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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Vector Bundles on Perfectoid Spaces

A Dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Shaunak Das

Committee in charge:

Professor Kiran Kedlaya, Chair
Professor Ioan Bejenaru
Professor Ronald Graham
Professor Julian McAuley
Professor Dragos Oprea

2016

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The Dissertation of Shaunak Das is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2016

DEDICATION

To anyone who believed in me. Thanks.

EPIGRAPH

I don't know if anybody's ever told you that half the time, this business comes down to

'I don't like that guy.'

— Roger Sterling

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VITA

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ABSTRACT OF THE DISSERTATION

Vector Bundles on Perfectoid Spaces

by

Shaunak Das

Doctor of Philosophy in Mathematics

University of California, San Diego, 2016

Professor Kiran Kedlaya, Chair

We explicitly compute the Picard groups of the projectivoid line and its corresponding characteristic p tilt. The desire to generalize this to higher-dimensional projectivoid spaces, as well as other perfectoid spaces, leads us to ask whether a GAGA principle holds between perfections of analytifiable schemes in characteristic p , and corresponding completed perfections of their adic analytifications.

Chapter 1

Introduction

Perfectoid spaces were introduced relatively recently, in order to provide a geometric framework for the following isomorphism from p -adic Hodge Theory, due to Fontaine-Wintenberger:

Theorem 1.0.1 (Fontaine-Wintenberger). *For fixed prime $p > 0$, there is a canonical isomorphism (of topological groups) between the absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))$.*

However, these objects have proven their worth beyond the above result and have been key to understanding a number of interesting problems in number theory and arithmetic geometry, for instance,

- providing new instances in which Deligne's Weight-Monodromy Conjecture holds
- classification of p -divisible groups

For this reason, the geometry of these perfectoid objects is worth studying, for its own sake, and such is carried out in this work. The main result of this thesis is the

following isomorphism:

Theorem 1.0.2. *Let $K = (K, K^\circ)$ be a perfectoid affinoid field with corresponding characteristic p tilt $K^\flat = (K^\flat, (K^\flat)^\circ)$, and consider the projectivoid line $\mathbb{P}_K^{1,ad,perf}$ over K . Then we have the following isomorphism:*

$$Pic(\mathbb{P}_K^{1,ad,perf}) \cong Pic(\mathbb{P}_{K^\flat}^{1,ad,perf}) \cong \mathbb{Z}[1/p]$$

Every line bundle on either of the spaces arises as a twisting sheaf $\mathcal{O}(\alpha)$, for $\alpha \in \mathbb{Z}[1/p]$, and each sufficiently large p -power twist of such arises as the pullback of a twisting sheaf on the corresponding regular adic projective line.

With this in hand, there are a number of natural questions one may ask, which go off in several directions. Such includes:

- *When will the Picard groups of a perfectoid space X and its tilt $X^\flat$ be isomorphic?*
- *Does every finite rank vector bundle on the projectivoid line split as a direct sum of line bundles?*

As we will see in this work, these all naturally lead to the following question.

Question 1.0.3. *For X/K proper, where K is a nontrivially-valued nonarchimedean field in characteristic p , does a GAGA principle hold for the associated perfection of X ? Namely, is there an equivalence of categories between coherent sheaves over the perfection of X and coherent sheaves over the completed perfection of the (adic) analytification of X ?*

Chapter 2

Geometric Preliminaries

2.1 The Rings

In this section, we introduce the class of topological rings that cover the nonarchimedean geometric objects that we will need in building towards perfectoid spaces. The recommended resource for much of this material is [8].

Definition 2.1.1. *Let R be a topological ring. We say that R is **adic** if there exists an ideal $I \subset R$ such that (I^n) forms a fundamental system of neighborhoods of zero; we call I an **ideal of definition** of R .*

Example 2.1.2. *Consider the localization of $\mathbb{Z}$ at prime ideal (p) (not the element p), $R = \mathbb{Z}_{(p)}$, equipped with the p -adic absolute value $|\cdot|_p$. It follows that the ideals $p^n \mathbb{Z}_{(p)}$ form a fundamental system of neighborhoods of 0, and so $\mathbb{Z}_{(p)}$ is adic, with ideal of definition $p\mathbb{Z}_{(p)}$.*

Every adic ring contains a natural subring of interest.

Definition 2.1.3. For R an adic ring and element $f \in R$, consider the sequence of increasing powers (f^n) . The element f is called **power-bounded** if this sequence is a bounded subset of R . We denote the corresponding ring of power-bounded elements by:

$$R^\circ := \{f \in R : f \text{ is power-bounded} \}$$

An element f is said to be **topologically nilpotent** if the sequence of powers $f^n \rightarrow 0$. The set of such elements forms an ideal inside of R° and is denoted

$$R^{\circ\circ} := \{f \in R : f \text{ is topologically nilpotent}\}$$

Example 2.1.4. If $R = \mathbb{Q}$ with the p -adic topology, we have that $\mathbb{Z}_{(p)}$ corresponds to the subring of power-bounded elements, and its corresponding maximal ideal $p\mathbb{Z}_{(p)}$ is the subring of topologically nilpotent elements in $\mathbb{Q}$.

Definition 2.1.5. A topological ring R is called **f -adic** if there exists an open subring $R_0 \subset R$ such that:

- R_0 with the subspace topology is adic
- R_0 admits an ideal of definition which is finitely-generated

The subring R_0 is called a **ring of definition** of R .

Example 2.1.6. For fixed prime p , consider the field of rational numbers $R = \mathbb{Q}$ equipped with the p -adic absolute value $|\cdot|_p$. It follows that $R_0 = \mathbb{Z}_{(p)}$ is an open

subring of $\mathbb{Q}$ with ideal of definition generated by a single element $p \in \mathbb{Z}_{(p)}$. Therefore, $\mathbb{Q}$ is f -adic when equipped with the p -adic topology.

Definition 2.1.7. A topological ring R is called a **Tate ring** if it is f -adic and contains a topologically nilpotent unit.

This definition of a Tate ring seems awfully unwieldy; however, there are many common and important examples of Tate rings. We present a few of the most important ones (at least for the purposes of this work).

Example 2.1.8. Since $p^n \rightarrow 0$ with respect to $|\cdot|_p$, it follows that $\mathbb{Q}$ equipped with the p -adic topology is a Tate ring.

Note that $\mathbb{Q}$ is not complete with respect to the p -adic norm. However, if we form its p -adic completion $\mathbb{Q}_p$, we still get a Tate ring; the completion of $\mathbb{Z}_{(p)}$ will remain open in $\mathbb{Q}_p$, and p of course remains a topologically nilpotent unit in $\mathbb{Q}_p$. The fact that $\mathbb{Q}_p$ holds more generally, and motivates our next definition.

Definition 2.1.9. A **nonarchimedean field** K is a field which is complete with respect to a non-trivial **absolute value** $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ which satisfies the following conditions:

- **Multiplicativity:** $|xy| = |x||y|$
- **Strong Triangle Inequality:** $|x + y| \leq \max(|x|, |y|)$
- **Positive-definiteness:** $|x| = 0 \iff x = 0$

for all $x, y \in K$. Inside of K , there is the open (with respect to the topology induced by the absolute value) valuation ring

$$\mathcal{O}_K := \{f \in K : |f| \leq 1\}$$

Remark 2.1.10. *Note that the valuation ring of a nonarchimedean field coincides with the subring of power-bounded elements, due to the fact that we demand the absolute value be multiplicative. Thus, we use valuation ring and ring of power-bounded elements interchangeably in this case.*

Example 2.1.11. *The p -adic completion of $\mathbb{Q}$, denoted $\mathbb{Q}_p$, is a nonarchimedean field. Its valuation ring is the p -adic integers $\mathbb{Z}_p$.*

Example 2.1.12. *Let $R = K$ be a nonarchimedean field; as mentioned, we have an open valuation ring $R_0 = K^\circ$; For any $\varpi \in K^\circ$ with absolute value $|\varpi| < 1$ (such exists since K is non-trivially valued), we have that (ϖ^n) forms a fundamental system of neighborhoods of zero inside of K° . Since ϖ is a topologically nilpotent unit of K , it follows that K is a Tate ring.*

Example 2.1.13. *For nonarchimedean field K , consider the univariate polynomial ring $K[T]$; since K has an absolute value, we may equip $K[T]$ with the following multiplicative **Gauss norm**:*

$$\left\| \sum_{i=0}^N a_i T^i \right\| := \max_i |a_i|$$

This ring contains the open subring

$$K^\circ[T] := \{f \in K[T] : \|f\| \leq 1\}$$

It is clear that we can extend the ideal of definition on K° from the previous example, to an ideal of definition of $K^\circ[T]$; we thus have that $K[T]$, equipped with the Gauss norm is also an example of a Tate ring.

Note that, just like $\mathbb{Q}$ for the p -adic absolute value, $K[T]$ is not complete with respect to the Gauss norm. If we complete this ring, we obtain the univariate **Tate algebra** over K , defined as the following subring of formal power series

$$K\langle T \rangle := \left\{ \sum_{i=0}^{\infty} a_i T^i : \lim_{n \rightarrow \infty} |a_i| = 0 \right\}$$

Once more, it is easy to verify that this ring will also be a Tate ring.

Example 2.1.14 (Banach algebras). *More generally, for nonarchimedean field K , let R be a K -Banach algebra with norm $\|\cdot\| : R \rightarrow \mathbb{R}_{\geq 0}$ satisfying*

- $\|1\| = 1$
- **Positive Definite:** $\|f\| = 0 \iff f = 0$
- **Scalar Multiplicativity:** $\|cf\| = |c|\|f\|$ for all $c \in K, f \in R$
- **Submultiplicativity:** $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in R$
- **Strong Triangle Inequality:** $\|f + g\| \leq \max\{\|f\|, \|g\|\}$

It follows from the Strong Triangle Inequality that the subring $R^+ := \{f \in R : \|f\| \leq 1\}$ is open; moreover, for any topologically nilpotent $\varpi \in K$, we have that the ideals (ϖ^n) will necessarily form a fundamental system of neighborhoods of $0 \in R^+$, and so we get that R will be a Tate ring.

Example 2.1.15. Let R be a Tate ring, which is complete with respect to its adic topology. Consider the subring of the formal power series ring

$$R\langle T \rangle := \left\{ \sum_{i=0}^{\infty} a_i T^i : \lim_{i \rightarrow \infty} a_i = 0 \right\}$$

with coefficients in R such that $a_i \rightarrow 0$ with respect to the adic topology. This ring will also be a Tate ring. This can be generalized to arbitrary number of variables.

Definition 2.1.16. An **affinoid field** (K, K^+) consists of a nonarchimedean field K , and an open, integrally-closed subring $K^+ \subset K^\circ$.

For the relative version, we define the following:

Definition 2.1.17. For (K, K^+) an affinoid field, we define an **affinoid** (K, K^+) -**algebra** as a pair of rings (R, R^+) such that

- R is a K -algebra that is f -adic (hence Tate, since K will contain a topologically nilpotent unit)
- R^+ is a subring of R° , which contains K^+ , and is open and integrally-closed inside of R

Moreover, a morphism of affinoid (K, K^+) -algebras is simply a continuous morphism $\phi : (R, R^+) \longrightarrow (S, S^+)$ such that $\phi(R^+) \subset S^+$, and the following diagram commutes:

$$\begin{array}{ccc} (K, K^+) & \longrightarrow & (R, R^+) \\ & \searrow & \downarrow \phi \\ & & (S, S^+) \end{array}$$

Example 2.1.18. *Going back to the example of the Tate algebra $K\langle T \rangle$, we have that the subring of power-bounded elements*

$$K\langle T \rangle^\circ := \{f \in K\langle T \rangle : \|f\| \leq 1\} = K^\circ\langle T \rangle$$

is open, and moreover integrally-closed (use multiplicativity of Gauss norm!) inside of $K\langle T \rangle$. The pair

$$(R, R^+) := (K\langle T \rangle, K^\circ\langle T \rangle)$$

is a (K, K°) -affinoid algebra. Moreover, the inclusion $(K, K^\circ) \longrightarrow (K\langle T \rangle, K^\circ\langle T \rangle)$ is a continuous/bounded ring map.

Remark 2.1.19. *Since the notation (K, K^+) and (R, R^+) is a bit much, we will typically refer to such affinoid pairs by the ambient field K or ring R , for convenience.*

2.2 The Points of the Space

We have defined the notion of an affinoid K -algebra, and we now want to consider these as rings of regular functions on some topological space, similar to how we associate a prime spectrum to any commutative ring, in the algebraic category. What is the corresponding notion of prime spectrum in the analytic category?

Definition 2.2.1. *Let Γ be a totally-ordered abelian group. A subgroup Δ of Γ is **convex** if for $\gamma_1, \gamma_2 \in \Delta$ such that $\gamma_1 \leq \gamma_2$, then $\gamma \in \Delta$ for all $\gamma_1 \leq \gamma \leq \gamma_2$. The **rank** of Γ is the number of non-trivial convex subgroups of Γ .*

Definition 2.2.2. *Let R be a ring. A **valuation** on R is a map $|\cdot| : R \rightarrow \Gamma \cup \{0\}$, where Γ is a totally-ordered abelian group (whose group law is written multiplicatively) of arbitrary rank, and 0 is an element satisfying $0 < \gamma$, for all $\gamma \in \Gamma$), which satisfies for all $f, g \in R$:*

- $|1| = 1$
- **Multiplicativity:** $|fg| = |f||g|$
- **Strong Triangle Inequality:** $|f + g| \leq \max(|f|, |g|)$
- **Positive-semidefiniteness:** $|0| = 0$

Since a valuation is merely required to satisfy positive-*semidefiniteness*, there may be nontrivial elements in the ring which have valuation 0. The multiplicative property encoded in the definition of valuations forces the set of such elements, called

the **support of the valuation**,

$$\text{supp}(| \cdot |) := \{f \in R : |f| = 0\}$$

to be a prime ideal inside of R .

The valuation $| \cdot |$ on R therefore induces an **absolute value** on the integral domain $R/\text{supp}(| \cdot |)$, and we can further extend $| \cdot |$ uniquely to its fraction field, again using multiplicativity to do such a thing. This fraction field $\text{Frac}(R/\text{supp}(| \cdot |))$ now has a topology.

Definition 2.2.3. Let $| \cdot | : R \longrightarrow \Gamma \cup \{0\}$ and $| \cdot |' : R \longrightarrow \Gamma' \cup \{0\}$ be two valuations on ring R . The valuations $| \cdot |$ and $| \cdot |'$ are called **equivalent** if there exists an isomorphism $\phi : \Gamma \longrightarrow \Gamma'$ of totally-ordered abelian groups, such that the resulting diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{| \cdot |} & \Gamma \cup \{0\} \\ & \searrow | \cdot |' & \downarrow \phi \\ & & \Gamma' \cup \{0\} \end{array}$$

Definition 2.2.4. Let R be a ring, and $| \cdot | : R \longrightarrow \Gamma \cup \{0\}$ a valuation on R . If R carries a topology, then the valuation $| \cdot |$ is called **continuous** if the induced morphism of topological rings

$$R \longrightarrow \text{Frac}\left(\frac{R}{\text{supp}(| \cdot |)}\right)$$

is a continuous map.

Now that we have the necessary notion of continuous valuations, and equivalence on valuations, we can define our spaces of interest. Let (R, R^+) be an affinoid K -algebra, as defined before. Consider the set

$$\mathrm{Spa}(R, R^+) := \{ |\cdot| : R \rightarrow \Gamma \cup \{0\} \text{ continuous} \mid |f| \leq 1, \forall f \in R^+ \} / \cong$$

where $\cong$ is the equivalence relation of equivalence of valuations. This will constitute our set of points on which we evaluate our rings of functions (R, R^+) . We will call this the **adic spectrum** of (R, R^+) .

For a given point $x = |\cdot|_x \in X := \mathrm{Spa}(R, R^+)$ and function $f \in R$, we denote evaluation at the point x by

$$f \mapsto |f|_x$$

Example 2.2.5. *For nonarchimedean field K , with open valuation ring K° , consider the space $\mathrm{Spa}(K, K^\circ)$. We claim that this space consists of a single point: the absolute value that comes for free with K .*

Suppose $x = |\cdot|_x$ is some point that does not correspond to the absolute value with respect to which we are topologizing K . Then we know that $|f|_x \leq |f| \leq 1$ for all $f \in K^\circ$, and so

$$K^\circ \subset \mathcal{O}_x := \{f \in K : |f|_x \leq 1\}$$

Moreover, we have the inclusion of maximal ideals (of local rings K° and $\mathcal{O}_x$, respectively):

$$\mathfrak{m} := \{f \in K^\circ : |f| < 1\} \subset \{f \in \mathcal{O}_x : |f|_x < 1\} =: \mathfrak{m}_x$$

To show that $\mathfrak{m} = \mathfrak{m}_x \cap K^\circ$, we need the other inclusion, and so argue by contradiction. Suppose $|f|_x < 1$, but $|f| \geq 1$, for some $f \in K^\circ$. It follows that $|1/f| \leq 1$, hence $1/f \in K^\circ$, and so $|1/f|_x \leq |1/f| \leq 1$. This is a contradiction, since $|\cdot|_x$ is multiplicative, hence $|f| < 1$, giving us $\mathfrak{m} = \mathfrak{m}_x \cap K^\circ$.

Since we have the inclusion of local subrings of K

$$(K^\circ, \mathfrak{m}) \subset (\mathcal{O}_x, \mathfrak{m}_x)$$

such that $\mathfrak{m} = \mathfrak{m}_x \cap K^\circ$, it necessarily follows (from the fact that valuation rings are maximal amongst local rings, with respect to the relation of domination) that $K^\circ = \mathcal{O}_x$. Two valuations $|\cdot|_x$ and $|\cdot|$ are equivalent (in the above defined sense) if and only if they have the same valuation rings.

Remark 2.2.6. The above situation changes if K° is replaced by a ring K^+ which is not necessarily a valuation ring inside of K . All we demand is that K^+ be open, integrally closed in K , and bounded (trivial, since $K^+ \subset K^\circ$).

