamma^1_\rho(J\times\mathbb{R}^d)$ satisfy*

$$Re\,p(z,x,\xi)\geq c|\xi|$$

for some positive constant c. Then for any $f\in Y^r(J)$ and $v_0\in H^r(\mathbb{R}^d)$, there exists $v\in X^r(J)$ solving the parabolic evolution equation

$$(\partial_z+T_p)v=f,\qquad v|_{z=z_0}=v_0,$$

satisfying

$$\|v\|_{X^r(J)}\lesssim\|v_0\|_{H^r}+\|f\|_{Y^r(J)}$$

with a constant depending only on r,ρ,c, and $\mathcal{M}^1_\rho(p)$.

First, we make $\partial_z \approx T_A$ precise in a (global) Sobolev norm, on which we will induct for the local counterpart:

Lemma B.11. *Let $0 \leq \sigma \leq s - \frac{1}{2}$ and $-1 < z_1 < z_0 < 0$. Denote $J_0 = [z_0, 0], J_1 = [z_1, 0]$. Consider θ solving (A.14). Then*

$$\|(\partial_z - T_A)\tilde{\theta}\|_{X^\sigma(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^{\sigma+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J_1)}).$$

Proof. We insert a smooth vertical cutoff $\chi(z)$ vanishing on $[-1, z_1]$ with $\chi = 1$ on $J_0 = [z_0, 0] \subseteq (-1, 0]$, so that we have

$$(\partial_z - T_a)\chi(\partial_z - T_A)\tilde{\theta} = \chi(F_1 + F_2 + F_3) + \chi'(\partial_z - T_A)\tilde{\theta} =: F'.$$

We estimate $\chi'(\partial_z - T_A)\tilde{\theta}$ directly, obtaining

$$\|\chi'(\partial_z - T_A)\tilde{\theta}\|_{L^1(J_1;H^\sigma)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^\sigma)}.$$

Combining this estimate with the Sobolev estimates of Proposition B.9, we have

$$\|F'\|_{Y^\sigma(J_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^\sigma)} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J_1)}).$$

By Proposition A.12 with $\sigma - \frac{1}{2}$ in the place of σ,

$$\|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^\sigma)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\sigma+\frac{1}{2}}}.$$

Lastly, applying the Sobolev parabolic estimate of Proposition B.10, we obtain the result. □

We are now ready to establish a local Sobolev counterpart to Lemma B.11:

Lemma B.12. *Let $0 \leq \sigma \leq s - \frac{1}{2}$ and $z_0 \in (-1, 0]$, $J = [z_0, 0]$. Consider θ solving (A.14). Then*

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu(\partial_z - T_A)\tilde{\theta}\|_{X^{\sigma'}(J)} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^{\sigma+\frac{1}{2}}}, \|f\|_{H^{1+}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*}) \\ &\cdot (\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}} + \mu^{\sigma'}\|f\|_{LS^{\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}).\end{aligned}$$

Proof. By Proposition B.9, we have

$$w_{x_0,\lambda}S_\mu(\partial_z - T_a)(\partial_z - T_A)\tilde{\theta} = w_{x_0,\lambda}S_\mu(F_1 + F_2 + F_3).$$

We will commute

$$[w_{x_0,\lambda}S_\mu, T_a](\partial_z - T_A)\tilde{\theta}.$$

As in the proof of Lemma B.11, we also insert a smooth vertical cutoff $\chi(z)$ vanishing on $[-1, z_1]$ with $\chi = 1$ on $J = [z_0, 0] \subseteq (-1, 0]$. Writing $W = (\partial_z - T_A)\tilde{\theta}$, we then have

$$\begin{aligned}(\partial_z - T_a)\chi w_{x_0,\lambda}S_\mu W = \chi w_{x_0,\lambda}S_\mu(F_1 + F_2 + F_3) + \chi[w_{x_0,\lambda}S_\mu, T_a]W \\ + \chi' w_{x_0,\lambda}S_\mu W =: F'.\end{aligned} \tag{B.3}$$

Our goal is to estimate $F' \in Y^{\sigma'}(J_1)$ to apply the parabolic estimate. First, we consider the commutator on the right hand side of (B.3). By Proposition B.3,

$$\|[w_{x_0,\lambda}S_\mu, T_a]W\|_{H^{\sigma'-\frac{1}{2}}} \lesssim M_0^1(a)\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|W\|_{H^{\sigma'-\frac{1}{2}+1-\frac{1}{8}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|W\|_{H^{\sigma+\frac{1}{2}}}.$$

Applying Lemma B.11 on W, we conclude

$$\|[w_{x_0,\lambda}S_\mu, T_a]W\|_{L^2(J_1;H^{\sigma'-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^{\sigma+\frac{1}{2}}})\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}} \cdot (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J_1)}).$$

Likewise, for the term $\chi' w_{x_0,\lambda}S_\mu W$ on the right hand side of (B.3), we again apply Lemma B.11 (without using any of the localization), obtaining a bound with the same right hand side.

Collecting these estimates with the local Sobolev estimates of Proposition B.9, we have

$$\begin{aligned}\|F'\|_{Y^{\sigma'}(J_1)} \leq{}& \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^{\sigma+\frac{1}{2}}})(\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J_1)})\\ &+ (\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{\sigma-s+\frac{1}{2}+}(J_1)}\\ &+ (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(\mu^{\sigma'}\|\nabla_{x,z}w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^2(J_1;L^2)} + \|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{L^2(J_1;H^{\sigma'})}\\ &+ \|\nabla_{x,z}\tilde{\theta}\|_{L^2(J_1;H^\sigma)})).\end{aligned}$$

On the left hand side, applying the parabolic estimate of Proposition B.10 to (B.3) on J_1, we may exchange $\|F'\|$ with

$$\|\chi w_{x_0,\lambda}S_\mu(\partial_z - T_A)\tilde{\theta}\|_{X^{\sigma'}(J)}$$

as desired. Here we may drop the χ, as on on J, we have $\chi \equiv 1$.

We may bound the right hand side using Propositions A.12 and A.13 to estimate the Sobolev norms, and Proposition A.14 for the Hölder norms,

$$\begin{aligned}&\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^{\sigma+\frac{1}{2}}}, \|f\|_{H^{1+}})(\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*})\\ &+ (\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|\eta\|_{LS^{s-\sigma+\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*})\\ &+ (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{\sigma'}\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}f\|_{H^{\frac{1}{2}}} + \|w_{x_0,\mu}S_\mu f\|_{H^{\sigma'+\frac{1}{2}}})).\end{aligned}$$

Further rearranging, and recalling the definition of $LS^{\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}$, yields the desired estimate. □

B.5. **Local Sobolev paralinearization.** In this subsection we perform a paralinearization of the Dirichlet to Neumann map, measuring the error locally in $H^{s'-\frac{1}{2}}$. For comparison, the error was measured (globally) in $H^{s-\frac{1}{2}}$ in [ABZ14a], [ABZ14b].

