PREPRINT – Published as
Krause, C. M. & Wille, A. M. (2021). Sign language in light of mathematics education: an exploration within semiotic and embodiment theories of learning mathematics. *American Annals of the Deaf, 166*(3) *(Special issue 'Critical Topics in Mathematics Education: Research to Practice with Deaf/Hard-of-Hearing Students'),* 358–383.

Sign language in light of mathematics education: an exploration within semiotic and embodiment theories of learning mathematics

C.M. Krause, University of Duisburg-Essen (Germany) / University of California, Berkeley (USA)

Annika M. Wille, Universität Klagenfurt (Austria)

Abstract: *Learning mathematics, as reported by numerous studies from deaf education, is different for deaf and hard-of-hearing students (DHH). However, rarely is the research focused on the different ways DHH students encounter mathematical ideas and how they deal with them in process of learning mathematics, for example, considering the unique complexities related to using sign languages (SL). While this includes the use and challenges of SL in the mathematics classroom, it also involves the opportunities that come with learning mathematics in this gestural-somatic medium. We will examine this issue within mathematics education, considering deaf students first and foremost as learners of mathematics and their use of SL as a special case of language in the mathematics classroom. More specifically, we will explore the use of sign language in teaching and learning mathematics within semiotic and embodiment perspectives – how their use might matter for the development, conceptualization, and representation of mathematical meaning in signs. While there are many sign languages, we focus our theoretical discussion on aspects found across sign languages that we illustrate with examples from our work and research with Deaf German and Austrian learners and experts, related to topics in geometry, arithmetic, and fraction concepts. The examples serve to inform the context of mathematics teaching and learning, more generally, by illuminating features of SL that distinguish learning mathematics for deaf learners in comparison to their hearing peers.*

Introduction

Ever since the paradigmatic shift from a behaviorist to a more constructivist understanding of learning, research in mathematics education has shown a strong emphasis on understanding better the processes of making meaning—on understanding better *how* students come to know what they know—and how different components shape learning processes in mathematics. In this, deaf learners constitute a specific, albeit crucially under-researched population. Historically, the focus has been set on assessing deaf students' competencies and comparing test results to those of hearing students resulting in the identification of lower mathematics achievement scores and a delay in mathematical performance, starting already prior to formal school education (Traxler, 2000; Kritzer, 2009). These comparisons have often been measured quantitatively: explorations of the qualitative characteristics of deaf students' processes of learning mathematics—for example, describing the ways in which they approach mathematical content and their strategies when solving mathematical problems, but also the obstacles and pitfalls as they might be related to their specific practices in the learning process—have rarely been reported (for exceptions e.g., Titus, 1995 for working with fractions or Zevenberg et al., 2003; Pagliaro & Ansell, 2008 for linguistic strategies when solving arithmetic word problems). It is, however, through understanding better these practices that we can be able to align teaching material and methods to the strengths and needs of deaf learners, with our understanding always depending on the theoretical lenses we choose and the focus of our observations.

This contribution centers around sign language as a specific practice significantly shaping teaching and learning for deaf mathematics learners as primary modes of meaning making from theoretical perspectives of semiotics and embodiment. In this, the semiotic lens considers sign languages as an important semiotic resources—that is, signs in the conventional, non-linguistic sense as representations of something—interacting with other semiotic resources such as gestures and written signs, and how this contributes to the development of shared mathematical meaning. The embodiment perspective concerns how bodily and cultural experiences underlying mathematical thinking interact with signs used to refer to mathematical ideas—both in representing these experiences and shaping mathematical thought as embodied modes of learning. To capture our theoretical exploration, we adopt and adapt a conceptual framework developed within mathematics education that distinguishes different roles of language (so far only spoken) in teaching and learning mathematics. With this, we refrain from a deficit perspective but rather highlight Deaf learners as learners of mathematics—and their use of sign language a specific case of language in mathematics education.

Literature review: The role of sign language in mathematics thinking and learning

Researchers increasingly emphasize the role of sign language (SL) not only an indirect predictive factor related to mathematical skills (Hyde et al., 2003; Nunes, 2004), but also directly related as practice specific to the Deaf, crucially contributing to shaping their learning process (see Kurz & Pagliaro, 2020). For example, signed algorithms considered to help carry out mental calculations have been found to be commonly used among Deaf users of ASL (Nunes & Moreno, 1998) and Finnish SL (Rainò et al., 2018)[1]; similarly Healy and colleagues (2016) describe a Brazilian Deaf learner's individual strategy for mental multiplication as supported by the use of LIBRAS (Brazilian SL). The use of the counting string in ASL is also mentioned by Pagliaro and Ansell (2008) as ASL-related strategy of students solving story problems. As reported in Kurz and Pagliaro (2020), two other successful strategies observed concerned the "use of the inherent cardinality of the numbers signs 1-5" (p. 93) and the organization of the signing space such that it can be used "like a third device (after their hands) on which to keep track of the counting strings or manipulatives" (p. 93).

Healy and colleagues investigated gestural and signed expression of Deaf learners in geometric and algebraic contexts, and in the context of engaging with educational technology, setting a focus on the use of sign languages in the mathematics discourse (Healy, 2015; Fernandes & Healy, 2014; Healy et al., 2016; Magalhães & Healy, 2007). In a study carried out in a bilingual Brazilian classroom with five deaf and three hearing students, Healy (2015) described the development and use of signs and gestures in a mixed collaboration of a hearing and a deaf student (where "the hearing student in this pair spoke some LIBRAS and the deaf student was partially oralised" (p. 296)) when exploring and expressing symmetry and reflection through a Logo-programmed 'microworld' (expressive digital media based on principles such as invention, play and discovery, Papert, 1980). Similarly, Fernandes and Healy (2014), in a study in Brazil with six Deaf students, using a microworld "designed to encourage students to produce a variable procedure" (p. 51), observed the creation of a signed denotation of a variable *n* as fixed unknown value by one of the students. Using LIBRAS, she refers to *n* as the "secret number" (p. 53) and shares this interpretation of the variable to become adopted as used among the six students. Healy and colleagues point out the "process of coordinating bodily resources with visual, dynamic and linguistic signs in order to attribute meanings to mathematical objects" (p. 55) as one main aspect of the successful collaborative coordination of mathematical meaning in their teaching

[1] Interestingly, these are both SLs with one-handed signs for numbers. Whether there are similar algorithms in SLs that use two hands for number-signs, like German SL, and how they might look like, is an open question.

experiments. Inventing and negotiating ad-hoc signs in order to collaborate, the students embody their experiences in signs that reflect not only the specific case, but the students' shared conceptual understanding of the mathematical idea they encountered. Healy (2015) terms signs recalling an action through which the signed mathematical concept has been explored "imagined re-enactments" (p. 305).

In studies involving a German grade 5 geometry classroom of nine Deaf students and a Deaf teacher using German SL (*Deutsche Gebärdensprache*: DGS), Krause (2018; 2019) found such iconic re-enactments while focusing on the iconic aspects of the mathematical signs used. In particular, she traced how the teacher explicitly grounded his signs for 'axial symmetry' and 'point symmetry' in the respective actions of folding and reflecting (axial symmetry), and rotating around a point (point symmetry). Krause argued that in establishing the link between manual activity and sign, the teacher as a heritage DGS user and mathematics professional might provide a scaffold by using language as conceptual support. Although this practice is considered crucial (Kurz & Pagliaro, 2020), both its theorization and application are still in their infancy and classroom observations of processes of learning mathematics and of teaching practices are scarce.

Investigations of the specific features of Deaf students' processes of learning mathematics and working mathematically is especially in need as connected to the different ways of thinking that are considered to be related to Deaf learners' use of SL (Emmorey et al., 1993; Marschark, 2003; Marschark & Hauser, 2008). These can be seen as linked to the different affordances of SLs in comparison to spoken languages, as summarized by Grote and colleagues (2018), concerning, for example, the language modality as gestural-visual vs. vocal-auditive, or the degree of iconicity as strongly iconic in SL and less onomatopoeic in spoken language. The differences of Deaf learners' mathematics are hence more than just a matter of translation. They moreover concern the structuration of information, caused by the modalities of signed and spoken languages differing in articulation, perception, and processing, and guided by their linguistic features and rules. For example, the spatial-visual articulation of SLs enables simultaneous representation of information where spoken language expresses the same information linearly. Also, literature suggests that early exposure to SL leads to enhanced recall of visuospatial information and that signers have generally enhanced visuospatial skills, a preference for spatial coding, and less developed sequential cuing (Hall & Bavelier, 2010).

