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Sandine, Ely

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On Hunter-type Gravitational Collapse Profiles for the Euler-Poisson System

By

Ely Sandine

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Sung-Jin Oh, Chair

Professor Ruixiang Zhang

Professor Maciej Zworski

Spring 2026

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by

Ely Sandine

Abstract

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Professor Sung-Jin Oh, Chair

We rigorously construct a family of smooth self-similar solutions to the isentropic gravitational Euler-Poisson system with a polytropic equation of state $p = \rho^\gamma$. The problem can be expressed as that of finding global solutions to a system of non-linear, non-autonomous ODEs which feature singular points.

The isothermal case, $\gamma = 1$, is simpler. In this case self-similar solutions were found numerically by Larson, Penston and Hunter in their investigations of spherical gravitational collapse. For $\gamma = 1$, we construct a sequence of solutions which corresponds to a subset of the Hunter family. The proof is based on a bifurcation-type argument developed in the study of blow-up for semilinear PDEs and adapts this framework to the quasilinear setting.

The polytropic case, $\gamma > 1$, is more challenging. In this case, we extend our isothermal construction to the energy-supercritical range of polytropic indices $1 < \gamma < \frac{6}{5}$. As an ingredient in the proof, we introduce a general method for proving the local analyticity of the solution near the singular points of the ODE system.

This thesis is dedicated to my parents, who put me on this road, and also to all the amazing people that I have met along the way.

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Chapter 1

Introduction

The Euler-Poisson system of partial differential equations describes the motion of a self-gravitating compressible gas. Physically, it is an elementary model for the formation, evolution and collapse of stars. Mathematically, the system is classified as quasilinear and hyperbolic. In this thesis, we prove the existence of certain gravitational collapse profiles for the system. These profiles are self-similar and the existence proof is based on a bifurcation-type argument inspired by works on related semilinear models.

This introductory section is expository in nature. In Section 1.1 we introduce the Euler-Poisson system, the main semilinear models and several associated “toy” equations. In Section 1.2, we compare the literature on blow-up for three main equations, emphasizing methods based on self-similar analyses. In Section 1.2.4 we discuss the method we use to construct self-similar solutions and compare the techniques to others in the literature.

Chapter 2, consisting of the paper [100], proves the existence of certain gravitational collapse profiles for polytropic index $\gamma = 1$. Chapter 3, consisting of the paper [101], generalizes the previous construction to the range of polytropic indices $1 < \gamma < \frac{6}{5}$.

1.1 On Blow-up for Several PDEs

We begin with a PDE which does not blow up for smooth data: the linear wave equation for $\varphi: \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}$. The equations are

$$\begin{cases} \partial_t^2 \varphi - \Delta_x \varphi &= 0 \\ \varphi(0, x) &= \varphi_0(x) \\ \partial_t \varphi(0, x) &= \varphi_1(x) \end{cases} \quad (\text{LW})$$

We begin by reviewing a standard proof that if $\varphi_0, \varphi_1 \in C^\infty(\mathbb{R}_x^3)$ then $\varphi \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3)$.

To begin the proof, we recall that by conservation of energy, we have

$$\|\nabla_x \varphi(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_t \varphi(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 = \|\nabla_x \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_1\|_{L^2(\mathbb{R}^3)}^2.$$

By commuting the equation with ∂_x^α for multi-indices α , we obtain that for any $k \in \mathbb{Z}_{\geq 0}$,

$$\|\varphi(t, \cdot)\|_{\dot{H}^{k+1}(\mathbb{R}^3)}^2 + \|\partial_t \varphi(t, \cdot)\|_{\dot{H}^k(\mathbb{R}^3)}^2 = \|\varphi_0\|_{\dot{H}^{k+1}(\mathbb{R}^3)}^2 + \|\varphi_1\|_{\dot{H}^k(\mathbb{R}^3)}^2,$$

where $\dot{H}^k$ denotes the homogeneous Sobolev spaces for $k \geq 0$:

$$\|f\|_{\dot{H}^k} = \| |\xi|^k \hat{f} \|_{L^2(\mathbb{R}^3)}.$$

We next recall the following Sobolev embedding estimate for $f \in C_0^\infty(\mathbb{R}^3)$:

$$\|f\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{1}{2}} \|f\|_{\dot{H}^2(\mathbb{R}^3)}^{\frac{1}{2}}.$$

Combining the Sobolev embedding with energy conservation, we have

$$\begin{aligned} \|\varphi\|_{L^\infty} &\lesssim \|\varphi(t, \cdot)\|_{\dot{H}^2}^{\frac{1}{2}} \|\varphi(t, \cdot)\|_{\dot{H}^1}^{\frac{1}{2}} \\ &\lesssim (\|\varphi_0\|_{\dot{H}^2} + \|\varphi_1\|_{\dot{H}^1})^{\frac{1}{2}} (\|\varphi_0\|_{\dot{H}^1} + \|\varphi_1\|_{L^2})^{\frac{1}{2}}. \end{aligned}$$

This gives boundedness of φ , and higher derivatives can be controlled by commuting with derivatives, yielding $\varphi \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3)$. We comment that this argument requires some integrability of $(\nabla\varphi, \varphi_1)$ and their derivatives, but these conditions can be removed due to finite speed of propagation, concluding the proof.

We comment that while there is a global-in-time bound on $\|\varphi\|_{L^\infty}$, in contrast to a transport or heat-type equation, we do not have L^∞ boundedness of the wave flow. In particular the following estimate fails:

$$\|\partial_t \varphi\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^3)} + \|\nabla_x \varphi\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^3)} \lesssim \|\varphi_1\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \varphi_0\|_{L^\infty(\mathbb{R}^3)}.$$

This lack of boundedness is due to focusing. One way to think about focusing is as the time-reversal of dispersion. It is well-known that given smooth and localized data for the wave equation, after sufficiently long time, the solution will be concentrated near $|x| = t$ and obey $|\partial_t \varphi|, |\nabla_x \varphi| \lesssim t^{-1}$. If one then time reverses the flow, we see that initial data which is $O(1)$ on an annulus of radius R can focus to be $O(R)$ in amplitude after $t = O(R)$. In the rest of this thesis, we will consider PDEs which, unlike the linear wave equation, can blow up in finite time. Focusing will be one possible growth mechanism to be considered.

We now discuss blow-up for three toy models, beginning with the Riccati ODE for $u: I_t \rightarrow \mathbb{R}$ (here and throughout I_t denotes an open time interval):

$$\begin{cases} \dot{u} = |u|^{p-1}u \\ u(0) = u_0 \end{cases}, \quad (\text{R1})$$

where $p > 1$. This ODE can be solved explicitly as

$$u(t) = \frac{u_0}{(1 - (p-1)|u_0|^{p-1}t)^{\frac{1}{p-1}}}. \quad (1.1)$$

In particular, if $u_0 \neq 0$, we have that the solution will be unbounded as t approaches the blow-up time $t = ((p-1)|u_0|^{p-1})^{-1}$. This is the simplest example of what we mean by a differential equation blowing up and will serve as a model for more complicated PDEs. We comment that a key difference between (LW) and (R1) is that while the former has a conserved energy which constrains the solution, one should think of the latter as the gradient

flow associated to the potential $V(u) = -|u|^{p+1}/(p+1)$. This potential has a maximum, not a minimum, at the origin and so solutions are pushed away from equilibrium.

A variation on the above ODE is the following second-order Riccati equation for $u: I_t \rightarrow \mathbb{R}$:

$$\ddot{u} = |u|^{p-1}u. \quad (\text{R2})$$

Unlike first equation, there is no solution formula for general p . Nonetheless, if one defines the energy to be

$$E = \frac{1}{2}\dot{u}^2 - \frac{1}{p+1}|u|^{p+1},$$

then it will be conserved. Using the energy, one can easily sketch integral curves in the $(u, \dot{u})$ -plane. A rough sketch is provided in Figure 1.1. In particular, $E = 0$ consists of four branches, which correspond to the explicit solutions

$$u_{\pm,s}(t) = \pm \frac{c_p}{(-t)^{\frac{2}{p-1}}} \quad \text{and} \quad u_{\pm,u}(t) = \pm \frac{c_p}{t^{\frac{2}{p-1}}}, \quad (1.2)$$

(the subscripts s, u stand for stable and unstable) where

$$c_p = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}.$$

In particular, unlike (R1), there exists solutions which do not blow up, namely $u_{\pm,u}(t)$. However, we may see from the phase portrait that these solutions are unstable, and general data will blow up along $u_{\pm,s}(t)$. In comparison to (R1), the equation (R2) is closer to (NLW) due to its time-reversibility. The energy does not suppress blow-up though as the second term is non-positive. We also comment that in this case the blow-up is due to a self-amplification of the solution. The growth mechanism is different than the focusing effect of (LW).

The third elementary blow-up model we shall mention is the inviscid Burger's equation for $u: I_t \times \mathbb{R}_x \rightarrow \mathbb{R}$:

$$\begin{cases} \partial_t u + u \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases}. \quad (\text{IB})$$

In this case, given Lipschitz data the equation can be solved locally via the method of characteristics. In this case (as u is constant along characteristics), one obtains

$$\|u(t, \cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

In particular, the amplitude of u cannot blow up and there are no wave-like focusing effects. In fact, one can see by integrating by parts that $\|u\|_{L^p}$ is bounded for all $1 \leq p < \infty$, and so at first glance one might expect smooth solutions to remain smooth. This is false however, and one way to see this is to apply ∂_x to (IB), which results in

$$(\partial_t + u \partial_x)(\partial_x u) + (\partial_x u)^2 = 0.$$

From this equation, one sees that if $u'_0(x_0) < 0$, then along the x_0 characteristic, $x(t)$, we have that $(\partial_x u)(t, x(t))$ will obey the Riccati equation (R1) with $p = 2$ and hence blow up in

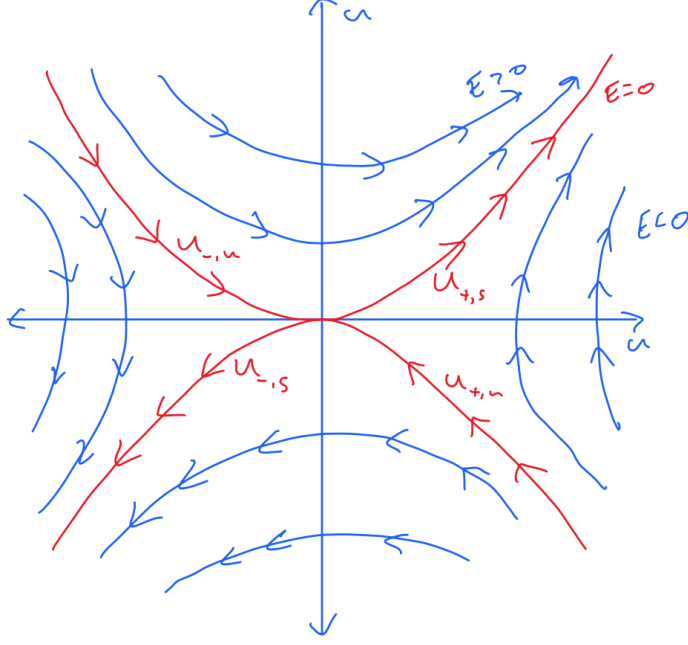


Figure 1.1: Phase Portrait of (R2)

finite time. Since u remains bounded at the blow-up time, this can be thought of as a mild singularity. The equation can be formulated as the conservation law

$$\partial_t u + \partial_x \frac{u^2}{2},$$

Invoking the entropy condition allows one to uniquely continue the PDE locally past a smooth pre-shock as a discontinuous shock solution. This PDE shows that quasilinearity can introduce another blow-up mechanism that only appears after differentiation. It also introduces the concept that a mild blow-up need not be the end of an equation's predictability.

The three blow-up examples discussed above, (R1), (R2) and (IB), are all solvable in closed form and feature no effects to counteract the singularity formation mechanism. We now introduce the three main PDEs to be discussed in this introduction. The main PDEs are the *focusing nonlinear heat equation* for $\varphi: I_t \times \mathbb{R}_x^d \rightarrow \mathbb{R}$:

$$\begin{cases} \partial_t \varphi - \Delta \varphi - |\varphi|^{p-1} \varphi = 0 \\ \varphi(0, x) = \varphi_0(x) \end{cases}, \quad (\text{NLH})$$

the *focusing nonlinear wave equation* for $\varphi: I_t \times \mathbb{R}_x^d \rightarrow \mathbb{R}$

$$\begin{cases} \partial_t^2 \varphi - \Delta \varphi - |\varphi|^{p-1} \varphi = 0 \\ \varphi(0, x) = \varphi_0(x), \quad \partial_t \varphi(0, x) = \varphi_1(x) \end{cases}, \quad (\text{NLW})$$

and the *irrotational polytropic compressible Euler-Poisson equations* for $\rho: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}_{>0}$,

$u: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$ and $F: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + u \cdot \nabla u + \rho^{-1} \nabla \rho^\gamma + F = 0 \\ \nabla \times u = 0, \quad \nabla \times F = 0, \quad \operatorname{div} F = 4\pi \rho \\ \rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x) \end{cases} \quad (\text{EP})$$

The first two equations can be thought of as model problems where one couples the linear heat and wave equations with (R1) and (R2) respectively. We remark that (at least formally), one can view (NLH) as the gradient flow of the energy

$$E_{NLH} = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{p+1} |\varphi|^{p+1} dx. \quad (1.3)$$

Similarly, one can view (NLW) as the Hamiltonian flow of the energy

$$E_{NLW} = \int_{\mathbb{R}^3} \frac{1}{2} |\partial_t \varphi|^2 + \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{p+1} |\varphi|^{p+1} dx. \quad (1.4)$$

A non-trivial question for these model PDEs is whether, given generic initial data, the diffusive or dispersive effects respectively will beat out the Ricatti amplification effect. We remark that title of “focusing” in these equation describes the sign of the nonlinear term and is different from the focusing effect described previously.

The Euler-Poisson system requires more explanation and motivation. In this case, ρ and u describe the density and velocity of a gas respectively. The gas is assumed to be isentropic and the particular pressure law of ρ^γ is chosen for convenience. One should think of $\gamma \geq 1$ as a parameter, analogous to the choice of p for (NLH) and (NLW). The irrotationality condition $\nabla \times u = 0$ is chosen to simplify the problem. Indeed, in this case, assuming no external force on the gas, conservation of mass and momentum lead to the *irrotational polytropic compressible Euler equations* for $\rho: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}_{>0}$ and $u: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + u \cdot \nabla u + \rho^{-1} \nabla \rho^\gamma = 0 \\ \nabla \times u = 0 \\ \rho(0, \cdot) = \rho_0(\cdot), \quad u(0, \cdot) = u_0(\cdot) \end{cases}, \quad (\text{CE})$$

This system has several conserved quantities, including the mass

$$M = \int_{\mathbb{R}^3} \rho dx \quad (1.5)$$

and the energy

$$E_{CE} = \begin{cases} \int_{\mathbb{R}^3} \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma-1} \rho^\gamma dx & \gamma > 1 \\ \int_{\mathbb{R}^3} \frac{1}{2} \rho |u|^2 + \rho \log \rho & \gamma = 1 \end{cases}. \quad (1.6)$$

The irrotationality condition allows one to formulate (CE) as a quasilinear wave equation. One can also consider it to be a multi-dimensional generalization of (IB). Indeed, if one

considers plane-symmetry, rewrites the equation in terms of Riemann invariants and sets one Riemann invariant to be constant (the simple wave ansatz), one obtains (IB) exactly. As such, it is physically unsurprising that (CE) can form shocks from smooth data. In addition to shock singularities, in contrast to (IB), the system (CE) allows for smooth solutions which implode ($\rho \rightarrow \infty$) in finite time, as discovered recently ([78]). This is surprising as there is no analog for (LW), and the conserved mass and energy are both coercive. Heuristically, one way to think about this as a focusing of acoustic waves which interact nonlinearly. The conserved quantities constrain the possible scalings of this blow-up, but do not prevent it. One expects these solutions to be unstable (see, for instance, [5]), but understanding this instability rigorously remains an open problem.

To derive (EP), we add to (CE) an assumption that the gas self-gravitates according to Newtonian mechanics. In this case F represents the gravitational force. The gravitational potential solves a Poisson equation and acts on the gas. The equations arise in astrophysics when one models an isolated body (such as a cloud of dust or a star), evolving in a regime in which relativistic effects are negligible. In contrast to (CE), there is a natural instability mechanism introduced by the gravity, and so it is unsurprising that the density can become infinite in finite time, a situation referred to as implosion or gravitational collapse. Nonetheless, pressure combats this Riccati mechanism, and the equations also allow for shocks. As such, rigorously understanding the stability of gravitational collapse for this equation presents serious mathematical challenges. We comment that (EP) still conserves the mass, (1.5), but the energy should be modified to

$$E_{EP} = \begin{cases} \int_{\mathbb{R}^3} \frac{1}{2} \rho u^2 + \frac{1}{\gamma-1} \rho^\gamma - \frac{1}{8\pi} |F|^2 dx & \gamma > 1 \\ \int_{\mathbb{R}^3} \frac{1}{2} \rho u^2 + \frac{1}{\gamma-1} \rho^\gamma - \frac{1}{8\pi} |F|^2 dx & \gamma = 1 \end{cases} \quad (1.7)$$

We also note that if one drops the pressure term from (EP), one obtains the *irrotational Euler-Dust equations* for $\rho: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}_{>0}$, $u: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$ and $F: \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + u \cdot \nabla u + F = 0 \\ \nabla \times u = 0, \quad \nabla \times F = 0, \quad \operatorname{div} F = 4\pi \rho \\ \rho(0, \cdot) = \rho_0(\cdot), \quad u(0, \cdot) = u_0(\cdot) \end{cases}, \quad (\text{ED})$$

which we will view as a toy model for (EP). Roughly, (ED) is to (EP) what the ODEs (R1) and (R2) are to (NLH) and (NLW) respectively. In addition, as the wave equation structure is lost, (ED) can be solved by the method of characteristics in radial symmetry, allowing one to understand pressure-less gravitational collapse quite explicitly.

1.2 Self-Similar Techniques

Both (NLH) and (NLW) can exhibit singularities in finite time for large data. A basic mathematical problem is to identify initial data which definitively lead to blow-up. In addition, one would like to know the local behavior of the blowing-up solution at the time of singularity formation. One motivation for this is the problem of continuing the solution past the time of blow-up.

For (NLW), in the case of $p > 1$ an odd integer, it is particularly easy to construct blowing up solutions. Firstly, one notes that the Ricatti solution $\varphi(t, x) = u_{+,s}(t-1)$, defined in (1.2), solves (NLW) and blows up at $t = 0$. In this case, $\partial_t \varphi$ will be constant, and not lie in L^2 , so the energy integral diverges. This can be remedied by multiplying the initial conditions by a compactly supported cutoff χ so that $\chi(x) = 1$ for $|x| \leq 1$. By finite speed of propagation, the cutoff data's development will agree with $u_{+,s}(t-1)$ inside the backwards light cone of $(t, x) = (1, 0)$ and hence blow up. Thus, we see that C_0^∞ initial data can lead to blow-up. We mention three undesirable features of this construction. Firstly, it is possible that the cutoff introduces earlier singularities. Secondly, at the point of blow-up, we have $\partial_x^\alpha u = 0$ for all multi-indices α which is non-generic. In particular, since the blow-up is dominated by (R2), one might expect the first singularity to occur at a local maximum of φ with $\partial_x^2 \varphi < 0$, and in theory the Laplacian could then prevent blow-up. Thirdly, it uses finite speed of propagation in a very strong way, and does not generalize to the heat equation (NLH), or to quasilinear problems, such as (EP), where for large data the speed of propagation of outside disturbances can be large.

A way to get around these issues is to “zoom in” on the singularity using self-similar coordinates. Mathematically, the idea is that one can refine the choice of $u_{+,s}(t-1)$ in the above argument with a solution to the PDE (or part of the PDE) which is invariant with respect to a scaling symmetry. This is called the self-similar profile. One can then replace the cutoff part of the argument with a proof that the self-similar profile is locally stable in a suitable sense to either perturbations of the initial data or the equation.

We remark that similarity solutions have been studied in physics, engineering and mathematical contexts for a long time, see, for instance, [49]. In the recent PDE literature, self-similar variables were introduced by Giga and Kohn to study the stability of blow-ups to (NLH) in the series of works [43], [44] and [45]. A nice introduction to the use of self-similar profiles for studying singularities is the book [39].

We comment that while we focus on self-similar methods, there is a long history of studying blow-up for (NLH) and (NLW). For (NLW), we refer interested readers to the books [107], [2] and [108].

1.2.1 Self-Similar Solutions for the Toy Models

To begin our discussion of self-similar solutions, we begin with the four toy models, (R1) (R2) (IB) and (ED), where it turns out that one can find explicit self-similar solutions. We derive these solutions in this section, and in the next we describe how they can be used to construct blow-ups for the more complicated PDEs.

We begin by considering (R1), which we now view as a PDE for $u: I_t \times \mathbb{R}_x^d \rightarrow \mathbb{R}$. The PDE is invariant under the two-parameter family of scalings given by

$$u(t, x) \rightarrow u_{\lambda_1, \lambda_2} := \frac{1}{\lambda_1^{\frac{1}{p-1}}} u\left(\frac{t}{\lambda_1}, \frac{x}{\lambda_2}\right).$$

In particular, given $\beta \in \mathbb{R}_{>0}$, the PDE is invariant under the 1-parameter subgroup given by

$$u(t, x) \rightarrow u_\lambda := \frac{1}{\lambda^{\frac{1}{p-1}}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda^\beta}\right). \quad (1.8)$$

As we shall see, there is a discrete set of $\beta \in \mathbb{R}$ which admit non-constant, radial, positive, smooth solutions of (R1) on $I_t = (-\infty, 0)$ which are invariant with respect to (1.8). Indeed, if u is such a solution, it can be written as

$$u(t, r) = \frac{1}{(-t)^{\frac{1}{p-1}}} \tilde{u}\left(\frac{r}{(-t)^\beta}\right), \quad (1.9)$$

where

$$\tilde{u}(y) = u(-1, y).$$

If we substitute (1.9) into (R1), we obtain the self-similar ODE

$$\beta y \partial_y \tilde{u} + \frac{1}{p-1} \tilde{u} = \tilde{u}^p. \quad (1.10)$$

This is a Bernoulli differential equation, and the general solution is (for C an integration constant)

$$\tilde{u}_{C,\beta}(y) = \frac{\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}}{(1 + C y^{\frac{1}{\beta}})^{\frac{1}{p-1}}}.$$

In order for this to be defined globally in y , we need $C \geq 0$. If $C = 0$ we obtain the constant-in-space solution, and if $C > 0$, the solution will be decaying in y . If $C > 0$, the solution will be smooth at the origin if and only if $\beta = \frac{1}{2k}$, for $k \in \mathbb{Z}_{\geq 1}$. We note that if we consider $u(-1, y) = \tilde{u}_{C,\beta}(y)$ to be the initial data, then locally near $y = 0$, we have

$$u(-1, y) = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} - \left(\frac{1}{p-1}\right)^{\frac{p}{p-1}} C y^{\frac{1}{\beta}} + o(y^{\frac{1}{\beta}}).$$

In particular, $\beta = \frac{1}{2}$ corresponds to a non-degenerate maximum, and $\beta = \frac{1}{2k}$ for $k \geq 2$ corresponds to increasingly degenerate maxima at the origin. These profiles are universal in the sense that if $u(t, x)$ is any solution that blows up at the origin at $t = 0$ whose initial data is

$$u(-1, x) = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} - C \left(\frac{1}{p-1}\right)^{\frac{p}{p-1}} x^{\frac{1}{\beta}} + o(x^{\frac{1}{\beta}}),$$

then in self-similar coordinates it will converge to $\tilde{u}_{C,\beta}$. To clarify in what sense the convergence takes place, we use the change of variables

$$u(t, x) = \frac{1}{(-t)^{\frac{1}{p-1}}} \tilde{u}(-\ln(-t), \frac{x}{(-t)^\beta}), \quad s = -\ln(-t), \quad y = \frac{x}{(-t)^\beta},$$

and note that the the initial conditions become

$$\tilde{u}(0, y) = u(-1, y).$$

The claim is then that $\tilde{u}$ converges to $\tilde{u}_{C,\beta}$ (at least locally in y). One way to understand this is through linearizing the self-similar equations (see [39], for instance). An alternate way to think about this is to use the solution formula (1.1), which gives

$$\tilde{u}(s, y) = e^{-\frac{s}{p-1}} \frac{u_0(e^{-s\beta} y)}{(1 - (p-1)u_0^{p-1}(e^{-s\beta} y)(1 - e^{-s})^{\frac{1}{p-1}})}.$$

We consider $y \leq A$. We use that as $s \rightarrow \infty$,

$$\tilde{u}(e^{-\beta s}y) = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \left(1 - \frac{C}{p-1} y^{\frac{1}{\beta}} e^{-s}\right) + o(e^{-s}),$$

and derive

$$\tilde{u}(s, y) = \frac{\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}}{(1 + Cy^{\frac{1}{\beta}})} + o_{s \rightarrow \infty}(1)$$

We comment that in this case, there is a particular self-similar solution ($\beta = \frac{1}{2}$) that corresponds to generic data. The other self-similar solutions ($\beta = \frac{1}{2k}$, $k > 1$) are increasingly non-generic. We note that the condition $u_0(0) = (p-1)^{-1/(p-1)}$ was chosen so that the blow-up time would be exactly $t = 0$, but if $u_0(0) \neq (p-1)^{-1/(p-1)}$, a similar expansion is achieved by modulating the blow-up time.

One can also construct a discrete sequence of self-similar profiles for the second order equation (R2) (viewed as a PDE). In this case, for each $\beta > 0$, the equation is invariant with respect to the scaling

$$u(t, x) \rightarrow \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda^\beta}\right). \quad (1.11)$$

In this case, substituting the ansatz

$$u(t, x) = \frac{1}{(-t)^{\frac{2}{p-1}}} \tilde{u}\left(\frac{x}{(-t)^\beta}\right)$$

leads to a tricky second-order ODE. A better approach (which the author learned from his advisor), is to consider a subclass of solutions which pointwisely blow up as time-translates of the special solution $u_{+,s}$, defined in (1.2). In other words, consider the solutions of the form

$$u(t, x) = \frac{u_0}{\left(1 - \left(\frac{u_0}{c_p}\right)^{\frac{p-1}{2}} (t+1)\right)^{\frac{2}{p-1}}},$$

where $u_0(x) = u(-1, x)$. Assuming that $u(t, x)$ is invariant under the scaling symmetry (1.11) gives

$$u = \frac{u_0\left(\frac{x}{\lambda^\beta}\right)}{\left(\lambda - \left(\frac{u_0(x/\lambda)}{c_p}\right)^{\frac{p-1}{2}} (t + \lambda)\right)^{\frac{2}{p-1}}}.$$

Differentiating the right-hand-side of the above equation with respect to λ and evaluating at $\lambda = 1, t = -1$ gives

$$-\beta x u'_0 - \frac{2}{p-1} u_0 + \left(\frac{2}{p+1}\right)^{\frac{1}{2}} u_0^{\frac{p+1}{2}} = 0.$$

This is again a Bernoulli ODE and can be solved to find self-similar solutions. In this case, we have

$$u(t, x) = \frac{1}{(-t)^{\frac{2}{p-1}}} \tilde{u}_{C,\beta}\left(\frac{x}{(-t)^\beta}\right)$$

where the solutions are

$$\tilde{u}_{C,\beta}(y) = \frac{c_p}{(1 + Cy^{\frac{1}{\beta}})^{\frac{2}{p-1}}}.$$

Again, we see that the allowed values of β are $\beta = \frac{1}{2k}$, for $k \in \mathbb{Z}_{\geq 1}$. In this case, we once again expect that $\beta = \frac{1}{2}$ corresponds to generic blow-ups, and $\beta = \frac{1}{2k}$ corresponds to increasingly unstable blow-ups, although this is harder to prove for general data without an explicit solution formula.

In the first two examples, there are discrete families of self-similar profiles describing self-amplification. Similarly, for (IB) there is a discrete family of self-similar profiles which describe shock formation. Indeed, in this case, there is again a two-parameter scaling symmetry, and considering a 1-parameter subgroup leads to the ansatz

$$u(t, x) = (-t)^{\beta-1} \tilde{u}\left(\frac{x}{(-t)^\beta}\right),$$

which gives the self-similar ODE

$$\beta y \partial_y \tilde{u} + (1 - \beta) \tilde{u} + \tilde{u} \partial_y \tilde{u} = 0.$$

This ODE can be solved by considering y as a function of $\tilde{u}$, and noting that we have

$$\frac{dy}{d\tilde{u}} - \frac{\beta y + \tilde{u}}{(\beta - 1)\tilde{u}} = 0.$$

This linear ODE can then be solved via integrating factors to yield

$$y = -\tilde{u} - C\tilde{u}^{\frac{\beta}{\beta-1}}.$$

This implicit equation will define $\tilde{u}_{C,\beta}$ as a smooth function of y provided that $C \geq 0$. Again, we see that $C = 0$ corresponds to a degenerate self-similar solution. If $C > 0$, the self-similar solution will be smooth if and only if $\beta = \frac{2k+1}{2k}$, for $k \in \mathbb{Z}_{\geq 1}$. In this case, the self-similar solution blows up at $t = 0$, and at $t = -1$, the initial data is of the form

$$u(-1, y) = \tilde{u}_{C,\beta}(y) = -y + Cy^{\frac{\beta}{\beta-1}} + o(y^{\frac{\beta}{\beta-1}}).$$

In particular, we note that in this case, $\tilde{u}'(y)$ has a non-degenerate maximum at the origin in the stable case, $\beta = \frac{3}{2}$. The profiles for $k \geq 2$ are increasingly degenerate.

We finally consider the system (ED) in radial symmetry and explicitly derive self-similar solutions. Our treatment is related to that of [51], although the authors do not employ exact self-similar profiles in this work. In radial symmetry, (ED) can be written as (under the assumption $4\pi = 1$)

$$\begin{cases} \partial_t \rho + \frac{1}{r^2} \partial_r (r^2 \rho u) = 0 \\ \partial_t u + u \partial_r u + \frac{1}{r^2} \int_0^r (r')^2 \rho(r') dr' = 0 \end{cases}.$$

We define the local mass to be $m = \int_0^r (r')^2 \rho(r') dr'$, and note that these equations can be rewritten as

$$\begin{cases} \partial_t m + u \partial_r m = 0 \\ \partial_t u + u \partial_r u + \frac{m}{r^2} = 0 \end{cases}.$$

This system can now be solved via the method of characteristics. We initialize the problem at $t = -1$, and define $r(t, z)$ to be the solution to the ODE

$$\begin{cases} \dot{r} = u(t, r(t)) \\ r(-1) = z \end{cases}$$

Along the characteristics, we then have

$$\begin{cases} \dot{m} = 0 \\ \dot{u} = -\frac{m}{r^2} \\ \dot{r} = u \end{cases}.$$

In particular, m is constant along characteristics, and (ED) is reduced to the well-known inverse-square law ODE:

$$\ddot{r} = -\frac{m}{r^2}. \quad (1.12)$$

Thus, solving (ED) in radial symmetry reduces to standard ODEs. We note that (1.12) has the conserved energy

$$E = \frac{1}{2}\dot{r}^2 - \frac{m}{r}.$$

As can be seen from the phase portrait, Figure 1.2, the $E = 0$ integral curves divide the dynamics into several regimes. The $E = 0$ solutions can be written explicitly as

$$r = (C + 3\sqrt{\frac{m}{2}}t)^{\frac{2}{3}}$$

or

$$r = (C - 3\sqrt{\frac{m}{2}}t)^{\frac{2}{3}}. \quad (1.13)$$

The former solutions go to $r = \infty$ with zero asymptotic velocity and the latter are the time-reversal of the former.

Returning to the PDE, one can derive self-similar solutions as done for (R2). In particular, one can first assume that each trajectory is of the type (1.13) (note this simplification is also made in [51]). If one considers the initial local mass to be $m_0(r)$, then

$$r(t, z) = (z^{\frac{3}{2}} - 3\sqrt{\frac{m_0(z)}{2}}(t+1))^{\frac{2}{3}}. \quad (1.14)$$

To express the solution in the standard (t, r) coordinates we implicitly define (for each fixed t) $z(t, r)$ to be the inverse of $z \mapsto r(t, z)$. We then have

$$m(t, r) = m_0(z(t, r)). \quad (1.15)$$

We then note that for any $\beta > 0$, the PDE (ED) obeys the scaling symmetry

$$\rho(t, x), u(t, x) \rightarrow \frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda}, \frac{r}{\lambda^\beta}\right), \lambda^{\beta-1}u\left(\frac{t}{\lambda}, \frac{r}{\lambda^\beta}\right).$$

With respect to this scaling, the mass transforms as

$$m(t, r) \rightarrow \lambda^{3\beta-2}m\left(\frac{t}{\lambda}, \frac{r}{\lambda^\beta}\right).$$

If we assume that $m_0(r)$ corresponds to a self-similar solution, then from the solution formula (1.15), we have

$$m(t, r) = \lambda^{3\beta-2}m_0\left(z\left(\frac{t}{\lambda}, \frac{r}{\lambda^\beta}\right)\right)$$

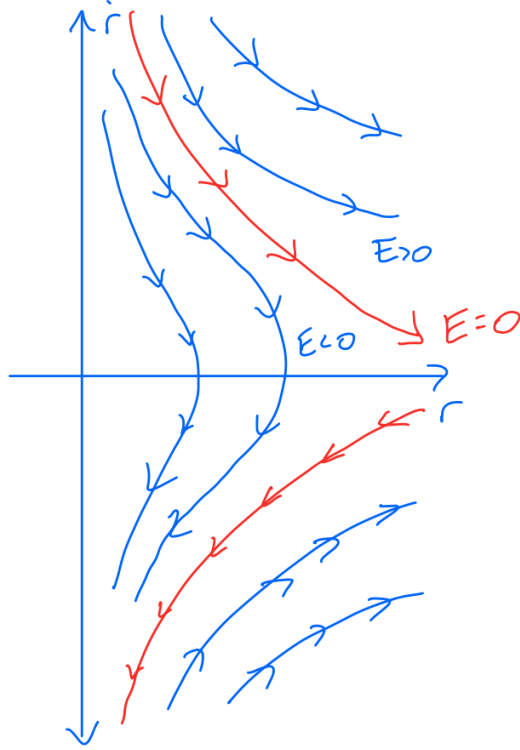


Figure 1.2: Phase Portrait of (1.12)

One can then differentiate with respect to λ and evaluate at $\lambda = 1, t = -1$ to obtain an ODE for m_0 . It is useful to note that as a consequence of (1.14), we have that at $t = -1$,

$$\partial_r z|_{t=-1} = 1, \quad \partial_t z|_{t=-1} = \sqrt{\frac{2m}{r}}.$$

Using this, we have that if $m_0(r)$ leads to a self-similar solution, then

$$\begin{aligned} 0 &= (3\beta - 2)m_0 + (\partial_r m_0)(\partial_t z - \beta r \partial_r z)|_{t=-1} \\ &= (3\beta - 2)m_0 + \partial_r m_0 \left(\sqrt{\frac{2m_0}{r}} - \beta r \right). \end{aligned}$$

It remains to solve this ODE for m_0 . To do this, we let $M = \sqrt{\frac{2m_0}{r}}$, which gives

$$M' = \frac{2(1 - \beta)rM - M^2}{2r(M - \beta r)}.$$

We then let $M = rf$, and obtain

$$rf' = \frac{f(2 - 3f)}{2(f - \beta)}.$$

We let $r = e^z$, and separate variables which gives

$$z + C = \int \frac{2(f - \beta)}{f(2 - 3f)} df$$

$$\begin{aligned}
&= \int \frac{-\beta}{f} + \frac{2-3\beta}{2-3f} df \\
&= \ln \left(\frac{(2-3f)^{\beta-\frac{2}{3}}}{f^\beta} \right)
\end{aligned}$$

Thus, we see that f is the implicit solution to the equation

$$\frac{(2-3f)^{\beta-\frac{2}{3}}}{f^\beta} = Cr.$$

We comment that if we assume $f(0) = \frac{2}{3}$, and $C > 0$, then this gives a smooth solution if $\beta = \frac{2}{3} + \frac{1}{2k}$, $k \in \mathbb{Z}_{\geq 1}$. In this case, f will be a radially decreasing function of r . One can then recover m_0 to be $m_0(r) = \frac{r^3 f^2}{2}$. Thus, just as in (R1), (1.2) and (IB), we see that (ED) has a discrete set of scalings for which there is self-similar solution that can be solved for rather explicitly.

In summary, the equations (R1), (R2), (IB) and (ED) each admit a two-parameter scaling symmetry. For these equations, for a discrete sequence β of spatial scalings (depending on the equation), one can easily derive self-similar solutions which blow up at the origin.

1.2.2 Treating Diffusive and Dissipative Effects Perturbatively

We now discuss the feasibility of perturbing the above solutions to obtain blow-ups for more complicated PDE which feature competition between blow-up mechanisms and regularizing effects. There are many works in this direction. The author's introduction to this topic was the paper [90]. In this work, the authors show the existence of blow-ups for several perturbations of (IB), including the *fractional KdV equation*, for $u: \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{R}$:

$$\partial_t u + u \partial_x u + |D_x|^{\alpha-1} \partial_x u = 0, \quad (1.16)$$

where $|D_x|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$, and $0 \leq \alpha < 1$. The blow-ups are constructed by perturbing the self-similar Burger's profiles described in the previous section. In this work, the authors give a powerful heuristic to test when this will work, namely to assume a self-similar Burger's blow-up and perform a dimensional analysis for the perturbation. Indeed, we recall that for (IB), if $[t], [x]$ are the dimensions of length and space, then we have the dimensional relationship, expressed in the author's advisor's notation,

$$[u] = \frac{[x]}{[t]},$$

from which we derive that for (1.16)

$$[\partial_t u], [u \partial_x u] = \frac{[x]}{[t]^2}, \quad [|D_x|^{\alpha-1} \partial_x u] = \frac{[x]^{1-\alpha}}{[t]}.$$

For the self-similar profile, we have that $[x] = [t]^\beta$, so these dimensions become

$$[\partial_t u], [u \partial_x u] = [t]^{\beta-2}, \quad [|D_x|^{\alpha-1} \partial_x u] = [t]^{\beta(1-\alpha)-1}.$$

As $t \rightarrow 0$, the dispersive term is perturbative if it has more factors of t , which leads to the condition

$$\beta(\alpha - 1) - 1 > \beta - 2,$$

or equivalently,

$$\alpha < \frac{1}{\beta}.$$

Recalling that the allowed β are $\beta = \frac{2k+1}{2k}$, we see that if $\alpha < 1$, there exists a k such that the above inequality will be satisfied. The authors of [90], emphasize that as $\alpha \rightarrow 1^-$, one must use increasingly unstable self-similar profiles, leading to the conclusion that increasingly unstable Burger's profiles, from the perspective of initial data, are more stable with respect to perturbations of the equation.

We now consider a similar dimensional analysis for (NLW). In this case, for (R2), we have

$$[\varphi] = [t]^{-\frac{2}{p-1}}.$$

Considering now a self-similar scaling, we have

$$[\partial_t^2 \varphi], [\varphi^p] = [t]^{-2-\frac{2}{p-1}}, \quad [\Delta \varphi] = [t]^{-\frac{2}{p-1}-2\beta}.$$

The second term will be perturbative if

$$\beta < 1.$$

We remark that in this case, the condition does not depend on p and also that the allowed β are $\beta = \frac{1}{2k}$. In particular, we see that for all β , *including the stable profile*, $\beta = \frac{1}{2}$, we expect the ODE blow-up to dominate the dispersive effects.

Indeed, the stability of ODE-type blow-up for (NLW) has been studied by a wide-range of authors using a variety of tools. In one spatial dimension, the uniqueness and stability of ODE-type blow-up was studied by Merle-Zaag in the series of papers [84], [85], [86] and [87]. Their work also extended to several spatial dimensions in the subconformal range, see [81], [82], [83], [88] and [89]. In the case of energy-supercritical nonlinearities, where one does not have uniqueness of the ODE-type blow-up, an alternate proof of stability of self-similar blow-ups has been developed using semigroup theory, see [21], [32], [33], [34], [35], [36], [38], [96] and [111]. A related problem that has received much attention is that of constructing data which blow up on a prescribed curve. In this direction, we mention the works [15], [63], [64], [65], [18], [19] and [62].

If we apply the same heuristics to (NLH), we have have that from (R1)

$$[\varphi] = [t]^{-\frac{1}{p-1}},$$

and so we expect that near a self-similar blow-up

$$[\partial_t \varphi], [\varphi^p] = [t]^{-1-\frac{1}{p-1}}, \quad [\Delta \varphi] = [t]^{-\frac{1}{p-1}-2\beta}.$$

In particular, in this case, we see that the diffusive effects will be perturbative if

$$\beta < \frac{1}{2}.$$

Main Equation	Model	β For Model	Condition
(1.16)	(IB)	$\frac{2k+1}{2k}$	$\alpha < \frac{1}{\beta}$
(NLW)	(R2)	$\frac{1}{2k}$	$\beta < 1$
(NLH)	(R1)	$\frac{1}{2k}$	$\beta < \frac{1}{2}$
(EP)	(ED)	$\frac{2}{3} + \frac{1}{2k}$	$\beta < 2 - \gamma$

Figure 1.3: Scaling Heuristics for Treating Diffusion/Dispersion/Pressure Perturbatively

In particular, recalling that the allowed β are $\beta = \frac{1}{2k}$, we see that the stable self-similar ODE blow-up barely fails the above inequality. In [80], for $d = 1$, following numerics of [4], and earlier works, including [10], it was shown that there is a logarithmic correction to the blow-up dynamics. This type of criticality is sometimes termed “slow blow-up,” and a discussion of its ubiquity in many parabolic models is discussed in the extensive survey article [40].

One can also apply this heuristic to understand dust-like collapses for (EP), i.e. when one can treat the pressure term perturbatively. In this case, one has that from (ED)

$$[\rho] = [t]^{-2}, \quad [u] = \frac{[x]}{[t]^2},$$

and so if one considers the second equation of (EP) near a self-similar dust collapse, we expect

$$[\partial_t u] = [u \partial_x u] = [F] = [t]^{\beta-2}, \quad [\rho^{-1} \nabla \rho^\gamma] = [t]^{2-2\gamma-\beta},$$

which gives the condition

$$\beta < 2 - \gamma \tag{1.17}$$

In this case, the allowed values of β are $\frac{2}{3} + \frac{1}{2k}$. In particular, if $1 \leq \gamma < \frac{4}{3}$, one can find a sufficiently large k so that the k th self-similar dust solution should dominate the pressure term. We see in this case that as $\gamma \rightarrow \frac{4}{3}^-$, one must consider increasingly unstable dust solutions. We comment that dust-like collapses have been rigorously constructed in [51]. In this work the authors do not use the self-similar dust solutions described above, but instead perturb a larger class of dust profiles which satisfy the vanishing condition at the origin that $\rho = \rho_0 - cr^n + o(r^n)$. Their proof requires a restriction on n , their equation (1.60),

$$\frac{4}{3} - \gamma - \frac{1}{n} > 0$$

which in this discussion corresponds to (1.17). We also comment that even for the least critical case $\gamma = 1$, one cannot take the stable dust blow-up ($\beta = \frac{7}{6}$, $n = 2$ respectively). As such, unlike (NLW), one *does not* expect generic collapse to locally converge to a dust profile.

The scaling heuristics for the models discussed so far are summarized in Figure 1.3.

As a final comment, we remark that unlike (NLH), (NLW) and (EP), the system (CE) does not feature an obvious blow-up mechanism (Ricatti blow-up and dust-like collapse for the other models). Another difference is that it features a two-parameter scaling symmetry, instead of a 1-parameter scaling symmetry. It turns out, that for $d \geq 2$, for certain values of β , there do exist self-similar implosion profiles ([78]). Under scaling conditions dictated

by dimensional analyses similar to the ones described in this section, these can in turn be perturbed to construct blow-up for other models, such as the compressible Navier-Stokes and defocusing nonlinear Schrödinger equations, as shown in the works [79, 77]. We comment that there have been several subsequent works on these type of smooth implosions. In particular, the parameter range for which one can construct blow-ups to (CE) has been generalized in the works [11] and [103]. In addition, non-isentropic stable blow-ups have been recently constructed in the work [24]. Recent works have also explored various settings in which these blow-ups can be perturbed to create singularities for related equations, see [3], [12], [16], [17], [22], [23], [104] and [105].

1.2.3 Self-Similar Blow-ups for the Full Equations

The heuristics in the previous section show that one expects the explicit stable self-similar profile of (R2) to dominate the dynamics of (NLW) when the latter experiences blow-up. The analogous heuristic (approximating by dust) fails for (EP). In this case, it turns out that there is still a universal object which describes the generic blow-up dynamics: the Larson-Penston solution. As we shall see, instead of being a self-similar solution to toy model (ED), it is a self-similar solution to the *full* equation (EP).

Indeed, while previously we considered constructing blow-ups to the main equations by neglecting certain terms, another approach to constructing blow-ups begins by noting that (NLH), (NLW) and (EP) each contain a 1-parameter scaling symmetry. For (NLH), the symmetry is given by

$$\varphi \mapsto \varphi_\lambda := \frac{1}{\lambda^{\frac{1}{p-1}}} \varphi\left(\frac{t}{\lambda}, \frac{x}{\lambda^{\frac{1}{2}}}\right).$$

For (NLW), the scaling symmetry is given by

$$\varphi \mapsto \varphi_\lambda := \frac{1}{\lambda^{\frac{2}{p-1}}} \varphi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right).$$

For (EP), the scaling symmetry is given by

$$(\rho, u) \mapsto (\rho_\lambda, u_\lambda),$$

where

$$\rho_\lambda = \frac{1}{\lambda^2} \rho\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right), \quad u_\lambda = \lambda^{1-\gamma} u\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right).$$

We comment that with respect to this scaling, the mass (1.5), scales as

$$M \rightarrow \lambda^{4-3\gamma} M,$$

and in particular, the range $\gamma < \frac{4}{3}$, where one can construct (unstable) implosions by perturbing dust, corresponds to the *mass-supercritical* range. Heuristically, for this range of γ , one should understand that the mass does not obstruct blow-up.

We comment that the energies (1.3) and (1.4) scale as

$$E_{NLH} \rightarrow \lambda^{\frac{1}{2}-\frac{2}{p-1}} E_{NLH}, \quad E_{NLW} \rightarrow \lambda^{1-\frac{4}{p-1}} E_{NLW}.$$

In particular, for both equations, the exponent $p = 5$ is energy-critical. Unlike the mass-critical exponent for Euler-Poisson, this is not a threshold for collapse, and its relevance to blow-up dynamics is more subtle (see Subsection 1.2.4). The energy (1.7) scales as

$$E_{EP} \rightarrow \lambda^{6-5\gamma} E_{EP},$$

and so we see that in this case $\lambda = \frac{6}{5}$ is the energy-critical exponent.

Motivated by this scaling symmetry, we now perform a self-similar change of variables. For (NLH), this is achieved by considering the ansatz,

$$\varphi(t, x) = \frac{1}{(-t)^{\frac{1}{p-1}}} \tilde{\varphi}(s, y), \quad s = -\ln(-t), \quad y = \frac{x}{(-t)^{\frac{1}{2}}},$$

resulting in the PDE

$$\partial_s \tilde{\varphi} + \frac{1}{2} y \partial_y \tilde{\varphi} + \frac{1}{p-1} \tilde{\varphi} - \partial_y^2 \tilde{\varphi} - \frac{2}{y} \partial_y \tilde{\varphi} - \tilde{\varphi}^p = 0. \quad (1.18)$$

For (NLW), one considers the ansatz

$$\varphi(t, x) = \frac{1}{(-t)^{\frac{2}{p-1}}} \tilde{\varphi}(s, y), \quad s = -\ln(-t), \quad y = \frac{x}{-t},$$

resulting in the PDE

$$\begin{aligned} \partial_s^2 \tilde{\varphi} + 2y \partial_y \partial_s \tilde{\varphi} + (y^2 - 1) \partial_y^2 \tilde{\varphi} + \frac{p+3}{p-1} \partial_s \tilde{\varphi} + \left(\frac{2(p+1)}{p-1} y - \frac{2}{y} \right) \partial_y \tilde{\varphi} \\ + \frac{2(p+1)}{(p-1)^2} \tilde{\varphi} - \tilde{\varphi}^p = 0. \end{aligned} \quad (1.19)$$

For (EP), one considers the ansatz

$$\rho(t, x) = \frac{1}{(-t)^2} \tilde{\rho}(s, y), \quad u(t, x) = (-t)^{1-\gamma} \tilde{u}(s, y), \quad s = -\ln(-t), \quad y = \frac{x}{(-t)^{2-\gamma}},$$

leading to the system

$$\begin{cases} \partial_s \tilde{\rho} + 2\tilde{\rho} + (2-\gamma)y \partial_y \tilde{\rho} + \operatorname{div}_y(\tilde{\rho} \tilde{u}) = 0 \\ \partial_s \tilde{u} + (\gamma-1)\tilde{u} + (2-\gamma)y \partial_y \tilde{u} + \tilde{u} \partial_y \tilde{u} + \gamma \tilde{\rho}^{\gamma-2} \partial_y \tilde{\rho} + 4\pi \operatorname{div}_y^{-1}(\tilde{\rho}) = 0. \end{cases} \quad (1.20)$$

We comment that s -independent solutions to the above equations correspond to radially symmetric solutions of the main equations which are invariant with respect to the scaling symmetry. Thus, finding scale-invariant solutions corresponds to solving the following ODEs in y (the *SS* tag stands for self-similar):

$$\frac{1}{2} y \partial_y \tilde{\varphi} + \frac{1}{p-1} \tilde{\varphi} - \partial_y^2 \tilde{\varphi} - \frac{2}{y} \partial_y \tilde{\varphi} - \tilde{\varphi}^p = 0, \quad (\text{NLH-SS})$$

$$(y^2 - 1)\partial_y^2 \tilde{\varphi} + \left(\frac{2(p+1)}{p-1}y - \frac{2}{y}\right)\partial_y \tilde{\varphi} + \frac{2(p+1)}{(p-1)^2}\tilde{\varphi} - \tilde{\varphi}^p = 0, \quad (\text{NLW-SS})$$

and

$$\begin{cases} 2\tilde{\rho} + (2 - \gamma)y\partial_y \tilde{\rho} + \text{div}_y(\tilde{\rho}\tilde{u}) = 0 \\ (\gamma - 1)\tilde{u} + (2 - \gamma)y\partial_y \tilde{u} + \tilde{u}\partial_y \tilde{u} + \gamma\tilde{\rho}^{\gamma-2}\partial_y \tilde{\rho} + 4\pi \text{div}_y^{-1}(\tilde{\rho}) = 0 \end{cases}.$$

We note that this last system contains a nonlocal term, but this can be removed by rewriting the first equation as

$$\tilde{\rho} = \frac{1}{4 - 3\gamma} \text{div}_y(\tilde{\rho}(\tilde{u} + (2 - \gamma)y)),$$

resulting in the system

$$\begin{cases} 2\tilde{\rho} + (2 - \gamma)y\partial_y \tilde{\rho} + \text{div}_y(\tilde{\rho}\tilde{u}) = 0 \\ (\gamma - 1)\tilde{u} + (2 - \gamma)y\partial_y \tilde{u} + \tilde{u}\partial_y \tilde{u} + \gamma\tilde{\rho}^{\gamma-2}\partial_y \tilde{\rho} + \frac{4\pi\tilde{\rho}(\tilde{u} + (2 - \gamma)y)}{4 - 3\gamma} = 0 \end{cases}. \quad (\text{EP-SS})$$

Unlike the ODEs considered in Section 1.2.1, the ODEs (NLH-SS), (NLW-SS) and (EP-SS) cannot be solved via elementary methods. Nonetheless, each equation admits two trivial solutions. The first trivial solution, which we will call the *Friedman solution*, taking the name from the literature on (EP-SS), is given for the three equations by

$$\begin{cases} \tilde{\varphi}_F = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} & \text{for (NLH-SS)} \\ \tilde{\varphi}_F = c_p & \text{for (NLW-SS)} \\ \tilde{\rho}_F = \frac{1}{6\pi}, \tilde{u}_F = -\frac{2}{3}y & \text{for (EP-SS)} \end{cases}. \quad (1.21)$$

These solutions are all degenerate forms of the blow-ups for the toy models which are homogeneous in space, and so the diffusion/dispersion/pressure vanish trivially. As such, they do not tell us anything new, but can serve as useful benchmarks when constructing solutions to the self-similar equations. The other trivial solution is the *far-field solution*,

$$\begin{cases} \tilde{\varphi}_f = \left(\frac{2(p-3)}{(p-1)^2}\right)^{\frac{1}{p-1}} \frac{1}{y^{\frac{2}{p-1}}} & \text{for (NLH-SS)} \\ \tilde{\varphi}_f = \left(\frac{2(p-3)}{(p-1)^2}\right)^{\frac{1}{p-1}} \frac{1}{y^{\frac{2}{p-1}}} & \text{for (NLW-SS)} \\ \tilde{\rho}_f = \left(\frac{\gamma(4-3\gamma)}{2\pi(2-\gamma)^2}\right)^{\frac{1}{2-\gamma}} \frac{1}{y^{\frac{2}{2-\gamma}}}, \tilde{u}_F = 0 & \text{for (EP-SS)} \end{cases}. \quad (1.22)$$

The far-field solutions correspond, in physical variables, to singular stationary solutions of the main equations. In particular, they are all singular at the origin, $y = 0$, and do not blow up at all. Again, as with the Friedman solutions, one can use them to gain information about the self-similar ODEs.

Refocusing attention on (EP-SS), one can ask if there is a smooth solution to (EP-SS) with ρ decaying as $y \rightarrow \infty$ (to rule out the Friedman solution). This problem was first considered by Larson ([70]) and Penston ([99]) who, for the simplest case $\gamma = 1$, independently used numerics to find a non-trivial solution to (EP-SS), now called the Larson-Penston solution. We remark that these numerical investigations are non-trivial as the ODE degenerates at the origin and when the solution experiences sonic points (see Section 1.2.4). Indeed, there are in

fact additional smooth self-similar solutions, as noticed later by Hunter ([59]), who for $\gamma = 1$ found a discrete sequence solutions for which the velocity exhibits increasing oscillations.

Numerical simulations of the Euler-Poisson system suggest that the Larson-Penston solution indeed serves as a leading order blow-up profile for (EP), see for instance [59], [41] and [9]. We comment that the study [9] observed that dust-like collapse is unstable to a “Jeans-like” instability, in rough agreement with the heuristics in Section 1.2.2.

The Larson-Penston solution has been recently constructed rigorously by Guo-Hadžić-Jang ([52]). Moreover, their proof has been extended to the relativistic setting [53], following the investigations of Ori-Piran ([92, 94, 95]). Guo-Hadžić-Jang also proved that the solution may be truncated to exhibit a naked singularity, shedding light on cosmic censorship for this matter model. The authors, along with Schrecker, were also able to extend their construction to the full mass-supercritical polytropic range $1 < \gamma < \frac{4}{3}$, [54], following numerical work of Yahil, [113]. In Chapter 2 of this thesis, a subclass of the solutions found by Hunter is constructed for $\gamma = 1$. In Chapter 3, the previous result is extended to the energy-supercritical range $1 < \gamma < \frac{6}{5}$. These works adapt a bifurcation approach that had been successfully applied to construct solutions to NLH-SS and (NLW-SS), see, for instance [8], [26] and [66].

1.2.4 Thresholds and Mode Stability

Given that the Larson-Penston solution describes generic collapse for (EP), a natural question is: what role do the Hunter solutions play in describing the dynamics?

To prepare for this discussion, we first derive the mode stability ODEs for the main self-similar equations of interest. For (NLH), we consider $\tilde{\varphi}_0$ to be an exact solution of (NLH-SS). We then substitute the ansatz $\tilde{\varphi}(s, y) = \tilde{\varphi}_0(y) + \epsilon e^{\lambda s} \tilde{\varphi}_1(y)$ into (1.18) and formally take $\epsilon \rightarrow 0$. This yields the following linear ODE for $\tilde{\varphi}_1$:

$$-\partial_y^2 \tilde{\varphi}_1 + \left(\frac{1}{2}y - \frac{2}{y}\right) \partial_y \tilde{\varphi}_1 + \left(\frac{1}{p-1} - p\tilde{\varphi}_0^{p-1}\right) \tilde{\varphi}_1 + \lambda \tilde{\varphi}_1 = 0. \quad (1.23)$$

For the case of (NLW), substituting a similar ansatz into (1.19) yields

$$\begin{aligned} (y^2 - 1) \partial_y^2 \tilde{\varphi}_1 + \left(\frac{2(p+1)}{p-1} + 2\lambda\right) - \frac{2}{y} \partial_y \tilde{\varphi}_1 \\ + \left(\lambda^2 + \frac{p+3}{p-1} \lambda + \frac{2(p+1)}{(p-1)^2}\right) \tilde{\varphi}_1 - p\tilde{\varphi}_0^{p-1} \tilde{\varphi}_1 = 0. \end{aligned} \quad (1.24)$$

We comment that while (1.23) resembles a standard eigenvalue problem, the equation (1.24) is more complicated as λ appears as a coefficient of $\partial_y \tilde{\varphi}_1$ and moreover the system is singular at $y = 1$. Following a similar derivation as that of (EP-SS), one can write the mode equation for (EP), for $\lambda \neq 1$, as the system

$$\begin{cases} (\lambda - 1) \tilde{\rho}_1 + \operatorname{div}((\tilde{u}_0 + y) \tilde{\rho}_1 + \tilde{u}_1 \tilde{\rho}_0) = 0 \\ \gamma \tilde{\rho}_0^{\gamma-2} \partial_y \tilde{\rho}_1 + (\tilde{u}_0 + y) \partial_y \tilde{u}_1 + \gamma(\gamma - 2) \tilde{\rho}_0^{\gamma-3} (\partial_y \tilde{\rho}_0) \tilde{\rho}_1 \\ \quad + (\lambda + (\gamma - 1) + \partial_y \tilde{u}_0) \tilde{u}_1 + \frac{4\pi}{1-\lambda} ((\tilde{u}_0 + y) \tilde{\rho}_1 + \tilde{u}_1 \tilde{\rho}_0) = 0 \end{cases}. \quad (1.25)$$

For these problems, given a self-similar solution $\tilde{\varphi}_0$ or $(\tilde{\rho}_0, \tilde{u}_0)$, we may look for analytic modes. If a mode exists whose corresponding eigenvalue has positive real part, we say that the self-similar solution is linearly unstable.

We now discuss self-similar blow-ups to (NLW). Critical phenomena for this equation explored numerically in [7]. In this work, for several odd values of p , the authors numerically solve (NLW) for a 1-parameter family of radial initial data. The family is chosen so that the solutions disperse for small values of the parameter and blow up for large values of the parameter. The blow-ups they observe are modeled by (R2), which is consistent with the previous heuristics. Surprisingly, they found that the threshold dynamics depend on the exponent p , with a qualitative transition taking place at the energy-critical threshold $p = 5$. For $p > 5$, the authors observe that the critical dynamics are well-approximated by a solution to (NLW-SS), which they solve for numerically. They observe that this solution, which they call U_1 “appears as the codimension-one attractor in the evolution of specially prepared initial data.” In addition to U_1 , they find a discrete sequence $U_2, U_3, \dots$ of solutions. We emphasize the similarity between the structure of the set of solutions of (EP-SS) found by Hunter and the set of solutions to (NLW-SS) found by these authors. Moreover, by considering (1.24) they numerically observe that U_n has n unstable modes.

There has been much recent activity on self-similar blow-up for (NLW) and related equations. As shall be discussed in Section 1.2.4, self-similar solutions $U_1, U_2, \dots$ are rigorously constructed in [8]. We remark in particular the surprising result [47], which demonstrated that for $d \geq 5$, the first non-trivial self-similar solution U_1 to the energy-supercritical cubic NLW admits a closed form expression. Moreover, in the case of $d = 7$, they were also able to find a closed form for the unstable mode and hence prove non-linear codimension-1 stability of U_1 . This solution’s role as a threshold was studied numerically in [46]. For the quadratic NLW, in the energy-supercritical case of $d \geq 7$, closed form self-similar solutions were found in [27]. In the case of $d = 9$, the unstable mode is again explicit, allowing the authors to show codimension-1 stability. This work was extended beyond the past-light cone of the singularity in [25]. We mention the recent work [61] which studies the stability of self-similar solutions up to the Cauchy horizon.

In the case of (NLW), Giga and Kohn have proved the non-existence of solutions to (NLH-SS) for $p < 5$ ([43]). For $p > 5$, self-similar solutions do exist. Early constructions are due to [110], [14] and [71]. The work [26] gives an alternate construction of solutions analogous to Hunter solutions, for N sufficiently large. This work additionally invokes Sturm oscillation to prove that the N th solution has exactly N unstable modes. This is used to show that the N th solution is nonlinearly codimension- N stable. We comment that the analogous statement for (NLW) remains out of reach. A recent work in this direction is [66], which shows finite-codimension stability without determining the codimension.

In the Euler-Poisson case, the linearized spectrum of the Larson-Penston and Hunter solutions has been numerically analyzed by [57] and [73]. In this case, the authors find that the Larson-Penston solution is linearly stable, and that the i th Hunter solution is codimension- i unstable. As with (NLW), the first unstable self-similar solution is understood to describe critical phenomena ([58]). For an extensive review of self-similarity and critical phenomena we recommend the survey [50]. We comment that in both the cases of (NLW) and (EP), attention has been primarily focused on the first unstable self-similar solution; the higher ones can be naturally considered to describe critical phenomena with increasingly

high codimension.

We comment that even when one has found a solution to a self-similar equation, such as (NLH-SS), (NLW-SS) and (EP-SS), the mode stability problem is typically hard, and can only be solved in a few cases. One situation in which one can solve this problem is for the Friedman-type solutions, (1.21), where there exists polynomial modes. In this case, for (NLH), (NLW) and (EP), if one restricts to radial symmetry, one can find that the solutions to (1.23), (1.24) and (1.25) respectively are given by (for $k \in \mathbb{Z}_{\geq 0}$)

$$\begin{cases} \lambda = 1 - k & (\text{NLH}) \\ \lambda = 1 - 2k, -2 - \frac{4}{p-1} - 2k & (\text{NLW}) . \\ \lambda = 1 - 2k(\frac{4}{3} - \gamma), -\frac{2}{3} - 2k(\frac{4}{3} - \gamma) & (\text{EP}) \end{cases}$$

For (NLW), this computation can be found in [7], and for the other equations the computations are similar. In each of these three problems, the $\lambda = 1$ instability corresponds to time-translation symmetry and should be ignored. For (NLW), we see that the rest of the eigenvalues are negative, reflecting the fact that ODE-type blow-up is stable. For (NLH), we see that $\lambda = 0$ is an eigenfunction, again reflecting that this problem is critical from a scaling perspective. Finally, for (EP), we see that $2\gamma - \frac{4}{3}$ will always be an unstable eigenvalue, consistent with the instability of dust-like collapse for this system.

We remark that the stability of the Larson-Penston solution was recently solved in the breakthrough work [55]. The corresponding stability problem for the Hunter solutions remains open.

1.3 Overview of Construction of Hunter-type Solutions

We now discuss in greater detail how we construct Hunter-type solutions to (EP-SS). The author learned these methods through reading the paper [26], in which solutions to (NLH-SS) are constructed. We will contrast the proof with the construction of the Larson-Penston-type solutions to (EP-SS) by Guo-Hadžić-Jang-Schrecker ([52], [54]) and the construction of solutions to (NLW-SS) of Bizoń-Maison-Wasserman ([8]).

We begin by discussing the ODE (NLW-SS). We see that, as a byproduct of working in radial symmetry, the ODE is singular at $y = 0$. As a consequence, if one considers $\tilde{\varphi}$ to be a C^2 solution to (NLW-SS), then necessarily one must have $\tilde{\varphi}'(0) = 0$. By Taylor expanding in y^2 , one can formally observe that there should be a 1-parameter family of solutions regular at the origin determined by $\tilde{\varphi}(0)$.

The point $y = 1$ is also singular for (NLW-SS). In particular, substituting $y = 1$ gives the constraint

$$\frac{4}{p-1}\tilde{\varphi}'(1) = (\tilde{\varphi}(1))^p - \frac{2(p+1)}{(p-1)^2}\tilde{\varphi}(1). \quad (1.26)$$

Again, we see that there is now a 1-parameter family of smooth solutions. As an aside, we give a geometric explanation for this singularity. If we consider $y_0 \in \mathbb{R}$, imposing $\tilde{\varphi}(y_0), \tilde{\varphi}'(y_0)$ as data for the self-similar equations is equivalent to imposing Cauchy data $(\varphi, \partial_r \varphi)$ along

the curve $r = y_0(-t)$ at the level of the $1+1$ radial semilinear wave equation. When $y_0 = 1$, this curve is characteristic, and so one cannot freely impose zero and first order data. We also comment that $\varphi^{(k)}(1)$ for $k \geq 2$ can be solved for by expanding (NLW-SS) about $y = 1$. For instance, if we impose (1.26) and expand the equation to $o(y-1)$, we obtain

$$(2 + \frac{4}{p-1})\tilde{\varphi}''(1) = p\tilde{\varphi}^{p-1} - \frac{2(p+1)}{(p-1)^2}\tilde{\varphi}'(1) - (2 + \frac{2(p+1)}{(p-1)})\tilde{\varphi}'(1) = 0.$$

One sees that higher derivatives are similarly determined, and so, at least formally, there is a unique solution at this point once $\tilde{\varphi}(1)$ is specified.

We now discuss the basic algorithm for constructing self-similar solutions to (NLW-SS). The first step is to pick a matching point $y_0 \in [0, 1]$. The next step is to “shoot” from $y = 0$ to the right and “shoot” from $y = 1$ to the left using the parameters $\tilde{\varphi}(0)$ and $\tilde{\varphi}(1)$ respectively. One then tries to match $\tilde{\varphi}(y_0)$ and $\tilde{\varphi}'(y_0)$. If successful, this gives a smooth solution on $[0, 1]$. The final step is to check if this solution extends to $y = \infty$. This strategy was numerically implemented in [7], resulting in a discrete set of solutions in the supercritical range.

The known analytically rigorous techniques for constructing solutions proceed roughly along the same lines. There are three main components. The first is solving the local problem of showing that there are indeed 1-parameter families of solutions which smoothly pass through $y = 0$ and $y = 1$. The second step is solving the global problem, whereby one must show that, at least for sufficient ranges of the parameters, the solutions extend from $y = 0$ and $y = 1$ to y_0 . This is non-trivial as the ODEs are non-linear and so a Ricatti-type blow-up could occur. In addition, as the system is non-autonomous, one can not understand global dynamics by examining a 2 dimensional phase portrait. The final piece is solving the matching problem, which consists of showing that the left and right solutions actually connect for certain parameter values. We comment that one must also solve the extension problem of verifying that the smooth solution through the sonic point extends to $y = \infty$, but this in practice turns out to be easier and will not be discussed further in this introduction.

Before discussing how these steps are handled for (NLW-SS), we bring up the ODE system (EP-SS). For simplicity, we restrict to the case $\gamma = 1$. In this case, one can again read off that the equation is singular at $y = 0$ which gives $\tilde{u}(0) = 0$. There is also an analog of $y = 1$. To see this, one can rewrite (EP-SS) as the matrix system

$$\begin{pmatrix} \tilde{u} + y & \tilde{\rho} \\ \frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} \begin{pmatrix} \tilde{\rho}' \\ \tilde{u}' \end{pmatrix} + \begin{pmatrix} \frac{2\tilde{\rho}(\tilde{u}+y)}{y} \\ 4\pi(\tilde{u}+y)\tilde{\rho} \end{pmatrix} = 0. \quad (1.27)$$

We remark that the coefficient matrix degenerates when

$$(\tilde{u} + y)^2 - 1 = 0. \quad (1.28)$$

This condition is the analog of $y = 1$ and it now depends on the solution. The y values which saturate (1.28) are called *sonic points*. All the self-similar solutions we shall study will have a sonic point satisfying

$$\tilde{u}(y_*) = 1 - y_*.$$

For instance, the Friedman and far-field solutions have sonic points at

$$y_F = 3, \quad y_f = 1$$

respectively. For any solution, one can read off from the equation that if $(\tilde{\rho}, \tilde{u})$ are a smooth solution (see Chapter 2) then

$$\tilde{\rho}(y_*) = \frac{1}{2\pi y_*}.$$

In contrast to (NLW-SS), we see that for a given sonic point, we now have that $\tilde{\rho}(y_*)$ and $\tilde{u}(y_*)$ are completely determined. Nonetheless, there is still a 1-parameter family of solutions depending now on y_* . Another difference between (NLW-SS) and (EP-SS) occurs when one takes an additional derivative. For (NLW-SS), we had $\tilde{\varphi}''(1)$ was determined uniquely. In the case of (EP-SS), one can derive the equations (see Chapter 2)

$$\begin{aligned} \tilde{\rho}'(y_*) + \frac{1}{2\pi y_*} \tilde{u}'(y_*) + \frac{1}{\pi y_*^2} &= 0, \\ 2\pi y_* (\tilde{\rho}'(y_*))^2 + 2\tilde{\rho}'(y_*) \tilde{u}'(y_*) - \frac{1}{2\pi y_*} (\tilde{u}'(y_*))^2 + \tilde{\rho}'(y_*) - \frac{1}{2\pi y_*} \tilde{u}'(y_*) - \frac{1}{\pi y_*^3} &= 0. \end{aligned}$$

In particular, due to the quasilinear structure, the second equation is quadratic. These equations give

$$(\tilde{u}'(y_*))^2 + \tilde{u}'(y_*) + \frac{1}{y_*} - \frac{1}{y_*^2} = 0.$$

This quadratic has two roots, called the Larson-Penston and Hunter branches as follows:

$$\tilde{u}(y_*) = \begin{cases} -\frac{1}{y_*} & \text{Larson-Penston} \\ \frac{1}{y_*} - 1 & \text{Hunter} \end{cases}.$$

Once a branch is chosen, then the higher derivatives of $\tilde{\rho}$ and $\tilde{u}$ at y_* can then be solved for recursively.

The general strategy for finding solutions to (EP-SS) is then similar to that of (NLW-SS), with the free parameters now being $\tilde{\rho}(0)$ and y_* and the matching conditions being that $\tilde{\rho}$ and $\tilde{u}$ should be continuous at y_* . This was implemented numerically by [70], [99] and [59].

We now discuss, in parallel, the rigorous constructions of solutions to (NLW-SS) and (EP-SS). For both (NLW-SS) and (NLW-SS), the first analytic step is to show that the singular points locally give rise to 1-parameter families of initial data. For (NLW-SS), since the singularities are semilinear in nature, one can appeal to existing literature. For (EP-SS), the well-posedness is more subtle. In [52], the local well-posedness is established by using a combinatorial argument to prove that the Taylor series has a positive radius of convergence. This type of argument is also used in the polytropic case, [54], and in this case rigorous interval arithmetic was used to justify certain bounds. In contrast, in our construction of Hunter solutions, the singular ODE is solved in two ways. Firstly, at the level of C^1 solutions, the fundamental matrix of the singular linearized operator is used for the nonlinear argument, being careful to not lose derivatives. This allows one to obtain smooth solutions passing through $y_* = y_f + \epsilon$ for ϵ sufficiently small, where y_f is the far-field sonic point. Secondly, one demonstrates that the C^1 solution is in fact real analytic. In the isothermal case, our proof is based on that of [52]. In the polytropic case, a more general argument is introduced that relies only on the analyticity of the coefficients.

The second step is to show a global existence-type result that rules out Ricatti-type blow-ups. This is achieved in [8] by multiplying (NLW-SS) by $\partial_y \tilde{\varphi}$ and integrating by parts, which leads to the equation

$$H' = \left(\frac{p+3}{p-1}y - \frac{2}{y}\right)(\tilde{\varphi}')^2, \quad (1.29)$$

where the Lyapunov functional H is defined as

$$H = (1 - y^2) \frac{(\tilde{\varphi}')^2}{2} + V(\tilde{\varphi}), \quad (1.30)$$

where

$$V(\tilde{\varphi}) = \frac{\tilde{\varphi}^{p+1}}{p+1} - \frac{(p+1)}{(p-1)^2} \tilde{\varphi}^2.$$

The authors then observe that since $p > 5$, the right-hand-side of (1.29) is negative on $[0, 1]$. In addition, the Hamiltonian H has the confining property that if

$$c \geq c_p = \tilde{\varphi}_F(0)$$

then

$$V(u) \leq V(c) \implies |u| \leq c.$$

In particular, if $\tilde{\varphi}(0) \geq \tilde{\varphi}_F(0)$, then $\tilde{\varphi}(y) \leq \tilde{\varphi}(0)$ for all $0 \leq y \leq 1$. One also has control of $(1 - y^2)(\tilde{\varphi}')^2$, and so if one shoots from the left then one automatically has the existence of a 1-parameter family of solutions on $[0, 1)$. There is a similar argument using the same Lyapunov functional which allows one to shoot from the right.

In the case of (EP-SS), there seems to be no analogous Lyapunov functional to (1.30). In the construction of the Larson-Penston solution, [52], the authors instead uncover monotonicities in the equation which allow them to bootstrap boundedness of the solution when shooting from y_* to the left (see section 4 of their paper). Their technique is refined in the polytropic construction [54], in which case the monotonicity of the relative velocity plays a key role. We comment that these arguments are highly tuned to the features of the Larson-Penston solution and seem hard to generalize to the construction of Hunter solutions.

In the author's construction of Hunter solutions, the global boundedness is established directly from a perturbative analysis. The key is to, for some y_0 fixed, split $[0, \infty)$ into the exterior $[y_0, \infty)$ and the interior $[0, y_0]$.

