UNIVERSITY OF CALIFORNIA

Los Angeles

Equilibrium and non-equilibrium aspects of Gibbs measures

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Mathematics

by

Younghak Kwon

2021

ABSTRACT OF THE DISSERTATION

Equilibrium and non-equilibrium aspects of Gibbs measures

by

Younghak Kwon

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2021

Professor Marek Biskup, Co-Chair

Professor Georg Menz, Co-Chair

We consider a one-dimensional lattice system of unbounded, real-valued spins with arbitrary strong, quadratic, finite-range interaction. The first area of interest concerns the equivalence of the grand canonical ensemble and the canonical ensemble: on the level of thermodynamic functions, on the level of observables, and on the level of correlations. More precisely, in the thermodynamic limit (size N of the system goes to ∞), we show that the free energy, expectation of intensive observable, and correlation of two intensive functions are the same for the grand canonical ensemble and canonical ensemble.

The second area of interest concerns the decay of correlations and uniqueness of infinite-volume Gibbs measure of the canonical ensemble. It is shown that the correlations of the canonical ensemble decay exponentially plus a volume correction term. As a consequence, we verify a conjecture that the infinite-volume Gibbs measure of the canonical ensemble is unique on the one-dimensional lattice, extending results that are known for the case of weak interaction.

The third area of interest concerns the logarithmic Sobolev inequality (LSI). It is shown that the canonical ensemble satisfies a uniform LSI. The LSI constant is uniform in the boundary data, the external field and scales optimally in the system size. We deduce the LSI by combining two different methods, the two-scale approach and the Zegarlinski method.

The last area of interest concerns the hydrodynamic limit. We deduce the hydrodynamic limit of Kawasaki dynamics. The main ingredients are uniform LSI and decay of correlations for the canonical ensemble. The proof is based on a method invented by Grunewald, Otto, Villani and Westdickenberg.

The dissertation of Younghak Kwon is approved.

Timothy Austin

Christina Kim

Marek Biskup, Committee Co-Chair

Georg Menz, Committee Co-Chair

University of California, Los Angeles

2021

Contents

ACKNOWLEDGEMENTS

This work is dedicated to Professor Thomas M. Liggett (March 29, 1944 - May 12, 2020). He was a great friend, mentor, teacher and colleague to us all at UCLA.

First, I would like to thank Professor Georg Menz, my advisor, for introducing me to the wonderful world of probability theory and mentoring me. Georg has always been a role model and he has been giving me valuable advice on countless occasions. Professor Marek Biskup deserves my gratitude for discussing the problem and helping to improve the work. Marek gave me inspiration which significantly broadened my horizon in probability. Huge thanks go out to Professors Tim Austin and Inwon Kim for making the time to be on my committee. I would also like to thank Professor Guido Montúfar for discussions on the machine learning.

I would like to thank Sangchul Lee for teaching me analysis, probability and LaTeX. Mathematical discussions with Sangchul were very helpful. Thanks also go to Vadim Kaushansky and Hanbaek Lyu for introducing me machine learning and sampling methods. I would also like to thank Jaehun Lee for discussions on equivalence of ensembles. The material in Chapter 4 is the work done jointly with Georg and Jaehun. Kyeongsik Nam also deserves my thanks for teaching me new techniques in analysis which was crucial in proving the hydrodynamic limit of Kawasaki dynamics. The material in Chapter 7 is the work done jointly with Georg and Kyeongsik.

I would also like to thank my friends Bumsu Kim, Gene Kim, Dohyun Kwon, Jaehoon Lee, Sangjin Lee, Wonjun Lee, and Bonsoon Lin. Our regular conversations in UCLA were meaningful and provided inspiration. Huge thanks to Sanghyeok Jeong, Hangbin Lee, and Sihwan Yang. Our friendship has been crucial in my life. I would like to thank Korea Foundation For Advanced Studies (KFAS) for supporting my graduate study.

I would like to thank my wonderful wife Junok, who has always encouraged me. I could not have finished my degree without her support. Last but not the least, I would also like to thank my family for their support.

Chapters 3, 5 and 6 are reprinted versions of the work [KM18], [KM19] and [KM20], respectively. These works are done by Georg Menz and myself. As one of the authors of these articles, I retain the right to include these papers in a thesis or dissertation, provided it is not published commercially. In all works, Menz provided insight into the Gibbs measures that serve as the starting point of these works, and I performed main computations and carried out the proof thereof.

Chapter 4 is a reprinted version of the work [KLM21] by Lee, Menz and myself. As one of the authors of these articles, I retain the right to include this paper in a thesis or dissertation, provided it is not published commercially. My primary contribution in this work is the proof of the equivalence of ensembles on the level of observables and correlations.

Chapter 7 is a *verbatim* transcript of the preprint [KMN20] by Menz, Nam and myself. In this work, Menz provided insight into the stochastic differential equations and hydrodynamic limits. Nam worked on the quantitative limit of the coarse-grained Hamiltonian and optimal scaling moment estimates. I generalized the uniform logarithmic Sobolev inequality and strict convexity of the coarse-grained Hamiltonian from one block to multi blocks which made adapting two-scale approach possible.

Some of the works reported in this thesis has been done with financial support from Korea Foundation For Advanced Studies (KFAS) and the National Science Foundation grant DMS-1712632.

VITA Spring 2021

Younghak Kwon

EDUCATION

Ph.D. candidate in Mathematics, Department of Mathematics, University of California Los Angeles, CA. Major field: probability. Research interest: probability, statistical physics, functional inequalities. Advisor: Georg Menz. (Aug. 2015 – present)

Bachelor of Science in Mathematics, Seoul National University. (Mar. 2009 – Feb. 2015)

PUBLICATIONS AND PREPRINTS

Y. Kwon, and G. Menz, Strict convexity of the free energy of the canonical ensemble under decay of correlations, *Journal of Statistical Physics* (2018).

Y. Kwon, and G. Menz, Decay of correlations and uniqueness of the infinite-volume Gibbs measure of the canonical ensemble of 1d-lattice systems, *Journal of Statistical Physics* (2019)

Y. Kwon, and G. Menz, Uniform LSI for the canonical ensemble on the 1D-lattice with strong, finite-range interaction, *ESAIM: PS* (2020).

Y. Kwon, J. Lee, and G. Menz, Equivalence of the grand canonical ensemble and the canonical ensemble on 1d-lattice systems, Markov Processes and Related Fields (2021).

Y. Kwon, G. Menz, and K. Nam, Hydrodynamic limit of the Kawasaki dynamics on the 1D-lattice with strong, finite-range interaction, submitted (2021).

AWARDS AND FELLOWSHIPS

(2015–2020) Doctoral Study Abroad Program Fellowship of Korea Foundation for Advanced Studies

(2009–2012) Presidential Science Scholarship of Korea Student Aid Foundation

Chapter 1

Introduction

Conservation laws are fundamental to physics. One of the most well-known conservation laws is mass conservation. In a closed system, the total mass of the system is preserved over time. Many physical and mathematical models have conservation laws. However, studying systems with conservation laws is non-trivial because every mathematical analysis becomes quite subtle.

One way to study a system with a conservation law is to find an equivalent system without conservation laws. For example, n-dimensional Euclidean space with a linear constraint forms a $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, which is equivalent to $\mathbb{R}^{n-1}$. Once the equivalence is established, it is expected that they will share common mathematical properties. In this thesis, we study one particular form of the equivalence, namely, the equivalence of ensembles. It means that the canonical ensemble, a Gibbs measure with mean spin conservation, is equivalent to the grand canonical ensemble, a Gibbs measure without conservation laws.

The equivalence of ensembles in one-dimensional lattice system was well studied for discrete spin values (see [DT77] or [CM00]). We consider this question for real-valued, unbounded spins with strong interactions. Considering unbounded spins is harder because we lose compactness and one cannot transfer the arguments from the discrete case. Strong interactions makes the problem even harder because we lose the independence of random variables distributed according to the grand canonical ensemble. We accomplish this by generalizing the method that is based on characteristic functions and Fourier inversion (see [Fel71] and [GOVW09]).

One interesting property of the grand canonical ensemble is the exponential decay of correlations (cf. [Yos01]). This is a very useful tool when studying dependent system because it implies, in a large scale, the measure behaves like a product measure. As a direct consequence of the equivalence of ensembles, the decay of correlations of the canonical ensemble is deduced. More precisely, we prove that the correlations of the canonical ensemble decay exponentially fast plus a volume correction term.

Decay of correlations is a classical tool in deducing the absence of phase transitions. More precisely, in this work, we say a system has no phase transitions if the infinite-volume Gibbs measure of the system is unique. It is well known that on the one-dimensional lattice, the grand canonical ensemble does not have a phase transition if the interaction decays fast enough (see for example [Isi25, Dob68, Dob74, Rue68, MN14]). It is a natural question if the canonical ensemble also does not have a phase transition on the one-dimensional lattice. In this thesis, we prove that the answer is positive.

Another important property of the grand canonical ensemble is the uniform logarithmic Sobolev inequality (LSI). The LSI is a powerful tool for studying spin systems. For example, the LSI implies Gaussian concentration via Herbst's argument, is equivalent to hypercontractivity and characterizes the exponential rate of convergence to equilibrium of the naturally associated diffusion process. It is known that on the one-dimensional lattice, a uniform LSI holds for the grand canonical ensemble even for infinite-range interactions, given the interaction decays fast enough (see [MN14]). With the help of equivalence of ensembles and decay of correlations, we prove that the canonical ensemble satisfies a uniform LSI on the one-dimensional lattice with strong interactions.

Deducing a uniform LSI for the canonical ensemble has a special importance. Because the uniform LSI controls the entropy production, it plays an integral role in deducing the hydrodynamic limit. The hydrodynamic limit is the law of large numbers for the processes.

It states that under the correct scaling, a stochastic process converges to the solution of a deterministic partial differential equation.

Hydrodynamic limits have been actively studied in different settings. One interesting example is the Kawasaki dynamics, a stochastic dynamics which preserves the mean spin of the system. In [Fri87, GPV88a, GOVW09], this hydrodynamic limit problem was solved in the case of absent interactions. Examples of interacting lattices are [Rez90] and [Yau91], where the strong dominance of single-site potentials to interaction was assumed. In this thesis, we generalize the results to the lattice system where the interaction is not dominated by the single-site potential but of the same order.

1.1 The Gibbs measures

1.1.1 Precise settings and definitions

The Gibbs measures we consider throughout the paper are canonical ensembles with strong interactions. The simplest case of canonical ensembles, where all of the interactions are removed, is considered in [GOVW09] and [DMOW18a]. The interactions we consider are strong in the sense that they are beyond the perturbative regime.

More precisely, we consider a system of unbounded continuous spins on the lattice $\mathbb{Z}$. The formal Hamiltonian $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ of the system is defined as

$$\begin{aligned} H(x) &= \sum_{i\in\mathbb{Z}} \left(\psi(x_i) + s_i x_i + \frac{1}{2} \sum_{j:\ 1\leq|j-i|\leq R} M_{ij} x_i x_j \right) \\ &= \sum_{i\in\mathbb{Z}} \left(\psi_b(x_i) + s_i x_i + \frac{1}{2} \sum_{j:|j-i|\leq R} M_{ij} x_i x_j \right), \end{aligned} \tag{1.1.1}$$

where $\psi(z) := \frac{1}{2} z^2 + \psi_b(z)$ and $M_{ii} := 1$. We assume the following:

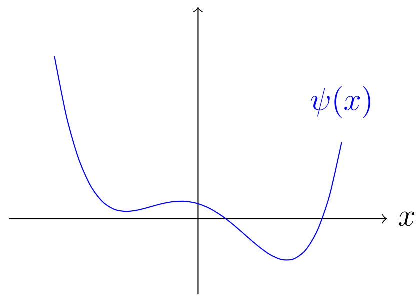

Figure 1.1: Example of a single-site potential ψ

- The function $\psi_b : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\psi_b|_\infty + |\psi_b'|_\infty + |\psi_b''|_\infty < \infty. \tag{1.1.2}$$

  It is best to imagine $\psi(z) = \frac{1}{2}z^2 + \psi_b(z)$ as a double-well potential (see Figure 1.1).

- The interaction is symmetric i.e.

$$M_{ij} = M_{ji} \qquad \text{for all distinct } i, j \in \mathbb{Z}.$$

- The fixed, finite number $R \in \mathbb{N}$ models the range of interactions between the particles in the system i.e. it holds that $M_{ij} = 0$ for all i, j such that $|i - j| > R$.

- The matrix $M = (M_{ij})$ is strictly diagonal dominant i.e. for some $\delta > 0$, it holds for any $i \in \mathbb{Z}$ that

$$\sum_{1 \le |j-i| \le R} |M_{ij}| + \delta \le M_{ii} = 1. \tag{1.1.3}$$

- The vector $s = (s_i) \in \mathbb{R}^{\mathbb{Z}}$ is arbitrary. It models the interaction with an inhomogeneous external field. Because the interaction is quadratic, this term also models the interaction of the system with the boundary.

Let us consider a finite sublattice $\Lambda \subset \mathbb{Z}$. Given boundary values $x^{\mathbb{Z}\setminus\Lambda} \in \mathbb{R}^{\mathbb{Z}\setminus\Lambda}$ we define the finite volume Hamiltonian $H_\Lambda : \mathbb{R}^\Lambda \to \mathbb{R}$ as (with a slight abuse of notation)

$$H_\Lambda(x^\Lambda) := H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda})$$

$$= \sum_{i \in \Lambda} \left(\psi(x_i) + s_i x_i + \frac{1}{2} \sum_{j:\ 1 \le |j-i| \le R} M_{ij} x_i x_j \right)$$

$$= \sum_{i \in \Lambda} \left(\psi(x_i) + \left(s_i + \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \Lambda:\ 1 \le |j-i| \le R} M_{ij} x_j \right) x_i + \frac{1}{2} \sum_{j \in \Lambda:\ 1 \le |j-i| \le R} M_{ij} x_i x_j \right). \tag{1.1.4}$$

We want to point out that in (1.1.4) the boundary values $x^{\mathbb{Z} \setminus \Lambda}$ just modify the external field that is seen by a particular spin x_i.

Definition 1.1.1. *The grand canonical ensemble (gce) μ_Λ^σ associated to the Hamiltonian H with boundary values $x^{\mathbb{Z} \setminus \Lambda}$ is the probability measure on $\mathbb{R}^\Lambda$ given by the Lebesgue density*

$$\mu_\Lambda^\sigma \left(dx^\Lambda \right) := \frac{1}{Z} \exp \left(\sigma \sum_{i \in \Lambda} x_i - H(x^\Lambda, x^{\mathbb{Z} \setminus \Lambda}) \right) dx^\Lambda, \tag{1.1.5}$$

where dx^Λ denotes the Lebesgue measure on $\mathbb{R}^\Lambda$. The canonical ensemble (ce) $\mu_{\Lambda,m}$ is the probability measure on

$$X_{\Lambda,m} := \left\{ x^\Lambda \in \mathbb{R}^\Lambda :\ \frac{1}{|\Lambda|} \sum_{i \in \Lambda} x_i = m \right\} \subset \mathbb{R}^\Lambda \tag{1.1.6}$$

with density

$$\mu_{\Lambda,m}(dx^\Lambda) := \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{|\Lambda|} \sum_{i \in \Lambda} x_i = m \right\}} \left(x^\Lambda \right) \exp \left(-H(x^\Lambda, x^{\mathbb{Z} \setminus \Lambda}) \right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda). \tag{1.1.7}$$

Here $\mathcal{L}^{|\Lambda|-1}(dx)$ denotes the $(|\Lambda|-1)$-dimensional Hausdorff measure supported on $X_{\Lambda,m}$.

Remark 1. The ce $\mu_{\Lambda,m}$ emerges from the gce μ_Λ^σ by conditioning on the mean spin

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} x_i = m. \tag{1.1.8}$$

More precisely, given (1.1.8), the term $\sigma \sum_{i \in \Lambda} x_i$ inside the exponential in (1.1.5) acts like a constant and hence is cancelled out with the normalization constant Z as follows:

$$\mu_\Lambda^\sigma \left(dx^\Lambda \mid \frac{1}{|\Lambda|} \sum_{i \in \Lambda} x_i = m \right)$$

$$= \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{|\Lambda|} \sum_{i \in \Lambda} x_i = m \right\}} \left(x^\Lambda \right) \exp \left(\sigma m |\Lambda| - H(x^\Lambda, x^{\mathbb{Z} \setminus \Lambda}) \right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda)$$

$$= \frac{1}{\widetilde{Z}} \mathbb{1}_{\left\{ \frac{1}{|\Lambda|} \sum_{i\in\Lambda} x_i = m \right\}} \left(x^\Lambda \right) \exp\left(-H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda)$$

$$= \mu_{\Lambda,m}(dx^\Lambda).$$

Note that the ce $\mu_{\Lambda,m}$ does not depend on σ, even though it emerged from the gce μ_Λ^σ.

To relate the external field σ of μ_Λ^σ and the mean spin m of $\mu_{\Lambda,m}$ we make the following definition which will be justified in Chapter 2 (see Definition 2.1.6)

Definition 1.1.2. *For each $m \in \mathbb{R}$, we choose $\sigma = \sigma(m) \in \mathbb{R}$ such that*

$$\frac{1}{|\Lambda|} \sum_{i\in\Lambda} m_i = m,$$

where $m_i := \int x_i \mu_\Lambda^\sigma(dx^\Lambda)$.

Notations

- The symbol $T_{(k)}$ denotes the term that is given by the line (k).
- We denote with $0 < C < \infty$ a generic uniform constant. This means that the actual value of C might change from line to line or even within a line.
- Uniform means that a statement holds uniformly in the system size $|\Lambda|$, the mean spin m, the boundary $x^{\mathbb{Z}\setminus\Lambda}$ and the external field s.
- $a \lesssim b$ denotes that there is a uniform constant C such that $a \leq Cb$.
- $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$.
- $\mathcal{L}^k$ denotes the k-dimensional Hausdorff measure. If there is no cause of confusion we write $\mathcal{L}$.
- Z is a generic normalization constant. It denotes the partition function of a measure.
- For each $N \in \mathbb{N}$, $[N]$ denotes the set $\{1, \ldots N\}$.
- For a vector $x \in \mathbb{R}^\mathbb{Z}$ and a set $A \subset \mathbb{Z}$, $x^A \in \mathbb{R}^A$ denotes the vector $(x^A)_i = x_i$ for all $i \in A$.
- For a function $f : \mathbb{R}^\mathbb{Z} \to \mathbb{C}$, denote $\operatorname{supp} f$ by the minimal subset of $\mathbb{Z}$ with $f(x) = f\left(x^{\operatorname{supp} f}\right)$.
- A function $f : \mathbb{R}^\mathbb{Z} \to \mathbb{C}$ is said to be local if $\operatorname{supp} f$ is finite.

Conventions

- In Chapter 2, 3, 4, 6 and 7 we assume $\Lambda = [N] = \{1, 2, \cdots, N\}$ for simplicity. However, all arguments can be generalized to arbitrary sub-lattice $\Lambda \subset \mathbb{R}$.
- With slight abuse of notations, we denote

$$\mu_N^\sigma(dx) = \mu_{[N]}^\sigma(dx^{[N]}), \qquad \mu_{N,m}(dx) = \mu_{[N],m}(dx^{[N]}),$$

$$A_N^{gce} = A_{[N]}^{gce}, \qquad \text{and} \qquad A_N^{ce} = A_{[N]}^{ce}.$$

- For notational simplicity, we often write $H(x) = H_\Lambda(x^\Lambda)$ if it does not cause any confusion. Indeed, the boundary values just modify the external field that is seen by a particular spin x_i.

1.2 Main Results

1.2.1 Equivalence of ensembles

There are many levels of defining equivalence of ensembles. In this thesis we follow the presentation of [CM00], where three levels of equivalence of ensembles were introduced: on the level of thermodynamic functions, on the level of observables, and on the level of correlations. That is, in the thermodynamic limit (size $|\Lambda|$ of the system goes to ∞), free energy, expectation of intensive observable, and correlation of two intensive functions are the same for the gce and ce.

To begin with, let us define the free energies of the gce μ_Λ^σ and ce $\mu_{\Lambda,m}$.

Definition 1.2.1 (The free energy of the gce and ce). *The free energy $A_\Lambda^{gce} : \mathbb{R} \to \mathbb{R}$ of the gce μ_Λ^σ is defined as*

$$A_\Lambda^{gce}(\sigma) := \frac{1}{|\Lambda|} \log \int_{\mathbb{R}^\Lambda} \exp\left(\sigma \sum_{i \in \Lambda} x_i - H_\Lambda(x^\Lambda)\right) dx^\Lambda. \tag{1.2.1}$$

Similarly, the free energy $A_\Lambda^{ce} : \mathbb{R} \to \mathbb{R}$ *of the ce* $\mu_{\Lambda,m}$ *is defined as*

$$A_\Lambda^{ce}(\sigma) := \frac{1}{|\Lambda|} \log \int_{\left\{\frac{1}{|\Lambda|}\sum_{i\in\Lambda} x_i = m\right\}} \exp\left(\sigma \sum_{i\in\Lambda} x_i - H_\Lambda(x^\Lambda)\right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda).$$

The first level of equivalence of ensembles is the equivalence of the free energies.

Theorem 1.2.2 (Equivalence of the ce and gce on the level of free energy)**.** *It holds that*

$$\lim_{N\to\infty} |A_\Lambda^{gce} - A_\Lambda^{ce}|_{C^2} = 0,$$

where the convergence is uniform in the mean spin m*. More precisely, given a constant* $\varepsilon > 0$*, there is an integer* $N_0 \in \mathbb{N}$ *such that if* $|\Lambda| \geq N_0$*,*

$$\sup_{\sigma\in\mathbb{R}} |A_\Lambda^{gce}(\sigma) - A_\Lambda^{ce}(\sigma)| \lesssim \frac{1}{|\Lambda|},$$

$$\sup_{\sigma\in\mathbb{R}} \left|\frac{d}{d\sigma} A_\Lambda^{gce}(\sigma) - \frac{d}{d\sigma} A_\Lambda^{ce}(\sigma)\right| \lesssim \frac{1}{|\Lambda|^{1-\varepsilon}},$$

$$\sup_{\sigma\in\mathbb{R}} \left|\frac{d^2}{d\sigma^2} A_\Lambda^{gce}(\sigma) - \frac{d^2}{d\sigma^2} A_\Lambda^{ce}(\sigma)\right| \lesssim \frac{1}{|\Lambda|^{\frac{1}{2}-\varepsilon}}.$$

Before we proceed to the next level, let us introduce the notion of local, intensive and extensive functions.

Definition 1.2.3 (Local, intensive, and extensive functions/ observables)**.** *For a function* $f : \mathbb{R}^\mathbb{Z} \to \mathbb{C}$*, denote* $\operatorname{supp} f$ *by the minimal subset of* $\mathbb{Z}$ *with* $f(x) = f\left(x^{\operatorname{supp} f}\right)$*. We call* f *a local function if it has a finite support independent of the system size* $|\Lambda|$*. A function* f *is called intensive if there is a positive constant* ε *such that* $|\operatorname{supp} f| \lesssim |\Lambda|^{1-\varepsilon}$*. A function* f *is called extensive if it is not intensive.*

The second level of equivalence is the equivalence on the level of observables.

Theorem 1.2.4 (Equivalence of the ce and gce on the level of observables)**.** *Let* $f : \mathbb{R}^\Lambda \to \mathbb{R}$ *be an intensive function. There are constants* $C \in (0, \infty)$ *and* $N_0 \in \mathbb{N}$ *independent of the external field* s *and the mean spin* m *such that if* $|\Lambda| \geq N_0$*, it holds that*

$$\left|\mathbb{E}_{\mu_\Lambda^\sigma}[f] - \mathbb{E}_{\mu_{\Lambda,m}}[f]\right| \leq C \frac{|\operatorname{supp} f|}{|\Lambda|} \|\nabla f\|_\infty.$$

Lastly, the ce and gce are equivalent on the level of correlations.

Theorem 1.2.5 (Equivalence of the ce and gce on the level of correlations)**.** *Let $f, g : \mathbb{R}^\Lambda \to \mathbb{R}$ be intensive functions. There exist constants $C \in (0, \infty)$ and $N_0 \in \mathbb{N}$ independent of the external field s and the mean spin m such that if $|\Lambda| \geq N_0$, it holds that*

$$\begin{aligned}
&\left|\operatorname{cov}_{\mu_{\Lambda,m}}(f,g) - \operatorname{cov}_{\mu_\Lambda^\sigma}(f,g)\right| \\
&\qquad \leq C \; \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{|\Lambda|} + \exp\left(-C \, dist(\operatorname{supp} f, \operatorname{supp} g)\right) \right).
\end{aligned}$$

Remark 2. One should compare Theorem 1.2.4 and Theorem 1.2.5 with Theorem 4.1 and Proposition 7.3 in [CM00]. In [CM00], the equivalence of observables and correlations were deduced in the discrete spin system. The main difference is that we use L^∞ norm of ∇f, while [CM00] used L^∞ norm of f. However, such bounds with $\|f\|_\infty$ have limited use in the continuous settings. For example, one could not deduce the decay of the spin-spin correlation function, which will be described in Chapter 5.

1.2.2 Decay of correlations and uniqueness of the infinite-volume Gibbs measure of the canonical ensemble

There are many different ways to characterize phase transitions. In this thesis we use the convention that an ensemble has no phase transitions if the associated infinite-volume Gibbs measure is unique.

A classical ingredient for deducing the absence of phase transition is the decay of correlations. Let us first recall that the gce μ_Λ^σ satisfies the exponential decay of correlations (cf. [Yos01], [HM16]).

Theorem 1.2.6 (Decay of correlations of the gce)**.** *Let $f, g : \mathbb{R}^\Lambda \to \mathbb{R}$ be intensive functions. Then there is a positive constant $C > 0$ such that*

$$\left|\operatorname{cov}_{\mu_\Lambda^\sigma}(f,g)\right| \leq C \|\nabla f\|_{L^2(\mu_\Lambda^\sigma)} \|\nabla g\|_{L^2(\mu_\Lambda^\sigma)} \exp\left(-C \, dist(\operatorname{supp} f, \operatorname{supp} g)\right).$$

Similarly, we prove that the correlations of the ce decays exponentially fast plus a volume correction term.

Theorem 1.2.7 (Decay of correlations of the ce)**.** *Under the same assumptions as in Theorem 1.2.5, it holds that*

$$\left|\operatorname{cov}_{\mu_{\Lambda,m}}(f,g)\right| \leq C\ \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{|\Lambda|} + \exp\left(-C\, dist\left(\operatorname{supp} f, \operatorname{supp} g\right)\right)\right).$$

Let us turn to the next main result, namely the uniqueness of the infinite-volume Gibbs measure of the ce.

Definition 1.2.8 (Infinite-volume Gibbs measure)**.** *Let μ be a probability measure on $\mathbb{R}^{\mathbb{Z}}$ with standard product Borel sigma-algebra. For any finite subset $\Lambda \subset \mathbb{Z}$ we decompose the measure μ into the conditional measure $\mu(dx^\Lambda|x^{\mathbb{Z}\setminus\Lambda})$ and the marginal $\bar{\mu}(dx^{\mathbb{Z}\setminus\Lambda})$. This means that for any test function f it holds*

$$\int f(x)\mu(dx) = \int\int f(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda})\mu\left(dx^\Lambda|x^{\mathbb{Z}\setminus\Lambda}\right)\mu\left(dx^{\mathbb{Z}\setminus\Lambda}\right).$$

We say that μ is an infinite-volume Gibbs measure of the ce if the conditional measures $\mu\left(dx^\Lambda|x^{\mathbb{Z}\setminus\Lambda}\right)$ are given by the finite volume ce $\mu_{\Lambda,m}(dx^\Lambda)$ for any sublattice Λ, i.e.,

$$\mu\left(dx^\Lambda|x^{\mathbb{Z}\setminus\Lambda}\right) = \mu_{\Lambda,m}(dx^\Lambda).$$

The equations of the last identity are called Dobrushin-Lanford-Ruelle (DLR) equations.

Theorem 1.2.9 (Uniqueness of the infinite-volume Gibbs measure of the ce)**.** *There is only one infinite-volume Gibbs measure of the ce that satisfies the uniform bound*

$$\sup_{i\in\mathbb{Z}} \operatorname{var}_\mu(x_i) < \infty.$$

Remark 3. In this thesis, we only show the uniqueness of infinite-volume Gibbs measure of the ce, not the existence. However, the author of this thesis believe that with a cosmetic change, the existence should follow by a compactness argument (see for example [BHK82]).

1.2.3 Uniform logarithmic Sobolev inequality

The logarithmic Sobolev inequality (LSI) – introduced by Gross [Gro75]– is a powerful tool for studying spin systems. For example, the LSI implies Gaussian concentration via Herbst's argument, is equivalent to hypercontractivity and characterizes the exponential rate of convergence to equilibrium of the naturally associated diffusion process. By the equivalence of dynamic and static phase transitions, a uniform LSI also indicates the absence of a phase transition (see e.g. [Yos03, HM16]).

Definition 1.2.10 (Logarithmic Sobolev inequality (LSI))**.** *Let X be a Euclidean space. A Borel probability measure μ on X satisfies the LSI with constant $\varrho > 0$ (or LSI(ϱ)) if, for all nonnegative locally Lipschitz functions $f \in L^1(\mu)$,*

$$\int f \log f d\mu - \int f d\mu \log \left(\int f d\mu \right) \leq \frac{1}{2\varrho} \int \frac{|\nabla f|^2}{f} d\mu, \tag{1.2.2}$$

where ∇ denotes the gradient in the Euclidean space X.

It is well known that on the one-dimensional lattice, the gce satisfies a uniform LSI if the interaction decays fast enough (see for example [MN14, Theorem 1.6]).

Theorem 1.2.11. *The gce μ_Λ^σ satisfies a uniform LSI(ϱ), where $\varrho > 0$ is independent of the system size $|\Lambda|$ and the external fields σ, s.*

Another main result of this thesis is that the ce also satisfies on the one-dimensional lattice a uniform LSI for arbitrary strong, finite-range interaction.

Theorem 1.2.12. *The ce $\mu_{\Lambda,m}$ satisfies a uniform LSI(ϱ), where $\varrho > 0$ is independent of the system size $|\Lambda|$, the external field s and the mean spin $m \in \mathbb{R}$.*

Remark 4 (From Glauber to Kawasaki). We want to point out that the ce $\mu_{\Lambda,m}$ is defined on the space $X_{\Lambda,m}$ given by (1.1.6). Because the space $X_{\Lambda,m}$ is endowed with the standard Euclidean structure inherited from $\mathbb{R}^\Lambda$, the bound on the right-hand side of the LSI (see (1.2.2)) is given in terms of the Glauber dynamics. By the discrete Poincaré inequality one can recover the bound for the Kawasaki dynamics (cf. [Cap03] or [GOVW09, Remark 15]) in the sense that one endows $X_{\Lambda,m}$ with the Euclidean structure coming from the discrete H^{-1}-norm. The so obtained diffusive scaling of LSI constant for the Kawasaki dynamics is known to be optimal (see [Yau96] and also Remark 2 in [Men11]).

1.2.4 Hydrodynamic limit of Kawasaki dynamics

In this section, we assume that $\Lambda = [N] := \{1, 2, \cdots, N\}$ and there are no external fields $s = (s_i), x^{\mathbb{R}\setminus\Lambda}$ in the definition (1.1.1) of Hamiltonian H. We also assume that interactions M_{ij} are spacial homogeneous. That is, there exists a function $h : \mathbb{Z} \to (-1, 1)$ such that

$$M_{ij} = h(|i-j|), \qquad \forall i, j \in \Lambda.$$

A natural dynamics for the conservative system is the Kawasaki dynamics, which is defined as follows. Let A denote the second-order difference operator given by the $N \times N$ matrix

$$A_{ij} := N^2 \left(-\delta_{i,j-1} + 2\delta_{i,j} - \delta_{i,j+1}\right),$$

where we define $\delta_{i,0} = \delta_{i,N}$ and $\delta_{i,N+1} = \delta_{i,1}$. The Kawasaki dynamics is a stochastic process $X(t) \in \mathbb{R}^\Lambda$ satisfying the following stochastic differential equation:

$$dX(t) = -A\nabla H_\Lambda(X(t))dt + \sqrt{2A}dB(t),$$

where $B(t)$ denotes a standard Brownian motion on $\mathbb{R}^\Lambda$. The Kawasaki dynamics preserves its mean spin, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N} X_i(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(0) = m.$$

This implies that we can restrict the state space $\mathbb{R}^\Lambda$ to the hyperplane $X = X_{\Lambda,m}$ and consider the corresponding ce $\mu_{\Lambda,m}$ as an invariant measure. If the process X_t is distributed according to $f\mu_{\Lambda,m}$, then the time dependent probability density $f = f(t,x)$ satisfies

$$\frac{\partial}{\partial t}(f\mu_{\Lambda,m}) = \nabla \cdot \left(A\nabla f\mu_{\Lambda,m}\right). \tag{1.2.3}$$

In order to define a continuous counterpart of the configuration space $X_{\Lambda,m}$, let us define the space $\bar{X} = \bar{X}_{\Lambda,m}$ of piecewise constant functions on $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$ with mean m by

$$\bar{X} := \left\{ \bar{x} : \mathbb{T}^1 \to \mathbb{R};\ \bar{x} \text{ is constant on } \left(\frac{j-1}{N}, \frac{j}{N} \right] \text{ for } j = 1, \cdots, N, \text{ and has mean } m \right\}.$$

We shall identify the space X with $\bar{X}$ by the following relation:

- For each $x \in X$, the step function $\bar{x} \in \bar{X}$ associated to x is

$$\bar{x}(\theta) = x_j, \qquad \text{if } \theta \in \left(\frac{j-1}{N}, \frac{j}{N} \right].$$

- For each step function $\bar{x} \in \bar{X}$, the corresponding vector $x \in X$ is

$$x_j = \bar{x}\left(\frac{j}{N} \right), \qquad j = 1, \cdots, N.$$

We equip the space of locally integrable functions $f : \mathbb{T}^1 \to \mathbb{R}$ having a mean m with H^{-1} norm by

$$\|f\|_{H^{-1}}^2 = \int_{\mathbb{T}^1} \omega^2(\theta) d\theta,$$

where ω is a function such that

$$\omega' = f, \qquad \int_{\mathbb{T}^1} \omega(\theta) d\theta = 0.$$

Now we are ready to formulate our main result, namely the hydrodynamic limit of the Kawasaki dynamics for the ce. We establish that the evolution along the Kawasaki dynamics gets close to the solution to a certain nonlinear parabolic equation as $N \to \infty$.

Theorem 1.2.13. *Let $f = f(t,x)$ be a solution of the Kawasaki dynamics* (1.2.3) *with initial condition $f(0,\cdot) = f_0(\cdot)$. Assume that there exists a constant $C > 0$ such that for any $N \geq 1$,*

$$\int f_0(x) \log f_0(x) \mu_{\Lambda,m}(dx) \leq CN. \tag{1.2.4}$$

Assume also that there is a $\zeta_0 \in L^2(\mathbb{T}^1)$ such that $\int \zeta_0 d\theta = m$ and

$$\lim_{N\to\infty} \int \|\bar{x} - \zeta_0\|_{H^{-1}}^2 f_0(x) \mu_{\Lambda,m}(dx) = 0.$$

Let $\zeta = \zeta(t,\theta)$ be the unique weak solution of the nonlinear parabolic equation

$$\begin{cases} \frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2}\varphi'(\zeta), \\ \zeta(0,\cdot) = \zeta_0, \end{cases} \tag{1.2.5}$$

where φ is defined as

$$\varphi(m) := \lim_{N\to\infty} -\frac{1}{N}\log \int_{\{\frac{1}{N}\sum_{i=1}^N x_i = m\}} \exp\left(-H_N(x)\right)\mathcal{L}^{N-1}(dx). \tag{1.2.6}$$

Then, for any $T > 0$,

$$\lim_{N\to\infty}\sup_{0\le t\le T}\int \|\bar{x} - \zeta(t,\cdot)\|_{H^{-1}}^2 f(t,x)\mu_{\Lambda,m}(dx) = 0.$$

Here, we say that $\zeta = \zeta(t,\theta)$ is a weak solution of (1.2.5) on $[0,T]\times\mathbb{T}^1$ if

$$\zeta \in L_t^\infty(L_\theta^2), \qquad \frac{\partial\zeta}{\partial t} \in L_t^2(H_\theta^{-1}), \qquad \varphi'(\zeta) \in L_t^2(L_\theta^2),$$

and

$$\left\langle \xi, \frac{\partial\zeta}{\partial t}\right\rangle_{H^{-1}} = -\int_{\mathbb{T}^1}\xi\varphi'(\zeta)d\theta, \qquad \forall \xi\in L^2, \text{ for almost every } t\in[0,T].$$

Remark 5. There are some issues in Theorem 1.2.13 to be resolved. First, one has to verify that the pointwise limit of (1.2.6) exists and is differentiable. This will be established in Section 7.3.2. In addition, the existence and uniqueness of a weak solution of (1.2.5) for convex φ follows from the standard argument in the nonlinear PDE theory (see for example [GOVW09, Lemma 38]).

Remark 6. Although we assumed the initial relative entropy bound (1.2.4) in Theorem 1.2.13, we expect that the condition (1.2.4) can be relaxed to cover more general class of initial configurations including deterministic states. In fact, it was proved in [Lu95] that for the Kawasaki dynamics with non-interacting Hamiltonians starting from deterministic configurations, the relative entropy with respect to the canonical ensemble instantaneously becomes of order N.

Remark 7. The quantity inside the limit of (1.2.6), denoted by $\bar{H}_N(m)$, represents the distribution $f_N(m)dm$ of the mean value $(x_1 + \cdots + x_N)/N$ under $\mu_{N,m}$:

$$f_N(m)dm = \frac{1}{Z_N} e^{-N\bar{H}_N(m)} dm.$$

In the case of ce without interactions, i.e. $M_{ij} = 0$, as a consequence of local Cramèr theorem (see [GOVW09, Proposition 31]), φ in (1.2.6) is a Legendre transform of the logarithmic generating function of the distribution $\frac{1}{Z}e^{-\psi(x)}dx$. On the other hand, in the presence of interactions, φ in (1.2.6) can also be expressed in terms of the Legendre transform of the thermodynamic free energy. This point will be discussed in Section 7.3.2.

