UNIVERSITY OF CALIFORNIA
RIVERSIDE

# On a Notion of Cohen-Macaulay and the Non-vanishing of Čech Cohomology Modules

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Andrew James Walker

June 2017

Dissertation Committee:

Professor David Rush, Chairperson
Professor Wee Liang Gan
Professor José González

The Dissertation of Andrew James Walker is approved:

Committee Chairperson

University of California, Riverside

## Acknowledgments

First and foremost I give thanks to our Father in heaven for blessing me with the privilege of attending graduate school and guiding me along the way. All glory goes to Him. I would like to thank my advisor, Dr. David Rush, for his guidance in my research, as well as my other committee members Dr. José González and Dr. Wee Liang Gan. I am thankful for my mom and dad for all their love and encouragement they gave me throughout my education. I owe my big brother Tom for being someone I have looked up to since I was little for pushing me to pursue math and having my back. I am grateful to Hyun-Seung Choi for being a great collaborator and even greater friend to me. I would like to thank Paolo Mantero and Youngsu Kim for many insightful conversations about commutative algebra. Lastly, a shout out to the squad - Alex Sherbetjian, Frank Kloster, Donna Blanton, Jordan Tousignant, Brandon Coya, Andrea Arauza - almost definitely I would have been done sooner without you guys, but it was a blast and I could not imagine having gone through school without you all.

"Trust in the Lord with all your heart, and lean not on your own understanding. In all your ways acknowledge Him, and He shall direct your paths." Proverbs 3:5 (ESV)

ABSTRACT OF THE DISSERTATION

On a Notion of Cohen-Macaulay and the Non-vanishing of Čech Cohomology Modules

by

Andrew James Walker

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2017
Professor David Rush, Chairperson

In this paper, we study the Cohen-Macaulay property of a general commutative ring with unity defined by Hamilton and Marley. We give sufficient conditions on pullback constructions, fixed rings, and normal monoid rings to all be Cohen-Macaulay in this sense. We also exhibit a class of quasi-local rings where the top Čech cohomology module with respect to a sequence generating the maximal ideal up to radical is non-vanishing.

# Contents

# Introduction

The theory of Cohen-Macaulay (CM) rings has developed into a central topic of commutative algebra. While there are many characterizations of Noetherian CM rings, one of the simplest ways to introduce such rings is where grade and height coincide for any ideal in the ring (see [36, Chapter 3] for details). The study of such rings has needed the Noetherian assumption for much of the theory of CM rings to work nicely. Replacing grade with polynomial grade, a generalized notion of grade that works well outside of a Noetherian context, Sarah Glaz studied [21] non-Noetherian rings where $\mathrm{ht}(P) = \mathrm{p.grade}(PR_P)$ for each $P \in \mathrm{Spec}(R)$. However, there are examples of coherent regular rings which fail to enjoy this property. This would be undesirable in any definition for a non-Noetherian ring to be Cohen-Macaulay, since in the Noetherian case, regular rings are CM. It is also a classical result of Hochster and Eagon [14, Proposition 12] that for any $G$, a group of automorphisms of a Noetherian CM ring $R$, the fixed ring $R^G$ is CM provided $R^G$ is a module retract of $R$, and $R$ is a finitely generated $R^G$-module. With these facts in mind, Sarah Glaz proposed [19] the existence of a definition for a non-Noetherian CM ring that agrees with the original definition for Noetherian rings, and any fixed subring $R^G$ of a coherent regular ring $R$ satisfying the conditions in Hochster's theorem above should also be CM. As a consequence of such a definition, any coherent regular ring must be CM with this definition that Glaz expected to exist. Tom Marley and Tracy Hamilton [26] in 2006 produced a notion of Cohen-Macaulay that successfully answers most of Glaz's question. Namely, they showed that with their definition of CM, every coherent ring would be CM, and that up to dimension 2, any fixed subring $R^G$ of a coherent regular ring $R$ satisfying the conditions in Hochster's theorem above would also be CM.

While their definition answered most of Glaz's question, this class of rings that is CM in the sense of Hamilton and Marley is still mysterious. For instance, Hamilton proposed [24] that based off analogous results for Noetherian CM rings, it should be reasonable to expect in any definition for a (non-Noetherian) Cohen-Macaulay ring that $R$ is CM precisely when $R_P$ is CM for all $P \in \operatorname{Spec}(R)$, and that $R$ is CM precisely when $R[x]$ is CM, where $x$ is an indeterminate over $R$. However, even this is still fully unresolved.

The key insight with Hamilton and Marley's definition of a CM ring is their recognition of the fact that height can behave unexpectedly in the non-Noetherian context, for example Krull's Hauptidealsatz need not hold. To get around this, Hamilton and Marley's definition replaced a condition on height with a condition on Čech cohomology. Their reasons for using Čech cohomology were motivated for at least two key reasons. The first is that when the base ring $R$ is Noetherian, Čech cohomology agrees with local cohomology, a tool introduced by Alexander Grothendieck in the 1960s to resolve a conjecture of Pierre Samuel (See [22], [23]) that has more modernly become an area of active research. Using the work of Peter Schenzel [45] that describes when Čech cohomology and Local cohomology agree for non-Noetherian rings, Hamilton and Marley are able to make dual use of these cohomology theories for their definition of a non-Noetherian CM ring. A second key insight of theirs is that for Noetherian local rings $(R, \mathfrak{m})$, the condition of being Cohen-Macaulay has a simple characterization in terms of local cohomology. Specifically, $R$ is Cohen-Macaulay precisely when there is a unique non-vanishing local cohomology term $H_{\mathfrak{m}}^{\dim(R)}(R)$. This characterization of Noetherian local Cohen-Macaulay rings rests heavily on Grothendieck's non-vanishing theorem (GNVT) [22, Proposition 6.4] that says for a Noetherian local ring $(R, \mathfrak{m})$, $H_{\mathfrak{m}}^{\dim(R)}$ is always non-vanishing.

In this thesis, we set out to accomplish two main objectives. Chapter 2 covers our first goal: To provide examples of Cohen-Macaulay rings in the sense of Hamilton and Marley. We achieve this by using many popular constructions that appear in Non-Noetherian ring theory. Namely, we show that certain rings coming from the pullback construction of Gilmer, direct limits of fixed rings, and non-finitely generated monoid rings can all be Cohen-Macaulay under certain conditions. Our second goal is to find a sufficient condition on a ring that can generalize GNVT to a non-Noetherian setting. Rather than work with local cohomology though, we instead work with Čech cohomology which still works nicely in a general setting. We investigate this in Chapter 3, and find a certain class of rings (containing Noetherian ones) where GNVT holds for Čech cohomology using a proof by I. G. MacDonald and R. Y. Sharp [39, Theorem 2.2] of the classical GNVT and the machinery of weakly coassociated primes introduced by S. Yassemi [46]. We also give a few examples of non-Noetherian rings that satisfy our non-vanishing result.

# Chapter 1

# Preliminaries

## 1.1 Notation

All rings will be assumed to be commutative with unity and all $R$-modules are unitary. We will using the following notation throughout this paper:

- $\mathbb{N} = \{1, 2, \ldots\}$.
- $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
- $\mathrm{Spec}(R)$: The set of prime ideals of a ring $R$.
- $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I \subseteq \mathfrak{p}\}$ : The set of prime ideals of a ring $R$ that contain the ideal $I$ of $R$.
- $\mathrm{Min}(I)$: The set of prime ideals of $R$ that are minimal over $I$.
- $\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$. This is called the radical of $I$.

- $\operatorname{ara}(I)$: The arithmetic rank of an ideal $I$. That is,

$$\operatorname{ara}(I) = \inf\{i \in \mathbb{N}_0 : \exists x_1, \ldots, x_i \in R \text{ with } \sqrt{(x_1, \ldots, x_i)R} = \sqrt{I}\}.$$

- $\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : M_{\mathfrak{p}} \neq 0\}$: The set of prime ideals of a ring $R$ whose localizations at an $R$-module are nonzero.

- $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$: The residue field at $\mathfrak{p}$.

- $(A :_C B) = \{c \in C \mid cB \subseteq A\}$.

- $\operatorname{Ass}_R(M)$: The set of associated prime ideals of an $R$-module $M$. Recall that $\mathfrak{p}$ is an associated prime ideal of $M \Leftrightarrow$ for some $m \in M$, $\mathfrak{p} = (0 :_R m)$.

- A ring $R$ is called *quasi-local* if it has a unique maximal ideal, but is not necessarily Noetherian. If $R$ is Noetherian and quasi-local, we shall say $R$ is *local.* In either case, we will often denote the ring by $(R, \mathfrak{m})$ to refer to $R$ and its unique maximal ideal.

## 1.2 Classical and polynomial grade

Let $M$ be an $R$-module and $x \in R$. Then we say $x$ is *$M$-regular* to mean that $x$ is a non-zero divisor on $M$. In other words, $x$ is $M$-regular if and only if the natural map $M \to M$ given by multiplication by $x$ is injective. More generally, if $\underline{x} = x_1, \ldots, x_\ell \in R$, we say that $\underline{x}$ is a *weak $M$-sequence* if $x_i$ is $M/(x_1, \ldots, x_i)M$-regular for each $i = 1, \ldots, \ell$. If, in addition $(\underline{x})M \neq M$, then we say that $\underline{x}$ is an *$M$-sequence* or that $\underline{x}$ form a *regular sequence* on $M$.

**Example 1.** Let $R = k[x_1, \ldots, x_n]$, where $k$ is a field and $x_1, \ldots, x_n$ are indeterminates over $k$. Then $x_1, \ldots, x_n$ form a regular sequence on $R$.

For an ideal $I$ of a ring $R$ and $M$ an $R$-module, the *classical grade* of $I$ on $M$ is the least upper bound on all lengths of weak $M$-sequences contained in $I$, and is denoted by grade$(I, M)$. When $R$ is Noetherian, $M$ is finitely generated and $IM \neq M$, all maximal $M$-sequences contained in $I$ have common length equal to grade$(I, M)$, which is also equal to $\inf\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$. A frequently enjoyed property of classical grade in the Noetherian context is that if $M$ is finitely generated, then grade$(I, M) > 0$ exactly when $(0 :_M I) = 0$.

In the non-Noetherian case, this need no longer hold. The example below shows such an instance of this, and uses the technique of *idealization* of a module [1]. Indeed, if $S$ is a ring and $N$ is an $S$-module, let $R = S \times N$. Then $R$ becomes a commutative ring with unity, where the addition is inherited from the direct product, and the multiplication is defined by $(s, n) \cdot (s', n') = (ss', sn' + s'n)$ for $s, s' \in S$ and $n, n' \in N$. We denote this ring by $R = S \ltimes N$. For $I$ an ideal of $S$, we write $I \ltimes M$ for the ideal of $S$ generated by the images of $I$ and $M$ in $S$.

**Example 2.** ([43, Example p. 116]) Let $S = k[x, y]$, the polynomial in two variables $x, y$ over a field $k$. Set

$$N = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec}(R), \\ \operatorname{ht}(\mathfrak{p}) = 1}} \kappa(\mathfrak{p}),$$

where $\kappa(\mathfrak{p})$ denotes the residue field of $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $R$ be the idealization of the $S$-module $N$. Set $I = (X, Y)R$, where $X = (x, 0)$ and $Y = (y, 0)$. Then $I$ is a finitely generated ideal of $R$ with $(0 :_R I) = 0$, yet $I$ consists entirely of zero-divisors on $R$.

Northcott notes in [43, p. 131-132] that Hochster first observed that adjoining indeterminates to a ring can create the existence of non-zerodivisors, "almost as if by magic." With this in mind, the issue discussed above and many other problems that appear in the non-Noetherian setting can be resolved using a different notion of grade, which Northcott called *true grade* (we will use the more common notation of [13]): Let $R$ be a ring, $I$ an ideal of $R$ and $M$ an $R$-module. Then the *polynomial grade* of $I$ on $M$ is

$$\operatorname{p.grade}(I, M) = \lim_{m\to\infty} \operatorname{grade}\Big(IR[X_1, \ldots, X_m], R[X_1, \ldots, X_m] \otimes_R M\Big).$$

**Theorem 1.2.1.** *([43, Chapter 5]). Let $R$ be a ring, $I$ an ideal, and $M$ an $R$-module.*

*(i) If $I$ is finitely generated, then* $\operatorname{p.grade}(I, M) > 0 \Leftrightarrow (0 :_M I) = 0$. *The latter happens exactly when $f = a_0 + a_1X + \ldots + a_mX^m$ is a non-zerodivisor on $M \otimes_R R[X]$, where $a_0, \ldots, a_m$ is a set of generators for $I$.*

*(ii) If $I$ is finitely generated and $IM \neq M$ then* $\operatorname{p.grade}(I, M) \leq \mu(I)$.

*(iii) If $J \subseteq I$, then* $\operatorname{p.grade}(J, M) \leq \operatorname{p.grade}(I, M)$.

*(iv)* $\operatorname{p.grade}(I, M) = \sup_J\{\operatorname{p.grade}(J, M)\}$ *where $J$ ranges over all finitely generated ideals contained in $I$.*

*(v) If $\underline{x} = x_1, \ldots, x_\ell$ is an $M$-sequence contained in $I$, then*

$$\operatorname{p.grade}(I, M) = \operatorname{p.grade}(I, M/(\underline{x})M) + \ell.$$

*(vi) If $I$ is proper, there is some $P \in V(I)$ such that* $\operatorname{p.grade}(I, M) = \operatorname{p.grade}(P, M)$.

(vii) *Let $\varphi\colon R \to S$ be a ring homomorphism. If $I$ is any ideal of $R$ and $N$ is an $S$-module, then*

$$\text{p.grade}_R(I, N) = \text{p.grade}_S(IS, N).$$

Other theories of grade are discussed in [6].

## 1.3 Local cohomology

Local cohomology was introduced as a tool by Alexander Grothendieck in the 1960s to resolve a conjecture of Samuel, and has since been studied for its own sake. Let $I$ be an ideal of a ring $R$ and $M$ an $R$-module. The *$I$-torsion* submodule of $M$, denoted by $\Gamma_I(M)$, is the set of all $m \in M$ such that $I^n m = 0$ for some $n \in \mathbb{N}$. That is,

$$\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \in \mathbb{N}\}.$$

If $f\colon M \to N$ is a morphism of $R$-modules, then since $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$, $f$ restricted to $\Gamma_I(M)$ defines a morphism $R$-modules $\Gamma_I(f)\colon \Gamma_I(M) \to \Gamma_I(N)$ so that $\Gamma_I(-)$ is functorial. The $i^{th}$ *local cohomology functor with support in $I$* is defined as the $i^{th}$ right-derived functor of $\Gamma_I(-)$, and is denoted by $H^i_I(-)$. Since $\Gamma_I(-)$ is left-exact, we have that $H^0_I(-) \cong \Gamma_I(-)$. For $i > 0$, to compute $H^i_I(M)$, where $M$ is an $R$-module, first recall that an *injective resolution* $E^\bullet$ of $M$ is a sequence of injective $R$-modules

$$E^\bullet\colon 0 \to E^0 \to E^1 \to E^2 \to \cdots$$

that is exact at each $i \neq 0$ with $H^0(E^\bullet) = M$. After applying the $I$-torsion functor to $E^\bullet$, we get another complex $\Gamma_I(E^\bullet)$. Then $H^i_I(M) \cong H^i(\Gamma_I(E^\bullet))$.

**Example 1.3.1.** Let $R$ be a PID and $a \in R$. Then if $K$ is the quotient field of $R$, the natural sequence $0 \to K \to K/R \to 0$ is an injective resolution of $R$ where $K$ is in degree 0 and $K/R$ is in degree 1. Applying the functor $\Gamma_{aR}(-)$ to this sequence yields another sequence

$$0 \to \Gamma_{aR}(K) \to \Gamma_{aR}(K/R) \to 0.$$

Then $\Gamma_{aR}(K) = 0$, and $\Gamma_{aR}(K/R) = R_a/R$. Thus $H^i_{aR}(R) = 0$ for all $i \neq 1$, and $H^1_{aR}(R) \cong R_a/R$.

For $M$ an $R$-module and $I$ an ideal of $R$, the *cohomological dimension* of $M$ with respect to $I$, denoted by $\mathrm{cd}(I, M)$, is the maximal integer $i$ for which $H^i_I(M) \neq 0$.

**Theorem 1.3.2.** *[Grothendieck's Vanishing Theorem] Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$. Then $H^i_{\mathfrak{m}}(R) = 0$ for all $i > d$.*

**Theorem 1.3.3.** *[Grothendieck's Non-Vanishing Theorem] Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$. Then $H^d_{\mathfrak{m}}(R) \neq 0$.*

In other words, these two theorems simply say that $\mathrm{cd}(\mathfrak{m}, R) = \dim(R)$ when $(R, \mathfrak{m})$ is a Noetherian local ring. Local cohomology also has an alternative formulation in terms of direct limits. We first set up the following notation: By a *directed set* $\Lambda$, we mean a set $\Lambda$ equipped with a partial order $<$ with the following property: whenever $\alpha, \beta \in \Lambda$, there is some $\gamma \in \Lambda$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. Let $\mathcal{C}$ be any category and $\Lambda$ a directed set. A *direct system* over $\Lambda$ is the following data:

(1) For each $\lambda \in \Lambda$, there is an object $A_\lambda \in \mathrm{Ob}(\mathcal{C})$.

(2) For each $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, we have morphisms $\varphi_{\mu\lambda}\colon A_\lambda \to A_\mu$ that obey the following rules:

  (a) $\varphi_{\lambda\lambda} = \mathrm{id}_{A_\lambda}$ for each $\lambda \in \Lambda$.

  (b) $\varphi_{\gamma\lambda} = \varphi_{\gamma\mu} \circ \varphi_{\mu\lambda}$ whenever $\gamma \geq \mu \geq \lambda \in \Lambda$.

We will use the notation $\{A_\lambda; \varphi_{\mu\lambda}\}$ to denote a direct system over $\Lambda$. In certain categories, such as $\mathcal{C} = \mathrm{CommRing}$ or Grp, we can form an object called the *direct limit* of $\{A_\lambda; \varphi_{\mu\lambda}\}$ that has a certain universal property described below (see also, for instance, [42, Appendix A]). Indeed, suppose for a given direct system $\{A_\lambda; \varphi_{\mu\lambda}\}$, we have some $L \in \mathrm{Ob}(\mathcal{C})$ together with a collection of morphisms $f_\lambda\colon A_\lambda \to L$ such that whenever $\mu \geq \lambda$ in $\Lambda$, we have $f_\mu \circ \varphi_{\mu\lambda} = f_\lambda$. In other words, the following diagram commutes:

$$\begin{tikzcd} A_\lambda \arrow[r, "\varphi_{\mu\lambda}"] \arrow[dr, "f_\lambda"'] & A_\mu \arrow[d, "f_\mu"] \\ & L \end{tikzcd}$$

Suppose further that this pair $\{L; f_\lambda\}$ is universal with respect to this property, meaning that for any other collection $\{X, g_\lambda\}$ satisfying $g_\mu \circ \varphi_{\mu\lambda} = g_\lambda$, there is a unique map $h\colon L \to X$ such that $h \circ f_\lambda = g_\lambda$ for all $\lambda \in \Lambda$. That is, the following diagram commutes:

$$\begin{tikzcd} A_\lambda \arrow[rr, "\varphi_{\mu\lambda}"] \arrow[dr, "f_\lambda"] \arrow[ddr, bend right, "g_\lambda"'] & & A_\mu \arrow[dl, "f_\mu"'] \arrow[ddl, bend left, "g_\mu"] \\ & L \arrow[d, dotted, "h"] & \\ & X & \end{tikzcd}$$

In this case, $L$, together with the maps $f_\lambda$, is called the direct limit of the direct system $\{A_\lambda; \varphi_{\mu\lambda}\}$, and is denoted by $\varinjlim A_\lambda$. For $a \in A_\lambda$, we will let $[a]$ denote the image of $a$ in $\varinjlim A_\lambda$ via the morphism $A_\lambda \to \varinjlim A_\lambda$. Every element in $\varinjlim A_\lambda$ can be written as $[a]$ for some $a \in A_\lambda$. Moreover, $[b] = [a]$ for some $b \in A_\mu$ if and only if there is some $\gamma \in \Lambda$ with $\gamma \geq \mu, \lambda$ such that $\varphi_{\gamma\mu}(b) = \varphi_{\gamma\lambda}$.

**Example 1.3.4.** When $\mathcal{C} = \text{CommRing}$, for a given direct system $\{R_\lambda; \rho_{\mu\lambda}\}$ over a directed set $\Lambda$, we can always form its direct limit. Indeed, set $\varinjlim R_\lambda := \bigsqcup R_\lambda / \sim$, where $r_\lambda \sim r_\mu$ if and only if there is some $\gamma$ such that $\gamma \geq \mu, \lambda$ with $\rho_{\gamma\mu}(r_\mu) = \rho_{\gamma\lambda}(r_\lambda)$. We have canonical maps $R_\lambda \to \varinjlim R_\lambda$ that send $r \mapsto [r]$. Furthermore, this construction yields direct limits in other categories, such as $\mathcal{C} = \text{Grp}$.