The different ways of thinking caused by language modality not only can be expected to shape the individual learner's understanding, they furthermore also manifest in communicative situations as expression become structured based on how concepts are organized on a cognitive level. From a socio-constructivist perspective, this changes not only the quality of the learning processes, but potentially also of the mathematical knowledge as outcome as in this perspective, this knowledge becomes constructed through the students' ongoing negotiation and validation of mathematical meaning in social discourse about mathematics (Bauersfeld, 1992).

Mathematics education research increasingly acknowledges the relationship between individual and social dimensions of thinking, learning, and knowing. In this contribution, we therefore explore the potential of three lenses within mathematics education research we consider especially suitable and significant for an approach to describing and understanding better the role of SL in mathematical thinking and learning. In particular, we consider those branches not in their entirety (as this would hardly be possible) but with respect to conceptual and theoretical perspectives influenced by our prior research background and existing expertise in the field of mathematics education, and by how we found them intersecting and complementing each other in compatible and harmonious ways.

We will start with building a *conceptual frame* that embeds the case of SL within the larger body of research on language in the teaching and learning of mathematics. More concretely, this will capture the roles of SL as a learning medium, a learning goal, a potential obstacle, a prerequisite for learning, and as a resource in the mathematics classroom. We will then outline *theoretical perspectives* on aspects of semiotics and embodiment as relevant from a mathematics education perspective and discuss the several roles of SL as a language in teaching and learning mathematics in these lights. The theoretical explorations will be illustrated through examples from German SL (DGS) and Austrian SL (*Österreichische Gebärdensprache*: ÖGS). We will then link the research to potential implications for practice before closing with final remarks about potential extensions and future perspectives.

Sign language as a case of language in mathematics education: a conceptual framework

Research on aspects of (so far, spoken) language in the teaching and learning of mathematics was, and still is, increasingly gaining attention among mathematics education scholars in the last decades, certainly not unrelated to the growing linguistic, cultural, and socioeconomic diversity in mathematics classrooms. Foci have been set on what it means to learn mathematics in a second language, how language proficiency is related to mathematics learning, bi- or multilingual settings in the mathematics classroom, or which aspects of language might support or hinder the learning of mathematics. Based on this, current research on language in mathematics education distilled the roles of language as *learning medium* and as *learning goal* (Lampert & Cobb, 2003), as a *prerequisite for learning* (Prediger & Schüler-Meyer, 2017), as a potential *obstacle for learning* mathematics (Prediger et al., 2019), and as a *resource for learning* (Planas, 2018).

This frames mathematics learning “as a discursive practice: doing mathematics essentially entails speaking mathematically (or writing or using other communicational modes)” (Morgan et al. 2014, p. 846). Mathematical meaning can then be considered as constructed either *through* using language—understanding mathematical objects as non-tangible per se but only accessible through using representations, including language signs (like words in spoken languages or signs of sign languages) (Duval, 2006)— or as constructed *in* using language, understanding the mathematical discourse itself as the learning process (Sfard, 2008).

Deaf learners and their use of SLs have been neglected in the work on language in mathematics education so far. However, some have implicitly suggested the potential integration of non-spoken or visual languages as well, for example, when juxtaposing “verbal and visual” languages (Planas, 2018, p. 216) or included in a definition of language as “a system of communication used by a particular country or community” (Morgan et al., 2014, p. 844, quoting the Oxford Dictionary online). We deem the current discussion in mathematics education incomplete without considering the distinct characteristics of learning mathematics in SLs. This section provides a background for exploring SLs as a specific case of language in the learning of mathematics and sets the terminology to frame our theoretical investigation of the roles of SL in mathematical teaching and learning processes through different theoretical lenses.

The roles of language as a learning goal, a learning obstacle, a prerequisite for learning

Communication about mathematics and becoming proficient in engaging in mathematical discourse are considered crucial for working mathematically, as also reflected in the integration of communication and language in mathematics as major topic in the standards for school mathematics in the United States (NCTM, 1989; 2000) and similarly in other countries (e.g., KMK, 2004 in Germany). In this, language becomes a *learning goal* in the mathematics classroom, including the appropriate use of mathematical terminology and, more generally, building up Cognitive Academic Language Proficiency (CALP; Cummins, 2000). This can be seen as deeply linked to both *language as an obstacle* and a *prerequisite for learning mathematics* as one's lack of competence to engage in mathematical discourse—both passively and actively—can cause constraints for learning on the individual's level as well as affect the learning of the whole class on a social level. In this case, language needs to become a learning goal in order to harness the functions of language as a *learning medium*.

Language as an obstacle for learning mathematics, for example, has been widely described in the literature related to students in various settings, especially to those with low language proficiency (see Prediger et al., 2019). Prediger and colleagues single out a number of potential obstacles on the word, sentence, and text level that influence the mathematical learning process in different ways. While deaf students have to face these with respect to written language too—maybe even more extensively considering deaf students' reported difficulties with perceiving and processing word problems (Hyde et al., 2003)—their specificities and their relation to students' processes of learning mathematics need to be revisited for Deaf signers in the context of SL. For example, mathematical vocabulary can fall in different categories that depend on the linguistic features of a specific language, distinguished in eleven categories for English mathematical vocabulary by Riccomini et al. (2015, p. 238). Some of these categories certainly apply to mathematical signs too, for example, the dependence of the mathematical reference of some words/signs depending on the context and having discipline-specific technical meaning, different to the meaning in everyday life, or that mathematical words/signs can be semantically related but show no similarity on a morphological level (see Kurz & Pagliaro, 2020). Other categories have no analogue in SL—like irregularities in spelling (singular/plural)—and if mathematical meanings of math signs are more or less precise than their everyday meanings is open to speculation. In addition, there might be other features of SL that shape mathematical vocabulary as part of the *learning medium*. We will turn towards signed mathematical vocabulary throughout this contribution, for example, with respect to iconicity and phonological features of SL, and hence consider signed mathematical vocabulary not as an obstacle per se but also as a potential resource for learning mathematics with respect to what will be described in the following.

Language as a resource for learning mathematics

Planas (2018, 2019) describes the notion of language as a resource for learning mathematics in the context of multilingual settings, acknowledging its potential surplus in the mathematics classroom in terms of its *pedagogical,* its *epistemic,* and its *political value.* In its pedagogical value, it is a means for orchestrating and fostering teaching and learning, including instruction and development of learning material that takes language into account. In its epistemic value, it concerns how language contributed in "creation and exchange" (Planas 2019, p. 21), or rather construction and negotiation, of knowledge. The political value of language has been scarcely elaborated so far in the context of mathematics education. With the pedagogical value concerning general pedagogical aspects—not necessarily related to disciplinary learning—our elaboration of SL as a resource for learning mathematics will focus on its epistemic value.

A semiotic perspective on sign languages in the learning of mathematics

Theories of semiotics deal with signs in a general sense[2]—distinguished, for example, by the modality in which they are produced as written signs, spoken signs, gestural signs,...—how they are used and how they are endowed with meaning. These theories can provide powerful tools for understanding better students' learning of mathematics by considering the role of the signs as they are a constitutive part of communication, social interaction, and mathematical activity (see, e.g., Arzarello, 2006; Duval, 2006; Krause, 2016; Wille, 2019a). Understanding a communicative act in its categorical (speech or sign) and imagistic (gesture) components (Goldin-Meadow & Brentari, 2017), SL shapes the learning in distinctive ways and a semiotic lens might help us to better understand how learning processes of Deaf students and hearing students relate to each other.

In this section, we consider the semiotic features of SLs as combining characteristics of linguistic signs, that is, certain rules of sign production and an underlying meaning structure (Ernest, 2006, pp. 69-70), with those of gesture signs as holistic, compound and partly idiosyncratic means of expression. We will investigate the potential of SL as a semiotic resource in mathematical learning processes in the light of linguistic signs and of gesture signs.[3] In particular, we will explore the idea of seeing mathematics as a sign game where the meaning of mathematical signs arises from their use (Wittgenstein, RFM). In this, the core of learning mathematics lies in becoming proficient in both engaging in mathematical activity and communicating about this activity (Wille, 2019a). With signs playing a role in being the objects of the mathematical activity as well as the means to communicate, we will see how the use of SL signs influences both in specific ways.