1.3 Outline of the thesis

Chapter 2 is devoted to provide auxiliary estimates. The results are a bit exhaustive but standard ingredients when studying the gce and ce. In Section 2.1, we provide concentration inequalities for the gce. In Section 2.2, we prove that the first and second moment estimates of the gce and ce are bounded. The scalings are optimal in the system size Λ, external field σ, and the mean spin m. In Section 2.3 and Section 2.4, we provide auxiliary estimates which play central roles in proving the equivalence of ensembles.

Chapter 3 is a reprinted version of the work [KM18]. We prove the first level of equivalence of ensembles: on the level of free energy. Section 3.1 introduces the notion of equivalence of ensembles and relevant literature. In Section 3.2, we state the main results and outline the proof. In Section 3.3, we prove the main proposition that is needed in the proof of the main theorem.

Chapter 4 is a reprinted version of [KLM21]. Here, the second and third level of equivalence of ensembles are discussed: on the level of observables and on the level of correlations. Section 4.1 reviews literature and discuss main idea of the proof. In Section 4.2 the main results are stated and we provide the outline of the proof. The main computations are provided in Section 4.3, which complete the proof of the main theorems.

Chapter 5 is a reprinted version of [KM19]. Section 5.1 reviews decay of correlations, the notion of phase transitions, and historical backgrounds of the problem. The main results and outline of the proof are given in 5.2.

Chapter 6 is a reprinted version of [KM20]. In Section 6.1, we begin by briefly reviewing the background of the problem. The main results and the proof are outlined in Section 6.2. The proof is based on two scale criterion for the LSI. In Section 6.3, we provide the proof of main computations.

Chapter 7 is a reprint of [KMN20] so this is included *verbatim.* Section 7.1 reviews background of the problem and briefly explain main idea. The main result and its proof are stated in Section 7.2. In Section 7.3, auxiliary results are provided : the uniform LSI and strict convexity of coarse-grained Hamiltonian. These play integral roles in deducing the hydrodynamic limit of Kawasaki dynamics. The proofs of auxiliary results are given in Section 7.4 and Section 7.5. Lastly, Section 7.6 and Section 7.7 are dedicated to the proof of main propositions.

Chapter 2

Auxiliary Lemmas

In this chapter we provide several auxiliary results. These results are standard ingredients when studying the gce and ce.

2.1 Concentration inequalities

We begin with providing a general moment estimate for the gce.

Lemma 2.1.1. *For each $k \geq 1$, there is a constant $C = C(k)$ such that for any smooth function $f : \mathbb{R}^\Lambda \to \mathbb{R}$*

$$\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X) - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^k\right] \leq C(k)\|\nabla f\|_\infty^k. \tag{2.1.1}$$

Proof of Lemma 2.1.1. It is well known that the gce μ_Λ^σ satisfies a uniform LSI and Poincaré inequality (see for example [HM16]). The case $k = 2$ easily follows from an application of Poincaré inequality. More precisely, we have

$$\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X) - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^2\right] \leq \frac{1}{\rho}\mathbb{E}_{\mu_\Lambda^\sigma}\left[|\nabla f|^2\right] \leq \frac{1}{\rho}\|\nabla f\|_\infty^2,$$

where $\rho > 0$ is a uniform constant in Poincaré inequality. Thanks to the Schwarz inequality, (2.1.1) also holds for $k = 1$. Assume that (2.1.1) holds for some $k = 2n \geq 2$. Again, Poincaré inequality implies that

$$\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X) - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{2n+2}\right] - \left(\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X) - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{n+1}\right]\right)^2$$

$$\leq \frac{1}{\rho}\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|\nabla\left(\left|f(X)-\mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{n+1}\right)\right|^2\right]$$
$$\leq \frac{n+1}{\rho}\|\nabla f\|_\infty^2\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X)-\mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{2n}\right]$$
$$\leq \frac{n+1}{\rho}\|\nabla f\|_\infty^{2n+2}. \tag{2.1.2}$$

Because $n+1 \leq 2n$, Schwarz inequality implies

$$\left(\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X)-\mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{n+1}\right]\right)^2 \leq \mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|f(X)-\mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right|^{2n}\right]^{\frac{n+1}{2n}\cdot 2}$$
$$\leq \left(C(2n)\|\nabla f\|_\infty^{2n}\right)^{\frac{n+1}{n}} \lesssim \|\nabla f\|_\infty^{2n+1}. \tag{2.1.3}$$

A combination of (2.1.2) and (2.1.3) proves (2.1.1) for $k = 2n+2$. Schwarz inequality also implies that this holds for $k = 2n+1$. Then mathematical induction concludes the proof of Lemma 2.1.1. □

The next statement is a direct consequence of Lemma 2.1.1.

Corollary 2.1.2. *For each $i \in \Lambda$, we define*

$$m_i := \int x_i \mu_\Lambda^\sigma(dx).$$

Then for each $k \geq 1$, there is a constant $C = C(k)$ such that for each $i \in [N]$

$$\mathbb{E}_{\mu_\Lambda^\sigma}\left[|X_i - m_i|^k\right] \leq C(k).$$

Next, we prove that the single-site variance of the gce is uniformly bounded from below.

Lemma 2.1.3. *There is a universal constant $0 < C < \infty$ (depending only on the interaction matrix M and the nonconvexity ψ_b) such that for all $i \in \Lambda$*

$$\operatorname{var}_{\mu_\Lambda^\sigma}(X_i) \geq C. \tag{2.1.4}$$

Proof of Lemma 2.1.3. By conditioning it hols that

$$\operatorname{var}_{\mu_\Lambda^\sigma}(X_i) = \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i|(X_j)_{j\neq i}\right)\right] + \operatorname{var}_{\mu_\Lambda^\sigma}\left(\mathbb{E}_{\mu_\Lambda^\sigma}\left[X_i|(X_j)_{j\neq i}\right]\right)$$

$$\geq \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i|(X_j)_{j\neq i}\right)\right].$$

The desired estimate (2.1.4) will follow from the uniform bound

$$\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i|(X_j)_{j\neq i}\right) \geq C > 0.$$

Indeed, the conditional measure $\mu_\Lambda^\sigma\left(dx_i|(x_j)_{j\neq i}\right)$ has the Lebesgue density

$$\mu_\Lambda^\sigma\left(dx_i|(x_j)_{j\neq i}\right) = \frac{1}{Z}\exp\left(-\frac{1}{2}x_i^2 - \tilde{s}_i x_i - \psi_b(x_i)\right),$$

where $\tilde{s}_i$ is given by

$$\tilde{s}_i = s_i + \frac{1}{2}\sum_{j\neq i} M_{ij}x_j.$$

Let ν denote the one-dimensional measure given by the Lebesgue density

$$\nu(dz) = \frac{1}{Z_\nu}\exp\left(-\frac{1}{2}z^2 - \tilde{s}z\right)dz$$

Using the bound $|\psi_b|_\infty \leq C$ and the optimality of the mean for the variance, and the fact that $\operatorname{var}_\nu(Z) = 1$ we obtain the desired estimate

$$\begin{aligned}\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i|(X_j)_{j\neq i}\right) &\geq \exp\left(-2C\right)\frac{1}{Z_\nu}\int\left(z - \mathbb{E}_\nu\left[X_i\right]\right)^2\exp\left(-\frac{1}{2}z^2 - \tilde{s}z\right)dz \\ &\geq \exp\left(-2C\right)\operatorname{var}_\nu(Z) = \exp\left(-2C\right) = C > 0.\end{aligned}$$

□

A consequence of Lemma 2.1.1 and Lemma 2.1.3 is that the variance of the gce is well controlled.

Lemma 2.1.4. *There is a universal constant* $0 < C < \infty$ *such that*

$$\frac{1}{C} \leq \frac{1}{|\Lambda|}\operatorname{var}_{\mu_\Lambda^\sigma}\left(\sum_{i\in\Lambda}X_i\right) \leq C.$$

Proof of Lemma 2.1.4. The upper bound is a direct consequence of Lemma 2.1.1. Indeed, we have

$$\operatorname{var}_{\mu_\Lambda^\sigma}\left(\sum_{i\in\Lambda}X_i\right) = \mathbb{E}_{\mu_\Lambda^\sigma}\left[\left|\sum_{i\in\Lambda}X_i - \mathbb{E}_{\mu_\Lambda^\sigma}\left[\sum_{i\in\Lambda}X_i\right]\right|^2\right] \leq C\left\|\nabla\left(\sum_{i\in\Lambda}X_i\right)\right\|_\infty^2 = C|\Lambda|.$$

For the lower bound, we assume $\Lambda = \{1, 2, \cdots, N\}$ for simplicity. Recalling that R is the range of interactions within the Hamiltonian H, define Q as

$$\{R+1, 2(R+1), 3(R+1), \cdots\} \cap \Lambda.$$

By conditioning we get that

$$\begin{aligned}
\operatorname{var}_{\mu_\Lambda^\sigma}\left(\sum_{i\in\Lambda} X_i\right) &= \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(\sum_{i\in Q} X_i \middle| (X_j)_{j\in\Lambda\setminus Q}\right)\right] \\
&\quad + \operatorname{var}_{\mu_\Lambda^\sigma}\left(\mathbb{E}_{\mu_\Lambda^\sigma}\left[\sum_{i\in Q} X_i \middle| (X_j)_{j\in\Lambda\setminus Q}\right]\right) \\
&\geq \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(\sum_{i\in Q} X_i \middle| (X_j)_{j\in\Lambda\setminus Q}\right)\right] \\
&= \sum_{i\in Q} \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i \middle| (X_j)_{j\in\Lambda\setminus Q}\right)\right],
\end{aligned}$$

where we used in the last line the fact that because the interaction range is R, different sites in Q become independent after conditioning onto the spin values $(X_j)_{j\in\Lambda\setminus Q}$. Now, an application of Lemma 2.1.3 yields that

$$\begin{aligned}
\operatorname{var}_{\mu_\Lambda^\sigma}(X_i) &\geq \sum_{i\in Q} \mathbb{E}_{\mu_\Lambda^\sigma}\left[\operatorname{var}_{\mu_\Lambda^\sigma}\left(X_i | (X_j)_{j\in\Lambda\setminus Q}\right)\right] \\
&\geq C\,|Q| = \frac{C}{R+1}|\Lambda|,
\end{aligned}$$

which is the desired statement. □

Corollary 2.1.5. *There exists a constant $C \in (0, \infty)$ such that for all $\sigma \in \mathbb{R}$,*

$$\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_\Lambda^{gce}(\sigma) \leq C.$$

Proof of Corollary 2.1.5. Recall the definition (1.2.1) of A_Λ^{gce}. A straightforward calculation yields that

$$\frac{d}{d\sigma} A_\Lambda^{gce}(\sigma) = \frac{1}{|\Lambda|} \mathbb{E}_{\mu_\Lambda^\sigma}\left[\sum_{i\in\Lambda} X_i\right],$$

$$\frac{d^2}{d\sigma^2} A_\Lambda^{gce}(\sigma) = \frac{1}{|\Lambda|} \operatorname{var}_{\mu_\Lambda^\sigma} \left(\sum_{i \in \Lambda} X_i \right).$$

Then the desired estimate follows from Lemma 2.1.4. □

With the help of Corollary 2.1.5, we can now relate the external field σ of μ_Λ^σ and the mean spin m of $\mu_{\Lambda,m}$ as follows:

Definition 2.1.6. *We choose $\sigma = \sigma(m) \in \mathbb{R}$ and $m = m(\sigma) \in \mathbb{R}$ such that*

$$\frac{d}{d\sigma} A_\Lambda^{gce}(\sigma) = m. \tag{2.1.5}$$

Setting $m_i := \int x_i \mu_\Lambda^\sigma \left(dx^\Lambda\right)$ we equivalently get

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} m_i = \frac{1}{|\Lambda|} \int \left(\sum_{i \in \Lambda} x_i \right) \mu_\Lambda^\sigma \left(dx^\Lambda\right) = \frac{1}{|\Lambda|} \mathbb{E}_{\mu_\Lambda^\sigma} \left[\sum_{i \in \Lambda} X_i \right] = m. \tag{2.1.6}$$

By strict convexity of $A_\Lambda^{gce}(\sigma)$, for each $m \in \mathbb{R}$ there exists a unique $\sigma = \sigma(m)$ satisfying (2.1.5) *or vice versa.*

The next statement is an estimation of cubic moments.

Lemma 2.1.7. *It holds that*

$$\left| \mathbb{E}_{\mu_\Lambda^\sigma} \left[\left(\sum_{i \in \Lambda} (X_i - m_i) \right)^3 \right] \right| \lesssim |\Lambda|.$$

Proof of Lemma 2.1.7. Without loss of generality, we assume $\Lambda = [N] = \{1, 2, \cdots, N\}$. We denote $\mu_N^\sigma = \mu_{[N]}^\sigma$ for notational convenience.

Let $(X_1, \cdots, X_N)$ be a real-valued random variable distributed according to the gce μ_N^σ. For each $i \in \{1, \cdots, N\}$ denote $Y_i := X_i - m_i$. For each pair $(i, j, k) \subset \{1, \cdots, N\}$ with $i \le j \le k$, we have by Theorem 1.2.6 and Lemma 2.1.1 that

$$\left| \mathbb{E}_{\mu_N^\sigma} \left[Y_i Y_j Y_k \right] \right| = \left| \operatorname{cov}_{\mu_N^\sigma} \left(Y_i, Y_j Y_k \right) \right| \lesssim \exp\left(-C|i - j| \right).$$

Similarly, one also gets

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_iY_jY_k\right]\right| \lesssim \exp\left(-C|j-k|\right)$$

and conclude

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_iY_jY_k\right]\right| \lesssim \exp\left(-C\max\left(|i-j|,|j-k|\right)\right). \tag{2.1.7}$$

Combined with the triangle inequality, the estimate (2.1.7) yields

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N Y_i\right)^3\right]\right| \lesssim \sum_{i\le j\le N}\mathbb{E}_{\mu_N^\sigma}\left[Y_iY_jY_k\right] \lesssim \sum_{i\le j\le k}\exp\left(-C\max\left(|i-j|,|j-k|\right)\right).$$

For fixed $j\in\{1,\cdots,N\}$ and $d=\max\left(|i-j|,|j-k|\right)$, there are at most $2(d+1)$ pairs of (i,j,k) with $i\le j\le k$. Therefore we conclude that

$$\sum_{i\le j\le N}\exp\left(-C\max\left(|i-j|,|j-k|\right)\right) \le \sum_{j=1}^N\sum_{d=0}^N 2(d+1)\exp\left(-Cd\right) \lesssim N.$$

This finishes the proof of Lemma 2.1.7. □

Lastly, we provide estimates for quartic moments.

Lemma 2.1.8. *It holds that*

$$\left|\mathbb{E}_{\mu_\Lambda^\sigma}\left[\left(\sum_{i\in\Lambda}\left(X_i-m_i\right)\right)^4\right]\right| \sim |\Lambda|^2.$$

Proof of Lemma 2.1.8. Again, we assume $\Lambda=\{1,\cdots,N\}$ and use the notation from proof of Lemma 2.1.7. For each pair $(i,j,k,l)\subset\{1,\cdots,N\}$ with $i\le j\le k\le l$ we have by Theorem 1.2.6 and Corollary 2.1.1 that

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_iY_jY_kY_l\right]\right| \lesssim \exp\left(-C\max\left(|i-j|,|k-l|\right)\right).$$

For fixed $j\le k$ and $d=\max(|i-j|,|k-l|)$, there are at most $2(d+1)$ pairs of (i,j,k,l) with $i\le j\le k\le l$. Thus we conclude

$$\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N Y_i\right)^4\right] \lesssim \sum_{i\le j\le k\le l}\mathbb{E}_{\mu_N^\sigma}\left[Y_iY_jY_kY_l\right] \lesssim \sum_{j\le N}\sum_{d=0}^{N-1}2(d+1)\exp\left(-Cd\right) \lesssim N^2.$$

For the lower bound, we apply Lemma 2.1.4 to get

$$\begin{aligned}\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N Y_i\right)^4\right] &= \mathrm{var}_{\mu_N^\sigma}\left(\left(\sum_{i=1}^N Y_i\right)^2\right) + \left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N Y_i\right)^2\right]\right)^2 \\ &\geq \left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N Y_i\right)^2\right]\right)^2 \gtrsim N^2.\end{aligned}$$

□

2.2 First and second moment estimates

In this section, we obtain sharp moment estimates for the gce and ce. We assume that there is no external field s in this section.

2.2.1 Moment estimates under grand canonical ensemble

Lemma 2.2.1. *For any $i \in \Lambda$, we have*

$$\left|\mathbb{E}_{\mu_\Lambda^\sigma}[X_i]\right| \lesssim |\sigma| + 1.$$

It is delicate to directly estimate the first moment of the gce. The key idea is that we compare the first moment under the gce and Gaussian ensemble using the method of interpolation. Since it is straightforward to compute the first moment under the Gaussian measure, one can finally deduce Lemma 2.2.1.

Proof of Lemma 2.2.1. As mentioned above, proof consists of the following two steps:

- Transfer from the gce to Gaussian ensembles using interpolation.
- Obtain a sharp estimate on the first moment under the Gaussian ensembles.

Step 1. Comparison with Gaussian ensembles.

For $t \in \mathbb{R}$, define a Hamiltonian

$$H_{\Lambda_t}(x) := \sum_{i \in \Lambda} \left(\frac{1}{2} x_i^2 + \sum_{j: 1 \le |j-i| \le R} M_{ij} x_i x_j + t \psi_b(x_i) \right)$$

and the corresponding gce with the external field

$$\nu^\sigma_{\Lambda_t}(dx) := \frac{1}{Z} \exp\left(\sigma \sum_{i \in \Lambda} x_i - H_{\Lambda_t}(x) \right) dx.$$

Note that since $\psi(x) = \frac{1}{2} x^2 + \psi_b(x)$,

$$\nu^\sigma_{\Lambda_1} = \mu^\sigma_\Lambda.$$

and $\nu^\sigma_{\Lambda_0}$ is a Gaussian ensemble.

We interpolate between two measures $\nu^\sigma_{\Lambda_0}$ and $\nu^\sigma_{\Lambda_1}$ for $i \in \Lambda$,

$$\begin{aligned}
\mathbb{E}_{\nu^\sigma_{\Lambda_1}}[X_i] - \mathbb{E}_{\nu^\sigma_{\Lambda_0}}[X_i] &= \int_0^1 \frac{d}{dt} \mathbb{E}_{\nu^\sigma_{\Lambda_t}}[X_i]\, dt \\
&= \int_0^1 \operatorname{cov}_{\nu^\sigma_{\Lambda_t}} \left(X_i, -\sum_{j \in \Lambda} \psi_b(X_j) \right) dt \\
&= -\int_0^1 \sum_{j \in \Lambda} \operatorname{cov}_{\nu^\sigma_{\Lambda_t}} (X_i, \psi_b(X_j))\, dt.
\end{aligned}$$

By Theorem 1.2.6, for some constant $C_t > 0$,

$$\begin{aligned}
\left| \operatorname{cov}_{\nu^\sigma_{\Lambda_t}} (X_i, \psi_b(X_j)) \right| &\le C_t \|\nabla X_i\|_{L^2(\nu^\sigma_{\Lambda_t})} \|\nabla \psi_b(X_j)\|_{L^2(\nu^\sigma_{\Lambda_t})} \exp\left(-C_t |i-j| \right) \\
&\le C \|\psi_b'\|_\infty \exp\left(-C_t |i-j| \right).
\end{aligned}$$

Note that the constant C_t can be chosen to be uniformly bounded for $t \in [0,1]$. Thus,

$$\left| \mathbb{E}_{\nu^\sigma_{\Lambda_1}}[X_i] - \mathbb{E}_{\nu^\sigma_{\Lambda_0}}[X_i] \right| \le C \sum_{j \in \Lambda} \exp\left(-C|i-j| \right) \lesssim 1.$$

This implies that

$$\left| \mathbb{E}_{\mu^\sigma_\Lambda}[X_i] \right| = \left| \mathbb{E}_{\nu^\sigma_{\Lambda_1}}[X_i] \right| \lesssim \left| \mathbb{E}_{\nu^\sigma_{\Lambda_0}}[X_i] \right| + 1. \tag{2.2.1}$$

Step 2. First moment estimate under Gaussian ensembles.

Let us denote a mean vector of the Gaussian measure $\nu^{\sigma}_{\Lambda_0}$ by $(\eta^{\sigma}_i)_{i\in\Lambda}$. Then, we have

$$(\eta^{\sigma}_i)^T_{i\in\Lambda} = M^{-1}(\sigma)^T_{i\in\Lambda}.$$

Since the Hamiltonian $H_{\Lambda_0}(x)$ is strictly convex, the measure $\nu^{\sigma}_{\Lambda_0}$ satisfies a uniform LSI independent of σ and thus we have the exponential decay of correlations (see [HM16] for example). For multivariate Gaussian distribution, the covariance matrix is given by the inverse of the quadratic coefficient matrix M, i.e. $\mathrm{cov}_{\nu^{\sigma}_{\Lambda_0}}(X) = M^{-1}$. In particular, the coefficient of M^{-1} decays exponentially in the sense that there is a constant $C > 0$ such that for any Λ,

$$|(M^{-1})_{ij}| \leq C \exp\left(-C|i-j|\right), \qquad \forall i, j.$$

Therefore there exists a constant $c'_1 > 0$ such that for any $i \in \Lambda$,

$$|\mathbb{E}_{\nu^{\sigma}_{\Lambda_0}}\left[X_i\right]| = |\eta^{\sigma}_i| \leq c'_1|\sigma|. \tag{2.2.2}$$

Thus, by (2.2.1) and (2.2.2), one can deduce that there exist constants $c_1, c_2 > 0$ such that for any $\sigma \in \mathbb{R}$,

$$\left|\mathbb{E}_{\mu^{\sigma}_{\Lambda}}\left[X_i\right]\right| \leq c_1|\sigma| + c_2.$$

□

The next statement is a direct consequence of Lemma 2.2.1 and Lemma 2.1.4.

Corollary 2.2.2. *It holds that*

$$\left|\mathbb{E}_{\mu^{\sigma}_{\Lambda}}\left[X_i^2\right]\right| \lesssim \sigma^2 + 1.$$

2.2.2 Moment estimates under canonical ensemble

In the previous section, we obtained moment estimates under gce. Combining this with the principle of equivalence of observables (Proposition 1.2.4), we finally obtain the moment estimate for the ce.

Lemma 2.2.3. *For any $i \in \Lambda$, it holds that*

$$\left|\mathbb{E}_{\mu_{\Lambda,m}}\left[X_i\right]\right| \lesssim |m| + 1.$$

Proof of Lemma 2.2.3. Recall the definition (1.2.1) of the free energy A_Λ^{gce}. We first prove that there exist constants $\gamma_1, \gamma_2 > 0$ such that for any σ and m satisfying

$$\frac{d}{d\sigma} A_\Lambda^{gce}(\sigma) = m, \tag{2.2.3}$$

we have

$$|\sigma| \leq \gamma_1 |m| + \gamma_2. \tag{2.2.4}$$

By Lemma 2.1.5, there exists a constant $C > 0$ such that for any $\sigma \in \mathbb{R}$,

$$\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_\Lambda^{gce}(\sigma) \leq C. \tag{2.2.5}$$

Also, note that

$$\left.\frac{d}{d\sigma} A_\Lambda^{gce}\right|_{\sigma=0} = \frac{1}{N}\mathbb{E}_{\mu_\Lambda^0}\left[\sum_{i\in\Lambda} X_i\right]. \tag{2.2.6}$$

This quantity is uniformly bounded in $|\Lambda|$ thanks to Lemma 2.2.1. Thus, by (2.2.3), (2.2.5) and (2.2.6), we have (2.2.4) for some constants $\gamma_1, \gamma_2 > 0$.

Note that by (2.2.3) and the equivalence of observable result (Theorem 1.2.4), we have

$$\left|\mathbb{E}_{\mu_{\Lambda,m}}\left[X_i\right] - \mathbb{E}_{\mu_\Lambda^\sigma}\left[X_i\right]\right| = O(\frac{1}{|\Lambda|}). \tag{2.2.7}$$

Thus, by Lemma 2.2.1, (2.2.4) and (2.2.7), proof is concluded.

□

Because the ce $\mu_{\Lambda,m}$ satisfies a uniform LSI and hence Poincaré inequality, the variance of the ce is well behaved. Therefore we have the following statement.

Corollary 2.2.4. *For any $i \in \Lambda$, it holds that*

$$\left|\mathbb{E}_{\mu_{\Lambda,m}}\left[X_i^2\right]\right| \lesssim m^2 + 1.$$

2.3 Covariance estimates

In this section, we provide auxiliary covariance estimates. We only consider the one-dimensional lattice $\Lambda = [N] = \{1, 2, \cdots, N\}$. To reduce our notational burden, we denote (with slight abuse of notations) $H_N = H_{[N]}$, $\mu_N^\sigma = \mu_{[N]}^\sigma$ and $\mu_{N,m} = \mu_{[N],m}$.

First of all, we provide estimates of covariances between an intensive function f and sum of spins.

Lemma 2.3.1. *Let f be an intensive function and A be any subset of $[N]$. Then it holds that*

$$\left| \operatorname{cov}_{\mu_N^\sigma}\left(f(X) \ , \ \sum_{i \in A} X_i \right) \right| \lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)} |\operatorname{supp} f|^{\frac{1}{2}}.$$

Proof of Lemma 2.3.1. As before we assume $A = [N]$. For each $d \in \mathbb{N}$ denote S_d by

$$S_d := \{k \in [N] : \operatorname{dist}(\operatorname{supp} f \ , \ k) = d\}. \tag{2.3.1}$$

We note that for each $d \in \mathbb{N}$, the cardinality of S_d is bounded from above by $2|\operatorname{supp} f|$. By triangle inequality we have

$$\left| \operatorname{cov}_{\mu_N^\sigma}\left(f(X), \sum_{i=1}^N X_i \right) \right| \leq \left| \operatorname{cov}_{\mu_N^\sigma}\left(f(X), \sum_{k \in \operatorname{supp} f} X_k \right) \right| + \sum_{d \geq 1} \left| \operatorname{cov}_{\mu_N^\sigma}\left(f(X), \sum_{k \in S_d} X_k \right) \right|. \tag{2.3.2}$$

Then a combination of Schwarz inequality, Poincaré inequality, Lemma 2.1.1, and Corollary 2.1.2 yields that the first term in (2.3.2) is bounded by

$$\begin{aligned} \left| \operatorname{cov}_{\mu_N^\sigma}\left(f(X), \sum_{k \in \operatorname{supp} f} X_k \right) \right| &\leq \| f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X) \right] \|_{L^2(\mu_N^\sigma)} \left\| \sum_{k \in \operatorname{supp} f} (X_k - m_k) \right\|_{L^2(\mu_N^\sigma)} \\ &\lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)} |\operatorname{supp} f|^{\frac{1}{2}}. \end{aligned}$$

Next, an application of Theorem 1.2.6 yields that the second term in (2.3.2) can be estimated as follows:

$$\sum_{d\geq 1}\left|\operatorname{cov}_{\mu_N^\sigma}\left(f(X),\sum_{k\in S_d}X_k\right)\right| \lesssim \sum_{d\geq 1}\|\nabla f\|_{L^2(\mu_N^\sigma)}|S_d|^{\frac{1}{2}}\exp\left(-Cd\right)$$
$$\lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)}|\operatorname{supp} f|^{\frac{1}{2}}\sum_{d\geq 1}\exp\left(-Cd\right)$$
$$\lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)}|\operatorname{supp} f|^{\frac{1}{2}}.$$

□

Lemma 2.3.2. *Let f be an intensive function and A be any subset of $[N]$. Then it holds that*

$$\left|\operatorname{cov}_{\mu_N^\sigma}\left(f(X),\left(\sum_{i\in A}(X_i-m_i)\right)^2\right)\right| \lesssim \|\nabla f\|_{L^4(\mu_N^\sigma)}|\operatorname{supp} f|. \tag{2.3.3}$$

Proof of Lemma 2.3.2. Proof of Lemma 2.3.2 is motivated by the proof of Lemma 2.1.7. We use similar idea accompanied with more careful estimate when applying Theorem 1.2.6 and Corollary 2.1.2. In this proof, the set S denotes $\operatorname{supp} f$. We first decompose the left hand side of (2.3.3) by

$$\operatorname{cov}_{\mu_N^\sigma}\left(f(X),\left(\sum_{i\in A}(X_i-m_i)\right)^2\right)$$
$$= \mathbb{E}_{\mu_N^\sigma}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(\sum_{i\in S}(X_i-m_i)\right)^2\right] \tag{2.3.4}$$
$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\sum_{i\in S}(X_i-m_i)\sum_{j\notin S}(X_j-m_j)\right] \tag{2.3.5}$$
$$+ \mathbb{E}_{\mu_N^\sigma}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(\sum_{j\notin S}(X_i-m_i)\right)^2\right]. \tag{2.3.6}$$

We estimate each term by term. Let us begin with estimating (2.3.4). An application of Schwarz inequality followed by Lemma 2.1.8 yields

$$T_{(2.3.4)} \leq \|f-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\|_{L^2(\mu_N^\sigma)}\left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i\in S}(X_i-m_i)\right)^4\right]\right)^{\frac{1}{2}}$$

$$\lesssim \|f - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\|_{L^2(\mu_N^\sigma)}|S| = \|f - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\|_{L^2(\mu_N^\sigma)}|\operatorname{supp} f|$$
$$\overset{Poincare}{\lesssim} \|\nabla f\|_{L^2(\mu_N^\sigma)}|\operatorname{supp} f| \le \|\nabla f\|_{L^4(\mu_N^\sigma)}|\operatorname{supp} f|.$$

Let us turn to the estimation of (2.3.5). As in the proof of Lemma 2.3.1, we denote

$$S_d := \{j \in [N] \ : \ \operatorname{dist}(S,j) = d\}, \qquad d = 1, 2, \cdots .$$

Let us recall that for each $d \ge 1$ we have $|S_d| \le 2|\operatorname{supp} f|$. We write $T_{(2.3.5)}$ as

$$T_{(2.3.5)} = 2\sum_{d\ge 1} \operatorname{cov}_{\mu_N^\sigma}\left(\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\sum_{i\in S}\left(X_i - m_i\right), \sum_{k\in S_d} X_k\right).$$

It holds by Theorem 1.2.6 that

$$\begin{aligned}
&\operatorname{cov}_{\mu_N^\sigma}\left(\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\sum_{i\in S}\left(X_i - m_i\right), \sum_{k\in S_d} X_k\right) \\
&\quad\lesssim \left\|\nabla\left(\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\sum_{i\in S}\left(X_i - m_i\right)\right)\right\|_{L^2(\mu_N^\sigma)} |S_d|^{\frac{1}{2}}\exp\left(-Cd\right) \\
&\quad\lesssim \left\|\left(\sum_{i\in S}\left(X_i - m_i\right)\right)\nabla f(X)\right\|_{L^2(\mu_N^\sigma)} |\operatorname{supp} f|^{\frac{1}{2}}\exp\left(-Cd\right) \qquad (2.3.7)\\
&\qquad + \left\|\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\right\|_{L^2(\mu_N^\sigma)} |\operatorname{supp} f|\exp\left(-Cd\right). \qquad (2.3.8)
\end{aligned}$$

Then a direct calculation yields

$$\begin{aligned}
\left\|\left(\sum_{i\in S}\left(X_i - m_i\right)\right)\nabla f(X)\right\|_{L^2(\mu_N^\sigma)}^2 &= \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i\in S}\left(X_i - m_i\right)\right)^2 |\nabla f(X)|^2\right] \\
&\le \|\nabla f\|_{L^4(\mu_N^\sigma)}^2 \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i\in S}\left(X_i - m_i\right)\right)^4\right]^{\frac{1}{2}} \\
&\overset{Corollary\ 2.1.2}{\lesssim} \|\nabla f\|_{L^4(\mu_N^\sigma)}^2 |S| = \|\nabla f\|_{L^4(\mu_N^\sigma)}^2 |\operatorname{supp} f|.
\end{aligned} \qquad (2.3.9)$$

Hence plugging the estimate (2.3.9) into (2.3.7) gives

$$T_{(2.3.7)} \lesssim \|\nabla f\|_{L^4(\mu_N^\sigma)}^2 |\operatorname{supp} f|\exp\left(-Cd\right).$$

It also holds from Poincaré inequality that

$$T_{(2.3.8)} \lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)} |\operatorname{supp} f| \exp\left(-Cd\right) \le \|\nabla f\|_{L^4(\mu_N^\sigma)} |\operatorname{supp} f| \exp\left(-Cd\right).$$

Thus we conclude

$$T_{(2.3.5)} \lesssim \sum_{d\ge 1} \left(T_{(2.3.7)} + T_{(2.3.8)}\right) \lesssim \sum_{d\ge 1} \|\nabla f\|_{L^4(\mu_N^\sigma)} |\operatorname{supp} f| \exp\left(-Cd\right) \lesssim \|\nabla f\|_{L^4(\mu_N^\sigma)} |\operatorname{supp} f|.$$

The estimation of (2.3.6) follows from similar calculations given in Lemma 2.1.7 and estimation of (2.3.5). □

For any $i \in [N]$ and $l > 0$, let us denote $B_l(i)$ by

$$B_l(i) := \{k \in [N] \ : \ |k - i| \le l\}. \tag{2.3.10}$$

The last statement of this section provides a lower bound onto block-block correlations.

Lemma 2.3.3. *There are positive constants C and L such that for any $i \in [N]$, $l \ge L$, and a set $T \supseteq B_l(i)$,*

$$\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j, \sum_{k\in T} X_k\right) \ge Cl.$$

Proof of Lemma 2.3.3. By Lemma 2.1.4, there is a constant $C_1 > 0$ such that

$$\operatorname{var}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j\right) \ge C_1 l.$$

Next, an application of Theorem 1.2.6 implies that for some $C_2 > 0$,

$$\begin{aligned}\sum_{k\in T\setminus B_l(i)} \left|\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j, X_k\right)\right| &\le C \sum_{k\in T\setminus B_l(i)} l^{\frac{1}{2}} \exp\left(-C\operatorname{dist}(k, B_l(i))\right) \\ &\le C l^{\frac{1}{2}} \sum_{d=1}^{\infty} \exp\left(-Cd\right) \le C_2 l^{\frac{1}{2}}.\end{aligned}$$

We note that the constants C_1 and C_2 are uniform. By choosing L large enough, there is a uniform constant C such that for any $l \ge L$,

$$\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j, \sum_{k\in T} X_k\right) = \operatorname{var}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j\right) - \sum_{k\in T\setminus B_l(i)} \operatorname{cov}_{\mu_N^\sigma}\left(\sum_{j\in B_l(i)} X_j, X_k\right)$$

$$\geq C_1 l - C_2 l^{\frac{1}{2}} \geq Cl.$$

□

2.4 Influence of boundary conditions onto observables and correlations.

In this section, f is a given intensive function and let us denote $S = \operatorname{supp} f$. We also use the notations from Section 2.3.

Let us decompose the sublattice $[N]$ into two sets E_S and F_S as follows:

$$E_S := \{1, \cdots, N\} \cap \{k : dist(k, S) \leq M \log N\}, \tag{2.4.1}$$

$$F_S := \{1, \cdots, N\} \cap \{k : dist(k, S) > M \log N\}, \tag{2.4.2}$$

where M is a sufficiently large constant which will be chosen later. Recalling the definition (2.3.1) of S_d, the sets E_S and F_S can be written as

$$E_S = \bigcup_{d=0}^{M \log N} S_d, \qquad F_S = \bigcup_{d=M \log N+1}^{N} S_d,$$

where we define S_0 to be S. We decompose the gce μ_N^σ into the conditional measure $\mu_N^\sigma \left(dx^{E_S} \mid y^{F_S}\right)$ and the marginal measure $\bar{\mu}^\sigma \left(dy^{F_S}\right)$. That is, for any test function ζ

$$\int \zeta \mu_N^\sigma = \int \int \zeta \left(x^{E_S}, y^{F_S}\right) \mu_N^\sigma \left(dx^{E_S} \mid y^{F_S}\right) \bar{\mu}^\sigma \left(dy^{F_S}\right). \tag{2.4.3}$$

To reduce our notational burden, we write $x = x^{E_S}$, $y = y^{F_S}$, and $z = z^{F_S}$ in this section.

The next two lemmas estimate the influence of the boundary conditions onto observables and correlations.

Lemma 2.4.1. *Let $y = y^{F_S}$ and $z = z^{F_S}$ be given. For N large enough, it holds that*

$$\begin{aligned}&\left|\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right] - \mathbb{E}_{\mu_N^\sigma(dx|z)}\left[f(X)\right]\right| \\ &\quad\lesssim \|\nabla f\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right).\end{aligned}$$

Proof of Lemma 2.4.1. By interpolation we have

$$\begin{aligned}&\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right] - \mathbb{E}_{\mu_N^\sigma(dx|z)}\left[f(X)\right] \\ &\quad= \int_0^1 \frac{d}{dt}\mathbb{E}_{\mu_N^\sigma(dx|ty+(1-t)z)}\left[f(X)\right]dt \\ &\quad= \int_0^1 \operatorname{cov}_{\mu_N^\sigma(dx|ty+(1-t)z)}\left(f, \sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}X_i(y_j - z_j)\right)dt. \qquad (2.4.4)\end{aligned}$$

With slight abuse of notation we denote $\mu_{c,t}^\sigma = \mu_N^\sigma(dx|ty + (1-t)z)$. We note that the conditional measure $\mu_{c,t}^\sigma$ is again a gce. First, we compute

$$\begin{aligned}\left\|\nabla\left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}X_i(y_j - z_j)\right)\right\|_{L^2(\mu_{c,t}^\sigma)}^2 &= \mathbb{E}_{\mu_{c,t}^\sigma}\left[\sum_{i\in E_S}\left(\sum_{\substack{j\in F_S\\|i-j|\le R}} M_{ij}(y_j - z_j)\right)^2\right] \\ &= \sum_{i\in E_S}\left(\sum_{\substack{j\in F_S\\|i-j|\le R}} M_{ij}(y_j - z_j)\right)^2 \\ &\le \sum_{i\in E_S}(2R)\sum_{\substack{j\in F_S\\|i-j|\le R}} M_{ij}^2(y_j - z_j)^2 \\ &\lesssim \sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}^2(y_j - z_j)^2, \qquad (2.4.5)\end{aligned}$$

where the first inequality follows from Cauchy's inequality and an observation that for each $i \in E_S$ there are at most $2R$ number of j's with $j \in F_S$ and $|i - j| \le R$.

