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On Lattice-Like Subgroups of Witt Group Schemes, and Associated Moduli Spaces

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ON LATTICE-LIKE SUBGROUPS OF WITT GROUP SCHEMES, AND ASSOCIATED MODULI SPACES

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by

Frederic Theodore Nitz

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Abstract

On Lattice-Like Subgroups of Witt Group Schemes, and Associated Moduli Spaces

by

Frederic Theodore Nitz

In this work we define a moduli problem for subgroups of powers of Witt group schemes which we believe to be a good analogue for the lattices used in the construction of the classical affine Grassmannian. We then develop some tools for describing these subgroups. With these tools in hand we are able to show that the moduli problem is representable in general, and we construct the representing scheme in some cases.

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Chapter 1

Algebraic Geometry

In this thesis we will make extensive use of Grothendieck's language of schemes. This chapter should by no means be considered a complete exposition on the topic, our goal here is only to fix notation used throughout the remainder of this text, and to provide some less well-known results which will be essential to the remainder of the text. For more information on this topic we would encourage the reader to seek out Hartshorne's text [Har77], and of course the foundational works of Grothendieck [FGA; EGAI; EGAII; EGAIII1; EGAIII2; EGAIV1; EGAIV2; EGAIV3; EGAIV4].

Although we will only need a small amount, we will freely use the language of category theory where appropriate. For background information in this area we suggest the texts [ML98] and [KS06].

1.1 Essential Definitions

For our purposes, rings will be commutative with identity, and ring morphisms will be required to preserve the identity. If p is a prime and $q = p^f$ is a power of that prime, we will denote the field with q elements as $\mathbb{F}_q$. As is standard, we will refer to the ring of integers by $\mathbb{Z}$, and the field of rational numbers by $\mathbb{Q}$. For a fixed prime p we will call the p-adic integers $\mathbb{Z}_p$, and the p-adic numbers $\mathbb{Q}_p$.

We will call the category of sets by the name $\mathfrak{Set}$, the category of groups $\mathfrak{Grp}$, the category of rings $\mathfrak{Ring}$, and the category of (locally noetherian) schemes $\mathfrak{Sch}$. Given a fixed scheme $\mathbf{S}$ we will often have call for the slice category over $\mathbf{S}$ which we will call $\mathfrak{Sch}_{\mathbf{S}}$. In the case that $\mathbf{S}$ is an affine scheme, $\mathbf{S} = \mathrm{Spec}(R)$ for some commutative ring

R, we will call the slice category $\mathfrak{Sch}_R$; in context there will be no chance for confusion.

Definition 1.1.1. A *geometric point* of a scheme $\mathbf{S}$ is a map $\bar{\mathbf{s}} \to \mathbf{S}$ with $\bar{\mathbf{s}} = \mathrm{Spec}(k)$, for an algebraically closed field k.

Definition 1.1.2. A *geometric fiber* of a map $f\colon \mathbf{X} \to \mathbf{S}$ is the fiber $\mathbf{X}_{\bar{\mathbf{s}}} = \mathbf{X} \times_{\mathbf{S}} \bar{\mathbf{s}}$ over a geometric point $\bar{\mathbf{s}} \to \mathbf{S}$ of $\mathbf{S}$.

Definition 1.1.3. A map between schemes $f\colon \mathbf{X} \to \mathbf{S}$ is called *smooth* if

(i) it is locally of finite type,

(ii) it is flat,

(iii) and it has regular geometric fibers.

We say a scheme $\mathbf{X} \in \mathfrak{Sch}_{\mathbf{S}}$ is *smooth over* $\mathbf{S}$ if the map $\mathbf{X} \to \mathbf{S}$ is smooth.

1.2 Group Schemes

Given a scheme $\mathbf{S}$, a group scheme over $\mathbf{S}$ is a functor $\mathcal{G}\colon \mathfrak{Sch}_{\mathbf{S}}^{\mathrm{op}} \to \mathfrak{Grp}$ which is representable, that is to say there exists a scheme $\mathbf{G}$, over $\mathbf{S}$, such that we have a natural isomorphism of the functors $\mathcal{G}(-) \cong \mathrm{Hom}_{\mathfrak{Sch}_{\mathbf{S}}}(-, \mathbf{G})$. Although there are many interesting examples of group schemes which are not affine, we will concern ourselves only with affine group schemes in this text. That is, we will demand that there be a sheaf of $\mathcal{O}_{\mathbf{S}}$-algebras R such that $\mathbf{G} = \mathrm{Spec}(R)$. Then as $\mathbf{G}$ is a group scheme, we will have that R is a sheaf of (commutative) Hopf algebras over $\mathbf{S}$.

For our purposes there are a few important examples of group schemes defined over $\mathrm{Spec}(\mathbb{Z})$ that are worth discussing briefly. First there is the additive group which we denote $\mathbf{G}_{\mathrm{a}}$. This group has the Hopf algebra $\mathbb{Z}[x]$, and for a ring R, we have that $\mathbf{G}_{\mathrm{a}}(\mathrm{Spec}(R)) = R$, where R is thought of as a group under addition. When we want to work with $\mathbf{G}_{\mathrm{a}}$ as a group over an arbitrary base $\mathbf{S}$, we will denote it $\mathbf{G}_{\mathrm{a}/\mathbf{S}}$, unless $\mathbf{S} = \mathrm{Spec}(A)$ is affine, in which case we will call it $\mathbf{G}_{\mathrm{a}/A}$.

We will also need to consider the multiplicative group $\mathbf{G}_{\mathrm{m}}$. This is the group with Hopf algebra $\mathbb{Z}[x, y]/(xy - 1)$, and for a ring R we have $\mathbf{G}_{\mathrm{m}}(\mathrm{Spec}(R)) = R^{\times}$, the unit group of R. When we wish to work with $\mathbf{G}_{\mathrm{m}}$ over an arbitrary base $\mathbf{S}$, we will denote it $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$, and, similarly to $\mathbf{G}_{\mathrm{a}}$, if $\mathbf{S} = \mathrm{Spec}(A)$ is affine, we will write $\mathbf{G}_{\mathrm{m}/A}$.

Considering for a moment the characteristic p setting we have that $\mathbf{G}_{\mathrm{m}/\mathbb{F}_p}$ is affine, with Hopf algebra $\mathbb{F}_p[x,y]/(xy-1)$, and the endomorphism which sends $x \mapsto x^p$, $y \mapsto y^p$, is a Hopf algebra map, which induces a group homomorphism $F\colon \mathbf{G}_{\mathrm{m}/\mathbb{F}_p} \to \mathbf{G}_{\mathrm{m}/\mathbb{F}_p}$ which we will call the relative Frobenius map for $\mathbf{G}_{\mathrm{m}/\mathbb{F}_p}$.

A wealth of examples of algebraic groups can be found under the umbrella of matrix groups. In the work at hand we will restrict ourselves to considering the general linear groups. As is standard we will give the name $\mathbf{GL}_n$ to the algebraic group which for any ring R we have $\mathbf{GL}_n(\operatorname{Spec} R)$ is the group of $n \times n$ invertible matrices with coefficients in R. The Hopf algebra in this case is $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}, y]/(\det(x_{i,j})y-1)$. If we need to consider $\mathbf{GL}_n$ as a group over an arbitrary base scheme $\mathbf{S}$, we will denote it $\mathbf{GL}_{n/\mathbf{S}}$, and if $\mathbf{S} = \operatorname{Spec}(A)$ is affine we will call it $\mathbf{GL}_{n/A}$. The ring homomorphism $\mathbb{Z}[x,y]/(xy-1) \to \mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}, y]/(\det(x_{i,j})y-1)$ which sends $x \mapsto \det(x_{i,j})$ and $y \mapsto y$ is a Hopf algebra morphism, and induces an algebraic group homomorphism $\det\colon \mathbf{GL}_n \to \mathbf{G}_{\mathrm{m}}$.

Theorem 1.2.1. *If $\mathbf{G}$ is an affine group scheme over a field k, then $\mathbf{G}$ is connected if and only if it is irreducible.*

This result is proven in [Wat79, §6.6]. Unfortunately the proof requires a fair bit of machinery that would be too complicated to recreate here.

1.3 Moduli Problems

A common theme in algebraic geometry is the task of finding moduli spaces. A good first example is the idea of classifying lines through the origin in the plane. From the perspective of smooth geometry the answer is simple enough, the projective line $\mathbb{RP}^1$ classifies all of the lines through the origin in $\mathbb{R}^2$, in that there is a bijection between the points of $\mathbb{RP}^1$ and the lines in $\mathbb{R}^2$. From the perspective of schemes things become somewhat more complicated, first because we no longer have a single object which we might call the plane, but instead many candidates. Changing our idea of what constitutes a plane will also require that we change our idea of what constitutes a line. There are several reasonable choices for both, but perhaps the best is to replace the idea of the plane with the idea of a plane for each scheme $\mathbf{S}$, what we will use is the sheaf $\mathcal{O}_{\mathbf{S}}^2$, and we will replace lines in the plane with rank 1 $\mathcal{O}_{\mathbf{S}}$-submodules that have a locally free quotient. This definition fits well with our intuition, but allows us to define a projective line not just for fields, but in fact for any scheme. In this way we see that the idea of a projective line can be thought of not just as an object, but as a functor $\mathbb{P}^1 \colon \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Set}$ where we send a scheme $\mathbf{S}$ to the set of rank 1 $\mathcal{O}_{\mathbf{S}}$-submodules of $\mathcal{O}_{\mathbf{S}}^2$ with locally free quotient and functions act by pullback. Now the question of understanding the space of lines is transformed into the question of understanding this functor. The best way to understand such a functor is to first hope that it is representable, and if it is, you can examine the space that represents it.