For $I$ an ideal of a ring $R$, local cohomology can be thought of as a direct limit of Ext functors:

**Theorem 1.3.5.** *([9, Theorem 1.38]) Let $R$ be a ring and $I$ an ideal of $R$. Then for any $i \in \mathbb{Z}$, $H^i_I(-) \cong \varinjlim_n Ext^i_R(R/I^n, -)$.*

## 1.4 Čech cohomology

Let $R$ be a ring and $M$ an $R$-module. By $M_a$, we mean the localization of $M$ at the multiplicative set $\{a^n\}_{n \in \mathbb{N}_0}$. So $M_a = \{\frac{m}{a^n} \mid m \in M \text{ and } n \in \mathbb{N}_0\}$. For any $a, b \in R$, we have a natural map

$$M_a \to M_{ab}, \text{ where } \frac{m}{a^n} \mapsto \frac{mb^n}{(ab)^n}.$$

Let $\underline{x} = x_1, \ldots, x_\ell$ denote a sequence of elements of $R$ and $M$ an $R$-module. Define a complex

$$\check{\mathrm{C}}^{\bullet}_{\underline{x}}(M) : 0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} \bigoplus_i M_{x_i} \xrightarrow{d^1} \bigoplus_{i<j} M_{x_i x_j} \xrightarrow{d^2} \ldots \xrightarrow{d^{\ell-1}} M_{x_1 \cdots x_\ell} \xrightarrow{d^\ell} 0$$

where the maps $d^i$ are induced from the natural maps defined above, along with a sign convention [See Example 1.4.1]. Indeed, $d^{i+1}d^i = 0$ for each $i \in \mathbb{Z}$, so that $\check{\mathrm{C}}^{\bullet}_{\underline{x}}(M)$ indeed forms a complex, called the *Čech complex* with respect to $M$ and the sequence $\underline{x}$. For each $i \in \mathbb{Z}$, the $i^{th}$ *Čech cohomology* module with respect to $M$ and the sequence $\underline{x}$ is the $i^{th}$ cohomology module of $\check{\mathrm{C}}^{\bullet}_{\underline{x}}(M)$. That is,

$$H^i_{\underline{x}}(M) = H^i(\check{\mathrm{C}}^{\bullet}_{\underline{x}}(M)) = \frac{\ker(d^i)}{\operatorname{im}(d^{i-1})}.$$

**Example 1.4.1.** Let $R$ be a ring and $x_1, x_2 \in R$. If $M$ is an $R$-module, then the maps in $\check{\mathrm{C}}^{\bullet}_{x_1,x_2}(M)$ are determined by the sign convention on the following natural maps below:

$$\begin{tikzcd}
 & & M_{x_1} \arrow[dr, "-"] & & \\
0 \arrow[r] & M \arrow[ur, "+"] \arrow[dr, "+"'] & & M_{x_1x_2} \arrow[r] & 0 \\
 & & M_{x_2} \arrow[ur, "+"'] & &
\end{tikzcd}$$

So explicitly,

$$\begin{aligned}
d^0(m) &= \left(\frac{m}{1}, \frac{m}{1}\right). \\
d^1\left(\frac{m_1}{x_1^n}, \frac{m_2}{x_2^n}\right) &= \frac{m_2 x_1^n}{(x_1x_2)^n} - \frac{m_1 x_2^n}{(x_1x_2)^n}.
\end{aligned}$$

If $f \colon M \to N$ is a morphism of $R$-modules $M$ and $N$, then for any $a \in R$, there is an induced morphism of $R$-modules

$$f_a \colon M_a \to N_a \text{ given by } \frac{m}{a^n} \mapsto \frac{f(m)}{a^n}.$$

Thus, for any finite sequence $\underline{x}$ of elements of $R$, $f$ induces a morphism of $R$-complexes $\check{C}^{\bullet}_{\underline{x}}(f) \colon \check{C}^{\bullet}_{\underline{x}}(M) \to \check{C}^{\bullet}_{\underline{x}}(N)$ so that $\check{C}^{\bullet}_{\underline{x}}(-)$, and thus $H^{\bullet}_{\underline{x}}(-)$ are both functorial. Below we list some useful properties of Čech cohomology:

**Theorem 1.4.2.** *[26, Propositions 2.1, 2.4] Let $R$ be a ring, $\underline{x} = x_1, \ldots, x_\ell$ a finite sequence of elements of $R$ and $M$ an $R$-module.*

*(i)* $H^i_{\underline{x}}(M) = 0$ *for all* $i > \ell$.

*(ii) If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of $R$-modules, there is a natural long exact sequence in* Čech *cohomology:*

$$\ldots \to H^i_{\underline{x}}(A) \to H^i_{\underline{x}}(B) \to H^i_{\underline{x}}(C) \to H^{i+1}_{\underline{x}}(A) \to \ldots$$

*(iii) If* $\underline{x} = x_1, \ldots, x_\ell$, *then there is a long exact sequence*

$$\cdots \to H^i_{\underline{x}}(M) \to H^i_{\underline{x}'}(M) \stackrel{\pm x_\ell}{\to} H^i_{\underline{x}'}(M)_{x_\ell} \to H^{i+1}_{\underline{x}}(M) \to \cdots,$$

*where* $\underline{x}' = x_1, \ldots, x_{\ell-1}$.

*(iv) Let* $I = (\underline{x})R$. *Then for any* $a \in H^i_{\underline{x}}(M)$, $I^n a = 0$ *for* $n >> 0$.

*(v) If* $\underline{y}$ *is a finite sequence of elements with* $\sqrt{(\underline{y})R} = \sqrt{(\underline{x})R}$, *then* $H^i_{\underline{x}}(M) \cong H^i_{\underline{y}}(M)$ *for all* $i$.

*(vi) (Change of Rings) Let $N$ be an $S$-module and* $f \colon R \to S$ *a ring homomorphism. Then for all* $i$, $H^i_{\underline{x}}(N) \cong H^i_{f(\underline{x})}(N)$.

*(vii) (Flat Base Change) Let* $f \colon R \to S$ *be a flat ring homomorphism and $M$ an $R$-module. Then for any* $i$, $H^i_{\underline{x}}(M) \otimes_R S \cong H^i_{f(\underline{x})}(M \otimes_R S)$.

*(viii)* $H_{\underline{x}}^{\ell(\underline{x})}(M) \cong H_{\underline{x}}^{\ell(\underline{x})}(R) \otimes_R M$ *and* $\operatorname{Supp}\Big(H_{(\underline{x})}^{\ell(\underline{x})}(M)\Big) \subseteq \operatorname{Supp}\Big(M/\underline{x}M\Big)$.

*(ix)* *If* $\dim(R) = d < \infty$*, then* $H_{\underline{x}}^i(M) = 0$ *for all* $i > d$*.*

Let $I = (\underline{x})R = (x_1, \ldots, x_\ell)R$ be a finitely generated ideal of a ring $R$ and $M$ an $R$-module. Then we make the following observation:

$$\begin{aligned} H_{\underline{x}}^0(M) &= \ker(M \to \oplus_{i=1}^{\ell} M_{x_i}) \\ &= \{m \in M : m/1 = 0 \text{ in } M_{x_i} \text{ for each } i = 1, \ldots, \ell\} \\ &= \{m \in M : x_i^v m = 0 \text{ for each } i = 1, \ldots, \ell \text{ and some } v \in \mathbb{N}\} \\ &= \{m \in M : I^u m = 0 \text{ for some } u \in \mathbb{N}\} \\ &= H_I^0(M). \end{aligned}$$

In general, for $i > 0$, local cohomology and Čech cohomology need not coincide. For any $i \in \mathbb{Z}$, when the base ring $R$ is Noetherian though, we have that for any $R$-module $M$, $H_{\underline{x}}^i(M) \cong H_I^i(M)$ (see [10, Theorem 3.5.6] for a proof). In [45], P. Schenzel characterized the sequences $\underline{x}$ where Čech cohomology and local cohomology coincide in terms of Koszul homology: For $x \in R$, let $K_\bullet(x)$ denote the complex

$$K_\bullet(x) : 0 \to R \xrightarrow{x} R \to 0,$$

where the $R$ on the left appears in degree zero. More generally, for $\underline{x} = x_1, \ldots, x_\ell \in R$, we set $K_\bullet(\underline{x}) := K_\bullet(x_1) \otimes K_\bullet(x_2) \otimes \cdots \otimes K_\bullet(x_\ell)$. Then $K_\bullet(\underline{x})$ is called the *Koszul complex* with respect to $\underline{x}$. Now, given $x \in R$ and $n, m \in \mathbb{N}$ with $n \geq m$, we have a natural morphism of complexes $K_\bullet(x^n) \to K_\bullet(x^m)$ induced from multiplication by $x^{n-m}$ on $R$. Indeed, the following diagram commutes:

$$\begin{array}{ccccccc} K_\bullet(x^n): 0 & \longrightarrow & R & \xrightarrow{x^n} & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle x^{n-m}} & & \Big\downarrow{\scriptstyle 1} & & \\ K_\bullet(x^m): 0 & \longrightarrow & R & \xrightarrow{x^m} & R & \longrightarrow & 0 \end{array}$$

If $\underline{x} = x_1, \ldots, x_\ell \in R$, then whenever $n, m \in \mathbb{N}$ with $n \geq m$, the above morphism on $K_\bullet(x_i^n) \to K_\bullet(x_i^m)$ for each $i = 1, \ldots, \ell$ induces a morphism on $K_\bullet(\underline{x}^n) \to K_\bullet(\underline{x}^m)$, where $\underline{x}^j := x_1^j, \ldots, x_\ell^j$ ($j \in \mathbb{N}$). This in turn yields morphisms for each $i \in \mathbb{Z}$:

$$H_i(K_\bullet(\underline{x}^n)) \to H_i(K_\bullet(\underline{x}^m))$$

If, for each $m \in \mathbb{N}$ there is an $n \geq m$ so that the above morphisms are zero for all $i \geq 1$, then Schenzel calls such a sequence *weakly proregular*. These sequences completely characterize when local cohomology and Čech cohomology agree.

**Theorem 1.4.3.** *([45, Theorem 3.2]) Let $R$ be a ring, $\underline{x} = x_1, \ldots, x_\ell$ a sequence of elements of $R$ and $I = (\underline{x})R$. Then the following are equivalent:*

*(i) $\underline{x}$ is weakly proregular.*

*(ii) $H^i_{\underline{x}}(E) = 0$ for each $i \neq 0$ and each injective $R$-module $E$.*

*(iii) There are natural isomorphisms $H^i_{\underline{x}}(M) \cong H^i_I(M)$ for each $i \in \mathbb{Z}$.*

We conclude this section with a remark on how polynomial grade relates to the Čech complex. For $\underline{x} = x_1, \ldots, x_\ell \in R$ and $I = (\underline{x})R$ and $M$ an $R$-module, let

$$\check{\mathrm{C}}.\mathrm{grade}(I, M) = \inf\{i \mid H^i_{\underline{x}}(M) \neq 0\}.$$

It was shown in [26, Proposition 2.7] that $\mathrm{p.grade}(I, M) = \check{\mathrm{C}}.\mathrm{grade}(I, M)$, and that this number is finite iff $IM \neq M$.

## 1.5 Classical Cohen-Macaulay Rings

Let $R$ be a Noetherian ring. If whenever each sequence $\underline{x} = x_1, \ldots, x_\ell \in R$ with $\mathrm{ht}((x_1, \ldots, x_i)R) = i$ for each $i = 1, \ldots, \ell$ is a regular sequence on $R$, we say that the ring $R$ is *Cohen-Macaulay.* Hochster and Huneke write that for many theorems [29]:

"The Cohen-Macaulay condition (possibly on the local rings of a variety) is just what is needed to make the theory work."

Historically, the study of Cohen-Macaulay rings goes back to a little over a century ago. In 1916, Macaulay published a book *The algebraic theory of modular systems.* A *modular system* is just an ideal in the polynomial ring $R = \mathbb{C}[X_1, \ldots, X_n]$. He was concerned with the properties of the solution to

$$f_1 = f_2 = \ldots = f_k = 0,$$

where $f_k$ are homogeneous polynomials (that is, polynomials where all terms have the same degree), i.e., what we now call *varieties* in $\mathbb{P}^n_{\mathbb{C}}$. Most of his book deals with how the solution set of these equations behaves with respect to the ideal generated by this family of polynomials. In his work, he proved what is now known as an *unmixedness theorem*:

**Theorem 1.5.1.** *Let $R = \mathbb{C}[X_1, \ldots, X_n]$ and $I = (f_1, \ldots, f_k)$ an ideal of $R$ with $\mathrm{ht}(I) = k$. Then for any $\mathfrak{p} \in \mathrm{Ass}(R/I)$, we have $\mathrm{ht}(\mathfrak{p}) = k$.*

Later on in 1946, I.S. Cohen showed in his book, *On the Structure and Ideal Theory of Complete Local Rings*, that this same unmixedness theorem holds for any ideal in a power series ring over an arbitrary field $k$ (he actually did so in a slightly more general case - a class of rings called *regular local rings*). The results of Cohen and Macaulay launched an investigation into a class of rings where an unmixedness theorem holds. It turns out these are precisely the Cohen-Macaulay rings defined at the beginning of this section.

Cohen-Macaulay rings have ties to all sorts of branches of mathematics, such as algebraic topology, combinatorics, and algebraic geometry. For instance, Cohen-Macaulay rings appear naturally in *invariant theory* (see [34, Section 10.3]). Namely, let $G$ be a group that acts on a polynomial ring $R = k[x_1, \ldots, x_n]$ by degree-preserving $k$-algebra automorphisms, where $k$ is any field. We write $R^G$ to denote the *fixed ring* or *ring of invariants*, which is the subring

$$R^G := \{r \in R \mid g(r) = r \text{ for all } r \in R\}.$$

To describe $R^G$, one would like to find generators as a $k$-algebra for $R^G$. This corresponds to providing a surjection $S \to R^G$, where $S$ is another polynomial ring over $k$, assuming $R^G$ is a finitely generated $k$-algebra. To describe $R^G$ more completely as an $S$-module though, one would like to find a free resolution of $R^G$:

$$\cdots \to S^{n_2} \to S^{n_1} \to S \to R^G \to 0.$$

The minimal length of a free resolution of $R^G$ as an $S$-module is called the *projective dimension* of $R^G$, and is denoted $\operatorname{pd}_S(R^G)$. It turns out that when $R^G$ is a finitely generated $k$-algebra, there is a simple formula for $\operatorname{pd}_S(R^G)$ that holds precisely when $R^G$ is Cohen-Macaulay:

$$\operatorname{pd}_S(R^G) = \dim(S) - \dim(R^G).$$

## 1.6 A question of Glaz

Sarah Glaz proposed in [19] the existence of a definition for a non-Noetherian Cohen-Macaulay ring that meets these three conditions:

(G1): When $R$ is Noetherian, the new definition of a Cohen-Macaulay ring coincides with the original definition.

(G2): All coherent regular rings will be Cohen-Macaulay, where a ring $R$ is *coherent* if every finitely generated ideal is of finite presentation, and is *regular* if every finitely generated ideal of $R$ has finite projective dimension.

(G3): If $R$ is coherent regular and $G$ a group of ring automorphisms of $R$, then $R^G$ will be Cohen-Macaulay as long as $R^G$ is a direct summand of $R$ as an $R^G$-module and $R$ is a finitely generated $R^G$-module.

Glaz [21] studied rings where for each prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, $\operatorname{ht}(\mathfrak{p}) = \operatorname{p.grade}(\mathfrak{p}R_{\mathfrak{p}}, R_p)$ and while this appears suitable as a definition of Cohen-Macaulay, found examples showing condition (G2) fails. Later on, Tracy Hamilton in [24] studied Glaz's question and proposed two other properties that would be desirable in a definition of a non-Noetherian Cohen-Macaulay ring:

(H1): $R$ is Cohen-Macaulay $\Leftrightarrow$ $R[X]$ is Cohen-Macaulay, where $X$ is an indeterminate over $R$.

(H2): $R$ is Cohen-Macaulay $\Leftrightarrow$ $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Hamilton approached Glaz's conjecture from the perspective that Macaulay initially had about these rings: ones that satisfy some type of unmixedness theorem. That is, a Noetherian ring satisfies the *unmixedness theorem* if for any ideal $I$ generated by $\mathrm{ht}(I)$ elements, $\mathrm{Ass}(R/I) = \mathrm{Min}(I)$. Unfortunately, simple examples show that in the non-Noetherian case, the theory of associated primes can behave poorly. For example, if $R$ is an arbitrary non-zero ring, it's possible that $\mathrm{Ass}(R) = \varnothing$, while over a non-Noetherian ring, this never happens.

**Example 1.6.1.** Let $R = k[x_1, x_2, x_3, \ldots]$ where $k$ is a field and the $x_i$ are countably many indeterminates over $k$. Set $I = (x_1^2, x_2^2, x_3^2, \ldots)$ and consider the ring $S = R/I$. Then $S$ is non-Noetherian, and $\mathrm{Ass}(S) = \varnothing$.

This problem, and many others, can be often be remedied by dealing instead with the weakly associated primes: Let $R$ be a ring and $M$ an $R$-module. Then we say that $\mathfrak{p} \in \mathrm{Spec}(R)$ is a *weak-Bourbaki* or *weakly associated* prime of $M$ if $\mathfrak{p}$ is minimal over $(0 :_R m)$ for some $m \in M$. The set of weak-Bourbaki associated primes of $M$ is denoted by $\mathrm{wAss}(M)$.

It can be shown that when $R$ is Noetherian and $M$ an $R$-module, the weakly associated primes and the associated primes of $M$ coincide. With this in mind, Hamilton proposed studying a class of rings that satisfy a variation of the unmixedness theorem, instead using the weakly associated primes: A ring $R$ will be called *weak-Bourbaki unmixed* (or *wB-unmixed*) if for any finitely generated ideal $I$ that may be generated by $\mathrm{ht}(I)$ elements or less, $\mathrm{Min}(I) = \mathrm{wAss}(R/I)$.

In [24], Hamilton determined that this wB-unmixed rings satisfy condition (G1) and the '$\Leftarrow$' implication in conditions (H1) and (H2). It's unknown currently if the reverse implications hold or if (G2) and (G3) are true. We record a few other notions of Cohen-Macaulayness that other authors have produced:

- M. Sakaguchi [44] in 1980 came up with a definition that relates polynomial grade and valuative dimension, in place of Krull dimension. His notion of Cohen-Macaulay (at least) satisfies conditions (G1) and (H2), and fails condition (G2) in general. Including him here is a little anachronistic, as he studied these rings before Glaz even proposed her conjecture in 1994.

- T. Marley and T. Hamilton [26] in 2006 used Čech cohomology to produce a notion of Cohen-Macaulay that satisfies conditions (G1), (G2), at least part of (G3), and the '$\Leftarrow$' implication of conditions (H1) and (H2). It's currently unknown if the reverse implications are true and what more can be said about (G3).

- M. Asgharzadeh and M. Tousi [6] in 2009 modified many of the various definitions of a non-Noetherian Cohen-Macaulay ring discussed thus far (aside from Sakaguchi's) and performed an analysis of the relationships amongst all of these notions of Cohen-Macaulayness. In particular, they produced a notion of Cohen-Macaulay, which they call *weakly Cohen-Macaulay*, and showed that it satisfies conditions (G1), (G2), almost all of (G3) and the '$\Leftarrow$' implication of (H2).

The notion of Cohen-Macaulay introduced by Hamilton and Marley mentioned above was the first significant step towards resolving Glaz's question, and is currently an active area of study (see for example, [3, 4, 5, 6, 32, 40]). Recognizing that the height of an ideal may behave strangely over non-Noetherian rings (for instance, Krull's PIT may fail), they replace height with a condition on Čech cohomology. Since in the Noetherian case it is often desirable to have the flexibility to work with local cohomology or Čech cohomology, they also propose to make use of the notion of weakly proregular sequences. With this in mind, we have their definitions: Let $\underline{x} = x_1, \ldots, x_\ell \in R$ be a sequence of elements of a ring $R$. We shall say that $\underline{x}$ is a *parameter sequence* on $R$ if the following hold:

(i) $(\underline{x})R \neq R$.

(ii) $\underline{x}$ is weakly proregular.

(iii) $H^{\ell}_{\underline{x}}(R)_{\mathfrak{p}} \neq 0$ for each $\mathfrak{p} \in V((\underline{x})R)$.

Moreover, if for each $i = 1, \ldots, \ell$, $x_1, \ldots, x_i$ is a parameter sequence on $R$, we shall that $\underline{x}$ is a *strong parameter sequence on* $R$. The ring $R$ is said to be *Cohen-Macaulay* if every strong parameter sequence is a regular sequence on $R$. It is shown in [26, Remark 3.2] that when $R$ is a Noetherian ring, $x_1, \ldots, x_\ell$ is a parameter sequence if and only if $\operatorname{ht}((x_1, \ldots, x_\ell)R) = \ell$.

**Example 1.6.2.** The following are several examples of Cohen-Macaulay rings:

(i) ([26, Proposition 4.4(a)]) Any zero-dimensional ring is Cohen-Macaulay.

(ii) ([26, Proposition 4.4(b)]) A one-dimensional integral domain is Cohen-Macaulay.

(iii) ([26, Example 4.9]) Let $x, y$ be indeterminates over $\mathbb{C}$. Then $R = \mathbb{C} + x\mathbb{C}[\![x, y]\!]$ is Cohen-Macaulay.

(iv) ([26, Theorem 4.11]) Let $R$ be an excellent Noetherian domain of characteristic $p > 0$. Then $R^+$, the *absolute integral closure* of $R$, is Cohen-Macaulay.

(v) ([5, Corollary 3.6]) Let $k$ be a field and $R$ a pure $k$-subalgebra of $k[x_1, x_2, x_3, \ldots]$, where the $x_i$ are indeterminates over $k$. Then $R$ is Cohen-Macaulay.

(vi) ([4, Theorem 4.10]) Let $R = k + xk[x, y]$, where $k$ is a field and $x, y$ are indeterminates over $R$. Then $R$ is Cohen-Macaulay.

# Chapter 2

# Examples of Cohen-Macaulay Rings

## 2.1 Pullback construction

We will fix some notation for this section: Suppose that $V$ is a *valuation domain* (an integral domain where the set of ideals is totally ordered) with unique maximal ideal $M$ and $V$ can be written in the form $V = k + M$ for some field $k$. For $D$ a subring of $k$, consider the following diagram:

$$\begin{tikzcd} & V \arrow[d] \\ D \arrow[r] & k \end{tikzcd}$$

The pullback of this diagram is the ring $R = D + M$. Gilmer [17] first organized many of the facts about rings of this type, which are now typically called *pullback rings*. In this section, we will look at which of these rings have the Cohen-Macaulay property defined by Hamilton and Marley. First, we need some results from Gilmer and Bastida:

**Theorem 2.1.1.** *([7, Theorem 2.1]) Let $V$ be a nontrivial valuation domain, and assume $V$ is of the form $k+M$, where $k$ is a field and $M$ is the maximal ideal of $V$. Let $D$ be a domain with identity that is a proper subring of $k$, and let $R = D+M$. Then the following hold:*

*(i)* $\dim(R) = \dim(D) + \dim(V)$.

