Title
COURSE IN THE THEORY AND DESIGN OF PARTICLE ACCELERATORS - LECTURE 11

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Course in the Theory and Design
of Particle Accelerators

LECTURE 11
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(Notes by: Ralph Peters)

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FOCUSSING

Focussing bears about the same relation to Particle Accelerators as a steering gear bears to automobiles. Some focussing, or at least an absence of defocussing, is always necessary for a Particle Accelerator, especially the circular varieties. At high velocities particles tend to stay centered in a Linear Accelerator.

Focussing is done by electrostatic and magnetic fields. In circular machines practically all the focussing is done by the magnetic field. The magnetic equipotential lines are concave toward the center of the machine. Since a particle moving across a magnetic field is pushed at right angles to the lines of flux a concave curved field always pushes a circulating particle toward the median plane of the machine. These curved lines, required for focussing, require that the magnetic field get weaker as the radius increases; but the field must not drop too rapidly with radius or the particle will spiral out rapidly and be lost.

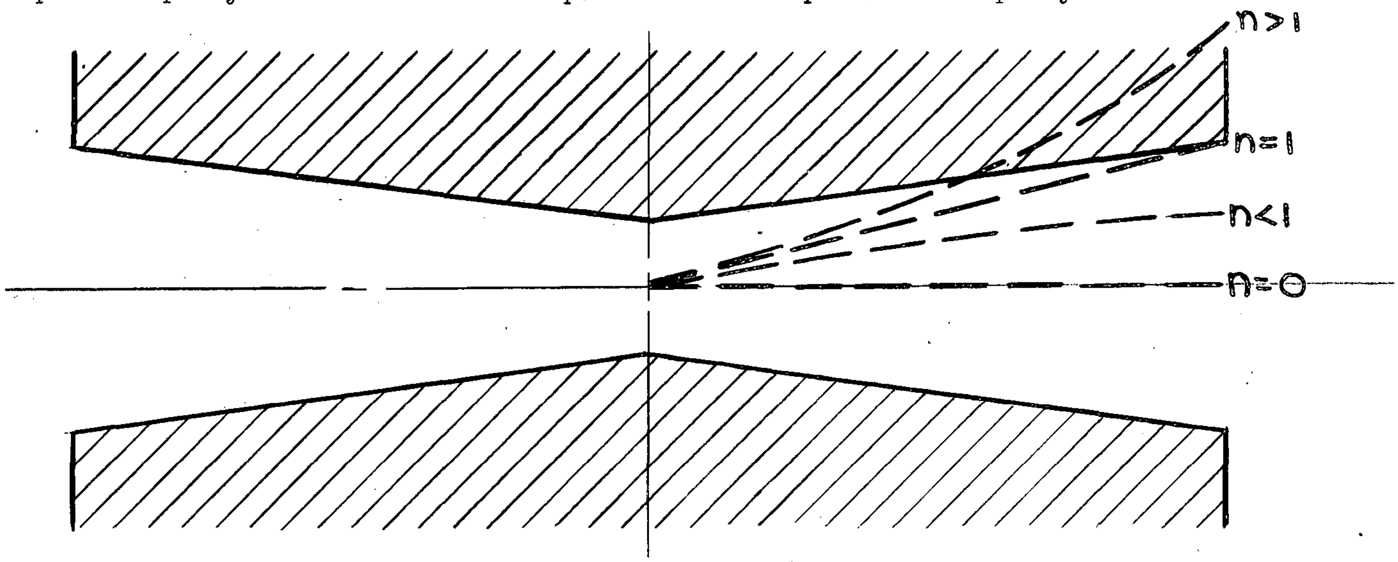

Figure I

Slope of Magnetic Field = $\frac{dB}{dR}$ Gauss/cm. This is the rate at which the field changes with radius.

DISCLAIMER

The field exponent generally called n is defined as:

$$n = -\frac{\frac{dB}{B}}{\frac{dR}{R}} \qquad (1)$$

This is dimensionless and $= \dfrac{\text{change in field over field}}{\text{change in radius over radius}}$

n is positive if B decreases as R increases, which is the case illustrated in Figure II, so in Equation (1) dB would be negative. n is not necessarily constant but if it is, then over a small region:

$$\int_{B_o}^{B} \frac{dB}{B} = -n \int_{R_o}^{R} \frac{dR}{R}$$

Integrating:

$$\log \frac{B}{B_o} = -n \log \frac{R}{R_o}$$

$$\frac{B}{B_o} = \left(\frac{R}{R_o}\right)^{-n} = \left(\frac{R_o}{R}\right)^{n}$$

$$B = B_o R_o^{\ n} \left(\frac{1}{R}\right)^{n}$$

We see that B is inversely proportional to R to some power of n. If a magnet of steel with infinite permeability is assumed, then approximately:

$$B \propto \frac{1}{gap}$$

and

$$\frac{1}{gap} = \left(\frac{1}{R}\right)^{n}$$

or

$$gap = R^{n}$$

If $n = 1$ then steel pole faces would touch at the center of the machine. See Figure I. A Cyclotron, of course, with a solid core cannot be built with $n = 1$ (there would be zero gap at the center), but a hollow center machine like a Synchrotron, for example, does not have this inherent limitation.

n generally varies slowly so over a small region assume it to be constant. Also assume iron of infinite permeability so that the field is inversely proportional to the gap. Bg = constant. From Figure II we can say