This is a common theme in moduli problems, first you must state the problem as a functor from $\mathfrak{Sch}^{\mathrm{op}}$ to $\mathfrak{Set}$, and then you look for an object which represents your functor. At times finding the correct statement of the problem is just as hard as finding the moduli space which represents the functor. One important aspect to keep in mind when searching for a moduli problem is how the functor treats morphisms, often a morphism of schemes becomes a morphism of sets by pullback. In this case it is important that any properties of schemes that are used be stable under base change, so that the moduli problem is actually a functor.

For our purposes in this text there are a couple of classic moduli problems that will show up, and so we will define them now. First is the Grassmannian: we will call $\mathcal{G}r_r^d$ (resp. $\mathcal{G}r^d$), is the functor which takes a scheme $\mathbf{S}$ to the set of rank r (resp. arbitrary rank) $\mathcal{O}_S$-submodules of $\mathcal{O}_S^d$ with locally free quotient. Let $\mathbf{Gr}_r^d$ (resp. $\mathbf{Gr}^d$) be the scheme that represents $\mathcal{G}r_r^d$ (resp. $\mathcal{G}r^d$); as a special case we at times call $\mathbf{Gr}_{d-1}^d$ by the alternate name $\mathbb{P}^{d-1}$.

The other is the famous Hilbert scheme. If $\mathbf{S}$ is a noetherian scheme, and $\mathbf{X} \in \mathfrak{Sch}_{\mathbf{S}}$ is of finite type over $\mathbf{S}$, then we have a functor $\mathcal{H}ilb_{\mathbf{X}/\mathbf{S}} \colon \mathfrak{Sch}_{\mathbf{S}} \to \mathfrak{Set}$ which sends a scheme $\mathbf{T}$ to the set of all closed subschemes $\mathbf{Y} \subset \mathbf{T} \times_{\mathbf{S}} \mathbf{X}$ that are proper and flat over $\mathbf{T}$. This functor is represented by the Hilbert scheme of $\mathbf{X}$, denoted $\mathbf{Hilb}_{\mathbf{X}/\mathbf{S}}$. For more details see [Fan+05], and [FGA].

Chapter 2

Witt Vectors

Much of the exposition in this chapter follows the presentation in Serre's text [Ser79]. For this chapter fix a prime number p.

If $\sigma \colon \mathbb{F}_p \to \mathbb{Z}_p$ is a section of the quotient map $\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, then we call the image of σ a set of representatives for $\mathbb{F}_p$ in $\mathbb{Z}_p$. Let $S \subset \mathbb{Z}_p$ be such a set of representatives; then we can write any $x \in \mathbb{Z}_p$ as a power series in p with coefficients in S

$$x = \sum_{n \geq 0} a_n p^n \quad \text{with} \quad a_n \in S.$$

A common choice is $S = \{0, 1, \ldots, p-1\}$, but if we chose instead the Teichmüller representatives

$$S = \{w \in \mathbb{Z}_p \mid w^p - w = 0\}$$

then there is a description of the ring operations of $\mathbb{Z}_p$ purely in terms of algebra in the residue field due to Witt [Wit37], and Teichmüller [Tei36] [Tei37]. Their insight was to consider the polynomials

$$W_n(X_0, \ldots, X_n) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}},$$

which we now call the Witt polynomials, whose usefulness is found through the following theorem.

Theorem 2.0.1. *For any $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \ldots, \varphi_n)$ of polynomials in $\mathbb{Z}[X_0, \ldots X_n, Y_0, \ldots, Y_n]$ such that:*

$$W_n(\varphi_0, \ldots, \varphi_n) = \Phi(W_n(X_0, \ldots, X_n), W_n(Y_0, \ldots, Y_n)).$$

Proof. The proof of this theorem can be found in Serre [Ser79], or Witt [Wit37]. □

Applying this theorem to the polynomial $\Phi(X, Y) = X + Y$ we obtain a sequence we call $(S_0, S_1, \dots)$, and applying it to $\Phi(X, Y) = XY$ we obtain a sequence we call $(P_0, P_1, \dots)$. These polynomials can be computed recursively and the first few are given in Tables 2.1 and 2.2. As you can see from looking at these tables, the complexity of the polynomials in question grows rapidly as n increases, although when p is small it is easy to work out the coefficients explicitly as in Table 2.3.

These polynomials can be used to place a novel ring structure on the set A^n for any ring A, where for two elements $(x_0, \dots, x_{n-1}), (y_0, \dots, y_{n-1}) \in A^n$ the sum and product are given by

$$(x_0, \dots, x_{n-1}) + (y_0, \dots, y_{n-1}) = (S_0(x_0, y_0), \dots, S_{n-1}(x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}))$$
$$(x_0, \dots, x_{n-1}) \cdot (y_0, \dots, y_{n-1}) = (P_0(x_0, y_0), \dots, P_{n-1}(x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}))$$

We call the resulting ring the truncated (p-typical) Witt vectors of length n over A, which we denote $\mathbf{W}_n(A)$.

In the event that p is an invertible element of A, this ring structure is isomorphic to the standard structure on A^n, but our interest lies in the other direction, as we are most interested in the situation where A is an $\mathbb{F}_p$-algebra.

Also of interest is the ring of Witt vectors over A which we denote $\mathbf{W}(A)$, and is equal to $\varprojlim \mathbf{W}_n(A)$, which we can describe explicitly. As a set $\mathbf{W}(A) = A^{\mathbb{N}}$, the set of sequences of elements of A, and we add and multiply elements essentially as in the case of the truncated Witt vectors; that is to say, given two sequences (x_i), and (y_i),

n	S_n
0	$X_0 + Y_0$
1	$X_1 + Y_1 + \frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p)$
2	$X_2 + Y_2 + \frac{1}{p}\left(X_1^p + Y_1^p - \left(X_1 + Y_1 + \frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p)\right)^p\right) + \frac{1}{p^2}\left(X_0^{p^2} + Y_0^{p^2} - (X_0 + Y_0)^{p^2}\right)$

Table 2.1: The first few polynomials S_n.

n	P_n
0	X_0Y_0
1	$Y_0^pX_1 + Y_1X_0^p + pX_1Y_1$
2	$X_0^{p^2}Y_2 + X_2Y_0^{p^2} + pX_1^pY_2 + pX_2Y_1^p + p^2X_2Y_2$ $+ \frac{1}{p}\left(X_0^{p^2}Y_1^p + X_1^pY_0^{p^2} + pX_1^pY_1^p - (Y_0^pX_1 + Y_1X_0^p + pX_1Y_1)^p\right)$

Table 2.2: The first few polynomials P_n.

n	S_n
0	$X_0 + Y_0$
1	$X_1 + Y_1 + X_0Y_0$
2	$X_2 + Y_2 + X_1Y_1 + X_1X_0Y_0 + Y_1X_0Y_0 + X_0^3Y_0 + X_0Y_0^3$
3	$X_3 + Y_3 + X_0^7Y_0 + 7X_0^5Y_0^3 + 7X_0^3Y_0^5 + X_0Y_0^7 + X_0^4X_1Y_0^2 + X_0^4Y_0^2Y_1$ $+ X_0^3X_1Y_0^3 + X_0^3Y_0^3Y_1 + X_0^2X_1Y_0^4 + X_0^2Y_0^4Y_1 + X_0^3X_1Y_0Y_1 + X_0X_1Y_0^3Y_1$ $+ X_0^3X_2Y_0 + X_0^3Y_0Y_2 + X_0X_1^3Y_0 + X_0X_2Y_0^3 + X_0Y_0^3Y_2 + X_0Y_0Y_1^3$ $+ X_0X_1X_2Y_0 + X_0X_1Y_0Y_2 + X_0X_2Y_0Y_1 + X_0Y_0Y_1Y_2 + X_1^3Y_1 + X_1Y_1^3$ $+ X_1X_2Y_1 + X_1Y_1Y_2 + X_2Y_2$

Table 2.3: The first few polynomials S_n with $p = 2$.

the i^{th} coefficient of their sum will be $S_i(x_0, \ldots, x_i, y_0, \ldots, y_i)$, and the i^{th} coefficient of their product will be $P_i(x_0, \ldots, x_i, y_0, \ldots, y_i)$.

2.1 Important Functions

There are several functions related to Witt vectors that are of essential importance. The first is truncation: given two natural numbers m, n, with $m > n$, we have a map $Tr_m^{m-n} \colon \mathbf{W}_m(A) \to \mathbf{W}_n(A)$ which takes the element $(x_0, \ldots, x_{m-1})$ and truncates it to $(x_0, \ldots, x_{n-1})$, this is a ring homomorphism. When $m - n = 1$ we will omit the superscript, and when the domain is clear from context we will omit the subscript.