Example 2.2.7 (Adic Unit Disc). For nonarchimedean field K , recall the affinoid (K, K°) -algebra $(K\langle T \rangle, K^\circ\langle T \rangle)$ from Example 2.1.18. We form the open unit disc via taking the corresponding adic spectrum:

$$\mathbb{B}_K^{1,ad} := \text{Spa}(K\langle T \rangle, K^\circ\langle T \rangle)$$

A description of the points of this space requires a little more investigation into the

rings in question. For point $|\cdot|_x = x \in \mathbb{B}_K^{1,ad}$, we have corresponding $|\cdot|_x$ -valued field

$$K(x) := \text{Frac} \left(\frac{K\langle T \rangle}{\text{supp}(x)} \right)$$

is an extension of the base field K ; moreover, since $K^\circ \subset K^\circ\langle T \rangle$, we obtain that $K^\circ \subset \mathcal{O}_{K(x)}$, and thus $|\cdot|_x$ restricts to an absolute value on K° such that K° is contained within $\mathcal{O}_{K(x)} \cap K$; it follows by the previous example, that $|\cdot|_x$ is in fact an extension of the absolute value on K to $K(x)$. Thus $K(x)/K$ will necessarily be an **extension of valued fields**. Both contain valuation rings, hence have corresponding residue fields $\kappa(x)$ and κ , respectively. What is not immediately obvious, is that $K(x)/K$ will necessarily be of finite transcendence degree; the strategy is to find an upper bound on the transcendence degree of these extensions, and then combining with Abhyankar's Inequality

$$\text{tr.deg}(\kappa(x)/\kappa) + \text{rank}_{\mathbb{Q}}(\Gamma_x/\Gamma) \leq \text{tr.deg}(K(x)/K)$$

where Γ_x denotes the value group of $|\cdot|_x$ and Γ (necessarily a subgroup of Γ_x) the value group of $|\cdot|$ on K

Using the above, one can figure out a classification of the various types of points; a good resource for such is [2]. When K is algebraically closed, the set of such points has the following tree-like structure (rank 2 points not included):

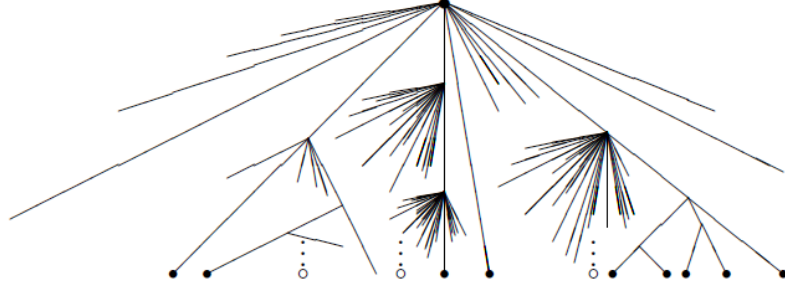


Figure 2.8: A depiction of the rank 1 points of the adic unit disc, corresponding to Berkovich points. Image taken from [2].

2.3 The Topology and Presheaf

We have discussed the set of points for the spaces that we will be interested in working with. For affinoid field (K, K^+) , we now restrict to affinoid (K, K^+) -algebras (R, R^+) , and endow $\mathrm{Spa}(R, R^+)$ with the following topology:

For $f_1, \dots, f_n \in R$ generating the unit ideal and $g \in R$, we consider subsets of $\mathrm{Spa}(R, R^+)$ of the form

$$U\left(\frac{f_1, \dots, f_n}{g}\right) := \{x \in \mathrm{Spa}(R, R^+) : |f_i|_x \leq |g|_x, i = 1, \dots, n\}$$

These are called **rational subsets** of $\mathrm{Spa}(R, R^+)$, and will form a basis for the topology for the adic spectrum. These are analogous to the distinguished opens $D(g) = \mathrm{Spec}(R[1/g]) \subset \mathrm{Spec}(R)$ used in constructing affine schemes. This analogy will be reflected at the level of rings, in fact.

Any continuous morphism $\phi : (R, R^+) \longrightarrow (S, S^+)$ of K -affinoid algebras naturally induces a continuous morphism $\phi^* : \mathrm{Spa}(S, S^+) \longrightarrow \mathrm{Spa}(R, R^+)$ at the level of topological spaces. Namely, $\phi^*(|\cdot|_y) := |\phi(\cdot)|_y$, for $y \in Y$:

$$\begin{array}{ccc}
(R, R^+) & \xrightarrow{\phi} & (S, S^+) \\
& \searrow |\phi(\cdot)| & \downarrow |\cdot| \\
& & \Gamma \cup \{0\}
\end{array}$$

This is analogous to what occurs in algebraic geometry; a ring morphism induces a Zariski-continuous morphism on prime spectra, which goes in the opposite direction (since prime ideals pull back to prime ideals). The fact that this morphism is well-defined is a consequence that we demand our ring map ϕ is continuous.

We have suitably topologized the adic spectrum of an affinoid K -algebra, and now want to equip it with a structure sheaf of regular functions. We, of course, want R to be the ring of global sections (in general, this does not occur, as we will see below). What about for other opens?

Associated to each rational subset

$$U\left(\frac{f_1, \dots, f_n}{g}\right)$$

there is an affinoid K -algebra

$$\left(R\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle, R\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle^+\right)$$

Let me briefly describe how it is constructed from the original affinoid pair (R, R^+) . Consider the subring $R[f_1/g, \dots, f_n/g]$ of the localization of R_g . We know that the topology on K^+ can be generated by powers of a topologically-nilpotent unit ϖ , and so the ring of definition $R_0 \subset R$ has topology that is principally-generated by ϖ .

We then have $R_0[f_1/g, \dots, f_n/g]$ is a ring of definition of $R[f_1/g, \dots, f_n/g]$, and (ϖ^n) forms a fundamental system of neighborhoods of 0. Letting B be the integral closure of $R^+[f_1/g, \dots, f_n/g]$ inside of $R[f_1/g, \dots, f_n/g]$, and define the appropriate completions of these rings

- $R\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle := R\left[\widehat{\frac{f_1}{g}, \dots, \frac{f_n}{g}}\right]$
- $R\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle^+ := \widehat{B}$

We use the shorthand $\underline{f} := (f_1, \dots, f_n)$. The pair $(R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+)$ is also an affinoid (K, K^+) -algebra, and is called a **rational localization** of (R, R^+) . Such comes with a map

$$\psi_{\underline{f}/g} : (R, R^+) \longrightarrow (R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+)$$

which together satisfy a universal property:

Theorem 2.3.1. *Let (R, R^+) be an affinoid K -algebra, $V = U\left(\frac{f}{g}\right) \subset \text{Spa}(R, R^+)$ a rational subset, for $\underline{f} = (f_1, \dots, f_n)$. The following are satisfied:*

- *For any morphism $\phi : (R, R^+) \longrightarrow (S, S^+)$ of affinoid K -algebras such that the image of the induced map on topological spaces $\text{Spa}(S, S^+) \rightarrow \text{Spa}(R, R^+)$ lands inside of V , we have a unique morphism of (K, K^+) -affinoid algebras*

$$\left(R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+\right) \longrightarrow (S, S^+)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
(R, R^+) & \xrightarrow{\psi_{\underline{f}/g}} & (R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+) \\
& \searrow \phi & \downarrow \exists! \\
& & (S, S^+)
\end{array}$$

- Any valuation $|\cdot|_x \in V$ uniquely extends to a continuous valuation on the ring $R\langle \underline{f}/g \rangle$ that is bounded above by 1 on the subring $R\langle \underline{f}/g \rangle^+$; that is, we can find a continuous valuation $|\cdot|_{x'} : R\langle \underline{f}/g \rangle \rightarrow \Gamma_x \cup \{0\}$ such that the following diagram commutes:

$$\begin{array}{ccc}
(R, R^+) & \xrightarrow{\psi_{\underline{f}/g}} & (R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+) \\
& \searrow |\cdot|_x & \downarrow |\cdot|_{x'} \\
& & \Gamma_x \cup \{0\}
\end{array}$$

- The corresponding adic spectrum of $(R\langle \underline{f}/g \rangle, R\langle \underline{f}/g \rangle^+)$ is homeomorphic to V .

The above universal property should remind one of the analogous universal property that localization satisfies, within the category of commutative rings:

$$\begin{array}{ccc}
R & \xrightarrow{f \mapsto f/1} & R[1/g] \\
& \searrow \phi & \downarrow \exists! \\
& & S
\end{array}$$

As the name suggests, rational localizations are the nonarchimedean world's analogue of algebraic localizations; any morphism of prime spectra whose image

lands in the non-vanishing locus of function g (on $\text{Spec}(R)$), necessarily factors, as a ring map, through the localization $R[1/g]$. The non-vanishing locus of a function is replaced by the set of points where certain functions $(f_1, \dots, f_n, g)$ satisfy appropriate inequalities, giving us a much richer topological space to work with.

Thus, for $V = U \left(\frac{f_1, \dots, f_n}{g} \right)$ a basic rational open subset of $X = \text{Spa}(R, R^+)$, we have defined the topological ring of functions

$$\mathcal{O}_X(V) := R \left\langle \frac{f_1, \dots, f_n}{g} \right\rangle$$

From this, we do the natural operation for arbitrary opens: for any open subset $U \subset X = \text{Spa}(R, R^+)$, we define

$$\mathcal{O}_X(U) := \varprojlim_{V \subset U} \mathcal{O}_X(V)$$

where the above limit is taken over the directed system of rational open subsets V contained inside of U .

Since each of these rings in the inverse system has a topology, we then give the limit ring the **inverse limit/Frechet topology**. This is the weakest topology on this ring for which the restriction maps will be continuous ring maps:

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{i} & \prod_{V \subset U} \mathcal{O}_X(V) \\ & \searrow \rho_{U,W} & \downarrow \pi_W \\ & & \mathcal{O}_X(W) \end{array}$$

Remark 2.3.2. *In the case that the topology on all the rational localizations is induced by a power-multiplicative norm (such a ring is called **stably uniform**), we claim that this topology on the rings of functions corresponds to the topology of uniform convergence on compact subsets. We let $(f_n) \subset \mathcal{O}_X(U)$ be a sequence of local sections over some arbitrary U , we know then by definition of the inverse limit topology, that $f_n \rightarrow f \in \mathcal{O}_X(U)$ if and only if for each basic rational open $V \subset U$:*

$$f_n|_V \xrightarrow{\|\cdot\|_V} f|_V \in \mathcal{O}_X(V)$$

*However, by power-multiplicativity, we get that the **spectral seminorm** coincides with the norm on each rational localization:*

$$\sup_{x \in V} |f(x)| =: \|f\|_{sp,V} = \lim_{n \rightarrow \infty} \|f^n\|_V^{1/n} = \|f\|_V$$

and since the rational opens form a basis of quasicompact opens for the adic spectrum, our claim follows. This should remind the reader of how spaces of holomorphic functions are topologized in complex analysis, yielding that holomorphic functions form a sheaf (analytic continuation) of complete, topological $\mathbb{C}$ -algebras.

It follows from the universal property satisfied by rational localizations combined with the universal property satisfied by inverse limits implies that the assignment

$$U \mapsto \mathcal{O}_X(U)$$

yields a presheaf of topological K -algebras (where all restriction morphisms are continuous). In general, this will not be a sheaf (more on this later).

Example 2.3.3. Consider the (K, K°) -affinoid algebra $(K[T], K^\circ[T])$; if

$$X = \mathrm{Spa}(K[T], K^\circ[T])$$

then we note that

$$\mathcal{O}_X(X) = \mathcal{O}_X(U(1/1)) = \widehat{K[T]} = K\langle T \rangle$$

and so we see, in general, we cannot expect to recover our original rings from the adic spectrum, by way of taking global sections.

What is true, in general, is that for a K -affinoid algebra (R, R^+) , we have $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(\widehat{R}, \widehat{R}^+)$ as (pre-)adic spaces. We will need this important fact later.

We briefly look at the stalk rings for this presheaf:

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U)$$

Since valuation x can be *uniquely* extended to a continuous valuation to any rational localization corresponding to a basic rational open U containing x , the following assignment at the level of germs is well-defined:

$$|(U, f)|_x := |f|_x \in \Gamma_x \cup \{0\}$$

(the notation should not be ambiguous either, due to uniqueness of extensions). The stalk rings are in fact local with maximal ideal corresponding to $\mathrm{supp}(x)$ on $\mathcal{O}_{X,x}$.

What else can we say about the underlying topological space $\mathrm{Spa}(R, R^+)$? It

turns out these spaces are in fact **spectral spaces**, i.e. are homeomorphic to the Zariski spectrum of some commutative ring. Additionally, every point of these adic spectra is the specialization (i.e. belongs in the closure) of a unique rank-1 point. The **Krull dimension** of such a space (and more generally, any locally-spectral space) is defined as the supremum of the lengths n of chains of such specializations:

$$x_0 \succ x_1 \succ \dots \succ x_n$$

where we use $x \succ y$ to denote that y is a specialization of x . For more on spectral spaces, the reader is encouraged to read [7].

Thus, associated to any affinoid (K, K^+) -algebra (R, R^+) we have a triple $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$ where

- X is a topological space corresponding to the adic spectrum of (R, R^+)
- $\mathcal{O}_X$ is a presheaf of complete topological K -algebras
- for each $x \in X$, we have a continuous valuation $|\cdot|_x : \mathcal{O}_{X,x} \longrightarrow \Gamma_x \cup \{0\}$

Such a triple will be referred to as an **affinoid pre-adic space**; when $\mathcal{O}_X$ is a sheaf, we call such an **affinoid adic space**. A morphism of affinoid (pre-)adic spaces $(f, f^\#) : (X, \mathcal{O}_X, (|\cdot|_x)_{x \in X}) \longrightarrow (Y, \mathcal{O}_Y, (|\cdot|_y)_{y \in Y})$ consists of:

- $f : X \longrightarrow Y$ a continuous map on underlying topological spaces
- $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ a *continuous* morphism of sheaves of topological K -algebras over Y , satisfying the following compatibility at the level of stalk rings:

$$\begin{array}{ccc}
\mathcal{O}_{Y,f(x)} & \xrightarrow{f_x^\#} & \mathcal{O}_{X,x} \\
& \searrow & \downarrow \\
& & \Gamma_x \cup \{0\}
\end{array}
\begin{array}{c}
| \cdot |_{f(x)} \\
| \cdot |_x
\end{array}$$

The above compatibility, of course, guarantees that the maps between stalk rings are in fact local rings (supports of valuations are our maximal ideals). In order to suppress notation, from henceforth $\mathrm{Spa}(R, R^+)$ will be used to denote such triples (just as $\mathrm{Spec}(R)$ implies both a topological space and sheaf of rings).

Now that we have constructed the affinoid objects of interest, it is time to globalize. For the sake of formality, we define the ambient category in which these global objects will live in. For a fixed nonarchimedean base field K , let $\mathcal{V}$ denote the category whose objects that are triples $(X, \mathcal{O}_X, (| \cdot |_x)_{x \in X})$, where:

- $(X, \mathcal{O}_X)$ is a locally-ringded space
- $| \cdot |_x : \mathcal{O}_{X,x} \longrightarrow \Gamma_x \cup \{0\}$ a continuous valuation, for each $x \in X$

Morphism of such triples will correspond to morphisms of locally-ringded spaces, which satisfy the above compatibility on valuations, at the level of stalk rings. In particular, affinoid adic spaces are objects in $\mathcal{V}$.

Definition 2.3.4. An *adic space* is an object in $\mathcal{V}$ that is locally an affinoid adic space.

Remark 2.3.5. If we do not demand that $\mathcal{O}_X$ be a sheaf (but merely a presheaf), then we define a pre-adic space analogously.

Chapter 3

Perfectoid Spaces

I have given a cursory overview of the geometric framework we will be working in, i.e. I have introduced affinoid adic spaces associated to an affinoid K -algebra, and globalized to adic spaces. We now specialize to a particular class of nonarchimedean fields and Banach rings, which are of particular interest to arithmetic geometers: perfectoid fields and algebras. We then introduce the main players of this work, perfectoid spaces, and describe what has been termed *the tilting equivalence*.

3.1 Perfectoid Fields

Throughout this section, let K be a nonarchimedean field of arbitrary characteristic. We also fix a pseudo-uniformizer $\varpi \in K^\times$ such that $|p| \leq |\varpi| < 1$.

Definition 3.1.1. *A nonarchimedean field K is a **perfectoid field** if*

- *value group $|K^\times|$ is **non-discrete** of rank 1*
- *residue field $\kappa := K^\circ/K^{\circ\circ}$ is of characteristic $p > 0$*

- Frobenius map $\Phi : K^\circ/p \longrightarrow K^\circ/p$ is surjective.

Example 3.1.2. Recall the field of norms isomorphism (Fontaine-Wintenberger) involving the direct limit of totally-ramified extensions of $\mathbb{Q}_p$:

$$\mathbb{Q}_p(p^{1/p^\infty}) := \varinjlim_n \mathbb{Q}_p(p^{1/p^n})$$

We can uniquely extend the p -adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ to this infinite extension, and then complete the field with respect to this extended norm; denote this completion by K . Then K is a perfectoid field.

Non-example 3.1.3. The field $\mathbb{Q}_p$ is not perfectoid. Its value group $\mathbb{Z}$ is **discrete** of rank 1.

The above conditions seem very specific; however, with these hypotheses, we can naturally associate a second field in characteristic p to each perfectoid field. The construction is the following. We may quotient out by ϖ first, to get a ring in characteristic p . We can then form the inverse system with respect to Frobenius:

$$K^\circ/\varpi \xleftarrow{x \mapsto x^p} K^\circ/\varpi \xleftarrow{x \mapsto x^p} K^\circ/\varpi \xleftarrow{x \mapsto x^p} \dots$$

We then take the inverse limit; it turns out that this ring is a perfect integral domain in characteristic p . We then define the **tilt** of K as the perfect field

$$K^b := \text{Frac} \left(\varprojlim_{x \mapsto x^p} K^\circ/\varpi \right)$$

Thus, for any perfectoid field K , we have associated to it a perfect field $K^\flat$ in characteristic p ; we record some important properties that this new field satisfies.

Proposition 3.1.4. *Let K be a perfectoid field with corresponding tilt $K^\flat$. The following are true*

- *There is a multiplicative isomorphism of abelian groups*

$$K^{\flat\times} \cong \varprojlim_{x \mapsto x^p} K^\times$$

- *$K^\flat$ is a nonarchimedean field, with the same value group as $|K^\times|$.*

We briefly describe how one puts a norm on the tilt $K^\flat$. Let $f = (f_0, f_1, \dots) \in \varprojlim_{x \mapsto x^p} K^\circ / \varpi$ be a coherent sequence, and $\tilde{f}_n \in K^\circ$ be a lift of f_n , for each n . We define the element

$$f^\sharp := \lim_{n \rightarrow \infty} \tilde{f}_n^{p^n}$$

It turns out that this limit does exist in K , and is independent of all our choices of lifts. The mapping $f \mapsto f^\sharp$ is clearly multiplicative, so we may extend this to a map from $K^\flat \longrightarrow K$. We then define the norm $|\cdot|^\flat : K^\flat \longrightarrow |K^\times| \cup \{0\}$ by the formula

$$|f|^\flat := |f^\sharp|$$

This will be multiplicative, of course; the fact that this satisfies the Strong Triangle Inequality is a consequence of Binomial Formula. The multiplicative isomorphism

from $K^{\flat \times} \longrightarrow \varprojlim_{x \mapsto x^p} K^\times$ is given by

$$f \mapsto (f^\sharp, (f^{1/p})^\sharp, (f^{1/p^2})^\sharp, \dots)$$

Example 3.1.5. *For K the completion of $\mathbb{Q}_p(p^{1/p^\infty})$, we have corresponding tilt*

$$K^\flat = \mathbb{F}_p(\widehat{(T)}(T^{1/p^\infty}))$$

where we are completing with respect to the T -adic valuation.

3.2 Tilting Equivalence

It turns out that a perfectoid field K and its tilt $K^\flat$ share much more in common than just their value groups. Perhaps one of the big results about perfectoid fields is the following.