*(ii) Every ideal of $R$ compares with $M$ under inclusion.*

*(iii) The set of ideals of $R$ containing $M$ is $\{A_\alpha + M\}$, where $A_\alpha$ an ideal of $D$. Moreover,*

$$R/(A_\alpha + M) \cong D/A_\alpha,$$

*so that $A_\alpha + M$ is maximal, prime or $(P_\alpha + M)$-primary in $R$ precisely when $A_\alpha$ is maximal, prime or $P_\alpha$-primary respectively in $D$.*

*(iv) If $A$ is an ideal of $R$ contained in $M$, then either $A$ is an ideal of $V$, or $AV$ is a principal ideal of $V$. In this case, if $AV = aV$ for some $a \in A$, then $A = Wa + Ma$, where $W$ is a $D$-submodule of $K$ such that $D \subseteq W \subset K$.*

*(v) The finitely generated ideals of $R$ properly containing $M$ are those of the form $A_\alpha + M$, where $A_\alpha$ is a finitely generated ideal of $R$. On the other hand, the finitely generated ideals of $R$ contained in $M$ are of the form $I = Wx + Mx$, where $W$ is a finitely generated $D$-submodule of $K$ containing $D$ and $x$ is a nonzero element of $M$.*

Before we proceed further we shall need some simple results about Čech cohomology:

**Lemma 2.1.2.** *Let $I$ be an ideal of a ring $R$ and $M$ an $R$-module. If $IM = 0$ and $\underline{x}$ is any finite sequence contained in $I$, then $H^i_{\underline{x}}(M) = 0$ for all $i > 0$.*

*Proof.* The Čech complex $\check{\mathrm{C}}^{\bullet}_{\underline{x}}(M)$ vanishes in all terms bigger than degree 0, since $M_a = 0$ for all $a \in M$. □

**Lemma 2.1.3.** *Let $R \hookrightarrow S$ be an inclusion of rings and suppose $\underline{x}$ is a finite sequence of elements contained in $R$. If $\underline{x} \in (R :_R S)$, then for all $i \geq 2$, $H^i_{\underline{x}}(R) = H^i_{\underline{x}}(S)$.*

*Proof.* We have a short-exact sequence of $R$-modules $0 \to R \to S \to S/R \to 0$, and since $\underline{x}(S/R) = 0$, we have that $H^i_{\underline{x}}(S/R) = 0$ for any $i > 0$ by the above lemma. Therefore by the long exact sequence in Čech cohomology, we have that $H^i_{\underline{x}}(R) \cong H^i_{\underline{x}}(S)$ for all $i > 1$. □

**Theorem 2.1.4.** *Let $V$ be a valuation domain of the form $V = k + M$ and $D$ a one-dimensional subring of $k$, where $k$ is a field. Suppose that every finite sequence of elements in $D$ is weakly proregular in $D$. Then for $R = D + M$, we have that $R$ is Cohen-Macaulay.*

*Proof.* Let $\underline{x} = x_1, \ldots, x_\ell$ be a strong parameter sequence of $R$ and set $I = (\underline{x})R$. Then by Theorem 2.1.1, either $I \subseteq M$ or $I \supsetneq M$. Suppose first that $I \subseteq M$. Then we have an inclusion of rings $R \to V$ and $\underline{x} \in (R :_R V)$, since $\underline{x} \in M$, so that $(\underline{x})V \subseteq MV = M \subset R$. Thus by the above lemma, $H^i_{\underline{x}}(R) \cong H^i_{\underline{x}}(V) = 0$ for all $i > 1$. Now since $(\underline{x})V$ is a principal ideal of $V$, we must have that $H^i_{\underline{x}}(V) = 0$ for all $i > 1$. So in this case, the only possibility is that $\ell = 1$, and since $R$ is a domain, we have $\underline{x} = x_1$ is a regular element.

On the other hand, suppose now that $I \supsetneq M$. Then $I = J + M$, where $J = Dy_1 + \ldots + Dy_k$ for some $y_1, \ldots, y_k \in D$ by Theorem 2.1.1. We claim that $\underline{y} = y_1, \ldots, y_k$ is also a strong parameter sequence in $R$. Indeed, this is immediate from [26, Proposition 3.3(b)] once we show $(\underline{y})R = I = (\underline{x})R$. The inclusion $(\underline{y})R \subseteq I$ is immediate. For the other inclusion, just note that since $I$ properly contains $M$, we can assume each of the $y_i \notin M$. Then $y_i$ is a unit in $V$, so that $M = My_i \subseteq Ry_i \subseteq (\underline{y})R$, and hence $I = J + M \subseteq (\underline{y})R$. So $I = (\underline{y})R$, and it thus remains to show that p.grade$(I) = k$.

We have an inclusion of rings $D \to R$, and $(\underline{y})D$ is a proper ideal of $D$ since $I$ is a proper ideal of $R$. Moreover, say $\mathfrak{p} \in V(\underline{y}D)$. Then $(D - \mathfrak{p}) \cap (\underline{y}) = \varnothing \Rightarrow (D - \mathfrak{p}) \cap I = \varnothing$, since $I \subseteq \mathfrak{p}+M$. Thus we have $(\underline{y})R_{\mathfrak{p}} \neq R_{\mathfrak{p}}$, so that we conclude $\underline{y}$ is a strong parameter sequence in $D$ by [26, Proposition 3.3(d)]. Since $D$ is one-dimensional, the maximal length of a parameter sequence is 1. Thus $k = 1$, and since we're in a domain, it follows p.grade$(I) = k = 1$. $\square$

**Corollary 2.1.5.** *Let $D$ be a 1-dimensional Noetherian domain and $K$ its quotient field. Then $R = D + XK[[X]]$ is a 2-dimensional Cohen-Macaulay ring.*

*Proof.* The ring $K[[X]]$ is a DVR, and since $D$ is Noetherian, any finite sequence of elements in $D$ is weakly proregular. Thus the claim follows by the above theorem. $\square$

In the previous section it was mentioned that Hamilton and Marley's definition of a Cohen-Macaulay ring is unknown to satisfy the '$\Rightarrow$' implication of condition (H1). That is, it is unresolved if $R$ Cohen-Macaulay implies $R[X]$ is Cohen-Macaulay, where $X$ is an indeterminate over $R$. We show this holds in low dimension in certain cases. First we need a couple preparatory lemmas (see [36, Theorems 150,151] for proofs with similar ideas).

**Lemma 2.1.6.** *Let $R$ be a ring. Suppose that $\underline{X} = X_1, \ldots, X_n$ and $t$ are indeterminates over $R$. If $J$ is any ideal of $R[\underline{X}]$ and $Q$ is a maximal ideal of $R[t]$, then*

$$\Big(JR[t] :_{R[\underline{X},t]} QR[\underline{X}]\Big) = JR[t].$$

*Proof.* We'll set

$$I = \Big(JR[t] :_{R[\underline{X},t]} QR[\underline{X}]\Big).$$

The one inclusion is clear. For the other inclusion, we consider two cases. First, assume that $t \in Q$. If this is the case, then for any $f \in I$, we have that $tf \in JR[t]$. Now write

$$f = \sum_{i=0}^{n} a_i t^i, a_i \in R[\underline{X}].$$

Then if

$$tf = \sum_{i=0}^{n} a_i t^{i+1} \in JR[t],$$

we must have that $a_i \in J$ for each $i$, so that $f \in JR[t]$. This handles the first case. For the second case, where $t \notin Q$, we note that since $Q$ is a maximal ideal of $R[t]$, there is some $g \in Q$ and $h \in R[t]$ so that $1 = g + th$. Now again, if $f \in I$, then $fg = f(1-th) = f - fth \in JR[t]$. We write

$$f = \sum_{i=0}^{n} a_i t^i, a_i \in R[\underline{X}] \text{ and } h = \sum_{j=0}^{m} b_j t^j, b_j \in R.$$

Thus we have that

$$f - fth = \Big(\sum_{i=0}^{n} a_i t^i\Big) - \Big(\sum_{i=0}^{n} a_i t^{i+1}\Big)\Big(\sum_{j=0}^{m} b_j t^j\Big) \in JR[t].$$

Now viewed as a polynomial in $(R[\underline{X}])[t]$, the constant term of $f - fth$ is just $a_0$, hence is in $J$. In general the degree $k > 0$ coefficient of $f + fth$ is just

$$a_k - \Big(\sum_{i+j=k-1} a_i b_j\Big) \in J.$$

Now the sum $\sum_{i+j=k-1} a_i b_j \in J$ since each $a_i \in J$ for $i < k$ by induction, so that $a_k \in J$. In any case, all of the coefficients of $f$ are in $J$, so that $f \in JR[t]$. $\square$

**Lemma 2.1.7.** *Let $R$ be a ring and $t$ an indeterminate over $R$. Then for any maximal ideal $Q$ of $R[t]$, let $\mathfrak{q} = Q \cap R$. If p.grade$(\mathfrak{q}, R) < \infty$, then*

$$p.grade(\mathfrak{q}, R) < p.grade(Q, R[t]).$$

*Proof.* Say that p.grade$(\mathfrak{q}, R) = k$. Then after possibly adjoining a finite number of indeterminates $\underline{X}$ to $R$, we have that $\mathfrak{q}[\underline{X}]$ contains a $R[\underline{X}]$-sequence of length $k$, call it $\underline{f} = f_1, \ldots, f_k$. Let $J = (\underline{f})R[\underline{X}] \subseteq \mathfrak{q}[\underline{X}]$. Evidently, after adjoining $t$, $\underline{f}$ remains a $R[\underline{X}, t]$-sequence of length $k$ that lives in $\mathfrak{q}[\underline{X}, t] \subseteq Q[\underline{X}]$. Then

$$\text{p.grade}(Q, R[t]) = \text{p.grade}(Q[\underline{X}], R[\underline{X}, t]) = k + \text{p.grade}\Big(Q[\underline{X}], \frac{R[\underline{X}, t]}{JR[t]}\Big).$$

Thus it remains to show that p.grade$\Big(Q[\underline{X}], \dfrac{R[\underline{X}, t]}{JR[t]}\Big) > 0$. But this happens precisely when

$$\Big(JR[t] :_{R[\underline{X},t]} QR[\underline{X}]\Big) = JR[t].$$

$\square$

**Theorem 2.1.8.** *Let $R$ be a 0-dimensional ring. Then $R[X]$ is Cohen-Macaulay, where $X$ is an indeterminate.*

*Proof.* We have that $R[X]$ is a 1-dimensional ring, so the length of a strong parameter sequence can only be one. So suppose that $f$ is a strong parameter on $R[X]$. Write $f = \sum_{i=0}^{n} a_i X^i$, and let $I = (a_0, \ldots, a_n)R$. We must show that $f$ is a $R[X]$-regular element. By Theorem 1.2.1, this happens precisely when $(0 :_R I) = 0$.

So suppose otherwise. Then there is some $c \neq 0$ with $cI = 0$. That is, $I \subseteq (0 :_R c)$. Since $c$ is nonzero, $(0 :_R c)$ is a proper ideal of $R$. Thus there is some $\mathfrak{p} \in \operatorname{Spec}(R)$ with $(0 :_R c) \subseteq \mathfrak{p}$. But then $f \in IR[X] \subseteq (0 :_{R[X]} c) \subseteq \mathfrak{p}R[X]$. But since $\operatorname{ht}(\mathfrak{p}) = 0$, we have $\operatorname{ht}(\mathfrak{p}R[X]) = 0$ as well. This is a contradiction though, since $f$ is a strong parameter means that $\operatorname{ht}(fR[X]) \geq 1$. $\square$

While we are unable to resolve (H1) in general when $\dim(R) > 0$, we show that under the restriction that $\dim(R) = 1$ and $R$ is a Jaffard domain, condition (H1) holds. Recall that an integral domain is *Jaffard* if $\dim(R[X_1, \ldots, X_n]) = \dim(R) + n$, whenever $X_1, \ldots, X_n$ are indeterminates over $R$.

**Lemma 2.1.9.** *Let $R$ be a Jaffard domain with $\dim(R) = 1$. Then if $R$ is Cohen-Macaulay, $R[X]$ is also Cohen-Macaulay.*

*Proof.* Since $R$ is a Jaffard domain, $\dim(R[X]) = 2$. So the maximal length of a parameter sequence in $R[X]$ is 2. It's clear that since $R[X]$ is a domain, every parameter $f$ in $R[X]$ is a regular element. Thus let $f, g$ be a strong parameter sequence in $R[X]$. We must show that $f, g$ is a regular sequence, or equivalently, if $J = (f, g)R[X]$, then $\text{p.grade}(J, R[X]) = 2$. Thus by [26, Proposition 3.6], $\operatorname{ht}(J) = 2$. Choose $Q \in V(J)$ so that $\text{p.grade}(Q, R[X]) = \text{p.grade}(J, R[X])$. This prime ideal $Q$ must actually be maximal in $R[X]$. Now set $\mathfrak{q} = Q \cap R$. Since $\text{p.grade}(\mathfrak{q}, R) \leq \text{p.grade}(Q, R[X]) \leq 2$, we must actually have that by Lemma 2.1.7,

$$\text{p.grade}(\mathfrak{q}, R) < \text{p.grade}(Q, R[X]).$$

Now $\mathfrak{q} \neq 0$ by [36, Theorem 37], so that $\text{p.grade}(\mathfrak{q}, R) \geq 1$, hence $\text{p.grade}(Q, R[X]) = 2$, and we're done. $\square$

While the above lemma shows that a 1-dimensional Jaffard domain satisfies (H1), we use the pullback construction to give an example of a 1-dimensional domain $R$ that is not Jaffard, yet satisfies (H1).

**Theorem 2.1.10.** *Let $V = \mathbb{C}(t)[\![X]\!]$, where $t$ is an indeterminate over $\mathbb{C}$ and $X$ an indeterminate over $\mathbb{C}(t)$ and $R = \mathbb{C} + X\mathbb{C}(t)[\![X]\!]$. Then $R$ and $R[Y]$ are Cohen-Macaulay, where $Y$ is an indeterminate over $R$.*

*Proof.* We have that $\dim(R) = 1$ by the above theorem, and since any 1-dimensional domain is Cohen-Macaulay, $R$ is Cohen-Macaulay. Even more, $\operatorname{Spec}(R) = \{0, \mathfrak{p}\}$, where $\mathfrak{p} = X\mathbb{C}(t)[\![X]\!]$ by [17, Theorem A(c)]. On the other hand, if $Y$ is an indeterminate over $R$, then $\dim(R[Y]) = 3$ by [11, Proposition 2.1], and thus $\operatorname{ht}(\mathfrak{p}R[Y]) = 2$. We claim next that $R[Y]$ is Cohen-Macaulay. We achieve this by showing that for each $Q \in \operatorname{Spec}(R[Y])$, $R[Y]_Q$ is Cohen-Macaulay. Set $T = R[Y]_Q$. Since $\dim(R[Y]) = 3$, this leaves us with 3 possibilities (the case when $Q = 0$ is simple, since then $R[Y]_Q$ is a field):

(1) $\operatorname{ht}(Q) = 1$. Then $T$ is a 1-dimensional integral domain, which is always Cohen-Macaulay.

(2) $\operatorname{ht}(Q) = 2$. In fact, we must have that $Q = \mathfrak{p}R[Y]$. Indeed, this follows since no three prime ideals of $R[Y]$ can contract to the same prime ideal of $R$. Since $R$ has only two prime ideals, 0 and $\mathfrak{p}$, we must have $Q \cap R = \mathfrak{p}$. But then $\mathfrak{p}R[Y]$ has height 2 in $R[Y]$ by our previous remarks, and since $\mathfrak{p}R[Y] \subseteq Q$, we must have that $Q = \mathfrak{p}R[Y]$. So we must show that $T$ is Cohen-Macaulay. Since $\dim(T) = 2$, a (strong) parameter sequence can be at most length 2. So suppose $f$ is a parameter in $T$. Since $T$ is a domain and $f$ is non-zero, we must have $f$ is a regular element. So suppose now $f, g$ is a strong

parameter sequence in $T$ and let $J = (f,g)T$. Then $\mathrm{ht}(J) \geq 2$, and in fact we must have that $\mathrm{ht}(J) = 2$, and so $\sqrt{J} = \mathfrak{p}T$. On the other hand, we observe that $\sqrt{XR} = \mathfrak{p}$, so that $\sqrt{XR[Y]} = \mathfrak{p}R[Y]$, and thus $\sqrt{XT} = \mathfrak{p}T$. But since Čech cohomology is computed up to radical, we have that $H^2_{f,g}(T) \cong H^2_X(T) = 0$, contradicting that $f,g$ is a strong parameter sequence. Thus this situation is impossible. So we conclude $T$ is Cohen-Macaulay.

(3) $\mathrm{ht}(Q) = 3$. We note that as before, we must have $Q \cap R = \mathfrak{p}$, so that we have $\mathfrak{p}R[Y] \subset Q$. Now the length of a parameter sequence can be at most length 3. If $f$ is a parameter, then as before, $f$ must be regular. Now suppose that $f,g$ is a strong parameter sequence in $T$. Set $J = (f,g)T$. Then there are two subcases to consider:

(a) $\mathrm{ht}(J) = 2$. In this case, we must have that $\sqrt{J} = \mathfrak{p}T$, and as before $H^2_{f,g}(T) \cong H^2_X(T) = 0$, contradicting that $f,g$ is a strong parameter sequence. So this situation can't happen.

(b) $\mathrm{ht}(J) = 3$. In this case, we must have that $\sqrt{J} = QT$, and so $\mathrm{p.grade}(J,T) = \mathrm{p.grade}(QT,T) \leq 2$. Now $\mathrm{p.grade}(Q,R[Y]) \leq \mathrm{p.grade}(QT,T)$, and so by Lemma 2.1.7, we have that $1 = \mathrm{p.grade}(\mathfrak{p},R) < \mathrm{p.grade}(Q,R[Y])$, hence $\mathrm{p.grade}(J,T) = 2$.

Thus we have any strong parameter sequence $f,g$ is a regular sequence on $T$. Lastly, say $f,g,h$ is a strong parameter sequence on $R[Y]$ and set $I = (f,g,h)T$. Then $\mathrm{ht}((f,g,h)T) = 3$, and so we have $\sqrt{(f,g,h)T} = QT$. Now $f,g,h$ is a strong parameter sequence means that $f,g$ is also a strong parameter sequence. By our previous comments, this can only happen if $\sqrt{(f,g)T} = QT$. But then since Čech cohomology is computed up to radical, we must have $H^3_{f,g,h}(T) \cong H^3_{f,g}(T) = 0$, contradicting that $f, g, h$ is a strong parameter sequence. So there are no strong parameter sequences of length 3. So $R[Y]_Q$ is Cohen-Macaulay.

□

Hamilton and Marley suggest in [26] there might be some additional assumptions on a ring needed to fully resolve conditions (H1) and (H2). With this in mind, they propose studying FGFC rings (see [41] for more details): A ring $R$ is said to be *FGFC* if every finitely generated $R$-ideal has finitely many minimal primes. We have a nice property of FGFC rings (this was proved only for pure subrings of $k[x_1, x_{2,\ldots}]$ where $k$ is a field and the $x_i$ are indeterminates in [5, Lemma 2.4], but the statement and the proof extend easily to any FGFC ring):

**Lemma 2.1.11.** *([5, Lemma 2.4]) Let $R$ be a FGFC ring and $I$ an ideal with ht$(I) \geq k$. Then there are $\{x_i\}_{i=1}^k$ in $I$ such that ht$((x_1, \ldots, x_i)R) \geq i$ for each $i = 1, \ldots, k$.*

**Theorem 2.1.12.** *Suppose that $R$ is a FGFG wB-unmixed ring one-dimensional ring. Then $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in$ Spec$(R)$.*

*Proof.* Let $\underline{x}$ be a strong parameter sequence in $R_\mathfrak{p}$. Since $R_\mathfrak{p}$ is at most one dimensional, $\underline{x}$ can only have length 1. So write $\underline{x} = x \in R_\mathfrak{p}$. We must show now that

$$\text{p.grade}(xR_\mathfrak{p}, R_\mathfrak{p}) = 1.$$

There is some $Q \in V(xR_\mathfrak{p})$ by Theorem 1.2.1 such that

$$\text{p.grade}(Q, R_\mathfrak{p}) = \text{p.grade}(xR_\mathfrak{p}, R_\mathfrak{p}).$$

Write $Q = \mathfrak{q}R_\mathfrak{p}$, so that $\mathfrak{p} \in V(\mathfrak{q})$. Since $x$ is a parameter, we must have that $\mathrm{ht}(xR_\mathfrak{p}) = 1$ by [26, Proposition 3.6], so $Q = \mathfrak{p}R_\mathfrak{p}$ and $\mathfrak{q} = \mathfrak{p}$. Thus $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}R_\mathfrak{p}) = \mathrm{ht}(xR_\mathfrak{p}) = 1$. Then by the above lemma, there is some $y \in \mathfrak{p}$ such that $\mathrm{ht}(yR) \geq 1$. If we can show that $y$ is regular, this completes the proof, since then

$$1 \leq \text{p.grade}(\mathfrak{p}, R) \leq \text{p.grade}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}) = \text{p.grade}(xR_\mathfrak{p}, R_\mathfrak{p}) \leq 1,$$

so that we must have equality. Now $\mathrm{ht}(yR) \geq 1$, so that $y \notin \mathfrak{r}$ for any $\mathfrak{r} \in \mathrm{Min}(R) = \mathrm{wAss}(R)$. Thus $y_1 \notin \bigcup\{\mathfrak{r} \mid \mathfrak{r} \in \mathrm{wAss}(R)\}$, where equality holds since $R$ is wB-unmixed, which by [33, Theorem 6.2(i)] is equal to $Z(R)$. Thus $y$ is regular. $\square$

## 2.2 Direct Limits of Fixed Rings

Fix a directed set $\Lambda$ and let $\{R_\lambda; \rho_{\mu\lambda}\}$ denote a direct system of rings. Suppose further that we have a direct system of groups $\{G_\lambda; f_{\mu\lambda}\}$ over $\Lambda$ so that the following two conditions hold:

(i) For each $\lambda \in \Lambda$, $G_\lambda$ is a group of ring automorphisms of $R_\lambda$.