In the exterior, the key observation is that if one linearizes (EP-SS) about the far-field solution (1.22), the linear system one obtains is equivalent to the following hypergeometric equation for $g(\xi)$:

$$g'' + \frac{\xi - 2}{2\xi(\xi - 1)}g' + \frac{1}{2\xi(\xi - 1)}g. \quad (1.31)$$

We comment that a change of coordinates has taken place so that the sonic point y_f corresponds to $z = 0$, the origin $y = 0$ corresponds to $z = -\infty$ and $y = \infty$ corresponds to $z = 1$. One can write down a solution to (1.31) which smoothly passes through the origin using Gauss's hypergeometric function and read off the asymptotics as $y \rightarrow 0, \infty$. Using the hypergeometric function to construct the fundamental matrix to the ODE, the nonlinear

problem is solved using a fixed-point argument in function spaces which capture the desired asymptotics.

In the interior region $[0, y_0]$, after rescaling to unit scale, one can write the Euler-Poisson system as a perturbation of the well-known isothermal sphere equation

$$\Delta Q + 2e^Q = 0.$$

The solutions to this ODE are not explicit, but can be directly analyzed through a phase-portrait analysis. In addition, due to the scaling symmetry, one immediately sees that

$$y\partial_y Q + 2$$

is a kernel element of the linearized operator

$$\Delta + 2e^Q.$$

This linearized solution is global and has well-understood dynamics as $y \rightarrow 0, \infty$. Expressing another homogeneous solution by solving the Wronskian ODE, one can again read off explicit asymptotics for the fundamental matrix to the ODE and perform a nonlinear fixed point argument.

The third step in constructing self-similar solutions is to match the inner and outer solutions. We start by discussing (EP-SS). In the case of Larson-Penston-type solutions, a topological argument is used (see [52], [54]). In contrast, in the case of Hunter-type solutions the argument is based on the implicit function theorem. This approach is more quantitative, but (as with the construction of the inner and outer solutions) requires smallness. This need for smallness is why we are only able to construct the N th Hunter solution for N sufficiently large.

We comment that in comparison to these two constructions, the earlier work [8] on (NLW-SS) uses a combination of these methods. In particular, the N th solution for N sufficiently large is constructed using perturbative techniques. The authors of [8] are then able to leverage the Lyapunov functional (1.30) to show that by decreasing $\tilde{\varphi}(0)$ from this regime down to $\tilde{\varphi}_F(0)$, one must pass through the entire discrete family of self-similar solutions. We conclude by mentioning that the analogous construction of Hunter solutions for N small remains an open problem. Perhaps adapting this descent method could prove fruitful.

Chapter 2

Hunter self-similar implosion profiles for the gravitational Euler-Poisson system

2.1 Introduction

We begin by describing the setup of the problem. We consider a time interval I_t , Euclidean space $\mathbb{R}_x^3$, a gas with density $\rho > 0$ and its velocity vector field u . The (non-relativistic) gravitational force per unit mass generated by the gas is $-2 \operatorname{grad}(\Delta^{-1}\rho)$, units having been chosen so that the gravitational constant is $(2\pi)^{-1}$. We assume that the gas is ideal and isothermal, and choose units so that the ratio of its pressure to its density is 1. If we model the evolution of this gas by coupling the isentropic compressible Euler equations with self-gravitation, we get the following system of equations for $\rho: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}_{>0}$ and $u: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$, which we refer to as the Euler-Poisson system,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + u \cdot \operatorname{grad} u + \frac{\operatorname{grad} \rho}{\rho} + 2 \operatorname{grad}(\Delta^{-1}\rho) = 0 \end{cases} \quad (2.1)$$

We will focus on solutions with the property that $\rho(0) \rightarrow \infty$ in finite time, which we refer to as implosion. We will make the highly simplifying assumption that both ρ and u are radially symmetric. For notational convenience we define

$$\operatorname{div}_r f(r) = (\partial_r + \frac{2}{r})f \quad \text{and} \quad \operatorname{div}_r^{-1} f(r) = \frac{1}{r^2} \int_0^r (r')^2 f(r') dr',$$

so that under radial symmetry (2.1) becomes the following system of two equations on $I_t \times \mathbb{R}_r$:

$$\begin{cases} \partial_t \rho + \operatorname{div}_r(\rho u) = 0 \\ \partial_t u + u \partial_r u + \frac{\partial_r \rho}{\rho} + 2 \operatorname{div}_r^{-1}(\rho) = 0 \end{cases} \quad .$$

Since the equations are invariant with respect to time translations, we may assume that the implosion occurs at $t = 0$. We then note that we have the following dynamical scaling

symmetry:

$$(\rho(t, r), u(t, r)) \text{ solves (2.1)} \iff \left(\frac{1}{\lambda^2} \rho\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), u\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right) \right) \text{ solves (2.1).} \quad (2.2)$$

A natural first step in understanding singularity formation is to understand the self-similar solutions, namely those which are fixed by this symmetry.

We now summarize the rest of the introduction. In Subsection 2.1.1, we will derive the main system of ODEs, state the theorem and discuss related works. We then compare the system to a class of semilinear focusing heat equations in Subsection 2.1.2, highlighting connections that motivate our proof of the theorem. In Subsection 2.1.3 we give an outline of the proof and describe the organization of the rest of the paper. Subsection 2.1.4 lists some of the notation used in the paper for the reader's reference.

2.1.1 Derivation of equations and statement of the theorem

As a first comment, we shall only study classical solutions of (2.1), by which we mean that ρ and u are C^1 and ρ has sufficient decay for the gravitational force to be well-defined. As a comment, we do not require that either the total mass $\int_{\mathbb{R}^3} \rho dx$ or the total energy,

$$E(\rho, u) = \int_{\mathbb{R}^3} \frac{1}{2} \rho |u|^2 + \rho \log(\rho) - |\text{grad } \Delta^{-1}(\rho)|^2 dx,$$

are finite. In fact, the solutions we discuss will have infinite mass and energy.

We will study the system through the technique of similarity variables, by which we mean switching to the variables $(\tilde{\rho}, \tilde{u}, s, y)$ defined by

$$\rho(t, r) = \frac{1}{t^2} \tilde{\rho}(-\log(-t), \frac{r}{-t}), \quad u(t, r) = \tilde{u}(-\log(-t), \frac{r}{-t}) \quad s = -\log(-t), \quad y = \frac{r}{-t}.$$

Performing this change of variables leads to the following system:

$$\begin{cases} \partial_s \tilde{\rho} + y \partial_y \tilde{\rho} + 2\tilde{\rho} + \text{div}_y(\tilde{\rho} \tilde{u}) = 0 \\ \partial_s \tilde{u} + y \partial_y \tilde{u} + \tilde{u} \partial_y \tilde{u} + \frac{\partial_y \tilde{\rho}}{\tilde{\rho}} + 2 \text{div}_y^{-1}(\tilde{\rho}) = 0 \end{cases} \quad (2.3)$$

We see that (ρ, u) is a solution to (2.1) on $(-\infty, 0) \times \mathbb{R}_x^3$ which is both radial and invariant under the scaling symmetry (2.2) if and only if $(\tilde{\rho}, \tilde{u}) := (\rho(-1, y), u(-1, y))$ is an s -independent solution to (2.3). In this case, the equations for $(\tilde{\rho}(y), \tilde{u}(y))$ become (from now on $\text{div} = \text{div}_y$)

$$\begin{cases} y \partial_y \tilde{\rho} + 2\tilde{\rho} + \text{div}(\tilde{\rho} \tilde{u}) = 0 \\ y \partial_y \tilde{u} + \tilde{u} \partial_y \tilde{u} + \frac{\partial_y \tilde{\rho}}{\tilde{\rho}} + 2 \text{div}^{-1}(\tilde{\rho}) = 0 \end{cases} \quad (2.4)$$

We comment that the use of self-similar solutions to study compressible fluid implosion has an extensive history. Existence of self-similar solutions to the compressible Euler equations (without gravity) was shown by Guderley [49] (see also Sedov [102]). Recently, Merle-Raphaël-Rodnianski-Szeftel [78] constructed smooth self-similar solutions. In the Euler-Poisson system, the presence of gravity removes a degree of scaling freedom from the Euler

system studied by these authors, leading to a different type of ODE problem. These authors used stability properties of their smooth profiles to rigorously describe implosion for the defocusing non-linear Schrödinger, compressible Euler and compressible Navier-Stokes equations ([77], [79]). For further developments, we refer readers to the recent works of Buckmaster-Cao-Labora-Gómez-Serrano [11], Jenssen-Tsikkou [60] and Cao-Labora-Gómez-Serrano-Shi-Staffilani [17], and their discussions.

To understand the structure of the system (2.4) we eliminate the non-local term by rewriting the first equation as

$$\tilde{\rho} = \operatorname{div}(y\tilde{\rho} + \tilde{u}\tilde{\rho})$$

and substitute this into the second equation. This leads to the following formulation of the *self-similar Euler-Poisson system* as a system of quasilinear, non-autonomous ODEs:

$$\begin{pmatrix} \tilde{u} + y & \tilde{\rho} \\ \frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} \begin{pmatrix} \tilde{\rho}' \\ \tilde{u}' \end{pmatrix} + \begin{pmatrix} \frac{2\tilde{\rho}(\tilde{u}+y)}{y} \\ 2\tilde{\rho}(\tilde{u}+y) \end{pmatrix} = 0. \quad (2.5)$$

We now discuss some essential features of the system. We adopt the terminology and discussion of Guo-Hadzic-Jang [52], the only differences being that we work with $\tilde{u}$ instead of their variable $\tilde{\omega}$ and that our derivations only require C^1 regularity of the solutions. Firstly, we note that the system is singular at $y = 0$, where the equations impose the boundary conditions

$$\tilde{u}(0) = 0 \quad \text{and} \quad \tilde{u}'(0) = -\frac{2}{3}. \quad (2.6)$$

Secondly, if for some point y_* we have $(\tilde{u}(y_*) + y_*)^2 = 1$ then the system is singular (the coefficient matrix has determinant zero). Such a point y_* is referred to as a *sonic point*.

We note that if $\tilde{u}(0) = 0$ and $\tilde{u}$ is bounded, then by continuity there exists at least one sonic point y_* for which $\tilde{u}(y_*) + y_* = 1$. In this case, multiplying the system (2.5) on the left by $\begin{pmatrix} 1 & -\tilde{\rho}(y_*) \end{pmatrix}$ gives $\tilde{\rho}(y_*) = y_*^{-1}$. Assuming now that both $\tilde{u}(y_*) = 1 - y_*$ and $\tilde{\rho}(y_*) = y_*^{-1}$, multiplying the system (2.5) on the left by $\begin{pmatrix} y_* & 1 \end{pmatrix}$ gives the condition

$$2y_*\tilde{\rho}'(y_*) + 2\tilde{u}'(y_*) + \frac{4}{y_*} = 0.$$

Also, assuming again $\tilde{u}(y_*) = 1 - y_*$, $\tilde{\rho}(y_*) = y_*^{-1}$ and $\tilde{\rho}, \tilde{u}$ are C^1 , we can multiply (2.5) by $\begin{pmatrix} 1 & -y_*^{-1} \end{pmatrix}$, and substitute

$$\tilde{\rho}(y) = y_*^{-1} + \tilde{\rho}'(y_*)(y - y_*) + o(y - y_*), \quad \tilde{u}(y) = 1 - y_* + \tilde{u}'(y_*)(y - y_*) + o(y - y_*)$$

which gives

$$\left(y_*\tilde{\rho}'(y_*)^2 + 2\tilde{\rho}'(y_*)\tilde{u}'(y_*) - \frac{1}{y_*}\tilde{u}'(y_*)^2 + \tilde{\rho}'(y_*) - \frac{1}{y_*}\tilde{u}'(y_*) - \frac{2}{y_*^3} \right) (y - y_*) + o(y - y_*) = 0.$$

which combined with the previous equation gives the following quadratic for $\tilde{u}'(y_*)$:

$$\tilde{u}'(y_*)^2 + \tilde{u}'(y_*) + \frac{1}{y_*} - \frac{1}{y_*^2} = 0$$

which has two solutions, $-y_*^{-1}$, $y_*^{-1} - 1$. In summary, a C^1 solution at a sonic point y_* where $\tilde{u}(y_*) + y_* = 1$ must satisfy either

$$\begin{aligned} \tilde{\rho}(y_*) = \frac{1}{y_*}, \quad \tilde{\rho}'(y_*) = -\frac{1}{y_*^2}, \quad \tilde{u}(y_*) = 1 - y_*, \quad \tilde{u}'(y_*) = -\frac{1}{y_*} \quad \text{or} \\ \tilde{\rho}(y_*) = \frac{1}{y_*}, \quad \tilde{\rho}'(y_*) = \frac{1}{y_*} - \frac{3}{y_*^2}, \quad \tilde{u}(y_*) = 1 - y_*, \quad \tilde{u}'(y_*) = \frac{1}{y_*} - 1. \end{aligned} \quad (2.7)$$

We use the convention that the first type of first order Taylor expansion is called a Larson-Penston expansion and the second type is called a Hunter expansion.

Two explicit solutions to (2.5) are the Friedman solution $(\tilde{\rho}_F, \tilde{u}_F) = (\frac{1}{3}, -\frac{2}{3}y)$ and the far-field solution $(\tilde{\rho}_f, \tilde{u}_f) = (y^{-2}, 0)$. Neither are classical solutions as for the former the gravitational field is not well-defined and for the latter there is a singularity at $y = 0$.

The historically first non-trivial solution to (2.5) to receive attention is called the Larson-Penston solution, which we now label $(\tilde{\rho}_0, \tilde{u}_0)$, named for the authors who discovered it independently through numerical integration [70, 99]. The solution is analytic and satisfies the qualitative properties $\tilde{\rho}_0 \lesssim \langle y \rangle^{-2}$, $\tilde{u}_0 \lesssim 1$ and $\tilde{u}_0 > 0$. Moreover, the solution has a unique sonic point $y_* \in (2, 3)$ where it has a Larson-Penston expansion. This solution was rigorously proved to exist by Guo-Hadžić-Jang [52]. Further solutions were found numerically by Hunter [59], which we label as $\{(\tilde{\rho}_k, \tilde{u}_k)\}_{k=1}^\infty$. These are still analytic and obey $\tilde{\rho}_N \lesssim_N \langle y \rangle^{-2}$ and $\tilde{u}_N \lesssim_N 1$, but the velocity is no longer positive. These solutions each have a unique sonic point $y_* \in (0, 2)$ where $\tilde{\rho}_k$ has a Hunter-type expansion. From the plots of the Larson-Penston and Hunter solutions (see Figure 4 of Maeda-Harada [73] for instance) we infer that for $k \geq 0$, $\tilde{\rho}_k$ intersects the far-field density y^{-2} precisely $k + 1$ times on $(0, \infty)$.

We also comment that Shu [106] criticized the Larson-Penston solution on physical grounds and proposed instead a certain “expanding wave” self-similar solution. Further, non-smooth, self-similar solutions were introduced by Summers-Whitworth [112] and were later argued to be unstable by Ori-Piran [93]. We focus on the Larson-Penston and Hunter solutions for simplicity and also because their stability has been investigated numerically through both linear mode analyses and simulations of the full nonlinear equation. Some references for computations of unstable eigenvalues are Hanawa-Nakayama [57] and Maeda-Harada [73]. The numerics seem to suggest that for $k \geq 0$ the operator obtained by formally linearizing (2.3) about $(\tilde{\rho}_k, \tilde{u}_k)$ has $k + 1$ unstable modes, with the least unstable mode coming from time-translation symmetry. This suggests that the Larson-Penston and Hunter solutions fit into a sort of “non-linear spectrum” with the Larson-Penston solution playing the role of the ground state. We also mention that there have also been numerical simulations of the radially-symmetric Euler-Poisson system indicating a self-similar collapse regime. Some sources include Hunter [59], Foster-Chevalier [41] and Brenner-Witelski [9].

In this paper we prove the following result which are consistent with an extrapolation of the numerical works on the Hunter solutions.

Theorem 2.1.1. *For some integer $N \gg 1$ there exists real analytic solutions $\{(\tilde{\rho}_k, \tilde{u}_k)\}_{k=N}^\infty$ to (2.5) on $[0, \infty)$ subject to the boundary conditions (2.6) and the requirement $\tilde{\rho}_k > 0$. In addition, $\tilde{\rho}_k$ intersects y^{-2} exactly $k + 1$ times. For these solutions, the sonic point condition $(\tilde{u}_k(y) + y)^2 = 1$ is true at exactly one point $y = y_* \in (0, 2)$, where $\tilde{\rho}$ admits the Hunter-type*

expansion (2.7). Finally, these solutions satisfy the asymptotic bounds $\tilde{\rho}_k \lesssim_k \langle y \rangle^{-2}$, $\tilde{u}_k \lesssim_k 1$.

The proof of the theorem is contained in Section 2.4 and the structure is based on an approach in Collot-Raphaël-Szeftel [26] (which we discuss more in the next section) adapted to the quasilinear nature of the Euler-Poisson system. While we solve the same system of ODEs as considered in [52], our method of proof is very different, resulting in different types of solutions. In their work, the authors use nonlinear invariances of the flow to perform a shooting argument resulting in the Larson-Penston solution, which is expected to be the stable profile (and hence the most physically relevant). In contrast, our argument perturbs off of the far-field solution and the isothermal sphere (a solution to the static Euler-Poisson system). Relative to the Larson-Penston solution, the solutions we construct are expected to be highly unstable. In particular, we do not construct the first Hunter solution, which is believed to play a role in describing critical collapse (as argued by Harada-Maeda-Semelin [58] for instance). Another technical difference is that in [52] they work with real analytic functions, while in our case the bulk of the argument is concerned with C^1 functions. To show the solutions we construct are analytic at 0 and y_* we use the analytic local-wellposedness theory of [52] along with a singular ODE argument at the C^1 -level. As a final point of comparison, we note that the methods of [52] have been adapted to more complex systems. In particular, the relativistic analog of the Larson-Penston solution was first studied numerically by Ori-Piran [92, 94, 95] leading to a prediction of the formation of a naked singularity. Guo-Hadžić-Jang [53] were able to both rigorously construct this profile and show the existence of a spacetime exhibiting a naked singularity. Returning to the Newtonian setting, these authors, along with Schrecker, [54] proved the existence of polytropic analogs of the Larson-Penston solution discovered numerically by Yahil [113]. It is currently unclear whether the methods in this work extend to either the polytropic or relativistic scenarios.

We finally comment that while we have so far only discussed self-similar implosions, other dynamical regimes for the Euler-Poisson system have also been investigated mathematically. For pressure laws $p = \rho^\gamma$ for $1 < \gamma < \frac{4}{3}$, Guo-Hadžić-Jang [51] have shown existence of gravitational collapse which is dust-like rather than self-similar. For $\gamma = \frac{4}{3}$, there exists a class of homologous collapsing solutions found by Goldreich-Weber [48] and further investigated mathematically by Makino [74], Fu-Lin [42], Deng-Liu-Yang-Yao [30] and Deng-Xiang-Yang [31]. For a further introduction to the literature we mention the survey article [56].

2.1.2 Comparison with a class of focusing semilinear heat equations

In this work, we will view (2.1) to be modeled by the following equation on $I_t \times \mathbb{R}_x^3$,

$$\partial_t \varphi = \Delta \varphi + \varphi^p, \quad p > 5. \quad (2.8)$$

While this equation is in many respects different from (2.1), there are also a few similarities.

At a very heuristic level, the $\rho^{-1} \text{grad } \rho$ term in (2.1) and the $\Delta \varphi$ term in (2.8) correspond in the sense that they work to prevent the formation of bell-shaped singularities. In contrast, the $2 \text{grad } \Delta^{-1} \rho$ term in (2.1) and the φ^p term in (2.8) accelerate this type of blow up. In

each case, the dynamics are driven by this competition of forces, and implosion results from inertia and the latter force winning out over the former.

More formally, each equation admits a 1-parameter scaling symmetry, and in each case the equation is energy-supercritical. When we apply the scaling symmetry of the Euler-Poisson system, the energy transforms as

$$E\left(\frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right) = \lambda E(\rho(t, r), u(t, r)).$$

As a result, rescaling initial data to have smaller initial energy shortens the dynamical timescale.

The nonlinear heat equation also has a 1-dimensional scaling symmetry, namely

$$\varphi(t, x) \text{ solves (2.8)} \iff \frac{1}{\lambda^{2/p-1}}\varphi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \text{ solves (2.8)}.$$

In this case, an energy which is dissipated by the flow is

$$E(\varphi) = \int_{\mathbb{R}^3} |\nabla \varphi|^2 - \frac{1}{p+1} \varphi^{p+1} dx.$$

As $p > 5$, we see that this equation too is energy-supercritical.

Another shared quality of these equations is the existence of a 1-parameter family of time-independent radial solutions. For the Euler-Poisson system, if one imposes $u = 0$, and lets $\rho = e^w$, then taking a divergence of the second equation in (2.1) leads to the equation

$$\Delta w + 2e^w = 0. \tag{2.9}$$

Under the assumption of radial symmetry this is the well-studied isothermal sphere ODE and there is a family of smooth, decreasing solutions parameterized by

$$\lambda \mapsto Q_\lambda = Q(\cdot/\lambda) - 2\log(\lambda),$$

where Q is the solution to (2.9) with $Q(0) = 0$. As $\lambda \rightarrow 0$, the family Q_λ converges pointwise on $\mathbb{R}^3 \setminus \{0\}$ to the far-field solution,

$$Q_f = -2\log(r), \quad e^{Q_f} = \frac{1}{r^2}.$$

which is both stationary and self-similar. For the nonlinear heat equation, imposing $\partial_t \varphi = 0$ and radial symmetry leads to the classical Lane-Emden ODE

$$\varphi'' + \frac{2}{r}\varphi' + \varphi^p = 0.$$

For $p > 5$, this ODE may be studied in the same way as for the isothermal sphere equation, and there is a 1-parameter family of smooth, positive, decreasing solutions parameterized by

$$\lambda \mapsto \Phi_\lambda = \lambda^{-\frac{2}{p-1}} \Phi(\cdot/\lambda),$$

where Φ satisfies $\Phi(0) = 1$. In this case, as $\lambda \rightarrow 0$, the limiting far-field solution is

$$\Phi_f = \frac{c_\infty}{r^{\frac{2}{p-1}}}, \quad c_\infty = \left(\frac{2}{p-1} \left(1 - \frac{2}{p-1} \right) \right)^{\frac{1}{p-1}}.$$

One can consult Chandrasekhar [20], for instance, for a treatment of both the isothermal sphere and Lane-Emden equations.

In comparison to the Euler-Poisson system, self-similar singularity formation for (2.8) is better understood mathematically. For these equation, self-similar variables were used to study the stability of ODE-type blowups by Giga-Kohn [43, 44, 45], Merle-Zaag [80] and Bricmont-Kupiainen [10]. The first constructions of self-similar solutions to (2.8) are due to Troy [110], Budd-Qi [14] and Lepin [71]. Collot-Raphaël-Szeftel [26] proved finite-codimension stable self-similar blowup for the equation outside of radial symmetry. An ingredient of their proof was a new construction of a discrete set of highly oscillatory self-similar solutions to (2.8) using a perturbative argument based on the Lane-Emden and far-field solutions (Proposition 2.5 of their monograph). Our proof of Theorem 2.1.1 is an adaptation of their argument. We comment that while we learned the method from this paper, there is a long history of using bifurcation-type methods in related contexts. A few sources in this direction are the works of Budd-Norbury [13], Biernat-Bizoń [6], Dancer-Guo-Wei [29] and Koch [67].

We note that in addition to proving existence of self-similar profiles, [26] proves the linear and non-linear stability of the associated blowups, questions which are not addressed in this work. Their results also fit into the general theory of self-similar blowup for heat equations, which is much more mathematically advanced than the corresponding theory for the Euler-Poisson system. In particular, in the radial setting there is a classification of self-similar blowups due to Matano-Merle [75, 76].

The fact that we are still able to construct solutions in the hyperbolic, quasilinear setting is evidence for the strength of the approach of [26] and its potential applicability to other equations which are focusing, energy-supercritical and have a 1-parameter scaling symmetry.

2.1.3 Proof outline

We now present a more detailed outline of the proof of Theorem 2.1.1.

We consider a fixed $y_0 \ll 1$ and perturbatively construct C^1 solutions defined on $[y_0, \infty)$ and $[0, y_0]$ respectively. We call these the “exterior” and “interior” regions. Matching these solutions at y_0 results in a discrete family. Figure 2.1 is a sketch illustrating the qualitative locations of y_0, y_* with respect to the features of the solutions we construct (one should imagine many more oscillations and $y_0 \ll y_*$). We note that this picture is similar to the numerical solutions plotted in Figure 4 of [73] (their independent variable is the reciprocal of ours so one should horizontally flip the image). We also comment that we shall see oscillations in the linear analyses for both the exterior and interior regions as $y \rightarrow y_0$.

The exterior solutions are modeled on the far-field solution, $(\tilde{\rho}_f, \tilde{u}_f) = (y^{-2}, 0)$. We let ϵ be a small parameter, require the solution have a Hunter-type sonic point at $1 + \epsilon$ and then rescale the variables to put the degeneracy at 1 (a similar technique is employed in [52]). We then study a linearization of the problem. In comparison to [26], our linearized operator L is

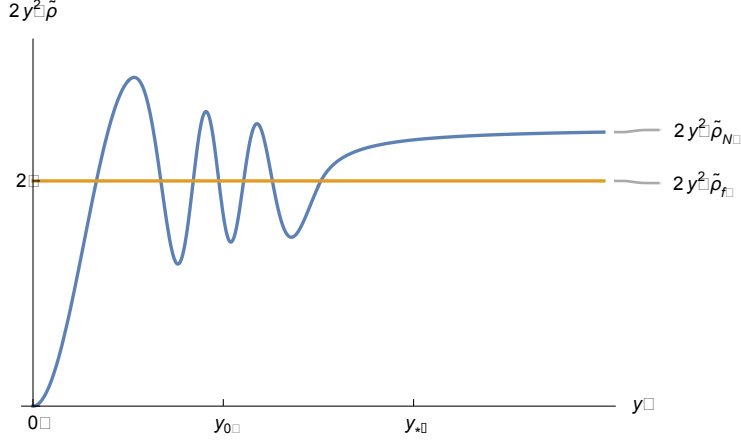


Figure 2.1: Qualitative sketch.

analogous to their operator $\mathcal{L}_\infty$. In their case, a unique (up to scaling) homogeneous solution ψ_1 is identified by imposing decay conditions at ∞ . In our case, requiring the homogeneous solution be smooth at the singular point 1 identifies a unique (up to scaling) homogeneous solution (p_{hom}, ω_{hom}) .

To explain what we mean by this, and to illustrate some of the difficulties which do not appear in the problem of [26], we now discuss the following model ODE on $(-\infty, \infty)$:

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \quad (2.10)$$

In this case, we see that two linearly independent homogeneous solutions for (f, g) are

$$(1, 0) \quad \text{and} \quad (\log |x|, 1).$$

In particular, within the space of homogeneous solutions, there is a 1-D subspace of distinguished solutions which pass through the origin continuously. Returning to the inhomogeneous problem, if we assume that f, g are a C^1 solutions to this equation and plug in $x = 0$, we see that from the first line that we must have $g(0) = F(0)$. We also see from inverting the first matrix, that

$$f' + \frac{g - g(0)}{x} = \frac{F(x) - F(0)}{x}.$$

Thus, if f, g are C^1 , the righthand side must be continuous, so F must be differentiable at 0. Assuming F, G are continuous, and F is differentiable at 0, we can write down a 1-D space of solutions to (2.10), namely

$$g(x) = F(0) + \int_0^x G(x') dx', \quad f(x) = \int_0^x \frac{F(x') - g(x')}{x'} dx' + C.$$

A key takeaway is that we need to assume that a component of our non-linearity is differentiable at 0 in order to have a C^1 solution. The linearized operator we study is similar, and the iteration scheme works because under the assumption that our solution is C^1 , it can be

shown that a component of the non-linearity, instead of being merely continuous, is in fact differentiable at 0.

As another point of comparison, in the problem considered in [26], their homogeneous solution ψ_1 can be written down using the Tricomi solution to Kummer's equation whereas in our case we can use Gauss's hypergeometric function to express our homogeneous solutions $(p_{hom}(y), \omega_{hom}(y))$. The connection formulae for hypergeometric functions allow us to see that the asymptotics of $(p_{hom}(y), \omega_{hom}(y))$ as $y \rightarrow 0$ are given by

$$\begin{aligned} p_{hom}(y) &= \frac{c_1}{y^{1/2}} \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_1\right) + O(y^{3/2}), \\ \omega_{hom}(y) &= \frac{c_1}{y^{1/2}} \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_1 + \theta_0\right) + O(y^{3/2}), \end{aligned}$$

for some explicit $c_1 > 0$, $d_1 \in \mathbb{R}/2\pi\mathbb{Z}$ and

$$\theta_0 := \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi. \quad (2.11)$$

We comment that the frequency $\frac{\sqrt{7}}{2}$ is the limit of the frequency ω used in the argument of [26] as the power in the non-linear heat equation tends to infinity. To briefly explain where the numerology comes from in our problem (in the exterior region), we note that if one linearizes (2.5) about $(\tilde{\rho}_f, \tilde{u}_f)$, the resulting equation, written in matrix form, is

$$\partial_y \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} \frac{2y^2-4}{y^3-y} & \frac{-2}{y^6-y^4} \\ \frac{2y^2}{y^2-1} & \frac{2}{y^3-y} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} = 0. \quad (2.12)$$

This system, viewed as a complex ODE in y , has a pole of order 4 at $y = 0$. One obtains a simple pole at $y = 0$ if one changes variables to, for instance, $\tilde{p} = y^2 \tilde{\rho}$, $\tilde{v} = y^{-1} \tilde{u}$. In this case, the resulting system is

$$\partial_y \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} -\frac{2}{y(y^2-1)} & -\frac{2}{y(y^2-1)} \\ \frac{2}{y(y^2-1)} & \frac{y^2+1}{y(y^2-1)} \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} = 0.$$

As the pole at $y = 0$ is simple, one can locally construct homogeneous solutions by means of the method of Frobenius. In particular, the leading order asymptotics of these homogeneous solutions are governed by the residue of the coefficient of the zeroth order term. We may diagonalize (the negative of) this residue as

$$\begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ e^{i\theta_0} & e^{-i\theta_0} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{7}}{2} & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ e^{i\theta_0} & e^{-i\theta_0} \end{pmatrix}^{-1},$$

and in doing so we see the constant θ_0 and frequency $\frac{\sqrt{7}}{2}$.

We use a contractive mapping argument to construct a family of exterior solutions $(\tilde{\rho}_{ext}[\epsilon](y), \tilde{u}_{ext}[\epsilon](y))$ for $\epsilon \ll 1$ which are C^1 on $[y_0, \infty)$, solving (2.5), with a Hunter-type sonic point at $y = 1 + \epsilon$ of the form

$$\begin{cases} \tilde{\rho}_{ext} = \frac{1}{y^2} + \frac{\epsilon}{y^2} p_{hom} + \epsilon \rho_{ext} \\ \tilde{u}_{ext} = \epsilon y \omega_{hom} + \epsilon u_{ext} \end{cases},$$

where ρ_{ext} and u_{ext} are suitably small, (see lemma 2.2.7 for details).

In the interior region, after a rescaling, we approximate $\tilde{\rho}$ by e^Q , where Q is the radial solution to (2.9) with $Q(0) = 0$. We define u_* by solving the self-similar conservation of mass equation

$$\begin{cases} (y\partial_y + 2)e^Q + \operatorname{div}(e^Q u_*) = 0 \\ u_*(0) = 0, \quad u'_*(0) = -\frac{2}{3} \end{cases}. \quad (2.13)$$

By studying the isothermal sphere equation, it is shown that e^Q and u_* have the following asymptotic behavior as $y \rightarrow \infty$, for some $c_2 > 0$ and $d_2 \in \mathbb{R}/2\pi\mathbb{Z}$,

$$e^Q = \frac{1}{y^2} + \frac{c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_2\right)}{y^{5/2}} + O\left(\frac{1}{y^3}\right).$$

To understand the asymptotic behavior of u_* as $y \rightarrow \infty$, we can use the isothermal sphere equation (2.9), to rewrite (2.13) as

$$-\frac{1}{2}(y\partial_y + 2)(\Delta Q) + \operatorname{div}(e^Q u_*) = 0,$$

and since $(y\partial_y + 2)\Delta = \Delta y\partial_y$, we may rewrite this as

$$-\frac{1}{2} \operatorname{div} \partial_y (y\partial_y Q) + \operatorname{div}(e^Q u_*) = 0.$$

Applying now $\operatorname{div}_y^{-1}$, we see that explicitly

$$u_* = \frac{1}{2e^Q} \partial_y (y\partial_y Q) \quad (2.14)$$

from which we may read off the asymptotic behavior

$$u_* = c_2 y^{1/2} \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_2 + \theta_0\right) + O(1).$$

We note that this numerology is consistent with the previous section. In this case, one can read off the frequency $\frac{\sqrt{7}}{2}$ from the isothermal sphere ODE by either looking associated phase portrait in homologous variables, or by computing the homogeneous solutions to $\Delta + \frac{2}{y^2}$ and the constant θ_0 arises from integrating when solving (2.13). It is shown (again through an iterative construction) that for every $\lambda \leq \frac{y_0}{10}$ one can construct an interior solution

$(\tilde{\rho}_{int}[\lambda](y), \tilde{u}_{int}[\lambda](y))$ which is C^1 on $[0, y_0]$ (we actually construct a C^2 solution, but this distinction doesn't matter for the matching), solving (2.5), with $\tilde{\rho}(0) = \lambda^{-2}$, of the form

$$\begin{cases} \tilde{\rho}_{int} = \frac{1}{\lambda^2} e^Q(\cdot/\lambda) + \rho_{int}(\cdot/\lambda) \\ \tilde{u}_{int} = \lambda u_*(\cdot/\lambda) + \lambda^3 u_{int}(\cdot/\lambda) \end{cases},$$

where ρ_{int}, u_{int} are suitably small (see Lemma 2.3.6 for more details).

With both regimes having been studied, the remaining step is to implement the matching using the parameters ϵ and λ . Heuristically this matching is possible because of the fact that the interior and exterior asymptotic behaviors match as we approach y_0 (both regimes witness the oscillation). To simplify the algebra, we note that without loss of generality, $y_0 \ll 1$ can be chosen so that

$$\frac{\sqrt{7}}{2} \log(y_0) + d_1 = \frac{\pi}{2} \pmod{2\pi}. \quad (2.15)$$

We then consider the exterior solutions defined for $\epsilon \lesssim y_0^{5/2} \ll y_0^{1/2}$. We have the following asymptotics uniform with respect to ϵ, y_0 :

$$\begin{cases} \tilde{\rho}_{ext}(y_0) - \frac{1}{y_0^2} &= \epsilon \frac{1}{y_0^{5/2}} c_1 \sin\left(\frac{\sqrt{7}}{2} \log(y_0) + d_1\right) + O(\epsilon y_0^{-1/2}) \\ \frac{\tilde{u}_{ext}(y_0)}{y_0^3} &= \epsilon \frac{1}{y_0^{5/2}} c_1 \sin\left(\frac{\sqrt{7}}{2} \log(y_0) + d_1 + \theta_0\right) + O(\epsilon y_0^{-1/2}) \end{cases}. \quad (2.16)$$

Similarly, for the same fixed y_0 , we consider the interior solutions in the regime $\lambda^{1/2} \lesssim y_0^{5/2} \ll y_0^{1/2}$, we have the following asymptotics uniform with respect to λ, y_0 :

$$\begin{cases} \tilde{\rho}_{int}(y_0) - \frac{1}{y_0^2} &= \lambda^{1/2} \frac{1}{y_0^{5/2}} c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2\right) + O(\lambda^{1/2} y_0^{-1/2}) \\ \frac{\tilde{u}_{int}(y_0)}{y_0^3} &= \lambda^{1/2} \frac{1}{y_0^{5/2}} c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2 + \theta_0\right) + O(\lambda^{1/2} y_0^{-1/2}) \end{cases}. \quad (2.17)$$

If we match (2.16) with (2.17), and implement (2.15), then we get

$$\begin{cases} \epsilon c_1 + O(\epsilon y_0^2) &= c_2 \lambda^{1/2} \cos\left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2\right) + O(\lambda^{1/2} y_0^2) \\ \epsilon c_1 \cos(\theta_0) + O(\epsilon y_0^2) &= c_2 \lambda^{1/2} \cos\left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2 + \theta_0\right) + O(\lambda^{1/2} y_0^2) \end{cases}.$$

We may then solve the first equation for ϵ , substitute it into the second equation and simplify, so that the matching condition becomes

$$G[\lambda](y_0) = 0$$

for some function $G[\lambda](y_0)$ such that

$$G[\lambda](y_0) = \sin\left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2\right) + O(y_0^2). \quad (2.18)$$

We then define $\lambda_{k,\pm} = \frac{2}{\sqrt{7}} \exp(-k\pi - d_1 + d_2 \pm \frac{1}{10})$. Assuming y_0 has been chosen sufficiently small, and $k > N$ for some large N (so that $\lambda_{k,\pm}$ are small) we see that by the intermediate

value theorem there will exist $\lambda_k \in (\lambda_{k,-}, \lambda_{k,+})$ which solve $G[\lambda](y_0) = 0$. Each λ_k will correspond to a classical solution obtained by gluing together $(\tilde{\rho}_{ext}, \tilde{u}_{ext})$ and $(\tilde{\rho}_{int}, \tilde{u}_{int})$. The perturbative nature of this argument also allows us to conclude positivity of $\tilde{\rho}$ and also exactly count the zeroes of $y^2\tilde{\rho}_k - 1$ and show that they increase by 1 as k increases. This oscillation-type result allows us to enumerate the solutions.

This argument (done rigorously) finishes the bulk of the proof of Theorem 2.1.1. Two things remain: to show each solution has a unique sonic point and to show that the solutions are real analytic. As a first step in counting the sonic points, we note that by construction, any matched solution will have one sonic point near $y = 1$. To show that there are no other sonic points, separate arguments are used on $[0, y_0]$ and $[y_0, \infty)$. For the interior, we show $\tilde{u} + y = O(y)$, which rules out sonic points assuming y_0 is sufficiently small. For the exterior, we expect $\tilde{u} + y$ to be close to y as we are perturbing off of the far-field solution. Motivated by this, we then show that for the exterior solutions $(\tilde{u} + y)' \geq \frac{1}{2}$, which combined with the absence of sonic points in the interior suffices to show that there is at most one sonic point.

In regards to regularity of the solutions, by standard local ODE theory, it follows that the C^1 solutions are locally real analytic at all points except possibly $y \in \{y_*, 0\}$. For each of these points, by the methods of [52], it follows that given the appropriate boundary conditions there exists locally a unique real analytic Hunter solution. We then prove that this local solution is the unique one of regularity C^1 by linearizing the system about it and using its analyticity to derive asymptotics for the principal matrix solution.

The rest of the paper is organized as follows. Sections 2.2 and 2.3 detail the construction of the exterior and interior solutions respectively. The proofs of several lemmas are deferred. Section 2.4 details the matching argument and contains the proof of the main theorem. Finally, Sections 2.5 and 2.6 contain the proofs deferred from Sections 2.2 and 2.3 respectively.

2.1.4 Notation

To aid the reader we informally describe a few notational conventions.

- (t, x) are the standard (Euclidean) time and space coordinates.
- r is the Euclidean radial coordinate.
- (ρ, u) are the mass density and velocity. In the radial setting, u is the radial velocity.
- y is the self-similar radial coordinate, defined in Subsection 2.1.1.
- $\tilde{\rho}(y), \tilde{u}(y)$ are the self-similar mass density and velocity, defined in Subsection 2.1.1.
- y_* denotes a sonic point, defined in Subsection 2.1.1.
- $\tilde{\rho}_f, \tilde{u}_f$ denotes the far-field solution, defined in Subsection 2.1.1.
- Q is representative isothermal sphere solution, defined in Subsection 2.1.2.
- Q_f is the far-field isothermal sphere solution, defined in Subsection 2.1.2.
- y_0 is the matching point, defined in Subsection 2.4.

- p_{hom}, ω_{hom} are homogeneous solutions to a linear ODE, defined in Subsection 2.2.2.
- c_1, d_1 are constants appearing in the asymptotic expansions of p_{hom}, ω_{hom} , defined in Subsection 2.2.2.
- θ_0 is an explicit constant appearing in asymptotic expansions, defined in Subsection 2.1.3.
- $\tilde{p}, \tilde{v}$ are convenient rescalings of $\tilde{\rho}, \tilde{u}$, defined in Subsection 2.1.3.
- u_* is a function appearing in a linearized analysis, defined in Subsection 2.1.3.
- c_2, d_2 are constants appearing in the asymptotic expansions of Q, u_* , defined in Subsection 2.3.1.
- $\tilde{p}, \tilde{\omega}$ are convenient variables for the exterior analysis, defined in Subsection 2.2.1. This definition of $\tilde{p}$ is consistent with that in Subsection 2.1.3.
- z is a rescaling of y that maps the sonic point to $z = 1$, defined in Subsection 2.2.1.
- $\underline{p}, \underline{\omega}$ are rescalings of $\tilde{p}, \tilde{\omega}$, defined in Subsection 2.2.1.
- $\hat{p}, \hat{\omega}$ are renormalized differences of $\underline{p}, \underline{\omega}$ and the far-field solution, defined in Subsection 2.2.1.
- In Sections 2.2 and 2.5, p, ω are the differences of $\hat{p}, \hat{\omega}$ and p_{hom}, ω_{hom} , defined in Subsection 2.2.3.
- $\tilde{w}$ is a convenient variable for the interior analysis, defined in Subsection 2.3.1.
- $\bar{w}, \bar{u}$ are rescalings of $\tilde{w}, \tilde{u}$, defined in Subsection 2.3.2.
- Q_λ, u_λ are rescalings of Q, u_* , defined in Subsection 2.3.3.
- In Sections 2.3 and 2.6, w, u are differences of $\tilde{w}, \tilde{u}$ and Q_λ, u_λ , defined in Subsection 2.3.3. As a comment, this usage of u should not be confused with u as used in the introduction for the physical velocity.
- In Sections 2.3 and 2.6, y_1, r are the rescalings of y_0, λ , as defined in Subsection 2.3.3. As a comment, this usage of r should not be confused with r as used in the introduction to for the Euclidean radial coordinate.

2.2 Solving the self-similar equations on $[y_0, \infty)$

2.2.1 Choice of variables

In the exterior region we will first change variables from $\tilde{\rho}, \tilde{u}$ to $\tilde{p}, \tilde{\omega}$ defined implicitly by

$$\tilde{\rho} = \frac{\tilde{p}}{y^2} \quad \text{and} \quad \tilde{u} = y(\tilde{\omega} - 1).$$

In this case, (2.5) with the Hunter boundary conditions become

$$\begin{cases} (\tilde{p}\tilde{\omega}y)' - \tilde{p} = 0 \\ \tilde{\omega}(y(\tilde{\omega} - 1))' + \frac{\tilde{p}'}{y\tilde{p}} + \frac{2}{y^2}(\tilde{p}\tilde{\omega} - 1) = 0 \\ \tilde{p}(y_*) = y_*, \quad \tilde{p}'(y_*) = y_* - 1, \quad \tilde{\omega}(y_*) = \frac{1}{y_*}, \quad \tilde{\omega}'(y_*) = 0 \end{cases}. \quad (2.19)$$

The principal motivation for these variables is that the far-field solution takes the relatively simple form $(\tilde{p}, \tilde{\omega}) = (1, 1)$.

We now perform a y_* -dependent change of variables that will fix the domain $[y_0, \infty)$, move the sonic point from y_* to 1 and simplify the values of p, ω at the sonic point (a similar change of variables is done in [52]). In particular, we define

$$\alpha = \frac{1 - y_0}{y_* - y_0}, \quad \beta = \frac{(y_* - 1)y_0}{y_* - y_0}, \quad z = \alpha y + \beta, \\ \tilde{p} = y_* \underline{p}(\alpha y + \beta), \quad \tilde{\omega} = \frac{1}{y_*} \underline{\omega}(\alpha y + \beta).$$

Under this change of variables, (2.19) becomes the following system for $\underline{p}(z), \underline{\omega}(z)$,

$$\begin{cases} (\underline{p}\underline{\omega}(z - \beta))' - y_* \underline{p} = 0 \\ \underline{\omega}((\underline{\omega} - y_*)(z - \beta))' + (\alpha y_*)^2 \frac{\underline{p}'}{\underline{p}(z - \beta)} + 2 \frac{(\alpha y_*)^2}{(z - \beta)^2} (\underline{p}\underline{\omega} - 1) = 0. \\ \underline{p}(1) = 1, \quad \underline{p}'(1) = \frac{y_* - 1}{\alpha y_*}, \quad \underline{\omega}(1) = 1, \quad \underline{\omega}'(1) = 0. \end{cases} \quad (2.20)$$

We next let $y_* = 1 + \epsilon$ for some parameter ϵ small. As a note, for notational convenience we continue to use both ϵ and y_* in the equations. We then switch variables to $\hat{p}(z), \hat{\omega}(z)$ defined by

$$\underline{p} = 1 + \frac{\epsilon}{\alpha y_*} (\hat{p} - 1), \quad \underline{\omega} = 1 + \epsilon (\hat{\omega} + 1).$$

We then define the following quantities (which depend on ϵ as well as on $\hat{p}, \hat{\omega}$):

$$A_1(\epsilon, \hat{p}, \hat{\omega}) = \epsilon(1 + \frac{\epsilon}{y_* \alpha} (\hat{p} - 1)) (\hat{\omega}(z - \beta))' + \frac{\epsilon}{y_* \alpha} \hat{p}'(y_* + \epsilon \hat{\omega})(z - \beta), \\ A_2(\epsilon, \hat{p}, \hat{\omega}) := \epsilon(y_* + \epsilon \hat{\omega}) (\hat{\omega}(z - \beta))' + \epsilon \hat{p}' \frac{\alpha y_*}{z - \beta} \frac{1}{(1 + \frac{\epsilon}{y_* \alpha} (\hat{p} - 1))} \\ + \frac{2(y_* \alpha)^2}{(z - \beta)^2} \left((1 + \frac{\epsilon}{y_* \alpha} (\hat{p} - 1)) (y_* + \epsilon \hat{\omega}) - 1 \right).$$

We also define the linear operator

$$L \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} L_1(\hat{p}, \hat{\omega}) \\ L_2(\hat{p}, \hat{\omega}) \end{pmatrix} := \begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix} \partial_z \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \frac{2}{z^2} & \frac{2}{z^2} + 1 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix},$$

which can be extended to acting on matrices by acting column-wise. We then note that formally $(A_1(\epsilon, \hat{p}, \hat{\omega}), A_2(\epsilon, \hat{p}, \hat{\omega})) = \epsilon L + O(\epsilon^2)$. We can thus define

$$\hat{N}_i(\epsilon \hat{p}, \hat{\omega}) := \frac{1}{\epsilon} (\epsilon L_i(\hat{p}, \hat{\omega}) - A_i(\hat{p}, \hat{\omega})), \quad \hat{N}(\epsilon, \hat{p}, \hat{\omega}) := (\hat{N}_1(\epsilon, \hat{p}, \hat{\omega}), \hat{N}_2(\epsilon, \hat{p}, \hat{\omega})).$$

With this notation, (2.20) becomes

$$\begin{cases} L(\hat{p}, \hat{\omega}) = \hat{N}(\epsilon, \hat{p}, \hat{\omega}) \\ \hat{p}(1) = 1, \quad \hat{p}'(1) = 1, \quad \hat{\omega}(1) = -1, \quad \hat{\omega}'(1) = 0 \end{cases} \quad (2.21)$$

We view this system as a family of ODEs parameterized by ϵ . In the next section we will study the linearized operator. In the subsequent section we will use this information to show that the equation is well-posed for ϵ close to zero.

2.2.2 Properties of the linearized operator

We now consider the following equation on $(0, \infty)$:

$$L(p, \omega) = (P, \Omega). \quad (2.22)$$

We remark that L is a variable coefficient operator which is singular at $z = 1$. We shall see that when solving from $z = 1$ the structure is similar to the model problem (2.10) discussed in the introduction. In comparison with that problem, the constraints are similarly easy to derive, but due to the variable coefficients it is harder to identify the distinguished subspace of homogeneous solutions and to show the existence of solutions for the inhomogeneous problem.

In particular, if we assume $p, \omega \in C^1$, then multiplying the system on the left by $\begin{pmatrix} 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \end{pmatrix}$ respectively, and substituting in $z = 1$ give

$$\begin{cases} p(1) + \omega(1) = \frac{P(1) - \Omega(1)}{-2} \\ p'(1) + \omega'(1) + p(1) + 2\omega(1) = \frac{P(1) + \Omega(1)}{2} \end{cases} \quad (2.23)$$

The first equation yields the constraint $p(1) + \omega(1) = \frac{P(1) - \Omega(1)}{-2}$. We next note that for $z \neq 1$ we may invert the highest order coefficient in (2.22) to obtain

$$\begin{pmatrix} p \\ \omega \end{pmatrix}' + \frac{1}{z(z^2 - 1)} \begin{pmatrix} -2 & -2 \\ 2 & z^2 + 1 \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = \begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix}. \quad (2.24)$$

Multiplication by $\begin{pmatrix} 1 & -1 \end{pmatrix}$ then yields that as $z \rightarrow 1$

$$p' = \frac{(P - \Omega) + 2(p + \omega)}{2(z - 1)} + O(1).$$

In particular, if $p, \omega \in C^1$ we must have that $\frac{(P - \Omega) - (P(1) - \Omega(1))}{z - 1}$ is continuous, or in other words $P - \Omega$ is differentiable at $z = 1$. In this case, we again multiply (2.22) by $\begin{pmatrix} 1 & -1 \end{pmatrix}$, and this time expand to order $o(z - 1)$. In this case, we obtain

$$2(z - 1)p'(z) - 2(1 - 2(z - 1))(p(z) - \omega(z)) = P(1) - \Omega(1) + (P - \Omega)'(1)(z - 1) + o(z - 1).$$

Simplifying this using (2.23) gives the equation

$$\omega'(1) = -(P(1) + \Omega(1)) - \frac{1}{2}(P - \Omega)'(1).$$

This equation, along with (2.23) imply that if $P, \Omega, \frac{P-\Omega}{z-1}$ are continuous and

$$P(1) = \Omega(1) = (P - \Omega)'(1) = 0. \quad (2.25)$$

then if a solution to $L(p, \omega) = (P, \Omega)$ exists, then up to scaling by a constant, its Taylor expansion at $z = 1$ must be

$$p = 1 + (z - 1) + o(z - 1), \quad \omega = -1 + o(z - 1). \quad (2.26)$$

In particular, any homogeneous solution must have such an expansion.

Our next goal is to find the distinguished subspace of homogeneous solutions and to solve the inhomogeneous problem for data subject to the constraints. A key observation is that the equation $L(p, \omega) = 0$ can be transformed into a rational-coefficient 2nd order ODE with three simple singularities. This transformation allows one to write down homogeneous solutions using Gauss's hypergeometric function.

We use the terminology that a fundamental matrix associated to L on an interval I is a smooth matrix-valued function $U(z)$ defined on I such that $LU = 0$. We recall that if U is a fundamental matrix, and M is a constant invertible matrix then $U \cdot M$ is also a fundamental matrix, and moreover any two fundamental matrices are related in this way.

Lemma 2.2.1. *(a) There exists unique $(p_{hom}(z), \omega_{hom}(z))$ analytic on $(0, \infty)$ such that $L(p_{hom}, \omega_{hom}) = 0$ and such that their first order Taylor polynomials at $z = 1$ agree with (2.26).*

(b) There exists analytic fundamental matrices $U_\infty(z), U_0(z)$ for L defined on $(1, \infty)$ and $(0, 1)$ respectively. Moreover, these can be chosen such that as $z \rightarrow \infty$

$$U_\infty(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^2}) & O(\frac{1}{z^3}) \\ O(\frac{1}{z^2}) & O(\frac{1}{z^3}) \end{pmatrix}, \quad U_\infty^{-1}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^2}) & O(\frac{1}{z^2}) \\ O(\frac{1}{z}) & O(\frac{1}{z}) \end{pmatrix}.$$

and as $z \rightarrow 0$

$$U_0(z) = \frac{1}{z^{1/2}} \begin{pmatrix} \cos\left(\frac{\sqrt{7}}{2} \log(z)\right) & \sin\left(\frac{\sqrt{7}}{2} \log(z)\right) \\ \cos\left(\frac{\sqrt{7}}{2} \log(z) + \theta_0\right) & \sin\left(\frac{\sqrt{7}}{2} \log(z) + \theta_0\right) \end{pmatrix} + O(z^{3/2}).$$

$$U_0^{-1}(z) = \frac{z^{1/2}}{\sin(\theta_0)} \begin{pmatrix} \sin\left(\frac{\sqrt{7}}{2} \log(z) + \theta_0\right) & -\sin\left(\frac{\sqrt{7}}{2} \log(z)\right) \\ -\cos\left(\frac{\sqrt{7}}{2} \log(z) + \theta_0\right) & \cos\left(\frac{\sqrt{7}}{2} \log(z)\right) \end{pmatrix} + O(z^{5/2}).$$

(c) The asymptotics of the fundamental solution (p_{hom}, ω_{hom}) as $z \rightarrow \infty, 0$ respectively are, for some $\mu_3, \mu_4, c_1 \in \mathbb{R}$, $d_1 \in \mathbb{R}/2\pi\mathbb{Z}$ with $\sqrt{\mu_3^2 + \mu_4^2}, c_1 \neq 0$,

$$\begin{pmatrix} p_{hom} \\ \omega_{hom} \end{pmatrix} = \begin{pmatrix} \mu_3 \\ \mu_4 \\ z \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^2}) \\ O(\frac{1}{z^3}) \end{pmatrix} \quad (2.27)$$

$$\begin{pmatrix} p_{hom} \\ \omega_{hom} \end{pmatrix} = \frac{c_1}{z^{1/2}} \begin{pmatrix} \sin\left(\frac{\sqrt{7}}{2} \log(z) + d_1\right) \\ \sin\left(\frac{\sqrt{7}}{2} \log(z) + d_1 + \theta_0\right) \end{pmatrix} + O(z^{3/2}). \quad (2.28)$$

(d) Also, there exists constant invertible matrices M_0, M_∞ such that as $z \rightarrow 1^\pm$, we have respectively

$$U_\infty M_\infty, U_0 M_0 = \begin{pmatrix} 1 & \log|z-1| + \frac{1}{2} \\ -1 & -\log|z-1| + \frac{1}{2} \end{pmatrix} + \begin{pmatrix} O(z-1) & O(|z-1| \log|z-1|) \\ O(z-1) & O(|z-1| \log|z-1|) \end{pmatrix} \quad (2.29)$$

$$(U_\infty M_\infty)^{-1}, (U_0 M_0)^{-1} = \begin{pmatrix} -\log|z-1| + \frac{1}{2} & -\log|z-1| - \frac{1}{2} \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} O(|z-1| \log|z-1|) & O(|z-1| \log|z-1|) \\ O(z-1) & O(z-1) \end{pmatrix}$$

(e) If we define the principal matrix solution on $(0, 1) \times (0, 1) \cup (1, \infty) \times (1, \infty)$ to be

$$S(z, z') = \begin{cases} U_0(z)U_0^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1}, & (z, z') \in (0, 1) \times (0, 1) \\ U_\infty(z)U_\infty^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1}, & (z, z') \in (1, \infty) \times (1, \infty) \end{cases}.$$

then if $P(z), \Omega(z), \frac{P-\Omega-(P(1)-\Omega(1))}{z-1}$ are continuous on $(0, \infty)$ and satisfy the constraints (2.25) then

$$(p, \omega) := R(P(z), \Omega(z)) := \int_1^z S(z, z') \begin{pmatrix} P(z') \\ \Omega(z') \end{pmatrix} dz' \quad (2.30)$$

defines an element of $C^1((0, \infty))$ which solves $L(p, \omega) = (P, \Omega)$ and

$$p(1) = p'(1) = \omega(1) = \omega'(1) = 0.$$

For the proof, see Subsection 2.5.1.

2.2.3 Difference equations and iteration scheme

We now consider (2.21), and substitute the following ansatz, where (p, ω) are now unknown,

$$(\hat{p}, \hat{\omega}) = (p_{hom}, \omega_{hom}) + (p, \omega).$$

This gives the equation

$$\begin{cases} L(p, \omega) = N(\epsilon, p, \omega) \\ p(1) = p'(1) = \omega(1) = \omega'(1) = 0 \end{cases} \quad (2.31)$$

where we have defined

$$N(\epsilon, p, \omega) = \hat{N}(\epsilon, p_{hom} + p, \omega_{hom} + \omega).$$

The solution operator defined in Lemma 2.2.1.e allows us to rewrite (2.31) as the integral equation

$$(p, \omega) = RN(\epsilon, p, \omega). \quad (2.32)$$

As a note, for $N(\epsilon, p, \omega)$ to be well-defined, we impose the mass-positivity condition

$$1 + \frac{\epsilon}{y_* \alpha} (p_{hom} + p - 1) > 0.$$

on solutions to (2.32). The integral equation can then be solved on $[y_0, \infty)$ using a classical fixed point method with suitable Banach spaces. Before defining these Banach spaces, we first introduce the notation

$$D_\infty f(z) = -z^2 \partial_z f.$$

Definition 2.2.2. We define our “solution” norm to be

$$\|(p, \omega)\|_{X_{y_0}} := \sup_{z \geq 1} |p| + |z\omega| + |D_\infty p| + |D_\infty(z\omega)| + \sup_{y_0 \leq z \leq 1} z^{1/2}(|p| + |\omega|) + z^{3/2}(|p'| + |\omega'|).$$

We define our “solution” Banach space to be

$$X_{y_0} := \{(p, \omega) \in C^1([y_0, \infty)) : \lim_{z \rightarrow \infty} (D_\infty(p), D_\infty(z\omega)) \text{ exists}, \\ \|(p, \omega)\|_{X_{y_0}} < \infty \quad \text{and} \quad p(1) = \omega(1) = p'(1) = \omega'(1) = 0\}.$$

Definition 2.2.3. We define our “nonlinearity” norm to be

$$\|(P, \Omega)\|_{N_{y_0}} := \sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{P - \Omega}{z - 1} \right| + \sup_{z \geq 1} |zP| + |z^2 \Omega| + \sup_{y_0 \leq z \leq 1} (|zP| + |z^3 \Omega|).$$

We define our “nonlinearity” Banach space by letting

$$N_{y_0} := \{(P, \Omega) \in C^0([y_0, \infty)) : \frac{P - \Omega}{z - 1} \in C^0([y_0, \infty)), \lim_{z \rightarrow \infty} (zP, z^2 \Omega) \text{ exists}, \\ \|(P, \Omega)\|_{N_{y_0}} < \infty \quad \text{and} \quad P(1) = \Omega(1) = (P - \Omega)'(1) = 0\}.$$

As a comment, the first term in the norm is non-standard and it requires a computation to show that it is, in fact, finite for non-linearities we consider.

We now state the three lemmas which will lead to our construction of exterior solutions.

Lemma 2.2.4. For $y_0 \leq \frac{1}{2}$ there exists a $C_1 \geq 1$ (independent of y_0) so that if $(P, \Omega) \in N_{y_0}$ then

$$(p, \omega) := R(P, \Omega) \in X_{y_0} \quad \text{and} \quad \|(p, \omega)\|_{X_{y_0}} \leq C_1 y_0^{-1/2} \|(P, \Omega)\|_{N_{y_0}}.$$

For the proof, see Subsection 2.5.2.

Lemma 2.2.5. If $y_0 \leq \frac{1}{2}$, then for all $C > 0$ there exists a $C_0(C) \ll 1$ (independent of y_0) such that if $\epsilon \leq C_0(C) y_0^{1/2}$ (ensuring mass-positivity) then there exists $C_2(C) \geq 1$ (independent of y_0, ϵ), such that,

(a) If $(p, \omega) \in X_{y_0}$, $\|(p, \omega)\|_{X_{y_0}} \leq C$ then

$$(P, \Omega) := N(\epsilon, p, \omega) \in N_{y_0} \quad \text{and} \quad \|(P, \Omega)\|_{N_{y_0}} \leq |\epsilon|C_2. \quad (2.33)$$

(b) We have that $(\epsilon, p, \omega) \mapsto N(\epsilon, p, \omega)$ is C^1 as a function of (ϵ, p, ω) with domain

$$\mathcal{D}_C := (-C_0(C)y_0^{1/2}, C_0(C)y_0^{1/2}) \times \{(p, \omega) \in X_{y_0} : \|p, \omega\|_{X_{y_0}^1} < C\}$$

valued in N_{y_0} , with

$$\left\| \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) \right\|_{L(X_{y_0}, N_{y_0})} \leq C_2 |\epsilon_0|, \quad \left\| \frac{\partial N}{\partial \epsilon} \right\|_{N_{y_0}} \leq C_2 \quad (2.34)$$

for all $(\epsilon_0, p_0, \omega_0) \in \mathcal{D}_C$. In addition, if $(\epsilon_1, p_1, \omega_1) \in \mathcal{D}_C$ we have the following Lipschitz control of the derivative

$$\left\| \frac{\partial N}{\partial \epsilon} \Big|_{(\epsilon_1, p_1, \omega_1)} - \frac{\partial N}{\partial \epsilon} \Big|_{(\epsilon_0, p_0, \omega_0)} \right\|_{N_{y_0}} \leq C_2 (|\epsilon_1 - \epsilon_0| + \|(p_1, \omega_1) - (p_0, \omega_0)\|_{X_{y_0}}), \quad (2.35)$$

$$\begin{aligned} & \left\| \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) \Big|_{(\epsilon_1, p_1, \omega_1)} - \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) \Big|_{(\epsilon_0, p_0, \omega_0)} \right\|_{L(X_{y_0}, N_{y_0})} \\ & \leq C_2 \left(|\epsilon_1 - \epsilon_0| + \max(|\epsilon_0|, |\epsilon_1|) \|(p_1, \omega_1) - (p_0, \omega_0)\|_{X_{y_0}} \right). \end{aligned} \quad (2.36)$$

For the proof, see Subsection 2.5.3.

Lemma 2.2.6. (a) There exists a universal constant C_3 so that if one fixes $y_0 \leq 1$, then for $|\epsilon| \ll y_0^{1/2}$, there exists a unique solution $(p[\epsilon](y), \omega[\epsilon](y)) \in X_{y_0}$ to (2.32) with (all inequalities independent of y_0, ϵ)

$$\|(p[\epsilon], \omega[\epsilon])\|_{X_{y_0}} \leq C_3 |\epsilon| y_0^{-1/2}.$$

(b) Moreover, for this range of ϵ , we have that $\epsilon \mapsto (p[\epsilon], \omega[\epsilon])$ is C^1 valued in X_{y_0} . In particular, we may define $p_{lin}[\epsilon], \omega_{lin}[\epsilon]$ to be the unique solution to

$$(p_{lin}, \omega_{lin}) = R \frac{\partial N}{\partial \epsilon} \Big|_{(\epsilon, p[\epsilon], \omega[\epsilon])} + R \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) \Big|_{(\epsilon, p[\epsilon], \omega[\epsilon])} \cdot (p_{lin}, \omega_{lin})$$

and

$$\frac{\partial p[\epsilon]}{\partial \epsilon}, \frac{\partial \omega[\epsilon]}{\partial \epsilon} = (p_{lin}[\epsilon], \omega_{lin}[\epsilon]).$$

In addition, we have

$$(p[0], \omega[0]) = 0, \quad \left\| \frac{\partial p[\epsilon]}{\partial \epsilon}, \frac{\partial \omega[\epsilon]}{\partial \epsilon} \right\|_{X_{y_0}} \lesssim y_0^{-1/2}.$$

For the proof, see Subsection 2.5.4.

In preparation for the matching, we convert this statement to the original variables. In particular, we define for each ϵ ,

$$\begin{aligned}\rho_{ext}[\epsilon](y) &:= \frac{1}{\epsilon y^2} \left(y_* \left(1 + \frac{\epsilon}{y_* \alpha} (p_{hom}(\alpha y + \beta) + p[\epsilon](\alpha y + \beta) - 1) \right) - 1 - \epsilon p_{hom}(y) \right), \\ &= \frac{1}{y^2} \left(\left(1 - \frac{1}{\alpha} \right) + \left(\frac{1}{\alpha} p_{hom}(\alpha y + \beta) - p_{hom}(y) \right) + \frac{1}{\alpha} p[\epsilon](\alpha y + \beta) \right), \\ u_{ext}[\epsilon](y) &= \frac{y}{\epsilon} \left(\frac{1}{y_*} (1 + \epsilon(\omega_{hom} + \omega + 1)) - 1 - \epsilon \omega_{hom} \right), \\ &= y \left(\left(\frac{1}{y_*} \omega_{hom}(\alpha y + \beta) - \omega_{hom}(y) \right) + \frac{1}{y_*} \omega[\epsilon](\alpha y + \beta) \right).\end{aligned}$$

We can then compute

$$\begin{aligned}\rho_{ext}[\epsilon](y_0) &= \frac{1}{y_0^2} \left(1 - \frac{1}{\alpha} + \left(\frac{1}{\alpha} - 1 \right) p_{hom}(y_0) + \frac{1}{\alpha} p[\epsilon](y_0) \right), \\ u_{ext}[\epsilon](y_0) &= y_0 \left(\left(\frac{1}{y_*} - 1 \right) \omega_{hom}(y_0) + \frac{1}{y_*} \omega[\epsilon](y_0) \right).\end{aligned}$$

In particular, we see that if $p[\epsilon](y_0), \omega[\epsilon](y_0)$ are C^1 then so are $\rho[\epsilon](y_0)$ and $u[\epsilon](y_0)$. Moreover, as $\|(y^{-2}, 0)\|_{X_{y_0}}, \|(y^{-2} p'_{hom}, y \omega'_{hom})\|_{X_{y_0}} < \infty$, we have that

$$\|y^2 \rho_{ext}, y^{-1} u_{ext}\|_{X_{y_0}} \lesssim \epsilon + \|(p[\epsilon], \omega[\epsilon])\|_{X_{y_0}}.$$

Using these variables leads to the following corollary of Lemma 2.2.6.

Lemma 2.2.7. *For a fixed $y_0 \leq 1$, $\epsilon \ll y_0^{1/2}$, there exists functions $\tilde{\rho}, \tilde{u}$ which are C^1 on $[y_0, \infty)$ with $\tilde{\rho} > 0$ satisfying (2.5) with a Hunter-type sonic point at $y_* = 1 + \epsilon$ of the form*

$$\begin{cases} \tilde{\rho}_{ext}(y) = \frac{1}{y^2} + \frac{\epsilon}{y^2} p_{hom}(y) + \epsilon \rho_{ext}[\epsilon](y) \\ \tilde{u}_{ext}(y) = \epsilon y \omega_{hom}(y) + \epsilon u_{ext}[\epsilon](y) \end{cases},$$

where p_{hom}, ω_{hom} are defined in Lemma 2.2.1. We have the following bounds (independent of y_0, ϵ):

$$\sup_{y_0 \leq y \leq 1} |y^{5/2} \rho_{ext}|, |y^{-1/2} u_{ext}|, |y^{7/2} \rho'_{ext}|, |y^{1/2} u'_{ext}| \lesssim \epsilon y_0^{-1/2} \quad \text{and} \quad (2.37)$$

$$\sup_{1 \leq y \leq \infty} |y^2 \rho_{ext}|, |u_{ext}|, |y^3 \rho'_{ext}|, |y u'_{ext}| \lesssim \epsilon y_0^{-1/2}. \quad (2.38)$$

Moreover, we have that $\epsilon \mapsto (\epsilon \rho_{ext}[\epsilon], \epsilon u_{ext}[\epsilon])(y_0)$ is C^1 on a neighborhood of 0. In addition, as $\epsilon \rightarrow 0$ we have (inequality independent of y_0)

$$\epsilon \rho[\epsilon](y_0) = O(\epsilon^2 y_0^{-3}), \quad \epsilon u[\epsilon] = O(\epsilon^2). \quad (2.39)$$

Finally, we have the following bound on the velocity (independent of y_0, ϵ)

$$(\tilde{u} + y)' \geq \frac{1}{2}. \quad (2.40)$$

Proof. We start with Lemma 2.2.6 and implement the change of variables. This yields proves the lemma up to and including (2.39). To prove (2.40) one computes

$$\tilde{u}' = \epsilon(yw'_{hom} + w_{hom} + u').$$

We then use the asymptotics for ω_{hom} given in Lemma 2.2.1 and (2.37), (2.38) for u' to derive the following bounds on $[y_0, \infty)$:

$$w_{hom} \lesssim \frac{1}{y} \left\langle \frac{1}{y} \right\rangle^{-1/2}, \quad yw'_{hom} \lesssim \frac{1}{y} \left\langle \frac{1}{y} \right\rangle^{-1/2}, \quad u' \lesssim (\epsilon y_0^{-1/2}) \frac{1}{y} \left\langle \frac{1}{y} \right\rangle^{-1/2}.$$

If we then use $y \geq y_0$, we get the global bound $\tilde{u}' \lesssim \epsilon y_0^{-1/2}$. As we assuming $\epsilon y_0^{-1/2} \ll 1$, by shrinking the range of admissible ϵ if necessary, we obtain $|\tilde{u}'| \leq 1/2$, yielding (2.40). $\square$

2.3 Solving the self-similar equations on $[0, y_0]$

2.3.1 Choice of variables and review of the isothermal sphere equation

In the interior region, we will change variables from $(\tilde{\rho}, \tilde{u})$ to $(\tilde{w}, \tilde{u})$, where $\tilde{w}$ is defined by the relation

$$\tilde{\rho} = e^{\tilde{w}}.$$

In this region, we will start with the formulation (2.4) and take a divergence of the second equation, which loses no generality under the assumption that $\tilde{\rho}, \tilde{u}$ are C^2 on $[0, y_0]$. When we do this, the system becomes

$$\begin{cases} (2 + y\partial_y)(e^{\tilde{w}}) + \text{div}(e^{\tilde{w}}\tilde{u}) = 0 \\ \text{div}((y + \tilde{u})\partial_y\tilde{u}) + \Delta\tilde{w} + 2e^{\tilde{w}} = 0 \\ \tilde{u}(0) = 0, \quad \tilde{u}'(0) = -\frac{2}{3} \end{cases} \quad (2.41)$$

The motivation for this formulation of the equation is that it highlights the connection with the isothermal sphere equation, (2.9). We next review properties of (2.9). We switch the radial coordinate from “ r ” to “ y ” to maintain consistency. As a further comment, although we will study (2.41) on $[0, y_0]$, with $y_0 \ll 1$, this will involve Q_λ for $\lambda \lesssim y_0^{\frac{5}{2}}$, so the asymptotics of our representative solution Q (and u_*) for large values of its argument will be relevant.

As mentioned in the introduction there exists a ground state solution Q , uniquely determined within the scaling symmetry class by requiring

$$Q(0) = 0. \quad (2.42)$$

The asymptotics of this ground state as $y \rightarrow \infty$ are

$$Q = -2\log(y) + O(y^{-1/2}). \quad (2.43)$$

Converting to the mass-density variable gives

$$e^Q = \frac{1}{y^2} + O\left(\frac{1}{y^{5/2}}\right). \quad (2.44)$$

Moreover, from the equation (2.9), it follows that the asymptotic relations (2.43) and (2.44) can be differentiated arbitrarily many times. We also note that the linearized operator associated to (2.9) about $Q_f = -2\log(y)$,

$$\tilde{H} := -(\Delta + \frac{2}{y^2}),$$

has a two-dimensional kernel spanned by the real and imaginary parts of $y^{-\frac{1}{2}+i\frac{\sqrt{7}}{2}}$.

We now discuss homogeneous solutions for the linearization about Q of the isothermal sphere equation. We define an operator

$$H := -(\Delta + 2e^Q).$$

From the scaling symmetry of (2.9), we see that one element of the kernel is

$$v_1 := y\partial_y Q + 2.$$

If one takes any solution v to $Hv = 0$, and defines the Wronskian with v_1 to be $W = v_1 v' - v v_1'$, then differentiating the Wronskian identity yields $W = Cy^{-2}$. By choosing $C = 1$ and solving the ODE $v_1 v' - v v_1' = W$ for v , we can pick another homogeneous solution, linearly independent from v_1 , to be (here Y_0 is an arbitrary constant)

$$v_2 := -v_1 \left(\int_y^{Y_0} \frac{1}{(v_1)^2 (y')^2} dy' \right).$$

We note that $v_1, v_2 = O(\frac{1}{y^{1/2}})$ as $y \rightarrow \infty$. In fact, one can obtain the following more detailed asymptotics of v_1, v_2 .

Lemma 2.3.1. *As $y \rightarrow \infty$, the functions v_1, v_2 satisfy, for some $c_3, c_4 \in \mathbb{R}^*$ and $d_3, d_4 \in \mathbb{R}/2\pi\mathbb{Z}$,*

$$(v_1, v_2) = \left(c_3 \frac{\sin\left(\frac{\sqrt{7}}{2} \log(y) + d_3\right)}{y^{1/2}}, c_4 \frac{\sin\left(\frac{\sqrt{7}}{2} \log(y) + d_4\right)}{y^{1/2}} \right) + O\left(\frac{1}{y}\right), \quad (2.45)$$

and one can differentiate these asymptotic relations arbitrarily many times.

This lemma is proved in Subsection 2.6.1.

As $y\partial_y Q + 2 = v_1$, by (2.43) we can express

$$Q(y) = -2\log(y) - \int_y^\infty \frac{v_1}{y'} dy'.$$

This allows us to refine the error in (2.43) to, for some $c_2 > 0$ and $d_2 \in \mathbb{R}/2\pi\mathbb{Z}$,

$$Q = -2\log(y) + \frac{c_2 \sin\left(\frac{\sqrt{7}}{2}\log(y) + d_2\right)}{y^{1/2}} + O\left(\frac{1}{y}\right).$$

Taking an exponential yields

$$e^Q = \frac{1}{y^2} + \frac{c_2 \sin\left(\frac{\sqrt{7}}{2}\log(y) + d_2\right)}{y^{5/2}} + O\left(\frac{1}{y^3}\right), \quad (2.46)$$

and we can differentiate these asymptotic relations.