For convenience, we record Lemma B.12 with $\sigma = s - \frac{1}{2}$, and Proposition A.14 with $\sigma = 0+$:

Corollary B.13. *Let $z_0 \in (-1, 0]$, $J = [z_0, 0]$. Consider θ solving (A.14). Then*

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu(\partial_z - T_A)\tilde{\theta}\|_{X^{s'-\frac{1}{2}}(J)} \leq{}& \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*})\\ &\cdot (\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}} + \mu^{s'-\frac{1}{2}}\|f\|_{LS^{\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}),\end{aligned}$$

$$\|\nabla_{x,z}\tilde{\theta}\|_{L^\infty(J;C^{0+}_*)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*}).$$

Recall that we write Λ for the principal symbol of the Dirichlet to Neumann map.

Proposition B.14. *Write*

$$\Lambda(t,x,\xi) = \sqrt{(1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2}.$$

Then

$$\|w_{x_0,\lambda}S_\mu(G(\eta)-T_\Lambda)f\|_{H^{s'-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*})$$
$$\cdot(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}} + \mu^{s'-\frac{1}{2}}\|f\|_{LS^{\frac{1}{2}}_{x_0,\xi_0,\lambda,\mu}}).$$

Proof. Recall

$$G(\eta)f = \left(\frac{1+|\nabla\rho|^2}{\partial_z\rho}\partial_z\tilde{\theta} - \nabla\rho\cdot\nabla\tilde{\theta}\right)\Big|_{z=0} =: \left(\zeta\partial_z\tilde{\theta} - \nabla\rho\cdot\nabla\tilde{\theta}\right)\Big|_{z=0}.$$

First we reduce to paraproducts. Write

$$\zeta\partial_z\tilde{\theta} - \nabla\rho\cdot\nabla\tilde{\theta} = T_\zeta\partial_z\tilde{\theta} - T_{\nabla\rho}\nabla\tilde{\theta} + T_{\partial_z\tilde{\theta}}\zeta - T_{\nabla\tilde{\theta}}\cdot\nabla\rho + R(\zeta,\partial_z\tilde{\theta}) - R(\nabla\rho,\nabla\tilde{\theta}).$$

First we estimate $T_{\partial_z\tilde{\theta}}\zeta$ (note $T_{\nabla\tilde{\theta}}\cdot\nabla\rho$ is similar, with the same estimates). By Proposition B.1,

$$\|w_{x_0,\lambda}S_\mu T_{\partial_z\tilde{\theta}}\zeta\|_{H^{s'-\frac{1}{2}}} \lesssim \|w_{x_0,\mu}S_\mu T_{\partial_z\tilde{\theta}}\zeta\|_{H^{s'-\frac{1}{2}}} \lesssim \|\partial_z\tilde{\theta}\|_{L^\infty}(\|w_{x_0,\mu}S_\mu\zeta\|_{H^{s'-\frac{1}{2}}} + \|\zeta\|_{H^{s-\frac{1}{2}}}).$$

Use Corollary B.13 to estimate the $\partial_z\tilde{\theta}$ term. Also note that ζ satisfies the same local estimate as α in Corollary B.7 by using the same argument,

$$\|w_{x_0,\mu}S_\mu\zeta\|_{X^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|w_{x_0,\mu}S_\mu\eta\|_{H^{s'+\frac{1}{2}}}).$$

Likewise, ζ satisfies the same estimate as α in Corollary A.2, and in particular $\zeta\in X^{s-\frac{1}{2}}(J)$. Using the L^∞_z component of the $X^{s-\frac{1}{2}}(J)$ norms, the left hand sides bound the corresponding terms evaluated at $z=0$.

Second, we consider the $R(\cdot,\cdot)$ terms. We again consider only the typical $R(\zeta,\partial_z\tilde{\theta})$. By Proposition 11.9,

$$\|w_{x_0,\lambda}S_\mu R(\zeta,\partial_z\tilde{\theta})\|_{H^{s'-\frac{1}{2}}} \lesssim \mu^{s'-\frac{1}{2}}\|w_{x_0,\lambda}S_\mu R(\zeta,\partial_z\tilde{\theta})\|_{L^2}$$
$$\lesssim \mu^{s'-\frac{1}{2}}(\|\zeta\|_{LS^0_{x_0,\xi_0,\lambda,\mu}}\|\partial_z\tilde{\theta}\|_{C^{0+}_*} + \|S_\lambda\zeta\|_{C^{\frac{1}{2}}_*}\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\partial_z\tilde{\theta}\|_{H^{-\frac{1}{2}}}).$$

Again noting that ζ satisfies the same estimate as α in Corollaries B.8 and A.2 by using the same arguments, and using Corollary B.13 for $\partial_z\tilde{\theta}$, we conclude

$$\|w_{x_0,\lambda}S_\mu R(\zeta,\partial_z\tilde{\theta})\|_{L^\infty(J;H^{s'-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})$$
$$\cdot((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C^r_*})$$
$$+ (1+\|\eta\|_{W^{r+\frac{1}{2},\infty}})\mu^{s'-\frac{1}{2}}(\|\partial_z w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^\infty(J;H^{-\frac{1}{2}})})).$$

Lastly, estimate using Proposition A.13,

$$\|\partial_z w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^\infty(J;H^{-\frac{1}{2}})} \lesssim \lambda^{\frac{1}{2}-s'}\|\partial_z w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}\tilde{\theta}\|_{L^\infty(J;H^{s'-1})}$$
$$\leq \lambda^{\frac{1}{2}-s'}\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}f\|_{H^{s'}} + 1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})$$
$$\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(\|w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}f\|_{H^{\frac{1}{2}}} + \lambda^{\frac{1}{2}-s'}(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}))$$

which estimates $R(\zeta,\partial_z\tilde{\theta})$ by the desired right hand side.

Third, we may replace the vertical derivative $\partial_z\tilde{\theta}$ with $T_A\tilde{\theta}$ as a consequence of Corollary B.13. Note that to handle the multiplication of the paralinearization error with ζ, we use a straightforward generalization of Proposition B.1:

$$\|w_{x_0,\lambda}S_\mu T_\zeta(\partial_z - T_A)\tilde{\theta}\|_{H^{s'-\frac{1}{2}}} \lesssim \|\nabla\eta\|_{L^\infty}(\|w_{x_0,\lambda}S_\mu(\partial_z - T_A)\tilde{\theta}\|_{H^{s'-\frac{1}{2}}} + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}\|(\partial_z - T_A)\tilde{\theta}\|_{H^{s-\frac{1}{2}}}).$$

Here, the last term on the right may be estimated using Lemma B.11, as in the proof of Lemma B.12. Thus

$$G(\eta)f = (T_\zeta T_A\tilde{\theta} - T_{\nabla\rho}\cdot\nabla\tilde{\theta})|_{z=0} + R,$$

with the error R satisfying the desired estimate.