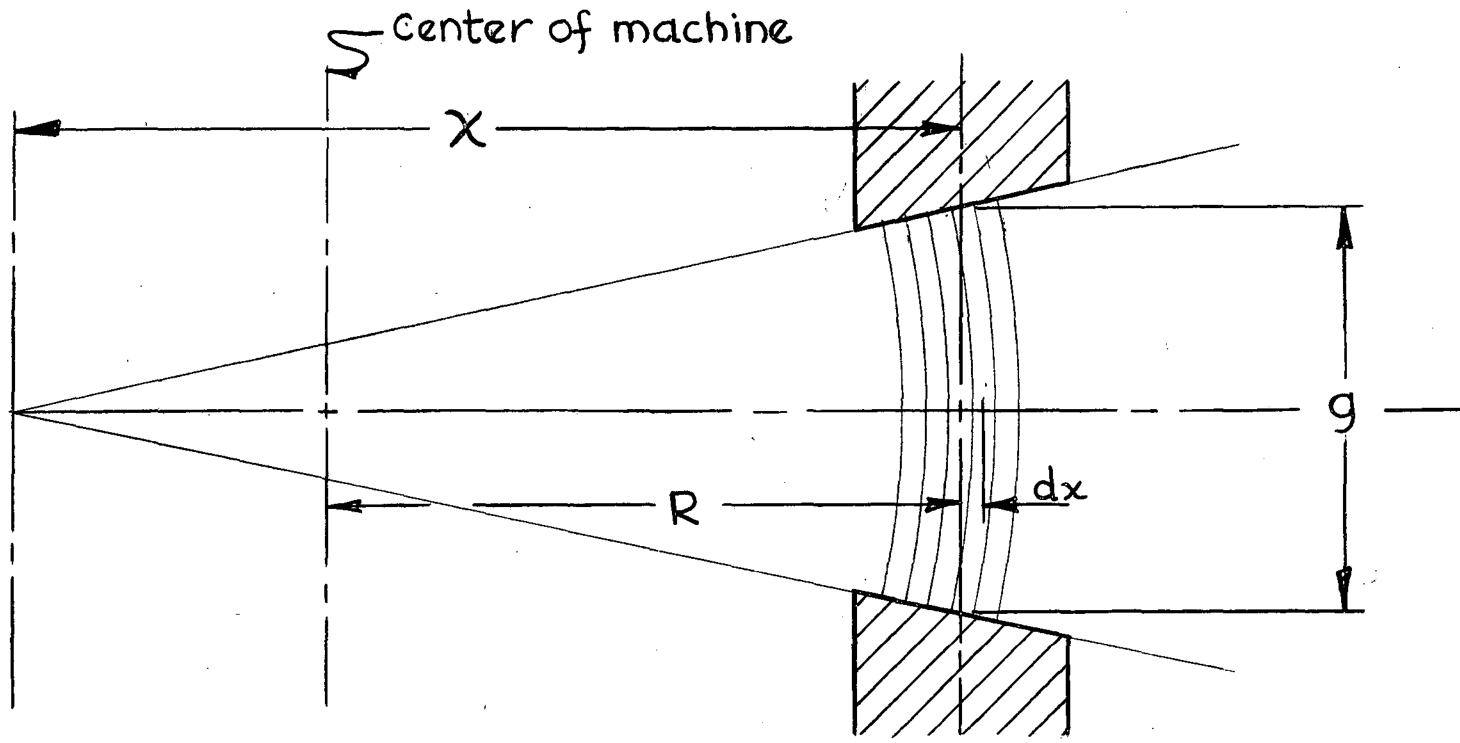

Figure II

B is the field strength at radius R. at radius R + dx the field is B + dB and the gap is g + dg.

$$\frac{B+dB}{B} = \frac{g}{g+dg} = \frac{x}{x+dx} = \frac{x-dx}{x}$$

or

$$1 + \frac{dB}{B} = 1 - \frac{dx}{x}$$

$$\frac{dB}{B} = - \frac{dx}{x}$$

From (1) we get $\frac{dB}{B} = - n \frac{dR}{R}$

so

$$\frac{dx}{x} = n \frac{dR}{R} \text{ or } x = \frac{Rdx}{ndR}$$

dx and dR are the same thing so

$$x = \frac{R}{n}$$

For the Bevatron n = .6 and R is about 600". So $x = \frac{600}{.6}$ = 1000". This is the distance from the center of the gap to the point where the slopes of the pole faces would intersect. Actual magnetic fields, due to such things as fringing fields and iron losses, are not exactly as predicted by this analysis so that measurements of actual fields have to be made for accurate information.

If the particle is oscillating radially, F_M is not equal to F_c. Then the unbalance in F_M and F_c is what produces radial acceleration and velocity.

For a particle with radial oscillation:

$$F_c - F_M = Ma = M\frac{d^2x}{dt^2} \tag{2}$$

$$\frac{Mv^2}{R} - \frac{Bev}{c} = \frac{Md^2x}{dt^2} \tag{3}$$

From Figure 3

$$R = R_o + x$$

and

$$B = B_o + \frac{dB}{dR}x$$

Rewriting

$$B = B_o \left[1 + \left(\frac{dB}{dR}\right)\frac{x}{B_o}\right]$$

$$= B_o \left[1 + \frac{dB}{dR}\frac{x}{B_o}\frac{R_o}{R_o}\right]$$

$$= B_o \left[1 + \frac{dB}{dR}\frac{R_o}{B_o}\frac{x}{R_o}\right]$$

But from Equation (1)

$$\frac{dB}{dR}\frac{R_o}{B_o} = -n$$

So

$$B = B_o \left[1 - \frac{nx}{R_o}\right] \tag{4}$$

$$\frac{Mv^2}{R} = \frac{Mv^2}{R_o+x} = \frac{Mv^2}{R_o(1+\frac{x}{R_o})}$$

If $\frac{x}{R} \ll 1$ then approximately this is also equal to

$$\frac{Mv^2}{R_o}\left(1 - \frac{x}{R_o}\right) \tag{5}$$

Combining Equations (3), (4) and (5) we may write

$$\frac{Mv^2}{R_o}\left(1 - \frac{x}{R_o}\right) - B_o\left[1 - \frac{nx}{R_o}\right]\frac{ev}{c} = \frac{Md^2x}{dt^2}$$

$$\frac{Mv^2}{R_o} - \frac{Mv^2x}{R_o^2} - \frac{B_oev}{c} + \frac{B_onxev}{R_oc} = \frac{Md^2x}{dt^2} \tag{6}$$

