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# Zero Mach Number Limit of the Compressible Primitive Equations Part I: Well-prepared Initial Data

Xin Liu\* and Edriss S. Titi†

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## Abstract

This work concerns the zero Mach number limit of the compressible primitive equations. The primitive equations with the incompressibility condition are identified as the limiting equations. The convergence with well-prepared initial data (i.e., initial data without acoustic oscillations) is rigorously justified, and the convergence rate is shown to be of order  $\mathcal{O}(\varepsilon)$ , as  $\varepsilon \rightarrow 0^+$ , where  $\varepsilon$  represents the Mach number. As a byproduct, we construct a class of global solutions to the compressible primitive equations, which are close to the incompressible flows.

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# 1 Introduction

## 1.1 The compressible primitive equations

The compressible primitive equations (see (CPE), below) are used by meteorologists to perform theoretical investigations and practical weather predictions (see, e.g., [57]). In comparison with the general hydrodynamic and thermodynamic equations, the vertical component of the momentum equations is missing in the compressible primitive equations. Instead, it is replaced by the hydrostatic balance equation (see (CPE)<sub>3</sub>, below), which is also known as the quasi-static equilibrium equation. From the meteorologists' point of view, such an approximation is reliable and useful for two reasons: the balance of gravity and pressure dominates the dynamics in the vertical direction; and the vertical velocity is usually hard to observe in reality (see, e.g., [74, Chapter 4]). On the other hand, by formally taking the zero limit of the aspect ratio between the vertical scale and the planetary horizontal scale, the authors in [21] derive the compressible primitive equations from the compressible hydrodynamic equations. Such a deviation is very common in planetary scale geophysical models, which represents the fact that the vertical scale of the atmosphere (or ocean) is significantly smaller than the planetary horizontal scale. We refer, for more comprehensive meteorological studies, to [74, 81]. See also, [26, 42–45, 64] for more general discussions on multi-scale analysis.

As far as we know, there are very few mathematical studies concerning the compressible primitive equations (referred to as CPE hereafter). Lions, Temam, and Wang first introduced CPE into the mathematical community in [57]. They formulated the commonly known primitive equations (referred

to as PE hereafter) with the incompressibility condition as the representation of the compressible primitive equations in the pressure coordinates ( $p$ -coordinates) instead of the physical ones with the vertical spatial coordinate. On the other hand, as mentioned before, the authors in [21] introduce these equations with a formal deviation, and a rigorous justification is still an open question for now. In [21], the stability of weak solutions is also investigated (see also [78]). The stability is meant in the sense that a sequence of weak solutions, satisfying some entropy conditions, contains a subsequence converging to another weak solution, i.e., a very weak sense of stability. The existence of such weak solutions is recently constructed in [61, 80] (see also [20, 29] for the existence of global weak solutions to some variant of compressible primitive equations in two spatial dimension). In [62], we also construct local strong solutions to CPE in two cases: with gravity but no vacuum; with vacuum but no gravity.

In analogy to the low Mach number limit in the study of compressible hydrodynamic equations, this and our subsequent works are aiming at studying the low Mach number limit of the compressible primitive equations without gravity and Coriolis force. It is worth mentioning that while taking into account the Coriolis force would not change much our proof, considering gravity in our system causes challenging difficulties. Let  $\varepsilon$  denote the Mach number, and let  $\rho^\varepsilon \in \mathbb{R}$ ,  $v^\varepsilon \in \mathbb{R}^2$ ,  $w^\varepsilon \in \mathbb{R}$  be the density, the horizontal and the vertical velocities, respectively. System (CPE), below, is obtained by rescaling the original CPE, which is similar to the rescaling of the compressible Navier–Stokes equations (see, e.g., [26]):

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}_h (\rho^\varepsilon v^\varepsilon) + \partial_z (\rho^\varepsilon w^\varepsilon) = 0 & \text{in } \Omega_h \times (0, 1), \\ \partial_t (\rho^\varepsilon v^\varepsilon) + \operatorname{div}_h (\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \partial_z (\rho^\varepsilon w^\varepsilon v^\varepsilon) \\ \quad + \frac{1}{\varepsilon^2} \nabla_h P(\rho^\varepsilon) = \operatorname{div}_h \mathbb{S}(v^\varepsilon) + \partial_{zz} v^\varepsilon & \text{in } \Omega_h \times (0, 1), \\ \partial_z P(\rho^\varepsilon) = 0 & \text{in } \Omega_h \times (0, 1), \end{cases} \quad (\text{CPE})$$

where  $P(\rho^\varepsilon) = (\rho^\varepsilon)^\gamma$  and  $\mathbb{S}(v^\varepsilon) = \mu(\nabla_h v^\varepsilon + \nabla_h^\top v^\varepsilon) + (\lambda - \mu) \operatorname{div}_h v^\varepsilon \mathbb{I}_2$  represent the pressure potential and the viscous stress tensor, respectively, with the shear and bulk viscosity coefficients  $\mu$  and  $\lambda - \mu + \frac{2}{3}\mu = \lambda - \frac{1}{3}\mu$ . The physical requirements of  $\mu, \lambda, \gamma$  are  $\lambda - \frac{1}{3}\mu > 0$ ,  $\mu > 0$  and  $\gamma > 1$ . Moreover, we focus our study on the case when  $\Omega_h := \mathbb{T}^2 \subset \mathbb{R}^2$ . (CPE) is complemented with the following stress-free and impermeability physical boundary conditions:

$$\partial_z v^\varepsilon|_{z=0,1} = 0, \quad w^\varepsilon|_{z=0,1} = 0. \quad (\text{BC-CPE})$$

Hereafter, we have and will use  $\nabla_h$ ,  $\operatorname{div}_h$  and  $\Delta_h$  to represent the horizontal gradient, the horizontal divergence, and the horizontal Laplace operator,

respectively; that is,

$$\nabla_h := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad \operatorname{div}_h := \nabla_h \cdot, \quad \Delta_h := \operatorname{div}_h \nabla_h.$$

Notice that if we consider system (CPE) subject to periodic boundary conditions, with fundamental periodic domain  $\Omega_h \times 2\mathbb{T}$ , where  $2\mathbb{T}$  denotes the periodic domain in  $\mathbb{R}$  with period 2; and subject to the following symmetry:

$$v^\varepsilon \text{ and } w^\varepsilon \text{ are even and odd, respectively, in the } z\text{-variable,} \quad (\text{SYM-CPE})$$

then its solutions obey the above symmetry. Moreover, when such solutions are restricted to the physical domain  $\Omega_h \times (0, 1)$  they automatically satisfy the physical boundary conditions (BC-CPE).

We recall the incompressible primitive equations:

$$\begin{cases} \operatorname{div}_h v_p + \partial_z w_p = 0 & \text{in } \Omega_h \times (0, 1), \\ \rho_0(\partial_t v_p + v_p \cdot \nabla_h v_p + w_p \partial_z v_p) + \nabla_h(c_s^2 \rho_1) \\ \quad = \mu \Delta_h v_p + \lambda \nabla_h \operatorname{div}_h v_p + \partial_{zz} v_p & \text{in } \Omega_h \times (0, 1), \\ \partial_z(c_s^2 \rho_1) = 0 & \text{in } \Omega_h \times (0, 1), \end{cases} \quad (\text{PE})$$

complemented with stress-free and impermeability physical boundary conditions:

$$\partial_z v_p|_{z=0,1} = 0, \quad w_p|_{z=0,1} = 0. \quad (\text{BC-PE})$$

We observe again that if we consider system (PE) subject to periodic boundary conditions, with fundamental periodic domain  $\Omega_h \times 2\mathbb{T}$ , and subject to the following symmetry

$$v_p \text{ and } w_p \text{ are even and odd respectively in the } z \text{ variable,} \quad (\text{SYM-PE})$$

then its solutions obey the above symmetry. Moreover, when such solutions are restricted to the physical domain  $\Omega_h \times (0, 1)$  they automatically satisfy the physical boundary conditions (BC-PE).

We aim at investigating the asymptotic behavior, as  $\varepsilon \rightarrow 0^+$ , of the solutions to (CPE). Owing to the symmetry property in (SYM-CPE) and the boundary conditions in (BC-CPE), it suffices to study the following system:

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}_h(\rho^\varepsilon v^\varepsilon) + \partial_z(\rho^\varepsilon w^\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}_h(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \partial_z(\rho^\varepsilon w^\varepsilon v^\varepsilon) \\ \quad + \frac{1}{\varepsilon^2} \nabla_h P(\rho^\varepsilon) = \mu \Delta_h v^\varepsilon + \lambda \nabla_h \operatorname{div}_h v^\varepsilon + \partial_{zz} v^\varepsilon & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \rho^\varepsilon = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \quad (1.1)$$

subject to the periodic boundary condition and symmetry (SYM-CPE). We will show that the asymptotic system of (1.1), as  $\varepsilon \rightarrow 0^+$ , is the incompressible primitive equations, subject to the periodic boundary condition and symmetry (SYM-PE):

$$\begin{cases} \operatorname{div}_h v_p + \partial_z w_p = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho_0(\partial_t v_p + v_p \cdot \nabla_h v_p + w_p \partial_z v_p) + \nabla_h(c_s^2 \rho_1) \\ \quad = \mu \Delta_h v_p + \lambda \nabla_h \operatorname{div}_h v_p + \partial_{zz} v_p & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z(c_s^2 \rho_1) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \quad (1.2)$$

with  $c_s^2 = \gamma \rho_0^{\gamma-1}$  and  $\rho_0 = \text{constant}$ . Here  $\rho_1$  is the Lagrangian multiplier for the constraint (1.2)<sub>1</sub> satisfying

$$\int_{\Omega_h \times 2\mathbb{T}} \rho_1 d\vec{x} = 0. \quad (1.3)$$

In addition, due to the conservation of linear momentum of (1.2), we can impose the following condition for  $v_p$ :

$$\int_{\Omega_h \times 2\mathbb{T}} v_p dx dy dz = 0, \quad (1.4)$$

for any time  $t \geq 0$  as long as the solution exists.

We remark again that the restrictions of the solutions of (1.1) and (1.2) to the physical domain  $\Omega_h \times (0, 1)$  solve the original system (CPE), with physical boundary conditions (BC-CPE), and system (PE), with the physical boundary conditions (BC-PE), respectively, due to symmetry (SYM-CPE) and (SYM-PE), provided the solutions exist and are regular enough.

Historically, the limit system (1.2), besides being as the representation of the CPE in the  $p$ -coordinates, is introduced as the limit system of Boussinesq equations (referred to as BE hereafter) when the aspect ratio between the vertical scale and the horizontal scale is very small, while the Boussinesq equations are the limit equations of the full compressible hydrodynamic equations with small Mach number and low stratification (see, e.g., [56]). That is to say, starting from the compressible hydrodynamic equations, by taking the low Mach number limit and then the small aspect ratio limit (referred to as LMSAR), one will arrive, formally, at the BE and then at the PE. On the other hand, by taking the small aspect ratio limit and then the low Mach number limit (referred to as SARLM), at least formally, the limit system of the compressible hydrodynamic equations is also the PE with the CPE as a middle state. Depending on the order of asymptotic limits, this

gives us two directions from the hydrodynamic equations to the PE, which we will refer to as the PE diagram (see Figure 1, below). The LMSAR part of the PE diagram has been shown to hold on solid ground in various settings (see, e.g., [4, 27, 52, 70, 72, 83]). However, the validity of the SARLM part is relatively open. In order to fully justify the PE diagram, we investigate the low Mach number limit of the CPE in this work, which, as mentioned above, leads to the PE as the limit system. We remark that, the stratification effect of the gravity has been neglected in this work.

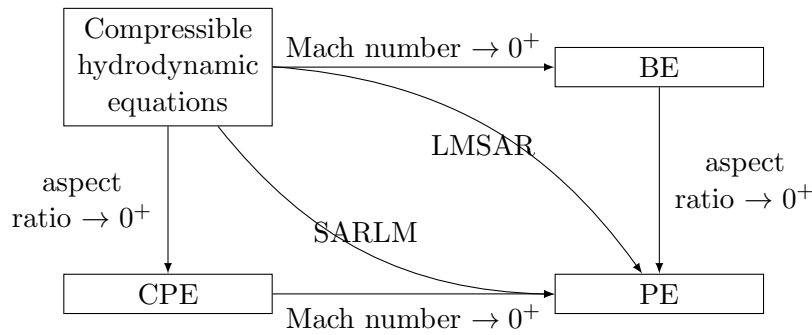


Figure 1: The PE diagram

Each of the equations in the PE diagram have its own significances and have been studied separately in a large number of literatures. It will be certainly too ambitious to review all of those works. We refer readers to the study of compressible hydrodynamic equations in, e.g., the books [22, 59, 60, 65]. As the limit system of the PE diagram, the primitive equations (PE) have been investigated intensively since they are introduced in [54–57]. For instance, the global weak solutions are established in [56]. Local well-posedness with general data and global well-posedness with small data of strong solutions to the PE in three spatial dimensions have been studied in [32] by Guillén-González, Masmoudi and Rodríguez-Bellido. Petcu and Wirosoetisno in [71] investigate the Sobolev and Gevrey regularity of the solutions to the PE. In [36], in a domain with small depth, the authors address the global existence of strong solutions to PE. The well-posedness of unique global strong solutions is obtained by Cao and Titi in [12] (see, also, [7–11, 13–15, 30, 33, 34, 37, 46, 47, 49–51] and the references therein for related studies). Considering the inviscid primitive equations, or hydrostatic incompressible Euler equations, the existence of solutions in the analytic function space and in the  $H^s$  space are established in [5, 48, 68]. Renardy in [73] shows that the linearization of the equations at certain shear flows is

ill-posed in the sense of Hadamard. Recently, the authors in [6,82] construct finite-time blowup for the inviscid PE in the absence of rotation (i.e., without the Coriolis force).

In this work, we show that the PE can be viewed as the limit system of the CPE with the zero Mach number limit. The zero Mach number limit of the compressible hydrodynamic equations is a vast subject which has been studied for decades. Fruitful results have been obtained since the early works of Klainerman and Majda in [40, 41], where the authors investigate the vanishing Mach number limit of compressible Euler equations with well-prepared initial data (see also [76, 77]). Later by Ukai [79], the theory of low Mach number limit of compressible Euler equations is extended to ill-prepared initial data (or called general data in some literatures). We remark here that the difference between the well-prepared and ill-prepared initial data, as in [69], is that the well-prepared initial data have excluded the acoustic waves, while the ill-prepared initial data allow the interaction of the solutions with the high-frequency acoustic waves. In  $\mathbb{R}^n$ ,  $n = 2, 3$ , such high-frequency acoustic waves disperse as shown in [79], which implies strong convergences as the Mach number goes to zero. This can be also proved by applying the Strichartz's estimate (see, e.g., [31, 39, 53]) for linear wave equations to the acoustic equations (see, e.g., [19]). In  $\mathbb{T}^n$ ,  $n = 2, 3$ , the high-frequency acoustic waves interact with each other and lead to fast oscillations and weak convergences when taking the low Mach number limit. Such a fast oscillation phenomenon was first systematically studied in [28, 75] for hyperbolic and parabolic systems, and by Lions and Masmoudi for compressible Navier-Stokes equations in [58]. We refer, for the comparison of the whole space case, i.e., in  $\mathbb{R}^n$  and the periodic domain case, i.e., in  $\mathbb{T}^n$ , to [66]. See also, [16–18] for the studies in the Besov spaces. We acknowledge that the discussion here barely unveils the theory of the low Mach number limit, and we refer the reader to [1–3, 23–27, 38, 67, 69, 70, 72, 83] and the references therein for more comprehensive studies and recent progress.

In this work, we will focus on investigating the low Mach number limit of (1.1) with well-prepared initial data. The convergence of the solutions of the CPE to the solution of the PE is in the strong sense. Furthermore, we are also able to obtain explicit convergence rate (see Theorem 1.2, below). In particular, we obtain a class of global large solutions to (1.1) with  $\varepsilon$  small enough.

We remark that in an upcoming paper [63], we will consider the low Mach number limit of (1.1) with ill-prepared initial data, i.e., initial data with large, high-frequency acoustic waves.



## 1.2 The low Mach number limit problem and main theorem

In order to describe the aforementioned asymptotic limit, we study (1.1) with  $(\rho^\varepsilon, v^\varepsilon, w^\varepsilon)$  close to an asymptotic state  $(\rho_0, v_p, w_p)$ , where  $(\rho_0, v_p, w_p)$  satisfies (1.2). For any  $\varepsilon > 0$ , the following ansatz is imposed:

$$\begin{cases} \rho^\varepsilon := \rho_0 + \varepsilon^2 \rho_1 + \xi^\varepsilon, \\ v^\varepsilon := v_p + \psi^{\varepsilon, h}, \\ w^\varepsilon := w_p + \psi^{\varepsilon, z}. \end{cases} \quad (1.5)$$

Recall that  $\rho_1$  has zero average in the domain (see (1.3)). This is motivated by [35]. In addition, we shall employ the notation

$$\zeta^\varepsilon := \varepsilon^2 \rho_1 + \xi^\varepsilon = \rho^\varepsilon - \rho_0.$$

For the sake of convenience, from time to time hereafter, we may drop the superscript  $\varepsilon$  from the functions. That is, we employ the notations  $\rho = \rho^\varepsilon, v = v^\varepsilon, w = w^\varepsilon, \xi = \xi^\varepsilon, \psi^h = \psi^{\varepsilon, h}, \psi^z = \psi^{\varepsilon, z}, \zeta = \zeta^\varepsilon$  throughout this work whenever there is no confusion. Consequently, from (1.1) and (1.2), the new unknown  $(\xi, \psi^h, \psi^z)$  is governed by the following system:

$$\begin{cases} \begin{cases} \partial_t \xi + \rho_0 (\operatorname{div}_h \psi^h + \partial_z \psi^z) = -(\operatorname{div}_h (\xi v) + \partial_z (\xi w)) \\ \quad - \varepsilon^2 (\partial_t \rho_1 + \operatorname{div}_h (\rho_1 v) + \partial_z (\rho_1 w)) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho \partial_t \psi^h + \rho v \cdot \nabla_h \psi^h + \rho w \partial_z \psi^h + \nabla_h (\varepsilon^{-2} (\rho^\gamma - \rho_0^\gamma - \varepsilon^2 c_s^2 \rho_1)) \\ = \mu \Delta_h \psi^h + \lambda \nabla_h \operatorname{div}_h \psi^h + \partial_{zz} \psi^h + \rho_0^{-1} (\varepsilon^2 \rho_1 + \xi) \\ \quad \times (\nabla_h (c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p) \\ \quad - \rho \psi^h \cdot \nabla_h v_p - \rho \psi^z \partial_z v_p & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi = 0 & \text{in } \Omega_h \times 2\mathbb{T}. \end{cases} \end{cases} \quad (1.6)$$

Observe that owing to the symmetry in (SYM-CPE) and (SYM-PE), the following conditions hold automatically

$$(\partial_z v, \partial_z v_p, \partial_z \psi^h)|_{z \in \mathbb{Z}} = 0, \quad (w, w_p, \psi^z)|_{z \in \mathbb{Z}} = 0, \quad (1.7)$$

for smooth enough functions, i.e., whenever one can make sense of these traces for  $z \in \mathbb{Z}$ . Recalling that  $c_s^2 = \gamma \rho_0^{\gamma-1}$ , we note that

$$\begin{aligned} \rho^\gamma - \rho_0^\gamma - \varepsilon^2 c_s^2 \rho_1 &= \gamma \rho_0^{\gamma-1} (\rho - \rho_0) + \gamma(\gamma - 1) \int_{\rho_0}^{\rho} (\rho - y) y^{\gamma-2} dy - \varepsilon^2 c_s^2 \rho_1 \\ &= c_s^2 \xi + \mathcal{R}, \end{aligned}$$

where

$$\mathcal{R} = \mathcal{R}(\zeta) := \gamma(\gamma - 1) \int_{\rho_0}^{\rho} (\rho - y) y^{\gamma-2} dy \leq C\zeta^2 \leq C(\varepsilon^4 \rho_1^2 + \xi^2), \quad (1.8)$$

with  $C = C(\|\rho^{\gamma-2}\|_{L^\infty}, \|\rho_0^{\gamma-2}\|_{L^\infty})$ . Therefore, by denoting

$$\begin{aligned} Q_p &:= \rho_0^{-1}(\nabla_h(c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p), \\ \mathcal{F}_1 &:= \zeta Q_p, \\ \mathcal{F}_2 &:= -\rho \psi^h \cdot \nabla_h v_p - \rho \psi^z \partial_z v_p, \\ \mathcal{G}_1 &:= -(\operatorname{div}_h(\xi v) + \partial_z(\xi w)), \\ \mathcal{G}_2 &:= -\varepsilon^2(\partial_t \rho_1 + \operatorname{div}_h(\rho_1 v) + \partial_z(\rho_1 w)), \end{aligned} \quad (1.9)$$

we can write system (1.6) as,

$$\begin{cases} \partial_t \xi + \rho_0(\operatorname{div}_h \psi^h + \partial_z \psi^z) = \mathcal{G}_1 + \mathcal{G}_2 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho \partial_t \psi^h + \rho v \cdot \nabla_h \psi^h + \rho w \partial_z \psi^h + \nabla_h(\varepsilon^{-2} c_s^2 \xi) = \mu \Delta_h \psi^h \\ \quad + \lambda \nabla_h \operatorname{div}_h \psi^h + \partial_{zz} \psi^h + \mathcal{F}_1 + \mathcal{F}_2 - \nabla_h(\varepsilon^{-2} \mathcal{R}) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi = 0 & \text{in } \Omega_h \times 2\mathbb{T}. \end{cases} \quad (1.10)$$

In order to recover the vertical velocity perturbation  $\psi^z$  from (1.1), we introduce the following notations, for any function  $f$  in  $\Omega_h \times 2\mathbb{T}$ , which is also even in the  $z$ -variable:

$$\bar{f}(x, y, t) := \int_0^1 f(x, y, z', t) dz' \quad \text{and} \quad \tilde{f} := f - \bar{f}.$$

The periodicity and symmetry of  $f$  imply that  $\bar{f}(x, y, t) = \int_k^{k+1} f(x, y, z', t) dz'$  for any  $k \in \mathbb{Z}$ . Notice that from (1.1)<sub>3</sub>,  $\rho$  is independent of the vertical variable  $z$ . Then by averaging (1.1)<sub>1</sub> over the vertical direction, thanks to (1.7), one will get

$$\begin{aligned} \partial_t \rho + \operatorname{div}_h(\rho \bar{v}) &= 0, \quad \text{and after comparing with (1.1)<sub>1</sub>,} \\ \operatorname{div}_h(\rho \tilde{v}) + \partial_z(\rho w) &= 0. \end{aligned}$$

In particular, from the above, the vertical velocity  $w$  is determined through  $\rho, v$  by the formula, thanks to (1.7):

$$\rho w = - \int_0^z \operatorname{div}_h(\rho \tilde{v}) dz' = - \int_0^z (\rho \operatorname{div}_h v_p + \rho \operatorname{div}_h \psi^h - \rho \operatorname{div}_h \bar{\psi}^h)$$

$$+ v \cdot \nabla_h \rho - \bar{v} \cdot \nabla_h \rho) dz', \quad \text{and therefore} \quad (1.11)$$

$$\begin{aligned} \rho \psi^z &= \rho w - \rho w_p = - \int_0^z (\rho \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \rho) dz' \\ &= - \int_0^z (\operatorname{div}_h (\rho \widetilde{\psi}^h) + \widetilde{v}_p \cdot \nabla_h \rho) dz', \end{aligned} \quad (1.12)$$

where we have substituted the following identity thanks to (1.2)<sub>1</sub> and (1.7),

$$w_p = - \int_0^z \operatorname{div}_h v_p dz'. \quad (1.13)$$

Such facts imply that in (1.1), (1.2) and (1.10), the vertical velocities and the vertical perturbation, i.e.,  $w^\varepsilon$ ,  $w_p^\varepsilon$ ,  $\psi^{\varepsilon,z}$ , are fully determined by  $v^\varepsilon$ ,  $v_p^\varepsilon$ ,  $\rho^\varepsilon$ . Therefore, there is no need to impose initial data for  $w^\varepsilon$ ,  $w_p^\varepsilon$ ,  $\psi^{\varepsilon,z}$ .

System (1.10) (or equivalently (1.6)) is complemented with initial data,

$$\begin{aligned} (\xi, \psi^h)|_{t=0} &= (\xi_{in}, \psi_{in}^h) \in H^2(\Omega_h \times 2\mathbb{T}; \mathbb{R}) \times H^2(\Omega_h \times 2\mathbb{T}; \mathbb{R}^2), \text{ where} \\ \partial_z \xi_{in} &= 0, \text{ and } \psi_{in}^h \text{ is even in the } z\text{-variable,} \end{aligned} \quad (1.14)$$

with the compatibility conditions:

$$\begin{aligned} \xi_{in,1} &= -\rho_0(\operatorname{div}_h \psi_{in}^h + \partial_z \psi_{in}^z) - (\operatorname{div}_h (\xi_{in} v_{in}^\varepsilon) + \partial_z (\xi_{in} w_{in}^\varepsilon)) \\ &\quad - \varepsilon^2 (\rho_{1,in,1} + \operatorname{div}_h (\rho_{1,in} v_{in}^\varepsilon) + \partial_z (\rho_{1,in} w_{in}^\varepsilon)) \quad \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho_{in}^\varepsilon \psi_{in,1}^h + \rho_{in}^\varepsilon v_{in}^\varepsilon \cdot \nabla_h \psi_{in}^h + \rho_{in}^\varepsilon w_{in}^\varepsilon \partial_z \psi_{in}^h \\ &\quad + \nabla_h (\varepsilon^{-2} ((\rho_{in}^\varepsilon)^\gamma - \rho_0^\gamma - \varepsilon^2 c_s^2 \rho_{1,in})) = \mu \Delta_h \psi_{in}^h + \lambda \nabla_h \operatorname{div}_h \psi_{in}^h \\ &\quad + \partial_{zz} \psi_{in}^h + \rho_0^{-1} (\varepsilon^2 \rho_{1,in} + \xi_{in}) (\nabla_h (c_s^2 \rho_{1,in}) - \mu \Delta_h v_{p,in} \\ &\quad - \lambda \nabla_h \operatorname{div}_h v_{p,in} - \partial_{zz} v_{p,in}) - \rho_{in}^\varepsilon \psi_{in}^h \cdot \nabla_h v_{p,in} - \rho_{in}^\varepsilon \psi_{in}^z \partial_z v_{p,in} \\ &\quad \text{in } \Omega_h \times 2\mathbb{T}, \quad \partial_z \xi_{in} = 0 \text{ and } \partial_z \xi_{in,1} = 0 \quad \text{in } \Omega_h \times 2\mathbb{T}, \end{aligned} \quad (1.15)$$

with  $(\xi_{in,1}, \psi_{in,1}^h) \in L^2(\Omega_h \times 2\mathbb{T}) \times L^2(\Omega_h \times 2\mathbb{T})$ ,

where  $\rho_{in}^\varepsilon = \rho_0 + \varepsilon^2 \rho_{1,in} + \xi_{in}$ ,  $v_{in}^\varepsilon = v_{p,in} + \psi_{in}^h$ ,  $w_{in}^\varepsilon = w_{p,in} + \psi_{in}^z$ , and  $\psi_{in}^z$  is given by

$$\rho_{in}^\varepsilon \psi_{in}^z = - \int_0^z (\operatorname{div}_h (\rho_{in}^\varepsilon \widetilde{\psi}_{in}^h) + \widetilde{v}_{p,in} \cdot \nabla_h \rho_{in}^\varepsilon) dz'.$$

Here  $v_{p,in}$ ,  $\rho_{1,in}$ ,  $\rho_{1,in,1}$ ,  $w_{p,in}$  are initial values of  $v_p$ ,  $\rho_1$ ,  $\partial_t \rho_1$ ,  $w_p$ , respectively, while  $w_{p,in}$  is given by  $w_{p,in} = - \int_0^z \operatorname{div}_h v_{p,in} dz'$ .