Lastly, by Proposition B.3, Propositions A.1 and A.4, and Proposition A.12 and Proposition A.13,

$$\begin{aligned}
\|w_{x_0,\mu}S_\mu(T_\zeta T_A - T_{\zeta A})\tilde{\theta}\|_{H^{s'-\frac{1}{2}}} &\lesssim (M_0^0(\zeta)M_{\frac{1}{2}}^1(A) + M_{\frac{1}{2}}^0(\zeta)M_0^1(A))(\|w_{x_0,\mu}S_\mu\tilde{\theta}\|_{H^{s'}} + \|\tilde{S}_\mu\tilde{\theta}\|_{H^s}) \\
&\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}) \\
&\quad \cdot (\|\nabla w_{x_0,\mu}S_\mu\tilde{\theta}\|_{H^{s'-1}} + \|\nabla\tilde{\theta}\|_{H^{s-1}}) \\
&\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}) \\
&\quad \cdot (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|w_{x_0,\mu}S_\mu f\|_{H^{s'}}).
\end{aligned}$$

We thus may exchange $T_\zeta T_A\tilde{\theta}$ for $T_{\zeta A}\tilde{\theta}$ in the expression for $G(\eta)f$, with an error R' satisfying the same estimates as R. We conclude, using that $\tilde{\theta}(0) = f$,

$$G(\eta)f = T_{\zeta A - i\nabla\rho\cdot\xi}f + R + R'.$$

A routine computation shows that $(\zeta A - i\nabla\rho\cdot\xi)|_{z=0} = \Lambda$ as desired. □

For comparison and later use, we also recall the following (global) Sobolev estimates:

Proposition B.15. [ABZ14b, Theorem 1.4] *We have*

$$\|G(\eta)f - T_\Lambda f\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}). \tag{B.4}$$

Proposition B.16. [ABZ14a, Proposition 3.13] *We have*

$$\|G(\eta)f - T_\Lambda f\|_{H^{s-1}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}). \tag{B.5}$$

Remark B.17. The Sobolev estimate here was stated for $s > \frac{d}{2} + \frac{3}{4}$, but using sharper elliptic estimates, this can be reduced to $s > \frac{d}{2} + \frac{1}{2}$.

Appendix C. Local Estimates on the Taylor Coefficient

In this section, we establish local Sobolev estimates on the Taylor coefficient

$$a = -(\partial_y P)|_{y=\eta(t,x)}.$$

Recall that the pressure is given by

$$-P = \partial_t\phi + \frac{1}{2}|\nabla_{x,y}\phi|^2 + gy.$$

This immediately implies an identity involving a derivative along the velocity field $v = \nabla_{x,y}\phi$,

$$(\partial_t + v\cdot\nabla_{x,y})\partial_y\phi = -\partial_y P - g, \tag{C.1}$$

as well as an elliptic equation,

$$\Delta_{x,y} P = -\nabla^2_{x,y}\phi \cdot \nabla^2_{x,y}\phi =: F, \qquad P|_{y=\eta(x)} = 0. \tag{C.2}$$

C.1. **Elliptic estimates on the pressure.** First we recall standard elliptic estimates on the pressure P:

Proposition C.1. *Let* $z_0 \in (-1, 0]$, $J = [z_0, 0]$. *Then*

$$\|\nabla_{x,z}\tilde{P}\|_{X^{s-\frac{1}{2}}(J)} \leq \mathcal{F}(M(t)).$$

Proof. Apply Proposition A.9 to (C.2) with $\sigma = s - \frac{1}{2}$:

$$\|\nabla_{x,z}\tilde{P}\|_{X^{s-\frac{1}{2}}(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|\alpha\tilde{F}\|_{Y^{s-\frac{1}{2}}(J_1)} + \|\nabla_{x,z}\tilde{P}\|_{X^{-\frac{1}{2}}(J_1)}).$$

Then, using [ABZ14a, (4.22)], we have

$$\|\nabla_{x,z}\tilde{P}\|_{X^{-\frac{1}{2}}(J_1)} \leq \mathcal{F}(M(t)).$$

Further, by [ABZ14a, Lemma 4.7] (which one easily checks only requires $s > \frac{d}{2} + \frac{1}{2}$),

$$\|\alpha F\|_{Y^{s-\frac{1}{2}}(J_1)} \leq \mathcal{F}(M(t)).$$

□

C.2. **Local elliptic estimates on the pressure.** Here we establish local Sobolev estimates on the pressure P. First, we apply Proposition B.9 to P:

Proposition C.2. *Let* $z_0 \in [-1, 0]$, $J = [z_0, 0]$. *Then we can write*

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{P} = F_0 + F_1 + F_2 + F_3$$

where for $i \geq 1$,

$$\|F_i\|_{Y^{s-\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}),$$
$$\|w_{x_0,\lambda}S_\mu F_i\|_{Y^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

Proof. Apply Proposition B.9 except with $F_0 \neq 0$ and $\sigma = s - \frac{1}{2}$:

$$\|F_i\|_{Y^{s-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{P}\|_{L^2(J;H^{s-\frac{1}{2}})} + \|\nabla_{x,z}\tilde{P}\|_{U^{0+}(J)}),$$

$$\begin{aligned}\|w_{x_0,\lambda}S_\mu F_i\|_{Y^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})((\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}})\|\nabla_{x,z}\tilde{P}\|_{U^{0+}(J)}\\ + (1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(\mu^{s'-\frac{1}{2}}\|\nabla_{x,z}w_{x_0,\lambda}S_{\xi_0,\lambda,\mu}P\|_{L^2(J;L^2)} + \|\nabla w_{x_0,\mu}S_\mu\tilde{P}\|_{L^2(J;H^{s'-\frac{1}{2}})}\\ + \|\nabla_{x,z}\tilde{P}\|_{L^2(J;H^{s-\frac{1}{2}})})).\end{aligned}$$

Then applying Sobolev embedding along with Proposition C.1 to the right hand sides (without using the localization), one obtains the estimate. □

Next, we obtain the counterparts of Lemmas B.11 and B.12:

Lemma C.3. *Let* $z_0 \in (-1, 0]$, $J = [z_0, 0]$. *Then*

$$\|(\partial_z - T_A)\tilde{P}\|_{X^{s-\frac{1}{2}}(J_0)} \leq \mathcal{F}(M(t)).$$

Proof. This is immediate from Proposition C.1 by estimating $\partial_z\tilde{P}$ and $T_A\tilde{P}$ separately. □

Lemma C.4. *Let $z_0 \in (-1, 0]$, $J = [z_0, 0]$. Then*

$$\|w_{x_0,\lambda} S_\mu(\partial_z - T_A)\tilde{P}\|_{X^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

Proof. The proof is the same as that of Lemma B.12, except in place of the inhomogeneous bounds of Proposition B.9, we apply Proposition C.2; in place of the global counterpart Lemma B.11, apply Lemma C.3; in place of the elliptic estimates of Propositions A.12 and A.13, apply Proposition C.1. For the Hölder estimates, Sobolev embedding suffices. □

Finally, we perform a second parabolic estimate to obtain the full elliptic estimate:

Proposition C.5. *Let $z_0 \in (-1, 0]$, $J = [z_0, 0]$. Then*

$$\|\nabla_{x,z} w_{x_0,\lambda} S_\mu \tilde{P}\|_{X^{s'-\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

Proof. First we estimate the commutator using Proposition B.3,

$$\|[w_{x_0,\lambda} S_\mu, T_A]\tilde{P}\|_{H^{s'}} \lesssim M^1_0(A)\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|\tilde{S}_\mu \tilde{P}\|_{H^{s'+1-\frac{1}{8}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\lambda^{\frac{3}{8}}\mu^{-\frac{3}{8}}\|\nabla \tilde{P}\|_{H^s}.$$

Then integrating L^2_z and applying Proposition C.1, we see the commutator satisfies the same estimate as $w_{x_0,\lambda} S_\mu(\partial_z - T_A)\tilde{P} \in Y^{s'+\frac{1}{2}}(J)$ obtained by Lemma C.4.

We conclude

$$(\partial_z - T_A) w_{x_0,\lambda} S_\mu \tilde{P} = F' \tag{C.3}$$

where

$$\|F'\|_{Y^{s'+\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

Then applying the parabolic estimate of Proposition B.10,

$$\|w_{x_0,\lambda} S_\mu \tilde{P}\|_{X^{s'+\frac{1}{2}}(J)} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

We thus obtain the desired estimate on $\nabla w_{x_0,\lambda} S_\mu \tilde{P}$. Then using the equation (C.3) and applying our estimate on F', we also obtain the estimate on $\partial_z w_{x_0,\lambda} S_\mu \tilde{P}$. □

C.3. **Local smoothing estimates on the Taylor coefficient.** To conclude this section, we state a local Sobolev estimate on the Taylor coefficient. This is almost immediate from the previous subsection, but we need to restore the change of coordinates:

Corollary C.6. *We have*

$$\|w_{x_0,\lambda} S_\mu a\|_{H^{s'-\frac{1}{2}}} \leq \mathcal{F}(M(t))(\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}} + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \mu^{s'-\frac{1}{2}}\|\eta\|_{LS^1_{x_0,\xi_0,\lambda,\mu}}).$$

Proof. Observe that

$$\widetilde{\partial_y \theta} = \frac{1}{\partial_z \rho}\partial_z \tilde{\theta}.$$

Thus, it suffices to apply the product estimate, Proposition B.2, combined with Proposition C.5 and the reciprocal estimate Proposition B.6 (it suffices to use Sobolev embedding on the L^∞ estimates). □

Appendix D. Hölder Estimates

We collect Hölder estimates on unknowns appearing in the water waves system.

D.1. **Paralinearization of the Dirichlet to Neumann map.** In this section we establish the Hölder analogue to Appendix B. We similarly begin with the paralinearization of ∂_z. We have from (A.13) in the proof of Proposition A.8 with $\sigma = 1/2$,

Lemma D.1. *Let $-1 < z_1 < z_0 < 0$. Denote $J_0 = [z_0, 0], J_1 = [z_1, 0]$. Consider θ solving (A.14). Then*

$$\|(\partial_z - T_A)\tilde{\theta}\|_{C^0(J_0;C_*^{\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^0(J_1)}.$$

We can remove the instance of $\tilde{\theta}$ on the right hand side of the lemma, working our way down to the "base case":

Corollary D.2. *Let $z_0 \in (-1, 0], J_0 = [z_0, 0]$. Consider θ solving (A.14). Then*

$$\|(\partial_z - T_A)\tilde{\theta}\|_{C^0(J_0;C_*^{\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C_*^r})$$
$$\|\nabla_{x,z}\tilde{\theta}\|_{U^0(J_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C_*^r}).$$

Proof. To estimate the right hand side of Lemma D.1, recall Proposition A.8 with $\sigma = \frac{1}{2}$, δ chosen small enough so that $\delta < r - 1$, and $F = 0$:

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^0(J_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^{r-\frac{3}{2}}(J_2)} + \|f\|_{C_*^r}.$$

In turn, to estimate the right hand side, we apply Sobolev embedding and Proposition A.9 with $\sigma = s - 1$ and $F = 0$:

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^{r-\frac{3}{2}}(J_2)} \lesssim \|\nabla_{x,z}\tilde{\theta}\|_{X^{s-1}(J_3)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|f\|_{H^s} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_3)}).$$

Lastly, recalling Proposition A.11 yields the desired estimate.

□

We now perform the paralinearization:

Proposition D.3. *Write*

$$\Lambda(t, x, \xi) = \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2}.$$

Then

$$\|G(\eta)f - T_\Lambda f\|_{W^{\frac{1}{2},\infty}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}).$$

Proof. Recall

$$G(\eta)f = \left(\frac{1 + |\nabla\rho|^2}{\partial_z\rho}\partial_z\tilde{\theta} - \nabla\rho \cdot \nabla\tilde{\theta}\right)\bigg|_{z=0} =: \left(\zeta\partial_z\tilde{\theta} - \nabla\rho \cdot \nabla\tilde{\theta}\right)\bigg|_{z=0}.$$

First we reduce to paraproducts. Write

$$\zeta\partial_z\tilde{\theta} - \nabla\rho \cdot \nabla\tilde{\theta} = T_\zeta\partial_z\tilde{\theta} - T_{\nabla\rho}\nabla\tilde{\theta} + T_{\partial_z\tilde{\theta}}\zeta - T_{\nabla\tilde{\theta}} \cdot \nabla\rho + R(\zeta, \partial_z\tilde{\theta}) - R(\nabla\rho, \nabla\tilde{\theta}).$$