2.3.2 Derivation of the linearized operators

The goal in this section will be to view (2.41) as a perturbation of the isothermal sphere equation, (2.9). In particular, we assume $\tilde{w}(0) = -2\log(\lambda)$ and scale by defining

$$\tilde{w} = \bar{w}(\cdot/\lambda) - 2\log(\lambda), \quad \tilde{u} = \lambda\bar{u}(\cdot/\lambda).$$

Substituting these relations into (2.41) yields

$$\begin{cases} (2 + y\partial_y)e^{\tilde{w}} + \operatorname{div}(e^{\tilde{w}}\tilde{u}) = 0 \\ \Delta\tilde{w} + 2e^{\tilde{w}} = -\lambda^2 \operatorname{div}((y + \tilde{u})\partial_y\tilde{u}) \\ \tilde{w}(0) = 0, \quad \tilde{u}(0) = 0, \quad \tilde{u}'(0) = -\frac{2}{3} \end{cases}. \quad (2.47)$$

We now consider λ to be a small parameter, and formally do an asymptotic expansion, $\tilde{w} = \sum_{n=0}^{\infty}(\lambda^2)^j w_j$, $\tilde{u} = \sum_{n=0}^{\infty}(\lambda^2)^j u_j$. The zeroth order equation in λ is

$$\begin{cases} (2 + y\partial_y)e^{w_0} + \operatorname{div}(e^{w_0}u_0) = 0 \\ \Delta w_0 + 2e^{w_0} = 0 \\ w_0(0) = 0, \quad u_0(0) = 0, \quad u_0'(0) = -\frac{2}{3}. \end{cases},$$

We note that the second equation is exactly (2.9), and so we can put $w_0 = Q$. With this being known, one can write the first equation as

$$\begin{cases} (2 + y\partial_y)e^Q + Ju_0 = 0 \\ u_0(0) = 0, \quad u_0'(0) = -\frac{2}{3} \end{cases} \quad (2.48)$$

where

$$Ju := \operatorname{div}(e^Q u).$$

We recall for further use that the ODE

$$\begin{cases} Ju + f = 0 \\ u(0) = 0, \quad u'(0) = -\frac{f(0)}{3} \end{cases}$$

is solved by $u = T(f)$, where

$$T(f) := -\frac{1}{y^2 e^Q} \int_0^y (y')^2 f dy'. \quad (2.49)$$

In particular, in this case the solution is

$$u_0 = u_* := T((2 + y\partial_y)e^Q). \quad (2.50)$$

Recalling (2.14), u_* has asymptotics

$$u_* = c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_2 + \theta_0\right) y^{1/2} + O(1) \quad (2.51)$$

and we can differentiate the asymptotics (the remainder decreasing in order by 1 each time).

To motivate definitions for the iteration scheme, we expand (2.47) to order λ^2 obtaining the following inhomogeneous linear ode for w_1, u_1 :

$$\begin{cases} Lw_1 + \operatorname{div}(e^Q u_1) = 0 \\ -Hw_1 + \operatorname{div}((y + u_*)\partial_y u_*) = 0 \\ w_1(0) = u_1(0) = u_1'(0) = 0 \end{cases} \quad (2.52)$$

where

$$Lw := (2 + y\partial_y)(e^Q w) + \operatorname{div}(e^Q u_* w).$$

We observe that this system admits a hierarchical structure. To solve the second equation for w_1 , we may perform variation of constants using the homogeneous solutions v_1, v_2 studied above. In particular, we define a solution operator satisfying $H \circ S = \operatorname{id}$ by

$$S(f)(y) := \left(\int_0^y f(y') v_2(y') \cdot (y')^2 dy' \right) v_1(y) - \left(\int_0^y f(y') v_1(y') \cdot (y')^2 dy' \right) v_2(y). \quad (2.53)$$

For f smooth up to the origin the formula gives the unique solution to $Hw = f$ which is smooth at the origin and satisfies $w(0) = 0$. Once w_1 has been found, we can solve the first equation for u_1 with the solution operator T .

When we use the heuristic that applying S multiplies asymptotic expansions by y^2 , we see that as $y \rightarrow \infty$,

$$w_1 = O(y^{3/2}). \quad (2.54)$$

If we then solve for u_1 , using the heuristic that T multiplies asymptotic expansions by y^3 , we see that as $y \rightarrow \infty$,

$$u_1 = O(y^{5/2}). \quad (2.55)$$

In the next section we study the full difference equation. These last two heuristic growth rates help motivate the weights for the function spaces.

2.3.3 Difference equations and iteration scheme

We now discuss the full construction. We begin by establishing the notation

$$Q_\lambda = Q(\cdot/\lambda) - 2\log(\lambda), \quad u_\lambda = \lambda u_*(\cdot/\lambda), \quad v_{i,\lambda} = v_i(\cdot/\lambda). \quad (2.56)$$

We consider a fixed $\lambda_0 \ll y_0$, and show existence of solutions with $\tilde{\rho}(0) = \frac{1}{\lambda^2}$ for λ close to λ_0 . To do this, we introduce variables w, u defined implicitly by

$$\tilde{w} = Q_\lambda + \lambda_0^2 w(\cdot/\lambda_0), \quad \tilde{u} = u_\lambda + \lambda_0^3 u(\cdot/\lambda_0).$$

We adopt the notations

$$y_1 = y_0/\lambda_0, \quad r = \lambda/\lambda_0$$

and note the scaling identities

$$Q_\lambda = Q_r(\cdot/\lambda_0) - 2\log(\lambda_0), \quad u_\lambda = \lambda_0 u_r(\cdot/\lambda_0).$$

We then compute that (2.41), with $w(0) = -2\log r \lambda_0$, on $[0, y_0]$ is equivalent to the following ODE system on $[0, y_1]$,

$$\begin{cases} (2 + y\partial_y)(e^{Q_r + \lambda_0^2 w}) + \operatorname{div}(e^{Q_r + \lambda_0^2 w}(u_r + \lambda_0^2 u)) = 0 \\ \lambda_0^2 \operatorname{div}((y + u_r + \lambda_0^2 u)\partial_y(u_r + \lambda_0^2 u)) + \Delta(Q_r + \lambda_0^2 w) + 2e^{Q_r + \lambda_0^2 w} = 0 \\ w(0) = u(0) = u'(0) = 0 \end{cases} \quad (2.57)$$

We next define scaled versions of operators defined in the previous subsection.

$$\begin{aligned} H_r w &:= -r^2(\Delta + 2e^{Q_r})w, \\ S_r f &:= \frac{1}{r^3} \left(\left(\int_0^y f v_{2,r}(y')^2 dy' \right) v_{1,r} - \left(\int_0^y f v_{1,r}(y')^2 dy' \right) v_{2,r} \right), \\ J_r u &:= r^2 \operatorname{div}(e^{Q_r} u), \\ T_r f &:= -\frac{1}{r^2 y^2 e^{Q_r}} \int_0^y f(y')^2 dy'. \end{aligned}$$

We note that $H_r S_r = \operatorname{id}$, $-J_r T_r = \operatorname{id}$, $J u_r + r^2(2 + y\partial_y)e^{Q_r} = 0$. We also define

$$L_r w := r^2 \left((2 + y\partial_y)(e^{Q_r} w) + \operatorname{div}(u_r e^{Q_r} w) \right), \quad K_r := T_r \circ L_r \circ S_r.$$

We then use Taylor's theorem, in particular

$$\begin{aligned} e^{Q_r + \lambda_0^2 w} &= e^{Q_r} + \lambda_0^2 w \int_0^1 e^{Q_r + t\lambda_0^2 w} dt \\ &= e^{Q_r} + \lambda_0^2 w e^{Q_r} + \lambda_0^4 w^2 \int_0^1 (1-t) e^{Q_r + t\lambda_0^2 w} dt, \end{aligned}$$

to write (2.57) as the difference equations

$$\begin{cases} 0 = L_r w + J_r u + F_r(w, u) \\ 0 = -H_r w + G_r(w, u) \\ w(0) = u(0) = u'(0) = 0 \end{cases}, \quad (2.58)$$

where we define nonlinear functions

$$\begin{aligned} F_r(w, u) &:= \lambda_0^2 r^2 \left((2 + y \partial_y)(w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt) \right. \\ &\quad \left. + \operatorname{div}(uw \int_0^1 e^{Q_r + t \lambda_0^2 w} dt) + \operatorname{div}(u_r w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt) \right), \\ G_r(w, u) &:= r^2 \operatorname{div}((y + u_r) \partial_y u_r) + \lambda_0^2 r^2 \left(2w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt \right. \\ &\quad \left. + \operatorname{div}(u \partial_y u_r + (y + u_r) \partial_y u + \lambda_0^2 u \partial_y u) \right). \end{aligned}$$

We then note that (2.58) can be written in integral form as

$$\begin{cases} u = T_r(F_r(w, u)) + K_r(G_r(w, u)) \\ w = S_r(G_r(w, u)) \end{cases}. \quad (2.59)$$

As done in the exterior region, we solve this difference equation by showing a contraction in a suitable space. In particular, we adopt the following definitions.

Definition 2.3.2. *We define the “solution” Banach spaces Z_{y_1}, Y_{y_1} and “nonlinearity” Banach spaces $\mathcal{G}_{y_1}, \mathcal{F}_{y_1}$ by*

$$\begin{aligned} Z_{y_1} &:= \{f \in C^2([0, y_1]) : \|f\|_{Z_{y_1}} < \infty, \quad f(0) = 0\}, \\ Y_{y_1} &:= \{f \in C^2([0, y_1]) : \|f\|_{Y_{y_1}} < \infty, \quad f(0) = f'(0) = 0\}, \\ \mathcal{G}_{y_1} &:= \{f \in C^0([0, y_1]) : \|f\|_{\mathcal{G}_{y_1}} < \infty\}, \\ \mathcal{F}_{y_1} &:= \{f \in C^1([0, y_1]) : \|f\|_{\mathcal{F}_{y_1}} < \infty, \quad f(0) = 0\}, \end{aligned}$$

where

$$\begin{aligned} \|f\|_{Z_{y_1}} &:= \sup_{y \in [0, y_1]} \langle y \rangle^{-3/2} (|f| + \langle y \rangle |\partial_r f| + \langle y \rangle^2 |\partial_r^2 f|), \\ \|f\|_{Y_{y_1}} &:= \sup_{y \in [0, y_1]} \langle y \rangle^{-5/2} (|f| + \langle y \rangle |\partial_r f| + \langle y \rangle^2 |\partial_r^2 f|), \\ \|f\|_{\mathcal{G}_{y_1}} &:= \sup_{y \in [0, y_1]} \langle y \rangle^{1/2} |f|, \\ \|f\|_{\mathcal{F}_{y_1}} &:= \sup_{y \in [0, y_1]} \langle y \rangle^{1/2} (|f| + \langle y \rangle |\partial_y f|). \end{aligned}$$

We then solve for $w \in Z_{y_1}, u \in Y_{y_1}$ by putting G, F in $\mathcal{G}_{y_1}$ and $\mathcal{F}_{y_1}$ respectively. As a note, the weights are consistent with the heuristics (2.54), (2.55). In addition, the number of

derivatives in each norm is motivated by the heuristic that inverting the divergence operator gains 1 derivative, and inverting H gains 2 derivatives.

As in the corresponding arguments for the exterior region, we summarize our construction by means of three lemmas.

Lemma 2.3.3. *If $y_0 \ll 1$ and $\lambda_0 \leq \frac{1}{10}y_0$ is fixed, then for $r \in (\frac{1}{2}, 2)$ if $G, F \in \mathcal{G}_{y_1}, \mathcal{F}_{y_1}$ respectively we have that $T_r(F), K_r(G) \in Y_{y_1}$ and $S_r(G) \in Z_{y_1}$. Moreover, we have*

$$\|T_r\|_{L(\mathcal{F}_{y_1}, Y_{y_1})}, \|S_r\|_{L(\mathcal{G}_{y_1}, Z_{y_1})}, \|K_r\|_{L(\mathcal{G}_{y_1}, Y_{y_1})} \leq C_1$$

In addition, we have that $r \mapsto (T_r, S_r, K_r)$ is C^1 with domain $(\frac{1}{2}, 2)$. For all $r_0 \in (\frac{1}{2}, 2)$,

$$\left\| \frac{\partial T_r}{\partial r} \Big|_{r_0} \right\|_{L(\mathcal{F}_{y_1}, Y_{y_1})}, \left\| \frac{\partial S_r}{\partial r} \Big|_{r_0} \right\|_{L(\mathcal{G}_{y_1}, Z_{y_1})}, \left\| \frac{\partial K_r}{\partial r} \Big|_{r_0} \right\|_{L(\mathcal{G}_{y_1}, Y_{y_1})} \leq C_1.$$

In addition, if r_1 is also in $(\frac{1}{2}, 2)$, we have the bounds

$$\left\| \frac{\partial S_r}{\partial r} \Big|_{r_1} - \frac{\partial S_r}{\partial r} \Big|_{r_0} \right\|_{L(\mathcal{G}_{y_1}, Z_{y_1})} \leq C_1 |r_1 - r_0|, \quad (2.60)$$

and analogous difference bounds for T_r, K_r .

For the proof, see Subsection 2.6.2.

Lemma 2.3.4. *For all $C > 0$ there exists $C_2(C) > 0$ (independent of y_0, λ, r) such that for $y_0 \ll 1$ and fixed $\lambda_0 \leq \frac{1}{10}y_0$, $r \in (\frac{1}{2}, 2)$,*

(a) If $(w, u) \in Z_{y_1} \times Y_{y_1}$ and $\|u\|_{Y_{y_1}} + \|w\|_{Z_{y_1}} \leq C$ then

$$(G_r(w, u), F_r(w, u)) \in \mathcal{G}_{y_1} \times \mathcal{F}_{y_1} \quad \text{and} \quad \|F_r(w, u)\|_{\mathcal{F}_{y_1}} + \|G_r(w, u)\|_{\mathcal{G}_{y_1}} \leq C_2.$$

(b) We have that $(r, w, u) \mapsto (G_r(w, u), F_r(w, u))$ is C^1 as a function of r, w, u with domain

$$\left(\frac{1}{2}, 2\right) \times \{(w, u) \in Z_{y_1} \times Y_{y_1} : \|w\|_{Z_{y_1}} + \|u\|_{Y_{y_1}} < C\}$$

valued in $\mathcal{G}_{y_1} \times \mathcal{F}_{y_1}$ with the uniform bounds

$$\left\| \frac{\partial G_r}{\partial r} \right\|_{\mathcal{G}_{y_1}} + \left\| \frac{\partial F_r}{\partial r} \right\|_{\mathcal{F}_{y_1}} \leq C_2, \quad (2.61)$$

$$\left\| \frac{\partial G_r}{\partial w} \right\|_{L(Z_{y_1}, \mathcal{G}_{y_1})} + \left\| \frac{\partial G_r}{\partial u} \right\|_{L(Y_{y_1}, \mathcal{G}_{y_1})} + \left\| \frac{\partial F_r}{\partial w} \right\|_{L(Z_{y_1}, \mathcal{F}_{y_1})} + \left\| \frac{\partial F_r}{\partial u} \right\|_{L(Y_{y_1}, \mathcal{F}_{y_1})} \leq C_2 y_0^2 \quad (2.62)$$

for all r_0, w_0, u_0 in this domain. In addition, if (r_1, w_1, u_1) is also in this domain, we have the following Lipschitz control of the derivative

$$\left\| \frac{\partial G}{\partial r} \Big|_{r_1, w_1, u_1} - \frac{\partial G}{\partial r} \Big|_{r_0, w_0, u_0} \right\| \leq C_1 (|r_1 - r_0| + \|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}}), \quad (2.63)$$

$$\begin{aligned} & \left\| \frac{\partial G}{\partial w} \Big|_{r_1, p_1, w_1} - \frac{\partial G}{\partial w} \Big|_{r_0, p_0, w_0} \right\|_{L(Z_{y_1}, \mathcal{G}_{y_1})} + \left\| \frac{\partial G}{\partial u} \Big|_{r_1, p_1, w_1} - \frac{\partial G}{\partial u} \Big|_{r_0, p_0, w_0} \right\|_{L(Y_{y_1}, \mathcal{G}_{y_1})} \\ & \leq y_0^2 C_1 (|r_1 - r_0| + \|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}}) \end{aligned} \quad (2.64)$$

with analogs of (2.61), (2.62), (2.63) and (2.64) holding for F as well.

For proof see Subsection 2.6.3.

Lemma 2.3.5. (a) *There exists a universal constant C_3 such that for fixed $y_0 \ll 1$, $\lambda_0 < \frac{1}{10}y_0$, $r \in (\frac{1}{2}, 2)$, there exists a C^2 solution to (2.41) of the form*

$$\tilde{w} = Q_{r\lambda_0} + \lambda_0^2 w[\lambda_0; r](\cdot/\lambda_0), \quad \tilde{u} = u_{r\lambda_0} + \lambda_0^3 u[\lambda_0; r](\cdot/\lambda_0)$$

with $(w, u) \in Z_{y_1} \times Y_{y_1}$, and $\|w\|_{Z_{y_1}} + \|u\|_{Y_{y_1}} \leq \frac{C_3}{100}$ (inequality independent of y_0, λ_0, r) and it is the unique solution such that $\|u\|_{Y_{y_1}} + \|w\|_{Z_{y_1}} \leq C_3$.

(b) *Moreover, we have that the map $r \mapsto (u[\lambda_0; r], w[\lambda_0; r])$ is C^1 with respect to $r \in (\frac{1}{2}, 2)$ valued in $Z_{y_1} \times Y_{y_1}$.*

For proof see Subsection 2.6.4.

In preparation for the matching, we convert this statement to the original variables. As a note, a consequence of the uniqueness we have that

$$w[r\lambda_0; 1] = \frac{1}{r^2} w[\lambda_0; r](r\cdot), \quad u[r\lambda_0; 1] = \frac{1}{r^3} u[\lambda_0; r](r\cdot). \quad (2.65)$$

In particular, we define for a fixed λ_0, r ,

$$\begin{aligned} \rho_{int}[\lambda_0; r] &:= e^{Q_{r\lambda_0} + \lambda_0^2 w[\lambda_0; r](\cdot/\lambda_0)} - e^{Q_{r\lambda_0}} \\ &= \lambda_0^2 w[r; \lambda_0](\cdot/\lambda) e^{Q_\lambda} \int_0^1 e^{t\lambda_0^2 w[r; \lambda_0](\cdot/\lambda)} dt, \\ u_{int}[\lambda_0; r] &:= \frac{\lambda_0^3}{\lambda^3} u[\lambda_0; r](\cdot/\lambda_0). \end{aligned}$$

By (2.65), the righthand side depends only on $\lambda := r\lambda_0$. In addition, we note that the C^1 -dependence $w[\lambda_0; r]$ on r implies (by keeping λ_0 fixed), C^1 dependence of $(\rho_{int}[\lambda](y_0), u_{int}[\lambda](y_0))$ on λ . This change of variables leads to the following corollary of Lemma 2.3.5.

Lemma 2.3.6. *If $y_0 \ll 1$ and $\lambda < \frac{1}{10}y_0$, there exists C^2 functions $\tilde{\rho}, \tilde{u}$ defined on $[0, y_0]$ with $\tilde{\rho} > 0$ which solve (2.5) of the form*

$$\begin{cases} \tilde{\rho}_{int}[\lambda](y) = e^{Q_\lambda} + \rho_{int}[\lambda](y) \\ \tilde{u}_{int}[\lambda](y) = u_\lambda + \lambda^3 u_{int}[\lambda](y) \end{cases}$$

where Q_λ, u_λ are defined by (2.56), ρ_{int}, u_{int} satisfy

$$\rho_{int}(0) = u_{int}(0) = u'_{int}(0) = 0$$

and they obey the following bounds on $[0, y_0]$ (uniform with respect to y_0, λ)

$$|\rho| \lesssim \frac{y}{\lambda} \left\langle \frac{y}{\lambda} \right\rangle^{-3/2}, \quad |u| \lesssim \frac{y^2}{\lambda^2} \left\langle \frac{y}{\lambda} \right\rangle^{1/2}, \quad (2.66)$$

$$|\rho'| \lesssim \frac{1}{\lambda} \left\langle \frac{y}{\lambda} \right\rangle^{-3/2}, \quad |u'| \lesssim \frac{y}{\lambda^2} \left\langle \frac{y}{\lambda} \right\rangle. \quad (2.67)$$

Moreover,

$$\lambda \mapsto (\rho_{int}(y_0), u_{int}(y_0))$$

is C^1 with domain $(0, \frac{1}{10}y_0)$. We also have the following velocity bound on $[0, y_0]$:

$$|\tilde{u}_{int} + y| \leq \frac{1}{2}. \quad (2.68)$$

Proof. We start with Lemma 2.3.5. Converting variables, and representing

$$\rho_{int}[\lambda] = \lambda^2 w[\lambda; 1](\cdot/\lambda) e^{Q_\lambda} \int_0^1 e^{t\lambda^2 w[\lambda; 1](\cdot/\lambda)} dt, \quad u_{int}[\lambda] = u[\lambda; 1](\cdot/\lambda),$$

yields the lemma up to and including (2.67). To show (2.68), we note that $|u_\lambda| \lesssim \lambda \frac{y}{\lambda} \left\langle \frac{y}{\lambda} \right\rangle^{-1/2}$, so along with (2.66), and the fact that $z \langle z \rangle^{-1/2}$, $z^2 \langle z \rangle^{1/2}$ are increasing,

$$|u_\lambda| \lesssim \lambda \left\langle \frac{y_0}{\lambda} \right\rangle^{-1/2} \lesssim y_0, \quad |\lambda^3 u_{int}| \lesssim \lambda^3 \left\langle \frac{y_0}{\lambda} \right\rangle^{5/2} \lesssim y_0,$$

so by choosing y_0 smaller if necessary we see $|y + \tilde{u}| \lesssim y_0 \leq \frac{1}{2}$. $\square$

2.4 Proof of the theorem

2.4.1 Existence of C^1 Hunter solutions

We now prove Theorem 2.1.1, except for showing that the solution is real analytic, which is covered by an independent argument in the next subsection.

Proof. We follow the outline presented in Subsection 2.1.3 and follow the proof of Proposition 2.5 of [26]. The first 8 steps correspond to the ones in their proof, and in the last step we count the sonic points.

1) We start by picking $y_0 \ll 1$. We then wiggle y_0 if necessary so that (2.15) holds. Then, for every $\epsilon \leq \epsilon_0 \ll y_0$ we apply Lemma 2.2.7 to obtain functions $\tilde{\rho}_{ext}[\epsilon], \tilde{u}_{ext}[\epsilon]$ defined on $[y_0, \infty)$. Then, for each $\lambda \leq \frac{1}{10}y_0$, we apply Lemma 2.3.6 to obtain functions $\tilde{\rho}_{int}[\lambda], \tilde{u}_{int}[\lambda]$ on $[0, y_0]$.

2) We next define the density matching function by

$$\begin{aligned}\mathcal{F}[y_0](\epsilon, \lambda) &= \tilde{\rho}_{ext}[\epsilon](y_0) - \tilde{\rho}_{int}[\lambda](y_0) \\ &= \left(\frac{1}{y^2} + \frac{\epsilon}{y^2} p_{hom}(y) - e^{Q_\lambda} \right) + (\epsilon \rho_{ext}[\epsilon](y_0) - \tilde{\rho}_{int}[\lambda](y_0)).\end{aligned}$$

It is clear from the asymptotic expansions given by (2.28), (2.46), (2.37), (2.66),

$$\mathcal{F}[y_0](0, 0) = 0.$$

Similarly, by (2.39) we have

$$\partial_\epsilon \mathcal{F}[y_0](0, 0) = \frac{p_{hom}(y_0)}{y_0^2} = \frac{c_1}{y_0^{5/2}} + O\left(\frac{1}{y_0^2}\right) \neq 0.$$

We note that $\mathcal{F}[y_0](0, \lambda) = O(y_0^{-5/2} \lambda^{1/2})$, is not C^1 with respect to λ , so in order to apply the implicit function theorem we change variables by picking a positive $\delta < \frac{1}{2}$, and defining $\mu = \lambda^\delta$, $\tilde{\mathcal{F}}(\epsilon, \mu) = \mathcal{F}(\epsilon, \mu^{\frac{1}{\delta}})$, so

$$\mathcal{F}(0, \lambda) = \mathcal{F}(0, \mu^{\frac{1}{1/2-\delta}}) := \tilde{\mathcal{F}}(0, \mu) \sim \mu^{\frac{1}{2\delta}}.$$

Thus, by the C^1 dependence of the solutions on the parameters in Lemmas 2.2.7 and 2.3.6, we have that $\tilde{\mathcal{F}}(\epsilon, \mu)$ will be C^1 on some set of the form $(-\epsilon_0, \epsilon_0) \times \mu_0$. Thus, by the implicit function theorem, assuming $\mu \leq \mu_1$ for some constant μ_0 (depending on y_0), we have that $\tilde{\mathcal{F}}[y_0](\epsilon, \mu) = 0$ determines ϵ as a C^1 function of μ . Converting back to λ , we have a function $\epsilon(\lambda)$ (of regularity $C^{\frac{1}{2}-}$) defined on $[0, \lambda_1)$ so that

$$\epsilon(\lambda) = O(\lambda^{1/2}) \tag{2.69}$$

and

$$\tilde{\rho}_{ext}[\epsilon(\lambda)](y_0) = \tilde{\rho}_{int}[\lambda](y_0).$$

3) We note that by definition, we have that

$$\epsilon(\lambda) \left(\frac{p_{hom}(y_0)}{y_0^2} + \rho_{ext}(y_0) \right) = (e^{Q_\lambda} - \frac{1}{y^2})(y_0) + \rho_{int}(y_0).$$

We let $q(y) = e^Q - \frac{1}{y^2} - \frac{c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y) + d_2\right)}{y^{5/2}}$ (it is of size $O(\frac{1}{y^3})$ by (2.46)) and have

$$\epsilon(\lambda) \left(\frac{p_{hom}(y_0)}{y_0^2} + \rho_{ext}(y_0) \right) = \lambda^{1/2} \frac{c_2 \sin\left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2\right)}{y_0^{5/2}} + \frac{1}{\lambda^2} q\left(\frac{y_0}{\lambda}\right) + \rho_{int}(y_0).$$

If we substitute in asymptotics, we get that if $\lambda \ll y_0$,

$$\epsilon(\lambda) \frac{c_1}{y_0^{5/2}} \left(1 + O(y_0^2) + O(\epsilon y_0^{-1/2}) \right)$$

$$= \frac{c_2 \lambda^{1/2}}{y_0^{5/2}} \left(\sin \left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2 \right) + O\left(\frac{\lambda^{1/2}}{y_0^{1/2}}\right) + O(y_0^2) \right).$$

We now without loss of generality assume $\lambda^{1/2} \lesssim y_0^{5/2}$, and hence by (2.69), $\epsilon \lesssim y_0^{5/2}$. This allows us to simplify the errors to

$$\epsilon(\lambda) \frac{c_1}{y_0^{5/2}} (1 + O(y_0^2)) = \frac{c_2 \lambda^{1/2}}{y_0^{5/2}} \left(\sin \left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2 \right) + O(y_0^2) \right).$$

Isolating ϵ gives that as $\lambda \rightarrow 0$

$$\epsilon(\lambda) = \frac{c_2}{c_1} \lambda^{1/2} \left(\sin \left(\frac{\sqrt{7}}{2} \log(y_0/\lambda) + d_2 \right) + O(y_0^2) \right),$$

which can be rewritten via (2.15) as

$$\epsilon(\lambda) = \frac{c_2}{c_1} \lambda^{1/2} \left(\cos \left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2 \right) + O(y_0^2) \right). \quad (2.70)$$

4) We then define the second matching function as

$$\mathcal{G}[y_0](\lambda) = \frac{1}{c_2 y_0^{1/2} \lambda^{1/2} \sin(\theta_0)} (\tilde{u}_{ext}[\epsilon(\lambda)](y_0) - \tilde{u}_{int}[\lambda](y_0)).$$

By the C^1 dependence of the solutions on their parameters, and the regularity from the implicit function theorem, we see that $\mathcal{G}[y_0](\lambda)$ will be C^1 on $(0, y_1)$. We can then decompose $\mathcal{G}[y_0](\lambda)$ as

$$\mathcal{G}[y_0](\lambda) = \frac{1}{\sin(\theta_0)} (\mathcal{G}_1[y_0](\lambda) + \mathcal{G}_2[y_0](\lambda)),$$

where

$$\begin{aligned} \mathcal{G}_1[y_0](\lambda) &:= \frac{y_0^{1/2} \epsilon(\lambda)}{c_2 \lambda^{1/2}} \omega_{hom}(y_0) - \frac{1}{c_2 y_0^{1/2} \lambda^{1/2}} u_\lambda(y_0), \\ \mathcal{G}_2[y_0](\lambda) &:= \frac{\epsilon(\lambda)}{c_2 y_0^{1/2} \lambda^{1/2}} u_{ext}(y_0) - \frac{\lambda^{5/2}}{c_2 y_0^{1/2}} u_{int}(y_0). \end{aligned}$$

We can compute these using (2.15), (2.28), (2.37), (2.51), (2.66), (2.70)

$$\begin{aligned} \mathcal{G}_1(y_0) &= \cos(\theta_0) \cos \left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2 \right) - \cos \left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2 + \theta_0 \right) + O(y_0^2) \\ &= \sin(\theta_0) \sin \left(\frac{\sqrt{7}}{2} \log(\lambda) + d_1 - d_2 \right) + O(y_0^2), \\ \mathcal{G}_2(y_0) &= O(y_0^2). \end{aligned}$$

Combining gives

$$\mathcal{G}[y_0](\lambda) = \sin\left(\frac{\sqrt{7}}{2}\log(\lambda) + d_1 - d_2\right) + O(y_0^2). \quad (2.71)$$

5) As mentioned in the outline, we now consider

$$\lambda_{k,\pm} = \exp\left(\frac{2}{\sqrt{7}}(-k\pi - d_1 + d_2 \pm \frac{1}{10})\right).$$

Assuming that k is sufficiently large so that $\lambda_{k,\pm} \leq \lambda_0$, $\lambda_{k,\pm} \lesssim y_0^{5/2}$, we have that

$$G[\lambda](y_0) = (-1)^k \sin\left(\pm \frac{1}{10}\right) + O(y_0^2).$$

Thus, as $y_0 \ll 1$, by continuity, for each $k \geq N$ for some N , we have that there exists $\lambda \in (\lambda_{k,-}, \lambda_{k,+})$ such that $G[\lambda](y_0) = 0$. These matching conditions ensure that we can glue the exterior and interior solutions together to get a continuous functions which solves the equation on $[0, y_0]$ and $[y_0, \infty]$. Moreover, as a consequence of Lemma 2.2.7 and Lemma 2.3.6 we have $|\tilde{u} + y| \geq \frac{1}{2}$ on $[y_0, \infty)$ and $|\tilde{u} + y| \leq \frac{1}{2}$ on $[0, y_0]$. In particular, y_0 is not a sonic point, so by nondegeneracy of the equation, the matched solutions are classical solutions on $[0, \infty)$. It remains to count the zeroes of $y^2\tilde{\rho} - 1$.

6) We first claim that for all $\epsilon \ll y_0^{1/2}$ (so that Lemma 2.2.7 applies) $y^2\tilde{\rho}_{ext}[\epsilon](y) - 1$ and $\frac{p_{hom}}{y^2}$ have the same number of zeroes on $[y_0, \infty)$. Indeed, to see this, for convenience, we introduce new variables by

$$\psi(t) = e^{t/2}p_{hom}(e^t), \quad \tilde{\psi}(t) = e^{5t/2}\tilde{\rho}_{ext}(e^t).$$

so the problem becomes that of equating the number of zeroes of ψ and $\psi + \tilde{\psi}$ on $[\log(y_0), \infty)$. We then note that by (2.27) we have that as $t \rightarrow \infty$

$$\psi = \mu_3 e^{t/2} + O(e^{-3t/2}).$$

and hence has no roots. Similarly, by (2.28) we have that as $t \rightarrow -\infty$,

$$\psi(t) = c_1 \sin\left(\frac{\sqrt{7}}{2}t + d_1\right) + O(e^{2t}),$$

in particular, by (2.15)

$$\psi(\log(y_0)) = c_1 + O(y_0^2)$$

so given that $y_0 \ll 1$ this is bounded away from zero. Thus, we see that ψ will only have discretely many roots, and they will all be in $[\log(y_0), Y_0]$ for some large Y_0 independent of ϵ, y_0 . In addition, there exists a $\delta > 0$ so that if $\{t_i\}$ are the roots of ψ , the $|\psi'| > \delta$ on

$\cup_i[t_i - \frac{1}{100}, t_i + \frac{1}{100}]$ and $|\psi| > \delta$ on $[\log(y_0), \infty) \setminus (\cup_i[t_i - \frac{1}{100}, t_i + \frac{1}{100}])$. Moreover, we have that by (2.37) and (2.38)

$$|\tilde{\psi}|, |\tilde{\psi}'| \lesssim \epsilon y_0^{-1/2} \langle e^{t/2} \rangle.$$

Thus, as we may assume $\epsilon \ll y_0^{1/2} e^{-Y_0/2}$, $|\tilde{\psi}|, |\tilde{\psi}'| \leq \frac{\delta}{100}$, which, along with the boundary conditions at $\log(y_0)$ and ∞ , imply that ψ and $\psi + \tilde{\psi}$ have the same number of roots.

7) We next claim that if λ_k is as in step 5, then $y^2 \tilde{\rho}_{int}[\lambda_k] - 1$ and $y^2 e^{Q\lambda} - 1$ intersect 0 the same number of times on $[0, y_0]$. To show this, we again, for convenience make a change of variables. We define

$$\begin{aligned} \chi(t) &:= \langle e^t \rangle^{\frac{5}{2}} (\lambda_k^2 e^{Q\lambda_k} (\lambda_k e^t) - e^{-2t}) = \\ &= \langle e^t \rangle^{\frac{5}{2}} (e^Q(e^t) - e^{-2t}) \end{aligned}$$

$$\tilde{\chi}(t) = \lambda_k^2 \langle e^t \rangle^{\frac{5}{2}} \rho_{int}(\lambda_k e^t).$$

and it suffices to show that $\chi(t)$ and $\chi(t) + \tilde{\chi}(t)$ have the same number of zeroes on $(-\infty, \log(y_0/\lambda_k)]$. We then see that as $t \rightarrow -\infty$,

$$\chi(t) = -e^{-2t} + O(1)$$

and is thus not vanishing in this regime. Similarly, as $t \rightarrow \infty$, we have that, by (2.46),

$$\chi(t) = c_2 \sin\left(\frac{\sqrt{7}}{2}t + d_2\right) + O(e^{-t/2}). \quad (2.72)$$

If we use that $\lambda_k^{1/2} \lesssim y_0^2$ and change independent variables from t to x defined by

$$t = \log(y_0) - \frac{2}{\sqrt{7}}(x - d_1 + d_2)$$

we have that by (2.15)

$$\chi(t) = c_2 \cos(x) + O(y_0^2)$$

and $t_{k,\pm} = \log(y_0/\lambda_{k,\pm})$ corresponds to $x_{k,\pm} = -k\pi \pm \frac{1}{10}$. Thus, as $|\cos(x)|$ is uniformly bounded away from zero on $[-k\pi - \frac{1}{5}, -k\pi + \frac{1}{5}]$ we can assume $|\chi(t)|$ is bounded away from zero on a neighborhood of $\log(y_0/\lambda_k)$. Similarly, near the roots of $\chi(t)$, the derivative of $\chi(t)$ is bounded away from zero. By Lemma 2.3.6, we have that on $(-\infty, \log(y_0/\lambda))$,

$$|\tilde{\chi}(t)|, |\tilde{\chi}'(t)| \lesssim \lambda^2 e^t \langle e^t \rangle \lesssim y_0^2.$$

Thus, as with before, we may conclude that χ and $\tilde{\chi}$ have the same number of zeroes.

8) By the previous two steps, the number of zeroes of $y^2 \tilde{\rho} - 1$ will be equal to the number of zeroes of $\frac{p_1}{y^2}$ on $[y_0, \infty)$, a fixed number with respect to k , plus the number of zeroes of $y^2 e^{Q\lambda_k} - 1$ on $[0, y_0]$, which changes with k . Thus, to show that the number of zeroes of $y^2 \tilde{\rho} - 1$ increases by 1 with each increment of k , it suffices to show that $y^2 e^Q - 1$ has exactly one zero

on $[y_0/\lambda_k], [y_0/\lambda_{k+1}]$ for k sufficiently large. Doing the same change of variables as in the previous step, it suffices to show that $\chi(t)$ has exactly one root in $[\log(y_0/\lambda_k), \log(y_0/\lambda_{k+1})]$. To do this, we define

$$\bar{\chi}(t) = \chi(t) - c_2 \sin\left(\frac{\sqrt{7}}{2}t + d_2\right).$$

By (2.46), we have that as $t \rightarrow \infty$,

$$\bar{\chi}(t) = O(e^{-t/2}) = O(y_0^2), \quad \bar{\chi}'(t) = O(e^{-t/2}) = O(y_0^2).$$

so $\chi(t)$ has the same number of roots as $c_2 \sin\left(\frac{\sqrt{7}}{2}t + d_2\right)$ on this interval. By changing variables from t to x , we see that $c_2 \sin\left(\frac{\sqrt{7}}{2}t + d_2\right)$ will have exactly one zero on $[\log(y_0/\lambda_k), \log(y_0/\lambda_{k+1})]$ because $\cos(x)$ has at least one zero in $[\frac{1}{10}, \pi - \frac{1}{10}]$ and at most one zero in $[-\frac{1}{10}, \pi + \frac{1}{10}]$.

Thus, for k sufficiently large we have that $y^2\tilde{\rho}_k - 1$ has a finite number of zeroes and $y^2\tilde{\rho}_{k+1} - 1$ has exactly one more zero than $y^2\tilde{\rho}_k - 1$. Thus, by shifting the labels if necessary we get that $y^2\tilde{\rho}_N - 1$ has exactly $N + 1$ zeroes.

9) All that remains is to count the sonic points. As $|\tilde{u} + y| \leq \frac{1}{2}$, we have that there are no sonic points on $[0, y_0]$. As $\tilde{u} + y$ is increasing past y_0 , we conclude that on $[0, \infty)$ there is a unique point where the sonic point condition $\tilde{u} + y = \pm 1$ is satisfied. From the construction in the exterior region, we know that $y_* = 1 + \epsilon(\lambda_k)$ is a sonic point where the Hunter conditions are satisfied.

□

2.4.2 Proof that C^1 Hunter solutions are real analytic.

It remains to show that the C^1 Hunter solution is analytic. As a note, this subsection is independent from the rest of the paper and some notations are redefined.

Analyticity on $(0, \infty) \setminus y_*$ is easy as we can locally rewrite (2.5) as

$$(\tilde{\rho}', \tilde{u}') = f(y, \tilde{\rho}, \tilde{u})$$

where

$$f(y, a, b) = \left(-\frac{2ab}{(y+b)^2 - 1} \left(\frac{b+y}{y} - a \right), -\frac{2ab}{(b+y)^2 - 1} \left(-\frac{1}{ya} + b + y \right) \right)$$

is analytic on $\mathbb{C}^3 \setminus \{y = 0\}, \{y + b = \pm 1\}$. Thus, by standard local ODE theory we have uniqueness of C^1 solutions and existence of real analytic solutions, so any C^1 solution is real analytic.

We next show that if $(\tilde{\rho}, \tilde{u})$ be a C^1 solution to the self-similar Euler-Poisson system defined on an interval containing $y_* \in (\frac{1}{2}, \frac{3}{2})$ satisfying the Hunter boundary conditions

$$\tilde{\rho}_1(y_*) = \frac{1}{y_*}, \quad \tilde{\rho}'_1(y_*) = \frac{1}{y_*} - \frac{3}{y_*^2}, \quad \tilde{u}_1(y_*) = 1 - y_*, \quad \tilde{u}'_1(y_*) = \frac{1}{y_*} - 1. \quad (2.73)$$

then $(\tilde{\rho}, \tilde{u})$ is analytic at y_* . This is a corollary of the following Lemma.

Lemma 2.4.1. (a) If one fixes $y_* \in (\frac{1}{2}, \frac{3}{2})$ then there exists a real analytic solution to the self-similar Euler-Poisson system, $(\tilde{\rho}_0, \tilde{u}_0)$, defined on a neighborhood of y_* satisfying the boundary conditions (2.73).

(b) Let $(\tilde{\rho}_1, \tilde{u}_1)$ be a C^1 solution to the self-similar Euler-Poisson system defined on an interval containing y_* satisfying the Hunter boundary conditions (2.73) then there exists an interval containing y_* s.t. $(\tilde{\rho}_1, \tilde{u}_1) = (\tilde{\rho}_0, \tilde{u}_0)$.

The proof of part (a) essentially follows from the proof of Theorem 2.10 of [52]. The only difference is that the matrix $\mathcal{A}_N$ must be replaced by $\mathcal{A}_N^H$ (defined in Remark 2.5 of [52]), leading to slight changes in inequalities (2.49) and (2.50). The rest of the proof carries over.

To prove part (b), we start with the following formulation of the ODEs:

$$\begin{pmatrix} \tilde{u} + y & \tilde{\rho} \\ \frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} \begin{pmatrix} \tilde{\rho}' \\ \tilde{u}' \end{pmatrix} + 2\tilde{\rho}(\tilde{u} + y) \begin{pmatrix} \frac{1}{y} \\ 1 \end{pmatrix} = 0.$$

We then linearize the equation about $(\tilde{\rho}_0, \tilde{u}_0)$ to get the operator

$$L_0(\rho, u) = \begin{pmatrix} \tilde{u}_0 + y & \tilde{\rho}_0 \\ \frac{1}{\tilde{\rho}_0} & \tilde{u}_0 + y \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}' + \begin{pmatrix} u & \rho \\ -\frac{\rho}{\tilde{\rho}_0^2} & u \end{pmatrix} \begin{pmatrix} \tilde{\rho}_0' \\ \tilde{u}_0' \end{pmatrix} + 2(\rho(\tilde{u}_0 + y) + \tilde{\rho}_0 u) \begin{pmatrix} \frac{1}{y} \\ 1 \end{pmatrix}.$$

which we may rewrite as

$$L_0(\rho, u) = \begin{pmatrix} \tilde{u}_0 + y & \tilde{\rho}_0 \\ \frac{1}{\tilde{\rho}_0} & \tilde{u}_0 + y \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}' + \begin{pmatrix} \tilde{u}_0' + 2\frac{\tilde{u}_0 + y}{y} & \tilde{\rho}_0' + \frac{2\tilde{\rho}_0}{y} \\ -\frac{\tilde{\rho}_0'}{\tilde{\rho}_0^2} + 2(\tilde{u}_0 + y) & \tilde{u}_0' + 2\tilde{\rho}_0 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}.$$

We then write the equation satisfied by their difference $(\rho, u) = (\tilde{\rho}_1 - \tilde{\rho}_0, \tilde{u}_1 - \tilde{u}_0)$ as

$$L_0(\rho, u) + N_0(\rho, u) = 0. \quad (2.74)$$

where

$$N_0(\rho, u) = \begin{pmatrix} u\rho' + \rho u' \\ -\frac{\rho'\rho}{\tilde{\rho}_0(\tilde{\rho}_0 + \rho)} + \frac{\rho^2\tilde{\rho}_0'}{\tilde{\rho}_0^2(\tilde{\rho}_0 + \rho)} + uu' \end{pmatrix} + 2\rho u \begin{pmatrix} \frac{1}{y} \\ 1 \end{pmatrix}. \quad (2.75)$$

We consider the solvability of

$$L_{\tilde{\rho}_0, \tilde{u}_0} \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} R \\ U \end{pmatrix}.$$

To simplify the ensuing linear algebra we multiply on the left by $\begin{pmatrix} y_* & -1 \\ y_* & 1 \end{pmatrix}$, and then Taylor expand the coefficients in $y - y_*$. When we do this, we get

$$\begin{pmatrix} (y_* - 2)(y - y_*) + O((y - y_*)^2) & (1 - \frac{4}{y_*})(y - y_*) + O((y - y_*)^2) \\ 2y_* + O(y - y_*) & 2 + O(y - y_*) \end{pmatrix} \begin{pmatrix} \rho' \\ u' \end{pmatrix} \\ + \begin{pmatrix} -2 + O(y - y_*) & 2 - \frac{4}{y_*} + O(y - y_*) \\ 8 - 2y_* + O(y - y_*) & \frac{2}{y_*} + O(y - y_*) \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} y_* R - U \\ y_* R + U \end{pmatrix}$$

From the first line, if we substitute $y = y_*$, we get the constraint $-2\rho(y_*) + 2 - \frac{4}{y_*}u(y_*) = y_*R(y_*) - U(y_*)$. If we divide by $y - y_*$, we then see that in order to have $\rho, u \in C^1$ we need

$$\frac{y_*(R - R(y_*)) - (U - U(y_*))}{y - y_*}$$

to be continuous at y_* . This motivates the following spaces.

Definition 2.4.2. (*function spaces*) We let $I_\epsilon = (y_* - \epsilon, y_* + \epsilon)$. We then define our “non-linearity” space to be

$$\mathcal{N}_\epsilon = \{(R, U) \in C(I_\epsilon) : \frac{y_*R - U}{y - y_*} \in C(I_\epsilon), \\ R(y_*) = U(y_*) = (y_*R - U)'(y_*) = 0, \quad \|(R, U)\|_{\mathcal{N}_\epsilon} < \infty\}$$

where

$$\|(R, U)\|_{\mathcal{N}_\epsilon} = \sup_{y \in I_\epsilon} |R| + |U| + \left| \frac{y_*R - U}{y - y_*} \right|.$$

We then define our “solution” space to be

$$\mathcal{X}_\epsilon = \{(\rho, u) \in C^1(I) : \rho(y_*) = \rho'(y_*) = \rho(y_*) = \rho'(y_*) = 0, \quad \|(\rho, u)\|_{\mathcal{X}_\epsilon} < \infty\}$$

where

$$\|(\rho, u)\|_{\mathcal{X}_\epsilon} = \sup_{y \in I_\epsilon} |\rho| + |\rho'| + |u| + |u'|$$

Lemma 2.4.3. (*linear continuity*) If ϵ is sufficiently small then there exists a resolvent T_0 and constant $C_1 > 0$ (independent of ϵ) so that $L_0T_0 = \text{id}$ and if $(R, U) \in \mathcal{N}_\epsilon$ then

$$(\rho, u) := T_0(R, U) \in \mathcal{X}_\epsilon \quad \text{and} \quad \|(\rho, u)\|_{\mathcal{X}} \leq C_1 \|(R, U)\|_{\mathcal{N}_\epsilon}.$$

Proof. As a first step towards defining T_0 , we construct the homogeneous solutions. In particular, we rewrite $L_0(\rho, u) = 0$ (dropping the subscripts of $\tilde{\rho}_0, \tilde{u}_0$) as

$$\begin{pmatrix} \rho \\ u \end{pmatrix}' + \frac{1}{(\tilde{u} + y)^2 - 1} \begin{pmatrix} \tilde{u} + y & -\tilde{\rho} \\ -\frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} \begin{pmatrix} \tilde{u}' + 2\frac{\tilde{u}+y}{y} & \tilde{\rho}' + \frac{2\tilde{\rho}}{y} \\ -\frac{\tilde{\rho}'}{\tilde{\rho}^2} + 2(\tilde{u} + y) & \tilde{u}' + 2\tilde{\rho} \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix} = 0. \quad (2.76)$$

We then let $\eta = y - y_*$ and Taylor expand in powers of η ,

$$\begin{aligned} \frac{1}{(\tilde{u} + y)^2 - 1} &= \frac{y_*}{2\eta} + O(1) \\ \begin{pmatrix} \tilde{u} + y & -\tilde{\rho} \\ -\frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} &= \begin{pmatrix} 1 & -\frac{1}{y_*} \\ -y_* & 1 \end{pmatrix} + O(\eta) \\ \begin{pmatrix} \tilde{u}' + 2\frac{\tilde{u}+y}{y} & \tilde{\rho}' + \frac{2\tilde{\rho}}{y} \\ -\frac{\tilde{\rho}'}{\tilde{\rho}^2} + 2(\tilde{u} + y) & \tilde{u}' + 2\tilde{\rho} \end{pmatrix} &= \begin{pmatrix} \frac{3}{y_*} - 1 & \frac{1}{y_*} - \frac{1}{y_*^2} \\ 5 - y_* & \frac{3}{y_*} - 1 \end{pmatrix} + O(\eta) \end{aligned}$$

Thus, we have that

$$\begin{pmatrix} \rho \\ u \end{pmatrix}' + \left(\frac{1}{\eta} \begin{pmatrix} -1 & 1 - \frac{2}{y_*} \\ y_* & 2 - y_* \end{pmatrix} + O(1) \right) \begin{pmatrix} \rho \\ u \end{pmatrix} = 0.$$

We note that the coefficient of $\frac{1}{y-y_*}$ has eigenvalues $0, y_* - 1$, in particular if $y_* \neq 1$ (the generic case), the linearized system has a pair of homogeneous solutions which are $1, |y - y_*|^{y_*-1}$ near $y = y_*$. The residue of the (negative of) the coefficient of the zeroth order term is

$$\begin{pmatrix} 1 & \frac{2}{y_*} - 1 \\ -y_* & y_* - 2 \end{pmatrix} = \begin{pmatrix} y_* - 2 & -1 \\ y_* & y_* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y_* - 1 \end{pmatrix} \begin{pmatrix} y_* - 2 & -1 \\ y_* & y_* \end{pmatrix}^{-1}.$$

By analyticity of $(\tilde{\rho}, \tilde{u})$ (here we assume ϵ is sufficiently small so that I_ϵ is contained in the radius of convergence of the series at y_*), it follows from the Frobenius method that there exists homogeneous solutions $(\rho_{h,0}, u_{h,0}), (\rho_{h,1}, u_{h,1})$ whose asymptotics as $y \rightarrow y_*$ are given by

$$U(y) := \begin{pmatrix} \rho_{h,0} & \rho_{h,1} \\ u_{h,0} & u_{h,1} \end{pmatrix} = \begin{pmatrix} y_* - 2 + O(\eta) & -|\eta|^{y_*-1} + O(|\eta|^{y_*}) \\ y_* + O(\eta) & y_* |\eta|^{y_*-1} + O(|\eta|^{y_*}) \end{pmatrix}.$$

We can then represent the principal matrix solution to $L_0(\rho, u) = 0$ as

$$S_0(y, y') := U(y)U^{-1}(y') \begin{pmatrix} \tilde{u}(y') + y' & \tilde{\rho}(y') \\ \frac{1}{\tilde{\rho}(y')} & \tilde{u}(y') + y' \end{pmatrix}^{-1}$$

We may then define our desired resolvent as

$$T_0 \begin{pmatrix} R \\ U \end{pmatrix} (y) := \int_{y_*}^y S_0(y, y') \begin{pmatrix} R \\ U \end{pmatrix} (y') dy'$$

We can asymptotically expand

$$U^{-1}(y) = \frac{1}{y_*^2 - y_*} \begin{pmatrix} y_* + O(\eta) & 1 + O(\eta) \\ -y_* |\eta|^{1-y_*} + O(|\eta|^{2-y_*}) & (y_* - 2) |\eta|^{1-y_*} + O(|\eta|^{2-y_*}) \end{pmatrix},$$

$$U^{-1} \begin{pmatrix} \tilde{u}(y') + y' & \tilde{\rho}(y') \\ \frac{1}{\tilde{\rho}(y')} & \tilde{u}(y') + y' \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ -\frac{y_*}{2} |\eta|^{-y_*} & \frac{1}{2} |\eta|^{-y_*} \end{pmatrix} + \begin{pmatrix} O(1) & O(1) \\ O(|\eta|^{1-y_*}) & O(|\eta|^{1-y_*}) \end{pmatrix}.$$

In particular, if $\frac{y_* R - U}{\eta}$ is continuous, then we have that as $\eta \rightarrow 0$,

$$U^{-1} \begin{pmatrix} \tilde{u}(y') + y' & \tilde{\rho}(y') \\ \frac{1}{\tilde{\rho}(y')} & \tilde{u}(y') + y' \end{pmatrix}^{-1} \begin{pmatrix} R \\ U \end{pmatrix} = \begin{pmatrix} o(\eta) \\ o(|\eta|^{2-y_*}) \end{pmatrix}$$

and hence $(\rho, u) := T_0(R, U)$ are $o(\eta)$ as desired, and so we have that

$$\rho(y_*) = \rho'(y_*) = u(y_*) = u'(y_*) = 0.$$

Using the asymptotics of the fundamental solutions it is straightforward to also check that $\|(\rho, u)\|_{\mathcal{X}_\epsilon} \lesssim \|(R, U)\|_{\mathcal{N}_\epsilon}$. $\square$

Lemma 2.4.4. *(nonlinear estimate) There exists $C_2 > 0$ (independent of ϵ sufficiently small) so that if $(\rho, u) \in \mathcal{X}$ then*

$$(R, U) := N_0(\rho, u) \in \mathcal{N} \quad \text{and} \quad \|(R, U)\|_{\mathcal{X}} \leq C_2 \|(\rho, u)\|^2.$$

Proof. We recall (2.75), the definition of N_0 , and decompose (dropping the subscripts of $\tilde{\rho}_0, \tilde{u}_0$)

$$\begin{aligned} \frac{R}{y - y_*} &= \frac{u}{y - y_*} \rho' + \frac{\rho}{y - y_*} u' + 2 \frac{\rho}{y - y_*} \frac{u}{y}, \\ \frac{U}{y - y_*} &= -\frac{\rho}{y - y_*} \frac{\rho'}{\tilde{\rho}_0(\tilde{\rho} + \rho)} + \frac{\tilde{\rho}}{y - y_*} \frac{\rho \tilde{\rho}'}{\tilde{\rho}^2(\tilde{\rho} + \rho)} + \frac{u}{y - y_*} u' + 2 \frac{\rho}{y - y_*} u. \end{aligned}$$

As $\rho, u \in C^1$, $\frac{\rho}{y - y_*}, \frac{u}{y - y_*}$ converge as $y \rightarrow y_*$, so we see that $\frac{R}{y - y_*}, \frac{U}{y - y_*}$ are continuous at y_* . Similarly, we may bound $\frac{\rho}{y - y_*}, \frac{u}{y - y_*}$ by ρ', u' respectively and as each term is quadratic in ρ, u , we obtain

$$\left| \frac{R}{y - y_*} \right|, \left| \frac{U}{y - y_*} \right| \lesssim \|(\rho, u)\|_{X_\epsilon}^2$$

which implies the desired continuity bound. $\square$

We now prove Lemma 2.4.1.b.

Proof. We rewrite (2.74) as

$$(\rho, u) = -T_0 N_0(\rho, u).$$

As $(\rho, u) \in C^1$ and their first order Taylor coefficients vanish at y_* , there exists some small ϵ such that $\|(\rho, u)\|_{\mathcal{X}_\epsilon} \leq \frac{1}{2C_1 C_2}$. We then have that

$$\|(\rho, u)\|_{\mathcal{X}} \leq C_1 C_2 \|(\rho, u)\|_{\mathcal{X}}^2 \leq \frac{1}{2} \|(\rho, u)\|_{\mathcal{X}}$$

which implies that $\|(\rho, u)\|_{\mathcal{X}_\epsilon} = 0$. $\square$

We finally show analyticity at $y = 0$. This is easier, and we proceed in a manner which is similar to the sonic point case. In particular, we show the following lemma.

Lemma 2.4.5. *(a) If one fixes $\rho_0 > 0$, then there exists a real analytic solution to the self-similar Euler-Poisson system, $\tilde{\rho}_0, \tilde{u}_0$, defined on a neighborhood of 0 satisfying the boundary conditions*

$$\tilde{\rho}_0(0) = \rho_0, \quad \tilde{\rho}'_0(0) = 0, \quad \tilde{u}_0(0) = 0, \quad \tilde{u}_0'(0) = -\frac{2}{3}y. \quad (2.77)$$

(b) Let $(\tilde{\rho}_1, \tilde{u}_1)$ be a C^1 solution to the self-similar Euler-Poisson system defined on an interval containing 0 satisfying the boundary conditions (2.77). Then there exists an interval containing 0 such that $(\tilde{\rho}_1, \tilde{u}_1) = (\tilde{\rho}_0, \tilde{u}_0)$.

Proof. In this case part (a) follows from Theorem 2.12 of [52]. For part (b), we consider the operators L_0, N_0 defined previously, and define the following spaces.

$$\tilde{\mathcal{N}}_\epsilon = \{(R, U) \in C([0, \epsilon]): R(0) = U(0) = 0, \quad \|(R, U)\|_{\tilde{\mathcal{N}}_\epsilon} < \infty\},$$

$$\|(R, U)\|_{\tilde{\mathcal{N}}_\epsilon} = \sup_{y \in [0, \epsilon]} |R| + |U|,$$

$$\tilde{\mathcal{X}}_\epsilon = \{(\rho, u) \in C^1([0, \epsilon]): \rho(0) = \rho'(0) = u(0) = u'(0) = 0, \quad \|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon} < \infty\},$$

$$\|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon} = \sup_{y \in [0, \epsilon]} |\rho| + |\rho'| + |u| + |u'|.$$

We then note that near $y = 0$, we can Taylor expand the coefficient in (2.76) as

$$\frac{1}{(\tilde{u} + y)^2 - 1} \begin{pmatrix} \tilde{u} + y & -\tilde{\rho} \\ -\frac{1}{\tilde{\rho}} & \tilde{u} + y \end{pmatrix} \begin{pmatrix} \tilde{u}' + 2\frac{\tilde{u}+y}{y} & \tilde{\rho}' + \frac{2\tilde{\rho}}{y} \\ -\frac{\tilde{\rho}'}{\tilde{\rho}^2} + 2(\tilde{u} + y) & \tilde{u}' + 2\tilde{\rho} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{y} \end{pmatrix} + O(1)$$

and hence we may use the Frobenius method to construct fundamental solutions whose leading order asymptotics as $y \rightarrow 0$ are given by

$$\tilde{U} = \begin{pmatrix} \rho_{h,2} & \rho_{h,3} \\ u_{h,2} & u_{h,3} \end{pmatrix} = \begin{pmatrix} 1 + O(y) & 0 + O(\frac{1}{y}) \\ 0 + O(y) & \frac{1}{y^2} + O(\frac{1}{y}) \end{pmatrix}.$$

We then define $\tilde{S}_0, \tilde{T}_0$ by

$$\begin{aligned} \tilde{S}_0(y, y') &:= \tilde{U}(y) \tilde{U}^{-1}(y') \begin{pmatrix} \tilde{u}(y') + y' & \tilde{\rho}(y') \\ \frac{1}{\tilde{\rho}(y')} & \tilde{u}(y') + y' \end{pmatrix}^{-1} \\ \tilde{T}_0 \begin{pmatrix} R \\ U \end{pmatrix}(y) &:= \int_0^y \tilde{S}_0(y, y') \begin{pmatrix} R \\ U \end{pmatrix}(y') dy'. \end{aligned}$$

We can then asymptotically expand

$$\tilde{U}^{-1} \begin{pmatrix} \tilde{u}(y) + y & \tilde{\rho}(y) \\ \frac{1}{\tilde{\rho}(y)} & \tilde{u}(y) + y \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{y^2 \rho_0} \\ y^2 \rho_0 & 0 \end{pmatrix} + \begin{pmatrix} O(y) & O(y) \\ O(y^3) & O(y^3) \end{pmatrix}$$

from which it is evident that $\tilde{T}_0$ is bounded from $\tilde{\mathcal{N}}_\epsilon$ to $\tilde{\mathcal{X}}_\epsilon$ and there exists $\tilde{C}_1$ independent of ϵ such that if $(\rho, u) = \tilde{T}_0(R, U)$,

$$\|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon} \leq \tilde{C}_1 \|(R, U)\|_{\tilde{\mathcal{N}}_\epsilon}.$$

Again, straightforward estimates using (2.75) show that if $(\rho, u) \in \tilde{\mathcal{X}}_\epsilon$, then $(R, U) := N_0(\rho, u) \in \tilde{\mathcal{N}}_\epsilon$ and that there exists $\tilde{C}_2 > 0$ such that

$$\|(R, U)\|_{\tilde{\mathcal{N}}_\epsilon} \leq \tilde{C}_2 \|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon}^2.$$

We may then conclude the proof of part (b) as before, namely by defining $(\rho, u) = (\tilde{\rho}_1 - \tilde{\rho}_0, \tilde{u}_1 - \tilde{u}_0)$, noting that it satisfies

$$(\rho, u) = -\tilde{T}_0 N_0(\rho, u)$$

then choosing ϵ sufficiently small so that $\|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon} < \frac{1}{2\tilde{C}_1\tilde{C}_2}$, and then deriving

$$\|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon} \leq \frac{1}{2} \|(\rho, u)\|_{\tilde{\mathcal{X}}_\epsilon}$$

and concluding $(\rho, u) = 0$. □

2.5 Deferred proofs from Section 2.2

2.5.1 Proof of Lemma 2.2.1

Proof. We discuss how to explicitly solve the homogeneous ODE

$$\begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix} \begin{pmatrix} p' \\ \omega' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \frac{2}{z^2} & \frac{2}{z^2} + 1 \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = 0.$$

Our first step is to temporarily assume $z \neq 1$, and invert the first matrix to obtain

$$\begin{pmatrix} p' \\ \omega' \end{pmatrix} + \frac{1}{z(z^2 - 1)} \begin{pmatrix} -2 & -2 \\ 2 & z^2 + 1 \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = 0.$$

We then let $q = p + \omega$ and note that we have

$$\begin{pmatrix} p' \\ q' \end{pmatrix} + \begin{pmatrix} 0 & -\frac{2}{z(z^2 - 1)} \\ \frac{-1}{z} & \frac{1}{z} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 0.$$

If we then substitute $q = \frac{z(z^2 - 1)}{2}p'$ into the second row we get that p solves

$$p'' + \frac{4z^2 - 2}{z(z^2 - 1)}p' - \frac{2}{z^2(z^2 - 1)}p = 0. \quad (2.78)$$

We then let $p(z) = f(z^2)$. In this case, if we let $x = z^2$, we have that $f(x)$ solves

$$f'' + \frac{5x - 3}{2x(x - 1)}f' - \frac{1}{2x^2(x - 1)}f = 0.$$

We observe that this ODE is singular at $x = \{0, 1, \infty\}$ with simple singularities, and so by classical ODE theory we can solve it using hypergeometric functions. It is convenient at this point to switch variables to $f(x) = g(1 - \frac{1}{x})$, $g(\xi)$ satisfies

$$g'' + \frac{\xi - 2}{2\xi(\xi - 1)}g' + \frac{1}{2\xi(\xi - 1)}g = 0. \quad (2.79)$$

where $\xi = 1 - \frac{1}{x}$. We recognize this as the hypergeometric equation with $a = -\frac{\gamma}{2}, b = -\frac{\bar{\gamma}}{2}, c = 1$, where

$$\gamma = \frac{1}{2} + i\frac{\sqrt{7}}{2}.$$

The unique solution to (2.79) which is smooth on $(-\infty, 1)$ and satisfies the normalization condition $g_1(0) = 1$ is

$$g_1(\xi) = F(-\frac{\gamma}{2}, -\frac{\bar{\gamma}}{2}, 1, \xi), \quad (2.80)$$

where $F(a, b, c, \xi)$ denotes the principal branch of Gauss's hypergeometric function. By inverting the changes of variables, we recover the following solution to $L(p, \omega) = 0$:

$$\begin{pmatrix} p_{hom}(z) \\ \omega_{hom}(z) \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ -1 & \frac{z^2 - 1}{z^2} \end{pmatrix} \begin{pmatrix} g_1(1 - \frac{1}{z^2}) \\ g_1'(1 - \frac{1}{z^2}) \end{pmatrix}.$$

From this representation, (2.26) follows immediately. This yields (a).

To construct U_∞ , we start by defining the following analytic functions for $\xi \in (0, 1)$:

$$g_3(\xi) = F(-\frac{\gamma}{2}, -\frac{\bar{\gamma}}{2}, -\frac{1}{2}, 1 - \xi), \quad g_4(\xi) = -\frac{2}{3}(1 - \xi)^{3/2}F(1 + \frac{\gamma}{2}, 1 + \frac{\bar{\gamma}}{2}, \frac{5}{2}, 1 - \xi).$$

We use these fundamental solutions to construct a fundamental matrix for (2.79),

$$V_\infty(\xi) := \begin{pmatrix} g_3(\xi) & g_4(\xi) \\ g_3'(\xi) & g_4'(\xi) \end{pmatrix}.$$

We note that the Wronskian $W_\infty := \det V_\infty$ satisfies the following ODE and asymptotic expansion as $\xi \rightarrow 1$

$$W_\infty' + \frac{\xi - 2}{2\xi(\xi - 1)}W_\infty = 0, \quad W_\infty = (1 - \xi)^{1/2} + O((1 - \xi)^{3/2}).$$

Solving this ODE gives

$$W_\infty(\xi) = \frac{\sqrt{1 - \xi}}{\xi}, \quad V_\infty^{-1}(\xi) = \frac{\xi}{\sqrt{1 - \xi}} \begin{pmatrix} g_4'(\xi) & -g_4(\xi) \\ -g_3'(\xi) & g_3(\xi) \end{pmatrix}.$$

We note that these are two linearly independent solutions to (2.79). Thus, we may define the following for $z \in (1, \infty)$

$$U_\infty(z) := \begin{pmatrix} 1 & 0 \\ -1 & \frac{z^2 - 1}{z^2} \end{pmatrix} V_\infty(1 - \frac{1}{z^2}).$$

and it satisfies $LU_\infty = 0$, and as $z \rightarrow \infty$. Taking the limit as $z \rightarrow \infty$ yields the stated asymptotics for U_∞ . Similarly, using the formula for $V_\infty^{-1}(\xi)$, one can straightforwardly compute the asymptotics for U_∞^{-1} . To construct U_0 , we start by defining the following functions for $\xi \in (-\infty, 0)$,

$$g_a(\xi) = (-\xi)^{\gamma/2}F(-\frac{\gamma}{2}, -\frac{\gamma}{2}, 1 - i\frac{\sqrt{7}}{2}, \frac{1}{\xi}), \quad g_b = (-\xi)^{\bar{\gamma}/2}F(-\frac{\bar{\gamma}}{2}, -\frac{\bar{\gamma}}{2}, 1 + i\frac{\sqrt{7}}{2}, \frac{1}{\xi}).$$

Then, one lets

$$g_5 = \frac{g_a + g_b}{2}, \quad g_6 = -\frac{g_a - g_b}{2i}.$$

These are now real valued. One then defines for $\xi \in (-\infty, 0)$,

$$V_0(\xi) = \begin{pmatrix} g_5(\xi) & g_6(\xi) \\ g_5'(\xi) & g_6'(\xi) \end{pmatrix}, \quad W_0 = \det V_0.$$

We obtain the same ODE for the Wronskian with the boundary condition that as $\xi \rightarrow -\infty$,

$$W_0 = \frac{\sqrt{7}}{4}(-\xi)^{-1/2} + O((-\xi)^{-3/2}).$$

Solving the ODE for with this boundary condition gives

$$W_0 = -\frac{\sqrt{7}}{4} \frac{\sqrt{1-\xi}}{\xi}, \quad V_0^{-1}(\xi) = \frac{-4}{\sqrt{7}} \frac{\xi}{\sqrt{1-\xi}} \begin{pmatrix} g'_6(\xi) & -g_6(\xi) \\ -g'_5(\xi) & g_5(\xi) \end{pmatrix}.$$

Thus, we may define for $z \in (0, 1)$

$$U_0(z) := \begin{pmatrix} 1 & 0 \\ -1 & \frac{z^2-1}{z^2} \end{pmatrix} V_0 \left(1 - \frac{1}{z^2}\right).$$

which satisfies $LU_0(z) = 0$. As with before, we obtain the asymptotics for U_0 and U_0^{-1} . This concludes the proof of (b).

For part (c), we first note that by the connection formulae for hypergeometric functions (equation 5.10.12 of [91] for instance) we see that for $\xi \in (0, 1)$,

$$g_1(\xi) = \frac{\Gamma(\frac{3}{2})}{\Gamma(1+\frac{\gamma}{2})\Gamma(1+\frac{\bar{\gamma}}{2})} g_3(\xi) - \frac{3}{2} \frac{\Gamma(-\frac{3}{2})}{\Gamma(-\frac{\gamma}{2})\Gamma(-\frac{\bar{\gamma}}{2})} g_4(\xi).$$

If we let $\mu_3 = \frac{\Gamma(\frac{3}{2})}{\Gamma(1+\frac{\gamma}{2})\Gamma(1+\frac{\bar{\gamma}}{2})}$ and $\mu_4 = -\frac{3}{2} \frac{\Gamma(-\frac{3}{2})}{\Gamma(-\frac{\gamma}{2})\Gamma(-\frac{\bar{\gamma}}{2})}$, then by linearity we have

$$\begin{pmatrix} p_1 \\ \omega_1 \end{pmatrix} = U_\infty \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix},$$

yielding the stated asymptotics for (p_{hom}, ω_{hom}) . By the another connection formula (equation 5.10.14 of [91] for instance), for $\xi \in (-\infty, 0)$, we have

$$g_1(\xi) = \frac{\Gamma(1)\Gamma(-\frac{\bar{\gamma}-\gamma}{2})}{\Gamma(-\frac{\bar{\gamma}}{2})\Gamma(1+\frac{\gamma}{2})} g_a(\xi) + \frac{\Gamma(1)\Gamma(-\frac{\gamma-\bar{\gamma}}{2})}{\Gamma(-\frac{\gamma}{2})\Gamma(1+\frac{\bar{\gamma}}{2})} g_b(\xi).$$

We define real constants μ_5, μ_6 by

$$\frac{\Gamma(1)\Gamma(-\frac{\bar{\gamma}-\gamma}{2})}{\Gamma(-\frac{\bar{\gamma}}{2})\Gamma(1+\frac{\gamma}{2})} = \frac{\mu_5}{2} + i \frac{\mu_6}{2},$$

so this becomes

$$g_1(\zeta) = \mu_5 g_5(\zeta) + \mu_6 g_6(\zeta).$$

By linearity, we have

$$\begin{pmatrix} p_{hom} \\ \omega_{hom} \end{pmatrix} = U_0 \begin{pmatrix} \mu_5 \\ \mu_6 \end{pmatrix}.$$

We then compute that if $c_1 := \sqrt{\mu_5^2 + \mu_6^2}$, then $c_1 \neq 0$, which yields the stated asymptotics for (p_{hom}, ω_{hom}) for some $d_1 \in \mathbb{R}$ finishing the proof of (c).

For part (d), we start by defining, for $\xi \in (0, 1)$,

$$g_{2,\infty}(\xi) = \begin{pmatrix} 1 & 0 \end{pmatrix} V_\infty \begin{pmatrix} -\frac{1}{2\mu_4} \\ \frac{1}{2\mu_3} \end{pmatrix}.$$

We note that

$$\begin{pmatrix} g_1 & g_{2,\infty} \\ g'_1 & g'_{2,\infty} \end{pmatrix} = V_\infty \begin{pmatrix} \mu_3 & -\frac{1}{2\mu_4} \\ \mu_4 & \frac{1}{2\mu_3} \end{pmatrix}.$$

and take a determinant to derive

$$g_1 g'_{2,\infty} - g'_1 g_{2,\infty} = \frac{\sqrt{1-\xi}}{\xi}.$$

We then note that because $g_1 = 1 + O(\xi)$ as $\xi \rightarrow 0$, in this regime we can rearrange the equation as

$$\left(\frac{g_{2,\infty}}{g_1}\right)' = \frac{1}{\xi} + \left(\frac{\sqrt{1-\xi}}{\xi g_1^2} - \frac{1}{\xi}\right).$$

which is equivalent to

$$\left(\frac{g_{2,\infty} - \log|\xi|g_1}{g_1}\right)' = \frac{\sqrt{1-\xi} - g_1^2}{\xi g_1^2}.$$

Thus, for some constant C , we have

$$g_{2,\infty} = g_1(\log|\xi| + C + \int_0^\xi \frac{\sqrt{1-\xi'} - g_1(\xi')^2}{\xi' g_1(\xi')^2} d\xi').$$

In particular, as $\xi \rightarrow 0$, for some constant C

$$g_{2,\infty} = \log|\xi| + C + O(|\xi| \log|\xi|), \quad g'_{2,\infty} = \frac{1}{\xi} + O(\log|\xi|).$$

Thus, as $z \rightarrow 1$, we have

$$V_\infty \begin{pmatrix} 1 - \frac{1}{z^2} \\ \mu_4 \end{pmatrix} \begin{pmatrix} \mu_3 & -\frac{1}{2\mu_4} \\ \frac{1}{2\mu_3} \end{pmatrix} = \begin{pmatrix} 1 + O(z-1) & \log|z-1| + C + \log(2) + O(|z-1| \log|z-1|) \\ \frac{1}{2} + O(z-1) & \frac{1}{2(z-1)} + O(\log|z-1|) \end{pmatrix}.$$

Thus, as $z \rightarrow 1$, we have that

$$\begin{aligned} U_\infty \begin{pmatrix} \mu_3 & -\frac{1}{2\mu_4} \\ \mu_4 & \frac{1}{2\mu_3} \end{pmatrix} \\ = \begin{pmatrix} 1 & \log|z-1| + C + \log(2) \\ -1 & -\log|z-1| + 1 - C - \log(2) \end{pmatrix} + \begin{pmatrix} O(z-1) & O(|z-1| \log|z-1|) \\ O(z-1) & O(|z-1| \log|z-1|) \end{pmatrix}. \end{aligned}$$

Thus, if

$$M_\infty = \begin{pmatrix} \mu_3 & -\frac{1}{2\mu_4} \\ \mu_4 & \frac{1}{2\mu_3} \end{pmatrix} \begin{pmatrix} 1 & -C - \log(2) + \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

we obtain the asymptotics for $U_\infty M_\infty$ claimed in (2.29). One can similarly define

$$g_{2,0}(\xi) = \begin{pmatrix} 1 & 0 \end{pmatrix} V_\infty \begin{pmatrix} -\frac{1}{2\mu_6} \\ \frac{1}{2\mu_5} \end{pmatrix} \frac{-4}{\sqrt{7}},$$

and through a similar process construct M_0 . In these cases, we have that

$$\det(U_\infty M_\infty) = \det(U_0 M_0) = \frac{1}{z}$$

and asymptotically expanding the cofactor form of the inverse gives (d).