A Peircean understanding of signs and the role for diagrams in mathematical activity

Underlying is an understanding of signs following the American mathematician, logician, semiotician, and philosopher Charles Sanders Peirce (1839–1914), as "something which stands to somebody for something in some respect or capacity" (Peirce, CP, 2.228) and concerns the relationship between *that what represents* (the representamen, or sign-vehicle), *that what it represents* (the object) and *the respective way* in which the representamen is representing the object (the interpretant). In this triadic relation, this understanding differs from theories that assume predetermined meaning of signs in a signifier-signified relationship (de Saussure, 1995) and instead considers the meaning of signs as depending on interpretation. In particular, the sign can have the form of an index, an icon or a symbol in the ways the object determines the sign (CP, Peirce; for further reading e.g., Atkin, 2013): Indexes direct attention to something, like an arrow or a pointing finger, an icon reflects the relational structure within an object and thus creates an impression of similarity to features of the object, and a symbol is interpreted on the basis of habits or conventionalized rules. This distinction makes it obvious that a sign can be a symbol for one person and an icon or an index for somebody else, and can even mean different things for different people.

[2] The 'signs' in semiotics and those being linguistic entities of SL share the same written referent, causing complication in both writing and reading this section. The signs of SL being specific kinds of signs in the semiotic sense does not make this any easier. To minimize ambiguity, we will use 'SL signs' when referring to the signs of SL and 'signs' in the general semiotic context throughout this section.

[3] We distinguish gesture and sign language signs following the discussion raised in (Goldin-Meadow & Brentari, 2017), based on whether they are integrated in the communicative act rather in their categorical or imagistic component, and acknowledging that at times there cannot be a clear and objective distinction between both. We use the 'signed expression' to integrate expressions that include sign language as well as situated mathematical signs and gestures. However, both SL signs and gestures are clearly seen as discriminable from manipulative actions.

For example, the signs of spoken and signed languages are symbols following certain linguistic conventions that form the respective language. However, SL signs often evoke an iconic relationship to an action or object, some of them even transparent to persons with no prior experience with the language (Taub, 2001, p. 19).

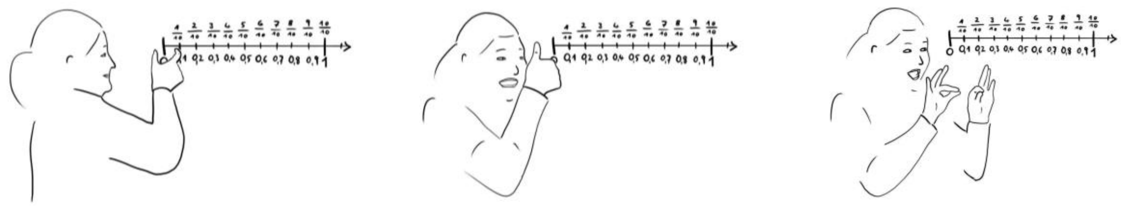
With respect to mathematics, the Peircean semiotics takes specific interest in *diagrams*, defined as “a representamen which is predominantly an icon of relations and is aided to be so by conventions. […] It should be carried out upon a perfectly consistent system of representation, founded upon a simple and easily intelligible basic idea" (Peirce, CP 4.418, 1903) and defining the rules for production and manipulation of diagrams. Note that Peirce's notion of diagram differs from an everyday-understanding of the term, not necessarily referring to a geometric context. Examples of diagrams in mathematics are mathematical notations such as variables, algebraic terms, equations, and also function graphs or geometric figures. It is through constructing diagrams, experimenting with, and manipulating on them—through *diagrammatic activity*—that it becomes possible "to discover unnoticed and hidden relations among the parts" (CP 3.363, 1885) and to gain new insights through observation (see Hoffmann, 2007). That is, dealing with diagrams and looking at them from different perspectives can give new ideas about the relations it can represent.

Diagrammatic activity in SL signs: examples from explanations in Austrian SL (ÖGS)

Coming back to the idea of mathematics as a sign game with diagrammatic activity in the center of learning mathematics, where mathematical meaning of signs arises in their use, diagrammatic activity is usually thought of as performed with mathematical inscriptions, for example, on paper. While diagrammatic activity can also be carried out purely through the use of spoken language, we claim that this is possible even to a much greater extent through the use of SL, as we will illustrate in an example of an explanation on decimal numbers, produced as video in the context of inclusive mathematics education for Deaf students (Wille, 2019b). The baseline for the video was the following German text, adapted to ÖGS by two native ÖGS signers: “Imagine we divide each section again into 10 parts. How many parts did you divide it into?” (English translation by AMW) (https://tinyurl.com/OegsVideoBrueche). Both the text as well as the video showed a number line as a mathematical inscription, the explanatory background was to get from tenths to hundredths. In this video it is discussed that a section on the number line should be divided into ten parts, talking about activity on the mathematical diagram. The video shows the ambiguity of talking *about* diagrammatic activity and the diagrammatic activity itself: In the beginning, two different ÖGS expressions are used to refer to “one tenth”: one works with indicating the section on the number line—a diagram of a geometric system of representation (Fig. 1a)—in the other one the SL sign resembles spatial aspects of the diagram 1/10 of a symbolic system of representations (Fig. 1b, c).

Fig. 1:

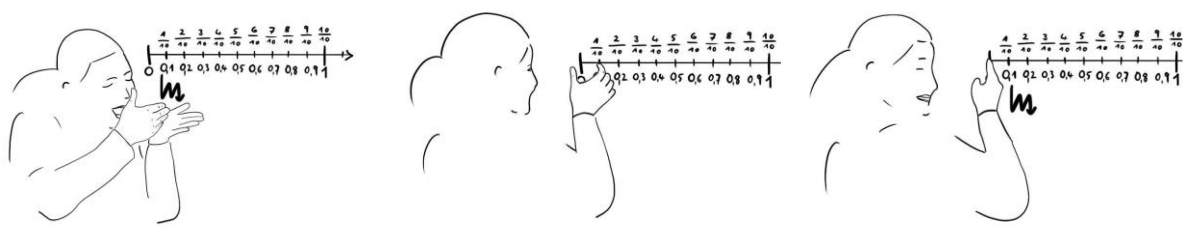
Indication of (a) the section “one tenth” on the number line, and the ÖGS signs for symbolic number (b) “one” (c) “tenth”



a b c

The diagrammatic activity of dividing a tenth into ten parts is first talked about using the general ÖGS sign for “dividing” (*teilen*), which resembles cutting into pieces with a knife (see Fig. 2a). Then, a different signed expression for dividing the tenths part further is accomplished directly on the number line, first indicating the segment of one tenth, combined with the ÖGS sign used before together with the number line (see Fig. 2b, c).

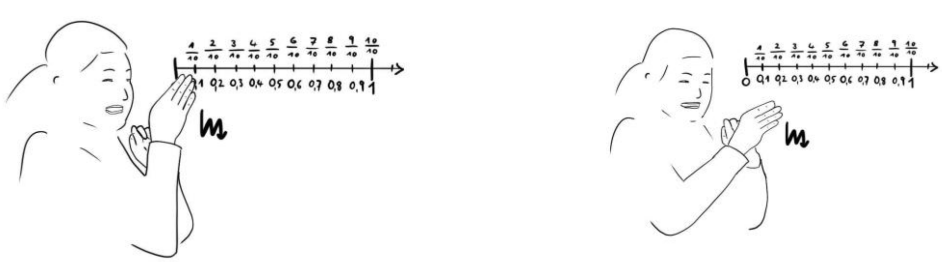
Fig. 2:
ÖGS sign for (a) “divide” (*teilen*), (b) indicating the section “one tenth” and (c) ÖGS sign for “divide” on a number line



a b c

Finally, this ÖGS sign for dividing moves from the written number line into the signing space (see Fig. 3a, b).