The next is the Verschiebung, or shift, map. Again given two natural numbers m, n with $m > n$ we have a map $V_n^{m-n} \colon \mathbf{W}_n(A) \to \mathbf{W}_m(A)$ which takes an element $(x_0, \ldots, x_{n-1})$, and shifts it to the right, prepending 0s as necessary to fill, to give us $(0, \ldots, 0, x_0, \ldots, x_{n-1})$; when there is no chance for confusion, we will omit the subscript and call the function simply V^{m-n}, furthermore when $m - n = 1$ we will omit the superscript as well, and call the function simply V. Verschiebung is not a ring homomorphism, but it is an additive map between rings; it does have some good properties with respect to multiplication which we will discuss in Section 2.2.1. At times we will need to consider preimages of subsets under Verschiebung, and in these cases we will use the alternate name V^{-n} for $(V^n)^{-1}$.

For any natural number n, we also have the multiplicative representative map $r \colon A \to \mathbf{W}_n(A)$ which sends an element a to the Witt vector $(a, 0, \ldots, 0)$. This is not an additive map (unless $n = 1$), but it is multiplicative, and so it is sometimes useful to think of it as a group homomorphism on the unit groups: $r \colon A^\times \to \mathbf{W}_n(A)^\times$.

In the case that A is an $\mathbb{F}_p$-algebra we have one more useful ring homomorphism, the relative Frobenius homomorphism $F \colon \mathbf{W}_n(A) \to \mathbf{W}_n(A)$ which sends $(x_0, \ldots, x_{n-1})$ to $(x_0^p, \ldots, x_{n-1}^p)$.

Lastly, if A and B are two rings, and $f \colon A \to B$ a ring homomorphism, then we have an induced homomorphism $\mathbf{W}_n(f) \colon \mathbf{W}_n(A) \to \mathbf{W}_n(B)$ which sends the element $(x_0, \ldots, x_{n-1})$ to $(f(x_0), \ldots, f(x_{n-1}))$. This is a ring homomorphism as our multiplication and addition are defined by polynomial maps with integer coefficients, and ring homomorphisms commute with polynomials. With this it is easy to see that $\mathbf{W}_n$ is in fact an endofunctor on $\mathfrak{Ring}$.

Interestingly, for any $\mathbb{F}_p$-algebra A, when $n > 1$ the ring $\mathbf{W}_n(A)$ is not of characteristic p. In fact multiplication by p on $\mathbf{W}_n(A)$ is given by $Tr \circ V \circ F$, a map we will call m_p.

2.2 Witt Vectors as Group Schemes

In order to frame Witt vectors in the language of algebraic group schemes, the best approach is to consider the functor $\mathbf{W}_n$ defined in the previous section, and recognize that it is representable, in fact $\mathbf{W}_n(R) = \mathrm{Hom}_{\mathfrak{Ring}}(\mathbb{Z}[x_0, \ldots, x_{n-1}], R)$ with a ring structure coming from extra functions that make $\mathbb{Z}[x_0, \ldots, x_{n-1}]$ into a Hopf algebra, and also functions that encode the multiplicative structure. From this we can build a functor $\mathbf{W}_n \colon \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Ring}$ by calling the scheme $\mathbf{W}_n = \mathrm{Spec}(\mathbb{Z}[x_0, \ldots, x_{n-1}])$ and, for any scheme $\mathbf{S}$, having $\mathbf{W}_n(\mathbf{S}) = \mathrm{Hom}_{\mathfrak{Sch}}(\mathbf{W}_n, \mathbf{S})$. This is more than an algebraic group scheme, it's an algebraic ring scheme, but we can apply the forgetful functor $\mathcal{U} \colon \mathfrak{Ring} \to \mathfrak{Grp}$ to it to get a functor $\mathbf{W}_n \colon \mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Grp}$.

When we wish to work in the category of schemes over a fixed base scheme $\mathbf{S}$ we will use instead the functor defined by $\mathbf{W}_{n/\mathbf{S}} = \mathbf{W}_n \times_{\mathbb{Z}} \mathbf{S}$.

As $\mathbf{W}_{n/\mathbf{S}}$ is a ring scheme, we can construct new group schemes by composing other group schemes with it. For our purposes the most interesting of these is $\mathbf{GL}_{d/\mathbf{S}} \circ \mathbf{W}_{n/\mathbf{S}}$, because of its natural action on $\mathbf{W}_{n/\mathbf{S}}^d$.

It is important to note that every function discussed in Section 2.1 can be viewed not just as functions on points $\mathbf{W}_n(A)$. In fact each can be defined as coming from a Hopf algebra morphism on $\mathbb{Z}[x_0, \ldots, x_{n-1}]$. We will call these Hopf algebra morphisms by the same names as the ring morphisms. We have for $m > n$ that

$$Tr_m^{m-n} \colon \mathbb{Z}[x_0, \ldots, x_{n-1}] \to \mathbb{Z}[x_0, \ldots, x_{m-1}]$$

is the inclusion homomorphism. For Verschiebung we have

$$V^{m-n} \colon \mathbb{Z}[x_0, \ldots, x_{m-1}] \to \mathbb{Z}[x_0, \ldots, x_{n-1}]$$

is reduction modulo the ideal $(x_0, \ldots, x_{m-n-1})$, followed by the relabeling $x_i \mapsto x_{i-m+n}$. Using these Hopf algebra homomorphisms we can obtain algebraic group homomorphisms $V^{m-n} \colon \mathbf{W}_n \to \mathbf{W}_m$, and $Tr_m^{m-n} \colon \mathbf{W}_m \to \mathbf{W}_n$.

In this context it is best to view the multiplicative representative map as a map

$$r\colon \mathbb{Z}[x_0, \ldots, x_{n-1}] \to \mathbb{Z}[x, y]/(xy - 1)$$

which sends $x_i \mapsto x^{p^i}$, and thus gives a scheme theoretic map $r\colon \mathbf{G}_{\mathrm{m}} \to \mathbf{W}_n$, that restricts to a map of group schemes $\mathbf{G}_{\mathrm{m}} \to \mathbf{W}_n^{\times}$, which we can use to define an action of $\mathbf{G}_{\mathrm{m}}$ on $\mathbf{W}_n$

$$\alpha_n\colon \mathbf{G}_{\mathrm{m}} \times_{\mathbb{Z}} \mathbf{W}_n \to \mathbf{W}_n$$

which can be thought of either as being $r \times Id$ followed by the ring multiplication, or as coming from the ring theoretic map

$$\mathbb{Z}[x_0, \ldots, x_{n-1}] \to \mathbb{Z}[x, y](xy - 1) \otimes \mathbb{Z}[x_0, \ldots, x_{n-1}]$$

which sends $x_i \mapsto x^{p^i} \otimes x_i$. In either case this induces a grading on $\mathbb{Z}[x_0, \ldots, x_{n-1}]$ where x_i is given degree p^i.

Lastly, if we work in $\mathfrak{Sch}_{\mathbb{F}_p}$ then we have a map

$$F\colon \mathbf{W}_{n/\mathbb{F}_p} \to \mathbf{W}_{n/\mathbb{F}_p},$$

called the relative Frobenius homomorphism, induced by the ring homomorphism

$$F\colon \mathbb{F}_p[x_0, \ldots, x_{n-1}] \to \mathbb{F}_p[x_0, \ldots, x_{n-1}]$$

that sends $x_i \mapsto x_i^p$.

2.2.1 Basic Facts

For any natural numbers r, n, and d these functions give us a $\mathbf{G}_{\mathrm{m}}$-intertwining short exact sequence of group schemes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{W}_r^d & \xhookrightarrow{V^n} & \mathbf{W}_{r+n}^d & \xrightarrow{Tr_{r+n}^r} & \mathbf{W}_n^d & \longrightarrow & 0 \\
 & \alpha_r \circ (F^n \times Id) & \uparrow & & \uparrow \alpha_{r+n} & & \uparrow \alpha_n & & \\
 & & \mathbf{G}_{\mathrm{m}} \times \mathbf{W}_r^d & \xrightarrow{Id \times V^n} & \mathbf{G}_{\mathrm{m}} \times \mathbf{W}_{r+n}^d & \xrightarrow{Id \times Tr_{r+n}^r} & \mathbf{G}_{\mathrm{m}} \times \mathbf{W}_n^d, & &
\end{array} \tag{2.2.1}$$

where the first row is the short exact sequence, and the $\mathbf{G}_{\mathrm{m}}$ actions are given by the vertical arrows.

Lemma 2.2.1. *For natural numbers* $r < n \leq m$ *we have* $Tr^n_{m+r} \circ V^r_m = V^r_{m-n} \circ Tr^n_m$*, as functions from* $\mathbf{W}_m$ *to* $\mathbf{W}_{m+r-n}$.

Proof. This is simply the statement that we can either shift the indices of our indeterminates down by d and then add $m-n$ more to the end, or add $m-n$ more indeterminates and then shift the indices. This is clear as long as we have at least d indeterminates to begin with. □

For more details on Witt vectors see [Ser79, Chapter 2 §6], or [Haz78].