For part (e), we note that for $z, z' > 1$ we may express

$$S(z, z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} = U_\infty(z) M_\infty (U_\infty M_\infty)^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1},$$

and as $z' \rightarrow 1$, we can asymptotically expand using part (d) to obtain

$$(U_\infty M_\infty)^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2(z'-1)} & -\frac{1}{2(z'-1)} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O(\log|z'-1|) & O(\log|z'-1|) \\ O(1) & O(1) \end{pmatrix}. \quad (2.81)$$

Thus, as $P, \Omega = o(1)$ and $P - \Omega = o(z - 1)$, we have that as $z \rightarrow 1^+$

$$\int_1^z (U_\infty M_\infty)^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} (z') = \begin{pmatrix} o(|z-1| \log|z-1|) \\ o(z-1) \end{pmatrix}.$$

and hence we see that $p, \omega = o(|z-1| \log|z-1|)$ as $z \rightarrow 1^+$ and $p + \omega = o(z-1)$ (to be used shortly). In particular, p, ω are continuous on $[1, \infty)$. One argues similarly using U_0, M_0 for $(0, 1]$. Moreover, the function is C^1 on $(0, \infty) \setminus 1$ and satisfies $L(p, \omega) = P, \Omega$. For $z \neq 1$, we can invert the leading coefficient of L to obtain

$$\begin{pmatrix} p' \\ \omega' \end{pmatrix} = \begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} - \frac{1}{z(z^2-1)} \begin{pmatrix} -2 & -2 \\ 2 & z^2+1 \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} \quad (2.82)$$

and taking the limit as $z \rightarrow 1$ (using $p + \omega = o(z-1)$ to control the latter term) shows that $(p, \omega) \in C^1(0, \infty)$ and

$$p(1) = p'(1) = \omega(1) = \omega'(1) = 0.$$

□

2.5.2 Proof of Lemma 2.2.4

Proof. We suppose $(P, \Omega) \in N_{y_0}$. We note that by Lemma 2.2.1.e, it follows that $(p, \omega) := R(P, \Omega)$ is well-defined, is in $C^1[y_0, \infty)$ and satisfies

$$p(1) = p'(1) = \omega(1) = \omega'(1) = 0.$$

By (2.81), and the asymptotics of $U_\infty M_\infty$ as $z \rightarrow 0$, we from the same arguments as in the proof of Lemma 2.2.1.e, now done quantitatively that

$$\sup_{1 \leq |z| \leq 2} |p| + |\omega| + \left| \frac{p + \omega}{z - 1} \right| \lesssim \sup_{1 \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right|.$$

One then uses (2.82) and these bounds to conclude

$$\sup_{1 \leq |z| \leq 2} |p'|, |\omega'| \lesssim \sup_{1 \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right|.$$

As with before, the argument on $[\frac{1}{2}, 1]$ is the same, so we get

$$\sup_{\frac{1}{2} \leq |z| \leq 2} |p'|, |\omega'| \lesssim \sup_{\frac{1}{2} \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right|.$$

For $z \geq 2$, we express (2.30) as

$$\begin{aligned} \begin{pmatrix} p \\ \omega \end{pmatrix} &= U_\infty(z) \left(M_\infty \int_1^2 (U_\infty M_\infty)^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} \right. \\ &\quad \left. + \int_2^z U_\infty^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} (z') dz' \right). \end{aligned}$$

As in the control near $z = 1$, the first integral is controlled by $\sup_{1 \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right|$. For the second integral, we then note that by Lemma 2.2.1.b, we have that as $z \rightarrow \infty$,

$$U_\infty^{-1}(z) \begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{z} & -\frac{1}{z} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^3}) & O(\frac{1}{z^3}) \\ O(\frac{1}{z^2}) & O(\frac{1}{z^2}) \end{pmatrix}$$

so that for $z' \geq 2$,

$$\left| U_\infty^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} \right| \lesssim \frac{1}{(z')^2} \left(\sup_{z \geq 2} |zP| + |z^2\Omega| \right),$$

so the second integral appearing above converges as $z \rightarrow \infty$ by $\sup_{z \geq 2} |zP| + |z^2\Omega|$. By the asymptotics of $U_\infty(z)$ as $z \rightarrow \infty$, we see that

$$\sup_{z \geq 2} |p| + |z\omega| \lesssim \sup_{1 \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right| + \sup_{z \geq 2} |zP| + |z^2\Omega|,$$

as well as the qualitative observation that $\lim_{z \rightarrow \infty} p$, $\lim_{z \rightarrow \infty} z\omega$ exist. For $z \leq \frac{1}{2}$, we use a similar integral formulation, namely

$$\begin{aligned} \begin{pmatrix} p \\ \omega \end{pmatrix} &= -U_0(z) \left(M_0 \int_{\frac{1}{2}}^1 (U_0 M_0)^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} \right. \\ &\quad \left. + \int_z^{\frac{1}{2}} U_0^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} (z') dz' \right). \end{aligned}$$

and to control the second integral we compute that as $z \rightarrow 0$,

$$U_0^{-1}(z) \begin{pmatrix} z & z \\ \frac{1}{z} & z \end{pmatrix}^{-1} = \frac{1}{\sin(\theta_0)} \begin{pmatrix} -z^{-1/2} \sin\left(\frac{\sqrt{7}}{2} \log(z)\right) & 0 \\ z^{-1/2} \cos\left(\frac{\sqrt{7}}{2} \log(z)\right) & 0 \end{pmatrix} + \begin{pmatrix} O(z^{\frac{3}{2}}) & O(z^{\frac{3}{2}}) \\ O(z^{\frac{3}{2}}) & O(z^{\frac{3}{2}}) \end{pmatrix}$$

so that for $y_0 \leq z' \leq \frac{1}{2}$

$$\left| U_0^{-1}(z') \begin{pmatrix} z' & z' \\ \frac{1}{z'} & z' \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} \right| \lesssim \frac{1}{(z')^{3/2}} \left(\sup_{y_0 \leq z \leq \frac{1}{2}} |zP| + |z^3\Omega| \right).$$

By considering the asymptotics of $U_0(z)$ as $z \rightarrow 0$, and using $z^{-1/2} \leq y_0^{-1/2}$, we obtain

$$\sup_{y_0 \leq z \leq \frac{1}{2}} |z^{1/2}p| + |z^{1/2}\omega| \lesssim y_0^{-1/2} \left(\sup_{\frac{1}{2} \leq y \leq 1} |P| + |\Omega| + \left| \frac{P - \Omega}{z - 1} \right| + \sup_{y_0 \leq z \leq \frac{1}{2}} |zP| + |z^3\Omega| \right).$$

As a note, this regime, $z \leq \frac{1}{2}$, is where the factor of $y_0^{-1/2}$ in the Lemma arises.

In summary, we have shown so far that $p, \omega \in C^1([y_0, \infty))$, they satisfy the correct boundary conditions at the origin, $\lim_{z \rightarrow \infty}(p, z\omega)$ exists and

$$\sup_{z \geq 1} |p| + |z\omega| + \sup_{y_0 \leq z \leq 1} z^{1/2}(|p| + |\omega|) \lesssim y_0^{-\frac{1}{2}} \|P, \Omega\|_{N_{y_0}}.$$

We then note that for $z \neq 1$, we may solve for (p', ω') using (2.82). Taking the limit as $z \rightarrow \infty$ shows that

$$\sup_{z \geq 1} |D_\infty p| + |D_\infty(z\omega)| \lesssim y_0^{-\frac{1}{2}} \|P, \Omega\|_{N_{y_0}}$$

and $\lim_{z \rightarrow \infty}(D_\infty p, D_\infty \omega)$ exists. Looking at the limit as $z \rightarrow 0$ shows the final desired estimate, namely

$$\sup_{y_0 \leq z \leq 1} z^{3/2}(|p'| + |\omega'|) \lesssim y_0^{-\frac{1}{2}} \|P, \Omega\|_{N_{y_0}}.$$

□

2.5.3 Proof of Lemma 2.2.5

Proof. As a first step, we will check that if $p, \omega \in C^1[y_0, \infty)$ with vanishing first order Taylor polynomials at $z = 1$ then $P, \Omega := N(p, \omega)$ satisfy $\frac{P - \Omega}{z - 1} \in C^0[y_0, \infty)$. As a note, this is a bit subtle as $\frac{p'}{z - 1}, \frac{\omega'}{z - 1}$ are not necessarily continuous. Before demonstrating this, we first rewrite $P - \Omega$ to make more explicit the smallness as we take ϵ to zero. Indeed, we start by computing

$$\begin{aligned} A_1 - A_2 &= \hat{\omega}'(z - \beta)\epsilon^2 \left(\frac{\hat{p}}{y_*\alpha} - \frac{1}{y_*\alpha} - \hat{\omega} - 1 \right) \\ &\quad + \epsilon \hat{p}' \left(\frac{(z - \beta)}{y_*\alpha} (y_* + \epsilon\hat{\omega}) - \frac{y_*\alpha}{(z - \beta)} \frac{1}{1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1)} \right) \\ &\quad + \epsilon^2 \left(\frac{\hat{p}}{y_*\alpha} - \frac{1}{y_*\alpha} - \hat{\omega} - 1 \right) \hat{\omega} - \frac{2(y_*\alpha)^2}{(z - \beta)^2} \left(\left(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1) \right) (y_* + \epsilon\hat{\omega}) - 1 \right), \\ \epsilon(L_1 - L_2) &= \epsilon \left(z - \frac{1}{z} \right) \hat{p}' - \frac{2\epsilon}{z^2} (\hat{p} + \hat{\omega}), \\ -\epsilon(P - \Omega) &= \hat{\omega}'(z - \beta)\epsilon^2 \left(\frac{\hat{p}}{y_*\alpha} - \frac{1}{y_*\alpha} - \hat{\omega} - 1 \right) + \hat{p}'\epsilon \left(\frac{(z - \beta)}{y_*\alpha} (y_* + \epsilon\hat{\omega}) \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{y_*\alpha}{(z-\beta)}\frac{1}{1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1)}-\frac{(z^2-1)}{z}\Bigg)+\epsilon^2\Big(\frac{\hat{p}}{y_*\alpha}-\frac{1}{y_*\alpha}-\hat{\omega}-1\Big)\hat{\omega} \\
& -\frac{2(y_*\alpha)^2}{(z-\beta)^2}\left((1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1))(y_*+\epsilon\hat{\omega})-1\right)+\frac{2\epsilon}{z^2}(\hat{p}+\hat{\omega}).
\end{aligned}$$

We then compute

$$\begin{aligned}
& \frac{z-\beta}{y_*\alpha}(y_*+\epsilon\hat{\omega})-\frac{y_*\alpha}{z-\beta}\frac{1}{1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1)}-\frac{z^2-1}{z} \\
& = (z-1)\left(\frac{\beta}{y_*\alpha}+\frac{\beta}{z(z-\beta)}\right)+\epsilon\left(1+\frac{z-1}{y_*\alpha}\right)(\hat{\omega}+1)+\frac{\epsilon}{y_*\alpha}\left(1-\frac{z-1}{z-\beta}\right)\frac{\hat{p}-1}{1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1)} \\
& = \epsilon\left((z-1)\left(\frac{\beta}{\epsilon y_*\alpha}+\frac{\beta}{\epsilon z(z-\beta)}\right)+(1+\frac{z-1}{y_*\alpha})(\hat{\omega}+1)+\frac{1}{y_*\alpha}\left(1-\frac{z-1}{z-\beta}\right)\frac{\hat{p}-1}{1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1)}\right)
\end{aligned}$$

and also

$$\begin{aligned}
& -\frac{2(y_*\alpha)^2}{(z-\beta)^2}\left((1+\frac{\epsilon}{y_*\alpha}(\hat{p}-1))(y_*+\epsilon\hat{\omega})-1\right)+\frac{2\epsilon}{z^2}(\hat{p}+\hat{\omega}) \\
& = -\frac{2(y_*\alpha)^2}{(z-\beta)^2}\left(\epsilon(\hat{\omega}+1)+\frac{\epsilon}{y_*\alpha}(\hat{p}-1)+\frac{\epsilon^2}{y_*\alpha}(\hat{p}-1)(\hat{\omega}+1)\right)+\frac{2\epsilon}{z^2}(\hat{p}+\hat{\omega}) \\
& = \epsilon^2\left(\frac{-2(y_*\alpha)}{(z-\beta)^2}\left((\hat{p}-1)(\hat{\omega}+1)-\frac{\beta}{\epsilon}\right)\right. \\
& \quad \left.+2\hat{p}\frac{\beta}{\epsilon}\frac{z^2-2z+\beta}{z^2(z-\beta)^2}+2\hat{\omega}\frac{\beta}{\epsilon}\frac{(z-1)(2z-\beta-\beta z)}{z^2(z-\beta)^2}\right).
\end{aligned}$$

Thus, if we define

$$\begin{aligned}
M_1(a,b) &:= -(z-\beta)\left(\frac{a-1}{y_*\alpha}-(b+1)\right), \\
M_2(a,b) &:= -\left((z-1)\left(\frac{\beta}{\epsilon y_*\alpha}+\frac{\beta}{\epsilon z(z-\beta)}\right)+(1+\frac{z-1}{y_*\alpha})(b+1)\right. \\
& \quad \left.+\frac{1}{y_*\alpha}\left(1-\frac{z-1}{z-\beta}\right)\frac{a-1}{1+\frac{\epsilon}{y_*\alpha}(a-1)}\right), \\
M_3(a,b) &:= -\left(\frac{a-1}{y_*\alpha}-(b+1)\right)b, \\
M_4(a,b) &:= \frac{2(y_*\alpha)}{(z-\beta)^2}\left((a-1)(b+1)-\frac{\beta}{\epsilon}\right) \\
& \quad -2a\left(\frac{\beta}{\epsilon}\frac{z^2-2z+\beta}{z^2(z-\beta)^2}\right)-2b\frac{\beta}{\epsilon}\left(\frac{(z-1)(2z-\beta-\beta z)}{z^2(z-\beta)^2}\right).
\end{aligned}$$

we have

$$P-\Omega=\epsilon\hat{\omega}'M_1(\hat{p},\hat{\omega})+\epsilon\hat{p}'M_2(\hat{p},\hat{\omega})+\epsilon M_3(\hat{p},\hat{\omega})+\epsilon M_4(\hat{p},\hat{\omega}). \quad (2.83)$$

In preparation for considering the behavior as $z \rightarrow \infty$, we can similarly factor a power of ϵ out from P and Ω individually. To do this, we first compute

$$\begin{aligned} \epsilon(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))(\hat{\omega}(z - \beta))' - \epsilon(z\hat{\omega})' &= \epsilon^2 \left(\frac{1}{y_*\alpha}(\hat{p} - 1)(\hat{\omega}z)' - \frac{\beta}{\epsilon}(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))\hat{\omega}' \right) \\ \frac{\epsilon}{y_*\alpha}\hat{p}'(y_* + \epsilon\hat{\omega})(z - \beta) - \epsilon z\hat{p}' &= \frac{\epsilon^2\hat{p}'}{y_*\alpha}(\frac{z\beta}{\epsilon} - \frac{\beta}{\epsilon} + (\hat{\omega} + 1)(z - \beta)), \\ \epsilon(1 + \epsilon(\hat{\omega} + 1))(\hat{\omega}(z - \beta))' - \epsilon(\hat{\omega}z)' &= \epsilon^2 \left((1 + \epsilon(\hat{\omega} + 1))\frac{-\beta}{\epsilon}\hat{\omega}' + (\hat{\omega} + 1)(\hat{\omega}z)' \right), \\ \epsilon\hat{p}'\frac{\alpha y_*}{1 - \beta}\frac{1}{1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1)} - \frac{\epsilon}{z}\hat{p}' &= \epsilon^2\hat{p}' \left(-\frac{\beta}{\epsilon}\frac{z - 1}{(z - \beta)z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))} \right. \\ &\quad \left. - \frac{\hat{p} - 1}{y_*\alpha z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))} \right). \end{aligned}$$

We may thus define

$$\begin{aligned} M_5 &:= - \left(\frac{1}{y_*\alpha}(\hat{p} - 1)(\hat{\omega}z)' - \frac{\beta}{\epsilon}(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))\hat{\omega}' \right) - \frac{\hat{p}'}{y_*\alpha}(\frac{z\beta}{\epsilon} - \frac{\beta}{\epsilon} + (\hat{\omega} + 1)(z - \beta)), \\ M_6 &:= - \left((y_* + \epsilon\hat{\omega})\frac{-\beta}{\epsilon}\hat{\omega}' + (\hat{\omega} + 1)(\hat{\omega}z)' \right) \\ &\quad - \hat{p}' \left(-\frac{\beta}{\epsilon}\frac{z - 1}{(z - \beta)z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))} - \frac{\hat{p} - 1}{y_*\alpha z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))} \right) - M_4(\hat{p}, \hat{\omega}), \end{aligned}$$

and then we can write

$$P = \epsilon M_5, \quad \Omega = \epsilon M_6. \quad (2.84)$$

We now let $\zeta = z - 1$, and note that as $\zeta \rightarrow 0$, we have

$$\hat{p} = 1 + \zeta + o(\zeta), \quad \hat{\omega} = -1 + o(\zeta), \quad \hat{p}' = 1 + o(1), \quad \hat{\omega}' = o(1).$$

We then use these expansions to compute that as $\zeta \rightarrow 0$

$$\begin{aligned} \epsilon M_1 &= O(\zeta), \quad \epsilon M_2 = -\frac{2\zeta}{y_*\alpha} - \frac{\epsilon}{y_*\alpha}\zeta + 2\zeta + o(\zeta), \\ \epsilon M_3 &= \epsilon\frac{\zeta}{y_*\alpha} + o(\zeta), \quad \epsilon M_4 = \frac{2}{y_*\alpha}\zeta - 2\zeta + o(\zeta). \end{aligned} \quad (2.85)$$

If we substitute these into (2.83), we see that $\frac{P - \Omega}{z - 1}$ is continuous near $z = 1$ and

$$(P - \Omega)(1) = (P - \Omega)'(1) = 0. \quad (2.86)$$

We note that as $\zeta \rightarrow 0$, we have

$$M_5 = o(1)$$

so by (2.84) $P(1) = 0$, which along with (2.86) ensures $P(1) = \Omega(1) = 0$.

We next consider the behavior near $z = \infty$. In this case, we note that assume $p, \omega \in X_{y_0}$, we have that $\|\hat{p}, \hat{\omega}\| < \infty$, so there exists constants $p_0, p_1, \omega_0, \omega_1$ such that

$$\begin{aligned}\hat{p}(z) &= p_0 + \frac{p_1}{z} + o\left(\frac{1}{z}\right), & \hat{\omega}(z) &= \frac{\omega_0}{z} + \frac{\omega_1}{z^2} + o\left(\frac{1}{z^2}\right), \\ \hat{p}'(z) &= -\frac{p_1}{z^2} + o\left(\frac{1}{z^2}\right), & \hat{\omega}'(z) &= -\frac{\omega_0}{z^2} - \frac{2\omega_1}{z^3} + o\left(\frac{1}{z^3}\right).\end{aligned}$$

Using this, one computes that as $z \rightarrow \infty$,

$$\begin{aligned}M_5 &= \frac{p_1}{(1 - y_0)} \frac{1}{z} + o\left(\frac{1}{z}\right), \\ M_6 &= -\frac{y_* \beta \omega_0}{\epsilon z^2} + \frac{\omega_1}{z^2} - \frac{2y_* \alpha (p_0 - 1 - \frac{\beta}{\epsilon}) + 2p_0 \frac{\beta}{\epsilon}}{z^2} + o\left(\frac{1}{z^2}\right).\end{aligned}$$

In summary, these computations show we have shown so far that if $(p, \omega) \in X_{y_0}$ then (P, Ω) satisfies the correct boundary conditions at $z = 1, \infty$ for (P, Ω) to be in N_{y_0} . Moreover, it is clear that (P, Ω) are continuous away from $z = 1$. We next show the inequality (2.33). As a note, we use “ $\lesssim$ ” to denote inequalities which are independent of y_0, ϵ .

To show (2.33), we first note that if $\|(p, \omega)\|_{X_{y_0}} \lesssim 1$, then we have that $\|\hat{p}, \hat{\omega}\|_{X_{y_0}} \lesssim 1$. Thus, by making ϵ sufficiently small, we have that

$$\left| \frac{1}{1 + \frac{\epsilon}{y_* \alpha} (\hat{p} - 1)} \right| \lesssim 1$$

and so the denominator causes no problems. Near zero it is useful to note that

$$\sup_{\frac{1}{2} \leq z \leq 2} |\hat{p}| + |\hat{p}'| + |\hat{\omega}| + |\hat{\omega}'| \lesssim 1. \quad (2.87)$$

We then compute

$$\frac{P - \Omega}{z - 1} = \epsilon \frac{M_1}{z - 1} \hat{\omega}' + \epsilon \frac{M_2}{z - 1} \hat{p}' + \epsilon \frac{M_3}{z - 1} + \epsilon \frac{M_4}{z - 1}.$$

We then note that as, for $i \in 1, 2, 3, 4$ $M_i(1) = 0$, we have

$$\sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{M_i}{z - 1} \right| \lesssim \sup_{\frac{1}{2} \leq z \leq 2} |M'_i|.$$

Moreover, for these indices M_i does not contain $\hat{p}'$ or $\hat{\omega}'$, so by (2.87)

$$\sup_{\frac{1}{2} \leq z \leq 2} |M'_i| \lesssim 1.$$

This gives

$$\sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{P - \Omega}{z - 1} \right| \lesssim |\epsilon|.$$

Similarly, we have by (2.87)

$$|M_5| \lesssim 1$$

so (by writing $\Omega = P - (z - 1)\frac{P - \Omega}{z - 1}$)

$$\sup_{\frac{1}{2} \leq z \leq 2} |P| + |\Omega| \lesssim |\epsilon|.$$

We now consider the regime $z \geq 2$ and note that as $\|(\hat{p}, \hat{\omega})\| \lesssim 1$ we have for some constants p_0, ω_0 ,

$$|\hat{p} - p_0| \lesssim \frac{1}{z}, \quad |\hat{p}'| \lesssim \frac{1}{z^2}, \quad \left| \hat{\omega} - \frac{\omega_0}{z} \right| \lesssim \frac{1}{z^2}, \quad |(z\hat{\omega})'| \lesssim \frac{1}{z^2}.$$

This gives $|M_5| \lesssim \frac{1}{z}$ and $|M_6| \lesssim \frac{1}{z^2}$, so we have

$$\sup_{z \geq 2} |zP| + |z^2\Omega| \lesssim |\epsilon|.$$

Finally, for $z \leq \frac{1}{2}$, we have from $\|(\hat{p}, \hat{\omega})\|_{X_{y_0}} \lesssim 1$ that

$$|\hat{p}|, |\hat{\omega}| \lesssim z^{-1/2}, \quad |\hat{p}'|, |\hat{\omega}'| \lesssim z^{-3/2}.$$

We also note by choosing ϵ small,

$$|\beta| = \frac{\epsilon}{1 + \epsilon - y_0} y_0 \leq \frac{y_0}{2} \leq \frac{|z|}{2}$$

So we may also estimate $\left| \frac{1}{z - \beta} \right| \lesssim \frac{1}{z}$. We also record $\left| \frac{\beta}{\epsilon} \right| \lesssim z$. We thus can compute

$$|M_5| \lesssim \frac{1}{z}, \quad |M_6| \lesssim \frac{1}{z^3}.$$

Thus,

$$\sup_{y_0 \leq z \leq \frac{1}{2}} |zP| + |z^3\Omega| \lesssim \epsilon$$

concluding the proof of inequality (2.33).

We now consider differentiating the nonlinearity with respect to $\hat{p}$ and $\hat{\omega}$. Near $z = 1$, we represent the formal derivative of $P - \Omega$ with respect to p , by computing

$$\begin{aligned} \left(\frac{\partial M_1}{\partial a}, \frac{\partial M_1}{\partial b} \right) &= \left(-(z - \beta) \frac{1}{y_* \alpha}, (z - \beta) \right) \\ \left(\frac{\partial M_2}{\partial a}, \frac{\partial M_2}{\partial b} \right) &= \left(\frac{1}{y_* \alpha} \left(1 - \frac{z - 1}{z - \beta} \right) \frac{1}{(1 + \frac{\epsilon}{y_* \alpha} (a - 1))^2}, 1 + \frac{z - 1}{y_* \alpha} \right) \\ \left(\frac{\partial M_3}{\partial a}, \frac{\partial M_3}{\partial b} \right) &= \left(-\frac{b}{y_* \alpha}, -\frac{a - 1}{y_* \alpha} + 2b + 1 \right) \\ \left(\frac{\partial M_4}{\partial a}, \frac{\partial M_4}{\partial b} \right) &= \left(\frac{2y_* \alpha (b + 1)}{(z - \beta)^2} - \frac{2\beta}{\epsilon} \frac{z^2 - 2z + \beta}{z^2 (z - \beta)^2}, \right. \end{aligned}$$

$$\frac{2y_*\alpha(a-1)}{(z-\beta)^2} - \frac{2\beta}{\epsilon} \frac{(z-1)(2z-\beta-\beta z)}{z^2(z-\beta)^2} \Bigg).$$

This then gives the linear operators (partial derivatives of M_i evaluated at $\hat{p}_0, \hat{\omega}_0$)

$$\begin{aligned} \frac{\partial P - \Omega}{\partial p} &:= \epsilon M_2 \frac{d}{dz} + \epsilon \hat{\omega}'_0 \frac{\partial M_1}{\partial a} + \epsilon \hat{p}'_0 \frac{\partial M_2}{\partial a} + \epsilon \frac{\partial M_3}{\partial a} + \epsilon \frac{\partial M_4}{\partial a} \\ \frac{\partial P - \Omega}{\partial \omega} &:= \epsilon M_1 \frac{d}{dz} + \epsilon \hat{\omega}'_0 \frac{\partial M_1}{\partial b} + \epsilon \hat{p}'_0 \frac{\partial M_2}{\partial b} + \epsilon \frac{\partial M_3}{\partial b} + \epsilon \frac{\partial M_4}{\partial b}. \end{aligned}$$

We then note that if $\|\hat{p}, \hat{\omega}\|_{X_{y_0}} \lesssim 1$, we have that

$$\begin{aligned} \sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{M_2}{z-1} \right| + \left| \hat{\omega}_0 \frac{\partial M_1}{\partial a} \right| + \left| \hat{p}_0' \frac{\partial M_2}{\partial a} \right| + \left| \frac{\partial M_3}{\partial a} \right| + \left| \frac{\partial M_4}{\partial a} \right| \\ + \left| \frac{M_1}{z-1} \right| + \left| \hat{\omega}_0 \frac{\partial M_1}{\partial b} \right| + \left| \hat{p}_0' \frac{\partial M_2}{\partial b} \right| + \left| \frac{\partial M_3}{\partial b} \right| + \left| \frac{\partial M_4}{\partial b} \right| \lesssim 1. \end{aligned}$$

so for any $(p, \omega) \in X_{y_0}$,

$$\begin{aligned} \sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{\partial P - \Omega}{\partial p} p \right| + \left| \frac{\partial P - \Omega}{\partial \omega} \omega \right| &\lesssim \epsilon \sup_{\frac{1}{2} \leq z \leq 2} |p'| + \left| \frac{p}{z-1} \right| + |\omega'| + \left| \frac{\omega}{z-1} \right| \\ &\lesssim \epsilon \sup_{\frac{1}{2} \leq z \leq 2} |p'| + |\omega'| \\ &\lesssim \epsilon \|(p, \omega)\|_{X_{y_0}}. \end{aligned}$$

We next estimate the derivative away from $z = 1$. This is done by formally differentiating the formulae for M_5, M_6 . When we do this, we obtain

$$\begin{aligned} \frac{\partial P}{\partial p} &= \epsilon \left(-\frac{1}{y_*\alpha} \left(\frac{(z-1)\beta}{\epsilon} + (\hat{\omega}_0 + 1)(z-\beta) \right) \frac{d}{dz} + \frac{(\hat{\omega}_0(z-\beta))'}{y_*\alpha} \right) \\ \frac{\partial P}{\partial \omega} &= \epsilon \left(\left(\frac{z-\beta}{y_*\alpha} (\hat{p}_0 - 1) - \frac{\beta}{\epsilon} \right) \frac{d}{dz} + \frac{\hat{p}_0 - 1}{y_*\alpha} - \frac{(z-\beta)\hat{p}'_0}{y_*\alpha} \right). \\ \frac{\partial \Omega}{\partial p} &= \epsilon \left(\left(\frac{\beta}{\epsilon} \frac{z-1}{(z-\beta)z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p}_0 - 1))} + \frac{\hat{p}_0 - 1}{y_*\alpha z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p}_0 - 1))} \right) \frac{d}{dz} - \right. \\ &\quad \left. - \hat{p}'_0 \left(\frac{\beta(z-1)}{y_*\alpha(z-\beta)z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p}_0 - 1))^2} + \frac{1 + (-1 + \frac{\epsilon}{y_*\alpha}(\hat{p} - 1))}{y_*\alpha z(1 + \frac{\epsilon}{y_*\alpha}(\hat{p}_0 - 1))^2} \right) - \frac{\partial M_4}{\partial a} \right) \\ \frac{\partial \Omega}{\partial \omega} &= -\epsilon \left(\left(-\frac{\beta}{\epsilon} + (z-\beta)(\hat{\omega}_0 + 1) \right) \frac{d}{dz} + (\hat{\omega}_0(z-\beta))' + \hat{\omega}_0 + 1 - \frac{\hat{p}'_0}{y_*\alpha} (z-\beta) \right). \end{aligned}$$

Thus, under the hypotheses of the lemma, for $(p, \omega) \in X_{y_0}$, for $z \geq 2$, we have the desired bounds

$$\left| \frac{\partial P}{\partial p} p \right| + \left| \frac{\partial P}{\partial \omega} \omega \right| \lesssim \epsilon (|zp'| + |z\omega'| + \left| \frac{p}{z} \right| + |\omega|) \lesssim \frac{\epsilon}{z} \|(p, \omega)\|_{X_{y_0}}$$

and (noting that $|z\omega' + \omega| \lesssim |\frac{1}{z^2}D_\infty z\omega|$)

$$\left| \frac{\partial \Omega}{\partial p} p \right| + \left| \frac{\partial \Omega}{\partial \omega} \omega \right| \lesssim \epsilon \left(\left| \frac{p'}{z} \right| + \left| \frac{p}{z^2} \right| + \frac{1}{z^2} |D_\infty \omega| + \frac{1}{z^2} |z\omega| \right) \lesssim \frac{\epsilon}{z^2} \|(p, \omega)\|_{X_{y_0}}.$$

It finally remains to examine the region $y_0 \leq z \leq \frac{1}{2}$. In this case, we have again

$$\left| \frac{\partial P}{\partial p} p \right| + \left| \frac{\partial O}{\partial \omega} \omega \right| \lesssim \epsilon (|z^{1/2} p'| + |z^{1/2} \omega'| + |z^{-1/2} p| + |z^{-1/2} \omega|) \lesssim \frac{\epsilon}{z} \|(p, \omega)\|_{X_{y_0}}$$

and

$$\left| \frac{\partial \Omega}{\partial p} p + \frac{\partial \Omega}{\partial \omega} \omega \right| \lesssim \epsilon (|z^{-3/2} p'| + |z^{-5/2} p| + |z^{1/2} \omega'| + |z^{-1/2} \omega|) \lesssim \frac{\epsilon}{z^3} \|(p, \omega)\|_{X_{y_0}}.$$

These estimates prove the first part of (2.34). As the coefficients of N vary smoothly with respect to ϵ , the first statement follows from the boundedness of N , and the fact that when we differentiate the representations (2.83) and (2.84), we can no longer factor out a factor of ϵ , resulting in the second inequality of (2.34) not having a factor of ϵ in front. To show that the formal derivatives truly approximate N to first order, it suffices to show

$$\begin{aligned} & \left\| N(\epsilon_1, p_1, \omega_1) - N(\epsilon_0, p_0, \omega_0) - \left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{|(\epsilon_0, p_0, \omega_0)} \cdot (\epsilon_1 - \epsilon_0, p_1 - p_0, \omega_1 - \omega_0) \right\|_{N_{y_0}} \\ & \leq C_2 \left(|\epsilon_1 - \epsilon_0|^2 + |\epsilon_1 - \epsilon_0| \|(p_1, \omega_1) - (p_0, \omega_0)\|_{X_{y_0}} \right. \\ & \quad \left. + \max(|\epsilon_0|, |\epsilon_1|) \|(p_1, \omega_1) - (p_0, \omega_0)\|_{X_{y_0}}^2 \right). \end{aligned}$$

which follows from straightforward estimates. Once this is done, (2.35) and (2.36) follow similarly. $\square$

2.5.4 Proof of Lemma 2.2.6

This is a classical fixed point argument.

Proof. We begin by proving (a). We let $(p_0, \omega_0) = (0, 0)$. We suppose

$$|\epsilon| \leq \min\left(\frac{y_0^{1/2}}{C_2(\frac{1}{2})C_1} \frac{1}{10}, y_0^{1/2} C_0(\frac{1}{2})\right). \quad (2.88)$$

For $n \geq 1$, we define $(P_n, \Omega_n) = N(\epsilon, p_{n-1}, \omega_{n-1})$, $(p_n, \omega_n) = R(P_n, \Omega_n)$. We inductively suppose $\|(p_n, \omega_n)\|_{X_{y_0}} \leq y_0^{-1/2} \epsilon C_2(\frac{1}{2}) C_1$. If this is true for $n = k-1$, then $\|(p_{k-1}, \omega_{k-1})\|_{X_{y_0}} < \frac{1}{2}$, so by Lemma 2.2.5.a we have $\|(P_k, \Omega_k)\|_{N_{y_0}} \leq |\epsilon| C_2(\frac{1}{2})$. By Lemma 2.2.4, $\|(p_k, \omega_k)\| \leq |\epsilon| C_1 C_2(\frac{1}{2}) y_0^{-1/2}$ as desired.

In addition, for $k \geq 1$, we have that

$$\|(p_{k+1}, \omega_{k+1}) - (p_k, \omega_k)\|_{X_{y_0}} \leq C_1 y_0^{-1/2} \|N(\epsilon, p_k, \omega_k) - N(\epsilon, p_{k-1}, \omega_{k-1})\|_{N_{y_0}}$$

$$\begin{aligned}
&\leq C_1 y_0^{1/2} C_2 \left(\frac{1}{2}\right) |\epsilon| \| (p_k, \omega_k) - (p_{k-1}, \omega_{k-1}) \|_{X_{y_0}} \\
&\leq \frac{1}{2} \| (p_k, \omega_k) - (p_{k-1}, \omega_{k-1}) \|_{X_{y_0}},
\end{aligned}$$

where to get from the first line to the second we used the uniform bound on $\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega}$ and the fundamental theorem of calculus. Thus, this iteration converges to a limit, $(p[\epsilon], \omega[\epsilon])$, with the desired bound. Uniqueness follows similarly from the contractive estimate, concluding the proof of part (a). We note that if $\epsilon = 0$, as $N(0, 0, 0) = 0$, we obtain $(p[0], \omega[0]) = 0$.

To prove part (b), we consider ϵ_0, ϵ_1 satisfying (2.88). We let $(p_0, \omega_0) = (p[\epsilon_0], \omega[\epsilon_0])$, $(p_1, \omega_1) = (p[\epsilon_1], \omega[\epsilon_1])$, which are well defined by part (a) and satisfy

$$(p_0, \omega_0) = RN(\epsilon_0, p_0, \omega_0), \quad (p_1, \omega_1) = RN(\epsilon_1, p_1, \omega_1).$$

If we subtract these identities we obtain, by the bounds on the partial derivatives of N ,

$$\begin{aligned}
\| (p_1, \omega_1) - (p_0, \omega_0) \|_{X_{y_0}} &= \| R(N(\epsilon_1, p_1, \omega_1) - N(\epsilon_0, p_0, \omega_0)) \|_{X_{y_0}} \\
&\leq C_1 y_0^{-1/2} \| N(\epsilon_1, p_1, \omega_1) - N(\epsilon_0, p_0, \omega_0) \| \\
&\leq C_1 y_0^{-1/2} C_2 |\epsilon_1 - \epsilon_0| \\
&\quad + C_1 y_0^{-1/2} C_2 \max(|\epsilon_1|, |\epsilon_0|) \| (p_1, \omega_1) - (p_0, \omega_0) \|_{X_{y_0}} \\
&\leq C_1 y_0^{-1/2} C_2 (\epsilon_1 - \epsilon_0) + \frac{1}{2} \| (p_1, \omega_1) - (p_0, \omega_0) \|_{X_{y_0}}.
\end{aligned}$$

By absorbing the second term into the right-hand side we conclude

$$\| (p_1, \omega_1) - (p_0, \omega_0) \|_{X_{y_0}} \leq 2C_1 C_2 y_0^{-1/2} |\epsilon_1 - \epsilon_0|. \quad (2.89)$$

Next, for each ϵ_0 , we consider the linearization of (2.32), namely

$$(p_{lin}, \omega_{lin}) = R \frac{\partial N}{\partial \epsilon} |_{(\epsilon, p[\epsilon], \omega[\epsilon])} + R \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) |_{(\epsilon, p[\epsilon], \omega[\epsilon])} \cdot (p_{lin}, \omega_{lin}). \quad (2.90)$$

and note that

$$\left\| R \frac{\partial N}{\partial \epsilon} \right\|_{X_{y_0}} \leq C_2 \left(\frac{1}{2}\right) C_1 y_0^{-1/2}, \quad \left\| R \left(\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right) |_{(\epsilon, p[\epsilon], \omega[\epsilon])} \right\|_{X_{y_0} \rightarrow X_{y_0}} \leq C_2 \left(\frac{1}{2}\right) C_1 y_0^{-1/2} \epsilon \leq \frac{1}{2}$$

so we can iteratively solve (2.90) resulting in a solution satisfying

$$\| (p_{lin}, \omega_{lin}) \|_{X_{y_0}} \leq 2C_2 \left(\frac{1}{2}\right) C_1 y_0^{-1/2}. \quad (2.91)$$

We next define

$$(p_{diff}, \omega_{diff}) = (p_1, \omega_1) - (p_0, \omega_0) - (\epsilon_1 - \epsilon_0)(p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0]).$$

We note that

$$(p_{diff}, \omega_{diff}) = R(N(p_1, \omega_1) - N(p_0, \omega_0))$$

$$\begin{aligned}
& -\left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega}\right)(\epsilon_1 - \epsilon_0, (\epsilon_1 - \epsilon_0)p_{lin}[\epsilon_0], (\epsilon_1 - \epsilon_0)\omega_{lin}[\epsilon_0]) \\
& = R(N(p_1, \omega_1) - N(p_0, \omega_0) - \left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega}\right)|_{(\epsilon_0, p_0, \omega_0)}(\epsilon_1 - \epsilon_0, p_1 - p_0, \omega_1 - \omega_0)) \\
& + R\left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega}\right)|_{(\epsilon_0, p_0, p_1)}(0, p_{diff}, \omega_{diff})
\end{aligned}$$

We use (2.35), (2.36) and (2.89) to control the first term and absorb the second term on the left hand side to obtain

$$\|(p_{diff}, \omega_{diff})\|_{N_{y_0}} \leq 10(C_1 C_2 (\frac{1}{2}) y_0^{-1/2})^2 |\epsilon_1 - \epsilon_0|^2.$$

Thus, we see that $(p[\epsilon], \omega[\epsilon])$ is differentiable with derivative $(p_{lin}[\epsilon], \omega_{lin}[\epsilon])$.

To show continuity of $(p_{lin}[\epsilon], \omega_{lin}[\epsilon])$ with respect to ϵ , we express

$$\begin{aligned}
& (p_{lin}[\epsilon_1], \omega_{lin}[\epsilon_1]) - (p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0]) \\
& = R \left(\left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{(\epsilon_1, p_1, \omega_1)} \cdot (1, p_{lin}[\epsilon_1], \omega_{lin}[\epsilon_1]) \right. \\
& \quad \left. - \left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{(\epsilon_0, p_0, \omega_0)} \cdot (1, p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0]) \right) \\
& = R \left(\left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{(\epsilon_1, p_1, \omega_1)} \cdot (0, p_{lin}[\epsilon_1] - p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_1] - \omega_{lin}[\epsilon_0]) \right) \\
& + R \left(\left(\left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{(\epsilon_1, p_1, \omega_1)} - \left(\frac{\partial N}{\partial \epsilon}, \frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega} \right)_{(\epsilon_0, p_0, \omega_0)} \right) \cdot (1, p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0]) \right)
\end{aligned}$$

We bound the first term using the control of the derivative of $\frac{\partial N}{\partial p}, \frac{\partial N}{\partial \omega}$ and bound the second term using (2.35), (2.36), (2.89) to obtain

$$\begin{aligned}
\|(p_{lin}[\epsilon_1], \omega_{lin}[\epsilon_1]) - (p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0])\|_{X_{y_0}} & \leq \frac{1}{2} \|(p_{lin}[\epsilon_1], \omega_{lin}[\epsilon_1]) - (p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0])\|_{X_{y_0}} \\
& + 5(C_2(\frac{1}{2})C_1 y_0^{-1/2})^2 |\epsilon_1 - \epsilon_0|.
\end{aligned}$$

Absorbing the first term shows the desired continuity of the derivative, namely

$$\|(p_{lin}[\epsilon_1], \omega_{lin}[\epsilon_1]) - (p_{lin}[\epsilon_0], \omega_{lin}[\epsilon_0])\|_{X_{y_0}} \leq 10(C_1 C_2 (\frac{1}{2}) y_0^{-1/2})^2 |\epsilon_1 - \epsilon_0|.$$

□

2.6 Deferred proofs from Section 2.3

2.6.1 Proof of Lemma 2.3.1

Proof. We let v denote v_1 or v_2 . For $y > 0$ we then rewrite $Hv = 0$ as $\tilde{H}v = f$, where

$$f := 2(e^Q - \frac{1}{y^2})v$$

We now choose the following basis of homogeneous solutions for $\tilde{H}$,

$$\tilde{v}_1 := \frac{\cos\left(\frac{\sqrt{7}}{2} \log(y)\right)}{y^{1/2}}, \quad \tilde{v}_2 := \frac{\sin\left(\frac{\sqrt{7}}{2} \log(y)\right)}{y^{1/2}}.$$

Their Wronskian is $\frac{\sqrt{7}}{y^2}$. We introduce now the following integral operator so that $\tilde{H} \circ \tilde{S}_\infty = \text{id}$:

$$\tilde{S}_\infty(f) := \tilde{v}_1 \int_y^\infty \frac{(y')^2}{\frac{\sqrt{7}}{2}} \tilde{v}_2 f dy' - \tilde{v}_2 \int_y^\infty \frac{(y')^2}{\frac{\sqrt{7}}{2}} \tilde{v}_1 f dy'.$$

We now recall that we have $f = O(y^{-3})$ as $y \rightarrow \infty$, so $\tilde{S}_\infty \circ \tilde{H}v = O(\frac{1}{y})$ as $y \rightarrow \infty$. We then note that

$$v - \tilde{S}_\infty \circ \tilde{H}v \in \ker \tilde{H},$$

so we have for some constants a_1, a_2

$$\begin{aligned} v &= a_1 \tilde{v}_1 + a_2 \tilde{v}_2 + \tilde{S}_\infty \circ \tilde{H}v \\ &= a_1 \tilde{v}_1 + a_2 \tilde{v}_2 + O\left(\frac{1}{y}\right). \end{aligned}$$

This, up to a linear change of variables, yields the claimed asymptotic expansions. To see why $c_3, c_4 \neq 0$, we have by (2.45), $W = \frac{c_3 c_4 \frac{\sqrt{7}}{2} \sin(d_1 - d_2)}{y^2}$, and also $W = \frac{1}{y^2}$. $\square$

2.6.2 Proof of Lemma 2.3.3

Proof. We first consider the operator S_r . We make the qualitative observation that if $f(x)$ is continuous then $S_r f$ is C^2 and $S_r f(0) = 0$. We next assume that $f \in \mathcal{G}_{y_1}$ and $\|f\|_{\mathcal{G}_{y_1}} \leq 1$. To show that $S_r f \in Z_{y_1}$ we argue differently for $0 \leq y \leq 1$ and $1 \leq y \leq y_1$. For $y \leq 1$, we note that as $y \rightarrow 0$, we have (uniformly in r),

$$v_{1,r} = O(y), \quad v_{2,r} = O\left(\frac{1}{y}\right) \tag{2.92}$$

and so we have (all inequalities independent of y_0, λ_0, r)

$$\sup_{0 \leq y \leq 1} |Sf| \lesssim \sup_{0 \leq y \leq 1} |f| \lesssim \|f\|_{\mathcal{G}_{y_1}}.$$

Next, for $y \geq 1$, we recall that the homogeneous solutions $v_{1,r}, v_{2,r}$ have the following asymptotics as $y \rightarrow \infty$,

$$v_{1,r}, v_{2,r} = O\left(\frac{1}{y^{1/2}}\right). \tag{2.93}$$

We then use the expression

$$S_r f(y) = \frac{1}{r^3} \left(\left(\int_0^1 f v_{2,r}(y')^2 dy' \right) v_{1,r} - \left(\int_0^1 f v_{1,r}(y')^2 dy' \right) v_{2,r} \right)$$

$$\begin{aligned}
& + \frac{1}{r^3} \left(\left(\int_1^y f v_{2,r}(y')^2 dy' \right) v_{1,r} - \left(\int_1^y f v_{1,r}(y')^2 dy' \right) v_{2,r} \right) \\
& = S_1(y) + S_2(y).
\end{aligned}$$

We then use the asymptotics of the fundamental solutions near zero and infinity to bound

$$S_1(y) \leq \langle y \rangle^{-1/2} \|f\|_{L^\infty[0,1]} \lesssim \langle y \rangle^{3/2} \|f\|_{\mathcal{G}_{y_1}},$$

and the asymptotics of the fundamental solutions near infinity to bound

$$S_2(y) \lesssim \langle y \rangle^{3/2} \left\| \langle y \rangle^{1/2} f \right\|_{L^\infty[1,y_1]} \lesssim \langle y \rangle^{3/2} \|f\|_{\mathcal{G}_{y_1}},$$

giving the desired pointwise control of f . To show control of the derivatives, we note that when we differentiate the definition of S we get a cancellation, so

$$(Sf)' = \frac{1}{r^2} \left(\left(\int_0^y f v_{2,r}(y')^2 dy' \right) v'_{1,r} - \left(\int_0^y f v_{1,r}(y')^2 dy' \right) v'_{2,r} \right).$$

This allows one to control $(Sf)'$ using only pointwise control of f . Similarly,

$$\begin{aligned}
(Sf)'' &= -(HS)f - \frac{2}{y} \partial_y(Sf) - 2e^Q(Sf) \\
&= -f - \frac{2}{y} \partial_y(Sf) - 2e^Q(Sf).
\end{aligned}$$

As the RHS can be controlled with pointwise control of f , we see that $\partial_r^2(Sf)$ can also be controlled with pointwise control of f . Using these observations, the control of the derivatives of S works in the same way as the pointwise control. Thus, S_r is bounded as a map from $\mathcal{G}_{y_1}$ to Z_{y_1} .

We then define the formal derivative of S with respect to r as

$$\begin{aligned}
S_{lin,r}f &:= -\frac{3}{r} S_r f + \frac{1}{r^3} \left(\left(\int_0^y f \frac{\partial v_{2,r}}{\partial r} (y')^2 dy' \right) v_{1,r} + \left(\int_0^y f v_{2,r} (y')^2 dy' \right) \frac{\partial v_{1,r}}{\partial r} + \right. \\
&\quad \left. - \left(\int_0^y f \frac{\partial v_{1,r}}{\partial r} (y')^2 dy' \right) v_{2,r} - \left(\int_0^y f v_{1,r} (y')^2 dy' \right) \frac{\partial v_{2,r}}{\partial r} \right)
\end{aligned}$$

where we note that

$$\frac{\partial v_{i,r}}{\partial r} = -\frac{y}{r^2} v'_i\left(\frac{y}{r}\right).$$

and as we may differentiate the asymptotics of v_i , we see that (2.92) and (2.93) continue to hold for $\frac{\partial v_{i,r}}{\partial r}$, so the same arguments showing the boundedness of S show that this operator is bounded. Straightforward difference estimates prove that

$$\|S_{r_1} - S_{r_0} - (r_1 - r_0)S_{lin,r_0}\|_{L(\mathcal{G}_{y_1}, Z_{y_1})} \lesssim |r_1 - r_0|^2$$

and

$$\|S_{lin,r_1} - S_{lin,r_0}\|_{L(\mathcal{G}_{y_1}, Z_{y_1})} \lesssim |r_1 - r_0|$$

which show that $S_{lin,r}$ is the derivative of S_r and it is continuous with respect to r as desired.

We next consider T_r . In this case, it is clear that if $f \in C^1$ and $f(0) = 0$ then $T_r f \in C^2$ and $T_r f(0) = (T_r f)'(0) = 0$. If one supposes $\|f\|_{\mathcal{F}_{y_1}} \leq 1$, then for $0 \leq y \leq 1$, we use that e^{-Q_r} goes to r^2 at zero to compute

$$T_r f(y) = -\frac{1}{r^2 y^2 e^Q} \int_0^y (y')^2 f dy' \lesssim \|f\|_{L^\infty[0,1]} \lesssim \|f\|_{\mathcal{F}_{y_1}}.$$

For $y \geq 1$, we use that $e^{-Q} = y^2 + O(y^{3/2})$ as $y \rightarrow \infty$ to compute

$$\begin{aligned} T_r f(y) &= -\frac{1}{r^2 y^2 e^Q} \int_0^1 (y')^2 f dy' - \frac{1}{r^2 y^2 e^Q} \int_1^y (y')^2 f dr' \\ &\lesssim \|f\|_{L^\infty[0,1]} + \langle y \rangle^{5/2} \left\| \langle y \rangle^{1/2} f \right\|_{L^\infty[1,y_1]} \lesssim \langle y \rangle^{5/2} \|f\|_{\mathcal{F}_{y_1}}. \end{aligned}$$

To control a derivative, we use the identity

$$(T_r f)' = \left(e^Q \left(\frac{1}{e^Q} \right)' - \frac{2}{y} \right) T_r f - \frac{1}{e^Q} f.$$

and note that the right side involves no derivatives of f . This allows one to control 1 derivative of $T_r f$ using pointwise control of f . Control of the second derivative of $T_r f$ requires pointwise control of the first derivative of f , which is supplied by the norm of $\mathcal{F}_{y_1}$. From this one obtains that T_r is bounded from $\mathcal{F}_{y_1}$ to Y_{y_1} . One can then compute the formal derivative of T_r with respect to r to be

$$T_{lin,r} f = -\frac{2}{r} T_r f + \frac{1}{r^2 y^2 e^{Q_r}} \frac{\partial Q_r}{\partial r} \int_0^y f(y') (y')^2 dy'$$

where one can compute

$$\frac{\partial Q_r}{\partial r} = -\frac{y}{r^2} Q' \left(\frac{y}{r} \right) - \frac{2}{r}.$$

It is then straightforward to check that this operator is bounded, approximates T_r to first order in r and is Lipschitz with respect to r valued in $L(\mathcal{F}_{y_1}, Y_{y_1})$.

Finally, to show that K_r has the desired property, we note that it suffices to check that $L_r: Z_{y_1} \rightarrow \mathcal{F}_{y_1}$ is bounded, and has the desired regularity when r is varied. It is clear that, if f is C^1 , $f(0) = 0$ then Tf is C^2 and $Tf(0) = (Tf)'(0) = 0$. Finally, if $f(0) = 0$ and f is C^2 , then $L_r f$ is C^1 . The boundedness follows from the asymptotics of Q_r . Finally, one can compute the formal derivative

$$L_{lin,r} w = \frac{2}{r} L_{lin,r} w + r^2 \left((2 + y \partial_y) (e^{Q_r} \frac{\partial Q_r}{\partial r} w) + \operatorname{div}(u_r e^{Q_r} \frac{\partial Q_r}{\partial r} w) + \operatorname{div} \left(\frac{\partial u_r}{\partial r} e^{Q_r} w \right) \right).$$

Once again, straightforward computations show that $L_{lin,r}$ is a bounded map from Z_{y_1} to $\mathcal{F}_{y_1}$, $L_{lin,r}$ approximates L_r to first order and $L_{lin,r}$ is Lipschitz with respect to r . $\square$

2.6.3 Proof of Lemma 2.3.4

Proof. We note that qualitatively if $w, u \in C^2([0, y_0])$ and $w(0) = u(0) = u'(0) = 0$ then $G_r(w, u) \in C^1([0, y_1])$, $G_r(w, u) \in C^0([0, y_1])$ and $F_r(w, u) = 0$. As a first step in showing boundedness of the linear operator, we define

$$\begin{aligned} F_{r,1}(w, u) &= \lambda_0^2 r^2 (2 + y \partial_y) (w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt), \\ F_{r,2}(w, u) &:= \lambda_0^2 r^2 \operatorname{div}(uw \int_0^1 e^{Q_r + t \lambda_0^2 w} dt), \\ F_{r,3}(w, u) &:= \lambda_0^2 r^2 \operatorname{div}(u_r w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt), \\ G_{r,1}(w, u) &= r^2 \operatorname{div}((y + u_r) \partial_y u_r), \\ G_{r,2}(w, u) &= 2 \lambda_0^2 r^2 w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt, \\ G_{r,3}(w, u) &= \lambda_0^2 r^2 \operatorname{div}(u \partial_y u_r + (y + u_r) \partial_y u + \lambda_0^2 u \partial_y u), \end{aligned}$$

so that we can express the nonlinearities as

$$F_r = F_{r,1} + F_{r,2} + F_{r,3} \quad \text{and} \quad G_r = G_{r,1} + G_{r,2} + G_{r,3}.$$

We now assume that $\|w\|_{Z_{y_1}}, \|u\|_{Y_{y_1}} \lesssim 1$ and discuss bounding $F_{r,i}, G_{r,i}$. As a first step, we note that as $y \leq \frac{y_0}{\lambda_0}$, and $\lambda \leq \frac{y_0}{10}$, we have $\langle y \rangle \lesssim \frac{y_0}{\lambda_0}$, or in other words

$$\lambda_0 \lesssim y_0 \langle y \rangle^{-1}. \quad (2.94)$$

Similarly, by (2.46), (2.51) we have

$$|e^{Q_r}, y \partial_y (e^{Q_r})| \lesssim \langle y \rangle^{-2}, \quad |u_r| \lesssim \langle y \rangle^{1/2} \quad (2.95)$$

These pointwise estimates allow us to conclude

$$\begin{aligned} |F_1| &\lesssim y_0^2 \langle y \rangle^{-1}, & |F_2| &\lesssim y_0^2 \langle y \rangle^{-1}, & |F_3| &\lesssim y_0^2 \langle y \rangle^{-3/2}, \\ |G_1| &\lesssim \langle y \rangle^{-1/2}, & |G_2| &\lesssim y_0^2 \langle y \rangle^{-1}, & |G_3| &\lesssim y_0^2 \langle y \rangle^{-1/2}. \end{aligned}$$

These bounds, along with (2.94), show that $|F|, |G| \lesssim \langle y \rangle^{1/2}$ as desired. We can similarly see that $|\partial_y F| \lesssim \langle y \rangle^{-1/2}$, finishing the proof of (a).

To show differentiability with respect to r , one formally computes

$$\begin{aligned} \frac{\partial F_{r,1}}{\partial r}|_{r_0} &= \lambda_0^2 r_0 (2 + y \partial_y) (w^2 (2 + r_0 \frac{\partial Q_r}{\partial r}|_{r_0}) \int_0^1 (1-t) e^{Q_{r_0} + t \lambda_0^2 w} dt), \\ \frac{\partial F_{r,2}}{\partial r}|_{r_0} &= \lambda_0^2 r_0 \operatorname{div}(uw (2 + r_0 \frac{\partial Q_r}{\partial r}|_{r_0}) \int_0^1 e^{Q_r + t \lambda_0^2 w}), \\ \frac{\partial F_{r,3}}{\partial r}|_{r_0} &= \lambda_0^2 r_0 \operatorname{div}(((2 + r_0 \frac{\partial Q_r}{\partial r}|_{r_0}) u_{r_0} + r_0 \frac{\partial u_r}{\partial r}|_{r_0}) w^2 \int_0^1 (1-t) e^{Q_r + t \lambda_0^2 w} dt), \end{aligned}$$

$$\begin{aligned}
\frac{\partial G_{r,1}}{\partial r}|_{r_0} &= r_0 \operatorname{div}(2(y + u_{r_0})\partial_y u_{r_0} + r_0 \frac{\partial u_r}{\partial r}|_{r_0} \partial_y u_{r_0} + r_0(y + u_{r_0})\partial_y \frac{\partial u_r}{\partial r}|_{r_0}), \\
\frac{\partial G_{r,2}}{\partial r}|_{r_0} &= 2\lambda_0^2 r_0 w^2 (2 + r_0 \frac{\partial Q_r}{\partial r}|_{r_0}) \int_0^1 (1-t) e^{Q_r + t\lambda_0^2 w} dt, \\
\frac{\partial G_{r,3}}{\partial r}|_{r_0} &= \lambda_0^2 r \operatorname{div}(2(u\partial_y u_{r_0} + (y + u_{r_0})\partial_y u + \lambda_0^2 u\partial_y u) + r_0 u \partial_y \frac{\partial u_r}{\partial r}|_{r_0} + r_0 \frac{\partial u_r}{\partial r}|_{r_0} \partial_y u_{r_0}).
\end{aligned}$$

From these formulas it is then straightforward to deduce (2.61).

To compute the derivatives of F_r, G_r with respect to w and u , it is convenient to express

$$\begin{aligned}
F_r(w, u) &= \frac{r^2}{\lambda_0^2} \left((2 + y\partial_y)(e^{Q_r + \lambda_0^2 w}) + \operatorname{div}(e^{Q_r + \lambda_0^2 w}(u_r + \lambda_0^2 u)) \right) - L_r w - J_r u, \\
G_r(w, u) &= r^2 \operatorname{div}((y + u_r + \lambda_0^2 u)\partial_y(u_r + \lambda_0^2 u)) + \frac{r^2}{\lambda_0^2} \Delta(Q_r + \lambda_0^2 w) + \frac{r^2}{\lambda_0^2} 2e^{Q_r + \lambda_0^2 w} + H_r w.
\end{aligned}$$

from which one computes the formal derivatives

$$\begin{aligned}
\frac{\partial F_r}{\partial w}|_{w_0, u_0} w &= \lambda_0^2 r^2 (2 + y\partial_y) \left(w_0 \left(\int_0^1 e^{Q_r + t\lambda_0^2 w_0} dt \right) w \right) \\
&\quad + \lambda_0^2 r^2 \operatorname{div} \left(\left(\int_0^1 e^{Q_r + t\lambda_0^2 w_0} dt \right) w_0 u_r w + e^{Q_r + \lambda_0^2 w_0} u_0 w \right), \\
\frac{\partial F_r}{\partial u}|_{w_0, u_0} u &= \lambda_0^2 r^2 \operatorname{div} \left(\left(\int_0^1 e^{Q_r + t\lambda_0^2 w} dt \right) w_0 u_0 \right), \\
\frac{\partial G_r}{\partial w}|_{w_0, u_0} w &= \lambda_0^2 r^2 2 \left(\int_0^1 e^{Q_r + t\lambda_0^2 w} dt \right) w_0 w, \\
\frac{\partial G_r}{\partial u}|_{w_0, u_0} u &= \lambda_0^2 r^2 \operatorname{div}((y + u_r + \lambda_0^2 u_0)\partial_y u + u\partial_y(u_r + \lambda_0^2 u_0)).
\end{aligned}$$

We note that each term contains a factor of λ_0^2 , and if we use (2.94) and (2.95) to estimate the rest of the terms, we see that if $|w|, |w_0| \lesssim \langle y \rangle^{3/2}$, $|u|, |u_0| \lesssim \langle y \rangle^{5/2}$,

$$\begin{aligned}
\left| \frac{\partial F_r}{\partial w}|_{w_0, u_0} w \right| &\lesssim y_0^2 \langle y \rangle^{-1}, & \left| \frac{\partial F_r}{\partial u}|_{w_0, u_0} u \right| &\lesssim y_0^2 \langle y \rangle^{-1} \\
\left| \frac{\partial G_r}{\partial w}|_{w_0, u_0} w \right| &\lesssim y_0^2 \langle y \rangle^{-1}, & \left| \frac{\partial G_r}{\partial u}|_{w_0, u_0} u \right| &\lesssim y_0^2 \langle y \rangle^{-1/2}.
\end{aligned}$$

which is the boundedness described in (2.62).

With these computations done, straightforward difference estimates allow one to prove that the formal derivatives so far computed are indeed first order approximations, and that one has the Lipschitz continuity statements (2.63) and (2.64). $\square$

2.6.4 Proof of Lemma 2.3.5

Proof. This is a standard fixed point argument using the integral formulation (2.59). The proof is structured similarly to the proof of Lemma 2.2.6. We let $(u_0, w_0) = (0, 0)$ and we shall inductively define, for $n \geq 1$,

$$(W_n, U_n) = (G_r(w_{n-1}, u_{n-1}), F_r(w_{n-1}, u_{n-1})), \quad (w_n, u_n) = (S_r(W_n), T_r(U_n), K_r(W_n))$$

We claim that this sequence will converge in $Z_{y_1} \times Y_{y_1}$. Indeed, we see that $\|w_1\|_{Z_{y_1}}, \|u_1\|_{Y_{y_1}} \leq C_1 C_2(1)$. We let $\tilde{C} = 2 \sup(C_1 C_2(1), 1)$. We then require that y_0 be sufficiently small so that $4C_1 C_2(\tilde{C})y_0^2 \leq \frac{1}{10}$. Assuming that for $k \geq 1$,

$$\|w_k\|_{Z_{y_1}}, \|w_{k-1}\|_{Z_{y_1}}, \|u_k\|_{Y_{y_1}}, \|u_{k-1}\|_{Y_{y_1}} \leq \tilde{C},$$

we obtain

$$\begin{aligned} \|w_{k+1} - w_k\|_{Z_{y_1}} + \|u_{k+1} - u_k\|_{Y_{y_1}} &\leq C_1 \left(\|W_{k+1} - W_k\|_{G_{y_1}} + \|U_{k+1} - U_k\|_{F_{y_1}} \right) \\ &\leq C_1 C_2(\tilde{C})y_0^2 (\|w_k - w_{k-1}\|_{Z_{y_1}} + \|u_k - u_{k-1}\|_{Y_{y_1}}) \\ &\leq \frac{1}{10} (\|w_k - w_{k-1}\|_{Z_{y_1}} + \|u_k - u_{k-1}\|_{Y_{y_1}}). \end{aligned}$$

where to get from the first line to the second line we use (2.62) and the fundamental theorem of calculus. Thus, as $w_0 = u_0 = 0$ and $\|w_1\|_{Z_{y_1}}, \|u_1\|_{Y_{y_1}} \leq \frac{\tilde{C}}{2}$, we obtain inductively that

$$\|u_n\|_{Y_{y_1}}, \|w_n\|_{Z_{y_1}} \leq \tilde{C} \left(\frac{1}{2} + \frac{1}{10} + \cdots + \frac{1}{10^n} \right) \leq \tilde{C}.$$

$$\|w_n - w_{n-1}\|_{Y_{y_1}} + \|u_n - u_{n-1}\|_{Z_{y_1}} \leq \frac{1}{10^{n-1}} \left(\|u_1\|_{Y_{y_1}} + \|w_1\|_{Z_{y_1}} \right)$$

Thus, if we let $C_3 = 100\tilde{C}$ the sequence converges to a fixed point with norm $\|w\|_{Z_{y_1}}, \|u\|_{Y_{y_1}} \leq \frac{C_3}{100}$. We note that if w', u' is a fixed point with $\|w'\|_{Z_{y_1}}, \|u'\|_{Y_{y_1}} \leq C_3$, then by the Lipschitz bounds we have

$$\|w - w'\|_{Z_{y_1}} + \|u - u'\|_{Y_{y_0}} \leq C_2(C_3)C_1 y_0^2 (\|w - w'\|_{Z_{y_1}} + \|u - u'\|_{Y_{y_0}})$$

and hence by imposing $y_0^2 C_2(C_3)C_1 \leq \frac{1}{2}$ we obtain the desired uniqueness concluding the proof of part (a).

To prove part (b), we consider $r_0, r_1 \in (\frac{1}{2}, 2)$. We let

$$(w_0, u_0) = (w[\lambda_0; r_0], u[\lambda_0; r_0]), \quad (w_1, u_1) = (w[\lambda_0; r_1], u[\lambda_0; r_1]).$$

As they are solutions to (2.59), we have

$$\begin{aligned} w_1 - w_0 &= S_1(G_{r_1}(w_1, u_1) - G_{r_0}(w_1, u_1)) \\ &\quad + (S_1 - S_0)(G_{r_0}(w_1, u_1)) + S_0(G_{r_0}(w_1, u_1) - G_{r_0}(w_0, u_0)). \end{aligned}$$

□

Expanding $u_1 - u_0$ similarly, and using the differentiability with respect to r , we obtain

$$\begin{aligned} \|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}} &\leq C_1 C_2(\tilde{C})|r_0 - r_1| \\ &\quad + 2C_1 C_2(\tilde{C})y_0^2 \left(\|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}} \right) \end{aligned}$$

$$\leq C_1 C_2(\tilde{C})|r_0 - r_1| + \frac{1}{2} \left(\|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}} \right)$$

and hence by absorbing the second term into the left-hand side, we conclude

$$\|w_1 - w_0\|_{Z_{y_1}} + \|u_1 - u_0\|_{Y_{y_1}} \leq 2C_1 C_2(\tilde{C})|r_0 - r_1|. \quad (2.96)$$

Next, for each $r \in (\frac{1}{2}, 2)$, we consider the linearization of (2.59), namely

$$\begin{cases} u_{lin} = \frac{\partial T_r}{\partial r} F_r + T_r \frac{\partial F_r}{\partial r} + \frac{\partial K_r}{\partial r} G_r + K_r \frac{\partial G_r}{\partial r} + \left(T_r \left(\frac{\partial F_r}{\partial w}, \frac{\partial F_r}{\partial u} \right) + K_r \left(\frac{\partial G_r}{\partial w}, \frac{\partial G_r}{\partial u} \right) \right) \cdot (w_{lin}, u_{lin}) \\ w_{lin} = \frac{\partial S_r}{\partial r} G_r + S_r \frac{\partial G_r}{\partial r} + S_r \left(\frac{\partial G_r}{\partial w}, \frac{\partial G_r}{\partial u} \right) \cdot (w_{lin}, u_{lin}) \end{cases} \quad (2.97)$$

with T, S, K, F, G and their derivatives evaluated at $(r, w[\lambda_0; r], u[\lambda_0; r])$. By the Lemmas 2.3.3 and 2.3.4,

$$\begin{aligned} \left\| \frac{\partial S_r}{\partial r} G_r + S_r \frac{\partial G_r}{\partial r} \right\|_{Z_{y_1}} &\leq 5C_1 C_2(C_3), \\ \left\| S_r \frac{\partial G_r}{\partial w} \right\|_{Z_{y_1} \rightarrow Z_{y_1}} + \left\| S_r \frac{\partial G_r}{\partial u} \right\|_{Y_{y_1} \rightarrow Z_{y_1}} &\leq 5y_0^2 C_1 C_2(C_3). \end{aligned}$$

There are similar estimates for the first equation. Thus, as we may assume $10y_0^2 C_1 C_2(C_3) \leq \frac{1}{2}$, we may iteratively solve (2.97) resulting in a solution $(w_{lin}[\lambda_0; r], u_{lin}[\lambda_0; r])$ satisfying

$$\|w_{lin}\|_{Z_{y_1}} + \|u_{lin}\|_{Y_{y_1}} \leq 20C_1 C_2(C_3).$$

We next define (recall $w_i, u_i = w[\lambda_0; r_i], u[\lambda_0; r_i]$)

$$(w_{diff}, u_{diff}) = (w_1, u_1) - (w_0, u_0) - (r_1 - r_0)(w_{lin}[\lambda_0; r_0], u_{lin}[\lambda_0; r_0]).$$

We note that

$$\begin{aligned} w_{diff} &= S_{r_1} G_{r_1}(w_1, u_1) - S_{r_0} G_{r_0}(w_0, u_0) - (r_1 - r_0) \left(\frac{\partial S_r}{\partial r} G_r + S_r \frac{\partial G_r}{\partial r} \right) \\ &\quad - S_{r_0} \frac{\partial G_{r_0}}{\partial w} (w_1 - w_0) - \frac{\partial G_{r_0}}{\partial u} (u_1 - u_0) + S_{r_0} \frac{\partial G_{r_0}}{\partial w} w_{diff} + S_{r_0} \frac{\partial G_{r_0}}{\partial u} u_{diff} \\ &= (S_{r_1} - S_{r_0})(G_{r_1} - G_{r_0})(w_1, u_1) + (S_{r_1} - S_{r_0})(G_{r_0}(w_1, u_1) - G_{r_0}(w_0, u_0)) \\ &\quad + \left(S_{r_1} - S_{r_0} - (r_1 - r_0) \frac{\partial S_r}{\partial r} \Big|_{r_0} \right) G_{r_0}(w_0, u_0) \\ &\quad + S_{r_0} ((G_{r_1} - G_{r_0})(w_1, u_1) - (G_{r_1} - G_{r_0})(w_0, u_0)) \\ &\quad + S_{r_0} (G_{r_1} - G_{r_0} - (r_1 - r_0) \frac{\partial G_r}{\partial r} \Big|_{r_0})(w_0, u_0) \\ &\quad + S_{r_0} \left(G_{r_0}(w_1, u_1) - G_{r_0}(w_0, u_0) - \left(\frac{\partial G_{r_0}}{\partial w} (w_1 - w_0) + \frac{\partial G_{r_0}}{\partial u} (u_1 - u_0) \right) \right) \\ &\quad + S_{r_0} \left(\frac{\partial G_{r_0}}{\partial w}, \frac{\partial G_{r_0}}{\partial u} \right) (w_{diff}, u_{diff}) \end{aligned}$$

We absorb the last term into the left-hand side, and use (2.60), (2.63), (2.64) and (2.96) to obtain

$$\|w_{diff}\|_{Z_{y_1}} \leq 100(C_1 C_2(\tilde{C}))^2 |r_1 - r_0|^2.$$

One can estimate u_{diff} similarly, and taking $r_1 \rightarrow r_0$ we see that $(w[\lambda_0; r], u[\lambda_0; r])$ is differentiable with respect to r at r_0 with derivative $(w_{lin}[\lambda_0; r_0], u_{lin}[\lambda_0; r_0])$.

Finally, to show continuity of $(w_{lin}[\lambda_0; r], u_{lin}[\lambda_0; r])$ with respect to r , we introduce the notation $w_{lin,i} = w_{lin}[\lambda_0; r_i]$ and compute

$$\begin{aligned} & w_{lin,1} - w_{lin,0} \\ &= \frac{\partial S_r}{\partial r}|_{r_1} G_{r_1}(w_1, u_1) + S_{r_1} \frac{\partial G_r}{\partial r}|_{r_1}(w_1, u_1) - \frac{\partial S_r}{\partial r}|_{r_0} G_{r_0}(w_0, u_0) - S_{r_0} \frac{\partial G_r}{\partial r}|_{r_0}(w_0, u_0) \\ &+ S_{r_1} \left(\frac{\partial G_{r_1}}{\partial w}, \frac{\partial G_{r_1}}{\partial u} \right)(w_{lin,1}, u_{lin,1}) - S_{r_0} \left(\frac{\partial G_{r_0}}{\partial w}, \frac{\partial G_{r_0}}{\partial u} \right)(w_{lin,0}, u_{lin,0}) \\ &= \left(\frac{\partial S_r}{\partial r}|_{r_1} - \frac{\partial S_r}{\partial r}|_{r_0} \right) G_{r_1}(w_1, u_1) \\ &+ \frac{\partial S_r}{\partial r}|_{r_0} (G_{r_1} - G_{r_0})(w_1, u_1) + \frac{\partial S_r}{\partial r}|_{r_0} (G_{r_0}(w_1, u_1) - G_{r_0}(w_0, u_0)) \\ &+ (S_{r_1} - S_{r_0}) \frac{\partial G}{\partial r}|_{r_1}(w_1, u_1) + S_{r_0} \left(\frac{\partial G_r}{\partial r}|_{r_1} - \frac{\partial G_r}{\partial r}|_{r_0} \right)(w_1, u_1) + S_{r_0} \frac{\partial G_r}{\partial r}|_{r_0}(w_1 - w_0, u_1 - u_0) \\ &+ (S_{r_1} - S_{r_0}) \frac{\partial G_{r_1}}{\partial w} w_{lin,1} + S_{r_0} \left(\frac{\partial G_{r_1}}{\partial w} - \frac{\partial G_{r_0}}{\partial w} \right) w_{lin,1} + S_{r_0} \frac{\partial G_{r_0}}{\partial w} (w_{lin,1} - w_{lin,0}) \\ &+ (S_{r_1} - S_{r_0}) \frac{\partial G_{r_1}}{\partial u} u_{lin,1} + S_{r_0} \left(\frac{\partial G_{r_1}}{\partial u} - \frac{\partial G_{r_0}}{\partial u} \right) u_{lin,1} + S_{r_0} \frac{\partial G_{r_0}}{\partial u} (u_{lin,1} - u_{lin,0}). \end{aligned}$$

There is a similar decomposition for $u_{lin,1} - u_{lin,0}$ we may then absorb the terms containing $w_{lin,1} - w_{lin,0}$ and $u_{lin,1} - u_{lin,0}$ to the left side, and bound the rest using (2.60), (2.63), (2.64) and (2.96), allowing one to show

$$\|w_{lin,1} - w_{lin,0}\|_{Z_{y_1}} + \|u_{lin,1} - u_{lin,0}\|_{Z_{y_1}} \leq 1000 C_1 C_2(\tilde{C}) |r_1 - r_0|$$

and hence continuity.