For a pair (i,j) with $i \in E_S$, $j \in F_S$ and $|i-j| \le R$, the triangle inequality implies that for large enough N

$$\operatorname{dist}(i,S) \ge \operatorname{dist}(j,S) - |i-j| \ge M\log N - R \ge \frac{1}{2}M\log N.$$

Because $\mu^{\sigma}_{c,t}$ is also a gce and the result of Theorem 1.2.6 does not depend on boundary values, and hence the constant C is independent of t. Therefore we get the desired estimate for the integrand in (2.4.4) by applying Theorem 1.2.6 as follows:

$$\begin{aligned}
&\left|\operatorname{cov}_{\mu^{\sigma}_{c,t}}\left(f, \sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}X_i(y_j - z_j)\right)\right| \\
&\quad\lesssim \|\nabla f\|_{L^2(\mu^{\sigma}_{c,t})}\left\|\nabla\left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}X_i(y_j - z_j)\right)\right\|_{L^2(\mu^{\sigma}_{c,t})} \exp\left(-C\frac{1}{2}M\log N\right) \\
&\quad\overset{(2.4.5)}{\lesssim} \|\nabla f\|_{\infty}\left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}^2(y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right). \qquad (2.4.6)
\end{aligned}$$

Plugging the estimate (2.4.6) into (2.4.4) finishes the proof of Lemma 2.4.1. □

Lemma 2.4.2. *Under the same assumptions as in Lemma 2.4.1, we have for each $k \in E_S$ with dist$(S,k) \ge \frac{1}{2}M\log N$,*

$$\left|\operatorname{cov}_{\mu^{\sigma}_N(dx|y)}\left(f(X), X_k\right) - \operatorname{cov}_{\mu^{\sigma}_N(dx|z)}\left(f(X), X_k\right)\right| \lesssim \|\nabla f\|_{\infty}\exp\left(-C\, dist(S,k)\right), \qquad (2.4.7)$$

and for $k \in E_S$ with dist$(S,k) < \frac{1}{2}M\log N$,

$$\begin{aligned}
&\left|\operatorname{cov}_{\mu^{\sigma}_N(dx|y)}\left(f(X), X_k\right) - \operatorname{cov}_{\mu^{\sigma}_N(dx|z)}\left(f(X), X_k\right)\right| \\
&\quad\lesssim \|\nabla f\|_{\infty}\left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}^2(y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right). \qquad (2.4.8)
\end{aligned}$$

Remark 8. The first estimate (2.4.7) implies that if the distance between S and k are far enough, the covariances are uniformly bounded from above. If S and k are close, (2.4.8) implies that the difference of covariances can be measured in terms of boundary spin values.

Proof of Lemma 2.4.2. Because conditional measures $\mu_N^\sigma(dx|y)$ and $\mu_N^\sigma(dx|z)$ are again gces, the case when $\operatorname{dist}(S,k) \geq \frac{1}{2}M\log N$ directly follows from Theorem 1.2.6. Let us assume $\operatorname{dist}(S,k) < \frac{1}{2}M\log N$. As in the proof of Lemma 2.4.1 we use interpolation to get

$$\begin{aligned}
&\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(f(X), X_k\right) - \operatorname{cov}_{\mu_N^\sigma(dx|z)}\left(f(X), X_k\right) \\
&\quad = \int_0^1 \frac{d}{dt}\operatorname{cov}_{\mu_{c,t}^\sigma}\left(f(X), X_k\right) dt,
\end{aligned}$$

where $\mu_{c,t}(dx) = \mu_N^\sigma(dx|ty + (1-t)z)$. A straightforward calculation gives

$$\begin{aligned}
&\frac{d}{dt}\operatorname{cov}_{\mu_{c,t}^\sigma}\left(f(X), X_t\right) \\
&= \frac{d}{dt}\mathbb{E}_{\mu_{c,t}^\sigma}\left[\left(f(X) - \mathbb{E}_{\mu_{c,t}^\sigma}\left[f(X)\right]\right)\left(X_k - \mathbb{E}_{\mu_{c,t}^\sigma}\left[X_k\right]\right)\right] \\
&= \operatorname{cov}_{\mu_{c,t}^\sigma}\left(\left(f(X) - \mathbb{E}_{\mu_{c,t}^\sigma}\left[f(X)\right]\right)\left(X_k - \mathbb{E}_{\mu_{c,t}^\sigma}\left[X_k\right]\right), \sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}X_i(y_j - z_j)\right) && (2.4.9) \\
&\quad - \operatorname{cov}_{\mu_{c,t}^\sigma}\left(f(X) - \mathbb{E}_{\mu_{c,t}^\sigma}\left[f(X)\right], \sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}X_i(y_j - z_j)\right)\mathbb{E}_{\mu_{c,t}^\sigma}\left[X_k - \mathbb{E}_{\mu_{c,t}^\sigma}\left[X_k\right]\right] && (2.4.10) \\
&\quad - \operatorname{cov}_{\mu_{c,t}^\sigma}\left(X_k - \mathbb{E}_{\mu_{c,t}^\sigma}\left[X_k\right], \sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}X_i(y_j - z_j)\right)\mathbb{E}_{\mu_{c,t}^\sigma}\left[f(X) - \mathbb{E}_{\mu_{c,t}^\sigma}\left[f(X)\right]\right]. \\
& && (2.4.11)
\end{aligned}$$

Then Theorem 1.2.6, Corollary 2.1.2 and Lemma 2.1.1 imply (see also estimations of (2.3.7), (2.3.8), and (2.4.5))

$$\begin{aligned}
\left|T_{(2.4.9)}\right| &\lesssim \left(\|\nabla f\|_\infty + \|f(X) - \mathbb{E}_{\mu_{c,t}}\left[f(X)\right]\|_{L^2(\mu_{c,t}^\sigma)}\right) \\
&\qquad \times \left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}^2(y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right) \\
&\lesssim \|\nabla f\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}^2(y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right).
\end{aligned}$$

Similar calculation also yields

$$\left|T_{(2.4.10)}\right|, \left|T_{(2.4.11)}\right| \lesssim \|\nabla f\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \exp\left(-CM\log N\right).$$

Hence we get

$$\begin{aligned} &\left|\mathrm{cov}_{\mu_N^\sigma(dx|y)}\left(f(X)\ ,\ X_k\right) - \mathrm{cov}_{\mu_N^\sigma(dx|z)}\left(f(X)\ ,\ X_k\right)\right| \\ &\lesssim \int_0^1 \|\nabla f\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \exp\left(-CM\log N\right) dt \\ &= \|\nabla f\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \exp\left(-CM\log N\right). \end{aligned}$$

□

Corollary 2.4.3. *Under the same assumptions as in Lemma 2.4.1, we have for each $k \in E_S$ with $dist(S,k) \geq \frac{1}{2}M\log N$,*

$$\left|\mathrm{cov}_{\mu_N^\sigma(dx|y)}\left(f(X)\ ,\ X_k\right) - \mathrm{cov}_{\mu_N^\sigma}\left(f(X)\ ,\ X_k\right)\right| \lesssim \|\nabla f\|_\infty \exp\left(-C\,dist(S,k)\right),$$

and for $k \in E_S$ with $dist(S,k) < \frac{1}{2}M\log N$,

$$\begin{aligned} &\left|\mathrm{cov}_{\mu_N^\sigma(dx|y)}\left(f(X)\ ,\ X_k\right) - \mathrm{cov}_{\mu_N^\sigma}\left(f(X)\ ,\ X_k\right)\right| \\ &\lesssim \|\nabla f\|_\infty \exp\left(-CM\log N\right) \int \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz) \\ &\quad + \|\nabla f\|_\infty |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM\log N\right). \end{aligned}$$

Proof of Corollary 2.4.3. The first case follows from Theorem 1.2.6 and triangle inequality. To prove the case when $k \in E_S$, $\mathrm{dist}(S,k) < \frac{1}{2}M\log N$, we use the law of total covariance and write

$$\left|\mathrm{cov}_{\mu_N^\sigma(dx|y)}\left(f(X), X_k\right) - \mathrm{cov}_{\mu_N^\sigma}\left(f(X), X_k\right)\right|$$

$$\leq \left| \mathrm{cov}_{\mu_N^\sigma(dx|y)} \left(f(X), X_k \right) - \int \mathrm{cov}_{\mu_N^\sigma(dx|z)} \left(f(X), X_k \right) \bar{\mu}^\sigma(dz) \right| \tag{2.4.12}$$

$$+ \left| \mathrm{cov}_{\mu_N^\sigma} \left(\mathbb{E}_{\mu_N^\sigma(dx|y)} \left[f(X) \right] \ , \ \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[X_k \right] \right) \right| . \tag{2.4.13}$$

Then Lemma 2.4.2 implies

$$\begin{aligned} T_{(2.4.12)} &\leq \int \left| \mathrm{cov}_{\mu_N^\sigma(dx|y)} \left(f(X), X_k \right) - \mathrm{cov}_{\mu_N^\sigma(dx|z)} \left(f(X), X_k \right) \right| \bar{\mu}^\sigma(dz) \\ &\leq \|\nabla f\|_\infty \exp\left(-CM \log N \right) \int \left(\sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz). \end{aligned}$$

A similar calculations using Lemma 2.4.1 gives

$$\begin{aligned} T_{(2.4.13)} &\leq \int \int \left| \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[f(X) \right] - \mathbb{E}_{\mu_N^\sigma(dx|z)} \left[f(X) \right] \right| \\ &\qquad \times \left| \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[X_k \right] - \mathbb{E}_{\mu_N^\sigma(dx|z)} \left[X_k \right] \right| \bar{\mu}^\sigma(dy) \bar{\mu}^\sigma(dz) \\ &\lesssim \|\nabla f\|_\infty \exp\left(-2CM \log N \right) \\ &\quad \times \int \int \left(\sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{i \in E_k, j \in F_k \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \bar{\mu}^\sigma(dy) \bar{\mu}^\sigma(dz), \end{aligned}$$

where E_k and F_k are defined by

$$E_k := \{1, \cdots, N\} \cap \{l : dist(l,k) \leq M \log N\},$$

$$F_k := \{1, \cdots, N\} \cap \{l : dist(l,k) > M \log N\}.$$

Now we apply Schwarz inequality followed by Corollary 2.1.2 and get, as desired,

$$\begin{aligned} &\int \int \left(\sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{i \in E_k, j \in F_k \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \bar{\mu}^\sigma(dy) \bar{\mu}^\sigma(dz) \\ &\quad \leq \left(\int \int \sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \bar{\mu}^\sigma(dy) \bar{\mu}^\sigma(dz) \right)^{\frac{1}{2}} \end{aligned}$$

$$
\times \left(\int \int \sum_{\substack{i \in E_k, j \in F_k \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \bar{\mu}^\sigma(dy) \bar{\mu}^\sigma(dz) \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(\sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 \operatorname{var}(X_j) \right)^{\frac{1}{2}} \left(\sum_{\substack{i \in E_k, j \in F_k \\ |i-j| \leq R}} M_{ij}^2 \operatorname{var}(X_j) \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(2R^2 |\operatorname{supp} f| \right)^{\frac{1}{2}} \left(2R^2 \right)^{\frac{1}{2}} \sim |\operatorname{supp} f|^{\frac{1}{2}}.
$$

□

Chapter 3

Equivalence of ensembles part 1: On the level of free energy

We consider a one-dimensional lattice system of unbounded, real-valued spins. We allow arbitrary strong, attractive, nearest-neighbor interaction. We show that the free energy of the canonical ensemble converges uniformly in C^2 to the free energy of the grand canonical ensemble. The error estimates are quantitative. A direct consequence is that the free energy of the canonical ensemble is uniformly strictly convex for large systems. Another consequence is a quantitative local Cramér theorem which yields the strict convexity of the coarse-grained Hamiltonian. With small adaptations, the argument could be generalized to systems with finite-range interaction on a graph, as long as the degree of the graph is uniformly bounded and the associated grand canonical ensemble has uniform decay of correlations.

3.1 Introduction

There are many levels of defining equivalence of ensembles. In this thesis we follow the presentation of [CM00], where three levels of equivalence of ensembles were introduced: on the level of thermodynamic functions, on the level of observables, and on the level of correlations. That is, in the thermodynamic limit (size $|\Lambda|$ of the system goes to ∞), free energy, expectation of intensive observable, and correlation of two intensive functions are independent of the ensemble used. In this chapter, we study equivalence of the grand canonical ensemble and the canonical ensemble on the level of free energy.

In equivalent ensembles, properties usually transfer from one ensemble to the other. The equivalence of ensembles in one-dimensional lattice system was deduced in [DT77] for discrete (or bounded) spin values or in [Geo95] for quadratic Hamiltonians. However, our case where the spin values are unbounded and the Hamiltonian is not quadratic is still open. In the main result of this chapter, i.e. in Theorem 3.2.1, we show that the grand canonical and canonical ensemble are equivalent on the level of free energy. In fact, we show that free energies converge uniformly in C^2 as the system size goes to infinity. The rate of convergence in Theorem 3.2.1 is explicit. We therefore extend and refine the results of Dobrushin [DT77] and Georgii [Geo95].

Our argument is quite general and should apply to more general situations. The argument does not use that the lattice is one-dimensional. Instead, it only uses that the grand canonical ensemble on a one-dimensional lattice has an uniform exponential decay of correlations (see for example [MN14] and [Zeg96]). Under the assumption of an uniform exponential decay of correlation, one should be able to use similar calculations to deduce the local Cramér theorem for spin systems on arbitrary graphs, as long as the degree is uniformly bounded and the interaction has finite range. However, we only consider the one-dimensional lattice with nearest-neighbor interaction because less notational burden is better for explaining ideas and presenting the calculations.

A consequence of Theorem 3.2.1 is that the free energy of the canonical ensemble is uniformly strictly convex and quadratic for large enough systems (see Corollary 3.2.2). Strict convexity of the free energy rules out phase coexistence which corresponds to flat parts in the free energy. The most prominent example of phase coexistence is that under ordinary pressure water and ice can coexist at 0 degree Celsius. We want to point out that our result already applies to large but finite systems. In the infinite-volume limit, ordinary equivalence of ensembles (and not equivalence in C^2) would suffice to conclude that the free energy of the canonical ensemble is strictly convex. We refer to Chapter 5 for more details.

Closely related to the free energy A_Λ^{ce} of the canonical ensemble is the notion of the coarse-grained Hamiltonian $\bar{H}_\Lambda$ (cf. is (3.2.4) and [GOVW09]). As in [GOVW09], we derive from Theorem 3.2.1 a local Cramér Theorem (see Theorem 3.2.4). The local Cramér theorem shows that the coarse-grained Hamiltonian converges in C^2 to the Legendre transform of the free energy of the grand canonical ensemble. It is a direct consequence of the C^2-local Cramér theorem that the coarse-grained Hamiltonian $\bar{H}_\Lambda$ is also uniformly strictly convex for large enough system size $|\Lambda|$ (cf. Corollary 3.2.5).

The coarse-grained Hamiltonian $\bar{H}_\Lambda$ plays an important role when studying the Kawasaki dynamics. The Kawasaki dynamics is natural drift diffusion process on our lattice system that conserves the mean spin of the system. The canonical ensemble is the stationary and ergodic distribution of the Kawasaki dynamics. The strict convexity of $\bar{H}_\Lambda$ is a central ingredient for deducing a uniform logarithmic Sobolev inequality (LSI) for the canonical ensemble via the two-scale approach [GOVW09]. The LSI characterizes the speed of convergence of the Kawasaki dynamics to the canonical ensemble. With the equivalence of dynamic and static phase transitions (see [MN14] or [Yos03] a uniform LSI would also yield the absence of a phase transition and verify our conjecture (i.e. that the infinite-volume Gibbs measure is unique). Additionally, a uniform LSI is one of the main ingredients when deducing hydrodynamic limit of the Kawasaki dynamic via the two scale approach (see next paragraph). The uniform LSI for the canonical ensemble with no interaction is a well-known result (see for example [Cha03, LPY02, GOVW09]). For weak interaction the uniform LSI was deduced in [Men11]. In Chapter 6, we prove the uniform LSI for the canonical ensemble with strong interaction.

The strict convexity of the coarse-grained Hamiltonian also plays a crucial role when deducing the hydrodynamic limit of the Kawasaki dynamic. The hydrodynamic limit is a law of large numbers for processes. It states that under the correct scaling the Kawasaki dynamics (which is a stochastic process) converges to the solution of a non-linear heat equation (which is deterministic). It is conjectured by H.T. Yau that the hydrodynamic limit also holds for strong finite-range interactions on a one-dimensional lattice. In Chapter 7, we prove that this is indeed the case.

Let us now comment on how the C^2-equivalence of ensembles is deduced. The motivation for our approach comes from the proof of the local Cramér theorem in [GOVW09] and [Men11]. By using Cramér's trick of an exponential shift is suffices to show C^2-bounds on the density of a sum of random variables X_i (see also Proposition 3.2.7 below). Those desired bounds were derived [GOVW09] and [Men11] via a local central limit theorem (clt) for independent random variables. Our situation is a lot more subtle: Instead of deducing a local clt for independent random variables we would have to deduce a local clt for dependent random variables. At this point one could hope to use existing methods to deduce the local clt. Let us mention for example the approach of Dobrushin [Dob74], the approach of Bender [Ben73] or the approach of Wang and Woodroofe [WW90]. Unfortunately this does not help. All methods –at least the ones that are known to the author– use the following principle (see also [DM16]):

$$\text{integral clt} \quad + \quad \text{regularity} \ \Rightarrow \ \text{local clt.}$$

The first ingredient, namely the integral clt for the dependent random variables X_i is relatively easy to deduce. There are a lot of methods available. Let us mention for example Stein's method (see for example [Rai03]), methods that are based on mixing, or methods that are based on Donsker's theorem (see for example [Dur10]). Deducing the second ingredient is tricky, not to mention that Dobrushin [Dob74] carried out that step only for discrete or bounded random variables.

All in all, this approach has two fundamental problems. The first one is that we need not only to control the density itself but also the first and second derivative. As a consequence, one would need very detailed information about the regularity of the density. We also believe that showing this regularity is as hard as directly deducing the local central limit theorem. Let us turn to the second problem. In order to deduce Theorem 3.2.1 the local central limit theorem must be quantitative. Using the principle from above yields sub-optimal rates of convergence. For deducing Theorem 3.2.1 one has to iteratively apply the principle three times; and in each iteration the convergence rate gets worse. One would have to hope that in the end the convergence rate is still good enough for deducing Theorem 3.2.1.

Instead of using the principle from above, we generalize a well-known method for proving the local clt for independent random variables to dependent ones. We generalize the method that is based on characteristic functions and Fourier inversion (see [Fel71] and [GOVW09]). Calculations get quite evolved and lengthy. We do not deduce a local clt for dependent random variables in this work. Instead, we only deduce bounds that are needed to deduce Theorem 3.2.1 (cf. Proposition 3.2.7 below). However, one could use our calculations as a guideline for deducing a quantitative, local clt for dependent random variables. When doing so, one would have to substitute some of our arguments that use the specific structure of our lattice model. We use the following special structure:

- Exponential decay of correlations (see Theorem 1.2.6).

- The interaction has finite range R. More precisely, we use that two spins x_i and x_j become independent if $|i-j| > R$ and one conditions on the spin values $(x_k)_{i<k<j}$

between them (see Section 1.1.1).

- The Hamiltonian is quadratic. More precisely, we use the following consequence. For all $i \in \Lambda$ the conditional variances $\operatorname{var}_{\mu_\Lambda^\sigma}(X_i|X_j,\ |j-i| \leq l)$ is bounded from above and below uniformly in the values $X_j,\ |j-i| \leq l$. (see Section 1.1.1)

- Higher moments of X_i conditioned on $X_j,\ j \neq i$ are uniformly controlled by lower moments. This fact is used to show that the characteristic functions of X_i conditioned on $X_j,\ j \neq i$ have a uniform decay (see Corollary 2.1.2 and Lemma 2.1.4).

As mentioned before, the C^2-local Cramér theorem (see Theorem 3.2.4) is deduced by generalizing the argument of [GOVW09] for independent random variables to dependent random variables. This adds a lot more complexity to the task. We overcome the technical challenges of considering dependent random variables by using two strategies. The first strategy is to induce artificial independence by conditioning on even or odd random variables. The second strategy is to handle dependencies as a perturbation. We morally treat large blocks as single sites of a coarse-grained system. Because there is a big distance between the blocks, the blocks are only weakly dependent. Then, the error term can be controlled by using the decay of correlations. For more details we refer to the comments after Proposition 3.2.7 and at the beginning of Section 3.3.

3.2 Main results and outline of the proof

In the remaining sections of this chapter, we only consider the one-dimensional lattice $\Lambda = [N] = \{1, 2, \cdots, N\}$ with nearest-neighbor interactions for better explanation of our ideas and calculations. We follow the conventions made at the end of Section 1.1.1.

3.2.1 Main Results

The first main result of this chapter is the equivalence of the free energies in C^2.

Theorem 3.2.1 (Equivalence of the ce and gce on the level of free energy)**.** *It holds that*

$$\lim_{N\to\infty} |A_N^{gce} - A_N^{ce}|_{C^2} = 0,$$

where the convergence is uniform in the mean spin m. More precisely, given a constant $\varepsilon > 0$, there is an integer $N_0 \in \mathbb{N}$ such that if $N \geq N_0$,

$$\sup_{\sigma\in\mathbb{R}} |A_N^{gce}(\sigma) - A_N^{ce}(\sigma)| \lesssim \frac{1}{N}, \tag{3.2.1}$$

$$\sup_{\sigma\in\mathbb{R}} \left| \frac{d}{d\sigma} A_N^{gce}(\sigma) - \frac{d}{d\sigma} A_N^{ce}(\sigma) \right| \lesssim \frac{1}{N^{1-\varepsilon}}, \tag{3.2.2}$$

$$\sup_{\sigma\in\mathbb{R}} \left| \frac{d^2}{d\sigma^2} A_N^{gce}(\sigma) - \frac{d^2}{d\sigma^2} A_N^{ce}(\sigma) \right| \lesssim \frac{1}{N^{\frac{1}{2}-\varepsilon}}. \tag{3.2.3}$$

The proof of Theorem 3.2.1 is stated in Section 3.2.2. A direct consequence of Corollary 2.1.5 and Theorem 3.2.1 is that the free energy A_N^{ce} is uniformly strictly convex for large enough systems.

Corollary 3.2.2. *There is a uniform constant $0 < C < \infty$ and an integer $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and all $\sigma \in \mathbb{R}$*

$$\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_N^{ce}(\sigma) \leq C.$$

Let us turn to the second main result of this section, the local Cramér theorem. For that purpose let us introduce $\mathcal{H}_N$ which denotes the Legendre transform of the free energy A_N^{gce} i.e.

$$\mathcal{H}_N(m) = \sup_{\sigma\in\mathbb{R}} \left(\sigma m - A_N^{gce}(\sigma)\right).$$

It follows from elementary observations that $\mathcal{H}_N$ is uniformly strictly convex.

Lemma 3.2.3. *For any $m \in \mathbb{R}$*

$$\mathcal{H}_N(m) = \sigma(m)m - A_N^{gce}(\sigma(m)).$$

Additionally, it holds that $\mathcal{H}_N$ is uniformly strictly convex in the sense that there is a uniform constant $0 < C < \infty$ such that for all $\sigma \in \mathbb{R}$

$$\frac{1}{C} \leq \frac{d^2}{dm^2}\mathcal{H}_N(m) \leq C.$$

The coarse-grained Hamiltonian $\bar{H}_N : \mathbb{R} \to \mathbb{R}$ is defined as

$$\bar{H}_N(m) = -\frac{1}{N} \log \int_{\left\{\frac{1}{N}\sum_{i=1}^N x_i = m\right\}} \exp(-H_N(x))\mathcal{L}^{N-1}(dx).$$

Hence, we can rewrite the free energy of the ce as

$$A_N^{ce}(\sigma) = \sigma m - \bar{H}_N(m). \tag{3.2.4}$$

It follows that the difference of the free energies A_N^{gce} and A_N^{ce} can be expressed as

$$\begin{aligned} A_N^{gce}(\sigma) - A_N^{ce}(\sigma) &= A_N^{gce}(\sigma) - \sigma m + \bar{H}_N(m) \\ &= \bar{H}_N(m) - \mathcal{H}_N(m). \end{aligned} \tag{3.2.5}$$

From Theorem 3.2.1 we deduce the following local Cramér theorem.

Theorem 3.2.4 (C^2-local Cramér theorem)**.** *It holds that*

$$\lim_{N\to\infty} \left|\bar{H}_N(m) - \mathcal{H}_N(m)\right|_{C^2} = 0,$$

where the convergence is uniform in the mean spin m and the external field s. More precisely, given a constant $\varepsilon > 0$, there is an integer $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\sup_{m\in\mathbb{R}} \left|\bar{H}_N(m) - \mathcal{H}_N(m)\right| \lesssim \frac{1}{N}, \tag{3.2.6}$$

$$\sup_{m\in\mathbb{R}} \left|\frac{d}{dm}\bar{H}_N(m) - \frac{d}{dm}\mathcal{H}_N(m)\right| \lesssim \frac{1}{N^{1-\varepsilon}}, \tag{3.2.7}$$

$$\sup_{m\in\mathbb{R}} \left|\frac{d^2}{dm^2}\bar{H}_N(m) - \frac{d^2}{dm^2}\mathcal{H}_N(m)\right| \lesssim \frac{1}{N^{\frac{1}{2}-\varepsilon}}. \tag{3.2.8}$$

Theorem 3.2.4 is an extension of the local Cramér theorems that were deduced in Proposition 31 in [GOVW09], Theorem 4 in [Men11] and [MO13]. The proof of Theorem 3.2.4 is stated in Section 3.2.2. The main ingredient is Theorem 3.2.1.

An important consequence of Lemma 3.2.3 and of Theorem 3.2.4 is that for large enough systems the coarse-grained Hamiltonian $\bar{H}_N$ is uniformly strictly convex.

Corollary 3.2.5. *There is a positive integer N_0 such that for all $N \geq N_0$ the coarse-grained Hamiltonian $\bar{H}_N : \mathbb{R} \to \mathbb{R}$ is uniformly strictly convex. More precisely, there is a uniform constant $0 < C < \infty$ such that for all $m \in \mathbb{R}$*

$$\frac{1}{C} \leq \frac{d^2}{dm^2} \bar{H}_N(m) \leq C.$$

3.2.2 Outline of the proof of main results

In this section we outline the proof of the main results of this chapter. Let us begin with the Cramér's trick of exponential shift of measures.

Lemma 3.2.6. *It holds that*

$$\begin{aligned} g_{N,m}(0) &= \exp\left(N A_N^{ce}(\sigma) - N A_N^{gce}(\sigma)\right) \\ &\overset{(3.2.5)}{=} \exp\left(N\mathcal{H}_N(m) - N\bar{H}_N(m)\right). \end{aligned} \tag{3.2.9}$$

Here, $g_{N,m}$ denotes the distribution of

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - m),$$

where $X = (X_1, \cdots, X_N)$ is the random variable distributed according to μ_N^σ.

Proof of Lemma 3.2.6. The lemma follows from a direct computation:

$$\begin{aligned} &K\mathcal{H}_N(m) - N\bar{H}_N(m) \\ &\quad = N\sigma(m)m - \log \int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^{N} \sigma(m) x_i - H_N(x)\right) dx \\ &\qquad + \log \int_{\left\{\frac{1}{N}\sum_{i=1}^{N} x_i = m\right\}} \exp\left(-H_N(x)\right) \mathcal{L}^{N-1}(dx) \end{aligned}$$

$$
\begin{aligned}
&= \log \int_{\left\{\frac{1}{N}\sum_{i=1}^N x_i = m\right\}} \exp\left(N\sigma(m)m - H_N(x)\right) \mathcal{L}^{N-1}(dx) \\
&\quad - \log \int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^N \sigma(m)x_i - H_N(x)\right) dx \\
&= \log \frac{\int_{\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^N (x_i - m) = 0\right\}} \exp\left(\sigma(m)\sum_{i=1}^N x_i - H_N(x)\right) \mathcal{L}^{N-1}(dx)}{\int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^N \sigma(m)x_i - H_N(x)\right) dx} \\
&= \log g_{N,m}(0).
\end{aligned}
$$

Taking the exponential function and using (3.2.5) yields the lemma as desired. □

Now, we get to the core estimates needed for the proof of Theorem 3.2.1.

Proposition 3.2.7. *For each $\alpha > 0$ and $\beta > \frac{1}{2}$, there exists a uniform constant $0 < C < \infty$ and an integer $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and all $\sigma \in \mathbb{R}$*

$$
\frac{1}{C} \leq g_{N,m}(0) \leq C, \tag{3.2.10}
$$

$$
\left| \frac{d}{d\sigma} g_{N,m}(0) \right| \lesssim N^{\alpha}, \tag{3.2.11}
$$

$$
\left| \frac{d^2}{d\sigma^2} g_{N,m}(0) \right| \lesssim N^{\beta}. \tag{3.2.12}
$$

The statement of Proposition 3.2.7 should be compared to Proposition 31 in [GOVW09] or Proposition 3.1 in [MO13]. The main difference is that in our situation the random variables $X_1, \ldots, X_N$ are dependent. This also makes the proof of Proposition 3.2.7 a lot harder.

The estimates of Proposition 3.2.7 are motivated from deducing a quantitative local central limit theorem for the properly normalized sum of the random variables $X_1, \ldots, X_N$. For example, if the random variables $X_1, \ldots, X_N$ are iid, the estimate (3.2.10) is a weaker version of the quantitative local clt estimate

$$
\left| g_{N,m}(0) - \frac{1}{\sqrt{2\pi}} \right| \lesssim \frac{1}{\sqrt{N}}.
$$

The last inequality states that the density of the normalized sum at point 0 converges to the density of the normal distribution. As we mentioned in the introduction, we believe that one could strengthen the estimates of Proposition 3.2.7 to get a local central limit theorem for dependent random variables. However, we choose to derive weaker bounds instead because they are sufficiently strong for deducing our main results (see Theorem 3.2.1 and Theorem 3.2.4). Deducing those weaker estimates is already quite subtle and challenging.

We deduce Proposition 3.2.7 in Section 3.3. There, we also comment on how to overcome the problem of considering dependent random variables and not independent ones. Now, we are prepared for the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. Let us begin with the estimate (3.2.1). From (3.2.9),

$$A_N^{ce}(\sigma) - A_N^{gce}(\sigma) = \frac{1}{N} \log g_{N,m}(0). \tag{3.2.13}$$

Then a combination of (3.2.10) and (3.2.13) yields, as desired,

$$|A_N^{ce}(\sigma) - A_N^{gce}(\sigma)| \lesssim \frac{1}{N}.$$

Let us turn to the estimate (3.2.2). Taking a derivative with respect to σ in (3.2.13) yields

$$\frac{d}{d\sigma} A_N^{ce}(\sigma) - \frac{d}{d\sigma} A_N^{gce}(\sigma) = \frac{1}{N} \frac{1}{g_{N,m}(0)} \frac{d g_{N,m}(0)}{d\sigma}. \tag{3.2.14}$$

Let us choose $\alpha = \varepsilon$. Then a combination of (3.2.10), (3.2.11) and (3.2.14) implies

$$\left| \frac{d}{d\sigma} A_N^{ce}(\sigma) - \frac{d}{d\sigma} A_N^{gce}(\sigma) \right| \lesssim \frac{1}{N} N^{\alpha} = \frac{1}{N^{1-\varepsilon}}.$$

Let us turn to the estimate (3.2.3). Differentiating (3.2.14) again, we get

$$\frac{d^2}{d\sigma^2} A_N^{ce}(\sigma) - \frac{d^2}{d\sigma^2} A_N^{gce}(\sigma) = -\frac{1}{N} \frac{1}{(g_{N,m}(0))^2} \left(\frac{d g_{N,m}(0)}{d\sigma} \right)^2 + \frac{1}{N} \frac{1}{g_{N,m}(0)} \frac{d^2 g_{N,m}(0)}{d\sigma^2}.$$

Then after choosing $\beta = \frac{1}{2} + \varepsilon$, a combination of (3.2.10), (3.2.11) and (3.2.12) yields

$$\left| \frac{d^2}{d\sigma^2} A_N^{ce}(\sigma) - \frac{d^2}{d\sigma^2} A_N^{gce}(\sigma) \right| \lesssim \frac{1}{N^{1-2\alpha}} + \frac{1}{N^{1-\beta}} \lesssim \frac{1}{N^{\frac{1}{2}-\varepsilon}}.$$

□

Let us proceed to the proof of Lemma 3.2.3.

PROOF OF LEMMA 3.2.3. Recall the definition $m_i = \mathbb{E}_{\mu_N^\sigma}[X_i]$. Since $\mathcal{H}_N(m)$ is the Legendre transform of the strict convex function $A_N^{gce}(\sigma)$, there exists a unique $\tilde{\sigma} = \tilde{\sigma}(m)$ such that

$$\mathcal{H}_N(m) = \tilde{\sigma}(m)m - A_N^{gce}(\tilde{\sigma}).$$

Moreover, for each m, $\tilde{\sigma}$ satisfies

$$\frac{d}{d\sigma}\left(\sigma m - A_N^{gce}(\sigma)\right)\Big|_{\sigma=\tilde{\sigma}} = 0,$$

which is equivalent to

$$m = \frac{d}{d\sigma}A_N^{gce}(\sigma)\Big|_{\sigma=\tilde{\sigma}}.$$

Therefore, $\tilde{\sigma}(m) = \sigma(m)$ from definition 2.1.6. Now it follows that

$$\begin{aligned}
\frac{d}{dm}\mathcal{H}_N(m) &= \frac{d}{dm}\left(\sigma(m)m - A_N^{gce}(\sigma(m))\right) \\
&= \frac{d\sigma(m)}{dm}m + \sigma(m) - \frac{d}{d\sigma}A_N^{gce}(\sigma)\cdot\frac{d\sigma}{dm} \\
&= \frac{d\sigma}{dm}m + \sigma - \frac{1}{N}\mathbb{E}_{\mu_N^\sigma}\left[\sum_{i=1}^N X_i\right]\cdot\frac{d\sigma}{dm} = \sigma.
\end{aligned}$$

We also note that

$$\frac{d}{d\sigma}m = \frac{d}{d\sigma}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{\mu_N^\sigma}[X_i]\right) = \frac{1}{N}\operatorname{var}_{\mu_N^\sigma}\left(\sum_{i=1}^N X_i\right). \tag{3.2.15}$$

Thus Lemma 2.1.4 implies there exists a constant $C > 0$ with

$$\frac{1}{C} \le \frac{d^2}{dm^2}\mathcal{H}_N(m) = \frac{d\sigma}{dm} = \left(\frac{dm}{d\sigma}\right)^{-1} \le C.$$

□

Before we move on to the proof of Theorem 3.2.4, let us deduce some auxiliary results.

Lemma 3.2.8. *It holds that*

$$\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right] = \mathbb{E}_{\mu_N^\sigma}\left[\frac{df}{d\sigma}(X)\right] + \mathbb{E}_{\mu_N^\sigma}\left[f(X)\left(\sum_{i=1}^N (X_i - m_i)\right)\right]. \tag{3.2.16}$$

In particular, we have

$$\frac{d}{d\sigma}m_i = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_i)\left(\sum_{j=1}^N (X_j - m_j)\right)\right] \tag{3.2.17}$$

Proof of Theorem 3.2.8. A straightforward calculation yields

$$\begin{aligned}
\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right] &= \frac{d}{d\sigma}\int f(x)\frac{\exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)}{\int \exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)dx}dx \\
&= \int \frac{df}{d\sigma}(x)\frac{\exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)}{\int \exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)dx}dx \\
&\quad + \int f(x)\left(\sum_{i=1}^N (x_i - m_i)\right)\frac{\exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)}{\int \exp\left(\sigma\sum_{i=1}^N x_i - H_N(x)\right)dx}dx \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\frac{df}{d\sigma}(X)\right] + \mathbb{E}_{\mu_N^\sigma}\left[f(X)\left(\sum_{i=1}^N (X_i - m_i)\right)\right].
\end{aligned}$$

As a consequence, it also holds that

$$\frac{d}{d\sigma}m_i = \frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[X_i\right] = \mathbb{E}_{\mu_N^\sigma}\left[X_i\left(\sum_{j=1}^N (X_j - m_j)\right)\right] = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_i)\left(\sum_{j=1}^N (X_j - m_j)\right)\right].$$

□

Now we are ready to give a proof of Theorem 3.2.4.