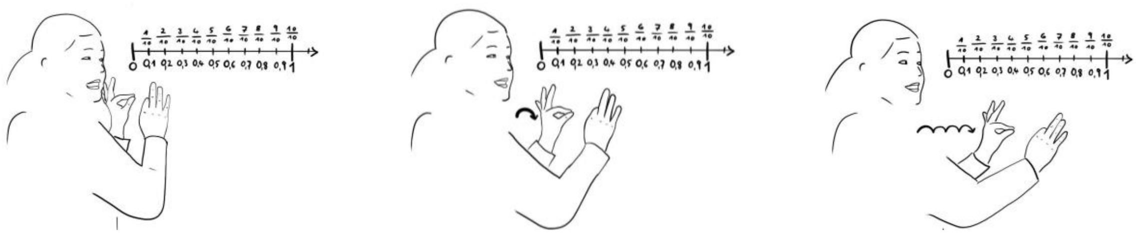
Fig. 3:
ÖGS sign for “divide” with a turning movement of the body



a b

Later a third way of referring to dividing into ten parts in ÖGS is used, combining the ÖGS sign for ten and a movement from left to right in the signing space (see Fig. 4a, b, c).

Fig. 4:
ÖGS signed expression for “each division into ten parts”



a b c

Hence, the signed explanation references the written diagram and its segmentation, the ÖGS signs for ‘dividing’ resemble the diagrammatic activity that would be carried out on paper (or on a whiteboard) on the number line.[4] This example shows that in some cases where diagrammatic activity and speaking about it can be clearly distinguished in spoken languages (leaving accompanying gestures aside for now), in SLs the line between both can be vague.

Iconicity and indexicality of mathematical SL signs in the development of meaning

Considering the meaning of signs as emerging in and through their use in Wittgenstein’s sign game as described previously, the question arises how the specific characteristics of mathematical SL signs might impact this meaning. For example, research in psycholinguistics provides evidence that “those features that are reflected in the iconic moment of sign language get a specific relevance for the whole semantic concept” (Grote, 2010, p. 316, translation by CK), meaning that this aspect might be associated stronger with the concept than those not reflected in the sign. The iconic aspects of SL signs used to talk about mathematical activity might hence influence the perceived meaning of these SL signs in certain ways (Krause, 2017a: 2017b; Wille, 2020). Similarities with mathematical diagrams, for example, could highlight some properties of the diagrams or the activity with it and leave others rather hidden. In the examples given previously, we can see this in the different SL signs referring to the subdivision / equal segmenting (hence, *what* in terms of activity) of the number line in parts. While the SL signs in Fig. 1, 2 and 3 emphasize the aspect of segmenting, the signed expression in Fig. 4 integrates and highlights the value “10” (*how many*).

[4] Both handshapes and their integration in the signed mathematical explanation—and more in general in the mathematical discourse—might become further discussed against the background of the idea of classifiers (see, e.g., Emmorey, 2003). While we see great potential in investigating further this connection in the context of mathematical thinking, learning, and teaching (within this semiotic approach and beyond), we acknowledge that this cannot be done in a sufficient way in this paper—considering both the authors’ currently still developing expertise on this matter as well as the space that would be needed to do justice to an introduction of classifier constructions and their potential integration in signed expression as either categorical (linguistic) or imagistic (gesture). To date, there is no empirical foundation to build on and only very few links between classifier handshapes and mathematical signs made in the existing literature (see (Kurz & Pagliaro, 2020) for an exception that mentions the use of the bent-L handshape adopted also in Fig. 1a and 2b as a classifier used to refer to numbers and quantities.). We therefore acknowledge that this topic would be served best in a future paper that puts its focus on classifiers in mathematical signs and signed mathematics discourse.

Wille (2020) furthermore explores the role of indexicality of mathematical SL signs and claims that this can influence the meaning of the signs as developing in their use when talking about mathematical activity. As one form of indexicality she refers to semantic fields as described in a categorisation of indexical signs in DGS suggested by Kutscher (2010). Following this, SL signs can indicate semantic fields through location of performance, indicating the reference to a certain class of signed concepts, like signs associated with cognitive processes (like thinking, forgetting, knowing) performed on the forehead. A mathematical example are the SL signs used for "minus", "times" (in the sense of multiplication), and "divided by" in DGS and ÖGS, indicating the symbolic notation for the operations "$-,\ \cdot, :$ " with the dominant hand in the palm of the non-dominant hand.

Wille extends this category of indexicality to resemblance in hand form, hand position or movement, claiming that each can be interpreted as indices that direct the attention to the semantic field (Wille, 2020). Krause refers to this as shades of *innerlinguistic iconicity* (2017, p. 93; 2019, p. 91). For example, the ÖGS signs for "formula" (*Formel*), "complicated" (*kompliziert*) and "crafting" (*basteln*) (see Fig. 5) differ only in the viseme, with simultaneous mouthing of the German word (not captured in Fig. 5). This can be interpreted as a reference to the semantic field of 'complicated things'. If a learner now uses such an indexical SL sign, this should have an influence on the meaning that emerges from it. For example, the innerlinguistic iconicity between the signs for 'formula' and 'complicated' might potentially lead the signer to perceive formulas as complicated and influencing a certain mindset towards mathematics.

Fig 5:
ÖGS signs for (a) "formula" (*Formel*), (b) "complicated" (*kompliziert*) and (c) "crafting" (*basteln*)

a b c

Kurz and Pagliaro (2020) support this claim and concretize this in the words of SL linguistics, referring to *phonological patterns*. These "often consist of one or more similar parameters (handshape, location, palm orientation, movement, and nonmanual markers) to portray a category of vocabulary or phrases that share similar characteristics, actions, or classifications" (p. 90). Kurz and Pagliaro argue that such patterns in spoken language "help the receiver to break down a word and make connections to its meaning." (p. 90) For SL users, this means that it might then become more difficult for a signer to break such patterns and link a mathematical SL sign to another representation not related to the semantic field.

The different roles of sign language in learning mathematics from a semiotic perspective

In this section about semiotics we had to distinguish SL signs from a more general notion of signs. This was not only an issue of terminology, but essential to understanding better the roles of *SL as a learning medium* and *as a resource for learning mathematics* as compared to spoken language in the hearing classroom. However, from what we have discussed we cannot identify any significant differences between SL and spoken *language as a prerequisite for learning*: In both cases, language is essential for talking about mathematical activity and hence, for learning mathematics.

One main characteristic of *SL as a learning medium* within the semiotic perspective concerns the interaction of signed expression and inscriptive (written and drawn) signs. In the Peirce-Wittgenstein approach, diagrammatic activities and talking about them are key components of learning mathematics and the examples show how SL signs—much more so than spoken signs—can actively take part in experimenting with and manipulate on existing diagrams and bear the potential to function as (visual) diagrams themselves. While one can argue that for spoken language, this role can be fulfilled by gestures accompanying spoken expression, an important difference is the way in which conventional meaning of SL signs can be complemented with idiosyncratic integration of gesture signs. This causes a potential inseparability of diagrammatic activity and talking about this activity, while the same requires two rather distinct processes in the spoken classroom. In Krause and Wille (*to appear*), we extend this semiotic perspective by a multimodal approach, describing how diagrammatic activity of hypothetically manipulating an inscriptive diagram through gesturally-simulated action becomes part of a mathematical sign eventually used in the classroom. The sign itself arises from diagrammatic activity and talking about it, and it becomes a diagram itself by incorporating aspects of this diagrammatic activity represented iconically.

Furthermore, two features of SL considered in psycholinguistic literature become important in the context of SL as a learning medium also from the semiotic perspective: iconicity and indexicality of signs, the latter congruent with so-called phonological patterns. In particular, these aspects concern an important feature of mathematical vocabulary specific to SLs as they might implicitly or explicitly influence the meaning that emerges from its use. In adding a semiotic perspective, we extend Kurz and Pagliaro's (2020) discussion of how this can be used with respect to *SL's potential as a resource for learning mathematics*.