Chapter 3

Hunter-type Implosion Profiles for Energy-supercritical Polytropic Equations of State

3.1 Introduction

The (gravitational) Euler-Poisson system of partial differential equations models the evolution of a compressible gas. We will work under the assumptions that the gas is supported on the Euclidean space $\mathbb{R}_x^3$ and that there is no vacuum. We will let I_t denote a time interval. We shall also assume that the gas is isentropic, and described completely by its density, $\rho: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}_{>0}$, and velocity vector field, $u: I_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$. It is also assumed that the gas self-gravitates according to Newtonian gravity, and we shall choose units so that the gravitational constant is 1. We shall also assume that equation of state is polytropic with polytropic index $\gamma \geq 1$. We shall choose units so that the pressure is given by ρ^γ . The laws of local conservation of mass and momentum are equivalent (for smooth solutions) to the following closed equations of motion

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + u \cdot \operatorname{grad} u + \frac{\operatorname{grad}(\rho^\gamma)}{\rho} + 4\pi \operatorname{grad}(\Delta^{-1}(\rho)) = 0 \end{cases} \quad (3.1)$$

From now on we shall refer to this system as the Euler-Poisson system.

We now motivate the critical polytropic indices. From the first equation it is easy to see that the total mass

$$M(\rho, u) := \int_{\mathbb{R}_x^3} \rho dx$$

is conserved. One can also check that the physical energy of this system, given by

$$E(\rho, u) := \begin{cases} \int_{\mathbb{R}_x^3} \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma-1} \rho^\gamma - 2\pi |\operatorname{grad} \Delta^{-1} \rho|^2 dx & \gamma > 1 \\ \int_{\mathbb{R}_x^3} \frac{1}{2} \rho |u|^2 + \rho \log(\rho) - 2\pi |\operatorname{grad} \Delta^{-1} \rho|^2 dx & \gamma = 1 \end{cases},$$

is also conserved.

In addition, the set of solutions to the equation is invariant under the scaling symmetry ($\lambda > 0$)

$$(\rho(t, x), u(t, x)) \mapsto \left(\frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right), \lambda^{1-\gamma}u\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right)\right). \quad (3.2)$$

We note that under this scaling

$$\begin{aligned} M\left(\frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right), \lambda^{1-\gamma}u\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right)\right) &= \lambda^{4-3\gamma}M(\rho(t, x), u(t, x)), \\ E\left(\frac{1}{\lambda^2}\rho\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right), \lambda^{1-\gamma}u\left(\frac{t}{\lambda}, \frac{x}{\lambda^{2-\gamma}}\right)\right) &= \lambda^{6-5\gamma}E(\rho(t, x), u(t, x)). \end{aligned}$$

In particular, we see that if $\gamma = \frac{4}{3}$, then the mass is invariant under this scaling. If $1 \leq \gamma < \frac{4}{3}$, then the equation is mass-supercritical, in the sense that we do not expect a norm at the same scaling as mass to control small-scale behavior. Similarly, $\gamma = \frac{6}{5}$ is the energy-critical exponent, and $1 \leq \gamma < \frac{6}{5}$ is the energy-supercritical regime.

We also shall assume that the motion is radially symmetric and let $r = |x|$. We will use the notations

$$\operatorname{div}_r := \partial_r + \frac{2}{r}, \quad \operatorname{div}_r^{-1} f(r) := \frac{1}{r^2} \int_0^r (r')^2 f(r') dr'.$$

Thus, in radial symmetry, we obtain the following system of 2 partial differential equations in 2 unknowns:

$$\begin{cases} \partial_t \rho + \operatorname{div}_r(\rho u) = 0 \\ \partial_t u + u \partial_r u + \gamma \rho^{\gamma-2} \partial_r \rho + 4\pi \operatorname{div}_r^{-1}(\rho) = 0 \end{cases} \quad (3.3)$$

We shall also restrict attention to self-similar evolutions, i.e. solutions to (3.3) which are invariant under the symmetry (3.2). This is done by considering the ansatz ($I_t = (-\infty, 0)$)

$$\rho = \frac{1}{(-t)^2} \tilde{\rho}\left(\frac{r}{(-t)^{2-\gamma}}\right), \quad u = (-t)^{1-\gamma} \tilde{u}\left(\frac{r}{(-t)^{2-\gamma}}\right).$$

We introduce the self-similar coordinate

$$y = \frac{r}{(-t)^{2-\gamma}}.$$

With these conventions, (3.3) takes the form

$$\begin{cases} 2\tilde{\rho} + (2-\gamma)y\partial_y \tilde{\rho} + \operatorname{div}_y(\tilde{\rho}\tilde{u}) = 0 \\ (\gamma-1)\tilde{u} + (2-\gamma)y\partial_y \tilde{u} + \tilde{u}\partial_y \tilde{u} + \gamma\tilde{\rho}^{\gamma-2}\partial_y \tilde{\rho} + 4\pi \operatorname{div}_y^{-1}(\tilde{\rho}) = 0. \end{cases} \quad (3.4)$$

Fortunately, for $\gamma \neq \frac{4}{3}$, this non-local system admits a reduction to a (local) system of ODEs. In particular, the first equation can be rewritten as

$$\tilde{\rho} = \frac{1}{4-3\gamma} \operatorname{div}_y(\tilde{\rho}(\tilde{u} + (2-\gamma)y)).$$

Substituting this into the non-local term of the second equation yields the main system of ODEs we study

$$\begin{pmatrix} \tilde{u} + (2-\gamma)y \\ \gamma\tilde{\rho}^{\gamma-2} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u} + (2-\gamma)y \end{pmatrix} \begin{pmatrix} \tilde{\rho}' \\ \tilde{u}' \end{pmatrix} + \begin{pmatrix} \frac{2\tilde{\rho}(\tilde{u}+y)}{y} \\ (\gamma-1)\tilde{u} + \frac{4\pi}{4-3\gamma}\tilde{\rho}(\tilde{u} + (2-\gamma)y) \end{pmatrix} = 0. \quad (3.5)$$

As a comment, requiring smoothness at the origin imposes the boundary conditions

$$\tilde{\rho}'(0) = 0, \quad \tilde{u}(0) = 0, \quad \tilde{u}'(0) = -\frac{2}{3}. \quad (3.6)$$

This system has two explicit solutions. We follow the conventions of [52], and use the name “Friedman solution” to refer to

$$(\tilde{\rho}_F, \tilde{u}_F) = \left(\frac{1}{6\pi}, -\frac{2}{3}y\right).$$

Similarly, we use the label “far-field solution” to refer to

$$(\tilde{\rho}_f, \tilde{u}_f) = \left(ky^{-\frac{2}{2-\gamma}}, 0\right), \quad (3.7)$$

where the constant k is defined as

$$k := \left(\frac{\gamma(4-3\gamma)}{2\pi(2-\gamma)^2}\right)^{\frac{1}{2-\gamma}}. \quad (3.8)$$

Neither of these models implosion at a point. The Friedman solution models mass-density blowing up at all locations simultaneously. The mass-density of the far-field solution is singular at the origin and does not evolve. In this work we study smooth self-similar solutions to (3.5) with the mass density radially decaying, in pursuit of the larger goal of gaining a more detailed understanding of the local structure of the type of singularities which would develop from regular and localized initial data.

3.1.1 Review of literature and statement of the theorem

We now discuss the existing literature on non-explicit smooth solutions to (3.5). Some sources are omitted, and further discussions can be found, for instance, in [52], [54] and [100].

Historically, the isothermal case, $\gamma = 1$, has received the most attention. In this case, the self-similar equations were first derived by Larson ([70]) and Penston ([99]) independently. Moreover, through the use of numerical integration they identified a smooth solution, known now as the Larson-Penston solution. Hunter ([59]) numerically found an additional discrete family of solutions. Numerical mode stability analyses of the Larson-Penston and Hunter solutions were carried out in [57] and [73]. These numerics support the hypothesis that the Larson-Penston solution is a stable description of gravitational collapse and the i th Hunter solution ($i \in \mathbb{Z}_{\geq 1}$) is codimension i stable. In particular, the first Hunter solution is believed to play a role in critical collapse theory (see, for instance, [58] for a discussion).

The first rigorous mathematical result on the isothermal self-similar system is [52], in which the authors proved existence of a solution whose properties match the Larson-Penston solution. We comment that this result was preceded by [51], a study of a dust-like, non-self-similar, regime of gravitational collapse for the Euler-Poisson system. In [100], for each i *sufficiently large* a solution whose properties matches the i th Hunter solution was constructed. We comment that constructing the i th Hunter solutions for small i (in particular $i = 1$) remains an open problem.

We now discuss the polytropic, mass-supercritical regime. The first numerical work in this area was by Yahil ([113]) who numerically constructed analogs of the Larson-Penston solution for $\frac{6}{5} < \gamma < \frac{4}{3}$. This work was in part motivated by the observation that for the isothermal case, the Larson-Penston solution has infinite mass and energy, but for $\frac{6}{5} < \gamma < \frac{4}{3}$ the analog would have finite energy (but still infinite mass).

These Larson-Penston-type solutions studied by Yahil were constructed in the full mass-supercritical range ($1 < \gamma < \frac{4}{3}$) in [54]. A difficulty in the construction of self-similar solutions for $\gamma > 1$ is that the algebra governing the local behavior of the solution near sonic points complicates significantly. The authors of [54] made the novel observation that for $\gamma > 1$ the algebraic Larson-Penston and Hunter sonic-point conditions merge into a single solution branch, termed the Larson-Penston-Hunter-type Taylor expansion. In this work, we use this insight to extend the result of [100] to the energy-supercritical range $1 < \gamma < \frac{6}{5}$. In particular, we establish the following theorem (where $\langle y \rangle = \sqrt{1 + y^2}$ denotes the Japanese bracket).

Theorem 3.1.1. *For some integer $N \gg 1$ there exists real analytic solutions $\{(\tilde{\rho}_i, \tilde{u}_i)\}_{i=N}^{\infty}$ to (3.4) on $[0, \infty)$ subject to the boundary conditions (3.6) and the requirement $\tilde{\rho}_i > 0$. In addition, the graph of $\tilde{\rho}_i(y)$ intersects that of the far-field solution $\tilde{\rho}_f(y)$, (3.7), exactly $i + 1$ times. For these solutions, the sonic point condition, $(\tilde{u} + (2 - \gamma)y)^2 - \gamma\tilde{\rho}^{\gamma-1} = 0$, is satisfied at exactly one point $y = y_* \in (0, 2y_f)$, where y_f is the sonic point for the far-field solution, (3.14). At y_* , the solution $\tilde{\rho}, \tilde{u}$ admits the Larson-Penston-Hunter-type Taylor expansion (as defined in [54]). Finally, these solutions satisfy the global pointwise bounds $\tilde{\rho}_i \lesssim_i \langle y \rangle^{-\frac{2}{2-\gamma}}, \tilde{u}_i \lesssim \langle y \rangle^{-\frac{\gamma-1}{2-\gamma}}$.*

This theorem is proved in Subsection 3.4.2. In contrast to the isothermal case, as far as the author is aware, there are no numerical constructions of these Hunter-like solutions in the existing literature for $\gamma > 1$. It is not hard to generate the first few Hunter-type solutions numerically by straightforwardly generalizing the approach of [59] to $1 < \gamma < \frac{6}{5}$.

We also mention that in contrast to [54], we are not able to rigorously construct solutions for $\frac{6}{5} \leq \gamma < \frac{4}{3}$. It is unclear whether Hunter-type solutions would exist in this range or not. This point is discussed further in Subsection 3.1.3 via analogy with a scalar wave equation.

We also comment that the existence proof in [54] used rigorous interval arithmetic to control the signs of certain quantities. One feature of the approach we use is that due to its more perturbative nature, we do not need to use computer-assisted techniques.

We mention two other classes of self-similar solutions for the Euler-Poisson system in the literature. Firstly, if one considers dynamic entropy, then a new regime of solutions has been found in [1]. Secondly, for the mass-critical polytropic index $\gamma = \frac{4}{3}$, there is a family of self-similar solutions to (3.1) found by Goldreich and Weber ([48]). These solutions differ in character from the type of solutions mentioned so far in that they are semi-explicit and compactly supported. We refer interested readers to [56] for an introduction to the mathematical literature regarding these solutions and their stability.

We also comment that one source of motivation for studying these types of blow-ups for the Euler-Poisson system is the cosmic censorship conjectures of general relativity originally formulated by Penrose ([98, 97]). Indeed, for the Einstein-Euler system (describing a relativistic self-gravitating gas) a relativistic Larson-Penston solution was numerically constructed by Ori-Piran for an isothermal equation of state and sufficiently small values of the

(constant) speed of sound ([92, 94, 95]). These authors argued that the solution could be suitably truncated to exhibit a naked singularity. A rigorous construction of the Ori-Piran solution and a dynamically forming naked singularity was established in [53], which also contains further references on related literature. It is currently unclear if our methods would extend to the Einstein-Euler system to prove existence of relativistic Hunter solutions (or even, as far as the author is aware, if such solutions exist in the first place). For additional discussions on self-similarity, gravitational collapse and related models, we refer readers to the (living) review [50].

Another related problem is constructing imploding solutions to the compressible Euler equations in the absence of a gravitational force. In this case, the system admits an additional scaling symmetry and the self-similar equations can be expressed as a system of *autonomous* ODEs. Existence of (not necessarily smooth) solutions was first established by Guderley and Sedov ([49], [102]). Smooth self-similar solutions were first constructed by Merle-Raphaël-Rodnianski-Szeftel ([78]). The (finite-codimension) stability of these smooth profiles allowed the authors to construct blow-ups for the compressible Navier-Stokes and defocusing Schrödinger equations ([79, 77]). Their results on singularity formation have been generalized to additional regimes. These include, in particular, implosions with mass density bounded from below ([11]), implosions outside of radial symmetry ([16, 17]) and also implosions where the vorticity simultaneously blows up ([23, 22]). Moreover, the instability of these blow-ups have been studied numerically in [5]. We emphasize that in the non-gravitational problem, the additional scaling symmetry leads to a choice in scaling parameter (labeled r in the above-mentioned works). There is no analog of this choice for the isentropic Euler-Poisson system, and so one must use non-autonomous methods.

We conclude this subsection by mentioning that, even in the isothermal case, the stability of the Larson-Penston and Hunter solutions has not been rigorously established. One would like to show that, in analogy with the numerics, the Larson-Penston solution is stable and that the i th Hunter solution is codimension- i unstable. One aspect of this problem that is difficult is that the linearized problem is non-self-adjoint and non-explicit. A speculative comparison of this problem with the related problem for a scalar wave equation is discussed in Subsection 3.1.3.

3.1.2 Sonic point boundary conditions and local solvability

We see that the ODE system (3.5) is singular when

$$(\tilde{u} + (2 - \gamma)y)^2 - \gamma\tilde{\rho}^{\gamma-1} = 0.$$

Following the literature, we refer to a point y where this occurs as a sonic point. We comment that in terms of the physical coordinates (t, x) , a sonic point corresponds to an integral curve of the scaling vector-field $t\partial_t + (2 - \gamma)\sum_{i=1}^3 x^i\partial_{x^i}$ being null with respect to the acoustical Lorentzian metric, $-c_s^2 dt^2 + \sum_{i=1}^3 (dx^i - u^i dt)^2$, where $c_s = \gamma^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}}$ is the speed of sound. When we transition to the y coordinate, we quotient along this curve introducing a singularity into the equations.

We now discuss the local theory of smooth solutions to (3.5) near a singular point. In particular, we shall provide a derivation of the Larson-Penston-Hunter-type expansion. In

contrast to previous, more direct, treatments, such as [54] and [100], we will treat (3.5) as a special case of a more general class of singular quasilinear ODEs. This treatment is the most significant development from [100], and we emphasize it as we believe the methods could be useful for studying self-similar solutions to other quasilinear systems.

The class of ODEs we consider will be those that can be put into the following normal form. Note that we abuse notation and the (u_1, u_2) below do not refer to velocity.

Definition 3.1.2. *We say that a singular quasilinear ODE system for $(u_1(t), u_2(t))$ is in normal form if it is of the form*

$$\begin{cases} A(t, u_1, u_2) \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + B(t, u_1, u_2) = 0 \\ u_1(0) = u_2(0) = 0 \end{cases}$$

for smooth functions $A(z_0, z_1, z_2)$, $B(z_0, z_1, z_2)$ valued in $\mathbb{R}^{2 \times 2}$, $\mathbb{R}^2$ respectively with

$$A(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For such a system, we define four characteristic parameters to be

$$a := \frac{\partial A_{22}}{\partial z_0}(0), \quad b := \frac{\partial A_{22}}{\partial z_2}(0), \quad c := \frac{\partial B_2}{\partial z_0}(0), \quad d = \frac{\partial B_2}{\partial z_2}(0).$$

The author was unable to find a reference for the theory of smooth solutions to ODEs of this form on a neighborhood of $z = 0$ (in particular in the quasilinear case $b \neq 0$), so the necessary lemmas are proved in the Appendix using elementary methods.

To motivate these lemmas, we now suppose that $(u_1(t), u_2(t))$ is a C^1 solution to an equation of the form described by in Definition 3.1.2. From evaluating the first line of the matrix equation at $t = 0$, we see that $u_1'(0) = 0$. Now suppose that $u_2'(0) = U$. By expanding the second line to $o(t)$, we see that U must satisfy the quadratic equation

$$(a + bU)U + c + dU = 0. \quad (3.9)$$

Thus, we see that, depending on the parameters a, b, c, d , there will generically be at most two possible choices for U . Indeed, for our problem one of these choices will correspond to the Larson-Penston-Hunter Taylor expansion.

Going further, supposing now that (u_1, u_2) is $C^{(k)}$ for $k \geq 2$ and $u_2'(0) = U$, if we expand the first equation to order $o(t^{k-1})$ and the second equation to $o(t^k)$, we obtain an equation schematically of the form (suppressing the dependence on the coefficients A, B)

$$\begin{cases} u_1^{(k)}(0) = F_{1,k}(u_1^{\leq(k-1)}(0), u_2^{\leq(k-1)}(0)) \\ (a + dU + (a + bU)k) u_2^{(k)}(0) = F_{2,k}(u_1^{\leq(k)}(0), u_2^{\leq(k-1)}(0)) \end{cases}.$$

In particular, we can solve the first equation for $u_1^{(k)}(0)$ and substitute it into the second equation. The second equation is a linear equation for $u_2^{(k)}(0)$ and it is solvable if

$$a + dU + (a + bU)k \neq 0. \quad (3.10)$$

Thus, assuming this non-degeneracy condition on the characteristic parameters, the higher derivatives are uniquely determined. Heuristically one should be able to solve the ODE exactly if one has a solution to the quadratic (3.9) and if for $k \geq 2$ one has the conditions (3.10). Justifying these heuristics rigorously is the content of the following lemmas. The proofs of these lemmas (which are elementary in terms of tools) are provided in the Appendix. As a comment, the statements of these lemmas do not give quantitative bounds on the intervals of existence, although in principle a closer examination of the proof would yield such bounds, which would depend on the coefficients. The first lemma deals with solutions of regularity C^1 .

Lemma 3.1.3. *Consider a singular ODE system for $u(t) = (u_1(t), u_2(t))$ as in Definition 3.1.2 with characteristic indices a, b, c, d . Suppose that the quadratic equation (3.9) has a root U . Suppose that $a + bU \neq 0$ and define $\kappa = \frac{d+bU}{a+bU}$. Suppose that $\kappa > -1$. Then, for T sufficiently small the ODE system subject to the condition $u'_2(0) = U$ admits a unique C^1 solution $u(t)$ defined on $(-T, T)$.*

For the proof, see Subsection 3.7.1. The second lemma deals with analytic regularity.

Lemma 3.1.4. *Consider a singular ODE system for $u(t) = (u_1(t), u_2(t))$ as in Definition 3.1.2 with characteristic indices a, b, c, d . Suppose that the coefficients $A(z), B(z)$ are real analytic. Suppose that the quadratic equation (3.9) has a root U . Suppose that $a + bU \neq 0$ and define $\kappa = \frac{d+bU}{a+bU}$. Suppose that $\kappa \notin \mathbb{Z}_{\leq -2}$. Then, for T sufficiently small the ODE system subject to the condition $u'_2(0) = U$ admits a unique real-analytic solution $u(t)$ defined on $(-T, T)$.*

For the proof, see Subsection 3.7.2.

A lengthy but straightforward computation shows that the system (3.5) can be placed into the desired form near a sonic point. We summarize the desired information in the following lemma.

Lemma 3.1.5. *Consider a solution $(\tilde{\rho}, \tilde{u})$ to (3.5) with a sonic point at y_* . Define*

$$\rho_0 := \tilde{\rho}(y_*), \quad u_0 = \tilde{u}(y_*), \quad \omega_0 := \frac{u_0}{y_*} + 2 - \gamma.$$

Suppose that the following two equations in (ρ_0, ω_0, y_) are satisfied;*

$$y_* \omega_0 = \gamma^{\frac{1}{2}} \rho_0^{\frac{\gamma-1}{2}}, \tag{3.11}$$

$$\frac{4\pi \rho_0 \omega_0}{4 - 3\gamma} = 2\omega_0^2 + (\gamma - 1)\omega_0 + (2 - \gamma)(\gamma - 1). \tag{3.12}$$

Then, there exists an invertible change of variables from $(\tilde{\rho}, \tilde{u})$ to $(\check{\rho}, \check{u})$ so that if $(\tilde{\rho}, \tilde{u})$ is a C^1 solution of (3.5) then $(\check{\rho}, \check{u})$ is a C^1 solution to an equation in the normal form of Definition 3.1.2 with characteristic parameters

$$a = \frac{2(2 - \gamma) + (\gamma - 3)(\omega_0 + (\gamma - 1))}{4y_* \omega_0},$$

$$\begin{aligned}
b &= -\frac{(\gamma + 1)}{4\rho_0}, \\
c &= \frac{\rho_0}{4(y_*\omega_0)^2} \left(-(\gamma + 3)\omega_0^2 + (-2\gamma^2 + \gamma + 3)\omega_0 + (2 - \gamma)(\gamma - 1)^2 \right) \\
&\quad - \frac{\rho_0^2}{4(y_*\omega_0)^2} \left(\frac{4\pi}{(4 - 3\gamma)} (4 - 3\gamma - 2\omega_0) \right), \\
d &= \frac{1}{4y_*\omega_0} (3(\gamma - 1) + (\gamma - 3)(\omega_0 + (\gamma - 1))).
\end{aligned}$$

The conditions $\tilde{\rho}'(0) = 0$, $\tilde{u}'(0) = U$ are equivalent to the equations for $\tilde{\rho}'(y_*)$ and $\tilde{u}'(y_*)$ determined by letting

$$R = \frac{y_*\tilde{\rho}'(y_*)}{\rho_0}, \quad W = \tilde{u}'(y_*) - y_*^{-1}u_0$$

and solving

$$\begin{cases} U - \frac{\rho_0}{y_*\omega_0}(\omega_0 R + (\omega_0 + (\gamma - 1))) = 0 \\ \omega_0 R + W + 3\omega_0 - (4 - 3\gamma) = 0 \end{cases}.$$

The Larson-Penston-Hunter branch corresponds to the solution of (3.9) given by

$$U = \frac{-(a + d) + \sqrt{(a + d)^2 - 4bc}}{2b}. \quad (3.13)$$

Moreover, the far-field solution obeys the Larson-Penston-Hunter sonic point conditions at

$$y_f := \frac{\gamma^{\frac{1}{2}}}{2 - \gamma} \left(\frac{4 - 3\gamma}{2\pi} \right)^{\frac{\gamma - 1}{2}} = k^{\frac{(\gamma - 1)(2 - \gamma)}{2}} \gamma^{\frac{2 - \gamma}{2}}. \quad (3.14)$$

Finally, for the far-field solution we have

$$a + bU = \frac{1}{2(2 - \gamma)y_f}, \quad \kappa = \frac{5(\gamma - 1)}{2}.$$

The proof of this lemma is deferred to Appendix 3.7.3. As a consistency check, we mention that if one substitutes $U = \frac{\rho_0}{y_*}(\omega_0 R + (\omega_0 + (\gamma - 1)))$ into the quadratic (3.9) then one obtains the quadratic

$$\begin{aligned}
&-(1 + \gamma)\omega_0^2 R^2 + (9 - 7\gamma - 8\omega_0)\omega_0 R \\
&- 6\omega_0^2 + (8 - 6\gamma)\omega_0 + (-3\gamma^2 + 9\gamma - 6) + \frac{(4 - 3\gamma)(2 - \gamma)(1 - \gamma)}{\omega_0} = 0, \quad (3.15)
\end{aligned}$$

which is consistent with [54] (their equation (2.69)).

We finally comment that the proof of local analyticity given by [54] for this system (their Theorem 2.15) is similar in terms of techniques to our proof of Lemma 3.1.4 and could in principle be modified for use with Hunter-type solutions. We treat the problem more abstractly to clarify precisely the assumptions needed and to allow future works to “black box” this part of the argument.

3.1.3 Analogous results and open problems for a focusing scalar wave equation

In the isothermal case, the author's previous construction of Hunter solutions was motivated by similarities between the Euler-Poisson system and a class of semilinear heat equations (see [100] for more details and references). In this section, we will compare the theory of self-similar solutions of (3.1) with that of solutions to the following semilinear focusing wave equation (NLW) for $\varphi: I_t \times \mathbb{R}_x^d \rightarrow \mathbb{R}$:

$$\partial_t^2 \varphi - \Delta \varphi - \varphi^p = 0. \quad (3.16)$$

One advantage of using this equation as a model, instead of a heat equation, is that it is hyperbolic, so the theory of non-linear stability of self-similar solutions to (3.16) should be a good model for the Euler-Poisson system. As a first step towards deriving the self-similar ODE, we note that this equation is invariant under the scaling transformation ($\lambda > 0$)

$$\varphi(t, x) \mapsto \lambda^{-\frac{2}{p-1}} \varphi(t/\lambda, x/\lambda).$$

If a solution is invariant with respect to this symmetry, its profile will satisfy the ODE

$$(1 - y^2) \partial_y^2 \varphi + \left(\frac{d-1}{y} - \frac{2(p+1)}{p-1} y \right) \partial_y \varphi - \frac{2(p+1)}{(p-1)^2} \varphi + \varphi^p. \quad (3.17)$$

In particular, we see that this equation admits (in our language) a sonic point, but unlike the Euler-Poisson problem, the location of the sonic point is independent of the solution. We also comment that, in the case $d = 3$, the energy-supercritical range of p for NLW is $p > 5$ (this was $\gamma < \frac{6}{5}$ for Euler-Poisson) and the conformal-supercritical range of p for NLW is $p > 3$ (this corresponds to $\gamma < \frac{4}{3}$ for Euler-Poisson).

For (3.17) there also exist versions of the Friedman and far-field solutions. The Friedman (or ODE blow-up solution) is given explicitly by

$$\varphi_F = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}.$$

The far-field solution (in the case $p > \frac{d}{d-2}$) is given by

$$\varphi_F = \left(\frac{2((p-1)(d-2)-2)}{(p-1)^2} \right)^{\frac{1}{p-1}} \cdot y^{-\frac{2}{p-1}}.$$

Uniqueness and stability of the Friedman solution for (3.17) in 1 spatial dimension was shown by Merle and Zaag ([81, 82, 84, 85, 86, 87]). In higher dimensions the stability problem is more delicate and has been the subject of much recent work ([21, 32, 33, 34, 35, 36, 37, 38, 72, 88, 89, 96, 111]). We comment however, that for the purposes of modeling gravitational collapse from isolated bodies, we expect the relevant profiles to decay as $|x| \rightarrow \infty$. Consequentially, we shall focus on the non-Friedman self-similar solutions to (3.17).

We now briefly discuss some literature on the existence of such solutions. To rule out the Friedman solution, we will restrict attention to self-similar solutions whose leading order

decay rate as $y \rightarrow \infty$ is at least that of the far-field solution, namely $y^{-\frac{2}{p-1}}$. This can be viewed as a localization requirement. A numerical study of several exponents is carried out in [7] in $d = 3$. In particular, for the energy-supercritical exponent $p = 7$ a family of self-similar solutions was found (with decay rate $y^{-\frac{2}{p-1}}$), including a solution with a single unstable mode which was observed to play a role in describing the threshold between blow-up and dispersion. Self-similar solutions in $d = 3$ for p which are odd and energy-supercritical were rigorously constructed in [8]. We emphasize the close resemblance of Figure 1 of [8], depicting their shooting construction of self-similar solutions to NLW, and Figure 2 of [59], depicting the shooting construction of Hunter solutions. In higher dimensions, energy-supercritical self-similar blow-up profiles were analyzed numerically in [69] and constructed rigorously in [28]. One surprising feature of NLW is that in higher spatial dimensions, $d \geq 5$, the (energy-supercritical) cubic NLW admits an explicit self-similar solution found and studied by Glogić-Schörkhuber ([47]). We also remark that it decays faster ($y^{-\frac{2}{p-1}-1}$) than the far-field solution ($y^{-\frac{2}{p-1}}$). This solution's role as a threshold was studied numerically in [46]. For the quadratic NLW, in the energy-supercritical dimensions $d \geq 7$, there is again an explicit self-similar solution found and analyzed in [27], which differs from the previously discussed solutions in that it is non-positive. This work was extended beyond the past-light cone of the singularity in [27].

One key difference between NLW and Euler-Poisson is that for NLW, each self-similar solution known to exist has at least one linear instability direction, whereas for the Euler-Poisson system, the Larson-Penston-type solutions are expected to be linearly stable.

We also comment that as far as the author is aware, for $d = 3$, there is currently no theoretical or numerical evidence of existence of self-similar blow-up profiles to NLW for the conformal-supercritical, energy-subcritical range $3 < p < 5$ with the far-field decay rate $\varphi = O(y^{-\frac{2}{p-1}})$. This is a bit surprising as for the more complicated Euler-Poisson system, this corresponds to the range of exponents $\frac{6}{5} < \gamma < \frac{4}{3}$ for which a solution is known to exist ([54] following numerics of [113]).

The work on NLW which seems most potentially applicable for understanding the stability of Hunter-type solutions to the Euler-Poisson system is [66]. In this work (for $d = 3$), the author constructed self-similar solutions in the full energy-supercritical range, and showed finite-codimensional stability of the solutions by adapting the framework of [79]. As our construction of solutions is very similar to the approach taken in this work, one could hope that the author's framework could be used to show the finite-codimension stability of the Hunter-type solutions we construct. We comment that unlike the preceding work on the non-linear heat equation ([26]), or the scenerio of the explicit solutions ([47], [27]), the dimension of instability is not revealed in [66].

Indeed, if one is either interested in whether Hunter-type solutions exist for $\frac{6}{5} < \gamma < \frac{4}{3}$ or interested in if one can show non-linear stability with the expected codimension, the related, probably easier, open problems for the NLW equation (3.16) would be the following.

- For $d = 3$ and $3 < p < 5$ do there exist any smooth self-similar blow-up profiles of (3.16) such that $\varphi = O(y^{-\frac{2}{p-1}})$ as $y \rightarrow \infty$? If yes, do there exist infinitely many solutions?
- Can one rigorously determine the exact codimension of instability for the solutions

constructed by Kim in [66]?

3.1.4 Comparison with the isothermal problem, numerology and proof outline

At a broad level, the proof of the theorem is an adaptation of the isothermal proof ([100]), which was in turn based on techniques from simpler models (see, in particular, [26]). In fact, the methods we use are largely insensitive to the choice of γ in $[1, \frac{6}{5})$ and the main challenges of the polytropic case are that the algebraic expressions get more complicated. We comment that in [54], the exponent $\gamma = \frac{10}{9}$ played a special role. One point in their argument where this exponent appears is when the authors parameterize the curve of triples (ρ_0, ω_0, y_*) which satisfy (3.11) and (3.12) by y_* . We parameterize this curve by ω_0 and consequentially do not see $\gamma = \frac{10}{9}$ at this step.

This work contains several technical improvements over [100]. The most significant of these is in the treatment of local analyticity, as outlined in Subsection 3.1.2 and carried out in Subsection 3.4.2. Moreover, the derivation of difference equations in Subsections 3.2 and 3.3 is streamlined. In particular, in Section 3.2, our linear lemma now accomodates compatible non-zero Taylor coefficients. Moreover, in Section 3.3 we do not rescale the coordinate y . In addition, in this work (through the use of Lemma 3.7.3) we perform the matching using only a Lipschitz dependence of the interior and exterior solutions on λ, ϵ , which reduces the number of estimates required. As these improvements are all technical, the paper is organized identically to [100], which the reader is recommended to consult as a guide since the algebraic expressions are much cleaner and there is more commentary in the proof.

We will now present some heuristics which allow relatively quick derivations of the numerology. In particular, this section should illuminate why our methods cannot work for $\gamma \geq \frac{6}{5}$.

The basic structure of the argument is to consider a fixed, small y_0 , construct a 1-parameter family of “exterior solutions” on $[y_0, \infty)$, construct a 1-parameter family of “interior” solutions on $[0, y_0]$ and then match them to obtain an infinite, discrete family. These solutions are a priori C^1 and higher regularity is obtained through application of the theory outlined in Subsection 3.1.2.

We begin with the (non-self-similar) Euler-Poisson system, (3.1). We shall define the enthalpy to be

$$w = \frac{\gamma}{\gamma - 1} \rho^{\gamma-1}.$$

Setting $u = 0$, taking the divergence of the second equation and changing variables from ρ to w gives the Lane-Emden equation

$$\Delta w + 4\pi \left(\frac{\gamma - 1}{\gamma} \right)^{\frac{1}{\gamma-1}} w^{\frac{1}{\gamma-1}} = 0. \quad (3.18)$$

We now summarize properties of solutions to this equation that can be found, for instance, in [20]. We shall define Q to be the unique solution such that

$$Q(0) = \frac{\gamma}{\gamma - 1}.$$

The family of regular solutions to (3.18) is parameterized by

$$\lambda \mapsto Q_\lambda = \frac{1}{\lambda^{\frac{2(\gamma-1)}{2-\gamma}}} Q(\cdot/\lambda).$$

The far-field solution in this case is

$$Q_f = \frac{\gamma}{\gamma-1} k^{\gamma-1} r^{-\frac{2(\gamma-1)}{2-\gamma}}$$

which, if we change back to ρ , is

$$\rho_f = \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} Q_f^{\frac{1}{\gamma-1}} = k r^{-\frac{2}{2-\gamma}}.$$

We comment that for $\gamma < \frac{6}{5}$, the Lane-Emden solutions are positive for all r and their spatial decay matches the far-field solution ($r^{-\frac{2}{2-\gamma}}$). This is not true for $\gamma \geq \frac{6}{5}$. In contrast to [54], the Lane-Emden equation and its linearization plays a key role in our analysis, so it would be unable to alter our proof to deal with $\gamma \geq \frac{6}{5}$.

We now return to the self-similar system (3.5). We recall that the far-field solution, (3.7), has a sonic point at $y = y_f$ (defined in (3.14)). If we linearize the self-similar system about the far-field solution $(\tilde{\rho}_f, \tilde{u}_f)$, we obtain the following linear ODE which is singular at $y \in \{0, y_f\}$,

$$\begin{aligned} \partial_y \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} + \frac{1}{(2-\gamma)^2 y^2 - k^{\gamma-1} \gamma y^{2-\frac{2}{2-\gamma}}} \cdot \begin{pmatrix} 2(2-\gamma)y - k^{\frac{4\pi(2-\gamma)(3-\gamma)}{4-3\gamma}} y^{1-\frac{2}{2-\gamma}} & -3(\gamma-1)ky^{-\frac{2}{2-\gamma}} - \frac{4\pi}{4-3\gamma} k^2 y^{-\frac{4}{2-\gamma}} \\ (-2k^{\gamma-2}\gamma + \frac{4\pi(2-\gamma)^2(3-\gamma)}{4-3\gamma})y^2 & (2\frac{\gamma-1}{2-\gamma} k^{\gamma-1}\gamma + \frac{4\pi(2-\gamma)k}{4-3\gamma})y^{1-\frac{2}{2-\gamma}} + (2-\gamma)(\gamma-1)y \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} = 0. \end{aligned}$$

If we then define $\tilde{p} = y^{\frac{2}{2-\gamma}} \tilde{\rho}$, $\tilde{v} = y^{-1} \tilde{u}$, then we get the system

$$\begin{aligned} \partial_y \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} + \frac{1}{y} \begin{pmatrix} -\frac{2}{2-\gamma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} + \frac{1}{(2-\gamma)^2 y^2 - k^{\gamma-1} \gamma y^{2-\frac{2}{2-\gamma}}} \cdot \begin{pmatrix} 2(2-\gamma)y - k^{\frac{4\pi(2-\gamma)(3-\gamma)}{4-3\gamma}} y^{1-\frac{2}{2-\gamma}} & -3(\gamma-1)ky - \frac{4\pi}{4-3\gamma} k^2 y^{1-\frac{2}{2-\gamma}} \\ (-2k^{\gamma-2}\gamma + \frac{4\pi(2-\gamma)^2(3-\gamma)}{4-3\gamma})y^{1-\frac{2}{2-\gamma}} & (2\frac{\gamma-1}{2-\gamma} k^{\gamma-1}\gamma + \frac{4\pi(2-\gamma)k}{4-3\gamma})y^{1-\frac{2}{2-\gamma}} + (2-\gamma)(\gamma-1)y \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} = 0. \end{aligned}$$

We can simplify this to

$$\partial_y \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} + \frac{1}{(2-\gamma)^2 y^2 - k^{\gamma-1} \gamma y^{2-\frac{2}{2-\gamma}}} \cdot$$

$$\begin{pmatrix} -\frac{4\pi k(2-\gamma)^2}{4-3\gamma} y^{1-\frac{2}{2-\gamma}} & -3(\gamma-1)ky - \frac{4\pi}{4-3\gamma} k^2 y^{1-\frac{2}{2-\gamma}} \\ \frac{4\pi(2-\gamma)^3}{4-3\gamma} y^{1-\frac{2}{2-\gamma}} & (2-\gamma)y + \frac{2\pi(2-\gamma)k(3\gamma-2)}{4-3\gamma} y^{1-\frac{2}{2-\gamma}} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{v} \end{pmatrix} = 0.$$

We can then simplify the residue at $y = 0$ to be

$$\begin{pmatrix} \frac{4\pi k^{2-\gamma}(2-\gamma)^2}{\gamma(4-3\gamma)} & \frac{4\pi k^2}{(4-3\gamma)k^{\gamma-1}\gamma} \\ 2k - \frac{4\pi(2-\gamma)^2(3-\gamma)}{(4-3\gamma)k^{\gamma-1}\gamma} & \frac{-2\pi(2-\gamma)k^{2-\gamma}(3\gamma-2)}{2(4-3\gamma)} \end{pmatrix} = \begin{pmatrix} 2 & \frac{2k}{(2-\gamma)^2} \\ -\frac{2(2-\gamma)}{k} & -\frac{3\gamma-2}{2-\gamma} \end{pmatrix}.$$

We define

$$\mu := \frac{6-5\gamma}{2(2-\gamma)}, \quad \nu := \frac{\sqrt{-\gamma^2-20\gamma+28}}{2(2-\gamma)}. \quad (3.19)$$

We may then diagonalize (the negative of) this residue to be

$$\begin{pmatrix} -2 & -\frac{2k}{(2-\gamma)^2} \\ \frac{2(2-\gamma)}{k} & \frac{3\gamma-2}{2-\gamma} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{(2-\gamma)^{\frac{3}{2}}}{k} e^{i\theta_0} & \frac{(2-\gamma)^{\frac{3}{2}}}{k} e^{-i\theta_0} \end{pmatrix} \cdot \begin{pmatrix} -\mu + i\nu & 0 \\ 0 & -\mu - i\nu \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{(2-\gamma)^{\frac{3}{2}}}{k} e^{i\theta_0} & \frac{(2-\gamma)^{\frac{3}{2}}}{k} e^{-i\theta_0} \end{pmatrix}^{-1}.$$

where we define

$$\theta_0 = \arctan\left(\frac{\sqrt{-\gamma^2-20\gamma+28}}{2+\gamma}\right) + \pi. \quad (3.20)$$

We comment that this angle satisfies

$$\sin(\theta_0) = -\frac{\sqrt{-\gamma^2-20\gamma+28}}{4\sqrt{2-\gamma}} \neq 0.$$

From this, we may read off the asymptotics of the homogenous solution to the linearized equations to be

$$p_{hom} = \frac{c_1}{y^\mu} \sin(\nu \log(y) + d_1) + O(y^{-\mu+\frac{2}{2-\gamma}}),$$

$$\omega_{hom} = \frac{c_1}{y^\mu} \frac{(2-\gamma)^{\frac{3}{2}}}{k} \sin(\nu \log(y) + d_1 + \theta_0) + O(y^{-\mu+\frac{2}{2-\gamma}}).$$

The exterior solutions we construct (for small ϵ) will then be of the form

$$\begin{cases} \tilde{\rho}_{ext} = \frac{k}{y^{\frac{2}{2-\gamma}}} + \frac{\epsilon}{y^{\frac{2}{2-\gamma}}} p_{hom} + \dots \\ \tilde{u}_{ext} = \epsilon y \omega_{hom} + \dots \end{cases}. \quad (3.21)$$

We comment that in order for the $O(\epsilon)$ terms in the expansion for $\tilde{\rho}_{ext}$ to have faster decay than the principal term, we need $\mu > 0$ which is equivalent to $\gamma < \frac{6}{5}$. We also comment that when $\gamma = 1$, $(\mu, \nu) = (\frac{1}{2}, \frac{\sqrt{7}}{2})$, which recovers the numerology observed in the isothermal problem.

Turning now to the interior problem, we shall define a universal profile u_* by solving the linear equation

$$\begin{cases} ((2-\gamma)y\partial_y + 2)(\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} + \operatorname{div}((\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}}u_*) = 0 \\ u_*(0) = 0, \quad u'_*(0) = -\frac{2}{3} \end{cases} \quad (3.22)$$

where Q is the solution to the Lane-Emden Equation (3.18) with $Q(0) = \frac{\gamma}{\gamma-1}$. When we linearize (3.18) about the far-field solution, the linearized operator is (after flipping signs)

$$\tilde{H} := -(\Delta + 4\pi(\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} Q_f^{\frac{2-\gamma}{\gamma-1}}),$$

which is simplified to

$$\tilde{H} := -(\Delta + \frac{2(4-3\gamma)}{(2-\gamma)^2} \frac{1}{y^2}). \quad (3.23)$$

In this case, the fundamental solutions of $\tilde{H}$ are given by $y^{-\frac{1}{2} \pm i\nu}$. This is used to justify the asymptotics

$$Q = \frac{\gamma}{\gamma-1} \frac{k^{\gamma-1}}{y^{\frac{2(\gamma-1)}{2-\gamma}}} + \frac{\gamma}{k^{2-\gamma}} \frac{c_2 \sin(\nu \log(y) + d_2)}{y^{\frac{2(\gamma-1)}{2-\gamma} + \mu}} + O(\frac{1}{y^{\frac{2(\gamma-1)}{2-\gamma} + 2\mu}}).$$

We comment again, that here the condition $\gamma < \frac{6}{5}$ plays a role to ensure that the first term is in fact of leading order as $y \rightarrow \infty$.

This is in terms of enthalpy variables, converting to density yields

$$(\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} Q^{\frac{1}{\gamma-1}} = \frac{k}{y^{\frac{2}{2-\gamma}}} + \frac{c_2 \sin(\nu \log(y) + d_2)}{y^{\frac{2}{2-\gamma} + \mu}} + O(\frac{1}{y^{\frac{2}{2-\gamma} + 2\mu}})$$

To solve for u_* it is useful to use the Lane-Emden equation to rewrite $(\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} = -\frac{1}{4\pi}\Delta Q$ and then to use the identity

$$((2-\gamma)y\partial_y + 2) \circ \Delta = \Delta \circ ((2-\gamma)y\partial_y + 2(\gamma-1))$$

This allows us to solve for u_* directly to be

$$u_* = (\frac{\gamma-1}{\gamma}Q)^{-\frac{1}{\gamma-1}} \frac{1}{4\pi} \partial_y ((2-\gamma)y\partial_y Q + 2(\gamma-1)Q).$$

From this, we see that as $y \rightarrow \infty$, we have that

$$u_*(y) = k^{-1}(2-\gamma)^{\frac{3}{2}} c_2 \sin(\nu \log(y) + d_2 + \theta_0) y^{1-\mu} + O(y^{1-2\mu}). \quad (3.24)$$

For completeness, we also mention that from the Lane-Emden equation and $Q(0) = \frac{\gamma}{\gamma-1}$, we see that as $y \rightarrow 0$, $Q(y) = \frac{\gamma}{\gamma-1} - \frac{4\pi}{6} y^2 + o(y^2)$ which gives $u'_*(0) = -\frac{2}{3}$.

For λ sufficiently small, we will construct an interior solution of the form

$$\begin{cases} \tilde{\rho}_{int} = \frac{1}{\lambda^{\frac{2}{2-\gamma}}} (\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} (\cdot/\lambda) + \dots \\ \tilde{u}_{int} = \lambda u_*(\cdot/\lambda) + \dots \end{cases} \quad (3.25)$$

Turning our attention to the matching, when we evaluate the exterior solutions, (3.21), at y_0 we will obtain

$$\begin{cases} \tilde{\rho}_{ext}(y_0) - \frac{k}{y_0^{\frac{2}{2-\gamma}}} &= \epsilon \frac{1}{y_0^{\frac{2}{2-\gamma}+\mu}} c_1 \sin(\nu \log(y_0) + d_1) + O(\epsilon y_0^{-\mu}) \\ \frac{\tilde{u}_{ext}(y_0)}{y_0^{\frac{4-\gamma}{2-\gamma}}} &= \frac{\epsilon}{y_0^{\frac{2}{2-\gamma}+\mu}} \frac{(2-\gamma)^{\frac{3}{2}}}{k} c_1 \sin(\nu \log(y_0) + d_1 + \theta_0) + O(\epsilon y_0^{-\mu}) \end{cases}.$$

Similarly, evaluating the interior solutions, (3.25), at y_0 will give

$$\begin{cases} \tilde{\rho}_{int}(y_0) - \frac{k}{y_0^{\frac{2}{2-\gamma}}} &= \frac{\lambda^\mu}{y_0^{\frac{2}{2-\gamma}+\mu}} c_2 \sin(\nu \log(y_0/\lambda) + d_2) + O(\lambda^\mu y_0^{-\mu}) \\ \frac{\tilde{u}_{int}(y_0)}{y_0^{\frac{4-\gamma}{2-\gamma}}} &= \frac{\lambda^\mu}{y_0^{\frac{2}{2-\gamma}+\mu}} \frac{c_2(2-\gamma)^{\frac{3}{2}}}{k} \sin(\nu \log(y_0/\lambda) + d_2 + \theta_0) + O(\lambda^\mu y_0^{-\mu}) \end{cases}.$$

For simplicity, by the flexibility in choosing y_0 , we may impose

$$\nu \log(y_0) + d_1 = \frac{\pi}{2} \mod 2\pi. \quad (3.26)$$

Matching the interior and exterior solutions will give the equations

$$\begin{cases} \epsilon c_1 + O(\epsilon y_0^{\frac{2}{2-\gamma}}) &= c_2 \lambda^\mu \cos(\nu \log(\lambda) + d_1 - d_2) + O(\lambda^\mu y_0^{\frac{2}{2-\gamma}}) \\ \epsilon c_1 \cos(\theta_0) + O(\epsilon y_0^{\frac{2}{2-\gamma}}) &= c_2 \lambda^\mu \cos(\nu \log(\lambda) + d_1 - d_2 - \theta_0) + O(\lambda^\mu y_0^{\frac{2}{2-\gamma}}) \end{cases}.$$

Solving the first equation for ϵ as a function of λ , and then substituting it into the second equation will yield

$$0 = \sin(\nu \log(\lambda) + d_1 - d_2) + O(y_0^{\frac{2}{2-\gamma}}).$$

By choosing y_0 sufficiently small, we see that this equation will have an infinite discrete family of solutions which will correspond to the Hunter-type solutions. Finally, non-trivial task of verifying analyticity of the solutions is accomplished through application of the framework described in Subsection 3.1.2.

3.1.5 Organization and notations

We now summarize the structure of the paper (which is parallel with [100]). Section 3.2 details the construction of the exterior family of solutions. Section 3.3 contains the construction of the interior family of solutions. The matching is performed in Section 3.4 culminating in the proof of the main theorem. Sections 3.5, 3.6 contain proofs deferred from Sections 3.2 and 3.3 respectively. Finally, Appendix 3.7 contains the relatively self-contained proofs of lemmas used in Section 3.4.

For the reader's convenience we also provide a partial dictionary of notational conventions used in the paper. The meanings of some notations are depend on the section of the paper in which they occur.

Selected section-independent notations:

- γ : polytropic index for the equation of state.

- $\tilde{\rho}, \tilde{u}$: self-similar mass density and velocity.
- y : self-similar coordinate.
- $\langle y \rangle$: $\sqrt{1 + y^2}$.
- $\tilde{\rho}_F, \tilde{u}_F$: Friedman solution.
- $\tilde{\rho}_f, \tilde{u}_f$: far-field solution.
- k : an explicit constant, see (3.8).
- y_* : a sonic point.
- y_f : the sonic point for the far-field solution, see (3.14).
- Q : Lane-Emden solution (at the level of enthalpy).
- Q_f : far-field enthalpy.
- u_* : a leading order velocity derived from the Lane-Emden solution, see (3.22).
- μ, ν : exponents arising from linearized analysis, see (3.19).
- θ_0 : angle arising from linearized analysis, see (3.20).
- y_0 : self-similar coordinate at which the matching occurs.
- ϵ : smallness parameter for the exterior region.
- λ : used for scaling, used as a smallness parameter for the interior region.

Selected notations used in Section 3.1:

- (t, x) : physical time and space coordinates.
- (ρ, u) : the physical (i.e. non-self-similar) mass density and velocity.
- $r = |x|$: physical radial coordinate.
- (u_1, u_2) : variables for the ODE normal form definition.
- z_0, z_1, z_2 : dummy variables for t, u_1, u_2 for the ODE normal form definition.
- a, b, c, d : characteristic parameters for the ODE normal form definition.
- U : solution to a quadratic equation associated to the ODE normal form.
- κ : parameter for the ODE lemmas.
- $(\check{\rho}, \check{u})$: variables for putting the self-similar Euler-Poisson system in normal form.

Selected notations used in Sections 3.2, 3.5:

- $(\tilde{p}, \tilde{\omega})$: convenient variables for the exterior analysis.
- α, β : parameters for changing coordinates to z .
- z : choice of coordinates which fixes the location of the sonic point.
- $(\underline{p}, \underline{\omega})$: a convenient choice of variables in z coordinates.
- $(\hat{p}, \hat{\omega})$: variables for the difference equation.
- $(\tilde{E}, \tilde{F})$, $(\underline{E}, \underline{F})$ and $(\hat{E}, \hat{F})$: coefficient functions for the equations involving $(\tilde{p}, \tilde{\omega})$, $(\underline{p}, \underline{\omega})$ and $(\hat{p}, \hat{\omega})$ respectively.
- (a, b) dummy variables for $(\tilde{p}, \tilde{\omega})$, $(\underline{p}, \underline{\omega})$ or $(\hat{p}, \hat{\omega})$.
- $\mathring{p}_0, \mathring{\omega}_0, \mathring{y}_*, \mathring{p}_1, \mathring{\omega}_1$: smooth functions of ϵ that describe the sonic point conditions.
- $\mathring{p}_{0,d}, \mathring{\omega}_{0,d}, \mathring{p}_{1,d}, \mathring{\omega}_{1,d}$, difference quotients associated to the above functions.
- $\mathring{p}_{0,e}, \mathring{\omega}_{0,e}, \mathring{p}_{1,e}, \mathring{\omega}_{1,e}$, further “error” difference quotients associated to the above functions.
- $\mathring{\sigma}_1, \mathring{\sigma}_2, \mathring{\sigma}_3$: smooth functions of ϵ that describe compatibility conditions.
- E_{lin}, F_{lin} : linearizations of $\hat{E}, \hat{F}$.
- L and L_1, L_2 , the linearized operator and its components.
- $\hat{N}$: the non-linearity in the difference equation.
- $\sigma_1, \sigma_2, \sigma_3$: variables used for describing compatibility conditions.
- (p_{hom}, ω_{hom}) : homogeneous solution for L .
- U_0, U_∞ : fundamental matrices for L on $(0, 1)$ and $(1, \infty)$.
- R : left-inverse to L constructed in Lemma 3.2.1.
- X_{y_0}, N_{y_0} : Banach spaces used for the iteration.
- $X_{y_0}[\epsilon], N_{y_0}[\epsilon], X_{y_0}[\epsilon_1, \epsilon_2], N_{y_0}[\epsilon_1, \epsilon_2]$: affine subspaces of X_{y_0}, N_{y_0} .
- (p, ω) : final difference variables.

Selected notations used in Sections 3.3, 3.6:

- $\tilde{\omega}$: self-similar enthalpy.
- H : linearization of Lane-Emden equation about Q .
- $\tilde{H}$: linearization of Lane-Emden equation about Q_f .
- $\bar{w}, \bar{u}$: a change of variables used for a heuristic.

- J, T, L, S : linear operators arising from the asymptotic expansion.
- $J_\lambda, L_\lambda, H_\lambda, S_\lambda, T_\lambda, K_\lambda$: rescaled linear operators.
- $F_\lambda, G_{\lambda,1}, G_{\lambda,2}, G_\lambda$: rescaled non-linear operators.
- $Z, Y, \mathcal{G}, \mathcal{F}$: Banach spaces used for the iteration.
- (w, u) : difference variables.

Selected notations used in Section 3.4:

- $\mathcal{F}, \mathcal{G}$: functions describing the matching.

Selected notations used in Appendix 3.7.

- ϕ, ψ : preliminary variables for putting the equations in normal form.

3.2 Solving in the exterior region

We begin by choosing convenient variables for the exterior problem. In this case, we define the variables $\tilde{p}, \tilde{\omega}$ implicitly via

$$\tilde{\rho} = \frac{\tilde{p}}{y^{\frac{2}{2-\gamma}}}, \quad \tilde{u} = y(\tilde{\omega} - (2 - \gamma)).$$

In these variables, the self-similar equations (3.5) take the form

$$\begin{cases} (\tilde{p}\tilde{\omega}y)' + 2\frac{1-\gamma}{2-\gamma}\tilde{p}\tilde{\omega} - (4 - 3\gamma)\tilde{p} & = 0 \\ \tilde{\omega}(y(\tilde{\omega} - (2 - \gamma)))' + \frac{\gamma\tilde{p}'}{y^{\frac{\gamma}{2-\gamma}}\tilde{p}^{2-\gamma}} + (\gamma - 1)(\tilde{\omega} - (2 - \gamma)) & \\ + \frac{1}{y^{\frac{2}{2-\gamma}}}\left(\frac{4\pi\tilde{p}\tilde{\omega}}{4-3\gamma} - \frac{2\gamma}{2-\gamma}\tilde{p}^{\gamma-1}\right) & = 0 \end{cases}. \quad (3.27)$$

In these variables, the far-field solution is

$$(\tilde{p}, \tilde{\omega}) = (k, 2 - \gamma).$$

We will be considering a perturbation of the far-field solution, and so we note that we may write the system more abstractly as

$$\tilde{E}(y, \tilde{p}, \tilde{\omega}) \begin{pmatrix} \tilde{p}' \\ \tilde{\omega}' \end{pmatrix} + \tilde{F}(y, \tilde{p}, \tilde{\omega}) = 0. \quad (3.28)$$

where (using dummy variables a, b for $\tilde{p}, \tilde{\omega}$)

$$\tilde{E}(y, a, b) = \begin{pmatrix} by & ay \\ \frac{\gamma}{y^{\frac{\gamma}{2-\gamma}}} \frac{1}{a^{2-\gamma}} & by \end{pmatrix},$$

$$\tilde{F}(y, a, b) = \left(b(b - (2 - \gamma)) + (\gamma - 1)(b - (2 - \gamma)) + \frac{1}{y^{\frac{2}{2-\gamma}}} \left(\frac{4\pi}{4-3\gamma} ab - \frac{2\gamma}{2-\gamma} a^{\gamma-1} \right) \right).$$

We note that $\tilde{E}, \tilde{F}$ are smooth with respect to a, b in a neighborhood of $\tilde{p}, \tilde{\omega} = (k, 2 - \gamma)$ and we have

$$\tilde{F}(y, k, 2 - \gamma) = 0.$$

We now discuss boundary conditions at the sonic points in these variables. We comment that in terms of p, ω , the condition (3.11) becomes

$$y_* = \frac{\gamma^{\frac{2-\gamma}{2}} \tilde{p}(y_*)^{\frac{(\gamma-1)(2-\gamma)}{2}}}{\tilde{\omega}(y_*)^{2-\gamma}}. \quad (3.29)$$

If we substitute this into (3.12) we get the equation

$$\tilde{p}(y_*) = \frac{\gamma^{\frac{1}{2-\gamma}}}{\tilde{\omega}(y_*)^{\frac{2}{2-\gamma}}} \left(\frac{4 - 3\gamma}{4\pi} \frac{2\tilde{\omega}(y_*)^2 + (\gamma - 1)\tilde{\omega}(y_*) + (2 - \gamma)(\gamma - 1)}{\tilde{\omega}(y_*)} \right)^{\frac{1}{2-\gamma}}. \quad (3.30)$$

Indeed, we see that if $\tilde{\omega}(y_*)$ is determined, then by (3.30), $\tilde{p}(y_*)$ is determined, and these together determine y_* by (3.29).

To compute the Larson-Penston-Hunter-type conditions for the derivatives of $\tilde{p}, \tilde{\omega}$, we see that R , defined in Lemma 3.1.5, takes the form

$$R = \frac{y_* \tilde{p}'(y_*)}{\tilde{p}(y_*)} - \frac{2}{2 - \gamma}$$

and recall that it has to satisfy a quadratic equation, (3.15). Once R is determined, then we can solve for $W = y_* \tilde{\omega}'(y_*)$ by recalling that it must satisfy

$$\tilde{\omega}(y_*)R + W = 4 - 3\gamma - 3\tilde{\omega}(y_*). \quad (3.31)$$

We comment that from (3.14), we may derive the useful identities

$$ky_f^{-\frac{2}{2-\gamma}} = \frac{4 - 3\gamma}{2\pi}, \quad \gamma k^{\gamma-2} = \frac{(2 - \gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k}.$$

We parameterize the exterior solutions by the value $\tilde{\omega}(y_*)$ which we set to be $\dot{\omega}_0[\epsilon] = \frac{2-\gamma}{1+\epsilon}$. One then defines $\dot{p}_0[\epsilon]$ by substituting $\tilde{\omega}(y_*) = \dot{\omega}_0[\epsilon]$ into (3.30) and solving for $\tilde{p}(y_*)$. Similarly, we then define $\dot{y}_*[\epsilon]$ via (3.29). One then defines, for ϵ close to 0, that $\dot{R}[\epsilon]$ is the solution branch near $\dot{R}[0] = -\frac{2}{2-\gamma}$ of (3.15). Finally, one defines $\dot{W}[\epsilon]$ via (3.31). We then define

$$\dot{p}_1[\epsilon] := \frac{\dot{p}[\epsilon]}{\dot{y}_*[\epsilon]} \left(\dot{R}[\epsilon] + \frac{2}{2 - \gamma} \right), \quad \dot{\omega}_1[\epsilon] := \frac{\dot{W}[\epsilon]}{\dot{y}_*[\epsilon]}.$$

Thus, in terms of the variables $(\tilde{p}, \tilde{\omega})$, Larson-Penston-Hunter-type expansion conditions at $y_* = \dot{y}_*[\epsilon]$ become

$$\tilde{p}(y_*) = \dot{p}_0[\epsilon], \quad \tilde{\omega}(y_*) = \dot{\omega}_0[\epsilon], \quad \tilde{p}'(y_*) = \dot{p}_1[\epsilon], \quad \tilde{\omega}'(y_*) = \dot{\omega}_1[\epsilon]. \quad (3.32)$$

As a comment, we see from the above identities that as $\epsilon \rightarrow 0$, we have

$$\dot{\omega}_0[\epsilon] = (2 - \gamma) - (2 - \gamma)\epsilon + O(\epsilon^2), \quad (3.33)$$

$$\dot{p}_0[\epsilon] = k + \frac{3\gamma - 1}{2(2 - \gamma)}k\epsilon + O(\epsilon^2), \quad (3.34)$$

$$\dot{y}_*[\epsilon] = y_f + \frac{3\gamma^2 - 8\gamma + 9}{4}y_f\epsilon + O(\epsilon^2),$$

$$\dot{R}[\epsilon] = -\frac{2}{2 - \gamma} + \frac{-9\gamma^2 + 9\gamma + 2}{(5\gamma - 3)(2 - \gamma)}\epsilon + O(\epsilon^2),$$

$$\dot{W}[\epsilon] = \frac{2(7 - 3\gamma)(\gamma - 1)}{5\gamma - 3}\epsilon + O(\epsilon^2).$$

In particular, these imply that

$$\dot{p}_1[\epsilon] = \frac{(-9\gamma^2 + 9\gamma + 2)}{(5\gamma - 3)(2 - \gamma)} \frac{k}{y_f} \epsilon + O(\epsilon^2), \quad (3.35)$$

$$\dot{\omega}_1[\epsilon] = \frac{2(7 - 3\gamma)(\gamma - 1)}{(5\gamma - 3)y_f} \epsilon + O(\epsilon^2). \quad (3.36)$$

We now do an ϵ -dependent change of variables that will move the sonic point from y_0 to y_f . In particular, we will define

$$\begin{aligned} \alpha &:= \frac{y_f - y_0}{\dot{y}_* - y_0}, & \beta &:= \frac{\dot{y}_* - y_f}{\dot{y}_* - y_0} y_0, & z &:= \alpha y + \beta, \\ \tilde{p} &:= \underline{p}(\alpha y + \beta), & \tilde{\omega} &:= \underline{\omega}(\alpha y + \beta). \end{aligned} \quad (3.37)$$

As a comment, α, β are chosen precisely so that

$$\alpha y_0 + \beta = y_0, \quad \alpha y_* + \beta = y_f.$$

We note that as $\epsilon \rightarrow 0$, we have

$$\alpha = 1 + O(\epsilon), \quad \beta = O(\epsilon).$$

This gives the system

$$\underline{E}(z, \underline{p}, \underline{\omega}, y_0, \epsilon) \left(\frac{\underline{p}'}{\underline{\omega}'} \right) + \underline{F}(z, \underline{p}, \underline{\omega}, y_0, \epsilon) = 0$$

where (in terms of the dummy variables a, b for $\tilde{p}, \tilde{\omega}$)

$$\underline{E}(z, a, b, y_0, \epsilon) := \alpha \tilde{E}\left(\frac{z - \beta}{\alpha}, a, b\right), \quad \underline{F}(z, a, b, y_0, \epsilon) := \tilde{F}\left(\frac{z - \beta}{\alpha}, a, b\right).$$

We note that in these variables, the sonic point boundary conditions become

$$\underline{p}(y_f) = \dot{p}_0[\epsilon], \quad \underline{\omega}(y_f) = \dot{\omega}_0[\epsilon], \quad \underline{p}'(y_f) = \frac{1}{\alpha} \dot{p}_0[\epsilon], \quad \underline{\omega}'(y_f) = \frac{1}{\alpha} \dot{\omega}_1[\epsilon].$$

We then define $\hat{p}, \hat{\omega}$ to be first order difference quotients, which are defined implicitly by

$$\begin{cases} \underline{p} = k + \epsilon \hat{p} \\ \underline{\omega} = 2 - \gamma + \epsilon \hat{\omega} \end{cases}. \quad (3.38)$$

To translate the boundary conditions to these variables, we define

$$\begin{cases} \mathring{p}_{0,d} := \frac{\mathring{p}[\epsilon] - k}{\epsilon} \\ \mathring{\omega}_{0,d} := \frac{\mathring{\omega}[\epsilon] - (2 - \gamma)}{\epsilon} \\ \mathring{p}_{1,d} := \frac{1}{\alpha \epsilon} \mathring{p}_1[\epsilon] \\ \mathring{\omega}_{1,d} := \frac{1}{\alpha \epsilon} \mathring{\omega}_1[\epsilon] \end{cases}.$$

and note that the conditions for $\hat{p}, \hat{\omega}$ are

$$\hat{p}(y_f) = \mathring{p}_{0,d}[\epsilon], \quad \hat{\omega}(y_f) = \mathring{\omega}_{0,d}[\epsilon], \quad \hat{p}'(y_f) = \mathring{p}_{1,d}[\epsilon], \quad \hat{\omega}'(y_f) = \mathring{\omega}_{1,d}[\epsilon].$$

We also note that

$$\mathring{p}_{0,d}[\epsilon] = \frac{(3\gamma - 1)}{2(2 - \gamma)} k + O(\epsilon), \quad \mathring{p}_{1,d}[\epsilon] = \frac{-9\gamma^2 + 9\gamma + 2}{(5\gamma - 3)(2 - \gamma)} \frac{k}{y_f} + O(\epsilon), \quad (3.39)$$

$$\mathring{\omega}_{0,d}[\epsilon] = -(2 - \gamma) + O(\epsilon), \quad \mathring{\omega}_{1,d}[\epsilon] = \frac{2(\gamma - 1)(7 - 3\gamma)}{5\gamma - 3} \frac{1}{y_f} + O(\epsilon). \quad (3.40)$$

As a comment, unlike [100], we do not perform additional manipulations to make these constants independent of ϵ , since this seems to add additional complications in the case of $\gamma > 1$. Substituting the change of variables (3.37) into (3.28) yields

$$\underline{E}(z, \underline{p}, \underline{\omega}, y_0, \epsilon) \partial_z \left(\frac{\underline{p}}{\underline{\omega}} \right) + \underline{F}(z, \underline{p}, \underline{\omega}, y_0, \epsilon) = 0. \quad (3.41)$$

where

$$\underline{E}(z, a, b, y_0, \epsilon) = \alpha \cdot \tilde{E}\left(\frac{z - \beta}{\alpha}, a, b\right), \quad \underline{F}(z, \underline{p}, \underline{\omega}, y_0, \epsilon) = \tilde{F}\left(\frac{z - \beta}{\alpha}, a, b\right).$$

We then substitute change of variables (3.38) into (3.41) and obtain the equation

$$\hat{E}(z, \hat{p}, \hat{\omega}, y_0, \epsilon) \partial_z \left(\frac{\hat{p}}{\hat{\omega}} \right) + \hat{F}(z, \hat{p}, \hat{\omega}, y_0, \epsilon) = 0, \quad (3.42)$$

where

$$\begin{aligned} \hat{E}(z, a, b, y_0, \epsilon) &:= \underline{E}(z, k + \epsilon a, 2 - \gamma + \epsilon b, y_0, \epsilon), \\ \hat{F}(z, a, b, y_0, \epsilon) &:= \frac{1}{\epsilon} \underline{F}(z, k + \epsilon a, 2 - \gamma + \epsilon b, y_0, \epsilon). \end{aligned}$$

3.2.1 First order difference quotients in ϵ

We now perform several Taylor expansions, beginning by expanding the symbols in terms of the dummy variables a, b . We start by noting that

$$\tilde{E}(y, a, b) = \begin{pmatrix} (2-\gamma)y & ky \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} \frac{y}{y^{\frac{2}{2-\gamma}}} & (2-\gamma)y \end{pmatrix} + O(a-k) + O(b-(2-\gamma)),$$

It follows that uniformly for $z \in (\frac{1}{2}, 2)$, we have that

$$\underline{E}(z, a, b, y_0, \epsilon) = \begin{pmatrix} (2-\gamma)z & kz \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} \frac{z}{z^{\frac{2}{2-\gamma}}} & (2-\gamma)z \end{pmatrix} + O(a-k) + O(b-(2-\gamma)) + O(\epsilon).$$

We then have that

$$\hat{E}(z, a, b, y_0, \epsilon) = \begin{pmatrix} (2-\gamma)z & kz \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} \frac{z}{z^{\frac{2}{2-\gamma}}} & (2-\gamma)z \end{pmatrix} + O(\epsilon).$$

We similarly Taylor expand

$$\begin{aligned} \tilde{F}(y, a, b) &= \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma}k \\ \frac{2(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{y^{\frac{2}{2-\gamma}}k} & 1 + \frac{2y_f^{\frac{2}{2-\gamma}}}{y^{\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} a-k \\ b-(2-\gamma) \end{pmatrix} \\ &\quad + O((a-k)^2) + O((a-k)(b-(2-\gamma))) + O((b-(2-\gamma))^2). \end{aligned}$$

From this it follows that

$$\begin{aligned} \underline{F}(z, a, b, y_0, \epsilon) &= \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma}k \\ \frac{2(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}k} & 1 + \frac{2y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} a-k \\ b-(2-\gamma) \end{pmatrix} \\ &\quad + O((a-k)^2) + O((a-k)(b-(2-\gamma))) + O((b-(2-\gamma))^2) + O(\epsilon). \end{aligned}$$

It then follows that

$$\hat{F}(z, a, b, y_0, \epsilon) = \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma}k \\ \frac{2(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}k} & 1 + \frac{2y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + O(\epsilon).$$

Identitfyng the leading terms for $\hat{E}$, $\hat{F}$, we introduce the notations (note that F_{lin} is a 2×2 matrix while $\hat{F}$ is a vector)

$$E_{lin}(z) = \begin{pmatrix} (2-\gamma)z & kz \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} \frac{z}{z^{\frac{2}{2-\gamma}}} & (2-\gamma)z \end{pmatrix}, \quad F_{lin}(z) = \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma}k \\ \frac{2(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}k} & 1 + \frac{2y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}} \end{pmatrix}.$$

We then define the linear operator

$$L(\hat{p}, \hat{\omega}) = \begin{pmatrix} L_1(\hat{p}, \hat{\omega}) \\ L_2(\hat{p}, \hat{\omega}) \end{pmatrix} := E_{lin} \frac{d}{dz} \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix} + F_{lin} \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix}.$$

and note formally that as $\epsilon \rightarrow 0$, the linearization of (3.42) is $L(\hat{p}, \hat{\omega}) = 0$.