Theorem 3.2.1 (Scholze). *Let K be a perfectoid field with corresponding tilt $K^\flat$. Then the categories of finite-étale covers of K and finite-étale covers of $K^\flat$ are equivalent.*

An immediate consequence of the above is the proof of field of norms isomorphism, mentioned in the Introduction, attributed to Fontaine and Wintenberger:

Corollary 3.2.2 (Fontaine-Wintenberger). *The absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((T))$ are isomorphic.*

Sketch of proof. We denote the absolute Galois group of a field K by G_K . From Krasner's Lemma, we necessarily have that

$$G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_K$$

for K the completion of $\mathbb{Q}_p(p^{1/p^\infty})$. Since adjoining p th-power roots of T does nothing (in terms of automorphisms) on the characteristic p side, we have that

$$G_{\mathbb{F}_p((T))} \cong G_{\mathbb{F}_p((T))(T^{1/p^\infty})} \cong G_{K^\flat}$$

for $K^\flat = \widehat{\mathbb{F}_p((T))(T^{1/p^\infty})}$ where again, we have used Krasner's Lemma to get the second isomorphism.

Since by the previous result, we have the category of finite étale covers of K is equivalent to the category of finite étale covers of $K^\flat$, then upon fixing a geometric base-point, the two fields necessarily have isomorphic étale fundamental groups. In the case of fields, we know that étale fundamental groups are precisely absolute Galois groups (where choice of a geometric base-point is equivalent to choosing an embedding of our base field into a separable closure). Thus, we arrive at the desired isomorphism:

$$G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_K \cong G_{K^\flat} \cong G_{\mathbb{F}_p((T))}$$

□

The previous theorem suggests that the notion of perfectoid is something which

should be relativized and then globalized. This, in some sense, is what is meant by providing a geometric framework to the field of norms isomorphism. We now provide the relative notion of being perfectoid.

Definition 3.2.3. Fix a perfectoid field K , and pseudo-uniformizer $\varpi \in K^\times$ such that $|p| \leq |\varpi| < 1$. A **perfectoid K -algebra** R is a Banach K -algebra, such that

- subring R° of power-bounded elements is **open** and **bounded** inside of R
- Frobenius map $\Phi : R^\circ/\varpi \rightarrow R^\circ/\varpi$ is surjective map.

The morphisms between perfectoid K -algebras are precisely continuous/bounded morphisms of K -Banach algebras.

Example 3.2.4 (Perfectoid Tate Algebra). For perfectoid field K , the ring

$$K\langle T^{1/p^\infty} \rangle := \varprojlim_i K^\circ[T^{1/p^\infty}]/\varpi^i$$

where $\varpi \in K^\times$ satisfies $0 \leq |p| \leq |\varpi| < 1$. Its ϖ -adic yields a multiplicative Gauss norm, as defined

$$\left\| \sum_{\alpha \in \mathbb{Z}[1/p]_{\geq 0}} f_\alpha T^\alpha \right\| := \sup_\alpha |f_\alpha|$$

(well-defined due to decay condition on constants). This is a perfectoid K -algebra, call it the **perfectoid Tate algebra** over K .

We now fix a base affinoid field (K, K^+) such that K is perfectoid. Such a pair will be called a **perfectoid affinoid field**. For such a pair, we can associate another affinoid field, namely its tilt $(K^\flat, K^{\flat+})$.

Definition 3.2.5. A *perfectoid affinoid (K, K^+) -algebra* is an affinoid (K, K^+) -algebra (R, R^+) such that R is a perfectoid K -algebra.

The big result is that the tilting equivalence which we observe at the level of fields, does relativize.

Theorem 3.2.6 (Scholze). *The category of perfectoid (K, K^+) -algebra is equivalent to the category of $(K^\flat, K^{\flat+})$ -algebras*

Proof. Refer to [12] Theorem 5.2.

Thus, to each perfectoid affinoid K -algebra (R, R^+) , we can associate to it its tilt $(R^\flat, R^{\flat+})$, which is a $K^\flat$ -affinoid algebra.

Example 3.2.7. For perfectoid Tate algebra $R = K\langle T^{1/p^\infty} \rangle$ over K , let $R^+ = R^\circ = K^\circ\langle T^{1/p^\infty} \rangle$ be the open ring of functions with Gauss norm less than/equal to 1. It follows that the pair (R, R^+) will be a perfectoid affinoid (K, K°) -algebra.

As we discussed back in the previous chapter, such affinoid algebras give rise to affinoid pre-adic spaces, sitting in the category $\mathcal{V}$. That is, for (R, R^+) we have $X = \mathrm{Spa}(R, R^+)$, which has a structure presheaf $\mathcal{O}_X$.

It is of independent interest to find conditions on the ring R that will give a structure sheaf, and there are a few sufficient conditions (e.g. being strongly Noetherian, stably uniform) that guarantee that the sheaf property holds. One such condition is the following hard result.

Theorem 3.2.8 (Tate’s Acyclicity Theorem). *Let $X = \mathrm{Spa}(R, R^+)$, for (R, R^+) an affinoid (K, K^+) -algebra, where R is a **perfectoid K -algebra**. Let $\{U_i\}_{i=1}^n$ be an open cover of X by finitely many rational subsets. Then the augmented Čech-complex*

$$0 \rightarrow \mathcal{O}_X(X) \rightarrow \prod_i \mathcal{O}_X(U_i) \rightarrow \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \rightarrow \dots$$

is exact. In particular, $\mathcal{O}_X$ is a sheaf.

Since each perfectoid affinoid K -algebra (R, R^+) in characteristic 0 will give rise to a corresponding perfectoid affinoid $K^\flat$ -algebra $(R^\flat, R^{\flat+})$ in characteristic p , we have two adic spectra to look at, one in characteristic 0 and the other (corresponding to tilted rings) in characteristic p . The big result which we have to relate the two affinoid objects is the following.

Theorem 3.2.9 (Scholze). *For fixed perfectoid field K in characteristic 0, let $X = \mathrm{Spa}(R, R^+)$ for (R, R^+) a perfectoid affinoid K -algebra, and $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$ the corresponding $K^\flat$ -affinoid space.*

- *There is a homeomorphism of underlying topological spaces $|X| \cong |X^\flat|$, naturally identifying rational subsets of both spaces.*
- *For rational subset $U \subset X$ with corresponding tilt $U^\flat \subset X^\flat$ (under above homeomorphism), $\mathcal{O}_X(U)$ is a perfectoid K -algebra satisfying:*

$$\mathcal{O}_X(U)^\flat = \mathcal{O}_{X^\flat}(U^\flat)$$

- *The corresponding presheaf on $X^\flat$ is a sheaf.*

Proof. Refer to [12], Theorem 6.3.

The second statement is telling us two things. First, the property of being a perfectoid algebra is stable under formation of rational localizations. Second, the rings we obtain in characteristic p after tilting are the ones we should hope to get: the tilt of a rational localization is the (expected) rational localization of the tilt. Thus, tilting and formation of rational localization commute.

Because of the homeomorphism $X \cong X^\flat$ which naturally identifies rational subsets, if we wish, we can think of there being one underlying topological space equipped with sheaves of rings in both characteristic 0 and characteristic p . It should be emphasized that the above is only a homeomorphism of *topological spaces*, not an isomorphism of locally-ringed spaces. One of the sheaves is rings of characteristic 0, the other sheaf is of rings of characteristic $p > 0$.

It turns out that the process of tilting glues, i.e. we can glue affinoid perfectoid spaces on the characteristic 0 side along open sets, and we get a corresponding gluing of affinoid objects along open sets on the characteristic p side. Thus, it makes sense to globalize.

Definition 3.2.10. *A **perfectoid space** is an adic space over perfectoid affinoid field (K, K^+) that is locally isomorphic in $\mathcal{V}$ to an affinoid perfectoid space. Morphisms between perfectoid spaces correspond to morphisms between adic spaces over (K, K^+) .*

Theorem 3.2.11. *The category of perfectoid spaces over K and perfectoid spaces over $K^\flat$ are equivalent.*

3.3 Some Examples of Perfectoid Spaces

We now present a couple of the basic, but very important, examples of perfectoid spaces, for the reader's benefit.

Example 3.3.1 (Perfectoid Disc). *We mentioned the perfectoid affinoid (K, K°) -algebra $(K\langle T^{1/p^\infty} \rangle, K^\circ\langle T^{1/p^\infty} \rangle)$ in Example 3.2.4. The corresponding perfectoid space is known as the perfectoid disc*

$$\mathbb{B}_K^{1,ad,perf} := Spa(K\langle T^{1/p^\infty} \rangle, K^\circ\langle T^{1/p^\infty} \rangle)$$

Since we know that the corresponding tilted rings are:

$$(K^\flat\langle T^{1/p^\infty} \rangle, (K^\flat)^\circ\langle T^{1/p^\infty} \rangle)$$

it follows that

$$(\mathbb{B}_K^{1,ad,perf})^\flat := Spa(K\langle T^{1/p^\infty} \rangle, K^\circ\langle T^{1/p^\infty} \rangle) = \mathbb{B}_{K^\flat}^{1,ad,perf}$$

Example 3.3.2 (Projectivoid Line). *Analogous to how the normal projective line in algebraic geometry can be constructed by gluing appropriately two copies of the affine line $\mathbb{A}_K^1$ along the punctured plane $\mathbb{A}_K^1$, we can construct the projectivoid line $\mathbb{P}_K^{1,ad,perf}$ by gluing two copies of the above perfectoid disc along the perfectoid (open) unit circle:*

$$\mathbb{T}_K^{1,ad,perf} := Spa(K\langle T^{\pm 1/p^\infty} \rangle, K^\circ\langle T^{\pm 1/p^\infty} \rangle)$$

The continuous ring maps which yield the gluing are the obvious inclusions (mirroring the construction in algebraic geometry, as mentioned):

$$\begin{array}{ccc}
 & K\langle T^{\pm 1/p^\infty} \rangle & \\
 T \mapsto T & \nearrow & \nwarrow T \mapsto T^{-1} \\
 K\langle T^{1/p^\infty} \rangle & & K\langle T^{1/p^\infty} \rangle
 \end{array}$$

Since we understand tilting on affinoid pieces of this space, we have that

$$(\mathbb{P}_K^{1,ad,perf})^\flat \cong \mathbb{P}_{K^\flat}^{1,ad,perf}$$

Chapter 4

The Picard Group of the Projectivoid Line

We have covered the appropriate background on Huber's adic spaces, and Scholze's perfectoid fields/algebras, and introduced the geometric objects of interest: perfectoid spaces. Just as well, we have introduced some very basic (but very important) examples of perfectoid spaces, the perfectoid analogues of the unit disc and projective line (referred to as the *projectivoid line*). In this chapter, we look to classify all isomorphism classes of line bundles on the projectivoid line, by way of computing its Picard group. As we will see, the computation relies heavily on Čech theory and hence is very explicit; however, the desire to generalize this result leads to some very natural and interesting questions, which are asked and investigated in later chapters.

4.1 Principal Units and Nonarchimedean Logarithm

Let R be a nonarchimedean Banach ring of characteristic 0, with *multiplicative* norm $\|\cdot\| : R \longrightarrow \mathbb{R}_{\geq 0}$, such that $\|p\| < 1$ for a fixed prime number p . Within its multiplicative group of units is the subset

$$U^{(1)}(R) := \{f \in R^\times : \|f - 1\| < 1\}$$

We will call this the **group of principal units** of R . We have that the roots of unity of R

$$\mu(R) := \{f \in R : f^n = 1, \text{ some } n\}$$

are contained inside of $U^{(1)}(R)$. We can define the nonarchimedean logarithm (working in characteristic 0) on $U^{(1)}(R)$, via the usual series formula:

$$\log(f) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(f-1)^n}{n}$$

For an appropriate $\varepsilon > 0$, we get a surjective group homomorphism

$$\log : U^{(1)}(R) \longrightarrow B_{\|\cdot\|}(0, \varepsilon) := \{f \in R : \|f\| < \varepsilon\}$$

What is its kernel? Since the norm on R is multiplicative, we have that R is necessarily an integral domain. Now we know that if f is a p -power root of unity, i.e. $f^{p^n} = 1$, then

$$0 = \log(1) = \log(f^{p^n}) = p^n \log(f) \in R$$

Since R is of characteristic 0 and a domain, it follows that $\log(f) = 0$. Conversely, let f belong in the kernel of $\log$. We are interested in what radius will guarantee that

$$\|\log(f)\| = \|f - 1\|$$

By the nonarchimedean property of $\|\cdot\|$, we are guaranteed this if

$$\|f - 1\| > \frac{\|(f - 1)^n\|}{\|n\|}$$

that is, if all higher terms for the series expansion of $\log(f)$ have norm strictly less than that of the first term $f - 1$. If we let $n = yp^x + z$, where p does not divide y and $0 \leq z < p$, this becomes:

$$\frac{\|(f - 1)^n\|}{\|n\|} = \frac{\|f - 1\|^{yp^x + z}}{\|p\|^x} < \|f - 1\|$$

and after doing the algebra,

$$\|f - 1\| < \|p\|^{\frac{x}{yp^x + z - 1}}$$

Since $\|p\| < 1$, we are interested in finding the maximum of the function $f(x, y, z) = x/yp^x + z - 1$, to ensure that this inequality holds uniformly for all such x, y, z ; We are guaranteed this for $x = 1, y = 1, z = 0$, and so when $\|f - 1\| < \|p\|^{1/p-1}$, we know that $\|\log(f)\| = \|f - 1\|$.

Since $\|f - 1\| < 1$, then $\|(f - 1)^{p^k}\| = \|f - 1\|^{p^k} < \|p\|^{1/p-1}$ for k sufficiently

large. For such k , the Binomial Formula yields

$$\begin{aligned} \|f^{p^k} - 1\| &= \left\| (f - 1)^{p^k} - \sum_{i=1}^{p^k-1} \binom{p^k}{i} (-1)^i f^{p^k-i} \right\| \\ &\leq \max_{1 \leq i \leq p^k-1} \left(\|f - 1\|^{p^k}, \left\| \binom{p^k}{i} \right\| \right) \\ &< \|p\|^{1/p-1} \end{aligned}$$

and so we have that

$$\|\log(f^{p^k})\| = \left\| \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (f^{p^k} - 1)^n}{n} \right\| = \|f^{p^k} - 1\|$$

Using that f belongs in the kernel of $\log$, it follows that

$$f^{p^k} - 1 = 0$$

that is, f is a p -power root of unity. We denote the group of p -power roots of unity in R by

$$\mu_{p^\infty}(R) := \{f \in R : f^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

We therefore have the following short exact sequence of groups associated to R :

$$1 \longrightarrow \mu_{p^\infty}(R) \longrightarrow U^{(1)}(R) \xrightarrow{\log(\cdot)} B_{\|\cdot\|}(0, \varepsilon) \longrightarrow 1$$

4.2 Reductions and Roots of Unity

I earlier defined the Tate K -algebra $K\langle T \rangle$ from rigid-analytic geometry. Within this ring, we have the subring of power-bounded elements:

$$K^\circ\langle T \rangle := \{f \in K\langle T \rangle : \|f\| \leq 1\}$$

which contains the ideal of topologically nilpotent elements

$$K^{\circ\circ}\langle T \rangle := \{f \in K^\circ\langle T \rangle : \|f\| < 1\}$$

If we let $\kappa = K^\circ/K^{\circ\circ}$ denote the residue field of K , then we have a short exact sequence of abelian groups:

$$1 \longrightarrow K^{\circ\circ}\langle T \rangle \longrightarrow K^\circ\langle T \rangle \longrightarrow \kappa[T] \longrightarrow 1$$

The ring $\kappa[T]$ is referred to as the **reduction** of $K\langle T \rangle$. If we only consider map $K\langle T \rangle \longrightarrow \kappa[T]$ as a multiplicative map on the group of units of the two rings, we get the following short exact sequence of abelian groups:

$$1 \longrightarrow U^{(1)}(K\langle T \rangle) \longrightarrow K^\circ\langle T \rangle^\times \longrightarrow \kappa[T]^\times \longrightarrow 1$$

If we work with the ring of convergent Laurent series:

$$K\langle T^{\pm 1} \rangle := \left\{ \sum_{n \in \mathbb{Z}} f_n T^n : |f_n| \longrightarrow 0, \text{ as } n \longrightarrow \pm\infty \right\}$$

we obtain a similar short exact sequence of abelian groups:

$$1 \longrightarrow U^{(1)}(K\langle T^{\pm 1} \rangle) \longrightarrow K^{\circ}\langle T^{\pm 1} \rangle^{\times} \longrightarrow \kappa[T^{\pm 1}]^{\times} \longrightarrow 1$$

We get the analogous short exact sequence when we adjoin all p -power roots of the variable T .

We quickly verify that the only roots of unity in $K\langle T \rangle$ (and similarly $K\langle T^{\pm 1} \rangle$) are those that lie in the base field K . Suppose $f^n = 1$. We have the natural inclusion of

$$\text{Frac}(K\langle T \rangle) \hookrightarrow \text{Frac}(\overline{K}\langle T \rangle)$$

for some fixed algebraic closure $\overline{K}$. We write $f = \sum_{i=0}^{\infty} f_i T^i$, for $f_i \in K$. Since $f^n = 1$, this forces $f \in \overline{K}$, i.e. $f_i = 0$ for $i = 1, 2, \dots$. Now since f had coefficients in K , it follows that $f_0 \in K$, and so $\mu(K\langle T \rangle) = \mu(K)$.

This same argument works when considering $K\langle T, T^{-1} \rangle$, and also when adjoining all p -power roots of variable T , to obtain the perfectoid Tate algebra. It simply uses the fact that these rings are integral domains, and so everything becomes an inclusion of fields.