Chapter 3

Results

3.1 Lattices

Inspired by the work of Haboush in [Hab05], we would like to describe the space of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p$ vector spaces, and hopefully be able to obtain a geometric Satake like correspondence in this mixed characteristic case. However Kreidl raises some objections to Haboush's methods in [Kre10], and so we need to be cautious about the representability of the moduli problem we construct. To that end we will make a slightly unconventional definition of what constitutes a (truncated) lattice, describe some nice properties of our definition through some easy results, and in Section 3.2 prove that with this definition our moduli problem is indeed representable.

Definition 3.1.1. Let $\mathbf{S}$ be a scheme over $\mathbb{F}_p$. A lattice in $\mathbf{W}^d_{r/\mathbf{S}}$ is a closed $\mathbf{S}$-subscheme $\mathbf{X} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}}$, such that

(i) $\mathbf{X}$ is a subgroup scheme of $\mathbf{W}^d_{r/\mathbf{S}}$.

(ii) $\mathbf{X}$ is smooth as a scheme over $\mathbf{S}$.

(iii) $\mathbf{X}$ is stable under the action of $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$ on $\mathbf{W}^d_{r/\mathbf{S}}$.

Proposition 3.1.2. *Let* $\mathbf{S}' \to \mathbf{S}$ *be a morphism of schemes. If* $\mathbf{X} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}}$ *is a lattice, then base extension to* $\mathbf{S}'$ *gives a lattice* $\mathbf{X}_{\mathbf{S}'} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}'}$

Proof. That $\mathbf{X}_{\mathbf{S}'}$ is smooth over $\mathbf{S}'$ is well known, [EGAIV4, 17.3.3(iii)], and it is a $\mathbf{G}_{\mathrm{m}}$-stable subgroup scheme of $\mathbf{W}^d_{r/\mathbf{S}'}$ by transport of structure. □

Definition 3.1.3. Let $\mathcal{L}at_r^d$ be the covariant functor from $\mathfrak{Sch}_{\mathbb{F}_p}$ to $\mathfrak{Set}$, sending any scheme $\mathbf{S}$ over $\mathbb{F}_p$ to the set of lattices in $\mathbf{W}^d_{r/\mathbf{S}}$.

Proposition 3.1.4. *$\mathcal{L}at_r^d$ is a sheaf on the big Zariski site over $\mathbb{F}_p$.*

Proof. What needs to be shown is that for any scheme $\mathbf{S}$ over $\mathbb{F}_p$, affine open cover $\{U_i = \mathrm{Spec}(R_i)\}_{i\in I}$ of $\mathbf{S}$, and collection $\{\mathbf{X}_i \in \mathcal{L}at_r^d(R_i)\}$ such that

$$\forall i, j,\ \mathbf{X}_i \times_{U_i} (U_i \cap U_j) = \mathbf{X}_j \times_{U_j} (U_i \cap U_j)\,;$$

then the scheme $\mathbf{X}$ obtained by gluing the $\mathbf{X}_i$ is a lattice, that is to say $\mathbf{X} \in \mathcal{L}at_r^d(\mathbf{S})$.

We have that $\mathbf{X}$ is a $\mathbf{G}_{\mathrm{m}}$-stable subgroup scheme of $\mathbf{W}^d_{r/\mathbf{S}}$ by transport of structure, and the map $\mathbf{X} \to \mathbf{S}$ is smooth since smoothness is a local condition on the target. Therefore $\mathbf{X} \in \mathcal{L}at_r^d(\mathbf{S})$ as desired. □

Our definition of a lattice can be difficult to work with at times, and the properties we chose may seem to be lacking, but in fact they are quite restrictive, as they imply many other desirable properties for our lattices. In Proposition 3.1.7 we will see an equivalent set of properties that define a lattice that can be easier to work with in practice.

Proposition 3.1.5. *Let $\mathbf{S}$ be a scheme over $\mathbb{F}_p$. Let $\mathbf{X} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}}$ be a closed subgroup scheme over $\mathbf{S}$, stable under the action of $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$. Then $\mathbf{X}$ has connected geometric fibers.*

Proof. The $\mathbf{G}_{\mathrm{m}}$-orbit of any point x in a geometric fiber $\mathbf{X}_{\bar{s}}$ is the image of a connected set under a continuous map, and is therefore connected. Taking the closure of such an orbit gives a Zariski closed, $\mathbf{G}_{\mathrm{m}}$-stable subset of $\mathbf{X}_{\bar{s}}$, which is therefore the zero locus of a graded ideal $I = (p_1, \ldots, p_t)$, with each p_i homogeneous. As the grading is positive, and $x \in V(I)$, we must have that the degree of each $p_i > 0$, and so we also have $0 \in V(I)$. Thus the closure of the orbit is a connected set that contains both x and 0. So every point x must lie in the connected component of 0, and therefore the entire geometric fiber must be connected. □

Corollary 3.1.6. *Let $\mathbf{S}$ be a scheme over $\mathbb{F}_p$. Let $\mathbf{X} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}}$ be a closed subgroup scheme over $\mathbf{S}$, stable under the action of $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$. Then $\mathbf{X}$ has irreducible geometric fibers.*

Proof. By Theorem 1.2.1, an affine algebraic group scheme over a field is irreducible if and only if it is connected. □

Proposition 3.1.7. *Let* $\mathbf{S}$ *be a scheme over* $\mathbb{F}_p$*. Let* $\mathbf{X} \hookrightarrow \mathbf{W}^d_{r/\mathbf{S}}$ *be a closed subgroup scheme over* $\mathbf{S}$*. The following conditions are equivalent:*

(i) $\mathbf{X}$ *is smooth over* $\mathbf{S}$*.*

(ii) $\mathbf{X}$ *is flat over* $\mathbf{S}$ *and the morphism* $\mathbf{X} \to \mathbf{S}$ *has reduced geometric fibers.*

Proof. First, if $\mathbf{X} \to \mathbf{S}$ is smooth, then it is also flat with geometric fibers that are smooth varieties, and are therefore reduced.

Conversely, if $\mathbf{X}$ is flat over $\mathbf{S}$ and $\mathbf{X} \to \mathbf{S}$ has reduced geometric fibers, then for any geometric point $\mathrm{Spec}(\bar{k}) \to \mathbf{S}$, we have that $\mathbf{X}_{\bar{k}}$ is a reduced group scheme over a field, which by [Wat79, §11.6] is smooth, therefore $\mathbf{X}$ is smooth, as smoothness can be checked at geometric points. □

Proposition 3.1.8. *If* $\mathbf{X}$ *is a lattice in* $\mathbf{W}^d_{r/\mathbf{S}}$*, then for all* $m < r$*,* $V^{-m}(\mathbf{X})$ *is a lattice in* $\mathbf{W}^d_{r-m/\mathbf{S}}$*.*

Proof. As V^m is a smooth group homomorphism we have that $V^{-m}(\mathbf{X})$ is a subgroup of $\mathbf{W}^d_{r-m/\mathbf{S}}$, and is also smooth over $\mathbf{S}$. What remains is to ensure that $V^{-m}(\mathbf{X})$ is stable under the action of $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$ on $\mathbf{W}^d_{r-m/\mathbf{S}}$. The $\mathbf{G}_{\mathrm{m}}$-action is given by $\alpha_r \colon \mathbf{G}_{\mathrm{m}/\mathbf{S}} \times_{\mathbf{S}} \mathbf{W}^d_{r/\mathbf{S}} \to \mathbf{W}^d_{r/\mathbf{S}}$, then following subsection 2.2.1 we have $\alpha_r \circ (F^m \times V^m) = V^m \circ \alpha_{r-m}$. Then

$$\begin{aligned}\alpha_{r-m}(\mathbf{G}_{\mathrm{m}/\mathbf{S}} \times_{\mathbf{S}} V^{-m}(\mathbf{X})) &= V^{-m}(\alpha_r([F^m \times V^m](\mathbf{G}_{\mathrm{m}/\mathbf{S}} \times_{\mathbf{S}} V^{-m}(\mathbf{X})))) \\ &= V^{-m}(\alpha_r(F^m(\mathbf{G}_{\mathrm{m}/\mathbf{S}}) \times_{\mathbf{S}} \mathbf{X})) \\ &\subseteq V^{-m}(\mathbf{X}),\end{aligned}$$

So $V^{-m}(\mathbf{x})$ is stable under the $\mathbf{G}_{\mathrm{m}}$-action. □

Corollary 3.1.9. *We have that* V^{-m} *is a natural transformation from* $\mathcal{L}at^d_r$ *to* $\mathcal{L}at^d_{r-m}$*.*

Proof. This follows immediately from Proposition 3.1.8. □

As a last stop before moving on to some results about this functor, there is a technical result in algebraic geometry we will need about subgroup schemes of Witt schemes.