RADIAL MAGNETIC FOCUSSING IN UNINTERRUPTED CIRCULAR MACHINES

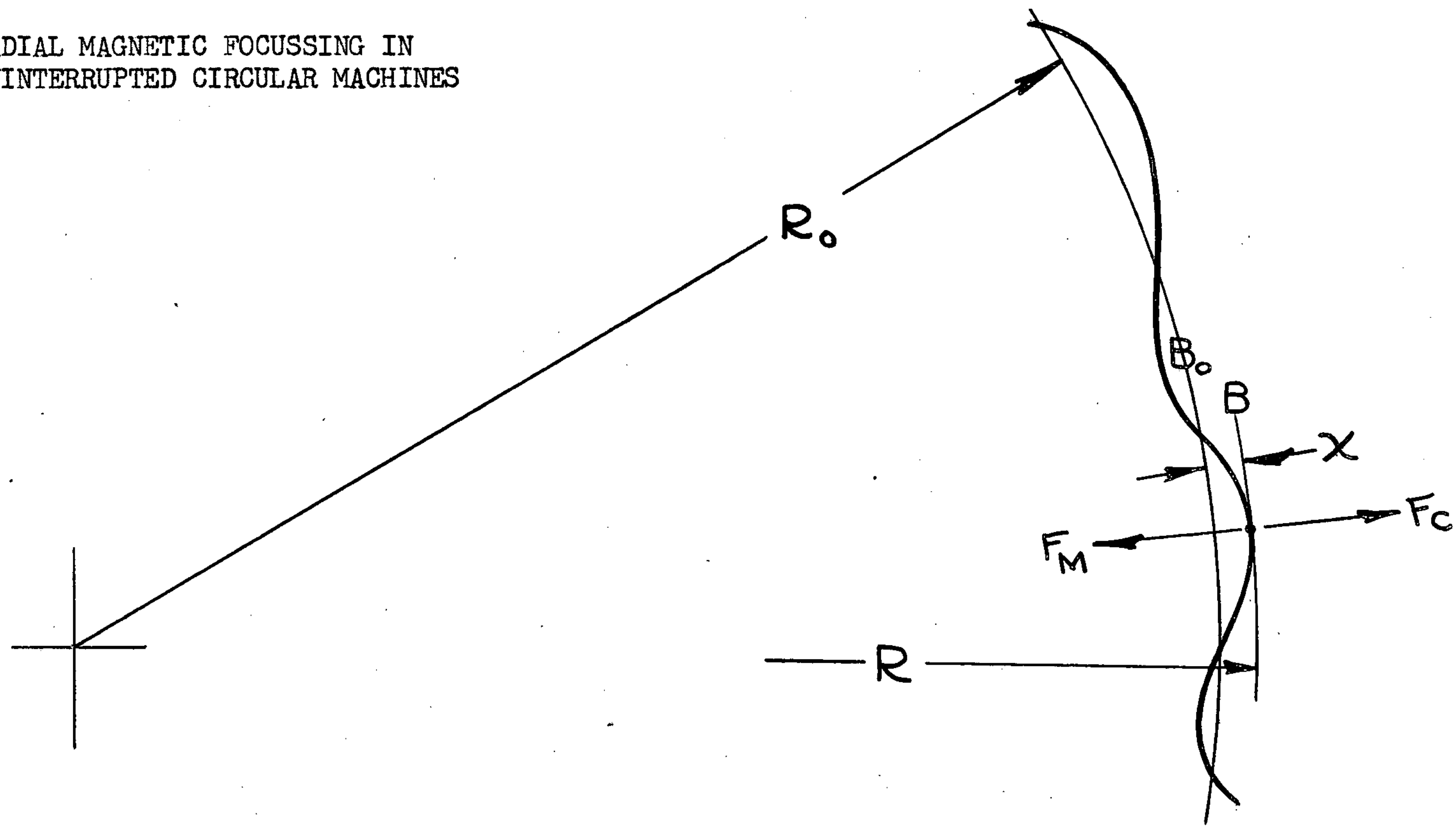

Figure III

F_c = centrifugal force on particle

$$= Ma = \frac{Mv^2}{R}$$

Here M is mass of particle, a is the radial acceleration of the particle, v is the velocity of the particle, and R is its radius of curvature. The mass used here is the relativistic mass that goes with the orbital velocity, v. The radial velocity is small and introduces no further change in mass.

F_M = force of magnetic field on particle at right angles to its velocity direction.

$$F_M = \frac{Bev}{c}$$

If the particle is moving on its equilibrium orbit

$$F_M = F_c$$

or

$$\frac{B_o ev}{c} = \frac{Mv^2}{R_o}$$

B_o and R_o are field and radius on the equilibrium orbit.

But $\frac{Mv^2}{R_o} = \frac{B_o ev}{c}$ since at the moment the particle crosses the equilibrium orbit it is not being accelerated radially because the centrifugal and centripetal forces are equal. So rewrite (6)

$$-\frac{Mv^2 x}{R_o^2} + \frac{B_o ev}{c} \quad \frac{nx}{R_o} = \frac{Md^2x}{dt^2}$$

$$-\frac{Mv^2 x}{R_o^2} + \left(\frac{Mv^2}{R_o^2}\right) nx = \frac{Md^2x}{dt^2}$$

Canceling M

$$\frac{v^2}{R_o^2} (-1 + n) x = \frac{d^2x}{dt^2}$$

$$\frac{v^2}{R_o^2} = \omega_o^2$$

ω_o = rotational frequency of particle

$$\omega_o^2 (n - 1) x = \frac{d^2x}{dt^2}$$

or also

$$-\omega_o^2 (1 - n) x = \frac{d^2x}{dt^2}$$

This is a differential equation of simple undamped harmonic motion with a solution

$$x = A \sin \left[\omega_o \sqrt{1 - n}\right] t$$

If we call the bracketed quantity ω_R then $x = A \sin \omega_R t$ where ω_R is the angular velocity of the vector representing radial oscillation. $\frac{F_R}{F_o}$ is the ratio of the frequency of radial oscillations to the frequency of the rotation of the particle in its orbit.

$$\frac{\omega_R}{\omega_o} = \frac{F_R}{F_o} = \sqrt{1 - n}$$

n	$\frac{F_R}{F_o}$	
1	0	This is the case with a flat magnetic field where there is no change in field with radius. The orbit can exist anywhere in the field. There is no force tending to center the orbit over the field center. This is the limit of stability.
>> 1	-	The field decreases rapidly with radius. The particle spirals outward rapidly. It is an unstable arrangement and must be avoided except for deflectors or in "strong focussing" where it exists for a short distance only.
< 0	> 1	The field increases with increasing radius. This is strongly focussing radially but cannot be used except under special conditions because it is vertically or axially defocussing and particles are lost up and down.
2/3	.58	This is Berkeley Synchrotron. Particle makes 1/.58 or 1.7 revolutions per cycle of radial oscillation.
.6	.63	This would be the Bevatron if it had no straight sections. The actual ratio $\frac{F_o}{F_R}$ with straight sections is 1.47. Brookhaven has the same n and 4 quadrants but since R and the length of straight section is different there $\frac{F_o}{F_R}$ is 1.43. Note in Figure IV that this theoretical number of revolutions per radial oscillation cycle is fairly well observed by measurement.