As a comment, before writing down the difference equations, we will let

$$S[\epsilon] = \begin{pmatrix} 1 & 1 \\ -\frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} & \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{k}{2-\gamma} & \frac{k}{2-\gamma} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2-\gamma}{k} \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \end{pmatrix}.$$

and multiply on the left by S^T (3.42). This symmetrization is done to fix a loss of derivatives that occurs when doing difference estimates, and we note that to zeroth order in ϵ , S is the identity matrix. Using this matrix we define

$$\begin{aligned} \hat{N}(\epsilon, \hat{p}, \hat{\omega}) &= \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix} \\ &:= - \left(S^T \hat{E}(y, \hat{p}, \hat{\omega}, y_0, \epsilon) - E_{lin} \right) \begin{pmatrix} \hat{p}' \\ \hat{\omega}' \end{pmatrix} - \left(S^T \hat{F}(y, \hat{p}, \hat{\omega}, y_0, \epsilon) - F_{lin} \begin{pmatrix} \hat{p} \\ \hat{\omega} \end{pmatrix} \right). \end{aligned} \quad (3.43)$$

We rewrite the non-linear system as

$$\begin{cases} L(\hat{p}, \hat{\omega}) := \hat{N}(\epsilon, \hat{p}, \hat{\omega}) \\ \hat{p}(y_f) = \dot{p}_{0,d}[\epsilon], \quad \hat{\omega}(y_f) = \dot{\omega}_{0,d}[\epsilon], \quad \hat{p}'(y_f) = \dot{p}_{1,d}[\epsilon], \quad \hat{\omega}'(y_f) = \dot{\omega}_{1,d}[\epsilon] \end{cases}. \quad (3.44)$$

3.2.2 Properties of the linearized operator

We consider the linearized equation with an inhomogeneity,

$$\begin{pmatrix} (2-\gamma)z & kz \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} z^{1-\frac{2}{2-\gamma}} & (2-\gamma)z \end{pmatrix} \begin{pmatrix} p' \\ \omega' \end{pmatrix} + \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma} k \\ \frac{2(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{kz^{\frac{2}{2-\gamma}}} & 1 + \frac{2y_f^{\frac{2}{2-\gamma}}}{z^{\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = \begin{pmatrix} P \\ \Omega \end{pmatrix}. \quad (3.45)$$

We note that at $z = y_f$, we can simplify these matrices to zeroth order as

$$\begin{pmatrix} (2-\gamma)y_f & ky_f \\ \frac{(2-\gamma)^2}{k} y_f & (2-\gamma)y_f \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma} k \\ \frac{2(2-\gamma)^2}{k} & 3 \end{pmatrix}$$

and in particular the coefficient matrix has eigenvalues $\{0, 2(2-\gamma)y_f\}$. Left-eigenvectors corresponding to each of these are

$$\left(1 \quad -\frac{k}{2-\gamma}\right), \quad \left(1 \quad \frac{k}{2-\gamma}\right)$$

When we multiply the equation (evaluated at $y = y_f$) on the left by each of these respectively, we obtain the following constraints:

$$\begin{cases} -2(2-\gamma)p(y_f) - \frac{3\gamma-1}{2-\gamma} k\omega(y_f) & = P(y_f) - \frac{k}{2-\gamma} \Omega(y_f) \\ 2(2-\gamma)y_f p'(y_f) + 2ky_f \omega'(y_f) & \\ +2(2-\gamma)p(y_f) + \frac{7-3\gamma}{2-\gamma} k\omega(y_f) & = P(y_f) + \frac{k}{2-\gamma} \Omega(y_f) \end{cases}. \quad (3.46)$$

If we start with (3.45), invert the coefficient matrix, and multiply on the left by $(1 - \frac{k}{2-\gamma})$, we obtain

$$-2(2-\gamma)\frac{p}{z-y_f} - \frac{k(3\gamma-1)}{2-\gamma}\frac{\omega}{z-y_f} = \frac{(P - \frac{k}{2-\gamma}\Omega)}{z-y_f} + o(\frac{1}{z-y_f}).$$

We see that if $(p, \omega) \in C^1$ then

$$\frac{P - \frac{k}{2-\gamma}\Omega - (P(y_f) - \frac{k}{2-\gamma}\Omega(y_f))}{z-y_f}$$

must be continuous on a neighborhood of $z = y_f$. Multiplying the original equation by $(1 - \frac{k}{2-\gamma})$ and expanding to $o(z - y_f)$ gives

$$\begin{aligned} & 2(z - y_f)p'(y_f) + \left(-2(2-\gamma)p(y_f) - \frac{3\gamma-1}{2-\gamma}\omega(y_f)\right) \\ & + \left(-2(2-\gamma)p'(y_f) - \frac{(3\gamma-1)k\omega'(y_f)}{2-\gamma}\right)(z - y_f) + \left(\frac{4p(y_f)}{y_*} + \frac{4k\omega(y_f)}{(2-\gamma)^2 y_f}\right)(z - y_f) \\ & = P(y_f) - \frac{k}{2-\gamma}\Omega(y_f) + (P - \frac{k}{2-\gamma}\Omega)'(y_f) \cdot (z - y_f) + o(z - y_f). \end{aligned}$$

By (3.46), the equations agree to order 0. Comparing the terms of order 1, we get the following constraint

$$2(\gamma-1)y_f p'(y_f) - \frac{(3\gamma-1)}{2-\gamma}ky_f \omega'(y_f) + 4p(y_f) + \frac{4}{(2-\gamma)^2}k\omega(y_f) = y_f(P - \frac{k}{2-\gamma}\Omega)'(y_f). \quad (3.47)$$

We now suppose that

$$P = \sigma_1 + o(1), \quad \Omega = \sigma_2 + o(1), \quad \frac{P - \frac{k}{2-\gamma}\Omega}{z-y_f} = \frac{(\sigma_1 - \frac{k}{2-\gamma}\sigma_2)}{z-y_f} + \sigma_3 + o(1). \quad (3.48)$$

We will also assume

$$p = p_0 + p_1(z - y_f) + o(z - y_f), \quad \omega = \omega_0 + \frac{1}{y_f}\omega_1(z - y_f) + o(z - y_f). \quad (3.49)$$

In terms of these notations, the system (3.46), (3.47) becomes

$$\begin{cases} -2(2-\gamma)p_0 - \frac{3\gamma-1}{2-\gamma}k\omega_0 = \sigma_1 - \frac{k}{2-\gamma}\sigma_2 \\ 2(2-\gamma)y_f p_1 + 2ky_f \omega_1 + 2(2-\gamma)p_0 + \frac{7-3\gamma}{2-\gamma}k\omega_0 = \sigma_1 + \frac{k}{2-\gamma}\sigma_2 \\ 2(\gamma-1)y_f p_1 - \frac{(3\gamma-1)}{2-\gamma}y_f k\omega_1 + 4p_0 + \frac{4}{(2-\gamma)^2}k\omega_0 = y_f \sigma_3 \end{cases}.$$

We will consider ω_0 to be a free variable. From the first equation, we have that

$$p_0 = -\frac{(3\gamma-1)}{2(2-\gamma)^2}k\omega_0 - \frac{\sigma_1}{2(2-\gamma)} + \frac{k\sigma_2}{2(2-\gamma)^2}. \quad (3.50)$$

We may then substitute this into the next two equations to obtain the system

$$\begin{cases} 2(2-\gamma)y_fp_1 + 2y_fk\omega_1 + \frac{2(4-3\gamma)}{(2-\gamma)}k\omega_0 = 2\sigma_1 \\ 2(\gamma-1)y_fp_1 - \frac{3\gamma-1}{2-\gamma}y_fk\omega_1 - \frac{6(\gamma-1)}{(2-\gamma)^2}k\omega_0 = \frac{2}{2-\gamma}\sigma_1 - \frac{2k\sigma_2}{(2-\gamma)^2} + y_f\sigma_3 \end{cases}.$$

We may multiply the first of these equations by $\frac{\gamma-1}{2-\gamma}$, and then subtract the second equation from it to obtain

$$\omega_1 = -\frac{2(\gamma-1)(7-3\gamma)}{(2-\gamma)(5\gamma-3)}\frac{1}{y_f}\omega_0 - \frac{2(2-\gamma)}{5\gamma-3}\frac{1}{y_fk}\sigma_1 + \frac{2}{(5\gamma-3)(2-\gamma)}\frac{1}{y_f}\sigma_2 - \frac{2-\gamma}{5\gamma-3}\frac{1}{k}\sigma_3. \quad (3.51)$$

$$\begin{aligned} p_1 = & -\frac{-9\gamma^2+9\gamma+2}{(2-\gamma)^2(5\gamma-3)}\frac{k}{y_f}\omega_0 + \frac{3\gamma+1}{(2-\gamma)(5\gamma-3)}\frac{1}{y_f}\sigma_1 \\ & - \frac{2}{(5\gamma-3)(2-\gamma)^2}\frac{k}{y_f}\sigma_2 + \frac{1}{5\gamma-3}\sigma_3. \end{aligned} \quad (3.52)$$

We comment that if $\sigma_1 = \sigma_2 = \sigma_3 = 0$, then, by plugging in $\omega_0 = -(2-\gamma)$, the homogeneous solution has first-order Taylor expansion

$$\begin{cases} p &= \frac{(3\gamma-1)}{2(2-\gamma)}k + \frac{-9\gamma^2+9\gamma+2}{(5\gamma-3)(2-\gamma)}\frac{k}{y_f}(z-y_f) + o(z-y_f) \\ \omega &= -(2-\gamma) + \frac{2(\gamma-1)(7-3\gamma)}{5\gamma-3}\frac{1}{y_f}(z-y_f) + o(z-y_f) \end{cases}. \quad (3.53)$$

These Taylor coefficients are consistent with those computed in (3.39), (3.40). We now state the main estimate regarding the linearized operator. As a comment, in comparison to the analogous statement in [100], part (e) is now modified, as in this problem it is more convenient to allow for non-zero values of $\sigma_1, \sigma_2, \sigma_3$.

Lemma 3.2.1. (a) *There exists a unique homogeneous solution $(p_{hom}(z), \omega_{hom}(z))$ such that $L(p_{hom}, \omega_{hom}) = 0$ and such that its first order Taylor polynomials at $z = y_f$ agree with (3.53).*

(b) *There exists analytic fundamental matrices $U_\infty(z), U_0(z)$ for L defined on $(1, \infty)$ and $(0, 1)$ respectively. Moreover, these can be chosen such that as $z \rightarrow \infty$*

$$\begin{aligned} U_\infty(z) &= \begin{pmatrix} 1 & -\frac{3(\gamma-1)k}{2-\gamma}\frac{1}{z^{\frac{1}{2-\gamma}}} \\ 0 & \frac{1}{z^{\frac{1}{2-\gamma}}} \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^{\frac{2}{2-\gamma}}}) & O(\frac{1}{z^{\frac{3}{2-\gamma}}}) \\ O(\frac{1}{z^{\frac{2}{2-\gamma}}}) & O(\frac{1}{z^{\frac{3}{2-\gamma}}}) \end{pmatrix}, \\ U_\infty^{-1} &= \begin{pmatrix} 1 & \frac{3(\gamma-1)k}{2-\gamma}\frac{1}{z^{\frac{1}{2-\gamma}}} \\ 0 & z^{\frac{1}{2-\gamma}} \end{pmatrix} + \begin{pmatrix} O(\frac{1}{z^{\frac{2}{2-\gamma}}}) & O(\frac{1}{z^{\frac{2}{2-\gamma}}}) \\ O(\frac{1}{z^{\frac{2}{2-\gamma}}}) & O(\frac{1}{z^{\frac{2}{2-\gamma}}}) \end{pmatrix}, \end{aligned}$$

and as $z \rightarrow 0$,

$$\begin{aligned} U_0(z) &= \frac{1}{z^\mu} \begin{pmatrix} \cos(\nu \log(z)) & \sin(\nu \log(z)) \\ \cos(\nu \log(z) + \theta_0) & \sin(\nu \log(z) + \theta_0) \end{pmatrix} + O(z^{-\mu+\frac{2}{2-\gamma}}), \\ U_0^{-1}(z) &= \frac{z^\mu}{\sin(\theta_0)} \begin{pmatrix} \sin(\nu \log(z) + \theta_0) & -\sin(\nu \log(z)) \\ -\cos(\nu \log(z) + \theta_0) & \cos(\nu \log(z)) \end{pmatrix} + O(z^{\mu+\frac{2}{2-\gamma}}). \end{aligned}$$

(c) There exists μ_3, μ_4, c_1, d_1 with $\sqrt{\mu_3^2 + \mu_4^2} \neq 0$, $c_1 > 0$, $d_1 \in \mathbb{R}/2\pi\mathbb{Z}$ such that on $(1, \infty)$ and $(0, 1)$ respectively we have

$$\begin{pmatrix} p_{hom} \\ \omega_{hom} \end{pmatrix} = U_\infty \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix}, \quad \begin{pmatrix} p_{hom} \\ \omega_{hom} \end{pmatrix} = U_0 \begin{pmatrix} c_1 \sin(d_1) \\ c_1 \cos(d_1) \end{pmatrix}. \quad (3.54)$$

(d) Also, there exists constant invertible matrices M_0, M_∞ such that as $z \rightarrow 1^\pm$, we have

$$\begin{aligned} U_\infty M_\infty, U_0 M_0 = & \begin{pmatrix} \frac{(3\gamma-1)k}{2(2-\gamma)} & \frac{(3\gamma-1)k}{2(2-\gamma)} |z - y_f|^{-\frac{5(\gamma-1)}{2}} \\ -(2-\gamma) & -\frac{3\gamma-1}{2} |z - y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix} \\ & + \begin{pmatrix} \frac{(-9\gamma^2+9\gamma+2)}{(2-\gamma)(5\gamma-3)} \frac{k}{y_f} (z - y_f) & \frac{(3\gamma-1)(25\gamma^3-136\gamma^2+191\gamma-72)k}{8(2-\gamma)^2(7-5\gamma)y_f} (z - y_f) |z - y_f|^{-\frac{5(\gamma-1)}{2}} \\ \frac{2(\gamma-1)(7-3\gamma)}{y_f(5\gamma-3)} (z - y_f) & -\frac{(\gamma-1)(3\gamma-1)(25\gamma^2-111\gamma+104)}{8(2-\gamma)(7-5\gamma)y_f} (z - y_f) |z - y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix} \\ & + \begin{pmatrix} o(z - y_f) & o(z - y_f) |z - y_f|^{-\frac{5(\gamma-1)}{2}} \\ o(z - y_f) & o(z - y_f) |z - y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix}. \end{aligned}$$

(e) If we define the principal matrix solution as

$$S(z, z') = \begin{cases} U_0(z)U_0^{-1}(z') \begin{pmatrix} (2-\gamma)z' & kz' \\ \frac{2\pi(2-\gamma)^2}{4-3\gamma} (z')^{1-\frac{2}{2-\gamma}} & (2-\gamma)z' \end{pmatrix}^{-1}, & z, z' \in (0, 1) \\ U_\infty(z)U_\infty^{-1}(z') \begin{pmatrix} (2-\gamma)z' & kz' \\ \frac{2\pi(2-\gamma)^2}{4-3\gamma} (z')^{1-\frac{2}{2-\gamma}} & (2-\gamma)z' \end{pmatrix}^{-1}, & z, z' \in (1, \infty) \end{cases},$$

then if $P(z), \Omega(z), \frac{P - \frac{k}{2-\gamma}\Omega - P(y_f) - \frac{k}{2-\gamma}\Omega(y_f)}{y - y_f}$ are continuous on $(0, \infty)$ and satisfy (3.48) then

$$(p, \omega) := R(P(z), \Omega(z)) := \int_1^z S(z, z') \begin{pmatrix} P(z') \\ \Omega(z') \end{pmatrix} dz'$$

defines an element of $C^1((0, \infty))$ which solves $L(p, \omega) = (P, \Omega)$ and the coefficients (defined by (3.49)) satisfy (3.50), (3.51), (3.52) and also

$$\omega_0 = -\frac{\frac{(2-\gamma)}{k}\sigma_1 - \sigma_2}{5(\gamma-1)}.$$

For the proof see Section 3.5.1.

3.2.3 Sonic point conditions at the difference equation level

We now note that if p, ω are C^1 then as $y \rightarrow y_*$, we have

$$\begin{aligned} \tilde{E}(y_*, \tilde{p}, \tilde{\omega}) &:= \tilde{E}(y_*, \tilde{p}(y_*), \tilde{\omega}(y_*)) + \tilde{E}_1(y_*, \tilde{p}(y_*), \tilde{\omega}(y_*), \tilde{p}'(y_*), \tilde{\omega}'(y_*))(y - y_*) + o(y - y_*), \\ \tilde{F}(y_*, \tilde{p}, \tilde{\omega}) &:= \tilde{F}(y_*, \tilde{p}(y_*), \tilde{\omega}(y_*)) + \tilde{F}_1(y_*, \tilde{p}(y_*), \tilde{\omega}(y_*), \tilde{p}'(y_*), \tilde{\omega}'(y_*))(y - y_*) + o(y - y_*), \end{aligned}$$

where

$$\begin{aligned}\tilde{E}_1(y, a, b, a_1, b_1) &= \left(\frac{\partial \tilde{E}}{\partial y} + \frac{\partial \tilde{E}}{\partial a} a_1 + \frac{\partial \tilde{E}}{\partial b} b_1 \right) |_{y, a, b}, \\ \tilde{F}_1(y, a, b, a_1, b_1) &= \left(\frac{\partial \tilde{F}}{\partial y} + \frac{\partial \tilde{F}}{\partial a} a_1 + \frac{\partial \tilde{F}}{\partial b} b_1 \right) |_{y, a, b}.\end{aligned}$$

We now note that by (3.29), (3.30), (3.31) and (3.15) that the following equations hold

$$\left\{ \begin{aligned} \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{E}(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon]) &= 0 \\ \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{F}(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon]) &= 0 \\ \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{E}(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon]) \cdot \begin{pmatrix} \dot{p}_1[\epsilon] \\ \dot{\omega}_1[\epsilon] \end{pmatrix} + \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{F}(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon]) &= 0 \\ \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{E}_1(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon], \dot{p}_1[\epsilon], \dot{\omega}_1[\epsilon]) \begin{pmatrix} \dot{p}_1[\epsilon] \\ \dot{\omega}_1[\epsilon] \end{pmatrix} \\ + \left(1 - \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \tilde{F}_1(\dot{y}_*[\epsilon], \dot{p}_0[\epsilon], \dot{\omega}_0[\epsilon], \dot{p}_1[\epsilon], \dot{\omega}_1[\epsilon]) &= 0 \end{aligned} \right. \quad (3.55)$$

We next transition variables. We define $\underline{E}_1, \underline{F}_1, \hat{E}_1, \hat{E}_2$ by requiring

$$\begin{aligned}\underline{E}(y_f, \underline{p}, \underline{\omega}) &= \underline{E}(y_f, \underline{p}(y_f), \underline{\omega}(y_f)) + \underline{E}_1(y_f, \underline{p}(y_f), \underline{\omega}(y_f), \underline{p}'(y_f), \underline{\omega}'(y_f))(z - y_f) + o(z - y_f), \\ \underline{F}(y_f, \underline{p}, \underline{\omega}) &= \underline{F}(y_f, \underline{p}(y_f), \underline{\omega}(y_f)) + \underline{F}_1(y_f, \underline{p}(y_f), \underline{\omega}(y_f), \underline{p}'(y_f), \underline{\omega}'(y_f))(z - y_f) + o(z - y_f), \\ \hat{E}(y_f, \hat{p}, \hat{\omega}) &= \hat{E}(y_f, \hat{p}(y_f), \hat{\omega}(y_f)) + \hat{E}_1(y_f, \hat{p}(y_f), \hat{\omega}(y_f), \hat{p}'(y_f), \hat{\omega}'(y_f))(z - y_f) + o(z - y_f), \\ \hat{F}(y_f, \hat{p}, \hat{\omega}) &= \hat{F}(y_f, \hat{p}(y_f), \hat{\omega}(y_f)) + \hat{F}_1(y_f, \hat{p}(y_f), \hat{\omega}(y_f), \hat{p}'(y_f), \hat{\omega}'(y_f))(z - y_f) + o(z - y_f)\end{aligned}$$

and note that these are given in terms of the previous functions by

$$\begin{aligned}\underline{E}_1(z, a, b, a_1, b_2, y_0, \epsilon) &= \tilde{E}_1\left(\frac{z - \beta}{\alpha}, a, b, \alpha a_1, \alpha b_1\right), \\ \underline{F}_1(z, a, b, a_1, b_2, y_0, \epsilon) &= \frac{1}{\alpha} \tilde{F}_1\left(\frac{z - \beta}{\alpha}, a, b, \alpha a_1, \alpha b_1\right), \\ \hat{E}_1(z, a, b, a_1, b_2, y_0, \epsilon) &= \underline{E}_1(z, k + \epsilon a, 2 - \gamma + \epsilon b, \epsilon a_1, \epsilon b_1, y_0, \epsilon), \\ \hat{F}_1(z, a, b, a_1, b_2, y_0, \epsilon) &= \frac{1}{\epsilon} \underline{F}_1(z, k + \epsilon a, 2 - \gamma + \epsilon b, \epsilon a_1, \epsilon b_1, y_0, \epsilon).\end{aligned}$$

We then define the “difference” equation boundary condition functions

$$\begin{cases} \dot{p}_{0,d} = \frac{\dot{p}[\epsilon] - k}{\epsilon} \\ \dot{\omega}_{0,d} = \frac{\dot{\omega}[\epsilon] - (2 - \gamma)}{\epsilon} \\ \dot{p}_{1,d} = \frac{1}{\alpha \epsilon} \dot{p}_1[\epsilon] \\ \dot{\omega}_{1,d} = \frac{1}{\alpha \epsilon} \dot{\omega}_1[\epsilon] \end{cases}.$$

From this we obtain the identities

$$\left\{ \begin{array}{l} \left(1 \quad -\frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{E}(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon]) = 0 \\ \left(1 \quad -\frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{F}(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon]) = 0 \\ \left(1 \quad \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{E}(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon]) \cdot \begin{pmatrix} \dot{p}_{1,d}[\epsilon] \\ \dot{\omega}_{1,d}[\epsilon] \end{pmatrix} + \left(1 \quad \frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{F}(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon]) = 0 \\ \left(1 \quad -\frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{E}_1(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon], \dot{p}_{1,d}[\epsilon], \dot{\omega}_{1,d}[\epsilon]) \begin{pmatrix} \dot{p}_{1,d}[\epsilon] \\ \dot{\omega}_{1,d}[\epsilon] \end{pmatrix} \\ + \left(1 \quad -\frac{\dot{p}_0[\epsilon]}{\dot{\omega}_0[\epsilon]} \right) \hat{F}_1(y_f, \dot{p}_{0,d}[\epsilon], \dot{\omega}_{0,d}[\epsilon], \dot{p}_{1,d}[\epsilon], \dot{\omega}_{1,d}[\epsilon]) = 0 \end{array} \right. \quad (3.56)$$

In particular, we return now to the difference equation (3.44). We suppose that the input functions, $(\hat{p}_{in}, \hat{\omega}_{in})$, satisfy

$$\hat{p}_{in}(y_f) = \dot{p}_{0,d}[\epsilon], \quad \hat{\omega}_{in}(y_f) = \dot{\omega}_{0,d}[\epsilon], \quad \hat{p}'_{in}(y_f) = \dot{p}_{1,d}[\epsilon], \quad \hat{\omega}'_{in}(y_f) = \dot{\omega}_{1,d}[\epsilon]. \quad (3.57)$$

We may then note that

$$\begin{aligned} \left(1 \quad -\frac{k}{2-\gamma} \right) \hat{N}(\epsilon, \hat{p}_{in}, \hat{\omega}_{in}) &= - \left(1 \quad -\frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{E} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} + \left(1 \quad -\frac{k}{2-\gamma} \right) E_{lin} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} \\ &\quad - \left(1 \quad -\frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{F} + \left(1 \quad -\frac{k}{2-\gamma} \right) F_{lin} \begin{pmatrix} \hat{p}_{in} \\ \hat{\omega}_{in} \end{pmatrix}, \\ \left(1 \quad \frac{k}{2-\gamma} \right) \hat{N}(\epsilon, \hat{p}_{in}, \hat{\omega}_{in}) &= - \left(1 \quad \frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{E} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} + \left(1 \quad \frac{k}{2-\gamma} \right) E_{lin} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} \\ &\quad - \left(1 \quad \frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{F} + \left(1 \quad \frac{k}{2-\gamma} \right) F_{lin} \begin{pmatrix} \hat{p}_{in} \\ \hat{\omega}_{in} \end{pmatrix}. \end{aligned}$$

We comment that by (3.57), (3.56) we have

$$\begin{aligned} \left(1 \quad -\frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{E} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} + \left(1 \quad -\frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{F} &= o(z - y_f), \\ \left(1 \quad -\frac{k}{2-\gamma} \right) E_{lin} &= \begin{pmatrix} 2 & 0 \end{pmatrix} (z - y_f) + o(z - y_f), \\ \left(1 \quad -\frac{k}{2-\gamma} \right) F_{lin} &= \begin{pmatrix} -2(2-\gamma) & -\frac{(3\gamma-1)k}{2-\gamma} \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{4}{y_f} & \frac{4k}{(2-\gamma)^2 y_f} \end{pmatrix} (z - y_f) + o(z - y_f), \\ \left(1 \quad \frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{E} \begin{pmatrix} \hat{p}'_{in} \\ \hat{\omega}'_{in} \end{pmatrix} + \left(1 \quad \frac{\dot{p}_0}{\dot{\omega}_0} \right) \hat{F} &= o(1), \\ \left(1 \quad \frac{k}{2-\gamma} \right) E_{lin} &= \begin{pmatrix} 2(2-\gamma)y_f & 2ky_f \end{pmatrix} + o(1), \\ \left(1 \quad \frac{k}{2-\gamma} \right) F_{lin} &= \begin{pmatrix} 2(2-\gamma) & (7-3\gamma)\frac{k}{2-\gamma} \end{pmatrix} + o(1). \end{aligned}$$

In particular, $\hat{N}$ satisfies (3.48) with coefficients

$$\begin{cases} \hat{\sigma}_1[\epsilon] := (2-\gamma)y_f \dot{p}_{1,d} + y_f k \dot{\omega}_{1,d} + \frac{4-3\gamma}{(2-\gamma)} k \dot{\omega}_{0,d} \\ \hat{\sigma}_2[\epsilon] := \frac{(2-\gamma)^2 y_f}{k} \dot{p}_{1,d} + (2-\gamma)y_f \dot{\omega}_{1,d} + \frac{2(2-\gamma)^2}{k} \dot{p}_{0,d} + 3\dot{\omega}_{0,d} \\ \hat{\sigma}_3[\epsilon] := 2(\gamma-1)\dot{p}_{1,d} - \frac{(3\gamma-1)}{2-\gamma} k \dot{\omega}_{1,d} + \frac{4}{y_f} \dot{p}_{0,d} + \frac{4k}{(2-\gamma)^2 y_f} \dot{\omega}_{0,d} \end{cases}.$$

In this case, if one substitutes $\dot{\sigma}_1[\epsilon], \dot{\sigma}_2[\epsilon], \dot{\sigma}_3[\epsilon], \dot{\omega}_{0,d}$ along with into the equations (3.50), (3.51), (3.52) then they are consistent with

$$(p_0, p_1, \omega_1) = (\dot{p}_{0,d}, \dot{p}_{1,d}, \dot{\omega}_{1,d}).$$

3.2.4 Difference equations and iteration scheme

We now define the “error” boundary condition parameterization functions to be

$$\begin{aligned} \dot{p}_{0,e}[\epsilon] &= \dot{p}_{0,d}[\epsilon] - \frac{(3\gamma - 1)}{2(2 - \gamma)}k, & \dot{\omega}_{0,e}[\epsilon] &= \dot{\omega}_{0,d}[\epsilon] + (2 - \gamma) \\ \dot{p}_{1,e}[\epsilon] &= \dot{p}_{1,d}[\epsilon] - \frac{-9\gamma^2 + 9\gamma + 2}{(5\gamma - 3)(2 - \gamma)}\frac{k}{y_f}, & \dot{\omega}_{1,e} &= \dot{\omega}_{1,d}[\epsilon] - \frac{2(\gamma - 1)(7 - 3\gamma)}{5\gamma - 3}\frac{1}{y_f}. \end{aligned}$$

We note that

$$\dot{p}_{0,e}, \dot{\omega}_{0,e}, \dot{p}_{1,e}, \dot{\omega}_{1,e} = O(\epsilon).$$

We also have that in terms of these coefficients

$$\begin{cases} \dot{\sigma}_1[\epsilon] = (2 - \gamma)y_f\dot{p}_{1,e} + y_fk\dot{\omega}_{1,e} + \frac{4-3\gamma}{(2-\gamma)}k\dot{\omega}_{0,e} \\ \dot{\sigma}_2[\epsilon] = \frac{(2-\gamma)^2y_f}{k}\dot{p}_{1,e} + (2 - \gamma)y_f\dot{\omega}_{1,e} + \frac{2(2-\gamma)^2}{k}\dot{p}_{0,d} + 3\dot{\omega}_{0,e} \\ \dot{\sigma}_3[\epsilon] = 2(\gamma - 1)\dot{p}_{1,e} - \frac{(3\gamma-1)}{2-\gamma}k\dot{\omega}_{1,e} + \frac{4}{y_f}\dot{p}_{0,e} + \frac{4k}{(2-\gamma)^2y_f}\dot{\omega}_{0,e} \end{cases},$$

so in particular,

$$\dot{\sigma}_1[\epsilon], \dot{\sigma}_2[\epsilon], \dot{\sigma}_3[\epsilon] = O(\epsilon).$$

We now consider (3.44) and substitute

$$(\hat{p}, \hat{\omega}) = (p_{hom}, \omega_{hom}) + (p, \omega)$$

which gives the following difference equations:

$$\begin{cases} L(p, \omega) = N(\epsilon, p, \omega) \\ p(y_f) = \dot{p}_{0,e}, & \omega(y_f) = \dot{\omega}_{0,e}, & p'(y_f) = \dot{\omega}_{1,e}, & \omega'(y_f) = \dot{\omega}_{1,e} \end{cases}, \quad (3.58)$$

where

$$N(\epsilon, p, \omega) := \hat{N}(\epsilon, p_{hom} + p, \omega_{hom} + \omega).$$

We then rewrite this as

$$(p, \omega) = RN(\epsilon, p, \omega) + \left(\frac{\frac{2-\gamma}{k}\dot{\sigma}_1 - \dot{\sigma}_2}{5(\gamma - 1)} + \dot{\omega}_{0,e} \right) \frac{1}{-(2 - \gamma)}(p_{hom}, \omega_{hom}). \quad (3.59)$$

As a comment, to ensure that we are not dividing by zero in the original equation, we need solutions to satisfy

$$k + \epsilon(p_{hom} + p) > 0$$

The integral equation will be solved using a fixed point method. We first introduce

$$D_\infty f(z) = -z^{1+\frac{1}{2-\gamma}}\partial_z f$$

Definition 3.2.2. We define the “solution” norm to be

$$\|(p, \omega)\|_{X_{y_0}} := \sup_{z \geq y_f} |p| + |z^{\frac{1}{2-\gamma}} \omega| + |D_\infty p| + |D_\infty z^{\frac{1}{2-\gamma}} \omega| + \sup_{y_0 \leq z \leq y_f} z^\mu (|p| + |\omega|) + z^{\mu+1} (|p'| + |\omega'|).$$

We define our “solution” Banach space to be

$$X_{y_0} = \{(p, \omega) \in C^1([y_0, \infty)) : \lim_{z \rightarrow \infty} D_\infty(p), D_\infty(z^{\frac{1}{2-\gamma}} \omega) \text{ exists, } \|(p, \omega)\|_{X_{y_0}} < \infty\}.$$

Definition 3.2.3. We define our “nonlinear” norm to be

$$\begin{aligned} \|(P, \Omega)\|_{N_{y_0}} := & \sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{P - \frac{k}{2-\gamma} \Omega - (P(0) - \frac{k}{2-\gamma} \Omega(0))}{z - y_f} \right| \\ & + \sup_{z \geq y_f} |z^{\frac{1}{2-\gamma}} P| + |z^{\frac{2}{2-\gamma}} \Omega| + \sup_{y_0 \leq z \leq y_f} |z^{2\mu} P| + |z^{2\mu + \frac{2}{2-\gamma}} \Omega|. \end{aligned}$$

We define our “nonlinear” Banach space to be

$$\begin{aligned} N_{y_0} = \{(P, \Omega) \in C^0([y_0, \infty)) : \frac{P - \frac{k}{2-\gamma} \Omega}{z - y_f} \in C^0([y_0, \infty)) \\ , \lim_{z \rightarrow \infty} (z^{\frac{1}{2-\gamma}} P, z^{\frac{2}{2-\gamma}} \Omega) \text{ exists, } \|(P, \Omega)\|_{N_{y_0}} < \infty\}. \end{aligned}$$

Now for each $\epsilon > 0$, we shall define affine subspaces on which the iteration will take place. We define the following ϵ -dependent affine spaces,

$$\begin{aligned} X_{y_0}[\epsilon] = \{(p, \omega) \in X_{y_0} : p(y_f) = \dot{p}_{0,e}[\epsilon], \\ \omega(y_f) = \dot{\omega}_{0,e}[\epsilon], \quad p'(y_f) = \dot{p}_{1,e}[\epsilon], \quad \omega'(y_f) = \dot{\omega}_{1,e}[\epsilon]\}, \end{aligned}$$

$$N_{y_0}[\epsilon] = \{(P, \Omega) \in N_{y_0} : P(y_f) = \dot{\sigma}_1[\epsilon], \quad \Omega(y_f) = \dot{\sigma}_2[\epsilon], \quad (P - \frac{k}{2-\gamma} \Omega)'(y_f) = \dot{\sigma}_3[\epsilon]\}.$$

We also define the difference spaces

$$\begin{aligned} X_{y_0}[\epsilon_1, \epsilon_2] = \{(p, \omega) \in X_{y_0} : p(y_f) = \dot{p}_{0,e}[\epsilon_1] - \dot{p}_{0,e}[\epsilon_2], \quad \omega(y_f) = \dot{\omega}_{0,e}[\epsilon_1] - \dot{\omega}_{0,e}[\epsilon_2] \\ p'(y_f) = \dot{p}_{1,e}[\epsilon_1] - \dot{p}_{1,e}[\epsilon_2], \quad \omega'(y_f) = \dot{\omega}_{1,e}[\epsilon_1] - \dot{\omega}_{1,e}[\epsilon_2]\}, \end{aligned}$$

$$\begin{aligned} N_{y_0}[\epsilon_1, \epsilon_2] = \{(P, \Omega) \in N_{y_0} : P(y_f) = \dot{\sigma}_1[\epsilon_1] - \dot{\sigma}_1[\epsilon_2], \quad \Omega(y_f) = \dot{\sigma}_2[\epsilon_1] - \dot{\sigma}_2[\epsilon_2] \\ (P - \frac{k}{2-\gamma} \Omega)'(y_f) = \dot{\sigma}_3[\epsilon_1] - \dot{\sigma}_3[\epsilon_2]\}. \end{aligned}$$

Lemma 3.2.4. For $y_0 \leq \frac{y_f}{2}$ there exists $C_1 > 1$ such that if $(P, \Omega) \in N_{y_0}[\epsilon]$ then

$$(p, \omega) := R(P, \Omega) + \left(\frac{\frac{2-\gamma}{k} \dot{\sigma}_1[\epsilon] - \dot{\sigma}_2[\epsilon]}{5(\gamma-1)} + \dot{\omega}_{0,e}[\epsilon] \right) \frac{1}{-(2-\gamma)} (p_{hom}, \omega_{hom}) \in X_{y_0}[\epsilon]$$

and

$$\|(p, \omega)\|_{X_{y_0}} \leq \frac{1}{2} C_1 y_0^{-\mu} \|(P, \Omega)\|_{N_{y_0}} + \frac{1}{2} C_1 |\epsilon|. \quad (3.60)$$

Moreover, if for some ϵ_1, ϵ_2 , we have that $(P_i, \Omega_i) \in N_{y_0}[\epsilon_i]$ and we define

$$(p_i, \omega_i) = R(P_i, \Omega_i) + \left(\frac{\frac{2-\gamma}{k} \dot{\sigma}_1 - \dot{\sigma}_2}{5(\gamma-1)} + \dot{\omega}_{0,e} \right) \frac{1}{-(2-\gamma)} (p_{hom}, \omega_{hom})$$

then $(p_1 - p_2, \omega_1 - \omega_2) \in X_{y_0}[\epsilon_1, \epsilon_2]$ and

$$\|(p_1 - p_2, \omega_1 - \omega_2)\|_{X_{y_0}} \leq \frac{1}{2} C_1 y_0^{-\mu} \|P_1 - P_2, \Omega_1 - \Omega_2\|_{N_{y_0}} + \frac{1}{2} C_1 |\epsilon_1 - \epsilon_2|. \quad (3.61)$$

For a proof see Subsection 3.5.2.

Also, we have that the non-linearity is bounded as follows.

Lemma 3.2.5. *If $y_0 \leq \frac{y_f}{2}$, then for all $C > 0$ there exists $C_0(C) \ll 1$ such that if $\epsilon \leq C_0(C) y_0^\mu$ then there exists $C_2(C) \geq 1$ such that if $(p, \omega) \in X_{y_0}[\epsilon]$, $\|(p, \omega)\|_{X_{y_0}} \leq C$ then*

$$(P, \Omega) := N(\epsilon, p, \omega) \in N_{y_0}[\epsilon], \quad \|(P, \Omega)\|_{N_{y_0}} \leq |\epsilon| C_2. \quad (3.62)$$

Moreover, if for some ϵ_1, ϵ_2 , we have $(p_i, \omega_i) \in X_{y_0}[\epsilon_i]$ then $N(\epsilon_1, p_1, \omega_1) - N(\epsilon_2, p_2, \omega_2) \in N_{y_0}[\epsilon_1, \epsilon_2]$ and

$$\|N(\epsilon_1, p_1, \omega_1) - N(\epsilon_2, p_2, \omega_2)\|_{N_{y_0}} \leq C_2 |\epsilon| \|(p_1, \omega_1) - (p_2, \omega_2)\|_{X_{y_0}} + C_2 |\epsilon_1 - \epsilon_2|. \quad (3.63)$$

For a proof see Subsection 3.5.3.

Lemma 3.2.6. *There exists a universal constant C_3 so that if one fixes $y_0 \leq \frac{y_f}{2}$ then for $|\epsilon| \ll y_0^\mu$ there exists a unique solution $(p[\epsilon](y), \omega[\epsilon](y)) \in X_{y_0}[\epsilon]$ to (3.58) with*

$$\|(p[\epsilon], \omega[\epsilon])\|_{X_{y_0}} \leq C_3 |\epsilon| y_0^{-\mu}.$$

Moreover, if $|\epsilon_1|, |\epsilon_2| \ll y_0^\mu$, then we have the Lipschitz estimate

$$\|(p[\epsilon_1], \omega[\epsilon_1]) - (p[\epsilon_2], \omega[\epsilon_2])\|_{X_{y_0}} \leq C_3 y_0^{-\mu} |\epsilon_1 - \epsilon_2|. \quad (3.64)$$

For a proof see Subsection 3.5.4.

If we convert this back to the original variables we obtain

$$\tilde{\rho} = \frac{k}{y^{\frac{2}{2-\gamma}}} + \epsilon \frac{p_{hom}(\alpha y + \beta)}{y^{\frac{2}{2-\gamma}}} + \epsilon \frac{p[\epsilon](\alpha y + \beta)}{y^{\frac{2}{2-\gamma}}}, \quad \tilde{u} = \epsilon y (\omega_{hom}(\alpha y + \beta) + \omega[\epsilon](\alpha y + \beta)).$$

We may express this as

$$\tilde{\rho} = \frac{k}{y^{\frac{2}{2-\gamma}}} + \epsilon \frac{p_{hom}(y)}{y^{\frac{2}{2-\gamma}}} + \epsilon \rho_{ext}[\epsilon](y), \quad \tilde{u} = \epsilon y \omega_{hom}(y) + \epsilon u_{ext}[\epsilon](y)$$

where

$$\rho_{ext}[\epsilon](y) := \frac{p_{hom}(\alpha y + \beta) - p_{hom}(y)}{y^{\frac{2}{2-\gamma}}} + \frac{p[\epsilon](\alpha y + \beta)}{y^{\frac{2}{2-\gamma}}}, \quad (3.65)$$

$$u_{ext}[\epsilon](y) := y(\omega_{hom}(\alpha y + \beta) - \omega_{hom}(y)) + y\omega[\epsilon](\alpha y + \beta). \quad (3.66)$$

Converting the lemma to these variables gives the following result.

Lemma 3.2.7. For $y_0 \leq 1$, $\epsilon \ll y_0^\mu$, there exists $\tilde{\rho}[\epsilon], \tilde{u}[\epsilon] \in C^1([y_0, \infty))$ which satisfy (3.4) and may be decomposed as

$$\begin{cases} \tilde{\rho}_{ext}[\epsilon](y) = \frac{k}{y^{\frac{2}{2-\gamma}}} + \frac{\epsilon}{y^{\frac{2}{2-\gamma}}} p_{hom}(y) + \epsilon \rho_{ext}[\epsilon](y) \\ \tilde{u}_{ext}[\epsilon](y) = \epsilon y \omega_{hom}(y) + \epsilon u_{ext}[\epsilon](y) \end{cases},$$

where we have the bounds

$$\sup_{y_0 \leq y \leq 1} |y^{\mu+\frac{2}{2-\gamma}} \rho_{ext}|, |y^{\mu-1} u_{ext}|, |y^{\mu+\frac{2}{2-\gamma}+1} \rho'_{ext}|, |y^\mu u'_{ext}| \lesssim \epsilon y_0^{-\mu} \quad (3.67)$$

and

$$\sup_{1 \leq y \leq \infty} |y^{\frac{2}{2-\gamma}} \rho_{ext}|, |y^{\frac{\gamma-1}{2-\gamma}} u_{ext}|, |y^{\frac{2}{2-\gamma}+1} \rho'_{ext}|, |y^{\frac{\gamma-1}{2-\gamma}+1} u'_{ext}| \lesssim \epsilon y_0^{-\mu}. \quad (3.68)$$

Also we have that $\epsilon \mapsto \rho_{ext}[\epsilon](y_0), u_{ext}[\epsilon](y_0)$ is Lipschitz on a neighborhood of zero, with

$$\rho_{ext}[0](y_0) = u_{ext}[0](y_0) = 0, \quad (3.69)$$

$$\frac{|\rho_{ext}[\epsilon_1](y_0) - \rho_{ext}[\epsilon_2](y_0)|}{\epsilon_1 - \epsilon_2} \lesssim y_0^{-\frac{8-5\gamma}{2-\gamma}}, \quad \left| \frac{u_{ext}[\epsilon_1](y_0) - u_{ext}[\epsilon_2](y_0)}{\epsilon_1 - \epsilon_2} \right| \lesssim y_0^{\frac{4(\gamma-1)}{2-\gamma}}. \quad (3.70)$$

Finally, we have the monotonicities

$$(u + (2 - \gamma)y)' \geq \frac{1}{2}(2 - \gamma), \quad \tilde{\rho}' \leq \frac{1}{2} \frac{2}{2 - \gamma} \frac{k}{y^{\frac{2}{2-\gamma}+1}}. \quad (3.71)$$

Proof. The bounds (3.67)-(3.70) follow from Lemma 3.2.6, and then the definitions (3.65), (3.66). We then compute

$$\tilde{u}' = \epsilon y \omega'_{hom} + \epsilon \omega_{hom} + \epsilon u', \quad \left(\tilde{\rho} - \frac{k}{y^{\frac{2}{2-\gamma}}} \right)' = \epsilon \frac{p'_{hom}}{y^{\frac{2}{2-\gamma}}} - \frac{2}{2 - \gamma} \frac{\epsilon p_{hom}}{y^{\frac{2}{2-\gamma}+1}} + \epsilon \rho'.$$

We then use the asymptotics for p_{hom}, ω_{hom} from Lemma 3.2.4 to get

$$\begin{aligned} |\omega_{hom}| &\lesssim y^{-\mu} \langle y \rangle^{\mu - \frac{1}{2-\gamma}}, & |y \omega'_{hom}| &\lesssim y^{-\mu} \langle y \rangle^{\mu - \frac{1}{2-\gamma}}, \\ |y^{-\frac{2}{2-\gamma}-1} p_{hom}| &\lesssim y^{-\frac{2}{2-\gamma}-1-\mu} \langle y \rangle^\mu, & |y^{-\frac{2}{2-\gamma}} p'_{hom}| &\lesssim y^{-\frac{2}{2-\gamma}-1-\mu} \langle y \rangle^\mu. \end{aligned}$$

We also note that by (3.67) and (3.68) we have

$$|u'| \lesssim \epsilon y_0^{-\mu} y^{-\mu} \langle y \rangle^{\mu - \frac{1}{2-\gamma}}, \quad |\rho'| \lesssim \epsilon y_0^{-\mu} y^{-\frac{2}{2-\gamma}-1-\mu} \langle y \rangle^\mu.$$

Combining these gives

$$|\tilde{u}'| \lesssim \epsilon y^{-\mu} \langle y \rangle^{\mu - \frac{1}{2-\gamma}}, \quad |y^{\frac{2}{2-\gamma}+1} (\tilde{\rho} - \frac{k}{y^{\frac{2}{2-\gamma}}})'| \lesssim \epsilon y^{-\mu} \langle y \rangle^\mu.$$

We then note that as $y \geq y_0$, we have that $\langle y \rangle \lesssim y y_0^{-1}$, so we have

$$|\tilde{u}'| \lesssim \epsilon y_0^{-\mu} \langle y \rangle^{-\frac{1}{2-\gamma}}, \quad |y^{\frac{2}{2-\gamma}+1} (\tilde{\rho} - \frac{k}{y^{\frac{2}{2-\gamma}}})'| \lesssim \epsilon y_0^{-\mu}.$$

Since we may without loss of generality choose $\epsilon y_0^{-\mu} \ll 1$, we may assume that these are bounded by $\frac{1}{2} \frac{2}{2-\gamma}$ and $\frac{1}{2} \frac{2}{2-\gamma} k$, which proves (3.71). $\square$

3.3 Solving in the interior region

3.3.1 Preliminaries on the Lane-Emden Equation

We introduce the variable $\tilde{w}$ (self-similar enthalpy) by

$$\tilde{\rho} = \left(\frac{\gamma-1}{\gamma}\tilde{w}\right)^{\frac{1}{\gamma-1}}.$$

We then take the divergence of the second equation of (3.4) and obtain

$$\begin{cases} (2 + (2 - \gamma)y\partial_y) \left(\frac{\gamma-1}{\gamma}\tilde{w}\right)^{\frac{1}{\gamma-1}} + \operatorname{div} \left(\left(\frac{\gamma-1}{\gamma}\tilde{w}\right)^{\frac{1}{\gamma-1}} \tilde{u} \right) = 0 \\ \operatorname{div} \left(((\gamma - 1) + (\tilde{u} + (2 - \gamma)y)\partial_y) \tilde{u} \right) + \Delta \tilde{w} + 4\pi \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} \tilde{w}^{\frac{1}{\gamma-1}} = 0 \\ \tilde{\rho}'(0) = 0, \quad \tilde{u}(0) = 0, \quad \tilde{u}'(0) = -\frac{2}{3} \end{cases} \quad (3.72)$$

We then recall that in this case the far-field solution is given by

$$Q_f(y) = \frac{\gamma}{\gamma-1} k^{\gamma-1} y^{-\frac{2(\gamma-1)}{2-\gamma}}, \quad \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} Q_f^{\frac{1}{\gamma-1}} = k y^{-\frac{2}{2-\gamma}}.$$

The asymptotics of the ground state (normalized by $Q(0) = \frac{\gamma}{\gamma-1}$) as $y \rightarrow \infty$ can then be seen to be

$$Q = \frac{\gamma}{\gamma-1} k^{\gamma-1} y^{-\frac{2(\gamma-1)}{2-\gamma}} (1 + O(y^{-\mu})).$$

In particular, we have the global pointwise bounds

$$\langle y \rangle^{-\frac{2(\gamma-1)}{2-\gamma}} \lesssim Q \lesssim \langle y \rangle^{-\frac{2(\gamma-1)}{2-\gamma}}. \quad (3.73)$$

At the level of the mass-density, we have

$$\left(\frac{\gamma-1}{\gamma}Q\right)^{\frac{1}{\gamma-1}} = \frac{k}{y^{\frac{2}{2-\gamma}}} (1 + O(y^{-\mu})).$$

We now define

$$H := -(\Delta + \frac{4\pi}{\gamma-1} \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} Q^{\frac{2-\gamma}{\gamma-1}}).$$

We may then compute the leading asymptotics of the potential to be

$$\left(\frac{4\pi}{\gamma-1} \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} Q^{\frac{2-\gamma}{\gamma-1}}\right) = \frac{2(4-3\gamma)}{(2-\gamma)^2} \frac{1}{y^2} + O\left(\frac{1}{y^{2+\mu}}\right) \quad (3.74)$$

From the scaling symmetry of the Lane Emden equation (3.18) we see that one element of the kernel of H is

$$v_1 := y\partial_y Q + \frac{2(\gamma-1)}{2-\gamma} Q.$$

We then define v_2 by imposing that $v_1 v_2' - v_1' v_2 = y^{-2}$, which gives

$$v_2 := -v_1 \left(\int_y^{Y_0} \frac{1}{(v_1)^2 (y')^2} dy' \right). \quad (3.75)$$

We note that $v_1, v_2 = O(\frac{1}{y^{\frac{1}{2}}})$ as $y \rightarrow \infty$. The following lemma describes the next order asymptotics.

Lemma 3.3.1. *As $y \rightarrow \infty$, the functions v_1, v_2 satisfy, for some $c_3, c_4 > 0$ and $d_3, d_4 \in \mathbb{R}/2\pi\mathbb{Z}$,*

$$(v_1, v_2) = (c_3 \frac{\sin(\nu \log(y) + d_3)}{y^{\frac{1}{2}}}, c_4 \frac{\sin(\nu \log(y) + d_4)}{y^{\frac{1}{2}}}) + O(\frac{1}{y^{\frac{1}{2}+\mu}})$$

and one can differentiate these asymptotics arbitrarily many times. Moreover, although $v_2 = O(\frac{1}{y})$ as $y \rightarrow 0$, v_1, yv_2 are smooth up to $y = 0$. As a consequence, we have that on $[0, \infty)$, we have the pointwise bounds that for $k \geq 0$

$$|\partial_y^k v_1| \lesssim \langle y \rangle^{-\frac{1}{2}-k}, \quad |v_2| \lesssim \frac{1}{y^{1+k}} \langle y \rangle^{\frac{1}{2}}.$$

This lemma is proved in Subsection 3.6.1. It then follows that for some $c_2 > 0$ and $d_2 \in \mathbb{R}/2\pi\mathbb{Z}$ we have that as $y \rightarrow \infty$,

$$Q = \frac{\gamma}{\gamma-1} k^{\gamma-1} \frac{1}{y^{\frac{2(\gamma-1)}{2-\gamma}}} + \frac{\gamma}{k^{2-\gamma}} \frac{c_2 \sin(\nu \log(y) + d_2)}{y^{\frac{1}{2}}} + O(\frac{1}{y^{\frac{1}{2}+\mu}})$$

At the level of the mass density we obtain that as $y \rightarrow \infty$

$$(\frac{\gamma-1}{\gamma} Q)^{\frac{1}{\gamma-1}} = \frac{k}{y^{\frac{2}{2-\gamma}}} + c_2 \frac{\sin(\nu \log(y) + d_2)}{y^{\frac{5}{2}}} + O(\frac{1}{y^{\frac{8-5\gamma}{2-\gamma}}}). \quad (3.76)$$

3.3.2 Motivating asymptotic expansion

We now assume $\tilde{w}(0) = \frac{\gamma}{\gamma-1} \lambda^{-\frac{2(\gamma-1)}{2-\gamma}}$ and perform the scaling

$$\tilde{w} = \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} \bar{w}(\cdot/\lambda), \quad \tilde{u} = \lambda \bar{u}(\cdot/\lambda).$$

We then substitute this into (3.72) to obtain

$$\begin{cases} (2 + (2-\gamma)y\partial_y)(\frac{\gamma-1}{\gamma}\bar{w})^{\frac{1}{\gamma-1}} + \operatorname{div} \left((\frac{\gamma-1}{\gamma}\bar{w})^{\frac{1}{\gamma-1}} \bar{u} \right) = 0 \\ \Delta \bar{w} + 4\pi(\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} \bar{w}^{\frac{1}{\gamma-1}} = -\lambda^{\frac{2}{2-\gamma}} \operatorname{div} \left(((\gamma-1) + (\bar{u} + (2-\gamma)y)\partial_y) \bar{u} \right) \\ \bar{w}(0) = \frac{\gamma}{\gamma-1}, \quad \bar{w}'(0) = 0, \quad \bar{u}(0) = 0, \quad \bar{u}'(0) = -\frac{2}{3}. \end{cases} \quad (3.77)$$

In this case one can consider doing a formal expansion in powers of $\lambda^{\frac{2}{2-\gamma}}$. The zeroth order contribution for u is the equation

$$\begin{cases} (2 + (2-\gamma)y\partial_y)(\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} + Ju_0 = 0 \\ u_0(0) = 0, \quad u'_0(0) = -\frac{2}{3} \end{cases}.$$

where

$$Ju := \operatorname{div} \left((\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} u \right).$$

We then recall that the system

$$\begin{cases} Ju + f = 0 \\ u(0) = 0, \quad u'(0) = -\frac{f(0)}{3} \end{cases}$$

is solved by $u = T(f)$, where

$$T(f) := -\frac{1}{y^2(\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}}} \int_0^y f(y')(y')^2 dy'.$$

One then defines

$$u_* := T((2 + (2 - \gamma)y\partial_y)(\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}}).$$

We then recall that from (3.24) that as $y \rightarrow \infty$

$$u_* = k^{-1}(2 - \gamma)^{\frac{3}{2}}c_2 \sin(\nu \log(y + d_2 + \theta_0)) y^{1-\mu} + O(y^{\frac{4(\gamma-1)}{2-\gamma}}).$$

In particular, u_* obeys the global pointwise bound

$$|u_*| \lesssim |y|\langle y \rangle^{-\mu}. \quad (3.78)$$

As a comment, in this case, the next order equations one obtains are

$$\begin{cases} \operatorname{div}((\frac{\gamma}{\gamma-1}Q)^{\frac{1}{\gamma-1}}u_1) + Lw_1 = 0 \\ -Hw_1 + \operatorname{div}(((\gamma-1) + (u_* + (2-\gamma)y)\partial_y)u_*) = 0 \\ w_1(0) = w'_1(0) = u_1(0) = u'_1(0) = 0 \end{cases},$$

where

$$Lw := (2 + (2 - \gamma)y\partial_y)(\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} (Q^{\frac{2-\gamma}{\gamma-1}}w) + \operatorname{div}((\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} Q^{\frac{2-\gamma}{\gamma-1}}u_*w).$$

We then note that since S will again multiply expansions by y^2 , we will get that

$$w_1 = O(y^{2-\mu}).$$

Since T will multiply expansions by $y^{1+\frac{2}{2-\gamma}}$, we will get that

$$u_1 = O(y^{1+\frac{2}{2-\gamma}-\mu}).$$

These growth rates motivate the function spaces, described in the next section, in which the iteration will take place.

3.3.3 Difference equations and iteration scheme

As a comment, in contrast to the previous section, in this section we will not rescale space and all the functions will be defined on $[0, y_0]$. This is to make more clear the continuity of the solutions with respect to changing λ . We now define

$$Q_\lambda = \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} Q(\cdot/\lambda), \quad u_\lambda = \lambda u(\cdot/\lambda), \quad v_{i,\lambda} = v_i(\cdot/\lambda).$$

We now show the existence of solutions with $\tilde{\rho}(0) = \lambda^{-\frac{2}{2-\gamma}}$ (equivalently $\tilde{w}(0) = \frac{\gamma}{\gamma-1} \lambda^{-\frac{2(\gamma-1)}{2-\gamma}}$). We define difference variables (w, u) implicitly by

$$\tilde{w}(y) = Q_\lambda(y) + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w(y), \quad \tilde{u}(y) = u_\lambda(y) + \lambda^{1+\frac{2}{2-\gamma}} u(y).$$

We then have that (3.72) with the boundary condition $\tilde{w}(0) = \frac{\gamma}{\gamma-1} \lambda^{-\frac{2(\gamma-1)}{2-\gamma}}$ is equivalent to the following system on $[0, y_0]$:

$$\begin{cases} (2 + (2 - \gamma)y\partial_y) \left(\frac{\gamma-1}{\gamma} (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w) \right)^{\frac{1}{\gamma-1}} \\ + \operatorname{div} \left(\left(\frac{\gamma-1}{\gamma} (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w) \right)^{\frac{1}{\gamma-1}} (u_\lambda + \lambda^{1+\frac{2}{2-\gamma}} u) \right) = 0 \\ \operatorname{div} \left(\left((\gamma - 1) + (u_\lambda + \lambda^{1+\frac{2}{2-\gamma}} u + (2 - \gamma)y)\partial_y \right) (u_\lambda + \lambda^{1+\frac{2}{2-\gamma}} u) \right) \\ + \Delta(Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w) + 4\pi \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w)^{\frac{1}{\gamma-1}} = 0 \\ w(0) = w'(0) = u(0) = u'(0) = 0 \end{cases}.$$

We then define

$$\begin{aligned} J_\lambda u &:= \lambda^{1+\frac{2}{2-\gamma}} \operatorname{div} \left(\left(\frac{\gamma-1}{\gamma} Q_\lambda \right)^{\frac{1}{\gamma-1}} u \right), \\ L_\lambda w &:= \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} \left((2 + (2 - \gamma)y\partial_y) \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} (Q_\lambda^{\frac{2-\gamma}{\gamma-1}} w) \right. \\ &\quad \left. + \operatorname{div} \left(\left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} Q_\lambda^{\frac{2-\gamma}{\gamma-1}} u_\lambda w \right) \right), \\ H_\lambda &:= -\lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} \left(\Delta + \frac{4\pi}{\gamma-1} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} Q_\lambda^{\frac{2-\gamma}{\gamma-1}} \right), \\ S_\lambda f &:= \frac{1}{\lambda^{1-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}}} \left(\left(\int_0^y f v_{2,\lambda}(y')^2 dy' \right) v_{1,\lambda} - \left(\int_0^y f v_{1,\lambda}(y')^2 dy' \right) v_{2,\lambda} \right), \\ T_\lambda f &:= -\frac{1}{\lambda^{1+\frac{2}{2-\gamma}} y^2 \left(\frac{\gamma-1}{\gamma} Q_\lambda \right)^{\frac{1}{\gamma-1}}} \int_0^y f(y') (y')^2 dy', \\ K_\lambda &:= T_\lambda \circ L_\lambda \circ S_\lambda. \end{aligned}$$

We note that these operators satisfy

$$H_\lambda S_\lambda = \operatorname{id}, \quad -J_\lambda T_\lambda = \operatorname{id}, \quad \frac{1}{\lambda} J_\lambda u_\lambda + \lambda^{\frac{2}{2-\gamma}} (2 + (2 - \gamma)y\partial_y) \left(\frac{\gamma-1}{\gamma} Q_\lambda \right)^{\frac{1}{\gamma-1}} = 0.$$

We will use the expansion

$$\begin{aligned} & \left(\frac{\gamma-1}{\gamma} (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w) \right)^{\frac{1}{\gamma-1}} \\ &= \left(\frac{\gamma-1}{\gamma} Q_\lambda \right)^{\frac{1}{\gamma-1}} + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} \int_0^1 (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} wt)^{\frac{2-\gamma}{\gamma-1}} dt \\ &= \left(\frac{\gamma-1}{\gamma} Q_\lambda \right)^{\frac{1}{\gamma-1}} + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} w \frac{1}{\gamma-1} Q_\lambda^{\frac{2-\gamma}{\gamma-1}} \end{aligned}$$

$$+ \lambda^{-\frac{4(\gamma-1)}{2-\gamma} + \frac{4}{2-\gamma}} w^2 \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{2-\gamma}{(\gamma-1)^2} \int_0^1 (1-t) (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} wt)^{\frac{3-2\gamma}{\gamma-1}} dt$$

to define

$$\begin{aligned} F_\lambda(w, u) &= \lambda^{-\frac{4(\gamma-1)}{2-\gamma} + \frac{4}{2-\gamma}} (2 + (2-\gamma)y\partial_y) \left(w^2 \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{2-\gamma}{(\gamma-1)^2} \right. \\ &\quad \cdot \int_0^1 (1-t) (Q_\lambda + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w)^{\frac{3-2\gamma}{\gamma-1}} dt \\ &\quad + \lambda^{1-\frac{2(\gamma-1)}{2-\gamma} + \frac{4}{2-\gamma}} \operatorname{div} \left(\left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} uw \int_0^1 (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} wt)^{\frac{2-\gamma}{\gamma-1}} dt \right) \\ &\quad + \lambda^{-\frac{4(\gamma-1)}{2-\gamma} + \frac{4}{2-\gamma}} \operatorname{div} \left(u_\lambda w^2 \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{2-\gamma}{(\gamma-1)^2} \right. \\ &\quad \cdot \left. \int_0^1 (1-t) (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} wt)^{\frac{3-2\gamma}{\gamma-1}} dt \right), \\ G_{\lambda,1}(w, u) &= \operatorname{div} ((\gamma-1) + (u_\lambda + (2-\gamma)y)\partial_y) u_\lambda \\ &\quad + \lambda^{1+\frac{2}{2-\gamma}} \operatorname{div} \left(u\partial_y u_\lambda + ((\gamma-1) + (u_\lambda + (2-\gamma)y)\partial_y) u + \lambda^{1+\frac{2}{2-\gamma}} u\partial_y u \right), \\ G_{\lambda,2}(w, u) &= 4\pi\lambda^{-\frac{4(\gamma-1)}{2-\gamma} + \frac{4}{2-\gamma}} w^2 \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} \frac{2-\gamma}{(\gamma-1)^2} \int_0^1 (1-t) (Q_\lambda + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} wt)^{\frac{3-2\gamma}{\gamma-1}} dt, \\ G_\lambda(w, u) &= G_{\lambda,1}(w, u) + G_{\lambda,2}(w, u). \end{aligned}$$

we then obtain the system

$$\begin{cases} 0 = L_\lambda w + J_\lambda u + F_\lambda(w, u) \\ 0 = -H_\lambda w + G_\lambda(w, u) \\ w(0) = w'(0) = u(0) = u'(0) = 0 \end{cases}$$

which we can solve using the linear solution operators as

$$\begin{cases} u = T_\lambda(F_\lambda(w, u)) + K_\lambda(G_\lambda(w, u)) \\ w = S_\lambda(G_\lambda(w, u)) \end{cases}. \quad (3.79)$$

We will now define the spaces in which our iteration will take place in and the norms which we will use.

Definition 3.3.2. *We have the following spaces*

$$\begin{aligned} Z &:= \{f \in C^2([0, y_0]): f(0) = f'(0) = 0\}, \\ Y &:= \{f \in C^2([0, y_0]): f(0) = f'(0) = 0\}, \\ \mathcal{G} &:= C^0([0, y_0]), \\ \mathcal{F} &:= \{f \in C^1([0, y_0]): f(0) = 0\}. \end{aligned}$$

On each of these spaces, we define a λ -parameterized set of norms as follows.

$$\|f\|_{Z_\lambda} := \sup_{y \in [0, y_0]} \left\langle \frac{y}{\lambda} \right\rangle^\mu \left(\frac{\lambda^2}{y^2} |f| + \frac{\lambda}{y} |\partial_y f| + \lambda^2 |\partial_y^2 f| \right),$$

$$\begin{aligned}
\|f\|_{Y_\lambda} &:= \sup_{y \in [0, y_0]} \left\langle \frac{y}{\lambda} \right\rangle^{1+\mu-\frac{2}{2-\gamma}} \left(\frac{\lambda^2}{y^2} |f| + \frac{\lambda}{y} |\partial_y f| + \lambda^2 |\partial_y^2 f| \right), \\
\|f\|_{\mathcal{G}_\lambda} &:= \sup_{y \in [0, y_0]} \left\langle \frac{y}{\lambda} \right\rangle^\mu |f|, \\
\|f\|_{\mathcal{F}_\lambda} &:= \sup_{y \in [0, y_0]} \left\langle \frac{y}{\lambda} \right\rangle^{1+\mu} \left(\frac{\lambda}{y} |f| + \lambda |\partial_y f| \right).
\end{aligned}$$

As a comment, to read off these exponents, one can first motivate the growth rates of Z_{y_1}, Y_{y_1} by considering w_1, u_1 in the previous section. From these, one can get the rates for $\mathcal{G}_{y_1}, \mathcal{F}_{y_1}$ by requiring that S, T be bounded.

We have the following lemma regarding the linearized operators.

Lemma 3.3.3. *If $y_0 \ll 1$, $\lambda_0 \leq \frac{1}{10}y_0$, we have that if $G, F \in \mathcal{G}, \mathcal{F}$, then $T_\lambda(F), K_\lambda(G) \in Y$, and $S_\lambda(G) \in Z$. Moreover, we have that there exists a constant C_1 independent of y_0, λ such that*

$$\begin{aligned}
\|T_\lambda(F)\|_{Y_\lambda} &\leq C_1 \|F\|_{\mathcal{F}_\lambda}, \\
\|S_\lambda(G)\|_{Z_\lambda} &\leq C_1 \|G\|_{\mathcal{G}_\lambda}, \\
\|K_\lambda(G)\|_{Y_\lambda} &\leq C_1 \|G\|_{\mathcal{G}_\lambda}.
\end{aligned}$$

Moreover, these family of operators have a Lipschitz dependence on λ , so that in particular if $\lambda_1 < \lambda_2 < \frac{y_0}{10}$, then we have

$$\begin{aligned}
\max_i \|T_{\lambda_2}(F) - T_{\lambda_1}(F)\|_{Y_{\lambda_i}} &\leq C_1 \frac{\lambda_2 - \lambda_1}{\lambda_1} \min_i \|F\|_{\mathcal{F}_{\lambda_i}}, \\
\max_i \|S_{\lambda_2}(G) - S_{\lambda_1}(G)\|_{Z_{\lambda_i}} &\leq C_1 \frac{\lambda_2 - \lambda_1}{\lambda_1} \min_i \|G\|_{\mathcal{G}_{\lambda_i}}, \\
\max_i \|K_{\lambda_2}(G) - K_{\lambda_1}(G)\|_{Y_{\lambda_i}} &\leq C_1 \frac{\lambda_2 - \lambda_1}{\lambda_1} \min_i \|G\|_{\mathcal{G}_{\lambda_i}}.
\end{aligned}$$

For the proof see Subsection 3.6.2

The analog of the nonlinear bound is the following.

Lemma 3.3.4. *There exists $C_2 > 0$ (independent of y_0, λ) such that if $u, w \in Y, Z$ respectively and if $\|u\|_{Y_\lambda} + \|w\|_{Z_\lambda} \leq C$ then if $C \cdot y_0^{\frac{2}{2-\gamma}} \ll 1$, $y_0^{\frac{2}{2-\gamma}} C^2 \leq 1$ and $\lambda \leq \frac{y_0}{10}$, then we have $F_\lambda(w, u), G_\lambda(w, u)$ are well-defined, lie in $\mathcal{F}$ and $\mathcal{G}$ respectively, and satisfy*

$$\|F_\lambda(w, u)\|_{\mathcal{F}_\lambda} + \|G_\lambda(w, u)\|_{\mathcal{G}_\lambda} \leq C_2.$$

Moreover we have the Lipschitz estimate that

$$\begin{aligned}
&\|F_\lambda(w_2, u_2) - F_\lambda(w_1, u_1)\|_{\mathcal{F}_\lambda} + \|G_\lambda(w_2, u_2) - G_\lambda(w_1, u_1)\|_{\mathcal{G}_\lambda} \\
&\leq C_2 \cdot (1 + C) y_0^{\frac{2}{2-\gamma}} (\|u_2 - u_1\|_{Y_\lambda} + \|w_2 - w_1\|_{Z_\lambda}). \quad (3.80)
\end{aligned}$$

Finally, we have the following estimate on how these operators depend on λ , for $0 < \lambda_1 < \lambda_2 < \frac{y_0}{10}$,

$$\max_i \|F_{\lambda_2}(w, u) - F_{\lambda_1}(w, u)\|_{\mathcal{F}_{\lambda_i}} + \max_i \|G_{\lambda_2}(w, u) - G_{\lambda_1}(w, u)\|_{\mathcal{G}_{\lambda_i}} \leq C_2 \frac{\lambda_2 - \lambda_1}{\lambda_1}.$$

For the proof see Subsection 3.6.3.

Lemma 3.3.5. *There exists a constant $C_3 > 0$ (independent of y_0, λ) such that for $y_0^{\frac{2}{2-\gamma}} \ll 1$, $\lambda \leq \frac{y_0}{10}$, there exists a unique C^2 solution to (3.72) with $\tilde{w} > 0$ and the property that it can be decomposed as*

$$\tilde{w}(y) = Q_\lambda(y) + \lambda^{-\frac{2(\gamma-1)}{2-\gamma} + \frac{2}{2-\gamma}} w[\lambda](y), \quad \tilde{u} = u_\lambda + \lambda^{1+\frac{2}{2-\gamma}} u[\lambda](y)$$

with $\|w\|_{Z_\lambda} + \|u\|_{Y_\lambda} \leq \frac{C_3}{100}$. Moreover, if $0 < \lambda_1 < \lambda_2 < \frac{y_0}{10}$, then we have

$$\max_i \|w[\lambda_2] - w[\lambda_1]\|_{Z_\lambda} + \|u[\lambda_2] - u[\lambda_1]\|_{Y_\lambda} \leq \frac{\lambda_2 - \lambda_1}{\lambda_1} C_3. \quad (3.81)$$

For the proof see Subsection 3.6.4

As a comment, the analog of Lemma 3.6 is then the following.

Lemma 3.3.6. *If $y_0^{\frac{2}{2-\gamma}} \ll 1$ is fixed, then for $\lambda \leq \frac{y_0}{10}$ there exists C^2 functions $\tilde{\rho}_{int}, \tilde{u}_{int}$ defined on $[0, y_0]$ which solve (3.4) of the form*

$$\begin{cases} \tilde{\rho}_{int}[\lambda](y) = (\frac{\gamma-1}{\gamma} Q_\lambda)^{\frac{1}{\gamma-1}} + \rho_{int}[\lambda](y) \\ \tilde{u}_{int}[\lambda](y) = u_\lambda + \lambda^{1+\frac{2}{2-\gamma}} u_{int}[\lambda](y) \\ \rho_{int}(0) = \rho'_{int}(0) = u_{int}(0) = u'_{int}(0) = 0 \end{cases}$$

and they obey the following bounds on $[0, y_0]$,

$$|\rho_{int}| \lesssim \left(\frac{y}{\lambda}\right)^2 \left\langle \frac{y}{\lambda} \right\rangle^{-2-\mu}, \quad |u_{int}| \lesssim \frac{y^2}{\lambda^2} \left\langle \frac{y}{\lambda} \right\rangle^{\frac{2}{2-\gamma}-1-\mu}. \quad (3.82)$$

$$|\rho'_{int}| \lesssim \frac{y}{\lambda^2} \left\langle \frac{y}{\lambda} \right\rangle^{-2-\mu}, \quad |u'_{int}| \lesssim \frac{y}{\lambda^2} \left\langle \frac{y}{\lambda} \right\rangle^{\frac{2}{2-\gamma}-1-\mu}. \quad (3.83)$$

Moreover, $\lambda \rightarrow 0$, $(\rho_{int}[\lambda](y_0), u_{int}[\lambda](y_0))$ satisfies the following on $(0, \frac{1}{10}y_0)$

$$\lim_{\lambda \rightarrow 0} \rho_{int}[\lambda] = 0, \quad \lim_{\lambda \rightarrow 0} \left(\frac{\lambda}{y_0}\right)^{1+\frac{2}{2-\gamma}} u_{int}[\lambda] = 0 \quad (3.84)$$

as well as the following Lipchitz estimates with respect to λ for $0 < \lambda_1 < \lambda_2 < \frac{y_0}{10}$:

$$\begin{cases} |\rho_{int}[\lambda_2](y_0) - \rho_{int}[\lambda_1](y_0)| & \lesssim \frac{\lambda_2 - \lambda_1}{\lambda_1} \left(\frac{\lambda_2}{y_0}\right)^\mu \\ |(\frac{\lambda_2}{y_0})^{1+\frac{2}{2-\gamma}} u_{int}[\lambda_2](y_0) - (\frac{\lambda_1}{y_0})^{1+\frac{2}{2-\gamma}} u_{int}[\lambda_1](y_0)| & \lesssim \frac{\lambda_2 - \lambda_1}{\lambda_1} \left(\frac{\lambda_2}{y_0}\right)^\mu \end{cases} \quad (3.85)$$

We also have the following bounds on $[0, y_0]$, for some absolute constants independent of y_0, λ ,

$$|\tilde{u} + (2 - \gamma)y| \lesssim y_0, \quad \tilde{\rho} \gtrsim y_0^{-\frac{2}{\gamma-1}}. \quad (3.86)$$

Proof. We begin with Lemma 3.3.5 and then define

$$\rho_{int}[\lambda] := \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} \frac{1}{\gamma-1} \lambda^{\frac{2}{2-\gamma} - \frac{2(\gamma-1)}{2-\gamma}} w[\lambda] \cdot \int_0^1 \left(Q_\lambda + t \lambda^{\frac{2}{2-\gamma} - \frac{2(\gamma-1)}{2-\gamma}} w[\lambda]\right)^{\frac{2-\gamma}{\gamma-1}} dt.$$

The estimates (3.82)-(3.85) then follow directly. We may then use

$$|u_\lambda| \lesssim |y| \leq y_0, \quad |Q_\lambda| \gtrsim \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} \left\langle \frac{y}{\lambda} \right\rangle^{-\frac{2(\gamma-1)}{2-\gamma}} \geq 2^{\frac{-2(\gamma-1)}{2-\gamma}} y_0^{\frac{-2(\gamma-1)}{2-\gamma}}.$$

Moreover, from (3.82) we have

$$|\lambda^{1+\frac{2}{2-\gamma}} u_{int}| \lesssim |y| \cdot \left\langle \frac{y}{\lambda} \right\rangle^{-\mu} \leq |y_0|, \quad |\rho_{int}| \lesssim \left\langle \frac{y}{\lambda} \right\rangle^{-\mu} \leq 1.$$

As we impose $y_0^{\frac{2}{2-\gamma}} \ll 1$, applying the triangle inequality yields (3.86). $\square$

3.4 Proof of the theorem

3.4.1 Matching and existence of C^1 polytropic Hunter solutions

In this section the following lemma is proved.

Lemma 3.4.1. *For some integer $N \gg 1$ there exists C^1 solutions to (3.5) on $[0, \infty)$ subject to the boundary conditions (3.6) and also $\tilde{\rho}_i > 0$. In addition, $\tilde{\rho}_i$ intersects the far-field solution, $\tilde{\rho}_f$, exactly $i + 1$ times. Moreover, the sonic point condition $(\tilde{u} + (2 - \gamma)y)^2 - \gamma\tilde{\rho}^{\gamma-1} = 0$ is satisfied at exactly one point $y = y_* \in (0, 2y_f)$, where y_f is the sonic point for the far-field solution, (3.14).*

Proof. We follow the same eight steps as in [100]. As a comment, it is easier technically to use a variant of the implicit function theorem which only requires (quantitative) Lipschitz regularity. This variant is non-standard, and so the statement and (elementary) proof are provided in Subsection 3.7.4.

1) We pick $y_0 \ll 1$ (dictated by the assumptions of Lemma 3.3.6). Without loss of generality, we may require that (3.26) is satisfied. Then, we pick $\epsilon_0 \ll y_0^\mu$ (dictated by the assumptions of Lemma 3.2.7). For $\epsilon \in [0, \epsilon_0]$ we apply Lemma 3.2.7 to obtain a family of solutions $(\tilde{\rho}_{ext}[\epsilon], \tilde{u}_{ext}[\epsilon])$ on $[y_0, \infty)$. Then, for $\lambda \leq \frac{y_0}{10}$, we apply Lemma 3.3.6 to obtain a family of solutions $(\tilde{\rho}_{int}[\lambda], \tilde{u}_{int}[\lambda])$ defined on $[0, y_0]$.