It is worth stressing that we will chose the initial time derivatives of the perturbations, i.e.,  $(\varepsilon^{-1} \xi_{in,1}, \psi_{in,1}^h)$  in (1.15), to be bounded, uniformly in  $\varepsilon$

(see (1.16) and Theorem 1.2, below). The reason for such choices of initial data is to exclude the high-frequency acoustic waves which corresponds to the fact that our initial data are well-prepared.

We denote the initial energy functional by

$$\mathcal{E}_{in} := \|\psi_{in}^h\|_{H^2}^2 + \|\varepsilon\psi_{in,1}^h\|_{L^2}^2 + \|\varepsilon^{-1}\xi_{in}\|_{H^2}^2 + \|\xi_{in,1}\|_{L^2}^2. \quad (1.16)$$

*Remark 1.*  $(\rho_{in}^\varepsilon, v_{in}^\varepsilon)$  is the corresponding initial datum of  $(\rho^\varepsilon, v^\varepsilon)$  for system (1.1).  $v_{p,in}$  is the initial datum of  $v_p$  for system (1.2). Accordingly,  $\rho_{1,in} = \rho_{1,in}(x, y)$ ,  $\rho_{1,in,1} = \rho_{1,in,1}(x, y)$  are determined by the following elliptic problems,

$$\begin{cases} -c_s^2 \Delta_h \rho_{1,in} = \rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v_{p,in} \otimes v_{p,in})) dz & \text{in } \Omega_h, \\ \int_{\Omega_h} \rho_{1,in} dx dy = 0; \\ -c_s^2 \Delta_h \rho_{1,in,1} = 2\rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v_{p,in} \otimes v_{p,in,1})) dz \\ \text{in } \Omega_h, \quad \int_{\Omega_h} \rho_{1,in,1} dx dy = 0, \end{cases}$$

where  $v_{p,in,1}$  is the initial value of  $\partial_t v_p$  determined by

$$\begin{aligned} \rho_0 v_{p,in,1} = & -\rho_0 (v_{p,in} \cdot \nabla_h v_{p,in} + w_{p,in} \partial_z v_{p,in}) - \nabla_h (c_s^2 \rho_{1,in}) \\ & + \mu \Delta_h v_{p,in} + \lambda \nabla_h \operatorname{div}_h v_{p,in} + \partial_{zz} v_{p,in} \quad \text{in } \Omega_h \times 2\mathbb{T}. \end{aligned}$$

As we stated before, we focus in this work on the asymptotic limit as  $\varepsilon \rightarrow 0^+$ . We have the following global regularity of the limit system (1.2):

**Theorem 1.1** (Global regularity of the PE). *For  $\lambda < 4\mu < 12\lambda$ , suppose that (1.2) is complemented with initial data  $v_{p,in} \in H^1(\Omega_h \times 2\mathbb{T})$ , which is even in the  $z$ -variable, and satisfies the compatibility conditions:*

$$\int_{\Omega_h \times 2\mathbb{T}} v_{p,in} d\vec{x} = 0, \quad \operatorname{div}_h \bar{v}_{p,in} = 0. \quad (1.17)$$

*Then there exists a solution  $(v_p, \rho_1)$ , with  $\int_0^1 \operatorname{div}_h v_p dz = 0$  and  $w_p$  given by (1.13), to the primitive equations (1.2). Moreover, there is a constant  $C_{p,in}$  depending only  $\|v_{p,in}\|_{H^1}$  such that*

$$\sup_{0 \leq t < \infty} \|v_p(t)\|_{H^1}^2 + \int_0^\infty \left( \|\nabla v_p(t)\|_{H^1}^2 + \|\partial_t v_p(t)\|_{L^2}^2 \right) dt \leq C_{p,in}.$$

Moreover,

$$\|v_p(t)\|_{L^2}^2 \leq C e^{-ct} \|v_{p,in}\|_{L^2}^2,$$

for some positive constants  $c, C$ . In addition, if  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$  for any integer  $s \geq 2$ , there is a positive constant  $C_{p,in,s}$ , depending only on  $\|v_{p,in}\|_{H^s}$  such that

$$\begin{aligned} & \sup_{0 \leq t < \infty} (\|v_p(t)\|_{H^s}^2 + \|\partial_t v_p(t)\|_{H^{s-2}}^2) + \int_0^\infty \left( \|v_p(t)\|_{H^{s+1}}^2 \right. \\ & \left. + \|\partial_t v_p(t)\|_{H^{s-1}}^2 \right) dt \leq C_{p,in,s}. \end{aligned}$$

*Proof.* The local well-posedness of solutions to (1.2) in the function space  $H^s$  has been established in [71]. What is left is the global regularity estimate, which is a direct consequence of Proposition 3, below.  $\square$

*Remark 2.* We only have to be careful about the different estimates caused by the viscosity tensor. In particular, it is the  $L^q$  estimate of  $v_p$ , below in section 5, that requires the constraints on the viscosity coefficients. For solutions with  $H^2$  initial data, the result can be found in [52]. The new thing we treat here is the case when  $v_{p,in} \in H^s$ , for  $s > 2$ .

Now we can state our main theorem in this work.

**Theorem 1.2** (Low Mach number limit of the CPE). *For  $\lambda < 4\mu < 12\lambda$ , suppose  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$ , with integer  $s \geq 3$ , and it satisfies the compatibility conditions (1.17). Also, we complement (1.10) with initial data  $(\xi^\varepsilon, \psi^{\varepsilon,h})|_{t=0} = (\xi_{in}, \psi_{in}^h) \in H^2(\Omega_h \times 2\mathbb{T}) \times H^2(\Omega_h \times 2\mathbb{T})$  as in (1.14), which satisfies the compatibility conditions (1.15). Recall that we also require  $v_{p,in}, \psi_{in}^h$  to be even in the  $z$ -variable. Then there exists a positive constant  $\varepsilon_0 \in (0, 1)$  small enough, such that if  $\varepsilon \in (0, \varepsilon_0)$  and  $\mathcal{E}_{in} \leq \varepsilon^2$ , there exists a global unique strong solution  $(\xi^\varepsilon, \psi^{\varepsilon,h})$  to system (1.10). In particular, the following regularity is satisfied:*

$$\begin{aligned} & \xi^\varepsilon \in L^\infty(0, \infty; H^2(\Omega_h \times 2\mathbb{T})), \\ & \partial_t \xi^\varepsilon \in L^\infty(0, \infty; L^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0, \infty; L^2(\Omega_h \times 2\mathbb{T})), \\ & \nabla_h \xi^\varepsilon \in L^2(0, \infty; H^1(\Omega_h \times 2\mathbb{T})), \quad \psi^{\varepsilon,h} \in L^\infty(0, \infty; H^2(\Omega_h \times 2\mathbb{T})), \\ & \partial_t \psi^{\varepsilon,h} \in L^\infty(0, \infty; L^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0, \infty; H^1(\Omega_h \times 2\mathbb{T})), \\ & \nabla \psi^{\varepsilon,h} \in L^2(0, \infty; H^2(\Omega_h \times 2\mathbb{T})). \end{aligned}$$

In addition, we have the following estimate:

$$\begin{aligned} & \sup_{0 \leq t < \infty} \left\{ \|\psi^{\varepsilon, h}(t)\|_{H^2}^2 + \|\varepsilon \partial_t \psi^{\varepsilon, h}(t)\|_{L^2}^2 + \|\varepsilon^{-1} \xi^\varepsilon(t)\|_{H^2}^2 \right. \\ & \quad \left. + \|\partial_t \xi^\varepsilon(t)\|_{L^2}^2 \right\} + \int_0^\infty \left\{ \|\nabla \psi^{\varepsilon, h}(t)\|_{H^2}^2 + \|\varepsilon \partial_t \psi^{\varepsilon, h}(t)\|_{H^1}^2 \right. \\ & \quad \left. + \|\varepsilon^{-1} \nabla_h \xi^\varepsilon(t)\|_{H^1}^2 + \|\partial_t \xi^\varepsilon(t)\|_{L^2}^2 \right\} dt \leq C\varepsilon^2. \end{aligned} \quad (1.18)$$

for some positive constant  $C$  depending only on  $\|v_{p, in}\|_{H^3}$ , which is independent of  $\varepsilon$ . In particular,  $(\rho^\varepsilon, v^\varepsilon, w^\varepsilon)$  as in (1.5) is a globally defined strong solution to (1.1) and the following asymptotic estimate holds:

$$\begin{aligned} & \sup_{0 \leq t < \infty} \left\{ \|v^\varepsilon(t) - v_p(t)\|_{H^2}^2 + \|\rho^\varepsilon(t) - \rho_0\|_{H^2}^2 \right. \\ & \quad \left. + \|w^\varepsilon(t) - w_p(t)\|_{H^1}^2 \right\} \leq C\varepsilon^2, \end{aligned} \quad (1.19)$$

for some positive constant  $C$  depending only on  $\|v_{p, in}\|_{H^3}$ , which is independent of  $\varepsilon$ , where  $w^\varepsilon, w_p$  are given as in (1.11), (1.13), respectively.

*Remark 3.* According to (1.18), the time derivatives, in comparison to the spatial derivatives, have larger perturbations. However, thanks to the well-prepared data setting, they are bounded.

*Remark 4.* In addition to (1.4), thanks to the conservation of mass and momentum in (1.1), we can impose the following conditions for  $\rho, v$ :

$$\int_{\Omega_h \times 2\mathbb{T}} \rho \, dx dy dz = \rho_0 |\Omega_h \times 2\mathbb{T}|, \quad \int_{\Omega_h \times 2\mathbb{T}} \rho v \, dx dy dz = 0. \quad (1.20)$$

Then with a little more effort, under the same assumptions as in Theorem 1.2, one can further conclude that the perturbation energy  $\mathcal{E}(t)$  exponentially decays as time grows. This can be shown as follows. Consider any integer  $s \geq 1$  and initial data  $v_{p, in} \in H^s(\Omega)$ , satisfying the compatibility condition (1.17), as in Theorem 1.1. Then the estimates in Theorem 1.1 hold. Next, in the proof of Proposition 3, below, the differential inequalities (for example, (5.26) with  $s = s - 1$ ) yield,

$$\frac{d}{dt} \|v_p\|_{H^s}^2 + c' \|v_p\|_{H^{s+1}}^2 \leq \mathfrak{P} \|v_p\|_{H^s}^2, \quad (1.21)$$

for any  $s \geq 1$  and some positive constant  $c'$  with  $\int_0^\infty \mathfrak{Y} dt < \infty$ . This implies,

$$\frac{d}{dt} \left\{ e^{c't} \|v_p\|_{H^s}^2 \right\} \leq \mathfrak{Y} e^{c't} \|v_p\|_{H^s}^2.$$

Thus after applying Grönwall's inequality, one can obtain that

$$\|v_p(t)\|_{H^s}^2 \lesssim e^{-c't}.$$

Furthermore, multiply (1.21) with  $e^{c''t}$ , for some  $c'' \in (0, c')$ , and integrate the resultant in the time variable; we deduce that, for arbitrary positive time  $T \in (0, \infty)$ ,

$$\begin{aligned} \int_0^T e^{c''t} \|v_p(t)\|_{H^{s+1}}^2 dt &\leq \int_0^T \mathfrak{Y}(t) e^{c''t} \|v_p(t)\|_{H^s}^2 dt + \|v_p(0)\|_{H^s}^2 \\ &- e^{c''T} \|v_p(T)\|_{H^s}^2 + c'' \int_0^T e^{c''t} \|v_p(s)\|_{H^s}^2 dt < \infty. \end{aligned}$$

Then, we have the following estimate of the  $H^s$  norm of  $v_p$ :

$$\sup_{0 \leq t < \infty} e^{c_s t} \|v_p(t)\|_{H^s}^2 + \int_0^\infty e^{c_s t} \|v_p(t)\|_{H^{s+1}}^2 dt < \infty,$$

for  $s \geq 1$  and some positive constant  $c_s$ . With such estimates, similar arguments as in the proof of Proposition 3 (see, for example, (5.29)), one can conclude that the time derivative of  $v_p$  will have similar estimates. That is:

$$\begin{cases} \int_0^\infty e^{c_1 t} \|\partial_t v_p(t)\|_{L^2}^2 dt < \infty, & \text{for } s = 1; \\ \sup_{0 \leq t < \infty} e^{c_s t} \|\partial_t v_p(t)\|_{H^{s-2}}^2 + \int_0^\infty e^{c_s t} \|\partial_t v_p(t)\|_{H^{s-1}}^2 dt < \infty, & \text{for } s \geq 2. \end{cases}$$

Then, following the same lines as in Corollary 1, below, will yield

$$\int_0^\infty e^{c_p t} \mathfrak{H}_p(t) dt < \infty, \quad (1.22)$$

for some positive constant  $c_p$ , where  $\mathfrak{H}_p(t)$  is defined in (3.4), below.

On the other hand, from the conservation of mass and momentum (1.20), one can derive the Poincaré type inequalities,

$$\|\psi^{\varepsilon, h}\|_{L^2} \lesssim \|\nabla \psi^{\varepsilon, h}\|_{L^2} + \tilde{Q}_2, \quad \|\xi^\varepsilon\|_{L^2} \lesssim \|\nabla_h \xi^\varepsilon\|_{L^2},$$

where  $\tilde{Q}_2$  is at least quadratic of the perturbations. After applying these inequalities, the inequality in (3.51), below, can be written as,

$$\frac{d}{dt}\mathcal{E}_{LM} + C_1\mathcal{E}_{LM} \lesssim \mathfrak{H}_p\varepsilon^2,$$

for some positive constant  $C_1$ , provided  $\mathcal{E} \lesssim \varepsilon^2$  and  $\varepsilon$  small enough. Here,  $\mathcal{E}$  and  $\mathcal{E}_{LM}$  are defined in (2.2) and (3.48), respectively. Thus together with (1.22), this inequality implies

$$e^{C_2t}\mathcal{E}_{LM}(t) < \infty,$$

for some positive constant  $C_2$ . Relation (3.50) yields the decay of the perturbation energy  $\mathcal{E}$  as we claim. We leave the details to readers.

This work will be organized as follows. In section 2, we summarize the notations which will be commonly used in later paragraphs. Section 3 focuses on the  $\varepsilon$ -independent a priori estimates, which are the foundation of the low Mach number limit. In section 4, we focus on the proof of Theorem 1.2. This will be shown through a continuity argument. Finally in section 5, we summarize the proof of Theorem 1.1.

## 2 Preliminaries

We use the notations

$$\int \cdot d\vec{x} = \int_{\Omega_h \times 2\mathbb{T}} \cdot d\vec{x} := \int_{\Omega_h \times 2\mathbb{T}} \cdot dx dy dz, \quad \int_{\Omega_h} \cdot dx dy,$$

to represent the integrals in  $\Omega$  and  $\Omega_h$  respectively. Hereafter,  $\partial_h \in \{\partial_x, \partial_y\}$  represents the horizontal derivatives, and  $\partial_z$  represents the vertical derivative.

We will use  $|\cdot|$ ,  $\|\cdot\|$  to denote norms in  $\Omega_h \subset \mathbb{R}^2$  and  $\Omega_h \times 2\mathbb{T} \subset \mathbb{R}^3$ , respectively. After applying Ladyzhenskaya's and Agmon's inequalities in  $\Omega_h$  and  $\Omega$ , directly we have

$$\begin{aligned} |f|_{L^4} &\leq C|f|_{L^2}^{1/2}|\nabla_h f|_{L^2}^{1/2} + |f|_{L^2}, \quad |f|_{\infty} \leq C|f|_{L^2}^{1/2}|f|_{H^2}^{1/2}, \\ \|f\|_{L^3} &\leq C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} + \|f\|_{L^2}, \end{aligned} \quad (2.1)$$

for the function  $f$  with bounded right-hand sides. Also, applying Minkowski's and Hölder's inequalities yields

$$|\bar{f}|_{L^q} \leq \int_0^1 |f(z)|_{L^q} dz \leq C\|f\|_{L^q},$$



and hence  $\|\tilde{f}\|_{L^q} \leq C\|f\|_{L^q}$ ,  $q \in [1, \infty)$ .

We use  $\delta > 0$  to denote a arbitrary constant which will be chosen later adaptively small. Correspondingly,  $C_\delta$  is some positive constant depending on  $\delta$ . In addition, for any quantities  $A$  and  $B$ ,  $A \lesssim B$  is used to denote that there exists a positive constant independent of the solutions such that  $A \leq CB$ .

The following energy and dissipation functionals will be employed

$$\mathcal{E}(t) := \|\psi^h(t)\|_{H^2}^2 + \|\varepsilon\psi_t^h(t)\|_{L^2}^2 + \|\varepsilon^{-1}\xi(t)\|_{H^2}^2 + \|\xi_t(t)\|_{L^2}^2, \quad (2.2)$$

$$\begin{aligned} \mathcal{D}(t) := & \|\nabla\psi^h(t)\|_{H^2}^2 + \|\varepsilon\psi_t^h(t)\|_{H^1}^2 + \|\varepsilon^{-1}\nabla_h\xi(t)\|_{H^1}^2 \\ & + \|\xi_t(t)\|_{L^2}^2. \end{aligned} \quad (2.3)$$

Then  $\mathcal{E}(0) = \mathcal{E}_{in}$ , where  $\mathcal{E}_{in}$  is as in (1.16). In this work, we shall use  $Q(\mathcal{E})$  to denote a polynomial quantity, with positive coefficients, of  $\sqrt{\mathcal{E}}$  and  $Q(0) = 0$ . In general,  $Q(\cdot)$  is a generic polynomial quantity, with positive coefficients, of the arguments and  $Q(0) = 0$ .

### 3 $\varepsilon$ -independent *a priori* estimate

This section is devoted to show the following:

**Proposition 1.** *For any  $T > 0$ ,  $t \in [0, T]$ , suppose that the solution  $(v_p, \rho_1)$  with  $w_p$  given by (1.13) to (1.2) satisfies*

$$\begin{aligned} & \|v_p(t)\|_{H^3}, \|v_{p,t}(t)\|_{L^2}, \|\rho_1(t)\|_{H^2}, \|\rho_{1,t}(t)\|_{L^2}, \|w_p(t)\|_{H^1} \leq C, \\ & \int_0^T (\|v_p(t)\|_{H^3}^2 + \|v_{p,t}(t)\|_{H^1}^2 + \|\rho_{1,t}(t)\|_{L^2}^2 + \|\rho_{1,t}(t)\|_{H^1}^2 \\ & \quad + \|\rho_1(t)\|_{H^2}^2 + \|w_p(t)\|_{H^2}^2 + \|v_p(t)\|_{H^2}) dt \leq C, \end{aligned} \quad (3.1)$$

for some positive constant  $C$ , and

$$\frac{1}{2}\rho_0 < \rho < 2\rho_0 \quad \text{in } (\Omega_h \times 2\mathbb{T}) \times [0, T]. \quad (3.2)$$

Then any solution  $(\psi^h, \xi)$  to (1.10), with initial data as in (1.14), provided that it exists in the time interval  $[0, T]$ , with  $\psi^z$  given by (1.12), satisfies,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathcal{E}(t) + \int_0^T \mathcal{D}(t) dt \leq C' e^{C' + Q(\sup_{0 \leq t \leq T} \mathcal{E}(t))} \left\{ \varepsilon^2 + \mathcal{E}_{in} \right. \\ & \quad \left. + \left( \varepsilon^2 + (\varepsilon^2 + 1)Q(\sup_{0 \leq t \leq T} \mathcal{E}(t)) \right) \int_0^T \mathcal{D}(t) dt \right\}, \end{aligned} \quad (3.3)$$

for some positive constant  $C'$  depending only on the bounds in (3.1). In particular,  $C'$  is independent of  $\varepsilon$  and  $T$ .

*Remark 5.* We remark here that, from the definition of  $\mathcal{E}(t)$  in (2.2) and (1.5), (3.2) automatically holds for  $\varepsilon$  small enough if  $\sup_{0 \leq t \leq T} \mathcal{E}(t) < \infty$  and (3.1) holds.

Throughout the rest of this section, it is assumed that  $(\xi, \psi^h)$  with  $\psi^z$  given by (1.12) is a solution to (1.10) which is smooth enough such that the estimates below can be established. To justify the arguments, one can employ the local well-posedness theory and the standard different quotient method to the corresponding lines below (replaced the differential operators by different quotients, for example). See, for instance, similar arguments in [61, 62].

We denote by  $\mathfrak{G}_p(t)$  a polynomial, with positive coefficients, quantity of the arguments

$$\|v_p(t)\|_{H^3}, \|v_{p,t}(t)\|_{L^2}, \|\rho_1(t)\|_{H^2}, \|\rho_{1,t}(t)\|_{L^2}, \|w_p(t)\|_{H^1},$$

and

$$\begin{aligned} \mathfrak{H}_p(t) := & \|v_p(t)\|_{H^3}^2 + \|v_{p,t}(t)\|_{H^1}^2 + \|\rho_{1,tt}(t)\|_{L^2}^2 + \|\rho_{1,t}(t)\|_{H^1}^2 \\ & + \|\rho_1(t)\|_{H^2}^2 + \|w_p(t)\|_{H^2}^2 + \|v_p(t)\|_{H^2}^2. \end{aligned} \quad (3.4)$$

In particular, (3.1) of Proposition 1, is equivalent to

$$\sup_{0 \leq t \leq T} \mathfrak{G}_p(t) + \int_0^T \mathfrak{H}_p(t) dt < C,$$

for some positive constant  $C$ . For the sake of convenience, we will shorten the notations  $\mathfrak{G}_p = \mathfrak{G}_p(t)$ ,  $\mathfrak{H}_p = \mathfrak{H}_p(t)$ , below. We also remind the reader that we have assumed that (3.1) and (3.2) hold throughout this section.

### 3.1 Temporal derivatives

We start by performing the time derivative estimate to the solutions to system (1.10). Applying  $\partial_t$  to system (1.10) we will have the following system:

$$\begin{cases} \partial_t \xi_t + \rho_0(\operatorname{div}_h \psi_t^h + \partial_z \psi_t^z) = \mathcal{G}_{1,t} + \mathcal{G}_{2,t} & \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho \partial_t \psi_t^h + \rho v \cdot \nabla_h \psi_t^h + \rho w \partial_z \psi_t^h + \nabla_h(\varepsilon^{-2} c_s^2 \xi_t) = \mu \Delta_h \psi_t^h \\ \quad + \lambda \nabla_h \operatorname{div}_h \psi_t^h + \partial_{zz} \psi_t^h + \mathcal{F}_{1,t} + \mathcal{F}_{2,t} - \nabla_h(\varepsilon^{-2} \mathcal{R}_t) + \mathcal{H}_t & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \quad (3.5)$$

where

$$\mathcal{H}_t := -\rho_t \psi_t^h - (\rho_t v + \rho v_t) \cdot \nabla_h \psi^h - (\rho_t w + \rho w_t) \partial_z \psi^h. \quad (3.6)$$

We will show the following:

**Lemma 1.** *In addition to the assumptions in Proposition 1, suppose that  $(\xi, \psi^h)$ , with  $\psi^z$  given by (1.12), is a smooth solution to (1.10) in the time interval  $[0, T]$ . We have*

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \varepsilon \psi_t^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\xi_t\|_{L^2}^2 \right\} + \mu \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 \\ & + \lambda \|\varepsilon \operatorname{div}_h \psi_t^h\|_{L^2}^2 + \|\varepsilon \partial_z \psi_t^h\|_{L^2}^2 \leq \delta (\|\xi_t\|_{L^2}^2 + \|\varepsilon \nabla \psi_t^h\|_{L^2}^2) \\ & + \|\varepsilon \psi_t^h\|_{L^2}^2 + \|\nabla \psi^h\|_{H^2}^2 + \varepsilon^2 C_\delta (Q(\mathcal{E}) + \mathfrak{G}_p) \|\partial_z \psi^h\|_{H^2}^2 \\ & + C_\delta Q(\mathcal{E}) (\|\varepsilon \psi_t^h\|_{L^2}^2 + \|\nabla \psi^h\|_{H^1}^2 + \|\xi_t\|_{L^2}^2) \\ & + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi^h\|_{H^2}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2). \end{aligned} \quad (3.7)$$

*Proof.* Take the  $L^2$ -inner product of (3.5)<sub>2</sub> with  $\varepsilon^2 \psi_t^h$ . After applying integration by parts and substituting (1.1)<sub>1</sub>, we have the following:

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int \varepsilon^2 \rho |\psi_t^h|^2 d\vec{x} \right\} - \int c_s^2 \xi_t \operatorname{div}_h \psi_t^h d\vec{x} + \int \left( \mu |\varepsilon \nabla_h \psi_t^h|^2 \right. \\ & \quad \left. + \lambda |\varepsilon \operatorname{div}_h \psi_t^h|^2 + |\varepsilon \partial_z \psi_t^h|^2 \right) d\vec{x} = \int \varepsilon^2 \mathcal{F}_{1,t} \cdot \psi_t^h d\vec{x} \\ & \quad + \int \varepsilon^2 \mathcal{F}_{2,t} \cdot \psi_t^h d\vec{x} - \int \nabla_h \mathcal{R}_t \cdot \psi_t^h d\vec{x} + \int \varepsilon^2 \mathcal{H}_t \cdot \psi_t^h d\vec{x}. \end{aligned}$$

We substitute (3.5)<sub>1</sub> into the second term on the left-hand side of the above equation and compute the identity, thanks to (1.7),

$$\begin{aligned} & - \int c_s^2 \xi_t \operatorname{div}_h \psi_t^h d\vec{x} = \int \rho_0^{-1} c_s^2 \xi_t (\partial_t \xi_t + \rho_0 \partial_z \psi_t^z - \mathcal{G}_{1,t} - \mathcal{G}_{2,t}) d\vec{x} \\ & = \frac{d}{dt} \left\{ \frac{c_s^2}{2\rho_0} \int |\xi_t|^2 d\vec{x} \right\} - \rho_0^{-1} c_s^2 \int \xi_t \mathcal{G}_{1,t} d\vec{x} - \rho_0^{-1} c_s^2 \int \xi_t \mathcal{G}_{2,t} d\vec{x}. \end{aligned}$$

Consequently, we have,

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \varepsilon \psi_t^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\xi_t\|_{L^2}^2 \right\} + \mu \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 \\
& + \lambda \|\varepsilon \operatorname{div}_h \psi_t^h\|_{L^2}^2 + \|\varepsilon \partial_z \psi_t^h\|_{L^2}^2 = \int \varepsilon^2 \mathcal{F}_{1,t} \cdot \psi_t^h d\vec{x} \\
& + \int \varepsilon^2 \mathcal{F}_{2,t} \cdot \psi_t^h d\vec{x} + \rho_0^{-1} c_s^2 \int \xi_t \mathcal{G}_{1,t} d\vec{x} + \rho_0^{-1} c_s^2 \int \xi_t \mathcal{G}_{2,t} d\vec{x} \\
& + \int \varepsilon^2 \mathcal{H}_t \cdot \psi_t^h d\vec{x} - \int \nabla_h \mathcal{R}_t \cdot \psi_t^h d\vec{x} =: \sum_{i=1}^6 I_i.
\end{aligned} \tag{3.8}$$

Next we estimate the right-hand side of (3.8). After substituting (3.6) into  $I_5$ , it can be written as

$$\begin{aligned}
I_5 &= - \int \zeta_t |\varepsilon \psi_t^h|^2 d\vec{x} - \int (\varepsilon^2 \zeta_t v \cdot \nabla_h \psi_t^h \cdot \psi_t^h + \varepsilon^2 \rho v_t \cdot \nabla_h \psi_t^h \cdot \psi_t^h) d\vec{x} \\
& - \int (\varepsilon^2 \zeta_t w \partial_z \psi_t^h \cdot \psi_t^h + \varepsilon^2 \rho w_t \partial_z \psi_t^h \cdot \psi_t^h) d\vec{x} =: I'_5 + I''_5 + I'''_5.
\end{aligned}$$