Furthermore, we provide a new perspective on past research that considers *SL as a potential learning obstacle*, supporting these observations from a theoretical perspective on learning mathematics. While these have mainly focused on iconicity (e.g., Bryant, 1995), seeing phonological patterns as indexical feature of signs integrates them into the larger discourse of a semiotic understanding of learning mathematics in SL. In our example, associating functions with "complicated things" might have affective consequences on a student's approach to mathematics we need to be sensitive for. All the more, this highlights the role of *SL as a learning goal*: Students do not only need to be able to use SL to talk about mathematics, similar to hearing students in spoken language. In order to seize the potential of SL as a resource for learning, a further goal in the mathematics classroom needs to be to foster the development and discussion of meaning in and of signed mathematical vocabulary. However, more research needs to be done to get a better understanding of how we can leverage the semiotic potential of mathematical SL signs.

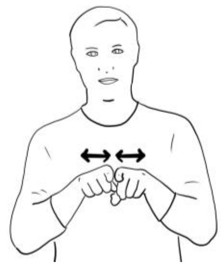
Emphasis on sign language as an embodied mode of learning

SL with its meaningful integration of hand-movements and bodily expression also becomes relevant from the perspective of embodiment theories of learning, as will be described and explored more in detail in this section. The relationship between thinking and learning as being grounded in bodily experience and using sign language has been considered by researchers in psycholinguistics (e.g., Grote et al., 2018; Inoue, 2006), the first author's previous work in mathematics education (Krause, 2017; 2018; 2019) and is also highlighted in another contribution in this special issue (Thom & Hallenbeck, *this issue*). This section aims at providing grounds to frame the embodied nature of SLs in the context of learning mathematics. In particular, we will look at how embodied experience with the world can shape mathematical thinking and links these to iconic and metaphoric features of signs and signed and gestural expression in mathematical discourse.

The role of metaphors

Embodiment theories root cognition in the body (Shapiro, 2014; Lakoff & Núñez, 2000; Nemirovsky, 2003; Varela et al., 1991). They build on the assumption that our bodily experience in the physical and cultural world grounds our cognitive processes. That is, the way we think and reason about mathematics emerges from the way we experience the world. Different scholars in mathematics education consider the body in mathematics from different perspectives and with different foci: From the standpoint of metaphors, bodily experiences allow us to understand mathematical ideas in terms of concrete physical actions (Lakoff & Núñez, 2000), *grounding* fundamental mathematical ideas in real world experience. For example, for understanding equality via the balance model—as balancing out the two sides as having equal value—and use it to reason about representing, manipulating, and solving equations (Filloy & Rojano, 1989), one needs to have experienced states of equilibrium and disequilibrium. Vice versa, experiences in the real world allow us to express individual mathematical approaches and mathematical understanding in terms of metaphors, consciously or not. While metaphors are originally a linguistic concept, they can be reflected in gestures (Edwards, 2009) as well as spoken language. In her studies of "Iconicity and Metaphor in American Sign Language", Taub (2001) described the relationship between iconicity and metaphorics as expressed in metaphorical-iconic signs in ASL, that is, in signs that express complex ideas through visual-spatial metaphor. While the sign itself reflects an idea in iconic similarity, it does not refer to this idea concretely but uses this idea in a transferred, metaphoric way, using a double mapping—first between concrete source idea and iconic referent, then between iconic referent and metaphoric goal idea (Taub, 2001, pp. 96-113). As Taub wrote, "if a metaphorical mapping exists that connects the abstract domain to a concrete domain, and if that concrete domain can be represented iconically by the language in question, the language user is in luck: He or she can construct a metaphorical–iconic linguistic item to represent the concept" (p. 110). This seems to be the case for mathematical SL: With SLs known to be rich in their iconicity, the link between a mathematical idea and the mathematical discourse about it can be much closer than is possible in spoken language. For example, the ÖGS sign for "equal to" used in mathematical contexts can be found to reflect the idea of the equilibrium in the scale in a metaphorical way (Fig. 11):

Fig. 11:
Sign for "equal" (*gleich*) in ÖGS und DGS



As it can also be seen as iconically representing the two bars of the equal sign in the extended index fingers, this sign can provide a link between the symbol, the concept, and the grounding metaphor of balance, as it is necessary to understand the equality of terms on two sides of the equal signs, an essential precondition for algebra learning. Integrating this in learning might provide a conceptual bridge that might help to tackle students' well-known struggle of understanding the concept of equality and the multifarious meaning of the equal-sign—often reduced to a signal for computation (Kieran, 2006) —and consequently, with algebra. While it is (yet) open to speculation if and how such signs actually influence and guide students' understanding of mathematical ideas, this shows how SL sign can provide a potentially more conceptually accessible representation as compared to spoken/written language.

Understanding and thinking as perceptuo-motor activities

Nemirovsky (2003) claims the origin of mathematical ideas lying in bodily activities, "having the potential to refer to things and events as well as to be self-referential" (p. 106), encompassing many mathematical ideas, like, for example, the idea of measuring, originally done with body parts like feet or forearms. This considers both the cultural-historically developed mathematics and the individual conceptualization of mathematical ideas as deeply rooted in the body, considering "understanding and thinking [as] perceptuo-motor activities" (p. 108)—for example, bodily actions, gestures, manipulation of materials—and "that of which we think emerg[ing] from and in these activities themselves" (p. 109). This resonates with an enactivist stance on embodiment focusing on the loops of perception and action that *situate* cognition as core of thinking. Mathematical thinking and learning is then considered as shaped by the body in that it both grounds and situates mathematical thinking and the understanding of mathematical concepts through building up fundamental sensorimotor patterns and navigating them in the moment. While theories of the embodied mind certainly encompass and emphasize more aspects, we consider the described framework that embeds situated enacted mathematical cognition in grounded mathematical cognition central for the aim of understanding SL as an embodied mode of learning.

Gestures as embodied resources

Gestures as embodied resource in mathematics have been fascinating mathematics educators both as a means to access mathematical thought, and considering their roles in mathematical thinking and learning (e.g., Alibali et al., 2014; Edwards, 2009; Gerofsky, 2010; Hall & Nemirovsky, 2012; Krause & Salle, 2019). Signs of SLs are certainly different from gestures, but both modes of expression share the same spatial-somatic modality. In this, the more comprehensive system of language as including the spontaneous production of idiosyncratic gestures next to signed expression leads to an observed hybridity of gesture and sign in mathematical discourse, potentially also grounded in action. For example, Krause (2018) described how a German Deaf mathematics teacher had his students explore the idea of axial symmetry in activities of folding and cutting paper and how he moved from these activities via gestures to his mathematical sign for axial symmetry (Fig. 12). In this, part of the sign reflects the action of unfolding, with the hand embodying the two parts of the paper on both sides of the folding line/axis. Transitioning from action to sign, the gestures simulate the action in combination with the paper as an artifact, leading to the sign as a situationally-conventionalized iconic model of the activity at hand. However, in this example the focus is more on the teacher, less on the learner. The gesture as it unfolds is used for explanation.

Fig. 12:
German teacher's DGS sign for 'axial symmetry' (*Achsensymmetrie*) (see Krause, 2018)

This example connects to a framework that considers representational gestures—for example, the teacher's gestures as representing the action of folding—as *simulated actions* (Hostetter & Alibali, 2008; 2018). It states that gestures depicting action, movement or shape, or that indicate location or trajectory "reflect the motor activity that occurs automatically when people think about and speak about mental simulations of motor actions and perceptual states" (2018, p. 721). Simulation here is understood as "the activation of motor and perceptual systems in the absence of external input" (p. 722), gesture production is linked to the activation of mental images of actions and perceptual states. Representational gestures then embody actions considered related to the task at hand by the person producing the gesture. This can also occur in metaphorical ways, such as simulating the action of grasping and putting when elaborating the solution to a mathematical task involving substitution (Krause, 2016). Although this framework was developed in the context of co-speech gestures, its tenets make it applicable also beyond, for example, for co-thought gestures (Hostetter & Alibali, 2018) and gestures produced while signing.