4.3 Some Calculations

Recall that the projectivoid line $X = \mathbb{P}_K^{1, \text{ad}, \text{perf}}$ over a perfectoid field K was constructed via gluing two copies of the perfectoid unit disc $U = V = \mathbb{B}_K^{1, \text{ad}, \text{perf}}$ over K along the perfectoid unit circle $U \cap V = \mathbb{T}_K^{1, \text{ad}, \text{perf}}$. Letting $\mathfrak{U} = \{U, V\}$ denote this

cover, we obtain the following Čech complex for the sheaf of units $\mathcal{O}_X^\times$:

$$1 \longrightarrow K^\circ \langle T_{p^\infty}^{\frac{1}{p^\infty}} \rangle^\times \oplus K^\circ \langle T_{p^\infty}^{\frac{-1}{p^\infty}} \rangle \longrightarrow K^\circ \langle T_{p^\infty}^{\pm \frac{1}{p^\infty}} \rangle^\times \longrightarrow 1$$

Using information from the previous two sections, we obtain the following commutative diagram of abelian groups:

$$\begin{array}{ccccc} K^\circ \langle T_{p^\infty}^{\frac{1}{p^\infty}} \rangle^\times \oplus K^\circ \langle T_{p^\infty}^{\frac{-1}{p^\infty}} \rangle^\times & \longrightarrow & K \langle T_{p^\infty}^{\frac{1}{p^\infty}} \rangle^\times \oplus K \langle T_{p^\infty}^{\frac{-1}{p^\infty}} \rangle^\times & \xrightarrow{\|\cdot\|} & |K^\times| \oplus |K^\times| \\ (f,g) \mapsto f/g \downarrow & & (f,g) \mapsto f/g \downarrow & & (f,g) \mapsto f/g \downarrow \\ K^\circ \langle T_{p^\infty}^{\pm \frac{1}{p^\infty}} \rangle^\times & \longrightarrow & K \langle T_{p^\infty}^{\pm \frac{1}{p^\infty}} \rangle^\times & \xrightarrow{\|\cdot\|} & |K^\times| \end{array}$$

where the two rows are short exact. The Snake Lemma yields the following exact sequence:

$$1 \longrightarrow (K^\circ)^\times \longrightarrow K^\times \xrightarrow{|\cdot|} |K^\times| \longrightarrow \check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\circ \times}) \longrightarrow \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \longrightarrow 1$$

It is clear that the map $K^\times \xrightarrow{|\cdot|} |K^\times|$ is surjective, and so

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\circ \times}) \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$$

We have reduced our task to understanding the sheaf of units for the power-bounded elements. As we saw in the previous section, we have the following commutative

diagram in which both rows are exact:

$$\begin{array}{ccccc}
U^{(1)}(K\langle T^{\frac{1}{p^\infty}} \rangle) \oplus U^{(1)}(K\langle T^{\frac{-1}{p^\infty}} \rangle) & \longrightarrow & K^\circ\langle T^{\frac{1}{p^\infty}} \rangle^\times \oplus K^\circ\langle T^{\frac{-1}{p^\infty}} \rangle^\times & \longrightarrow & \kappa[T^{\frac{1}{p^\infty}}]^\times \oplus \kappa[T^{\frac{-1}{p^\infty}}]^\times \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
U^{(1)}(K\langle T^{\pm \frac{1}{p^\infty}} \rangle) & \longrightarrow & K^\circ\langle T^{\pm \frac{1}{p^\infty}} \rangle^\times & \longrightarrow & \kappa[T^{\pm \frac{1}{p^\infty}}]^\times
\end{array}$$

where the rows are short exact (I have omitted the 1's due to how large the diagram becomes). Again applying the Snake Lemma, we have the following exact sequence of abelian groups:

$$\begin{array}{ccccccc}
1 & \longrightarrow & U^{(1)}(K) & \longrightarrow & (K^\circ)^\times & \longrightarrow & \kappa^\times \\
& & & & \delta & & \\
\downarrow & & U^{(1)}(K\langle T^{\pm 1/p^\infty} \rangle)/\text{image}(\alpha) & \longrightarrow & \check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\circ \times}) & \longrightarrow & \kappa[T^{\pm 1/p^\infty}]^\times/\text{image}(\gamma) \longrightarrow 1
\end{array}$$

Now note that

$$\kappa[T^{\pm 1/p^\infty}]^\times := \{uT^k : u \in \kappa^\times, k \in \mathbb{Z}[1/p]\} \cong \kappa^\times \oplus \mathbb{Z}[1/p]$$

and the map $\gamma : \kappa[T^{1/p^\infty}]^\times \oplus \kappa[T^{-1/p^\infty}]^\times \longrightarrow \kappa[T^{\pm 1/p^\infty}]^\times$, given by

$$\gamma(u_1, u_2) = \frac{u_1}{u_2}$$

naturally maps onto the first direct summand $\kappa^\times$. Thus, we have that

$$\frac{\kappa[T^{\pm 1/p^\infty}]^\times}{\text{image}(\gamma)} \cong \mathbb{Z}[1/p]$$

We now claim that

$$\frac{U^{(1)}(K\langle T^{\pm 1/p^\infty} \rangle)}{\text{image}(\alpha)} = 1$$

In characteristic 0, this is an easy consequence of the following diagram:

$$\begin{array}{ccccc}
\mu_{p^\infty}(K) \oplus \mu_{p^\infty}(K) & \longrightarrow & U^{(1)}(K\langle T^{\frac{1}{p^\infty}} \rangle) \oplus U^{(1)}(K\langle T^{\frac{-1}{p^\infty}} \rangle) & \xrightarrow{\log(\cdot)} & K^{(\varepsilon)}\langle T^{\frac{1}{p^\infty}} \rangle \oplus K^{(\varepsilon)}\langle T^{\frac{-1}{p^\infty}} \rangle \\
\downarrow & & \downarrow \alpha & & \downarrow \\
\mu_{p^\infty}(K) & \longrightarrow & U^{(1)}(K\langle T^{\pm \frac{1}{p^\infty}} \rangle) & \xrightarrow{\log(\cdot)} & K^{(\varepsilon)}\langle T^{\pm \frac{1}{p^\infty}} \rangle
\end{array}$$

where $K^{(\varepsilon)}\langle T \rangle := \{f \in K\langle T^{1/p^\infty} \rangle : \|f\| < \varepsilon\}$ and similarly for $K^{(\varepsilon)}\langle T^{\pm 1/p^\infty} \rangle$ (here ε is chosen appropriately small to ensure a surjection). Again, the two rows of this diagram are short exact. the Snake Lemma then gives us,

$$1 \longrightarrow \mu_{p^\infty}(K) \longrightarrow U^{(1)}(K) \longrightarrow K^{(\varepsilon)} \xrightarrow{\delta} 1 \longrightarrow \frac{U^{(1)}(K\langle T^{\pm 1/p^\infty} \rangle)}{\text{image}(\alpha)} \longrightarrow 1$$

and so exactness gives the triviality of the group we desire. Plugging this into the second long exact sequence,

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\circ \times}) \cong \frac{\kappa[T^{\pm 1/p^\infty}]^\times}{\text{image}(\gamma)} \cong \mathbb{Z}[1/p]$$

More generally for any characteristic, the surjectivity of

$$\alpha : U^{(1)}(K\langle T^{1/p^\infty} \rangle) \oplus U^{(1)}(K\langle T^{-1/p^\infty} \rangle) \longrightarrow U^{(1)}(K\langle T^{\pm 1/p^\infty} \rangle)$$

will follow from repeated application of the following master factorization result.

Lemma 4.3.1 (Christoll). *Let R be a ring that is complete for a nonarchimedean norm, Suppose the nonzero elements $a, b, c \in R$ and additive subgroups $U, V, W \subset R$ satisfy the following conditions.*

- U, V are complete under the norm, and $UV \subset W$

- The map $f(u, v) = av + ub$ is a surjection of $U \oplus V$ onto W
- There exists $\lambda > 0$ such that $f(u, v) \geq \lambda \cdot \max(|a||v|, |b||u|)$, for all pairs $(u, v) \in U \oplus V$
- $ab - c \in W$ and $|ab - c| < \lambda^2|c|$

Then there exists a pair $(x, y) \in U \oplus V$ such that $c = (a + x)(b + y)$

Proof. Refer to [9], Theorem 2.2.1.

Putting everything we have obtained thus far together:

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\circ\times}) \cong \mathbb{Z}[1/p]$$

Our goal now is to show that we can use the cover $\mathfrak{U}$ to compute derived functor cohomology for the sheaf of units $\mathcal{O}_X^\times$, i.e. that $H^1(X, \mathcal{O}_X^\times) \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. For this, we will appeal to the following:

Theorem 4.3.2 (Leray). *Let X be a topological space, with covering $\mathfrak{U}$ by open sets, and $\mathcal{F}$ a sheaf of abelian groups over X . If $\mathcal{F}$ restricted to any finite intersection of opens inside of $\mathfrak{U}$ is acyclic, then we have for all $i \geq 0$,*

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F})$$

If we can show that the hypotheses for Leray's Theorem hold for our cover $\mathfrak{U} = \{U, V\}$, we will have established the following isomorphisms:

$$\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times) \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \cong \mathbb{Z}[1/p]$$

The next two sections are dedicated precisely to establishing that $\mathfrak{U}$ is a Leray cover for the sheaf of units $\mathcal{O}_X^\times$.

4.4 Vanishing in Degree 1: Weierstrass Division, Bezout Domains

We record a Weierstrass Division result for perfectoid Tate algebras (and corresponding perfectoid Laurent series rings). It is essentially a copy of the proof for standard Tate algebras, which one can see in [5].

Definition 4.4.1. *An element $f \in K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty} \rangle$ is regular in the variable T_n of degree $d \geq 0$ if $\|f\| = 1$ and its reduction $\bar{f} \in \kappa[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$ is of the form*

$$\bar{f} = \lambda T_n^d + g(T_1, \dots, T_n)$$

for $\lambda \in \kappa^\times$, and $g \in \kappa[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$ satisfying $\deg_{T_n}(g) < d$.

Proposition 4.4.2 (Weierstrass Division). *Let f be regular in T_n of degree d . Then for any g , there exists unique $q, r \in K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty} \rangle$ such that*

$$g = q \cdot f + r$$

and $\deg_{T_n}(r) < d$. Moreover, $\|g\| = \max(\|q\|, \|r\|)$.

Proof. We first verify the claim that $\|g\| = \max(\|q\|, \|f\|)$, we always have that

$$\|g\| \leq \max(\|q\|\|f\|, \|r\|) = \max(\|q\|, \|r\|)$$

with equality in the case that $\|q\| \neq \|r\|$. Now suppose that $\|g\| < \max(\|q\|, \|r\|)$, so that necessarily $\|q\| = \|r\| > 0$. We choose a constant $\alpha \in K^\times$ such that $|\alpha| = \|q\|^{-1}$, and scale through to obtain $\alpha g = (\alpha q)f + \alpha r$. Upon passing to the reduction $\kappa[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$, we observe that

$$\bar{0} = \overline{\alpha q} \bar{f} + \overline{\alpha r}$$

which is a contradiction, since αq is nonzero as a polynomial, the degree in T_n of f is d , but degree in T_n of αr is less than d . It follows then that we have

$$\|g\| = \max(\|q\|, \|r\|)$$

Now for division. We first treat the case that

$$f = \sum_{i=0}^d f_i T_n^i \in K\langle T_1^{1/p^\infty}, \dots, T_{n-1}^{1/p^\infty} \rangle [T_n^{1/p^\infty}]$$

We express arbitrary element g as

$$g = \sum_i g_i T_n^i \quad g_i \in K\langle T_1^{1/p^\infty}, \dots, T_{n-1}^{1/p^\infty} \rangle$$

then we have that $T_n^i, f \in K\langle T_1^{1/p^\infty}, \dots, T_{n-1}^{1/p^\infty} \rangle [T_n^{1/p^{N(i)}}]$ for some $N(i) \geq 0$. Since the leading coefficient of f is a unit, we may apply the ordinary division algorithm for

each i in order to obtain unique elements $q_i, r_i \in K\langle T_1^{1/p^\infty}, \dots, T_{n-1}^{1/p^\infty} \rangle [T_n^{1/p^{N(i)}}]$ such that

$$T_n^i = q_i f + r_i$$

and $\deg_{T_n}(r_i) < d$. Consider the two infinite series $q = \sum_i g_i q_i$, $r = \sum_i g_i r_i$; the fact that these actually converge is a consequence of

$$1 = \|T_n^i\| = \max(\|q_i\|, \|r_i\|)$$

and so $\|g_i q_i\|, \|g_i r_i\| \leq \|g_i\| \longrightarrow 0$.

Now for general f regular in T_n , we may express $f = f_0 - D$, where f_0 is a polynomial in T_n of degree d (as in the previous case) and $\|D\| < 1$. We may then express $g = q_0 f_0 + r_0$, hence

$$g = q_0(f + D) + r_0 = q_0 f + r_0 + q_0 D$$

We set $g_1 = q_0 D$, so that $\|g_1\| = \|q_0\| \|D\| \leq \|g\| \|D\|$. Applying the division algorithm to g_1, f_0 , we obtain

$$g_1 = q_1 f_0 + r_1 = q_1(f + D) + r_1$$

By what we showed about norms,

$$\|q_1\|, \|r_1\| \leq \|g_1\| \leq \|g\| \|D\|$$

Setting $g_2 = q_1 D$, we obtain $\|g_2\| = \|q_1\| \|D\| \leq \|g_1\| \|D\| \leq \|g\| \|D\|^2$. Again, we have

$$\|q_2\|, \|r_2\| \leq \|g_2\| \leq \|g\| \|D\|^2$$

In general, if we proceed in this fashion

$$g_{i+1} = q_i D$$

and continue to iterate the division algorithm, we obtain elements q_{i+1}, r_{i+1} which satisfy

$$\|q_{i+1}\|, \|r_{i+1}\| \leq \|g\| \|D\|^{i+1}$$

We form the two convergent series

$$q = \sum_{i=0}^{\infty} q_i \quad r = \sum_{i=0}^{\infty} r_i$$

These will satisfy $g = qf + r$.

We verify uniqueness of the elements q and r . Let q', r' denote another pair of such elements, with $\deg_{T_n}(r') < d$ such that

$$g = qf + r = q'f + r'$$

By assumption, $q - q', r - r' \neq 0$; after scaling by a constant, we may assume that $\|q - q'\| = 1$, hence $\|r - r'\| = 1$ (since $\|f\| = 1$). Passing to the reduction, we obtain that $\overline{r - r'} = \overline{q - q'} \cdot \overline{f} \neq 0$. However, $\deg_{T_n}(\overline{f}) = d$, and $\deg_{T_n}(\overline{r - r'}) < d$, which is a contradiction. We thus have uniqueness. $\square$

Remark 4.4.3. *The proof of the above Weierstrass Division Theorem follows through for the Laurent ring $K\langle T^{\pm 1/p^\infty} \rangle$; this is a consequence of the fact that the localization of a Euclidean domain is again a Euclidean domain, i.e. $\kappa[T^{\pm 1}]$ is a Euclidean domain.*

Note, we also could have proven the appropriate Weierstrass Preparation statement, but it really was not that important to the author.

Corollary 4.4.4. *Any element $f \in K\langle T^{1/p^\infty} \rangle$ can be uniquely expressed as*

$$f = u \cdot g$$

where $g \in K[T^{1/p^\infty}]$ and u is a unit inside of $K\langle T^{1/p^\infty} \rangle$.

Proof. Let $f \in K\langle T^{1/p^\infty} \rangle$. After scaling by a suitable constant in the base field K , we may assume that $\|f\| = 1$. We are in the univariate case, so f is already regular in T of some degree d . We apply the division algorithm for T^d and f :

$$T^d = qf + r$$

where $\deg_T(r) < d$. It follows then that $T^d - r = qf$ is also regular in T of degree d , and so application of the division algorithm once more yields

$$f = u(T^d - r) + s$$

with $\deg_T(s) < d$. Combining the two, we have

$$f = u(T^d - r) + s = u(qf) + s = (uq)f + s$$

However, due to the uniqueness of the division, this implies that $s = 0$, and $uq = 1$.

In particular, we have $u \in K\langle T^{1/p^\infty} \rangle^\times$. We have established

$$f = u \cdot g$$

where u is a unit and $g = T^d - r \in K[T^{1/p^\infty}]$. □

Definition 4.4.5. *A ring R is called a Bezout domain if any finitely-generated ideal is principally-generated.*

Corollary 4.4.6. *The rings $K\langle T^{1/p^\infty} \rangle, K\langle T^{\pm 1/p^\infty} \rangle$ are Bezout domains.*

Proof. It suffices to prove that any ideal generated by two elements is in fact principal. Let $I = (f_1, f_2) \subset K\langle T^{1/p^\infty} \rangle$. By Corollary 4.4.4, $f_i = u_i g_i$, where u_i are units, and $g_i \in K[T^{1/p^\infty}]$, for $i = 1, 2$. Thus, we have $I = (f_1, f_2) = (g_1, g_2)$. We know that $g_1, g_2 \in K[T^{1/p^N}]$ for N sufficiently large. Let $J = (g_1, g_2) \subset K[T^{1/p^N}]$; since $K[T^{1/p^N}]$ is a PID, we have that J is principal, and therefore the extension to the ambient ring

$$I = J \cdot K\langle T^{1/p^\infty} \rangle$$

is also principal.

Since the proof of the Weierstrass Division Theorem holds for the Laurent ring $K\langle T^{\pm 1/p^\infty} \rangle$, so does Corollary 4.4.4 holds for this ring. Using the fact that the localization of a PID is a PID (so $K[T^{\pm 1}]$ is a PID) allows us to mimic the above argument and obtain the Bezout property for this ring as well. □

We will need the following two results in order to establish the vanishing of first cohomology groups on the perfectoid disc and perfectoid circle.

Lemma 4.4.7. *Every finite projective module over a Bezout domain is a free module.*

Proof. Refer to [4] for results on Bezout domains.

Additionally, we have available to us the following adic version of the famous Serre-Swan Theorem.

Theorem 4.4.8. *For $X = \mathrm{Spa}(R, R^+)$ an affinoid adic space, such that R is a Tate ring, the category of finite projective R -modules is equivalent to the category of finite, locally-free sheaves of $\mathcal{O}_X$ -modules.*

Proof. Refer to [10] Theorem 8.2.22.

Combining these two with the fact that $K\langle T^{1/p^\infty} \rangle$ and $K\langle T^{\pm 1/p^\infty} \rangle$ are Bezout domains, we have then that

- $H^1(\mathbb{B}_K^{1,\mathrm{ad},\mathrm{perf}}, \mathcal{O}^\times) \cong \mathrm{Pic}(\mathbb{B}_K^{1,\mathrm{ad},\mathrm{perf}}) = 1$
- $H^1(\mathbb{T}_K^{1,\mathrm{ad},\mathrm{perf}}, \mathcal{O}^\times) \cong \mathrm{Pic}(\mathbb{T}_K^{1,\mathrm{ad},\mathrm{perf}}) = 1$

and so we have vanishing of the first cohomology groups of $\mathcal{O}^\times$ over the two affinoid opens in $\mathfrak{U}$ and their intersection.

4.5 Vanishing in Degrees 2 and Up: Completed Perfections

We have used the Bezout property of the univariate perfectoid Tate algebra (and its corresponding Laurent series analogue) in order to establish the vanishing

of first cohomology groups for the sheaf of units. We now need to show vanishing of all higher cohomology groups. Recall from Section 2.3, that affinoid adic spaces are spectral spaces (i.e. their underlying topological spaces are homeomorphic to Spec of some commutative ring), whose Krull dimensions can be determined by computing the supremum of chains of specializations of points. The following result is available for us to apply to $\mathbb{B}_K^{1,\mathrm{ad},\mathrm{perf}}$ and $\mathbb{T}_K^{1,\mathrm{ad},\mathrm{perf}}$.

Proposition 4.5.1. *Let X be a spectral space of Krull dimension d . Given an abelian sheaf $\mathcal{F}$ over X , then for all $q > d$,*

$$H^q(X, \mathcal{F}) = 0$$

Proof. Refer to [11] Tag 0A3G.

For the rest of this section, unless otherwise stated, all rings will be of characteristic $p > 0$. We record some basic properties about perfect closures. Let R be a ring in characteristic $p > 0$. R is, of course, equipped with a Frobenius homomorphism,

$$x \mapsto x^p$$

Associated to R is its *perfect closure*, denoted R^{perf} , which satisfies the following universal mapping problem:

Theorem 4.5.2. *Let R be a ring in characteristic $p > 0$. There is a pair $(R^{\mathrm{perf}}, \phi)$, where R^{perf} is a perfect ring of characteristic p , and $\phi : R \rightarrow R^{\mathrm{perf}}$ is a ring map such that the following is satisfied: For any other such pair (S, ψ) , there exists a unique map $\psi' : R^{\mathrm{perf}} \rightarrow S$ such that the following diagram commutes*

$$\begin{array}{ccc}
 R & \xrightarrow{\phi} & R^{perf} \\
 & \searrow \psi & \downarrow \psi' \\
 & & S
 \end{array}$$

Moreover, the pair (R^{perf}, ϕ) is unique up to unique isomorphism.