Notice that  $\zeta = \varepsilon^2 \rho_1 + \xi$  is independent of the  $z$  variable. Then, for every  $\delta > 0$  there exists a positive constant  $C_\delta$  such that

$$\begin{aligned}
I'_5 &\lesssim |\zeta_t|_{L^2} \int_0^1 \|\varepsilon \psi_t^h\|_{L^2}^2 dz = \|\zeta_t\|_{L^2} \int_0^1 \|\varepsilon \psi_t^h\|_{L^4}^2 dz \\
&\lesssim \|\zeta_t\|_{L^2} \int_0^1 \left( \|\varepsilon \psi_t^h\|_{L^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}^2 \right) dz \\
&\lesssim \|\zeta_t\|_{L^2} (\|\varepsilon \psi_t^h\|_{L^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}^2) \lesssim \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 \\
&\quad + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta (\varepsilon^4 \|\rho_{1,t}\|_{L^2}^2 + \mathcal{E}) \|\varepsilon \psi_t^h\|_{L^2}^2,
\end{aligned}$$

where we have applied the Minkowski, Hölder's, the Sobolev embedding and Young's inequalities. On the other hand,  $I''_5$  can be estimated directly using Hölder's, the Sobolev embedding and Young's inequalities:

$$\begin{aligned}
I''_5 &\lesssim \varepsilon \|\zeta_t\|_{L^2} \|v\|_{L^\infty} \|\nabla_h \psi_t^h\|_{L^6} (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\quad + \|\rho\|_{L^\infty} (\varepsilon \|v_{p,t}\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \|\nabla_h \psi_t^h\|_{L^6} \\
&\quad \times (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\lesssim \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) (\|\nabla_h \psi_t^h\|_{H^1}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2) \\
&\quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi_t^h\|_{H^2}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2).
\end{aligned}$$

On the other hand, from (1.11) (or (1.12)), we have the identities:

$$\begin{aligned}
w &= w_p - \int_0^z \left( \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho \right) dz' = w_p + \psi^z, \\
w_t &= w_{p,t} - \int_0^z \left( \operatorname{div}_h \widetilde{\psi}_t^h + \widetilde{v}_t \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_t \right) dz' \\
&= w_{p,t} + \psi_t^z.
\end{aligned} \tag{3.9}$$

Therefore,  $I_5'''$  can be written as

$$\begin{aligned}
I_5''' &= - \int \varepsilon^2 (\zeta_t w_p + \rho w_{p,t}) \partial_z (\psi^h \cdot \psi_t^h) d\vec{x} + \int \left[ \varepsilon^2 \int_0^z (\operatorname{div}_h \widetilde{\psi}^h \right. \\
&\quad \left. + \widetilde{v} \cdot \nabla_h \log \rho) dz' \times (\zeta_t \partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} + \int \left[ \varepsilon^2 \int_0^z (\rho \operatorname{div}_h \widetilde{\psi}_t^h \right. \\
&\quad \left. + \widetilde{v}_t \cdot \nabla_h \rho + \rho \widetilde{v} \cdot \nabla_h (\log \rho)_t) dz' \times (\partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} =: \sum_{i=1}^3 I_{5,i}'''.
\end{aligned}$$

Then, we plug in identity (1.13) and apply the Hölder, Minkowski and Young inequalities to infer,

$$\begin{aligned}
I_{5,1}''' &= \int \left[ \varepsilon^2 (\zeta_t \int_0^z \operatorname{div}_h v_p dz' + \rho \int_0^z \operatorname{div}_h v_{p,t} dz') \times (\partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} \\
&= \int \left[ \varepsilon^2 \zeta_t \left( \int_0^z \operatorname{div}_h v_p dz' \right) \times (\partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} \\
&\quad - \int \left[ \varepsilon^2 \left( \int_0^z v_{p,t} dz' \right) \cdot \nabla_h (\rho \partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} \\
&\lesssim \varepsilon \int_0^1 \left[ \left( \int_0^z |\nabla_h v_p|_{L^8} dz' \right) \times (|\zeta_t|_{L^2} |\partial_z \psi^h|_{L^8} |\varepsilon \psi_t^h|_{L^4}) \right] dz \\
&\quad + \varepsilon \int_0^1 \left[ \left( \int_0^z |v_{p,t}|_{L^2} dz' \right) \times (|\nabla_h \zeta|_{L^8} |\partial_z \psi^h|_{L^8} |\varepsilon \psi_t^h|_{L^4}) \right] dz \\
&\quad + \varepsilon \int_0^1 \left[ \left( \int_0^z |v_{p,t}|_{L^4} dz' \right) \times (|\rho|_{L^\infty} |\partial_z \nabla_h \psi^h|_{L^2} |\varepsilon \psi_t^h|_{L^4} \right. \\
&\quad \left. + |\rho|_{L^\infty} |\partial_z \psi^h|_{L^4} |\varepsilon \nabla_h \psi_t^h|_{L^2}) \right] dz \lesssim \varepsilon (\|\zeta_t\|_{L^2} \|v_p\|_{H^2} \\
&\quad + \|\nabla_h \zeta\|_{H^1} \|v_{p,t}\|_{L^2} + \|\rho\|_{L^\infty} \|v_{p,t}\|_{H^1}) \|\partial_z \psi^h\|_{H^1} \\
&\quad \times (\|\varepsilon \nabla \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 \\
&\quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

On the other hand, a straight forward estimate shows that

$$\begin{aligned}
I_{5,2}''' &\lesssim \varepsilon \int_0^1 \left( |\nabla_h \psi^h|_{L^8} + |v|_{L^\infty} |\nabla_h \zeta|_{L^8} \right) dz \times \int_0^1 |\zeta_t|_{L^2} |\partial_z \psi^h|_{L^8} |\varepsilon \psi_t^h|_{L^4} dz \\
&\lesssim \varepsilon \|\zeta_t\|_{L^2} \left( \|\nabla_h \psi^h\|_{H^1} + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1} \right) \|\partial_z \psi^h\|_{H^1} \\
&\quad \times \left( \|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2} \right) \lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 \\
&\quad + C_\delta Q(\mathcal{E}) \|\partial_z \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \|\psi^h\|_{H^2}^2.
\end{aligned}$$

To estimate  $I_{5,3}'''$ , we apply integration by parts as below,

$$\begin{aligned}
I_{5,3}''' &= \int \left[ \varepsilon^2 \int_0^z (\rho \operatorname{div}_h \widetilde{\psi}_t^h + \widetilde{v}_t \cdot \nabla_h \rho - \zeta_t \widetilde{v} \cdot \nabla_h \log \rho - \zeta_t \operatorname{div}_h \widetilde{v}) dz \right. \\
&\quad \left. \times (\partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x} - \int \left[ \varepsilon^2 \int_0^z \zeta_t \widetilde{v} dz \cdot \nabla_h (\partial_z \psi^h \cdot \psi_t^h) \right] d\vec{x},
\end{aligned}$$

from which, we infer

$$\begin{aligned}
I_{5,3}''' &\lesssim \int_0^1 \left( |\rho|_{L^\infty} |\varepsilon \nabla_h \psi_t^h|_{L^2} + |\varepsilon v_t|_{L^4} |\nabla_h \zeta|_{L^4} \right) dz \\
&\quad \times \int_0^1 |\partial_z \psi^h|_{L^4} |\varepsilon \psi_t^h|_{L^4} dz + \int_0^1 |\partial_z \psi^h|_{L^8} |\varepsilon \psi_t^h|_{L^4} dz \\
&\quad \times \int_0^1 \left( \varepsilon |\zeta_t|_{L^2} |v|_{L^\infty} |\nabla_h \zeta|_{L^8} + \varepsilon |\zeta_t|_{L^2} |\nabla_h v|_{L^8} \right) dz \\
&\quad + \int_0^1 \left( |\partial_z \psi^h|_{L^\infty} |\varepsilon \nabla_h \psi_t^h|_{L^2} + |\nabla_h \partial_z \psi^h|_{L^4} |\varepsilon \psi_t^h|_{L^4} \right) dz \\
&\quad \times \int_0^1 \varepsilon |\zeta_t|_{L^2} |v|_{L^\infty} dz \lesssim (\|\rho\|_{L^\infty} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} \\
&\quad + \|\varepsilon v_{p,t}\|_{H^1} \|\nabla_h \zeta\|_{H^1}) \|\psi_z^h\|_{H^1} (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^{1/2} \\
&\quad + \|\varepsilon \psi_t^h\|_{L^2}) + \|\nabla_h \zeta\|_{H^1} \|\psi_z^h\|_{H^1} (\|\varepsilon \psi_t^h\|_{L^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} \\
&\quad + \|\varepsilon \psi_t^h\|_{L^2}^2) + \varepsilon \|v\|_{H^2} (1 + \|\nabla_h \zeta\|_{H^1}) \|\psi_z^h\|_{H^2} \|\zeta_t\|_{L^2} \\
&\quad \times (\|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\lesssim \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2 C_\delta (Q(\mathcal{E}) + \mathfrak{G}_p) \|\psi_z^h\|_{H^2}^2 \\
&\quad + C_\delta Q(\mathcal{E}) (\|\varepsilon \psi_t^h\|_{L^2}^2 + \|\partial_z \psi^h\|_{H^1}^2) + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \|\psi^h\|_{H^2}^2 \\
&\quad \times (\|\psi^h\|_{H^2}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2).
\end{aligned}$$

Therefore, we have shown

$$\begin{aligned}
I_5 &\lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2 C_\delta (Q(\mathcal{E}) + \mathfrak{G}_p) \|\psi_z^h\|_{H^2}^2 \\
&\quad + C_\delta Q(\mathcal{E}) (\|\varepsilon \psi_t^h\|_{L^2}^2 + \|\nabla \psi^h\|_{H^1}^2) \\
&\quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi^h\|_{H^2}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2).
\end{aligned} \tag{3.10}$$

Next, after substituting (1.9) into  $I_2$ , it follows that

$$\begin{aligned}
I_2 &= - \int \varepsilon^2 (\rho \psi^h \cdot \nabla_h v_{p,t} + \zeta_t \psi^h \cdot \nabla_h v_p + \rho \psi_t^h \cdot \nabla_h v_p) \cdot \psi_t^h d\vec{x} \\
&\quad - \int \varepsilon^2 (\rho \psi^z \partial_z v_{p,t} + \zeta_t \psi^z \partial_z v_p + \rho \psi_t^z \partial_z v_p) \cdot \psi_t^h d\vec{x} =: I_2' + I_2''.
\end{aligned}$$

Similarly as before,

$$\begin{aligned}
I_2' &\lesssim \|\rho\|_{L^\infty} \|\nabla_h v_p\|_{L^6} \|\varepsilon \psi_t^h\|_{L^3} \|\varepsilon \psi_t^h\|_{L^2} + \varepsilon \|\rho\|_{L^\infty} \|\nabla_h v_{p,t}\|_{L^2} \\
&\quad \times \|\psi^h\|_{L^6} \|\psi_t^h\|_{L^3} + \varepsilon \|\nabla_h v_p\|_{L^\infty} \|\psi^h\|_{L^6} \|\zeta_t\|_{L^2} \|\varepsilon \psi_t^h\|_{L^3} \\
&\lesssim \|\rho\|_{H^2} \|\nabla_h v_p\|_{L^2} (\|\varepsilon \psi_t^h\|_{L^2}^{3/2} \|\varepsilon \nabla \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}^2) \\
&\quad + \varepsilon \|\psi^h\|_{H^1} \|\nabla_h v_{p,t}\|_{L^2} (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\quad + \varepsilon \|\nabla_h v_p\|_{H^2} \|\psi^h\|_{H^1} \|\zeta_t\|_{L^2} (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2} + C_\delta Q(\mathcal{E}) \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\varepsilon \psi_t^h\|_{L^2}^2 \\
&\quad + \varepsilon^2 C_\delta \mathfrak{H}_p.
\end{aligned}$$

On the other hand, after substituting (3.9),  $I_2''$  can be written as

$$\begin{aligned}
I_2'' &= \int \left[ \varepsilon^2 \int_0^z \left( \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho \right) dz \times \left( (\rho \partial_z v_{p,t} + \zeta_t \partial_z v_p) \cdot \psi_t^h \right) \right] d\vec{x} \\
&\quad + \int \left[ \varepsilon^2 \int_0^z \left( \operatorname{div}_h \widetilde{\psi}_t^h + \widetilde{v}_t \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_t \right) dz \right. \\
&\quad \left. \times \left( \rho \partial_z v_p \cdot \psi_t^h \right) \right] d\vec{x} =: I_{2,1}'' + I_{2,2}''.
\end{aligned}$$

Then we have the following estimate:

$$\begin{aligned}
I_{2,1}'' &\lesssim \int_0^1 \left( |\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4} \right) dz \times \int_0^1 \left( (\varepsilon |\partial_z v_{p,t}|_{L^2} \right. \\
&\quad \left. + \varepsilon |\zeta_t|_{L^2} |\partial_z v_p|_{L^\infty}) |\varepsilon \psi_t^h|_{L^4} \right) dz \lesssim (\|v\|_{H^2} \|\nabla_h \zeta\|_{H^1}
\end{aligned}$$

$$\begin{aligned}
& + \|\nabla_h \psi^h\|_{H^1}) (\varepsilon \|\partial_z v_{p,t}\|_{L^2} + \varepsilon \|\zeta_t\|_{L^2} \|\partial_z v_p\|_{H^2}) \\
& \times (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla \psi_t^h\|_{L^2}^{1/2} + \|\varepsilon \psi_t^h\|_{L^2}) \lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 \\
& + C_\delta Q(\mathcal{E}) \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2 C_\delta \mathfrak{H}_p.
\end{aligned}$$

To estimate  $I''_{2,2}$ , we first apply integration by parts as below,

$$\begin{aligned}
I''_{2,2} &= \int \left[ \varepsilon^2 \int_0^z \left( \rho \operatorname{div}_h \widetilde{\psi}_t^h + \widetilde{v}_t \cdot \nabla_h \rho - \zeta_t \widetilde{v}_t \cdot \nabla_h \log \rho - \zeta_t \operatorname{div}_h \widetilde{v}_t \right) dz \right. \\
& \quad \left. \times (\partial_z v_p \cdot \psi_t^h) \right] d\vec{x} - \int \left[ \varepsilon^2 \left( \int_0^z \zeta_t \widetilde{v}_t dz \right) \cdot \nabla_h (\partial_z v_p \cdot \psi_t^h) \right] d\vec{x},
\end{aligned}$$

which yields, similarly to the estimate of  $I'''_{5,3}$ ,

$$\begin{aligned}
I''_{2,2} &\lesssim \int_0^1 \left( |\rho|_{L^\infty} |\varepsilon \nabla_h \psi_t^h|_{L^2} + |\varepsilon v_t|_{L^4} |\nabla_h \zeta|_{L^4} \right) dz \\
& \quad \times \int_0^1 |\partial_z v_p|_{L^4} |\varepsilon \psi_t^h|_{L^4} dz + \int_0^1 |\partial_z v_p|_{L^8} |\varepsilon \psi_t^h|_{L^4} dz \\
& \quad \times \int_0^1 \left( \varepsilon |\zeta_t|_{L^2} |v|_{L^\infty} |\nabla_h \zeta|_{L^8} + \varepsilon |\zeta_t|_{L^2} |\nabla_h v|_{L^8} \right) dz \\
& + \int_0^1 \left( |\partial_z v_p|_{L^\infty} |\varepsilon \nabla_h \psi_t^h|_{L^2} + |\nabla_h \partial_z v_p|_{L^4} |\varepsilon \psi_t^h|_{L^4} \right) dz \\
& \quad \times \int_0^1 \varepsilon |\zeta_t|_{L^2} |v|_{L^\infty} dz \lesssim (\|\rho\|_{L^\infty} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} \\
& \quad + \|\varepsilon v_{p,t}\|_{H^1} \|\nabla_h \zeta\|_{H^1}) \|\partial_z v_p\|_{H^1} (\|\varepsilon \psi_t^h\|_{L^2}^{1/2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^{1/2} \\
& \quad + \|\varepsilon \psi_t^h\|_{L^2}) + (\|\varepsilon \psi_t^h\|_{L^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}^2) \\
& \quad \times \|\nabla_h \zeta\|_{H^1} \|\partial_z v_p\|_{H^1} + \varepsilon \|v\|_{H^2} (1 + \|\nabla_h \zeta\|_{H^1}) \\
& \quad \times \|\partial_z v_p\|_{H^2} \|\zeta_t\|_{L^2} (\|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
& \lesssim \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \\
& \quad \times (\|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_2 &\lesssim \delta \|\varepsilon \nabla \psi_t^h\|_2^2 + C_\delta Q(\mathcal{E}) \|\varepsilon \psi_t^h\|_2^2 \\
& \quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2).
\end{aligned} \tag{3.11}$$



Now, we will estimate  $I_3$ , which reads

$$\begin{aligned}
I_3 &= -\rho_0^{-1} c_s^2 \int \left( \xi_t \operatorname{div}_h (\xi_t v) + \xi_t \operatorname{div}_h (\xi v_t) + \xi_t \partial_z (\xi_t w) \right. \\
&\quad \left. + \xi_t \partial_z (\xi w_t) \right) d\vec{x} = -\frac{c_s^2}{2\rho_0} \int |\xi_t|^2 \operatorname{div}_h \psi^h d\vec{x} \\
&\quad - \rho_0^{-1} c_s^2 \int \left( \xi_t \xi \operatorname{div}_h \psi_t^h + \xi_t v_t \cdot \nabla_h \xi \right) d\vec{x} \\
&\lesssim \|\xi_t\|_{L^2} (\|\xi_t\|_{L^2} \|\nabla_h \psi^h\|_{H^2} + \|\varepsilon^{-1} \xi\|_{H^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2}) \\
&\quad + (\varepsilon \|v_{p,t}\|_{L^6} + \|\varepsilon \nabla \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \|\varepsilon^{-1} \nabla_h \xi\|_{H^1} \|\xi_t\|_{L^2} \\
&\lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + \delta \|\nabla_h \psi^h\|_{H^2}^2 + C_\delta Q(\mathcal{E}) \|\xi_t\|_2^2 \\
&\quad + \varepsilon^2 C_\delta \mathfrak{H}_p.
\end{aligned} \tag{3.12}$$

Here we have employed the facts that  $\xi$  is independent of the  $z$ -variable and that  $\int_0^1 \operatorname{div}_h v_p dz = \int_0^1 \operatorname{div}_h v_{p,t} dz = 0$ . The rest is straightforward. For instance, substituting (1.9) in  $I_1$  yields

$$\begin{aligned}
I_1 &= \rho_0^{-1} \int \varepsilon \zeta_t (\nabla_h (c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p) \cdot \varepsilon \psi_t^h d\vec{x} \\
&\quad + \rho_0^{-1} \int \varepsilon \zeta (\nabla_h (c_s^2 \rho_{1,t}) - \mu \Delta_h v_{p,t} - \lambda \nabla_h \operatorname{div}_h v_{p,t} - \partial_{zz} v_{p,t}) \cdot \varepsilon \psi_t^h d\vec{x} \\
&= \rho_0^{-1} \int \varepsilon \zeta_t (\nabla_h (c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p) \cdot \varepsilon \psi_t^h d\vec{x} \\
&\quad - \rho_0^{-1} \int \left( \varepsilon c_s^2 \rho_{1,t} \operatorname{div}_h (\zeta \varepsilon \psi_t^h) - \varepsilon \mu \nabla_h v_{p,t} : \nabla_h (\zeta \varepsilon \psi_t^h) \right. \\
&\quad \left. - \varepsilon \lambda \operatorname{div}_h v_{p,t} \operatorname{div}_h (\zeta \varepsilon \psi_t^h) - \varepsilon \partial_z v_{p,t} \cdot \partial_z (\zeta \varepsilon \psi_t^h) \right) d\vec{x} \\
&\lesssim \varepsilon (\|\nabla_h \rho_1\|_{H^1} + \|\nabla v_p\|_{H^2}) \|\zeta_t\|_{L^2} (\|\varepsilon \nabla \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\quad + \varepsilon (\|\rho_{1,t}\|_{L^2} + \|\nabla v_{p,t}\|_{L^2}) \|\zeta\|_{H^2} (\|\varepsilon \nabla \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + \varepsilon^2 C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p.
\end{aligned} \tag{3.13}$$

We list estimates for  $I_4, I_6$  below:

$$\begin{aligned}
I_4 &= -\rho_0^{-1} c_s^2 \varepsilon^2 \int \left( \xi_t (\rho_{1,tt} + \rho_{1,t} \operatorname{div}_h \psi^h + v \cdot \nabla_h \rho_{1,t}) \right. \\
&\quad \left. + \xi_t (\rho_1 \operatorname{div}_h \psi_t^h + v_t \cdot \nabla_h \rho_1) \right) d\vec{x} \lesssim \varepsilon^2 \|\xi_t\|_{L^2} \left( \|\rho_{1,tt}\|_{L^2} \right. \\
&\quad \left. + \|\rho_{1,t}\|_{L^2} \|\nabla_h \psi^h\|_{H^2} + \|\nabla_h \rho_{1,t}\|_{L^2} \|v\|_{H^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left( \|\rho_1\|_{H^2} \|\nabla_h \psi_t^h\|_{L^2} + \|\nabla_h \rho_1\|_{H^1} \|v_t\|_{H^1} \right) \lesssim \delta \|\xi_t\|_{L^2}^2 \\
& + \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 + \delta \|\nabla_h \psi^h\|_{H^2} + \varepsilon^2 C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p, \quad (3.14)
\end{aligned}$$

$$\begin{aligned}
I_6 & = \int \mathcal{R}_t \operatorname{div}_h \psi_t^h d\bar{x} \lesssim \|\varepsilon^{-1} \zeta\|_{L^\infty} \|\zeta_t\|_{L^2} \|\varepsilon \nabla_h \psi_t^h\|_{L^2} \lesssim \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 \\
& + C_\delta (\varepsilon^2 \|\rho_1\|_{H^2}^2 + \|\varepsilon^{-1} \xi\|_{H^2}^2) (\varepsilon^4 \|\rho_{1,t}\|_2^2 + \|\xi_t\|_2^2), \quad (3.15)
\end{aligned}$$

where we have used the fact that from (1.8)

$$|\mathcal{R}_t| = |\gamma(\gamma - 1) \int_{\rho_0}^{\rho} \rho_t y^{\gamma-2} dy| \leq |\gamma \rho_t (\rho^{\gamma-1} - \rho_0^{\gamma-1})| \leq C |\zeta_t| |\zeta|.$$

Summing up inequalities (3.10), (3.11), (3.12), (3.13), (3.14), (3.15) and (3.8) completes the proof.  $\square$

The next lemma follows directly from system (1.10), and it shows the estimates of the temporal derivatives of  $\xi, \psi^h$  in terms of the spatial derivatives.

**Lemma 2.** *Under the same assumptions as in Lemma 1,*

$$\begin{aligned}
\|\xi_t\|_{L^2}^2 & \leq C \|\nabla_h \psi^h\|_{L^2}^2 + \varepsilon^2 Q(\mathcal{E}) (\|\nabla_h \psi^h\|_{H^1}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2) \\
& + \varepsilon^2 C (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p, \quad (3.16)
\end{aligned}$$

$$\begin{aligned}
\|\varepsilon \rho \psi_t^h\|_{L^2}^2 & \leq C \|\varepsilon^{-1} \nabla_h \xi\|_{L^2}^2 + \varepsilon^2 C \|\nabla \psi^h\|_{H^1}^2 + \varepsilon^2 Q(\mathcal{E}) (\|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 \\
& + \|\nabla \psi^h\|_{H^1}^2) + \varepsilon^2 C (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p, \quad (3.17)
\end{aligned}$$

for some positive constant  $C$  independent of  $\varepsilon$ .

*Proof.* Indeed, after integrating (1.10)<sub>1</sub> in the  $z$  variable, we have, thanks to (1.7),

$$\partial_t \xi + \rho_0 \operatorname{div}_h \overline{\psi^h} = -\operatorname{div}_h (\xi \overline{v}) - \varepsilon^2 (\partial_t \rho_1 + \operatorname{div}_h (\rho_1 \overline{v})).$$

Then directly, we have, after applying the Minkowski, Hölder and Sobolev embedding inequalities,

$$\begin{aligned}
\|\xi_t\|_{L^2} & = |\xi_t|_{L^2} \lesssim \int_0^1 |\nabla_h \psi^h|_{L^2} dz + \int_0^1 (|\xi|_{L^4} |\nabla_h v|_{L^4} + |\nabla_h \xi|_{L^4} |v|_{L^4} \\
& + \varepsilon^2 (|\rho_{1,t}|_{L^2} + |\rho_1|_{L^4} |\nabla_h v|_{L^4} + |\nabla_h \rho_1|_{L^4} |v|_{L^4})) dz \lesssim \|\nabla_h \psi^h\|_{L^2} \\
& + \|\xi\|_{H^1} \|\nabla_h \psi^h\|_{H^1} + \|\nabla_h \xi\|_{H^1} \|\psi^h\|_{H^1} + \varepsilon \|v_p\|_{H^2} \|\varepsilon^{-1} \xi\|_{H^2}
\end{aligned}$$

$$+ \varepsilon^2 (\|\rho_{1,t}\|_{L^2} + \|\rho_1\|_{H^2} \|v_p\|_{H^2} + \|\rho_1\|_{H^2} \|\psi^h\|_{H^2}).$$

Taking square on both sides of the above inequality yields (3.16).