RADIAL AND VERTICAL OSCILLATIONS OF PROTON OBSERVED AFTER INJECTION ON BROOKHAVEN COSMOTRON

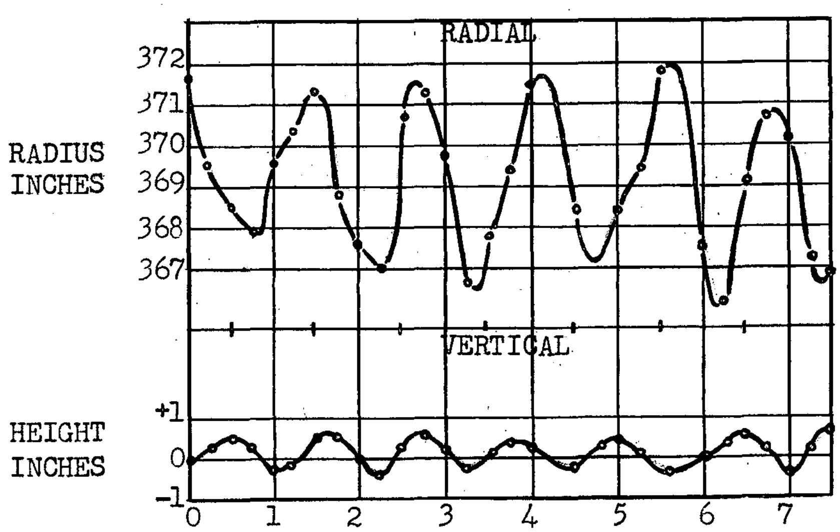

Figure IV

Generally, in circular accelerators, a particle does not move in circles whose center coincides with the center of the machine. Generally, F_R is not equal to F_o and the particle can be thought of as moving about a center which center is moving slowly in small circles about the center of the machine. The center of the orbit precesses about the center of the machine. ω_p is the angular velocity of the precession and

$$\omega_p = \omega_o - \omega_R$$
$$= \omega_o - \omega_o \sqrt{1 - n}$$
$$= \omega_o (1 - \sqrt{1 - n})$$

If $\omega_R = \omega_o$ there is no precession. If $n < 1$, ω_p comes out positive which means the center of the orbit is precessing about the center of the machine forwards, or in the same direction as the rotation of the beam.

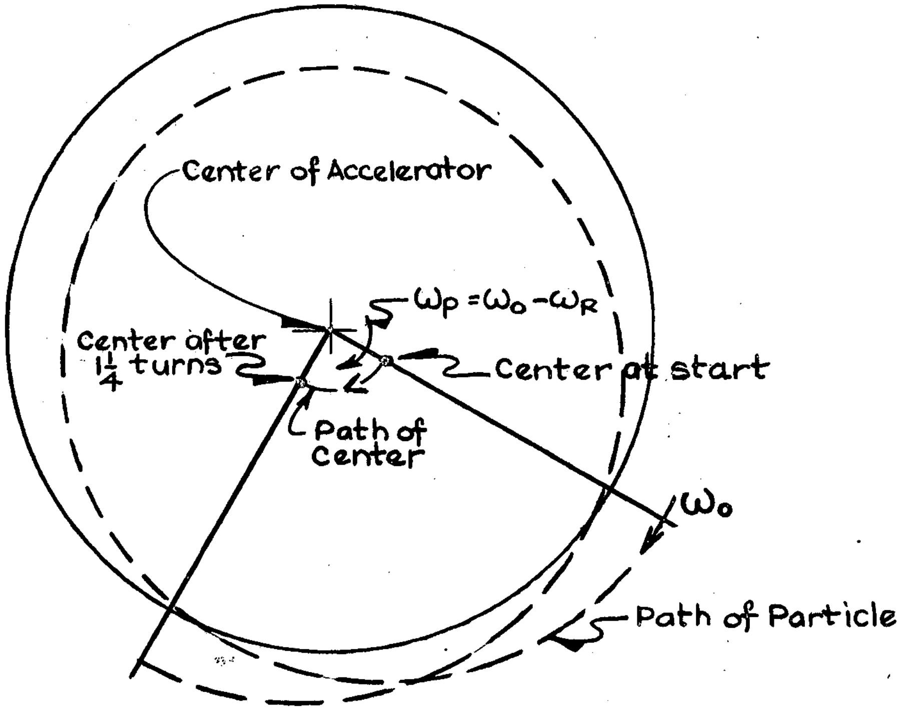

Figure V

Radial oscillations can be studied or predicted by plotting curves shown in Figures VI and VII.