Consequences for theorizing instructional strategies

This framework might have interesting implications for teaching mathematics in SLs and the grounding of mathematical signs in perceptuo-motor activity in which mathematical understanding emerges. It allows for a much closer link between formal mathematical terminology, the concepts, and the activities in which they are born and raised. This relates to another semiotic model that combines an enactive approach to learning with semiotic representation: Within Bruner's (1966) model of establishing a mathematical concept through moving between three representational modes—enactive, iconic, symbolic—the mathematical sign, like the teacher's sign for "axial symmetry" (Fig. 12) can be considered a dynamic symbol, further bearing iconic features that can capture an aspect of the enactive representation. The symbol can hence still be enacted and mathematical discourse about the concept can encapsulate key features of the action as enactive representation informally in iconic gestural expression. This way, 'intermodal transfer'—a transfer between the different modes that should not end once arriving at the symbolic modality—not only becomes natural in the gestural modality of SL. It might also provoke a closer link to the production of the mathematical sign and of representational iconic gesture as simulating the action, arguing for a potentially easier recall of perceptuo-motor activities and activation of related sensorimotor patterns through mathematical signs.

The different roles of sign language in learning mathematics from an embodied perspective

From what we have seen within the embodied perspective, SL as a *medium for learning mathematics* can be characterized by two main aspects: First, SLs are highly iconic and with that, bear the potential for capturing metaphors through which mathematical ideas can be understood. Second, SLs are dynamic-visual and live in a modal hybridity with non-conventionalized gestural expression. These idiosyncratic gestures can be seen as simulated actions, physically enacting a motor activity or a physical state when it becomes (unconsciously) relevant for the task at hand. This modal hybridity allows for a smooth transition between an enacted approach to a mathematical idea and its conventionalized sign as bridged through using representational gestures.

This in itself reflects the high potential of SL as a *resource for learning*, guided through instruction. As described, intermodal transfer between action, iconic gestural expression, and symbolic sign becomes much more natural in SLs. This process needs, however, to be initiated and supervised in order to exploit its epistemic value, calling for respective teaching methods. Same needs to be mentioned for enabling students to realize and use the representational potential of signs as conceptual bridge, as the one for “equal”, presented in Fig. 10. It can hence be considered a *learning goal* to understand the metaphoric potential of mathematical SL signs in order to use them as a benefit for learning mathematics.

The embodiment perspective as we discussed it in this section does not allow to make statements about *SL as a learning obstacle* and *as a prerequisite for learning*. However, the mere absence of these aspects can be seen as linked to the nature of embodiment: Within this approach, learning does not start with language but originates from the body (e.g., Nemirovsky, 2003). While language is still an important prerequisite for conceptualizing meaning, as far as it concerns embodiment, it does not seem to make a difference if this prerequisite has the form of spoken or signed language.

Theory to practice: Challenges and opportunities

This contribution adapted theoretical perspectives from mathematics education to understand better how SL might influence Deaf learners’ mathematical thinking and learning by focusing on aspects specific for learning the discipline. In particular, this concerned the semiotics of mathematical learning, the embodied processes underlying the understanding and learning of mathematical concepts, and the development of mathematical meaning in and of signs.

Iconicity appeared to be a common thread, with both theoretical perspectives emphasizing its role differently. While research in Deaf education and psycholinguistics already pointed out the influence of iconicity in SL on conceptual understanding, the theoretical discussions in this paper provided potential explanations as grounded in theories of learning specific for mathematics. With that, they also reframed learning goals, potential learning obstacles as related to SL, more in general the role of SL as a medium and as resource for learning mathematics, and provided a background for developing methods to navigate these roles in the mathematics classroom in beneficial ways.

Thoughts on the potential of sign language as a resource in the mathematics classroom

From what we have observed, SL offers great potential as a resource to be integrated beneficially both in the Deaf mathematics classroom as well as in inclusive settings. Within a semiotic perspective we saw how diagrammatic activity can literally go hand in hand with talking about diagrams. It might be interesting to follow further into how this might become implemented into teaching practice as it might provide a fruitful opportunity for fostering students’ diagrammatic activity which might then also open a door to diagrammatic activity with and on inscriptive diagrams. Furthermore, semantic fields related to mathematical SL signs can become an explicit focus of the interaction of talking about mathematics. As that, they can be identified and displayed in, for example, on posters exhibited in the classroom. These suggestions support Kurz and Pagliaro's (2020) idea of letting the students become "language experts" and letting them seek out "patterns of meaning in specialized vocabulary and discourse" (p. 90) as emphasizing the influence of their use on the social learning process. Both from a semiotic and an embodied perspective, a discussion about 'where the mathematical signs come from', how they might relate iconically or metaphorically to an underlying action, inscription—that is, in which respect they are "conceptually accurate" (Kurz & Pagliaro, 2020, p. 87)—can be considered fruitful. In addition, students might think about what could be alternative signs based on their understanding of the mathematical idea and discuss these. This would not only provide a diagnostic opportunity for the teacher to access the students' understandings, but also fosters the students' changing perspectives in the sense of learning from an "other knowledgeable other" (Krause, 2019, p. 95).

Making SL signs an explicit topic in the inclusive classroom can furthermore become a tool that potentially benefits all the learners while highlighting the Deaf learners' practice of signing as a strength. It can become an additional representational resource that widens access to mathematical topics. For example, Wille (2019b) implemented the use of videos in which some fraction concepts become explained in ÖGS in an inclusive classroom with two Deaf signers. While the (captioned) videos were primarily used to facilitate access to the content for the Deaf students, avoiding a problematic attentional switch between the teacher's explanation and the interpreter, Wille described positive feedback also from hearing students as well as the teacher. This also concerns the explication of the signed mathematical terms, presented by the Deaf students following the video and becoming a topic of discussion in class. As that, the mathematical SL signs can fulfill a representational function as gestural signs (Krause, 2016) even for the hearing students and can hereby facilitate mathematical interaction in the inclusive classroom. Signed videos as those developed currently in ÖGS and those elaborated as multi-step tool encompassing 'concept – lecture (explanation) – term – definition' in ASL in the project 'ASL-clear' for several disciplines (https://aslclear.org/app/#/) might thus become a classroom tool in the sense of basic principles of *universal design for learning* (Rose & Meyer, 2002, p. 69).

The approaches to SL for learning mathematics pointed at aspects important for teacher education: It is not only important to be aware of the mathematical signs used in the classroom, but also that students can endow them with mathematical meaning through action as activity with diagrams and also as embodied experience. The semiotic and the embodied lens—and generally theories that focus on what characterizes the learning of mathematics and how this is related to SL—enable us to understand better how Deaf teachers integrate this in their teaching potentially implicitly, opening the door for methods that can be learned and reflected on by future teachers.

Future directions

Of course, this article only provided a brief glimpse into how theories from mathematics education can enable alternative perspectives on SL in the mathematics classroom. Different theoretical approaches would shift the focus to other aspects of SL, for example, given the close link between language and culture, a socio-cultural approach would focus on the relationship between SL and Deaf culture in the learning of mathematics. As Barton (2008) remarks: "If mathematics is the way mathematicians talk, then the cultural influences on that talk (the language of discourse, the meanings of words and symbols at the time of talk) create different mathematics" (p. 129), and it would be worthwhile to understand better the mathematics created through signed mathematical discourse.

We also only provided very specific perspectives within the theories we chose for our conceptual and theoretical frameworks. For example, due to the limited space in this paper, we simplified the idea of 'language as a learning medium' to a commonsense understanding. Language as a learning medium also concerns the central role language plays through its communicative—being a tool for exchanging information through a conventualized linguistic system—and cognitive functions—a tool for thinking mathematically—in the development of mathematical meaning (Maier & Schweiger, 1999). While this provides again new substance for theoretical explorations and discussions from semiotic as well as embodiment sides (e.g., related to cognitive functions of gestures (Krause & Salle, 2019)), we leave this to future research. Other aspects have fallen short due to limited space in this contribution, such as further differences of mathematical explanations in SL compared to spoken language concerning structuration as related to the affordances of SL (see Wille & Schreiber, 2019), and the role of classifiers (see footnote 5).

Also, the Deaf mathematics classroom is inherently bilingual, as signed and written language needs to be coordinated by the Deaf learners (in inclusive classrooms this issue is more complex and the handling of its multilinguistic character needs further discussion). Considering the relationship between bilingualism and logical reasoning (Secada, 1988), argumentation structures of Deaf signers might be worth exploring further. Not only is argumentation closely related to language and communication, its importance for learning mathematics is explicated as its own competence to develop in, for example, the NCTM principles and standards (NCTM, 1989, 2000) or the German standards for school mathematics (e.g., KMK, 2004). Argumentation has therefore been a well-researched topic in mathematics education (Sriraman & Umland, 2014). However, the affordances of SL as a medium in learning mathematics might lead to different qualities of argumentation in the Deaf classroom, worth investigating as its own but also in relation to argumentation structure in the second (written) language in the Deaf-as-bilingual classroom.