On the other hand, from (1.10)<sub>2</sub>, we have

$$\begin{aligned} \rho \partial_t \psi^h &= -\rho v \cdot \nabla_h \psi^h - \rho w \partial_z \psi^h - \nabla_h (\varepsilon^{-2} c_s^2 \xi) + \mu \Delta_h \psi^h + \lambda \nabla_h \operatorname{div}_h \psi^h \\ &+ \partial_{zz} \psi^h + \mathcal{F}_1 + \mathcal{F}_2 - \nabla_h (\varepsilon^{-2} \mathcal{R}). \end{aligned}$$

Directly, we have

$$\begin{aligned} \|\rho \psi_t^h\|_{L^2} &\lesssim \|\rho\|_{L^\infty} \|v\|_{L^\infty} \|\nabla_h \psi^h\|_{L^2} + \varepsilon^{-1} \|\varepsilon^{-1} \nabla_h \xi\|_{L^2} + \|\nabla^2 \psi^h\|_{L^2} \\ &+ \|\varepsilon^{-1} \zeta\|_{L^3} \|\varepsilon^{-1} \nabla_h \zeta\|_{L^6} + \|\mathcal{F}_1\|_{L^2} + \|\mathcal{F}_2\|_{L^2} + \|\rho w \partial_z \psi^h\|_{L^2} \\ &\lesssim \varepsilon^{-1} \|\varepsilon^{-1} \nabla_h \xi\|_{L^2} + \|\nabla \psi^h\|_{H^1} + Q(\mathcal{E}) (\|\varepsilon^{-1} \nabla_h \xi\|_{H^1} + \|\nabla \psi^h\|_{L^2}) \\ &+ (Q(\mathcal{E}) + \varepsilon \|\rho_1\|_{H^1}) \varepsilon \|\rho_1\|_{H^2} + \|v_p\|_{H^2} \|\nabla_h \psi^h\|_{L^2} \\ &+ \|\mathcal{F}_1\|_{L^2} + \|\mathcal{F}_2\|_{L^2} + \|\rho w \partial_z \psi^h\|_{L^2}, \end{aligned}$$

where we have substituted the following inequality from (1.8),

$$|\nabla_h \mathcal{R}| = |\gamma(\gamma - 1) \int_{\rho_0}^{\rho} \nabla_h \rho y^{\gamma-2} dy| \leq |\gamma \nabla_h \rho (\rho^{\gamma-1} - \rho_0^{\gamma-1})| \lesssim |\nabla_h \zeta| |\zeta|. \quad (3.18)$$

From (1.11), (1.12), one has,

$$\begin{aligned} \rho w \partial_z \psi^h &= \rho w_p \partial_z \psi^h - \int_0^z \rho \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \rho dz \partial_z \psi^h, \\ \mathcal{F}_1 &= \zeta Q_p, \\ \mathcal{F}_2 &= -\rho \psi^h \cdot \nabla_h v_p + \left( \int_0^z (\rho \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \rho) dz \right) \partial_z v_p. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{F}_1\|_{L^2} + \|\mathcal{F}_2\|_{L^2} + \|\rho w \partial_z \psi^h\|_{L^2} &\lesssim \|\zeta\|_{\infty} \|Q_p\|_{L^2} \\ &+ \|\rho\|_{L^\infty} (\|w_p\|_{L^3} \|\partial_z \psi^h\|_{L^6} + \|\psi^h\|_{L^\infty} \|\nabla_h v_p\|_{L^2} \\ &+ \|\nabla_h \psi^h\|_{L^3} \|\partial_z v_p\|_{L^6} + \|\nabla_h \psi^h\|_{L^3} \|\partial_z \psi^h\|_{L^6}) \\ &+ \|v\|_{L^\infty} \|\nabla_h \zeta\|_{L^3} \|\partial_z \psi^h\|_{L^6} + \|v\|_{L^\infty} \|\nabla_h \zeta\|_{L^3} \|\partial_z v_p\|_{L^6} \\ &\lesssim (\varepsilon^2 \|\rho_1\|_{H^2} + \varepsilon \|\varepsilon^{-1} \xi\|_{H^2}) (\|\rho_1\|_{H^1} + \|v_p\|_{H^2}) \\ &+ (\|w_p\|_{H^1} + \|v_p\|_{H^2} + \|\nabla \psi^h\|_{H^1}) \|\psi^h\|_{H^2} \\ &+ (\|v_p\|_{H^2} + \|\psi^h\|_{H^2})^2 (\varepsilon^2 \|\rho_1\|_{H^2} + \varepsilon \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}). \end{aligned}$$

Summing the above inequalities yields (3.17).  $\square$

### 3.2 Horizontal derivatives

We derive the required estimates for the horizontal derivatives in this subsection. After applying  $\partial_{hh} = \partial_h^2$  to system (1.10), we obtain the following system:

$$\begin{cases} \partial_t \xi_{hh} + \rho_0 (\operatorname{div}_h \psi_{hh}^h + \partial_z \psi_{hh}^z) = \mathcal{G}_{1,hh} + \mathcal{G}_{2,hh} & \text{in } \Omega, \\ \rho \partial_t \psi_{hh}^h + \rho v \cdot \nabla_h \psi_{hh}^h + \rho w \partial_z \psi_{hh}^h + \nabla_h (\varepsilon^{-2} c_s^2 \xi_{hh}) \\ = \mu \Delta_h \psi_{hh}^h + \lambda \nabla_h \operatorname{div}_h \psi_{hh}^h + \partial_{zz} \psi_{hh}^h + \mathcal{F}_{1,hh} \\ + \mathcal{F}_{2,hh} - \nabla_h (\varepsilon^{-2} \mathcal{R}_{hh}) + \mathcal{H}_{hh} & \text{in } \Omega, \end{cases} \quad (3.19)$$

where

$$\begin{aligned} \mathcal{H}_{hh} := & -\rho_{hh} \psi_t^h - (\rho v)_{hh} \cdot \nabla_h \psi^h - (\rho w)_{hh} \partial_z \psi^h - 2\rho_h \psi_{ht}^h \\ & - 2(\rho v)_h \cdot \nabla_h \psi_h^h - 2(\rho w)_h \partial_z \psi_h^h. \end{aligned} \quad (3.20)$$

**Lemma 3.** *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_{hh}^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\varepsilon^{-1} \xi_{hh}\|_{L^2}^2 \right\} + C_{\mu,\lambda} \|\nabla \psi_{hh}^h\|_{L^2}^2 \\ & \leq \delta (\|\nabla \psi^h\|_{H^2}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 + \|\varepsilon \nabla \psi_t^h\|_{L^2}^2) \\ & + \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) (\|\nabla \psi^h\|_{H^1}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2) \\ & + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi^h\|_{H^2}^2 + \|\varepsilon^{-1} \xi\|_{H^2}^2 + \varepsilon^2), \end{aligned} \quad (3.21)$$

for some positive constant  $C_{\mu,\lambda}$ , which is independent of  $\varepsilon$ .

*Proof.* Take the inner product of (3.19)<sub>2</sub> with  $\psi_{hh}^h$  and integrate the resultant over  $\Omega$ . Similarly as before, we will have the following:

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_{hh}^h\|_{L^2}^2 \right\} - \int \varepsilon^{-2} c_s^2 \xi_{hh} \operatorname{div}_h \psi_{hh}^h d\vec{x} + \mu \|\nabla_h \psi_{hh}^h\|_{L^2}^2 \\ & + \lambda \|\operatorname{div}_h \psi_{hh}^h\|_{L^2}^2 + \|\partial_z \psi_{hh}^h\|_{L^2}^2 = \int \mathcal{F}_{1,hh} \cdot \psi_{hh}^h d\vec{x} \\ & + \int \mathcal{F}_{2,hh} \cdot \psi_{hh}^h d\vec{x} - \int \nabla_h (\varepsilon^{-2} \mathcal{R}_{hh}) \cdot \psi_{hh}^h d\vec{x} + \int \mathcal{H}_{hh} \cdot \psi_{hh}^h d\vec{x}. \end{aligned}$$

Again, we employ equation (3.19)<sub>1</sub> to derive that,

$$- \int \varepsilon^{-2} c_s^2 \xi_{hh} \operatorname{div}_h \psi_{hh}^h d\vec{x} = \int \left( \varepsilon^{-2} \rho_0^{-1} c_s^2 \xi_{hh} (\partial_t \xi_{hh} + \rho_0 \partial_z \psi_{hh}^z - \mathcal{G}_{1,hh}$$

$$\begin{aligned}
& -\mathcal{G}_{2,hh}) \Big) d\vec{x} = \frac{d}{dt} \left\{ \frac{c_s^2}{2\rho_0} \int \varepsilon^{-2} |\xi_{hh}|^2 d\vec{x} \right\} - \rho_0^{-1} c_s^2 \int \varepsilon^{-2} \xi_{hh} \mathcal{G}_{1,hh} d\vec{x} \\
& - \rho_0^{-1} c_s^2 \int \varepsilon^{-2} \xi_{hh} \mathcal{G}_{2,hh} d\vec{x}.
\end{aligned}$$

Therefore, the following equality holds:

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_{hh}^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\varepsilon^{-1} \xi_{hh}\|_{L^2}^2 \right\} + \mu \|\nabla_h \psi_{hh}^h\|_{L^2}^2 \\
& + \lambda \|\operatorname{div}_h \psi_{hh}^h\|_{L^2}^2 + \|\partial_z \psi_{hh}^h\|_{L^2}^2 = \int \mathcal{F}_{1,hh} \cdot \psi_{hh}^h d\vec{x} \\
& + \int \mathcal{F}_{2,hh} \cdot \psi_{hh}^h d\vec{x} + \rho_0^{-1} c_s^2 \int \varepsilon^{-2} \xi_{hh} \mathcal{G}_{1,hh} d\vec{x} \\
& + \rho_0^{-1} c_s^2 \int \varepsilon^{-2} \xi_{hh} \mathcal{G}_{2,hh} d\vec{x} + \int \mathcal{H}_{hh} \cdot \psi_{hh}^h d\vec{x} \\
& - \int \nabla_h (\varepsilon^{-2} \mathcal{R}_{hh}) \cdot \psi_{hh}^h d\vec{x} =: \sum_{i=7}^{12} I_i.
\end{aligned} \tag{3.22}$$

In the following, we shall estimate the terms on the right-hand side of (3.22).  $I_{11}$  can be written, after substituting (3.20), as

$$\begin{aligned}
I_{11} &= - \int \left( \zeta_{hh} \psi_t^h \cdot \psi_{hh}^h + 2\zeta_h \psi_{ht}^h \cdot \psi_{hh}^h \right) d\vec{x} - \int \left( (\rho v)_{hh} \cdot \nabla_h \psi^h \cdot \psi_{hh}^h \right. \\
& + 2(\rho v)_h \cdot \nabla_h \psi_h^h \cdot \psi_{hh}^h \Big) d\vec{x} - \int \left( (\rho w)_{hh} \partial_z \psi^h \cdot \psi_{hh}^h \right. \\
& + 2(\rho w)_h \partial_z \psi_h^h \cdot \psi_{hh}^h \Big) d\vec{x} =: I'_{11} + I''_{11} + I'''_{11}.
\end{aligned}$$

Straightforward estimates show the following inequalities:

$$\begin{aligned}
I'_{11} &\lesssim |\zeta_{hh}|_{L^2} \int_0^1 |\psi_t^h|_{L^4} |\psi_{hh}^h|_{L^4} dz + |\zeta_h|_{L^4} \int_0^1 |\psi_{ht}^h|_{L^2} |\psi_{hh}^h|_{L^4} dz \\
&\lesssim \|\varepsilon^{-1} \nabla \zeta\|_{H^1} (\|\varepsilon \nabla_h \psi_t^h\|_{L^2} + \|\varepsilon \psi_t^h\|_{L^2}) \\
&\quad \times (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \lesssim \delta \|\nabla_h \psi_{hh}^h\|_{L^2}^2 \\
&\quad + \delta \|\varepsilon \nabla_h \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 \\
&\quad + C_\delta (\mathfrak{G}_p + 1) \mathfrak{H}_p \|\psi^h\|_{H^2}^2, \\
I''_{11} &\lesssim (\|\zeta_{hh}\|_{L^2} \|v\|_{L^\infty} + \|\rho\|_{L^\infty} \|v_{hh}\|_{L^2} + \|\zeta_h\|_{L^3} \|v_h\|_{L^6}) \\
&\quad \times \|\nabla_h \psi^h\|_{L^6} \|\psi_{hh}^h\|_{L^3} + (\|\zeta_h\|_{L^6} \|v\|_{L^\infty} + \|\rho\|_{L^\infty} \|v_h\|_{L^6})
\end{aligned}$$

$$\begin{aligned}
& \times \|\nabla_h^2 \psi^h\|_{L^2} \|\psi_{hh}^h\|_{L^3} \lesssim (\|\nabla_h \zeta\|_{H^1} + 1) \|v\|_{H^2} \|\nabla \psi^h\|_{H^1} \\
& \times (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 \\
& + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

In order to estimate  $I_{11}'''$ , we first take  $\partial_h, \partial_{hh}$  in (1.11) to obtain the identities:

$$\begin{aligned}
w_h &= w_{p,h} - \int_0^z \left( \operatorname{div}_h \widetilde{\psi}_h^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h \right) dz \\
&= w_{p,h} + \psi_{hh}^z, \\
w_{hh} &= w_{p,hh} - \int_0^z \left( \operatorname{div}_h \widetilde{\psi}_{hh}^h + \widetilde{v}_{hh} \cdot \nabla_h \log \rho + 2\widetilde{v}_h \cdot \nabla_h (\log \rho)_h \right. \\
&\quad \left. + \widetilde{v} \cdot \nabla_h (\log \rho)_{hh} \right) dz = w_{p,hh} + \psi_{hh}^z.
\end{aligned} \tag{3.23}$$

Then  $I_{11}'''$  can be written as

$$\begin{aligned}
I_{11}''' &= - \int (\zeta_{hh} w_p + 2\zeta_h w_{p,h} + \rho w_{p,hh}) \partial_z \psi^h \cdot \psi_{hh}^h d\vec{x} \\
&\quad - 2 \int (\zeta_h w_p + \rho w_{p,h}) \partial_z \psi_h^h \cdot \psi_{hh}^h d\vec{x} + \int \left( \int_0^z \rho \widetilde{v} \cdot \nabla_h (\log \rho)_{hh} dz \right. \\
&\quad \times \left. (\partial_z \psi^h \cdot \psi_{hh}^h) \right) d\vec{x} + \int \left[ \int_0^z \left( \zeta_{hh} \operatorname{div}_h \widetilde{\psi}_h^h + \zeta_{hh} \widetilde{v} \cdot \nabla_h \log \rho \right. \right. \\
&\quad \left. \left. + 2\zeta_h \operatorname{div}_h \widetilde{\psi}_h^h + 2\zeta_h \widetilde{v}_h \cdot \nabla_h \log \rho + 2\zeta_h \widetilde{v} \cdot \nabla_h (\log \rho)_h + \rho \operatorname{div}_h \widetilde{\psi}_{hh}^h \right. \right. \\
&\quad \left. \left. + \rho \widetilde{v}_{hh} \cdot \nabla_h \log \rho + 2\rho \widetilde{v}_h \cdot \nabla_h (\log \rho)_h \right) dz \times (\partial_z \psi^h \cdot \psi_{hh}^h) \right] d\vec{x} \\
&\quad + 2 \int \left[ \int_0^z \left( \zeta_h \operatorname{div}_h \widetilde{\psi}_h^h + \zeta_h \widetilde{v} \cdot \nabla_h \log \rho + \rho \operatorname{div}_h \widetilde{\psi}_h^h + \rho \widetilde{v}_h \cdot \nabla_h \log \rho \right. \right. \\
&\quad \left. \left. + \rho \widetilde{v} \cdot \nabla_h (\log \rho)_h \right) dz \partial_z \times (\psi_h^h \cdot \psi_{hh}^h) \right] d\vec{x} =: \sum_{i=1}^5 I_{11,i}'''.
\end{aligned}$$

Similarly, after applying the Hölder, Sobolev embedding and Young inequalities, one has

$$\begin{aligned}
I_{11,1}''' + I_{11,2}''' &\lesssim (\|\zeta_{hh}\|_{L^2} \|w_p\|_{L^\infty} + \|\zeta_h\|_{L^6} \|w_{p,h}\|_{L^3} \\
&\quad + \|\rho\|_{L^\infty} \|w_{p,hh}\|_{L^2}) \|\partial_z \psi^h\|_{L^6} \|\psi_{hh}^h\|_{L^3} \\
&\quad + (\|\zeta_h\|_{L^6} \|w_p\|_{L^6} + \|\rho\|_{L^\infty} \|w_{p,h}\|_{L^3}) \|\partial_z \psi_h^h\|_{L^2} \|\psi_{hh}^h\|_{L^6} \\
&\lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

On the other hand, after applying integration by parts,  $I''_{11,3}$  is written as

$$\begin{aligned} I''_{11,3} &= - \int \left( \int_0^z (\log \rho)_{hh} (\rho \operatorname{div}_h \tilde{v} + \tilde{v} \cdot \nabla_h \zeta) dz \right) \times (\partial_z \psi^h \cdot \psi_{hh}^h) d\vec{x} \\ &\quad - \int \left( \int_0^z (\log \rho)_{hh} \rho \tilde{v} dz \right) \cdot \nabla_h (\partial_z \psi^h \cdot \psi_{hh}^h) d\vec{x}, \end{aligned}$$

which can be estimated as below:

$$\begin{aligned} I''_{11,3} &\lesssim \int_0^1 (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) (|\rho|_{L^\infty} |\nabla_h v|_{L^8} + |v|_{L^\infty} |\nabla_h \zeta|_{L^8}) dz \\ &\quad \times \int_0^1 |\partial_z \psi^h|_{L^8} |\psi_{hh}^h|_{L^4} dz + \int_0^1 (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) |\rho|_{L^\infty} |v|_{L^\infty} dz \\ &\quad \times \int_0^1 (|\partial_z \psi^h|_{L^\infty} |\nabla_h \psi_{hh}^h|_{L^2} + |\nabla_h \partial_z \psi^h|_{L^4} |\psi_{hh}^h|_{L^4}) dz \\ &\lesssim (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2) (\|\rho\|_{L^\infty} \|v\|_{H^2} + \|v\|_{H^2} \|\zeta\|_{H^2}) \\ &\quad \times \|\partial_z \psi^h\|_{H^1} (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \\ &+ (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2) \|\rho\|_{L^\infty} \|v\|_{H^2} (\|\partial_z \psi^h\|_{L^2}^{1/2} \\ &\quad \times \|\partial_z \psi^h\|_{H^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2} + (\|\nabla_h \partial_z \psi^h\|_{L^2}^{1/2} \|\nabla_h^2 \partial_z \psi^h\|_{L^2}^{1/2} \\ &\quad + \|\nabla_h \partial_z \psi^h\|_{L^2}) (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} \\ &\quad + \|\psi_{hh}^h\|_{L^2})) \lesssim \delta \|\nabla^3 \psi^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 \\ &\quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2. \end{aligned}$$

Applying the Hölder, Minkowski, Sobolev embedding and Young inequalities implies,

$$\begin{aligned} I''_{11,4} &\lesssim \int_0^1 \left( |\zeta_{hh}|_{L^2} |\nabla_h \psi^h|_{L^8} + |\zeta_{hh}|_{L^2} |v|_{L^\infty} |\nabla_h \zeta|_{L^8} + |\zeta_h|_{L^8} |\nabla_h \psi_h^h|_{L^2} \right. \\ &\quad + |\zeta_h|_{L^8} |v_h|_{L^4} |\nabla_h \zeta|_{L^4} + |\zeta_h|_{L^8} |v|_{L^\infty} (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2) \\ &\quad + |\rho|_{L^8} |\nabla_h \psi_{hh}^h|_{L^2} + |\rho|_{L^\infty} |v_{hh}|_2 |\nabla_h \zeta|_{L^8} + |\rho|_{L^\infty} |v_h|_{L^8} \\ &\quad \left. \times (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2) \right) dz \times \int_0^1 |\partial_z \psi^h|_{L^8} |\psi_{hh}^h|_{L^4} dz \\ &\lesssim \left( \|\nabla_h \zeta\|_{H^1} \|\nabla \psi^h\|_{H^1} + (\|\nabla_h \zeta\|_{H^1}^2 + \|\nabla_h \zeta\|_{H^1}^3) \|v\|_{H^2} \right. \\ &\quad \left. + \|\rho\|_{H^1} \|\nabla_h \psi_{hh}^h\|_{L^2} + \|\rho\|_{H^2} \|v\|_{H^2} (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2) \right) \end{aligned}$$

$$\begin{aligned}
& \times \|\partial_z \psi^h\|_{H^1} \times (\|\psi_{hh}^h\|_2^{1/2} \|\nabla_h \psi_{hh}^h\|_2^{1/2} + \|\psi_{hh}^h\|_2) \\
& \lesssim \delta \|\nabla \psi_{hh}^h\|_2^2 + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2, \\
I_{11,5}''' & \lesssim \int_0^1 \left( |\zeta_h|_{L^4} |\nabla_h \psi^h|_{L^4} + |\zeta_h|_{L^4} |v|_{L^\infty} |\nabla_h \zeta|_{L^4} + |\rho|_{L^\infty} |\nabla_h \psi_h^h|_{L^2} \right. \\
& \quad \left. + |\rho|_{L^\infty} |v_h|_{L^4} |\nabla_h \zeta|_{L^4} + |\rho|_{L^\infty} |v|_{L^\infty} (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2) \right) dz \\
& \quad \times \int_0^1 |\partial_z \psi_h^h|_{L^4} |\psi_{hh}^h|_{L^4} dz \lesssim \left( \|\zeta\|_{H^2} \|\psi^h\|_{H^2} + \|v\|_{H^2} \|\zeta\|_{H^2}^2 \right. \\
& \quad \left. + \|\rho\|_{L^\infty} \|\psi^h\|_{H^2} + \|\rho\|_{L^\infty} \|v\|_{H^2} \|\zeta\|_{H^2} + \|\rho\|_{L^\infty} \|v\|_{H^2} \right. \\
& \quad \left. \times (\|\zeta\|_{H^2} + \|\zeta\|_{H^2}^2) \right) (\|\partial_z \psi_h^h\|_{L^2}^{1/2} \|\nabla_h \partial_z \psi_h^h\|_{L^2}^{1/2} + \|\partial_z \psi_h^h\|_{L^2}) \\
& \quad \times (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \\
& \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

Therefore, we have established the following estimate

$$\begin{aligned}
I_{11} & \lesssim \delta \|\nabla^3 \psi^h\|_{L^2}^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 + \delta \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + \delta \|\varepsilon \psi_t^h\|_{L^2}^2 \\
& \quad + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned} \tag{3.24}$$

Other terms on the right-hand side of (3.22) can be estimated in the same manner. We will list them below. In order to estimate  $I_8$ , notice

$$\begin{aligned}
\mathcal{F}_{2,hh} & = -(\rho \psi^h)_{hh} \cdot \nabla_h v_p - (\rho \psi^z)_{hh} \partial_z v_p - 2(\rho \psi^h)_h \cdot \nabla_h v_{p,h} \\
& \quad - 2(\rho \psi^z)_h \partial_z v_{p,h} - \rho \psi^h \cdot \nabla_h v_{p,hh} - \rho \psi^z \partial_z v_{p,hh}.
\end{aligned}$$

Then  $I_8$  can be written as

$$\begin{aligned}
I_8 & = - \int ((\rho \psi^h)_{hh} \cdot \nabla_h v_p + 2(\rho \psi^h)_h \cdot \nabla_h v_{p,h} + \rho \psi^h \cdot \nabla_h v_{p,hh}) \cdot \psi_{hh}^h d\vec{x} \\
& \quad - \int ((\rho \psi^z)_{hh} \partial_z v_p + 2(\rho \psi^z)_h \partial_z v_{p,h} + \rho \psi^z \partial_z v_{p,hh}) \cdot \psi_{hh}^h d\vec{x} := I_8' + I_8'',
\end{aligned}$$

with the estimate

$$\begin{aligned}
I_8' & \lesssim \left( \|\zeta_{hh}\|_{L^2} \|\psi^h\|_{L^\infty} \|\nabla_h v_p\|_{L^6} + \|\zeta_h\|_{L^6} \|\psi_h^h\|_{L^3} \|\nabla_h v_p\|_{L^6} \right. \\
& \quad \left. + \|\rho\|_{L^\infty} \|\psi_{hh}^h\|_{L^2} \|\nabla_h v_p\|_{L^6} + \|\zeta_h\|_{L^6} \|\psi^h\|_{L^\infty} \|\nabla_h v_{p,h}\|_{L^2} \right. \\
& \quad \left. + \|\rho\|_{L^\infty} \|\psi_h^h\|_{L^6} \|\nabla_h v_{p,h}\|_{L^2} + \|\rho\|_{L^\infty} \|\psi^h\|_{L^6} \|\nabla_h v_{p,hh}\|_{L^2} \right)
\end{aligned}$$



$$\begin{aligned}
& \times \|\psi_{hh}^h\|_3 \lesssim (\|\nabla_h \zeta\|_{H^1} \|\psi^h\|_{H^2} \|v_p\|_{H^2} + \|\rho\|_{L^\infty} \|\psi^h\|_{H^2} \|v_p\|_{H^3}) \\
& \times (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \\
& \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

Next, after substituting (3.9) and (3.23) into the term  $I_8''$ , we have the following:

$$\begin{aligned}
I_8'' &= \int \left[ \int_0^z \left( \zeta_{hh} (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) + 2\zeta_h (\operatorname{div}_h \widetilde{\psi}_h^h \right. \right. \\
& \quad \left. \left. + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h \right) + \rho (\operatorname{div}_h \widetilde{\psi}_{hh}^h + \widetilde{v}_{hh} \cdot \nabla_h \log \rho \right. \\
& \quad \left. \left. + 2\widetilde{v}_h \cdot \nabla_h (\log \rho)_h \right) dz \times (\partial_z v_p \cdot \psi_{hh}^h) \right] d\vec{x} + \int \left[ \int_0^z \rho \widetilde{v} \cdot \nabla_h (\log \rho)_{hh} dz \right. \\
& \quad \left. \times (\partial_z v_p \cdot \psi_{hh}^h) \right] d\vec{x} + 2 \int \left[ \int_0^z \left( \zeta_h (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) + \rho (\operatorname{div}_h \widetilde{\psi}_h^h \right. \right. \\
& \quad \left. \left. + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h \right) dz \times (\partial_z v_{p,h} \cdot \psi_{hh}^h) \right] d\vec{x} \\
& \quad \left. + \int \left[ \int_0^z \rho (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) dz \times (\partial_z v_{p,hh} \cdot \psi_{hh}^h) \right] d\vec{x} \right. \\
& \quad \left. =: I_{8,1}'' + I_{8,2}'' + I_{8,3}'' + I_{8,4}'' \right.
\end{aligned}$$

Applying integration by parts in  $I_{8,2}''$  yields the following:

$$\begin{aligned}
I_{8,2}'' &= - \int \left[ \int_0^z \left( (\log \rho)_{hh} \rho \operatorname{div}_h \widetilde{v} + (\log \rho)_{hh} \widetilde{v} \cdot \nabla_h \rho \right) dz \right. \\
& \quad \left. \times (\partial_z v_p \cdot \psi_{hh}^h) \right] d\vec{x} - \int \int_0^z (\log \rho)_{hh} \rho \widetilde{v} dz \cdot \nabla_h (\partial_z v_p \cdot \psi_{hh}^h) d\vec{x} \\
& \lesssim \int_0^1 (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) (|\rho|_{L^\infty} |\nabla_h v|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) dz \\
& \quad \times \int_0^1 |\partial_z v_p|_{L^\infty} |\psi_{hh}^h|_{L^4} dz + \int_0^1 (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) |\rho|_{L^\infty} |v|_{L^\infty} dz \\
& \quad \times \int_0^1 \left( |\nabla_h \partial_z v_p|_{L^4} |\psi_{hh}^h|_{L^4} + |\partial_z v_p|_{L^\infty} |\nabla_h \psi_{hh}^h|_{L^2} \right) dz \\
& \lesssim (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2) \left( (\|\rho\|_{L^\infty} \|\nabla v\|_{H^1} + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1}) \right. \\
& \quad \left. \times \|\partial_z v_p\|_{L^\infty} + \|\rho\|_{L^\infty} \|v\|_{H^2} \|\nabla_h \partial_z v_p\|_{L^4} \right) \\
& \quad \times (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) + (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2)
\end{aligned}$$

$$\begin{aligned}
& \times \|\rho\|_{L^\infty} \|v\|_{H^2} \|\partial_z v_p\|_{L^\infty} \|\nabla_h \psi_{hh}^h\|_{L^2} \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 \\
& + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2).
\end{aligned}$$

The estimates of the rest of  $I_g''$  are listed below:

$$\begin{aligned}
I_{8,1}'' & \lesssim \int_0^1 \left( |\zeta_{hh}|_{L^2} (|\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) + |\zeta_h|_{L^4} (|\nabla_h \psi_{hh}^h|_{L^2} \right. \\
& + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} + |v|_{L^\infty} (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2)) + |\rho|_{L^4} |\nabla_h \psi_{hh}^h|_{L^2} \\
& \left. + |\rho|_{L^\infty} (|v_{hh}|_{L^2} |\nabla_h \zeta|_{L^4} + |v_h|_{L^4} (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2)) \right) dz \\
& \times \int_0^1 |\partial_z v_p|_{L^\infty} |\psi_{hh}^h|_{L^4} dz \lesssim \left( \|\nabla_h \zeta\|_{H^1} (\|\nabla \psi^h\|_{H^1} \right. \\
& + \|v\|_{H^2} (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2)) + \|\rho\|_{H^1} \|\nabla_h \psi_{hh}^h\|_{L^2} \\
& + \|\rho\|_{L^\infty} (\|\nabla v\|_{H^1} \|\nabla_h \zeta\|_{H^1} + \|\nabla v\|_{H^1} (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2)) \left. \right) \\
& \times \|\partial_z v_p\|_{L^\infty} (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 \\
& + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\|\psi^h\|_{H^2}^2 + \varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2), \\
I_{8,3}'' & \lesssim \int_0^1 \left( |\zeta_h|_{L^4} (|\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) + |\rho|_{L^\infty} (|\nabla_h^2 \psi^h|_{L^2} \right. \\
& \left. + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} + |v|_{L^\infty} (|\nabla_h^2 \zeta|_{L^2} + |\nabla_h \zeta|_{L^4}^2)) \right) dz \\
& \times \int_0^1 |\partial_z v_{p,h}|_{L^4} |\psi_{hh}^h|_{L^4} dz \lesssim \left( \|\nabla_h \zeta\|_{H^1} (\|\nabla \psi^h\|_{H^1} \right. \\
& + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1}) + \|\rho\|_{L^\infty} [\|\nabla \psi^h\|_{H^1} + \|v\|_{H^2} (\|\nabla_h \zeta\|_{H^1} \\
& \left. + \|\nabla_h \zeta\|_{H^1}^2)] \right) \|v_{p,hz}\|_{L^4} (\|\psi_{hh}^h\|_{L^2}^{1/2} \|\nabla_h \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \\
& \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\|\psi^h\|_{H^2}^2 \\
& + \varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2), \\
I_{8,4}'' & \lesssim \int_0^1 |\rho|_{L^\infty} (|\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) dz \times \int_0^1 |\partial_z v_{p,hh}|_{L^2} |\psi_{hh}^h|_{L^4} dz \\
& \lesssim \|\rho\|_{L^\infty} (\|\nabla \psi^h\|_{H^1} + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1}) \|v_{p,hhz}\|_{L^2} (\|\psi_{hh}^h\|_{L^2}^{1/2} \\
& \times \|\nabla \psi_{hh}^h\|_{L^2}^{1/2} + \|\psi_{hh}^h\|_{L^2}) \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 \\
& + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\|\psi^h\|_{H^2}^2 + \varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2).
\end{aligned}$$