Lemma 3.1.10. *If $\mathbf{X}$ is an affine algebraic group over an algebraically closed field k with a $\mathbf{G}_{\mathrm{m}}$-action, then $\mathbf{X}$ is reduced if and only if its projectivization with respect to the $\mathbf{G}_{\mathrm{m}}$-action, $\mathbf{P}$, is reduced.*

Proof. Let A be the graded K-algebra such that $\mathrm{Spec}(A) = \mathbf{X}$, then $\mathrm{Proj}(A) = \mathbf{P}$, and if $x_0 \in \mathbf{X}$ is the point corresponding to the irrelevant ideal of A, we have a morphism $\mathbf{X} \setminus x_0 \to \mathbf{P}$, as $\mathbf{X}$ is the affine cone over $\mathbf{P}$. If $\mathbf{X}$ is reduced, then so is $\mathbf{X} \setminus x_0$, and therefore so is $\mathbf{P}$, as it is the scheme-theoretic image of $\mathbf{X} \setminus x_0$ under this map.

For the converse, consider a point (closed or not) $p \in \mathbf{P}$ which corresponds to a homogeneous prime ideal in A. The stalk of the structure sheaf of P at p is just $\mathcal{O}_p = A_{(p)}$ – the localization of the ring A at the prime ideal p.

Thus if $\mathbf{P}$ is reduced, than there exists a point $p \in \mathbf{X}$ such that $A_{(p)}$ is reduced – just view the homogeneous prime ideal as a prime ideal. It follows that $A_{(\mathfrak{m})}$ is reduced for any maximal ideal $\mathfrak{m}$ containing p.

But now, since $\mathbf{X}$ is an algebraic group, we have a translation isomorphism $\alpha_x : \mathbf{X} \to \mathbf{X}$ for every $x \in \mathbf{X}(k)$. It follows that $\mathbf{X}$ is reduced at $\alpha_x(\mathfrak{m})$ as well. Hence $A_{(\mathfrak{m})}$ is reduced for all maximal ideals $\mathfrak{m}$. This implies that A is reduced, since the homomorphism:

$$\phi : A \to \prod_{\mathfrak{m}} A_{(\mathfrak{m})}$$

is injective; an element of A is determined by its images in all localizations at all maximal ideals. If $a \in A$ were nilpotent, then $\phi(a)$ would be nilpotent, hence zero. □

Theorem 3.1.11. *Let $\mathbf{S}$ be a Noetherian scheme over $\mathbb{F}_p$. Then for any $\mathbf{G}_{\mathrm{m}}$-stable subgroup scheme $\mathbf{X} \subset \mathbf{W}^d_{n/\mathbf{S}}$ with $\mathbf{X} \to \mathbf{S}$ flat, the subset $U \subset \mathbf{S}$ of points for which the fiber of $\mathbf{X}$ has reduced geometric fibers is open.*

Proof. It suffices to work affine locally on the base, so assume $\mathbf{S} = \mathrm{Spec}(A)$ with A a Noetherian ring of characteristic p; then $\mathbf{X}$ is affine, cut out by a homogeneous ideal J in $A[x_{1,0}, \ldots, x_{d,n-1}]$ with the appropriate Witt grading.

If $s \in \mathbf{S}$ (a scheme-theoretic point, i.e. a prime ideal $\mathfrak{p} \subset A$), write $\mathbf{X}_s$ for the fiber of $\mathbf{X}$ over s. We say $\mathbf{X}_s$ is geometrically reduced if for any $A/\mathfrak{p} \to k$ with k an algebraically closed field, the base change $\mathbf{X}_k$ is reduced. Equivalently, $\mathbf{X}_s$ is geometrically reduced if, for every geometric point of $\mathbf{S}$ in the closure of s, the fiber at the geometric point is reduced (i.e., reduced geometric fibers on the closure of s).

Let $\mathbf{P}$ denote the projectivization of $\mathbf{X}$ with respect to the $\mathbf{G}_{\mathrm{m}}$-action. In other words, locally on $\mathbf{S}$, take Proj of the graded ring $A[x_{1,0},\dots,x_{d,n-1}]/J$. Then as the $\mathbf{G}_{\mathrm{m}}$-action comes from a positive grading, $\mathbf{P}\to\mathbf{S}$ is proper, flat, and locally of finite presentation. Moreover, by Lemma 3.1.10, for any $s\in\mathbf{S}$, $\mathbf{P}_s$ is geometrically reduced if and only if $\mathbf{X}_s$ is geometrically reduced.

So consider the set $U\subset\mathbf{S}$ consisting of those s for which $\mathbf{X}_s$ (equivalently $\mathbf{P}_s$ by Lemma 3.1.10) is geometrically reduced. By [EGAIV3, Theorem 12.2.1 (viii)] (with $\mathbf{X}=\mathbf{P}$ and $\mathbf{Y}=\mathbf{S}$ and $\mathcal{F}$ equal to $\mathcal{O}_{\mathbf{P}}$), the set U is open. □

3.2 Representability

Theorem 3.2.4 is the main result of this section, but we will first prove the representability of a broader moduli problem, and see how our problem can be cut out of that broader problem as the intersection of an open and a closed subset.

Proposition 3.2.1. *Let $\mathcal{HS}^d_r\colon\mathfrak{Sch}^{\mathrm{op}}_{\mathbb{F}_p}\to\mathfrak{Set}$ be the functor that takes a scheme $\mathbf{S}$ to the set of closed, flat over $\mathbf{S}$, $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$-stable subschemes of $\mathbf{W}^d_{r/\mathbf{S}}$. There exists a scheme $\mathbf{HS}^d_r$ over $\mathbb{F}_p$ which represents the functor $\mathcal{HS}^d_r$.*

Proof. By the proof of Proposition 3.1.4, we can see that $\mathcal{HS}^d_r$ is a sheaf on the big Zariski site, and so it is enough to prove that it is representable on the subcategory of affine schemes. To show this we consider the related functor $\widehat{\mathcal{HS}^d_r}\colon\mathfrak{Ring}\to\mathfrak{Set}$, which sends a ring R to the set of homogeneous ideals

$$I\subset R\otimes\mathbb{Z}[x_{1,0},\dots,x_{d,r-1}],$$

such that the rank of $(R\otimes\mathbb{Z}[x_{1,0},\dots,x_{d,r-1}])_a/I_a$ is finite for each $a\in\mathbb{N}$, subject to the Witt grading on $\mathbb{Z}[x_{1,0},\dots,x_{d,r-1}]$ (which gives degree p^j to $x_{i,j}$); and show that it is representable, which establishes the result.

This is a direct consequence of Theorem 1.1 of [HS04], by taking $A=\mathbb{Z}$, $S=\mathbb{Z}[x_{1,0},\dots,x_{d,r-1}]$, and taking the union over all possible Hilbert functions. □

Lemma 3.2.2. *The subfunctor of $\mathcal{HS}^d_r$ consisting of the subschemes that are subgroups is represented by a closed subscheme of $\mathbf{HS}^d_r$.*

Proof. Consider the automorphism $\sigma\colon \mathbf{W}_r^d \times \mathbf{W}_r^d \to \mathbf{W}_r^d \times \mathbf{W}_r^d = \mathbf{W}_r^{2d}$ given by $(x, y) \mapsto (x - y, y)$. Note that the subgroup schemes of $\mathbf{W}_r^d$ are exactly the subschemes $\mathbf{X} \subset \mathbf{W}_r^d$ such that $\sigma(\mathbf{X} \times \mathbf{X}) \subset \mathbf{X} \times \mathbf{X}$. We have that σ respects the $\mathbf{G}_{\mathrm{m}}$ action (as $\mathbf{W}_r^d$ is a ring scheme), and therefore gives an automorphism $\bar{\sigma}$ of $\mathbf{HS}_r^{2d}$. The subscheme of $\mathbf{HS}_r^d$ we are interested in is exactly the preimage under the diagonal map $\Delta\colon \mathbf{HS}_r^d \to \mathbf{HS}_r^{2d}$ of the closed subscheme $\mathbf{Y} \subset \mathbf{HS}_r^{2d}$ of points fixed by $\bar{\sigma}$, which is closed. □

Lemma 3.2.3. *The subfunctor of $\mathcal{HS}_r^d$ consisting of the subschemes for which the structure has reduced geometric fibers is represented by an open subscheme of $\mathbf{HS}_r^d$.*

Proof. If we take $\mathbf{X}$ to be the universal family sitting over $\mathbf{HS}_r^d$, and apply Theorem 3.1.11, we get that the subset U of $\mathbf{HS}_r^d$ that corresponds to $\mathbf{X}$ having reduced geometric fibers is open. □

Theorem 3.2.4. *The functor $\mathcal{L}at_r^d$ is representable by a locally closed (i.e., intersection of a closed and open) subscheme of $\mathbf{HS}_r^d$. We will call this subscheme $\mathbf{Lat}_r^d$.*

Proof. By Lemma 3.2.2 we have a closed subscheme $\mathbf{X} \subset \mathbf{HS}_r^d$ corresponding to subgroups of $\mathbf{W}_r^d$, and by Lemma 3.2.3 we have an open subscheme $\mathbf{Y} \subset \mathbf{HS}_r^d$ which corresponds to those subschemes of $\mathbf{W}_r^d$ with reduced geometric fibers. Therefore the intersection $\mathbf{Lat}_r^d = \mathbf{X} \cap \mathbf{Y}$ corresponds to the $\mathbf{G}_{\mathrm{m}}$-stable, closed, flat over the base, subgroup schemes of $\mathbf{W}_r^d$ with reduced geometric fibers which, by Proposition 3.1.7, is exactly the set of lattices in $\mathbf{W}_r^d$. □

Note that the action of $\mathbf{GL}_d \circ \mathbf{W}_r$ on $\mathbf{W}_r^d$ commutes with the $\mathbf{G}_{\mathrm{m}}$-action, in fact $\mathbf{G}_{\mathrm{m}}$ acts as scalar matrices, which lie in the center of $\mathbf{GL}_d \circ \mathbf{W}_r$. Also the action is smooth, and therefore the image of a lattice under the action will again be a lattice, so we obtain an action of $\mathbf{GL}_d \circ \mathbf{W}_r$ on $\mathbf{Lat}_r^d$. In fact as lattices are stable under the $\mathbf{G}_{\mathrm{m}}$ action, the center of $\mathbf{GL}_d$ acts trivially, and so the $\mathbf{GL}_d$-action descends to an action of $\mathbf{PGL}_d \circ \mathbf{W}_r$ on $\mathbf{Lat}_r^d$.