Proof. Refer to [6] Section 1.

Theorem 4.5.3. *For the morphism $\phi : R \longrightarrow R^{perf}$, the kernel is precisely the set of p -power nilpotent elements in R*

$$R_{p^\infty} := \{f \in R : f^{p^n} = 0 \text{ for some } n\}$$

Just as well, for each $f \in R^{perf}$, $f^{p^n} \in \phi(R)$ for n sufficiently large.

Proof. Refer to [6] Section 5.

Remark 4.5.4. *One can argue via the universal property satisfied by perfect closures and the description of the kernel of $\phi : R \longrightarrow R^{perf}$ that any injection of rings induces an injection of perfect closures. I mention this so that there is no ambiguity as to how an inclusion of rings induces an inclusion of perfect closures.*

We let R be a ring equipped with a nonarchimedean norm $\|\cdot\| : R \longrightarrow \mathbb{R}_{\geq 0}$ that satisfies the usual properties:

- $\|1\| = 1$
- **Positive-definite:** $\|f\| = 0 \iff f = 0$

- **Sub-multiplicativity:** $\|fg\| \leq \|f\|\|g\|$
- **Strong Triangle Inequality:** $\|f + g\| \leq \max(\|f\|, \|g\|)$

In addition, we demand that the norm be *power-multiplicative*, i.e. $\|f^n\| = \|f\|^n$, for all $f \in R$, $n \in \mathbb{N}$. Note that this forces R to be a reduced ring, and hence R injects into its perfect closure R^{perf} .

We wish to extend the norm on R to one on R^{perf} . Let $f \in R^{\text{perf}}$, so that $f^{p^n} \in R$ for n sufficiently large. Since the norm on R is power-multiplicative, it is natural to simply have $\|\cdot\|' : R^{\text{perf}} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$\|f\|' := \|f^{p^n}\|^{1/p^n}$$

where n is again sufficiently large enough to land f^{p^n} in R . This is independent of the power n , and so is a well-defined extension of the norm on R . We have that the natural map $\phi : R \rightarrow R^{\text{perf}}$ is bounded/continuous; this is clear from the definition of the extended norm.

We record some basic results that will be useful later when discussing adic spaces.

Lemma 4.5.5. *If S is an integrally-closed subring of R , then S^{perf} is an integrally-closed subring of R^{perf} .*

Proof. Let $\alpha \in R^{\text{perf}}$ belong in the integral closure of S^{perf} , so that it is a root of

$$f(x) = f_0 + f_1x + \dots x^n \in S^{\text{perf}}[x]$$

Since all $f_i^{p^N} \in S$ and $\alpha^{p^N} \in R$ for sufficiently large value of N , we have that α^{p^N} is a root of the monic polynomial over S

$$f_0^{p^N} + f_1^{p^N}x + \dots + x^n \in S[x]$$

and so $\alpha^{p^N} \in S$ by the hypothesis that S is integrally-closed inside of R . It follows that $\alpha \in S^{\text{perf}}$. $\square$

Lemma 4.5.6. *Let S be an open subring of R . Then S^{perf} is an open subring of R^{perf} .*

Proof. It suffices to show that for some $\varepsilon > 0$,

$$B_{\|\cdot\|'}(0, \varepsilon) := \{f \in R^{\text{perf}} : \|f\|' < \varepsilon\} \subset S^{\text{perf}}$$

Since we have that S is open in R , then $B_{\|\cdot\|}(0, \varepsilon) \subset S$, for some $0 < \varepsilon < 1$. Let $f \in B_{\|\cdot\|'}(0, \varepsilon) \subset R^{\text{perf}}$, so that $\|f\|' < \varepsilon$. By definition, this means for some sufficiently large n for which $f^{p^n} \in R$, we have

$$\|f\|' := \|f^{p^n}\|^{1/p^n} < \varepsilon$$

and so we have that

$$\|f^{p^n}\| < \varepsilon^{p^n} < \varepsilon$$

where the last inequality is due to our choice of $\varepsilon < 1$. It follows that $f^{p^n} \in B_{\|\cdot\|}(0, \varepsilon) \subset S$, and so $f \in S^{\text{perf}}$. We have that $B_{\|\cdot\|'}(0, \varepsilon) \subset S^{\text{perf}}$. $\square$

Since the norm on R is power-multiplicative, we have that the subring of

power-bounded elements R° is precisely

$$R^\circ = \{f \in R : \|f\| \leq 1\}$$

We have the following behavior under taking perfect closures.

Lemma 4.5.7. *For a nonarchimedean ring R equipped with multiplicative norm, $(R^\circ)^{\text{perf}} = (R^{\text{perf}})^\circ$.*

Proof. Let $f \in (R^\circ)^{\text{perf}}$, so that $f^{p^n} \in R^\circ$ for some n . since $\|f^{p^n}\| \leq 1$,

$$\|f\|' := \|f^{p^n}\|^{1/p^n} \leq 1$$

and so we have that $f \in (R^{\text{perf}})^\circ$. Conversely, let $f \in R^{\text{perf}}$ satisfy $\|f\|' \leq 1$. Since $f^{p^n} \in R$ for large enough n , we have that

$$\|f^{p^n}\|^{1/p^n} =: \|f\|' \leq 1$$

and so f^{p^n} belongs in R° . It follows that $f \in (R^\circ)^{\text{perf}}$. □

Let R be a ring equipped with a topology now. Recall that a valuation $|\cdot| : R \longrightarrow \Gamma \cup \{0\}$ is called continuous if the induced map

$$|\cdot| : R \longrightarrow \text{Frac}(R/\text{supp}(|\cdot|))$$

is continuous; here, the target field is given the topology induced by the absolute value of $|\cdot|$.

Since valuations on rings are multiplicative (by definition), we would love to mimic the above process of extending any valuation $|\cdot| : R \longrightarrow \Gamma \cup \{0\}$ to one on the perfect closure R^{perf} . Moreover, if the ring R comes equipped with a power-multiplicative norm (which we can extend to a power-multiplicative norm on R^{perf} as before), then we would like to be able to extend any *continuous* valuation on R to a *continuous* valuation on R^{perf} . What gets into the way of the previous proof is the fact that the totally-ordered abelian group $(\Gamma, \leq)$ need not be p -divisible (unlike $\mathbb{R}_{\geq 0}$). We wish to introduce p -power roots via the following procedure. Since Γ is a $\mathbb{Z}$ -module, we may then form

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

We interpret symbols in this abelian group as follows:

$$\gamma^{a/p^i} := \gamma \otimes a/p^i = \gamma^a \otimes 1/p^i$$

However, we wish to impose the following ordering relation on this group, where $\gamma_1, \gamma_2 \in \Gamma$:

$$\gamma_1^{1/p^i} \leq_p \gamma_2^{1/p^j} \iff \gamma_1^{p^j} \leq \gamma_2^{p^i}$$

Reflexivity and totality of this relation are satisfied. The proof of transitivity is the same as for fractions, just written multiplicatively. Antisymmetry is not necessarily satisfied for this group. We must force relations, hence quotient out by the subgroup generated by all symbols that satisfy

$$\Gamma' := \left\langle (\gamma_1 \otimes 1/p^i)(\gamma_1 \otimes 1/p^j)^{-1} \mid \gamma_1^{p^j} = \gamma_2^{p^i} \right\rangle$$

Now the quotient group $\Gamma^{\text{perf}} := (\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[1/p])/\Gamma'$ inherits the relation $\leq_p$ in the obvious way; this is now a totally-ordered abelian group, by construction. Moreover, Γ naturally embeds into Γ^{perf} , via the map $\gamma \mapsto \gamma \otimes 1$, and we have that $\leq_p$ is an extension of the total order $\leq$ on Γ . We have that Γ^{perf} is, in fact, p -divisible.

We can now mimic the argument in the first section for any valuation $|\cdot| : R \longrightarrow \Gamma \cup \{0\}$, where R is equipped with a power-multiplicative norm. That is, if we consider R^{perf} with the extended norm, we may define for any $f \in R^{\text{perf}}$,

$$|f|' := |f^{p^n}|^{1/p^n} \in \Gamma^{\text{perf}}$$

where $f^{p^n} \in R$ for a sufficiently large n . Again, by multiplicativity, we have that this definition is independent of the value of n , hence is a well-defined value. We thus have the following commutativity:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & R^{\text{perf}} \\ \downarrow |\cdot| & & \downarrow |\cdot|' \\ \Gamma \cup \{0\} & \longrightarrow & \Gamma^{\text{perf}} \cup \{0\} \end{array}$$

We verify that this induces a continuous valuation on R^{perf} . If we let

$$\psi : R^{\text{perf}} \longrightarrow \text{Frac}(R^{\text{perf}}/\text{supp}(|\cdot|'))$$

denote the natural ring map, we observe

$$|\psi(f)|' := |\psi(f)^{p^n}|^{1/p^n} = |\psi(f^{p^n})|^{1/p^n} \leq |f^{p^n}|^{1/p^n} =: |f|'$$

where we have used that $\psi|_R : R \longrightarrow \Gamma \cup \{0\}$ is continuous. The continuity of $|\cdot|'$ is

really just a consequence of the continuity of $|\cdot|$.

Proposition 4.5.8. *Let R be a ring equipped with a power-multiplicative norm, and give R^{perf} the norm defined as before. Then there is a bijection between continuous valuations on R and R^{perf} . Moreover, if $|\cdot| : R \rightarrow \Gamma \cup \{0\}$ is a continuous valuation, and $S \subset R$ such that $|g| \leq 1$ for all $g \in S$, then $|f'| \leq 1$ for all $f \in S^{\text{perf}} \subset R^{\text{perf}}$, where $|\cdot|' : R^{\text{perf}} \rightarrow \Gamma^{1/p^\infty}$ is the extension of $|\cdot|$ to the perfect closure of R .*

Proof. Suppose we can extend any valuation $|\cdot|$ on R in two ways $|\cdot|', |\cdot|''$ to one on R^{perf} . Since for any $f \in R^{\text{perf}}$ we have $f^{p^n} \in R$ for n sufficiently large, and these two valuations are *extensions* of $|\cdot|$, then multiplicativity of valuations yields

$$(|f'|)^{p^n} = |f^{p^n}|' = |f^{p^n}| = |f^{p^n}|'' = (|f|'')^{p^n}$$

Taking p^n -th roots yields $|f'| = |f|''$, hence the extension of $|\cdot|$ is unique. Since every valuation on the perfect closure restricts to one on the original ring R , which may then be extended back to the perfect closure uniquely, our bijection follows.

Now suppose $f \in S^{\text{perf}}$ for $S \subset R$. Let $|\cdot| : R \rightarrow \Gamma \cup \{0\}$ satisfy $|g| \leq 1$ for all $g \in S$. Since $f^{p^n} = g \in R$ for large enough n , we have that

$$|f'| = |f^{p^n}|^{1/p^n} = |g|^{1/p^n} \leq 1$$

which proves the second statement of the proposition. $\square$

Again, we let R be a ring of characteristic $p > 0$ equipped with a power-multiplicative norm. Let (R, R^+) be an affinoid pair, by which we mean R^+ is open and integrally-closed subring of the ring R° of power-bounded elements of R . Also,

let R have a topologically nilpotent unit. It follows from Lemmas 1.1, 1.2, 1.3 that $(R^{\text{perf}}, (R^+)^{\text{perf}})$ will also be an affinoid pair. For pre-adic space $X := \text{Spa}(R, R^+)$, we can associate to it a corresponding pre-adic space $X^{\text{perf}} := \text{Spa}(R^{\text{perf}}, (R^+)^{\text{perf}})$. Moreover, since the ring inclusion $\phi : (R, R^+) \longrightarrow (R^{\text{perf}}, (R^+)^{\text{perf}})$ is continuous/bounded, it follows that this induces a morphism of pre-adic spaces, $\text{Spa}(\phi) : X^{\text{perf}} \longrightarrow X$.

Proposition 4.5.9. *The morphism $\text{Spa}(\phi) : X^{\text{perf}} \longrightarrow X$ of affinoid pre-adic spaces is a homeomorphism on underlying topological spaces, which takes rational opens to rational opens.*

Proof. Certainly the morphism is continuous, and bijectivity follows from Proposition 2.1. To show this is a homeomorphism, we show that $|\text{Spa}(\phi)|$ is an open map at the level of topological spaces. We let $U' = U \left(\frac{f_1, \dots, f_k}{g} \right) \subset X^{\text{perf}}$ be a rational open subset, where $f_1, \dots, f_k$ generate the unit ideal in R^{perf} . Since we are dealing with finitely many ring elements in the perfect closure, then for n sufficiently large $f_1^{p^n}, \dots, f_k^{p^n}, g^{p^n} \in R$, and $f_1^{p^n}, \dots, f_k^{p^n}$ generate the unit ideal in R . We claim that the image of U' under $\text{Spa}(\phi)$ is precisely

$$U := U \left(\frac{f_1^{p^n}, \dots, f_k^{p^n}}{g^{p^n}} \right)$$

Let $x = |\cdot|_x$ belong in this rational subset of X , so that $|f_i^{p^n}|_x \leq |g^{p^n}|_x$ for $i = 1, \dots, k$.

If we denote by $x' = |\cdot|_{x'}$ the unique preimage of x in X^{perf} , then it follows

$$|f_i|_{x'} := |f_i^{p^n}|_x^{1/p^n} \leq |g^{p^n}|_x^{1/p^n} =: |g|_{x'}$$

for all i , hence $x' \in U$, yielding

$$\text{Spa}(\phi)^{-1}(U) \subset U'$$

The other inclusion should be clear. $\square$

Remark 4.5.10. *If $X = \mathrm{Spa}(R, R^+)$ is an adic space, such that R is a Banach K -algebra for K a nonarchimedean field (hence contains a topologically nilpotent unit), then it in fact holds that the completed perfection X^{perf} becomes an adic space. This is in [10] Section 3.1.*

Example 4.5.11. *Consider the pair $(K\langle T \rangle, K^\circ\langle T \rangle)$, for K a perfectoid field. We know its tilt $K^\flat$ is perfect of characteristic $p > 0$. It follows then from the natural homeomorphism of perfectoid spaces with their corresponding tilts, and the fact that completion does not alter adic spaces,*

$$|\mathbb{B}_K^{1, \mathrm{ad}, \mathrm{perf}}| \cong |\mathbb{B}_{K^\flat}^{1, \mathrm{ad}, \mathrm{perf}}| \cong |\mathrm{Spa}(K^\flat[T^{1/p^\infty}], (K^\flat)^\circ[T^{1/p^\infty}])|$$

By the previous proposition, and again using that completion does not alter adic spaces

$$|\mathrm{Spa}(K^\flat[T^{1/p^\infty}], (K^\flat)^\circ[T^{1/p^\infty}])| \cong |\mathrm{Spa}(K^\flat[T], (K^\flat)^\circ[T])| \cong |\mathbb{B}_{K^\flat}^{1, \mathrm{ad}}|$$

Combining the two, we have $|\mathbb{B}_K^{1, \mathrm{ad}, \mathrm{perf}}| \cong |\mathbb{B}_{K^\flat}^{1, \mathrm{ad}}|$. By the same argument, we have that $|\mathbb{T}_K^{1, \mathrm{ad}, \mathrm{perf}}| \cong |\mathbb{T}_{K^\flat}^{1, \mathrm{ad}}|$. This holds for higher-dimensional perfectoid discs and tori, of course.

As a consequence of this previous example, we observe that the Krull dimensions of the perfectoid disc and circle (over K) are precisely the Krull dimensions of the regular adic disc and circle (over the tilt $K^\flat$), respectively. Since both of the latter

adic spaces have Krull dimension 1, so do the former two spaces. It follows then from Proposition 4.5.1 that for $q > 1$,

$$H^q(\mathbb{B}_K^{1,\text{ad},\text{perf}}, \mathcal{O}^\times) = H^q(\mathbb{T}_K^{1,\text{ad},\text{perf}}, \mathcal{O}^\times) = 1$$

We have established vanishing of cohomology in degrees 2 and higher for all intersections of opens in the cover $\mathfrak{U}$ of $\mathbb{P}_K^{1,\text{perf}}$, and thus have the following result:

Theorem 4.5.12. *For fixed perfectoid field K in characteristic 0, the Picard group of $\mathbb{P}_K^{1,\text{ad},\text{perf}}$ is isomorphic to $\mathbb{Z}[1/p]$.*

Since the above isomorphism holds independent of the base perfectoid field K , we get $\text{Pic}(\mathbb{P}_K^{1,\text{ad},\text{perf}}) \cong \text{Pic}(\mathbb{P}_{K^\flat}^{1,\text{ad},\text{perf}})$ for free.

4.6 Twisting Sheaves on the Projectivoid Line

It should be clear where our line bundles over the projectivoid line come from. If we trivialize $\mathcal{L}$ to our standard cover, then we may choose canonical local sections of the form T^α , for $\alpha \in \mathbb{Z}[1/p]$ (just factor out the lower power of T); it follows then when restricting to overlaps that $T^\alpha/T^\beta \in K\langle T^{\pm 1/p^\infty} \rangle^\times = K^\times \oplus T^{\pm 1/p^\infty}$. Modding out by $K^\times$ (local sections which disagree on the overlap by such a unit in the base field are necessarily in the same isomorphism class of line bundle), we obtain that the sections T^α , for $\alpha \in \mathbb{Z}[1/p]$, parametrize our Picard group.

Thus, we have that for any $\alpha \in \mathbb{Z}[1/p]$, we can naturally associate the isomorphism class of twisting sheaf $\mathcal{O}(\alpha) \in \text{Pic}(\mathbb{P}_K^{1,\text{ad},\text{perf}})$. Note that clearing denominators

for α corresponds to twisting the sheaf $\mathcal{O}(\alpha)$ to a high enough power of p ; in this case, this high p -power is in the same isomorphism class as the pullback of a regular twisting sheaf on the normal adic projective line over K (which has Picard group $\mathbb{Z}$). This simply follows from the fact that we know how $K\langle T \rangle \hookrightarrow K\langle T^{1/p^\infty} \rangle$ naturally injects.

Chapter 5

Some Natural Follow-up Questions

The previous methods for computing the Picard group of the projectivoid line were very ad hoc, i.e. many of the techniques can only be applied for the case of dimension 1. It would be worthwhile to come up with a more general proof that works for arbitrary dimensional projectivoid space, and maybe even for a wider class of perfectoid spaces. This chapter will be used to catalogue some natural questions that arise, when attempting to get at such generality.

5.1 A Comparison Theorem for Picard Groups?

We know that the perfection of an analytic (i.e. every ring contains a topologically nilpotent unit) adic space X in characteristic p , as mentioned (but not proven) in Remark 4.5.11, is also an adic space. In such a scenario, I am tempted to suggest that the following isomorphism holds:

$$\mathrm{Pic}(X^{\mathrm{perf}}) \cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

The only evidence I have for such a statement is for the regular adic projective line (which has Picard group $\mathbb{Z}$, a consequence of the nonarchimedean GAGA principle, which we discuss later) over a field of characteristic p ; this was proven in the last chapter.