Hence, we have got

$$\begin{aligned} I_8 &\lesssim \delta \|\nabla \psi_{hh}^h\|_2^2 + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \\ &\quad \times (\|\psi^h\|_{H^2}^2 + \varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1}\xi\|_{H^2}^2). \end{aligned} \quad (3.25)$$

Next, we establish the estimate of  $I_9$ . Notice, by making use of the facts that  $\xi_{hh}, \xi_h, \xi$  are independent of the  $z$ -variable, and that  $\int_0^1 \operatorname{div}_h v_{p,hh} dz = \int_0^1 \operatorname{div}_h v_{p,h} dz = \int_0^1 \operatorname{div}_h v_p dz = 0$ , thanks to (1.7), one can derive

$$\begin{aligned} I_9 &= -\rho_0^{-1} c_s^2 \int \varepsilon^{-2} \xi_{hh} \operatorname{div}_h (\xi_{hh} v + 2\xi_h v_h + \xi v_{hh}) d\vec{x} \\ &= -\rho_0^{-1} c_s^2 \int \left( \varepsilon^{-2} \xi_{hh} \left( \frac{1}{2} \xi_{hh} \operatorname{div}_h \psi^h + 2\xi_h \operatorname{div}_h \psi_h^h + 2v_h \cdot \nabla_h \xi_h \right. \right. \\ &\quad \left. \left. + \xi \operatorname{div}_h \psi_{hh}^h + v_{hh} \cdot \nabla_h \xi \right) d\vec{x} \lesssim |\varepsilon^{-1} \nabla_h \xi|_{H^1}^2 \int_0^1 \left( |\nabla_h \psi^h|_{L^\infty} \right. \right. \\ &\quad \left. \left. + |v_{p,h}|_{L^\infty} \right) dz + |\varepsilon^{-1} \xi_{hh}|_{L^2} \left( |\varepsilon^{-1} \xi_h|_{L^4} \int_0^1 (|\nabla_h^2 \psi^h|_{L^4} \right. \right. \\ &\quad \left. \left. + |v_{p,hh}|_{L^4}) dz + |\varepsilon^{-1} \xi|_{L^\infty} \int_0^1 |\nabla_h \psi_{hh}^h|_{L^2} dz \right) \lesssim \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 \\ &\quad \times (\|\nabla \psi^h\|_{H^2} + \|v_p\|_{H^3}) + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1} \|\varepsilon^{-1} \xi\|_{H^2} \\ &\quad \times \|\nabla_h \psi_{hh}^h\|_{L^2} \lesssim \delta \|\nabla^3 \psi^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 \\ &\quad + C_\delta Q(\mathcal{E}) \mathfrak{H}_p \|\varepsilon^{-1} \xi\|_{H^2}^2. \end{aligned} \quad (3.26)$$

Similarly,  $I_7, I_{10}, I_{12}$  can be estimated as follows:

$$\begin{aligned} I_7 &= \rho_0^{-1} \int \zeta_{hh} (\nabla_h (c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p) \cdot \psi_{hh}^h d\vec{x} \\ &\quad + 2\rho_0^{-1} \int \zeta_h (\nabla_h (c_s^2 \rho_{1,h}) - \mu \Delta_h v_{p,h} - \lambda \nabla_h \operatorname{div}_h v_{p,h} - \partial_{zz} v_{p,h}) \cdot \psi_{hh}^h d\vec{x} \\ &\quad - \rho_0^{-1} \int (\nabla_h (c_s^2 \rho_{1,h}) - \mu \Delta_h v_{p,h} - \lambda \nabla_h \operatorname{div}_h v_{p,h} - \partial_{zz} v_{p,h}) \cdot (\zeta \psi_{hh}^h)_h d\vec{x} \\ &\lesssim \|\zeta_{hh}\|_{L^2} \|\nabla_h (c_s^2 \rho_1) - \mu \Delta_h v_p - \lambda \nabla_h \operatorname{div}_h v_p - \partial_{zz} v_p\|_{L^6} \|\psi_{hh}^h\|_{L^3} \\ &\quad + \|\nabla_h (c_s^2 \rho_{1,h}) - \mu \Delta_h v_{p,h} - \lambda \nabla_h \operatorname{div}_h v_{p,h} - \partial_{zz} v_{p,h}\|_{L^2} \\ &\quad \times (\|\zeta_h\|_{L^6} \|\psi_{hh}^h\|_{L^3} + \|\zeta\|_{L^\infty} \|\psi_{hh}^h\|_{L^2}) \lesssim \delta \|\nabla \psi_{hh}^h\|_{L^2}^2 \\ &\quad + \delta \|\psi_{hh}^h\|_{L^2}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2), \end{aligned} \quad (3.27)$$

$$I_{10} = -\rho_0^{-1} c_s^2 \int \left( \xi_{hh} (\partial_t \rho_{1,hh} + \rho_{1,hh} \operatorname{div}_h \psi^h + 2\rho_{1,h} \operatorname{div}_h \psi_h^h + \rho_1 \operatorname{div}_h \psi_{hh}^h \right.$$

$$\begin{aligned}
& + v \cdot \nabla_h \rho_{1,hh} + 2v_h \cdot \nabla_h \rho_{1,h} + v_{hh} \cdot \nabla_h \rho_1) \Big) d\vec{x} \lesssim \|\nabla_h \xi\|_{H^1} \\
& \times \left( \|\partial_t \rho_{1,hh}\|_{L^2} + \|\rho_{1,hh}\|_{L^3} \|\nabla_h \psi^h\|_{L^6} + \|\rho_{1,h}\|_{L^\infty} \|\nabla_h \psi^h\|_{L^2} \right. \\
& + \|\rho_1\|_{L^\infty} \|\nabla_h \psi_{hh}^h\|_{L^2} + \|v\|_{H^2} \|\rho_1\|_{H^3} + \|v_h\|_{L^6} \|\nabla_h \rho_{1,h}\|_{L^3} \\
& \left. + \|v_{hh}\|_{L^2} \|\nabla_h \rho_1\|_{L^\infty} \right) \lesssim \delta \|\nabla_h \psi_{hh}^h\|_{L^2}^2 + \delta \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 \\
& + C_\delta (1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^2 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2 + \varepsilon^2 \|\psi^h\|_{H^2}^2), \tag{3.28}
\end{aligned}$$

$$\begin{aligned}
I_{12} & = \int \varepsilon^{-2} \mathcal{R}_{hh} \operatorname{div}_h \psi_{hh}^h d\vec{x} \lesssim \varepsilon^{-2} (\|\zeta_{hh}\|_{L^2} \|\zeta\|_{L^\infty} + \|\zeta_h\|_{L^4}^2) \\
& \times \|\nabla_h \psi_{hh}^h\|_{L^2} \lesssim \delta \|\nabla_h \psi_{hh}^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 \\
& + C_\delta (Q(\mathcal{E}) + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^4 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2). \tag{3.29}
\end{aligned}$$

Here we have applied (3.18) and that

$$|\mathcal{R}_{hh}| = |\gamma(\gamma-1) \int_{\rho_0}^\rho \rho_{hh} y^{\gamma-2} dy + \gamma(\gamma-1) \rho_h^2 \rho^{\gamma-2}| \lesssim |\zeta_{hh}| |\zeta| + |\zeta_h|^2. \tag{3.30}$$

Therefore, (3.21) follows from (3.22) and inequalities (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), above.  $\square$

Next, we will derive the required estimate of  $\xi_{hh}$ . After integrating (1.10)<sub>2</sub> over  $z \in (0, 1)$ , we have the following equation, thanks to (1.7):

$$\begin{aligned}
& \rho \partial_t \overline{\psi^h} + \int_0^1 (\rho v \cdot \nabla_h \psi^h - \rho w_z \psi^h) dz + \nabla_h (\varepsilon^{-2} c_s^2 \xi) \\
& = \mu \Delta_h \overline{\psi^h} + \lambda \nabla_h \operatorname{div}_h \overline{\psi^h} - \nabla_h (\varepsilon^{-2} \mathcal{R}) + \int_0^1 (\mathcal{F}_1 + \mathcal{F}_2) dz. \tag{3.31}
\end{aligned}$$

After applying  $\partial_h$  to (3.31), one has

$$\begin{aligned}
\varepsilon^{-2} c_s^2 \nabla_h \xi_h &= \underbrace{-\zeta_h \partial_t \overline{\psi^h} - \rho \partial_t \overline{\psi_h^h}}_{R_1} \\
&\quad - \underbrace{\int_0^1 \left( \zeta_h v \cdot \nabla_h \psi^h + \rho v_h \cdot \nabla_h \psi^h + \rho v \cdot \nabla_h \psi_h^h \right) dz}_{R_2} \\
&\quad + \underbrace{\int_0^1 \left( \zeta_h w_z \psi^h + \rho w_{hz} \psi^h + \rho w_z \psi_h^h \right) dz}_{R_3} \\
&\quad + \underbrace{\mu \Delta_h \overline{\psi^h} + \lambda \nabla_h \operatorname{div}_h \overline{\psi^h} - \nabla_h (\varepsilon^{-2} \mathcal{R}_h)}_{R_4} + \underbrace{\int_0^1 \left( \mathcal{F}_{1,h} + \mathcal{F}_{2,h} \right) dz}_{R_5}.
\end{aligned} \tag{3.32}$$

What we need is to estimate the  $L^2$ -norm of the terms on the right-hand side of (3.32). In fact, after applying the Minkowski, Hölder and Sobolev embedding inequalities, one has

$$\begin{aligned}
|R_1|_{L^2} &\lesssim \int_0^1 |\zeta_h|_{L^4} |\partial_t \psi^h|_{L^4} dz + \int_0^1 |\rho|_{L^\infty} |\partial_t \psi_h^h|_{L^2} dz \lesssim \|\zeta_h\|_{H^1} \\
&\quad \times (\|\partial_t \psi^h\|_{L^2}^{1/2} \|\nabla \partial_t \psi^h\|_{L^2}^{1/2} + \|\partial_t \psi^h\|_{L^2}) + \|\rho\|_{L^\infty} \|\partial_t \psi_h^h\|_{L^2} \\
&\lesssim \|\nabla \partial_t \psi^h\|_{L^2} + (\|\varepsilon^{-1} \xi\|_{H^2} + \mathfrak{G}_p \|\varepsilon \rho_1\|_{H^2}) \|\varepsilon \partial_t \psi^h\|_{L^2}, \\
|R_2|_{L^2} &\lesssim \int_0^1 \left( |\zeta_h|_{L^4} |v|_{L^\infty} |\nabla_h \psi^h|_{L^4} + |\rho|_{L^\infty} |v_h|_{L^4} |\nabla_h \psi^h|_{L^4} + |\rho|_{L^\infty} \right. \\
&\quad \left. \times |v|_{L^\infty} |\nabla_h \psi_h^h|_{L^2} \right) dz \lesssim (\|\zeta\|_{H^2} + \|\rho\|_{L^\infty}) \|v\|_{H^2} \|\nabla_h \psi^h\|_{H^1}, \\
|R_4|_{L^2} &\lesssim \int_0^1 |\nabla_h \psi^h|_{H^2} dz + \varepsilon^{-2} (|\zeta_{hh}|_{L^2} |\zeta|_{L^\infty} + |\zeta_h|_{L^4}^2) \lesssim \|\nabla_h \psi^h\|_{H^2} \\
&\quad + \varepsilon^2 \|\rho_1\|_{H^2}^2 + \|\varepsilon^{-1} \xi\|_{H^2} \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}.
\end{aligned}$$

On the other hand, after substituting (3.41) and (3.46), below, we have

$$\begin{aligned}
R_3 &= \int_0^1 \left( [w_{p,z} - (\operatorname{div}_h \widetilde{\psi^h} + \widetilde{v} \cdot \nabla_h \log \rho)] (\zeta_h \psi^h + \rho \psi_h^h) + \rho \psi^h [w_{p,hz} \right. \\
&\quad \left. - (\operatorname{div}_h \widetilde{\psi_h^h} + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h)] \right) dz.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
|R_3|_{L^2} &\lesssim \int_0^1 \left( (|w_{p,z}|_{L^4} + |\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) \right. \\
&\quad \times (|\zeta_h|_{L^4} |\psi^h|_{L^\infty} + |\rho|_{L^\infty} |\psi_h^h|_{L^4}) + |\rho|_{L^\infty} |\psi^h|_{L^\infty} \\
&\quad \times (|w_{p,hz}|_{L^2} + |\nabla_h \psi_h^h|_{L^2} + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} \\
&\quad \left. + |v|_{L^\infty} (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) \right) dz \\
&\lesssim Q(\mathcal{E}) (\|\nabla_h \psi^h\|_{H^1} + \varepsilon \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}) + (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p^{1/2}.
\end{aligned}$$

After substituting (1.9), we obtain

$$\begin{aligned}
R_5 &= \int_0^1 \left( \zeta_h Q_p + \zeta Q_{p,h} - \zeta_h \psi^h \cdot \nabla_h v_p - \rho \psi_h^h \cdot \nabla_h v_p - \rho \psi^h \cdot \nabla_h v_{p,h} \right. \\
&\quad \left. + \psi_z^z (\zeta_h v_p + \rho v_{p,h}) + \rho v_p \psi_{hz}^z \right) dz = \int_0^1 \left( \zeta_h Q_p + \zeta Q_{p,h} - \zeta_h \psi^h \cdot \nabla_h v_p \right. \\
&\quad \left. - \rho \psi_h^h \cdot \nabla_h v_p - \rho \psi^h \cdot \nabla_h v_{p,h} - (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) (\zeta_h v_p + \rho v_{p,h}) \right. \\
&\quad \left. - (\operatorname{div}_h \widetilde{\psi}_h^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h) \rho v_p \right) dz.
\end{aligned}$$

Hence, applying the Minkowski, Hölder and Sobolev embedding inequalities implies

$$\begin{aligned}
|R_5|_{L^2} &\lesssim \int_0^1 \left( |\zeta_h|_{L^4} |Q_p|_{L^4} + |\zeta|_{L^\infty} |Q_{p,h}|_{L^2} + |\zeta_h|_{L^4} |\psi^h|_{L^\infty} |\nabla_h v_p|_{L^4} \right. \\
&\quad + |\rho|_{L^\infty} |\psi_h^h|_{L^4} |\nabla_h v_p|_{L^4} + |\rho|_{L^\infty} |\psi^h|_{L^\infty} |\nabla_h v_{p,h}|_{L^2} \\
&\quad + (|\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) (|\zeta_h|_{L^4} |v_p|_{L^\infty} + |\rho|_{L^\infty} |v_{p,h}|_{L^4}) \\
&\quad + |\rho|_{L^\infty} |v_p|_{L^\infty} [|\nabla_h \psi_h^h|_{L^2} + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} \\
&\quad \left. + |v|_{L^\infty} (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) \right) dz \\
&\lesssim (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p^{1/2} (\|\psi^h\|_{H^2} + \varepsilon).
\end{aligned}$$

Summing up these estimates, we get the following inequality from (3.32),

$$\begin{aligned}
\|\varepsilon^{-1} \xi_{hh}\|_{L^2}^2 &\lesssim \varepsilon^2 |\varepsilon^{-2} \xi_{hh}|_{L^2}^2 \lesssim \sum_{i=1}^5 \varepsilon^2 |R_i|_{L^2}^2 \lesssim \|\varepsilon \nabla \partial_t \psi^h\|_{L^2}^2 \\
&\quad + \varepsilon^2 \|\nabla \psi^h\|_{H^2}^2 + \varepsilon^2 Q(\mathcal{E}) (\|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 + \|\nabla \psi^h\|_{H^1}^2) \\
&\quad + \|\varepsilon \partial_t \psi^h\|_{L^2}^2 + \varepsilon^2 (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p.
\end{aligned} \tag{3.33}$$

We summarize the result in the following:

**Lemma 4.** *Under the same assumptions as in Lemma 1, the following holds:*

$$\begin{aligned} \|\varepsilon^{-1}\nabla_h\xi\|_{H^1}^2 &\leq C\|\varepsilon\nabla\partial_t\psi^h\|_{L^2}^2 + \varepsilon^2\|\nabla\psi^h\|_{H^2}^2 \\ &+ \varepsilon^2CQ(\mathcal{E})(\|\varepsilon^{-1}\nabla_h\xi\|_{H^1}^2 + \|\nabla\psi^h\|_{H^1}^2 + \|\varepsilon\psi_t^h\|_{L^2}^2) \\ &+ C\varepsilon^2(Q(\mathcal{E}) + 1 + \mathfrak{G}_p)\mathfrak{H}_p, \end{aligned} \quad (3.34)$$

for some positive constant  $C$  independent of  $\varepsilon$ .

*Proof.* This is the direct consequence of (3.33) and the Poincaré inequality.  $\square$

### 3.3 Vertical derivatives estimates

Now we turn to the required estimates of vertical derivatives. To do so, we first apply  $\partial_z$  to system (1.10) and write down the resultant system as follows:

$$\begin{cases} \rho_0(\operatorname{div}_h\psi_z^h + \partial_z\psi_z^h) = \mathcal{G}_{1,z} + \mathcal{G}_{2,z} & \text{in } \Omega, \\ \rho\partial_t\psi_z^h + \rho v \cdot \nabla_h\psi_z^h + \rho w\partial_z\psi_z^h = \mu\Delta_h\psi_z^h + \lambda\nabla_h\operatorname{div}_h\psi_z^h \\ \quad + \partial_{zz}\psi_z^h + \mathcal{F}_{1,z} + \mathcal{F}_{2,z} + \mathcal{H}_z & \text{in } \Omega, \end{cases} \quad (3.35)$$

where

$$\mathcal{H}_z := -\rho v_z \cdot \nabla_h\psi^h - \rho w_z\partial_z\psi^h. \quad (3.36)$$

Then we apply  $\partial_z$  to system (3.35) again and obtain the following system:

$$\begin{cases} \rho_0(\operatorname{div}_h\psi_{zz}^h + \partial_z\psi_{zz}^h) = \mathcal{G}_{1,zz} + \mathcal{G}_{2,zz} & \text{in } \Omega, \\ \rho\partial_t\psi_{zz}^h + \rho v \cdot \nabla_h\psi_{zz}^h + \rho w\partial_z\psi_{zz}^h = \mu\Delta_h\psi_{zz}^h + \lambda\nabla_h\operatorname{div}_h\psi_{zz}^h \\ \quad + \partial_{zz}\psi_{zz}^h + \mathcal{F}_{1,zz} + \mathcal{F}_{2,zz} + \mathcal{H}_{zz} & \text{in } \Omega, \end{cases} \quad (3.37)$$

where

$$\mathcal{H}_{zz} := -\rho v_{zz} \cdot \nabla_h\psi^h - 2\rho v_z \cdot \nabla_h\psi_z^h - \rho w_{zz}\partial_z\psi^h - 2\rho w_z\partial_z\psi_z^h. \quad (3.38)$$

Notice, here we have employed the fact that  $\rho, \xi, \rho_1, \mathcal{R}$  are independent of the  $z$  variable. Also (3.37)<sub>2</sub> is a parabolic equation of  $\psi_{zz}^h$ . Now we perform standard  $L^2$  estimate on system (3.37).

**Lemma 5.** *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned} \frac{d}{dt} \|\rho^{1/2} \psi_{zz}^h\|_{L^2}^2 + C_{\mu,\lambda} \|\nabla \psi_{zz}^h\|_{L^2}^2 &\leq \delta \|\nabla \psi_{zz}^h\|_{L^2}^2 + \delta \|\psi_{zz}^h\|_{L^2}^2 \\ &+ C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi^h\|_{H^2}^2 + \varepsilon^2), \end{aligned} \quad (3.39)$$

for some positive constant  $C_{\mu,\lambda}$  independent of  $\varepsilon$ .

*Proof.* After taking the  $L^2$ -inner product of (3.37)<sub>2</sub> with  $\psi_{zz}^h$ , we have the following:

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_{zz}^h\|_{L^2}^2 \right\} + \mu \|\nabla_h \psi_{zz}^h\|_{L^2}^2 + \lambda \|\operatorname{div}_h \psi_{zz}^h\|_{L^2}^2 \\ + \|\partial_z \psi_{zz}^h\|_{L^2}^2 = \int \mathcal{F}_{1,zz} \cdot \psi_{zz}^h d\vec{x} + \int \mathcal{F}_{2,zz} \cdot \psi_{zz}^h d\vec{x} \\ + \int \mathcal{H}_{zz} \cdot \psi_{zz}^h d\vec{x} =: I_{13} + I_{14} + I_{15}. \end{aligned} \quad (3.40)$$

Again, we shall estimate the terms on the right-hand side of (3.40). We begin with the term  $I_{15}$ . Notice first, after taking  $\partial_z, \partial_{zz}$  to (1.11), we have the following identities:

$$\begin{aligned} w_z &= w_{p,z} - (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) = w_{p,z} + \psi_z^z, \\ w_{zz} &= w_{p,zz} - (\operatorname{div}_h \widetilde{\psi}_z^h + \widetilde{v}_z \cdot \nabla_h \log \rho) = w_{p,zz} + \psi_{zz}^z. \end{aligned} \quad (3.41)$$

Consequently, after substituting (3.41) in  $I_{15}$ , it can be estimated as follows:

$$\begin{aligned} I_{15} &= - \int \left( \rho (v_{zz} \cdot \nabla_h \psi^h + 2v_z \cdot \nabla_h \psi_z^h + w_{p,zz} \partial_z \psi^h \right. \\ &\quad - \operatorname{div}_h \widetilde{\psi}_z^h \partial_z \psi^h - \widetilde{v}_z \cdot \nabla_h \log \rho \partial_z \psi^h + 2w_{p,z} \partial_z \psi_z^h \\ &\quad \left. - 2 \operatorname{div}_h \widetilde{\psi}^h \partial_z \psi_z^h - 2\widetilde{v} \cdot \nabla_h \log \rho \partial_z \psi_z^h) \cdot \psi_{zz}^h \right) d\vec{x} \\ &\lesssim \|\rho\|_{L^\infty} \|\psi_{zz}^h\|_{L^3} \left( \|v_{zz}\|_{L^2} \|\nabla_h \psi^h\|_{L^6} + \|v_z\|_{L^6} \|\nabla_h \psi_z^h\|_{L^2} \right. \\ &\quad + \|w_{p,zz}\|_{L^2} \|\partial_z \psi^h\|_{L^6} + \|\nabla_h \psi_z^h\|_{L^2} \|\partial_z \psi^h\|_{L^6} \\ &\quad + \|\nabla_h \zeta\|_{L^3} \|v_z\|_{L^6} \|\partial_z \psi^h\|_{L^6} + \|w_{p,z}\|_{L^6} \|\partial_z \psi_z^h\|_{L^2} \\ &\quad \left. + \|\nabla_h \psi^h\|_{L^6} \|\partial_z \psi_z^h\|_{L^2} + \|\nabla_h \zeta\|_{L^6} \|v\|_{L^\infty} \|\partial_z \psi_z^h\|_{L^2} \right) \\ &\lesssim \|\rho\|_{L^\infty} (\|\psi_{zz}^h\|_{L^2}^{1/2} \|\nabla \psi_{zz}^h\|_{L^2}^{1/2} + \|\psi_{zz}^h\|_{L^2}) \end{aligned}$$



$$\begin{aligned}
& \times (\|v_p\|_{H^2} + \|\nabla\psi^h\|_{H^1} + \|\partial_z w_p\|_{H^1} + \|\nabla_h \zeta\|_{H^1} \|v\|_{H^2}) \\
& \times \|\nabla\psi^h\|_{H^1} \lesssim \delta \|\nabla\psi_{zz}^h\|_{L^2}^2 + \delta \|\psi_{zz}^h\|_{L^2}^2 \\
& + C_\delta Q(\mathcal{E}) \|\nabla\psi^h\|_{H^1}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p \|\psi^h\|_{H^2}^2.
\end{aligned}$$

The estimates of the terms  $I_{13}, I_{14}$  are listed below:

$$\begin{aligned}
I_{14} &= - \int \left( \rho(\psi_{zz}^h \cdot \nabla_h v_p + 2\psi_z^h \cdot \nabla_h v_{p,z} + \psi^h \cdot \nabla_h v_{p,zz} + \psi_{zz}^z \partial_z v_p \right. \\
& \quad \left. + 2\psi_z^z \partial_z v_{p,z} + \psi^z \partial_z v_{p,zz}) \cdot \psi_{zz}^h \right) d\vec{x} = - \int \left( \rho(\psi_{zz}^h \cdot \nabla_h v_p \right. \\
& \quad \left. + 2\psi_z^h \cdot \nabla_h v_{p,z} + \psi^h \cdot \nabla_h v_{p,zz} - \operatorname{div}_h \widetilde{\psi}_z^h \partial_z v_p - \widetilde{v}_z \cdot \nabla_h \log \rho \partial_z v_p \right. \\
& \quad \left. - 2\operatorname{div}_h \widetilde{\psi}^h \partial_z v_{p,z} - 2\widetilde{v} \cdot \nabla_h \log \rho \partial_z v_{p,z}) \cdot \psi_{zz}^h \right) d\vec{x} \\
& \quad + \int \left[ \int_0^z \left( \rho \operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \rho \right) dz \times (\partial_z v_{p,zz} \cdot \psi_{zz}^h) \right] d\vec{x} \\
& \lesssim \|\rho\|_{L^\infty} \|\psi_{zz}^h\|_{L^3} (\|\psi_{zz}^h\|_{L^2} \|\nabla_h v_p\|_{L^6} + \|\psi_z^h\|_{L^6} \|\nabla_h v_{p,z}\|_{L^2} \\
& \quad + \|\psi^h\|_{L^6} \|\nabla_h v_{p,zz}\|_{L^2} + \|\nabla_h \psi_z^h\|_{L^2} \|\partial_z v_p\|_{L^6} \\
& \quad + \|v_z\|_{L^6} \|\nabla_h \zeta\|_{L^3} \|\partial_z v_p\|_{L^6} + \|\nabla_h \psi^h\|_{L^6} \|v_{p,zz}\|_{L^2} \\
& \quad + \|v\|_{L^\infty} \|\nabla_h \zeta\|_{L^6} \|v_{p,zz}\|_{L^2}) + \int_0^1 \left( |\rho|_{L^\infty} |\nabla_h \psi^h|_{L^4} \right. \\
& \quad \left. + |v|_{L^\infty} |\nabla_h \zeta|_{L^4} \right) dz \times \int_0^1 |\partial_z v_{p,zz}|_{L^2} |\psi_{zz}^h|_{L^4} dz \\
& \lesssim (\|\psi_{zz}^h\|_{L^2}^{1/2} \|\nabla\psi_{zz}^h\|_{L^2}^{1/2} + \|\psi_{zz}^h\|_{L^2}) \|\rho\|_{L^\infty} (\|\nabla\psi^h\|_{H^1} \|v_p\|_{H^2} \\
& \quad + \|\psi^h\|_{H^1} \|v_p\|_{H^3} + \|\nabla_h \zeta\|_{H^1} \|v\|_{H^2} \|v_p\|_{H^2}) \\
& \quad + (\|\psi_{zz}^h\|_{L^2}^{1/2} \|\nabla\psi_{zz}^h\|_{L^2}^{1/2} + \|\psi_{zz}^h\|_{L^2}) (\|\rho\|_{L^\infty} \|\nabla_h \psi^h\|_{H^1} \\
& \quad + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1}) \|v_p\|_{H^3} \lesssim \delta \|\nabla\psi_{zz}^h\|_{L^2}^2 + \delta \|\psi_{zz}^h\|_{L^2}^2 \\
& \quad + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\|\psi^h\|_{H^2}^2 + \varepsilon^2 \|\varepsilon^{-1} \xi\|_{H^2}^2 + \varepsilon^4), \\
I_{13} &= \int \zeta Q_{p,zz} \cdot \psi_{zz}^h d\vec{x} = - \int \zeta Q_{p,z} \cdot \psi_{zzz}^h d\vec{x} \lesssim \delta \|\psi_{zzz}^h\|_{L^2}^2 \\
& \quad + C_\delta(Q(\mathcal{E}) + \mathfrak{G}_p) \mathfrak{H}_p \varepsilon^2.
\end{aligned}$$

After summing the estimates for  $I_{13}, I_{14}, I_{15}$ , above, and (3.40), we conclude (3.39).  $\square$

### 3.4 Mixed horizontal and vertical derivatives estimates

What is left is to estimate the  $L^2$  norm of  $\partial_{hz}\psi^h$ . We apply  $\partial_h$  to (3.35) and write down the resultant system:

$$\begin{cases} \rho_0(\operatorname{div}_h \psi_{hz}^h + \partial_z \psi_{hz}^h) = \mathcal{G}_{1,hz} + \mathcal{G}_{2,hz} & \text{in } \Omega, \\ \rho \partial_t \psi_{hz}^h + \rho v \cdot \nabla_h \psi_{hz}^h + \rho w \partial_z \psi_{hz}^h = \mu \Delta_h \psi_{hz}^h \\ \quad + \lambda \nabla_h \operatorname{div}_h \psi_{hz}^h + \partial_{zz} \psi_{hz}^h + \mathcal{F}_{1,hz} + \mathcal{F}_{2,hz} + \mathcal{H}_{hz} & \text{in } \Omega, \end{cases} \quad (3.42)$$

where

$$\begin{aligned} \mathcal{H}_{hz} := & -\zeta_h \partial_t \psi_z^h - \zeta_h v_z \cdot \nabla_h \psi^h - \rho(v_z \cdot \nabla_h \psi^h)_h - \zeta_h w_z \partial_z \psi^h \\ & - \rho(w_z \partial_z \psi^h)_h - \zeta_h v \cdot \nabla_h \psi_z^h - \rho v_h \cdot \nabla_h \psi_z^h - \zeta_h w \partial_z \psi_z^h \\ & - \rho w_h \partial_z \psi_z^h. \end{aligned} \quad (3.43)$$

**Lemma 6.** *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned} \frac{d}{dt} \|\rho^{1/2} \psi_{hz}^h\|_{L^2}^2 + C_{\mu,\lambda} \|\nabla \psi_{hz}^h\|_{L^2}^2 &\leq \delta (\|\nabla^3 \psi^h\|_{L^2}^2 + \|\psi_{hz}^h\|_{L^2}^2 \\ &+ \|\varepsilon \nabla \partial_t \psi^h\|_{L^2}^2) + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 \\ &+ C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\varepsilon^2 + \|\psi^h\|_{H^2}^2), \end{aligned} \quad (3.44)$$

for some positive constant  $C_{\mu,\lambda}$ , which is independent of  $\varepsilon$ .