(ii) For any $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, if $g \in G_\lambda$, then $f_{\mu\lambda}(g) \circ \rho_{\mu\lambda} = \rho_{\mu\lambda} \circ g$. In other words, the following diagram commutes:

$$\begin{tikzcd} R_\lambda \arrow[r, "\rho_{\mu\lambda}"] \arrow[d, "g"'] & R_\mu \arrow[d, "f_{\mu\lambda}(g)"] \\ R_\lambda \arrow[r, "\rho_{\mu\lambda}"] & R_\mu \end{tikzcd}$$

In this case, we will say that the two systems are *compatible*.

**Lemma 2.2.1.** *Let $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ be compatible direct systems of rings and groups respectively over a directed set $\Lambda$. Whenever $\gamma \geq \mu \geq \lambda$ in $\Lambda$, for any $g \in G_\lambda$ we have the following square commutes:*

$$\begin{tikzcd} R_\mu \arrow[r, "\rho_{\gamma\mu}"] \arrow[d, "f_{\mu\lambda}(g)"'] & R_\gamma \arrow[d, "f_{\gamma\lambda}(g)"] \\ R_\mu \arrow[r, "\rho_{\gamma\mu}"] & R_\gamma \end{tikzcd}$$

*Proof.* We have that if $g' = f_{\mu\lambda}(g)$, then since $f_{\gamma\mu}(g') = f_{\gamma\lambda}(g)$, the claim follows by compatibility.

□

**Theorem 2.2.2.** *Let $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ be compatible direct systems of rings and groups respectively over a directed set $\Lambda$. Then $G = \varinjlim_\lambda G_\lambda$ is a group of ring automorphisms of $R = \varinjlim_\lambda R_\lambda$.*

*Proof.* Suppose that $[g] \in G$, where $g \in G_\lambda$ for some $\lambda \in \Lambda$. We'll define a map $[g]\colon R \to R$ as follows: For $[r] \in R$, where $r \in R_\alpha$, let $[g]([r]) = [f_{\mu\lambda}(g)(\rho_{\mu\alpha}(r))]$, where $\mu$ is any $\mu \in \Lambda$ with $\mu \geq \alpha, \lambda$. To check that $[g]$ is well-defined, we must check that this map doesn't depend on the choice of $g, r$ or $\mu$:

(i) Independence of $\mu$: Suppose that $\mu'$ is another element of $\Lambda$ with $\mu' \geq \alpha, \lambda$. We want to show that $[f_{\mu\lambda}(g)(\rho_{\mu\alpha}(r))] = [f_{\mu'\lambda}(g)(\rho_{\mu'\alpha}(r))]$. Now choose $\gamma \in \Lambda$ with $\gamma \geq \mu, \mu'$. Since the systems are compatible, by the above lemma, the following diagram commutes:

$$\begin{tikzcd} R_\mu \arrow[r, "\rho_{\gamma\mu}"] \arrow[d, "f_{\mu\lambda}(g)"'] & R_\gamma \arrow[d, "f_{\gamma\lambda}(g)"] & R_{\mu'} \arrow[l, "\rho_{\gamma\mu'}"'] \arrow[d, "f_{\mu'\lambda}(g)"] \\ R_\mu \arrow[r, "\rho_{\gamma\mu}"'] & R_\gamma & R_{\mu'} \arrow[l, "\rho_{\gamma\mu'}"] \end{tikzcd}$$

Thus, we have that

$$\begin{aligned}
\rho_{\gamma\mu}\Big(f_{\mu\lambda}(g)(\rho_{\mu\alpha}(r))\Big) &= f_{\gamma\lambda}(g)\Big(\rho_{\gamma\mu}(\rho_{\mu\alpha}(r))\Big) \\
&= f_{\gamma\lambda}(g)(\rho_{\gamma\alpha}(r)) \\
&= f_{\gamma\lambda}(g)\Big(\rho_{\gamma\mu'}(\rho_{\mu'\alpha}(r))\Big) \\
&= \rho_{\gamma\mu'}\Big(f_{\mu'\lambda}(g)(\rho_{\mu'\alpha}(r))\Big),
\end{aligned}$$

so that the claim follows.

(ii) Independence of $r$: Say $[r] = [r']$ for some $r' \in R_{\alpha'}$. Choose $\beta \geq \alpha, \alpha'$ with $\rho_{\beta\alpha}(r) = \rho_{\beta\alpha'}(r')$. By the independence of $\mu$, we can assume without loss of generality that $\mu \geq \beta, \lambda$, so that $\rho_{\mu\alpha}(r) = \rho_{\mu\alpha'}(r')$. Thus it's easy to see $[g]([r]) = [g]([r'])$.

(iii) Independence of $g$: Say $[g] = [g']$ for some $g' \in G_{\lambda'}$. Then there is some $\gamma \geq \lambda, \lambda'$ with $f_{\gamma\lambda}(g) = f_{\gamma\lambda'}(g')$. By the independence of $\mu$, choose $\mu$ with $\mu \geq \gamma, \alpha$. Then $f_{\mu\lambda}(g) = f_{\mu\lambda'}(g')$, from which it follows that $[g]([r]) = [g']([r])$.

Thus $[g]$ is a well-defined map on $R$. In fact, by the independence above, when evaluating $[g]([r])$, where $[r] \in R$, we may assume that $g \in G_\lambda$ and $r \in R_\lambda$ for the same $\lambda \in \Lambda$. Thus $[g]([r]) = [g(r)]$, from which it's clear that $[g]$ defines a ring homomorphism on $R$. Lastly, we will show that $[g]$ is an automorphism of $R$. So suppose that $[g]([r]) = [g(r)] = 0$ for some $[r] \in R$. Then there is some $\mu \geq \lambda$ with $\rho_{\mu\lambda}(g(r)) = f_{\mu\lambda}(g)(\rho_{\mu\lambda}(r)) = 0$ by compatibility of the systems. Now $f_{\mu\lambda}(g)$ is an automorphism of $R_\mu$, so that we must have $\rho_{\mu\lambda}(r) = 0$, and thus $[r] = 0$, so that $[g]$ is monic. It is clear that $[g]$ is epic from the fact that $[g]([r]) = [g(r)]$ and $g$ is a ring automorphism of $R_\lambda$. □

From the above theorem, it then makes sense to consider the fixed ring $R^G$ of $R$, where $G = \varinjlim_\lambda G_\lambda$ and $R = \varinjlim_\lambda R_\lambda$. The question of what conditions on $R$ and $G$ are necessary in order for the fixed ring to be Cohen-Macaulay have been well-studied in the (classical) Noetherian case (e.g., see [31]), particularly when $R = k[X_1, \dots, X_n]$ is a polynomial ring over a field and $G$ is a linear algebraic group. Recently in [5], the Cohen-Macaulayness of $R^G$ has been studied for when $R = k[X_1, X_2, \dots]$ is a polynomial ring in infinitely many variables over a field. The rest of our work in this section builds up to a set of sufficient conditions (Theorem 2.2.9) on the direct systems $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ that give the Cohen-Macaulayness of $\left(\varinjlim R_\lambda\right)^{\varinjlim G_\lambda}$.

**Lemma 2.2.3.** *Let $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ be compatible direct systems of rings and groups respectively over a directed set $\Lambda$, and suppose the transition maps $f_{\mu\lambda}$ are surjective whenever $\mu \geq \lambda$ in $\Lambda$. Then $\{R_\lambda^{G_\lambda}; \psi_{\mu\lambda}\}$ is a direct system of rings, where $\psi_{\mu\lambda}\colon R_\lambda^{G_\lambda} \to R_\mu^{G_\mu}$ is the natural map induced by restriction of $\rho_{\mu\lambda}\colon R_\lambda \to R_\lambda$.*

*Proof.* Since $\psi_{\mu\lambda} = \rho_{\mu\lambda}\big|_{R_\lambda^{G_\lambda}}$, we need only check that the image of $\psi_{\mu\lambda}$ is contained in $R_\mu^{G_\mu}$. Indeed, suppose that $g \in G_\mu$ and $r \in R_\lambda^{G_\lambda}$. Since $f_{\mu\lambda}$ is epic, there is some $h \in G_\lambda$ with $g = f_{\mu\lambda}(h)$. Thus, by compatibility, we have that $g(\rho_{\mu\lambda}(r)) = f_{\mu\lambda}(h)(\rho_{\mu\lambda}(r)) = \rho_{\mu\lambda}(h(r)) = \rho_{\mu\lambda}(r)$, so that $\rho_{\mu\lambda}(r) \in R_\mu^{G_\mu}$, completing our claim. □

**Lemma 2.2.4.** *Let $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ be compatible direct systems of rings and groups respectively over a directed set $\Lambda$, and suppose the transition maps $f_{\mu\lambda}$ are surjective and $\rho_{\mu\lambda}$ are injective whenever $\mu \geq \lambda$ in $\Lambda$. Set $G = \varinjlim_\lambda G_\lambda$ and $R = \varinjlim_\lambda R_\lambda$. Then the natural map $T\colon \varinjlim R_\lambda^{G_\lambda} \to R^G$, where $[r] \mapsto [r]$ with $r \in R_\lambda^{G_\lambda}$, is an isomorphism of rings.*

*Proof.* First, we remark that $T$ is well-defined. This holds, provided that whenever $r \in R_\lambda^{G_\lambda}$, we have $[r] \in R^G$. But if $[g] \in G$, say $g \in G_\lambda$ without loss of generality. Then $[g]([r]) = [g(r)] = [r]$, so that indeed, $[r] \in R^G$ and thus $T$ is well-defined. We define an inverse map $S\colon R^G \to \varinjlim R_\lambda^{G_\lambda}$ as follows: For $[r] \in R^G$, say $r \in R_\lambda$. Then let $S([r]) = [r]$. We need to check that this map is well-defined; that is, that $r \in R_\lambda^{G_\lambda}$. So suppose $g \in G_\lambda$. Now in $R^G$, $[(g(r))] = [g]([r]) = [r]$. Thus there is some $\mu \geq \lambda$ with $\rho_{\mu\lambda}(g(r)) = \rho_{\mu\lambda}(r)$, so that by injectivity, $g(r) = r$. Since $S$ is then well-defined, it's easy to see that $S$ and $T$ are inverses of each other. $\square$

An extension of rings $R \to S$ is said to have the *lying over* property, if whenever $\mathfrak{p} \in \operatorname{Spec}(R)$, there is a $\mathfrak{q} \in \operatorname{Spec}(S)$ with $\mathfrak{q} \cap R = \mathfrak{p}$.

**Lemma 2.2.5.** *Suppose that*

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ T & \xrightarrow{g} & U \end{array}$$

*is a commuting square of rings and ring homomorphisms. If $g$ and $\alpha$ have the lying over property, then so does $f$.*

*Proof.* Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then there is some $\mathfrak{t} \in \operatorname{Spec}(T)$ so that $\mathfrak{t} \cap R = \mathfrak{p}$, and likewise there is some $\mathfrak{u} \in \operatorname{Spec}(U)$ so that $\mathfrak{u} \cap T = \mathfrak{t}$. Thus $\mathfrak{p} = \mathfrak{u} \cap T \cap R = \mathfrak{u} \cap S \cap R$, and so if $\mathfrak{q} = \mathfrak{u} \cap S \in \operatorname{Spec}(S)$, then $\mathfrak{q} \cap R = \mathfrak{p}$. $\square$

An inclusion of rings $R \subseteq S$ is said to be *pure* if for any $R$-module $M$, the sequence $0 \to M \otimes_R R \to M \otimes_R S$ is exact. If $R$ is a direct summand of $S$, then it is easy to see that the extension is pure. We record the following simple fact for referral later.

**Lemma 2.2.6.** *Every pure inclusion of rings $R \subseteq S$ satisfies the lying over property.*

*Proof.* Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\mathfrak{p}S \cap R = \mathfrak{p}$ since our extension is pure by [31, Corollary 6.3]. Thus $\mathfrak{p}S \in \operatorname{Spec}(S)$ and contracts to $\mathfrak{p}$, so that the claim follows. □

We shall need two key results, the first due to Asgharzadeh, Dorreh, and Tousi [4], and the second due to Hochster and Roberts [31].

**Theorem 2.2.7.** *([4, Proposition 4.2]) Suppose that $\{R_\lambda; \rho_{\mu\lambda}\}$ is a direct system of Noetherian Cohen-Macaulay rings with transition maps satisfying lying over (e.g., pure or integral extensions). Then $R = \varinjlim R_\lambda$ is Cohen-Macaulay.*

**Theorem 2.2.8.** *([31, Section 7]) If $S$ is a regular Noetherian ring of characteristic $p > 0$, and $R \to S$ is a pure extension of rings, then $R$ is Noetherian Cohen-Macaulay.*

Recall that if $R \subseteq S$ is an inclusion of rings, we say $R$ is a *module retract* of $S$ if there is an $R$-linear map $\sigma\colon S \to R$ so that $\sigma(r) = r$ for all $r \in R$. In this case we call the map $\sigma$ above a *module retraction*. It is easy to see that if $R$ is a module retract of $S$, $R$ is a direct summand of $S$. Indeed, the module retraction $\sigma$ gives a splitting of the natural short exact sequence $0 \to R \to S \to S/R \to 0$. For a group $G$ acting on a ring $R$ by ring automorphisms, Bergman [8, Proposition 1.1] gives two instances when $R^G$ is a module retract of $R$:

(i) If $G$ is finite and the order of $G$ is a unit in $R$, we have a module retraction called the *Reynolds operator* $\sigma\colon R \to R^G$, (see [**?**, Propositions 9-12]), where for each $x \in R$, $\sigma(x)$ is defined by

$$\sigma(x) = \frac{1}{|G|} \sum_{g \in G} g(x).$$

(ii) If $G$ is *locally* finite, i.e. if for each $x \in R$, the orbits $\mathcal{O}_x = \{gx \mid g \in G\}$ are finite, and $|\mathcal{O}_x|$ is a unit in $R$ for each $x \in R$, then a module retraction $\rho\colon R \to R^G$ is defined for each $x \in R$ by

$$\rho(x) = \frac{1}{|\mathcal{O}_x|} \sum_{r \in \mathcal{O}_x} r.$$

.

**Theorem 2.2.9.** *Suppose that $\{R_\lambda; \rho_{\mu\lambda}\}$ and $\{G_\lambda; f_{\mu\lambda}\}$ are compatible direct systems of rings and groups respectively over a directed set $\Lambda$. Let $R = \varinjlim R_\lambda$ and $G = \varinjlim G_\lambda$. Suppose further that for some prime $p > 0$, and for any $\mu, \lambda \in \Lambda$ with $\mu \geq \lambda$ the following hold:*

*(i) $R_\lambda$ is a Noetherian regular ring of characteristic $p$.*

*(ii) $R_\lambda^{G_\lambda}$ is a module retraction of $R_\lambda$.*

*(iii) The transitions maps $\rho_{\mu\lambda}$ are injective and satisfy lying over, and the $f_{\mu\lambda}$ are surjective.*

*Then $R^G$ is Cohen-Macaulay.*

*Proof.* We first claim that for each $\lambda \in \Lambda$, the extension $R_\lambda^{G_\lambda} \to R_\lambda$ is pure. This follows immediately from the fact that $R_\lambda^{G_\lambda}$ is a module retract of $R_\lambda$, hence a direct summand of $R_\lambda$, so that the extension $R_\lambda^{G_\lambda} \to R_\lambda$ is pure. Next, by Lemma 2.2.3, $\{R_\lambda^{G_\lambda}; \psi_{\mu\lambda}\}$ form a direct system of rings, where $\psi_{\mu\lambda} := \rho_{\mu\lambda}\Big|_{R_\lambda^{G_\lambda}} : R_\lambda^{G_\lambda} \to R_\mu^{G_\mu}$. We claim that for any $\mu \geq \lambda$ in $\Lambda$, the maps $\psi_{\mu\lambda}$ satisfy lying over. Indeed, we have a commuting square

$$\begin{array}{ccc} R_\lambda^{G_\lambda} & \xrightarrow{\psi_{\mu\lambda}} & R_\mu^{G_\mu} \\ \downarrow & & \downarrow \\ R_\mu & \xrightarrow[\rho_{\mu\lambda}]{} & R_\gamma \end{array}$$

where $\rho_{\mu\lambda}$ has lying over by assumption, and the inclusion $R_\lambda^{G_\lambda} \to R_\lambda$ is a pure extension as shown above, so that $R_\lambda^{G_\lambda} \to R_\lambda$ also has the lying over property by Lemma 2.2.6. From Lemma 2.2.5, we see then that the extension $R_\lambda^{G_\lambda} \overset{\psi_{\mu\lambda}}{\to} R_\mu^{G_\mu}$ satisfies lying over. Lastly, from Theorem 2.2.8, each $R_\lambda^{G_\lambda}$ is Noetherian Cohen-Macaulay. Thus, we conclude that by Theorem 2.2.7, $\varinjlim R_\lambda^{G_\lambda} \cong R^G$ is Cohen-Macaulay, where the isomorphism holds by Lemma 2.2.4. □

In [5, Example 4.3], it is shown that for a field $k$, the *Veronese subring* $S = k[\{x_i x_j\}_{i,j\geq 1}]$ of $R = k[x_1, x_2, \ldots]$ is Cohen-Macaulay, where the $x_1, x_2, \ldots$ are countably many indeterminates over $k$. We obtain a similar result, except in positive characteristic $\neq 2$ and for power series.

**Example 2.2.10.** Let $k$ be a field and $p > 0$ a prime with $\mathrm{char}(k) = p \neq 2$. For each $n \in \mathbb{N}$, suppose that $x_1, x_2, \ldots$ are indeterminates over $k$ and let

$$R_n = k[\![x_1, \ldots, x_n]\!],$$

$$G_n = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}.$$

Then we have a natural automorphism

$$-1\colon R_n \to R_n, \text{ where}$$

$$x_i \longmapsto -x_i \text{ for each } i = 1, \ldots, n,$$

so that each $\langle -1 \rangle = \mathbb{Z}/2\mathbb{Z} = G_n$ defines a group of ring automorphisms of $R_n$. Moreover, for any $n \geq m$, we have a natural inclusion $\rho_{n,m}\colon R_m \to R_n$ and identity map $f_{n,m}\colon G_m \to G_n$ that form direct systems $\{R_n; \rho_{n,m}\}$ and $\{G_n; f_{n,m}\}$ of rings and groups respectively.

The limit of these direct systems is thus

$$R := \varinjlim_n R_n = k[\![x_1, x_2, \dots]\!] \text{ and } G = \varinjlim_n G_n = \mathbb{Z}_2.$$

It's clear these two direct systems are compatible, since for any $n \geq m \in \mathbb{N}$ and any $g \in \mathbb{Z}_2$, we have the following square commutes:

$$\begin{tikzcd} R_m \arrow[r, hook] \arrow[d, "g"'] & R_n \arrow[d, "g"] \\ R_m \arrow[r, hook] & R_n \end{tikzcd}$$

Moreover, for any $n \geq m \in \mathbb{N}$, $\rho_{n,m} \colon R_m \to R_n$ has the lying over property. Indeed, it's enough to show this for when $m \in \mathbb{N}$ and $n = m + 1$. In this case,

$$R_{m+1} = k[\![x_1, \dots, x_m, x_{m+1}]\!] = R_m[\![x_{m+1}]\!] \cong \prod_{i=0}^{\infty} R_m,$$

where the isomorphism holds as $R_m$-modules. Since $R_m$ is a Noetherian ring, an arbitrary product of flat $R_m$-modules is flat ([12, Theorem 2.1]), so that the extension $R_m \to R_{m+1}$ is flat. Even better, for each $\mathfrak{m} \in \operatorname{Max}(R_m)$, we can see from the natural surjection $R_{m+1} \to R_m$ that sends $x_{m+1} \mapsto 0$ that $\mathfrak{m}R_{m+1} \neq R_{m+1}$, so that the extension $R_m \to R_{m+1}$ is faithfully flat by [42, Theorem 7.2], hence has the lying over property. Thus we satisfy the conditions of Theorem 2.2.9, so that we see the ring

$$R^G = k[\![\{x_i x_j\}_{i,j \geq 1}]\!] \text{ is Cohen-Macaulay.}$$

## 2.3 Monoid Rings

Let $\mathbb{N}_0^n$ denote the $n$-fold product of $\mathbb{N}_0$, where $n \geq 1$. We equip $\mathbb{N}_0^n$ with component-wise addition, so that it becomes a semigroup (monoid even).

For $A, B \subseteq \mathbb{N}_0^n$, their *Minkowski sum* is the set $A + B = \{a + b \mid a \in A, b \in B\}$. By a $\mathbb{N}_0^n$-*ideal*, we mean a set $S \subseteq \mathbb{N}_0^n$ that is a subsemigroup of $\mathbb{N}_0^n$ such that $S + \mathbb{N}_0^n \subseteq S$. Furthermore, we say that $S$ is a *prime* $\mathbb{N}_0^n$-*ideal* when in addition $\mathbb{N}_0^n \backslash S$ is also a subsemigroup of $\mathbb{N}_0^n$. Let $I \subseteq \{1, \ldots, n\}$, and set $P_I := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \mid \alpha_i > 0 \text{ for some } i \in I\}$. Then $P_I$ is a prime $\mathbb{N}_0^n$-ideal, and in fact these are the only prime ideals of $\mathbb{N}_0^n$ [28, Lemma 2.4]. By a *square-free* $\mathbb{N}_0^n$-*ideal*, we mean an ideal $H$ of $\mathbb{N}_0^n$ of the form $H = P_{I_1} \cap P_{I_2} \cap \cdots \cap P_{I_k}$ for subsets $I_1, \ldots, I_k$ of $\{1, \ldots, n\}$. We can assume without loss of generality that the $I_j$ are incomparable.