Question 5.1.1. *If X is an adic space of characteristic $p > 0$, and X^{perf} is its corresponding perfection (also an adic space), is it true that $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \text{Pic}(X^{\text{perf}})$?*

If we can prove such a result, then we have a strategy for generalizing our result to arbitrary dimensional projectivoid space. We show that $\text{Pic}(\mathbb{P}_K^{n,\text{ad},\text{perf}}) \cong \text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{ad},\text{perf}})$ and then use

$$\text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{ad},\text{perf}}) \cong \text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{ad}}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \mathbb{Z}[1/p]$$

where the second to last isomorphism is a GAGA statement (more on this later).

5.2 Stabilization of Picard Groups Under Tilting?

It should be noted that I am assuming that $\text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{ad},\text{perf}}) \cong \text{Pic}(\mathbb{P}_{K^{\flat}}^{n,\text{ad}})$. We certainly have that this held in the dimension 1 case. In general, we should expect the Picard group for any projectivoid space should be insensitive to the ground field which we are working over (just as we see in algebraic geometry); an easy consequence of such would be then $\mathbb{P}_K^{n,\text{ad},\text{perf}} \cong \mathbb{P}_{K^{\flat}}^{n,\text{ad},\text{perf}}$.

Question 5.2.1. *For perfectoid space X with corresponding characteristic p tilt $X^{\flat}$, when can we expect $\text{Pic}(X) \cong \text{Pic}(X^{\flat})$?*

In general one should *not* expect to get an isomorphism between the Picard groups of a perfectoid space X and its tilt $X^\flat$. This follows due to the following argument. For perfectoid space X , we have the following Kummer sequences, for each n :

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{O}_X^\times \xrightarrow{f \mapsto f^{p^n}} \mathcal{O}_X^\times \longrightarrow 1$$

where μ_{p^n} denotes the sheaf of p^n th roots of unity. It is not clear why the above is right-exact; in general, it is not for the standard topology (generated by rational opens) on X ; however, it will be right-exact for the pro-étale topology on X . This is a consequence of the fact that all stalk rings of the sheaf $\mathcal{O}_X$ for this topology will be both henselian and corresponding residue fields contains all p th roots (thus, roots of $X^{p^n} - f$ over the residue field, which necessarily exist, lift to roots in the stalk rings, yielding surjectivity).

It follows that we can do this for all powers of p , and connect each exact sequence through the appropriate transition maps:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_{p^n} & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{f \mapsto f^{p^n}} & \mathcal{O}_X^\times \longrightarrow 1 \\ & & \uparrow f \mapsto f^p & & \uparrow f \mapsto f^p & & \parallel \\ 1 & \longrightarrow & \mu_{p^{n+1}} & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{f \mapsto f^{p^{n+1}}} & \mathcal{O}_X^\times \longrightarrow 1 \end{array}$$

Passing to the inverse limit, we arrive at the following sequence of sheaves on the pro-étale site of X :

$$1 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathcal{O}_{X^\flat}^\times \xrightarrow{[\cdot]} \mathcal{O}_X^\times \longrightarrow 1$$

where $[\cdot] : \mathcal{O}_{X^\flat}^\times \longrightarrow \mathcal{O}_X^\times$ denotes the Teichmüller map from characteristic p to charac-

teristic 0, and we use the fact that for a perfectoid algebra R with corresponding tilt $R^\flat$, we know

$$(R^\flat)^\times \cong \varprojlim_{f \mapsto f^p} R^\times$$

I will not go into the details of the pro-étale site, since I have not worked with it much. The key feature, for our purposes in this discussion, that we need is the following:

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times) \cong H^1(X_{\mathrm{pro-et}}, \mathcal{O}_X^\times)$$

and so we have the following long exact sequence of abelian groups:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_p(1)(X) & \longrightarrow & \mathcal{O}_{X^\flat}(X^\flat) & \longrightarrow & \mathcal{O}_X(X) \\ & & & & \searrow & & \\ & & & & H^1(X_{\mathrm{pro-et}}, \mathbb{Z}_p(1)) & \longrightarrow & \mathrm{Pic}(X^\flat) \longrightarrow \mathrm{Pic}(X) \longrightarrow \dots \end{array}$$

Thus, we have a sufficient condition for when $\mathrm{Pic}(X) \cong \mathrm{Pic}(X^\flat)$; namely, if the first and second cohomology groups for the **sheaf of Tate modules** $\mathbb{Z}_p(1)$ on the pro-étale site vanish, we get the desired isomorphism.

What is a necessary condition for such to vanishing of cohomology for this sheaf? We know that we can project onto the first factor, to obtain a surjection of sheaves $\mathbb{Z}_p(1) \longrightarrow \mu_p$; the kernel is precisely $\mathbb{Z}_p(1)$, and so we have the following short exact sequence of sheaves on the pro-étale site:

$$1 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mu_p \longrightarrow 1$$

Using the resulting long exact sequence in cohomology, it follows that necessarily $H^1(X_{\mathrm{pro-et}}, \mu_p) = 1$, in this case. Thus, vanishing of cohomology for the sheaf of p th roots of unity is definitely of interest for this problem.

This question was initially posed by the author of this work; Peter Scholze was able to convey the existence of the short exact sequence to the author's thesis advisor, who then conveyed it to the author. The interested reader is encouraged to read more about the pro-étale site in [13].

5.3 Classification of Higher Rank Vector Bundles?

The previous questions seek to generalize the result

$$\mathrm{Pic}(\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}) \cong \mathrm{Pic}(\mathbb{P}_{K^b}^{1,\mathrm{ad},\mathrm{perf}}) \cong \mathbb{Z}[1/p]$$

to higher-dimensional projectivoid spaces, as well as other perfectoid spaces. Another direction in which we can generalize this result, is to consider higher rank vector bundles on the projectivoid line. In the algebraic and complex-analytic categories is the following due to Grothendieck:

Theorem 5.3.1 (Grothendieck). *For base field K , every rank n vector bundle $\mathcal{E}$ over $\mathbb{P}_K^{1,\mathrm{alg}}$ is isomorphic to a direct sum of n twisting sheaves; that is*

$$\mathcal{E} \cong \mathcal{O}(\alpha_1) \oplus \dots \oplus \mathcal{O}(\alpha_n)$$

for $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$.

We ask then whether we get the analogous result in the perfectoid world.

Question 5.3.2. *Does every rank n vector bundle $\mathcal{E}$ over the projectivoid line split as*

a direct sum of n line bundles? That is, is it true that

$$\mathcal{E} \cong \mathcal{O}(\alpha_1) \oplus \dots \oplus \mathcal{O}(\alpha_n)$$

for $\alpha_1, \dots, \alpha_n \in \mathbb{Z}[1/p]$?

Going even more generally than this, we can maybe ask to what extent can we classify finite rank vector bundles on perfectoid spaces (I am intentionally stating this vaguely). The moduli problem of classifying finite rank vector bundles is typically a difficult problem, and one usually restrict to stable vector bundles. However, there are various notions of stability and slopes, which depend on which notion of degrees for line bundles is being used.

Question 5.3.3. *Does there exist a well-defined notion of ‘degree’ for line bundles on perfectoid spaces?*

Though we are not able to fully provide answers to all of these questions in this work, the rest of this work is an earnest attempt at such. We first make a diversion to the algebraic category, and investigate analogous questions.

Chapter 6

More on Perfections of Schemes and Adic Spaces

6.1 Picard Groups for Perfections of Schemes

In this section, we seek to address Question 5.1.1, in which we ask about a comparison isomorphism between the Picard group of an adic space in characteristic p and its associated completed perfection. An affirmative answer to the analogous question in the algebraic category was proven in [3] Lemma 3.5, when restricting to schemes that are quasi-compact and quasi-separated (they use that the Picard functor commutes with direct limits in this case). This result can be slightly generalized in two directions, which we quickly record here.

Definition 6.1.1. *Let p be a fixed prime, and A an abelian group (written additively).*

The *p*-power perfect closure of A is defined as

$$A^{perf} := \varinjlim_{x \mapsto px} A$$

Lemma 6.1.2. *For fixed prime p and abelian group A , we have the following isomorphism for p -power perfect closure:*

$$A^{perf} \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

Proof. We fix an indexing

$$A_0 \xrightarrow{x \mapsto px} A_1 \xrightarrow{x \mapsto px} A_2 \xrightarrow{x \mapsto px} \dots$$

where $A_i = A$. For equivalence class $[f] \in A^{perf}$, choose representative (f, i) for $f \in A_i$, and define the mapping

$$[f] \mapsto f \otimes \frac{1}{p^i}$$

If we chose another representative (g, j) (assume without loss of generality that $j \geq i$), then we know $p^{j-i}f = g$, and so

$$g \otimes \frac{1}{p^j} = p^{j-i}f \otimes \frac{1}{p^j} = f \otimes \frac{p^{j-i}}{p^j} = f \otimes \frac{1}{p^i}$$

and so this mapping is well-defined. It is easy to check that this is a group homomorphism. To verify surjectivity, based on the group multiplication rule, we merely need to consider elementary tensors of the form $f \otimes 1/p^i$ in the target; this should be clear. Injectivity follows from $f \otimes 1/p^i = 0$ if and only if $f = 0$. $\square$

Lemma 6.1.3. *Let R be a ring in characteristic p , R^{perf} its perfect closure, and $\phi : R \longrightarrow R^{\text{perf}}$ the natural ring map. Then the induced map on prime spectra*

$$\text{Spec}(\phi) : \text{Spec}(R^{\text{perf}}) \longrightarrow \text{Spec}(R)$$

is a homeomorphism on underlying topological spaces, which takes distinguished opens to distinguished opens.

Proof. Refer to [6].

The above should be clear, since the kernel of the map $R \longrightarrow R^{\text{perf}}$ is precisely p -power nilpotents, and by killing nilpotents we lose nothing at the level of topological spectra. We can globalize the above notion of taking perfect closures for any scheme X in characteristic p ; simply replace all rings with their corresponding perfect closure. This procedure glues, and so we obtain a scheme X^{perf} whose underlying topological space is naturally homeomorphic to X . If we want, we can simply identify X and X^{perf} as the same space, but with different sheaves of functions. For this reason, the notation in the following result should not be ambiguous, since we are working over the same topological space.

Proposition 6.1.4. *Let $(X, \mathcal{O}_X)$ be a ringed space, with $\mathcal{O}_X$ a sheaf of rings in characteristic $p > 0$, and let $\mathcal{O}_X^{\text{perf}}$ be the corresponding sheaf obtained by taking perfect closures:*

$$\mathcal{O}_X^{\text{perf}}(U) := \mathcal{O}_X(U)^{\text{perf}}$$

Then there is an isomorphism of sheaves of abelian groups

$$\mathcal{O}_{X^{\text{perf}}}^{\times} \cong \mathcal{O}_X^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

Proof. This simply follows from the fact that taking units of a ring will commute with taking perfect closures, and all restriction morphisms are naturally compatible with the isomorphism we exhibited previously:

$$\begin{array}{ccc} \mathcal{O}_{X^{\text{perf}}}(U)^{\times} & \xrightarrow{\cong} & \mathcal{O}_X(U)^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \\ \downarrow |_W & & \downarrow |_W \\ \mathcal{O}_{X^{\text{perf}}}(W)^{\times} & \xrightarrow{\cong} & \mathcal{O}_X(W)^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \end{array}$$

□

Proposition 6.1.5. *For X a quasicompact, quasi-separated scheme in characteristic p with perfection X^{perf} , we have the following isomorphisms of cohomology groups for sheaves of units:*

$$H^i(X, \mathcal{O}_{X^{\text{perf}}}^{\times}) \cong H^i(X, \mathcal{O}_X^{\times}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

In particular, we recover the isomorphism

$$\text{Pic}(X^{\text{perf}}) \cong \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

Proof. This follows from the fact that

$$\mathcal{O}_{X^{\text{perf}}}^{\times} = \varinjlim_{x \mapsto x^p} \mathcal{O}_X^{\times}$$

We know that cohomology on any quasi-compact, quasi-separated scheme commutes

with direct limits, so obtain

$$H^i(X, \mathcal{O}_{X^{\text{perf}}}^\times) \cong H^i(X, \varinjlim_{x \mapsto x^p} \mathcal{O}_X^\times) \cong \varinjlim_{x \mapsto x^p} H^i(X, \mathcal{O}_X^\times) \cong H^i(X, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

where the last isomorphism is Lemma 6.1.2. Restricting to degree 1, we obtain the isomorphism in [3] Lemma 3.5. $\square$

We quickly drop the quasi-separated assumption from [3] Lemma 3.5.

Proposition 6.1.6. *Let X be a quasicompact scheme in characteristic $p > 0$ with perfection X^{perf} . Then $\text{Pic}(X^{\text{perf}}) \cong \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.*

Proof. We take a finite (quasi-compact assumption) affine cover $\mathfrak{U} = \{U_i\}$ of X , and form the Čech-complex for the sheaf of units $\mathcal{O}_X^\times$ subordinate to this open affine cover:

$$1 \longrightarrow \prod_i \mathcal{O}_X(U_i)^\times \longrightarrow \prod_{i < j} \mathcal{O}_X(U_i \cap U_j)^\times \longrightarrow \dots$$

Since $\mathbb{Z}[1/p]$ is flat over $\mathbb{Z}$ (localizations are always flat), and tensor products commute with finite direct sums, we get the following diagram of complexes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \{\prod_i \mathcal{O}_X(U_i)^\times\} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] & \longrightarrow & \{\prod_{i < j} \mathcal{O}_X(U_i \cap U_j)^\times\} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] & \longrightarrow & \dots \\ & & \downarrow \cong & & \downarrow \cong & & \\ 1 & \longrightarrow & \prod_i \mathcal{O}_X(U_i)^\times \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] & \longrightarrow & \prod_{i < j} \mathcal{O}_X(U_i \cap U_j)^\times \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] & \longrightarrow & \dots \\ & & \downarrow \cong & & \downarrow \cong & & \\ 1 & \longrightarrow & \prod_i \mathcal{O}_{X^{\text{perf}}}(U_i)^\times & \longrightarrow & \prod_{i < j} \mathcal{O}_{X^{\text{perf}}}(U_i \cap U_j)^\times & \longrightarrow & \dots \end{array}$$

where the isomorphisms between the bottom two rows are from Lemma 6.1.4. We have a canonical isomorphism of complexes:

$$\check{C}^\bullet(\mathfrak{U}, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \check{C}^\bullet(\mathfrak{U}, (\mathcal{O}_{X^{\text{perf}}})^\times)$$

and so this induces an isomorphism at the level of Čech cohomology:

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \check{H}^1(\mathfrak{U}, (\mathcal{O}_{X^{\text{perf}}})^\times)$$

Realizing that we are working in degree 1, we may take the direct limit over all such finite, affine open covers of X (and correspondingly of X^{perf}), where open covers are partially-ordered with respect to refinement. We thus arrive at

$$\begin{aligned} H^1(X, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] &\cong \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \\ &\cong \varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, (\mathcal{O}_{X^{\text{perf}}})^\times) \cong H^i(X^{\text{perf}}, \mathcal{O}_{X^{\text{perf}}}^\times) \end{aligned}$$

and so we recover the isomorphism of Picard groups. $\square$

Remark 6.1.7. *I have abused notation here, and used $\mathfrak{U}$ to indicate covers of both X and X^{perf} ; this should not cause bewilderment, since distinguished opens are naturally identified between the two spaces.*

Remark 6.1.8. *Since everything we prove in the algebraic category is with an eye towards proving an analogous statement in the analytic category, the lifting of the condition of quasi-separatedness used in [3] Lemma 3.5 is really not much of a gain;*

when analytifying schemes, we will necessarily be dealing with objects that are proper over some nonarchimedean base field K . More on this later.

6.2 Picard Groups for Perfections of Adic Spaces

What makes proving any sort of analogous result in the adic setting more difficult is the fact that we are not merely performing the algebraic operation of taking perfect closures of rings; we are also performing the analytic operation of completion, afterwards. What is needed is an approximation result, that allows us to relate the completed perfect closure to some smaller, more manageable ring. This does not immediately appear to be a simple task.

In general, it is not reasonable to expect any density results between the units of some normed ring R and the units in its completion $\widehat{R}$. For instance if $R = K[T]$ for K a nonarchimedean field (equipped with the Gauss norm), we only get the units in our base field K . Completing this ring to get $K\langle T \rangle$, we get many more units (e.g. convergent geometric series). Even better, consider the units in $\mathbb{Z}$, equipped with the p -adic norm, and its completion $\mathbb{Z}_p$; the former has unit group that is cyclic of order 2, whereas the latter has a much richer unit group:

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \oplus 1 + p\mathbb{Z}_p$$

However, R and its completion $\widehat{R}$ do share one ring in common: their reductions.

Lemma 6.2.1. *Let R be a ring, and I an ideal in R . For any R -module M , define*

the I -adic completion as:

$$\widehat{M} := \varprojlim_n M/I^n M$$

If $I \subset J$ as ideals in R , then $\widehat{R}/\widehat{J} \cong R/J$.