*Proof.* Take the  $L^2$  inner produce of (3.42)<sub>2</sub> with  $\psi_{hz}^h$ . It follows,

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_{hz}^h\|_{L^2}^2 \right\} + \mu \|\nabla_h \psi_{hz}^h\|_{L^2}^2 + \lambda \|\operatorname{div}_h \psi_{hz}^h\|_{L^2}^2 \\ + \|\partial_z \psi_{hz}^h\|_{L^2}^2 = \int \mathcal{F}_{1,hz} \cdot \psi_{hz}^h d\vec{x} + \int \mathcal{F}_{2,hz} \cdot \psi_{hz}^h d\vec{x} \\ + \int \mathcal{H}_{hz} \cdot \psi_{hz}^h d\vec{x} =: I_{16} + I_{17} + I_{18}. \end{aligned} \quad (3.45)$$

As before, we shall estimate the terms on the right-hand side of (3.45). Notice that, we have the following identity after taking  $\partial_h$  to (3.41)<sub>1</sub>:

$$w_{hz} = w_{p,hz} - (\operatorname{div}_h \widetilde{\psi}_h^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h) = w_{p,hz} + \psi_{hz}^z. \quad (3.46)$$

Therefore, after substituting (3.43) and (3.46) into  $I_{18}$ , we have

$$I_{18} = - \int \left( \zeta_h \partial_t \psi_z^h + \zeta_h v_z \cdot \nabla_h \psi^h + \rho v_{hz} \cdot \nabla_h \psi^h + \rho v_z \cdot \nabla_h \psi_h^h \right)$$

$$\begin{aligned}
& + \zeta_h w_{p,z} \partial_z \psi^h + \rho w_{p,hz} \partial_z \psi^h + \rho w_{p,z} \partial_z \psi_h^h - (\operatorname{div}_h \widetilde{\psi}^h \\
& + \widetilde{v} \cdot \nabla_h \log \rho) (\zeta_h \partial_z \psi^h + \rho \partial_z \psi_h^h) - \rho (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v}_h \cdot \nabla_h \log \rho \\
& + \widetilde{v} \cdot \nabla_h (\log \rho)_h) \partial_z \psi^h + \zeta_h v \cdot \nabla_h \psi_z^h + \rho v_h \cdot \nabla_h \psi_z^h + \zeta_h w_p \partial_z \psi_z^h \\
& + \rho w_{p,h} \partial_z \psi_z^h) \cdot \psi_{hz}^h d\vec{x} + \int \left[ \int_0^z \zeta_h (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v} \cdot \nabla_h \log \rho) dz \right. \\
& \times (\partial_z \psi_z^h \cdot \psi_{hz}^h) + \int_0^z \rho (\operatorname{div}_h \widetilde{\psi}^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h) dz \\
& \left. \times (\partial_z \psi_z^h \cdot \psi_{hz}^h) \right] d\vec{x} \lesssim \|\psi_{hz}^h\|_{L^3} \times \left( \|\varepsilon^{-1} \zeta_h\|_{L^6} \|\varepsilon \partial_t \psi_z^h\|_{L^2} \right. \\
& + \|\zeta_h\|_{L^6} \|v_z\|_{L^4} \|\nabla_h \psi^h\|_{L^4} + \|\rho\|_{L^\infty} \|v_{hz}\|_{L^2} \|\nabla_h \psi^h\|_{L^6} \\
& + \|\rho\|_{L^\infty} \|v_z\|_{L^6} \|\nabla_h \psi_h^h\|_{L^2} + \|\zeta_h\|_{L^6} \|w_{p,z}\|_{L^3} \|\partial_z \psi^h\|_{L^6} \\
& + \|\rho\|_{L^\infty} \|w_{p,hz}\|_{L^2} \|\partial_z \psi^h\|_{L^6} + \|\rho\|_{L^\infty} \|w_{p,z}\|_{L^6} \|\partial_z \psi_h^h\|_{L^2} \\
& + (\|\nabla_h \psi^h\|_{L^6} + \|v\|_{L^\infty} \|\nabla_h \zeta\|_{L^6}) \times (\|\zeta_h\|_{L^6} \|\partial_z \psi^h\|_{L^3} \\
& + \|\rho\|_{L^\infty} \|\partial_z \psi_h^h\|_{L^2}) + \|\rho\|_{L^\infty} \|\partial_z \psi^h\|_{L^6} \times (\|\nabla_h \psi_h^h\|_{L^2} \\
& + \|v_h\|_{L^3} \|\nabla_h \zeta\|_{L^6} + \|v\|_{L^\infty} (\|\zeta_{hh}\|_{L^2} + \|\zeta_h\|_{L^4}^2)) \\
& + \|\zeta_h\|_{L^6} \|v\|_{L^\infty} \|\nabla_h \psi_z^h\|_{L^2} + \|\rho\|_{L^\infty} \|v_h\|_{L^6} \|\nabla_h \psi_z^h\|_{L^2} \\
& + \|\zeta_h\|_{L^6} \|w_p\|_{L^3} \|\partial_z \psi_z^h\|_{L^6} + \|\rho\|_{L^\infty} \|w_{p,h}\|_{L^2} \|\partial_z \psi_z^h\|_{L^6} \Big) \\
& + \int_0^1 \left( |\zeta_h|_{L^4} (|\nabla_h \psi^h|_{L^4} + |v|_{L^\infty} |\nabla_h \zeta|_{L^4}) + |\rho|_{L^\infty} (|\nabla_h \psi_h^h|_{L^2} \right. \\
& + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} + |v|_{L^\infty} (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2)) \Big) dz \\
& \times \int_0^1 |\partial_z \psi_z^h|_{L^4} |\psi_{hz}^h|_{L^4} dz \lesssim \left( \|\psi_{hz}^h\|_{L^2}^{1/2} \|\nabla \psi_{hz}^h\|_{L^2}^{1/2} + \|\psi_{hz}^h\|_{L^2} \right) \\
& \times \left( \|\varepsilon^{-1} \nabla_h \zeta\|_{H^1} \|\varepsilon \partial_t \psi_z^h\|_{L^2} + (\|\zeta\|_{H^2} + \|\zeta\|_{H^2}^2 + \|\rho\|_{L^\infty} \right. \\
& + \|\rho\|_{L^\infty} \|\zeta\|_{H^2}^2 + \|\rho\|_{L^\infty} \|\zeta\|_{H^2}) (\|\psi^h\|_{H^2} + \|v\|_{H^2} \\
& + \|w_{p,z}\|_{H^1}) \|\nabla \psi^h\|_{H^1} + (\|\rho\|_{L^\infty} \|\nabla_h \zeta\|_{H^1}^2 + \|\rho\|_{L^\infty} \|\nabla_h \zeta\|_{H^1} \\
& + \|\nabla_h \zeta\|_{H^1}^2 + \|\nabla_h \zeta\|_{H^1} + \|\rho\|_{L^\infty}) (\|w_p\|_{H^1} + \|\nabla \psi^h\|_{H^1} \\
& \left. + \|v\|_{H^2}) (\|\nabla^2 \psi^h\|_{L^2} + \|\nabla^3 \psi^h\|_{L^2}) \right) \\
& \lesssim \delta \|\nabla^3 \psi^h\|_{L^2}^2 + \delta \|\varepsilon \nabla \partial_t \psi^h\|_{L^2}^2 + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2
\end{aligned}$$

$$+ C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p)\mathfrak{H}_p \|\psi^h\|_{H^2}^2.$$

Similarly, after substituting (1.9), (3.23) and (3.41), we have

$$\begin{aligned}
I_{17} &= - \int \left[ \left( \zeta_h \psi_z^h \cdot \nabla_h v_p + \zeta_h \psi^h \cdot \nabla_h v_{p,z} - \zeta_h (\operatorname{div}_h \widetilde{\psi}^h \right. \right. \\
&\quad + \widetilde{v} \cdot \nabla_h \log \rho) \partial_z v_p + \rho \psi_{hz}^h \cdot \nabla_h v_p + \rho \psi_h^h \cdot \nabla_h v_{p,z} \\
&\quad - \rho (\operatorname{div}_h \widetilde{\psi}_h^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h) \partial_z v_p \\
&\quad + \rho \psi_z^h \cdot \nabla_h v_{p,h} + \rho \psi^h \cdot \nabla_h v_{p,hz} - \rho (\operatorname{div}_h \widetilde{\psi}^h \\
&\quad + \widetilde{v} \cdot \nabla_h \log \rho) \partial_z v_{p,h} \left. \right) \cdot \psi_{hz}^h \Big] d\vec{x} + \int \left[ \int_0^z \left( \operatorname{div}_h \widetilde{\psi}^h \right. \right. \\
&\quad + \widetilde{v} \cdot \nabla_h \log \rho \left. \right) dz \times (\zeta_h \partial_z v_{p,z} \cdot \psi_{hz}^h + \rho \partial_z v_{p,hz} \cdot \psi_{hz}^h) \\
&\quad + \int_0^z \left( \operatorname{div}_h \widetilde{\psi}_h^h + \widetilde{v}_h \cdot \nabla_h \log \rho + \widetilde{v} \cdot \nabla_h (\log \rho)_h \right) dz \\
&\quad \times (\rho \partial_z v_{p,z} \cdot \psi_{hz}^h) \Big] d\vec{x} \lesssim \left( \|\nabla_h \zeta\|_{H^1} (\|\psi^h\|_{H^2} \right. \\
&\quad + \|v\|_{H^2} \|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}) + \|\rho\|_{L^\infty} (\|\psi^h\|_{H^2} \\
&\quad + \|v\|_{H^2} (\|\nabla_h \zeta\|_{H^1} + \|\nabla_h \zeta\|_{H^1}^2)) \Big) \|v_p\|_{H^3} \\
&\quad \times \left( \|\psi_{hz}^h\|_{L^2}^{1/2} \|\nabla \psi_{hz}^h\|_{L^2}^{1/2} + \|\psi_{hz}^h\|_{L^2} \right) + \int_0^1 \left( |\nabla_h \psi^h|_{L^4} \right. \\
&\quad + \varepsilon |v|_{L^\infty} |\varepsilon^{-1} \nabla_h \zeta|_{L^4} \Big) dz \times \int_0^1 \left( |\zeta_h|_{L^4} |\partial_z v_{p,z}|_{L^4} |\psi_{hz}^h|_{L^4} \right. \\
&\quad + |\rho|_{L^\infty} |\partial_z v_{p,hz}|_{L^2} |\psi_{hz}^h|_{L^4} \Big) dz + \int_0^1 \left( |\nabla_h \psi_h^h|_{L^2} + |v_h|_{L^4} |\nabla_h \zeta|_{L^4} \right. \\
&\quad + |v|_{L^\infty} (|\zeta_{hh}|_{L^2} + |\zeta_h|_{L^4}^2) \Big) dz \cdot \int_0^1 |\rho|_{L^\infty} |\partial_z v_{p,z}|_{L^4} |\psi_{hz}^h|_{L^4} dz \\
&\lesssim \delta \|\nabla \psi_{hz}^h\|_{L^2}^2 + \delta \|\psi_{hz}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + 1 + \mathfrak{G}_p)\mathfrak{H}_p \\
&\quad \times (\|\psi^h\|_{H^2}^2 + \varepsilon^2). \\
I_{16} &= \int (\zeta_h Q_{p,z} \cdot \psi_{hz}^h + \zeta Q_{p,hz} \cdot \psi_{hz}^h) d\vec{x} = - \int (\zeta_h Q_p \cdot \psi_{hzz}^h \\
&\quad + \zeta Q_{p,h} \cdot \psi_{hzz}^h) d\vec{x} \lesssim \delta \|\psi_{hzz}^h\|_{L^2}^2 + C_\delta(Q(\mathcal{E}) + \mathfrak{G}_p)\mathfrak{H}_p \varepsilon^2.
\end{aligned}$$

Therefore after summing up the estimates of  $I_{16}, I_{17}, I_{18}$ , above, and (3.45), we conclude the proof of (3.44).  $\square$

### 3.5 Proof of Proposition 1

After applying similar arguments as in the proofs of Lemma 3, Lemma 5 and Lemma 6, one can easily check the following inequalities hold:

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\varepsilon^{-1} \xi\|_{L^2}^2 \right\} + C_{\mu, \delta} \|\nabla \psi^h\|_{L^2}^2 \\
& \leq \delta (\|\nabla \psi^h\|_{L^2}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2) + C_\delta Q(\mathcal{E}) (\|\nabla \psi^h\|_{H^1}^2 \\
& \quad + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2) + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^2 + \|\psi^h\|_{H^2}^2); \\
& \frac{d}{dt} \left\{ \frac{1}{2} \|\rho^{1/2} \psi_h^h\|_{L^2}^2 + \frac{c_s^2}{2\rho_0} \|\varepsilon^{-1} \xi_h\|_{L^2}^2 \right\} + C_{\mu, \lambda} \|\nabla \psi_h^h\|_{L^2}^2 \\
& \leq \delta (\|\nabla^2 \psi^h\|_{L^2}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 + \|\varepsilon \psi_t^h\|_{L^2}^2) \\
& \quad + C_\delta Q(\mathcal{E}) (\|\nabla \psi^h\|_{H^1}^2 + \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2) \\
& \quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^2 + \|\psi^h\|_{H^2}^2); \\
& \frac{d}{dt} \|\rho^{1/2} \psi_z^h\|_{L^2}^2 + C_{\mu, \lambda} \|\nabla \psi_z^h\|_{L^2}^2 \leq \delta (\|\nabla^2 \psi^h\|_{L^2}^2 + \|\psi_z^h\|_{L^2}^2 \\
& \quad + \|\varepsilon \partial_t \psi^h\|_{L^2}^2) + C_\delta Q(\mathcal{E}) \|\nabla \psi^h\|_{H^1}^2 \\
& \quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^2 + \|\psi^h\|_{H^2}^2).
\end{aligned}$$

Therefore, the above inequalities, together with (3.7), (3.16), (3.17), (3.21), (3.34), (3.39) and (3.44), imply that there exist positive constants  $c_i$ ,  $i \in \{1, 2, \dots, 10\}$ , such that

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}_{LM} + \mathcal{D}_{LM} & \leq (\delta + \varepsilon^2 C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) + (C_\delta + \varepsilon^2) Q(\mathcal{E})) \mathcal{D} \\
& \quad + C_\delta (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p (\varepsilon^2 + \mathcal{E}),
\end{aligned} \tag{3.47}$$

where we denote by

$$\begin{aligned}
\mathcal{E}_{LM} = \mathcal{E}_{LM}(t) & := \frac{c_1}{2} \|\rho^{1/2} \varepsilon \psi_t^h\|_{L^2}^2 + \frac{c_1 c_s^2}{2\rho_0} \|\xi_t\|_{L^2}^2 + \frac{c_2}{2} \|\rho^{1/2} \nabla_h^2 \psi^h\|_{L^2}^2 \\
& \quad + \frac{c_2 c_s^2}{2\rho_0} \|\varepsilon^{-1} \nabla_h^2 \xi\|_{L^2}^2 + c_3 \|\rho^{1/2} \psi_{zz}^h\|_{L^2}^2 + c_4 \|\rho^{1/2} \nabla_h \psi_z^h\|_{L^2}^2 \\
& \quad + \frac{c_5}{2} \|\rho^{1/2} \nabla_h \psi^h\|_{L^2}^2 + \frac{c_5 c_s^2}{2\rho_0} \|\varepsilon^{-1} \nabla_h \xi\|_{L^2}^2 + c_6 \|\rho^{1/2} \psi_z^h\|_{L^2}^2 \\
& \quad + \frac{c_7}{2} \|\rho^{1/2} \psi^h\|_{L^2}^2 + \frac{c_7 c_s^2}{2\rho_0} \|\varepsilon^{-1} \xi\|_{L^2}^2,
\end{aligned} \tag{3.48}$$

$$\mathcal{D}_{LM} = \mathcal{D}_{LM}(t) := c_1 \|\varepsilon \nabla \psi_t^h\|_{L^2}^2 + c_2 \|\nabla \nabla_h^2 \psi^h\|_{L^2}^2 + c_3 \|\nabla \psi_{zz}^h\|_{L^2}^2$$

$$\begin{aligned}
& + c_4 \|\nabla \nabla_h \psi_z^h\|_{L^2}^2 + c_5 \|\nabla \nabla_h \psi^h\|_{L^2}^2 + c_6 \|\nabla \psi_z^h\|_{L^2}^2 \\
& + c_7 \|\nabla \psi^h\|_{L^2}^2 + c_8 \|\xi_t\|_{L^2}^2 + c_9 \|\varepsilon^{-1} \nabla_h \xi\|_{H^1}^2 + C_{10} \|\varepsilon \rho \psi_t^h\|_{L^2}^2. \quad (3.49)
\end{aligned}$$

Under the assumption (3.2), it is easy to check that

$$\mathcal{E} \lesssim \mathcal{E}_{LM} \lesssim \mathcal{E}, \quad \mathcal{D} \lesssim \mathcal{D}_{LM} \lesssim \mathcal{D}, \quad (3.50)$$

where  $\mathcal{E} = \mathcal{E}(t)$  and  $\mathcal{D} = \mathcal{D}(t)$  are defined in (2.2) and (2.3), respectively. Therefore (3.47) can be written, after choosing  $\delta$  small enough, as

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}_{LM} + \mathcal{D}_{LM} & \leq (\varepsilon^2 C(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) + (1 + \varepsilon^2) Q(\mathcal{E})) \mathcal{D}_{LM} \\
& + C(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p(\varepsilon^2 + \mathcal{E}_{LM}). \quad (3.51)
\end{aligned}$$

Then after applying Grönwall's inequality to (3.51), one concludes that,

$$\begin{aligned}
\sup_{0 \leq t \leq T} \mathcal{E}_{LM}(t) + \int_0^T \mathcal{D}_{LM}(t) dt & \lesssim e^{C \int_0^T (Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \mathfrak{H}_p dt} \\
& \times \left\{ \varepsilon^2 + \mathcal{E}_{LM}(0) + \int_0^T \left[ \varepsilon^2 C(Q(\mathcal{E}) + 1 + \mathfrak{G}_p) \right. \right. \\
& \left. \left. + (1 + \varepsilon^2) Q(\mathcal{E}) \right] \mathcal{D}_{LM} dt \right\}.
\end{aligned}$$

Under the assumptions of Proposition 1, this completes the proof of (3.3).

## 4 Low Mach number limit

In this section, we will establish the asymptotic behavior of  $(\xi, \psi^h) = (\xi^\varepsilon, \psi^{\varepsilon, h})$  as  $\varepsilon \rightarrow 0^+$ . In particular, we prove Theorem 1.2 in this section.

First, as a consequence of Theorem 1.1, we have the following:

**Corollary 1.** *Under the same assumptions of Theorem 1.1, consider any integer  $s \geq 3$ . Then (3.1) holds true for  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$ , with the compatibility conditions as in (1.17).*

*Proof.* Directly from the conclusion of Theorem 1.1, for  $s \geq 3$  and  $v_{p,in} \in H^s(\Omega)$  as stated in the theorem, one has

$$\begin{aligned}
\sup_{0 \leq t < \infty} (\|v_p(t)\|_{H^3} + \|v_{p,t}(t)\|_{L^2}) & + \int_0^\infty \left( \|v_p(t)\|_{H^3}^2 + \|v_{p,t}(t)\|_{H^1}^2 \right. \\
& \left. + \|v_p(t)\|_{H^2} \right) dt \leq C_{p,in,3} + 1,
\end{aligned}$$

where we have used the Gagliardo-Nirenberg interpolation inequality:

$$\begin{aligned} \int_0^\infty \|v_p(t)\|_{H^2} dt &\lesssim \int_0^\infty \|v_p(t)\|_{L^2}^{1/3} \|v_p(t)\|_{H^3}^{2/3} dt \\ &\lesssim \int_0^\infty e^{-\frac{\varepsilon}{3}t} dt \times C_{p,in,3}^{1/2} \leq C_{p,in,3} + 1. \end{aligned}$$

Moreover, applying the Minkowski and Hölder inequalities to the expression of  $w_p$  as in (1.13) yields

$$\begin{aligned} \sup_{0 \leq t < \infty} \|w_p(t)\|_{H^1} &\leq \sup_{0 \leq t < \infty} \|v_p(t)\|_{H^2} \leq C_{p,in,2} + 1, \\ \int_0^\infty \|w_p(t)\|_{H^2}^2 dt &\leq \int_0^\infty \|v_p(t)\|_{H^3}^2 dt \leq C_{p,in,2}. \end{aligned}$$

What is left is to estimate

$$\|\rho_1\|_{H^2}, \|\rho_{1,t}\|_{L^2}, \int_0^\infty (\|\rho_{1,tt}\|_{L^2}^2 + \|\rho_{1,t}\|_{H^1}^2 + \|\rho_1\|_{H^2}^2) dt.$$

To do so, we write down the elliptic problem for  $\rho_1$ , which is obtained by taking average over the  $z$ -variable and then taking  $\operatorname{div}_h$  in (1.2)<sub>2</sub>, below:

$$-c_s^2 \Delta_h \rho_1 = \rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v_p \otimes v_p)) dz \quad \text{in } \Omega_h, \quad \int_{\Omega_h} \rho_1 dx dy = 0. \quad (4.1)$$

Then the  $L^p$  estimate of the Riesz transform implies, together with the Minkowski, Hölder and Sobolev embedding inequalities,

$$\|\rho_1\|_{H^2} \leq |\rho_1|_{H^2} \lesssim \| |v_p|^2 \|_{H^2} \lesssim \|v_p\|_{H^2}^2.$$

Consequently, for  $s \geq 2$ ,

$$\begin{aligned} \sup_{0 \leq t < \infty} \|\rho_1(t)\|_{H^2} + \int_0^\infty \|\rho_1(t)\|_{H^2}^2 dt &\lesssim \sup_{0 \leq t < \infty} \|v_p(t)\|_{H^2}^2 \\ + \sup_{0 \leq t < \infty} \|v_p(t)\|_{H^2}^2 \times \int_0^\infty \|v_p(t)\|_{H^2}^2 dt &\leq (1 + C_{p,in,1}) C_{p,in,2}. \end{aligned} \quad (4.2)$$

Furthermore, after taking time derivatives of (4.1), we have the following elliptic problems:

$$\begin{cases} -c_s^2 \Delta_h \rho_{1,t} = 2\rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v_p \otimes v_{p,t})) dz & \text{in } \Omega_h, \\ \int_{\Omega_h} \rho_{1,t} dx dy = 0; \end{cases} \quad (4.3)$$

$$\begin{cases} -c_s^2 \Delta_h \rho_{1,tt} = 2\rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v_p \otimes v_{p,tt} + v_{p,t} \otimes v_{p,t})) dz \\ \text{in } \Omega_h, \quad \int_{\Omega_h} \rho_{1,tt} dx dy = 0. \end{cases} \quad (4.4)$$

Thus similarly, one has, for  $s \geq 3$ ,

$$\begin{aligned} & \|\rho_{1,t}\|_{L^2} \lesssim \| |v_p| |v_{p,t}| \|_{L^2} \lesssim \|v_p\|_{H^2} \|v_{p,t}\|_{L^2} \leq C_{p,in,2}, \\ & \int_0^\infty \|\rho_{1,t}\|_{H^1}^2 dt \lesssim \int_0^\infty (\| |\nabla_h v_p| |v_{p,t}| \|_{L^2}^2 + \| |v_p| |\nabla_h v_{p,t}| \|_{L^2}^2) dt \\ & \lesssim \int_0^\infty \|v_p\|_{H^2}^2 \|v_{p,t}\|_{H^1}^2 dt \lesssim \sup_{0 \leq t < \infty} \|v_p\|_{H^2}^2 \\ & \quad \times \int_0^\infty \|v_{p,t}\|_{H^1}^2 dt \leq C_{p,in,2}^2, \\ & \int_0^\infty \|\rho_{1,tt}\|_{L^2}^2 \lesssim \int_0^\infty (\| |v_p| |v_{p,tt}| \|_{L^2}^2 + \| |v_{p,t}|^2 \|_{L^2}^2) dt \\ & \lesssim \int_0^\infty (\|v_p\|_{H^2}^2 \|v_{p,tt}\|_{L^2}^2 + \|v_{p,t}\|_{H^1}^4) dt \\ & \lesssim \sup_{0 \leq t < \infty} \|v_p\|_{H^2}^2 \times \int_0^\infty \|v_{p,tt}\|_{L^2}^2 dt + \sup_{0 \leq t < \infty} \|v_{p,t}\|_{H^1}^2 \\ & \quad \times \int_0^\infty \|v_{p,t}\|_{H^1}^2 \leq C_{p,in,2} \int_0^\infty \|v_{p,tt}\|_{L^2}^2 dt + C_{p,in,3}^2. \end{aligned} \quad (4.5)$$