By (E.10),

$$\|T_{\partial_z\tilde{\theta}}\zeta\|_{C_*^{r-\frac{1}{2}}} \lesssim \|\partial_z\tilde{\theta}\|_{L^\infty}\|\zeta\|_{C_*^{r-\frac{1}{2}}}$$
$$\|T_{\nabla\tilde{\theta}} \cdot \nabla\rho\|_{C_*^{r-\frac{1}{2}}} \lesssim \|\nabla\tilde{\theta}\|_{L^\infty}\|\nabla\rho\|_{C_*^{r-\frac{1}{2}}}.$$

Taking L_z^∞ and using Proposition A.1 and Corollary D.2, the right hand sides are bounded by the right hand side of the desired estimate. Similarly, by (E.6)

$$\|R(\zeta, \partial_z \tilde{\theta})\|_{C_*^{r-\frac{1}{2}}} \lesssim \|\partial_z \tilde{\theta}\|_{L^\infty} \|\zeta\|_{C_*^{r-\frac{1}{2}}}$$
$$\|R(\nabla\rho, \nabla\tilde{\theta})\|_{C_*^{r-\frac{1}{2}}} \lesssim \|\nabla\tilde{\theta}\|_{L^\infty} \|\nabla\rho\|_{C_*^{r-\frac{1}{2}}}.$$

Second, we may replace the vertical derivative $\partial_z \tilde{\theta}$ with $T_A \tilde{\theta}$ as a consequence of Corollary D.2 along $z = 0$:

$$\|((\partial_z - T_A)\tilde{\theta})|_{z=0}\|_{C_*^{\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C_*^r}).$$

Thus (using $\zeta \in L^\infty$)

$$G(\eta)f = (T_\zeta T_A \tilde{\theta} - T_{\nabla\rho} \cdot \nabla\tilde{\theta})|_{z=0} + R,$$

with the error R satisfying

$$\|R\|_{C_*^{\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}).$$

Lastly, by the symbolic calculus (E.3), Proposition A.1, and Proposition A.4,

$$\begin{aligned}\|(T_\zeta T_A \tilde{\theta} - T_{\zeta A})\tilde{\theta}\|_{C_*^{r-\frac{1}{2}}} &\lesssim (M^0_{r-\frac{1}{2}}(\zeta) M^1_0(A) + M^0_0(\zeta) M^1_{r-\frac{1}{2}}(A))\|\tilde{\theta}\|_{C_*^1} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{W^{r+\frac{1}{2},\infty}} \|\tilde{\theta}\|_{C_*^1}.\end{aligned}$$

We may exchange $\tilde{\theta}$ for $S_{>1/10}\tilde{\theta}$ in the previous inequalities by the inhomogeneous paradifferential calculus, and hence use Corollary D.2 to conclude

$$\|(T_\zeta T_A \tilde{\theta} - T_{\zeta A})\tilde{\theta}\|_{C_*^{r-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s})\|\eta\|_{W^{r+\frac{1}{2},\infty}}(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{C_*^r}).$$

We thus may exchange $T_\zeta T_A \tilde{\theta}$ for $T_{\zeta A}\tilde{\theta}$ in the expression for $G(\eta)f$, with an error R' satisfying the same estimates as R. We conclude, using that $\tilde{\theta}(0) = f$,

$$G(\eta)f = T_{\zeta A - i\nabla\rho\cdot\xi} f + R + R'.$$

A routine computation shows that $(\zeta A - i\nabla\rho \cdot \xi)|_{z=0} = \Lambda$ as desired. □

D.2. **General bottom estimates.** In this section we recall errors that arise due to the presence of a general bottom to our fluid domain. We recall the following identities from Propositions 4.3 and 4.5 in [ABZ14a]:

$$G(\eta)B = -\nabla \cdot V - \Gamma_y$$
$$L\nabla\eta = G(\eta)V + \nabla\eta G(\eta)B + \Gamma_x + \nabla\eta\Gamma_y$$

where

$$\|\Gamma_x\|_{H^{s-\frac{1}{2}}} + \|\Gamma_y\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}}).$$

Here, Γ_{x_i} is defined as follows: let θ_i be the solution to

$$\Delta_{x,y}\theta_i = 0, \qquad \theta_i|_{y=\eta(x)} = V_i, \qquad \partial_n \theta_i|_\Gamma = 0.$$

Then

$$\Gamma_{x_i} = ((\partial_y - \nabla\eta \cdot \nabla)(\partial_i \phi - \theta_i))|_{y=\eta(x)}. \tag{D.1}$$

Note that by the definition of V, $\partial_i\phi = \theta_i$ if $\Gamma = \emptyset$, so Γ_{x_i} is only nonzero in the presence of a bottom. Γ_y is defined in the analogous way with B in place of V.

We recall the following Sobolev estimate:

Proposition D.4. [ABZ14a, Propositions 4.3, 4.5] *We have*

$$\|\Gamma_y\|_{H^{s-\frac{1}{2}}} + \|\Gamma_x\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}}).$$

We also establish the corresponding estimates in Hölder norm. For this we apply the inhomogeneous elliptic Hölder estimates of Appendix A.

Lemma D.5. *Let $-1 < z_1 < z_0 < 0$. Denote $J_0 = [z_0, 0], J_1 = [z_1, 0]$. Consider $\tilde{\theta}$ solving (A.2) with $f = 0$. Then*

$$\begin{aligned}\|\nabla_{x,z}\tilde{\theta}\|_{C^0(J_0;C_*^{\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})(\|F_0\|_{Y^{s-\frac{1}{2}}(J_1)} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_1)}) \\ + \|F_0\|_{L^1(J_1;C_*^{\frac{1}{2}})}.\end{aligned}$$

Proof. Applying Proposition A.8 with $\sigma = 1$ and $f = 0$, we have

$$\|\nabla_{x,z}\tilde{\theta}\|_{C^0(J_0;C_*^{\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}})\|\nabla_{x,z}\tilde{\theta}\|_{U^0(J_0')} + \|F_0\|_{L^1(J_0';C_*^{\frac{1}{2}})}.$$

To estimate the right hand side, we apply Sobolev embedding and Proposition A.9 with $\sigma = s - \frac{1}{2}$ and $f = 0$:

$$\|\nabla_{x,z}\tilde{\theta}\|_{U^{r-1}(J_0')} \lesssim \|\nabla_{x,z}\tilde{\theta}\|_{X^{s-\frac{1}{2}}(J_0')} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})(\|F_0\|_{Y^{s-\frac{1}{2}}(J_1)} + \|\nabla_{x,z}\tilde{\theta}\|_{X^{-\frac{1}{2}}(J_1)}).$$

□

The rest of the proof closely follows the proof of Proposition 4.3 in [ABZ14a] with the appropriate modifications to replace Sobolev with Hölder norms, but we provide the details here for completeness. We will need the following estimate:

Lemma D.6 ([ABZ14a, Lemma 3.11])**.** *Let $-\frac{1}{2} \leq a < b \leq -\frac{1}{5}$. Then the strip*

$$S_{a,b} = \{(x, y) \in \mathbb{R}^{d+1} : ah < y - \eta(x) < bh\}$$

is contained in Ω and for any $k \geq 1$, we have for any ϕ solving

$$\Delta_{x,y}\phi = 0, \qquad \phi|_{y=\eta(x)} = \psi,$$

the estimate

$$\|\phi\|_{H^k(S_{a,b})} \lesssim \|\psi\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}.$$

Proposition D.7. *Consider Γ_x, Γ_y as defined in (D.1). Then*

$$\|\Gamma_x\|_{W^{\frac{1}{2},\infty}} + \|\Gamma_y\|_{W^{\frac{1}{2},\infty}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi, V, B)\|_{H^{\frac{1}{2}}})(1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}}).$$

Proof. We prove the case of Γ_x as the case of Γ_y is similar.

First we localize the problem near the surface Γ, away from the general bottom. Let $\chi_0 \in C^\infty(\mathbb{R}), \eta_1 \in H^\infty(\mathbb{R}^d)$ be such that $\chi_0(z) = 1$ if $z \geq 0$, $\chi_0(z) = 0$ if $z \leq -\frac{1}{4}$, and

$$\eta(x) - \frac{h}{4} \leq \eta_1(x) \leq \eta(x) - \frac{h}{5}.$$

Set

$$U_i(x, y) = \chi_0\left(\frac{y - \eta_1(x)}{h}\right)(\partial_i\phi - \theta_i)(x, y).$$

By construction, $\Gamma_{x_i} = ((\partial_y - \nabla\eta\cdot\nabla)U_i)|_{y=\eta(x)}$. Moreover, U_i satisfies

$$\Delta_{x,y}U_i = [\Delta_{x,y}, \chi_0\left(\frac{y-\eta_1(x)}{h}\right)](\partial_i\phi - \theta_i) =: F_i, \qquad U_i|_{y=\eta(x)} = 0.$$

where

$$\text{supp } F_i \subseteq S_{-\frac{1}{2},-\frac{1}{5}} = \left\{(x,y) : x\in\mathbb{R}^d, \eta(x) - \frac{h}{2} \le y \le \eta(x) - \frac{h}{5}\right\}.$$

By Lemma D.6, we have for arbitrary $\alpha \in \mathbb{N}^{d+1}$,

$$\|\partial^\alpha_{x,y}F_i\|_{L^\infty(S_{-\frac{1}{2},-\frac{1}{5}})\cap L^2(S_{-\frac{1}{2},-\frac{1}{5}})} \lesssim \|(V,B)\|_{H^{\frac{1}{2}}}$$

and in particular F_i is smooth.

Next we flatten the boundary defined by the graph of η. Set

$$\tilde{U}_i(x,z) = U_i(x,\rho(x,z)), \quad \tilde{F}_i = F_i(x,\rho(x,z))$$

where ρ is the Lipschitz diffeomorphism discussed in Appendix A. In particular, the image of the strip $S_{-\frac{1}{2},-\frac{1}{5}}$ is a wider strip, but still away from zero. In turn, this implies $\tilde{F}_i$ is still smooth. We also have

$$(\partial_z^2 + \alpha\Delta + \beta\cdot\nabla\partial_z - \gamma\partial_z)\tilde{U}_i = \frac{(\partial_z\rho)^2}{1+|\nabla\rho|^2}\tilde{F}_i, \qquad \tilde{U}_i|_{z=0} = 0.$$

Thus, apply Lemma D.5 with $\tilde{\theta} = \tilde{U}_i$ and smooth inhomogeneity $\tilde{F}_i$:

$$\|\nabla_{x,z}\tilde{U}_i\|_{C^0(J_0;C_*^{\frac{1}{2}})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V,B)\|_{H^{\frac{1}{2}}})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}})(1+\|\nabla_{x,z}\tilde{U}_i\|_{X^{-\frac{1}{2}}(J_1)}).$$

As noted in the proof of Proposition 4.3 in [ABZ14a],

$$\|\nabla_{x,z}\tilde{U}_i\|_{X^{-\frac{1}{2}}(I)} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}(\|\psi\|_{H^{\frac{1}{2}}} + \|V_i\|_{H^{\frac{1}{2}}})$$

so we conclude

$$\|\nabla_{x,z}\tilde{U}_i\|_{C^0(J_0;C_*^{\frac{1}{2}})} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi,V,B)\|_{H^{\frac{1}{2}}})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}).$$

From the definition of ρ in Section 3 of [ABZ14a],

$$\Gamma_{x_i} = \left(\left(\frac{1+|\nabla\eta|^2}{1+\delta\langle D_x\rangle\eta}\partial_z - \nabla\eta\cdot\nabla\right)\tilde{U}_i\right)\Big|_{z=0},$$

so we conclude by Sobolev embedding on $\nabla\eta$ that

$$\|\Gamma_{x_i}\|_{W^{\frac{1}{2},\infty}} \le \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(\psi,V,B)\|_{H^{\frac{1}{2}}})(1+\|\eta\|_{W^{r+\frac{1}{2},\infty}}).$$

□

D.3. **Estimates on the Taylor coefficient.** We recall Hölder estimates on the Taylor coefficient:

Proposition D.8 ([ABZ14b, Proposition C.1]). *Remain in the setting of Proposition 2.1. Let $0 < \epsilon < \min(r-1, s-\frac{d}{2}-\frac{3}{4})$. Then for all $t \in [0,T]$,*

$$\|a(t)-g\|_{L^\infty} \lesssim \|a(t)-g\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(M(t)),$$

$$\|a(t)\|_{W^{\frac{1}{2}+\epsilon,\infty}} + \|La(t)\|_{W^{\epsilon,\infty}} \le \mathcal{F}(M(t))Z(t).$$

Remark D.9. The Sobolev estimate here was stated for $s > \frac{d}{2}+1$, but the proof is actually valid for $s > \frac{d}{2} + \frac{1}{2}$. The Hölder estimates also hold for $s > \frac{d}{2} + \frac{1}{2}$, but require sharper arguments.