Proof. At each level n , we have the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & J/I^{n+1} & \longrightarrow & R/I^{n+1} & \longrightarrow & R/J & \longrightarrow & 1 \\ & & \downarrow \alpha_n & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & J/I^n & \longrightarrow & R/I^n & \longrightarrow & R/J & \longrightarrow & 1 \end{array}$$

where the map $\alpha_n : J/I^{n+1} \longrightarrow J/I^n$ is surjective. By [1] Proposition 10.2 we get inverse limits will be exact in this situation, and so we have $\widehat{R}/\widehat{J} \cong R/J$. $\square$

Proposition 6.2.2. *For K a nonarchimedean field, let R be a K -algebra equipped with a norm (that extends the norm on K), and $\widehat{R}$ its completion. Then*

$$R^\circ/R^{\circ\circ} \cong \widehat{R}^\circ/\widehat{R}^{\circ\circ}$$

Proof. We know that the completion $\widehat{R}$ is obtained via the inverse limit construction

$$\widehat{R} := \left(\varprojlim_n R^\circ/\varpi^n \right) [1/\varpi]$$

for $\varpi \in K^{\circ\circ}$ a topologically nilpotent unit. It is easy to check that $(\widehat{R})^{\circ\circ} = \widehat{R^{\circ\circ}}$ and also $(\widehat{R})^\circ = \widehat{R}^\circ$. Combining this with the previous corollary for $I = (\varpi)$ and $J = R^{\circ\circ}$, we obtain $R^\circ/R^{\circ\circ} \cong \widehat{R}^\circ/\widehat{R}^{\circ\circ}$. We have shown that reductions are preserved under the process of completion. $\square$

We now return to the context of adic spaces in characteristic p . Let us first treat the case that X is an analytic adic space covered by Banach rings. As mentioned, in such a case we can form the corresponding completed perfection X^{perf} , which will also be an adic space. We know that $\mathcal{O}_{X^{\text{perf}}}$ is the sheaf associated to the presheaf

$$U \mapsto \widehat{\mathcal{O}_X^{\text{perf}}(U)} := \left(\varprojlim_n (\mathcal{O}_X(U)^{\circ})^{\text{perf}} / \varpi^n \right) \left[\frac{1}{\varpi} \right]$$

We note that $\mathcal{O}_X^{\text{perf}}$, what we obtain from taking perfect closures only, will also be a sheaf of (not necessarily complete) topological K -algebras. The analogous statements can be said about the *subs*sheaves $\mathcal{O}_X^{\circ}, \mathcal{O}_X^{\circ\circ}$ of power-bounded and topologically nilpotent elements, respectively. We have that the following diagram of sheaves will commute (just sheafify the ring arguments we made previously):

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathcal{O}_X^{\text{perf}})^{\circ\circ} & \longrightarrow & (\mathcal{O}_X^{\text{perf}})^{\circ} & \longrightarrow & (\mathcal{O}_X^{\text{perf}})^{\circ} / (\mathcal{O}_X^{\text{perf}})^{\circ\circ} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{O}_{X^{\text{perf}}}^{\circ\circ} & \longrightarrow & \mathcal{O}_{X^{\text{perf}}}^{\circ} & \longrightarrow & \mathcal{O}_{X^{\text{perf}}}^{\circ} / \mathcal{O}_{X^{\text{perf}}}^{\circ\circ} \longrightarrow 0 \end{array}$$

Taking sheaves of units:

$$\begin{array}{ccccccc} 1 & \longrightarrow & 1 + (\mathcal{O}_X^{\text{perf}})^{\circ\circ} & \longrightarrow & (\mathcal{O}_X^{\text{perf}})^{\circ\times} & \longrightarrow & (\mathcal{O}_X^{\text{perf}})^{\circ\times} / [1 + (\mathcal{O}_X^{\text{perf}})^{\circ\circ}] \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & 1 + \mathcal{O}_{X^{\text{perf}}}^{\circ\circ} & \longrightarrow & \mathcal{O}_{X^{\text{perf}}}^{\circ\times} & \longrightarrow & \mathcal{O}_{X^{\text{perf}}}^{\circ} / [1 + \mathcal{O}_{X^{\text{perf}}}^{\circ\circ}] \longrightarrow 1 \end{array}$$

Passing to long exact sequences in cohomology, we have the following:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^1(1 + (\mathcal{O}_X^{\text{perf}})^{\circ\circ}) & \longrightarrow & H^1((\mathcal{O}_X^{\text{perf}})^{\circ\times}) & \longrightarrow & H^1((\mathcal{O}_X^{\text{perf}})^{\circ\times} / [1 + (\mathcal{O}_X^{\text{perf}})^{\circ\circ}]) \longrightarrow \dots \\ & & \downarrow f_1 & & \downarrow f_2 & & \cong \downarrow f_3 \\ \dots & \longrightarrow & H^1(1 + \mathcal{O}_{X^{\text{perf}}}^{\circ\circ}) & \longrightarrow & H^1(\mathcal{O}_{X^{\text{perf}}}^{\circ\times}) & \longrightarrow & H^1(\mathcal{O}_{X^{\text{perf}}}^{\circ} / [1 + \mathcal{O}_{X^{\text{perf}}}^{\circ\circ}]) \longrightarrow \dots \end{array}$$

Note that since all topological spaces we are dealing with are the same, there is no ambiguity in omitting the space X in our notation. We would like to be able to apply the four-lemma to the associated morphism between long exact sequences in cohomology, to say something about f_2 .

Question 6.2.3. *When will the map $f_2 : H^1(X, (\mathcal{O}_X^{\text{perf}})^{\circ\times}) \longrightarrow H^1(X, \mathcal{O}_{X^{\text{perf}}}^{\circ\times})$ be an isomorphism? Even stronger, when will the sheaf $1 + \mathcal{O}_X^{\circ\circ}$ be acyclic?*

If we can figure establish conditions for which we get such an isomorphism, then we have the following isomorphisms:

$$H^1(X, (\mathcal{O}_{X^{\text{perf}}}^{\circ})^{\times}) \cong H^1(X, (\mathcal{O}_X^{\text{perf}})^{\circ\times}) \cong H^1(X, (\mathcal{O}_X^{\circ})^{\times}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

where I have implicitly used that taking perfect closures commutes with taking power-bounded elements in our situation, and Proposition 6.1.4, to obtain the second isomorphism.

If we can then establish conditions on an adic space X , for which

$$H^1(X, \mathcal{O}_X^{\times}) \cong H^1(X, \mathcal{O}_X^{\circ\times})$$

then we will have a way to establish our result:

$$\begin{aligned}
\mathrm{Pic}(X^{\mathrm{perf}}) &\cong H^1(X, \mathcal{O}_{X^{\mathrm{perf}}}^\times) \\
&\cong^? H^1(X, (\mathcal{O}_{X^{\mathrm{perf}}}^\circ)^\times) \\
&\cong^? H^1(X, (\mathcal{O}_X^{\mathrm{perf}})^{\circ \times}) \\
&\cong H^1(X, (\mathcal{O}_X^\circ)^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \\
&\cong^? H^1(X, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \\
&\cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]
\end{aligned}$$

We record an analogous result in the algebraic setting.

Proposition 6.2.4. *Let X be an integral scheme, such that its function field $K(X) = \mathcal{O}_{X,\eta}$ is equipped with a nonarchimedean norm $|\cdot|$ and $|K(X)^\times| = |\Gamma(X, \mathcal{O}_X)^\times|$. Let $\mathcal{O}_X^\circ$ denote the subsheaf of power-bounded elements of $\mathcal{O}_X$:*

$$\mathcal{O}_X^\circ(U) := \{f \in \mathcal{O}_X(U) : |f_\eta| \leq 1\}$$

Then we have the isomorphism $H^i(X, \mathcal{O}_X^{\circ \times}) \cong H^i(X, \mathcal{O}_X^\times)$, for all $i > 0$.

Proof. We have the short exact sequence of sheaves on X

$$1 \longrightarrow \mathcal{O}_X^{\circ \times} \longrightarrow \mathcal{O}_X^\times \xrightarrow{|\cdot|} |K(X)^\times| \longrightarrow 1$$

Again, using that $|K(X)^\times|$ is flasque, we obtain the long exact sequence:

$$\dots \longrightarrow \Gamma(X, \mathcal{O}_X^\times) \xrightarrow{|\cdot|} |K(X)^\times| \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^{\circ \times}) \longrightarrow 1 \longrightarrow \dots$$

where the map $|\cdot| : \mathcal{O}_X(X)^\times \longrightarrow |K(X)^\times|$ takes a local section, sends it to its stalk at the generic point of X , and evaluates its norm there. Again, by assumptions on value groups, we get $H^i(X, \mathcal{O}_X^{\circ\times}) \cong H^i(X, \mathcal{O}_X^\times)$, for all $i > 0$. $\square$

It is then natural to ask whether $\text{Pic}(R) \cong \text{Pic}(R^\circ)$, under the above hypotheses for R .

Proposition 6.2.5. *Let R be a ring equipped with a nonarchimedean, multiplicative norm $|\cdot|$ (i.e. induced from a valuation), such that $|R^\times| = |\text{Frac}(R)^\times|$, and let R° denote the subring of power-bounded elements (i.e. elements with norm less than/equal to 1). Then the natural map $\text{Pic}(R^\circ) \longrightarrow \text{Pic}(R)$ induced by base extension is an isomorphism.*

Proof. Let $f : X := \text{Spec}(R) \longrightarrow \text{Spec}(R^\circ) =: Y$ be the morphism of affine schemes induced by the inclusion $R^\circ \hookrightarrow R$; then we have the following short exact sequence of sheaves on Y :

$$1 \longrightarrow f_*\mathcal{O}_X^{\circ\times} \longrightarrow f_*\mathcal{O}_X^\times \xrightarrow{|\cdot|} |R^\times| \longrightarrow 1$$

where $\mathcal{O}_X^\circ$ is the subsheaf of $\mathcal{O}_X$ of power-bounded elements:

$$\mathcal{O}_X^\circ(U) := \{f \in \mathcal{O}_X(U) : |f_\eta| \leq 1\}$$

Here η denotes the generic point of X .

We claim that sitting between $f_*\mathcal{O}_X^{\circ\times}$ and $f_*\mathcal{O}_X^\times$ is the sheaf $\mathcal{O}_Y^\times$. We verify

this, by showing the inclusion of rings:

$$R[1/g]^\circ \subset R^\circ[1/g] \subset R[1/g]$$

for each $g \in R^\circ$ (i.e. verify inclusion at the level of distinguished opens of Y). The second inclusion of rings is clear. For the first, let $h/g^n \in R[1/g]^\circ$, so that

$$|h/g^n| = |h|/|g|^n \leq 1$$

Since $|h| \leq |g|^n \leq 1$ (the second inequality due to $g \in R^\circ$), it follows that $h \in R^\circ$, and so $h/g^n \in R^\circ[1/g]$, proving the first inclusion. Thus, we have the following inclusions of sheaves on Y :

$$f_*\mathcal{O}_X^{\circ\times} \longrightarrow \mathcal{O}_Y^\times \xrightarrow{f^\#} f_*\mathcal{O}_X^\times$$

We know that $R = R^\circ[1/\varpi]$ for any ϖ that is a topologically nilpotent unit in R ; it is clear that X open immerses into Y , since it is a distinguished open inside of Y , and therefore the stalks of $\mathcal{O}_Y^\times$ and $f_*\mathcal{O}_X^\times$ agree at all points of X .

Now choose $p \in |Y - X|$ corresponding to some prime $\mathfrak{p} \in R^\circ$. Since $p \in Z(\varpi) = |Y - X|$ for all topologically nilpotent units ϖ , it follows that all such $\varpi \in \mathfrak{p}$. Let f/g be an invertible element in $\mathcal{O}_{Y,p} \cong (R^\circ)_\mathfrak{p}$, where $f, g \in R^\circ$, and $f, g \notin \mathfrak{p}$. In particular, $f, g \notin (\varpi)$ for all such topologically nilpotent units. We claim that $|f| = |g| = 1$. If $|f|, |g| < 1$, then by the condition that $|R^\times| = |\text{Frac}(R)^\times|$, we can find ϖ_1, ϖ_2 in $R^\times$ such that $|f| = |\varpi_1|$ and $|g| = |\varpi_2|$ (i.e. ϖ_1, ϖ_2 are topologically nilpotent units in R). Since $f, g, \varpi_1, \varpi_2 \in R$, and ϖ_1, ϖ_2 are invertible in R , we have that $\alpha_1 := f/\varpi_1, \alpha_2 := g/\varpi_2$ both belong in R and have norm 1; by definition, $\alpha_1, \alpha_2 \in R^\circ$, and so $f = \alpha_1\varpi_1, g = \alpha_2\varpi_2 \in R^\circ$, contradicting the fact that $f, g \notin (\varpi)$,

for all such topologically nilpotent units ϖ . Therefore, $|f| = |g| = 1$, and in particular, every such unit f/g in these stalks has norm 1.

By the definition of the stalk of pushforward sheaves, we know that

$$f_*\mathcal{O}_{X,p} := \left(\varinjlim_{p \in U} \mathcal{O}_X(f^{-1}(U)) \right) = \left(\bigcup_{p \in U} \mathcal{O}_X(f^{-1}(U)) \right)$$

where $\bigcup_{p \in U} \mathcal{O}_X(f^{-1}(U))$ is some ring sitting in between R and $\text{Frac}(R)$. The key point is that since $|R^\times| = |\text{Frac}(R)^\times|$, this intermediate ring will have no choice but to have the same value group. As mentioned previously, we are guaranteed this short exact sequence at all points $p \in Y$:

$$1 \longrightarrow f_*\mathcal{O}_{X,p}^{\circ\times} \longrightarrow f_*\mathcal{O}_{X,p}^\times \xrightarrow{|\cdot|} |R^\times| \longrightarrow 1$$

We know that $f_*\mathcal{O}_{X,p}^{\circ\times} \subset \mathcal{O}_{Y,p}^\times$ (direct limits are exact). As we showed, any $f \in \mathcal{O}_{Y,p}^\times \subset f_*\mathcal{O}_{X,p}^\times$ will have norm 1, so necessarily belongs in $f_*\mathcal{O}_{X,p}^{\circ\times}$, and so it follows that for points p away from X , we have $f_*\mathcal{O}_{X,p}^{\circ\times} = \mathcal{O}_{Y,p}^\times$. Therefore, for $p \notin X$, we have the following short exact sequence:

$$1 \longrightarrow \mathcal{O}_{Y,p}^\times \longrightarrow f_*\mathcal{O}_{X,p}^\times \xrightarrow{|\cdot|} |R^\times| \longrightarrow 1$$

We now can guess what the quotient sheaf $f_*\mathcal{O}_X^\times/\mathcal{O}_Y^\times$ should be. Let $j : Z := \text{Spec}(R^\circ/I) \longrightarrow Y$ be the natural closed immersion into $Y - X$ (where $I = \sqrt{\varpi}$ is the radical ideal of some topologically nilpotent unit, giving Z the reduced closed subscheme structure). If we let $|R^\times|$ be the locally-constant sheaf on Z , then by our previous investigation of stalks, we have the following short exact sequence of sheaves

on Y :

$$1 \longrightarrow \mathcal{O}_Y^\times \longrightarrow f_* \mathcal{O}_X^\times \xrightarrow{|\cdot|} j_* |R^\times| \longrightarrow 1$$

Passing to long exact sequences in cohomology, we obtain:

$$1 \longrightarrow R^{\circ \times} \longrightarrow R^\times \xrightarrow{|\cdot|} |R^\times|^N \longrightarrow \text{Pic}(R^\circ) \longrightarrow \text{Pic}(R) \xrightarrow{|\cdot|} H^1(Z, |R^\times|) \longrightarrow \dots$$

where N is the number of connected components of $|Z|$.

We quickly show that R°/I is an integral domain. Suppose $fg \in I$, so that $f^n g^n \in (\varpi)$ for any ϖ topologically nilpotent unit ϖ (easy to check they all have the same radical ideal), and $n = n(\varpi)$ sufficiently large. It follows from multiplicativity of $|\cdot|$ that either $|f| < 1$ or $|g| < 1$, and so either $f = \alpha\varpi$ or $g = \alpha\varpi$ for some ϖ (using that $f, g \in \text{Frac}(R)^\times$, and $|R^\times| = |\text{Frac}(R)^\times|$, again, we can choose such ϖ topologically nilpotent unit in R and α in R°). Since all topologically nilpotent units generate the same radical ideal (raise to high-enough powers and use norms, to show this), it follows that either $f \in I$ or $g \in I$, hence I is a prime ideal, and R°/I is a domain.

It follows that $Z = \text{Spec}(R^\circ/I)$ is irreducible, and so we have the number of connected components $N = 1$, and $|R^\times|$ is necessarily a (constant) flasque sheaf on this irreducible space. Since pushforwards of flasque sheaves are flasque, it will follow that $R^\times \xrightarrow{|\cdot|} |R^\times|$ is surjective, and $H^1(Z, |R^\times|) = 1$, yielding $\text{Pic}(R) \cong \text{Pic}(R^\circ)$. $\square$

Remark 6.2.6. *We can attempt to globalize this for arbitrary scheme X whose function field $K(X)$ is equipped with a nonarchimedean norm and $|\Gamma(X, \mathcal{O}_X)^\times| = |K(X)^\times|$, but unfortunately will not get very far. Letting $R = \Gamma(X, \mathcal{O}_X)$, we get a*

morphism $f : X \longrightarrow Y =: \operatorname{Spec}(R^\circ)$, and can again use the morphism $\mathcal{O}_Y^\times \xrightarrow{f^\#} f_*\mathcal{O}_X^\times$. However, we definitely are not guaranteed a surjection $\operatorname{Pic}(R^\circ) \xrightarrow{f^\#} \operatorname{Pic}(X)$. As an example, let K be a valued field, and $X = \mathbb{P}_K^n$. Equip its function field $K(T)$ with the multiplicative Gauss norm. In this case $R = \Gamma(X, \mathcal{O}_X) = K$, and so we get no jumps in value groups. R° is a valuation ring with trivial Picard group, but $\operatorname{Pic}(X) = \mathbb{Z}$.

What is nice about the above isomorphism of Picard groups is that it recovers a lot of instances of results of what we have observed before. For example, if R is a valued-field (not necessarily discrete) and R° its valuation ring, then we know that $\operatorname{Pic}(R) = \operatorname{Pic}(R^\circ) = 1$. We already knew this from commutative algebra: finite projective modules over fields are free, and valuation rings are Bezout domains. Just as well, for any valued field K with valuation ring K° , if we equip the polynomial ring $K[T_1, \dots, T_n]$ with the Gauss norm, we observe

$$\operatorname{Pic}(K^\circ[T_1, \dots, T_n]) = \operatorname{Pic}(K[T_1, \dots, T_n]) = 1 = \operatorname{Pic}(K^\circ)$$

That is, valuation rings are **seminormal** (recall, an integral domain D is seminormal if and only if $\operatorname{Pic}(D) \cong \operatorname{Pic}(D[T_1, \dots, T_n])$).

Corollary 6.2.7. *If R is equipped with a multiplicative norm such that $|R^\times| = |\operatorname{Frac}(R)^\times|$, then R is seminormal if and only if R° is seminormal.*

Proof. One direction holds independent of value group considerations; if R is seminormal (inside of $\operatorname{Frac}(R)$), then since R° is integrally-closed inside of R , it is also seminormal (using (2, 3)-closed condition). Now assume that R° is seminormal,

so that $\text{Pic}(R^\circ[T_1, \dots, T_n]) \cong \text{Pic}(R^\circ)$, for all $n \geq 0$. If we equip $R[T_1, \dots, T_n]$ with the multiplicative Gauss norm, we get no jumps in value groups, and moreover $R[T_1, \dots, T_n]^\circ = R^\circ[T_1, \dots, T_n]$. Applying Proposition 1, we obtain our result:

$$\text{Pic}(R[T_1, \dots, T_n]) \cong \text{Pic}(R^\circ[T_1, \dots, T_n]) \cong \text{Pic}(R^\circ) \cong \text{Pic}(R)$$

□

Question 6.2.8 (Nonarchimedean Homotopy Invariance). *Let R be a ring complete for a nonarchimedean, multiplicative norm. If R is seminormal, is it true that $\text{Pic}(R) \cong \text{Pic}(R\langle T_1, \dots, T_n \rangle)$?*

Remark 6.2.9. *The above question has an affirmative answer when R is a nonarchimedean field, due to a result of Lutkebohmert (a nonarchimedean version of the Quillen-Suslin Theorem from commutative algebra, so deals with higher-rank projective modules as well).*

To show that the last proposition has more value, we apply it to recover the following result about uniquely extending line bundles.

Corollary 6.2.10. *Let R be a Noetherian, integrally-closed domain, and $X = \text{Spec}(R)$. If $Y_1, \dots, Y_k$ correspond to principal prime divisors on X , then any line bundle defined away from $Y_1 \cup \dots \cup Y_k$ extends uniquely to all of X .*

Proof. Since the Y_i are principal, it follows that $[Y_i] = \text{div}(f_i) \in \text{Pic}(R)$ (i.e. f_i only vanishes along Y_i with order 1), for $f_i \in R$ (since R is Noetherian, integrally-closed domain). Therefore

$$X - \bigcup_{i=1}^k Y_i = \bigcap_{i=1}^k (X - Y_i) = \bigcap_{i=1}^k (X - Z(f_i)) = \bigcap_{i=1}^k D(f_i) = D(f_1 \dots f_k) = \text{Spec} \left(R \left[\frac{1}{f_1 \dots f_k} \right] \right)$$

We know that each Y_i defines a unique nonarchimedean, multiplicative norm $|\cdot|_{Y_i}$ on the function field $K(X) = \text{Frac}(R)$, each giving a different topology (and correspondingly different valuation ring). The strategy is to use the norms to work our way down from the larger ring $R[1/f_1 \dots f_k]$ to R itself.