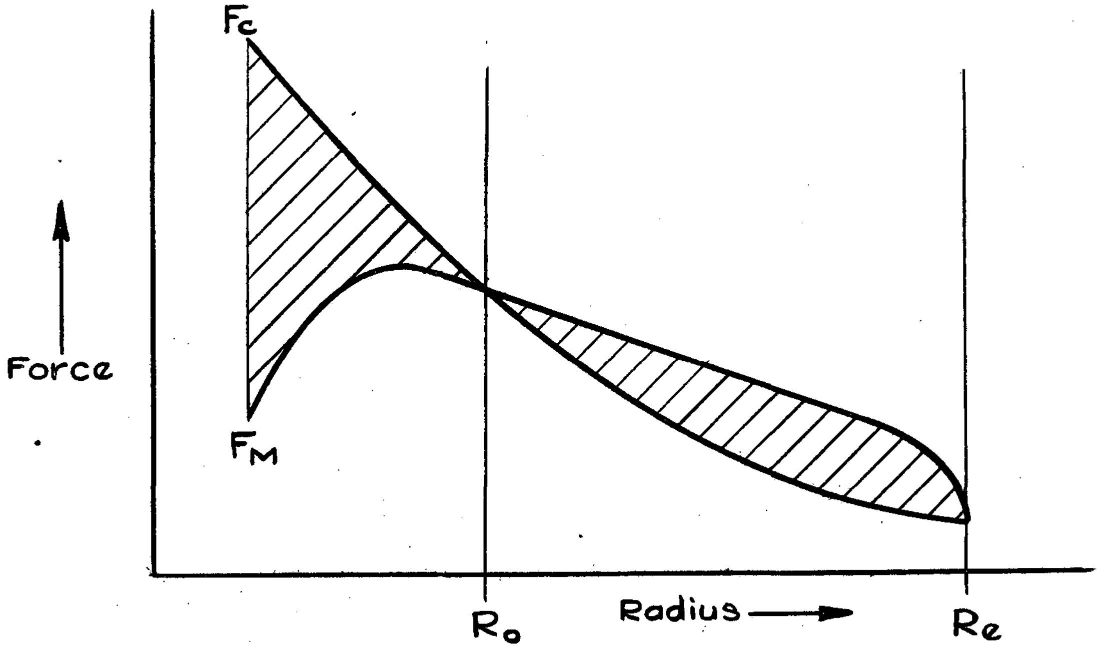

Figure VI

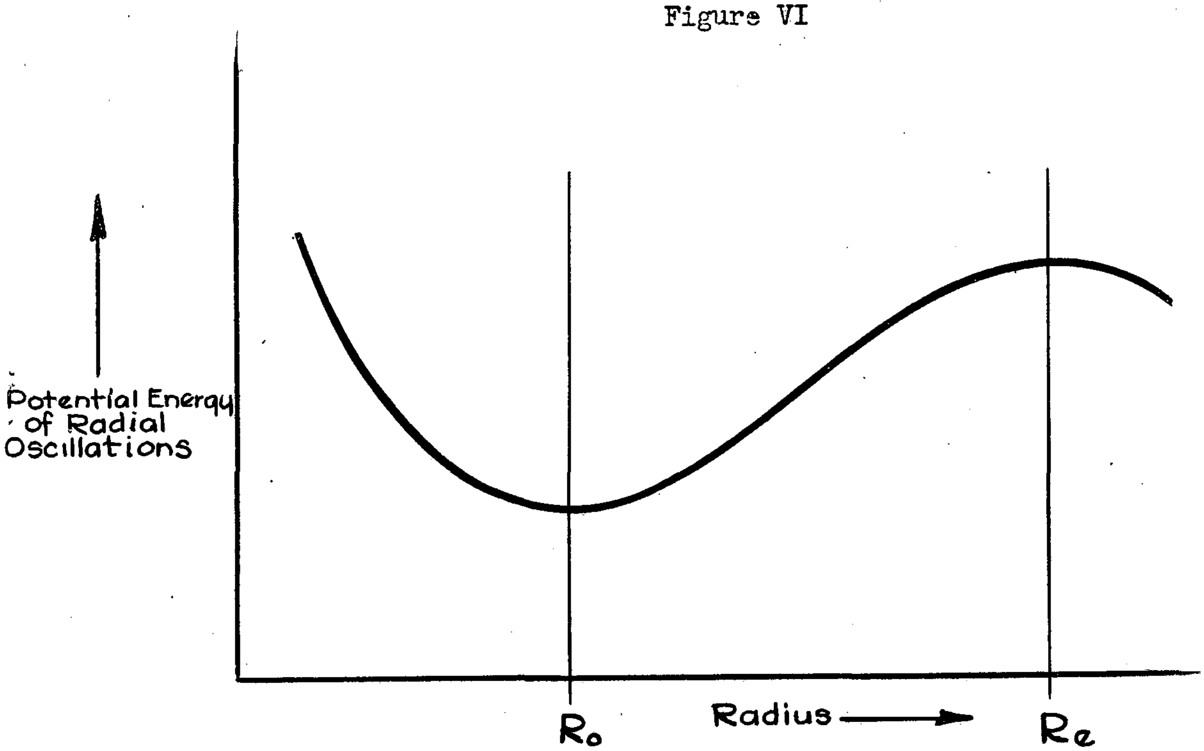

Figure VII

These curves would be plotted for a particular field shape, particle mass, and velocity. The shaded area in Figure VI is the stable region. R_o is the equilibrium orbit about which the particle makes radial oscillations. In Figure VI, a particle that finds itself somewhat to the right of R_o experiences an inward force due to B that is greater than the centrifugal force on it so it is accelerated inward. This accelerating force inward goes to zero as the particle crosses R_o. The particle overshoots R_o and finds itself in a region where F_c is greater than F_M so it starts back out again on its cycle of oscillations about R_o. If, however, for some reason on an outward swing of its

Mass is constant. It is not changing due to its small vertical velocity.

$$F_z = -\frac{z}{x} F_M = -\frac{z}{x}\frac{Bev}{c}$$

$$= -\frac{z}{\frac{R}{n}} \quad \frac{Bev}{c}$$

If the particle is on its equilibrium orbit this can also be written

$$F_z = -\frac{zn}{R}\frac{Mv^2}{R} = -\frac{znMv^2}{R^2}$$

$$= -znM\omega_o^2 \qquad (8)$$

Equations (7) and (8) are equal so

$$F_z = M\frac{d^2z}{dt^2} = -znM\omega_o^2$$

$$\frac{d^2z}{dt^2} = -n\omega_o^2 z$$

This is an equation of simple harmonic motion with a solution

$$z = A \sin \sqrt{n}\,\omega_o t$$

Set

$$\sqrt{n}\;\omega_o = \omega_z$$

This is the angular velocity of the vector representing vertical oscillation.

$$z = A \sin \omega_z t$$

n	$\frac{\omega_z}{\omega_o}$	
0	0	This is a flat magnetic field. There is no vertical restoring force. As a particle gets started so it keeps on going with constant vertical velocity.
1	1	The particle makes one complete vertical cycle for each revolution.
>1	>1	The particle is executing more than one vertical oscillation per revolution. The plane of the orbit is moving around.
<0	-	This unstable particle is lost vertically. It started up or down and kept right on going with increasing velocity.

Both radial and vertical oscillations are stable for n between 0 and 1.

oscillation it goes past R_e where $F_M = F_c$, it will be lost. It keeps right on spiraling out. The curve in Figure VII is called a potential curve. The particle oscillates radially about the low point in the curve which is the equilibrium orbit. If it goes over the hump at R_e it is lost. Particles are not lost to the inside radius, but if they go too far in, on the next outward swing they will overshoot R_e and be lost. Figure VII is a plot of the integral of the net radial force on a particle times its distance from the equilibrium orbit. It is the area between curves F_c and F_M between a radius R and R_o plotted at R.