2) We define

$$\begin{aligned} \mathcal{F}[y_0](\epsilon, \lambda) &:= \tilde{\rho}_{ext}[\epsilon](y_0) - \tilde{\rho}_{int}[\lambda](y_0) \\ &= \left(\frac{k}{y_0^{\frac{2}{2-\gamma}}} + \frac{\epsilon}{y_0^{\frac{2}{2-\gamma}}} p_{hom}(y_0) - \left(\frac{\gamma-1}{\gamma} Q_\lambda\right)^{\frac{1}{\gamma-1}} \right) + (\epsilon \rho_{ext}[\epsilon](y_0) - \rho_{int}[\lambda](y_0)). \end{aligned}$$

We let $q(y) = (\frac{\gamma-1}{\gamma}Q)^{\frac{1}{\gamma-1}} - \frac{k}{y^{\frac{2}{2-\gamma}}} - c_2 \frac{\sin(\nu \log(y)+d_2)}{y^{\frac{5}{2}}}$, and note that by (3.76) we have

$$q(y) = O\left(\frac{1}{y^{\frac{2}{2-\gamma}+2\mu}}\right).$$

From (3.54), (3.69), (3.70), (3.84), (3.85), we see that we may express

$$\mathcal{F}[y_0](\epsilon, \lambda) = \frac{\epsilon}{y_0^{\frac{5}{2}}} c_1 \sin(\nu \log(y_0) + d_1) - c_2 \frac{\lambda^{\frac{6-5\gamma}{2-\gamma}}}{y_0^{\frac{5}{2}}} \sin(\nu \log(y_0/\lambda) + d_2) + \frac{1}{y_0^{\frac{5}{2}}} q_1(\epsilon) + \frac{1}{y_0^{\frac{5}{2}}} q_2(\lambda)$$

where

$$\begin{aligned} q_1(0) &= 0, & \frac{q_1(\epsilon_1) - q_1(\epsilon_2)}{\epsilon_1 - \epsilon_2} &= O(y_0^{\frac{2}{2-\gamma}}) + O(\epsilon y_0^{-\mu}). \\ \lim_{\lambda \rightarrow 0^+} q_2(\lambda) &= 0, & \frac{\frac{q_2(\lambda_1) - q_2(\lambda_2)}{\lambda_1^{\frac{6-5\gamma}{2(2-\gamma)}} - \lambda_2^{\frac{6-5\gamma}{2(2-\gamma)}}}}{\lambda_1^{\frac{6-5\gamma}{2(2-\gamma)}} - \lambda_2^{\frac{6-5\gamma}{2(2-\gamma)}}} &= O(y_0^{\frac{2}{2-\gamma}}) + O(\lambda^\mu y_0^{-\mu}). \end{aligned}$$

We then recall (3.26) which gives

$$\mathcal{F}_{y_0} = y_0^{-\frac{5}{2}} (\epsilon c_1 - \lambda^\mu c_2 \cos(\nu \log(\lambda) + d_1 - d_2) + q_1(\epsilon) + q_2(\lambda)), \quad (3.87)$$

and note that if we consider the regime $\epsilon, \lambda^\mu \lesssim y_0^{\frac{2}{2-\gamma}+\mu}$ then

$$\left| \frac{q_1(\epsilon_1) - q_2(\epsilon_2)}{\epsilon_1 - \epsilon_2} \right| \lesssim y_0^{\frac{2}{2-\gamma}}, \quad \left| \frac{q_2(\lambda_1^\mu) - q_2(\lambda_2^\mu)}{\lambda_1^\mu - \lambda_2^\mu} \right| \lesssim y_0^{\frac{2}{2-\gamma}}. \quad (3.88)$$

We then note that if $y_0^{\frac{2}{2-\gamma}} \ll 1$, then by Lemma 3.7.3, we have a well-defined function $\epsilon(\lambda)$, Lipschitz in λ^μ , which satisfies $\lim_{\lambda \rightarrow 0^+} \epsilon(\lambda) = 0$ is locally uniquely determined for $\lambda > 0$ by

$$\mathcal{F}[y_0](\epsilon(\lambda), \lambda) = 0.$$

3) We observe that from (3.87) and (3.88) we may read off

$$\epsilon(\lambda) = \frac{c_2}{c_1} \lambda^\mu \left(\cos(\nu \log(\lambda) + d_1 - d_2) + O(y_0^{\frac{2}{2-\gamma}}) \right).$$

4) We now define the second matching function to be

$$\mathcal{G}[y_0](\lambda) := \frac{1}{c_2 y_0^{\frac{3\gamma-2}{2(2-\gamma)}} \lambda^\mu \sin(\theta_0)} \frac{k}{(2-\gamma)^{\frac{3}{2}}} (\tilde{u}_{ext}[\epsilon(\lambda)](y_0) - \tilde{u}_{int}[\lambda](y_0))$$

This will be a Lipschitz function of λ^μ . We then may compute that (by a similar computation as for the first matching function), using (3.54), (3.69), (3.70), (3.84), (3.85),

$$\mathcal{G}[y_0](\lambda) = -\sin(\nu \log(\lambda) + d_1 - d_2) + O(y_0^{\frac{2}{2-\gamma}}).$$

5) We then pick

$$\lambda_{i,\pm} = \exp\left(\frac{1}{-\nu}\left(\pm\frac{1}{10} - d_1 + d_2 + k\pi\right)\right)$$

and we have that if i is sufficiently large so that $\lambda_{i,\pm} \leq \lambda_0$ $\lambda_{i,\pm}^\mu \lesssim y_0^{\frac{2}{2-\gamma}+\mu}$ then $\mathcal{G}[y_0](\lambda_{i,\pm})$ is well-defined and

$$\mathcal{G}[y_0](\lambda_{i,\pm}) = (-1)^k \sin\left(\pm\frac{1}{10}\right) + O(y_0^{\frac{2}{2-\gamma}}).$$

We may choose $y_0^{\frac{2}{2-\gamma}} \ll 1$ if necessary to ensure that the remainder is bounded in magnitude by $\frac{1}{2}\sin(\frac{1}{10})$. Thus, for each $i > N$ for some N , we have that by the intermediate value theorem there exists $\lambda_i \in (\lambda_{i,+}, \lambda_{i,-})$ such that $\mathcal{G}[y_0](\lambda_i) = 0$. Thus, we may glue together the interior and exterior solutions to get a continuous solution on $[0, \infty)$. Moreover, by (3.86), we see that the determinant

$$D(y) := (u + (2 - \gamma)y)^2 - \gamma\tilde{\rho}^{\gamma-1}$$

is negative at y_0 (the negative contribution is $O(y_0^{\frac{-2(\gamma-1)}{2-\gamma}})$ while the positive contribution is $O(y_0)$). Thus, $D(y)$ does not vanish at the sonic point, so the ODE system is regular, and so the solutions (ρ_{int}, u_{int}) and (ρ_{ext}, u_{ext}) can be glued at y_0 to obtain a classical solution on $[0, \infty)$. We note that as previously mentioned, on $[0, y_0]$, we have that $D(y)$ is negative. Moreover, we see from (3.71) that $D(y)$ is increasing on $[y_0, \infty)$, and so there is exactly one sonic point. it remains to count the intersections with the far-field solution.

6) We first show that $y^{\frac{2}{2-\gamma}}\tilde{\rho} - k$ and p_{hom} have the same number of zeroes on $[y_0, \infty)$. To do this, we define

$$\psi(t) := e^{\mu t} p_{hom}(e^t), \quad \tilde{\psi}(t) := e^{(\mu + \frac{2}{2-\gamma})t} \tilde{\rho}_{ext}(e^t).$$

The problem then becomes showing that $\psi(t)$ and $\psi(t) + \tilde{\psi}(t)$ have the same number of zeroes on $[-\log(y_0), \infty)$. We then note that by Lemma 3.2.1 we have that as $t \rightarrow \infty$,

$$\psi(t) = \mu_3 e^{\mu t} + O(e^{(\mu - \frac{1}{2-\gamma})t})$$

and hence has no roots. Similarly, as $t \rightarrow -\infty$, we have

$$\psi(t) = c_1 \sin(\nu t + d_1) + O(e^{\frac{2}{2-\gamma}t}).$$

In particular, by (3.26) we have $\psi(\log(y_0)) = c_1 + O(y_0^{\frac{2}{2-\gamma}})$ which is bounded away from zero. It follows that ψ will have finitely many roots and they will all be contained in an interval $[\log(y_0), \log(Y_0)]$ for some large Y_0 . Moreover, there exists $\delta > 0$ so that if $\{t\}_l$ are these roots then $|\psi'| > \delta$ on $\cup_l [t_l - \frac{1}{100}, t_l + \frac{1}{100}]$ and $|\psi| > \delta$ on the complement of this set. It follows from (3.67) and (3.68) that

$$|\tilde{\psi}|, |\tilde{\psi}'| \lesssim \epsilon y_0^{-\mu} \langle e^t \rangle^\mu.$$

Since the roots occurred in a compact set, by choosing $\epsilon y_0^{-\mu} \ll 1$, $|\tilde{\psi}|$, we may ensure $|\tilde{\psi}'| \leq \frac{\delta}{100}$ which will ensure that ψ and $\psi + \tilde{\psi}$ have the same number of roots.

7) We next claim that if $y_0^{\frac{2}{2-\gamma}} \ll 1$ and $\lambda^\mu \lesssim y_0^{\frac{2}{2-\gamma} + \mu}$, then we will have that $y^{\frac{2}{2-\gamma}} \tilde{\rho} - k$ and $y^{\frac{2}{2-\gamma}} (\frac{\gamma-1}{\gamma} Q_\lambda)^{\frac{1}{\gamma-1}} - k$ will intersect 0 the same number of times on $[0, y_0]$. To see this, we define

$$\begin{aligned}\chi(t) &= \langle e^t \rangle^\mu \left((\lambda e^t)^{\frac{2}{2-\gamma}} (\frac{\gamma-1}{\gamma} Q_\lambda(\lambda e^t))^{\frac{1}{\gamma-1}} - k \right), \\ \tilde{\chi}(t) &= \langle e^t \rangle^\mu \left((\lambda e^t)^{\frac{2}{2-\gamma}} \tilde{\rho}(\lambda e^t) \right)\end{aligned}$$

and note that as $t \rightarrow -\infty$ we have

$$\chi(t) = -k + O(e^{\frac{2}{2-\gamma}t})$$

and as $t \rightarrow \infty$ we have, using (3.76),

$$\chi(t) = c_2 \sin(\nu t + d_2) + O(e^{-\mu t}).$$

We note that we will have $y \lesssim y_0$, so we have that, using $\lambda^\mu \lesssim y_0^{\frac{2}{2-\gamma} + \mu}$

$$e^{-\mu t} = \left(\frac{y}{\lambda}\right)^{-\mu} \leq \frac{\lambda^\mu}{y_0^\mu} \lesssim y_0^{\frac{2}{2-\gamma}}.$$

Thus, if we switch variables to x , which is defined implicitly by

$$t = \log(y_0) - \frac{1}{\nu}(x - d_1 + d_2)$$

we will have be (3.26) that

$$\chi(t) = c_2 \cos(x) + O(y_0^{\frac{2}{2-\gamma}}).$$

and $t_{k,\pm} = \log(y_0/\lambda_{k,\pm})$ corresponds to $x_{k,\pm} = -k\pi \pm \frac{1}{10}$. We can thus see that $|\chi(t)|$ is bounded away from zero on a neighborhood of $\log(y_0/\lambda_k)$. Similarly on neighborhoods of the roots of $\chi(t)$ one can show that the derivative is bounded away from zero. Finally, it follows from (3.82) and (3.83) that on $(-\infty, \log(y_0/\lambda))$ we have

$$|\chi(t)|, |\chi'(t)| \lesssim \langle e^t \rangle^\mu (\lambda e^t)^{\frac{2}{2-\gamma}} \langle e^t \rangle^{-2-\mu} \leq (\lambda e^t)^{\frac{2}{2-\gamma}} \leq y_0^{\frac{2}{2-\gamma}}.$$

From this, similarly to Step 6, we may conclude that χ and $\tilde{\chi}$ have the same number of zeroes. It then follows that $y^{\frac{2}{2-\gamma}} \tilde{\rho}_{int}[\lambda_i] - k$ and $y^2 e^{Q_{\lambda_i}} - k$ intersect 0 the same number of times.

8) We see that the number of zeroes of $y^{\frac{2}{2-\gamma}} \tilde{\rho} - k$ will be the sum of the number of zeroes of p_{hom} on $[y_0, \infty)$ (a finite number, independent of i) along with the number of zeroes of $y^{\frac{2}{2-\gamma}} (\frac{\gamma-1}{\gamma} Q_{\lambda_i})^{\frac{1}{\gamma-1}} - k$ on $[0, y_0]$. We now show that this latter number increases by 1 when $i \rightarrow i+1$. It suffices to show that the function $\chi(t)$ defined in the previous section has exactly one root in $[\log(y_0/\lambda_i), \log(y_0/\lambda_{i+1}))]$. We define

$$\bar{\chi}(t) = \chi(t) - c_2 \sin(\nu t + d_2).$$

and note that by (3.76) we have that as $t \rightarrow \infty$

$$|\bar{\chi}(t)|, |\bar{\chi}'(t)| \lesssim y_0^{\frac{2}{2-\gamma}}$$

and so we see that $\chi(t)$ has the same number of roots on this interval as $c_2 \sin(\nu t + d_2)$. By a similar argument as in the previous step, we may see that $c_2 \sin(\nu t + d_2)$ has precisely one root on this interval.

Thus, for i sufficiently large, the number of roots of $y^{\frac{2}{2-\gamma}} \tilde{\rho} - k$ increases by 1 as i increases by 1, so by re-enumerating if necessary we see that $y^{\frac{2}{2-\gamma}} \tilde{\rho}_i - k$ has precisely $i + 1$ zeroes.

□

3.4.2 Proof that C^1 polytropic Hunter solutions are real analytic

We recall that by Lemma 3.1.5 the Lane-Emden equation can be put into the desired normal form at the sonic point. We now state and proof an (easier) lemma regarding the singular ODE problem at the origin.

Lemma 3.4.2. *Near $y = 0$, assuming $\tilde{\rho}(0) = \rho_0$, $\tilde{u}(0) = 0$ (and $\rho_0 > 0$) we can put (3.5) in normal form. The parameters in this case are*

$$a = \rho_0, \quad b = 0, \quad c = 2\rho_0, \quad d = 2\rho_0.$$

In particular, $U = \tilde{u}'(0) = \frac{2}{3}$ is determined uniquely and we have the nondegeneracies

$$a + bU = \rho_0, \quad \kappa = 2.$$

Proof. We consider the ODE system (3.5). We assume that $\tilde{\rho}(0) = \rho_0 > 0$, and multiply the mass-conservation equation by y . We divide the u equation by $\gamma \rho_0^{\gamma-2}$. Finally, we substitute $\tilde{\rho} = \rho_0 + \dot{\rho}$. This gives the system

$$\begin{pmatrix} \frac{(\rho_0 + \dot{\rho})^{\gamma-2}}{\rho_0^{\gamma-2}} & \frac{\tilde{u} + (2-\gamma)y}{\gamma \rho_0^{\gamma-2}} \\ y(\tilde{u} + (2-\gamma)y) & y(\rho_0 + \dot{\rho}) \end{pmatrix} \begin{pmatrix} \dot{\rho}' \\ \tilde{u}' \end{pmatrix} + \begin{pmatrix} \frac{(\gamma-1)\tilde{u}}{\gamma \rho_0^{\gamma-2}} + \frac{4\pi}{(4-3\gamma)\gamma \rho_0^{\gamma-2}} (\rho_0 + \dot{\rho})(\tilde{u} + (2-\gamma)y) \\ 2(\rho_0 + \dot{\rho})(\tilde{u} + y) \end{pmatrix} = 0.$$

This equation is in the desired normal form. We may then read off

$$a = \rho_0, \quad b = 0, \quad c = 2\rho_0, \quad d = 2\rho_0.$$

In this case, the characteristic quadratic, (3.9), is linear and the unique solution is $U = -\frac{2}{3}$. We thus have

$$a + bU = \rho_0, \quad \kappa = 2.$$

□

We now prove Theorem 3.1.1.

Proof. By Lemma 3.4.1 there exists a C^1 solution with these properties. The solution is real analytic for $y \in (0, \infty) \setminus y_*$ by standard ODE theory (since the ODE is non-singular). We comment that by Lemma 3.1.5, in a neighborhood of the soinc point, the system (3.5) can be put in the form of Definition 3.1.2. We then comment that if we replace $\omega_0 = 2 - \gamma$ by $\omega_\epsilon = \dot{\omega}_0[\epsilon]$, then the characteristic parameters will change continuously. Thus, by Lemma 3.1.5, for ϵ sufficiently small we will have $a + bU \neq 0$, $\kappa > -\frac{1}{2}$. Thus, by the existence part of Lemma 3.1.4, by choosing the Larson-Penston-Hunter boundary condition there exists locally an analytic solution. By the uniqueness aspect of Lemma 3.1.3, this agrees with the previously constructed solution, establishing analyticity at y_f . Similarly, by Lemma 3.4.2, on a neighborhood of the origin there is a unique analytic solution. Thus, the solutions provided by Lemma 3.4.1 are in fact analytic on $[0, \infty)$ as desired. $\square$

3.5 Deferred proofs of Section 3.2

3.5.1 Proof of Lemma 3.2.1

Proof. In this case, the linearized ODE we are considering is

$$\begin{pmatrix} (2-\gamma)z & kz \\ \frac{2\pi(2-\gamma)^2}{4-3\gamma}z^{1-\frac{2}{2-\gamma}} & (2-\gamma)z \end{pmatrix} \begin{pmatrix} p' \\ \omega' \end{pmatrix} + \begin{pmatrix} 0 & \frac{4-3\gamma}{2-\gamma}k \\ \frac{4\pi(2-\gamma)^2}{(4-3\gamma)z^{\frac{2}{2-\gamma}}} & 1 + \frac{4\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = 0.$$

We comment that the coefficient matrix is singular when $kz^{\frac{-2}{2-\gamma}} = \frac{4-3\gamma}{2\pi}$. Assuming that this is not the case, we may invert the first matrix to obtain

$$\begin{pmatrix} p' \\ \omega' \end{pmatrix} + \frac{1}{1 - \frac{2\pi}{4-3\gamma}kz^{-\frac{2}{2-\gamma}}} \begin{pmatrix} \frac{-4\pi k}{(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} & -3\frac{(\gamma-1)k}{(2-\gamma)^2z} - \frac{4\pi k^2}{(4-3\gamma)(2-\gamma)^2z^{1+\frac{2}{2-\gamma}}} \\ \frac{4\pi(2-\gamma)}{(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} & \frac{1}{(2-\gamma)z} + \frac{2\pi k(3\gamma-2)}{(2-\gamma)(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} p \\ \omega \end{pmatrix} = 0.$$

We first let

$$\bar{p} = p + 3\frac{(\gamma-1)k}{2-\gamma}\omega$$

and then we have

$$\begin{pmatrix} \bar{p}' \\ \omega' \end{pmatrix} + \frac{1}{1 - \frac{2\pi}{(4-3\gamma)}kz^{-\frac{2}{2-\gamma}}} \begin{pmatrix} \frac{-4\pi k}{z^{1+\frac{2}{2-\gamma}}} & \frac{-2\pi(6\gamma^2-15\gamma+11)k^2}{(2-\gamma)^2z^{1+\frac{2}{2-\gamma}}} \\ \frac{4\pi(2-\gamma)}{(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} & \frac{1}{(2-\gamma)z} + \frac{2\pi k(6\gamma^2-15\gamma+10)}{(2-\gamma)(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} \bar{p} \\ \omega \end{pmatrix} = 0$$

We then let

$$q = \bar{p} + \frac{(6\gamma^2 - 15\gamma + 11)k}{2(2-\gamma)^2}\omega$$

and in terms of $(\bar{p}, q)$ we have

$$\begin{pmatrix} \bar{p}' \\ q' \end{pmatrix} + \frac{1}{1 - \frac{2\pi}{4-3\gamma}kz^{-\frac{2}{2-\gamma}}} \begin{pmatrix} 0 & -\frac{4\pi k}{z^{1+\frac{2}{2-\gamma}}} \\ -\frac{1}{(2-\gamma)z} + \frac{2\pi k}{(2-\gamma)(4-3\gamma)z^{1+\frac{2}{2-\gamma}}} & \frac{1}{(2-\gamma)z} - \frac{2\pi k(6-5\gamma)}{(4-3\gamma)(2-\gamma)z^{1+\frac{2}{2-\gamma}}} \end{pmatrix} \begin{pmatrix} \bar{p} \\ q \end{pmatrix} = 0.$$

From the first equation we have that

$$q = \left(\frac{1}{4\pi k}\right) z^{1+\frac{2}{2-\gamma}} \left(1 - \frac{2\pi}{4-3\gamma} k z^{-\frac{2}{2-\gamma}}\right) \bar{p}'$$

and we may substitute this into the second line to obtain the following second order equation for $\bar{p}$ (note a factorization is used to simplify the coefficient of $\bar{p}$)

$$\bar{p}'' + \frac{1}{1 - \frac{2\pi k}{(4-3\gamma)} z^{-\frac{2}{2-\gamma}}} \left(\frac{5-\gamma}{(2-\gamma)z} - \frac{4\pi k}{(2-\gamma)z^{1+\frac{2}{2-\gamma}}} \right) \bar{p}' - \frac{1}{1 - \frac{2\pi k}{(4-3\gamma)} z^{-\frac{2}{2-\gamma}}} \frac{4\pi k}{(2-\gamma)} z^{-2-\frac{2}{2-\gamma}} \bar{p} = 0.$$

We then let $\bar{p}(z) = f\left(\frac{4-3\gamma}{2\pi k} z^{\frac{2}{2-\gamma}}\right)$. We let $x = \frac{4-3\gamma}{2\pi k} z^{\frac{2}{2-\gamma}}$ and obtain the following equation for $f(x)$:

$$f'' + \frac{1}{2x(x-1)} (5x - (8-5\gamma)) f' - \frac{(2-\gamma)(4-3\gamma)}{2x^2(x-1)} f = 0.$$

We then let $f(x) = g(1 - \frac{1}{x})$, $\xi = 1 - \frac{1}{x}$ to obtain the following equation

$$g'' + \frac{(-4+5\gamma)\xi + 3 - 5\gamma}{2\xi(\xi-1)} g' + \frac{(2-\gamma)(4-3\gamma)}{2\xi(\xi-1)} g = 0. \quad (3.89)$$

We recall the definitions

$$\mu := \frac{6-5\gamma}{2(2-\gamma)}, \quad \nu := \frac{\sqrt{-\gamma^2 - 20\gamma + 28}}{2(2-\gamma)}.$$

We note that this is the hypergeometric equation with

$$a = \frac{2-\gamma}{2}(-\mu - i\nu), \quad b = \frac{2-\gamma}{2}(-\mu + i\nu), \quad c = \frac{5\gamma-3}{2}.$$

The unique solution (up to a constant multiple) which is smooth on a neighborhood of $\xi = 0$ is given by

$$g_1(\xi) = \frac{6\gamma^2 - 15\gamma + 11}{3\gamma - 1} F\left(\frac{2-\gamma}{2}(-\mu - i\nu), \frac{2-\gamma}{2}(-\mu + i\nu), \frac{5\gamma-3}{2}, \xi\right).$$

We recall that as $\xi \rightarrow 0$, we have

$$F\left(\frac{2-\gamma}{2}(-\mu - i\nu), \frac{2-\gamma}{2}(-\mu + i\nu), \frac{5\gamma-3}{2}, \xi\right) = 1 + \frac{(4-3\gamma)(2-\gamma)}{5-3\gamma} \xi + o(\xi).$$

where $F(a, b, c, \xi)$ denotes Gauss's hypergeometric function. By inverting the change of variables, we get the following homogeneous solution for L :

$$\begin{pmatrix} p_{hom,pre}(z) \\ \omega_{hom,pre}(z) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} \cdot \begin{pmatrix} g_1 \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ g'_1 \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix}.$$

One can then check that as $z \rightarrow y_f$, we have

$$\begin{cases} p_{hom,pre} &= 1 + \frac{-18\gamma^2+18\gamma+4}{(5\gamma-3)(3\gamma-1)} \frac{1}{y_f} (z - y_f) + o(z - y_f) \\ \omega_{hom,pre} &= -\frac{2(2-\gamma)^2}{(3\gamma-1)k} + \frac{4(2-\gamma)(\gamma-1)(7-3\gamma)}{(3\gamma-1)(5\gamma-3)} \frac{1}{ky_f} (z - y_f) + o(z - y_f) \end{cases}.$$

Finally, letting $(p_{hom}, \omega_{hom}) = \frac{k(3\gamma-1)}{2(2-\gamma)} (p_{hom,pre}, \omega_{hom,pre})$ and this finishes proving part (a).

For the next part, we start by considering

$$\begin{cases} g_3(\xi) &:= F\left(\frac{2-\gamma}{2}(-\mu - i\nu), \frac{2-\gamma}{2}(-\mu + i\nu), -\frac{1}{2}, 1 - \xi\right) \\ g_4(\xi) &:= -\frac{2}{3}(1 - \xi)^{\frac{3}{2}} F\left(\frac{15\gamma-12}{4} + i\frac{2-\gamma}{2}\nu, \frac{15\gamma-12}{4} - i\frac{2-\gamma}{2}\nu, \frac{5}{2}, 1 - \xi\right) \end{cases}.$$

This gives the following fundamental matrix for (3.89):

$$V_\infty(\xi) := \begin{pmatrix} g_3(\xi) & g_4(\xi) \\ g'_3(\xi) & g'_4(\xi) \end{pmatrix}.$$

In this case, the determinant satisfies the following ODE and asymptotic expansion as $\xi \rightarrow 1$:

$$W'_\infty + \frac{(-4 + 5\gamma)\xi + 3 - 5\gamma}{2\xi(\xi - 1)} W_\infty = 0, \quad W_\infty = (1 - \xi)^{\frac{1}{2}} + O((1 - \xi)^{\frac{3}{2}}).$$

Solving this ODE gives

$$W_\infty(\xi) = \frac{\sqrt{1-\xi}}{\xi^{\frac{5\gamma-3}{2}}}, \quad V_\infty^{-1}(\xi) = \frac{\xi^{\frac{5\gamma-3}{2}}}{\sqrt{1-\xi}} \begin{pmatrix} g'_4(\xi) & -g_4(\xi) \\ -g'_3(\xi) & g_3(\xi) \end{pmatrix}.$$

This leads to the following fundamental matrix

$$\tilde{U}_\infty(z) := \begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} V_\infty \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right).$$

This will satisfy $LU_\infty = 0$. Taking the limit as $z \rightarrow \infty$ gives

$$\tilde{U}_\infty(z) = \begin{pmatrix} 1 & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(\frac{2\pi k}{4-3\gamma}\right)^{\frac{1}{2}} \frac{1}{z^{\frac{1}{2-\gamma}}} \\ 0 & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(\frac{2\pi k}{4-3\gamma}\right)^{\frac{1}{2}} \frac{1}{z^{\frac{1}{2-\gamma}}} \end{pmatrix} + \begin{pmatrix} O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) & O\left(\frac{1}{z^{\frac{3}{2-\gamma}}}\right) \\ O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) & O\left(\frac{1}{z^{\frac{3}{2-\gamma}}}\right) \end{pmatrix}.$$

Finally, for the fundamental matrix of the lemma, we define

$$U_\infty(z) = \tilde{U}_\infty \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{(4-3\gamma)(6\gamma^2-15\gamma+11)k}{2(2-\gamma)} \left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{1}{2}} \end{pmatrix}.$$

To consider the asymptotics of the inverse, we first compute

$$\det \begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} = \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right),$$

which in particular gives

$$\begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{3(\gamma-1)k}{2-\gamma} \\ (2-\gamma)(4-3\gamma) \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right)^{-1} & \frac{(4-3\gamma)(3\gamma-1)k}{2(2-\gamma)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right)^{-1} \end{pmatrix}.$$

This allows us to compute the asymptotics for the inverse of $\tilde{U}_\infty$ to be

$$\tilde{U}_\infty^{-1} = \begin{pmatrix} 1 & \frac{3(\gamma-1)k}{2-\gamma} \\ 0 & \frac{(4-3\gamma)(6\gamma^2-15\gamma+11)k}{2(2-\gamma)} \left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{1}{2}} z^{\frac{1}{2-\gamma}} \end{pmatrix} + \begin{pmatrix} O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) & O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) \\ O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) & O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix}.$$

Finally, the asymptotics of U_∞^{-1} follow from

$$U_\infty^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(4-3\gamma)(6\gamma^2-15\gamma+11)k}{2(2-\gamma)} \left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{1}{2}} \end{pmatrix}^{-1} \tilde{U}_\infty^{-1}.$$

To construct U_0 , we start by defining

$$\begin{cases} g_a := (-\xi)^{\frac{2-\gamma}{2}(\mu+i\nu)} F\left(\frac{2-\gamma}{2}(-\mu-i\nu), \frac{2-\gamma}{2}(-\mu-i\nu) - \frac{5(\gamma-1)}{2}, 1-i(2-\gamma)\nu, \frac{1}{\xi}\right) \\ g_b := (-\xi)^{\frac{2-\gamma}{2}(\mu-i\nu)} F\left(\frac{2-\gamma}{2}(-\mu+i\nu), \frac{2-\gamma}{2}(-\mu+i\nu) - \frac{5(\gamma-1)}{2}, 1+i(2-\gamma)\nu, \frac{1}{\xi}\right) \end{cases}.$$

We then define

$$g_5 = \frac{g_a + g_b}{2}, \quad g_6 = -\frac{g_a - g_b}{2i}$$

which are real-valued. We then define for $\xi \in (-\infty, 0)$,

$$V_0(\xi) = \begin{pmatrix} g_5(\xi) & g_6(\xi) \\ g_5'(\xi) & g_6'(\xi) \end{pmatrix}, \quad W_0 = \det V_0.$$

In this case, the boundary condition is that as $\xi \rightarrow -\infty$,

$$W_0 = \frac{2-\gamma}{2} \nu (-\xi)^{\frac{4-5\gamma}{2}} + O((- \xi)^{\frac{2-5\gamma}{2}}).$$

Thus, we get

$$W_0 = \frac{(2-\gamma)\nu}{2} \frac{\sqrt{1-\xi}}{(-\xi)^{\frac{5\gamma-3}{2}}}, \quad V_0^{-1}(\xi) = \frac{2}{(2-\gamma)\nu} \frac{(-\xi)^{\frac{5\gamma-3}{2}}}{\sqrt{1-\xi}} \begin{pmatrix} g'_6(\xi) & -g_6(\xi) \\ -g'_5(\xi) & g_5(\xi) \end{pmatrix}.$$

We then get the following fundamental matrix for $z \in (0, 1)$:

$$\tilde{U}_0(z) := \begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} V_0 \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right).$$

We then note that as $\xi \rightarrow \infty$, we have

$$\tilde{U}_0\left(\left(\frac{2\pi k}{(1-\xi)(4-3\gamma)}\right)^{\frac{2-\gamma}{2}}\right) = (-\xi)^{\frac{6-5\gamma}{4}} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} + O\left((-\xi)^{\frac{2-5\gamma}{4}}\right)$$

where

$$\begin{aligned} T_1 &= \frac{(4-3\gamma)(3\gamma-1) - 3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} \cos\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right) \\ &\quad + \frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \sin\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right), \\ T_2 &= \frac{-(4-3\gamma)(3\gamma-1) + 3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} \sin\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right) \\ &\quad + \frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \cos\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right), \\ T_3 &= \frac{-2(2-\gamma)^2(4-3\gamma) + (2-\gamma)^2\mu}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \cos\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right) \\ &\quad + \frac{-(2-\gamma)^2\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \sin\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right), \\ T_4 &= \frac{2(2-\gamma)^2(4-3\gamma) - (2-\gamma)^2\mu}{2(4-3\gamma)(6\gamma^2-15\gamma+11)k} \sin\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right) \\ &\quad + \frac{-(2-\gamma)^2\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \cos\left(\frac{2-\gamma}{2}\nu \log(-\xi)\right). \end{aligned}$$

We then claim that we may factorize this as

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{2-\gamma}{2}\nu \log(-\xi)\right) & \sin\left(-\frac{2-\gamma}{2}\nu \log(-\xi)\right) \\ \frac{(2-\gamma)^{\frac{3}{2}}}{k} \cos\left(-\frac{2-\gamma}{2}\nu \log(-\xi) + \theta_0\right) & \frac{(2-\gamma)^{\frac{3}{2}}}{k} \sin\left(-\frac{2-\gamma}{2}\nu \log(-\xi) + \theta_0\right) \end{pmatrix} A$$

for some constant-coefficient matrix A . Indeed, by inspecting the upper two entries if this were to hold, we must have

$$A = \begin{pmatrix} \frac{(4-3\gamma)(3\gamma-1) - 3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} & \frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \\ -\frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} & \frac{(4-3\gamma)(3\gamma-1) - 3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} \end{pmatrix}.$$

From this ansatz, verifying this factorization for T_3 follows from showing the two equalities

$$(2-\gamma)^{\frac{3}{2}}(((4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu)\cos(\theta_0)-3(\gamma-1)(2-\gamma)\nu\sin(\theta_0)) \\ = -2(2-\gamma)^2(4-3\gamma)+(2-\gamma)^2\mu,$$

$$(2-\gamma)^{\frac{3}{2}}(((4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu)\sin(\theta_0)+3(\gamma-1)(2-\gamma)\nu\cos(\theta_0)) = -(2-\gamma)^2\nu.$$

Indeed, to verify these, we recall

$$\cos(\theta_0) = -\frac{2+\gamma}{4\sqrt{2-\gamma}}, \quad \sin(\theta_0) = -\frac{\sqrt{-\gamma^2-20\gamma-28}}{4\sqrt{2-\gamma}} = -\nu\frac{\sqrt{2-\gamma}}{2}$$

We can then simplify the first equation to be

$$(2+\gamma)(3\gamma^2+3\gamma-10)+3(\gamma-1)(-\gamma^2-20\gamma+28) = -16(2-\gamma)(4-3\gamma)+4(6-5\gamma).$$

This holds by direct computation. Similarly, the second equation can be rewritten as

$$-2(4-3\gamma)(3\gamma-1)+3(\gamma-1)(6-5\gamma)-3(\gamma-1)(2+\gamma) = -4$$

Another direct computation verifies this identity. In terms of the variable z , we have that

$$\tilde{U}_0(z) = \left(\frac{2\pi}{4-3\gamma}\right)^{\frac{6-5\gamma}{4}} z^{-\mu} \begin{pmatrix} \cos\left(\nu \log\left(\left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{2-\gamma}{2}} z\right)\right) & \sin\left(\nu \log\left(\left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{2-\gamma}{2}} z\right)\right) \\ \cos\left(\nu \log\left(\left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{2-\gamma}{2}} z\right) + \theta_0\right) & \sin\left(\nu \log\left(\left(\frac{4-3\gamma}{2\pi k}\right)^{\frac{2-\gamma}{2}} z\right) + \theta_0\right) \end{pmatrix} \\ \begin{pmatrix} \frac{(4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} & \frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \\ -\frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} & \frac{(4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} \end{pmatrix} + O(z^{\frac{5\gamma-2}{2(2-\gamma)}}).$$

Finally, we define

$$U_0(z) = \left(\frac{2\pi}{4-3\gamma}\right)^{-\frac{6-5\gamma}{4}} \tilde{U}_0(z) \cdot \begin{pmatrix} \frac{(4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} & \frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \\ -\frac{3(\gamma-1)(2-\gamma)\nu}{(4-3\gamma)(6\gamma^2-15\gamma+11)} & \frac{(4-3\gamma)(3\gamma-1)-3(\gamma-1)(2-\gamma)\mu}{(6\gamma^2-15\gamma+11)(4-3\gamma)} \end{pmatrix}^{-1} \\ \cdot \begin{pmatrix} \cos\left(\frac{2-\gamma}{2}\nu \log\left(\frac{4-3\gamma}{2\pi k}\right)\right) & -\sin\left(\frac{2-\gamma}{2}\nu \log\left(\frac{4-3\gamma}{2\pi k}\right)\right) \\ -\sin\left(\frac{2-\gamma}{2}\nu \log\left(\frac{4-3\gamma}{2\pi k}\right)\right) & \cos\left(\frac{2-\gamma}{2}\nu \log\left(\frac{4-3\gamma}{2\pi k}\right)\right) \end{pmatrix}$$

and it has the asymptotics stated in the Lemma. One can similarly compute the asymptotics for U_0^{-1} finishing the proof of part (b).

Part (c) follows from the standard existence and uniqueness for linear ODEs as there are no singularities on either of these intervals. In principle one could compute these constants exactly using connection formulae (as done for $\gamma = 1$ in [100]), but this is omitted in this case.

For part (d), we start by defining the following on $(-\infty, 0) \cup (0, 1)$:

$$g_2(\xi) = |\xi|^{-\frac{5(\gamma-1)}{2}} F\left(\frac{4-5\gamma}{4} - i\frac{\sqrt{-\gamma^2-20\gamma+28}}{4}, \frac{4-5\gamma}{4} + i\frac{\sqrt{-\gamma^2-20\gamma+28}}{4}, \frac{7-5\gamma}{2}, \xi\right).$$

We comment that this is a solution to (3.89) defined on a neighborhood of $\xi = 0$ which is linearly independent from g_1 (see section 5.10.1 of [91]). We now use (and g_1) this to construct a fundamental matrix for L on a neighborhood of $z = y_f$. We recall that as $\xi \rightarrow 0$, we have

$$F\left(\frac{4-5\gamma}{4} - i\frac{\sqrt{-\gamma^2-20\gamma+28}}{5}, \frac{4-5\gamma}{4} + i\frac{\sqrt{-\gamma^2-20\gamma+28}}{5}, \frac{7-5\gamma}{2}, \xi\right) = 1 + \frac{6\gamma^2-15\gamma+11}{2(7-5\gamma)}\xi + o(\xi).$$

We note that we have (assuming $\xi > 0$)

$$g_2 = \xi^{-\frac{5(\gamma-1)}{2}} + \frac{6\gamma^2-15\gamma+11}{2(7-5\gamma)}\xi^{\frac{7-5\gamma}{2}} + o(\xi^{\frac{7-5\gamma}{2}}),$$

$$g'_2 = -\frac{5(\gamma-1)}{2}\xi^{-\frac{5\gamma-3}{2}} + \frac{6\gamma^2-15\gamma+11}{4}\xi^{-\frac{5(\gamma-1)}{2}} + o(\xi^{\frac{5(\gamma-1)}{2}}).$$

We then recall that

$$1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}} = \frac{2}{(2-\gamma)y_f}(z-y_f) - \frac{4-\gamma}{(2-\gamma)^2}\frac{(z-y_f)^2}{y_f^2} + o(z-y_f)^2.$$

from which it follows that

$$g_2\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) = \left(\frac{2(z-y_f)}{(2-\gamma)y_f}\right)^{-\frac{5(\gamma-1)}{2}} - \frac{(25\gamma^2-61\gamma+32)(3-\gamma)}{8(7-5\gamma)}\left(\frac{2(z-y_f)}{(2-\gamma)y_f}\right)^{\frac{7-5\gamma}{2}} + o(z-y_f)^{\frac{7-5\gamma}{2}},$$

$$g'_2\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) = -\frac{5(\gamma-1)}{2}\left(\frac{2(z-y_f)}{(2-\gamma)y_f}\right)^{-\frac{5\gamma-3}{2}} + \frac{(25\gamma^3-116\gamma^2+115\gamma-16)}{16}\left(\frac{2(z-y_f)}{(2-\gamma)y_f}\right)^{-\frac{5(\gamma-1)}{2}} + o(z-y_f)^{-\frac{5(\gamma-1)}{2}}.$$

We then note that as $z \rightarrow y_f$, we have that

$$\begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)}\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k}\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} \begin{pmatrix} g_2\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ g'_2\left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{4-3\gamma} \\ -\frac{(2-\gamma)}{(4-3\gamma)k} \end{pmatrix} \left|\frac{2(z-y_f)}{(2-\gamma)y_f}\right|^{-\frac{5(\gamma-1)}{2}} + \begin{pmatrix} \frac{25\gamma^3-136\gamma^2+191\gamma-72}{8(4-3\gamma)(7-5\gamma)} \\ -\frac{(2-\gamma)(\gamma-1)(25\gamma^2-111\gamma+104)}{8(4-3\gamma)(7-5\gamma)k} \end{pmatrix} \frac{2(z-y_f)}{(2-\gamma)y_f} \left|\frac{2(z-y_f)}{(2-\gamma)y_f}\right|^{-\frac{5(\gamma-1)}{2}}$$

$$+ o((z-y_f)|z-y_f|^{-\frac{5(\gamma-1)}{2}}).$$

Thus, if we define the (prescaling) subordinate homogeneous solution by

$$\begin{pmatrix} p_{sub,pre} \\ \omega_{sub,pre} \end{pmatrix} = (4-3\gamma) \left(\frac{2}{2-\gamma} y_f^{-1} \right)^{\frac{5(\gamma-1)}{2}} \cdot \begin{pmatrix} \frac{3\gamma-1}{(6\gamma^2-15\gamma+11)} & -\frac{6(\gamma-1)}{(4-3\gamma)(6\gamma^2-15\gamma+11)} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ -\frac{2(2-\gamma)^2}{(6\gamma^2-15\gamma+11)k} & \frac{2(2-\gamma)}{(4-3\gamma)(6\gamma^2-15\gamma+11)k} \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix} \begin{pmatrix} g_2 \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \\ g_2' \left(1 - \frac{2\pi k}{(4-3\gamma)z^{\frac{2}{2-\gamma}}}\right) \end{pmatrix}.$$

We then define $(p_{sub}, \omega_{sub}) = \frac{k(3\gamma-1)}{2(2-\gamma)} (p_{sub,pre}, \omega_{sub,pre})$ then

$$\begin{pmatrix} p_{hom} & p_{sub} \\ \omega_{hom} & \omega_{sub} \end{pmatrix} = \frac{k(3\gamma-1)}{2(2-\gamma)} \begin{pmatrix} 1 & |z-y_f|^{-\frac{5(\gamma-1)}{2}} \\ -\frac{2(2-\gamma)^2}{k(3\gamma-1)} & -\frac{(2-\gamma)}{k} |z-y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix} \\ + \frac{k(3\gamma-1)}{2(2-\gamma)} \begin{pmatrix} \frac{2(-9\gamma^2+9\gamma+2)}{(3\gamma-1)(5\gamma-3)y_f} (z-y_f) & \frac{25\gamma^3-136\gamma^2+191\gamma-72}{4(2-\gamma)(7-5\gamma)y_f} (z-y_f) |z-y_f|^{-\frac{5(\gamma-1)}{2}} \\ \frac{4(2-\gamma)(\gamma-1)(7-3\gamma)}{ky_f(3\gamma-1)(5\gamma-3)} (z-y_f) & -\frac{(\gamma-1)(25\gamma^2-111\gamma+104)}{4(7-5\gamma)ky_f} (z-y_f) |z-y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix} \\ + \begin{pmatrix} o(z-y_f) & o(z-y_f) |z-y_f|^{-\frac{5(\gamma-1)}{2}} \\ o(z-y_f) & o(z-y_f) |z-y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix}.$$

Then, as in the previous section existence and uniqueness yield existence of M_∞, M_0 such that

$$U_\infty M_\infty = U_0 M_0 = \begin{pmatrix} p_{hom} & p_{sub} \\ \omega_{hom} & \omega_{sub} \end{pmatrix}$$

yielding the first asymptotics stated in part (d) of the Lemma.

For part (e), we note that we can factor

$$\begin{pmatrix} p_{hom} & p_{sub} \\ \omega_{hom} & \omega_{sub} \end{pmatrix} = (X_0 + X_1(z-y_f) + o(z-y_f)) \begin{pmatrix} 1 & 0 \\ 0 & |z-y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix}$$

where

$$X_0 = \begin{pmatrix} \frac{(3\gamma-1)k}{2(2-\gamma)} & \frac{(3\gamma-1)k}{2(2-\gamma)} \\ -(2-\gamma) & -\frac{3\gamma-1}{2} \end{pmatrix}, \quad X_1 = \begin{pmatrix} \frac{(-9\gamma^2+9\gamma+2)}{(2-\gamma)(5\gamma-3)} \frac{k}{y_f} & -\frac{(3\gamma-1)(25\gamma^3-136\gamma^2+191\gamma-72)k}{8(\gamma-2)^2(5\gamma-7)y_f} \\ \frac{2(\gamma-1)(7-3\gamma)}{y_f(5\gamma-3)} & -\frac{(\gamma-1)(3\gamma-1)(25\gamma^2-111\gamma+104)}{8(\gamma-2)(5\gamma-7)y_f} \end{pmatrix}.$$

We note that

$$\begin{pmatrix} p_{hom} & p_{sub} \\ \omega_{hom} & \omega_{sub} \end{pmatrix} = (X_0 + X_1(z-y_f) + o(z-y_f)) \begin{pmatrix} 1 & 0 \\ 0 & |z-y_f|^{-\frac{5(\gamma-1)}{2}} \end{pmatrix}.$$

We then write abstractly

$$\begin{pmatrix} (2-\gamma)z & kz \\ \frac{2\pi(2-\gamma)^2}{4-3\gamma} z^{1-\frac{2}{2-\gamma}} & (2-\gamma)z \end{pmatrix}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix} = I_0 \frac{1}{z-y_f} + I_1 + o(1).$$

It follows that the Taylor expansion of the solution will be given by (here Γ can be chosen to be arbitrary, it will cancel out)

$$\begin{pmatrix} p \\ \omega \end{pmatrix} = X_0 \begin{pmatrix} \Gamma & 0 \\ 0 & \frac{2}{5(\gamma-1)} \end{pmatrix} X_0^{-1} I_0 \left(X_1 \begin{pmatrix} \Gamma & 0 \\ 0 & \frac{2}{5(\gamma-1)} \end{pmatrix} X_0^{-1} I_0 + X_0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{5(\gamma-3)} \end{pmatrix} X_0^{-1} I_1 \right. \\ \left. - X_0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{5(\gamma-3)} \end{pmatrix} X_0^{-1} X_1 X_0^{-1} I_0 \right) (z - y_f) + o(z - y_f). \quad (3.90)$$

We thus let

$$\begin{aligned} A_0 &= X_0 \begin{pmatrix} \Gamma & 0 \\ 0 & \frac{2}{5(\gamma-1)} \end{pmatrix} X_0^{-1}, \\ A_1 &= X_0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{5\gamma-3} \end{pmatrix} X_0^{-1}, \\ A_2 &= X_1 X_0^{-1}, \end{aligned}$$

and (3.90) can be simplified to

$$\begin{pmatrix} p \\ \omega \end{pmatrix} = A_0 I_0 + (A_2 A_0 I_0 + A_1 I_1 - A_1 A_2 I_0) (z - y_f) + o(z - y_f). \quad (3.91)$$

We may then explicitly compute

$$\begin{aligned} A_0 &= \begin{pmatrix} \frac{5(3\gamma-1)(\gamma-1)\Gamma-4(2-\gamma)}{25(\gamma-1)^2} & \frac{k(5(3\gamma-1)(\gamma-1)\Gamma-2(3\gamma-1))}{25(2-\gamma)(\gamma-1)^2} \\ \frac{4(2-\gamma)^2-10(2-\gamma)^2(\gamma-1)\Gamma}{25(\gamma-1)^2 k} & \frac{2(3\gamma-1)-10(2-\gamma)(\gamma-1)\Gamma}{25(\gamma-1)^2} \end{pmatrix}, \\ A_1 &= \begin{pmatrix} \frac{3\gamma+1}{5\gamma-3} & \frac{(3\gamma-1)k}{(2-\gamma)(5\gamma-3)} \\ -\frac{2(2-\gamma)^2}{(5\gamma-3)k} & \frac{2(\gamma-1)}{5\gamma-3} \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -\frac{25\gamma^3-162\gamma^2+197\gamma-32}{2(7-5\gamma)(5\gamma-3)y_f} & -\frac{(75\gamma^4-331\gamma^3+321\gamma^2-81\gamma+88)k}{4(2-\gamma)^2(7-5\gamma)(5\gamma-3)y_f} \\ \frac{(2-\gamma)(25\gamma^3-102\gamma^2+81\gamma+16)}{2(7-5\gamma)(5\gamma-3)ky_f} & \frac{75\gamma^4-451\gamma^3+913\gamma^2-873\gamma+376}{4(2-\gamma)(7-5\gamma)(5\gamma-3)y_f} \end{pmatrix}, \\ I_0 &= \begin{pmatrix} \frac{1}{2} \left(\sigma_1 - \frac{k\sigma_2}{2-\gamma} \right) \\ -\frac{(2-\gamma)(\sigma_1 - \frac{k\sigma_2}{2-\gamma})}{2k} \end{pmatrix}, \\ I_1 &= \begin{pmatrix} -\frac{\sigma_2(\gamma k)}{4(2-\gamma)^2 y_f} + \frac{\gamma\sigma_1}{4(2-\gamma)y_f} + \frac{\sigma_3}{2} \\ \frac{(4-\gamma)\sigma_1}{4ky_f} + \frac{\gamma\sigma_2}{4(2-\gamma)y_f} - \frac{(2-\gamma)\sigma_3}{2k} \end{pmatrix}, \end{aligned}$$

and then simplifying (3.91) gives the formula stated in the lemma. $\square$

3.5.2 Proof of Lemma 3.2.4

We suppose that $(P, \Omega) \in N_{y_0}[\epsilon]$. It follows from Lemma 3.2.1 that $(\mathring{p}, \mathring{\omega})$ is in $C^1[y_0, \infty]$. It follows the computations in section 3.2.3 that

$$p(y_f) = \mathring{p}_{0,e}[\epsilon], \quad \omega(y_f) = \mathring{\omega}_{0,e}[\epsilon], \quad \mathring{p}'(y_f) = \mathring{p}_{1,e}[\epsilon], \quad \mathring{\omega}'(y_f) = \mathring{\omega}_{1,e}[\epsilon].$$

It will also follow from examining the integral kernel of R that

$$\sup_{\frac{1}{2} \leq z \leq 2} |p| + |\omega| + \left| \frac{p + \frac{3\gamma-1}{2(2-\gamma)^2} k\omega + \frac{\sigma_1}{2(2-\gamma)} - \frac{k\sigma_2}{2(2-\gamma^2)}}{z - y_f} \right| \lesssim \sup_{\frac{1}{2} \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \frac{k}{2-\gamma} \Omega}{z - y_f} \right| + O(\epsilon). \quad (3.92)$$

We may then rewrite $L(p, \omega) = (P, \Omega)$ as

$$E_{lin} \begin{pmatrix} p \\ \omega \end{pmatrix}' = (P, \Omega) - F_{lin} \begin{pmatrix} p \\ \omega \end{pmatrix},$$

We recall that as $z \rightarrow y_f$, we have that

$$\begin{aligned} E_{lin}^{-1} &= \begin{pmatrix} \frac{1}{2} & -\frac{k}{2(2-\gamma)} \\ -\frac{2-\gamma}{2k} & \frac{1}{2} \end{pmatrix} \frac{1}{z - y_f} + O(1) \\ E_{lin}^{-1} F_{lin} &= \begin{pmatrix} -(2-\gamma) & -\frac{(3\gamma-1)k}{2(2-\gamma)} \\ \frac{(2-\gamma)^2}{k} & \frac{3\gamma-1}{2} \end{pmatrix} \frac{1}{z - y_f} + O(1). \end{aligned}$$

and then if we invert E_{lin} and use (3.92) we obtain

$$\sup_{\frac{1}{2} \leq z \leq 2} |p'| + |\omega'| \lesssim \sup_{\frac{1}{2} \leq z \leq 2} |P| + |\Omega| + \left| \frac{P - \frac{k}{2-\gamma} \Omega}{z - y_f} \right| + O(\epsilon).$$

For $z \geq 2$, we recall that as $z \rightarrow \infty$,

$$\begin{aligned} U_\infty(z) &= \left(\begin{pmatrix} 1 & -\frac{3(\gamma-1)k}{2-\gamma} \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right) \right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z^{\frac{1}{2-\gamma}}} \end{pmatrix}, \\ E_{lin}(z) &= \begin{pmatrix} 2-\gamma & k \\ 0 & 2-\gamma \end{pmatrix} z + O(z^{1-\frac{2}{2-\gamma}}), \end{aligned}$$

from which it follows that

$$U_\infty^{-1} E_{lin}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & z^{\frac{1}{2-\gamma}} \end{pmatrix} \cdot \left(\begin{pmatrix} \frac{1}{2-\gamma} & -\frac{4-3\gamma}{(2-\gamma)^2} k \\ 0 & \frac{1}{2-\gamma} \end{pmatrix} z^{-1} + O(z^{-1-\frac{2}{2-\gamma}}) \right)$$

and so for $z \geq 2$

$$|U_\infty^{-1} E_{lin}^{-1} \begin{pmatrix} P \\ \Omega \end{pmatrix}| \lesssim \frac{1}{z^{\frac{3-\gamma}{2-\gamma}}} \left(\sup_{z \geq 2} |z^{\frac{1}{2-\gamma}} P| + |z^{\frac{2}{2-\gamma}} \Omega| \right).$$

Integrating this from 2 to ∞ gives the existence of the desired limits and bounds as $z \rightarrow \infty$.

We similarly compute that as $z \rightarrow 0$, we have that

$$E_{lin}(z) = \begin{pmatrix} z & 0 \\ 0 & z^{1-\frac{2}{2-\gamma}} \end{pmatrix} \left(\begin{pmatrix} 2-\gamma & k \\ \frac{(2-\gamma)^2 y_f^{\frac{2}{2-\gamma}}}{k} & 0 \end{pmatrix} + O(z^{\frac{2}{2-\gamma}}) \right)$$

so that

$$\begin{aligned} E_{lin}^{-1}(z) &= \begin{pmatrix} 0 & 0 \\ \frac{1}{k} & 0 \end{pmatrix} \frac{1}{z} + O(z^{\frac{2}{2-\gamma}-1}), \\ U_0^{-1}(z) E_{lin}^{-1} &= \begin{pmatrix} -\frac{\sin(\nu \log(z))}{k \sin(\theta_0)} & 0 \\ \frac{\cos(\nu \log(z))}{k \sin(\theta_0)} & 0 \end{pmatrix} z^{\mu-1} + O(z^{\mu-1+\frac{2}{2-\gamma}}). \end{aligned}$$

This gives that for $z \leq \frac{1}{2}$

$$|U_0^{-1}(z)E_{lin}^{-1}\left(\frac{P}{\Omega}\right)| \lesssim \left(z^{-\mu-1}|z^{2\mu}P| + |z^{2\mu+\frac{2}{2-\gamma}}\Omega|\right).$$

Integrating from y_0 to $\frac{1}{2}$ gives the desired limits and (3.60) (note this is where one picks up the factor of $y_0^{-\mu}$). A similar argument gives (3.61).

3.5.3 Proof of Lemma 3.2.5

We first check that p, ω do not vanish. Indeed, we see that if $(\hat{p}, \hat{\omega}) = (p_{hom}, \omega_{hom}) + (p, \omega)$ then

$$||(\hat{p}, \hat{\omega})||_{X_{y_0}} \leq C + ||(p_{hom}, \omega_{hom})||_{X_{y_0}}.$$

Examining the weights for $y \leq y_0, y \geq y_0$ respectively, we see that this is the same as (up to a universal constant)

$$|\hat{p}| \leq \frac{\langle y \rangle^\mu}{y^\mu} (C + ||(p_{hom}, \omega_{hom})||_{X_{y_0}}), \quad |\hat{\omega}| \leq \frac{\langle y \rangle^{\mu-\frac{1}{2-\gamma}}}{y^\mu} (C + ||(p_{hom}, \omega_{hom})||_{X_{y_0}}).$$

In particular, bounding the weights by $y_0^{-\mu}$, we have that

$$|\epsilon \hat{p}|, |\epsilon \hat{\omega}| \leq \epsilon y_0^{-\mu} (C + ||(p_{hom}, \omega_{hom})||_{X_{y_0}}).$$

and hence by choosing C_0 sufficiently small, we can ensure

$$|\epsilon \hat{p}| \leq \frac{1}{2}k, \quad |\epsilon \hat{\omega}| \leq \frac{1}{2}(2-\gamma). \quad (3.93)$$

In particular, we see that $\tilde{p} = k + \epsilon \hat{p}$ does not vanish.

We first comment that by the computations in Section 3.2.3, if $p, \omega \in X_{y_0}[\epsilon]$, then we have that $\frac{P-\frac{k}{2-\gamma}\Omega}{z-y_f} \in C^0([y_0, \infty])$, and the boundary conditions

$$P(y_f) = \mathring{\sigma}_1[\epsilon], \quad \Omega(y_f) = \mathring{\sigma}_2[\epsilon], \quad (P - \frac{k}{2-\gamma}\Omega)'(y_f) = \mathring{\sigma}_3[\epsilon]$$

are satisfied. One also obtains the proper boundedness of

$$\sup_{\frac{1}{2} \leq z \leq 2} \left| \frac{P - \frac{k}{2-\gamma}\Omega - (P(0) - \frac{k}{2-\gamma}\Omega(0))}{z - y_f} \right|.$$

We now consider the regime in which $z \rightarrow \infty$. It is helpful at this point to use the fundamental theorem of calculus to rewrite (3.43) as

$$\hat{N}(\epsilon, \hat{p}, \hat{\omega}) = \epsilon \left(\hat{E}_e(z, \hat{p}, \hat{\omega}, y_0, \epsilon) \left(\frac{\hat{p}'}{\hat{\omega}'} \right) + \hat{F}_e(z, \hat{p}, \hat{\omega}, y_0, \epsilon) \right)$$

where

$$\hat{E}_e(z, a, b, y_0, \epsilon) = - \int_0^1 \frac{\partial}{\partial \epsilon} (S^T \hat{E})|_{(z, a, b, y_0, t\epsilon)} dt, \quad \hat{F}_e = - \int_0^1 \frac{\partial}{\partial \epsilon} (S^T \hat{F})|_{(z, a, b, y_0, t\epsilon)} dt.$$

We note that in this case, since $(p, \omega) \in X_{y_0}$, $\|\hat{p}, \hat{\omega}\|_{X_{y_0}} < \infty$, so there exists constants $p_0, p_1, \omega_0, \omega_1$ such that

$$\hat{p}(z) = \hat{p}_{\infty,0} + \frac{\hat{p}_{\infty,1}}{z^{\frac{1}{2-\gamma}}} + o\left(\frac{1}{z^{\frac{1}{2-\gamma}}}\right), \quad \hat{\omega}(z) = \frac{\hat{\omega}_{\infty,0}}{z^{\frac{1}{2-\gamma}}} + \frac{\hat{\omega}_{\infty,1}}{z^{\frac{2}{2-\gamma}}} + o\left(\frac{1}{z^{\frac{2}{2-\gamma}}}\right). \quad (3.94)$$

$$\hat{p}'(z) = -\frac{1}{2-\gamma} \frac{\hat{p}_{\infty,1}}{z^{\frac{1}{2-\gamma}+1}} + o\left(\frac{1}{z^{\frac{1}{2-\gamma}+1}}\right), \quad \hat{\omega}'(z) = -\frac{1}{2-\gamma} \frac{\hat{\omega}_{\infty,0}}{z^{\frac{1}{2-\gamma}+1}} - \frac{2}{2-\gamma} \frac{\omega_{\infty,1}}{z^{\frac{2}{2-\gamma}+1}} + o\left(\frac{1}{z^{\frac{2}{2-\gamma}+1}}\right). \quad (3.95)$$

To do the difference estimates more cleanly, we introduce two symbol classes. In particular, we define

$$S^k = \{f(y, a, b) \in C^\infty([y_0, \infty) \times [-\frac{k}{2}, \frac{k}{2}] \times [-\frac{2-\gamma}{2}, \frac{2-\gamma}{2}]) : \\ |\partial_y^{l_1} \partial_a^{l_2} \partial_b^{l_3} f| \leq C_{l_1, l_2, l_3} |y|^{k-l_1}, \quad \lim_{y \rightarrow \infty} y^{l_1-k} \partial_y^{l_1} f \text{ exists and depends smoothly } a, b\},$$

$$\underline{S}^k = \{f(z, a, b, \epsilon) \in C^\infty([y_0, \infty) \times [\frac{k}{2}, 2k] \times [\frac{2-\gamma}{2}, 2(2-\gamma)] \times [0, 1]) : \\ |\partial_z^{l_1} \partial_a^{l_2} \partial_b^{l_3} \partial_\epsilon^{l_4} f| \leq C_{l_1, l_2, l_3, l_4} |z|^{k-l_1}, \\ \lim_{z \rightarrow \infty} z^{l_1-k} \partial_z^{l_1} \partial_a^{l_2} \partial_b^{l_3} \partial_\epsilon^{l_4} f \text{ exists and depends smoothly } a, b, \epsilon\},$$

$$\hat{S}^k = \{f(z, a, b, \epsilon) \in C^\infty([y_0, \infty) \times [-\frac{k}{2\epsilon}, \frac{k}{2\epsilon}] \times [-\frac{2-\gamma}{2\epsilon}, \frac{2-\gamma}{2\epsilon}] \times [0, 1]) : \\ |\partial_z^{l_1} \partial_a^{l_2} \partial_b^{l_3} \partial_\epsilon^{l_4} f| \leq C_{l_1, l_2, l_3, l_4} \epsilon^{l_2+l_3} (1 + |a| + |b|)^{l_4} |z|^{k-l_1}, \\ \lim_{z \rightarrow \infty} z^{l_1-k} \partial_z^{l_1} \partial_a^{l_2} \partial_b^{l_3} \partial_\epsilon^{l_4} f \text{ exists and depends smoothly } a, b, \epsilon\}.$$

We abbreviate $O_S(y^k)$ for an error in S^k (and analogously for $\underline{S}, \hat{S}$). We then note that if a, b are restricted to $(-\frac{k}{2}, \frac{k}{2}), (-\frac{2-\gamma}{2}, \frac{2-\gamma}{2})$, which will always be the case by (3.93), then

$$\tilde{E}(y, k+a, 2-\gamma+b) = \begin{pmatrix} O_S(y) & O_S(y) \\ O_S(y^{1-\frac{2}{2-\gamma}}) & (2-\gamma)y + bO_S(y) \end{pmatrix}.$$

It then follows that

$$\underline{E}(z, k+a, 2-\gamma+b, y_0, \epsilon) = \begin{pmatrix} O_{\underline{S}}(z) & O_{\underline{S}}(z) \\ O_{\underline{S}}(z^{1-\frac{2}{2-\gamma}}) & (2-\gamma)(z-\beta) + bO_{\underline{S}}(z) \end{pmatrix}.$$

It then follows that

$$\hat{E}(z, a, b, y_0, \epsilon) = \begin{pmatrix} O_{\hat{S}}(z) & O_{\hat{S}}(z) \\ O_{\hat{S}}(z^{1-\frac{2}{2-\gamma}}) & (2-\gamma)(z-\beta) + bO_{\hat{S}}(z) \end{pmatrix}$$

It then follows that

$$E_e(z, a, b, y_0, \epsilon) = \left(\begin{array}{c} O_{\hat{S}}(z) \\ O_{\hat{S}}(z^{1-\frac{2}{2-\gamma}}) \end{array} \quad \begin{array}{c} O_{\hat{S}}(z) \\ \frac{\sqrt{7}}{2}[\epsilon](2-\gamma)(z-\beta) + bO_{\hat{S}}(z) \end{array} \right) \cdot (1 + |a| + |b|)$$

where $\frac{\sqrt{7}}{2}[\epsilon] = -\frac{1}{\epsilon}(\frac{2-\gamma}{k} \frac{\hat{p}[\epsilon]}{\hat{\omega}[\epsilon]} - 1)$ (in fact there is an asymptotic expansion).

Similarly, we note that in the same regimes,

$$\begin{aligned} \tilde{F}(y, k+a, 2-\gamma+b) &= \left(\begin{array}{c} bO_S(1) \\ b + aO_S(y^{-\frac{2}{2-\gamma}}) + bO_S(y^{-\frac{2}{2-\gamma}}) + b^2O_S(1) \end{array} \right) \\ \underline{F}(z, k+a, 2-\gamma+b, y_0, \epsilon) &= \left(\begin{array}{c} bO_{\hat{S}}(1) \\ b + aO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + bO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + b^2O_{\hat{S}}(1) \end{array} \right) \\ \hat{F}(z, a, b, y_0, \epsilon) &= \left(\begin{array}{c} bO_{\hat{S}}(1) \\ b + aO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + bO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + b^2O_{\hat{S}}(1) \end{array} \right) \cdot \\ F_e(z, a, b, y_0, \epsilon) &= \left(\begin{array}{c} bO_{\hat{S}}(1) \\ \frac{\sqrt{7}}{2}[\epsilon]b + aO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + bO_{\hat{S}}(z^{-\frac{2}{2-\gamma}}) + b^2O_{\hat{S}}(1) \end{array} \right) \cdot (1 + |a| + |b|) \end{aligned}$$

As a comment, from this we see that as $z \rightarrow \infty$, substituting (3.94), (3.95), since the contributions of $(2-\gamma)z\partial_z\hat{\omega} + \hat{\omega}$ will cancel, we have that

$$\hat{N} = \epsilon \left(\begin{array}{c} O(z^{-\frac{1}{2-\gamma}}) \\ O(z^{-\frac{2}{2-\gamma}}) \end{array} \right).$$

From this we get the desired boundedness

$$\sup_{z \geq y_f} |z^{\frac{1}{2-\gamma}} P| + |z^{\frac{2}{2-\gamma}} \Omega| \leq C_2 |\epsilon|$$

By considering the asymptotic expansions more carefully, one sees that the desired limits as $z \rightarrow \infty$ exist.

Similarly, we note that as $|\hat{p}|, |\hat{\omega}| \lesssim y_0^{-\mu}, |\hat{p}'|, |\hat{\omega}'| \lesssim y_0^{-\mu-1}$ we have that

$$\sup_{y_0 \leq z \leq y_f} |z^{2\mu} P| + |z^{2\mu+\frac{2}{2-\gamma}} \Omega| \leq C_2 |\epsilon|.$$

This finishes the proof of (3.62). A straightforward application of the fundamental theorem of calculus and the symbol properties shows (3.63).

3.5.4 Proof of Lemma 3.2.6

Proof. We begin by supposing

$$|\epsilon| \leq \min\left(\frac{y_0^\mu}{C_2(\frac{1}{2}) \cdot C_1 \cdot 10}, y_0^\mu C_0(\frac{1}{2})\right).$$

We then pick $(p_0, \omega_0) \in X_{y_0}[\epsilon]$ so that $\|(p_0, \omega_0)\|_{X_{y_0}} \leq y_0^{-\mu} \cdot \epsilon \cdot C_2(\frac{1}{2}) \cdot C_1$. We then define $(P_n, \Omega_n) = N(\epsilon, p_{n-1}, \omega_{n-1})$, and then

$$(p_n, \omega_n) = R(P_n, \Omega_n) + \left(\frac{\frac{2-\gamma}{k} \hat{\sigma}_1[\epsilon] - \hat{\sigma}_2[\epsilon]}{5(\gamma-1)} + \hat{\omega}_{0,e}[\epsilon] \right) \frac{1}{-(2-\gamma)} (p_{hom}, \omega_{hom}).$$

We note that $(P_j, \Omega_j) \in N_{y_0}[\epsilon]$ and $(p_j, \omega_j) \in X_{y_0}[\epsilon]$. We assume inductively that $\|(p_k, \omega_k)\|_{X_{y_0}} \leq y_0^{-\mu} \cdot \epsilon \cdot C_2(\frac{1}{2}) \cdot C_1$. it then follows by choice of smallness of ϵ that $\|(p_k, \omega_k)\|_{X_{y_0}} \leq \frac{1}{2}$. It then follows from Lemma 3.2.5 that $\|(P_{k+1}, \Omega_{k+1})\|_{N_{y_0}} \leq |\epsilon| C_2(\frac{1}{2})$. From Lemma 3.2.4, we obtain $\|(p_{k+1}, \omega_{k+1})\|_{X_{y_0}} \leq y_0^{-\mu} \cdot |\epsilon| \cdot C_1 C_2(\frac{1}{2})$, completing the induction.

In addition, for $k \geq 1$, we have that

$$\begin{aligned} \|(p_{k+1}, \omega_{k+1}) - (p_k, \omega_k)\|_{X_{y_0}} &\leq \frac{1}{2} C_1 y_0^{-\mu} |N(\epsilon, p_k, \omega_k) - N(\epsilon, p_{k-1}, \omega_{k-1})| \\ &\leq \frac{1}{2} C_1 y_0^{-\mu} C_2(\frac{1}{2}) |\epsilon| \|(p_k, \omega_k) - (p_{k-1}, \omega_{k-1})\|_{X_{y_0}} \\ &\leq \frac{1}{2} \|(p_k, \omega_k) - (p_{k-1}, \omega_{k-1})\|_{X_{y_0}}. \end{aligned}$$

Thus, we have a contraction, giving us a unique limit point, concluding the proof of existence and uniqueness with $C_3 = C_1 C_2(\frac{1}{2})$.

To prove (3.64), we suppose $(p[\epsilon_1], \omega[\epsilon_1])$ and $(p[\epsilon_2], \omega[\epsilon_2])$ each solve (3.58). We then use (3.59) and Lemma 3.2.4 to estimate

$$\begin{aligned} \|(p[\epsilon_1], \omega[\epsilon_1]) - (p[\epsilon_2], \omega[\epsilon_2])\|_{X_{y_0}} &\leq \frac{1}{2} C_1 y_0^{-\mu} |N(\epsilon_1, p[\epsilon_1], \omega[\epsilon_1]) - N(\epsilon_2, p[\epsilon_2], \omega[\epsilon_2])| + \frac{1}{2} C_1 |\epsilon_1 - \epsilon_2|. \end{aligned}$$

We then use Lemma (3.2.5) and smallness of ϵ to get

$$\begin{aligned} \|(p[\epsilon_1], \omega[\epsilon_1]) - (p[\epsilon_2], \omega[\epsilon_2])\|_{X_{y_0}} &\leq \frac{1}{2} C_1 \cdot C_2(\frac{1}{2}) |\epsilon| y_0^{-\mu} \cdot \|(p[\epsilon_1], \omega[\epsilon_1]) - (p[\epsilon_2], \omega[\epsilon_2])\|_{X_{y_0}} \\ &\quad + \frac{2}{3} C_1 C_2(\frac{1}{2}) y_0^{-\mu} |\epsilon_1 - \epsilon_2| \\ &\leq \frac{1}{10} \|(p[\epsilon_1], \omega[\epsilon_1]) - (p[\epsilon_2], \omega[\epsilon_2])\|_{X_{y_0}} \\ &\quad + \frac{2}{3} C_1 C_2(\frac{1}{2}) y_0^{-\mu} |\epsilon_1 - \epsilon_2|. \end{aligned}$$

We may then absorb the first term on the RHS into the LHS, and when we do this we obtain the desired estimate. $\square$

3.6 Deferred proofs of Section 3.3

3.6.1 Proof of Lemma 3.3.1

Proof. We proceed analogously as in the proof of the corresponding lemma of [100]. We note that in this case, we may rewrite $Hv = 0$ as

$$\tilde{H}v = f$$

where

$$f := \left(\frac{4\pi}{\gamma-1} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} Q^{\frac{2-\gamma}{\gamma-1}} - \frac{2(4-3\gamma)}{(2-\gamma)^2} \frac{1}{y^2} \right) v.$$

Our basis of homogeneous solutions is then chosen to be

$$\tilde{v}_1 := \frac{\cos(\nu \log(y))}{y^{\frac{1}{2}}}, \quad \tilde{v}_2 := \frac{\sin(\nu \log(y))}{y^{\frac{1}{2}}}.$$

The Wronskian $(\tilde{v}_1 \tilde{v}_2' - \tilde{v}_1' \tilde{v}_2)$ is found to be $\frac{\nu}{y^2}$. A right-inverse to $\tilde{H}$ is found to be

$$\tilde{S}_\infty(f) := \tilde{v}_1 \int_y^\infty \frac{(y')^2}{\nu} \tilde{v}_2(y') \cdot f(y') dy' - \tilde{v}_2 \int_y^\infty \frac{(y')^2}{\nu} \tilde{v}_1(y') \cdot f(y') dy'.$$

We then recall that by (3.74) and $v_i = O(y^{2-\frac{2}{2-\gamma}-\mu})$, we have that $f = O(y^{-\frac{2}{2-\gamma}-2\mu})$ as $y \rightarrow \infty$. Thus, we have

$$\tilde{S}_\infty(f) = O(y^{2-\frac{2}{2-\gamma}-2\mu}).$$

Since $v - \tilde{S}_\infty \circ f \in \ker \tilde{H}$, we obtain

$$v = a_1 \tilde{v}_1 + a_2 \tilde{v}_2 + O(y^{2-\frac{2}{2-\gamma}-2\mu}).$$

This gives the desired asymptotic expansions. The non-vanishing of the Wronskian W ensures that $c_3, c_4 \neq 0$ (and so they can be taken to be positive). Moreover, it is clear that v_1 is smooth up to the origin. We then note that the equation (3.75) ensures that $y v_2$ is smooth up to the origin. Combining this regularity with the asymptotics as $y \rightarrow \infty$ yields the pointwise bounds stated in the lemma. $\square$

3.6.2 Proof of Lemma 3.3.3

Proof. We first note that by scaling (3.73) and (3.78) we have that

$$Q_\lambda^{-\frac{1}{\gamma-1}} \lesssim \lambda^{\frac{2}{2-\gamma}} \left\langle \frac{y}{\lambda} \right\rangle^{\frac{2}{2-\gamma}}, \quad |u_\lambda| \lesssim |y| \left\langle \frac{y}{\lambda} \right\rangle^{-\mu} \quad (3.96)$$

and (since we may differentiate the asymptotics) for $k \geq 0$

$$|\partial_y^k Q_\lambda^{-\frac{1}{\gamma-1}}| \lesssim \lambda^{-\frac{2}{2-\gamma}-k} \left\langle \frac{y}{\lambda} \right\rangle^{-\frac{2}{2-\gamma}-k}, \quad |\partial_y^{k+1} u_\lambda| \lesssim \lambda^{-k} \left\langle \frac{y}{\lambda} \right\rangle^{-\mu-k}. \quad (3.97)$$

Similarly, as a consequence of Lemma 3.3.1 and the rescaling, we have that for $k \geq 0$

$$|\partial_y^k v_{1,\lambda}| \lesssim \lambda^{-k} \left\langle \frac{y}{\lambda} \right\rangle^{-\frac{1}{2}-k}, \quad |\partial_y^k v_{2,\lambda}| \lesssim \lambda^{-k} \left(\frac{y}{\lambda} \right)^{-1-k} \left\langle \frac{y}{\lambda} \right\rangle^{\frac{1}{2}}.$$

We will use these bounds repeatedly in the following. We first consider S_λ , and note that from the regularity of v_i at the origin (Lemma 3.3.1) we see that if $f \in \mathcal{G}$ then we have $S_\lambda f \in Z$. We will then use

$$|f| \leq \left\langle \frac{y}{\lambda} \right\rangle^{-\mu} \|f\|_{\mathcal{G}} \quad (3.98)$$

and the pointwise bounds in Lemma 3.3.1 to estimate

$$\begin{aligned} |f \cdot v_{2,\lambda} y^2| &\lesssim y^2 \frac{\lambda}{y} \cdot \langle \frac{y}{\lambda} \rangle^{\frac{1}{2}} |f| \\ &\lesssim y^2 \frac{\lambda}{y} \cdot \langle \frac{y}{\lambda} \rangle^{\frac{1}{2}-\mu} \|f\|_{\mathcal{G}}. \end{aligned}$$

Integrating from 0 to y then gives

$$|\int_0^y f v_{2,\lambda} \cdot (y')^2 dy'| \lesssim y^3 \frac{\lambda}{y} \cdot \langle \frac{y}{\lambda} \rangle^{\frac{1}{2}-\mu} \|f\|_{\mathcal{G}}.$$

Combining this with the bound for $v_{1,\lambda}$ gives

$$|\lambda^{\frac{2(\gamma-1)}{2-\gamma}-1-\frac{2}{2-\gamma}} v_{1,\lambda} \int_0^y f v_{2,\lambda} \cdot (y')^2 dy'| \lesssim (\frac{y}{\lambda})^2 \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{\mathcal{G}}.$$

Estimating the second term in the variation of constants formula similarly gives

$$|S_\lambda f| \lesssim (\frac{y}{\lambda})^2 \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{\mathcal{G}}. \quad (3.99)$$

To bound the first derivative of $S_\lambda f$, we note that when we differentiate the variation of constants formula there is a cancellation, and so we may express

$$(S_\lambda f)' = \frac{1}{\lambda^{1-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}}} \left(\left(\int_0^y f v_{2,\lambda} \cdot (y')^2 dy' \right) v'_{1,\lambda} - \left(\int_0^y f v_{1,\lambda} \cdot (y')^2 dy' \right) v'_{2,\lambda} \right) \quad (3.100)$$

and then estimating the terms similarly yields

$$|(S_\lambda f)'| \lesssim \frac{y}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{\mathcal{G}}.$$

We then use $H_\lambda \circ S_\lambda = \text{id}$ to express

$$(S_\lambda f)'' = -\lambda^{-2} f - \frac{2}{y} (S_\lambda f)' - \frac{4\pi}{\gamma-1} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{\gamma-1}} Q_\lambda^{\frac{2-\gamma}{\gamma-1}} (S_\lambda f).$$

Estimating these three terms using (3.98), (3.100) and (3.99) respectively gives the desired estimate

$$|(S_\lambda f)''| \lesssim \lambda^{-2} \langle \frac{y}{\lambda} \rangle^{-\mu}.$$

This yields the desired boundedness of S_λ . Using that, as a consequence of Lemma 3.3.1

$$|y \partial_y v_1| \lesssim \langle y \rangle^{-\frac{1}{2}}, \quad |y \partial_y v_2| \lesssim \frac{1}{y} \langle y \rangle^{-\frac{1}{2}}$$

one can obtain the desired boundedness of $S_{\lambda_2} - S_{\lambda_1}$. We next consider T_λ . In this case, since Q_λ is smooth and non-vanishing at the origin, it is clear that if $f \in \mathcal{F}$ then $T_\lambda(f) \in Y$. We then estimate

$$|f| \lesssim \frac{y}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu} \|f\|_{\mathcal{F}}$$

so that

$$|\int_0^y f(y') \cdot (y')^2 dy'| \lesssim y^3 \frac{y}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu} \|f\|_{\mathcal{F}}$$

We then use that

$$|Q_{\lambda}^{-\frac{1}{\gamma-1}}| \lesssim \lambda^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}}$$

to obtain the desired bound

$$|T_{\lambda} f| \lesssim (\frac{y}{\lambda})^2 \langle \frac{y}{\lambda} \rangle^{-1-\mu} \|f\|_{\mathcal{F}}.$$

Differentiating T_{λ} allows one to similarly conclude

$$|(T_{\lambda} f)'| \lesssim \frac{y}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}-1-\mu} \|f\|_{\mathcal{F}}.$$

One can then use

$$|f'| \lesssim \frac{1}{\lambda} \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}-1-\mu} \|f\|_{\mathcal{F}}$$

to conclude the desired estimate

$$|(T_{\lambda} f)''| \lesssim \frac{1}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}-1-\mu} \|f\|_{\mathcal{F}}.$$

One then uses a straightforward argument, along with the estimate

$$|y \partial_y Q| \lesssim \langle y \rangle^{-\frac{2(\gamma-1)}{2-\gamma}}$$

to obtain the desired boundedness of $T_{\lambda_2} - T_{\lambda_1}$.