On the other hand, after taking time derivative of (1.2)<sub>2</sub>, we have the identity,

$$\begin{aligned} \rho_0 v_{p,tt} &= -\rho_0 (v_p \cdot \nabla_h v_p + w_p \partial_z v_p)_t - c_s^2 \nabla_h \rho_{1,t} \\ & \quad + \mu \Delta_h v_{p,t} + \lambda \nabla_h \operatorname{div}_h v_{p,t} + \partial_{zz} v_{p,t}. \end{aligned}$$

Therefore, directly one has,

$$\begin{aligned} & \|v_{p,tt}\|_{L^2} \lesssim \|\rho_{1,t}\|_{H^1} + \|v_{p,t}\|_{H^2} + \|v_p \cdot \nabla_h v_{p,t}\|_{L^2} + \|v_{p,t} \cdot \nabla_h v_p\|_{L^2} \\ & \quad + \|w_p \partial_z v_{p,t}\|_{L^2} + \|w_{p,t} \partial_z v_p\|_{L^2} \lesssim \|\rho_{1,t}\|_{H^1} + \|v_{p,t}\|_{H^2} \\ & \quad + \|v_p\|_{H^2} \|v_{p,t}\|_{H^1} + \|w_p\|_{H^2} \|v_{p,t}\|_{H^1} + \|w_{p,t}\|_{L^2} \|v_p\|_{H^3} \\ & \lesssim \|\rho_{1,t}\|_{H^1} + \|v_{p,t}\|_{H^2} + \|v_p\|_{H^3} \|v_{p,t}\|_{H^1}, \end{aligned}$$

where we have applied the Minkowski, Sobolev embedding and Hölder inequalities, and the following inequalities as the consequence of (1.13),

$$\|w_p\|_{H^2} \leq \|v_p\|_{H^3}, \quad \|w_{p,t}\|_{L^2} \leq \|v_{p,t}\|_{H^1}.$$



Consequently, one concludes that, for  $s \geq 3$ ,

$$\begin{aligned} \int_0^\infty \|v_{p,tt}\|_{L^2}^2 dt &\lesssim \int_0^\infty \|\rho_{1,t}\|_{H^1}^2 dt + (1 + \sup_{0 \leq t < \infty} \|v_p\|_{H^3}^2) \\ &\times \int_0^\infty \|v_{p,t}\|_{H^2}^2 dt \lesssim C_{p,in,2}^2 + (1 + C_{p,in,3})C_{p,in,3}, \end{aligned}$$

where we have substituted inequality (4.5)<sub>2</sub>. Thus (4.5)<sub>3</sub> yields

$$\int_0^\infty \|\rho_{1,tt}\|_{L^2}^2 \lesssim C_{p,in,3}^2 + C_{p,in,2}(C_{p,in,2}^2 + (1 + C_{p,in,3})C_{p,in,3}) < \infty.$$

This completes the proof.  $\square$

Now, given  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$ , for any integer  $s \geq 3$ , which is even in the  $z$ -variable and satisfies the compatibility conditions (1.17), one can apply the conclusion of Proposition 1 to establish the global bound of the perturbation energy  $\mathcal{E}$ , provided it is initially small. This is done through a continuity argument. We state first the proposition concerning the local well-posedness of solutions to system (1.10) with  $\mathcal{E}_{in}$  small enough:

**Proposition 2.** *Let  $v_p$  be the solution to system (1.2), as stated in Theorem 1.1 with initial data  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$ , for an integer  $s \geq 3$ . Consider the initial data  $(\xi_{in}, \psi_{in}^h) \in H^2(\Omega_h \times 2\mathbb{T}) \times H^2(\Omega_h \times 2\mathbb{T})$  as in (1.14) and satisfying the compatibility condition (1.15). There is a positive constant  $\bar{\varepsilon} \in (0, 1)$ , small enough, and a positive time  $T_{\bar{\varepsilon}} \in (0, \infty)$ , such that if  $\varepsilon \in (0, \bar{\varepsilon})$  and  $\mathcal{E}_{in} \leq \bar{\varepsilon}$ , there exists a unique strong solution  $(\xi^\varepsilon, \psi^{\varepsilon,h}) \in L^\infty(0, T_{\bar{\varepsilon}}; H^2(\Omega_h \times 2\mathbb{T}))$ , with  $\psi^{\varepsilon,z}$  as in (1.12), to system (1.10) in the time interval  $[0, T_{\bar{\varepsilon}}]$ . The existence time  $T_{\bar{\varepsilon}}$  depends only on  $\bar{\varepsilon}$  and  $\|v_{p,in}\|_{H^3}$  and is independent of  $\varepsilon$ . Here  $\mathcal{E}_{in}$  is as in (1.16). Moreover,  $(\partial_t \xi^\varepsilon, \partial_t \psi^{\varepsilon,h}) \in L^\infty(0, T_{\bar{\varepsilon}}; L^2(\Omega_h \times 2\mathbb{T}))$ ,  $\rho = \rho_0 + \varepsilon^2 \rho_1 + \xi^\varepsilon \in (\frac{1}{2}\rho_0, 2\rho_0)$  in  $\Omega \times [0, T_{\bar{\varepsilon}}]$ , and there is a constant  $C'' > 0$ , independent of  $\varepsilon$ , such that,*

$$\sup_{0 \leq t \leq T_{\bar{\varepsilon}}} \mathcal{E}(t) \leq C'' \mathcal{E}_{in},$$

where  $\mathcal{E}(t)$  is as in (2.2).

The proof of Proposition 2 can be done via a fixed point argument similar to that in our previous work [62] and it is omitted here.

Now we are ready to establish the proof of Theorem 1.2

*Proof of Theorem 1.2.* Consider  $s \geq 3$  and  $\varepsilon \in (0, \bar{\varepsilon})$  with  $\bar{\varepsilon} \in (0, 1)$  as given in Proposition 2. Let the initial data  $(\xi_{in}, \psi_{in}^h) \in H^2(\Omega_h) \times H^2(\Omega_h \times 2\mathbb{T})$  satisfy (1.14), the compatibility conditions (1.15), and  $\mathcal{E}_{in} \leq \varepsilon^2$ , where  $\mathcal{E}_{in}$  is as in (1.16). Then  $\mathcal{E}_{in} \leq \bar{\varepsilon}$ , and there is a strong solution to system (1.10) as stated by Proposition 2 in the time interval  $[0, T_{\bar{\varepsilon}}]$ , for some  $T_{\bar{\varepsilon}} \in (0, \infty)$ , independent of  $\varepsilon$ . The strong solution satisfies

$$\begin{aligned} \rho &\in \left(\frac{1}{2}\rho_0, 2\rho_0\right) \text{ in } (\Omega_h \times 2\mathbb{T}) \times [0, T_{\bar{\varepsilon}}], \\ \text{and } \sup_{0 \leq t \leq T_{\bar{\varepsilon}}} \mathcal{E}(t) &\leq C'' \mathcal{E}_{in} \leq C'' \varepsilon^2. \end{aligned}$$

Such estimates, together with Theorem 1.1 and Corollary 1, imply that the assumptions in Proposition 1 hold true in the time interval  $[0, T_{\bar{\varepsilon}}]$ . Therefore applying (3.3) yields

$$\begin{aligned} \sup_{0 \leq t \leq T_{\bar{\varepsilon}}} \mathcal{E}(t) + \int_0^{T_{\bar{\varepsilon}}} \mathcal{D}(t) dt &\leq C' e^{C' + Q(C'' \varepsilon^2)} \left\{ \varepsilon^2 + \varepsilon^2 \right. \\ &\quad \left. + \left( \varepsilon^2 + (\varepsilon^2 + 1)Q(C'' \varepsilon^2) \right) \int_0^{T_{\bar{\varepsilon}}} \mathcal{D}(t) dt \right\} \\ &\leq 2C' e^{2C'} \varepsilon^2 + \frac{1}{2} \int_0^{T_{\bar{\varepsilon}}} \mathcal{D}(t) dt \Big\}, \end{aligned}$$

provided  $\varepsilon \in (0, \varepsilon_1) \subset (0, \bar{\varepsilon})$ , where  $\varepsilon_1$  is small enough such that  $Q(C'' \varepsilon_1^2) \leq C'$  and  $C' e^{2C'} (\varepsilon_1^2 + (\varepsilon_1^2 + 1)Q(C'' \varepsilon_1^2)) \leq 1/2$ . This inequality yields that,

$$\sup_{0 \leq t \leq T_{\bar{\varepsilon}}} \mathcal{E}(t) + \int_0^{T_{\bar{\varepsilon}}} \mathcal{D}(t) dt \leq C''' \varepsilon^2 < \bar{\varepsilon}, \quad (4.6)$$

where  $C''' = 4C' e^{2C'}$ , and provided  $\varepsilon_1$  is small such that  $C''' \varepsilon_1^2 < \bar{\varepsilon}$ . In particular,  $\mathcal{E}(T_{\bar{\varepsilon}}) \leq \bar{\varepsilon}$ . We apply Proposition 2 again in the time interval  $[T_{\bar{\varepsilon}}, 2T_{\bar{\varepsilon}}]$ , which states that there exists a strong solution satisfying

$$\begin{aligned} \rho &\in \left(\frac{1}{2}\rho_0, 2\rho_0\right) \text{ in } (\Omega_h \times 2\mathbb{T}) \times [T_{\bar{\varepsilon}}, 2T_{\bar{\varepsilon}}], \\ \text{and } \sup_{T_{\bar{\varepsilon}} \leq t \leq 2T_{\bar{\varepsilon}}} \mathcal{E}(t) &\leq C'' \mathcal{E}(T_{\bar{\varepsilon}}) \leq C'' C''' \varepsilon^2 \end{aligned}$$

Together with (4.6), this implies

$$\sup_{0 \leq t \leq 2T_{\bar{\varepsilon}}} \mathcal{E}(t) \leq C'''' \varepsilon^2 \text{ with } C'''' = \max\{C'', C'' C'''\},$$

Consequently Proposition 1 applies. In particular, (3.3) yields

$$\begin{aligned} \sup_{0 \leq t \leq 2T_{\bar{\varepsilon}}} \mathcal{E}(t) + \int_0^{2T_{\bar{\varepsilon}}} \mathcal{D}(t) dt &\leq C' e^{C'+Q(C''''\varepsilon^2)} \left\{ \varepsilon^2 + \varepsilon^2 \right. \\ &\left. + \left( \varepsilon^2 + (\varepsilon^2 + 1)Q(C''''\varepsilon^2) \right) \int_0^{2T_{\bar{\varepsilon}}} \mathcal{D}(t) dt \right\}. \end{aligned}$$

As above, this implies

$$\sup_{0 \leq t \leq 2T_{\bar{\varepsilon}}} \mathcal{E}(t) + \int_0^{2T_{\bar{\varepsilon}}} \mathcal{D}(t) dt \leq C'''' \varepsilon^2 < \bar{\varepsilon}, \quad (4.7)$$

provided  $\varepsilon \in (0, \varepsilon_2) \subset (0, \varepsilon_1) \subset (0, \bar{\varepsilon})$ , for  $\varepsilon_2$  small enough such that  $Q(C''''\varepsilon_2^2) \leq Q(C''\varepsilon_1^2)$ . Then inductively, without needing to determine the smallness of  $\varepsilon$  again, the arguments from (4.6) to (4.7) hold true for  $T_{\bar{\varepsilon}}, 2T_{\bar{\varepsilon}}$  replaced by  $nT_{\bar{\varepsilon}}, (n+1)T_{\bar{\varepsilon}}, n \geq 2$ , respectively. In particular, (4.7) holds true for  $2T_{\bar{\varepsilon}}$  replaced by  $(n+1)T_{\bar{\varepsilon}}$ . Recall that  $T_{\bar{\varepsilon}}$  is independent of  $\varepsilon$ . This concludes the proof of (1.18). (1.19) is a direct consequence of (1.18), (1.5), (1.11), (1.13) and the fact that  $\|\rho_1\|_{H^2} < \infty$  as in (4.2). Therefore let  $\varepsilon_0 = \varepsilon_2$ , and we complete the proof of Theorem 1.2.  $\square$

## 5 Global regularity estimates of the solution to the primitive equations

Let us recall the primitive equations (1.2) first. We shorten the notations  $v = v_p, w = w_p$  in this section. Recall that

$$\begin{cases} \operatorname{div}_h v + \partial_z w = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \rho_0(\partial_t v + v \cdot \nabla_h v + w \partial_z v) + \nabla_h(c_s^2 \rho_1) = \mu \Delta_h v \\ \quad + \lambda \nabla_h \operatorname{div}_h v + \partial_{zz} v & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z(c_s^2 \rho_1) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \quad (1.2)$$

where  $\mu > 0, \lambda > \frac{1}{3}\mu > 0$ . Here the symmetry (SYM-PE) and the side condition (1.3) are imposed. We will make further assumptions on the viscous coefficients later.

In this section, we will study the global regularity of the solution  $(\rho_1, v, w)$  to (1.2) with initial data  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$  for arbitrary integer  $s \in \{1, 2, 3, \dots\}$ , with  $v_{p,in}$  being even in the  $z$ -variable and satisfying the compatible conditions:

$$\int_{\Omega_h \times 2\mathbb{T}} v_{p,in} d\vec{x} = 0, \quad \operatorname{div}_h \bar{v}_{p,in} = 0.$$

We will show the following proposition:

**Proposition 3.** *For  $0 < \lambda < 4\mu < 12\lambda$ , suppose (1.2) is complemented with initial data  $v_{p,in} \in H^1(\Omega_h \times 2\mathbb{T})$  as above. Then the unique solution  $v$  to the primitive equations (1.2), satisfies*

$$\sup_{0 \leq t < \infty} \|v(t)\|_{H^1}^2 + \int_0^\infty \left( \|\nabla v(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2 \right) dt \leq C_{p,in},$$

for some positive constant  $C_{p,in}$  depending only  $\|v_{p,in}\|_{H^1}$ . Furthermore,

$$\|v(t)\|_{L^2}^2 \leq C e^{-ct} \|v_{p,in}\|_{L^2}^2 \quad t \in [0, \infty),$$

for some positive constants  $c, C$ . Moreover, for any integer  $s \geq 2$ , if  $v_{p,in} \in H^s(\Omega_h \times 2\mathbb{T})$ , there is a constant  $C_{p,in,s}$  depending only on  $\|v_{p,in}\|_{H^s}$  such that

$$\begin{aligned} \sup_{0 \leq t < \infty} \left( \|v(t)\|_{H^s}^2 + \|v_t(t)\|_{H^{s-2}}^2 \right) + \int_0^\infty \left( \|v(t)\|_{H^{s+1}}^2 \right. \\ \left. + \|\partial_t v(t)\|_{H^{s-1}}^2 \right) dt \leq C_{p,in,s}. \end{aligned}$$

As in [52], we focus on the *a priori* estimates below. In fact, the local-in-time regularity in Sobolev and analytic function spaces has been studied in [71], and therefore following a continuity argument, one can check the validity of the proof below.

### Basic energy estimate

Take the  $L^2$ -inner product of (1.2)<sub>2</sub> with  $v$ . We have

$$\frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v\|_{L^2}^2 \right\} + \mu \|\nabla_h v\|_{L^2}^2 + \lambda \|\operatorname{div}_h v\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 = 0. \quad (5.1)$$

Integrating the above equation in the time variable yields

$$\begin{aligned} \sup_{0 \leq t < \infty} \left\{ \frac{\rho_0}{2} \|v(t)\|_{L^2}^2 \right\} + \int_0^\infty \left( \mu \|\nabla_h v(t)\|_{L^2}^2 + \lambda \|\operatorname{div}_h v(t)\|_{L^2}^2 \right. \\ \left. + \|\partial_z v(t)\|_{L^2}^2 \right) dt = \frac{\rho_0}{2} \|v_{p,in}\|_{L^2}^2. \end{aligned} \quad (5.2)$$

Moreover, under the assumption (1.4), after applying the Poincaré inequality in (5.1), we have the inequality

$$\frac{d}{dt} \|v\|_{L^2}^2 + c \|v\|_{L^2}^2 \leq 0,$$

for some positive constant  $c$ . Thus one can derive from above that

$$\|v(t)\|_{L^2}^2 \lesssim e^{-ct} \|v_{p,in}\|_{L^2}^2, \quad (5.3)$$

for all  $t \in [0, \infty)$ .

### $H^1$ estimate

After applying  $\partial_z$  to (1.2)<sub>2</sub>, we write down the following equation:

$$\begin{aligned} \rho_0(\partial_t v_z + v \cdot \nabla_h v_z + w \partial_z v_z) &= \mu \Delta_h v_z + \lambda \nabla_h \operatorname{div}_h v_z + \partial_{zz} v_z \\ &- \rho_0(v_z \cdot \nabla_h v + w_z \partial_z v). \end{aligned} \quad (5.4)$$

Then take the  $L^2$  inner product of (5.4) with  $v_z$ . It follows, after substituting (1.13), that

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_z\|_{L^2}^2 \right\} + \mu \|\nabla_h v_z\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_z\|_{L^2}^2 + \|\partial_z v_z\|_{L^2}^2 \\ = -\rho_0 \int (v_z \cdot \nabla_h v) \cdot v_z \, d\vec{x} + \rho_0 \int \operatorname{div}_h v (\partial_z v \cdot v_z) \, d\vec{x} =: L_1 + L_2. \end{aligned} \quad (5.5)$$

After applying integration by parts, one will have

$$\begin{aligned} L_1 &= \rho_0 \int (v_z \cdot \nabla_h v_z) \cdot v \, d\vec{x} + \rho_0 \int (v \cdot v_z) \operatorname{div}_h v_z \, d\vec{x}, \\ L_2 &= -2\rho_0 \int (v \cdot \nabla_h v_z) \cdot v_z \, d\vec{x}. \end{aligned}$$

Therefore, let  $p, q$  be some positive constants, to be determined later, satisfying

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 < p < 6 \quad (\text{equivalently } q > 3). \quad (5.6)$$

After applying Hölder's, the Gagliardo-Nirenberg interpolation and Young's inequalities, one has

$$\begin{aligned} L_1 + L_2 &\lesssim \|v_z\|_{L^p} \|\nabla_h v_z\|_{L^2} \|v\|_{L^q} \lesssim \|v_z\|_{L^2}^{3/p-1/2} \|\nabla v_z\|_{L^2}^{3/2-3/p} \\ &\quad \times \|\nabla_h v_z\|_{L^2} \|v\|_{L^q} \lesssim \delta \|\nabla v_z\|_{L^2}^2 + C_\delta \|v_z\|_{L^2}^2 \|v\|_{L^q}^{4p/(6-p)} \\ &= \delta \|\nabla v_z\|_{L^2}^2 + C_\delta \|v_z\|_{L^2}^2 \|v\|_{L^q}^{2q/(q-3)}. \end{aligned}$$

Hence, after choosing  $\delta > 0$ , sufficiently small, in the above estimate, we have

$$\frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_z\|_{L^2}^2 \right\} + \frac{\mu}{2} \|\nabla_h v_z\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_z\|_{L^2}^2 + \frac{1}{2} \|\partial_z v_z\|_{L^2}^2$$

$$\lesssim \|v_z\|_{L^2}^2 \|v\|_{L^q}^{\frac{2q}{q-3}}.$$

Integrating the above inequality in the time variable yields

$$\begin{aligned} \sup_{0 \leq t < \infty} \|v_z(t)\|_{L^2}^2 + \int_0^\infty \|\nabla v_z(t)\|_{L^2}^2 dt &\lesssim \sup_{0 \leq t < \infty} \|v(t)\|_{L^q}^{\frac{2q}{q-3}} \\ &\times \int_0^\infty \|\nabla v(t)\|_{L^2}^2 dt + \|v_{p,in,z}\|_{L^2}^2 \lesssim \sup_{0 \leq t < \infty} \|v(t)\|_{L^q}^{\frac{2q}{q-3}} \\ &\times \|v_{p,in}\|_{L^2}^2 + \|v_{p,in,z}\|_{L^2}^2, \end{aligned} \quad (5.7)$$

for  $q \in (3, \infty)$ , where we have substituted (5.2).

On the other hand, after applying  $\partial_h$  to (1.2)<sub>2</sub>, one gets the equation:

$$\begin{aligned} \rho_0(\partial_t v_h + v \cdot \nabla_h v_h + w \partial_z v_h) + \nabla_h(c_s^2 \rho_{1,h}) &= \mu \Delta_h v_h \\ &+ \lambda \nabla_h \operatorname{div}_h v_h + \partial_{zz} v_h - \rho_0(v_h \cdot \nabla_h v + w_h \partial_z v). \end{aligned} \quad (5.8)$$

Then after taking  $L^2$ -inner product of (5.8) with  $v_h$ , we have

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_h\|_{L^2}^2 \right\} + \mu \|\nabla_h v_h\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_h\|_{L^2}^2 + \|\partial_z v_h\|_{L^2}^2 \\ = -\rho_0 \int (v_h \cdot \nabla_h v) \cdot v_h d\vec{x} - \rho_0 \int w_h (\partial_z v \cdot v_h) d\vec{x} =: L_3 + L_4. \end{aligned} \quad (5.9)$$

Now we estimate the terms on the right-hand side of the above equality. As before, let  $q > 3$ ,  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ . After applying integration by parts and the Hölder, Gagliardo-Nirenberg interpolation and Young inequalities, one has

$$\begin{aligned} L_3 &= \rho_0 \int \left( (v_h \cdot \nabla_h v_h) \cdot v + (v \cdot v_h) \operatorname{div}_h v_h \right) d\vec{x} \lesssim \int |v| |\nabla_h v| |\nabla_h^2 v| d\vec{x} \\ &\lesssim \|\nabla_h v\|_{L^p} \|\nabla_h^2 v\|_{L^2} \|v\|_{L^q} \lesssim \|\nabla_h v\|_{L^2}^{3/p-1/2} \|\nabla \nabla_h v\|_{L^2}^{3/2-3/p} \\ &\times \|\nabla_h^2 v\|_{L^2} \|v\|_{L^q} \lesssim \delta \|\nabla \nabla_h v\|_{L^2}^2 + C_\delta \|\nabla_h v\|_{L^2}^2 \|v\|_{L^q}^{2q/(q-3)}. \end{aligned}$$

On the other hand, after substituting (1.13) in the term  $L_4$  and applying the Minkowski, Hölder's, the Gagliardo-Nirenberg interpolation and Young's inequalities, one has

$$\begin{aligned} L_4 &= \rho_0 \int \left( \int_0^z \operatorname{div}_h v_h dz \right) \times (\partial_z v \cdot v_h) d\vec{x} \lesssim \int_0^1 |\nabla_h^2 v|_{L^2} dz \\ &\times \int_0^1 |v_z|_{L^4} |v_h|_{L^4} dz \lesssim \|\nabla_h^2 v\|_{L^2} \|v_z\|_{L^2}^{1/2} \|\nabla_h v_z\|_{L^2}^{1/2} \end{aligned}$$

$$\begin{aligned}
& \times \|v_h\|_{L^2}^{1/2} \|\nabla_h v_h\|_{L^2}^{1/2} \lesssim \delta \|\nabla \nabla_h v\|_{L^2}^2 + C_\delta \|v_z\|_{L^2}^2 \\
& \times \|\nabla_h v_z\|_{L^2}^2 \|v_h\|_{L^2}^2.
\end{aligned}$$

Similarly, take the  $L^2$  inner product of (1.2)<sub>2</sub> with  $v_t$ . One has,

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{\mu}{2} \|\nabla_h v\|_{L^2}^2 + \frac{\lambda}{2} \|\operatorname{div}_h v\|_{L^2}^2 + \frac{1}{2} \|\partial_z v\|_{L^2}^2 \right\} + \rho_0 \|\partial_t v\|_{L^2}^2 \\
& = -\rho_0 \int (v \cdot \nabla_h v) \cdot v_t \, d\vec{x} - \rho_0 \int w (\partial_z v \cdot v_t) \, d\vec{x} =: L_5 + L_6.
\end{aligned} \tag{5.10}$$

As before, applying the Hölder, Minkowski, Gagliardo-Nirenberg interpolation and Young inequalities yield, for  $q > 3$ ,  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ ,

$$\begin{aligned}
L_5 & \lesssim \|\nabla_h v\|_{L^p} \|v_t\|_{L^2} \|v\|_{L^q} \lesssim \|\nabla_h v\|_{L^2}^{3/p-1/2} \|\nabla \nabla_h v\|_{L^2}^{3/2-3/p} \\
& \times \|v_t\|_{L^2} \|v\|_{L^q} \lesssim \delta (\|v_t\|_{L^2}^2 + \|\nabla \nabla_h v\|_{L^2}^2) \\
& + C_\delta \|\nabla_h v\|_{L^2}^2 \|v\|_{L^q}^{2q/(q-3)}, \\
L_6 & = \rho_0 \int \left( \int_0^z \operatorname{div}_h v \, dz \right) \times (\partial_z v \cdot v_t) \, d\vec{x} \lesssim \int_0^1 |\nabla_h v|_{L^4} \, dz \\
& \times \int_0^1 |\partial_z v|_{L^4} |v_t|_{L^2} \, dz \lesssim \|\nabla_h v\|_{L^2}^{1/2} \|\nabla_h^2 v\|_{L^2}^{1/2} \|v_z\|_{L^2}^{1/2} \|\nabla_h v_z\|_{L^2}^{1/2} \\
& \times \|v_t\|_{L^2} \lesssim \delta (\|v_t\|_{L^2}^2 + \|\nabla_h^2 v\|_{L^2}^2) + C_\delta \|v_z\|_{L^2}^2 \|\nabla_h v_z\|_{L^2}^2 \\
& \times \|\nabla_h v\|_{L^2}^2.
\end{aligned}$$

After summing (5.9), (5.10) and the estimates of  $L_3, L_4, L_5, L_6$  above with sufficiently small  $\delta$ , one has

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_h\|_{L^2}^2 + \frac{\mu}{2} \|\nabla_h v\|_{L^2}^2 + \frac{\lambda}{2} \|\operatorname{div}_h v\|_{L^2}^2 + \frac{1}{2} \|\partial_z v\|_{L^2}^2 \right\} \\
& + \frac{\mu}{2} \|\nabla_h v_h\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_h\|_{L^2}^2 + \frac{1}{2} \|\partial_z v_h\|_{L^2}^2 + \frac{\rho_0}{2} \|\partial_t v\|_{L^2}^2 \\
& \lesssim \|\nabla_h v\|_{L^2}^2 \|v\|_{L^q}^{2q/(q-3)} + \|v_z\|_{L^2}^2 \|\nabla_h v_z\|_{L^2}^2 \|\nabla_h v\|_{L^2}^2.
\end{aligned}$$

Then after applying the Grönwall's inequality, it follows

$$\begin{aligned}
& \sup_{0 \leq t < \infty} \|\nabla v(t)\|_{L^2}^2 + \int_0^\infty \left( \|\nabla v_h(t)\|_{L^2}^2 + \|\partial_t v(t)\|_{L^2}^2 \right) dt \\
& \lesssim e^C \int_0^\infty \|v_z(t)\|_{L^2}^2 \|\nabla_h v_z(t)\|_{L^2}^2 dt \left( \|\nabla v_{p,in}\|_{L^2}^2 \right. \\
& \quad \left. + \int_0^\infty \|\nabla_h v(t)\|_{L^2}^2 dt \times \sup_{0 \leq t < \infty} \|v(t)\|_{L^q}^{\frac{2q}{q-3}} \right), \tag{5.11} \\
& \lesssim e^{C(\sup_{0 \leq t < \infty} \|v(t)\|_{L^q}^{\frac{4q}{q-3}} \times \|v_{p,in}\|_{L^2}^4 + \|v_{p,in,z}\|_{L^2}^4)} \left( \|\nabla v_{p,in}\|_{L^2}^2 \right. \\
& \quad \left. + \|v_{p,in}\|_{L^2}^2 \times \sup_{0 \leq t < \infty} \|v(t)\|_{L^q}^{\frac{2q}{q-3}} \right),
\end{aligned}$$

for some positive constant  $C$  and  $q \in (3, \infty)$ , where we have substituted (5.2) and (5.7).