AXIAL OSCILLATIONS
(VERTICAL)

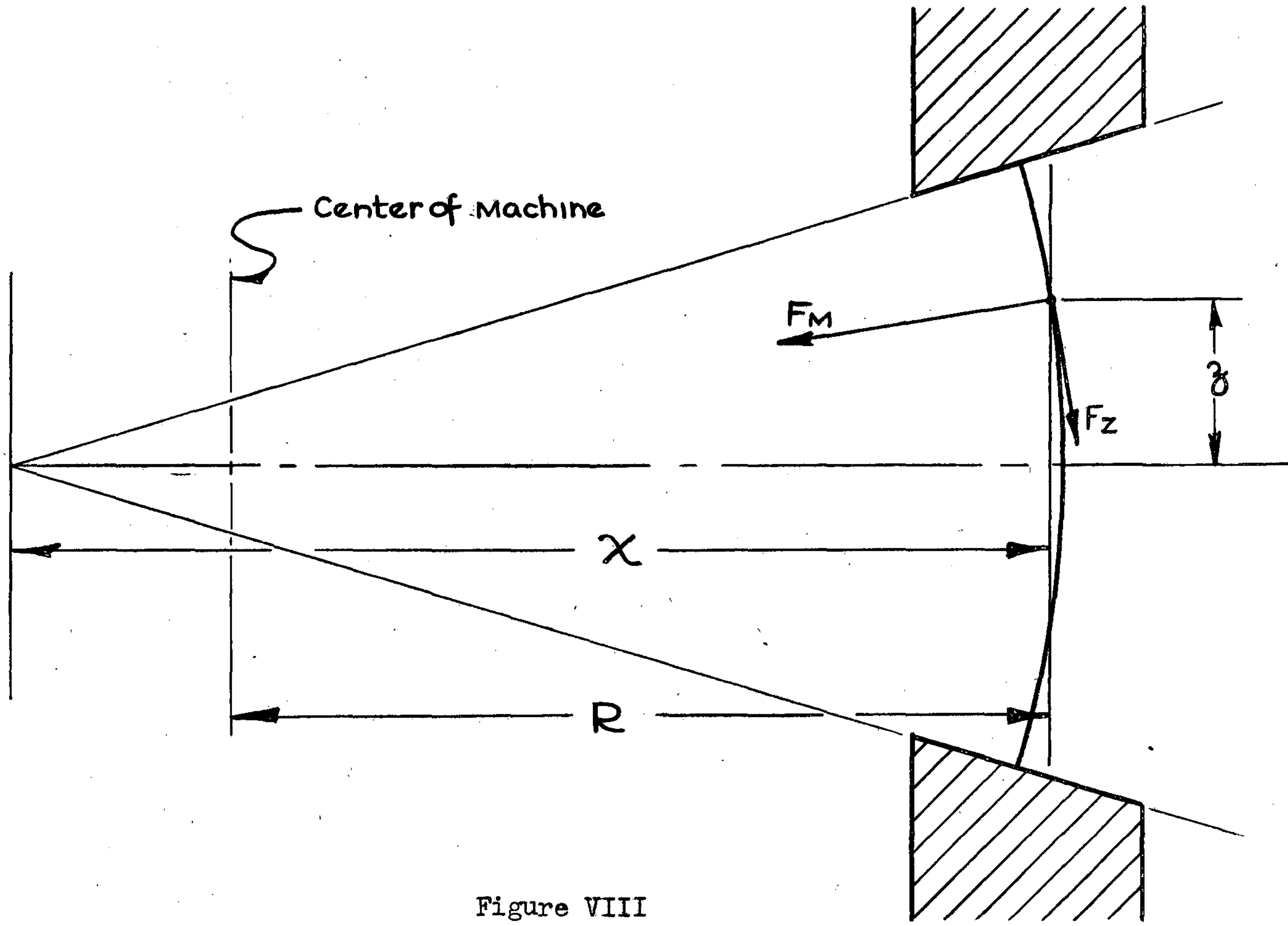

Figure VIII

From Geometry

$$F_z = -\frac{z}{x} F_M$$

The minus means force is toward center.

Also

$$x = \frac{R}{n}$$

$$F_z = Ma = \frac{Md^2z}{dt^2} \qquad (7)$$

RELATIONSHIP BETWEEN RADIAL
AND VERTICAL OSCILLATIONS

The formulae in this section apply to continuous ring accelerators like a Betatron, Synchrotron or Cyclotron.

Previously it was shown that:

$$n = -\frac{\frac{dB}{B}}{\frac{dR}{R}}$$

$$\omega_R = \omega_o \sqrt{1 - n}$$

$$\omega_z = \omega_o \sqrt{n}$$

Rational ratios among ω_R, ω_z, and ω_o need to be avoided. If a rational ratio exists between any two of these, energy will feed from one mode into the other and vice versa, possibly causing a disastrous amplitude of oscillation in one or the other mode. Rational ratios between ω_R and ω_z do not necessarily lose beam but generally do. Amplitude is limited but may be too great for available space. Resonance with ω_o is disastrous as no limit exists to energy that can be received. A mechanical analogy of this phenomenon is a weight on a coil spring which has both axial and rotational modes of oscillation. The energy from one mode can go into an increased amplitude in the other mode. In 184" no beam gets past the radius where n = .2 and $\frac{\omega_z}{\omega_R} = \frac{1}{2}$. All particles at this radius go into large vertical oscillations and are wiped off by the top or bottom of the dee.

n	$\frac{\omega_R}{\omega_o}$	$\frac{\omega_z}{\omega_o}$	$\frac{\omega_z}{\omega_R}$	
	$\sqrt{1-n}$	$\sqrt{n}$	$\sqrt{\frac{n}{1-n}}$	
0	1	0	0	This is the case of one disturbance per revolution of the particle. ω_R becomes large.
1/5	$\sqrt{4/5}$	$\sqrt{1/5}$	1/2	Here energy is transferred between radial and vertical oscillations. This point limits the maximum radius on the 184".
1/4	$\sqrt{3/4}$	1/2	$\sqrt{1/3}$	Here energy is transferred between vertical and rotational frequencies because of the simple ratio.
1/2	$\sqrt{1/2}$	$\sqrt{1/2}$	1	Here there is strong coupling between vertical and radial oscillations.
3/4	1/2	$\sqrt{3/4}$	$\sqrt{3}$	Radial oscillation couples with rotational frequency.
1/5	$\sqrt{1/5}$	$\sqrt{4/5}$	2	Another point where radial and axial oscillations couple.
1	0	1	∞	This n is the limit of stability, and here vertical and rotational frequency are rational.

n should be chosen, among other restrictions, to avoid any of the values in the table above. A particle from beginning to end of its acceleration should never be in, or pass thru, a region where n has any of these values. The 184" has n going from zero at the center to .2 near the circumference. The Synchrotron has $n = 2/3$ which is more than 1/2 and less than 3/4 which are bad values. The Bevatron has $n = .6$. The above table applies to magnets of an uninterrupted circle and does not exactly apply to the Bevatron, but it does nearly so.