Finally, to show boundedness of the operator K_{λ} we consider the operator L_{λ} and note that it maps Z to $\mathcal{F}$. We then note that using

$$|f| \lesssim \frac{y^2}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{Z_{\lambda}}, \quad |f'| \lesssim \frac{y}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{Z_{\lambda}}$$

along with the asymptotics for $Q_{\lambda}, Q'_{\lambda}, u_{\lambda}, \partial_y u_{\lambda}$, we obtain

$$\begin{aligned} |L_{\lambda} f| &\lesssim (\frac{y}{\lambda})^2 \langle \frac{y}{\lambda} \rangle^{-2-\mu} \|f\|_{Z_{\lambda}} \\ &\lesssim (\frac{y}{\lambda}) \langle \frac{y}{\lambda} \rangle^{-1-\mu} \|f\|_{Z_{\lambda}}. \end{aligned}$$

Similarly, if one differentiates $L_{\lambda} f$ and uses

$$|f''| \lesssim \frac{1}{\lambda^2} \langle \frac{y}{\lambda} \rangle^{-\mu} \|f\|_{Z_{\lambda}}$$

one obtains the desired estimate

$$|(L_{\lambda} f)'| \lesssim \frac{1}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu} \|f\|_{Z_{\lambda}}.$$

Once again, the bound for $L_{\lambda_2} - L_{\lambda_1}$ follows analogously, using that one may differentiate the asymptotics of Q, u . Combining the estimates for S_{λ}, T_{λ} and L_{λ} give the boundedness of K_{λ} and $K_{\lambda_2} - K_{\lambda_1}$. $\square$

3.6.3 Proof of Lemma 3.3.4

Proof. We first note that using the boundedness of $\|u\|_{Y_\lambda}$ we have (using $\lambda \langle \frac{y}{\lambda} \rangle \leq 2 \cdot y_0$ for the second inequality)

$$|\lambda u| \lesssim |y| \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}-\mu} C, \quad |\lambda^{1+\frac{2}{2-\gamma}} u| \lesssim |y| y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\mu} C.$$

Similarly, for $k \in \{0, 1\}$ we have

$$|\partial_y^{1+k}(\lambda u)| \lesssim \lambda^{-k} \langle \frac{y}{\lambda} \rangle^{\frac{2}{2-\gamma}-\mu-k} C, \quad |\partial_y^{1+k}(\lambda^{1+\frac{2}{2-\gamma}} u)| \lesssim \lambda^{-k} y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\mu-k} C.$$

Similarly, using boundedness of $\|w\|_{Z_\lambda}$, for $k \in \{0, 1, 2\}$ we have that

$$|\partial_y^k w| \lesssim \lambda^{-k} (\frac{y}{\lambda})^{2-k} \langle \frac{y}{\lambda} \rangle^{-\mu} C, \quad |\partial_y^k \lambda^{\frac{2}{2-\gamma}} w| \lesssim \lambda^{-k} (\frac{y}{\lambda})^{2-k} y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2}{2-\gamma}-\mu} C.$$

We then note that

$$Q + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}} w = \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} \left(Q(\cdot/\lambda) + t\lambda^{\frac{2}{2-\gamma}} w \right).$$

To compare this to Q , we compute

$$|\lambda^{\frac{2}{2-\gamma}} w| \lesssim (\frac{y}{\lambda})^2 y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2}{2-\gamma}-\mu} \leq \langle \frac{y}{\lambda} \rangle^{-\frac{2(\gamma-1)}{2-\gamma}-\mu} y_0^{\frac{2}{2-\gamma}} C$$

We note that since $y_0^{\frac{2}{2-\gamma}} C \ll 1$, we may use (3.73) and obtain that uniformly for $t \in [0, 1]$,

$$\lambda^{-\frac{2(\gamma-1)}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2(\gamma-1)}{2-\gamma}} \lesssim Q + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}} w \lesssim \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2(\gamma-1)}{2-\gamma}}.$$

In particular, this quantity is always positive, so we may take powers of it, and differentiate, so Z the nonlinearity is well-defined. We note that as a consequence we have that uniformly for $t \in [0, 1]$,

$$\begin{aligned} |(Q_\lambda + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}} w)^{\frac{1}{\gamma-1}}| &\lesssim \lambda^{-\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2}{2-\gamma}}, \\ \lambda^{-\frac{2(\gamma-1)}{2-\gamma}} |(Q_\lambda + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}} w)^{\frac{2-\gamma}{\gamma-1}}| &\lesssim \lambda^{-\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-2}, \\ \lambda^{-\frac{4(\gamma-1)}{2-\gamma}} |(Q_\lambda + t\lambda^{-\frac{2(\gamma-1)}{2-\gamma}+\frac{2}{2-\gamma}} w)^{\frac{3-2\gamma}{\gamma-1}}| &\lesssim \lambda^{-\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\frac{2(3-2\gamma)}{2-\gamma}}. \end{aligned}$$

From these bounds, along with (3.96) and (3.97) one sees directly that

$$\begin{aligned} |G_{\lambda,1}| &\lesssim \langle \frac{y}{\lambda} \rangle^{-\mu} (1 + y_0^{\frac{2}{2-\gamma}} C)^2, \\ |G_{\lambda,2}| &\lesssim y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-2\mu} C^2, \\ |F_\lambda| &\lesssim y_0^{\frac{2}{2-\gamma}} (\frac{y}{\lambda}) \langle \frac{y}{\lambda} \rangle^{-1-2\mu} C^2. \end{aligned}$$

We will then use $y_0^{\frac{2}{2-\gamma}} C^2 \lesssim 1$, $\langle \frac{y}{\lambda} \rangle^{-2\mu} \leq \langle \frac{y}{\lambda} \rangle^{-\mu}$ to obtain the estimate

$$|G_\lambda| \lesssim \langle \frac{y}{\lambda} \rangle^{-\mu}, \quad |F_\lambda| \lesssim \frac{y}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu}.$$

One can similarly estimate

$$|F'_\lambda| \lesssim \frac{1}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu}.$$

which shows the desired boundedness of the nonlinearities. We may similarly estimate

$$\begin{aligned} |G_{\lambda,1}(w_2, u_2) - G_{\lambda,1}(w_1, u_1)| &\lesssim y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\mu} (\|u_2 - u_1\|_{Y_\lambda} + \|w_2 - w_1\|_{Z_\lambda}), \\ |G_{\lambda,2}(w_2, u_2) - G_{\lambda,2}(w_1, u_1)| &\lesssim C y_0^{\frac{2}{2-\gamma}} \langle \frac{y}{\lambda} \rangle^{-\mu} (\|u_2 - u_1\|_{Y_\lambda} + \|w_2 - w_1\|_{Z_\lambda}), \\ |F_{\lambda,2}(w_2, u_2) - F_{\lambda,2}(w_1, u_1)| &\lesssim C y_0^{\frac{2}{2-\gamma}} \frac{y}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu} (\|u_2 - u_1\|_{Y_\lambda} + \|w_2 - w_1\|_{Z_\lambda}), \\ |F_{\lambda,2}(w_2, u_2)' - F_{\lambda,2}(w_1, u_1)'| &\lesssim C y_0^{\frac{2}{2-\gamma}} \frac{1}{\lambda} \langle \frac{y}{\lambda} \rangle^{-1-\mu} (\|u_2 - u_1\|_{Y_\lambda} + \|w_2 - w_1\|_{Z_\lambda}). \end{aligned}$$

Summing these gives the desired Lipschitz continuity with respect to w, u . Finally, the continuity with respect to λ follows straightforwardly. $\square$

3.6.4 Proof of Lemma 3.3.5

Proof. We consider the integral formulation (3.79). We let $(u_0, w_0) = (0, 0)$. We then inductively define that for $n \geq 1$,

$$\begin{aligned} (W_n, U_n) &:= (G_\lambda(w_{n-1}, u_{n-1}), F_\lambda(w_{n-1}, u_{n-1})), \\ (w_n, u_n) &:= (S_\lambda(W_n), T_\lambda(U_n) + K_\lambda(W_n)). \end{aligned}$$

We note that $\|w_1\|_{Z_\lambda}, \|u_1\|_{Y_\lambda} \leq C_1 \cdot C_2$. We then let $\tilde{C} = 2 \max(C_1 C_2, 1)$. We shall make the assumption

$$y_0^{\frac{2}{2-\gamma}} \tilde{C}^2 \ll 1. \tag{3.101}$$

We then will inductively assume that for $k \geq 1$,

$$\|w_k\|_{Z_\lambda}, \quad \|w_{k-1}\|_{Z_\lambda}, \quad \|u_k\|_{Y_\lambda}, \quad \|u_{k-1}\|_{Y_\lambda} \leq \tilde{C}.$$

It then follows that

$$\begin{aligned} \|w_{k+1} - w_k\|_{Z_\lambda} + \|u_{k+1} - u_k\|_{Y_\lambda} &\leq 2C_1 (\|W_{k+1} - W_k\|_{\mathcal{G}_\lambda} + \|U_{k+1} - U_k\|_{\mathcal{F}_\lambda}) \\ &\leq 2C_1 C_2 (1 + \tilde{C}) y_0^{\frac{2}{2-\gamma}} (\|w_k - w_{k-1}\|_{Z_\lambda} + \|u_k - u_{k-1}\|_{Y_\lambda}). \end{aligned}$$

By (3.101), we can conclude

$$\|w_{k+1} - w_k\|_{Z_\lambda} + \|u_{k+1} - u_k\|_{Y_\lambda} \leq \frac{1}{10} (\|w_k - w_{k-1}\|_{Z_\lambda} + \|u_k - u_{k-1}\|_{Y_\lambda})$$

and so we see that the sequence converges. We may choose $C_3 = 100\tilde{C}$. The uniqueness of the fixed point follows as we have constructed a contraction. For the Lipschitz dependence, we suppose that for $\in \{1, 2\}$

$$w[\lambda_i] = S_{\lambda_i}(G_{\lambda_i}(w[\lambda_i], u[\lambda_i])).$$

Subtracting these equations gives

$$\begin{aligned} w[\lambda_2] - w[\lambda_1] &= (S_{\lambda_2} - S_{\lambda_1})G_{\lambda_2}(w[\lambda_2], u[\lambda_2]) \\ &+ S_{\lambda_1}(G_{\lambda_2}(w[\lambda_2], u[\lambda_2]) - G_{\lambda_1}(w[\lambda_2], u[\lambda_2])) + S_{\lambda_1}(G_{\lambda_1}(w[\lambda_2], u[\lambda_2]) - G_{\lambda_1}(w[\lambda_1], u[\lambda_1])). \end{aligned}$$

We then use the dependencies on λ in the bounds of Lemmas 3.3.3 and 3.3.4 to estimate the first two terms, and then difference estimate (3.80) to bound the third term, which gives

$$\begin{aligned} \max_i \|w[\lambda_2] - w[\lambda_1]\|_{Z_{\lambda_i}} &\leq \frac{\lambda_2 - \lambda_1}{\lambda_1} C_1 C_2 + \frac{\lambda_2 - \lambda_1}{\lambda_1} C_1 C_2 \\ &+ C_1 C_2 y_0^{\frac{2}{2-\gamma}} \left(\min_i \|w[\lambda_2] - w[\lambda_1]\|_{Z_{\lambda_i}} + \min_i \|u[\lambda_2] - u[\lambda_1]\|_{Y_{\lambda_i}} \right). \end{aligned}$$

A similar bound holds for $u[\lambda_2] - u[\lambda_1]$, and then one may use (3.101) to absorb the last term into the left-hand side, which gives (3.81). $\square$

3.7 Appendix

3.7.1 Proof of Lemma 3.1.3

Proof. The key idea is that if $\kappa > -1$ then $\frac{d}{dt} + \frac{\kappa}{t}$ is invertible as a mapping from $\{f \in C^1 : f(0) = 0\}$ into C^0 .

We consider the ODE system

$$\begin{cases} A(t, u_1, u_2) \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} + B(t, u_1, u_2) = 0 \\ u_1(0) = u_2(0) = u'_1(0) = 0, \quad u'_2(0) = U \end{cases}, \quad (3.102)$$

with A, B as in the statement of the lemma. We first note that if we multiply the system on the left by

$$\begin{pmatrix} \frac{1}{A_{11}} & 0 \\ -\frac{A_{21}}{A_{11}} & 1 \end{pmatrix}$$

we obtain the system

$$\begin{pmatrix} 1 & \frac{A_{12}}{A_{11}} \\ 0 & A_{22} - \frac{A_{12}A_{21}}{A_{11}} \end{pmatrix} \cdot \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} + \begin{pmatrix} \frac{B_1}{A_{11}} \\ B_2 - \frac{A_{21}B_1}{A_{11}} \end{pmatrix} = 0.$$

then we note that this system is of the same form (and a, b, c, d do not change). Thus, without loss of generality, we may assume that

$$A_{11} = 1, \quad A_{21} = 0.$$

so that the system takes the form

$$\begin{pmatrix} 1 & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0. \quad (3.103)$$

We will also define

$$e := \frac{\partial A_{22}}{\partial z_1}(0), \quad f := \frac{\partial B_2}{\partial z_1}(0).$$

In preparation for carrying out a contraction mapping argument, we will perform finite Taylor expansions of the coefficients in z_0, z_1, z_2 . We will use multi-index notation, so that for $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $f(z_0, z_1, z_2)$,

$$|\alpha| = \sum_{i=0}^2 \alpha_i, \quad z^\alpha = z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}, \quad \partial^\alpha f = \partial_{z_0}^{\alpha_0} \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2}, \quad \alpha! = \alpha_0! \alpha_1! \alpha_2!.$$

We will also adopt the convention that

$$f_\alpha := \frac{1}{\alpha!} (\partial^\alpha f)(0).$$

We then recall that for any smooth $f(z_0, z_1, z_2)$, we may use the fundamental theorem of calculus to decompose the function to zero and first order as

$$\begin{aligned} f(z_0, z_1, z_2) &= f(0) + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^3 \\ |\alpha|=1}} f_{\alpha,rem}(z) \cdot z^\alpha \\ &= \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^3 \\ |\alpha| \leq 1}} f_\alpha z^\alpha + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^3 \\ |\alpha|=2}} f_{\alpha,rem}(z) \cdot z^\alpha, \end{aligned}$$

where the remainder terms are smooth functions defined by

$$f_{\alpha,rem}(z) = \begin{cases} \int_0^1 \sum_{|\alpha|=1} \partial_\alpha f(tz) dt & |\alpha| = 1 \\ \int_0^1 2(1-t) \left(\sum_{|\alpha|=2} \frac{\partial_\alpha f(tz)}{\alpha!} \right) dt & |\alpha| = 2 \end{cases}.$$

We will use the zeroth-order expansion for A_{12}, B_1 to rewrite the line of the matrix equation (3.103) as the equation (using multi-index notation and $u_0 = t$)

$$\partial_t u_1 = - \left(\sum_{|\alpha|=1} u^\alpha (A_{12})_{\alpha,rem} \right) \partial_t u_2 - \left(\sum_{|\alpha|=1} u^\alpha (B_2)_{\alpha,rem} \right).$$

We will then use the second-order expansion for A_{22}, B_2 to rewrite the second equation. We will also use the decompositions

$$bu_2 u'_2 = (bUt)u'_2 + bu_2 U + b(u_2 - Ut)(u'_2 - U) - bU^2 t, \quad eu_1 u'_2 = eu_1(u'_2 - U) + eu_1 U.$$

We may then write the second equation as

$$(a + bU)tu'_2 + (d + bU)u_2 + (f + eU)u_1 = (bU^2 - c)t \\ - b(u_2 - Ut)(u'_2 - U) - eu_1(u'_2 - U) - \left(\sum_{|\alpha|=2} u^\alpha (A_{22})_{\alpha,rem} \right) u'_2 - \left(\sum_{|\alpha|=2} u^\alpha (B_2)_{\alpha,rem} \right).$$

We will then divide this equation by $a + bU$ and also use the algebraic identity

$$\frac{bU^2 - c}{a + bU} = (\alpha + 1)U.$$

to rewrite the entire system as

$$\begin{cases} u'_1 = F_1(t, u) \\ tu'_2 + \kappa u_2 = -\frac{(f+eU)}{a+bU}u_1 + (\alpha + 1)Ut + F_2(t, u) \\ u_1(0) = u'_1(0) = u_2(0) = 0, \quad u'_2(0) = U. \end{cases} \quad (3.104)$$

where

$$F_1(t, u_1, u_2) = -\left(\sum_{|\alpha|=1} u^\alpha (A_{12})_{\alpha,rem} \right) \partial_t u_2 - \left(\sum_{|\alpha|=1} u^\alpha (B_2)_{\alpha,rem} \right) \\ F_2(t, u_1, u_2) := -\frac{b}{a + bU}(u_2 - Ut)(u'_2 - U) - \frac{e}{a + bU}u_1(u'_2 - U) \\ - \frac{1}{a + bU} \left(\sum_{|\alpha|=2} u^\alpha (A_{22})_{\alpha,rem} \right) u'_2 - \frac{1}{a + bU} \left(\sum_{|\alpha|=2} u^\alpha (B_2)_{\alpha,rem} \right).$$

We will now choose a large constant D so that

$$|U| + 1 \leq D.$$

We next chose R sufficiently larger than D so that for $\gamma \in \mathbb{Z}_{\geq 0}^3$, $|\gamma| \leq 3$, uniformly for z in a neighborhood of zero,

$$|(A_{12})_\gamma| \leq \frac{1}{100} \left(\frac{R}{2} \right)^{|\gamma|} D^{-\gamma_2}, \\ |(B_1)_\gamma| \leq \frac{1}{100} \left(\frac{R}{2} \right)^{|\gamma|} D^{1-\gamma_2}, \\ |(A_{22})_\gamma| \leq \frac{1}{100} |a + bU| \left(\frac{R}{2} \right)^{|\gamma|} D^{-\gamma_2}, \\ |(B_2)_\gamma| \leq \frac{1}{100} |a + bU| \left(\frac{R}{2} \right)^{|\gamma|} D^{1-\gamma_2}.$$

As a comment, this may be done since $A_{12}(0) = B_1(0) = A_{22}(0) = B_2(0) = 0$, and so each term will have a positive power of R or $\frac{R}{D}$. In particular, we have

$$\left| \frac{b}{a + bU} \right| \leq \frac{R}{D}, \quad \left| \frac{e}{a + bU} \right| \leq R, \quad \left| \frac{f + eU}{a + bU} \right| \leq DR.$$

We will now define spaces (depending on T)

$$\begin{cases} X_U := \{f \in C^1(-T, T) : f(0) = 0, f'(0) = U\}, & \|f\|_X = \|f'\|_{L^\infty} \\ N_{(\alpha+1)U} := \{f \in C^0(-T, T) : f(0) = (\alpha+1)U\}, & \|f\|_N := \|f\|_{L^\infty} \end{cases}$$

We comment that if $\alpha > -1$, then $\frac{d}{dt} + \frac{\alpha}{t}$ is invertible $X_U \rightarrow X_{(\alpha+1)U}$ with inverse given by

$$f \mapsto I_\alpha f := t^{-\alpha} \int_0^t (t')^\alpha f(t') dt'.$$

Motivated by this, we will consider the integral formulation

$$\begin{cases} u_1 = \int_0^t F_1(t', u(t')) dt' \\ u_2 = I_\alpha \left(-\frac{f+eU}{a+bU} u_1 + (\alpha+1)Ut + F_2(t, u) \right) \end{cases} \quad (3.105)$$

We will first describe boundedness of this iteration. Our goal will be to propagate

$$\|u_2 - Ut\|_X \leq D\epsilon.$$

We first note that in order to control the contribution of $\frac{-f+eU}{a+bU} u_1$, we will need the pointwise bound

$$\|u'_1\|_{L^\infty} \leq \frac{C_1}{R} \epsilon$$

for some universal small constant C_1 (depending on α). Assuming these bounds on $u'_1, u'_2 - U$, we can bound

$$\|F_1(t, u)\|_N \lesssim TRD, \quad \left\| \frac{F_2(t, u)}{t} \right\|_N \lesssim RD\epsilon^2 + TR^2D$$

and then choose ϵ sufficiently small so that Thus, to close, we may choose T sufficiently small so that

$$TR^2D \ll \epsilon,$$

which allows us to close the bound on u_1 and then choose ϵ sufficiently small so that

$$\epsilon R \ll 1$$

which allows us to close the bound on u_2 . Similarly, if we have that for u_a, u_b both in the ball $\|u_2 - U\|_X \leq D\epsilon$, $\|u_1\|_X \leq \frac{C_1}{R} \epsilon$ and we have the difference estimate

$$\|(u_a)_2 - (u_b)_2\|_X \leq \delta, \quad \|(u_a)_1 - (u_b)_1\|_X \leq C_1 \frac{\delta}{DR},$$

we can estimate (replacing u_a, u_b with the output of the the integral formulation (3.105))

$$\|(u_a)_2 - (u_b)_2\|_X \leq \frac{\delta}{2}, \quad \|(u_a)_1 - (u_b)_1\|_X \leq \frac{1}{2} C_1 \frac{\delta}{DR}.$$

It then follows from a standard fixed point argument that the integral formulation (3.105) has a unique solution $(u_1, u_2) \in C^1(-T, T)$ such that

$$\|u'_1\|_{L^\infty} \leq \frac{\alpha+1}{100R} \epsilon, \quad \|u'_2 - Ut\|_{L^\infty} \leq D\epsilon. \quad (3.106)$$

We note that such a solution will be a C^1 solution to (3.102) as desired, which gives the existence part of the lemma. Moreover, any solution to (3.102) will solve the integral formulation (3.105), and by continuity of the first derivative (by taking T smaller if necessary) we can ensure (3.106), which gives the uniqueness part of the theorem. $\square$

3.7.2 Proof of Lemma 3.1.4

Before presenting the main proof of this section, we will first establish the following related lemma, which will be used in the proof. The key idea (for both this lemma and the main one of the subsection) is majorization, see for instance the treatment of the analytic implicit function theorem in [68]. Moreover, dividing by $\kappa + i$ for successive integers i will not affect the convergence, as long as this factor never vanishes. Our approach to dealing with this is inspired by the method of the proof of Theorem 4.5 in [109].

Lemma 3.7.1. (*Scalar, semilinear singular ODE*) Suppose $P(z_0, z_1)$ is an analytic function on a neighborhood of 0 such that

$$P(0) = \partial_{z_1} P(0) = 0.$$

Suppose $\kappa \notin \mathbb{Z}_{\leq -1}$. Then, for T sufficiently small, the ODE

$$\begin{cases} tw' + \kappa w = P(t, w) \\ w(0) = 0 \end{cases}$$

has a unique real-analytic solution on $(-T, T)$.

Proof. We first show the existence of a formal Taylor series solution and then show that the Taylor series converges. We will use standard multi-index notation. In particular, if $\alpha = (\alpha_0, \alpha_1) \in \mathbb{Z}_{\geq 0}^2$ and $z = (z_0, z_1) \in \mathbb{R}^2$, then we define

$$\alpha! := \alpha_0! \alpha_1!, \quad z^\alpha := z_0^{\alpha_0} z_1^{\alpha_1}, \quad \partial_z^\alpha := \partial_{z_0}^{\alpha_0} \partial_{z_1}^{\alpha_1}.$$

We will also use the abbreviation that for $m \in \mathbb{Z}_{\geq 0}$ and multi-indices $\alpha \in \mathbb{Z}_{\geq 0}^2$,

$$w_m := \frac{w^{(m)}(0)}{m!}, \quad P_\alpha := \frac{P^{(\alpha)}(0)}{\alpha!}.$$

Thus, we can write the (formal) Taylor expansion of w and the (necessarily convergent) Taylor expansion of P as

$$w(t) = \sum_{m=1}^{\infty} w_m t^m, \quad P(z_0, z_1) = \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^2 \\ \alpha_0 + 2\alpha_1 \geq 2}} P_\alpha z^\alpha.$$

We then note that formally, by collecting terms with the same powers of t , we may express

$$P(t, w(t)) = \sum_{r=1}^{\infty} Q_r(P_\alpha, w_m) t^r,$$

where for each fixed r , Q_r can be written as, using the indexing $\beta = (\beta_1, \dots, \beta_M)$,

$$Q_r(P_\alpha, w_m) = \sum_{\substack{(\alpha_0, M) \in \mathbb{Z}_{\geq 0}^2 \\ \alpha_0 + M = r}} \sum_{\alpha_1 \leq M} P_{(\alpha_0, \alpha_1)} \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^M \\ |\beta| = \alpha_1 \\ \sum_{i=1}^M i\beta_i = M}} c_{\alpha_0, \beta} w^\beta,$$

where the coefficients $c_{\alpha_0, \beta}$ are positive combinatorial constants independent of P, w . We use here the notation that if $\beta \in \mathbb{Z}_{\geq 0}^M$,

$$w^\beta := w_1^{\beta_1} \cdots w_M^{\beta_M}.$$

We also comment, that since $P_{(0,1)} = 0$, we have that Q_r only depends on w_j for $j \leq r-1$. We note that the equation specifies that for $m \geq 1$

$$(\kappa + m)w_m = Q_m(P_\alpha, w_m).$$

Since by assumption we have that for all $m \geq 0$, $\kappa + m \neq 0$, we can uniquely solve for each w_m . Thus, the formal power series is uniquely determined by the equation, and if it can be shown to converge on an open interval containing zero we have a unique real-analytic solution.

To show convergence we will use the method of majorants. We shall let $R > 0$ be a sufficiently large constant to be determined later. We now define

$$\bar{P}(z_0, z_1) = \frac{1}{(1 - Rz_0)(1 - Rz_1)} - 1 - Rz_1.$$

We note that the Taylor coefficients of $\bar{P}$ are all non-negative. We comment that as P is analytic, by choosing R sufficiently large, we can ensure it is majorized by $\bar{P}$, i.e.

$$|P_\alpha| \leq \bar{P}_\alpha.$$

We then note that the implicit equation

$$V = \frac{1}{2}\bar{P}(t, V), \quad V(0) = 0 \tag{3.107}$$

is solved by

$$\mathring{V}(t) := \frac{2(1 - Rt) - \sqrt{4(1 - Rt)^2 - 4R^2t(2 + R)(1 - Rt)}}{2R(2 + R)(1 - Rt)}.$$

In particular, by Taylor expanding each side of (3.107), we have that for $r \geq 1$ (from now on we notationally omit the ranges of the summation indices)

$$\mathring{V}_r = \frac{1}{2}Q_r(P_\alpha, \mathring{V}_m). \tag{3.108}$$

We note that (by comparing the series solution with that of $V = \frac{1}{2}(\frac{1}{1-Rt} - 1)$)

$$\mathring{V}_r \geq \frac{1}{2}R^r. \tag{3.109}$$

We will now prove inductively that for R sufficiently large, for $k \geq 1$,

$$w_k \leq \frac{1}{k}\mathring{V}_k.$$

Indeed, as a first step, we note that there exists K sufficiently large such that for all $k \geq K$,

$$\frac{1}{|1 + \frac{\kappa}{k}|} \leq 2. \quad (3.110)$$

We then note that by the lower bound (3.109), by choosing R sufficiently large, we may assume without loss of generality that for all $k < K$, $|w_k| \leq \frac{\mathring{V}_k}{k}$.

For $k \geq K$, we will work inductively. In particular, we consider the equation

$$tw' + \kappa w = P(t, w).$$

and expand each side to order k . This gives

$$(k + \kappa)w_k = Q_k(P_\alpha, w_m).$$

where the RHS only depends on w_r for $r \leq k - 1$. We next take the absolute value of each side. We estimate the RHS using the triangle inequality, the fact that the constants $c_{\alpha_0, \beta}$ are positive and also (the second following from the inductive hypothesis)

$$|P_\alpha| \leq \mathring{P}_\alpha, \quad |w_m| \leq \mathring{V}_m.$$

This allows us to estimate

$$\begin{aligned} |Q_k(P_\alpha, w_m)| &\leq Q_k(|P_\alpha|, |w_m|) \\ &\leq Q_k(\mathring{P}_\alpha, \mathring{V}_m) \\ &= \frac{1}{2} \mathring{V}_k, \end{aligned}$$

with the last line coming from (3.108). This gives

$$|w_k| \leq \frac{1}{2} \frac{1}{|1 + \frac{\kappa}{k}|} \frac{\mathring{V}_k}{k} \leq \frac{\mathring{V}_k}{k},$$

with the second inequality following from (3.110). Thus, it follows that w is majorized by $\mathring{V}$, and since the latter is analytic on a neighborhood of zero, it follows that w is analytic on a neighborhood of 0. $\square$

We now prove Lemma 3.1.4.

Proof. As a comment, as in the proof of Lemma 3.1.3, we may assume without loss of generality that

$$A_{11} = 1, \quad A_{21} = 0.$$

Thus, we have the system

$$\begin{cases} u'_1 + A_{12}(t, u_1, u_2)u'_2 + B_1(t, u_1, u_2) = 0 \\ A_{22}(t, u_1, u_2)u'_2 + B_2(t, u_1, u_2) = 0 \end{cases}.$$

We then switch variables from u_1, u_2 to v_1, v_2 which are defined by

$$u_1 = v_1, \quad u_2 = v_2 + tU.$$

We then get the system

$$\begin{cases} v'_1 + \tilde{A}_{12}(t, v_1, v_2)v'_2 + \tilde{B}_2(t, v_1, v_2) = 0 \\ \tilde{A}_{22}(t, v_1, v_2)v'_2 + \tilde{B}_2(t, v_1, v_2) = 0 \end{cases}.$$

where (we also divide the second equation by $a + bU$)

$$\begin{aligned} \tilde{A}_{12}(z_0, z_1, z_2) &:= A_{12}(z_0, z_1, z_2 + Uz_0), \\ \tilde{B}_1(z_0, z_1, z_2) &:= B_1(z_0, z_1, z_2 + Uz_0) + U\tilde{A}_{12}(z_0, z_1, z_2), \\ \tilde{A}_{22}(z_0, z_1, z_2) &:= \frac{1}{a + bU}A_{22}(z_0, z_1, z_2 + Uz_0), \\ \tilde{B}_2(z_0, z_1, z_2) &:= \frac{1}{a + bU}B_2(z_0, z_1, z_2 + Uz_0) + U\tilde{A}_{22}(z_0, z_1, z_2). \end{aligned}$$

We note that (using (3.9) for the third equality) we have that

$$\frac{\partial \tilde{A}_{22}}{\partial z_0}(0) = 1, \quad \frac{\partial \tilde{A}_{22}}{\partial z_2} = \frac{b}{a + bU}, \quad \frac{\partial \tilde{B}_2}{\partial z_0} = 0, \quad \frac{\partial \tilde{B}_2}{\partial z_2} = \kappa.$$

We may rewrite the second equation as

$$\begin{aligned} tv'_2 + \kappa v_2 &= -(\tilde{A}_{22} - z_0)v'_2 - (\tilde{B}_2 - \kappa z_2) \\ &= \frac{-(\tilde{A}_{22} - z_0)|_{t,v}tv'_2 - (z_0\tilde{B}_2 - \kappa z_0 z_2)|_{(t,v)}}{t}. \end{aligned}$$

In particular, if we define (introducing a new dummy variable $\tilde{z}_2$ for tv'_2)

$$\begin{aligned} H(z_0, z_1, z_2, \tilde{z}_2) &:= -(\tilde{A}_{22} - z_0)\tilde{z}_2 - (z_0\tilde{B}_2 - \kappa z_0 z_2), \\ S(z_0, z_1, z_2) &:= -\tilde{B}_1(z_0, z_1, z_2), \\ T(z_0, z_1, z_2) &:= -\tilde{A}_{12}(z_0, z_1, z_2), \end{aligned}$$

then we can write the system as

$$\begin{cases} v'_1 &= S(t, v_1, v_2) + T(t, v_1, v_2)v'_2 \\ tv'_2 + \kappa v_2 &= \frac{H(t, v_1, v_2, tv'_2)}{t} \end{cases}. \quad (3.111)$$

We first examine the first equation. We recall that $S(0) = T(0) = 0$ and further that we can Taylor expand

$$S(t, v_1, v_2)(t, z_1, z_2) = \sum_{\gamma \in \mathbb{Z}_{\geq 0}^3} S_\gamma t^{\gamma_0} (v_1)^{\gamma_1} (v_2)^{\gamma_2}$$

$$= \sum_{r=1}^{\infty} P_r(S_\gamma, (v_1)_m, (v_2)_m) t^r,$$

where

$$\begin{aligned} P_r(S_\gamma, (v_1)_m, (v_2)_m) &= \sum_{\gamma_0 + M_1 + M_2 = r} \sum_{\gamma_1 \leq M_1} \sum_{\gamma_2 \leq M_2} S_\gamma \sum_{\substack{\beta_a \in \mathbb{Z}_{\geq 0}^{M_1} \\ |\beta_a| = \gamma_1 \\ \sum_{i=1}^{M_a} i(\beta_a)_i = M_1}} \sum_{\substack{\beta_b \in \mathbb{Z}_{\geq 0}^{M_2} \\ |\beta_b| = \gamma_2 \\ \sum_{i=1}^{M_b} i(\beta_b)_i = M_2}} c_{\gamma_0, \beta_a, \beta_b} (v_1)^{\beta_a} (v_2)^{\beta_b}. \end{aligned}$$

We use here the notation that, for instance, if $\beta_a = ((\beta_a)_1, \dots, (\beta_a)_M)$ then $(v_1)^{\beta_a} = (v_1)_1^{(\beta_a)_1} \dots (v_1)_M^{(\beta_a)_M}$. The constants $c_{\gamma_0, \beta_a, \beta_b}$ are combinatorial in nature and necessarily positive. We comment that P_r only depends on $(v_1)_j$ and $(v_2)_j$ such that $j \leq r$. We may similarly expand the term $T(t, v_1, v_2)v'_2$, and we see that we may expand the RHS of the first equation as

$$\begin{aligned} S(t, v_1, v_2) + T(t, v_1, v_2)v'_2 &= \sum_{r=1}^{\infty} \left(P_r(S_\gamma, (v_1)_m, (v_2)_m) + \sum_{l=1}^r \binom{r}{l} (r-l+1) P_r(T_\gamma, (v_1)_m, (v_2)_m) \cdot (v_1)_{r-l+1} \right) t^r \end{aligned}$$

and we see that the coefficient of t^r only depends on $(v_1)_j$ and $(v_2)_j$ for $j \leq r$.

We comment that the numerator of the second equation satisfies

$$H(0) = \partial_{z_0} H(0) = \partial_{z_1} H(0) = \partial_{z_2} H(0) = \partial_{\bar{z}_2} H(0) = 0,$$

$$\partial_{z_0}^2 H(0) = \partial_{z_0} \partial_{z_2} H(0) = \partial_{z_0} \partial_{\bar{z}_2} H(0) = 0.$$

We note that if we denote $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, we may Taylor expand

$$\begin{aligned} H(t, v_1, v_2, tv'_2) &= \sum_{\gamma \in \mathbb{Z}_{\geq 0}^4} H_\gamma t^{\gamma_0} v_1^{\gamma_1} v_2^{\gamma_2} (tv'_2)^{\gamma_3} \\ &= \sum_{r=3}^{\infty} Q_r(H_\gamma, (v_1)_m, (v_2)_m) t^r, \end{aligned}$$

where

$$\begin{aligned} Q_r(H_\gamma, (v_1)_m, (v_2)_m) &= \sum_{\gamma_0 + M_1 + M_2 + M_3 = r} \sum_{\gamma_1 \leq M_1} \sum_{\gamma_2 \leq M_2} \sum_{\gamma_3 \leq M_3} H_\gamma \\ &\cdot \sum_{\substack{\beta_a \in \mathbb{Z}_{\geq 0}^{M_1} \\ |\beta_a| = \gamma_1 \\ \sum_{i=1}^{M_a} i(\beta_a)_i = M_1}} \sum_{\substack{\beta_b \in \mathbb{Z}_{\geq 0}^{M_2} \\ |\beta_b| = \gamma_2 \\ \sum_{i=1}^{M_b} i(\beta_b)_i = M_2}} \sum_{\substack{\beta_c \in \mathbb{Z}_{\geq 0}^{M_3} \\ |\beta_c| = \gamma_3 \\ \sum_{i=1}^{M_c} i(\beta_c)_i = M_3}} c_{\gamma_0, \beta_a, \beta_b, \beta_c} (v_1)^{\beta_a} (v_2)^{\beta_b} (tv'_2)^{\beta_c}. \end{aligned}$$

We also note that we may conclude

$$Q_0(0) = Q_1(0) = Q_2(0) = 0$$

using the fact that $v_1(0) = v'_1(0) = v_2(0) = v'_2(0) = 0$ and also

$$H(0) = \partial_{z_0} H(0) = \partial_{z_1} H(0) = \partial_{z_2} H(0) = \partial_{\bar{z}_2} H(0) = \partial_{z_0}^2 H(0) = 0.$$

Moreover, using $\partial_{z_1} H(0) = 0$, we see that Q_r only depends on $(v_1)_j$ for $j \leq r-1$. Similarly, using the fact that

$$\partial_{z_2} H(0) = \partial_{z_0} \partial_{z_2} H(0) = \partial_{\bar{z}_2} H(0) = \partial_{z_0} \partial_{\bar{z}_2} H(0) = 0$$

and the first order vanishing of v_1, v_2 , we see that Q_r only depends on $(v_2)_j$ for $j \leq r-2$. We now return our attention to the system (3.111). We note that if we differentiate the first equation $k-1$ times and evaluate at zero, by the above analyses, we can solve for $(v_1)_k$ in terms of $(v_1)_j, (v_2)_j$ for $j \leq k-1$. Then, differentiating the second equation k times, we can solve for $(v_2)_k$ in terms of $(v_1)_i, (v_2)_j$ for $i \leq k, j \leq k-1$. Thus, the Taylor series is uniquely determined by the equation. It suffices to prove that the Taylor series constructed this way converges.

As a first step, motivated by the fact that v_2 and tv'_2 will have Taylor coefficients differing only by a factor of k , we define

$$\tilde{H}(z_0, z_1, z_2) = H(z_0, z_1, z_2, z_2).$$

We note that this will have an expansion

$$\tilde{H}(t, v_1, v_2) = \sum_{r=3}^{\infty} \tilde{Q}_r(\tilde{H}_\gamma, (v_1)_m, (v_2)_m) t^r,$$

where

$$\begin{aligned} \tilde{Q}_r(\tilde{H}_\gamma, (v_1)_m, (v_2)_m) = \\ \sum_{\gamma_0 + M_1 + M_2 = r} \sum_{\gamma_1 \leq M_1} \sum_{\gamma_2 \leq M_2} \tilde{H}_\gamma \sum_{\substack{\beta_a \in \mathbb{Z}_{\geq 0}^{M_1} \\ |\beta_a| = \gamma_1 \\ \sum_{i=1}^{M_1} i(\beta_a)_i = M_1}} \sum_{\substack{\beta_b \in \mathbb{Z}_{\geq 0}^{M_2} \\ |\beta_b| = \gamma_2 \\ \sum_{i=1}^{M_2} i(\beta_b)_i = M_2}} \tilde{c}_{\gamma_0, \beta_a, \beta_b} (v_1)^{\beta_a} (v_2)^{\beta_b}. \end{aligned}$$

Once again we have that the coefficients are positive, Q_r only depends on $(v_1)_j$ for $j \leq r-1$ and Q_r only depends on j for $j \leq r-1$. We note that we have the inequality that for any H_γ, v_1, v_2 , since $|(v_2)_m| \leq |(tv'_2)_m|$,

$$Q_r(|H_\gamma|, |(v_1)_m|, |(v_2)_m|) \leq \tilde{Q}_r(|\tilde{H}_\gamma|, |(v_1)_m|, |(tv'_2)_m|). \quad (3.112)$$

We will now perform a majorization argument. We will let D, R be sufficiently large constants to be determined later (in particular we will first pick D to be sufficiently large, then pick R large depending on D). We will pick these constants sufficiently large so that for $\gamma = (\gamma_0, \gamma_1, \gamma_2) \in \mathbb{Z}_{\geq 0}^3$,

$$|(A_{12})_\gamma| \leq \frac{1}{100} \left(\frac{R}{2}\right)^{|\gamma|} D^{-\gamma_2},$$

$$\begin{aligned}
|(B_1)_\gamma| &\leq \frac{1}{100} \left(\frac{R}{2}\right)^{|\gamma|} D^{1-\gamma_2}, \\
|(A_{22})_\gamma| &\leq \frac{1}{100} |a + bU| \left(\frac{R}{2}\right)^{|\gamma|} D^{-\gamma_2}, \\
|(B_2)_\gamma| &\leq \frac{1}{100} |a + bU| \left(\frac{R}{2}\right)^{|\gamma|} D^{1-\gamma_2}.
\end{aligned}$$

When we change variables from u to v , and assume $|U| + 1 \leq D$, this means that the coefficient functions satisfy

$$\begin{aligned}
|(\tilde{A}_{12})_\gamma| &\leq R^{|\gamma|} D^{-\gamma_2}, \\
|(\tilde{B}_1)_\gamma| &\leq R^{|\gamma|} D^{1-\gamma_2}, \\
|(\tilde{A}_{22})_\gamma| &\leq \frac{1}{2} R^{|\gamma|} D^{-\gamma_2}, \\
|(\tilde{B}_2)_\gamma| &\leq \frac{1}{2} R^{|\gamma|} D^{1-\gamma_2}.
\end{aligned}$$

From these, bounds, we obtain

$$|\tilde{H}_\gamma| \leq R^{|\gamma|-1} D^{1-\gamma_2-\gamma_3}, \quad |S_\gamma|, |T_\gamma| \leq R^{|\gamma|} D^{1-\gamma_2}.$$

We then define

$$\begin{aligned}
\bar{H}(z_0, z_1, z_2) &:= \frac{D}{R} \left(\frac{1}{(1 - Rz_0)(1 - Rz_1)(1 - \frac{R}{D}z_2)} \right. \\
&\quad \left. - \left(1 + Rz_0 + Rz_1 + \frac{R}{D}z_2 + R^2z_0^2 + \frac{R^2}{D}z_0z_2 \right) \right), \\
\bar{S}(z_0, z_1, z_2) &:= D \left(\frac{1}{(1 - Rz_0)(1 - Rz_1)(1 - \frac{R}{D}z_2)} - 1 \right).
\end{aligned}$$

We comment that these functions majorize the terms appearing in our equation in the sense that

$$|\tilde{H}_\gamma| \leq \bar{H}_\gamma, \quad |S_\gamma| \leq \bar{S}_\gamma, \quad |T_\gamma| \leq \bar{S}_\gamma. \quad (3.113)$$

We now consider the ODE

$$\begin{cases} v'_1 &= \bar{S}(t, v_1, v_2)(1 + v'_2) \\ v_2 &= \frac{1}{2} \frac{\bar{H}(t, v_1, v_2)}{t} \\ v_1(0) &= v'_1(0) = v_2(0) = v'_2(0) = 0 \end{cases}. \quad (3.114)$$

We will use an explicit solution to the ODE system to majorize the system (3.111). To start with, we comment that if t, v_1 are fixed, the second equation can be viewed as a quadratic for v_2 . In particular, if we define

$$A_2 := \frac{1}{D^2} (2R^2t + R^2 + R^3t) (1 - Rt)(1 - Rv_1),$$

$$A_1 := \frac{1}{D} (-2Rt + R(Rv_1 + R^2t^2)) (1 - Rt)(1 - Rv_1),$$

$$A_0 := 1 - (1 + Rt + Rv_1 + R^2t^2)(1 - Rt)(1 - Rv_1).$$

then the equation becomes

$$A_2v_2^2 + A_1v_2 + A_0 = 0.$$

After simplifying, this quadratic can be rewritten as

$$\tilde{A}_2\left(\frac{R}{Dt}v_2\right)^2 + \tilde{A}_1\left(\frac{R}{Dt}v_2\right) + \tilde{A}_0 = 0,$$

where

$$\begin{aligned}\tilde{A}_2 &:= (2t + 1 + Rt)(1 - Rt)(1 - Rv_1), \\ \tilde{A}_1 &:= \left(-2 + R\frac{v_1}{t} + R^2t\right)(1 - Rt)(1 - Rv_1), \\ \tilde{A}_0 &:= R^2\left(\frac{v_1}{t}\right)^2(1 - Rt) + \frac{v_1}{t}(R^2 - R^4t) + R^3t.\end{aligned}$$

We comment that these are each smooth functions of $t, \frac{v_1}{t}$. Introducing the dummy variables y_0, y_1 for $t, \frac{v_1}{t}$, we define

$$\begin{aligned}\mathring{A}_2(y_0, y_1) &:= (2y_0 + 1 + Ry_0)(1 - Ry_0)(1 - Ry_0y_1), \\ \mathring{A}_1(y_0, y_1) &:= (-2 + Ry_1 + R^2y_0)(1 - Ry_0)(1 - Ry_0y_1), \\ \mathring{A}_0(y_0, y_1) &:= R^2y_1^2(1 - Ry_0) + y_1(R^2 - R^4y_0^2) + R^3y_0.\end{aligned}$$

Thus, if we define

$$\mathring{Q}(y_0, y_1) = \frac{D}{R} \left(\frac{-\mathring{A}_1 - \sqrt{\mathring{A}_1^2 - 4\mathring{A}_0\mathring{A}_2}}{2\mathring{A}_1} \right).$$

Then we can write the solution to the quadratic as

$$v_2 = y_0\mathring{Q}(y_0, y_1)|_{t, \frac{v_1}{t}}.$$

We now define $w = \frac{v_1}{t}$, and we have that the second equation becomes

$$v_2 = t\mathring{Q}(t, w).$$

From this, when we differentiate in time, we get

$$v_2' = \mathring{Q}(t, w) + t\partial_{y_0}\mathring{Q}(t, w) + t\frac{\partial\mathring{Q}}{\partial y_1}(t, w)w'.$$

If we substitute this into the first equation of (3.114), we get, where $\bar{S}$ is evaluated at $(t, tw, t\mathring{Q})$ and $\mathring{Q}$ is evaluated at (t, w) ,

$$tw' + w = \bar{S}(1 + (\mathring{Q} + t\partial_{z_0}\mathring{Q})) + t\bar{S}\frac{\partial\mathring{Q}}{\partial z_1}w'.$$

We note that we can rewrite this as

$$(1 - \bar{S} \frac{\partial \dot{Q}}{\partial y_1}) tw' + w = \bar{S}(1 + (\dot{Q} + t \partial_{y_0} \dot{Q})).$$

We may then divide by the principal term, and write the resulting equation as

$$tw' + w = -\frac{\bar{S}}{(1 - \bar{S} \frac{\partial \dot{Q}}{\partial z_1})} w + \frac{\bar{S}}{(1 - \bar{S} \frac{\partial \dot{Q}}{\partial z_1})} (1 + (\dot{Q} + t \partial_{z_0} \dot{Q})).$$

This ODE has an analytic solution by Lemma 3.7.1. We call the analytic solution to this ODE system $\dot{W}(t)$. It follows that if we define

$$\begin{aligned} \dot{V}_1(t) &:= t \cdot \dot{W}(t), \\ \dot{V}_2(t) &:= t \dot{Q}(t, \dot{W}(t)), \end{aligned}$$

then $\dot{V}_1(t)$ and $\dot{V}_2(t)$ solve (3.114). In particular, we have the combinatorial identities

$$(\dot{V}_1)_r = P_r(\bar{S}_\gamma, (\dot{V}_1)_m, (\dot{V}_2)_m) + \sum_{l=1}^r \binom{r}{l} (r-l+1) P_r(\bar{S}_\gamma, (\dot{V}_1)_m, (\dot{V}_2)_m) \cdot (\dot{V}_1)_{r-l+1}, \quad (3.115)$$

$$(\dot{V}_2)_r = \frac{1}{2} \tilde{Q}_{r+1}(\bar{H}_\gamma, (\dot{V}_1)_m, (\dot{V}_2)_m). \quad (3.116)$$

We comment that since the right-hand-sides of the equations have non-negative Taylor series it follows that the Taylor coefficients of $\dot{V}_1, \dot{V}_2$ are non-negative. In fact, we see that

$$\begin{aligned} \left(\frac{d^k}{dt^k} \bar{S}(t, \dot{V}_1, \dot{V}_2) \right)(0) &\geq \left(\frac{d^k}{dt^k} D \left(\frac{1}{1-Rt} - 1 \right) \right)(0), \\ \left(\frac{d^k}{dt^k} \frac{\bar{H}(t, \dot{V}_1, \dot{V}_2)}{t} \right)(0) &\geq \left(\frac{d^k}{dt^k} \frac{D}{Rt} \left(\frac{1}{1-Rt} - 1 - Rt - R^2 t \right) \right)(0). \end{aligned}$$

from which we conclude the lower bounds that for $m \geq 2$,

$$(\dot{V}_1)_m \geq \frac{DR^{(m-1)}}{m}, \quad (\dot{V}_2)_m \geq \frac{1}{2} DR^m. \quad (3.117)$$

We will now prove that by choosing D, R sufficiently large, we have that

$$(v_1)_k \leq (\dot{V}_1)_k, \quad (v_2)_k \leq \frac{1}{k} (\dot{V}_2)_k. \quad (3.118)$$

As a first step, we note that we may choose K sufficiently large so that for all $k \geq K$,

$$\frac{1}{|1 + \frac{\kappa}{K}|} \leq 2. \quad (3.119)$$

We then note that using the lower bound (3.117) for the coefficients, by choosing D, R sufficiently large, we may assume that (3.118) holds for $k \leq K-1$. We now suppose $k \geq K$

and proceed inductively. We differentiate the first equation of (3.111) $k - 1$ times and evaluate at zero. We have inductively that

$$\begin{aligned}
|(v_1)_k| &\leq |P_k(S_\gamma, (v_1)_m, (v_2)_m) + \sum_{l=1}^k \binom{k}{l} (k-l+1) P_k(T_\gamma, (v_1)_m, (v_2)_m) \cdot (v_1)_{k-l+1}| \\
&\leq P_k(|S_\gamma|, |(v_1)_m|, |(v_2)_m|) + \sum_{l=1}^k \binom{k}{l} (k-l+1) P_k(|T_\gamma|, |(v_1)_m|, |(v_2)_m|) \cdot |(v_1)_{k-l+1}| \\
&\leq P_k(\bar{S}_\gamma, (\dot{V}_1)_m, (\dot{V}_2)_m) + \sum_{l=1}^k \binom{k}{l} (k-l+1) P_k(\bar{S}_\gamma, (\dot{V}_1)_m, (\dot{V}_2)_m) \cdot (\dot{V}_1)_{k-l+1},
\end{aligned}$$

where to obtain the third line we have used the inductive hypothesis and the upper bounds for $|S_\gamma|$ and $|T_\gamma|$ provided in (3.113). We then use (3.115) to obtain the desired estimate

$$|(v_1)_k| \leq (\dot{V}_1)_k.$$

Similarly, differentiate the second equation of (3.111) k times, evaluate at zero and divide by $k + \kappa$. This gives

$$\begin{aligned}
|(v_2)_k| &\leq \frac{1}{|1 + \frac{\kappa}{k}|} \frac{1}{2k} |Q_{k+1}(H_\gamma, (v_1)_m, (v_2)_m)|, \\
&\leq \frac{1}{|1 + \frac{\kappa}{k}|} \frac{1}{2k} (Q_{k+1}(|H_\gamma|, |(v_1)_m|, |(v_2)_m|)), \\
&\leq \frac{1}{|1 + \frac{\kappa}{k}|} \frac{1}{2k} \left(\tilde{Q}_{k+1}(|\tilde{H}_\gamma|, |(v_1)_m|, |(tv'_2)_m|) \right), \\
&\leq \frac{1}{|1 + \frac{\kappa}{k}|} \frac{1}{2k} \left(\tilde{Q}_{k+1}(\bar{H}_\gamma, |(\dot{V}_1)_m|, |(\dot{V}_2)_m|) \right),
\end{aligned}$$

where to obtain the third line we use the inequality (3.112), and to obtain the last line we use the inductive hypothesis and the upper bound for $|H_\gamma|$ provided by the inequality (3.113). We then use (3.116) and (3.119) to obtain the desired estimate

$$\leq \frac{1}{2} \frac{1}{|1 + \frac{\kappa}{k}|} \frac{(\dot{V}_2)_k}{k} \leq \frac{(\dot{V}_2)_k}{k}.$$

Thus, we have established (3.118), and since the majorizing functions are analytic, we have that the Taylor expansions of v_1, v_2 converge uniformly as desired. $\square$

3.7.3 Proof of Lemma 3.1.5

Proof. We consider the polytropic Euler-Poisson system (3.5). We note that the equation (3.11) is equivalent to

$$y_* \omega_0 = \gamma^{\frac{1}{2}} \rho_0^{\frac{\gamma-1}{2}}. \quad (3.120)$$

We also will define

$$A(y, \tilde{\rho}, \tilde{u}) := \begin{pmatrix} \tilde{u} + (2 - \gamma)y & \tilde{\rho} \\ \gamma \tilde{\rho}^{\gamma-2} & \tilde{u} + (2 - \gamma)y \end{pmatrix},$$

$$B(y, \tilde{\rho}, \tilde{u}) := \begin{pmatrix} \frac{2\tilde{\rho}(\tilde{u}+y)}{y} \\ (\gamma - 1)\tilde{u} + \frac{4\pi}{4-3\gamma}\tilde{\rho}(\tilde{u} + (2 - \gamma)y) \end{pmatrix}.$$

We note that

$$A_0 := A(y_*, \rho_0, u_0) = \begin{pmatrix} y_*\omega_0 & \rho_0 \\ \frac{y_*^2\omega_0^2}{\rho_0} & y_*\omega_0 \end{pmatrix},$$

$$B_0 := B(y_*, \rho_0, u_0) = \begin{pmatrix} 2\rho_0\omega_0 + 2(\gamma - 1)\rho_0 \\ (\gamma - 1)\omega_0 y_* - (2 - \gamma)(\gamma - 1)y_* + \frac{4\pi}{4-3\gamma}\rho_0 y_*\omega_0 \end{pmatrix}.$$

We also note that assuming $\tilde{\rho} - \rho_0, \tilde{u} - u_0 = O(y - y_*)$, we have that

$$A(y, \tilde{\rho}, \tilde{u}) = A_0 + \begin{pmatrix} (2 - \gamma)(y - y_*) + (u - u_0) & \tilde{\rho} - \rho_0 \\ (\gamma - 2)\frac{(y_*\omega_0)^2}{\rho_0^2}(\tilde{\rho} - \rho_0) & (2 - \gamma)(y - y_*) + (u - u_0) \end{pmatrix} + O(y - y_*)^2.$$

Similarly,

$$B(y, \tilde{\rho}, \tilde{u}) = B_0 + \begin{pmatrix} 2(\omega_0 + (\gamma - 1))(\tilde{\rho} - \rho_0) + \frac{2\rho_0}{y_*}(\tilde{u} - u_0) - \frac{2\rho_0}{y_*}(\omega_0 - (2 - \gamma))(y - y_*) \\ (\gamma - 1)(\tilde{u} - u_0) + \frac{4\pi}{4-3\gamma}(\tilde{\rho} - \rho_0)y_*\omega_0 + \frac{4\pi}{4-3\gamma}\rho_0(\tilde{u} - u_0) + \frac{4\pi(2-\gamma)}{4-3\gamma}\rho_0(y - y_*) \end{pmatrix} + O((y - y_*)^2).$$

We will then define the matrices

$$C := \begin{pmatrix} \frac{1}{4y_*\omega_0} & \frac{\rho_0}{4(y_*\omega_0)^2} \\ \frac{1}{4y_*\omega_0} & \frac{-\rho_0}{4(y_*\omega_0)^2} \end{pmatrix}, \quad D := \begin{pmatrix} \frac{1}{y_*\omega_0} & \frac{1}{\rho_0} \\ \frac{y_*\omega_0}{\rho_0} & -\frac{y_*\omega_0}{\rho_0} \end{pmatrix}.$$

We will now perform a first change of variables

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} + D \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

We will implement this change of variables, and then multiply the equation by C . We define

$$\mathcal{A} = C \cdot A(y, \rho_0 + D_{11}\phi + D_{12}\psi, u_0 + D_{21}\phi + D_{22}\psi),$$

$$\mathcal{B} = C \cdot B(y, \rho_0 + D_{11}\phi + D_{12}\psi, u_0 + D_{21}\phi + D_{22}\psi),$$

so that the system can be written as

$$\mathcal{A}(y, \phi, \psi) \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} + \mathcal{B}(y, \phi, \psi) = 0.$$

We will expand (assuming $\rho, u = O(y - y_*)$)

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + O((y - y_*)^2), \quad \mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + O((y - y_*)^2).$$

where

$$\mathcal{A}_1 = \begin{pmatrix} \frac{2(2-\gamma)}{4y_*\omega_0}(y - y_*) + \frac{(\gamma+1)}{4\rho_0}\phi + \frac{(\gamma-3)}{4\rho_0}\psi & \frac{(\gamma-3)}{4\rho_0}\phi + \frac{(\gamma-3)}{4\rho_0}\psi \\ -\frac{(\gamma-3)}{4\rho_0}\phi - \frac{(\gamma-3)}{4\rho_0}\psi & \frac{2(2-\gamma)}{4y_*\omega_0}(y - y_*) - \frac{(\gamma-3)}{4\rho_0}\phi - \frac{(\gamma+1)}{4\rho_0}\psi \end{pmatrix}.$$

We note that we have

$$\mathcal{A}_0 = C A_0 D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{B}_0 = C B_0 = \frac{\rho_0}{4y_*\omega_0^2} \begin{pmatrix} 2\omega_0^2 + 3(\gamma - 1)\omega_0 - (2 - \gamma)(\gamma - 1) + \frac{4\pi\rho_0\omega_0}{4-3\gamma} \\ 2\omega_0^2 + (\gamma - 1)\omega_0 + (2 - \gamma)(\gamma - 1) - \frac{4\pi\rho_0\omega_0}{4-3\gamma} \end{pmatrix}.$$

We then simplify using (3.12) to write

$$\mathcal{B}_0 = \begin{pmatrix} \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1)) \\ 0 \end{pmatrix}.$$

We also have

$$\mathcal{B}_1 = \frac{1}{4y_*\omega_0} \cdot \begin{pmatrix} (4\omega_0 + 3(\gamma - 1))\phi + (\gamma - 1)\psi - \frac{2\rho_0}{y_*}(\omega_0 - (2 - \gamma))(y - y_*) + \frac{4\pi\rho_0^2}{4-3\gamma}\left(\frac{2\phi}{\rho_0} + \frac{(2-\gamma)(y-y_*)}{y_*\omega_0}\right) \\ (4\omega_0 + (\gamma - 1))\phi + 3(\gamma - 1)\psi - \frac{2\rho_0}{y_*}(\omega_0 - (2 - \gamma))(y - y_*) - \frac{4\pi\rho_0^2}{4-3\gamma}\left(\frac{2\phi}{\rho_0} + \frac{(2-\gamma)(y-y_*)}{y_*\omega_0}\right) \end{pmatrix}.$$

We then comment that the first line, evaluated at $y = y_*$ gives

$$\phi'(y_*) + \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1)) = 0.$$

We then do an addition change of variables from ϕ, ψ to $\check{\rho}, \check{u}$ where

$$\phi = \check{\rho}(y - y_*) - \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1)) \cdot (y - y_*), \quad \psi = \check{u}(y - y_*).$$

We then define

$$\check{\mathcal{A}}(y, \check{\rho}, \check{u}) = \mathcal{A}(y, \check{\rho} - \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1))(y - y_*), \check{u}),$$

$$\check{\mathcal{B}}(y, \check{\rho}, \check{u}) = \mathcal{B}(y, \check{\rho} - \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1))(y - y_*), \check{u}) + \check{\mathcal{A}} \cdot \begin{pmatrix} -\frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma - 1)) \\ 0 \end{pmatrix}$$

and then note that we get

$$\check{\mathcal{A}} \begin{pmatrix} \check{\rho}' \\ \check{u}' \end{pmatrix} + \check{\mathcal{B}} = 0.$$

We note that now

$$\check{B}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We now compute

$$\check{A}_1 = \begin{pmatrix} \frac{2(2-\gamma)-(\gamma+1)(\omega_0+(\gamma-1))}{4y_*\omega_0}y + \frac{(\gamma+1)}{4\rho_0}\check{\rho} + \frac{(\gamma-3)}{4\rho_0}\check{u} & \frac{(\gamma-3)}{4\rho_0}\check{\rho} + \frac{(\gamma-3)}{4\rho_0}\check{u} - \frac{(\gamma-3)(\omega_0+(\gamma-1))}{4y_*\omega_0}y \\ -\frac{(\gamma-3)}{4\rho_0}\check{\rho} - \frac{(\gamma-3)}{4\rho_0}\check{u} + \frac{(\gamma-3)(\omega_0+(\gamma-1))}{4y_*\omega_0}y & \frac{2(2-\gamma)+(\gamma-3)(\omega_0+(\gamma-1))}{4y_*\omega_0}y - \frac{(\gamma-3)}{4\rho_0}\check{\rho} - \frac{(\gamma+1)}{4\rho_0}\check{u} \end{pmatrix}.$$

We then compute

$$\begin{aligned} \check{B}_1 = & \frac{1}{4y_*\omega_0} \begin{pmatrix} (4\omega_0 + 3(\gamma-1))\check{\rho} + (\gamma-1)\check{u} + \frac{\rho_0(-2\omega_0(\omega_0-(2-\gamma))-(4\omega_0+3(\gamma-1))(\omega_0+(\gamma-1)))}{\omega_0 y_*}y \\ (4\omega_0 + (\gamma-1))\check{\rho} + 3(\gamma-1)\check{u} + \frac{\rho_0(-2\omega_0(\omega_0-(2-\gamma))-(4\omega_0+(\gamma-1))(\omega_0+(\gamma-1)))}{\omega_0 y_*}y \end{pmatrix} \\ & + \frac{1}{4y_*\omega_0} \begin{pmatrix} \frac{4\pi\rho_0^2}{4-3\gamma}(\frac{2\check{\rho}}{\rho_0} + \frac{((2-\gamma)-2(\omega_0+(\gamma-1)))y}{y_*\omega_0}) \\ -\frac{4\pi\rho_0^2}{4-3\gamma}(\frac{2\check{\rho}}{\rho_0} + \frac{((2-\gamma)-2(\omega_0+(\gamma-1)))y}{y_*\omega_0}) \end{pmatrix} \\ & + \frac{(\omega_0 + (\gamma-1))}{4y_*\omega_0} \begin{pmatrix} -\frac{\rho_0}{y_*\omega_0}(2(2-\gamma) - (\gamma+1)(\omega_0 + (\gamma-1)))y - \frac{\gamma+1}{y_*\omega_0}\check{\rho} - \frac{\gamma-3}{y_*\omega_0}\check{u} \\ -\frac{\rho_0}{y_*\omega_0}(\gamma-3)(\omega_0 + (\gamma-1))y + (\gamma-3)\check{\rho} + (\gamma-3)\check{u} \end{pmatrix}. \end{aligned}$$

From these we can read off a, b, c, d . Then, examining the change of variables gives

$$\begin{cases} \check{\rho}'(y_*) = \check{\rho}'(0) + \check{u}'(0) - \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma-1)) \\ \check{u}'(y_*) = \frac{y_*\omega_0}{\rho_0}(\check{\rho}'(0) - \check{u}'(0) - \frac{\rho_0}{y_*\omega_0}(\omega_0 + (\gamma-1))) \end{cases},$$

which yields the stated equations for R, W in terms of U . Finally, according to [54], the Larson-Penston-Hunter branch corresponds to the left-most root of the quadratic (in R), which corresponds to the branch stated in Lemma 3.1.5 since $b < 0$. We next confirm that the far-field solution has the Larson-Penston-Hunter Taylor expansion. Indeed, if we substitute $\omega_0 = 2 - \gamma, \rho_0 = \frac{4-3\gamma}{2\pi}$ then we have We also note that we will have

$$a + d = \frac{3\gamma - 5}{4y_f\omega_0}, \quad b = \frac{-(\gamma+1)}{4\rho_0}, \quad c = \frac{4(\gamma-1)\rho_0}{4y_*^2\omega_0^2}.$$

if we substitute this into (3.13), we obtain

$$U_{farfield} = -\frac{\rho_0}{y_*\omega_0}.$$

which if we substitute back in terms of R , we obtain

$$\frac{y_f\check{\rho}'(y_f)}{\rho_0} = -\frac{2}{2-\gamma}.$$

which is consistent with the derivative of the far-field solution. We can then compute

$$a + bU = \frac{2}{4y_f\omega_0}, \quad d + bU = \frac{5(\gamma-1)}{4y_f\omega_0}.$$

□

3.7.4 On matching with Lipschitz dependence on parameters

We will first prove a variant of the Inverse Function Theorem (where we replace the usual assumption of C^1 differentiability with a quantitative lipschitz bound). This is then used to prove a corresponding variant of the Implicit function theorem which is used in Section 3.4.1.

Lemma 3.7.2. *Let $B_\epsilon(0) \subset \mathbb{R}^d$ denote the closed ball of radius ϵ . Consider $f: B_\epsilon(0) \rightarrow \mathbb{R}^d$ of the form*

$$f(x) = Ax + g(x)$$

where A is an invertible matrix and g is a Lipschitz function satisfying

$$g(0) = 0, \quad |g(x_1) - g(x_2)| \leq L_1|x_1 - x_2|.$$

We define

$$a = \|A^{-1}\|^{-1}.$$

Suppose that $L_1 < a$. Let $\delta = \epsilon a(1 - \frac{L_1}{a})$. Then for each $y \in B_\delta(0)$, there exists a unique x in $B_\epsilon(0)$ such that $f(x) = y$. Moreover, we have that f^{-1} is Lipschitz with

$$\left| \frac{f^{-1}(y_1) - f^{-1}(y_2)}{|y_1 - y_2|} \right| \leq \frac{1}{a} \frac{1}{1 - \frac{L_1}{a}}. \quad (3.121)$$

Proof. For a fixed $y_0 \in B_\delta(0)$, we define $T_{y_0}(x): B_\epsilon(0) \rightarrow \mathbb{R}^d$ by

$$T_{y_0}(x) = A^{-1}(y_0 - g(x)).$$

We comment that if $x_1, x_2 \in B_\epsilon(0)$ then

$$|T_{y_0}(x)| \leq \frac{|y_0|}{a} + \frac{L_1}{a}|x|, \quad |T_{y_0}(x_1) - T_{y_0}(x_2)| \leq \frac{L_1}{a}|x_1 - x_2|$$

In particular, we see that if $\delta = \epsilon a(1 - \frac{L_1}{a})$, then if $|y_0| \leq \delta$, $\{T_{y_0}^n(0)\}_n$ will remain in $B_\epsilon(0)$ and will converge. Thus, f^{-1} is well-defined. We then note that

$$y_1 - y_2 = A(x_1 - x_2) + g(x_1) - g(x_2).$$

We may then solve for $x_1 - x_2$ as

$$x_1 - x_2 = A^{-1}(y_1 - y_2) - A^{-1}(g(x_1) - g(x_2))$$

We thus have

$$|x_1 - x_2| \leq \frac{1}{a}|y_1 - y_2| + \frac{L_1}{a}|x_1 - x_2|$$

from which we obtain (3.121). □

Lemma 3.7.3. Let $B_\epsilon(0) \subset \mathbb{R}^2$ denote the closed ball of radius ϵ . Consider $f: B_\epsilon(0) \rightarrow \mathbb{R}$ of the form

$$f(x, y) = ax + by + g(x, y).$$

Suppose $a \neq 0$. Suppose that, on B_ϵ , $g(x, y)$ satisfies the Lipschitz estimate

$$|g(x_1, y_1) - g(x_2, y_2)| \leq L_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Suppose that L_2 satisfies

$$L_2 < \frac{|a|}{1 + |\frac{b}{a}|}.$$

Then, on a neighborhood of 0, the equation $f(x, y) = 0$ uniquely determines $x = \varphi(y)$ for a Lipschitz function $\varphi(y)$.

Proof. We let

$$\tilde{f}(x, y) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g(x, y) \\ 0 \end{pmatrix}.$$

We recall that

$$\left\| \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\| = \sqrt{a^2 + \frac{b^2}{2} + \frac{|b|}{2} \sqrt{4a^2 + b^2}}.$$

Similarly,

$$\left\| \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{-1} \right\| = \sqrt{a^{-2} + \frac{1}{2}b^2a^{-4} + \frac{|b|}{2a^2} \sqrt{\frac{4}{a^2} + \frac{b^2}{a^4}}}.$$

We define

$$\begin{aligned} \lambda_1 &= \frac{1}{\sqrt{a^{-2} + \frac{1}{2}b^2a^{-4} + \frac{|b|}{2a^2} \sqrt{\frac{4}{a^2} + \frac{b^2}{a^4}}}} \\ &= \frac{|a|}{\sqrt{1 + \frac{1}{2}(\frac{b}{a})^2 + |\frac{b}{a}| \sqrt{4 + (\frac{b}{a})^2}}} \\ &\geq \frac{1}{1 + |\frac{b}{a}|}. \end{aligned}$$

It follows that if $L_2 < \lambda_1$, then by Lemma 3.7.2 there is a Lipschitz inverse function

$$\tilde{f}^{-1}(z_1, z_2) = \begin{pmatrix} x(z_1, z_2) \\ y(z_1, z_2) \end{pmatrix}.$$

By $f \circ \tilde{f}^{-1} = \text{id}$, we have

$$f(x(z_1, z_2), y(z_1, z_2)) = z_1, \quad ay(z_1, z_2) = z_2$$

It follows that $y(z_1, z_2) = \frac{z_2}{a}$. We then have

$$f(x(z_1, z_2), \frac{z_2}{a}) = z_1.$$

If we substitute $z_1 = 0$, we have

$$f(x(0, z_2), \frac{z_2}{a}) = 0.$$

Thus, if we define

$$\varphi(y) := x(0, a \cdot y),$$

then we note that we have $f(\varphi(y), y) = 0$. Thus, we see that we may parameterize the curve with respect to y . $\square$

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