Write $k[\mathbb{N}_0^n] = k[x_1, \ldots, x_n]$, where $k$ is a field and $x_1, \ldots, x_n$ are indeterminates over $k$. So for $H \hookrightarrow \mathbb{N}_0^n$ a square-free ideal, we may view the ring $k[H \cup \{0\}] = k + k[H]$ as a subring of $k[x_1, \ldots, x_n]$.

**Example 2.3.1.** Let $H$ be the ideal of $\mathbb{N}_0^3$ generated by $(1, 1, 0), (0, 1, 1), (1, 0, 1)$. Then $H = P_{1,2} \cap P_{2,3} \cap P_{1,3}$, and if $k$ is any field, then

$$k + k[H] = k + (x_1x_2, x_2x_3, x_1x_3)k[x_1, x_2, x_3] \hookrightarrow k[x_1, x_2, x_3] = k[\mathbb{N}_0^3].$$

In [3], Asgharzadeh and Dorreh proved that all rings $R$ of the form $R = k + k[H]$, where $H$ is a square-free ideal of $\mathbb{N}_0^2$, are Cohen-Macaulay. Indeed, they showed that the rings

(i) $k + xk[x, y]$

(ii) $k + xyk[x, y]$

are both Cohen-Macaulay in the sense of Hamilton-Marley when $k$ is a field and $x, y$ are indeterminates over $k$.

**Lemma 2.3.2.** *Let $k$ be a field and $H \subseteq \mathbb{N}_0^n$. Then $k[H]$ is a square-free monomial ideal of $k[\mathbb{N}_0^n] = k[x_1, \ldots, x_n]$, where the $x_i$ are indeterminates over $k$, if and only if $H$ is a square-free ideal of $\mathbb{N}_0^n$.*

*Proof.* We have $H = \bigcap_{i=1}^{k} P_{I_i}$ if and only if $k[H] = \bigcap_{i=1}^{k} k[P_{I_i}]$. The square-free monomial ideals of $k[x_1, \ldots, x_n]$ are precisely the ideals that are the intersection of the $\mathbb{N}_0^n$-graded prime ideals, i.e., ideals of the form $k[P_I]$ for some $I \subseteq \{1, \ldots, n\}$. $\square$

**Lemma 2.3.3.** *Let $S = k[\mathbb{N}_0^n] = k[x_1, \ldots, x_n]$, where $k$ is a field and $x_1, \ldots, x_n$ are indeterminates over $k$. For $H \hookrightarrow \mathbb{N}_0^n$ a square-free ideal, say $k[H] = (m_1, \ldots, m_d)S$ for some $m_1, \ldots, m_d \in S$. Then the following hold in the ring $R = k + k[H]$:*

*(i)* $k[H]$ *is a maximal ideal of* $R$.

*(ii)* $V_R(\{m_1, \ldots, m_d\}) = \{k[H]\}$.

*(iii)* $R_{m_i}$ *is a regular* $n$*-dimensional Noetherian ring for any* $i \in \{1, \ldots, d\}$.

*(iv)* $R_{\mathfrak{p}}$ *is Noetherian regular for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *with* $\mathfrak{p} \neq k[H]$.

*(v)* $\dim(R) = n$.

*(vi)* $H^n_{\underline{m}}(R) = 0$ *unless* $\underline{m} = x_1, \ldots, x_n$.

*Proof.* (i) Since $R/k[H] = k$, this claim is clear. (ii) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $m_1, \ldots, m_d \in \mathfrak{p}$. Now $k[H]$ is a homogeneous ideal of $S$, and thus is generated as a $k$-vector space by elements of the form $mm_i$, where $i \in \{1, \ldots, d\}$ and $m \in S$ is an arbitrary monomial. It thus suffices to show that any element of the form $mm_i$ above lies in $\mathfrak{p}$ since then $k[H] \subseteq \mathfrak{p}$, so that by maximality of $k[H]$ we would have $\mathfrak{p} = k[H]$. Now $(mm_i)^2 = (m^2 m_i)m_i \in Rm_i \subseteq \mathfrak{p}$, so that

$mm_i \in \mathfrak{p}$. (iii) Let $j \in \{1, \ldots, d\}$. For any $i \in \{1, \ldots, n\}$, we have $\frac{x_i m_j}{m_j} = x_i \in R_{m_i}$. Now if $m_i = x_1^{i_1} \cdots x_n^{i_n}$, then assume without loss of generality that there is some $m \in \mathbb{N}$ so that $i_j > 0$ for all $1 \leq j \leq m$ and $i_j = 0$ for all $j > m$. Then for each $j \in \{1, \ldots, m\}$, we have

$$\frac{x_1^{i_1} \cdots x_{j-1}^{i_{j-1}} x_j^{i_j - 1} x_{j+1}^{i_{j+1}} \cdots x_m^{i_m}}{m_i} = \frac{1}{x_j} \in R_{m_i},$$

so that we must have $R_{m_i} \cong k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, x_{m+1}, \ldots, x_n]$, and thus is a regular Noetherian ring of dimension $n$. (iv) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{p} \neq k[H]$, then by (ii), we must have $m_i \notin k[H]$ for some $i = 1, \ldots, d$. Thus $R_{\mathfrak{p}} = (R_{m_i})_{\mathfrak{p}}$ is a localization of a Noetherian regular ring, hence is also Noetherian regular.

(v) We have $\dim(R) = \mathrm{rank}_{\mathbb{Z}}(H \cup \{0\})$. Since $\mathbb{Z}(H \cup \{0\}) = \mathbb{Z}^n$, we see $\dim(R) = n$ [18, Theorem 21.4]. (vi) We will let $I$ denote the ideal of $S$ generated by $\underline{m}$ (we use this notation to distinguish which ring we will work in, even though set-theoretically, $I = k[H]$). When $n = 1$, the only square-free monomial ideal of $S$ is just $I = (x)S$, and in that case $R = S$, where the claim is trivially true. So suppose $n \geq 2$. There is a short exact sequence of $R$-modules

$$0 \to R \to S \to S/R \to 0,$$

where $I(S/R) = 0$. Thus $H^i_{\underline{m}}(S/R) = 0$ for $i \geq 1$. From the long exact sequence in Čech cohomology, we have an exact sequence of the form

$$0 = H^{n-1}_{\underline{m}}(S/R) \to H^n_{\underline{m}}(R) \to H^n_{\underline{m}}(S) \to H^n_{\underline{m}}(S/R) = 0.$$

So $H^n_{\underline{m}}(R) \cong H^n_{\underline{m}}(S) \cong H^n_I(S)$ since $S$ is a Noetherian ring and all sequences are weakly proregular. From [38], we have that

$$\mathrm{pd}_S(S/I) = \mathrm{cd}(S, I) = \max\{i \mid H^i_I(S) \neq 0\}.$$

Since $H_I^i(S) = 0$ for any $i > \dim(S) = n$ by Grothendieck's Vanishing Theorem [9], $\mathrm{pd}_S(S/I) = n$ if and only if $H_I^n(S) \cong H_{\underline{m}}^n(R) \neq 0$. Now let $\mathfrak{m} = (x_1, \ldots, x_n)S$ and suppose $\mathrm{pd}_S(S/I) = n$. Then localizing at $\mathfrak{m}$, by [10, Proposition 15.15(e)], we have $\mathrm{pd}_S(S/I) = \mathrm{pd}_{S_\mathfrak{m}}(S_\mathfrak{m}/IS_\mathfrak{m}) = n$. From the Auslander-Buchsbaum formula, we have

$$\mathrm{pd}_{S_\mathfrak{m}}(S_\mathfrak{m}/IS_\mathfrak{m}) + \mathrm{depth}(S_\mathfrak{m}/IS_\mathfrak{m}) = \mathrm{depth}(S_\mathfrak{m}) = n,$$

so that $\mathrm{depth}(S_\mathfrak{m}/IS_\mathfrak{m}) = 0$. This implies $\mathfrak{m} \in \mathrm{Ass}_S(S/I)$. But since $I = \sqrt{I}$, a primary decomposition of $I$ is just an intersection of its minimal primes. Thus, $\mathfrak{m}$ is minimal over $I$, so that $I$ is $\mathfrak{m}$-primary and thus $I = \mathfrak{m}$. $\square$

**Lemma 2.3.4.** *Let $H \subseteq \mathbb{N}_0^n$ be a square-free ideal and $R = k + k[H]$. Then every strong parameter sequence on $R_{k[H]}$ has length less than $n$.*

*Proof.* Let $\underline{f} = f_1, \ldots, f_\ell$ be a strong parameter sequence on $R_{k[H]}$. Then since $\dim(R_{k[H]}) \leq \dim(R) = n$ by Lemma 2.3.3(v), we must have $\ell \leq n$. If $\ell = n$, then since $\mathrm{height}_{R_{k[H]}}((\underline{f})R_{k[H]}) = n$, we must have $\sqrt{(\underline{f})R_{k[H]}} = k[H]R_{k[H]}$. Write $k[H] = (\underline{m})k[\mathbb{N}_0^n]$ for some square-free monomials $\underline{m} = m_1, \ldots, m_d \in k[\mathbb{N}_0^n]$. Then

$$H_{\underline{f}}^n(R_{k[H]}) \cong H_{\underline{m}}^n(R_{k[H]}) \cong H_{\underline{m}}^n(R)_{k[H]} = 0$$

by Lemma 2.3.3(vi). So $\ell < n$. $\square$

As a consequence of this lemma, we obtain a different proof than the one in [3] of the fact that the rings $R = k + k[H]$ are Cohen-Macaulay when $\dim(R) = 2$ and $H$ is a square-free ideal of $\mathbb{N}_0^2$.

**Corollary 2.3.5.** *Let $H \subseteq \mathbb{N}_0^2$ be a square-free ideal and $R = k + k[H]$. Then $R$ is Cohen-Macaulay.*

*Proof.* We instead will show that $R$ is locally Cohen-Macaulay, i.e., $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$. This implies $R$ is Cohen-Macaulay by [26, Proposition 4.7]. First, write $k[H] = (m_1, \ldots, m_d)k[\mathbb{N}_0^2]$ for square-free monomials $m_1, \ldots, m_d \in k[H]$. Suppose $\mathfrak{p} \in \operatorname{Spec}(R)$. If $m_i \notin \mathfrak{p}$ for some $i = 1, \ldots, d$, then by Lemma 2.3.3(iv), $R_{\mathfrak{p}}$ is Noetherian regular, and thus is Cohen-Macaulay.

On the other hand, if $m_i \in \mathfrak{p}$ for all $i = 1, \ldots, d$, then by Lemma 2.3.3(ii) we must have $\mathfrak{p} = k[H]$. Now in the ring $R_{k[H]}$, the maximal length of a strong parameter sequence is one by the above lemma. Thus this must be a regular element since $R_{k[H]}$ is a domain. So $R_{k[H]}$ is Cohen-Macaulay, and thus $R$ is Cohen-Macaulay since $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$. $\square$

# Chapter 3

# Non-vanishing of Čech Cohomology

We will attempt to give a generalization of Grothendieck's non-vanishing theorem using Čech cohomology instead of local cohomology.

## 3.1 Čech stability

Hamilton and Marley showed that any coherent regular ring is Cohen-Macaulay with their definition. Any valuation domain is coherent regular (see [20]), so that any valuation domain will be Cohen-Macaulay. We will give a simple proof of this fact though that is independent of knowing all valuation domains are coherent regular:

**Lemma 3.1.1.** *If $V$ is a valuation domain, then $H^i_{\underline{x}}(R) = 0$ for all $i > 1$ and any finite sequence $\underline{x}$ in $V$. Moreover, $V$ is Cohen-Macaulay.*

*Proof.* Let $\underline{x} = x_1, \ldots, x_\ell$ be a finite sequence in $V$. Then $(\underline{x})V = (x_i)V$ for some $i \in \{1, \ldots, \ell\}$, so that $H^i_{\underline{x}}(V) \cong H^i_{x_i}(V) = 0$ for all $i > 1$. Thus, if $\underline{x}$ is a strong parameter sequence in $V$, we must have $\ell = 1$, from which it is clear $V$ is Cohen-Macaulay. $\square$

Recall that a *Prüfer domain* is an integral domain $R$ such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ is a valuation domain.

**Corollary 3.1.2.** *Any Prüfer domain $R$ is Cohen-Macaulay.*

*Proof.* For any $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ is a valuation domain, which is Cohen-Macaulay. Since $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $R$ is Cohen-Macaulay by [26, Proposition 4.7]. □

**Lemma 3.1.3.** *For any nonzero, proper ideal $I$ in a Prüfer domain $R$, p.grade$(I) = 1$.*

*Proof.* Since $R$ is a domain, p.grade$(I) \geq 1$. Moreover, p.grade$(I)$ is the supremum over all p.grade$(J)$ such that $J$ is a finitely generated ideal contained in $I$. Thus to prove the lemma it will suffice to show that if $I$ is finitely generated, then p.grade$(I) \leq 1$. Now if p.grade$(I) \geq 2$, then in some polynomial ring extension $R[\underline{X}]$, we have that $I[\underline{X}]$ contains a $R[\underline{X}]$-regular sequence $f, g$. It follows by [36, Exercise 3.1.2] that $(IR[\underline{X}])^{-1} = R[\underline{X}]$, from which we see $I^{-1} = R$. But since $R$ is Prüfer, we have $R = II^{-1} = I$. □

The above lemma presents an issue in establishing a connection between grade and height in a Prüfer domain, despite the fact that these rings are all Cohen-Macaulay in the sense of Hamilton and Marley. Indeed, there are Prüfer domains with prime ideals of arbitrarily large height (e.g., for any $n \in \mathbb{N}$, there are valuation domains of dimension $n$), yet ideals can only have polynomial grade equal to 1. However, in the Noetherian case, the notion of grade and height are supposed to coincide in a Cohen-Macaulay ring. The underlying issue is that when $R$ is any Noetherian ring, local cohomology (or equivalently, Čech cohomology) has a clear connection to height that disappears in the general, non-Noetherian case. We record this well-known relationship between height and local cohomology in the following lemma.

**Lemma 3.1.4.** *Let $R$ be a Noetherian ring and $I$ a proper ideal of $R$. Then*

$$\mathrm{ht}(I) = \inf_{\mathfrak{p}\in V(I)} \Big\{ \mathrm{cd}(I, R_{\mathfrak{p}}) \Big\} = \inf_{\mathfrak{p}\in V(I)} \Big\{ \sup\{k \in \mathbb{Z} \mid H^k_I(R)_{\mathfrak{p}} \neq 0\} \Big\}.$$

*Proof.* First we notice that if $\mathfrak{q} \in V(I)$, then $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{p} \in \mathrm{Min}(I)$ and so $H^k_I(R)_{\mathfrak{p}} \neq 0 \Rightarrow H^k_I(R)_{\mathfrak{q}} \neq 0$. Thus it's enough to show that

$$\mathrm{ht}(I) = \inf_{\mathfrak{p}\in \mathrm{Min}(I)} \Big\{ \sup\{k \in \mathbb{Z} \mid H^k_I(R)_{\mathfrak{p}} \neq 0\} \Big\}.$$

So suppose $\mathfrak{p} \in \mathrm{Min}(I)$, then $\sqrt{IR_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}$. By standard facts about local cohomology (for instance, in [9]), we have for any $k \in \mathbb{Z}$,

$$H^k_I(R)_{\mathfrak{p}} \cong H^k_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^k_{\sqrt{IR_{\mathfrak{p}}}}(R_{\mathfrak{p}}) = H^k_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

Moreover, we have that by Grothendieck's theorems on the vanishing of local cohomology modules over a local ring with support in the maximal ideal (see Theorems 1.3.2, 1.3.3),

$$\mathrm{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) = \mathrm{cd}(\mathfrak{p}R_{\mathfrak{p}}, R_{\mathfrak{p}}) = \sup\{k \mid H^k_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0\},$$

and thus the claim follows. □

Since local cohomology and Čech cohomology coincide in a Noetherian ring, the above claim would hold just as well if local cohomology was replaced with Čech cohomology. Čech cohomology has many formal properties though that hold even over non-Noetherian rings, for instance, see Theorem 1.4.2. With this in mind, we will instead focus our attention for most of this section on the relationship between height and Čech cohomology (rather than local cohomology) in the general case. The following example illustrates that the Noetherian condition was crucial in proving Lemma 3.1.4:

**Example 3.1.5.** Let $V$ be a valuation domain with $\dim(V) = k$, so that $\text{Spec}(V) = \{0, \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{k-1}, \mathfrak{p}_k\}$, where $\mathfrak{p}_\mathfrak{i} \subsetneq \mathfrak{p}_{i+1}$ for each $i$. Now suppose $I$ is a proper (non-zero) ideal of $V$. Then $\sqrt{I} = \mathfrak{p}_i$ for some $i$, and in fact $\text{ht}(I) = i$. Choose $x \in \mathfrak{p}_i - \mathfrak{p}_{i-1}$, so that $\sqrt{xV} = \sqrt{I}$. Now $V$ is a domain, so that $x$ is weakly proregular, and thus $H^k_I(V) \cong H^k_{xV}(V) \cong H^k_x(V) = 0$ for any $k > 1$. Thus, in this case, the height of an ideal $I$ plays no role in the non-vanishing of Čech cohomology.

In the Noetherian case, the key fact that makes the above situation impossible is the non-vanishing theorem of Grothendieck (Theorem 1.3.3). There's no reason to expect this theorem to be true in general, and all known proofs of this result use structure theorems that depend heavily on the Noetherian assumption - for example, Cohen's structure theorem or the theory of attached primes (see [9]). With this in mind, since results like Lemma 3.1.4 that link height and Čech cohomology may fail over non-Noetherian rings, we will investigate a modified definition of height defined in terms of Čech cohomology.

Let $R$ be a ring and $I$ a finitely generated ideal of $R$. Then we may write $I = (\underline{x})R$, where $\underline{x} = x_1, \dots, x_\ell \in R$. For $M$ an $R$-module, we define the *Čech cohomological dimension* of $M$ with respect to $I$ to be

$$\check{\text{C}}.\text{cd}(I, M) := \sup\{k \in \mathbb{Z} \mid H^k_{\underline{x}}(M) \neq 0\}.$$

Since Čech cohomology is independent of the generating set for $I$, this notation is justified. Moreover for finitely generated $I$, we define the *Cech height* of $I$ to be

$$\check{\text{C}}.\text{ht}(I) := \inf_{\mathfrak{p} \in V(I)} \left\{ \check{\text{C}}.\text{cd}(I, R_\mathfrak{p}) \right\}.$$

By convention, $\sup\{\varnothing\} = -\infty$ and $\inf\{\varnothing\} = \infty$.