Concluding remarks

The starting point of this contribution has been to look at SL from the perspective of mathematics education, testing how the integration of theories of mathematical thinking and learning might help us understand better the role of SL in teaching and learning mathematics. The phenomena that caught our attention might not have been new, but the angle from which we considered them certainly is. New perspectives like these allow for a more comprehensive reflection about and understanding of what such phenomena might mean for Deaf students' learning of mathematics.

References

Alibali, M. W., Church, R. B., Kita, S., & Hostetter, A. B. (2014). Embodied knowledge in the development of conservation of quantity. In L. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspective on gesture and embodiment in mathematics* (pp. 27–50). Information Age Publishing.

Atkin, A. (2013). Peirce's Theory of Signs. In E. N. Zalta (Ed.), *The Stanford encyclopedia of Philosophy*. URL = <https://plato.stanford.edu/archives/sum2013/entries/peirce-semiotics/

Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME, Numero Especial*, 267-299.

Barton, B. (2008). *The language of mathematics - Telling mathematical tales.* Springer. https://doi.org/ 10.1007/978-0-387-72859-9

Bauersfeld, H. (1992). Classroom cultures from a social constructivist's perspective. *Educational Studies in Mathematics, 23*, 467-481. https://doi.org/10.1007/BF00571468

Bruner, J. S. (1966). *Toward a theory of instruction*. Belkapp Press. https://doi.org/10.1177%2F019263656605030929

Bryant, J. (1995). Language and concepts in geometry: Implications for sign language research. *FOCUS on Learning Problems in Mathematics*, *17*(3), 41–56.

Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire.* Multilingual Matters. https://doi.org/10.21832/9781853596773

de Saussure, F. (1995). *Cours de linguistique générale*. Paris: Payot.

Duval, R. (2006). A cognitive analysis of problems of comprehension in the learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics*, *70*, 127–141. https://doi.org/10.1007/s10649-008-9124-6

Emmorey, K. (Ed.) (2003). *Perspectives on classifier constructions in sign languages.* Erlbaum. https://doi.org/10.4324/9781410607447

Emmorey, K., Kosslyn, S. M., & Bellugi, U. (1993). Visual imagery and visual-spatial language: Enhanced imagery abilities in deaf and hearing ASL signers. *Cognition*, *46*, 139–181. https://doi.org/ 10.1016/0010-0277(93)90017-p

Ernest, P. (2006). A Semiotic Perspective of Mathematical Activity. *Educational Studies in Mathematics*, *61*(1/2), 67–101. https://doi.org/10.1007/s10649-006-6423-7

Fernandes, S. H. A., & Healy, L. (2014). Algebraic expressions of deaf students: Connecting visuo-gestural and dynamic digital representations. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol 3, pp. 49–56). PME.

Filloy, E., & Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra. *For the Learning of Mathematics 9*(2), 19–25.

Gerofsky, S. (2010). Mathematical learning and gesture: Character viewpoint and observer viewpoint in students' gestured graphs of functions. *Gesture 10*(2/3), 321–343. https://psycnet.apa.org/doi/10.1075/gest.10.2-3.10ger

Goldin-Meadow, S., & Brentari, D. (2017). Gesture, sign, and language: The coming of age of sign language and gesture studies. *Behavioral and Brain Sciences, 40.* https://doi.org/10.1017/S0140525X15001247

Grote, K. (2010). Denken Gehörlose anders? Auswirkungen der gestisch-visuellen Gebärdensprache auf die Begriffsbildung [Do Deaf people think differently? Influence of the gestural–visual sign language on conceptualization]. *Das Zeichen – Zeitschrift für Sprache* *und Kultur Gehörloser [The Sign - Journal for language and culture of the Deaf]*, *85*, 310–319.

Grote, K., Sieprath, H., & Staudt, B. (2018). Deaf Didaktik? Weshalb wir eine spezielle Didaktik für den Unterricht in Gebärdensprache benötigen [Deaf didactics? Why we need a special didactics for the signing deaf classroom]. *Das Zeichen [The Sign]*, *110*, 426–437.

Hall, M. L., & Bavelier, D. (2010). Working memory, Deafness, and sign language. In M. Marschark, P. E. Spencer (Eds.), *The Oxford handbook of Deaf Studies, language, and education, Vol. 2*. (pp. 457–472). Oxford University Press. https://doi.org/10.1111/j.1467-9450.2009.00744.x

Hall, R., & Nemirovsky, R. (2012). Modalities of body engagement in mathematical activity and learning. *Journal of the Learning Sciences*, *21*, 207–215. https://doi.org/10.1080/10508406.2011.611447

Healy, L. (2015). Hands that see, hands that speak: Investigating relationships between sensory activity, forms of communicating and mathematical cognition. In S. J. Cho (Ed.), *Selected regular lectures from ICME 12* (pp. 298–316). Springer. https://doi.org/10.1007/978-3-319-17187-6_17

Healy, L., Jahn, A.P., & Bolite Frant, J. (2010). Digital technologies and the challenge of constructing an inclusive school mathematics. *ZDM Mathem. Education*, *42*(3/4), 393–404. http://dx.doi.org/10.1007/s11858-010-0252-y

Healy, L., Ramos, E. B., Fernandes, S. H. A. A. & Botelho Peixoto, J. L. (2016). Mathematics in the hands of deaf learners and blind learners: Visual–gestural–somatic means of doing and expressing mathematics. In R. Barwell, P. Clarkson, A. Halai, M. Kazima, J. Moschkovich, N. Planas, M. Setati-Phakeng, P. Valero, & M. V. Ubillús (Eds.), *Mathematics education and language diversity (The 21st ICMI Study)* (pp. 141–162). Springer. https://doi.org/10.1007/978-3-319-14511-2

Hoffmann, M. H. G. (2007). Cognitive Conditions of Diagrammatic Reasoning. *Georgia Tech's School of Public Policy Working Paper Series*, 24 [PDF file]. Retrieved from https://smartech.gatech.edu/bitstream/handle/1853/23809/wp24.pdf

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review*, *15*, 495–514. https://doi.org/10.3758/PBR.15.3.495

Hostetter, A. B., & Alibali, M. W. (2018). Gesture as simulated action: Revisiting the framework. *Psychonomic Bulletin & Review*, *26*, 721–752. https://doi.org/10.3758/s13423-018-1548-0

Hyde, M., Zevenbergen, R., & Power, D. (2003). Deaf and hard of hearing students' performance on arithmetic word problems. *American Annals of the Deaf*, *148*, 56–64. https://doi.org/10.1353/aad.2003.0003

Inoue, T. (2006). Memory in deaf signers and embodied cognition of sign languages. *Japanese Psychological Research 48*(3), 223–232. https://doi.org/10.1111/j.1468-5884.2006.00315.x

Kieran, C. (2006). Research on the learning and teaching of algebra. A broadening of sources of meaning. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 11–49). Sense Publisher. https://doi.org/10.1163/9789087901127_003

KMK (Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland) (2004). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss. Beschluss vom 4.12.2003*. Wolters-Kluwer, Luchterhand Verlag. https://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2003/2003_12_04-Bildungsstandards-Mathe-Mittleren-SA.pdf

Krause, C. M. (2016). *The mathematics in our hands: How gestures contribute to constructing mathematical knowledge*. Springer Spektrum. https://doi.org/ 10.1007/978-3-658-11948-5

Krause, C. M. (2017a). DeafMath: Exploring the influence of sign language on mathematical conceptualization. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Congress of ERME* (pp. 1316–1323). Inst. of Education, Dublin University, ERME.

Krause, C. M. (2017b). Iconicity in signed fraction talk of hearing-impaired sixth graders. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of PME* (Vol. 3, pp. 89–96). National Institute of Education, Nanyang Technological University.