We have the following straightforward consequence:

Corollary D.10. *Remain in the setting of Proposition D.8 and fix multi-index β. Then for all $t \in [0,T]$, uniformly on $\{|\xi| = 1\}$,*

$$\|\partial_\xi^\beta \gamma(t,\cdot,\xi)\|_{L^\infty} \leq \mathcal{F}(M(t)),$$

$$\|\partial_\xi^\beta \gamma(t,\cdot,\xi)\|_{W^{\frac{1}{2}+\epsilon,\infty}} \leq \mathcal{F}(M(t))Z(t).$$

D.4. **Vector field commutator estimate.** We record a version of Lemma 2.17 in [ABZ14a] for Hölder spaces. The proof is similar, exchanging Sobolev norms with Hölder norms.

Proposition D.11. *For any $t \in I$, $r \geq 0$, and $\epsilon > -r$,*

$$\begin{aligned} \| ((\partial_t + V\cdot\nabla)T_p - T_p(\partial_t + T_V\cdot\nabla))u(t)\|_{W^{r,\infty}} & \\ \lesssim M_0^m(p(t))\|V(t)\|_{W^{1,\infty}}\|u(t)\|_{B^{r+m}_{\infty,1}} & \\ + M_0^m(p(t))\|V(t)\|_{W^{r+\epsilon,\infty}}\|u(t)\|_{B^{1+m-\epsilon}_{\infty,1}} & \\ + M_0^m(\partial_t p(t) + V\cdot\nabla p(t))\|u(t)\|_{W^{r+m,\infty}}. & \end{aligned}$$

Appendix E. Paradifferential Calculus

For the reader's convenience, we provide an appendix of notation and estimates from Bony's paradifferential calculus. This is a subset of the appendix in [ABZ14b].

E.1. **Notation.** For $\rho = k+\sigma$, $k \in \mathbb{N}$, $\sigma \in (0,1)$, denote by $W^{\rho,\infty}(\mathbb{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ.

Definition E.1. *Given $\rho \in [0,1]$ and $m \in \mathbb{R}$, let $\Gamma_\rho^m(\mathbb{R}^d)$ denote the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$, which are C^∞ functions of ξ away from the origin and such that, for any $\alpha \in \mathbb{N}^d$ and any $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ is in $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant C_α such that on $\{|\xi| \geq \frac{1}{2}\}$,*

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbb{R}^d)} \leq C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

For $a \in \Gamma_\rho^m$, we define

$$M_\rho^m(a) = \sup_{|\alpha|\leq 1+2d+\rho}\ \sup_{|\xi|\geq 1/2} \|(1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbb{R}^d)}.$$

Given a symbol $a \in \Gamma_\rho^m(\mathbb{R}^d)$, define the (inhomogeneous) paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d}\int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\,d\eta,$$

where $\widehat{a}(\theta,\xi)$ is the Fourier transform of a with respect to the first variable, and χ and ψ are two fixed C^∞ functions satisfying, for small $0 < \epsilon_1 < \epsilon_2$,

$$\begin{cases} \psi(\eta) = 0 & \text{on } \{|\eta| \leq 1\} \\ \psi(\eta) = 1 & \text{on } \{|\eta| \geq 2\}, \end{cases}$$

$$\begin{cases}\chi(\theta,\eta)=1 & \text{on } \{|\theta|\le \epsilon_1|\eta|\}\\ \chi(\theta,\eta)=0 & \text{on } \{|\theta|\ge \epsilon_2|\eta|\},\end{cases}\qquad |\partial_\theta^\alpha\partial_\eta^\beta\chi(\theta,\eta)|\le c_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}.$$

The cutoff function χ can be chosen so that T_a coincides with the usual definition of a paraproduct (in terms of a Littlewood-Paley decomposition), where the symbol a depends only on x.

E.2. **Symbolic calculus.** We shall use results from [Mét08] about operator norm estimates for the pseudodifferential symbolic calculus.

Definition E.2. *Consider a dyadic decomposition of the identity:*

$$I = S_0 + \sum_{\lambda=1} S_\lambda.$$

If $s\in\mathbb{R}$, the Zygmund class $C_*^s(\mathbb{R}^d)$ is the space of tempered distributions u such that

$$\|u\|_{C_*^s} := \sup_\lambda \lambda^s \|S_\lambda u\|_{L^\infty} < \infty.$$

Remark E.3. If $s>0$ is not an integer, then $C_*^s(\mathbb{R}^d) = W^{s,\infty}(\mathbb{R}^d)$.

Definition E.4. *Let $m\in\mathbb{R}$. We say an operator T is of order m if for every $\mu\in\mathbb{R}$, it is bounded from H^μ to $H^{\mu-m}$ and from C_*^μ to $C_*^{\mu-m}$.*

The main features of the symbolic calculus for paradifferential operators are given by the following proposition:

Proposition E.5. *Let $m\in\mathbb{R}$ and $\rho\in[0,1]$.*
i) If $a\in\Gamma_0^m(\mathbb{R}^d)$, then T_a is of order m. Moreover, for all $\mu\in\mathbb{R}$,

$$\|T_a\|_{H^\mu\to H^{\mu-m}} \lesssim M_0^m(a), \quad \|T_a\|_{C_*^\mu\to C_*^{\mu-m}} \lesssim M_0^m(a). \tag{E.1}$$

ii) If $a\in\Gamma_\rho^m(\mathbb{R}^d)$ and $b\in\Gamma_\rho^{m'}(\mathbb{R}^d)$ then $T_aT_b - T_{ab}$ is of order $m+m'-\rho$. Moreover, for all $\mu\in\mathbb{R}$,

$$\|T_aT_b - T_{ab}\|_{H^\mu\to H^{\mu-m-m'+\rho}} \lesssim M_\rho^m(a)M_0^{m'}(b) + M_0^m(a)M_\rho^{m'}(b), \tag{E.2}$$

$$\|T_aT_b - T_{ab}\|_{C_*^\mu\to C_*^{\mu-m-m'+\rho}} \lesssim M_\rho^m(a)M_0^{m'}(b) + M_0^m(a)M_\rho^{m'}(b). \tag{E.3}$$

We also need to consider paradifferential operators with negative regularity. As a consequence, we need to extend our previous definition.

Definition E.6. *For $m\in\mathbb{R}$ and $\rho<0$, $\Gamma_\rho^m(\mathbb{R}^d)$ denotes the space of distributions $a(x,\xi)$ on $\mathbb{R}^d\times(\mathbb{R}^d\backslash 0)$ which are C^∞ with respect to ξ and such that, for all $\alpha\in\mathbb{N}^d$ and all $\xi\neq 0$, the function $x\mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $C_*^\rho(\mathbb{R}^d)$ and there exists a constant C_α such that on $\{|\xi|\ge \frac12\}$,*

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{C_*^\rho} \le c_\alpha(1+|\xi|)^{m-|\alpha|}.$$

For $a\in\Gamma_\rho^m$, we define

$$M_\rho^m(a) = \sup_{|\alpha|\le \frac{3d}{2}+\rho+1}\ \sup_{|\xi|\ge 1/2} \|(1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(\cdot,\xi)\|_{C_*^\rho(\mathbb{R}^d)}.$$

We recall Proposition 2.12 in [ABZ14a] which is a generalization of (E.1).