**Lemma 3.1.6.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. Then* $\check{\mathrm{C}}.\mathrm{ht}(I) \geq 0$.

*Proof.* If $I = R$, this is certainly true. So assume $I$ is proper. It's enough to show that for every prime ideal $\mathfrak{p} \in V(I)$, $\check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) \geq 0$. If not, then the only other possibility is $\check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) = -\infty$, i.e. for all $k \in \mathbb{Z}$, $H^k_{\underline{x}}(R)_\mathfrak{p} = 0$, where $\underline{x}$ is a finite generating set for $I$. But then this says $\mathrm{p.grade}(I, R_\mathfrak{p}) = \mathrm{p.grade}(IR_\mathfrak{p}, R_\mathfrak{p}) = \infty$, so that by [43, Chapter 5], we must have $IR_\mathfrak{p} = R_\mathfrak{p}$, i.e., $I \not\subseteq \mathfrak{p}$, a contradiction. $\square$

**Lemma 3.1.7.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. If $\mathfrak{p} \subseteq \mathfrak{q}$ in $V(I)$,* $\check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) \leq \check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{q})$. *Moreover,*

$$\check{\mathrm{C}}.\mathrm{ht}(I) = \inf_{\mathfrak{p} \in \mathrm{Min}(I)} \left\{ \check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) \right\}.$$

*Proof.* Let $\underline{x}$ be a finite generating set for $I$. Now if $H^k_{\underline{x}}(R)_\mathfrak{p} \neq 0$ for some $k \in \mathbb{Z}$, then $H^k_{\underline{x}}(R)_\mathfrak{p} = H^k_{\underline{x}}(R)_\mathfrak{q} \otimes_R R_\mathfrak{p}$, so that $H^k_{\underline{x}}(R)_\mathfrak{q} \neq 0$. This completes the proof of the first statement. The second statement follows from the first. $\square$

**Lemma 3.1.8.** *Let $R$ be a ring, $I$ a finitely generated ideal of $R$, and $\mathfrak{p} \in V(I)$. Then* $\check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) = \check{\mathrm{C}}.\mathrm{cd}(IR_\mathfrak{p}, R_\mathfrak{p})$. *Thus* $\check{\mathrm{C}}.\mathrm{ht}(I) \leq \check{\mathrm{C}}.\mathrm{ht}(IR_\mathfrak{p})$, *and in particular*

$$\check{\mathrm{C}}.\mathrm{ht}(I) = \inf_{\mathfrak{q} \in V(I)} \check{\mathrm{C}}.\mathrm{ht}(IR_\mathfrak{q}).$$

*Proof.* Suppose $I$ has a finite generating set $\underline{x}$. We have a flat map $f \colon R \to R_\mathfrak{p}$, and $H^k_{\underline{x}}(R)_\mathfrak{p} \cong H^k_{f(\underline{x})}(R_\mathfrak{p})$, where the latter module is isomorphic to $H^k_{f(\underline{x})}(R_\mathfrak{p})_{\mathfrak{p}R_\mathfrak{p}} \neq 0$. Since $f(\underline{x})$ is a generating set for $IR_\mathfrak{p}$, this shows $\check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) \leq \check{\mathrm{C}}.\mathrm{cd}(IR_\mathfrak{p}, R_\mathfrak{p})$. Conversely we can take the generating set for $IR_\mathfrak{p}$ to be in the image of $f$, so the reverse inequality holds for the same reasoning. This completes the first statement. The second statement follows from the first. $\square$

We have an easy version of Krull's Principal Ideal Theorem for Čech height:

**Lemma 3.1.9.** *Suppose $I$ is a finitely generated $R$-ideal. Then for each $\mathfrak{p} \in V(I)$, we have $\check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) \leq \mu(I)$. In particular, $\check{\mathrm{C}}.\mathrm{ht}(I) \leq \mu(I)$.*

*Proof.* Let $\underline{x} = x_1, \ldots, x_\ell$ be a generating set for $I$. Then $H^k_{\underline{x}}(R)_{\mathfrak{p}} = 0$ for any $k > \ell$, since the Čech complex vanishes in all degrees larger than $\ell$. $\square$

**Lemma 3.1.10.** *Let $R$ be a ring and $I$ a finitely generated $R$-ideal. Then for any $\mathfrak{p} \in V(I)$, we have $\check{\mathrm{C}}.\mathrm{cd}(I, R[X]_{\mathfrak{p}R[X]}) = \check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}})$, where $X$ is an indeterminate over $R$. Moreover, $\check{\mathrm{C}}.\mathrm{ht}(IR[X]) = \check{\mathrm{C}}.\mathrm{ht}(I)$.*

*Proof.* Let $\underline{x}$ be a finite set of generators for $I$ and say that $H^k_{\underline{x}}(R[X])_{\mathfrak{p}R[X]} \neq 0$ for some $k \in \mathbb{Z}$. Now $\underline{x}$ is also a generating set for $IR[X]$ in $R[X]$. Then we have

$$H^k_{\underline{x}}(R[X])_{\mathfrak{p}R[X]} = H^k_{\underline{x}}\Big(R[X]_{\mathfrak{p}R[X]}\Big) = H^k_{\underline{x}}\Big(R_{\mathfrak{p}}[X]_{\mathfrak{p}R[X]}\Big) = H^k_{\underline{x}}(R_{\mathfrak{p}}[X])_{\mathfrak{p}R[X]}$$

$$= \Big(H^k_{\underline{x}}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}[X]\Big)_{\mathfrak{p}R[X]},$$

so that $H^k_{\underline{x}}(R_{\mathfrak{p}}) \cong H^k_{\underline{x}}(R)_{\mathfrak{p}} \neq 0$. Conversely, suppose that $H^k_{\underline{x}}(R)_{\mathfrak{p}} \neq 0$. Then since $R \to R[X]$ is faithfully flat, we have that $R[X]_{\mathfrak{p}R[X]} \cong R_{\mathfrak{p}}[X]_{\mathfrak{p}R[X]}$ is faithfully flat over $R_{\mathfrak{p}}$. Thus

$$0 \neq H^k_{\underline{x}}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}[X]_{\mathfrak{p}R[X]} = \Big(H^k_{\underline{x}}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}[X]\Big) \otimes_{R_{\mathfrak{p}}[X]} R_{\mathfrak{p}}[X]_{\mathfrak{p}R[X]} = H^k_{\underline{x}}(R[X])_{\mathfrak{p}R[X]}$$

as in the lines above. This completes the proof of the first statement. For the second statement, just note that $Q \in \mathrm{Min}(IR[X]) \Leftrightarrow Q = \mathfrak{p}R[X]$ for some $\mathfrak{p} \in \mathrm{Min}(I)$. $\square$

**Lemma 3.1.11.** *Let $I$ be a finitely generated $R$-ideal. Then* $\check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) \le \mathrm{ht}(\mathfrak{p})$ *for any* $\mathfrak{p} \in V(I)$. *In particular,* $\check{\mathrm{C}}.\mathrm{ht}(I) \le \mathrm{ht}(I)$.

*Proof.* For each $\mathfrak{p} \in V(I)$, we have that $\check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) \le \mathrm{ht}(\mathfrak{p})$ since $H^i_{\underline{x}}(R)_{\mathfrak{p}} = 0$ for all $i > \dim(R_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p})$, regardless of the sequence $\underline{x}$. The second statement follows immediately. $\square$

**Lemma 3.1.12.** *Say $R$ is a ring and $I$ a finitely generated $R$-ideal. Then* $\mathrm{p.grade}(I, R) = \check{\mathrm{C}}.\mathrm{grade}(I, R) \le \check{\mathrm{C}}.\mathrm{ht}(I)$.

*Proof.* Write $I = (\underline{x})R$ for some finite sequence $\underline{x}$ in $R$ without loss of generality. Next, we assume that $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m}$ minimal over $I$. In this case, we have that $\check{\mathrm{C}}.\mathrm{ht}(I) = \check{\mathrm{C}}.\mathrm{cd}(I, R) = \sup\{k \mid H^k_{\underline{x}}(R) \neq 0\} \ge \inf\{k \mid H^k_{\underline{x}}(R) \neq 0\} = \check{\mathrm{C}}.\mathrm{grade}(I, R) = \mathrm{p.grade}(I, R)$. The general case follows from the fact that for any $\mathfrak{p} \in \mathrm{Min}(I)$, by the above argument, $\mathrm{p.grade}(I, R) \le \mathrm{p.grade}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) \le \check{\mathrm{C}}.\mathrm{ht}(IR_{\mathfrak{p}})$. Taking the inf over all $\mathfrak{p} \in \mathrm{Min}(I)$ then yields the claim. $\square$

**Lemma 3.1.13.** *Let $R$ be a ring and $I$ a proper finitely generated $R$-ideal. If $I = (\underline{x})R$, where $\underline{x} = x_1, \ldots, x_\ell \in R$, then* $\mathrm{Supp}(H^\ell_{\underline{x}}(R)) = V(I) \Leftrightarrow \check{\mathrm{C}}.\mathrm{ht}(I) = \ell$.

*Proof.* $\Rightarrow$: We have that by Lemma 3.1.9, $\check{\mathrm{C}}.\mathrm{ht}(I) \le \ell$ always holds. Thus it remains to show $\check{\mathrm{C}}.\mathrm{ht}(I) \ge \ell$; that is, $\check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) \ge \ell$ for each $\mathfrak{p} \in V(I)$. But by assumption $\mathfrak{p} \in \mathrm{Supp}(H^\ell_{\underline{x}}(R))$, so that this follows immediately. $\Leftarrow$: Since $\mathrm{Supp}(H^\ell_{\underline{x}}(R)) \subseteq V(I)$ always holds, it remains to show the other inclusion. So suppose $\mathfrak{p} \in V(I)$. Then $\ell = \check{\mathrm{C}}.\mathrm{ht}(I) \le \check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) \le \ell$, so that in fact, $\check{\mathrm{C}}.\mathrm{cd}(I, R_{\mathfrak{p}}) = \ell$, and thus $H^\ell_{\underline{x}}(R)_{\mathfrak{p}} \neq 0$. $\square$

This allows us to reformulate Hamilton and Marley's notion of Cohen-Macaulay in terms of Čech height:

**Corollary 3.1.14.** *Let $R$ be a ring. Then $R$ is Cohen-Macaulay $\Leftrightarrow$ every weakly proregular sequence $x_1, \ldots, x_\ell$ with $\check{\mathrm{C}}.\mathrm{ht}((x_1, \ldots, x_i)R) = i$ for all $i \in \{1, \ldots, \ell\}$ is a regular sequence.*

Let $R$ be a ring and $I$ a finitely generated ideal of $R$. In general, we have the following:

$$\mathrm{p.grade}(I, R) = \check{\mathrm{C}}.\mathrm{grade}(I, R) \leq \check{\mathrm{C}}.\mathrm{ht}(I) \leq \mathrm{ht}(I).$$

When $R$ is Noetherian, as a result of Grothendieck's non-vanishing theorem, we have that $\mathrm{ht}(I) = \check{\mathrm{C}}.\mathrm{ht}(I)$. With this in mind, we shall say that a finitely generated ideal $I$ of a ring $R$ is *Čech stable* if $\check{\mathrm{C}}.\mathrm{ht}(I) = \mathrm{ht}(I)$.

**Lemma 3.1.15.** *Let $R$ be a ring and suppose that every finitely generated ideal of $R_\mathfrak{p}$ is Čech stable whenever $\mathfrak{p} \in \mathrm{Spec}(R)$. Then every finitely generated ideal of $R$ is Čech stable.*

*Proof.* Let $I$ be a finitely generated $R$-ideal. Then for any $\mathfrak{p} \in \mathrm{Min}(I)$, we have that $\check{\mathrm{C}}.\mathrm{cd}(I, R_p) = \check{\mathrm{C}}.\mathrm{cd}(IR_\mathfrak{p}, R_\mathfrak{p}) = \check{\mathrm{C}}.\mathrm{ht}(IR_\mathfrak{p})$ since $\mathrm{Min}(IR_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$. Thus, since $IR_\mathfrak{p}$ is Čech stable in $R_\mathfrak{p}$, we have that $\check{\mathrm{C}}.\mathrm{ht}(IR_\mathfrak{p}) = \mathrm{ht}(IR_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$. So,

$$\check{\mathrm{C}}.\mathrm{ht}(I) = \inf_{\mathfrak{p} \in \mathrm{Min}(I)} \left\{ \check{\mathrm{C}}.\mathrm{cd}(I, R_\mathfrak{p}) \right\} = \inf_{\mathfrak{p} \in \mathrm{Min}(I)} \left\{ \mathrm{ht}(\mathfrak{p}) \right\} = \mathrm{ht}(I).$$

□

**Lemma 3.1.16.** *Suppose $S$ is a Noetherian ring, and let $R = S[X_1, X_2, \ldots]$, where the $X_i$ are indeterminates over $S$. Then every finitely generated ideal of $R$ is Čech stable.*

*Proof.* Let $I$ be a finitely generated ideal of $R$ and write $I = (\underline{f})R$, where $f_1, \ldots, f_m \in R$. We must show that $t := \mathrm{ht}(I)$ is equal to $\check{\mathrm{C}}.\mathrm{ht}(I)$. Since $t \geq \check{\mathrm{C}}.\mathrm{ht}(I)$ always holds, it remains to show $t \leq \check{\mathrm{C}}.\mathrm{ht}(I)$. To achieve this, by Lemma 3.1.7 it is enough to show that for each $P \in \mathrm{Min}_R(I)$ there is some $j \geq t$ so that $H^j_{\underline{f}}(R)_P \neq 0$.

So suppose $P \in \mathrm{Min}_R(I)$. Then there is $N$ sufficiently large so that $f_i \in S[X_1, \ldots, X_N]$ for each $i \in \{1, \ldots, m\}$. Consider the natural inclusion $S[X_1, \ldots, X_N] \to R$ and let $J = I \cap S[X_1, \ldots, X_N]$, so that we have $\mathfrak{p} := P \cap S[X_1, \ldots, X_N]$ is minimal over $J = (\underline{f})R$. Moreover, $\mathfrak{p}R$ is a prime ideal of $R$ such that $I \subseteq \mathfrak{p}R \subseteq P$. Thus we must have $P = \mathfrak{p}R$, and by [25, Lemma 4.1], we also have $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(P) \geq t$. Let $j = \mathrm{ht}(\mathfrak{p})$. Then since $S[X_1, \ldots, X_N] \to R$ is faithfully flat, we have $S[X_1, \ldots, X_N]_{\mathfrak{p}} \to R_P$ is also faithfully flat. Then $H^j_{\underline{f}}(R)_P \cong H^j_{\underline{f}}(S[X_1, \ldots, X_n])_{\mathfrak{p}} \otimes_{S[X_1,\ldots,X_n]_{\mathfrak{p}}} R_P \neq 0$ since $H^j_{\underline{f}}(S[X_1, \ldots, X_n])_{\mathfrak{p}} = H^j_{\mathfrak{p}}(S[X_1, \ldots, X_n]_{\mathfrak{p}}) \neq 0$. $\square$

In Fall 2016, Yves André [2] proved a famous conjecture of Hochster's called the *Direct Summand Conjecture* [30]: If $R$ is a Noetherian regular ring and $R \hookrightarrow S$ a module-finite extension of $R$, then $R$ is a direct summand of $S$. Hochster showed the Direct Summand Conjecture is equivalent to another conjecture of his, called the *Monomial Conjecture*: If $(R, \mathfrak{m})$ is a $n$-dimensional Noetherian local ring and $x_1, \ldots, x_n$ a sequence of elements of $R$ with $\sqrt{(x_1, \ldots, x_n)R} = \mathfrak{m}$, then for all $k \in \mathbb{N}_0$, $(x_1 \cdots x_n)^k \notin (x_1^{k+1}, \ldots, x_n^{k+1})R$ which he was able to use in characteristic $p$ using the Frobenius map. We show below (Theorem 3.1.17), that as a consequence of having Čech stability, the Monomial Conjecture holds in non-Noetherian quasi-local rings in characteristic $p$. First, we make a remark about the top Čech cohomology module $H^\ell_{\underline{x}}(R)$ with respect to a sequence $\underline{x} = x_1, \ldots, x_\ell \in R$, where $R$ is any ring. Observe that

$$H^\ell_{\underline{x}}(R) = \mathrm{coker}\Big(\bigoplus_{i=1}^{\ell} R_{x_1 \cdots \widehat{x_i} \cdots x_\ell} \to R_{x_1 \cdots x_\ell}\Big).$$

We'll use $[\,]$ to denote the natural image of an element of $R_{x_1 \cdots x_\ell}$ in $H^\ell_{\underline{x}}(R)$. Every element in $H^\ell_{\underline{x}}(R)$ can thus be written in the form $[a/(x_1 \cdots x_\ell)^i]$ for some $a \in R$ and $i \in \mathbb{N}_0$.

**Theorem 3.1.17.** *Suppose that $(R, \mathfrak{m})$ is a quasi-local ring of dimension $n$ with char$(R) = p$ and $\mathfrak{m} = \sqrt{I}$, where $I$ is a finitely generated Čech stable ideal. Then if $I = (x_1, \ldots, x_n)R$, for all $k \in \mathbb{N}_0$, $(x_1 \cdots x_n)^k \notin (x_1^{k+1}, \ldots, x_n^{k+1})R$.*

*Proof.* Since $I$ is Čech stable and $\mathrm{Min}(I) = \{\mathfrak{m}\}$, we have that $n = \mathrm{ht}(I) = \check{\mathrm{C}}.\mathrm{ht}(I) = \check{\mathrm{C}}.\mathrm{cd}(I, R)$, and so $H^n_{\underline{x}}(R) \neq 0$. Suppose by way of contradiction, that for some $k \in \mathbb{N}_0$ we have $(x_1 \cdots x_n)^k \in (x_1^{k+1}, \ldots, x_n^{k+1})R$. Then for some $r_1, \ldots, r_n \in R$, $(x_1 \cdots x_n)^k = \sum_{i=1}^n r_i x_i^{k+1}$. Consider now the element $\eta := [1/x_1 \cdots x_n] \in H^n_{\underline{x}}(R)$. Then

$$\eta = \Big[\frac{1}{x_1 \cdots x_n}\Big] = \Big[\frac{(x_1 \cdots x_n)^k}{(x_1 \cdots x_n)^{k+1}}\Big] = \Big[\frac{r_1 x_1^{k+1} + \ldots + r_n x_n^{k+1}}{(x_1 \cdots x_\ell)^{k+1}}\Big] = 0.$$

We have a natural homomorphism of abelian groups $F \colon H^n_{\underline{x}}(R) \to H^n_{\underline{x}}(R)$ where

$$\Big[\frac{s}{(x_1 \cdots x_n)^j}\Big] \mapsto \Big[\frac{s^p}{(x_1 \cdots x_n)^{jp}}\Big].$$

Now, let $[r/(x_1 \cdots x_n)^i] \in H^n_{\underline{x}}(R)$ be arbitrary. Choose $t$ sufficiently large so that $p^t \geq i$. Then

$$\begin{aligned}\Big[\frac{r}{(x_1 \cdots x_n)^i}\Big] &= \Big[\frac{r(x_1 \cdots x_n)^{p^t - i}}{(x_1 \cdots x_n)^{p^t}}\Big] \\ &= r(x_1 \cdots x_n)^{p^t - i}\Big[\frac{1}{(x_1 \cdots x_n)^{p^t}}\Big] \\ &= r(x_1 \cdots x_n)^{p^t - i} F^t(\eta) = 0.\end{aligned}$$

This contradicts our assumption that $H^n_{\underline{x}}(R) \neq 0$. ∎

**Lemma 3.1.18.** *Let* $(R, \mathfrak{m})$ *be a quasi-local FGFC ring with* $\dim(R) < \infty$. *Then*

*(i)* $\operatorname{ara}(\mathfrak{m}) < \infty$.

*(ii) If* $\mathfrak{m} = \sqrt{I}$ *with* $I$ *finitely generated Čech stable, then* $\dim(R) = \operatorname{ara}(\mathfrak{m})$.

*Proof.* (i) Since $R$ is FGFC, if $\operatorname{ht}(\mathfrak{m}) = n$, we know that there are $x_1, \ldots, x_n \in \mathfrak{m}$ with $\operatorname{ht}((x_1, \ldots, x_n)R) = n$. Thus we must have that $\operatorname{Min}((x_1, \ldots, x_n)R) = \{\mathfrak{m}\}$, so that $\sqrt{(x_1, \ldots, x_n)R} = \mathfrak{m}$, and hence $\operatorname{ara}(\mathfrak{m}) < \infty$. For (ii), the above work shows $\dim(R) = \operatorname{ht}(\mathfrak{m}) \geq \operatorname{ara}(\mathfrak{m})$. The other inequality follows by Lemma 3.1.9. $\square$

So if $(R, \mathfrak{m})$ is quasi-local FGFC and $\mathfrak{m}$ is generated up to radical by a finitely generated Čech stable ideal, then $R$ has a *system of parameters*, that is, a sequence $x_1, \ldots, x_n$ such that $\sqrt{(x_1, \ldots, x_n)R} = \mathfrak{m}$ where $n = \dim(R)$ is the smallest integer for which this happens. The Čech stability condition is actually needed, since any valuation domain $(V, \mathfrak{m})$ of dimension larger than 1 will be FGFC, yet $\operatorname{ara}(\mathfrak{m}) = 1$. It's easy to produce rings non-Noetherian that have systems of parameters.

**Example 3.1.19.** Let $R = k[\![x, y]\!]$, where $k$ is a field and $x, y$ are indeterminates over $k$. Then $R$ is local with unique maximal ideal $\mathfrak{m} = (x, y)R$. Suppose that $M$ is any $R$-module and let $S = R \ltimes M$ be the idealization of $M$. Then $S$ is a 2-dimensional quasi-local ring with maximal ideal $\mathfrak{m} \ltimes M$. Moreover, $\sqrt{\mathfrak{m} \ltimes M} = \sqrt{(X, Y)S}$, where $X = (x, 0)$ and $Y = (y, 0)$. We claim $X, Y$ is a system of parameters for $S$ that generates a Čech stable ideal in $S$. To show that $(X, Y)S$ is Čech stable, we must show that $H^2_{x,y}(S) \neq 0$. As $R$-modules, we have that $H^2_{X,Y}(S) = H^2_{x,y}(S) = H^2_{x,y}(R \oplus M) = H^2_{x,y}(R) \oplus H^2_{x,y}(M)$, which is non-zero since $H^2_{x,y}(R) \neq 0$ by Theorem 1.3.2. So we conclude that $(X, Y)S$

is Čech stable. Moreover, $S$ is FGFC. Indeed, suppose $J$ is a finitely generated ideal of $S$. Then $\sqrt{J} = \sqrt{\pi(J)} \rtimes M$, where $\pi\colon S \to R$ is the natural projection (a ring homomorphism). Since $R$ is Noetherian, $\pi(J)$ has only finitely many minimal primes $P_1, \ldots, P_k$. Thus $J$ has minimal primes $P_1 \rtimes M, \ldots, P_k \rtimes M$. So we see that by Lemma 3.1.18, $X, Y$ is a system of parameters for $S$. Moreover,if $M$ is not a finitely generated $R$-module, then $S$ will not be a Noetherian ring.

**Lemma 3.1.20.** *Say $(R, \mathfrak{m})$ is a quasi-local ring and $I = (y, x_1, \ldots, x_\ell)R$ is an ideal of $R$ with $\sqrt{I} = \mathfrak{m}$. Then $\check{\mathrm{C}}.\mathrm{ht}(I) \geq \check{\mathrm{C}}.\mathrm{ht}(I/yR) \geq \check{\mathrm{C}}.\mathrm{ht}(I) - 2$ if $y$ is regular.*

*Proof.* Set $\underline{x} = x_1, \ldots, x_\ell$. Let $t = \check{\mathrm{C}}.\mathrm{ht}(I/yR) = \sup\{i \mid H^i_{y,\underline{x}}(R/yR) \neq 0\}$. We have a short exact sequence $0 \to R \xrightarrow{y} R \to R/yR \to 0$ that yields a long exact sequence in Čech cohomology:

$$\cdots \to H^i_{y,\underline{x}}(R/yR) \to H^{i+1}_{y,\underline{x}}(R) \xrightarrow{y} H^{i+1}_{y,\underline{x}}(R) \to H^{i+1}_{y,\underline{x}}(R/yR) \to \cdots$$

Now for $i > t$, we must have $H^i_{y,\underline{x}}(R/yR) = 0$, so that multiplication by $y$ on $H^{i+1}_{y,\underline{x}}(R)$ is injective. But $H^{i+1}_{y,\underline{x}}(R)$ is $I$-torsion, so that for any $m \in H^{i+1}_{y,\underline{x}}(R)$, $y^N m = 0$ for $N \gg 0$, and so $H^{i+1}_{y,\underline{x}}(R) = 0$. Thus $\check{\mathrm{C}}.\mathrm{ht}(I) \leq t + 2$. For the other inequality, observe that for any $j > \check{\mathrm{C}}.\mathrm{ht}(I) = \sup\{i \mid H^i_{y,\underline{x}}(R) \neq 0\}$, by the above long exact sequence it follows that $H^j_{y,\underline{x}}(R/yR) =$, so that $t \leq \check{\mathrm{C}}.\mathrm{ht}(I)$. □

The following is a technical lemma we shall need later.