Proof of Theorem 3.2.4. Recall the difference of the free energies (3.2.5) of A_N^{gce} and A_N^{ce}

$$A_N^{gce}(\sigma) - A_N^{ce}(\sigma) = \bar{H}_N(m) - \mathcal{H}_N(m).$$

Then the first desired estimate (3.2.6) follows from a combination of (3.2.5) and (3.2.1) in Theorem 3.2.1. Let us turn to the estimate (3.2.7). A direct computation yields

$$\frac{d}{dm}\left(\mathcal{H}_N(m) - \bar{H}_N(m)\right) = \frac{d}{dm}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)$$

$$
\begin{aligned}
&= \frac{d}{d\sigma}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\frac{d\sigma}{dm} \\
&= \frac{d}{d\sigma}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\left(\frac{dm}{d\sigma}\right)^{-1}.
\end{aligned} \tag{3.2.18}
$$

Then (3.2.2), (3.2.15) and Lemma 2.1.4 implies, as desired,

$$
\left|\frac{d}{dm}\left(\mathcal{H}_N(m) - \bar{H}_N(m)\right)\right| \lesssim \frac{1}{N^{1-\varepsilon}}.
$$

Before we proceed to the proof of (3.2.8), we note

$$
\begin{aligned}
&\frac{d}{d\sigma}\left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^2\right]\right) \\
&\overset{(3.2.16)}{=} \mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}\left(\sum_{i=1}^N (X_i - m_i)\right)^2\right] + \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right] \\
&\overset{(3.2.17)}{=} \mathbb{E}_{\mu_N^\sigma}\left[2\left(\sum_{i=1}^N (X_i - m_i)\right)\left(-\sum_{i=1}^N \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_i)\sum_{k=1}^N (X_k - m_k)\right]\right)\right] \\
&\qquad + \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right] \\
&= -2\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{j=1}^N (X_j - m_j)\right)^2\right]\mathbb{E}_{\mu_N^\sigma}\left[\sum_{i=1}^N (X_i - m_i)\right] + \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right] \\
&= 0 + \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right] = \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right].
\end{aligned} \tag{3.2.19}
$$

and thus

$$
\begin{aligned}
\left|\frac{d}{d\sigma}\left(\frac{d\sigma}{dm}\right)\right| &\overset{(3.2.15)}{=} \left|\frac{d}{d\sigma}\left(\frac{N}{\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^2\right]}\right)\right| \\
&= \left|-\frac{N}{\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^2\right]^2}\frac{d}{d\sigma}\left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^2\right]\right)\right| \\
&\overset{(3.2.19)}{=} \left|-\frac{N}{\left(\operatorname{var}\left(\sum_{i=1}^N X_i\right)\right)^2}\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i=1}^N (X_i - m_i)\right)^3\right]\right|
\end{aligned}
$$

$$\overset{Lemma 2.1.4, Lemma\ 2.1.7}{\leq} \frac{N}{N^2} N = 1. \tag{3.2.20}$$

Let us turn to the estimate (3.2.8). We differentiate (3.2.18) to obtain

$$\begin{aligned}
&\frac{d^2}{dm^2}\left(\mathcal{H}_N(m) - \bar{H}_N(m)\right) \\
&\quad = \frac{d}{dm}\left(\frac{d}{d\sigma}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\frac{d\sigma}{dm}\right) \\
&\quad = \frac{d}{d\sigma}\left(\frac{d}{d\sigma}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\frac{d\sigma}{dm}\right)\frac{d\sigma}{dm} \\
&\quad = \frac{d^2}{d\sigma^2}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\left(\frac{d\sigma}{dm}\right)^2 + \frac{d}{d\sigma}\left(A_N^{ce}(\sigma) - A_N^{gce}(\sigma)\right)\frac{d}{d\sigma}\left(\frac{d\sigma}{dm}\right)\frac{d\sigma}{dm}.
\end{aligned}$$

Then a combination of (3.2.15), (3.2.20), Theorem 3.2.1 and Lemma 2.1.4 yields

$$\left|\frac{d^2}{dm^2}\left(\mathcal{H}_N(m) - \bar{H}_N(m)\right)\right| \lesssim \frac{1}{N^{\frac{1}{2}-\varepsilon}},$$

and this finishes the proof of Theorem 3.2.4. □

3.3 Proof of Proposition 3.2.7

The proof of Proposition 3.2.7 represents the core of our argument. Before turning to the precise argument let us motivate and explain our approach in more detail. We will especially emphasize on how the problem of considering dependent and not independent random variables X_i is solved.

As we mentioned before, the argument is inspired from deducing local central limit theorems via the Fourier inversion method (see [Fel71] or [MO13]). The main idea of this method is to write the the density of the random variable

$$Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$$

by Fourier inversion as an integral involving the characteristic function (see (3.3.23) below)

$$\varphi_Z(\xi) = \mathbb{E}_{\mu_N^\sigma}\left[\exp(i\xi Z)\right].$$

The next step is to split up the integral into an inner integral over the interval $|\xi| \leq \delta\sqrt{N}$ and an outer integral over the interval $|\xi| > \delta\sqrt{N}$. The outer integral usually is an error term and the main contribution comes from the inner integral.

The big advantage of considering independent random variables X_i is that the characteristic function φ_Z becomes a product of the characteristic functions φ_{X_i} i.e.

$$\varphi_Z(\xi) = \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_i\right)\right] = \prod_{i=1}^{N}\varphi_{X_i}\left(\frac{\xi}{\sqrt{N}}\right).$$

Then the outer integral is small because each characteristic function $\varphi_{X_i} < 1$ is small and decays at least of the order $|\xi|^{-1}$. For the inner integral, one applies a Taylor expansion onto the functions $\log \varphi_{X_i}$ and gets the correct contribution due to the normalization of the random variables. This strategy would yield the desired estimate (3.2.10) in the case of independent random variables (see also Section 3 in [MO13]).

For deducing the estimates (3.2.11) and (3.2.12) in the case of independent random variables one proceeds in a similar way. The obtained integral representation is split up into an inner and an outer integral. One shows that the outer integral is small by using decay of the characteristic functions. The inner integral is estimated again by Taylor expansion. However, the situation becomes more subtle when considering dependent random variables. The obtained integral representation involves several new terms that look like covariances i.e. they are covariances if $\xi = 0$. Therefore, we have to be a lot more careful when applying this strategy.

The following observation helps a lot when considering dependent random variables: Because we only consider nearest-neighbor interaction, the odd random variables X_{odd} become independent if we condition on the values of the even random variables X_{even}. Additionally, because our Hamiltonian is quadratic the variances of the conditioned random variables $X_{odd}|X_{even}$ are uniformly bounded from above and from below (see proof of Lemma 3.3.1).

Using this observation the outer integral can be estimated in a straight-forward manner. We condition on the even random variables X_{even}. By conditional independence we get that the conditional characteristic function becomes a product i.e.

$$\begin{aligned}
\varphi_Z(\xi) &= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_i\right)\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}\sum_{j:even} X_j\right)\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}\sum_{i:odd} X_i\right)|X_j, j:even\right]\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}\sum_{j:even} X_j\right)\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}X_i\right)|X_j, j:even\right]\right].
\end{aligned}$$

Because the variances of the conditional random variables $X_{odd}|X_{even}$ are controlled uniformly in the conditioned values X_{even} we have that the conditional characteristic functions

$$\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\xi\frac{1}{\sqrt{N}}X_i\right)|X_j, j:even\right]$$

decay uniformly (see Lemma 3.3.1 below). Over-simplifying the argument, this yields the correct bounds on the outer integrals.

The situation for the inner integrals is more tricky and one has to proceed differently for the estimate (3.2.10) and for the estimates (3.2.11) and (3.2.12). Let us first consider the argument for (3.2.10). In the inner integral, we condition on the even random variables X_{even}. We use the conditional independence and the control on the conditional variances to do a Taylor expansion just for the characteristic functions of the conditional random variables $X_{odd}|X_{even}$. Then we show that this suffices to get the desired estimate of (3.2.10).

Let us turn to (3.2.11) and (3.2.12) and explain how the inner integrals are estimated there. As mentioned above, the Taylor expansion becomes a lot more tricky than for (3.2.10). For each additional derivative, the argument becomes more and more elaborate. The reason is that whenever calculating the inner and outer integral one ends up with more and more error terms. The first step of the argument is to carefully group those terms such that certain terms cancel and other terms become covariance-like. This means that those terms are a covariance if $\xi = 0$. However, if $\xi \neq 0$ they are not covariances and cannot be estimated by the decay of correlations. We are able to estimate those terms using the following idea: For each error term, we partition the sites of the lattice system into blocks (see Figure 3.1 and Figure 3.2 below). Then we carry out a multivariate Taylor expansion. Let's say we expand the function $F(\xi_1, \xi_2)$ and after expanding we set $\xi_1 = \xi$ and $\xi_2 = \xi$. The variable ξ_1 corresponds to the sites within the block and the variable ξ_2 corresponds to terms outside of the block. We carry out the Taylor expansion with respect to ξ_1. The resulting terms are controlled either by the help of decay of correlations or by the size of the blocks.

The proof is organized in the following way. In Section 3.3.1 we deduce auxiliary estimates for the conditional characteristic functions. In Section 3.3.2 we deduce the estimate (3.2.10). In Section 3.3.3 we verify the estimate (3.2.12). The estimate (3.2.11) can be derived by similar arguments. In Section 3.3.4 we deduce auxiliary lemmas used in Section 3.3.3.

3.3.1 Auxiliary estimates

In this section we provide some auxiliary estimates that are needed in the proof of Proposition 3.2.7. Let us introduce the auxiliary sets (cf. Figure 3.1)

$$E_1^l := \{k \mid |k-l| \leq L\} \tag{3.3.1}$$

$$E_2^l := \{k \mid |k-l| > L\} \tag{3.3.2}$$

and (cf. Figure 3.2)

$$F_1^{n,l} := \{k \mid |k-n| \leq L \text{ or } |k-l| \leq L\} \tag{3.3.3}$$

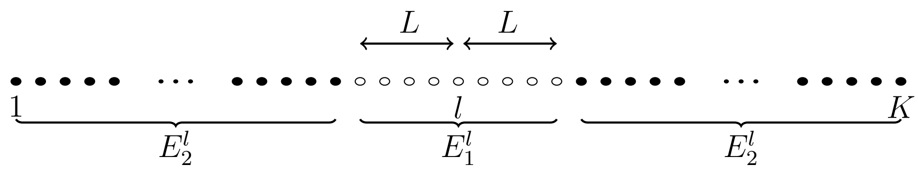

Figure 3.1: The sets E_1^l and E_2^l given by (3.3.1) and (3.3.2)

$$F_2^{n,l} := \{k \mid |k-n| > L \text{ and } |k-l| > L\}, \tag{3.3.4}$$

where $L \ll N$ is a positive integer that will be chosen later.

Let us also introduce the auxiliary notations. Let us denote $m_{i,2}$ and $s_{i,2}^2$ to be

$$m_{i,2} := \mathbb{E}_{\mu_N^\sigma}\left[X_i \mid X_j, j : even\right], \tag{3.3.5}$$

$$s_{i,2}^2 := \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^2 \mid X_j, j : even\right]. \tag{3.3.6}$$

Define the function $\hat{e} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{e}(\xi) := \exp\left(i\frac{1}{\sqrt{N}}\sum_{j:even}(X_j - m_j)\xi + i\frac{1}{\sqrt{N}}\sum_{i:odd}(m_{i,2} - m_i)\xi\right). \tag{3.3.7}$$

Then we have the following lemma:

Lemma 3.3.1. *For large enough N and $\delta > 0$ small enough, there exists a positive constant $C > 0$ such that the following inequalities hold for all $\xi \in \mathbb{R}$ with $\frac{|\xi|}{\sqrt{N}} \leq \delta$.*

$$\Bigg|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{k=1}^{N}(X_k - m_k)\frac{\xi}{\sqrt{N}}\right)\right] - \mathbb{E}_{\mu_N^\sigma}\left[\hat{e}(\xi)\exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)\right]\Bigg| \lesssim \frac{1}{\sqrt{N}}|\xi|^3\exp\left(-C\xi^2\right), \tag{3.3.8}$$

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{k\in E_2^l}(X_k - m_k)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l\right]\right| \lesssim \left(1+\xi^2\right)\exp\left(-C\xi^2\right), \tag{3.3.9}$$

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{k\in F_2^{n,l}}(X_k - m_k)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{G}_{n,l}\right]\right| \lesssim \left(1+\xi^2\right)\exp\left(-C\xi^2\right). \tag{3.3.10}$$

where $\mathscr{F}_l$, $\mathscr{G}_{n,l}$ *denote the sigma algebras defined by*

$$\mathscr{F}_l := \sigma\left(X_k,\ k \in E_1^l\right) \qquad \textit{and} \qquad \mathscr{G}_{n,l} := \sigma\left(X_k,\ k \in F_1^{n,l}\right).$$

Lemma 3.3.1 will be used in the estimation of the inner integrals when deriving (3.2.10), (3.2.11) and (3.2.12).

Proof of Lemma 3.3.1. We first deduce (3.3.8). Let us consider the conditional expectation with respect to $\{X_j \mid j : even\}$. In the case of nearest-neighbor interaction, the conditional Lebesgue density $\mu_N^\sigma(dx_1 dx_3 \cdots \mid x_j,\ j : even)$ can be written as

$$\begin{aligned}
&\mu_N^\sigma(dx_1 dx_3 \cdots \mid x_j,\ j : even) \\
&\qquad = \frac{1}{Z} \exp\left(\sum_{i:odd} -\psi(x_i) - (-M_{i,i-1}x_{i-1} - M_{i,i+1}x_{i+1} + s_i - \sigma)x_i\right) \\
&\qquad = \prod_{i:odd} \frac{1}{Z} \exp\left(-\psi(x_i) - (-M_{i,i-1}x_{i-1} - M_{i,i+1}x_{i+1} + s_i - \sigma)x_i\right), \qquad (3.3.11)
\end{aligned}$$

which implies that $\{\mathbb{E}_{\mu_N^\sigma}[X_i \mid X_j,\ j : even] \mid i : odd\}$ are independent and in particular,

$$\mu_N^\sigma\left(dx_i | x_2, x_4, \cdots\right) = \frac{1}{Z} \exp\left(-\psi(x_i) - (-M_{i,i-1}x_{i-1} - M_{i,i+1}x_{i+1} + s_i - \sigma)x_i\right). \qquad (3.3.12)$$

Note that

$$\begin{aligned}
&\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[X_i \mid X_j, j : even\right] \\
&\qquad = \frac{d}{d\sigma} \int x_i \cdot \frac{1}{Z} \exp\left(-\psi(x_i) - (-M_{i,i-1}x_{i-1} - M_{i,i+1}x_{i+1} + s_i - \sigma)x_i\right) dx \\
&\qquad = \int x_i \left(x_i - m_{i,2}\right) \cdot \frac{1}{Z} \exp\left(-\psi(x_i) - (-M_{i,i-1}x_{i-1} - M_{i,i+1}x_{i+1} + s_i - \sigma)x_i\right) dx \\
&\qquad = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^2 \mid X_j, j : even\right] = s_{i,2}^2, \qquad (3.3.13)
\end{aligned}$$

and a similar computation yields

$$\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[X_i \mid X_j, j : even\right] = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^3 \mid X_j, j : even\right] =: t_{i,2}. \qquad (3.3.14)$$

From the observations (3.3.11) and (3.3.12) from above, we get the product structure of conditional expectation

$$\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\left(X_k - m_k\right)\xi\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k:even}(X_k - m_k)\,\xi\right)\right.\\
&\qquad\qquad \left.\times\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i:odd}(X_i - m_i)\,\xi\right) \mid X_j, j : even\right]\right]\\
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k:even}(X_k - m_k)\,\xi + i\frac{1}{\sqrt{N}}\sum_{i:odd}(m_{i,2} - m_i)\,\xi\right)\right.\\
&\qquad\qquad \left.\times\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i:odd}(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right]\right]\\
&\overset{(3.3.7)}{=} \mathbb{E}_{\mu_N^\sigma}\left[\hat{e}\,(\xi)\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right]\right].
\end{aligned}
\tag{3.3.15}
$$

Let h_i denote the complex valued function

$$
h_i(\xi) := -\log\left(\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\,(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right]\right).
$$

Equivalently, denote

$$
F_i(\xi) : = \exp\left(-h_i(\xi)\right) = \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\,(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right].
$$

Differentiating both sides. we have

$$
\begin{aligned}
F_i'(\xi) &= -h_i'(\xi)\exp\left(-h_i(\xi)\right)\\
&\quad = i\mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})\exp\left(i\,(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right],
\end{aligned}
$$

$$
\begin{aligned}
F_i''(\xi) &= -h_i''(\xi)\exp\left(-h_i(\xi)\right) + h_i'(\xi)^2\exp\left(-h_i(\xi)\right)\\
&= -\mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^2\exp\left(i\,(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right],
\end{aligned}
$$

$$
\begin{aligned}
F_i'''(\xi) &= -h_i'''(\xi)\exp\left(-h_i(\xi)\right) + 3h_i''(\xi)h_i'(\xi)\exp\left(-h_i(\xi)\right)\\
&\quad - \left(h_i'(\xi)\right)^3\exp\left(-h_i(\xi)\right)\\
&= -i\mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^3\exp\left(i\,(X_i - m_{i,2})\,\xi\right) \mid X_j, j : even\right],
\end{aligned}
\tag{3.3.16}
$$

which implies $h_i(0) = h_i'(0) = 0$ and

$$
h_i''(0) = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^2 \mid X_j, j : even\right] \overset{(3.3.6)}{=} s_{i,2}^2.
$$

Before we proceed, let us note that analogue of Corollary 2.1.2 holds for the conditional expectation. More precisely, because the conditional measures $\mu_N^\sigma(dx_i|x_2, x_4, \cdots)$ are bounded perturbations of a one dimensional Gaussian measure, it holds that for each $n \in \mathbb{N}$ there is a constant $C(n)$ with

$$\mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}|^n \mid X_j, j : even\right] \leq C(n). \tag{3.3.17}$$

Note also that (3.3.17) implies

$$|F_i'(\xi)| \leq \mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}| \mid X_j, j : even\right] \lesssim 1.$$

Combined with the fact that $F_i(0) = 1$ we have for $|\xi|$ small enough

$$\frac{1}{2} \leq |F_i(\xi)| = |\exp(h_i(\xi))| \leq \frac{3}{2}.$$

Inserting this into (3.3.16), we obtain

$$\begin{aligned}
&|h_i'''(\xi)| \\
&\quad = \left|3h_i''(\xi)h_i'(\xi) - (h_i'(\xi))^3\, i\mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})^3 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j : even\right]\right| \\
&\qquad\quad \times |\exp(h_i(\xi))| \\
&\quad \lesssim \mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}|^2 \mid X_j, j : even\right] \cdot \mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}| \mid X_j, j : even\right] \\
&\qquad + \left(\mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}| \mid X_j, j : even\right]\right)^3 + \mathbb{E}_{\mu_N^\sigma}\left[|X_i - m_{i,2}|^3 \mid X_j, j : even\right] \\
&\quad \overset{(3.3.17)}{\lesssim} 1.
\end{aligned}$$

We thus have for $|\xi|$ small,

$$\left|h_i(\xi) - \frac{1}{2}s_{i,2}^2\xi^2\right| \lesssim |\xi|^3,$$

Summing up for all odd i's, we have

$$\left|\sum_{i:odd} h_i\left(\frac{\xi}{\sqrt{N}}\right) - \sum_{i:odd}\frac{1}{2N}s_{i,2}^2\xi^2\right| \lesssim \frac{1}{\sqrt{N}}|\xi|^3. \tag{3.3.18}$$

Note that for odd i's, (3.3.11), (3.3.12) and [Men11, Lemma 9] imply $s_{i,2}^2$ is uniformly bounded above and below. That is, there exists a uniform constant $C_{(3.3.19)} > 0$ such that

$$\frac{1}{C_{(3.3.19)}} \leq s_{i,2}^2 \leq C_{(3.3.19)}. \tag{3.3.19}$$

As a consequence, there is a constant $C > 0$ with

$$\frac{1}{C} \leq \sum_{i:odd} \frac{1}{2N} s_{i,2}^2 \leq C. \tag{3.3.20}$$

In particular for K large enough and $\left|\frac{\xi}{\sqrt{N}}\right| \leq \delta$, the estimate (3.3.18) yields

$$\mathrm{Re}\left(\sum_{i:odd} h_i\left(\frac{\xi}{\sqrt{N}}\right)\right) \geq \sum_{i:odd} \frac{s_{i,2}^2}{4N}\xi^2.$$

Furthermore, Lipschitz continuity of complex function $y \mapsto \exp(y) \in \mathbb{C}$ on $\mathrm{Re} y \leq -\sum_{i:odd} \frac{s_{i,2}^2}{4N}\xi^2$ yields

$$\begin{aligned}
&\left|\exp\left(-\sum_{i:odd} h_i\left(\frac{\xi}{\sqrt{N}}\right)\right) - \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)\right| \\
&\qquad \lesssim \frac{1}{\sqrt{N}}|\xi|^3 \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{4N}\xi^2\right).
\end{aligned} \tag{3.3.21}$$

Then a combination of (3.3.15), (3.3.20) and (3.3.21) implies the desired estimate (3.3.8).

Let us now address the estimate (3.3.9). The proof for this case is almost identical to (3.3.8). Let us denote $\mathscr{F}_l^2 := \sigma\left(X_k, k \in E_1^l \text{ or } k : even\right)$. Then we have

$$\begin{aligned}
&\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{k\in E_2^l}(X_k - m_k)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{\substack{k\in E_2^l\\ k:even}}(X_k - m_k)\frac{\xi}{\sqrt{N}} + i\sum_{\substack{i\in E_2^l\\ i:odd}}\left(\mathbb{E}_{\mu_N^\sigma}\left[X_i \mid \mathscr{F}_l^2\right] - m_i\right)\frac{\xi}{\sqrt{N}}\right)\right. \\
&\qquad \left.\times\, \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{\substack{i\in E_2^l\\ i:odd}}\left(X_i - \mathbb{E}_{\mu_N^\sigma}\left[X_i \mid \mathscr{F}_l^2\right]\right)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l^2\right] \mid \mathscr{F}_l\right].
\end{aligned} \tag{3.3.22}$$

Let us also denote $\tilde{s}_{i,2}^2 := \mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - \mathbb{E}_{\mu_N^\sigma}\left[X_i \mid \mathscr{F}_l^2\right]\right)^2 \mid \mathscr{F}_l^2\right]$. Then one can easily prove the analogue of (3.3.21). That is, for $\left|\frac{\xi}{\sqrt{N}}\right| \leq \delta$ and $L \ll N$, it holds that

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{\substack{i\in E_2^l\\ i:odd}}\left(X_i - \mathbb{E}_{\mu_N^\sigma}\left[X_i \mid \mathscr{F}_l^2\right]\right)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l^2\right] - \exp\left(-\sum_{\substack{i\in E_2^l\\ i:odd}}\frac{\tilde{s}_{i,2}^2}{2N}\xi^2\right)\right|$$
$$\lesssim \frac{1}{\sqrt{N}}|\xi|^3 \exp\left(-\sum_{\substack{i\in E_2^l\\ i:odd}}\frac{\tilde{s}_{i,2}^2}{4N}\xi^2\right).$$

Moreover, for $L \ll N$ and N large enough, there exists a uniform constant $C > 0$ with

$$\frac{1}{C} \leq \sum_{\substack{i\in E_2^l\\ i:odd}}\frac{\tilde{s}_{i,2}^2}{4N} \leq C.$$

This implies, as desired,

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{\substack{i\in E_2^l\\ i:odd}}\left(X_i - \mathbb{E}_{\mu_N^\sigma}\left[X_i \mid \mathscr{F}_l^2\right]\right)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l^2\right]\right| \lesssim \left(1 + \frac{|\xi|^3}{\sqrt{N}}\right)\exp\left(-\sum_{\substack{i\in E_2^l\\ i:odd}}\frac{\tilde{s}_{i,2}^2}{4N}\xi^2\right)$$
$$\lesssim \left(1 + \frac{|\xi|^3}{\sqrt{N}}\right)\exp\left(-C\xi^2\right).$$

In particular with (3.3.22),

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\sum_{k\in E_2^l}(X_k - m_k)\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{F}_l\right]\right| \lesssim \left(1 + \frac{|\xi|^3}{\sqrt{N}}\right)\exp\left(-C\xi^2\right)$$
$$\lesssim \left(1 + \xi^2\right)\exp\left(-C\xi^2\right),$$

which proves (3.3.9). The inequality (3.3.10) is deduced by same type of argument. □

3.3.2 Proof of (3.2.10) in Proposition 3.2.7

An application of the inverse Fourier transform with (2.1.6) yields the representation

$$2\pi g_{K,m}(0) = \int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k - m)\,\xi\right)\right]d\xi$$

$$\overset{(2.1.6)}{=} \int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\, \xi \right) \right] d\xi. \tag{3.3.23}$$

For $\delta > 0$, let us divide the integral (3.3.23) into two parts

$$\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\, \xi \right) \right] d\xi$$

$$= \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\, \xi \right) \right] d\xi \tag{3.3.24}$$

$$+ \int_{\{|(1/\sqrt{N})\xi| > \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\, \xi \right) \right] d\xi. \tag{3.3.25}$$

Argument for (3.3.24) **: Estimation of the inner integral** . Recall the definitions (3.3.5), (3.3.6) and (3.3.7) of $m_{i,2}$, $s_{i,2}^2$ and $\hat{e}(\xi)$. Choosing $\delta > 0$ small enough and for K large enough, (3.3.8) gives

$$\left| T_{(3.3.24)} - \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) \right] d\xi \right|$$

$$\lesssim \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \frac{1}{\sqrt{N}} |\xi|^3 \exp\left(-C\xi^2\right) d\xi \lesssim \frac{1}{\sqrt{N}}. \tag{3.3.26}$$

Note that we have by Fubini's theorem

$$\int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) \right] d\xi$$

$$= \mathbb{E}_{\mu_N^\sigma} \left[\int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) d\xi \right]. \tag{3.3.27}$$

We claim that (3.3.27) is uniformly bounded above and below by positive constants. To prove this, we divide the integral as follows

$$\mathbb{E}_{\mu_N^\sigma} \left[\int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) d\xi \right]$$

$$= \mathbb{E}_{\mu_N^\sigma} \left[\int_{\mathbb{R}} \hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) d\xi \right] \tag{3.3.28}$$

$$- \mathbb{E}_{\mu_N^\sigma} \left[\int_{\{|(1/\sqrt{N})\xi| > \delta\}} \hat{e}(\xi) \exp\left(- \sum_{i:odd} \frac{s_{i,2}^2}{2N} \xi^2 \right) d\xi \right]. \tag{3.3.29}$$

Our aim is to prove that the whole integral given by (3.3.28) is uniformly bounded above and below while the outer integral (3.3.29) is relatively small. Let us begin with the estimation of (3.3.28). The upper bound of (3.3.28) follows from (3.3.20) that

$$\left|T_{\,(3.3.28)}\right| \leq \mathbb{E}_{\mu_N^\sigma}\left[\int_{\mathbb{R}} \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)d\xi\right] \leq \mathbb{E}_{\mu_N^\sigma}\left[\int_{\mathbb{R}} \exp\left(-C\xi^2\right)d\xi\right] \lesssim 1.$$

To deduce the lower bound of (3.3.28), we compute the integral inside the expectation directly. It holds that

$$\begin{aligned}
&\int_{\mathbb{R}} \hat{e}\,(\xi)\exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)d\xi\\
&= \int_{\mathbb{R}} \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\left(\xi - i\frac{\sqrt{N}}{\sum_{i:odd}s_{i,2}^2}\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)\right)^2\right)\\
&\qquad \times \exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right)d\xi\\
&= \int_{\mathbb{R}} \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\left(\xi - i\frac{\sqrt{N}}{\sum_{i:odd}s_{i,2}^2}\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)\right)^2\right)d\xi\\
&\qquad \times \exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right).
\end{aligned} \tag{3.3.30}$$

Note that $\frac{\sqrt{N}}{\sum_{i:odd}s_{i,2}^2}\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right) \in \mathbb{R}$. Then complex contour integration implies

$$\begin{aligned}
T_{(3.3.30)} &= \int_{\mathbb{R}} \exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)d\xi\\
&\qquad \times \exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right)\\
&= \sqrt{\frac{2K\pi}{\sum_{i:odd}s_{i,2}^2}}\exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right)\\
&\overset{(3.3.20)}{\gtrsim} \exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right).
\end{aligned}$$

By taking the expectation and applying Jensen's inequality, it holds that

$$\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(-\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right)\right]$$

$$
\begin{aligned}
&\geq \exp\left(-\mathbb{E}_{\mu_N^\sigma}\left[\frac{\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2}{2\sum_{i:odd}s_{i,2}^2}\right]\right)\\
&\overset{(3.3.20)}{\geq} \exp\left(-\mathbb{E}_{\mu_N^\sigma}\left[\frac{C}{N}\left(\sum_{k:even}(X_k-m_k)+\sum_{i:odd}(m_{i,2}-m_i)\right)^2\right]\right)\\
&\geq \exp\left(-\mathbb{E}_{\mu_N^\sigma}\left[\frac{2C}{N}\left(\sum_{k:even}(X_k-m_k)\right)^2+\frac{2C}{N}\left(\sum_{i:odd}(m_{i,2}-m_l)\right)^2\right]\right).
\end{aligned}
$$

Now it follows from [Men11, Lemma 9] and Lemma 2.1.4 that as desired,

$$
\begin{aligned}
&\exp\left(-\mathbb{E}_{\mu_N^\sigma}\left[\frac{2C}{N}\left(\sum_{k:even}(X_k-m_k)\right)^2+\frac{2C}{N}\left(\sum_{i:odd}(m_{i,2}-m_i)\right)^2\right]\right)\\
&\quad= \exp\left(-\frac{2C}{N}\operatorname{var}\left(\sum_{k:even}X_k\right)-\frac{2C}{N}\operatorname{var}\left(\sum_{i:odd}m_{i,2}\right)\right)\\
&\quad= \exp\left(-\frac{2C}{N}\operatorname{var}\left(\sum_{k:even}X_k\right)-\frac{2C}{N}\sum_{i:odd}\operatorname{var}(m_{i,2})\right)\geq C. \qquad (3.3.31)
\end{aligned}
$$

Let us turn to the estimation of (3.3.29). Using (3.3.20), we have

$$
\begin{aligned}
\left|T_{(3.3.29)}\right| &\leq \mathbb{E}_{\mu_N^\sigma}\left[\int_{\{|(1/\sqrt{N})\xi|>\delta\}}\exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)d\xi\right]\\
&\leq \mathbb{E}_{\mu_N^\sigma}\left[\int_{\{|(1/\sqrt{N})\xi|>\delta\}}\exp\left(-C\xi^2\right)d\xi\right]\lesssim\frac{1}{\sqrt{N}}. \qquad (3.3.32)
\end{aligned}
$$

Therefore a combination of (3.3.31) and (3.3.32) implies that there exists a positive constant $C>0$ with

$$
\mathbb{E}_{\mu_N^\sigma}\left[\int_{\{|(1/\sqrt{N})\xi|\leq\delta\}}\hat{e}(\xi)\exp\left(-\sum_{i:odd}\frac{s_{i,2}^2}{2N}\xi^2\right)d\xi\right]\in\left(\frac{1}{C},C\right)
$$

for K large enough. Combined with (3.3.26), it holds that for K large enough,

$$
\frac{1}{C}\leq T_{(3.3.24)}\leq C. \qquad (3.3.33)
$$

Argument for (3.3.25) **: Estimation of the outer integral** . Let us recall the observations (3.3.11) and (3.3.12). We know the conditional random variable $X_i \mid X_j, j : even$ has the conditional Lebesgue density given by (3.3.12). We observe that this conditional density has the same form as the one dimensional measure considered in [MO13, Lemma 3.4] and [MO13, (47)]. Changing i or the conditioned spins X_j only changes the linear term in the Hamiltonian of (3.3.12). Because the estimates of [MO13, Lemma 3.4] and [MO13, (47)] are uniform in the linear term of the Hamiltonian, an application to the random variable $X_i \mid X_j, j : even$ and its distribution $\mu_N^\sigma (dx_i \mid x_j, j : even)$ yields the following: For any $\hat{\delta} > 0$ and odd i's, there exists $\lambda < 1$ with

$$\left| \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{X_i - m_{i,2}}{s_{i,2}} \xi \right) \mid X_j, j : even \right] \right| \leq \lambda \quad \text{for all } |\xi| \geq \hat{\delta} \tag{3.3.34}$$

and

$$\left| \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{X_i - m_{i,2}}{s_{i,2}} \xi \right) \mid X_j, j : even \right] \right| \lesssim |\xi|^{-1} . \tag{3.3.35}$$

Recall that $s_{i,2}^2$ is uniformly bounded from above and below (see (3.3.19)). By setting $\xi = \frac{\hat{\xi}}{s_{i,2}}$ and $\delta = \sqrt{C_{(3.3.19)}} \hat{\delta}$ the estimates (3.3.34) and (3.3.35) yield

$$\left| \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \left(X_i - m_{i,2} \right) \xi \right) \mid X_j, j : even \right] \right| \leq \lambda \quad \text{for all } |\xi| \geq \delta \tag{3.3.36}$$

and

$$\left| \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \left(X_i - m_{i,2} \right) \xi \right) \mid X_j, j : even \right] \right| \lesssim \frac{1}{s_{i,2} |\xi|} \overset{(3.3.19)}{\lesssim} \frac{1}{|\xi|}. \tag{3.3.37}$$

Now consider the conditional expectation with respect to $\{X_j \mid j : even\}$:

$$\begin{aligned}
&\left| \int_{\{|(1/\sqrt{N})\xi| > \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(X_k - m_k \right) \xi \right) \right] d\xi \right| \\
&\quad = \left| \int_{\{|(1/\sqrt{N})\xi| > \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{1}{\sqrt{N}} \sum_{k:even} \left(X_k - m_k \right) \xi \right) \right. \right. \\
&\qquad \left. \left. \times \, \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{1}{\sqrt{N}} \sum_{i:odd} \left(X_i - m_i \right) \xi \right) \mid X_j, j : even \right] \right] d\xi \right| \\
&\quad \leq \int_{\{|(1/\sqrt{N})\xi| > \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\left| \mathbb{E}_{\mu_N^\sigma} \left[\exp \left(i \frac{1}{\sqrt{N}} \sum_{i:odd} \left(X_i - m_i \right) \xi \right) \mid X_j, j : even \right] \right| \right] d\xi.
\end{aligned} \tag{3.3.38}$$

We apply (3.3.36) (on $\frac{N}{2}-2$ of $\frac{N}{2}$ factors) and (3.3.37) (on the remaining 2 factors) to obtain

$$\begin{aligned}\left|T_{(3.3.38)}\right| &\lesssim \int_{\{|(1/\sqrt{N})\xi|>\delta\}} \lambda^{\frac{N}{2}-2}\left(\frac{1}{1+\left(1/\sqrt{N}\right)|\xi|}\right)^2 d\xi \\ &\lesssim \int_{\{|(1/\sqrt{N})\xi|>\delta\}} N\lambda^{\frac{N}{2}-2}\frac{1}{N+\xi^2}d\xi \\ &\lesssim \int_{\{|(1/\sqrt{N})\xi|>\delta\}} N\lambda^{\frac{N}{2}-2}\frac{1}{1+\xi^2}d\xi \\ &\leq N\lambda^{\frac{N}{2}-2}\int_{\mathbb{R}}\frac{1}{1+\xi^2}d\xi \lesssim N\lambda^{\frac{N}{2}-2} \overset{\lambda<1}{\lesssim} \frac{1}{\sqrt{N}}.\end{aligned}$$

Therefore we conclude that

$$\left|T_{(3.3.25)}\right| \lesssim \frac{1}{\sqrt{N}}. \tag{3.3.39}$$

Choosing $K>0$ large enough, (3.3.33) and (3.3.39) imply the desired estimate (3.2.10). □

3.3.3 Proof of (3.2.12) in Proposition 3.2.7

Let us address the inequality (3.2.12). As before, we start with dividing the integral into inner and outer part.

$$\begin{aligned}2\pi\frac{d^2}{d\sigma^2}g_{K,m}(0) &= \frac{d^2}{d\sigma^2}\int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right]d\xi \\ &= \frac{d^2}{d\sigma^2}\int_{\{|(1/\sqrt{N})\xi|\leq\delta\}}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right]d\xi \qquad (3.3.40)\\ &\quad+\frac{d^2}{d\sigma^2}\int_{\{|(1/\sqrt{N})\xi|>\delta\}}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right]d\xi. \qquad (3.3.41)\end{aligned}$$

Recall the definition (3.3.3) and (3.3.4) of the auxiliary sets $F_1^{n,l}$ and $F_2^{n,l}$ (cf. Figure 3.2). We choose $L=N^\varepsilon \ll N$.

Argument for (3.3.40) : Estimation of the inner integral . We introduce a trick that allows to apply Taylor expansions for blocks. We consider the law of the random vector X.

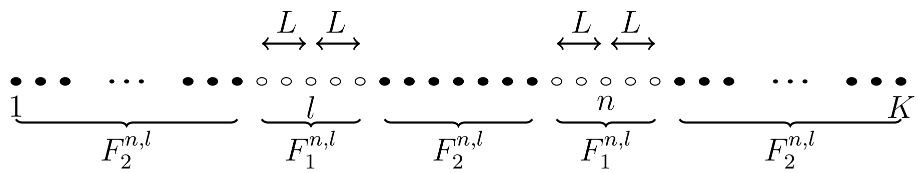

Figure 3.2: The sets $F_1^{n,l}$ and $F_2^{n,l}$ given by (3.3.3) and (3.3.4)

Instead of considering for the every site i the homogeneous linear term σx_i, we artificially introduce the heterogeneous linear term $\sigma_i x_i$. More precisely, it holds that

$$\mathbb{E}_{\mu_N^\sigma}[f(X)] = \int f(x) \frac{\exp\left(\sum_{i=1}^N \sigma_i x_i - H(x)\right)}{\int \exp\left(\sum_{i=1}^N \sigma_i x_i - H(x)\right) dx} dx, \tag{3.3.42}$$

where we introduced on every site i the variable σ_i and set $\sigma_i = \sigma$. Introducing the heterogeneous field σ_i gives more flexibility for Taylor expansions. We group sites together in blocks and then take advantage of the decay of correlations. Let us introduce an auxiliary notation

$$Y_k := X_k - m_k,$$

and define a function $G_{n,l} : \mathbb{R}^2 \to \mathbb{C}$ by

$$G_{n,l}(\xi_1, \xi_2) := \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i \sum_{i \in F_1^{n,l}} Y_i \xi_1 + i \sum_{j \in F_2^{n,l}} Y_j \xi_2\right)\right]. \tag{3.3.43}$$

Note that

$$\begin{aligned}
&\frac{d^2}{d\sigma^2} \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^N Y_k \xi\right)\right] d\xi \\
&\quad = \frac{d^2}{d\sigma^2} \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} G_{n,l}\left(\frac{\xi}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right) d\xi \\
&\quad = \sum_{l=1}^N \sum_{n=1}^N \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}\left(\frac{\xi}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right) \Bigg|_{\sigma_l = \sigma_n = \sigma} d\xi.
\end{aligned}$$

For convenience, we will write $\frac{d}{d\sigma_n} f$ to denote $\frac{d}{d\sigma_n} f \Big|_{\sigma_n = \sigma}$ for any index n. For deducing (3.3.40) it is sufficient to show that for given $\beta > \frac{1}{2}$ and K large enough

$$\sum_{l=1}^N \sum_{n=1}^N \left| \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}\left(\frac{\xi}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right) d\xi \right| \lesssim N^\beta. \tag{3.3.44}$$

To establish this, we take the 2nd order Taylor expansion

$$\begin{aligned}
&\int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\xi}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)d\xi \\
&\quad= \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)d\xi \qquad (3.3.45)\\
&\quad+ \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}d\xi \qquad (3.3.46)\\
&\quad+ \frac{1}{2}\int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\left(\frac{\xi}{\sqrt{N}}\right)^2 d\xi, \qquad (3.3.47)
\end{aligned}$$

where $\frac{\tilde{\xi}}{\sqrt{N}}$ is a real number between 0 and $\frac{\xi}{\sqrt{N}}$. In particular we have $\left|\frac{\tilde{\xi}}{\sqrt{N}}\right| \le \left|\frac{\xi}{\sqrt{N}}\right|$. The proof of (3.3.44) is based on the following three lemmas. The first lemma is an estimation of (3.3.45).