3.3 The Shape of a Subgroup

3.3.1 Definition and Basic Results

Let k be an algebraically closed field of characteristic p, and $\mathbf{G}$ be an algebraic subgroup in $\mathbf{W}_{n/k}^d$. For $i \leq n$ we give the name $\mathbf{G}_i$ to the subgroup $V^{i-n}(\mathbf{G})$ in $\mathbf{W}_{i/k}^d$, and set $\mathbf{G}_0 = 0$. If we identify $\mathbf{W}^d_{i/k}$ with its image under V^{n-i} in $\mathbf{W}^d_{n/k}$, then this is the same as defining $\mathbf{G}_i = \mathbf{G} \cap \mathbf{W}^d_{i/k}$. We will use this identification to consider $\mathbf{G}_i$ as a subgroup of $\mathbf{G}$, and at times $\mathbf{W}^d_{i/k}$ as a subset of $\mathbf{W}^d_{n/k}$. Then we define the *shape* of $\mathbf{G}$ to be the sequence $\vec{d} = (d_0, d_1, \ldots, d_{n-1})$ of natural numbers, given by

$$d_i = \dim(\mathbf{G}_{n-i}) - \dim(\mathbf{G}_{n-i-1}).$$

Considering k-points we have a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{W}^d_{i-1}(k) & \xhookrightarrow{V} & \mathbf{W}^d_i(k) & \xrightarrow{Tr_i^{i-1}} & k^d & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbf{G}_{i-1}(k) & \xrightarrow{V} & \mathbf{G}_i(k) & \xrightarrow{Tr_i^{i-1}} & S_i & \longrightarrow & 0. \end{array} \tag{3.3.1}$$

We have identified $\mathbf{W}^d_1(k) = k^d$, and defined S_i to be the image of $\mathbf{G}_i(k)$ under Tr_i^{i-1}. Note that S_i is closed under addition as it is a subgroup, and $\mathbf{G}_{\mathrm{m}}$-stable, therefore it is a vector subspace of k^d. Considering the second row we have that

$$\begin{aligned} \dim_k(S_i) &= \dim(\mathbf{G}_i) - \dim(\mathbf{G}_{i-1}) \\ &= d_{n-i}, \end{aligned}$$

and also as $S_i \subset k^d$ we have $0 \leq d_{n-i} \leq d$.

Proposition 3.3.1. *If $\mathbf{G}$ is a subgroup of $\mathbf{W}^d_{n/k}$ of shape $(d_0, \ldots, d_{n-1})$ and $\mathbf{H}$ is a subgroup of $\mathbf{G}$ of shape $(e_0, \ldots, e_{n-1})$, then $e_i \leq d_i$ for all $0 \leq i < n$.*

Proof. Since $\mathbf{H} \subset \mathbf{G}$, we also have $\mathbf{H}_i \subset \mathbf{G}_i$ for all i. Then we can considere the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{H}_{i-1}(k) & \xhookrightarrow{V} & \mathbf{H}_i(k) & \xrightarrow{Tr_i^{i-1}} & T_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & \mathbf{G}_{i-1}(k) & \xhookrightarrow{V} & \mathbf{G}_i(k) & \xrightarrow{Tr_i^{i-1}} & S_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{W}^d_{i-1}(k) & \xhookrightarrow{V} & \mathbf{W}^d_i(k) & \xrightarrow{Tr_i^{i-1}} & k^d & \longrightarrow & 0. \end{array} \tag{3.3.2}$$

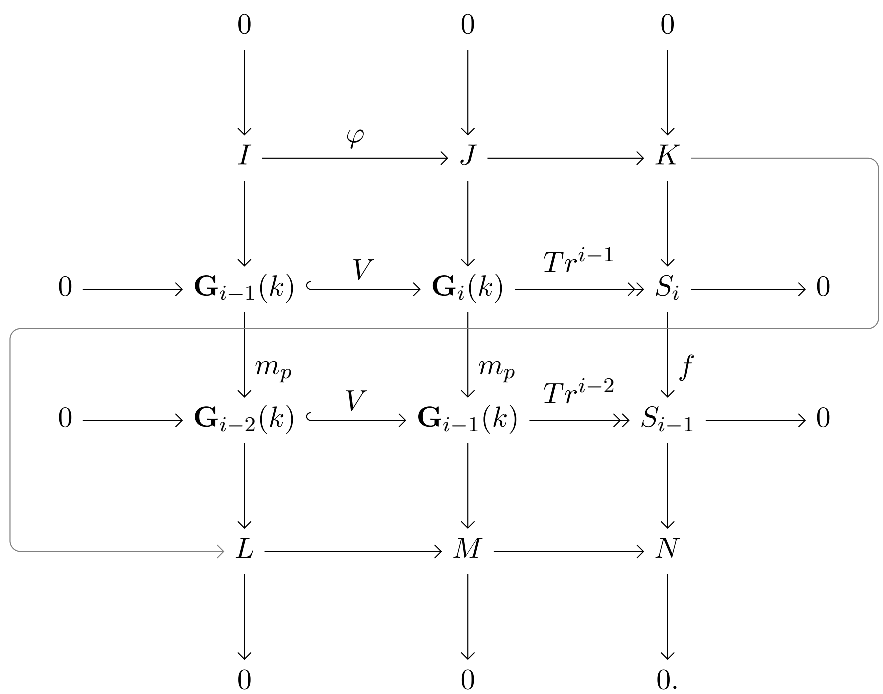
By identifying spaces with their image under V, we can see that

$$\begin{aligned}\mathbf{H}_{i-1}(k) &= \mathbf{H}_i(k) \cap \mathbf{W}^d_{i-1}(k) \\ &= \mathbf{H}_i(k) \cap \mathbf{G}_{i-1}(k).\end{aligned}$$

If $t \in T_i$ is mapped to 0 by α, then it comes from an element x of $\mathbf{H}_i(k)$, which after embedding in $\mathbf{G}_i(k)$, truncates to 0; and therefore is in the image of V. Then x is in $\mathbf{G}_{i-1}(k)$, and $\mathbf{H}_i(k)$, so it is in $\mathbf{H}_{i-1}(k)$, and then by exactness of the first row $t = 0$. Thus showing that α is injective, and therefore $e_i = \dim_k(T_i) \leq \dim_k(S_i) = d_i$, exactly as desired. □

Theorem 3.3.2. *For all $0 \leq i < n-1$ we have $d_i \leq d_{i+1}$.*

Proof. If we replicate the second row of Diagram 3.3.1 for $\mathbf{G}_i$, and $\mathbf{G}_{i-1}$, use the fact that m_p gives not just a map from $\mathbf{G}_k$ to itself, but has an image lying inside of the subgroup $\mathbf{G}_{k-1}$, and apply the snake lemma to the resulting map of short exact sequences we get the following diagram:



(3.3.3)

Here I, J, and K are the kernels of the relevant maps; and L, M, and N are their cokernels. The rows and columns are exact, and our goal is to show the map labeled f is an injection. We will do this by showing that $K = 0$, and for this it suffices to show that φ is a surjection and $L = 0$. First, over a perfect field m_p is surjective, and so $L = 0$; also as V is an injection, F is a bijection, and $m_p = F \circ V \circ Tr$: we have that the kernel of m_p in $\mathbf{G}_{i-1}(k)$ is the kernel of Tr on $[V \circ F](\mathbf{G}_{i-1}(k)) \subset \mathbf{G}_i(k)$ but the kernel of Tr sits in the image of V, so φ is in fact a surjection, thus $K = 0$, as desired. This gives that f is an injection, and therefore $\dim_k(S_{i-1}) \geq \dim_k(S_i)$, or $d_{n-i+1} \geq d_{n-i}$, exactly as desired. □

3.3.2 Applications to $\mathcal{L}at_r^d$

For $\vec{d}$ a shape of length r, we will give the name $\mathcal{L}at_{r,\vec{d}}^d$ to the subfunctor of $\mathcal{L}at_r^d$ of subgroups of shape $\vec{d}$ at every geometric fiber. We then have the following results.