**Lemma 3.1.21.** *Suppose $(R, \mathfrak{m})$ is FGFC quasi-local with $\dim(R) = \mathrm{ara}(\mathfrak{m}) = n < \infty$. If $x \in \mathfrak{m}$ is a regular element and $\mathfrak{m}/xR$ is generated up to radical by a Čech stable ideal of $R/xR$, then $\dim(R/xR) = n - 1$.*

*Proof.* Since $x$ is regular, we have that $\dim(R/xR) < \dim(R) = n$. Now suppose $\operatorname{ara}(R/xR) = k$. By Lemma 3.1.18, we have that $\dim(R/xR) = k$. Then there are $\overline{x_1}, \ldots, \overline{x_k} \in R/xR$ with $\sqrt{(\overline{x_1}, \ldots, \overline{x_k})R/xR} = \mathfrak{m}/xR$. Thus $\mathfrak{m} = \sqrt{(x_1, \ldots, x_k, x)R}$, so that $\operatorname{ara}(\mathfrak{m}) = n \leq k + 1$, i.e., $\dim(R/xR) \geq n - 1$. $\square$

## 3.2 Weakly coassociated primes

For a maximal ideal $\mathfrak{m}$ of a ring $R$, let $D_{\mathfrak{m}}(-) := \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$, where $E_R(R/\mathfrak{m})$ is the $R$-injective hull of $R/\mathfrak{m}$. Note that $D_{\mathfrak{m}}(-)$ is an exact, contravariant functor of $R$-modules. For an $R$-module $M$, we shall say $\mathfrak{p} \in \operatorname{Spec}(R)$ is a *weakly coassociated prime* of $M$ if there is a submodule $N \subseteq M$ so that $M/N \subseteq D_{\mathfrak{m}}(R)$ for some $\mathfrak{m} \in \operatorname{Max}(R)$ and $\mathfrak{p}$ is minimal over $(N : M)$. We shall need the following facts about weakly coassociated primes:

**Theorem 3.2.1.** *([46, Theorems 2.9,2.11,2.15]) Let $(R, \mathfrak{m})$ be a quasi-local ring and $M$ an $R$-module. Then the following hold:*

*(i)* $M = 0 \Leftrightarrow D_{\mathfrak{m}}(M) = 0$.

*(ii)* $\mathfrak{p}$ *is a weakly coassociated prime of* $M \Leftrightarrow \mathfrak{p} \in \operatorname{wAss}(D_{\mathfrak{m}}(M))$.

*(iii)* $\bigcup_{\mathfrak{p} \in wAss(D_{\mathfrak{m}}(M))} \mathfrak{p} = \{x \in R \colon M \xrightarrow{x} M \text{ is not onto}\}$.

Before we proceed, we need a few more technical lemmas.

**Lemma 3.2.2.** *([9, Lemma 10.1.16]) Let $R$ be any ring, and say $M$ is an $R$-module and $I$ an ideal of $R$ with $IM = 0$. Then we have an isomorphism of $R/I$-modules $(0 :_{E_R(M)} I) \cong E_{R/I}(M)$.*

**Lemma 3.2.3.** *Let $(R, \mathfrak{m})$ be a quasi-local ring and $M$ be an $R$-module. If $\mathrm{wAss}(D_{\mathfrak{m}}(M))$ is finite, then $\mathfrak{m} \notin \mathrm{wAss}(D_{\mathfrak{m}}(M)) \Leftrightarrow M \xrightarrow{b} M$ is onto for some $b \in \mathfrak{m}$.*

*Proof.* Follows immediately from prime avoidance. □

**Lemma 3.2.4.** *Let $(R, \mathfrak{m})$ be a quasi-local ring and $\underline{x}$ a finite sequence of elements of $R$. If $a$ is a regular element of $R$ and $n \in \mathbb{N}$, suppose that $H^n_{\underline{x}}(R) = 0$ and $H^{n-1}_{\underline{x}}(R/aR) \neq 0$. Then for some $\mathfrak{p} \in \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$, $a \in \mathfrak{p}$. In particular, $H^{n-1}_{\underline{x}}(R) \neq 0$.*

*Proof.* From the short exact sequence $0 \to R \xrightarrow{a} R \to R/aR \to 0$, we have an induced long exact sequence in Čech cohomology:

$$H^{n-1}_{\underline{x}}(R) \xrightarrow{a} H^{n-1}_{\underline{x}}(R) \to H^{n-1}_{\underline{x}}(R/aR) \to H^{n}_{\underline{x}}(R) = 0,$$

from which we see

$$H^{n-1}_{\underline{x}}(R/aR) \cong \frac{H^{n-1}_{\underline{x}}(R)}{aH^{n-1}_{\underline{x}}(R)} \neq 0.$$

Thus the map $H^{n-1}_{\underline{x}}(R) \xrightarrow{a} H^{n-1}_{\underline{x}}(R)$ is not surjective, and so $H^{n-1}_{\underline{x}}(R) \neq 0$. Now by Lemma 3.2.1(iii), there is some $\mathfrak{p} \in \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$ with $x \in \mathfrak{p}$. □

## 3.3 A sufficient condition for Čech stability

We would like to develop a sufficient condition on a quasi-local ring $(R, \mathfrak{m})$ of finite Krull dimension that guarantees $\mathfrak{m}$ is generated up to radical by a Čech stable ideal. In other words, we want to find a condition on $R$ so that if $\mathfrak{m} = \sqrt{(\underline{x})R}$ with $\underline{x} = x_1, \dots, x_\ell \in R$, then $H^{\dim(R)}_{\underline{x}}(R) \neq 0$. With this in mind, we introduce a class of quasi-local rings that will be sufficient for Čech stability.

**Definition 3.3.1.** Let $(R, \mathfrak{m})$ be a quasi-local ring. We will say that $R$ is in the class $\mathcal{C}_{\text{s.o.p.}}$, if for each ideal $I$ of $R$, the following conditions hold on the ring $S = R/I$ and its maximal ideal $\mathfrak{n} = \mathfrak{m}/I$:

(i) $\mathfrak{n} = \sqrt{(\underline{x})S}$ for some $\underline{x} = x_1, \ldots, x_\ell \in S$, i.e. $\operatorname{ara}(\mathfrak{n}) < \infty$.

(ii) $\operatorname{ara}(\mathfrak{n}) = \dim(S)$.

(iii) $|\operatorname{wAss}_S(S)| < \infty$.

(iv) $\left|\operatorname{wAss}_S\Big(D_{\mathfrak{n}}(H^k_{\underline{x}}(S))\Big)\right| < \infty$ for any $k \in \mathbb{N}$.

(v) If $\dim(S) > 0$, there is an ideal $J$ of $S$ such that $\dim(\frac{S}{\operatorname{ann} J}) < \dim(S)$ and via the natural morphism $\operatorname{Spec}(S/J) \to \operatorname{Spec}(S)$, we have

$$\operatorname{wAss}(S/J) = \{\mathfrak{p} \in \operatorname{Spec}(S) : \dim(S/\mathfrak{p}) = \dim(S)\}.$$

The proof of our result below is the same in spirit as the proof of Grothendeick's non-vanishing theorem by I.G. MacDonald and R.Y. Sharp in [39, Theorem 2.2]. However, this proof relied on the fact that the modules $H^i_{\mathfrak{m}}(R)$ are Artinian for any $i \in \mathbb{Z}$ and any Noetherian local ring $(R, \mathfrak{m})$. Since this need not be true in general, we replace the Noetherian assumption with the condition on the weakly associated (and coassociated) primes of the ring (and Čech cohomology modules, respectively) and arithmetic rank in the definition above.

**Theorem 3.3.2.** *Suppose $(R, \mathfrak{m})$ is a quasi-local ring in $\mathcal{C}_{\text{s.o.p.}}$. If $\mathfrak{m} = \sqrt{(\underline{x})R}$ for some finite sequence $\underline{x} = x_1, \ldots, x_\ell \in R$, then $(\underline{x})R$ is Čech stable. Moreover,*

$$\operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{\dim(R)}_{\underline{x}}(R))\Big) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \colon \dim(R/\mathfrak{p}) = \dim(R)\}.$$

*Proof.* Let $I = (\underline{x})R$. We'll prove our claim by induction on $n := \dim(R)$. When $n = 0$, we have that $\mathfrak{m}$ consists of nilpotent elements, so that since $I$ is finitely generated, $I^t = 0$ for $t \gg 0$, and thus $H^0_{\underline{x}}(R) = R \neq 0$. Since $\operatorname{Spec}(R) = \{\mathfrak{m}\}$, we have that necessarily $\operatorname{wAss}\Big(D_{\mathfrak{m}}(H^n_{\underline{x}}(R))\Big) = \{\mathfrak{m}\} = \{\mathfrak{p} \in \operatorname{wAss}(R) \colon \dim(R/\mathfrak{p}) = 0\}$. Thus the claim is complete when $n = 0$.

Suppose now $n = \dim(R) > 0$. Then by condition (v) above, there is an ideal $J$ of $R$ with $\dim(\frac{R}{\operatorname{ann} J}) < n$ and $\operatorname{wAss}(R/J) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim(R/\mathfrak{p}) = \dim(R)\}$. From the short exact sequence $0 \to J \to R \to R/J \to 0$, we get an induced long exact sequence in Čech cohomology:

$$H^n_{\underline{x}}(J) \to H^n_{\underline{x}}(R) \to H^n_{\underline{x}}(R/J) \to H^{n+1}_{\underline{x}}(J).$$

For any $i$, we may regard $H^i_{\underline{x}}(J)$ as a module over $\frac{R}{\operatorname{ann} J}$, and thus by Theorem 1.4.2(ix), $H^n_{\underline{x}}(J) = H^{n+1}_{\underline{x}}(J) = 0$ (or if $J = 0$, this follows trivially), so that $H^n_{\underline{x}}(R) \cong H^n_{\underline{x}}(R/J)$. Thus by replacing $R$ with $R/J$, which has the same dimension as $R$, we may assume $\operatorname{wAss}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \colon \dim(R/\mathfrak{p}) = n\}$.

Now say by way of contradiction that $H^n_{\underline{x}}(R) = 0$. By our assumptions on $R$. We claim next that $\mathfrak{m} \in \operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$. Since $R$ is in $\mathfrak{C}_{\text{s.o.p.}}$, $|\operatorname{wAss}(R)| < \infty$ and $\Big|\operatorname{wAss}_R\Big(D_{\mathfrak{m}}(H^k_{\underline{x}}(R))\Big)\Big| < \infty$, so that if $\mathfrak{m} \notin \operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$, then by prime avoidance, along with Lemma 3.2.1(iii) and the fact that $Z(R) = \bigcup_{\mathfrak{p} \in \operatorname{wAss}(R)} \mathfrak{p}$ ([33, Theorem 6.2]), we must have that

$$\mathfrak{m} \not\subseteq Z(R) \cup \{x \in R : H^{n-1}_{\underline{x}}(R) \xrightarrow{x} H^{n-1}_{\underline{x}}(R) \text{ not onto}\}.$$

Thus $\mathfrak{m}$ contains a non-zerodivisor $a$ on $R$ such that $D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R)) = aD_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))$. It follows $\dim(R/aR) \leq n - 1$, so that our induction hypothesis applies, and since $\mathfrak{m}/xR =$

$\sqrt{(\underline{x})R/aR}$, we have $(\underline{x})R/aR$ is Čech stable. Since $|\mathrm{wAss}(R/B)| < \infty$ for each finitely generated ideal $B$ of $R$, in particular $\mathrm{Min}(B)$ is finite, so that $R$ is FGFC. By Lemma 3.1.21 then, $\dim(R/aR) = n-1$ and hence $H^{n-1}_{\underline{x}}(R/aR) \neq 0$. Then Lemma 3.2.4 says that $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$. This means $H^{n-1}_{\underline{x}}(R) \xrightarrow{a} H^{n-1}_{\underline{x}}(R)$ is not onto, and since $a \in \mathfrak{p} \subseteq \mathfrak{m}$, we get a contradiction. Thus $\mathfrak{m} \in \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big)$ and so we write $\mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big) = \{\mathfrak{m}, \mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and $\mathrm{wAss}(R) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell\}$. Then by prime avoidance, $\mathfrak{m} \nsubseteq \left(\bigcup_{i=1}^k \mathfrak{p}_i\right) \cup \left(\bigcup_{j=1}^\ell \mathfrak{q}_j\right)$. Thus we may choose some $d \in \mathfrak{m} \backslash \bigcup_{i=1}^k \mathfrak{p}_i$ that is a non-zerodivisor on $R$. From the short exact sequence $0 \to R \xrightarrow{d} R \to R/dR \to 0$, the long exact sequence in Čech cohomology gives us the following exact sequence:

$$H^{n-1}_{\underline{x}}(R) \xrightarrow{d} H^{n-1}_{\underline{x}}(R) \to H^{n-1}_{\underline{x}}(R/dR) \to H^{n}_{\underline{x}}(R) = 0,$$

from which we conclude

$$H^{n-1}_{\underline{x}}(R/dR) \cong \frac{H^{n-1}_{\underline{x}}(R)}{dH^{n-1}_{\underline{x}}(R)} \neq 0.$$

Now from the short exact sequence

$$0 \to dH^{n-1}_{\underline{x}}(R) \to H^{n-1}_{\underline{x}}(R) \to H^{n-1}_{\underline{x}}(R/dR) \to 0,$$

after applying the exact functor $D_{\mathfrak{m}}(-)$, we get another exact sequence

$$0 \to D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R/dR)) \to D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R)) \to D_{\mathfrak{m}}(dH^{n-1}_{\underline{x}}(R)) \to 0,$$

from which we see $\mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R/dR))\Big) \subseteq \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R))\Big) = \{\mathfrak{m}, \mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. Now if $\mathfrak{q} \in \mathrm{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R/dR))\Big)$, then there is some $R$-linear map $f\colon H^{n-1}_{\underline{x}}(R/dR) \to E(R/\mathfrak{m})$ for which $\mathfrak{q}$ is minimal over $\mathrm{ann}_R(f)$. Since $dH^{n-1}_{\underline{x}}(R/dR) = 0$, we have

$$d \in \mathrm{ann}_R(H^{n-1}_{\underline{x}}(R/dR)) \subseteq \mathrm{ann}_R(f) \subseteq \mathfrak{q}.$$

But by our choice of $d$, the only member of $\{\mathfrak{m}, \mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ for which this is possible is $\mathfrak{m}$. Thus $\operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R/dR))\Big) = \{\mathfrak{m}\}$, which is a contradiction unless $n = 1$ since by induction hypothesis $\operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n-1}_{\underline{x}}(R/dR))\Big) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R/dR) \colon \dim(R/\mathfrak{p}) = n-1\}$. But if $n = 1$, observe that since $d \in \mathfrak{m}$, $1 \le \operatorname{p.grade}(\mathfrak{m}, R) = \operatorname{p.grade}(I, R) = \check{\mathrm{C}}.\operatorname{grade}(I, R) \le \dim(R) = n = 1$ by Theorem 1.2.1(vi), so that we actually must have $H^1_{\underline{x}}(R) \neq 0$. Thus for any $n \in \mathbb{N}_0$, we must have $H^n_{\underline{x}}(R) \neq 0$.

Lastly, we claim $\operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n}_{\underline{x}}(R))\Big) \subseteq \operatorname{wAss}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \colon \dim(R/\mathfrak{p}) = n\}$, which will complete the induction. So suppose $Q \in \operatorname{wAss}\Big(D_{\mathfrak{m}}(H^{n}_{\underline{x}}(R))\Big)$. Then $Q$ consists of zerodivisors on $R$. Indeed, suppose on the other hand that $f \in Q$ is a non-zerodivisor on $R$. From the short exact sequence $0 \to R \xrightarrow{f} R \to R/fR \to 0$ we get the following exact sequence in Čech cohomology:

$$H^n_{\underline{x}}(R) \xrightarrow{f} H^n_{\underline{x}}(R) \to H^n_{\underline{x}}(R/fR),$$

where the last term is zero since $\dim(R/fR) \le n-1$ along with Theorem 1.4.2(ix). Then $H^n_{\underline{x}}(R) \xrightarrow{f} H^n_{\underline{x}}(R)$ is surjective, meaning that $f \notin Q'$ for any $Q' \in \operatorname{wAss}(D_{\mathfrak{m}}(H^n_{\underline{x}}(R)))$, a contradiction. Thus $Q$ consists of zerodivisors on $R$, so by prime avoidance, $Q \subseteq \mathfrak{q}_{\mathfrak{i}}$ for some $i \in \{1, \ldots, \ell\}$. Since the $\mathfrak{q}_i$ are all necessarily minimal primes by our assumption on $\operatorname{wAss}(R)$, we must have that $Q = \mathfrak{q}_i$, and so our claim is complete.

$\square$

## 3.4 Examples of rings in $\mathcal{C}_{\text{s.o.p.}}$

**Example 3.4.1.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then $R$ is in $\mathcal{C}_{\text{s.o.p.}}$. Indeed, since any quotient of a Noetherian local ring is still Noetherian local, it is enough to show that $R$ satisfies conditions (i)-(v) of Definition 3.3.1. Every Noetherian local ring has a system of parameters, so that conditions (i) and (ii) are met. Moreover, since every ideal has a primary decomposition, $|\text{wAss}(R)| = |\text{Ass}(R)| < \infty$.

**Example 3.4.2.** Let $(R, \mathfrak{m})$ be a 0-dimensional quasi-local ring. Then $R$ is in $\mathcal{C}_{\text{s.o.p.}}$. Indeed, since every quotient of $R$ is still a 0-dimensional quasi-local ring, it's enough to just check that $R$ satisfies (i)-(iv) of Definition 3.3.1. We have $\mathfrak{m} = \sqrt{(0)R}$, so that $\text{ara}(\mathfrak{m}) = \dim(R) = 0$, and thus (i) and (ii) are satisfied. Since $|\text{Spec}(R)| = 1$, conditions (iii) and (iv) are met.

**Example 3.4.3.** Let $(R, \mathfrak{m})$ be a 1-dimensional quasi-local domain. with $|\text{Spec}(R)| < \infty$. Then $R$ is in $\mathcal{C}_{\text{s.o.p.}}$. Indeed, let $I$ be an ideal of $R$. We must check that $R/I$ satisfies conditions (i)-(v) of Definition 3.3.1. If $\dim(R/I) = 0$, then the above example shows $R/I$ is in $\mathcal{C}_{\text{s.o.p.}}$. If $\dim(R/I) = 1$, then $I = 0$, and since $|\text{Spec}(R)| = 2$, it is clear that $R$ satisfies conditions (i) - (v).

We would like to next show that every 2-dimensional quasi-local Krull domain with Noetherian spectrum is in $\mathcal{C}_{\text{s.o.p.}}$. Recall that an integral domain $R$ is a *Krull domain* if the following conditions on $R$ hold:

(i) For each $\mathfrak{p} \in X^{(1)} = \{\mathfrak{p} \in \text{Spec}(R) \mid \text{ht}(\mathfrak{p}) = 1\}$, $R_{\mathfrak{p}}$ is a discrete valuation ring.

(ii) $R = \bigcap_{\mathfrak{p} \in X^{(1)}} R_{\mathfrak{p}}$.

(iii) If $x \neq 0$, $|V(xR) \cap X^{(1)}| < \infty$.

Before we proceed, we shall need some facts about a certain closure operation on the ideals of an integral domain introduced by W. Fanggui and R.L. McCasland in [16]. Let $R$ be an integral domain $R$ with quotient field $K$. An ideal $J$ of $R$ is called a *Glaz-Vasconcelos* ideal of $R$ if $J$ is finitely generated and $J^{-1} := (R :_K J) = R$, which we denote by $J \in GV(R)$. For $I$ an ideal of $R$, the *$w$-envelope* of $I$, denoted by $I_w$, is the ideal

$$I_w = \bigcup_{J \in GV(R)} (I :_R J).$$

**Lemma 3.4.4.** *Let $J$ be a finitely generated ideal in a domain $R$. Then $J \in GV(R) \Leftrightarrow$ $p.grade(J, R) \geq 2$.*

*Proof.* If $J = R$, the claim is trivial, so we can assume throughout that $J$ is proper. So if $J^{-1} = R$, then since $J$ is proper, it's not principal. Thus we may write $J = Rb + I$ for some non-zero $b \in J$, where $I \not\subseteq Rb$ and $I$ is finitely generated. So write $I = (a_0, \ldots, a_n)R$, and let $f = a_0 + a_1X + \ldots + a_nX^n \in R[X]$. We claim that $b, f$ is a $R[X]$-sequence contained in $J[X]$.

Clearly $b$ is $R[X]$-regular. Next, we claim that $f$ is a non-zerodivisor on $R[X]/bR[X] \cong (R/bR)[X]$. By Lemma 1.2.1(i), this happens precisely when $(0 :_{R/bR} I) = 0$. So suppose $r + bR \in (0 :_{R/bR} I)$. Then $Ir + bR = bR$, so $Ir \subseteq bR \Rightarrow I(r/b) \subseteq R$. Then $r/b \in J^{-1} = R$, or in other words, $r \in bR$. Hence we have $(0 :_{R/bR} I) = 0$, so that $b, f$ forms a $R[X]$-sequence in $J[X]$. Thus p.grade$(J, R) \geq 2$.