Lemma 3.3.2. *Under the assumptions of Proposition 3.2.7, it holds that for* $\frac{|\xi|}{\sqrt{N}} \le \delta \le 1$ *and any* $n, l \in \{1, 2, \cdots N\}$,

$$\left|\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)\right| \lesssim \left(1+\xi^2\right)\exp\left(-CL\right).$$

The second lemma provides a bound on (3.3.46).

Lemma 3.3.3. *Under the assumptions of Proposition 3.2.7, it holds that for* $\frac{|\xi|}{\sqrt{N}} \le \delta \le 1$ *and any* $n, l \in \{1, 2, \cdots N\}$,

$$\left|\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)\right| \lesssim NL^2\left(1+\xi^2\right)\exp\left(-CL\right).$$

The last lemma is an estimation of (3.3.47).

Lemma 3.3.4. *Under the assumptions of Proposition 3.2.7, it holds that for* $\frac{|\tilde{\xi}|}{\sqrt{N}} \le \frac{|\xi|}{\sqrt{N}} \le \delta \le 1$ *and* n, l *with* $|n-l| > 2L$,

$$\begin{aligned}
&\left|\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right| \\
&\quad\lesssim L^3\left(1+\xi^2\right)\exp\left(-CL\right) + L^4\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right), \qquad (3.3.48)
\end{aligned}$$

and for n, l with $|n-l| \le 2L$,

$$\left|\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right|$$
$$\lesssim L^3\left(1+\xi^2\right)\exp\left(-CL\right)+L^4\left(1+\xi^2\right)\exp\left(-C\xi^2\right). \tag{3.3.49}$$

We give the proof of the lemmas from above in Section 3.3.4. We shall now see how it can be used to prove (3.3.44) and therefore (3.3.40).

Estimation of (3.3.44)**:** Using the lemmas from above and recalling that $\beta > \frac{1}{2}$ yield

$$\left|T_{(3.3.45)}\right| \overset{Lemma\ 3.3.2}{\lesssim} \int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\left(1+\xi^2\right)\exp\left(-CL\right)d\xi \overset{L=N^{\varepsilon}}{\lesssim} \frac{1}{N^{2-\beta}},$$

$$\left|T_{(3.3.46)}\right| \overset{Lemma\ 3.3.3}{\lesssim} \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} NL^2\left(1+\xi^2\right)\exp\left(-CL\right)\frac{|\xi|}{\sqrt{N}}d\xi \overset{L=N^{\varepsilon}}{\lesssim} \frac{1}{N^{2-\beta}},$$

Summing over all pairs (n, l), it holds that

$$\sum_{n=1}^{N}\sum_{l=1}^{N}\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)d\xi\right| \lesssim N^{\beta},$$
$$\sum_{n=1}^{N}\sum_{l=1}^{N}\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(0,\frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}d\xi\right| \lesssim N^{\beta}.$$

Let us consider (3.3.47). For a pair (n, l) with $|n-l| > 2L$, it holds that

$$\begin{aligned}\left|T_{(3.3.47)}\right| &\overset{(3.3.48)}{\lesssim} \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} L^3\left(1+\xi^2\right)\exp\left(-CL\right)\left(\frac{|\xi|}{\sqrt{N}}\right)^2 d\xi \\ &\quad + \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} L^4\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right)\left(\frac{|\xi|}{\sqrt{N}}\right)^2 d\xi \\ &\lesssim L^3\left(\sqrt{N}+N^{3/2}\right)\exp\left(-CL\right)+\frac{L^4}{N^{3/2}}.\end{aligned} \tag{3.3.50}$$

For (n, l) with $|n-l| \le 2L$,

$$\left|T_{(3.3.47)}\right| \overset{(3.3.49)}{\lesssim} \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} L^3\left(1+\xi^2\right)\exp\left(-CL\right)\left(\frac{|\xi|}{\sqrt{N}}\right)^2 d\xi$$

$$
\begin{aligned}
&+\int_{\{|(1/\sqrt{N})\xi|\le\delta\}} L^4\left(1+\xi^2\right)\exp\left(-C\xi^2\right)\left(\frac{|\xi|}{\sqrt{N}}\right)^2 d\xi \\
&\lesssim L^3\left(\sqrt{N}+N^{3/2}\right)\exp\left(-CL\right)+\frac{L^4}{N}. \qquad (3.3.51)
\end{aligned}
$$

A combination of (3.3.50) and (3.3.51) yields

$$
\begin{aligned}
&\sum_{n=1}^{N}\sum_{l=1}^{N}\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\left(\frac{\xi}{\sqrt{N}}\right)^2 d\xi\right| \\
&\quad=\sum_{n=1}^{N}\sum_{l:|n-l|>2L}\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\left(\frac{\xi}{\sqrt{N}}\right)^2 d\xi\right| \\
&\qquad+\sum_{n=1}^{N}\sum_{l:|n-l|\le 2L}\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}}\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\left(\frac{\xi}{\sqrt{N}}\right)^2 d\xi\right| \\
&\quad\lesssim N^2\left(L^3\left(\sqrt{N}+N^{3/2}\right)\exp\left(-CL\right)+\frac{L^4}{N^{3/2}}\right)+NL\left(L^3\left(\sqrt{N}+N^{3/2}\right)\exp\left(-CL\right)+\frac{L^4}{N}\right) \\
&\quad\overset{L=N^{\varepsilon}}{\lesssim} N^{\beta}.
\end{aligned}
$$

This gives the desired estimate (3.3.44). □

Argument for (3.3.41) **: Estimation of the outer integral.** Let us start with providing auxiliary ingredients. The first lemma provides auxiliary estimates:

Lemma 3.3.5. *Recall the definition* (3.3.5) *of* $m_{i,2}$*. It holds that*

$$
\left|\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right|\lesssim 1+|\xi|, \qquad (3.3.52)
$$

$$
\left|\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right|\lesssim 1+|\xi|^2. \qquad (3.3.53)
$$

Proof of Lemma 3.3.5. Let us begin with providing an auxiliary computation. Recalling (3.3.12), one can easily prove that (cf. see also (3.3.13), (3.3.14) and (3.2.16))

$$
\begin{aligned}
&\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[f(X_i)\mid X_j, j: even\right] \\
&\quad=\mathbb{E}_{\mu_N^\sigma}\left[\left(X_i-m_{i,2}\right)f(X_i)\mid X_j, j: even\right]+\mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}f(X_i)\mid X_j, j: even\right]. \qquad (3.3.54)
\end{aligned}
$$

This implies in particular

$$\begin{aligned}
&\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] \\
&\overset{(3.3.13)}{=} \mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - m_{i,2}\right)\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] && (3.3.55)\\
&\quad + \mathbb{E}_{\mu_N^\sigma}\left[-i\xi s_{i,2}^2 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]. && (3.3.56)
\end{aligned}$$

Note that (3.3.17) and (3.3.19) yields

$$\left|T_{(3.3.55)}\right| = \left|\mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - m_{i,2}\right)\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]\right| \overset{(3.3.17)}{\lesssim} 1, \tag{3.3.57}$$

$$\left|T_{(3.3.56)}\right| = \left|\mathbb{E}_{\mu_N^\sigma}\left[-i\xi s_{i,2}^2 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]\right| \overset{(3.3.19)}{\lesssim} |\xi|. \tag{3.3.58}$$

As desired, the estimates (3.3.57) and (3.3.58) yield

$$\left|\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]\right| \leq \left|T_{(3.3.55)}\right| + \left|T_{(3.3.56)}\right| \lesssim 1 + |\xi|.$$

Let us turn to the derivation of (3.3.53). The calculation from above yields that

$$\begin{aligned}
&\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] \\
&= \frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - m_{i,2}\right)\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] && (3.3.59)\\
&\quad + \frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[-i\xi s_{i,2}^2 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]. && (3.3.60)
\end{aligned}$$

Let us consider (3.3.59): A direct computation using (3.3.54) and (3.3.13) implies

$$\begin{aligned}
T_{(3.3.59)} &= \mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - m_{i,2}\right)^2 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] && (3.3.61)\\
&+ \mathbb{E}_{\mu_N^\sigma}\left[-s_{i,2}^2 \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right] && (3.3.62)\\
&+ \mathbb{E}_{\mu_N^\sigma}\left[\left(X_i - m_{i,2}\right)\left(-i\xi s_{i,2}^2\right) \exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j: even\right]. && (3.3.63)
\end{aligned}$$

The terms (3.3.61), (3.3.62) and (3.3.63) can be estimated similarly. Indeed, it holds that

$$\left|T_{(3.3.61)}\right|, \left|T_{(3.3.62)}\right| \lesssim 1 \qquad \text{and} \qquad \left|T_{(3.3.63)}\right| \lesssim |\xi|.$$

As a consequence, we have

$$\left|T_{(3.3.59)}\right| \leq \left|T_{(3.3.61)}\right| + \left|T_{(3.3.62)}\right| + \left|T_{(3.3.63)}\right| \lesssim 1 + |\xi| \lesssim 1 + |\xi|^2.$$

Let us consider (3.3.60): Recall the definition (3.3.14) of $t_{i,2}$. Then it holds from (3.3.54) that

$$T_{(3.3.60)} = \mathbb{E}_{\mu_N^\sigma}\left[(X_i - m_{i,2})\left(-i\xi s_{i,2}^2\right)\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j : even\right] \tag{3.3.64}$$

$$+ \mathbb{E}_{\mu_N^\sigma}\left[\left(-i\xi t_{i,2}\right)\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j : even\right] \tag{3.3.65}$$

$$+ \mathbb{E}_{\mu_N^\sigma}\left[\left(-i\xi s_{i,2}^2\right)^2\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j : even\right]. \tag{3.3.66}$$

Now (3.3.17) and (3.3.19) imply

$$\left|T_{(3.3.64)}\right|, \left|T_{(3.3.65)}\right| \lesssim |\xi| \qquad \text{and} \qquad \left|T_{(3.3.66)}\right| \lesssim |\xi|^2.$$

Therefore we obtain

$$\left|T_{(3.3.60)}\right| \leq \left|T_{(3.3.64)}\right| + \left|T_{(3.3.65)}\right| + \left|T_{(3.3.66)}\right| \lesssim |\xi| + |\xi|^2 \lesssim 1 + |\xi|^2.$$

To conclude, we sum up the estimates from above. As desired, we have

$$\left|\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\left(X_i - m_{i,2}\right)\xi\right) \mid X_j, j : even\right]\right| \leq \left|T_{(3.3.59)}\right| + \left|T_{(3.3.60)}\right| \lesssim 1 + |\xi|^2.$$

□

The second lemma is an auxiliary computation:

Lemma 3.3.6. *Recall the definition* (3.3.7) *and* (3.3.14) *of* $\hat{e}$ *and* $t_{i,2}$, *respectively. It holds that*

$$\frac{d}{d\sigma}\hat{e}(\xi) = \hat{e}(\xi)\left(-i\frac{1}{\sqrt{N}}\xi\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n - m_n)\right)^2\right] + i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}s_{i,2}^2\right), \tag{3.3.67}$$

$$\frac{d^2}{d\sigma^2}\hat{e}(\xi) = \hat{e}(\xi)\left(-i\frac{1}{\sqrt{N}}\xi\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n - m_n)\right)^2\right] + i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}s_{i,2}^2\right)^2$$
$$+ \hat{e}(\xi)\left(-i\frac{1}{\sqrt{N}}\xi\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n - m_n)\right)^3\right] + i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}t_{i,2}\right). \tag{3.3.68}$$

Proof of Lemma 3.3.6. We start with deducing (3.3.67). Let us recall the equations (3.2.17) and (3.3.13). It follows from a direct computation that

$$\frac{d}{d\sigma}\hat{e}(\xi) = \left(-i\frac{1}{\sqrt{N}}\xi\sum_{k:even}\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n - m_n)\right)X_k\right]\right.$$

$$+i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}\left(s_{i,2}^2-\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^N(X_n-m_n)\right)X_k\right]\right)\Bigg)$$

$$\times\exp\left(i\frac{1}{\sqrt{N}}\sum_{k:even}(X_k-m_k)\xi+i\frac{1}{\sqrt{N}}\sum_{i:odd}(m_{i,2}-m_i)\xi\right)$$

$$=\left(-i\frac{1}{\sqrt{N}}\xi\sum_{k=1}^N\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^N(X_n-m_n)\right)X_k\right]+i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}s_{i,2}^2\right)\hat{e}(\xi)$$

$$=\left(-i\frac{1}{\sqrt{N}}\xi\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^N(X_n-m_n)\right)^2\right]+i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}s_{i,2}^2\right)\hat{e}(\xi).$$

The equation (3.3.68) can be derived with a similar argument using (3.3.14). □

Now let us move on to the estimation of the outer integral (3.3.41). Recalling (3.2.16), we get that the integrand in (3.3.41) is

$$\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^N(X_k-m_k)\xi\right)\right]$$

$$=\frac{d^2}{d\sigma^2}\mathbb{E}_{\mu_N^\sigma}\left[\hat{e}(\xi)\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j,j:even\right]\right]$$

$$=\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^N(X_n-m_n)\right)\hat{e}(\xi)\right.$$

$$\left.\times\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j,j:even\right]\right] \tag{3.3.69}$$

$$+\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}(\hat{e}(\xi))\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j,j:even\right]\right] \tag{3.3.70}$$

$$+\frac{d}{d\sigma}\mathbb{E}_{\mu_N^\sigma}\left[\hat{e}(\xi)\frac{d}{d\sigma}\left(\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{l,2})\xi\right)\mid X_j,j:even\right]\right)\right]. \tag{3.3.71}$$

We estimate each term on the right hand side separately. We begin with estimating (3.3.69). Applying (3.2.16), we get

$$T_{(3.3.69)}=\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^N(X_n-m_n)\right)^2\hat{e}(\xi)\right.$$

$$\left.\times\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j,j:even\right]\right] \tag{3.3.72}$$

$$+\mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}\left(\sum_{n=1}^N(X_n-m_n)\right)\hat{e}(\xi)\right.$$

$$\times \prod_{i:odd} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : even \right] \Bigg] \tag{3.3.73}$$

$$+ \mathbb{E}_{\mu_N^\sigma} \Bigg[\left(\sum_{n=1}^{N} (X_n - m_n) \right) \frac{d}{d\sigma} (\hat{e}(\xi))$$

$$\times \prod_{i:odd} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : even \right] \Bigg] \tag{3.3.74}$$

$$+ \mathbb{E}_{\mu_N^\sigma} \Bigg[\left(\sum_{n=1}^{N} (X_n - m_n) \right) \hat{e}(\xi)$$

$$\times \frac{d}{d\sigma} \left(\prod_{i:odd} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : even \right] \right) \Bigg]. \tag{3.3.75}$$

Let us consider (3.3.72): Applying (3.3.36) to $\frac{N}{2} - 2$ many factors and (3.3.37) on the remaining 2 factors leads to

$$\left| T_{(3.3.72)} \right| \leq \mathbb{E}_{\mu_N^\sigma} \left[\left(\sum_{n=1}^{N} (X_n - m_n) \right)^2 \lambda^{\frac{N}{2}-2} \left(\frac{1}{1 + (1/\sqrt{N}) |\xi|} \right)^2 \right].$$

Then Corollary 2.1.2 implies

$$\left| T_{(3.3.72)} \right| \lesssim N^2 \lambda^{\frac{N}{2}-2} \left(\frac{1}{1 + (1/\sqrt{N}) |\xi|} \right)^2 \lesssim N^3 \lambda^{\frac{N}{2}-2} \frac{1}{N + \xi^2} \lesssim N^3 \lambda^{\frac{N}{2}-2} \frac{1}{1 + \xi^2}.$$

Let us consider (3.3.73): A similar estimation using (3.2.17) yields that

$$\begin{aligned} \left| T_{(3.3.73)} \right| = & \Bigg| \mathbb{E}_{\mu_N^\sigma} \Bigg[-\mathbb{E}_{\mu_N^\sigma} \left[\left(\sum_{n=1}^{N} (X_n - m_n) \right)^2 \right] \hat{e}(\xi) \\ & \times \prod_{i:odd} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} (X_i - m_{i,2}) \xi \right) \mid X_j, j : even \right] \Bigg] \Bigg| \\ \lesssim & N^3 \lambda^{\frac{N}{2}-2} \frac{1}{1 + \xi^2}. \end{aligned}$$

Let us consider (3.3.74): We first expand the term using (3.3.67). Then applying (3.3.36) to $\frac{N}{2} - 3$ many factors and (3.3.37) to 3 factors yields

$$\left| T_{(3.3.74)} \right| \overset{(3.3.67)}{=} \Bigg| \mathbb{E}_{\mu_N^\sigma} \Bigg[\left(\sum_{n=1}^{N} (X_n - m_n) \right) \hat{e}(\xi)$$

$$
\times\left(-i\frac{1}{\sqrt{N}}\xi\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n-m_n)\right)^2\right]+i\frac{1}{\sqrt{N}}\xi\sum_{i:odd}s_{i,2}^2\right)
$$

$$
\times\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j, j:even\right]\Bigg]\Bigg|
$$

$$
\overset{(2.1.1),(3.3.20),(3.3.36),(3.3.37)}{\lesssim} N\left(\frac{|\xi|}{\sqrt{N}}N^2+\frac{|\xi|}{\sqrt{N}}N\right)\lambda^{\frac{N}{2}-3}\left(\frac{1}{1+(1/\sqrt{N})|\xi|}\right)^3
$$

$$
\lesssim N^4\lambda^{\frac{N}{2}-3}|\xi|\left(\frac{1}{\sqrt{N}+|\xi|}\right)^3\lesssim N^4\lambda^{\frac{N}{2}-3}\frac{1}{1+\xi^2}.
$$

Let us consider (3.3.75): A combination of (2.1.1), (3.3.36), (3.3.37) and (3.3.52) yields

$$
\left|T_{(3.3.75)}\right|\lesssim\frac{N}{2}\cdot N\left(1+\left|\frac{\xi}{\sqrt{N}}\right|\right)\lambda^{\frac{N}{2}-4}\left(\frac{1}{1+\left(1/\sqrt{N}\right)|\xi|}\right)^3\lesssim N^3\lambda^{\frac{N}{2}-4}\frac{1}{1+\xi^2}.
$$

Hence we have

$$
\begin{aligned}
\left|T_{(3.3.69)}\right| &\le \left|T_{(3.3.72)}\right|+\left|T_{(3.3.73)}\right|+\left|T_{(3.3.74)}\right|+\left|T_{(3.3.75)}\right|\\
&\lesssim N^3\lambda^{\frac{N}{2}-2}\frac{1}{1+\xi^2}+N^4\lambda^{\frac{N}{2}-3}\frac{1}{1+\xi^2}+N^3\lambda^{\frac{N}{2}-4}\frac{1}{1+\xi^2}\lesssim N^4\lambda^{\frac{N}{2}-4}\frac{1}{1+\xi^2}.
\end{aligned}
$$

Let us turn to the estimation of (3.3.70). A direct computation using (3.2.16) yields

$$
T_{(3.3.70)}=\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}(X_n-m_n)\right)\frac{d}{d\sigma}(\hat{e}(\xi))\right.
$$

$$
\left.\times\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j, j:even\right]\right] \tag{3.3.76}
$$

$$
+\mathbb{E}_{\mu_N^\sigma}\left[\frac{d^2}{d\sigma^2}(\hat{e}(\xi))\right.
$$

$$
\left.\times\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j, j:even\right]\right] \tag{3.3.77}
$$

$$
+\mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}(\hat{e}(\xi))\right.
$$

$$
\left.\times\frac{d}{d\sigma}\left(\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}(X_i-m_{i,2})\xi\right)\mid X_j, j:even\right]\right)\right]. \tag{3.3.78}
$$

Let us estimate each term on right hand side step by step. We start with estimating (3.3.76). It holds that

$$
\left|T_{(3.3.76)}\right|=\left|T_{(3.3.74)}\right|\lesssim N^4\lambda^{\frac{N}{2}-3}\frac{1}{1+\xi^2}.
$$

Let us consider (3.3.77): An application of Corollary 2.1.2 to (3.3.68) yields

$$\left|\frac{d^2}{d\sigma^2}\left(\hat{e}\left(\xi\right)\right)\right| \overset{(2.1.1)}{\lesssim} \left(\frac{|\xi|}{\sqrt{N}}N^2+\frac{|\xi|}{\sqrt{N}}N\right)^2+\left(\frac{|\xi|}{\sqrt{N}}N^3+\frac{|\xi|}{\sqrt{N}}N\right)$$
$$\lesssim N^3\xi^2+N^{5/2}\left|\xi\right| \lesssim N^3\left(1+\xi^2\right). \tag{3.3.79}$$

Inserting (3.3.79) into (3.3.77) gives

$$\left|T_{(3.3.77)}\right| \lesssim N^3\left(1+\xi^2\right)\mathbb{E}_{\mu_N^\sigma}\left[\prod_{i:odd}\left|\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right|\right]$$
$$\overset{(3.3.36),(3.3.37)}{\lesssim N^3} \left(1+\xi^2\right)\lambda^{\frac{N}{2}-4}\left(\frac{1}{1+(1/\sqrt{N})\left|\xi\right|}\right)^4 \lesssim N^5\lambda^{\frac{N}{2}-4}\frac{1}{1+\xi^2}.$$

Let us consider (3.3.78): Using (3.3.67), (3.3.36), (3.3.37) and (3.3.52) yields

$$\left|T_{(3.3.78)}\right| \lesssim \frac{N}{2}\cdot\left(\frac{|\xi|}{\sqrt{N}}N^2+\frac{|\xi|}{\sqrt{N}}N\right)\left(1+\left|\frac{\xi}{\sqrt{N}}\right|\right)\lambda^{\frac{N}{2}-5}\left(\frac{1}{1+\left(1/\sqrt{N}\right)\left|\xi\right|}\right)^4$$
$$\lesssim N^4\lambda^{\frac{N}{2}-5}\frac{1}{1+\xi^2}.$$

Therefore it holds that

$$\left|T_{(3.3.70)}\right| \leq \left|T_{(3.3.76)}\right|+\left|T_{(3.3.77)}\right|+\left|T_{(3.3.78)}\right|$$
$$\lesssim N^4\lambda^{\frac{N}{2}-3}\frac{1}{1+\xi^2}+N^5\lambda^{\frac{N}{2}-4}\frac{1}{1+\xi^2}+N^4\lambda^{\frac{N}{2}-5}\frac{1}{1+\xi^2} \lesssim N^5\lambda^{\frac{N}{2}-5}\frac{1}{1+\xi^2}.$$

Let us turn to the estimation of (3.3.71), which is deduced by the same type of argument. More precisely, we get

$$T_{(3.3.71)}$$
$$=\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{n=1}^{N}\left(X_n-m_n\right)\right)\hat{e}\left(\xi\right)\right.$$
$$\left.\times\frac{d}{d\sigma}\left(\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right)\right] \tag{3.3.80}$$
$$+\mathbb{E}_{\mu_N^\sigma}\left[\frac{d}{d\sigma}\left(\hat{e}\left(\xi\right)\right)\frac{d}{d\sigma}\left(\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right)\right] \tag{3.3.81}$$
$$+\mathbb{E}_{\mu_N^\sigma}\left[\hat{e}\left(\xi\right)\frac{d^2}{d\sigma^2}\left(\prod_{i:odd}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\left(X_i-m_{i,2}\right)\xi\right)\mid X_j, j: even\right]\right)\right]. \tag{3.3.82}$$

Note that $T_{(3.3.80)} = T_{(3.3.75)}$ and $T_{(3.3.81)} = T_{(3.3.78)}$. Let us estimate (3.3.82). We have by using (3.3.52), (3.3.53), (3.3.36) and (3.3.37) that

$$\begin{aligned}
&\left| \frac{d^2}{d\sigma^2} \left(\prod_{i:odd} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \left(X_i - m_{i,2} \right) \xi \right) \mid X_j, j : even \right] \right) \right| \\
&\lesssim \frac{N}{2} \cdot \left(1 + \left| \frac{\xi}{\sqrt{N}} \right|^2 \right) \lambda^{\frac{N}{2}-5} \left(\frac{1}{1 + \left(1/\sqrt{N} \right) |\xi|} \right)^4 \\
&\quad + \frac{N}{2} \left(\frac{N}{2} - 1 \right) \left(1 + \left| \frac{\xi}{\sqrt{N}} \right| \right)^2 \lambda^{\frac{N}{2}-6} \left(\frac{1}{1 + \left(1/\sqrt{N} \right) |\xi|} \right)^4 \\
&\lesssim N^2 \lambda^{\frac{N}{2}-5} \frac{1}{1+\xi^2} + N^3 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2}.
\end{aligned}$$

Thus we conclude

$$\left| T_{(3.3.82)} \right| \lesssim N^3 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2}.$$

It follows that

$$\begin{aligned}
\left| T_{(3.3.71)} \right| &\leq \left| T_{(3.3.80)} \right| + \left| T_{(3.3.81)} \right| + \left| T_{(3.3.82)} \right| \\
&\lesssim N^3 \lambda^{\frac{N}{2}-4} \frac{1}{1+\xi^2} + N^4 \lambda^{\frac{N}{2}-5} \frac{1}{1+\xi^2} + N^3 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2} \lesssim N^4 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2}.
\end{aligned}$$

Overall, we have proven that

$$\begin{aligned}
&\left| \frac{d^2}{d\sigma^2} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(X_k - m_k \right) \xi \right) \right] \right| \\
&\leq \left| T_{(3.3.69)} \right| + \left| T_{(3.3.70)} \right| + \left| T_{(3.3.71)} \right| \\
&\lesssim N^4 \lambda^{\frac{N}{2}-4} \frac{1}{1+\xi^2} + N^5 \lambda^{\frac{N}{2}-5} \frac{1}{1+\xi^2} + N^4 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2} \\
&\lesssim N^5 \lambda^{\frac{N}{2}-6} \frac{1}{1+\xi^2} \overset{\lambda<1}{\lesssim} N^\beta \frac{1}{1+\xi^2}.
\end{aligned}$$

This implies, as desired,

$$\begin{aligned}
&\left| \frac{d^2}{d\sigma^2} \int_{\left\{ \left| \left(1/\sqrt{N} \right) \xi \right| > \delta \right\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(X_k - m_k \right) \xi \right) \right] d\xi \right| \\
&= \left| \int_{\left\{ \left| \left(1/\sqrt{N} \right) \xi \right| > \delta \right\}} \frac{d^2}{d\sigma^2} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(X_k - m_k \right) \xi \right) \right] d\xi \right|
\end{aligned}$$

$$\lesssim N^{\beta} \int_{\{|(1/\sqrt{N})\xi|>\delta\}} \frac{1}{1+\xi^2} d\xi$$

$$\lesssim N^{\beta}.$$

Combined with the argument for (3.3.40), we obtain (3.2.12) in Proposition 3.2.7. □

3.3.4 Proof of Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.4.

We shall only present the proof of Lemma 3.3.2 and Lemma 3.3.4. One can derive Lemma 3.3.3 by using similar arguments. The main ingredients are Corollary 2.1.2, Theorem 1.2.6 and Lemma 3.3.1. Additionally, we will need the following auxiliary observation. Recalling that $Y_l = X_l - \mathbb{E}_{\mu_N^\sigma}[X_l]$, a direct calculation using (3.3.42) yields

$$\frac{d}{d\sigma_l} \mathbb{E}_{\mu_N^\sigma}[f(X)] = \mathbb{E}_{\mu_N^\sigma}\left[Y_l f(X) + \frac{d}{d\sigma_l} f(X)\right]. \tag{3.3.83}$$

Hence, we have by additionally using that by definition $\mathbb{E}_{\mu_N^\sigma}[Y_k] = 0$ for any k

$$\frac{d}{d\sigma_l} Y_k = -\mathbb{E}_{\mu_N^\sigma}[Y_l X_k] = -\mathbb{E}_{\mu_N^\sigma}[Y_l Y_k], \text{ and } \frac{d}{d\sigma_n} \mathbb{E}_{\mu_N^\sigma}[Y_l Y_k] = \mathbb{E}_{\mu_N^\sigma}[Y_n Y_l Y_k]. \tag{3.3.84}$$

The last two formulas will be used many times in the upcoming arguments.

Proof of Lemma 3.3.2. Let $e(\xi_1, \xi_2)$ indicate

$$e(\xi_1, \xi_2) := \exp\left(i \sum_{i \in F_1^{n,l}} Y_i \xi_1 + i \sum_{j \in F_2^{n,l}} Y_j \xi_2 \right).$$

Recall the definition (3.3.43) of $G_{n,l}(\xi_1, \xi_2)$. A direct calculation using (3.3.83) yields

$$\frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}(\xi_1, \xi_2) = \frac{d}{d\sigma_n} \mathbb{E}_{\mu_N^\sigma}[Y_l e(\xi_1, \xi_2)] \tag{3.3.85}$$

$$+ \frac{d}{d\sigma_n} \mathbb{E}_{\mu_N^\sigma}\left[-i\xi_1 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i \in F_1^{n,l}} Y_i \right] e(\xi_1, \xi_2) \right] \tag{3.3.86}$$

$$+ \frac{d}{d\sigma_n} \mathbb{E}_{\mu_N^\sigma}\left[-i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j \in F_2^{n,l}} Y_j \right] e(\xi_1, \xi_2) \right]. \tag{3.3.87}$$

Let us consider (3.3.85). A combination of (3.3.83) and (3.3.84) yields

$$\frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[Y_l e(\xi_1,\xi_2)\right] = \mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l e(\xi_1,\xi_2)\right] \tag{3.3.88}$$

$$+\,\mathbb{E}_{\mu_N^\sigma}\left[-\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l\right] e(\xi_1,\xi_2)\right] \tag{3.3.89}$$

$$+\,\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(-i\xi_1\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i\in F_1^{n,l}} Y_i\right] e(\xi_1,\xi_2)\right] \tag{3.3.90}$$

$$+\,\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(-i\xi_2\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{j\in F_2^{n,l}} Y_j\right] e(\xi_1,\xi_2)\right]. \tag{3.3.91}$$

Setting $(\xi_1,\xi_2) = \left(0, \frac{\xi}{\sqrt{N}}\right)$ and using Corollary 2.1.2 and Theorem 1.2.6, we obtain

$$\left|T_{(3.3.88)} + T_{(3.3.89)}\right| = \left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n Y_l, e\left(0, \frac{\xi}{\sqrt{N}}\right)\right)\right| \lesssim |\xi| \exp\left(-CL\right),$$

$$T_{(3.3.90)} = 0, \qquad \text{and}$$

$$\left|T_{(3.3.91)}\right| \le \left|\frac{\xi}{\sqrt{N}}\right| \left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n, \sum_{j\in F_2^{n,l}} Y_j\right)\right| \mathbb{E}_{\mu_N^\sigma}\left|Y_l\right| \lesssim |\xi| \exp\left(-CL\right).$$

Therefore we conclude that for $(\xi_1,\xi_2) = \left(0, \frac{\xi}{\sqrt{N}}\right)$,

$$\begin{aligned}
\left|T_{(3.3.85)}\right| &= \left|T_{(3.3.88)} + T_{(3.3.89)} + T_{(3.3.90)} + T_{(3.3.91)}\right| \\
&\le \left|T_{(3.3.88)} + T_{(3.3.89)}\right| + \left|T_{(3.3.90)}\right| + \left|T_{(3.3.91)}\right| \\
&\lesssim |\xi| \exp\left(-CL\right) \lesssim \left(1+\xi^2\right) \exp\left(-CL\right).
\end{aligned} \tag{3.3.92}$$

We observe that when $(\xi_1,\xi_2) = \left(0, \frac{\xi}{\sqrt{N}}\right)$ we have

$$T_{(3.3.86)} = 0. \tag{3.3.93}$$

Let us turn to the estimation of (3.3.87). A direct computation using (3.3.83) and (3.3.84) yields

$$\frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[-i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j\in F_2^{n,l}} Y_j\right] e\left(\xi_1,\xi_2\right)\right]$$

$$= -i\xi_2 \frac{d}{d\sigma_n}\left(\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(\sum_{j\in F_2^{n,l}} Y_j\right)\right]\right)\mathbb{E}_{\mu_N^\sigma}\left[e\left(\xi_1,\xi_2\right)\right]$$

$$- i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j\in F_2^{n,l}} Y_j\right]\frac{d}{d\sigma_n}\left(\mathbb{E}_{\mu_N^\sigma}\left[e\left(\xi_1,\xi_2\right)\right]\right)$$

$$= -i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \sum_{j\in F_2^{n,l}} Y_j\right]\mathbb{E}_{\mu_N^\sigma}\left[e\left(\xi_1,\xi_2\right)\right] \tag{3.3.94}$$

$$- i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j\in F_2^{n,l}} Y_j\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n e\left(\xi_1,\xi_2\right)\right] \tag{3.3.95}$$

$$- i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j\in F_2^{n,l}} Y_j\right]\mathbb{E}_{\mu_N^\sigma}\left[\left(-i\xi_1\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i\in F_1^{n,l}} Y_i\right] e\left(\xi_1,\xi_2\right)\right] \tag{3.3.96}$$

$$- i\xi_2 \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{j_1\in F_2^{n,l}} Y_{j_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[\left(-i\xi_2\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{j_2\in F_2^{n,l}} Y_{j_2}\right] e\left(\xi_1,\xi_2\right)\right]. \tag{3.3.97}$$

Then plugging in $(\xi_1,\xi_2) = \left(0, \frac{\xi}{\sqrt{N}}\right)$ and applying Corollary 2.1.2 and Theorem 1.2.6 yield

$$\left|T_{(3.3.94)}\right| \le \left|\frac{\xi}{\sqrt{N}}\right|\left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n Y_l, \sum_{j\in F_2^{n,l}} Y_j\right)\right| \lesssim |\xi| \exp\left(-CL\right),$$

$$\left|T_{(3.3.95)}\right| \le \left|\frac{\xi}{\sqrt{N}}\right|\left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_l, \sum_{j\in F_2^{n,l}} Y_j\right)\right|\mathbb{E}_{\mu_N^\sigma}\left|Y_n\right| \lesssim |\xi| \exp\left(-CL\right),$$

$$T_{(3.3.96)} = 0, \qquad \text{and}$$

$$\left|T_{(3.3.97)}\right| \le \left(\frac{\xi}{\sqrt{N}}\right)^2\left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_l, \sum_{j_1\in F_2^{n,l}} Y_{j_1}\right)\right|\left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n, \sum_{j_2\in F_2^{n,l}} Y_{j_2}\right)\right| \lesssim \xi^2 \exp\left(-2CL\right).$$

Thus a combination of the bounds from above yields

$$\begin{aligned}\left|T_{(3.3.87)}\right| &= \left|T_{(3.3.94)} + T_{(3.3.95)} + T_{(3.3.96)} + T_{(3.3.97)}\right| \\ &\le \left|T_{(3.3.94)}\right| + \left|T_{(3.3.95)}\right| + \left|T_{(3.3.96)}\right| + \left|T_{(3.3.97)}\right|\end{aligned}$$

$$\lesssim |\xi| \exp(-CL) + \xi^2 \exp(-2CL)$$
$$\lesssim \left(|\xi| + \xi^2\right) \exp(-CL)$$
$$\lesssim \left(1 + \xi^2\right) \exp(-CL) . \tag{3.3.98}$$

We then sum up the bounds of (3.3.85), (3.3.86) and (3.3.87) given by (3.3.92), (3.3.93) and (3.3.98) respectively, and this finishes the proof of Lemma 3.3.2. □

The proof of Lemma 3.3.4 is similar, but it needs more careful estimation.