Proposition E.7. *Let $\rho<0$, $m\in\mathbb{R}$, and $a\in\dot\Gamma_\rho^m$. Then the operator T_a is of order $m-\rho$:*

$$\|T_a\|_{H^s\to H^{s-(m-\rho)}} \lesssim M_\rho^m(a), \quad \|T_a\|_{C_*^s\to C_*^{s-(m-\rho)}} \lesssim M_\rho^m(a). \tag{E.4}$$

E.3. **Paraproducts and product rules.** We recall here some properties of paraproducts, T_a where $a(x,\xi) = a(x)$. A key feature is that one can define paraproducts for rough functions a which do not belong to $L^\infty(\mathbb{R}^d)$ but merely $C_*^{-m}(\mathbb{R}^d)$ with $m > 0$.

Definition E.8. *Given two functions a, b defined on $\mathbb{R}^d$, we define the remainder*

$$R(a,u) = au - T_a u - T_u a.$$

We record here various estimates about paraproducts (see Chapter 2 in [BCD11]).

Proposition E.9. *i) Let $\alpha, \beta \in \mathbb{R}$. If $\alpha + \beta > 0$ then*

$$\|R(a,u)\|_{H^{\alpha+\beta-\frac{d}{2}}} \lesssim \|a\|_{H^\alpha} \|u\|_{H^\beta} \tag{E.5}$$

$$\|R(a,u)\|_{C_*^{\alpha+\beta}} \lesssim \|a\|_{C_*^\alpha} \|u\|_{C_*^\beta} \tag{E.6}$$

$$\|R(a,u)\|_{H^{\alpha+\beta}} \lesssim \|a\|_{C_*^\alpha} \|u\|_{H^\beta}. \tag{E.7}$$

ii) Let $m > 0$ and $s \in \mathbb{R}$. Then

$$\|T_a u\|_{H^{s-m}} \lesssim \|a\|_{C_*^{-m}} \|u\|_{H^s} \tag{E.8}$$

$$\|T_a u\|_{C_*^{s-m}} \lesssim \|a\|_{C_*^{-m}} \|u\|_{C_*^s} \tag{E.9}$$

$$\|T_a u\|_{C_*^s} \lesssim \|a\|_{L^\infty} \|u\|_{C_*^s}. \tag{E.10}$$

We have the following product estimates (for references and proofs, see [ABZ14b]):

Proposition E.10. *i) Let $s \geq 0$. Then*

$$\|u_1 u_2\|_{H^s} \lesssim \|u_1\|_{H^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{H^s} \tag{E.11}$$

$$\|u_1 u_2\|_{C_*^s} \lesssim \|u_1\|_{C_*^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{C_*^s}. \tag{E.12}$$

ii) Let

$$s_1 + s_2 > 0, \quad s_0 \leq s_1, s_2, \quad s_0 < s_1 + s_2 - \frac{d}{2}.$$

Then

$$\|u_1 u_2\|_{H^{s_0}} \lesssim \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}. \tag{E.13}$$

iii) Let $\beta > \alpha > 0$. Then

$$\|u_1 u_2\|_{C_*^{-\alpha}} \lesssim \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}}. \tag{E.14}$$

iv) Let $s_1 > d/2$ and $s_2 \geq 0$, and consider $F \in C^\infty(\mathbb{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(U)\|_{H^{s_1}} \leq \mathcal{F}(\|U\|_{L^\infty}) \|U\|_{H^{s_1}}, \quad \|F(U)\|_{C_*^{s_2}} \leq \mathcal{F}(\|U\|_{L^\infty}) \|U\|_{C_*^{s_2}}. \tag{E.15}$$

We have the following composition estimates:

Proposition E.11. *Let $0 < \alpha \leq 1$ and $f \in C^\alpha(\mathbb{R}^d)$. Let $\nabla g \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d)$. Then*

$$\|f(g(x))\|_{C^\alpha(\mathbb{R}^d)} \leq \|f\|_{C^\alpha} \|\nabla g\|_{L^\infty}^\alpha.$$

Proof. Compute

$$\frac{|f(g(x)) - f(g(y))|}{|x-y|^\alpha} = \frac{|f(g(x)) - f(g(y))|}{|g(x) - g(y)|^\alpha} \frac{|g(x) - g(y)|^\alpha}{|x-y|^\alpha}$$

Then

$$\sup_{x\neq y\in\mathbb{R}^d}\frac{|f(g(x))-f(g(y))|}{|x-y|^\alpha}\le\sup_{x\neq y}\frac{|f(g(x))-f(g(y))|}{|g(x)-g(y)|^\alpha}\left(\sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|}\right)^\alpha.$$

□

Proposition E.12. *Let $0<\alpha\le 1$ and $a(x,\zeta)\in L^\infty(\mathbb{R}^n\times\mathbb{R}^m)$, smooth in ζ, and $a(\cdot,\zeta)\in C^\alpha(\mathbb{R}^n)$ uniformly in ζ. Let $f\in C^\alpha(\mathbb{R}^n\to\mathbb{R}^m)$. Then*

$$\|a(x,f(x))\|_{C^\alpha_x(\mathbb{R}^n)}\lesssim\sup_y\|a(\cdot,y)\|_{C^\alpha(\mathbb{R}^n)}+\|\nabla_y a\|_{L^\infty_{x,y}(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{C^\alpha(\mathbb{R}^n\to\mathbb{R}^m)}.$$

Proof. Compute

$$\frac{a(x,f(x))-a(y,f(y))}{|x-y|^\alpha}=\frac{a(x,f(x))-a(y,f(x))}{|x-y|^\alpha}+\frac{a(y,f(x))-a(y,f(y))}{f(x)-f(y)}\frac{f(x)-f(y)}{|x-y|^\alpha}.$$

Then take the suprema of both sides in $x\neq y\in\mathbb{R}^n$ as before. □

We also need the following elementary composition estimate ([ABZ11b, Lemma 3.2]):

Proposition E.13. *Let $m\in\mathbb{N}$ and $0\le\sigma\le m$. Consider a diffeomorphism $\kappa:\mathbb{R}\to\mathbb{R}$. Then*

$$\|u\circ\kappa\|_{H^\sigma}\le\mathcal{F}(\|\kappa'\|_{W^{p-1,\infty}})\|\partial_x(\kappa^{-1})\|_{L^\infty}\|u\|_{H^\sigma}.$$

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