Krause, C. M. (2018). Embodied Geometry: Signs and gestures used in the deaf mathematics classroom – the case of symmetry. In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas, D. Wagner (Eds.), *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 171–193). Sense Publisher. http://doi.org/10.1163/9789004378735_009

Krause, C. M. (2019). What you see is what you get? – Sign language in the mathematics classroom. *Journal for Research in Mathematics Education, 50*(1), 84–97. https://doi.org/10.5951/jresematheduc.50.1.0084

Krause, C. M., & Salle, A. (2019). Towards cognitive functions of gestures – a case of mathematics. In M. Graven, H. Venkat, A. Essien, P. Vale (Eds.), *Proceedings of the 43rd Conference of PME* (Vol. 2, pp. 496–503). PME.

Krause, C. M. & Wille, A. M. (*to appear*). A semiotic lens on learning math in sign languages. In: *Contributions to TSG 60 (Semiotics in Mathematics Education) at the 14th International Congress on Mathematics Education (ICME-14).*

Kritzer, K. L. (2009). Barely Started and Already Left Behind: A Descriptive Analysis of the Mathematics Ability Demonstrated by Young Deaf Children. *The Journal of Deaf Studies and Deaf Education 14*(4). 409–421. https://doi.org/10.1093/deafed/enp015

Kurz, C., & Pagliaro, C. M. (2020). Using L1 sign language to teach mathematics. In: R. S. Rosen (Ed.), *The Routledge handbook of Sign Language pedagogy* (pp. 85–99). Routledge. https://doi.org/10.1093/deafed/enp015

Kutscher, S. (2010). Ikonizität und Indexikalität im gebärdensprachlichen Lexikon - Zur Typologie sprachlicher Zeichen. *Zeitschrift für Sprachwissenschaften 29*, 79–109. https://doi.org/10.1515/zfsw.2010.003

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Lampert, M., & Cobb, P. (2003). Communication and language. In J. Kilpatrick & D. Shifter (Eds.), *A research companion to the principles and standards for school mathematics* (pp. 237–249). National Council of Teachers of Mathematics.

Magalhães, G. R., & Healy, L. (2007). Questões de design de um micromundo para o estudo de concepções de provas produzidas por alunos surdos. In *Anais do IX Encontro Nacional de Educação Matemática (IX ENEM) ENEM)*. Belo Horizonte, Brazil.

Marschark, M. (2003). Cognitive functioning in deaf adults and children. In M. Marschark, & P. E. Spencer (Eds.), *Oxford handbook of Deaf Studies, language, and education* (pp. 464–477). Oxford University Press. https://doi.org/ 10.1093/oxfordhb/9780199750986.013.0034

Marschark, M., & Hauser, P. (2008). Cognitive underpinnings of learning by deaf and hard-of-hearing students: Differences, diversity, and directions. In M. Marschark, & P. C. Hauser (Eds.), *Deaf cognition: Foundations and outcomes* (pp. 3–23). Oxford University Press. https://psycnet.apa.org/doi/10.1093/acprof:oso/9780195368673.003.0001

Morgan, C., Craig, T., Schuette, M., & Wagner, D. (2014). Language and communication in mathematics education: an overview of research in the field. *ZDM Mathematics Education 46*, 843–853. https://doi.org/10.1007/s11858-014-0624-9

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. NCTM.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and evaluation standards for school mathematics*. NCTM.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In: N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.). *Proceedings of the 27th Conference of PME* (Vol. 1, pp. 105–109). PME.

Nunes, T. (2004). *Teaching Mathematics to Deaf Children*. Whurr Publishers.

Nunes, T., & Moreno, C. (1998). The signed algorithm and its bugs. *Educational Studies in Mathematics, 35*, 85–92. https://doi.org/10.1023/A:1003061009907

Pagliaro, C. M., & Ansell, E. (2008). Aspects of American Sign Language Used by Deaf Children to Facilitate Successful Problem Solving. *Paper presented at the 34th Annual Meeting of the Association of College Educators – Deaf and Hard of Hearing,* Monterey, CA.

Papert, S. (2002). The Turtle's long slow trip: Macro-educological perspectives on microworlds. *Journal of Educational Computing Research, 27*(1), 7–28. https://doi.org/10.2190%2FXG11-B72E-JK04-K8TA

Prediger, S., & Schüler-Meyer, A. (2017). Fostering the mathematics learning of language learners: introduction to trends and issues in research and professional development. *Eurasia Journal of Mathematics, Science & Technology Education, 13*, 4049–4056. https://doi.org/10.12973/eurasia.2017.00801a

Prediger S., Erath K., & Opitz E. M. (2019). The Language Dimension of Mathematical Difficulties. In A. Fritz, V. Haase, P. Räsänen (Eds.), *International handbook of mathematical learning difficulties* (pp.437–455). Springer. https://doi.org/10.1007/978-3-319-97148-3_27

Planas, N. (2018). Language as resource: a key notion for understanding the complexity of mathematics learning. *Educational Studies in Mathematics 98,* 215–229. https://doi.org/10.1007/s10649-018-9810-y

Planas, N. (2019). Transition zones in mathematics education research for the development of language as resource. In M. Graven, H. Venkat, A. Essien, P. Vale (Eds.), *Proceedings of the 43rd Conference of PME* (Vol. 1, pp. 17–32). PME.

Rainò, P., Ahonen, O., & Halkosaari, L. (2018). The deaf way of interpreting mathematical concepts. In C. Stone (Ed.), *Deaf Interpreting in Europe. Exploring best practice in the field* (pp. 10–20). Copenhagen: Danish Deaf Association. Available from the Erasmus+ project Developing Deaf Interpreting website: https://www.deaf-interpreters.com/publications

Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015). The Language of Mathematics. The Importance of Teaching and Learning Mathematical Vocabulary. *Reading & Writing Quarterly, 31*(3), 235–252. http://doi.org/10.1080/10573569.2015.1030995

Rose, D. H., & Meyer, A. (2002). *Teaching every student in the digital age: Universal design for learning*. Association for Supervision and Curriculum Development. http://doi.org/10.1007/s11423-007-9056-3

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Shapiro, L. (2014). *Routledge handbook of embodied cognition*. Routledge.

Sriraman B., & Umland K. (2014). Argumentation in Mathematics Education. In: Lerman S. (Ed.), *Encyclopedia of Mathematics Education*. Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_11

Taub, S. (2001). *Language from the body - Iconicity and metaphor in American Sign Language.* Cambridge University Press. https://psycnet.apa.org/doi/10.1017/CBO9780511509629

Thom, J., & Hallenbeck, T. (*this issue*). Spatial reasoning, embodied mathematics, and DHH education.

Traxler, C. B. (2000). The Stanford Achievement Test, 9th Edition: National Norming and Performance Standards for Deaf and Hard-of-Hearing Students. *Journal of Deaf Studies and Deaf Education*, *5*(4), 337–348. https://doi.org/10.1093/deafed/5.4.337

Varela, F. J., Thompson, E. T., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. MIT Press. https://doi.org/10.7551/mitpress/6730.001.0001

Wille, A. M. (2019a). Activity with signs and speaking about it: Exploring students' mathematical lines of thought regarding the derivative. *International Journal of Science and Mathematics Education.* https://doi.org/10.1007/s10763-019-10024-1

Wille, A. M. (2019b). Einsatz von Materialien zur Bruchrechnung für gehörlose Schülerinnen und Schüler im inklusiven Mathematikunterricht [Use of materials about fractions for deaf students in inclusive mathematics lessons]. In: *BzMU 2019* (pp. 901–904). WTM Verlag.

Wille, A. M. (2020). Mathematische Gebärden der Österreichischen Gebärdensprache aus semiotischer Sicht [Mathematical signs of Austrian Sign Language from a semiotic point of view]. In G. Kadunz (Ed.), *Zeichen und Sprache im Mathematikunterricht* (pp. 193–214). Springer. https://doi.org/10.1007/978-3-662-61194-4_9

Wille, A. M., & Schreiber, Ch. (2019). Explaining geometrical concepts in sign language and in spoken language – a comparison. In: U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.). *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (pp. 4609–4616). Freudenthal Grp. & Freudenthal Inst., Utrecht University & ERME. https://hal.archives-ouvertes.fr/hal-02435340

Wittgenstein, L. (RFM). *Remarks on the foundation of mathematics* (G. H. Wright, R. Rhees, G. E. M. Anscombe, Eds.) (G. E. M. Anscombe, Trans.). 1967. MIT Press.