PROOF OF LEMMA 3.3.4. A direct computation using (3.3.83) and (3.3.84) yields

$$\begin{aligned}
&\frac{d^2}{d\xi_1^2}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}(\xi_1,\xi_2) \\
&\quad = \frac{d}{d\sigma_n}\frac{d}{d\sigma_l}\frac{d^2}{d\xi_1^2}G_{n,l}(\xi_1,\xi_2) \\
&\quad = \frac{d}{d\sigma_n}\frac{d}{d\sigma_l}\mathbb{E}_{\mu_N^\sigma}\left[\left(i\sum_{i\in F_1^{n,l}} Y_i\right)^2 e(\xi_1,\xi_2)\right] \\
&\quad = \frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(i\sum_{i\in F_1^{n,l}} Y_i\right)^2 e(\xi_1,\xi_2)\right] && (3.3.99)\\
&\qquad + \frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[2\left(i\sum_{i_1\in F_1^{n,l}} Y_{i_1}\right)(-i)\,\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right] e(\xi_1,\xi_2)\right] && (3.3.100)\\
&\qquad + \frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[\left(i\sum_{i_1\in F_1^{n,l}} Y_{i_1}\right)^2 (-i\xi_1)\,\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right] e(\xi_1,\xi_2)\right] && (3.3.101)\\
&\qquad + \frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[\left(i\sum_{i\in F_1^{n,l}} Y_i\right)^2 (-i\xi_2)\,\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{j\in F_2^{n,l}} Y_j\right] e(\xi_1,\xi_2)\right]. && (3.3.102)
\end{aligned}$$

Estimating the terms (3.3.101) and (3.3.102) is a lot easier than estimating (3.3.99) and (3.3.100). The argument consists of a straightforward calculation and application of Corollary 2.1.2, Theorem 1.2.6 and Lemma 3.3.1. More precisely, for any $n, l \in \{1, 2, \cdots, N\}$ and $(\xi_1, \xi_2) =$

$\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)$ such that $\frac{|\tilde{\xi}|}{\sqrt{N}} \leq \frac{|\xi|}{\sqrt{N}} \leq \delta \leq 1$, it holds that

$$\left|T_{(3.3.101)}\right| \lesssim L^3 \left(1+\xi^2\right) \exp\left(-CL\right) + L^4 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right),$$

and

$$\left|T_{(3.3.102)}\right| \lesssim L^3 \left(1+\xi^2\right) \exp\left(-CL\right).$$

We leave the details to the reader and turn to the more subtle estimation of (3.3.99) and (3.3.100). The argument is more evolved because both terms have to be considered together. Only then, one sees covariances and can take advantage of the decay of correlations (cf. Theorem 1.2.6). Let us now turn to the details. Let us begin with expanding (3.3.99) and (3.3.100). A direct computation using (3.3.83) and (3.3.84) yields that (3.3.99) is

$$\frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[Y_l \left(i \sum_{i \in F_1^{n,l}} Y_i\right)^2 e\left(\xi_1, \xi_2\right)\right]$$

$$= -\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \left(\sum_{i \in F_1^{n,l}} Y_i\right)^2 e\left(\xi_1, \xi_2\right)\right] \tag{3.3.103}$$

$$- \mathbb{E}_{\mu_N^\sigma}\left[-\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l\right] \left(\sum_{i \in F_1^{n,l}} Y_i\right)^2 e\left(\xi_1, \xi_2\right)\right] \tag{3.3.104}$$

$$- \mathbb{E}_{\mu_N^\sigma}\left[Y_l 2 \left(\sum_{i_1 \in F_1^{n,l}} Y_{i_1}\right)\left(-\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_2 \in F_1^{n,l}} Y_{i_2}\right]\right) e\left(\xi_1, \xi_2\right)\right] \tag{3.3.105}$$

$$- \mathbb{E}_{\mu_N^\sigma}\left[Y_l \left(\sum_{i_1 \in F_1^{n,l}} Y_{i_1}\right)^2 \left(-i\xi_1\right) \mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_2 \in F_1^{n,l}} Y_{i_2}\right] e\left(\xi_1, \xi_2\right)\right] \tag{3.3.106}$$

$$- \mathbb{E}_{\mu_N^\sigma}\left[Y_l \left(\sum_{i \in F_1^{n,l}} Y_i\right)^2 \left(-i\xi_2\right) \mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{j \in F_2^{n,l}} Y_j\right] e\left(\xi_1, \xi_2\right)\right]. \tag{3.3.107}$$

and (3.3.100) is

$$\frac{d}{d\sigma_n}\mathbb{E}_{\mu_N^\sigma}\left[2\left(i \sum_{i_1 \in F_1^{n,l}} Y_{i_1}\right)\left(-i\right) \mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_2 \in F_1^{n,l}} Y_{i_2}\right] e\left(\xi_1, \xi_2\right)\right]$$

$$= 2\frac{d}{d\sigma_n}\left(\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\right)\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)e\left(\xi_1,\xi_2\right)\right]$$

$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\frac{d}{d\sigma_n}\left(\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)e\left(\xi_1,\xi_2\right)\right]\right)$$

$$= 2\mathbb{E}_{\mu_N^\sigma}\left[Y_nY_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)e\left(\xi_1,\xi_2\right)\right] \tag{3.3.108}$$

$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)e\left(\xi_1,\xi_2\right)\right] \tag{3.3.109}$$

$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[-\mathbb{E}_{\mu_N^\sigma}\left[Y_n\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right]e\left(\xi_1,\xi_2\right)\right] \tag{3.3.110}$$

$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)\left(-i\xi_1\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n\sum_{i_3\in F_1^{n,l}}Y_{i_3}\right]e\left(\xi_1,\xi_2\right)\right] \tag{3.3.111}$$

$$+ 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right)\left(-i\xi_2\right)\mathbb{E}_{\mu_N^\sigma}\left[Y_n\sum_{j\in F_2^{n,l}}Y_j\right]e\left(\xi_1,\xi_2\right)\right]. \tag{3.3.112}$$

We will divide into two cases. The first case is when $|n-l|>2L$ and the second case is when $|n-l|\leq 2L$. In the case when $|n-l|>2L$, we have at least $K\left(K-2L\right)$ pairs of (n,l). Hence in order to get the right estimate we have to apply the decay of correlations (cf. Theorem 1.2.6). The case when $|n-l|\leq 2L$ is much easier because there are at most $2NL$ pairs of (n,l). In this case, we just provide a rough estimate using Corollary 2.1.2, Theorem 1.2.6 and Lemma 3.3.1, which is still good enough for deducing Lemma 3.3.4.

Case 1 $|n-l|>2L$.

We first estimate the terms (3.3.104) to (3.3.112) except for the term (3.3.109). We postpone the estimation of the terms (3.3.103) and (3.3.109).

From now on, we set $(\xi_1,\xi_2)=\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)$ such that $\frac{|\tilde{\xi}|}{\sqrt{N}}\leq\frac{|\xi|}{\sqrt{N}}\leq\delta\leq 1$. Let us begin with the estimation of (3.3.104). We have by using Corollary 2.1.2 and Theorem 1.2.6 that

$$\begin{aligned}\left|T_{(3.3.104)}\right| &\lesssim \left|\mathrm{cov}_{\mu_N^\sigma}\left(Y_n, Y_l\right)\right| \mathbb{E}_{\mu_N^\sigma}\left|\left(\sum_{i\in F_1^{n,l}} Y_i\right)^2\right| \\ &\lesssim L^2 \exp\left(-CL\right).\end{aligned}$$

Let us turn to the estimation of (3.3.105) and (3.3.110). We combine these terms to make a covariance and do the Taylor expansion with respect to the first variable. Then we get

$$\begin{aligned}&T_{(3.3.105)} + T_{(3.3.110)} \\ &\quad = 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right] \mathrm{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2}, e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right) \\ &\quad = 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right] \mathrm{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2}, e\left(0, \frac{\xi}{\sqrt{N}}\right)\right) &(3.3.113) \\ &\qquad + 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right] \mathrm{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2}, \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right) i\frac{\tilde{\xi}}{\sqrt{N}}, \\ &&(3.3.114)\end{aligned}$$

where $\frac{\hat{\xi}}{\sqrt{N}}$ is a real number between 0 and $\frac{\tilde{\xi}}{\sqrt{N}}$. In particular, $\left|\frac{\hat{\xi}}{\sqrt{N}}\right| \leq \left|\frac{\tilde{\xi}}{\sqrt{N}}\right| \leq \frac{|\xi|}{\sqrt{N}}$. Let us consider (3.3.113): We have by using Corollary 2.1.2 and Theorem 1.2.6 that

$$\begin{aligned}\left|T_{(3.3.113)}\right| \leq &\left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right| \left|\mathrm{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_2:\ |i_2-l|\leq L/2} Y_{i_2}, e\left(0, \frac{\xi}{\sqrt{N}}\right)\right)\right| \\ &+ \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right| \left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{\substack{i_2\in F_1^{n,l} \\ |i_2-l|>L/2}} Y_{i_2} e\left(0, \frac{\xi}{\sqrt{N}}\right)\right]\right| \\ &+ \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right| \left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{\substack{i_2\in F_1^{n,l} \\ |i_2-l|>L/2}} Y_{i_2}\right] \mathbb{E}_{\mu_N^\sigma}\left[e\left(0, \frac{\xi}{\sqrt{N}}\right)\right]\right|\end{aligned}$$

$$
\begin{aligned}
&= \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right|\left|\mathrm{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_2:\ |i_2-l|\le L/2} Y_{i_2}, e\left(0,\frac{\xi}{\sqrt{N}}\right)\right)\right| \\
&\quad + \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right|\left|\mathrm{cov}_{\mu_N^\sigma}\left(Y_l, \sum_{\substack{i_2\in F_1^{n,l}\\ |i_2-l|>L/2}} Y_{i_2} e\left(0,\frac{\xi}{\sqrt{N}}\right)\right)\right| \\
&\quad + \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\right|\left|\mathrm{cov}_{\mu_N^\sigma}\left(Y_l, \sum_{\substack{i_2\in F_1^{n,l}\\ |i_2-l|>L/2}} Y_{i_2}\right)\right|\left|\mathbb{E}_{\mu_N^\sigma}\left[e\left(0,\frac{\xi}{\sqrt{N}}\right)\right]\right| \\
&\lesssim L^2 |\xi| \exp(-CL) + L^2(1+|\xi|)\exp(-CL) + L^2\exp(-CL) \\
&\lesssim L^2(1+|\xi|)\exp(-CL).
\end{aligned}
$$

Let us consider (3.3.114): By definition of covariances, it follows that

$$
T_{(3.3.114)} = 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2} \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\frac{\tilde{\xi}}{\sqrt{N}} \tag{3.3.115}
$$

$$
- 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_1\in F_1^{n,l}} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2}\right]\mathbb{E}_{\mu_N^\sigma}\left[\sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\frac{\tilde{\xi}}{\sqrt{N}}. \tag{3.3.116}
$$

We have by taking the conditional expectation with respect to $\mathscr{G}_{n,l} := \sigma\left(X_k, k\in F_1^{n,l}\right)$ and applying Corollary 2.1.2 and Lemma 3.3.1,

$$
\begin{aligned}
&\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2} \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\right| \\
&\quad = \left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2} \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, 0\right)\mathbb{E}_{\mu_N^\sigma}\left[e\left(0,\frac{\xi}{\sqrt{N}}\right) \mid \mathscr{G}_{n,l}\right]\right]\right| \\
&\quad \overset{Lemma\ 3.3.1}{\lesssim} \mathbb{E}_{\mu_N^\sigma}\left[\left|Y_l \sum_{i_2\in F_1^{n,l}} Y_{i_2} \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, 0\right)\right|\right]\left(1+\xi^2\right)\exp\left(-C\xi^2\right) \\
&\quad \overset{Corollary\ 2.1.2}{\lesssim} L^2\left(1+\xi^2\right)\exp\left(-C\xi^2\right),
\end{aligned} \tag{3.3.117}
$$

and similarly one gets

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right]\right| \lesssim L\left(1+\xi^2\right)\exp\left(-C\xi^2\right). \tag{3.3.118}$$

Plugging (3.3.117) and (3.3.118) into (3.3.115) and (3.3.116) and applying Corollary 2.1.2 yield

$$\left|T_{(3.3.114)}\right| \lesssim L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).$$

Therefore we have

$$\begin{aligned}\left|T_{(3.3.105)}+T_{(3.3.110)}\right| &\leq \left|T_{(3.3.113)}\right|+\left|T_{(3.3.114)}\right| \\ &\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right)+L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).\end{aligned}$$

Let us turn to the estimation of (3.3.106). This can also be estimated by taking the conditional expectation with respect to $\mathscr{G}_{n,l}$ and applying Corollary 2.1.2 and Lemma 3.3.1. Indeed,

$$\begin{aligned}\left|T_{(3.3.106)}\right| &= \left|\frac{\tilde{\xi}}{\sqrt{N}}\right|\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_n\sum_{i_2\in F_1^{n,l}}Y_{i_2}\right]\right|\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right]\right| \\ &\lesssim \frac{|\xi|}{\sqrt{N}}L\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(\sum_{i_1\in F_1^{n,l}}Y_{i_1}\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}},0\right)\mathbb{E}_{\mu_N^\sigma}\left[e\left(0,\frac{\xi}{\sqrt{N}}\right)\mid\mathscr{G}_{n,l}\right]\right]\right| \\ &\lesssim L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).\end{aligned}$$

Let us turn to the estimation of (3.3.107). We have by using Corollary 2.1.2 and Theorem 1.2.6 that

$$\begin{aligned}\left|T_{(3.3.107)}\right| &= \frac{|\xi|}{\sqrt{N}}\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_n\sum_{j\in F_2^{n,l}}Y_j\right]\right|\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(\sum_{i\in F_1^{n,l}}Y_i\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right]\right| \\ &\lesssim \frac{|\xi|}{\sqrt{N}}\left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n,\sum_{j\in F_2^{n,l}}Y_j\right)\right|\mathbb{E}_{\mu_N^\sigma}\left|Y_l\left(\sum_{i\in F_1^{n,l}}Y_i\right)^2\right| \\ &\lesssim L^2|\xi|\exp\left(-CL\right),\end{aligned}$$

Let us turn to the estimation of (3.3.108). This also follows from dividing into two parts and applying Corollary 2.1.2 and Theorem 1.2.6 together. More precisely, we have

$$\begin{aligned}\left|T_{(3.3.108)}\right| \leq & \left|2 \sum_{i_1:\ |i_1-n|\leq L} \mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l Y_{i_1}\right] \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right) e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\right| \\ & + \left|2 \sum_{i_1:\ |i_1-l|\leq L} \mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l Y_{i_1}\right] \mathbb{E}_{\mu_N^\sigma}\left[\left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right) e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\right| \\ \leq & \, 2 \sum_{i_1:\ |i_1-n|\leq L} \left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_l, Y_n Y_{i_1}\right)\right| \mathbb{E}_{\mu_N^\sigma}\left|\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right| \\ & + 2 \sum_{i_1:\ |i_1-l|\leq L} \left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n, Y_l Y_{i_1}\right)\right| \mathbb{E}_{\mu_N^\sigma}\left|\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right| \\ \lesssim & \, L^2 \exp\left(-CL\right).\end{aligned}$$

Let us turn to the estimation of (3.3.111). A similar argument as for (3.3.106) implies

$$\left|T_{(3.3.111)}\right| \lesssim L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right).$$

Let us turn to the estimation of (3.3.112). A similar argument as for (3.3.107) yields

$$\left|T_{(3.3.112)}\right| \lesssim L^2 \left|\xi\right| \exp\left(-CL\right).$$

It remains to estimate (3.3.103) and (3.3.109). Recalling the definition (3.3.3) of $F_1^{n,l}$, a direct computation yields

$$\begin{aligned}T_{(3.3.103)} = & -\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \left(\sum_{i_1:\ |i_1-l|\leq L} Y_{i_1} + \sum_{i_2:\ |i_2-n|\leq L} Y_{i_2}\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right] \\ = & -\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \left(\sum_{i_1:\ |i_1-l|\leq L} Y_{i_1}\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right] & (3.3.119) \\ & - \mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \left(\sum_{i_2:\ |i_2-l|\leq L} Y_{i_2}\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right] & (3.3.120) \\ & - 2\mathbb{E}_{\mu_N^\sigma}\left[Y_n Y_l \sum_{i_1:\ |i_1-n|\leq L} Y_{i_1} \sum_{i_2:\ |i_2-l|\leq L} Y_{i_2} e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right], & (3.3.121)\end{aligned}$$

and

$$
\begin{aligned}
T_{(3.3.109)} &= 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_1:\ |i_1-n|\le L} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n\left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right)e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right] \\
&\quad + 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n\left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right)e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right] \\
&= 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_1:\ |i_1-n|\le L} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n\left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right)e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right] \qquad (3.3.122) \\
&\quad + 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_2:\ |i_2-l|\le L} Y_{i_2} e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right] \qquad (3.3.123) \\
&\quad + 2\mathbb{E}_{\mu_N^\sigma}\left[Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right]\mathbb{E}_{\mu_N^\sigma}\left[Y_n \sum_{i_2:\ |i_2-n|\le L} Y_{i_2} e\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right]. \qquad (3.3.124)
\end{aligned}
$$

The terms (3.3.119), (3.3.120), (3.3.122) and (3.3.123) can be estimated using the arguments from above. Let us turn to the estimation of (3.3.119). Taylor expansion with respect to the first variable yields

$$
\begin{aligned}
T_{(3.3.119)} &= -\mathbb{E}_{\mu_N^\sigma}\left[Y_nY_l\left(\sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right)^2 e\left(0,\frac{\xi}{\sqrt{N}}\right)\right] \qquad (3.3.125) \\
&\quad - \mathbb{E}_{\mu_N^\sigma}\left[Y_nY_l\left(\sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right)^2\left(\sum_{i_2\in F_1} Y_{i_2}\right)e\left(\frac{\hat{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)\right] i\frac{\tilde{\xi}}{\sqrt{N}}. \qquad (3.3.126)
\end{aligned}
$$

Let us consider (3.3.125): Note that for i_1 with $|i_1-l|\le L$, we have $|i_1-n|\ge|n-l|-|i_1-l|>2L-L=L$. Thus applying Corollary 2.1.2 and Theorem 1.2.6 yields

$$
\left|T_{(3.3.125)}\right| = \left|\operatorname{cov}_{\mu_N^\sigma}\left(Y_n, Y_l\left(\sum_{i_1:\ |i_1-l|\le L} Y_{i_1}\right)^2 e\left(0,\frac{\xi}{\sqrt{N}}\right)\right)\right| \lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right).
$$

Let us consider (3.3.126): A similar argument as for (3.3.106) yields

$$
\left|T_{(3.3.126)}\right| \lesssim L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right),
$$

and thus

$$
\left|T_{(3.3.119)}\right| = \left|T_{(3.3.125)}+T_{(3.3.126)}\right|
$$

$$\le \left|T_{(3.3.125)}\right| + \left|T_{(3.3.126)}\right|$$
$$\lesssim L^2 (1+|\xi|) \exp(-CL) + L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right).$$

Let us turn to the estimation of (3.3.120). A similar argument as for (3.3.119) yields

$$\left|T_{(3.3.120)}\right| \lesssim L^2 (1+|\xi|) \exp(-CL) + L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right).$$

Let us turn to the estimation of (3.3.122). It follows from Corollary 2.1.2 and Theorem 1.2.6 that

$$\left|T_{(3.3.122)}\right| = \left|2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l, \sum_{i_1:\ |i_1-n|\le L} Y_{i_1}\right)\right| \left|\mathbb{E}_{\mu_N^\sigma}\left[Y_n \left(\sum_{i_2\in F_1^{n,l}} Y_{i_2}\right) e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right]\right|$$
$$\lesssim L^{3/2} \exp(-CL) \lesssim L^2 \exp(-CL).$$

Let us turn to the estimation of (3.3.123). A similar argument as for (3.3.119) yields

$$\left|T_{(3.3.123)}\right| \lesssim L^2 (1+|\xi|) \exp(-CL) + L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right).$$

Let us turn to the estimation of (3.3.121) and (3.3.124). Combining these terms and applying Taylor expansion with respect to the first variable yield

$$T_{(3.3.121)} + T_{(3.3.124)}$$
$$= -2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}, Y_n \sum_{i_2:\ |i_2-n|\le L} Y_{i_2} e\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right)$$
$$= -2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}, Y_n \sum_{i_2:\ |i_2-n|\le L} Y_{i_2} e\left(0, \frac{\xi}{\sqrt{N}}\right)\right) \tag{3.3.127}$$
$$- 2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_1:\ |i_1-l|\le L} Y_{i_1}, Y_n \sum_{i_2:\ |i_2-n|\le L} Y_{i_2} \sum_{i_3\in F_1^{n,l}} Y_{i_3} e\left(\frac{\hat{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right) i\frac{\tilde{\xi}}{\sqrt{N}}. \tag{3.3.128}$$

Let us consider (3.3.127): Dividing into two parts, we get

$$T_{(3.3.127)} = -2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_1:\ |i_1-l|\le L/2} Y_{i_1}, Y_n \left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right) e\left(0, \frac{\xi}{\sqrt{N}}\right)\right) \tag{3.3.129}$$

$$- 2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l \sum_{i_1:\ L/2\le|i_1-l|\le L} Y_{i_1}, Y_n \left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right) e\left(0,\frac{\xi}{\sqrt{N}}\right)\right). \qquad (3.3.130)$$

Note that for (i_1,i_2) with $|i_1-l|\le\frac{L}{2}$ and $|i_2-n|\le L$, it holds that

$$|i_1-i_2|\ge|l-n|-|i_1-l|-|i_2-n|\ge 2L-\frac{L}{2}-L\ge\frac{L}{2}.$$

Therefore Corollary 2.1.2 and Theorem 1.2.6 implies

$$\left|T_{(3.3.129)}\right|\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right),$$

and also by definition of covariances, it holds that

$$\begin{aligned}
\left|T_{(3.3.130)}\right| &\le \left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\left(\sum_{i_1:\ L/2\le|i_1-l|\le L} Y_{i_1}\right)Y_n\left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right)e\left(0,\frac{\xi}{\sqrt{N}}\right)\right]\right| \\
&\quad+\left|2\mathbb{E}_{\mu_N^\sigma}\left[Y_l\sum_{i_1:\ L/2\le|i_1-l|\le L} Y_{i_1}\right]\right|\left|\mathbb{E}_{\mu_N^\sigma}\left[Y_n\left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right)e\left(0,\frac{\xi}{\sqrt{N}}\right)\right]\right| \\
&=\left|2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l,\left(\sum_{i_1:\ L/2\le|i_1-l|\le L} Y_{i_1}\right)Y_n\left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right)e\left(0,\frac{\xi}{\sqrt{N}}\right)\right)\right| \\
&\quad+\left|2\operatorname{cov}_{\mu_N^\sigma}\left(Y_l,\sum_{i_1:\ L/2\le|i_1-l|\le L} Y_{i_1}\right)\right|\mathbb{E}_{\mu_N^\sigma}\left|Y_n\left(\sum_{i_2:\ |i_2-n|\le L} Y_{i_2}\right)\right| \\
&\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right)+L^{3/2}\exp\left(-CL\right)\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right).
\end{aligned}$$

Thus we have

$$\left|T_{(3.3.127)}\right|\le\left|T_{(3.3.129)}\right|+\left|T_{(3.3.130)}\right|\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right).$$

Let us consider (3.3.128): A similar argument as for (3.3.114) yields

$$\left|T_{(3.3.128)}\right|\lesssim L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).$$

Therefore we conclude that

$$\left|T_{(3.3.121)}+T_{(3.3.124)}\right|\le\left|T_{(3.3.127)}\right|+\left|T_{(3.3.128)}\right|$$

$$\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right)+L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).$$

To sum up the estimation of (3.3.103) and (3.3.109), we have

$$\begin{aligned}\left|T_{(3.3.103)}+T_{(3.3.109)}\right| &= \left|\left(T_{(3.3.119)}+T_{(3.3.120)}+T_{(3.3.121)}\right)+\left(T_{(3.3.122)}+T_{(3.3.123)}+T_{(3.3.124)}\right)\right| \\ &\leq \left|T_{(3.3.119)}\right|+\left|T_{(3.3.120)}\right|+\left|T_{(3.3.122)}\right|+\left|T_{(3.3.123)}\right|+\left|T_{(3.3.121)}+T_{(3.3.124)}\right| \\ &\lesssim L^2\left(1+|\xi|\right)\exp\left(-CL\right)+L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right) \\ &\lesssim L^2\left(1+\xi^2\right)\exp\left(-CL\right)+L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).\end{aligned}$$

Overall we have proven the desired estimates that for $|n-l|>2L$,

$$\begin{aligned}&\left|T_{(3.3.99)}+T_{(3.3.100)}\right| \\ &\leq \left|T_{(3.3.103)}+T_{(3.3.109)}\right|+\left|T_{(3.3.104)}\right|+\left|T_{(3.3.105)}+T_{(3.3.110)}\right|+\left|T_{(3.3.106)}\right| \\ &\quad+\left|T_{(3.3.107)}\right|+\left|T_{(3.3.108)}\right|+\left|T_{(3.3.111)}\right|+\left|T_{(3.3.112)}\right| \\ &\lesssim L^2\left(1+\xi^2\right)\exp\left(-CL\right)+L^3\frac{|\xi|}{\sqrt{N}}\left(1+\xi^2\right)\exp\left(-C\xi^2\right).\end{aligned}$$

Case 2 $|n-l|\leq 2L$.

Let us set $(\xi_1,\xi_2)=\left(\frac{\tilde{\xi}}{\sqrt{N}},\frac{\xi}{\sqrt{N}}\right)$ with $\left|\frac{\tilde{\xi}}{\sqrt{N}}\right|\leq\left|\frac{\xi}{\sqrt{N}}\right|\leq\delta\leq 1$. Except for (3.3.107) and (3.3.112), we may apply (3.3.10) in Lemma 3.3.1 followed by (2.1.1). For example, we have

$$\begin{aligned}\left|T_{(3.3.103)}\right| &= \left|\mathbb{E}_{\mu_N^\sigma}\left[Y_nY_l\left(\sum_{i\in F_1^{n,l}}Y_i\right)^2 e\left(\frac{\tilde{\xi}}{\sqrt{N}},0\right)\mathbb{E}_{\mu_N^\sigma}\left[e\left(0,\frac{\xi}{\sqrt{N}}\right)\mid\mathscr{G}_{n,l}\right]\right]\right| \\ &\lesssim \mathbb{E}_{\mu_N^\sigma}\left|Y_nY_l\left(\sum_{i\in F_1^{n,l}}Y_i\right)^2\right|\left(1+\frac{|\xi|^3}{\sqrt{N}}\right)\exp\left(-C\xi^2\right) \\ &\lesssim L^2\left(1+\xi^2\right)\exp\left(-C\xi^2\right).\end{aligned}$$

Similar computations yield

$$\left|T_{(3.3.104)}\right|,\left|T_{(3.3.105)}\right|,\left|T_{(3.3.108)}\right|,\left|T_{(3.3.109)}\right|,\left|T_{(3.3.110)}\right|\lesssim L^2\left(1+\xi^2\right)\exp\left(-C\xi^2\right),$$

$$\left|T_{(3.3.106)}\right|, \left|T_{(3.3.111)}\right| \lesssim L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right).$$

Let us turn to the estimation of (3.3.107) and (3.3.112): Because those terms involve a sum over $F_2^{n,l}$, one needs to apply the decay of correlations (cf. Theorem 1.2.6). It follows that (see also the computation for the case $|n-l| > 2L$)

$$\left|T_{(3.3.107)}\right|, \left|T_{(3.3.112)}\right| \lesssim L^2 |\xi| \exp(-CL).$$

Therefore we have for $|n-l| \leq 2L$,

$$\begin{aligned}
&\left|T_{(3.3.99)} + T_{(3.3.100)}\right| \\
&\leq \left|T_{(3.3.103)}\right| + \left|T_{(3.3.104)}\right| + \left|T_{(3.3.105)}\right| + \left|T_{(3.3.106)}\right| + \left|T_{(3.3.107)}\right| + \left|T_{(3.3.108)}\right| + \left|T_{(3.3.109)}\right| \\
&\quad + \left|T_{(3.3.110)}\right| + \left|T_{(3.3.111)}\right| + \left|T_{(3.3.112)}\right| \\
&\lesssim L^2 |\xi| \exp(-CL) + L^2 \left(1+\xi^2\right) \exp\left(-C\xi^2\right) + L^3 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right) \\
&\lesssim L^2 \left(1+\xi^2\right) \exp(-CL) + L^3 \left(1+\xi^2\right) \exp\left(-C\xi^2\right).
\end{aligned}$$

To conclude, we sum up all the bounds we have proven so far. That is, for $|n-l| > 2L$, we have

$$\begin{aligned}
&\left|\frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right| \\
&\qquad \leq \left|T_{(3.3.99)} + T_{(3.3.100)}\right| + \left|T_{(3.3.101)}\right| + \left|T_{(3.3.102)}\right| \\
&\qquad \lesssim L^3 \left(1+\xi^2\right) \exp(-CL) + L^4 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right)
\end{aligned}$$

and for $|n-l| \leq 2L$,

$$\begin{aligned}
&\left|\frac{d^2}{d\xi_1^2} \frac{d}{d\sigma_n} \frac{d}{d\sigma_l} G_{n,l}\left(\frac{\tilde{\xi}}{\sqrt{N}}, \frac{\xi}{\sqrt{N}}\right)\right| \\
&\qquad \leq \left|T_{(3.3.99)}\right| + \left|T_{(3.3.100)}\right| + \left|T_{(3.3.101)}\right| + \left|T_{(3.3.102)}\right| \\
&\qquad \lesssim L^3 \left(1+\xi^2\right) \exp(-CL) + L^3 \left(1+\xi^2\right) \exp\left(-C\xi^2\right) \\
&\qquad\quad + L^4 \frac{|\xi|}{\sqrt{N}} \left(1+\xi^2\right) \exp\left(-C\xi^2\right)
\end{aligned}$$

$$\lesssim L^3\left(1+\xi^2\right)\exp\left(-CL\right)+L^4\left(1+\xi^2\right)\exp\left(-C\xi^2\right).$$

This finishes the proof of Lemma 3.3.4. □

Remark 9. The proof of Lemma 3.3.3 is almost the same as the proof of Lemma 3.3.4. Carrying out a Taylor expansion with respect to the second variable in $G_{n,l}$ yields

$$\begin{aligned}&\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}(0,\xi_2)\xi_1\\&\quad=\underbrace{\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}(0,0)}_{=0}\xi_1+\frac{d}{d\xi_2}\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}(0,\tilde{\xi}_2)\xi_1\xi_2.\end{aligned}\tag{3.3.131}$$

The first term in (3.3.131) vanishes because

$$\frac{d}{d\xi_1}\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}G_{n,l}(0,0)=\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}\frac{d}{d\xi_1}G_{n,l}(0,0)=\frac{d}{d\sigma_n}\frac{d}{d\sigma_l}\mathbb{E}_{\mu_N^\sigma}\left[i\sum_{i\in F_1^{n,l}}Y_i\right]=0.$$

Then similar arguments applied to the second term in (3.3.131) gives the desired bound.

Chapter 4

Equivalence of ensembles part 2: On the level of observables and correlations

We consider a one-dimensional lattice system of unbounded, real-valued spins with arbitrary strong, quadratic, finite-range interaction. We show the equivalence of the grand canonical ensemble (gce) and the canonical ensemble (ce), in the sense of observables and correlations. A direct consequence is that the correlations of the ce decay exponentially plus a volume correction term. The volume correction term is uniform in the external field, the mean spin and scales optimally in the system size. This extends prior results of Cancrini & Martinelli for bounded discrete spins to unbounded continuous spins. The result is obtained by adapting Cancrini & Martinelli's method combined with auxiliary results developed in previous chapters.

4.1 Introduction

In this chapter, we study the second and third levels of equivalence of ensembles, namely on the level of observables and correlations. Cancrini & Martinelli [CM00] provided quantitative optimal upper bounds of the equivalence of the gce and the ce in the case of bounded discrete spins. The question whether quantitative optimal bounds can be obtained in our setting remained open. In this chapter, we show that this is indeed the case. The main results of this chapter, i.e., Theorem 4.2.2 and Theorem 4.2.3, state that the gce and ce are equivalent. The upper bounds are explicit and are scaling optimally. We therefore extend the results of [CM00] from bounded discrete spins to unbounded continuous spins.

In the proof of Theorem 4.2.2 we follow a Cancrini & Martinelli's method. Like in Cancrini & Martinelli's method, we use Fourier transform to write

$$\begin{aligned}
&\mathbb{E}_{\mu_{\Lambda,m}}\left[f(X)\right] - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right] \\
&= \frac{\int_{\mathbb{R}} \mathbb{E}_{\mu_\Lambda^\sigma}\left[\left(f(X) - \mathbb{E}_{\mu_\Lambda^\sigma}\left[f(X)\right]\right) \exp\left(i\frac{1}{\sqrt{|\Lambda|}}\sum_{i\in\Lambda}\left(X_i - m_i\right)\xi\right)\right] d\xi}{\int_{\mathbb{R}} \mathbb{E}_{\mu_\Lambda^\sigma}\left[\exp\left(i\frac{1}{\sqrt{|\Lambda|}}\sum_{i\in\Lambda}\left(X_i - m_i\right)\xi\right)\right] d\xi}.
\end{aligned}$$

Then we first prove the theorem for an intensive function f which is "almost orthogonal" to the random variable $\sum_{i\in\operatorname{supp} f} X_i$. That is, the covariance between f and $\sum_{i\in\operatorname{supp} f} X_i$ is of order $\frac{1}{|\Lambda|}$. For the general case, we decompose the intensive function f into "almost orthogonal" part and the remainder. We note that this can be done by subtracting $C\sum_{i\in\operatorname{supp} f} X_i$ from f, where C is a suitable constant depending on f. Then by further decomposing the remainder $C\sum_{i\in\operatorname{supp} f} X_i$ into "almost orthogonal remainder" and the rest, we obtain the desired result with a help of the identity

$$\sum_{i\in\Lambda} \mathbb{E}_{\mu_\Lambda^\sigma}\left[X_i\right] = \sum_{i\in\Lambda} \mathbb{E}_{\mu_{\Lambda,m}}\left[X_i\right].$$

The main results of this chapter (see Theorem 4.2.2 and Theorem 4.2.3) complement the recent results of Cancrini & Olla [CO17]. In [CO17], the equivalence of ensembles for *extensive* observables was deduced in particle systems via an Edgeworth expansion, whereas our result applies to *intensive* observables. In particular, the result of [CO17] is optimal proving the Lebowitz-Percus-Verlet formula and applies to a wider class of models. However, it is conditional, i.e., the assumptions are needed to be verified for any particular choice of observable f, while our result is unconditional. It might be possible that their method could be extended to intensive observables but to get an unconditional result, one would need to prove their assumptions for a class of intensive variables. This is equivalent to proving a weaker version of equivalence of ensembles. It is not clear if this is possible. It also might be that for certain intensive observables in non-Gaussian models those assumptions fail. Our results, Theorem 4.2.2 and Theorem 4.2.3, are unconditional and apply to wide class of intensive observables.

An important implication of the equivalence of ensemble is the decay of correlations of the ce (cf. Theorem 1.2.7). While the decay of correlation of an ensemble itself is a very interesting property, it also plays an integral role in deducing that there is no phase transition on the one-dimensional lattice (see Chapter 5). Moreover, it is also shown with the help of decay of correlation that the ce satisfies a uniform logarithmic Sobolev inequality (LSI) (see Chapter 6).

It is natural to ask whether the equivalence of ensembles can be extended to lattices of dimension 2 or higher. The author's answer to this question is optimistic. Our argument does not use that the lattice is one-dimensional. The only ingredient that relies on a one-dimensional lattice is the decay of correlations of the grand canonical ensemble (see Theorem 1.2.6). Under the extra assumption of exponential decay of correlations of the gce, one should be able to use similar calculations to deduce the results for higher dimensional cases. This extra assumption would rule out the presence of phase transitions. However, we only present the proof of one-dimensional case because the notational burden is already heavy and it is better for explaining ideas.

4.2 Main results and outline of the proof

In the remaining sections of this chapter, we only consider the one-dimensional lattice $\Lambda = [N] = \{1, 2, \cdots, N\}$ for better explanation of our ideas and calculations. We follow the conventions made at the end of Section 1.1.1.

4.2.1 Main Results

Let us recall the definition of local, intensive and intensive functions.

Definition 4.2.1 (Local, intensive, and extensive functions/ observables)**.** *For a function $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{C}$, denote* $\operatorname{supp} f$ *by the minimal subset of $\mathbb{Z}$ with $f(x) = f\left(x^{\operatorname{supp} f}\right)$. We call f a local function if it has a finite support independent of the system size N. A function f is called intensive if there is a positive constant ε such that $|\operatorname{supp} f| \lesssim N^{1-\varepsilon}$. A function f is called extensive if it is not intensive.*

The first main result of this section is the equivalence of the ce and gce on the level of observables.

Theorem 4.2.2 (Equivalence of the ce and gce on the level of observables)**.** *Let $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be an intensive function. There are constants $C \in (0, \infty)$ and $N_0 \in \mathbb{N}$ independent of the external field s and the mean spin m such that for all $N \geq N_0$, it holds that*

$$\left|\mathbb{E}_{\mu_N^\sigma}[f] - \mathbb{E}_{\mu_{N,m}}[f]\right| \leq C \frac{|\operatorname{supp} f|}{N} \|\nabla f\|_\infty.$$

We provide the proof of Theorem 4.2.2 in Section 4.2.2.

Let us now turn to the second main result of this chapter:

Theorem 4.2.3 (Equivalence of the ce and gce on the level of correlations)**.** *Let $f, g : \mathbb{R}^N \to \mathbb{R}$ be intensive functions. There exist constants $C \in (0, \infty)$ and $N_0 \in \mathbb{N}$ independent of the external field s and the mean spin m such that for all $N \geq N_0$, it holds that*

$$\begin{aligned}
&\left|\operatorname{cov}_{\mu_{N,m}}(f, g) - \operatorname{cov}_{\mu_N^\sigma}(f, g)\right| \\
&\quad \leq C \, \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C \, dist(\operatorname{supp} f, \operatorname{supp} g)\right)\right).
\end{aligned}$$

We give the proof of Theorem 4.2.3 in Section 4.2.2.

4.2.2 Outline of the proof of main results

As it is common when deducing equivalence of ensembles, we express the difference of observables and correlations between gce and ce using the inverse Fourier transform.

Lemma 4.2.4. *For any function $\zeta, \eta : \mathbb{R}^N \to \mathbb{R}$,*

$$\mathbb{E}_{\mu_{N,m}}\left[\zeta(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right] = \frac{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(\zeta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right]\right) \exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] d\xi}{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] d\xi}$$

and

$$\begin{aligned}
&\operatorname{cov}_{\mu_{N,m}}\left(\zeta(X), \eta(X)\right) - \operatorname{cov}_{\mu_N^\sigma}\left(\zeta(X), \eta(X)\right) \\
&= \frac{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(\zeta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right]\right)\left(\eta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\eta(X)\right]\right) \exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] d\xi}{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] d\xi} \\
&\quad - \left(\mathbb{E}_{\mu_{N,m}}\left[\zeta(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right]\right)\left(\mathbb{E}_{\mu_{N,m}}\left[\eta(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[\eta(X)\right]\right).
\end{aligned}$$

The proof of Lemma 4.2.4 is outlined in the Appendix.