ELECTROSTATIC FOCUSSING IN THE
CYCLOTRON IN THE VERTICAL DIRECTION

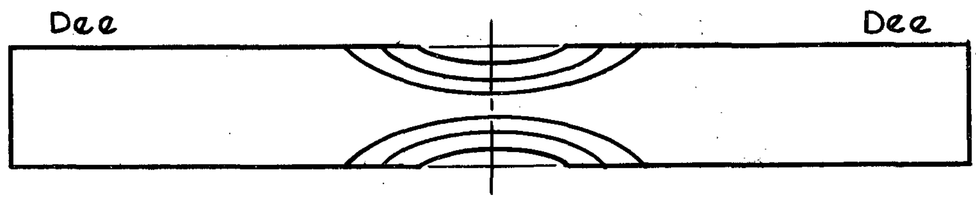

Figure IX

Figure IX represents the dees in a Cyclotron with opposite electrical charges. A particle crossing from one dee to the other is accelerated in a direction along the field lines shown. A particle crossing on the median plane receives no acceleration in the vertical direction. It is neither focussed nor defocussed. A particle crossing the gap above or below the median plane is driven toward the median plane during the first half of the traversal and away from the median plane during the last half. If it takes more time making the first half than it does making the last half of the crossing (which is the case for a particle gaining speed) and if the electric field remains constant while it is crossing, the particle will experience more focussing than defocussing simply because it was in the focussing field longer than it was in the defocussing field. Normally, the dee voltage is changing with time as the particle is making the crossing. If the voltage is falling as the particle crosses the gap, the particle experiences a focussing effect from this. In this case the electric field is stronger during the first half of the crossing, which is the focussing half, than it is during the second half.

So the focussing impulse is a function of phase, ϕ (voltage rising or falling) and ΔE, the amount of energy change during the crossing. For focussing, an effective value of n can be found which is the sum of the n of the magnet and the electrostatic focussing effect.

$$n = n_{mag} + n_{elec}$$

If

h = vertical dee height

V = peak voltage across dee gap

ϕ = phase angle (zero for no acceleration)

e = total energy of particle

R = radius of orbit

then

$$n_{elec} = \frac{1}{\pi\beta}\left[\underbrace{\frac{eV\cos\phi}{E}}_{\text{Term 1}} + \underbrace{\frac{2}{3}\left(\frac{R}{\beta^2 h}\right)\left(\frac{eV\sin\phi}{E}\right)^2}_{\text{Term 2}}\right]$$

Term 1 is the phase focussing component and Term 2 is the velocity focussing component. When enough things are known, this expression can be used to calculate the worst n_{elec} value that will be encountered. The magnet field shape must be so that it at least corrects by positive focussing the worst calculated electrostatic defocussing.

Radial focussing is generally not influenced by energy gain per turn nor by phase. There would be a radial effect if the voltage varied across the open face of the dee along the radius but it generally doesn't.

AMPLITUDE OF OSCILLATIONS

Amplitudes of oscillations cannot be calculated exactly. They are greatest at the beginning of acceleration and their amplitude during this period is considerably influenced by how the ions are injected into the machine.

Injection parameters are fairly well known for the Bevatron, less known for the Synchrotron and practically unknown for the 184".

Fortunately, both radial and vertical oscillation amplitudes diminish as the magnetic field strength goes up. This means that practically all the beam that gets thru the early stages of acceleration will not be lost.

$$A_{rad} \propto \frac{1}{B^{1/2}(1-n)^{1/4}}$$

$$A_z \propto \frac{1}{B^{1/2}\, n^{1/4}}$$

The change in B in the Cyclotron is so small as the particle moves out, that it doesn't help much in damping oscillations, but the change in n does help. In an Electron or Proton Synchrotron, the change in B is large and it is effective in damping oscillations.

EFFECT OF ERRORS IN THE MAGNETIC
FIELD ON OSCILLATIONS

(a) Azimuthal Variations:

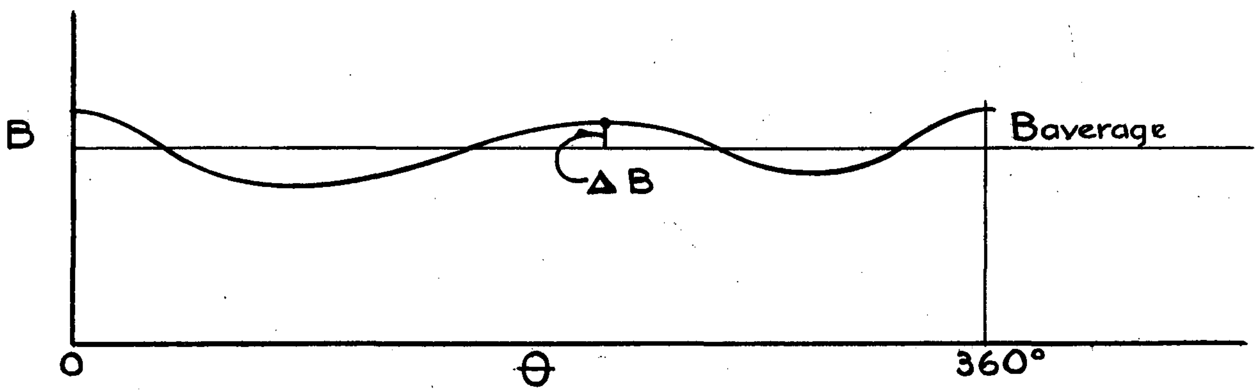

Figure X

Figure X represents what might be the actual magnetic field plot in a circular machine. $\frac{\Delta B}{B}$ can be represented by a Fourier series.

$$\frac{\Delta B}{B} = A_1 \cos(\theta + \alpha_1) + A_2 \cos(2\theta + \alpha_2)$$

$$+ A_3 \cos(3\theta + \alpha_3) + \ldots\ldots\ldots\ldots$$

$$= \sum_{\ell=1}^{\ell=\infty} A_\ell \cos(\ell\theta + \alpha_\ell)$$

The first terms in the series, represents the first harmonic, the second term, the second, etc.

It can be shown that:

$$\frac{\Delta R}{R} = \frac{A_1}{n} \cos(\theta + \alpha_1) + \frac{A_2}{3+n} \cos(2\theta + \alpha_2)$$

$$+ \frac{A_3}{8+n} \cos(3\theta + \alpha_3) .$$

$$= \sum_{\ell=1}^{\ell=\infty} \left(\frac{A_1}{\ell^2 + n-1}\right) \cos(\ell\theta + \alpha_\ell)$$

As a rule, not many terms in these series need to be considered because they get small rapidly. In a field as uniform as a Cyclotron, only the first term is significant. At weak fields in a Synchrotron, several terms need to be considered.