Theorem 3.3.3. *There is an isomorphism of functors given by V:*

$$\mathcal{L}at_{r,(0,d_1,\ldots,d_{r-1})}^d \cong \mathcal{L}at_{r-1,(d_1,\ldots,d_{r-1})}^d.$$

Proof. If $\mathbf{G}$ is a lattice in $\mathcal{L}at_{r,(0,d_1,\ldots,d_{r-1})}^d$, then V provides an isomorphism $\mathbf{G}_r \cong \mathbf{G}_{r-1}$, as $d_0 = 0$. Therefore V^{-1} is the desired isomorphism of functors. □

Theorem 3.3.4. *There is an isomorphism of functors given by Tr:*

$$\mathcal{L}at_{r,(d_0,\ldots,d_{r-2},d)}^d \cong \mathcal{L}at_{r,(d_0,\ldots,d_{r-2})}^d.$$

Proof. Any lattice in $\mathcal{L}at_{r,(d_0,\ldots,d_{r-2},d)}^d$ is the preimage under truncation of a unique lattice in $\mathcal{L}at_{r-1,(d_0,\ldots,d_{r-2})}^d$. □

3.4 Description of $\mathcal{L}at_1^d$

Theorem 3.4.1. *There is an isomorphism of functors $\mathcal{L}at_1^d \cong \mathcal{G}r^d$ which preserves dimension in such a way that $\mathcal{L}at_{1,(m)}^d \cong \mathcal{G}r_m^d$, and therefore $\mathbf{Lat}_{1,(m)}^d \cong \mathbf{Gr}_m^d$.*

Proof. When $r = 1$ we have that $\mathcal{HS}_1^d = \mathcal{H}ilb_{\mathbb{P}^{d-1}}$ and so by Theorem 3.2.4 $\mathcal{L}at_1^d$ is a locally closed subfunctor of $\mathcal{H}ilb_{\mathbb{P}^{d-1}}$. If we fix a noetherian $\mathbf{S}$ and take a subgroup scheme $\mathbf{X}$ corresponding to an element $x \in \mathcal{L}at_1^d(\mathbf{S})$ then if we call $\mathbf{Y} = (\mathbf{X} \setminus e(\mathbf{S}))/\mathbf{G}_\mathrm{m}$, where $e\colon \mathbf{S} \to \mathbf{X}$ is the unit of the group, and $e(\mathbf{S})$ is the closure of the scheme-theoretic image of $\mathbf{S}$ under e. We have $\mathbf{Y} \subset \mathbb{P}^{d-1}$, and as $\mathbf{Y}$ is flat over $\mathbf{S}$ we have that the Hilbert polynomial of $\mathbf{Y}$ considered as a family over $\mathbf{S}$ is constant on irreducible components. If we take a geometric point of $\mathbf{S}$, $\tau\colon \operatorname{Spec}(k) \to \mathbf{S}$ for some algebraically closed k, then we can consider the base change $\mathbf{Y}_k = \mathbf{Y} \times_{\mathbf{S}} \operatorname{Spec}(k)$, and detect the Hilbert polynomial of $\mathbf{Y}$ by considering $\mathbf{Y}_k$. As $\mathbf{Y}_k$ is smooth over k, we have that $\mathbf{Y}_k(k)$ is Zariski dense in $\mathbf{Y}_k$, but $\mathbf{Y}_k(k) = (\mathbf{X}_k(k) \setminus \{0\})/\mathbf{G}_{\mathrm{m}}(k)$, and $\mathbf{X}_k(k)$ is an additive subgroup of the vector space $\mathbf{G}^d_{\mathrm{a}/k}(k)$ which is stable under the action of $k^\times$, and is therefore a vector subspace of $\mathbf{G}^d_{\mathrm{a}/k}(k)$. This implies that $\mathbf{X}_k = \overline{\mathbf{X}_k(k)}$ is a linear subscheme of $\mathbf{G}^d_{\mathrm{a}/k}$. Thus the Hilbert polynomial of $\mathbf{Y}_k$ (and also $\mathbf{Y}$) is that of a linear subspace, and therefore $x \in \mathcal{G}r^d(S)$. Together this shows that $\mathcal{L}at^d_1(\mathbf{S}) \subset \mathcal{G}r^d(\mathbf{S})$ when $\mathbf{S}$ is noetherian, and so when restricted to the full subcategory of noetherian schemes, $\mathfrak{Sch}_N$, $\mathcal{L}at^d_1$ is a subfunctor of $\mathcal{G}r^d$.

Conversely for an element $x \in \mathcal{G}r^d(S)$ we have a subscheme $\mathbf{X} \subset \mathbf{G}^d_{\mathrm{a}/\mathbf{S}}$ cut out by linear homogeneous polynomials locally on the base. This is clearly a closed subgroup of $\mathbf{G}^d_{\mathrm{a}/\mathbf{S}}$, and $\mathbf{G}_{\mathrm{m}/\mathbf{S}}$-stable. It is smooth over $\mathbf{S}$ by the properties of the Grassmannian, therefore $\mathcal{G}r^d$ is a subfunctor of $\mathcal{L}at^d_1$.

It is clear from the construction that this isomorphism preserves dimension, and so gives an isomorphism $\mathcal{L}at^d_{1,(m)} \cong \mathcal{G}r^d_m$ for any m. □

Notice that the action of $\mathbf{PGL}_d$ on $\mathbf{Lat}^d_1$ is exactly the normal action of $\mathbf{PGL}_d$ on $\mathbf{Gr}^d$, and is therefore transitive on every component $\mathbf{Gr}^d_m$.

3.5 Description of $\mathcal{L}at^2_n$

The two theorems 3.3.3 and 3.3.4 allow us to reduce understanding of $\mathbf{Lat}^2_n$ to the case of understanding $\mathbf{Lat}^2_{1,(0)}$, $\mathbf{Lat}^2_{1,(2)}$, and the schemes $\mathbf{Lat}^2_{n,(1,1,\dots,1)}$. The first two, as well as the $n = 1$ case of the third are addressed by Theorem 3.4.1. All that remains is to describe the remaining schemes $\mathbf{Lat}^2_{n,(1,1,\dots,1)}$ for $n > 1$.

Lemma 3.5.1. *If we call the total space of the line bundle $\mathcal{O}(p)$ over $\mathbb{P}^1$ by the name $\mathcal{E}(p)$, then we have a morphism of schemes $\varphi\colon \mathcal{E}(p) \hookrightarrow \mathbf{Lat}^2_{2,(1,1)}$ such that the following* *diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{E}(p) & \xrightarrow{\ \varphi\ } & \mathbf{Lat}^2_{2,(1,1)} \\
\downarrow & & \downarrow V^{-1} \\
\mathbb{P}^1 & \longrightarrow & \mathbf{Lat}^2_{1,(1)} \\
& \searrow^{F} & \downarrow \wr \\
& & \mathbb{P}^1.
\end{array}
\tag{3.5.1}
$$

Here the arrow from $\mathcal{E}(p)$ *to* $\mathbb{P}^1$ *is the standard bundle map, and the arrow from* $\mathbb{P}^1$ *to* $\mathbf{Lat}^2_{1,(1)}$ *is defined to make the bottom triangle commute. Additionally for any* $\mathbb{F}_p$*-algebra* R *the induced map of sets* $\varphi\colon \mathcal{E}(p)(R) \to \mathbf{Lat}^2_{2,(1,1)}(R)$ *is an injection, therefore* φ *is a universally injective morphism of schemes.*

Proof. Every object in the picture is a sheaf on the big Zariski site, so it is enough to define φ on affine schemes. Let R be an $\mathbb{F}_p$ algebra, we will begin by defining coordinates on $\mathcal{E}(p)(R)$ which we will use to define the map φ. Let $\lambda \in \mathbb{P}^1(R)$, then λ defines a locally free rank 1 quotient of R^2; by passing to an open cover $\{\mathrm{Spec}(R_i)\}$ of $\mathrm{Spec}(R)$, we can assume that the quotient is free, rather than only locally free. In this case we can view λ as a map $\lambda\colon R^2 \to L$ where L is free of rank 1. By picking an isomorphism $L \cong R$ we obtain a pair $\alpha = \lambda(1,0)$, and $\beta = \lambda(0,1)$. These satisfy the property $(\alpha, \beta) = (1)$. Changing the isomorphism changes the pair α, β to a new pair $u\alpha, u\beta$ for some $u \in R^\times$. Then for any $g \in (R[x,y]/(\alpha x + \beta y))_p$ homogeneous of degree p we can identify

$$\mathcal{E}(p)(R) = \{(\alpha, \beta, g) \mid \alpha, \beta \in R \text{ such that } (\alpha,\beta) = 1, g \in (R[x,y]/(\alpha x + \beta y))_p\}/\sim,$$

where $\sim$ is the equivalence relation $(\alpha, \beta, g) \sim (u\alpha, u\beta, u^p g)$ for $u \in R^\times$.