Proof of Theorem 4.2.2 The proof of Theorem 4.2.2 is quite technical. The main technical difficulty comes from the estimation of the first order term in a Taylor expansion. Let us outline how we overcome this obstacle.

We decompose the sublattice $[N]$ into large blocks (cf. Lemma 2.3.3). Then for each intensive observable f, we carefully choose a linear approximation h_f in terms of block spins (cf. (4.2.5)). The key observation is that the difference $h = f - h_f$ satisfies the equivalence of observables of the right order because, by choosing h_f wisely, the problematic first order term in the Taylor expansion becomes small (see proof of Proposition 4.2.5 for more details). We then show that each summand in h_f also satisfies the equivalence of ensembles with the right scaling (cf. Lemma 4.2.7). This step takes advantage of the fact that h_f is linear and that the gce and ce have the same mean. Hence, together with elementary estimations, the linear function h_f can be decomposed into summands sharing a similar structure as $h = f - h_f$ (cf. (4.2.7)); and therefore also satisfy the equivalence of observables of the right order. We refer to Lemma 4.2.7 for more details.

Now let us turn to the detailed arguments. Let us begin with introducing auxiliary notations and definitions that are needed for the proof of Theorem 4.2.2. Recalling the definition (2.3.10) of $B_l(i)$, we decompose $[N]$ as

$$[N] = \bigcup_{j=1}^{M} B_{l_j}(w_j), \tag{4.2.1}$$

where $w_j \in [N]$, $l_j \geq L$ as in Lemma 2.3.3, and the union is disjoint. For notational simplicity, we denote for each $j \in [M]$,

$$B_j = B_{l_j}(w_j). \tag{4.2.2}$$

Then the decomposition (4.2.1) is rewritten as

$$[N] = \bigcup_{j=1}^{M} B_j. \tag{4.2.3}$$

Let us define a map $\varphi : [N] \to [M]$ that matches each site $i \in [N]$ with the block that contains it. More precisely, for each $i \in [N]$, there exists a unique $j(i) \in [M]$ such that $i \in B_{j(i)}$. Let us write

$$B_{\varphi(i)} = B_{j(i)} = B_{l_{j(i)}}(w_{j(i)}) \qquad \text{for each } i \in [N].$$

The first step towards to the proof of Theorem 4.2.2 is considering a special form of functions. Let us recall the definitions (2.4.1) and (2.4.2) of E_S and F_S, respectively. Let us fix an intensive function f. We denote $S = \operatorname{supp} f$ and define

$$c_f := \frac{\operatorname{cov}_{\mu_N^\sigma}\left(f(X), \sum_{j\in E_S} X_j\right)}{\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{j\in E_S} X_j\right)}.$$

Remark 10. By choosing l large enough, the denominator of c_f is bounded from below and hence c_f is well defined. More precisely, by Lemma 2.3.3 we have

$$\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{j\in E_S} X_j\right) \geq CL|S| \gtrsim |\operatorname{supp} f| > 0.$$

Moreover, combined with Lemma 2.3.1 we have the following estimate:

$$|c_f| \lesssim \frac{\|\nabla f\|_\infty}{|\operatorname{supp} f|^{\frac{1}{2}}}. \tag{4.2.4}$$

We define a linear approximation h_f of f as

$$h_f(x) = c_f \sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k \tag{4.2.5}$$

and write the difference h as

$$h(x) = f(x) - h_f(x) = f(x) - c_f \sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k. \tag{4.2.6}$$

The following proposition contains core estimate needed for the proof of Theorem 4.2.2.

Proposition 4.2.5. *There exist uniform constants $N_0 \in \mathbb{N}$ and $C > 0$ independent of the external field s and the mean spin m such that for all $N \geq N_0$,*

$$\left|\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(h(X) - \mathbb{E}_{\mu_N^\sigma}[h(X)]\right)\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] d\xi\right| \leq C\frac{|\operatorname{supp} f|}{N}\|\nabla f\|_\infty.$$

Remark 11. Proposition 4.2.5 was motivated by [CM00, Lemma 4.2]. The main difference is that [CM00] considers l-support while the definition of support of a function f in this paper is the minimal subset of $\mathbb{Z}$ with $f(x) = f(x^{\operatorname{supp} f})$. In [CM00], the assumption that l is large enough was used to guarantee the positiveness of c_f (cf. [CM00, Section 4] and Remark 10). To address this difference, we artificially introduce the block decomposition (4.2.1) of $[N]$ and additionally include a block summation in the definition of c_f (and consequently $h(x)$).

We present the proof of Proposition 4.2.5 in Section 4.3.1. The following is a direct consequence of Lemma 4.2.4, Proposition 3.2.7 and Proposition 4.2.5.

Corollary 4.2.6. *There exist uniform constants $N_0 \in \mathbb{N}$ and $C > 0$ independent of the external field s and the mean spin m such that for all $N \geq N_0$*

$$\left|\mathbb{E}_{\mu_{N,m}}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right| \leq C\frac{|\operatorname{supp} f|}{N}\|\nabla f\|_\infty.$$

We then prove Theorem 4.2.2 for $f = \sum_{k \in B_{\varphi(i)}} X_k$.

Lemma 4.2.7. *For each $i \in S$, it holds that*

$$\left|\mathbb{E}_{\mu_{N,m}}\left[\sum_{k \in B_{\varphi(i)}} X_k\right] - \mathbb{E}_{\mu_N^\sigma}\left[\sum_{k \in B_{\varphi(i)}} X_k\right]\right| \lesssim \frac{1}{N}.$$

Proof of Lemma 4.2.7. Let us fix $i \in S$ and recall the definition (4.2.2) of B_j and the decomposition (4.2.3). For each $j \in [M]$, we set $S_{ij} := B_{\varphi(i)} \cup B_j$. Recalling the definition (2.4.1) of E_S, we analogously denote E_{ij} and F_{ij} by

$$E_{ij} := [N] \cap \{k : \operatorname{dist}(k, S_{ij}) \leq M \log N\},$$
$$F_{ij} := [N] \cap \{k : \operatorname{dist}(k, S_{ij}) > M \log N\}.$$

Similar to the way we defined h from f, let us construct an auxiliary function h_{ij} from $\sum_{k \in B_j} X_j$ as follows:

$$h_{ij}(X) := \sum_{k \in B_j} X_k - c_{ij} \sum_{k \in B_{\varphi(i)}} X_k, \tag{4.2.7}$$

where

$$c_{ij} = \frac{\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{k \in B_j} X_k, \sum_{k \in E_{ij}} X_k\right)}{\operatorname{cov}_{\mu_N^\sigma}\left(\sum_{k \in B_{\varphi(i)}} X_k, \sum_{k \in E_{ij}} X_k\right)}.$$

A similar argument using Lemma 2.3.3 implies that the denominator of c_{ij} is positive, hence h_{ij} well defined (cf. Remark 10). Moreover, there is a positive constant C such that

$$\frac{1}{C} \leq c_{ij} \leq C. \tag{4.2.8}$$

A detailed analysis of the proofs show that the arguments for Proposition 4.2.5 still apply to h_{ij} (and hence Corollary 4.2.6) which implies (see Remark 12 in Section 4.3.1 for more details)

$$\left|\mathbb{E}_{\mu_{N,m}}\left[h_{ij}(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h_{ij}(X)\right]\right| \leq C\frac{1}{N}.$$

We additionally observe that (cf. (2.1.6))

$$\sum_{k=1}^{N} \mathbb{E}_{\mu_{N,m}}\left[X_k\right] = Nm = \sum_{k=1}^{N} \mathbb{E}_{\mu_N^\sigma}\left[X_k\right]. \tag{4.2.9}$$

Thus an application of triangle inequality yields

$$\begin{aligned}
\frac{M}{N} &\gtrsim \left|\sum_{j=1}^{M} \left(\mathbb{E}_{\mu_{N,m}}\left[h_{ij}(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h_{ij}(X)\right]\right)\right| \\
&= \Bigg|\left(\sum_{j=1}^{M}\sum_{k\in B_j} \mathbb{E}_{\mu_{N,m}}\left[X_k\right] - \sum_{j=1}^{M} c_{ij} \sum_{k\in B_{\varphi(i)}} \mathbb{E}_{\mu_{N,m}}\left[X_k\right]\right) \\
&\qquad - \left(\sum_{j=1}^{M}\sum_{k\in B_j} \mathbb{E}_{\mu_N^\sigma}\left[X_k\right] - \sum_{j=1}^{M} c_{ij} \sum_{k\in B_{\varphi(i)}} \mathbb{E}_{\mu_N^\sigma}\left[X_k\right]\right)\Bigg| \\
&\overset{(4.2.3),(4.2.9)}{=} \left|\left(\mathbb{E}_{\mu_{N,m}}\left[\sum_{k\in B_{\varphi(i)}} X_k\right] - \mathbb{E}_{\mu_N^\sigma}\left[\sum_{k\in B_{\varphi(i)}} X_k\right]\right)\sum_{j=1}^{M} c_{ij}\right| \\
&\overset{(4.2.8)}{=} \left|\mathbb{E}_{\mu_{N,m}}\left[\sum_{k\in B_{\varphi(i)}} X_k\right] - \mathbb{E}_{\mu_N^\sigma}\left[\sum_{k\in B_{\varphi(i)}} X_k\right]\right|\sum_{j=1}^{M} c_{ij}.
\end{aligned}$$

Now we conclude from (4.2.8) that, as desired,

$$\left|\mathbb{E}_{\mu_{N,m}}\left[\sum_{k\in B_{\varphi(i)}} X_k\right] - \mathbb{E}_{\mu_N^\sigma}\left[\sum_{k\in B_{\varphi(i)}} X_k\right]\right| \lesssim \frac{1}{N}.$$

∎

We are now ready to provide the proof of our first main result, Theorem 4.2.2.

Proof of Theorem 4.2.2. Let us recall the definition (4.2.6) of h. A combination of Corollary 4.2.6, Lemma 4.2.7, and (4.2.4) gives

$$\begin{aligned}
&\left|\mathbb{E}_{\mu_{N,m}}\left[f(X)\right]-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right| \\
&\qquad \le \left|\mathbb{E}_{\mu_{N,m}}\left[h(X)\right]-\mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right| + |c_f|\sum_{i\in S}\left|\mathbb{E}_{\mu_{N,m}}\left[\sum_{k\in B_{\varphi(i)}} X_k\right]-\mathbb{E}_{\mu_N^\sigma}\left[\sum_{k\in B_{\varphi(i)}} X_k\right]\right| \\
&\qquad \lesssim \frac{|\operatorname{supp} f|}{N}\|\nabla f\|_\infty + \frac{\|\nabla f\|_\infty}{|\operatorname{supp} f|^{\frac{1}{2}}}|\operatorname{supp} f|\frac{1}{N} \\
&\qquad \lesssim \frac{|\operatorname{supp} f|}{N}\|\nabla f\|_\infty.
\end{aligned}$$

∎

Proof of Theorem 4.2.3 The next proposition provides a core estimate that is needed in the proof of Theorem 4.2.3.

Proposition 4.2.8. *For any intensive functions $f, g : \mathbb{R}^N \to \mathbb{R}$, there exist constants $N_0 \in \mathbb{N}$ and $C > 0$ independent of the external field s and the mean spin m such that for all $N \ge N_0$,*

$$\begin{aligned}
&\left|\int \mathbb{E}_{\mu_N^\sigma}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right)\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^N\left(X_k-m_k\right)\xi\right)\right]d\xi\right| \\
&\qquad \le C\|\nabla f\|_\infty\|\nabla g\|_\infty\left(\frac{|\operatorname{supp} f|+|\operatorname{supp} g|}{N}+\exp\left(-dist\left(\operatorname{supp} f\ ,\ \operatorname{supp} g\right)\right)\right).
\end{aligned}$$

We present the proof of Proposition 4.2.8 in Section 4.3.2. Let us now provide the proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. A combination of Lemma 4.2.4, Theorem 4.2.2, Proposition 3.2.7

and Proposition 4.2.8 implies that

$$
\begin{aligned}
&\left|\operatorname{cov}_{\mu_N^\sigma}\left(f(X), g(X)\right)-\operatorname{cov}_{\mu_{N,m}}\left(f(X), g(X)\right)\right| \\
&\leq\left|\frac{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma}[f(X)]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma}[g(X)]\right) \exp \left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N\left(X_i-m_i\right) \xi\right)\right] d \xi}{\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\exp \left(i \frac{1}{\sqrt{N}} \sum_{i=1}^N\left(X_i-m_i\right) \xi\right)\right] d \xi}\right| \\
&\quad+\left|\mathbb{E}_{\mu_N^\sigma}[f(X)]-\mathbb{E}_{\mu_N^\sigma}[f(X)]\right|\left|\mathbb{E}_{\mu_N^\sigma}[g(X)]-\mathbb{E}_{\mu_N^\sigma}[g(X)]\right| \\
&\lesssim\|\nabla f\|_{\infty}\|\nabla g\|_{\infty}\left(\frac{|\operatorname{supp} f|+|\operatorname{supp} g|}{N}+\exp (-\operatorname{dist}(\operatorname{supp} f\ ,\ \operatorname{supp} g))\right) \\
&\quad+\frac{|\operatorname{supp} f||\operatorname{supp} g|}{N^2}\|\nabla f\|_{\infty}\|\nabla g\|_{\infty} \\
&\lesssim\|\nabla f\|_{\infty}\|\nabla g\|_{\infty}\left(\frac{|\operatorname{supp} f|+|\operatorname{supp} g|}{N}+\exp (-\operatorname{dist}(\operatorname{supp} f\ ,\ \operatorname{supp} g))\right) .
\end{aligned}
$$

□

4.3 Proof of Propositions

4.3.1 Proof of Proposition 4.2.5

The main argument for the proof of Proposition 4.2.5 follows a well known method for deducing local CLT. Like in the proof of Theorem 3.2.1, the integral is divided into inner and outer parts that are estimated separately. More precisely, let us fix $\delta > 0$ small enough and decompose the integral as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(h(X)-\mathbb{E}_{\mu_N^\sigma}[h(X)]\right) \exp \left(i \frac{1}{\sqrt{N}} \sum_{k=1}^N\left(X_k-m_k\right) \xi\right)\right] d \xi \\
&=\int_{\left\{\left|(1/\sqrt{N}) \xi\right| \leq \delta\right\}} \mathbb{E}_{\mu_N^\sigma}\left[\left(h(X)-\mathbb{E}_{\mu_N^\sigma}[h(X)]\right) \exp \left(i \frac{1}{\sqrt{N}} \sum_{k=1}^N\left(X_k-m_k\right) \xi\right)\right] d \xi \qquad (4.3.1)\\
&\quad+\int_{\left\{\left|(1/\sqrt{N}) \xi\right|>\delta\right\}} \mathbb{E}_{\mu_N^\sigma}\left[\left(h(X)-\mathbb{E}_{\mu_N^\sigma}[h(X)]\right) \exp \left(i \frac{1}{\sqrt{N}} \sum_{k=1}^N\left(X_k-m_k\right) \xi\right)\right] d \xi . \qquad (4.3.2)
\end{aligned}
$$

The estimation of the outer integral $T_{(4.3.2)}$ is the easy part. As it is usual when deducing local CLTs, the outer integral $T_{(4.3.2)}$ actually decays exponentially in the system size (see the proof of Theorem 3.2.1). Hence, it holds

$$\left|T_{(4.3.2)}\right| \lesssim \|\nabla f\|_{\infty} \frac{|\operatorname{supp} f|}{N}. \tag{4.3.3}$$

The subtle part of the argument is the estimation of the inner integral (4.3.1).

Lemma 4.3.1. *It holds that*

$$\left|T_{(4.3.1)}\right| \lesssim \|\nabla f\|_{\infty} \frac{|\operatorname{supp} f|}{N}.$$

Proposition 4.2.5 is a direct consequence of (4.3.3) and Lemma 4.3.1.

Proof of Proposition 4.2.5. A combination of (4.3.3) and Lemma 4.3.1 proves the Proposition 4.2.5. □

Let us see how we estimate the inner integral (4.3.1). We begin with introducing auxiliary definitions and notations for proof of Lemma 4.3.1. We set $S = \operatorname{supp} f$ and let us recall the definition (2.4.1) and (2.4.2) of the sets E_S and F_S and the decomposition (2.4.3) of the gce μ_N^σ. To reduce the notational burden we write

$$\mu_N^\sigma(dx|y) = \mu_N^\sigma(dx^{E_S}|y^{F_S}) \qquad \text{and} \qquad \bar{\mu}^\sigma(dy) = \bar{\mu}^\sigma(dy^{F_S}).$$

We observe that by the law of total covariance, the integrand in (4.3.1) can be written as

$$\begin{aligned}
&\mathbb{E}_{\mu_N^\sigma}\left[\left(h(X) - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right) \exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\,\xi\right)\right] \\
&= \operatorname{cov}_{\mu_N^\sigma}\left(h(X), \exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - m_i)\,\xi\right)\right) \\
&= \operatorname{cov}_{\mu_N^\sigma}\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right], \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - m_i)\,\xi\right)\right]\right) \qquad (4.3.4) \\
&\quad + \mathbb{E}_{\mu_N^\sigma}\left[\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - m_i)\,\xi\right)\right)\right]. \qquad (4.3.5)
\end{aligned}$$

We estimate (4.3.4) and (4.3.5) separately.

Lemma 4.3.2. *Under the same assumptions as in Lemma 4.3.1, it holds that*

$$\left|T_{(4.3.4)}\right| \lesssim \|\nabla f\|_{\infty} |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM\log N\right) \exp\left(-C\xi^2\right).$$

Proof of Lemma 4.3.2. Let us further decompose the set F_S into the boundary set (with respect to E_S) F_S^1 and the exterior set F_S^2 as follows:

$$F_S^1 := \{i \in F_S : \operatorname{dist}(i, E_S) \le R\}, \tag{4.3.6}$$

$$F_S^2 := \{i \in F_S : \operatorname{dist}(i, E_S) > R\}. \tag{4.3.7}$$

We note that $[N]$ is decomposed into

$$[N] = E_S \cup F_S^1 \cup F_S^2,$$

where the union is disjoint. We also denote for each $i \in E_S$

$$\tilde{m}_i := \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[X_i\right]. \tag{4.3.8}$$

We write

$$\begin{aligned}
&\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right] \\
&= \exp\left(i\frac{1}{\sqrt{N}}\sum_{j\in F_S^1}(X_j - m_j)\,\xi\right)\cdot\exp\left(i\frac{1}{\sqrt{N}}\sum_{j\in F_S^2}(X_j - m_j)\,\xi\right) \\
&\qquad \times \exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(\tilde{m}_i - m_i)\,\xi\right)\cdot\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i - \tilde{m}_i)\,\xi\right)\right] \\
&= A\cdot B\cdot C\cdot D.
\end{aligned} \tag{4.3.9}$$

Due to the finite range interaction (with interaction range R), the conditional expectations

$$\tilde{m}_i = \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[X_i\right],\ \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right],\ \text{and}\ \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i - \tilde{m}_i)\,\xi\right)\right]$$

are only dependent on spins at F_S^1 (and thus independent of spins at F_S^2). In particular, A, C, D from (4.3.9) and $\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right]$ are only dependent on spins at F_S^1. Thus we have

$$
\begin{aligned}
T_{(4.3.4)} &= \mathbb{E}_{\mu_N^\sigma}\left[\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right) A\cdot B\cdot C\cdot D\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\mathbb{E}_{\mu_N^\sigma}\left[\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right) A\cdot B\cdot C\cdot D \;\middle|\; X_i, i\in E_S\cup F_S^1\right]\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right) A\cdot C\cdot D\cdot \mathbb{E}_{\mu_N^\sigma}\left[B \mid X_i, i\in E_S\cup F_S^1\right]\right] \\
&= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{j\in F_S^1}\left(X_j - m_j\right)\xi + i\frac{1}{\sqrt{N}}\sum_{i\in E_S}\left(\tilde{m}_i - m_i\right)\xi\right)\right. \\
&\qquad \times \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}\left(X_i - \tilde{m}_i\right)\xi\right)\right]\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right) \\
&\qquad\qquad \left.\times\, \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k\in F_S^2}\left(X_j - m_j\right)\xi\right) \;\middle|\; X_i, i\in E_S\cup F_S^1\right]\right].
\end{aligned}
\tag{4.3.10}
$$

It holds by Lemma 2.4.1 that

$$
\begin{aligned}
&\left|\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[h(X)\right]\right| \\
&\quad = \left|\int\left(\mathbb{E}_{\mu_N^\sigma(dx^{E_S}|y^{F_S})}\left[h(X)\right] - \mathbb{E}_{\mu_N^\sigma(dx^{E_S}|z^{F_S})}\left[h(X)\right]\right)\bar{\mu}^\sigma(dz^{F_S})\right| \\
&\quad \lesssim \int \|\nabla h\|_\infty \left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \exp\left(-CM\log N\right)\bar{\mu}^\sigma(dz^{F_S}).
\end{aligned}
\tag{4.3.11}
$$

Then a combination of (4.3.10), (4.3.11) and Lemma 3.3.1 from below yields

$$
\begin{aligned}
\left|T_{(4.3.4)}\right| &\lesssim \|\nabla h\|_\infty \exp\left(-CM\log N\right)\exp\left(-C\xi^2\right) \\
&\qquad \times \mathbb{E}_{\mu_N^\sigma}\left[\int\left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}}\bar{\mu}^\sigma(dz^{F_S})\right].
\end{aligned}
\tag{4.3.12}
$$

Because there are at most $2R^2|\operatorname{supp} f| \sim |\operatorname{supp} f|$ many pairs of (i,j) with $i\in E_S$, $j\in F_S$

with $|i-j| \le R$, an application of Schwarz inequality implies, as desired,

$$
\begin{aligned}
&\mathbb{E}_{\mu_N^\sigma}\left[\int\left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}^2(y_j-z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz^{F_S})\right]\\
&\lesssim \left(\int\int \sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}^2(y_j-z_j)^2 \bar{\mu}^\sigma(dz^{F_S})\mu_N^\sigma(dy^{F_S})\right)^{\frac{1}{2}}\\
&= \left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} \int\int M_{ij}^2(y_j-z_j)^2 \mu_N^\sigma(dz^{F_S})\mu_N^\sigma(dy^{F_S})\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} 2\operatorname{var}_{\mu_N^\sigma}(X_j)\right)^{\frac{1}{2}} \overset{\text{Corollary 2.1.2}}{\lesssim} |\operatorname{supp} f|^{\frac{1}{2}}. \qquad (4.3.13)
\end{aligned}
$$

By the definition (4.2.6) and the inequality (4.2.4), we have

$$
\|\nabla h\|_\infty \le \|\nabla f\|_\infty + |c_f|\left\|\nabla\left(\sum_{i\in S} X_i\right)\right\|_\infty \lesssim \|\nabla f\|_\infty. \qquad (4.3.14)
$$

It also holds from definition (4.2.6) that $|\operatorname{supp} f| = |\operatorname{supp} h|$. Therefore we conclude from (4.3.12), (4.3.13) and (4.3.14) that

$$
\left|T_{(4.3.4)}\right| \lesssim \|\nabla f\|_\infty |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM\log N\right)\exp\left(-C\xi^2\right).
$$

□

The next statement is an estimation of (4.3.5).

Lemma 4.3.3. *Under the same settings as in Lemma 4.3.1, it holds that*

$$
\left|T_{(4.3.5)}\right| \lesssim \|\nabla f\|_\infty\left(1+\int\left(\sum_{\substack{i\in E_S, j\in F_S\\|i-j|\le R}} M_{ij}^2(y_j-z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz)\right)\frac{|\xi|}{N}
$$

$$+\|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}\xi^2 + \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}|\xi|^3.$$

Proof of Lemma 4.3.3. Let us recall the definition (4.3.6) and (4.3.7) of F_S^1 and F_S^2, respectively. As in Lemma 4.3.2, we write (4.3.5) as (see (4.3.10))

$$\begin{aligned} T_{(4.3.5)} = \mathbb{E}_{\mu_N^\sigma} \Bigg[&\exp\left(i\frac{1}{\sqrt{N}} \sum_{j\in F_S^1} (X_j - m_j)\,\xi + i\frac{1}{\sqrt{N}} \sum_{i\in E_S} (\tilde{m}_i - m_i)\,\xi \right) \\ &\times \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}} \sum_{i\in E_S} (X_i - \tilde{m}_i)\,\xi \right)\right) \\ &\times \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}} \sum_{k\in F_S^2} (X_j - m_j)\,\xi \right) \,\middle|\, X_i, i\in E_S \cup F_S^1 \right]\Bigg]. \end{aligned} \tag{4.3.15}$$

We then apply third order Taylor expansions to get

$$\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}} \sum_{i\in E_S} (X_i - \tilde{m}_i)\,\xi \right)\right) \tag{4.3.16}$$

$$= \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \sum_{i\in E_S} (X_i - \tilde{m}_i) \right) i\frac{1}{\sqrt{N}}\xi \tag{4.3.17}$$

$$+\frac{1}{2}\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \left(\sum_{i\in E_S} (X_i - \tilde{m}_i)\right)^2\right)\left(i\frac{1}{\sqrt{N}}\xi\right)^2 \tag{4.3.18}$$

$$+\frac{1}{6}\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \left(\sum_{i\in E_S} (X_i - \tilde{m}_i)\right)^3 \exp\left(i\frac{1}{\sqrt{N}} \sum_{i\in E_S} (X_i - \tilde{m}_i)\,\tilde{\xi} \right)\right)\left(i\frac{1}{\sqrt{N}}\xi\right)^3, \tag{4.3.19}$$

where $\tilde{\xi}$ is a real number between 0 and ξ. Let us begin with estimation of (4.3.17). Recalling the definition (4.2.6) of the function h, it holds that

$$\begin{aligned} \left|T_{(4.3.17)}\right| &= \left| \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(f, \sum_{i\in E_S} X_i \right) - c_f \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{i\in E_S} X_i \right)\right| \frac{|\xi|}{\sqrt{N}} \\ &\le \left| \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(f, \sum_{i\in E_S} X_i \right) - \operatorname{cov}_{\mu_N^\sigma}\left(f, \sum_{i\in E_S} X_i \right)\right| \frac{|\xi|}{\sqrt{N}} \\ &\quad + \left| \operatorname{cov}_{\mu_N^\sigma}\left(f, \sum_{i\in E_S} X_i \right) - c_f \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{i\in E_S} X_i \right)\right| \frac{|\xi|}{\sqrt{N}} \end{aligned}$$

$$= \left| \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(f, \sum_{i\in E_S} X_i\right) - \operatorname{cov}_{\mu_N^\sigma}\left(f, \sum_{i\in E_S} X_i\right)\right| \frac{|\xi|}{\sqrt{N}} \tag{4.3.20}$$

$$+ |c_f| \left| \operatorname{cov}_{\mu_N^\sigma}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{j\in E_S} X_j\right) - \operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(\sum_{i\in S}\sum_{k\in B_{\varphi(i)}} X_k, \sum_{j\in E_S} X_j\right)\right| \frac{|\xi|}{\sqrt{N}}. \tag{4.3.21}$$

Corollary 2.4.3 implies that

$$\begin{aligned} T_{(4.3.20)} &\lesssim \|\nabla f\|_\infty \exp\left(-CM\log N\right) \frac{|\xi|}{\sqrt{N}} \\ &\quad + |E_S|\, \|\nabla f\|_\infty \exp\left(-CM\log N\right) \int \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz) \frac{|\xi|}{\sqrt{N}} \\ &\quad + |E_S|\, \|\nabla f\|_\infty |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM\log N\right) \frac{|\xi|}{\sqrt{N}}. \end{aligned}$$

Because $|E_S| \le 2|\operatorname{supp} f| M \log N$, it holds for N large enough that

$$\left|T_{(4.3.20)}\right| \lesssim \|\nabla f\|_\infty \left(1 + \int \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz)\right) \frac{|\xi|}{N}.$$

Similarly, using Corollary 2.4.3 and (4.2.4), we get

$$\left|T_{(4.3.21)}\right| \lesssim \|\nabla f\|_\infty \left(1 + \int \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz)\right) \frac{|\xi|}{N}.$$

Therefore

$$\begin{aligned} \left|T_{(4.3.17)}\right| &\le \left|T_{(4.3.20)}\right| + \left|T_{(4.3.21)}\right| \\ &\lesssim \|\nabla f\|_\infty \left(1 + \int \left(\sum_{\substack{i\in E_S, j\in F_S \\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz)\right) \frac{|\xi|}{N}. \end{aligned}$$

The estimate for (4.3.18) follows from Lemma 2.3.2:

$$\left|T_{(4.3.18)}\right| \lesssim \|\nabla h\|_{L^4(\mu_N^\sigma(dx|y))} |\operatorname{supp} h| \frac{\xi^2}{N} \le \|\nabla h\|_\infty \frac{|\operatorname{supp} h|}{N} \xi^2 \overset{(4.3.14)}{\lesssim} \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N} \xi^2.$$

Let us turn to the estimation of (4.3.19). By applying Hölder's inequality we have

$$
\begin{aligned}
&\left|\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \left(\sum_{i\in E_S}(X_i-\tilde{m}_i)\right)^3 \exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i-\tilde{m}_i)\,\tilde{\xi}\right)\right)\right| \\
&=\left|\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(h(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right]\right)\left(\sum_{i\in E_S}(X_i-\tilde{m}_i)\right)^3 \exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i-\tilde{m}_i)\,\tilde{\xi}\right)\right]\right| \\
&\leq \|h(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right]\|_{L^4(\mu_N^\sigma(dx|y))}\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(\sum_{i\in E_S}(X_i-\tilde{m}_i)\right)^4\right]\right)^{\frac{3}{4}}. \qquad (4.3.22)
\end{aligned}
$$

A combination of (4.2.4) and Lemma 2.1.8 yields

$$
\begin{aligned}
&\|h(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right]\|_{L^4(\mu_N^\sigma(dx|y))} \\
&\qquad \leq \|f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\|_{L^4(\mu_N^\sigma(dx|y))}+|c_f|\left\|\sum_{i\in S}\sum_{k\in B_{\varphi(i)}}(X_k-\tilde{m}_k)\right\|_{L^4(\mu_N^\sigma(dx|y))} \\
&\qquad \lesssim \|\nabla f\|_\infty+\frac{\|\nabla f\|_\infty}{|\operatorname{supp} f|^{\frac{1}{2}}}|\operatorname{supp} f|^{\frac{1}{2}} \lesssim \|\nabla f\|_\infty. \qquad (4.3.23)
\end{aligned}
$$

Thus we conclude from (4.3.22), (4.3.23) and Lemma 2.1.8 that

$$
\left|T_{(4.3.19)}\right| \overset{(4.3.23),\ Lemma\ 2.1.8}{\lesssim} \|\nabla f\|_\infty |E_S|^{\frac{3}{2}}\frac{|\xi|^3}{N^{\frac{3}{2}}} \lesssim \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}|\xi|^3,
$$

where we used $|E_S|\leq 2|\operatorname{supp} f|M\log N$ and thus for N large,

$$
\frac{|E_S|^{\frac{3}{2}}}{N^{\frac{3}{2}}} \lesssim \frac{|\operatorname{supp} f|}{N}.
$$

Collecting all the estimates we have proven so far, we get

$$
\begin{aligned}
\left|T_{(4.3.16)}\right| &\leq \left|T_{(4.3.17)}\right|+\left|T_{(4.3.18)}\right|+\left|T_{(4.3.19)}\right| \\
&\lesssim \|\nabla f\|_\infty\left(1+\int\left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\leq R}} M_{ij}^2(y_j-z_j)^2\right)^{\frac{1}{2}}\bar{\mu}^\sigma(dz)\right)\frac{|\xi|}{N} \\
&\quad +\|\nabla f\|_\infty\frac{|\operatorname{supp} f|}{N}\xi^2+\|\nabla f\|_\infty\frac{|\operatorname{supp} f|}{N}|\xi|^3.
\end{aligned}
$$

$\square$

Lemma 4.3.4 is a direct consequence of Lemma 4.3.3.

Lemma 4.3.4. *Under the same settings as in Lemma 4.3.1, it holds that*

$$\left|\mathbb{E}_{\mu_N^\sigma}\left[\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right)\right]\right| \lesssim \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N} \exp\left(-C\xi^2\right).$$

Proof of Lemma 4.3.4. Let us recall the decomposition (4.3.15). We recall also estimation (4.3.13), which implies

$$\mathbb{E}_{\mu_N^\sigma}\left[\int \left(\sum_{\substack{i\in E_S, j\in F_S\\ |i-j|\le R}} M_{ij}^2 (y_j - z_j)^2\right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz)\right] \lesssim |\operatorname{supp} f|^{\frac{1}{2}},.$$

A combination of Lemma 3.3.1 and Lemma 4.3.3 yields

$$\begin{aligned}&\left|\mathbb{E}_{\mu_N^\sigma}\left[\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right)\right]\right| \\ &\quad\lesssim \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}\left(|\xi| + \xi^2 + |\xi|^3\right)\exp\left(-C\xi^2\right) \lesssim \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}\exp\left(-C\xi^2\right).\end{aligned}$$

□

Now we are ready to give a proof of Lemma 4.3.1.

Proof of Lemma 4.3.1. The law of total covariance implies

$$\begin{aligned}&\operatorname{cov}_{\mu_N^\sigma}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right) \\ &\quad= \operatorname{cov}_{\mu_N^\sigma}\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[h(X)\right], \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right]\right) \\ &\qquad+ \mathbb{E}_{\mu_N^\sigma}\left[\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right)\right].\end{aligned}$$

By Lemma 4.3.2 and Lemma 4.3.4 it holds that for M, N large enough,

$$\begin{aligned}&\left|\int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \operatorname{cov}_{\mu_N^\sigma}\left(h(X), \exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m_i)\,\xi\right)\right) d\xi\right| \\ &\quad\lesssim \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \|\nabla f\|_\infty |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM\log N\right)\exp\left(-C\xi^2\right) d\xi \\ &\qquad+ \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}\exp\left(-C\xi^2\right) d\xi\end{aligned}$$

$$\begin{aligned}
&\leq \int \|\nabla f\|_\infty |\operatorname{supp} f|^{\frac{1}{2}} \exp\left(-CM \log N\right) \exp\left(-C\xi^2\right) d\xi \\
&\quad + \int \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N} \exp\left(-C\xi^2\right) d\xi \\
&\lesssim \|\nabla f\|_\infty \frac{|\operatorname{supp} f|}{N}.
\end{aligned}$$

□

Remark 12. A detailed review show that the arguments in this section can be adapted to yield similar results applied to h_{ij} (see (4.2.7)) instead of the function h. The only place where one should check details is the proof of Lemma 4.3.3, especially the estimation of $T_{(4.3.17)}$. We choose not to outline the details because they would yield many redundancies.

4.3.2 Proof of Proposition 4.2.8

Proof of Proposition 4.2.8 follows the same idea of Proposition 4.2.5 with more careful estimation. We follow similar calculations as in the proof of Proposition 3.2.7. Instead of a second order Taylor expansion we use this time a third order Taylor expansion, which leads to improved estimates.

In this section, the set S denotes union of $\operatorname{supp} f$ and $\operatorname{supp} g$, i.e. $S = \operatorname{supp} f \cup \operatorname{supp} g$. Let us recall the definition (2.4.1) and (2.4.2) of the sets E_S and F_S and the decomposition (2.4.3) of the gce μ_N^σ. We write

$$\mu_N^\sigma(dx|y) = \mu_N^\sigma(dx^{E_S}|y^{F_S}), \qquad \bar{\mu}^\sigma(dy) = \bar{\mu}^\sigma(dy^{F_S}).$$

As before, the integral is divided into inner and outer parts and estimated separately. More precisely, let us fix $\delta > 0$ small enough and decompose the integral as

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}_{\mu_N^\sigma}\left[\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right) \exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^N (X_k - m_k)\,\xi\right)\right] d\xi \\
&\quad = \int_{\{|(1/\sqrt{N})\xi| \leq \delta\}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^N (X_k - m_k)\,\xi\right)\right.
\end{aligned}$$

$$\times \left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right)\Bigg] d\xi$$

$$+ \int_{\{|(1/\sqrt{N})\xi|>\delta\}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k - m_k)\xi\right)\right.$$

$$\left. \times \left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right)\right] d\xi \tag{4.3.24}$$

As in the proof of Proposition 4.2.5, the estimation of outer integral (4.3.24) follows from a slight modification of arguments in the proof of Proposition 3.2.7. More precisely, we have

$$\left|T_{(4.3.24)}\right| \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C d_{f,g}\right)\right). \tag{4.3.25}$$

Let us state and prove the following lemma, which corresponds to Lemma 4.3.1 in the proof of Proposition 4.2.3.