The effect of the first harmonic corresponds to a displacement of the center of the orbit from the center of the machine by an amount $\Delta R = \frac{R}{n} A_1$.

(b) Variations in Height of Median Plane:

If the median plane is $\perp$ to the axis, the orbit will be displaced the same amount as the median plane.

If the median plane is tilted, the orbit plane is tipped in the opposite direction at an angle $\frac{n}{1-n}$ times the angle of the median plane - proved below.

The lines of flux are approximately spheres centered on the intersection of the pole faces c. The median plane (which generally isn't a plane, but a surface) is the surface connecting the points where the field is parallel to the axis of the machine.

The particles make vertical oscillations about the median plane. The orbital plane contains the centers c.

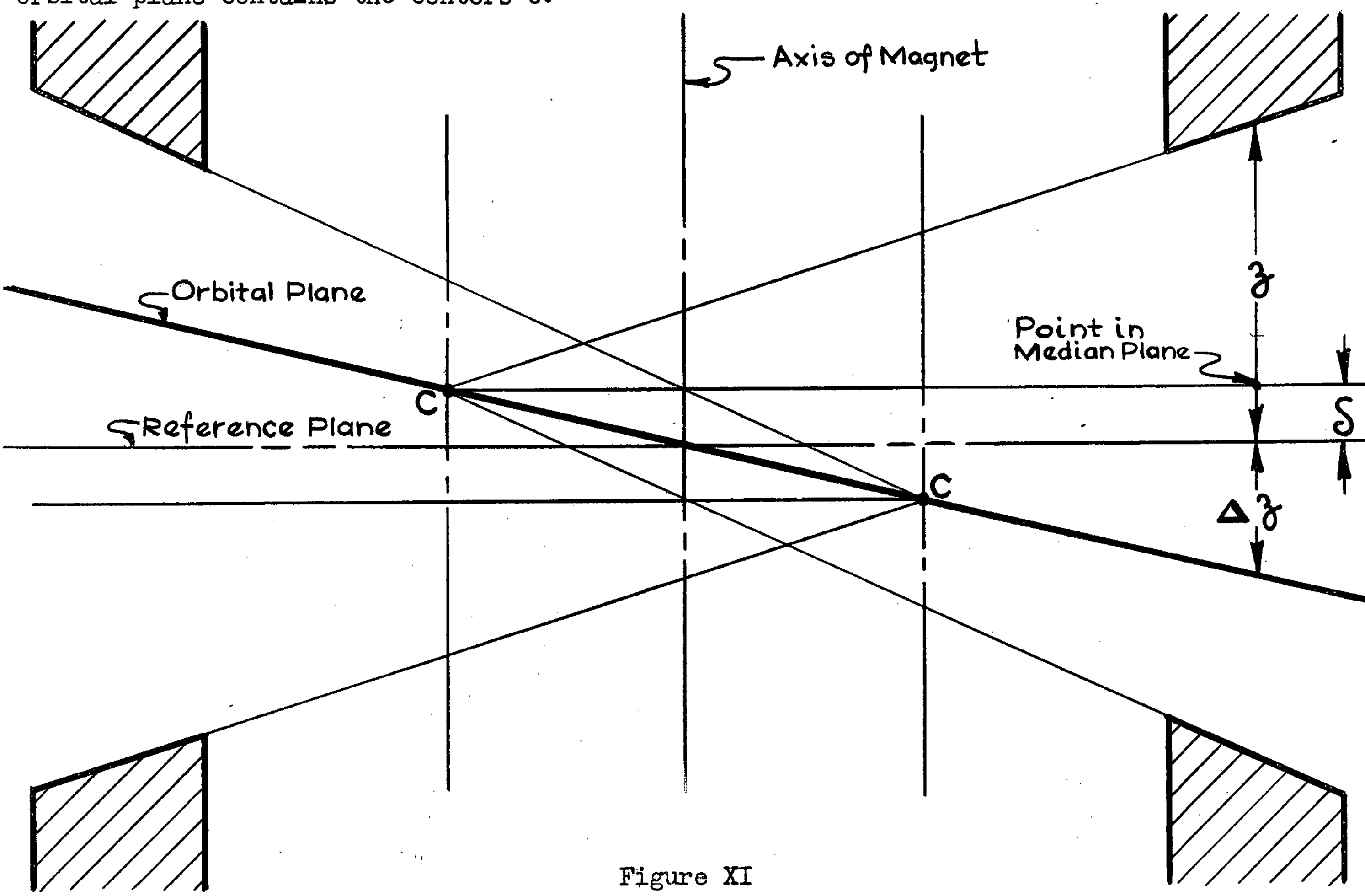

Figure XI

Reference plane is perpendicular to axis of machine. Reference plane intersects axis of machine at average aperture height. δ is distance from median plane to reference plane. Δz is distance from reference plant to orbital plane in gap.

$$\frac{\delta}{\frac{R}{n} - R} = \frac{\Delta z}{R}$$

$$\Delta z = \frac{R\delta}{\frac{R}{n} - R} = \frac{\delta n}{1 - n}$$

If n is small Δz is small and vertical stability of the beam is easier to achieve.

On the Bevatron δ should be 1/4" or less and $\frac{R}{n}$ is 1000 inches. $\frac{1}{4} \times \frac{1}{1000}$ is $\frac{1}{4000}$ of a radian which is about one minute of arc, which indicates the order of accuracy required in setting the slopes of the pole tips during assembly.

Note that in Figure XI the orbit plane tips in direction opposite to displacement of median plane.

Someone has shown that if the displacement of the "median plane", δ, is expressed as a Fourier series as

$$\delta = \delta_o + A_1 \cos(\theta_1 + a_1) + A_2 \cos(\theta_2 + a_2) + \dots .$$

Then

$$\Delta z = \delta_o - \frac{n}{1-n} A_1 \cos(\theta + a_1)$$

$$- \frac{n}{4-n} A_2 \cos(2\theta + a_2) - \dots \dots$$

$$= \delta_o - \sum_{\ell=1}^{\ell=\infty} \frac{n}{\ell^2 - n} A_\ell \cos(\ell\theta + a_\ell)$$

δ_o is average displacement of reference and median planes. The terms of this series decrease about as $\frac{1}{(\text{order})^2}$.