For any $(\alpha, \beta, g) \in \mathcal{E}(p)(R)$ we define the ideal

$$I_{\alpha,\beta,g} = (\alpha x_0 + \beta y_0,\ g(x_0, y_0) + \alpha^p x_1 + \beta^p y_1) \subset R[x_0, x_1, y_0, y_1].$$

Note first that if $(\alpha, \beta, g) \sim (u\alpha, u\beta, u^p g)$ then we have

$$\begin{aligned} I_{u\alpha,u\beta,u^p g} &= (u\alpha x_0 + u\beta y_0,\ u^p g(x_0,y_0) + (u\alpha)^p x_1 + (u\beta)^p y_1) \\ &= (u\,(\alpha x_0 + \beta y_0)\,,\ u^p\,(g(x_0,y_0) + \alpha^p x_1 + \beta^p y_1)) \\ &= (\alpha x_0 + \beta y_0,\ g(x_0,y_0) + \alpha^p x_1 + \beta^p y_1) \\ &= I_{\alpha,\beta,g}, \end{aligned}$$

so the definition makes sense. Next notice that for distinct elements $g, h \in (R[x,y]/(\alpha x + \beta y))_p$ we have that $I_{\alpha,\beta,g} \neq I_{\alpha,\beta,h}$.

We define φ to send the point $(\alpha, \beta, g) \in \mathcal{E}(p)(R)$ to

$$\mathbf{X}_{/R} = \operatorname{Spec}(R[x_0, x_1, y_0, y_1]/I_{\alpha,\beta,g}),$$

which we claim is an element of $\mathbf{Lat}^2_{2,(1,1)}(R)$. First note that as $I_{\alpha,\beta,g}$ is a homogeneous ideal for the Witt grading, $\mathbf{X}$ is $\mathbf{G}_{\mathrm{m}}$-stable. Also as $I_{\alpha,\beta,g}$ is prime, $\mathbf{X}$ is geometrically reduced. Also as $\dim \mathbf{X} = 2$, and $V^{-1}(\mathbf{X}) = \operatorname{Spec}(R[x_0, y_0]/(\alpha^p x_0, \beta^p y_0))$, all that remains is to show that $\mathbf{X}$ is a subgroup of $\mathbf{W}_2^2$; or equivalently that $I_{\alpha,\beta,g}$ is a Hopf ideal, that is to say

$$\Delta(I_{\alpha,\beta,g}) \subset I_{\alpha,\beta,g} \otimes R[x_0, x_1, y_0, y_1] + R[x_0, x_1, y_0, y_1] \otimes I_{\alpha,\beta,g}$$

for the Witt comultiplication function

$$\Delta \colon R[x_0, x_1, y_0, y_1] \to R[x_0, x_1, y_0, y_1] \otimes R[x_0, x_1, y_0, y_1].$$

As the $\mathbf{GL}_2 \circ \mathbf{W}_2$-action on $\mathbf{Lat}^2_{2,(1,1)}$ descends to a transitive action of $\mathbf{GL}_2 \circ \mathbf{W}_1$ on $\mathbf{Lat}^1_{2,(1)} \cong \mathbb{P}^1$, we can perform this calculation with $\alpha = 1$, and $\beta = 0$ without loss of generality.

In this setting we can take the representative $g(x_0, y_0) = c y_0^p$ for some $c \in R$. Then $I_{1,0,g} = (x_0, c y_0^p + x_1)$, and we only need compute

$$\begin{aligned} \Delta(x_0) &= x_0 \otimes 1 + 1 \otimes x_0 \\ &\subset I_{1,0,g} \otimes R[x_0, x_1, y_0, y_1] + R[x_0, x_1, y_0, y_1] \otimes I_{1,0,g}; \end{aligned}$$

and

$$\begin{aligned}\Delta\left(cy_0^p+x_1\right) &= c\left(y_0\otimes 1+1\otimes y_0\right)^p+x_1\otimes 1+1\otimes x_1\\ &\qquad +\frac{x_0^p\otimes 1+1\otimes x_0^p-(x_0\otimes 1+1\otimes x_0)^p}{p}\\ &= cy_0^p\otimes 1+1\otimes cy_0^p+x_1\otimes 1+1\otimes x_1+\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}x_0^i\otimes x_0^{p-i}\\ &= (cy_0^p\otimes 1+x_1\otimes 1)+\left(1\otimes cy_0^p+1\otimes x_1+\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}x_0^i\otimes x_0^{p-i}\right)\\ &\subset I_{1,0,g}\otimes R[x_0,x_1,y_0,y_1]+R[x_0,x_1,y_0,y_1]\otimes I_{1,0,g}.\end{aligned}$$

Thus $I_{\alpha,\beta,g}$ is a Hopf ideal, and $\mathbf{X}\in\mathbf{Lat}^2_{2,(1,1)}(R)$ as desired. □

Theorem 3.5.2. *The map $\varphi\colon\mathcal{E}(p)\to\mathbf{Lat}^2_{2,(1,1)}$ defined in Lemma 3.5.1 is surjective on geometric points.*

Proof. Let $\mathrm{Spec}(k)\to\mathbf{Lat}^2_{2,(1,1)}$ be a geometric point, then we wish to show that it factors through $\varphi\colon\mathcal{E}(p)\to\mathbf{Lat}^2_{2,(1,1)}$. That is to say let $\mathbf{X}\subset\mathbf{W}^2_{2/k}$ be a lattice in $\mathcal{L}at^2_{2,(1,1)}(k)$, and call by the name I the ideal in $k[x_0,x_1,y_0,y_1]$ which defines $\mathbf{X}$. Then if we call $V^{-1}(\mathbf{X})=\mathbf{X}'$, we have $\mathbf{X}'\subset\mathbf{W}^2_{1/k}$ in $\mathcal{L}at^2_{1,(1)}(k)$, and therefore, by Theorem 3.4.1 we have that $\mathbf{X}'$ is cut out of $k[x_0,y_0]$ by an ideal of the form $I'=(ax_0+by_0)$. As $\mathbf{X}$ is a lattice it is $\mathbf{G}_\mathrm{m}$-stable, and therefore irreducible by Corollary 3.1.6, so I is a homogeneous prime ideal. As $V^{-1}(\mathbf{X})=\mathbf{X}'$, we have that $V(I)=I'$, so it must be that

$$I=(f_i(x_0,x_1,y_0,y_1),g(x_0,y_0)+ax_1+by_1)$$

for some homogeneous polynomials f_i, and g, with g of degree p all contained in the ideal (x_0,y_0). As $\mathbf{X}$ is a subgroup, $p\cdot\mathbf{X}\subset\mathbf{X}$, and therefore

$$(ax_0^p+by_0^p)=(f_i(0,0,x_0^p,y_0^p),g(0,0)+ax_0^p+by_0^p)=p(I)\subset I.$$

As k is algebraically closed there exist α, and β such that $\alpha^p=a,\beta^p=b$. Then we have $ax_0^p+by_0^p=(\alpha x_0+\beta y_0)^p\in I$, and since I is prime we have $\alpha x_0+\beta y_0\in I$. As the dimension of $\mathbf{X}$ is 2, the height of I is 2, and so

$$I=(f_i(x_0,x_1,y_0,y_1))\mod(g(x_0,y_0)+ax_1+by_1))$$

must be of height one, and therefore principal. So it must be that $f_i - f_j \in (g(x_0, y_0) + ax_1 + by_1)$ for any pair i, j, therefore we can write $I = (f_1(x_0, x_1, y_0, y_1), g(x_0, y_0) + ax_1 + by_1)$. Now, as $\alpha x_0 + \beta y_0 \in I$, we must have polynomials P and Q such that

$$\alpha x_0 + \beta y_0 = P \cdot f_1 + Q \cdot (g + \alpha^p x_1 + \beta^p y_1)$$

As the left hand side is of degree 1, and both f_1 and $g + \alpha^p x_1 + \beta^p y_1$ are homogeneous, we must have that the degree of f_1 is either 1 or 0. If f_1 is of degree 0, then $I = k[x_0, x_1, y_0, y_1]$, and $\mathbf{X}$ is trivial, which cannot be. So we must have f_1 is homogeneous of degree 1, and then P must have degree 0, so we may as well replace f_1 with $\alpha x_0 + \beta y_0$. So $I = (\alpha x_0 + \beta y_0, g(x_0, y_0) + \alpha^p x_1 + \beta^p y_1)$ with $g \in (R[x_0, y_0]/(\alpha x_0 + \beta y_0))_p$ homogeneous of degree p. This is exactly the ideal $I_{\alpha,\beta,g}$ defined in the proof of Lemma 3.5.1, which is by definition in the image of φ.

This shows that all of the geometric points of $\mathbf{Lat}^2_{2,(1,1)}$ lie in the image of φ as desired. □

We suspect that the schemes $\mathbf{Lat}^d_n$ are normal, in which case the results of Lemma 3.5.1, and Theorem 3.5.2 would imply that φ is in fact an isomorphism of schemes.

The techniques in this Section may generalize to provide descriptions of the spaces $\mathbf{Lat}^2_{n,(1,1,\ldots,1)}$ with $n \geq 3$, but it seems that increasingly more complicated ideal computations are likely an inefficient way to describe those schemes. It is more likely that an approach through the theory of Dieudonné modules (see for example [Car62]) would prove more fruitful in order to describe these spaces.

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