Let $R_i := R[1/f_1 \dots f_i]$ for $i = 1, \dots, k$, and $R_0 := R$, so that $R_{i+1} = R_i[1/f_{i+1}]$ is an overring of R_i , for each i . We claim that $R_i = R_{i+1}^\circ$, when R_{i+1} is equipped with the $|\cdot|_{Y_{i+1}}$ norm. Let f be such a power-bounded element within R_{i+1} , so that $|f|_{Y_{i+1}} \leq 1$, and suppose that f is not contained in R_i . We may express $f = \alpha/f_{i+1}^j$, where $\alpha \in R_i$, and f_{i+1} does not divide α (i.e. we have cancelled out all powers of f_{i+1} from the numerator); our goal is to show the exponent $j = 0$. Since

$$|f|_{Y_{i+1}} = \frac{|\alpha|_{Y_{i+1}}}{(|f_{i+1}|_{Y_{i+1}})^j} \leq 1$$

It follows that $|\alpha|_{Y_{i+1}} \leq (|f_{i+1}|_{Y_{i+1}})^j$. Since f_{i+1} has Y_{i+1} -adic valuation 1, the Y_{i+1} -adic valuation of the right-hand side is precisely j . But this means that the Y_{i+1} -adic valuation of α must be greater than or equal to j . If $j > 0$, this means that α vanishes along Y_{i+1} , contradicting the fact that we cancelled out all powers of f_{i+1} from our numerator. Therefore, $j = 0$ and $f = \alpha/f_{i+1}^j = \alpha \in R_i$, giving us the inclusion $R_{i+1}^\circ \subset R_i$.

For the other inclusion, let $\alpha/f_1^{e_1}\dots f_i^{e_i}$ belong in R_i , for $\alpha \in R$. We know $|\alpha|_{Y_{i+1}} \leq 1$; since $\operatorname{div}(f_j) = [Y_j]$ for all j , then

$$|f_1^{e_1} \dots f_i^{e_i}|_{Y_{i+1}} = (|f_1|_{Y_{i+1}})^{e_1} \dots (|f_i|_{Y_{i+1}})^{e_i} = 1 \cdot \dots \cdot 1 = 1$$

and so $|\alpha/f_1^{e_1}\dots f_i^{e_i}|_{Y_{i+1}} \leq 1$. We have shown that $R_{i+1}^\circ = R_i$, when R_{i+1} is equipped with the $|\cdot|_{Y_{i+1}}$ norm. Moreover, since $\operatorname{div}(f_i) = [Y_i]$, each f_i is a uniformizing element for the $|\cdot|_{Y_i}$ norm, hence $|R_i^\times|_{Y_i} = \mathbb{Z} = |\operatorname{Frac}(R)^\times|_{Y_i}$ (since f_i is invertible in R_i , we fill out the entire value group by taking integer powers of this element). We can now repeatedly apply Proposition 1, to obtain our result:

$$\operatorname{Pic}(R_k) \cong \operatorname{Pic}(R_{k-1}) \cong \dots \cong \operatorname{Pic}(R_1) \cong \operatorname{Pic}(R_0) = \operatorname{Pic}(R)$$

□

If we globalize these arguments, we have the following.

Corollary 6.2.11. *Let X be a Noetherian, integral, normal scheme, and $Y_1, \dots, Y_k$ correspond to locally-principal prime divisors on X (i.e their corresponding sheaves of ideals are locally principally-generated). Then any line bundle defined away from $Y_1 \cup \dots \cup Y_k$ extends to all of X .*

Proof. Let $X_i := X - \bigcup_{j=1}^i Y_j$, and $X_0 := X$, so that $X_{i+1} = X_i - Y_i$, we have the natural open immersions of schemes:

$$X_k \xrightarrow{f_{k-1}} X_{k-1} \xrightarrow{f_{k-2}} \dots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = X$$

since each Y_i is cut out *locally* by a single regular function. Let $j_i : Y_i \hookrightarrow X_i$ be the natural closed immersion of this principal prime divisor. We know from previous discussions on valuations $|\cdot|_{Y_i}$ on $K(X)$, that at each step i of this tower, we have the following short exact sequence of sheaves on X_{i+1} :

$$1 \longrightarrow \mathcal{O}_{X_i}^\times \xrightarrow{f_i^\sharp} (f_i)_* \mathcal{O}_{X_{i+1}}^\times \longrightarrow (j_i)_* \mathbb{Z} \longrightarrow 1$$

Using that $\mathbb{Z}$ is flasque on the integral (hence irreducible) scheme Y_i , we obtain the following long exact sequence in cohomology:

$$1 \longrightarrow \mathcal{O}_{X_i}(X_i)^\times \xrightarrow{f_i^\sharp} \mathcal{O}_{X_{i+1}}(X_{i+1})^\times \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(X_i) \xrightarrow{f_i^\sharp} \mathrm{Pic}(X_{i+1}) \longrightarrow 1 \longrightarrow \dots$$

and so we get a surjection of Picard groups at each step i . Working our way down the tower, we get that $\mathrm{Pic}(X) = \mathrm{Pic}(X_0) \longrightarrow \dots \longrightarrow \mathrm{Pic}(X_k)$ is surjective. $\square$

Remark 6.2.12. *In general, one cannot expect uniqueness of this extension in the global setting. A very simple example is X the projective line over a field k , and Y a closed point. Then, $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{Pic}(X - Y) = 1$. In this example, $\Gamma(X, \mathcal{O}_X^\times) = k^\times$, and so we do not have any global uniformizer for $|\cdot|_Y$ (corresponding to a $(T - a)$ -adic valuation on function field $k(T)$).*

6.3 Extending Degrees to Perfections

As was alluded to in the previous chapter, it would be of interest to classify higher-rank vector bundles on the projectivoid line, and more general perfectoid spaces.

Classification of finite rank vector bundles (in particular, finding criterion for when such split as a direct sum of line bundles is a very interesting problem) is typically difficult; often one is forced to restrict themselves to classifying a subclass of such objects, namely stable vector bundles. In order to have a notion of stability of vector bundles, one needs a notion of degree for line bundles.

We let X be a quasicompact scheme in characteristic p . We know every line bundle on X^{perf} , when self-twisted to a high enough power of p , will come from X by way of pullback. This suggests a very natural answer to the following question.

Question 6.3.1. *If we have a notion of degree of a line bundle on X , what is the right notion of degree for line bundles on X^{perf} ?*

Here is the suggestion. if we are given a class $[\mathcal{L}] \in \text{Pic}(X^{\text{perf}})$, we tensor $\mathcal{L}$ to a sufficiently large enough power of p , so that we are now in the same isomorphism class as a line bundle that arises from pulling back to X^{perf} , that is

$$[\mathcal{L}^{\otimes p^N}] \cong [\mathcal{L}]^{\otimes p^N} \cong [\phi^*(\mathcal{E})]$$

for a unique isomorphism class $[\mathcal{E}] \in \text{Pic}(X)$. Since we assume that X has a notion of degree of line bundles, we simply define

$$\deg[\mathcal{L}] := \frac{\deg[\mathcal{E}]}{p^N}$$

The uniqueness of the class $[\mathcal{E}]$ guarantees that this map $\deg : \text{Pic}(X^{\text{perf}}) \rightarrow \deg(\text{Pic}(X))[1/p]$ is a well-defined class function. Moreover, we see that it is in

fact a group homomorphism: let $[\mathcal{L}_1], [\mathcal{L}_2] \in \text{Pic}(X^{\text{perf}})$ such that

$$[\mathcal{L}_1^{\otimes p^{N_1}}] \cong [\phi^* \mathcal{E}_1] \quad [\mathcal{L}_2^{\otimes p^{N_2}}] \cong [\phi^* \mathcal{E}_2]$$

Assuming without loss of generality that $N_1 \geq N_2$, we have then that

$$([\mathcal{L}_1] \otimes [\mathcal{L}_2])^{\otimes p^{N_1}} \cong [\phi^* \mathcal{E}_1] \otimes [\phi^* \mathcal{E}_2]^{p^{(N_1 - N_2)}}$$

Using our definition of degree, we obtain what we wanted:

$$\deg([\mathcal{L}_1] \otimes [\mathcal{L}_2]) = \frac{\deg[\mathcal{E}_1] + p^{N_1 - N_2} \deg[\mathcal{E}_2]}{p^{N_1}} = \frac{\deg[\mathcal{E}_1]}{p^{N_1}} + \frac{\deg[\mathcal{E}_2]}{p^{N_2}} = \deg[\mathcal{L}_1] + \deg[\mathcal{L}_2]$$

The above definition of degree also makes it clear how we should assign Weil divisors to global sections of the sheaf of meromorphic functions on X^{perf} , if X is suitably nice to support Weil divisors. Namely, we multiply a section to a high enough power of p such that it arises from a section on X , and then use valuations on the base X just like we used degrees of line bundles on the base space (dividing through by the power of p we needed to multiply by, afterwards).

Once we have degrees of line bundles, we can associate to any rank n vector bundle $\mathcal{E}$ a slope, defined as

$$\text{slope}(\mathcal{E}) := \frac{\deg(\bigwedge^n \mathcal{E})}{\text{rank}(\mathcal{E})}$$

The stable objects/vector bundles of $\mathcal{E}$ are precisely those which have all subobjects/vector bundles of slope strictly less than the slope of $\mathcal{E}$. This naturally raises the following question:

Question 6.3.2. *For X a scheme in characteristic p with corresponding perfection X^{perf} , where line bundles have a notion of degree on X (and consequently on X^{perf}), when will stable vector bundles pullback to stable vector bundles?*

Remark 6.3.3. *In general, we know stability typically is not preserved under pullback. Thus, any investigation into this question requires deeply studying the perfection map, and probably restricting to very specific rings.*

Chapter 7

A GAGA Principle for Perfections?

Our original goal was to say something about Picard groups in the analytic setting. Namely, we wanted to generalize the isomorphism $\mathrm{Pic}(\mathbb{P}_K^{1,\mathrm{ad},\mathrm{perf}}) \cong \mathbb{Z}[1/p]$ to higher-dimensional projectivoid spaces. This then naturally led us to wanting to show that for appropriately nice (i.e. those covered by Banach K -algebras, where K is non-trivially valued) adic spaces in characteristic p , we have

$$\mathrm{Pic}(X^{\mathrm{perf}}) \cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

The analogous result holds for quasicompact, quasi-separated schemes.

The main tool we typically have at our disposal for being able to move from the algebraic category to the analytic category is the nonarchimedean GAGA Theorem, whose proof follows the same spirit as Serre's in the complex algebraic/analytic setting. Here, we write down the statement of this result, for completeness:

Theorem 7.0.1 (GAGA). *For fixed nonarchimedean field K , let X be a proper K -scheme. There exists a unique adic space X^{ad} over $\mathrm{Spa}(K, K^\circ)$, and unique morphism*

$f : X^{\text{ad}} \longrightarrow X$ of locally-ringed spaces. Moreover, the category of coherent sheaves over X is naturally equivalent to the category of coherent sheaves over X^{ad} .

In particular, every vector bundle on X^{ad} arises as the pullback of a vector bundle on X . Assuming our base field K contains a topologically nilpotent unit, so that all of our rings corresponding to the analytification of X (which GAGA guarantees us) are quotients of Tate algebras (hence Banach K -algebras), we know that we can form its completed perfection as an adic spaces $X^{\text{ad,perf}}$. Putting everything together, we have the following diagram of locally-ringed spaces in characteristic p :

$$\begin{array}{ccc} X^{\text{perf}} & \xrightarrow{\phi} & X \\ & & \uparrow f \\ X^{\text{ad, perf}} & \xrightarrow{\psi} & X^{\text{ad}} \end{array}$$

We note that $\phi : X^{\text{perf}} \longrightarrow X$ satisfies a universal property in the category of locally-ringed spaces; this simply follows from the fact that perfect closures satisfy a universal property in the category of rings in characteristic p . Thus, we can make the above a Cartesian diagram in the category of locally-ringed spaces:

$$\begin{array}{ccc} X^{\text{perf}} & \xrightarrow{\phi} & X \\ \uparrow \exists! & & \uparrow f \\ X^{\text{ad, perf}} & \xrightarrow{\psi} & X^{\text{ad}} \end{array}$$

In particular, what we observe then, is that for any perfectoid field K in characteristic p (which is necessarily its own tilt), projectivoid space arises as the analytification of the perfection of regular projective space over K .

This now leads us to wonder whether a GAGA-style result holds on the perfect side of the above square. Let me be more precise.

Question 7.0.2. *Fix a nonarchimedean field K that contains a topologically nilpotent unit, and let X be a proper K -scheme. Is it true that the category of coherent sheaves over its perfection X^{perf} is equivalent to the category of coherent sheaves over the completed perfection $(X^{\text{ad}})^{\text{perf}}$ of its analytification X^{ad} ? If not, then what are sufficient conditions on the rings which cover X to ensure such an equivalence?*

I don't think this is an easy question to answer, and certainly not when stated at this level of generality. I am currently bereft of the tools needed to make any progress towards answering this question. Perhaps restricting to a more specific problem may be a better strategy. We can consider the perfect closure of a ring R in characteristic p as a direct limit of rings R^{1/p^n} in which we have adjoined all p^n th roots of elements in R . Since we are working with a proper K -scheme (locally of finite-type), it suffices that $R \cong K[T_1, \dots, T_l]/(f_1, \dots, f_m)$. For our purposes of dealing with perfectoid spaces, let us also assume that K is perfect; in this case it is a simple exercise to show that:

$$R^{1/p^n} \cong \frac{K[T_1^{1/p^n}, \dots, T_l^{1/p^n}]}{\langle f_1^{1/p^n}, \dots, f_m^{1/p^n} \rangle}$$

Notice that R^{1/p^n} is also finitely-generated over K (perfect) as an algebra, hence we have that it is necessarily finitely-generated as an R -algebra. Since R^{1/p^n} is an integral, finite-type extension of R , the corresponding map of prime spectra will necessarily be finite, and thus proper. This argument glues for X a proper K -scheme (again, K a perfect nonarchimedean field in characteristic p), and so X^{1/p^n} (the schemes corresponding to replacing all rings R of X with R^{1/p^n}) is therefore a proper K -scheme (composition of proper morphisms remain proper), and so we have that it is

analytifiable; denote it by $(X^{1/p^n})^{\text{ad}}$.

We can form perfection on the analytic side. For the adic space X^{ad} associated X , we can replace all rings with the completions of taking p^n th roots, to obtain an adic space $(X^{\text{ad}})^{1/p^n}$. By using the universal properties that analytification as well as taking p^n th roots satisfy, respectively, in the category of locally-ringed spaces, we obtain that $(X^{\text{ad}})^{1/p^n} \cong (X^{1/p^n})^{\text{ad}}$ as locally-ringed spaces (hence as objects in $\mathcal{V}$). Thus, at every finite level n , we have taking ‘ p^n th closures’ commutes with analytification.

$$\begin{array}{ccc} X^{1/p^n} & \longrightarrow & X \\ \uparrow & & \uparrow \\ (X^{1/p^n})^{\text{ad}} & \longrightarrow & X^{\text{ad}} \end{array}$$

Going back over to the algebraic side, we know all transition maps $X^{1/p^{n+1}} \rightarrow X^{1/p^n}$ are finite, hence necessarily affine. It follows from [11] Tag 01YX that the inverse limit scheme does in fact exist, and it is what we should hope it should be:

$$X^{\text{perf}} \cong \varprojlim_n X^{1/p^n}$$

However, the story on the analytic side is where we now stumble. In general, it does not make sense to talk about inverse limits of adic spaces; we know how to topologize rings at every finite level (of taking p^n th roots and then completing), but it’s not clear how to topologize the limit rings. That is, the category of affinoid K -algebras, in general, does not contain direct limits; this was an issue Huber was aware of quite some time ago, when he first introduced adic spaces.

At this point, it should be clear what we are trying to do: if (and this is a big if) we could make sense of inverse limits on the analytic side, then it would

necessarily follow from the universal property that inverse limits enjoy combined with the universal property that completed perfection enjoy, we have that

$$(X^{\text{ad}})^{\text{perf}} \cong \varprojlim_n (X^{\text{ad}})^{1/p^n}$$

as adic spaces. Thus, at every finite level n , we have the following projection maps:

$$\begin{array}{ccccc} X^{\text{perf}} & \xrightarrow{\pi_n^{\text{alg}}} & X^{1/p^n} & \xrightarrow{\pi_{n,0}^{\text{alg}}} & X \\ \uparrow & & \text{GAGA} \uparrow & & \text{GAGA} \uparrow \\ (X^{\text{ad}})^{\text{perf}} & \xrightarrow{\pi_n^{\text{ad}}} & (X^{1/p^n})^{\text{ad}} & \xrightarrow{\pi_{n,0}^{\text{ad}}} & X^{\text{ad}} \end{array}$$

And so we arrive at the following (only) slightly more specific question.

Question 7.0.3. *Given an inverse system of proper K -schemes, with finite transition maps, such that the inverse limit exists, and the corresponding inverse limit of associated adic spaces also exists, when can we expect a GAGA principle to hold between these inverse limits?*

Chapter 8

Conclusion

We were able to classify all line bundles on the projectivoid line and its characteristic p tilt, using methods that were specific to the case of dimension 1. By use of the standard cover $\mathfrak{U}$ of the projectivoid line by two perfectoid discs, we were able to show that $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) = \mathbb{Z}[1/p]$ via repeated use of the Snake Lemma. The goal was then to show acyclicity of the sheaf of units on the perfectoid disc and circle (i.e. all overlaps of the sets in $\mathfrak{U}$), so that we could appeal to Leray's Theorem, which for our purposes would produce the isomorphism $H^1(X, \mathcal{O}_X^\times) \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$.

To show vanishing of first cohomology groups for the sheaf of units, we used the fact that the rings in question were Bezout domains, and the equivalence of finite projective modules over the ring of global sections of an affinoid adic spaces to finite rank vector bundles over the affinoid space.

To show vanishing of cohomology in degrees 2 and higher, we showed the perfectoid disc and circle are homeomorphic to the regular adic disc and circle, respectively; both of the latter spaces are of Krull dimension 1. This relied on being able to extend continuous valuations to the perfect closure of certain rings (i.e those

endowed with a power-multiplicative norm) in characteristic $p > 0$. Using that cohomology of any abelian sheaf on a spectral space vanishes in degrees higher than the Krull dimension of the spectral space then yielded the vanishing of cohomology in degrees 2 and up, and so the hypotheses for Leray's Theorem hold.

The above isomorphism spurred on many natural questions, in particular leading us to ask about comparison of Picard group of appropriately nice adic spaces in characteristic p and their corresponding perfections. Such then led us to a natural way to extend degrees of line bundles to perfections of schemes X , a definition we would hope to be able to extend to the relevant adic settings. Perhaps most interesting of all questions that came about was whether a GAGA statement holds between perfections of proper K -schemes and the perfections of the corresponding analytification, when K was a nonarchimedean field of characteristic $p > 0$.

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