Conversely, suppose $\text{p.grade}(J,R) \geq 2$, so that for some $n \in \mathbb{N}_0$,

$$\text{grade}_{R[X_1,\dots,X_n]}(J[X_1,\dots,X_n], R[X_1,\dots,X_n]) \geq 2.$$

Now if $(J[X_1,\dots,X_n])^{-1} = R[X_1,\dots,X_n]$, then $J^{-1} = R$, so that we might as well assume $\text{grade}(J,R) \geq 2$. Then $J$ contains an $R$-sequence $x, y$. Now if $c \in J^{-1}$, then $cx = r$ and $cy = s$ for some $s, r \in R$. Hence we have $sx = cyx = ry \Rightarrow r = tx$ for some $t \in R$ since $y$ is a non-zerodivisor on $R/xR$. Thus $cx = r = tx \Rightarrow c = t \in R$. $\square$

**Lemma 3.4.5.** *Let $I$ be an ideal of a domain $R$. Then* $\text{p.grade}(I,R) = 1 \Leftrightarrow \text{p.grade}(I_w,R) = 1$.

*Proof.* If $\text{p.grade}(I_w,R) = 1$, then since $I \subseteq I_w$, we must have $\text{p.grade}(I,R) \leq 1$. Now $I_w \neq 0$ means $I \neq 0$, therefore $I$ contains a regular element, and so $\text{p.grade}(I,R) \geq 1$, and hence we actually get equality. Conversely, suppose that $\text{p.grade}(I,R) = 1$ and suppose that $\text{p.grade}(I_w,R)$ is bigger than 1. Then we may choose a finitely generated ideal $J \subseteq I_w$ with $\text{p.grade}(J,R) \geq 2$. Now by the above lemma, $J \in GV(R)$, and since $I_w$ contains $J$, we must have $R = (I_w)_w = I_w$. But then there is some $J' \in GV(R)$ with $J' \subseteq I$, which by the above lemma and the monotonicity of polynomial grade implies $\text{p.grade}(I,R) \geq 2$, a contradiction. $\square$

**Theorem 3.4.6.** *Let $R$ be a Krull domain, and $I = (a_1,\dots,a_n)_w$ a $w$-ideal of $R$. If $\mathfrak{p} \in \text{Spec}(R)$ is minimal over $I$, then $\text{ht}(\mathfrak{p}) \leq n$. In particular, for any $x \neq 0$ in $R$, any prime ideal minimal over $x$ has height one.*

*Proof.* This is just a special case of [15, Corollary 1.12], since every Krull domain is strong Mori. $\square$

**Lemma 3.4.7.** *Let $(R, \mathfrak{m})$ be a quasi-local Krull domain with $\dim(R) \geq 2$. Then the following hold:*

*(i)* $\mathfrak{m}_w = R$*, and therefore* $\text{p.grade}(\mathfrak{m}) \geq 2$.

*(ii)* $\text{ara}(\mathfrak{m}) \geq 2$.

*(iii)* $\mathfrak{m} = \bigcup_{\mathfrak{p} \in X^{(1)}} \mathfrak{p}$*, and so* $X^{(1)}$ *is infinite.*

*Proof.* (i): We have that since $\mathfrak{m}$ is maximal, either $\mathfrak{m}_w = \mathfrak{m}$ or $R$. In the former situation, then $\mathfrak{m}$ is a maximal $w$-ideal, so that by [15, Theorem 2.8(8)], we must have $\text{ht}(\mathfrak{m}) = 1$, a contradiction. Thus $\mathfrak{m}_w = R$, so there there is some $J \in GV(R)$ with $J \subseteq \mathfrak{m}$. But then $\text{p.grade}(\mathfrak{m}, R) \geq 2$. (ii): If $\mathfrak{m} = \sqrt{xR}$, then $\text{ht}(\mathfrak{m}) = 1$ by Theorem 3.4.6, which contradicts our assumption that $\dim(R) \geq 2$. (iii): If $\mathfrak{m} \neq \bigcup_{\mathfrak{p} \in X^{(1)}} \mathfrak{p}$, then there is some $x \in \mathfrak{m} \backslash \bigcup_{\mathfrak{p} \in X^{(1)}} \mathfrak{p}$. But then $\sqrt{xR} = \mathfrak{m}$, contradicting (ii). The fact that $X^{(1)}$ must be infinite follows immediately by prime avoidance. $\square$

**Lemma 3.4.8.** *Suppose that $(R, \mathfrak{m})$ is a quasi-local Krull domain with $\dim(R) = 2$. Then* $\text{ara}(\mathfrak{m}) = 2$*. In other words, there are $x, y \in R$ with $\sqrt{(x, y)R} = \mathfrak{m}$. Moreover, any such pair of elements must be an $R$-sequence.*

*Proof.* Suppose $x \neq 0$ is in $\mathfrak{m}$. Let $V(xR) \cap X^{(1)} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then $\mathfrak{m} \neq \bigcup_{i=1}^{t} \mathfrak{p}_i$. Thus there is some $y \in \mathfrak{m} \backslash \bigcup_{i=1}^{t} \mathfrak{p}_i$. Then $\sqrt{(x, y)R} = \mathfrak{m}$ as desired. For the moreover part, say $\sqrt{(c, d)R} = \mathfrak{m}$ for some $c, d \in R$. We claim this is a regular sequence on $R$. Indeed, $c \neq 0$ by part (ii) of the above lemma. Now if $d \in Z(R/cR)$, then $d \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{wAss}_R(R/cR) = \text{wAss}_{R/cR}(R/cR)$. We can't have $\text{ht}(\mathfrak{p}) = 1$, otherwise $c, d \in \mathfrak{p}$ contradicts $\sqrt{(c, d)R} = \mathfrak{m}$. Therefore $\text{ht}(\mathfrak{p}) = 2$, and so $\mathfrak{p} = \mathfrak{m}$. Now if $\mathfrak{m} \in \text{wAss}(R/cR)$, then $\text{p.grade}(\mathfrak{m}, R/cR) = 0$

by [26, Lemma 2.8]. But $\text{p.grade}(\mathfrak{m}, R/cR) = \text{p.grade}(\mathfrak{m}, R) - 1 = 2 - 1 = 1$ since $c$ is a regular element of $R$, a contradiction. Therefore $c, d$ is a regular sequence of $R$. $\square$

**Lemma 3.4.9.** *Let $(R, \mathfrak{m})$ be a quasi-local Krull domain with $\dim(R) = 2$. Then $R$ is Cohen-Macaulay.*

*Proof.* Let $f_1, \ldots, f_m \in R$ be a strong parameter sequence. Now if $m = 1$, then since $f_1 \neq 0$, we must have that $f_1$ is a regular element of $R$. If $m > 2$, then $H^m_{f_1,\ldots,f_m}(R) \neq 0$, which is impossible, since the Čech cohomology should vanish in degrees larger than $2 = \dim(R)$. So the only case left to check is when $m = 2$. In this case, we must have $\text{ht}((f_1, f_2)R) = 2$, so that $\sqrt{(f_1, f_2)R} = \mathfrak{m}$, and thus $f_1, f_2$ form a regular sequence by the previous lemma. $\square$

**Lemma 3.4.10.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. If $(0 :_R I) = 0$ and $\text{ht}(I) = 1$, then $I$ is Čech stable. More generally, if $I$ is any ideal with $\text{p.grade}(I) = \text{ht}(I)$, then $I$ is Čech stable.*

*Proof.* $(0 :_R I) = 0$ says that $\text{p.grade}(I, R) > 0$. The claim then follows immediately from the inequalities $\text{p.grade}(I, R) \leq \check{\text{C}}.\text{ht}(I) \leq \text{ht}(I)$. $\square$

**Lemma 1.** *Let $(R, \mathfrak{m})$ be a quasi-local Krull domain with $\dim(R) = 2$. Then every finitely generated ideal of $R$ is Čech stable.*

*Proof.* Let $I$ be a finitely generated ideal of $R$. If $\text{ht}(I) = 0$, $I = 0$, so there is nothing to do. If $\text{ht}(I) = 1$, then by the above lemma, since $R$ is a domain, $(0 :_R I) = 0$, and hence $I$ is Čech stable. If $\text{ht}(I) = 2$, then $\sqrt{I} = \mathfrak{m}$, so that $\text{p.grade}(\mathfrak{m}, R) = \text{p.grade}(I, R) = 2$, and thus by the above lemma we also conclude $I$ is Čech stable. $\square$

**Lemma 3.4.11.** *Let* $(R, \mathfrak{m})$ *be a 2-dimensional quasi-local Krull domain with Noetherian spectrum and* $I$ *a height one prime ideal of* $R$*. Then there is an ideal* $J$ *of* $R$ *with* $I \subseteq J$ *where the following hold:*

*(i)* $\dim\left(\dfrac{R}{I:J}\right) = 0.$

*(ii)* $\mathrm{wAss}(R/J) = \{\mathfrak{p} \in V(I) : \dim(R/\mathfrak{p}) = 1\}.$

*Proof.* Let $B = (x, y)R$ be a finitely generated ideal of $R$ with $\sqrt{B} = \mathfrak{m}$, where $x, y$ is a regular sequence on $R$. Since $R$ is Laskerian [27, remarks after Example 4.2], the ascending chain

$$(I :_R B) \subseteq (I :_R B^2) \subseteq (I :_R B^3) \subseteq \cdots$$

must terminate at some $n \in \mathbb{N}$ ([37, Proposition 3]). Let $J = (I :_R B^n)$. We claim that $J$ satisfies (i) and (ii) above. For (i), let $f \in \mathfrak{m} = \sqrt{B} = \sqrt{B^n}$, so that $f^k \in B^n$ for $k \gg 0$. Now if $x \in J$, then $xB^n \subseteq I \Rightarrow xf^k \in I$, and since $x$ was arbitrary, we must have $f^k J \subseteq I \Rightarrow f \in \sqrt{I :_R J}$. Thus $V(I :_R J) \subseteq \{\mathfrak{m}\}$, so that (i) holds. For (ii), we first note that $\mathfrak{m} \notin \mathrm{wAss}(R/J)$. Indeed, suppose by way of contradiction that $\mathfrak{m} \in \mathrm{wAss}(R/J)$. Then $\sqrt{J :_R f} = \mathfrak{m} = \sqrt{B^n}$ for some $f \notin J$. Since $B^n$ is finitely generated there is some $t > 0$ where

$$\begin{aligned} &B^{nt} \subseteq (J :_R f) \\ \Rightarrow &B^{nt} f \subseteq J = (I :_R B^n) \\ \Rightarrow &B^{n(t+1)} f \subseteq I \\ \Rightarrow &f \in (I :_R B^{n(t+1)}) = (I :_R B^n) = J. \end{aligned}$$

So $\mathfrak{m} \notin \mathrm{wAss}(R/J)$. We remark that $B \in GV(R) \Rightarrow B^n \subseteq GV(R)$ by [16, Lemma 1.1]. If $\mathfrak{p} \in \mathrm{Min}_R(I)$, then $\mathfrak{p}$ has height one, so that $I \subseteq \mathfrak{p} \Rightarrow J = (I :_R B^n) \subseteq I_w \subseteq \mathfrak{p}_w = \mathfrak{p}$, by [35, Corollaire 3]. Thus $V(I) = V(J)$. Moreover, $\{\mathfrak{p} \in V(I) : \dim(R/\mathfrak{p}) = 1\} = \mathrm{Min}_R(I) = \mathrm{Min}_R(J) = V(J)\backslash\{\mathfrak{m}\} = \mathrm{wAss}(R/J)$, so (ii) holds. $\square$

A ring $R$ is said to have *Noetherian spectrum* if it satisfies ACC on radical ideals, or equivalently if $\mathrm{Spec}(R)$ is a Noetherian topological space with the Zariski topology.

**Theorem 3.4.12.** *Let $(R, \mathfrak{m})$ be a 2-dimensional quasi-local Krull domain with Noetherian spectrum. Then $R$ is in $\mathfrak{C}_{\text{s.o.p.}}$.*

*Proof.* We must show that $R/I$ satisfies conditions (i)-(v) of Definition 3.3.1 for any ideal $I$ of $R$. We proceed into cases. Case I: If $\dim(R/I) = 2$, then $I = 0$, so we must show $R$ satisfies (i) - (v). As shown above, $\mathrm{ara}(\mathfrak{m}) = 2$, so that conditions (i) and (ii) are met above. Since $R$ is a domain, $\mathrm{wAss}(R) = \{(0)\} = \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim(R/\mathfrak{p}) = 2\}$, so that (iii) and (v) hold. Moreover, if $x, y \in R$ is such that $\sqrt{(x,y)R} = \mathfrak{m}$, then $x, y$ is a regular sequence on $R$ and so for all $k \neq 2$, $H^k_{x,y}(R) = 0$. If $c$ is any nonzero element of $R$, we also have a short exact sequence

$$0 \to R \xrightarrow{c} R \to R/cR \to 0$$

which gives an exact sequence in Čech cohomology:

$$H^2_{x,y}(R) \xrightarrow{c} H^2_{x,y}(R) \to H^2_{x,y}(R/cR) = 0$$

Thus multiplication by $c$ on $H^2_{x,y}(R)$ is surjective, so that $c \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{wAss}(D_{\mathfrak{m}}(H^2_{x,y}(R)))$. Since $c$ was arbitrary, we must have $\mathrm{wAss}(D_{\mathfrak{m}}(H^2_{x,y}(R))) = \{(0)\}$, and thus (iv) holds. Case II: If $\dim(R/I) = 1$, then $\mathrm{Spec}(R/I)$ is finite, so conditions (iii) and (iv) are satisfied.

If $V(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k, \mathfrak{m}\}$, then by prime avoidance there is some $b \in \mathfrak{m} \setminus \bigcup_{i=1}^{k} \mathfrak{p}_i$. Then $\sqrt{b(R/I)} = \mathfrak{m}/I$, so that $\dim(R/I) = \operatorname{ara}(\mathfrak{m}/I) = 1$ and thus (i) and (ii) are met. Lastly, (v) holds by Lemma 3.4.11. Case III: If $\dim(R/I) = 0$, then by Example 3.4.2, $R/I$ satisfies conditions (i)-(v). $\square$

# Bibliography


[1] Anderson, D.D. and Winders, M., *Idealization of a module*, J. Comm. Alg., **1**(1), 3–56 (2009). 6

[2] Y. André, *La conjecture du facteur direct*, arXiv:1609.00345. 54

[3] Asgharzadeh, M. and Dorreh, M., *Cohen-Macaulayness of non-affine normal semigroups,* arXiv.math AC:1302.5919. 20, 41, 44

[4] Asgharzadeh, M., Dorreh, M., and M. Tousi, *Direct limit of Cohen-Macaulay rings*, J. Pure Appl. Algebra, **218**, 1730–1744 (2014). 20, 21, 37

[5] Asgharzadeh, M., Dorreh, M., and Tousi, M., *Direct summands of infinite-dimensional polynomial rings*, J. of Comm. Alg., **9**(1), (2017). 20, 21, 31, 35, 39

[6] Asgharzadeh, M. and Tousi, M., *On the notion of Cohen-Macaulayness for non-Noetherian rings*, J. of Alg., **322**, 2297-2320 (2009). 8, 20

[7] Bastida, E. and Gilmer, R., *Overrings and divisorial ideals of rings of the form $D + M$*, Michigan Math. J., **20**, 79-95. 23

[8] Bergman G.M., *Groups Acting on Hereditary Rings*, Proceedings of the London Math. Soc., **23**, 70-82 (1971). 37

[9] Brodmann, M.P. and Sharp, R.Y., *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics no. **136**, Cambridge, Cambridge University Press, 2012. 11, 44, 48, 49, 58

[10] Bruns, W. and Herzog, J., *Cohen-Macaulay rings*, vol. 39 of Cambridge Studies in Advanced Mathe-matics, Cambridge University Press, Cambridge, 1993. 14, 44

[11] Bouvier, A. and Kabbaj, S., *Examples of Jaffard domains*, J. of Pure and Applied Alg., **54**, 155-165, (1988). 29

[12] Chase, Stephen U. *Direct products of modules*, Trans. Amer. Math. Soc., **97**(3), 457–473 (1960). 40

[13] Eagon, J. A. and Northcott, D. G., *On the Buchsbaum-Eisenbud theory of finite free resolutions*, J. Reine Angew. Math., **262/263**, 205–219 (1973). 7

[14] Eagon, J.A. and Hochster, M., *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math., **93** 1020–1058 (1971). 1

[15] Fanggui W., and McCasland, R.L., *On strong Mori domains*, J. of Pure and Applied Alg., **135**(2), 155-165 (1999). 66, 67

[16] Fanggui, W., and McCasland, R.L., *On w-modules over strong Mori domains*, Comm. in Alg., **25**(4), 1285-1306 (1997). 65, 70

[17] Gilmer, R.W., *Multiplicative ideal theory*, Queen's Papers in Pure and Applied Mathematics, No. 12. Queen's University, Kingston, Ontario, 1968. 22, 29

[18] Gilmer, R., *Commutative Semigroup Rings*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, 1984. 43

[19] Glaz, S. *Coherence, regularity, and homological dimensions of commutative fixed rings*, in Commutative Algebra (Ngô Viêt Trung, Aron Simis and Guiseppe Valla, eds.), World Scientific, Singapore, 89–106 (1992). 1, 18

[20] Glaz, S., *Homological characterizations of rings: the commutative case*, The Concise Handbook of Algebra, Kluwer, 505-508 (2002). 46

[21] Glaz, S., *Homological dimensions of localizations of polynomial rings*, in *Zero-Dimensional Commutative Rings*, John H. Barrett Memorial Lectures and Conf. on Commutative Ring Theory (David F. Anderson and David E. Dobbs, eds.), Lecture Notes in Pure and Appl. Math., vol. 171, Marcel Dekker, Inc., New York, pp. 209-222 (1994). 1, 18

[22] Grothendieck, A. (notes by Hartshorne, R.), *Local Cohomology, Springer Lecture Notes in Math.*, **41**, Springer-Verlag, 1966. 2, 3

[23] Grothendieck A., *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux SGA2*, North-Holland, Amsterdam, 1968. 2

[24] Hamilton, T.D., *Weak Bourbaki unmixed rings: A step towards Non-Noetherian Cohen-Macaulayness*, Rocky Mountain Journal of Mathematics, **34(3)**, 963-977 (2004). 2, 18, 19

[25] Hamilton, T.D., *Unmixedness and generalized principal ideal theorem*, Lect. Notes pure appl. Math., **241**, 282–292 (2008). 54

[26] Hamilton, T.D. and Marley, T. *Non-Noetherian Cohen-Macaulay rings*, J. of Alg., **307(1)**, 343-360 (2007). 2, 13, 15, 20, 21, 25, 28, 31, 32, 45, 47, 68

[27] Heinzer, W., and Lantz, D., *The Laskerian property in commutative Rings*, J. Alg., **72**, 101-114 (1981). 69

[28] Hoa, L.T. and Trung, N. V., *Affine semigroups and Cohen-Macaulay rings generated by monomials,* Trans. Amer. Math. Soc. 298, 145-167 (1986). 41

[29] Hochster, M. and Huneke, C., *Absolute integral closures are big Cohen-Macaulay algebras in characteristic $p$*, Bull. Amer. Math. Soc. (New Series), **24**, 137–143 (1991). 16

[30] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J., **51**, 25-43 (1973). 54

[31] Hochster, M. and Roberts, J. L., *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math., **13**, 115–175 (1974). 35, 37

[32] Hummel, L., and Marley, T., *The Auslander-Bridger formula and the Gorenstein property for coherent rings*, J. Commut. Algebra **1**, 283–314 (2009). 20

[33] Iroz, J. and Rush, D.E., *Associated prime ideals in Non-Noetherian rings*, Can. J. Math., **36(2)**, 344-360 (1984). 32, 61

[34] Iyengar, S.B., Leuschke, G.J., Leykin, A., Miller, C., Miller, E., Singh, A.K., and Walther, U., *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, **87**, American Mathematical Society, Providence, RI, 2007. MR 2355715 (2009a:13025) 17

[35] Jaffard, P., *Les Systeèmes d'Idèaux*, Dunod, Pari, 1960. 70

[36] Kaplansky, I., *Commutative Rings*, Allyn and Bacon, New York, 1970. 1, 25, 28, 47

[37] Lu, Chin-Pi *Modules satisfying ACC on a certain of colons*, Pacific Journal of Mathematics, **131**(2), 303–318 (1988). 69

[38] Lyubeznik, G., *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an $R$-sequence.* In: Greco, S., Strano, R. eds., Complete Intersections, Acireale, Lecture Notes in Mathematics No. 1092. Springer-Verlag, pp. 214–220 (1984). 43

[39] MacDonald, I.G. and Sharp, R.Y., *An elementary proof of the non-vanishing of certain local cohomology modules*, Quart. J. Math. Oxford, **2**(23), 197-204 (1972). 3, 60

[40] Mahdikhani, A. and Sahandi, P. and Shirmohammadi, N., *Cohen–Macaulayness of trivial extensions*, J. Algebra Appl. (2017). 20

[41] Marley, T. *A note on rings for which finitely generated ideals have only finitely many minimal components*, available on: arXiv:math/0601566, 2006. 31

[42] Matsumura, H., *Commutative Ring Theory*. Second edition. Translated from the Japanese. Cambridge Studies in Advanced Mathematics, Cambridge, UK: Cambridge University Press, 1989. 10, 40

[43] Northcott, D.G., *Finite Free Resolutions*, Cambridge Tracts in Mathematics no **71**, Cambridge, Cambridge University Press, 1976. 6, 7, 50

[44] Sakaguchi, M., *Generalized Cohen-Macaulay modules*, Hiroshima Math. J., **10**, 615-634 (1980). 20

[45] Peter Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand., **92(2)**, 271-289 (2003). 2, 14, 15

[46] Yassemi, S., *Coassociated primes of modules over a commutative ring*, Math. Scand., **80**, 175-187 (1997). 3, 58