### $L^q$ estimate

We take the  $L^2$ -inner product of (1.2)<sub>2</sub> with  $|v|^{q-2}v$ . It follows that,

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{\rho_0}{q} \|v\|_{L^q}^q \right\} + \mu \int \left( |v|^{q-2} |\nabla_h v|^2 + (q-2) |v|^{q-2} |\nabla_h |v||^2 \right) d\vec{x} \\
& \quad + \lambda \int |v|^{q-2} |\operatorname{div}_h v|^2 d\vec{x} + \int \left( |v|^{q-2} |\partial_z v|^2 \right. \\
& \quad \left. + (q-2) |v|^{q-2} |\partial_z |v||^2 \right) d\vec{x} = \int c_s^2 \rho_1 \operatorname{div}_h (|v|^{q-2} v) d\vec{x} \\
& \quad - \lambda(q-2) \int |v|^{q-3} (v \cdot \nabla_h |v|) \operatorname{div}_h v d\vec{x} =: L_7 + L_8. \tag{5.12}
\end{aligned}$$

By using the Cauchy-Schwarz inequality, it holds,

$$\begin{aligned}
L_8 & \leq \lambda(q-2) \int |v|^{q-2} |\nabla_h |v|| |\operatorname{div}_h v| d\vec{x} \leq \lambda \int |v|^{q-2} |\operatorname{div}_h v|^2 d\vec{x} \\
& \quad + \frac{\lambda(q-2)^2}{4} \int |v|^{q-2} |\nabla_h |v||^2 d\vec{x}.
\end{aligned}$$



Therefore, (5.12) implies

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{\rho_0}{q} \|v\|_{L^q}^q \right\} + \mu \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2}^2 + \| |v|^{\frac{q}{2}-1} \partial_z v \|_{L^2}^2 \\
& \lesssim \frac{d}{dt} \left\{ \frac{\rho_0}{q} \|v\|_{L^q}^q \right\} + \int \left( \mu |v|^{q-2} |\nabla_h v|^2 \right. \\
& \quad \left. + \left( \mu(q-2) - \frac{\lambda(q-2)^2}{4} \right) |v|^{q-2} |\nabla_h |v||^2 \right) d\vec{x} \\
& \quad + \int \left( |v|^{q-2} |\partial_z v|^2 + (q-2) |v|^{q-2} |\partial_z |v||^2 \right) d\vec{x} \leq L_7,
\end{aligned} \tag{5.13}$$

provided

$$q-2 \geq 0 \text{ and } \mu - \frac{\lambda(q-2)}{4} \geq 0, \text{ or equivalently, } 2 \leq q \leq \frac{4\mu}{\lambda} + 2. \tag{5.14}$$

In order to estimate  $L_7$ , we first derive an estimate for the ‘‘pressure’’  $\rho_1$ . Recall the elliptic problem (4.1),

$$-c_s^2 \Delta_h \rho_1 = \rho_0 \int_0^1 \operatorname{div}_h (\operatorname{div}_h (v \otimes v)) dz \text{ in } \Omega_h, \text{ with } \int_{\Omega_h} \rho_1 dx dy = 0.$$

Now we consider the  $L^{p_1}$  estimate of  $\rho_1$ . In fact, as the consequence of the  $L^p$  estimate of the Riesz transform, one has

$$\begin{aligned}
\|\rho_1\|_{L^{p_1}} &= |\rho_1|_{L^{p_1}} \lesssim \int_0^1 \| |v|^2 \|_{L^{p_1}} dz = \int_0^1 \| |v|^2 \|_{L^{2p_1}} dz \lesssim \int_0^1 \| |v| \|_{L^4} \| |v| \|_{L^{4p_1/(4-p_1)}} dz \\
&= \int_0^1 \| |v| \|_{L^4} \| |v|^{\frac{2}{q}} \|_{L^{\frac{8p_1}{q(4-p_1)}}} dz \lesssim \int_0^1 \| |v|^{\frac{1}{2}} \|_{L^2} \| |\nabla_h v|^{\frac{1}{2}} \|_{L^2} \| |v|^{\frac{q}{2}} \|_{L^2}^{\frac{2}{p_1}-\frac{1}{2}} \| |\nabla_h |v||^{\frac{q}{2}} \|_{L^2}^{\frac{1}{2}+\frac{2}{q}-\frac{2}{p_1}} dz \\
&\lesssim \| |v|^{\frac{1}{2}} \|_{L^2} \| |\nabla_h v|^{\frac{1}{2}} \|_{L^2} \| |v|^{\frac{q}{2}} \|_{L^2}^{\frac{2}{p_1}-\frac{1}{2}} \| |\nabla_h |v||^{\frac{q}{2}} \|_{L^2}^{\frac{1}{2}+\frac{2}{q}-\frac{2}{p_1}} \\
&\lesssim \| |v|^{\frac{1}{2}} \|_{L^2} \| |\nabla_h v|^{\frac{1}{2}} \|_{L^2} \| |v|^{\frac{q}{2}} \|_{L^q}^{\frac{2}{p_1}-\frac{q}{4}} \| |\nabla_h |v||^{\frac{q}{2}} \|_{L^2}^{\frac{1}{2}+\frac{2}{q}-\frac{2}{p_1}},
\end{aligned} \tag{5.15}$$

provided

$$p_1 > 2, \quad \frac{1}{4} < \frac{1}{p_1} < \frac{1}{q} + \frac{1}{4}, \quad q \geq 2, \tag{5.16}$$

where we have applied the Minkowski, Hölder and Gagliardo-Nirenberg interpolation inequalities. Let  $q_1 > 0$  be such that

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}.$$

Then we have, after applying the Minkowski, Hölder and Gagliardo-Nirenberg interpolation inequalities,

$$\begin{aligned}
L_7 &\lesssim \int |\rho_1| |v|^{\frac{q}{2}-1} |\nabla_h v| |v|^{\frac{q}{2}-1} d\vec{x} \lesssim |\rho_1|_{L^{p_1}} \\
&\times \int_0^1 \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2} \| |v|^{\frac{q}{2}-1} \|_{L^{q_1}} dz = \|\rho_1\|_{L^{p_1}} \\
&\times \int_0^1 \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2} \| |v|^{\frac{q}{2}} \|_{L^{\frac{q_1(q-2)}{q}}} dz \lesssim \|\rho_1\|_{L^{p_1}} \\
&\times \int_0^1 \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2} \| |v|^{\frac{2}{q_1}} \nabla_h |v|^{\frac{q}{2}} \|_{L^2}^{1-\frac{2}{q}-\frac{2}{q_1}} dz \lesssim \|\rho_1\|_{L^{p_1}} \quad (5.17) \\
&\times \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2} \| |v|^{\frac{q}{2}} \|_{L^2}^{1-\frac{2}{p_1}} \| \nabla_h |v|^{\frac{q}{2}} \|_{L^2}^{\frac{2}{p_1}-\frac{2}{q}} \\
&\lesssim \|\rho_1\|_{L^{p_1}} \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2} \| v \|_{L^q}^{\frac{q}{2}-\frac{q}{p_1}} \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2}^{\frac{2}{p_1}-\frac{2}{q}} \\
&\lesssim \| v \|_{L^2}^{1/2} \| \nabla_h v \|_{L^2}^{1/2} \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2}^{\frac{3}{2}} \| v \|_{L^q}^{\frac{q}{4}} \\
&\lesssim \delta \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2}^2 + C_\delta \| v \|_{L^2}^2 \| \nabla_h v \|_{L^2}^2 \| v \|_{L^q}^q,
\end{aligned}$$

provided

$$0 < \frac{1}{q_1} < \frac{1}{2} - \frac{1}{q}, \quad \text{or equivalently, } \frac{1}{q} < \frac{1}{p_1} \text{ and } q > 2, \quad (5.18)$$

where we have substituted (5.15) in the second but last inequality. Therefore after combining (5.14), (5.16), (5.18), for  $q$  satisfying

$$2 < q \leq \frac{4\mu}{\lambda} + 2, \quad (5.19)$$

we conclude from (5.13) and (5.17),

$$\begin{aligned}
&\frac{d}{dt} \left\{ \frac{\rho_0}{q} \| v \|_{L^q}^q \right\} + \frac{\mu}{2} \| |v|^{\frac{q}{2}-1} \nabla_h v \|_{L^2}^2 + \| |v|^{\frac{q}{2}-1} \partial_z v \|_{L^2}^2 \\
&\lesssim \| v \|_{L^2}^2 \| \nabla_h v \|_{L^2}^2 \| v \|_{L^q}^q,
\end{aligned}$$

after choosing  $\delta$  sufficiently small above. Applying Grönwall's inequality to the above inequality implies, for  $2 < q \leq \frac{4\mu}{\lambda} + 2$ ,

$$\begin{aligned}
&\sup_{0 \leq t < T} \| v(t) \|_{L^q}^q + \int_0^\infty \| |v|^{\frac{q}{2}-1} \nabla v(t) \|_{L^2}^2 dt \\
&\lesssim C_q e^C \int_0^\infty \| v(t) \|_{L^2}^2 \| \nabla_h v(t) \|_{L^2}^2 dt \| v_{p,in} \|_{L^q}^q \quad (5.20) \\
&\lesssim C_q e^C \| v_{p,in} \|_{L^2}^4 \| v_{p,in} \|_{L^q}^q,
\end{aligned}$$

for some positive constant  $C$  and  $C_q$  depending on  $q$ .

Therefore, after summing up the inequalities (5.2), (5.7), (5.11), (5.20), for  $\lambda < 4\mu < 12\lambda$ , one will have

$$\begin{aligned} & \sup_{0 \leq t < \infty} \|v(t)\|_{H^1}^2 + \int_0^\infty \left( \|\nabla v(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2 \right) dt \\ & \leq C_{p,in}(\|v_{p,in}\|_{H^1}, \|v_{p,in}\|_{L^q}), \end{aligned} \quad (5.21)$$

for some positive constant  $C_{p,in}$  depending on  $\|v_{p,in}\|_{H^1}, \|v_{p,in}\|_{L^q}$  with

$$3 < q \leq \frac{4\mu}{\lambda} + 2, \quad \frac{4\mu}{\lambda} \in (1, 12).$$

In particular, it suffices to take

$$q = \begin{cases} 4 & \text{if } \frac{4\mu}{\lambda} \in [4, 12), \\ \frac{4\mu}{\lambda} + 2 & \text{if } \frac{4\mu}{\lambda} \in (1, 4), \end{cases} \quad \text{such that } q \in [2, 6],$$

and therefore,

$$C_{p,in}(\|v_{p,in}\|_{H^1}, \|v_{p,in}\|_{L^q}) = C_{p,in}(\|v_{p,in}\|_{H^1}) \quad (5.22)$$

depends only on  $\|v_{p,in}\|_{H^1}$  by noticing that  $\|v_{p,in}\|_{L^q} \lesssim \|v_{p,in}\|_{H^1}$  in this case.

### $H^s$ estimates

Next, we will show the global regularity of the solution  $v$  to system (1.2) with more regular initial data  $v_{p,in}$ . That is, we complement (1.2) with the initial data  $v_{p,in} \in H^s(\Omega)$ , with  $s \geq 2$ . In fact, we will use the mathematical induction principle to show that for any  $s \geq 1, s \in \mathbb{Z}^+$ ,

$$\begin{cases} \sup_{0 \leq t < \infty} \|v(t)\|_{H^1}^2 + \int_0^\infty \left( \|v(t)\|_{H^2}^2 + \|\partial_t v(t)\|_{L^2}^2 \right) dt \\ \leq C_{p,in,1} \quad \text{if } s = 1, \\ \sup_{0 \leq t < \infty} \left( \|v(t)\|_{H^s}^2 + \|v_t(t)\|_{H^{s-2}}^2 \right) + \int_0^\infty \left( \|v(t)\|_{H^{s+1}}^2 \right. \\ \left. + \|\partial_t v(t)\|_{H^{s-1}}^2 \right) dt \leq C_{p,in,s} \quad \text{if } s \geq 2, \end{cases} \quad (5.23)$$

for some positive constant  $C_{p,in,s}$  depending on  $\|v_{p,in}\|_{H^s}$ . Notice, the case when  $s = 1$  has been shown in (5.21).

First, for any integer  $s \geq 1$ , it is assumed that (5.23) holds true. Our goal is to show that the same estimate is also true for  $s$  replaced by  $s + 1$ . In order to do so, we apply  $\partial^{s+1}$  to (1.2)<sub>2</sub> with  $\partial \in \{\partial_x, \partial_y, \partial_z\}$  and denote the  $k$ -order derivative by  $\cdot_k := \partial^k \cdot$  for any  $k \in \{0, 1, 2 \cdots s + 1\}$ . Then we have the following equation:

$$\begin{aligned} & \rho_0(\partial_t v_{s+1} + v \cdot \nabla_h v_{s+1} + w \partial_z v_{s+1}) + \nabla_h(c_s^2 \rho_{1,s+1}) \\ &= \mu \Delta_h v_{s+1} + \lambda \nabla_h \operatorname{div}_h v_{s+1} + \partial_{zz} v_{s+1} \\ & - \rho_0(\partial^{s+1}(v \cdot \nabla_h v) - v \cdot \nabla_h v_{s+1} + \partial^{s+1}(w \partial_z v) - w \partial_z v_{s+1}). \end{aligned} \quad (5.24)$$

Take the  $L^2$ -inner product of (5.24) with  $v_{s+1}$ . It follows,

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_{s+1}\|_{L^2}^2 \right\} + \mu \|\nabla_h v_{s+1}\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_{s+1}\|_{L^2}^2 \\ & + \|\partial_z v_{s+1}\|_{L^2}^2 = \rho_0 \int (v \cdot \nabla_h v_{s+1} - \partial^{s+1}(v \cdot \nabla_h v)) \cdot v_{s+1} d\vec{x} \\ & + \rho_0 \int (w \partial_z v_{s+1} - \partial^{s+1}(w \partial_z v)) \cdot v_{s+1} d\vec{x} =: K_1 + K_2. \end{aligned} \quad (5.25)$$

We estimate  $K_1, K_2$  on the right-hand side of (5.25) below. First, notice that  $K_1, K_2$  can be written as

$$\begin{aligned} K_1 &= -\rho_0 \sum_{i=0}^s \binom{s+1}{i} \int (v_{s+1-i} \cdot \nabla_h v_i) \cdot v_{s+1} d\vec{x} =: \sum_{i=0}^s K_{1,i}, \\ K_2 &= -\rho_0 \sum_{i=0}^s \binom{s+1}{i} \int w_{s+1-i} \partial_z v_i \cdot v_{s+1} d\vec{x} =: \sum_{i=0}^s K_{2,i}. \end{aligned}$$

We consider the estimates of  $K_{j,i}$ , for  $j \in \{1, 2\}$  and  $i \in \{0, 1, 2 \cdots s\}$  in three cases:

$$\begin{cases} 2 \leq i \leq s, \\ i = 1, \\ i = 0. \end{cases}$$

In the case when  $i \geq 2$ , we have

$$s + 1 - i \leq s - 1, \quad 3 \leq 1 + i \leq s + 1.$$

Therefore, applying the Hölder, Sobolev embedding and Young inequalities implies

$$K_{1,i} \lesssim \|v_{s+1-i}\|_{L^3} \|\nabla_h v_i\|_{L^6} \|v_{s+1}\|_{L^2} \lesssim \|v\|_{H^{s+2-i}} \|v\|_{H^{i+2}} \|v_{s+1}\|_{L^2}$$

$$\lesssim \|v\|_{H^s} \|v\|_{H^{s+2}} \|v_{s+1}\|_{L^2} \lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^s}^2 \|v_{s+1}\|_{L^2}^2.$$

Similarly, we apply the Minkowski, Hölder, Sobolev embedding and Young inequalities to estimate  $K_{2,i}$ . On the one hand, if  $w_{s+1-i} = \partial^{s+1-i} w = \partial_h^{s+1-i} w$ , we have, thanks to (1.13),

$$\begin{aligned} K_{2,i} &= \rho_0 \binom{s+1}{i} \int \left( \partial_h^{s+1-i} \left( \int_0^z \operatorname{div}_h v \, dz' \right) \times (\partial_z v_i \cdot v_{s+1}) \right) d\vec{x} \\ &\lesssim \int_0^1 |v_{s+2-i}|_{L^4} \, dz' \times \int_0^1 |\partial_z v_i|_{L^4} |v_{s+1}|_{L^2} \, dz \\ &\lesssim \int_0^1 |v_{s+2-i}|_{L^2}^{1/2} |v_{s+3-i}|_{L^2}^{1/2} \, dz' \times \int_0^1 |\partial_z v_i|_{L^2}^{1/2} |\partial_z v_{i+1}|_{L^2}^{1/2} |v_{s+1}|_{L^2} \, dz \\ &\lesssim \|v\|_{H^{s+2-i}}^{1/2} \|v\|_{H^{s+3-i}}^{1/2} \|v\|_{H^{i+1}}^{1/2} \|v\|_{H^{i+2}}^{1/2} \|v_{s+1}\|_2 \\ &\lesssim \|v\|_{H^s}^{1/2} \|v\|_{H^{s+1}} \|v\|_{H^{s+2}}^{1/2} \|v_{s+1}\|_2 \lesssim \delta \|v\|_{H^{s+2}}^2 \\ &\quad + C_\delta \|v\|_{H^{s+1}}^2 \|v_{s+1}\|_{L^2}^2. \end{aligned}$$

On the other hand, if  $w_{s+1-i} = \partial_z w_{s-i}$ , we have

$$\begin{aligned} K_{2,i} &= \rho_0 \binom{s+1}{i} \int \operatorname{div}_h v_{s-i} \partial_z v_i \cdot v_{s+1} \, d\vec{x} \lesssim \|v_{s+1-i}\|_{L^3} \|\partial_z v_i\|_{L^6} \|v_{s+1}\|_{L^2} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^s}^2 \|v_{s+1}\|_{L^2}^2. \end{aligned}$$

In the case when  $i = 1$ , direct application of the Hölder, Sobolev embedding and Young inequalities yields,

$$\begin{aligned} K_{1,1} &\lesssim \|v_s\|_{L^3} \|v_2\|_{L^2} \|v_{s+1}\|_{L^6} \lesssim \|v\|_{H^{s+1}} \|v\|_{H^2} \|v\|_{H^{s+2}} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^2}^2 \|v\|_{H^{s+1}}^2. \end{aligned}$$

Meanwhile, to estimate  $K_{2,1}$ , we will again apply the Minkowski, Hölder, Sobolev embedding and Young inequalities. If  $w_s = \partial_h^s w$ , we have, after substituting (1.13),

$$\begin{aligned} K_{2,1} &= \rho_0 \binom{s+1}{1} \int \left( \left( \int_0^z \operatorname{div}_h \partial_h^s v \, dz' \right) \times (\partial_z v_1 \cdot v_{s+1}) \right) d\vec{x} \\ &\lesssim \int_0^1 |\operatorname{div}_h v_s|_{L^4} \, dz' \times \int_0^1 |\partial_z v_1|_{L^2} |v_{s+1}|_{L^4} \, dz \\ &\lesssim \int_0^1 |\operatorname{div}_h v_s|_{L^2}^{1/2} |\nabla_h \operatorname{div}_h v_s|_{L^2}^{1/2} \, dz' \times \int_0^1 |\partial_z v_1|_{L^2} |v_{s+1}|_{L^2}^{1/2} |\nabla_h v_{s+1}|_{L^2}^{1/2} \, dz \end{aligned}$$

$$\lesssim \|v\|_{H^{s+1}} \|v\|_{H^{s+2}} \|v\|_{H^2} \lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^2}^2 \|v\|_{H^{s+1}}^2.$$

If  $w_s = \partial_z w_{s-1}$ , we have

$$\begin{aligned} K_{2,1} &= \rho_0 \binom{s+1}{1} \int \operatorname{div}_h v_{s-1} \partial_z v_1 \cdot v_{s+1} d\vec{x} \lesssim \|v_s\|_{L^3} \|v_2\|_{L^2} \|v_{s+1}\|_{L^6} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^2}^2 \|v\|_{H^{s+1}}^2. \end{aligned}$$

Finally, in the case when  $i = 0$ , we apply the Hölder, Sobolev embedding and Young inequalities to get

$$\begin{aligned} K_{1,0} &\lesssim \|v_{s+1}\|_{L^6} \|\nabla_h v\|_{L^3} \|v_{s+1}\|_{L^2} \lesssim \|v\|_{H^{s+2}} \|v\|_{H^2} \|v_{s+1}\|_{L^2} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^2}^2 \|v_{s+1}\|_{L^2}^2. \end{aligned}$$

When  $w_{s+1} = \partial_h^{s+1} w$ , applying the Minkowski, Hölder, Sobolev embedding and Young inequalities yields, after substituting (1.13),

$$\begin{aligned} K_{2,0} &= \rho_0 \binom{s+1}{0} \int \left( \partial_h^{s+1} \left( \int_0^z \operatorname{div}_h v dz' \right) \times (\partial_z v \cdot v_{s+1}) \right) d\vec{x} \\ &\lesssim \int_0^1 |v_{s+2}|_{L^2} dz' \times \int_0^1 |\partial_z v|_{L^4} |v_{s+1}|_{L^4} dz \lesssim \int_0^1 |v_{s+2}|_{L^2} dz' \\ &\quad \times \int_0^1 |\partial_z v|_{L^2}^{1/2} |\nabla_h \partial_z v|_{L^2}^{1/2} |v_{s+1}|_{L^2}^{1/2} |\nabla_h v_{s+1}|_{L^2}^{1/2} dz \\ &\lesssim \|v\|_{H^{s+2}}^{3/2} \|v_{s+1}\|_{L^2}^{1/2} \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^1}^2 \|v\|_{H^2}^2 \|v_{s+1}\|_{L^2}^2. \end{aligned}$$

When  $w_{s+1} = \partial_z w_s$ , we have, after substituting (1.13),

$$\begin{aligned} K_{2,0} &= \rho_0 \binom{s+1}{0} \int \operatorname{div}_h v_s \partial_z v \cdot v_{s+1} d\vec{x} \lesssim \|v_{s+1}\|_{L^6} \|\nabla v\|_{L^3} \|v_{s+1}\|_{L^2} \\ &\lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^2}^2 \|v_{s+1}\|_{L^2}^2. \end{aligned}$$

From the above estimates, one can conclude from (5.25) that for any integer  $s \geq 1$ ,

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{\rho_0}{2} \|v_{s+1}\|_{L^2}^2 \right\} + \mu \|\nabla_h v_{s+1}\|_{L^2}^2 + \lambda \|\operatorname{div}_h v_{s+1}\|_{L^2}^2 \\ &\quad + \|\partial_z v_{s+1}\|_{L^2}^2 \lesssim \delta \|v\|_{H^{s+2}}^2 + C_\delta \|v\|_{H^{s+1}}^2 \|v\|_{H^{s+1}}^2 \\ &\quad + C_\delta \|v\|_{H^1}^2 \|v\|_{H^2}^2 \|v_{s+1}\|_2^2 \lesssim \delta \|\nabla v_{s+1}\|_2^2 \\ &\quad + (\delta + C_\delta \|v\|_{H^s}^2) \|v\|_{H^{s+1}}^2 + C_\delta (\|v\|_{H^{s+1}}^2 + \|v\|_{H^1}^2 \|v\|_{H^2}^2) \\ &\quad \times \|v_{s+1}\|_{L^2}^2. \end{aligned} \tag{5.26}$$

Here we have used the notation  $\|v_{s+1}\|_{L^2}^2$  to denote  $\sum_{\partial \in \{\partial_x, \partial_y, \partial_z\}} \|\partial^{s+1} v\|_{L^2}^2$ . Then after taking  $\delta > 0$  small enough and applying Grönwall's inequality, together with the inequalities (5.23) and (5.21), we have

$$\begin{aligned}
& \sup_{0 \leq t < \infty} \|v(t)\|_{H^{s+1}}^2 + \int_0^\infty \|v(t)\|_{H^{s+2}}^2 dt \\
& \lesssim e^C \int_0^\infty (\|v(t)\|_{H^{s+1}}^2 + \|v(t)\|_{H^1}^2 \|v(t)\|_{H^2}^2) dt (\|v_{p,in}\|_{H^{s+1}}^2 \\
& \quad + \int_0^\infty (1 + \|v(t)\|_{H^s}^2) \|v(t)\|_{H^{s+1}}^2 dt) \lesssim e^{C_{p,in,s} + C_{p,in,1}^2} \\
& \quad \times (\|v_{p,in}\|_{H^{s+1}}^2 + C_{p,in,s} + C_{p,in,s}^2),
\end{aligned} \tag{5.27}$$

where  $C_{p,in,1} = C_{p,in}(\|v_{p,in}\|_{H^1})$  is as in (5.22).

On the other hand, after replacing  $s + 1$  by  $s$  in (5.24), we have the identity:

$$\begin{aligned}
\rho_0 \partial_t v_s &= -\nabla_h (c_s^2 \rho_{1,s}) + \mu \Delta_h v_s + \lambda \nabla_h \operatorname{div}_h v_s + \partial_{zz} v_s \\
&\quad - \rho_0 \partial^s (v \cdot \nabla_h v) - \rho_0 \partial^s (w \partial_z v).
\end{aligned} \tag{5.28}$$

After taking the  $L^2$ -inner product of (5.28) with  $\partial_t v_s$  and noticing the fact that

$$-\int \nabla_h (c_s^2 \rho_{1,s}) \cdot \partial_t v_s d\vec{x} = \int c_s^2 \rho_{1,s} \operatorname{div}_h \partial_t v_s d\vec{x} = 0,$$

this implies

$$\begin{aligned}
\|\partial_t v_s\|_{L^2}^2 &\lesssim \|\nabla v_{s+1}\|_{L^2}^2 + \|\partial^s (v \cdot \nabla_h v)\|_{L^2}^2 + \|\partial^s (w \partial_z v)\|_{L^2}^2 \\
&\lesssim \|\nabla v_{s+1}\|_{L^2}^2 + \sum_{i=0}^s (\|v_i \cdot \nabla_h v_{s-i}\|_{L^2}^2 + \|w_i \partial_z v_{s-i}\|_{L^2}^2) \\
&\lesssim (1 + \|v\|_{H^{s+1}}^2) \|v\|_{H^{s+2}}^2,
\end{aligned} \tag{5.29}$$

where we have applied the inequalities below: since  $s \geq 1$ ,

$$\begin{aligned}
\|v_i \cdot \nabla_h v_{s-i}\|_{L^2}^2 &\lesssim \|v_i\|_{L^3}^2 \|\nabla_h v_{s-i}\|_{L^6}^2 \lesssim \|v\|_{H^{i+1}}^2 \|v\|_{H^{s+2-i}}^2 \\
&\lesssim \|v\|_{H^{s+1}}^2 \|v\|_{H^{s+2}}^2, \\
\|w_i \partial_z v_{s-i}\|_{L^2}^2 &\lesssim \|w\|_{H^{i+1}}^2 \|v\|_{H^{s+2-i}}^2 \lesssim \|v\|_{H^{i+2}}^2 \|v\|_{H^{s+2-i}}^2 \\
&\lesssim \|v\|_{H^{s+1}}^2 \|v\|_{H^{s+2}}^2,
\end{aligned}$$

due to the fact that from (1.13),

$$\|w\|_{H^{i+1}} = \left\| \int_0^z \operatorname{div}_h v dz' \right\|_{H^{i+1}} \lesssim \|v\|_{H^{i+2}}.$$

Here we have applied the Minkowski, Hölder and Sobolev embedding inequalities. Similarly, taking  $s = s - 1$  in (5.29) yields

$$\|\partial_t v_{s-1}\|_{L^2}^2 \lesssim (1 + \|v\|_{H^s}^2) \|v\|_{H^{s+1}}^2. \quad (5.30)$$

Integrating (5.27) in the time variable, together with (5.29), (5.30), implies (5.23) with  $s$  replaced by  $s + 1$ . This finishes the mathematical induction. Hence, this concludes the proof of Proposition 3.

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