Lemma 4.3.5. *It holds that*

$$\Bigg| \int_{\{|(1/\sqrt{N})\xi|\le\delta\}} \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k - m_k)\xi\right)\right.$$

$$\left. \times \left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right)\right] d\xi \Bigg|$$

$$\lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C d_{f,g}\right)\right).$$

Proof of Proposition 4.2.8. This directly follows from Lemma 4.3.5 and (4.3.25). □

To prove Lemma 4.3.5, we first write

$$\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k - m_k)\xi\right)\left(f(X) - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma}\left[g(X)\right]\right)\right]$$

$$= \mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k - m_k)\xi\right)\right.$$

$$\left. \times \left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\right] \tag{4.3.26}$$

$$+\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right.$$
$$\left.\times\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}[f(X)]-\mathbb{E}_{\mu_N^\sigma}[f(X)]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}[g(X)]\right)\right] \tag{4.3.27}$$
$$+\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right.$$
$$\left.\times\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}[f(X)]\right)\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}[g(X)]-\mathbb{E}_{\mu_N^\sigma}[g(X)]\right)\right] \tag{4.3.28}$$
$$+\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(X_k-m_k)\,\xi\right)\right.$$
$$\left.\times\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}[f(X)]-\mathbb{E}_{\mu_N^\sigma}[f(X)]\right)\left(\mathbb{E}_{\mu_N^\sigma(dx|y)}[g(X)]-\mathbb{E}_{\mu_N^\sigma}[g(X)]\right)\right]. \tag{4.3.29}$$

Lemma 4.3.6. *It holds that*

$$\left|T_{(4.3.26)}\right|\lesssim\|\nabla f\|_\infty\|\nabla g\|_\infty\left(\frac{|\operatorname{supp} f|+|\operatorname{supp} g|}{N}+\exp\left(-Cd_{f,g}\right)\right)\exp\left(-C\xi^2\right).$$

Proof of Lemma 4.3.6 Let us recall the decomposition (4.3.6), (4.3.7) of F_S^1, F_S^2 and the definition (4.3.8) of $\tilde{m}_i$. It holds that (see for example (4.3.10) and (4.3.15))

$$\begin{aligned}T_{(4.3.26)}=\mathbb{E}_{\mu_N^\sigma}&\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{j\in F_S^1}(X_j-m_j)\,\xi+i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(\tilde{m}_i-m_i)\,\xi\right)\right.\\&\times\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i-\tilde{m}_i)\,\xi\right)\right.\\&\qquad\left.\times\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}[f(X)]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}[g(X)]\right)\right]\\&\left.\times\,\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{k\in F_S^2}(X_j-m_j)\,\xi\right)\middle|\;X_i,i\in E_S\cup F_S^1\right]\right],\end{aligned} \tag{4.3.30}$$

Taylor expansion implies that there is $\tilde{\xi}$ between 0 and ξ such that

$$\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}[f(X)]\right)\right.$$

$$\times \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i - \tilde{m}_i)\,\xi\right)\Bigg] \tag{4.3.31}$$

$$= \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\right] \tag{4.3.32}$$

$$+ \mathbb{E}_{\mu_N^\sigma(dx|y)}\Bigg[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)$$

$$\times \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{i\in E_S}(X_i - \tilde{m}_i)\Bigg]\left(i\frac{\xi}{\sqrt{N}}\right) \tag{4.3.33}$$

$$+ \frac{1}{2}\mathbb{E}_{\mu_N^\sigma(dx|y)}\Bigg[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)$$

$$\times \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\left(\sum_{i\in E_S}(X_i - \tilde{m}_i)\right)^2\Bigg]\left(i\frac{\xi}{\sqrt{N}}\right)^2 \tag{4.3.34}$$

$$+ \frac{1}{6}\mathbb{E}_{\mu_N^\sigma(dx|y)}\Bigg[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)$$

$$\times \left(\sum_{i\in E_S}(X_i - \tilde{m}_i)\right)^3 \exp\left(i\frac{1}{\sqrt{N}}\sum_{i\in E_S}(X_i - \tilde{m}_i)\,\tilde{\xi}\right)\Bigg]\left(i\frac{\xi}{\sqrt{N}}\right)^3. \tag{4.3.35}$$

Let us begin with estimation of (4.3.32). An application of Theorem 1.2.6 implies that

$$\left|T_{(4.3.32)}\right| \lesssim \|\nabla f\|_{L^2(\mu_N^\sigma(dx|y))}\|\nabla g\|_{L^2(\mu_N^\sigma(dx|y))}\exp\left(-Cd_{f,g}\right) \leq \|\nabla f\|_\infty\|\nabla g\|_\infty \exp\left(-Cd_{f,g}\right).$$

Next, to estimate (4.3.33), let us decompose E_S into two parts:

$$E_S^f := \{i \in E_S \ : \ \operatorname{dist}(i\ ,\ \operatorname{supp} f) \geq \frac{1}{2}d_{f,g}\}, \qquad E_S^g := E_S \backslash E_S^f.$$

We observe that for each $i \in E_S^g$,

$$\operatorname{dist}(i, \operatorname{supp} g) \geq \frac{1}{2}d_{f,g}.$$

We write

$$\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{i\in E_S}(X_i - \tilde{m}_i)\right]$$

$$= \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{i\in E_S^f}(X_i - \tilde{m}_i)\right] \tag{4.3.36}$$

$$+\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{i\in E_S^g}(X_i-\tilde{m}_i)\right] \tag{4.3.37}$$

Then it follows that (see estimations of (2.3.5) for example)

$$\begin{aligned}
\left|T_{(4.3.36)}\right| \leq \sum_{d\geq\frac{1}{2}d_{f,g}} &\Bigg|\mathbb{E}_{\mu_N^\sigma(dx|y)}\Bigg[\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right) \\
&\qquad\times\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{\substack{i\in E_S^f,\\ \mathrm{dist}(i,\operatorname{supp} f)=d}}(X_i-\tilde{m}_i)\Bigg]\Bigg| \\
&= \sum_{d\geq\frac{1}{2}d_{f,g}}\left|\mathrm{cov}_{\mu_N^\sigma(dx|y)}\left(f(X),\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\sum_{\substack{i\in E_S^f,\\ \mathrm{dist}(i,\operatorname{supp} f)=d}}(X_i-\tilde{m}_i)\right)\right| \\
&\overset{Theorem\ 1.2.6}{\lesssim} \sum_{d\geq\frac{1}{2}d_{f,g}}\Bigg(\|\nabla f\|_{L^2(\mu_N^\sigma(dx|y))}\|\nabla g\|_{L^4(\mu_N^\sigma(dx|y))} \\
&\qquad\times\left|\{i\in E_S^f \mid \mathrm{dist}(i,\operatorname{supp} f)=d\}\right|^{\frac{1}{2}}\exp\left(-Cd\right)\Bigg) \\
&\lesssim \|\nabla f\|_{L^2(\mu_N^\sigma(dx|y))}\|\nabla g\|_{L^4(\mu_N^\sigma(dx|y))}|\operatorname{supp} f|^{\frac{1}{2}}\sum_{d\geq\frac{1}{2}d_{f,g}}\exp\left(-Cd\right) \\
&\lesssim \|\nabla f\|_\infty\|\nabla g\|_\infty|\operatorname{supp} f|^{\frac{1}{2}}\exp\left(-Cd_{f,g}\right).
\end{aligned}$$

Similarly, one gets

$$\left|T_{(4.3.37)}\right| \lesssim \|\nabla f\|_\infty\|\nabla g\|_\infty|\operatorname{supp} g|^{\frac{1}{2}}\exp\left(-Cd_{f,g}\right),$$

and thus

$$\begin{aligned}
\left|T_{(4.3.33)}\right| &\lesssim \|\nabla f\|_\infty\|\nabla g\|_\infty\left(|\operatorname{supp} f|^{\frac{1}{2}}+|\operatorname{supp} g|^{\frac{1}{2}}\right)\exp\left(-Cd_{f,g}\right)\frac{|\xi|}{\sqrt{N}} \\
&\leq \|\nabla f\|_\infty\|\nabla g\|_\infty\exp\left(-Cd_{f,g}\right)|\xi|.
\end{aligned}$$

Let us turn to the estimation of (4.3.34). It holds that

$$\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\left(\sum_{i\in E_S}\left(X_i-\tilde{m}_i\right)\right)^2\right]$$

$$= \operatorname{cov}_{\mu_N^\sigma(dx|y)}\Bigg(\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right),$$

$$\left(\sum_{i\in E_S}\left(X_i-\tilde{m}_i\right)\right)^2\Bigg) \tag{4.3.38}$$

$$+\,\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\right]$$

$$\times\,\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(\sum_{i\in E_S}\left(X_i-\tilde{m}_i\right)\right)^2\right]. \tag{4.3.39}$$

Then Lemma 2.3.2 implies that

$$\begin{aligned}\left|T_{(4.3.38)}\right| &\lesssim \|\nabla\left(\left(f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right)\left(g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right)\right)\|_{L^4(\mu_N^\sigma(dx|y))}|S|\\ &\leq \Big(\|\nabla f\|_\infty\|g(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\|_{L^4(\mu_N^\sigma(dx|y))}\\ &\qquad +\|\nabla g\|_\infty\|f(X)-\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\|_{L^4(\mu_N^\sigma(dx|y))}\Big)\times|S|\\ &\overset{Lemma\ 2.1.1}{\lesssim} \|\nabla f\|_\infty\|\nabla g\|_\infty|S|.\end{aligned}$$

The second term (4.3.39) is estimated via Theorem 1.2.6 and Lemma 2.1.4 as follows:

$$\begin{aligned}\left|T_{(4.3.39)}\right| &= \left|\operatorname{cov}_{\mu_N^\sigma(dx|y)}\left(f(X),g(X)\right)\right|\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[\left(\sum_{i\in E_S}\left(X_i-\tilde{m}_i\right)\right)^2\right]\\ &\lesssim \|\nabla f\|_\infty\|\nabla g\|_\infty\exp\left(-Cd_{f,g}\right)|S|.\end{aligned}$$

Thus plugging the estimates for $T_{(4.3.38)}$ and $T_{(4.3.39)}$ into (4.3.34) yields

$$\begin{aligned}\left|T_{(4.3.34)}\right| &\lesssim \|\nabla f\|_\infty\|\nabla g\|_\infty\frac{|S|}{N}\xi^2\\ &\qquad +\|\nabla f\|_\infty\|\nabla g\|_\infty\exp\left(-Cd_{f,g}\right)\xi^2,\end{aligned}$$

where we used $|S|\leq N$ in the second term.

Lastly, we address the cubic term (4.3.35). Hölder's inequality followed by Lemma 2.1.8 gives

$$
\begin{aligned}
\left|T_{(4.3.35)}\right| &\lesssim \| \left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \|_{L^4(\mu_N^\sigma(dx|y))} \\
&\quad \times \left\| \sum_{i \in E_S} \left(X_i - \tilde{m}_i\right) \right\|_{L^4(\mu_N^\sigma(dx|y))}^3 \frac{|\xi|^3}{N^{\frac{3}{2}}} \\
&\lesssim \| \left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \|_{L^4(\mu_N^\sigma(dx|y))} \frac{|E_S|^{\frac{3}{2}}}{N^{\frac{3}{2}}} |\xi|^3.
\end{aligned}
$$

A combination of Schwarz inequality and Lemma 2.1.1 implies

$$
\| \left(f(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right]\right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \|_{L^4(\mu_N^\sigma(dx|y))} \leq \|\nabla f\|_\infty \|\nabla g\|_\infty.
$$

Because $|E_S| \leq 2\left(|\operatorname{supp} f| + |\operatorname{supp} g|\right) M \log N$, it holds for N large enough that

$$
\frac{|E_S|^{\frac{3}{2}}}{N^{\frac{3}{2}}} \lesssim \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N}.
$$

Therefore we conclude

$$
\left|T_{(4.3.35)}\right| \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} |\xi|^3.
$$

Lastly, we sum up all the estimates we have obtained so far. That is,

$$
\begin{aligned}
\left|T_{(4.3.31)}\right| &\leq \left|T_{(4.3.32)}\right| + \left|T_{(4.3.33)}\right| + \left|T_{(4.3.34)}\right| + \left|T_{(4.3.35)}\right| \\
&\lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C d_{f,g}\right) \right) \left(1 + |\xi| + \xi^2 + |\xi|^3\right).
\end{aligned}
$$

Putting this estimate and Lemma 3.3.1 into (4.3.30) implies

$$
\begin{aligned}
\left|T_{(4.3.26)}\right| &\lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C d_{f,g}\right) \right) \\
&\quad \times \left(1 + |\xi| + \xi^2 + |\xi|^3\right) \exp\left(-C\xi^2\right) \\
&\lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp\left(-C d_{f,g}\right) \right) \exp\left(-C\xi^2\right).
\end{aligned}
$$

□

Lemma 4.3.7. *It holds that*

$$\left|T_{(4.3.27)}\right|, \left|T_{(4.3.28)}\right|, \left|T_{(4.3.29)}\right| \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} \exp\left(-C\xi^2\right).$$

Proof of Lemma 4.3.7. Similar to the proof of Lemma 4.3.6 we decompose $T_{(4.3.27)}$ as follows:

$$\begin{aligned} T_{(4.3.27)} = \mathbb{E}_{\mu_N^\sigma} \Bigg[& \exp\left(i\frac{1}{\sqrt{N}} \sum_{j \in F_S^1} (X_j - m_j)\,\xi + i\frac{1}{\sqrt{N}} \sum_{i \in E_S} (\tilde{m}_i - m_i)\,\xi \right) \\ & \times \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[\exp\left(i\frac{1}{\sqrt{N}} \sum_{i \in E_S} (X_i - \tilde{m}_i)\,\xi \right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \right] \\ & \times \left(\mathbb{E}_{\mu_N^\sigma(dx|y)}\left[f(X)\right] - \mathbb{E}_{\mu_N^\sigma}\left[f(X)\right]\right) \\ & \times \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i\frac{1}{\sqrt{N}} \sum_{k \in F_S^2} (X_j - m_j)\,\xi \right) \,\middle|\, X_i, i \in E_S \cup F_S^1 \right] \Bigg], \end{aligned}$$

First of all, Lemma 2.1.1 implies

$$\begin{aligned} & \left| \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[\exp\left(i\frac{1}{\sqrt{N}} \sum_{i \in E_S} (X_i - \tilde{m}_i)\,\xi \right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right]\right) \right] \right| \\ & \leq \mathbb{E}_{\mu_N^\sigma(dx|y)} \left[\left| g(X) - \mathbb{E}_{\mu_N^\sigma(dx|y)}\left[g(X)\right] \right| \right] \lesssim \|\nabla g\|_\infty. \end{aligned}$$

Next, we apply Lemma 2.4.1 and Lemma 3.3.1 to obtain

$$\begin{aligned} \left|T_{(4.3.27)}\right| & \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \exp\left(-CM \log N\right) \exp\left(-C\xi^2\right) \\ & \quad \times \mathbb{E}_{\mu_N^\sigma} \left[\int \left(\sum_{\substack{i \in E_S, j \in F_S \\ |i-j| \leq R}} M_{ij}^2 (y_j - z_j)^2 \right)^{\frac{1}{2}} \bar{\mu}^\sigma(dz) \right] \\ & \overset{(4.3.13)}{\lesssim} \|\nabla f\|_\infty \|\nabla g\|_\infty \exp\left(-CM \log N\right) \left(|\operatorname{supp} f| + |\operatorname{supp} g|\right)^{\frac{1}{2}} \exp\left(-C\xi^2\right) \\ & \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} \exp\left(-C\xi^2\right). \end{aligned}$$

Similar calculations also imply

$$\left|T_{(4.3.28)}\right|, \left|T_{(4.3.29)}\right| \lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} \exp\left(-C\xi^2\right).$$

□

Now we are ready to give a proof of Lemma 4.3.5, which finishes the proof.

Proof of Lemma 4.3.5. By lemma 4.3.6 and Lemma 4.3.7 it holds that

$$\begin{aligned}
&\left| \int_{\{|(1/\sqrt{N})\xi| \le \delta\}} \mathbb{E}_{\mu_N^\sigma} \left[\exp\left(i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)\, \xi \right) \right. \right. \\
&\qquad \left. \left. \times \left(f(X) - \mathbb{E}_{\mu_N^\sigma}[f(X)] \right) \left(g(X) - \mathbb{E}_{\mu_N^\sigma}[g(X)] \right) \right] d\xi \right| \\
&\lesssim \int_{\mathbb{R}} \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp(-C d_{f,g}) \right) \exp\left(-C\xi^2\right) d\xi \\
&\lesssim \|\nabla f\|_\infty \|\nabla g\|_\infty \left(\frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N} + \exp(-C d_{f,g}) \right).
\end{aligned}$$

□

Chapter 5

Decay of correlations and uniqueness of the infinite-volume Gibbs measure of the canonical ensemble

We prove that on the one-dimensional lattice system of unbounded, real-valued spins with arbitrary strong, quadratic, finite-range interaction, the infinite-volume Gibbs measure of the canonical ensemble is unique. The main ingredient is the decay of correlations.

5.1 Introduction

A phase transition occurs if a microscopic change in a parameter leads to a fundamental change in one or more properties of the underlying physical system. The most well-known phase transition is when water becomes ice. Many physical systems, non-physical systems and mathematical models have phase transitions. For example, liquid to gas phase transitions are known as vaporization. Solid to liquid phase transitions are known as melting. Solid to gas phase transitions are known as sublimation. More examples are the phase transition in the 2-d Ising model (see for example [Sel16]), the Erdös-Renyi phase transition in random graphs (see for example [ER60], [ER61] or [KS13]) or phase transitions in social networks (see for example [FFH07]).

There are many different ways to characterize phase transitions. In this thesis we use the convention that an ensemble has no phase transition if the associated infinite-volume Gibbs measure is unique. On the two-dimensional lattice the gce has a phase transition (see for example [Pei36]). This is not the case if one considers the gce on the one-dimensional lattice. There, the gce does not have a phase transition if the interaction is finite-range or decays fast enough (see for example [Isi25, Dob68, Dob74, Rue68, MN14]). It is a natural question if those results extend from the gce to the ce. The results of Cancrini, Martinelli and Roberto [CMR02] suggest that if the spins are discrete, i.e. $\{0, 1\}$-valued, there is no phase transition for the ce on a one-dimensional lattice with nearest-neighbor interaction. In this thesis we consider this question for real-valued and unbounded spins. Considering unbounded spins is harder because we lose compactness and one cannot transfer the arguments from the discrete case. In this thesis, we show that this is indeed true: There is no phase transition for the ce on a one-dimensional lattice with finite-range interaction (see Theorem 5.2.4 from below).

For the proof of Theorem 5.2.4 we follow a standard argument (see e.g. [Yos03] or [Men14]) which is based on the decay of correlations. Decay of correlations is a classical ingredient when deducing the uniqueness of the infinite-volume Gibbs measure in lattice systems (see for example [DS87, Mar99, MN14] for the gce). In many lattice systems decay of correlations is one of many equivalent mixing conditions, including the Dobrushin-Sloshman mixing condition (see e.g. [DS85, DS87, DW90, Mar99, Yos03, HM16]). It is known that for one-dimensional lattice systems the correlations of the gce decay exponentially fast (see for example [Zeg96, MN14], references therein and Theorem 1.2.6 below). We use decay of correlations of the ce which was derived in Chapter 4 (cf. Theorem 5.2.1 and Corollary 5.2.2).

5.2 Main results and outline of the proof

5.2.1 Main Results

The first main result of this chapter is that the correlations of the ce decays exponentially fast plus a volume correction term.

Theorem 5.2.1 (Decay of correlations of the ce)**.** *Under the same assumptions as in Theorem 4.2.3, it holds that*

$$\left|\operatorname{cov}_{\mu_{\Lambda,m}}(f,g)\right| \leq C\ \|\nabla f\|_{\infty}\|\nabla g\|_{\infty}\left(\frac{|\operatorname{supp} f|+|\operatorname{supp} g|}{|\Lambda|}+\exp\left(-C\,dist(\operatorname{supp} f,\operatorname{supp} g)\right)\right).$$

We deduce Theorem 5.2.1 in Section 5.2.2. Let us now illustrate the use of Theorem 5.2.1 by deducing the decay of the spin-spin correlation function of the ce.

Corollary 5.2.2 (Decay of the spin-spin correlation function of the ce)**.** *There exist constants $N_0 \in \mathbb{N}$ and $C \in (0,\infty)$ independent of the external field s and the mean spin m such that if $|\Lambda| \geq N_0$, it holds that for any $i,j \in \Lambda$,*

$$\left|\operatorname{cov}_{\mu_{\Lambda,m}}(X_i,X_j)\right| \leq C\left(\frac{1}{|\Lambda|}+\exp\left(-C|i-j|\right)\right).$$

Remark 13. Compared to Theorem 1.2.6, there appears an additional volume correction term $\frac{1}{N}$ in Corollary 5.2.2. This term is due to the mean constraint $\frac{1}{|\Lambda|}\sum_{i\in\Lambda} x_i = m$ and is optimal. For example, assuming that $\Lambda = \{1,2,\cdots,N\}$ and the Hamiltonian H_Λ is symmetric, we have

$$\operatorname{cov}_{\mu_{\Lambda,m}}(X_1,X_2) = \operatorname{cov}_{\mu_{\Lambda,m}}(X_i,X_j) \qquad \text{for all distinct } i,j \in \{1,\cdots,N\}.$$

Thus we get

$$\begin{aligned}
\operatorname{cov}_{\mu_{\Lambda,m}}(X_1,X_2) &= \frac{1}{N-1}\operatorname{cov}_{\mu_{\Lambda,m}}(X_1,X_2+\cdots X_N) \\
&= \frac{1}{N-1}\operatorname{cov}_{\mu_{\Lambda,m}}(X_1,Nm-X_1) \\
&= -\frac{1}{N-1}\operatorname{var}_{\mu_{\Lambda,m}}(X_1).
\end{aligned}$$

Let us introduce the next main result of this chapter, namely the uniqueness of the infinite-volume Gibbs measure of the ce.

Definition 5.2.3 (Infinte-volume Gibbs measure)**.** *Let μ be a probability measure on $\mathbb{R}^{\mathbb{Z}}$ with standard product Borel sigma-algebra. For any finite subset $\Lambda \subset \mathbb{Z}$ we decompose the measure μ into the conditional measure $\mu(dx^{\Lambda}|x^{\mathbb{Z}\setminus\Lambda})$ and the marginal $\bar{\mu}(dx^{\mathbb{Z}\setminus\Lambda})$. This means that for any test function f it holds*

$$\int f(x)\mu(dx) = \int \int f(x^{\Lambda}, x^{\mathbb{Z}\setminus\Lambda})\mu\left(dx^{\Lambda}|x^{\mathbb{Z}\setminus\Lambda}\right)\mu\left(dx^{\mathbb{Z}\setminus\Lambda}\right).$$

We say that μ is an infinite-volume Gibbs measure of the ce if the conditional measures $\mu\left(dx^{\Lambda}|x^{\mathbb{Z}\setminus\Lambda}\right)$ are given by the finite volume ce $\mu_{\Lambda,m}(dx^{\Lambda})$ given by Definition 1.1.7 i.e.

$$\mu\left(dx^{\Lambda}|x^{\mathbb{Z}\setminus\Lambda}\right) = \mu_{\Lambda,m}(dx^{\Lambda}).$$

The equations of the last identity are called Dobrushin-Lanford-Ruelle (DLR) equations.

Theorem 5.2.4 (Uniqueness of the infinite-volume Gibbs measure of the ce)**.** *There is only one infinite-volume Gibbs measure of the ce that satisfies the uniform bound*

$$\sup_{i\in\mathbb{Z}} \operatorname{var}_{\mu}(x_i) < \infty. \tag{5.2.1}$$

We deduce Theorem 5.2.4 in Section 5.2.2. The main ingredient in the proof is the decay of correlations (cf. Theorem 5.2.1 and Corollary 5.2.2).

Remark 14. In this thesis, we only show the uniqueness of infinite-volume Gibbs measure of the ce, not the existence. However, the author believe that with a cosmetic change, the existence should follow by a compactness argument (see for example [BHK82]).

5.2.2 Outline of the proof of main results

The proof of Theorem 5.2.1 is a direct consequence of the equivalence of ensembles on the level of correlations (cf. Theorem 4.2.3).

Proof of Theorem 5.2.1. It follows directly from Theorem 4.2.3 and Theorem 1.2.6. □

Let us turn to the proof of Theorem 5.2.4.

Proof of Theorem 5.2.4. Suppose that there are two infinite-volume Gibbs measures μ and ν of the ce $\mu_{\Lambda,m}$. It suffices to prove that for any smooth function $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ with bounded support

$$\int f\mu = \int f\nu.$$

Let us fix a smooth function f with bounded support. For each $r > 0$ define $B_r \subset \mathbb{Z}$ by

$$B_r := \{k \in \mathbb{Z} \mid -r < k < r\}$$

and choose $K > 0$ so that

$$\operatorname{supp} f \subset B_K. \tag{5.2.2}$$

For each $r > K$ we decompose the measure μ into the conditional measure $\mu\left(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}\right)$ and the marginal measure $\bar{\mu}\left(dy^{\mathbb{Z}\setminus B_r}\right)$, i.e. for any test function g

$$\int g\mu = \int\int g\left(x^{B_r}, y^{\mathbb{Z}\setminus B_r}\right)\mu\left(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}\right)\bar{\mu}\left(dy^{\mathbb{Z}\setminus B_r}\right).$$

Similarly, decompose the measure ν into $\nu\left(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}\right)$ and $\bar{\nu}\left(dy^{\mathbb{Z}\setminus B_r}\right)$. Then it holds from (DLR) equations that

$$\mu\left(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}\right) = \nu\left(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}\right) = \mu_{B_r,m}(dx^{B_r}|y^{\mathbb{Z}\setminus B_r}). \tag{5.2.3}$$

For notational convenience we write

$$x = x^{B_r}, \qquad y = y^{\mathbb{Z}\setminus B_r} \qquad \text{and} \qquad z = z^{\mathbb{Z}\setminus B_r}. \tag{5.2.4}$$

Note that (5.2.3) implies

$$\left|\int f\mu - \int f\nu\right| = \left|\int\int f\mu\,(dx|y)\,\bar{\mu}\,(dy) - \int\int f\nu\,(dx|z)\,\bar{\nu}\,(dz)\right|$$

$$= \left| \int \int \left(\int f \mu_{B_r,m} (dx|y) - \int f \mu_{B_r,m} (dx|z) \right) \bar{\mu} (dy) \bar{\nu} (dz) \right|$$
$$\leq \int \int \left| \int f \mu_{B_r,m} (dx|y) - \int f \mu_{B_r,m} (dx|z) \right| \bar{\mu} (dy) \bar{\nu} (dz) . \qquad (5.2.5)$$

We claim that the right hand side of $T_{(5.2.5)}$ becomes small when choosing $r > 0$ large enough. More precisely, we have the following estimate.

Lemma 5.2.5. *Let ε be a fixed positive number. Then it holds that*

$$\int \int \left| \int f \mu_{B_r,m} (dx|y) - \int f \mu_{B_r,m} (dx|z) \right| \bar{\mu} (dy) \bar{\nu} (dz)$$
$$\lesssim R^2 \left(|f|_\infty + |\nabla f|_\infty \right) \left(\frac{|\operatorname{supp} f|}{r} + \exp\left(-C(r - R - K) \right) \right).$$

The statement from above finishes the proof of Theorem 5.2.4 by letting $r \to \infty$ and get

$$\left| \int f \mu - \int f \nu \right| = 0.$$

□

Now let us turn to the proof of Lemma 5.2.5.

Proof of Lemma 5.2.5. By interpolation it holds that (recall the convention (5.2.4))

$$\int f \mu_{B_r,m} (dx|y) - \int f \mu_{B_r,m} (dx|z) = \int_0^1 \left(\frac{d}{dt} \int f \mu_{B_r,m} (dx|ty + (1-t)z) \right) dt$$
$$= \int_0^1 \operatorname{cov}_{\mu_{B_r,m}(dx|ty+(1-t)z)} \left(f, \sum_{\substack{i \in B_r,\ j \notin B_r \\ |i-j| \leq R}} M_{ij} x_i (y_j - z_j) \right) dt. \qquad (5.2.6)$$

Let us consider the integrand in (5.2.6). To estimate the covariance with respect to the measure $\mu_{B_r,m} (dx|ty + (1-t)z)$, let us define the corresponding gce $\mu_{B_r}^\tau (dx|ty + (1-t)z)$ by

$$\mu_{B_r}^\tau (dx|ty + (1-t)z) = \frac{1}{Z} \exp\left(\tau \sum_{k \in B_r} x_k - H\left(x, ty + (1-t)z\right) \right) dx,$$

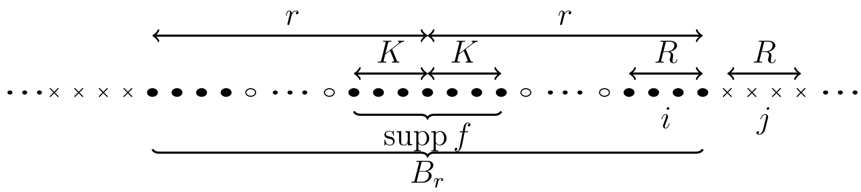

Figure 5.1: Distance between supp f and $\{i\}$

where we choose $\tau = \tau(m)$ such that (cf. Definition 2.1.6)

$$m = \frac{1}{|B_r|} \int \left(\sum_{k \in B_r} x_k \right) \mu_{B_r}^{\tau} \left(dx | ty + (1-t) z \right).$$

For a pair (i, j) with $i \in B_r$, $j \notin B_r$ and $|i - j| \le R$, the triangle inequality yields

$$|i| \ge |j| - |i - j| \ge r - R,$$

and in particular for $r > R + K$ (cf. (5.2.2) and Figure 5.1),

$$\operatorname{dist} \left(\operatorname{supp} f, \{i\} \right) \ge r - R - K.$$

Then a combination of Theorem 5.2.1 and Lemma 2.1.2 yields

$$\begin{aligned} &\left| \operatorname{cov}_{\mu_{B_r,m}(dx|ty+(1-t)z)} \left(f, \sum_{\substack{i \in B_r,\ j \notin B_r \\ |i-j| \le R}} M_{ij} x_i (y_j - z_j) \right) \right| \\ &\quad \le \sum_{\substack{i \in B_r,\ j \notin B_r \\ |i-j| \le R}} |M_{ij}| |y_j - z_j| \left| \operatorname{cov}_{\mu_{B_r,m}(dx|ty+(1-t)z)} (f, x_i) \right| \\ &\quad \lesssim \left(|f|_\infty + |\nabla f|_\infty \right) \left(\frac{|\operatorname{supp} f|}{r} + \exp\left(-C(r - R - K) \right) \right) \left(\sum_{\substack{i \in B_r,\ j \notin B_r \\ |i-j| \le R}} |y_j - z_j| \right). \end{aligned} \tag{5.2.7}$$

Hence (5.2.6) and (5.2.7) imply

$$\begin{aligned} &\int \int \left| \int f \mu_{B_r,m} \left(dx | y \right) - \int f \mu_{B_r,m} \left(dx | z \right) \right| \bar{\mu} \left(dy \right) \bar{\nu} \left(dz \right) \\ &\quad \le \left(|f|_\infty + |\nabla f|_\infty \right) \left(\frac{|\operatorname{supp} f|}{r} + \exp\left(-C(r - R - K) \right) \right) \end{aligned}$$

$$\times \sum_{\substack{i\in B_r,\ j\notin B_r \\ |i-j|\leq R}} \int\int |y_j - z_j| \bar{\mu}\,(dy)\, \bar{\nu}\,(dz)\,.$$

Note that Cauchy's inequality implies

$$\begin{aligned}\int\int |y_j - z_j|\bar{\mu}(dy)\bar{\mu}(dz) &\leq \left(\int\int (y_j - z_j)^2\, \bar{\mu}(dy)\bar{\mu}(dz)\right)^{\frac{1}{2}} \\ &= \left(2\operatorname{var}_{\bar{\mu}}(y_j)\right)^{\frac{1}{2}} \overset{(5.2.1)}{\lesssim} 1.\end{aligned}$$

Because there are at most $2R^2$ many pairs of (i, j) with $i \in B_r$, $j \notin B_r$ and $|i - j| \leq R$, we have

$$\begin{aligned}&\int\int \left|\int f\mu_{B_r,m}\,(dx|y) - \int f\mu_{B_r,m}\,(dx|z)\right| \bar{\mu}\,(dy)\,\bar{\nu}\,(dz) \\ &\quad\lesssim R^2\left(|f|_\infty + |\nabla f|_\infty\right)\left(\frac{|\operatorname{supp} f|}{r} + \exp\left(-C(r - R - K)\right)\right).\end{aligned}$$

□

Chapter 6

Logarithmic Sobolev inequality

We show that the canonical ensemble (ce) satisfies a uniform logarithmic Sobolev inequality (LSI). The LSI constant is uniform in the boundary data, the external field and scales optimally in the system size. This extends a classical result of H.T. Yau from discrete to unbounded, real-valued spins. It also extends prior results of Landim, Panizo & Yau or Menz for unbounded, real-valued spins from absent- or weak- to strong-interaction. We deduce the LSI by combining two different methods, the two-scale approach and the Zegarlinski method. Main ingredients are the strict convexity of the coarse-grained Hamiltonian, the equivalence of ensembles and the decay of correlations in the ce.

6.1 Introduction

The logarithmic Sobolev inequality (LSI) – introduced by Gross [Gro75]– is a powerful tool for studying spin systems. For example, the LSI implies Gaussian concentration via Herbst's argument, is equivalent to hypercontractivity and characterizes the exponential rate of convergence to equilibrium of the naturally associated diffusion process. By the equivalence of dynamic and static phase transitions, a uniform LSI also indicates the absence of a phase transition (see e.g. [Yos03, HM16]). For an introduction to the LSI we refer the reader to [Led01b, Led01a, Roy99, BGL14].

On the one-dimensional lattice, a uniform LSI holds for the gce μ_Λ^σ even for infinite-range interactions, given the interaction decays fast enough (see [MN14]). Deducing a uniform LSI becomes a lot harder when considering the ce $\mu_{\Lambda,m}$ instead of the gce μ_Λ^σ. Even if there is no interaction term in the Hamiltonian H, the ce $\mu_{\Lambda,m}$ is not a product measure. There are long-range interactions due to the conditioning onto the mean spin $m = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} x_i$. For $\{0,1\}$-valued spins on arbitrary lattices, a classical result of Yau [Yau96] states that the ce $\mu_{\Lambda,m}$ satisfies a uniform LSI as soon as the correlations of the gce μ_Λ^σ decay exponentially, which is the case on the one-dimensional lattice. The original proof by Yau [Yau96] is based on the Lu-Yau Martingale Method [LY93]. Later, Cancrini, Martinelli & Roberto [CMR02] gave an alternative, self-contained proof of the same statement.

Extending Yau's result [Yau96] to unbounded, real-valued spins is still an open problem. The main result of this chapter (cf. Theorem 6.2.3 below) solves this problem. Considering unbounded, real-valued spins instead of $\{0,1\}$-valued spins yields a technical challenge: Because spins are unbounded compactness is lost and many arguments do not carry over from the discrete case. Therefore, it was already quite a challenge to establish the LSI for the canonical ensemble in the case of a non-interacting Hamiltonian (cf. [Cha03, GOVW09, LPY02, MO13] for unbounded, real-valued spins). The main difficulty was to obtain the optimal scaling behavior of the LSI constant in the system size. More recently, the uniform LSI was deduced for arbitrary weak interaction in [Men11]. The method used in [Men11] is of perturbative nature and different ideas are needed when considering strong interaction. Therefore it is not surprising that deducing the LSI for the ce $\mu_{\Lambda,m}$ under a mixing condition for the gce μ_Λ^σ remained an open problem.

A major breakthrough is accomplished in Chapter 3 and Chapter 4 where a non-perturbative results on the equivalence of ensembles of the gce and the ce were deduced. A consequence is that the coarse-grained Hamiltonian of a single block is uniformly strictly convex (see

Corollary 3.2.5). This rises hope that the uniform LSI for the ce could be deduced via the two-scale approach [GOVW09]. It turns out that the situation is more complicated and the results of Chapter 3 and Chapter 4 are not sufficient to directly apply the two-scale approach. The strict convexity of the coarse-grained Hamiltonian was deduced in Corollary 3.2.5 for only one block. One would need strict convexity for all blocks simultaneously. This is a lot harder to show due to the strong interaction between blocks. However, we still manage to build up on the results of equivalence of Chapter 3 and Chapter 4 and deduce the uniform LSI for ce under a mixing condition for the gce (see Theorem 6.2.3). We overcome the obstacle of strong interactions between blocks by combining several ideas and methods from the two-scale approach (see [OR07, GOVW09, Men14]), the Zegarlinski method (see [Zeg96]), decay of correlations (cf. Theorem 1.2.6 and Theorem 5.2.1) and a decomposition method for Hamiltonians introduced in [Men14]. For more details on the argument we refer to Section 6.2.

Deducing a uniform LSI for the ce $\mu_{\Lambda,m}$ has another special importance: It is one of the main ingredients when deducing the hydrodynamic limit of the Kawasaki dynamics via the two-scale approach [GOVW09]. Because the uniform LSI controls the entropy production, it also plays an implicit role in other approaches to the hydrodynamic limit via the entropy method, the martingale method or the gradient flow method (see for example [GPV88b, Yau91, KL99, DF16]). The Kawasaki dynamics is a natural drift diffusion process on the lattice system that conserves the mean spin of the system. The ce $\mu_{\Lambda,m}$ is the stationary and ergodic distribution of the Kawasaki dynamics. The hydrodynamic limit is a dynamic manifestation of the law of large numbers. It states that under the correct scaling the Kawasaki dynamics (which is a stochastic process) converges to the solution of a non-linear heat equation (which is deterministic). It is conjectured by H.T. Yau that the hydrodynamic limit also holds for strong finite-range interactions on a one-dimensional lattice. In Chapter 7, we prove that this is indeed the case.

6.2 Main results and outline of the proof

In the remaining sections of this chapter, we only consider the one-dimensional lattice $\Lambda = [N] = \{1, 2, \cdots, N\}$ with nearest-neighbor interactions for better explanation of our ideas and calculations. We follow the conventions made at the end of Section 1.1.1.

6.2.1 Main Results

Let us begin with recalling the definition of logarithmic Sobolev inequality (LSi).

Definition 6.2.1 (Logarithmic Sobolev inequality (LSI)). *Let X be a Euclidean space. A Borel probability measure μ on X satisfies the LSI with constant $\varrho > 0$ (or LSI(ϱ)) if, for all nonnegative locally Lipschitz functions $f \in L^1(\mu)$,*

$$\int f \log f d\mu - \int f d\mu \log\left(\int f d\mu\right) \leq \frac{1}{2\varrho} \int \frac{|\nabla f|^2}{f} d\mu, \tag{6.2.1}$$

where ∇ denotes the gradient in the Euclidean space X.

It is well known that on the one-dimensional lattice, the gce satisfies a uniform LSI if the interaction decays fast enough.

Theorem 6.2.2 (Theorem 1.6 in [MN14]). *Let $H : \mathbb{R}^N \to \mathbb{R}$ be the formal Hamiltonian defined as in (1.1.1). Assume the interaction is symmetric and the interaction range is infinite. That is, M_{ij} is not necessarily 0 when $|i - j| > R$. Assume further that the matrix $M = (M_{ij})$ is strictly diagonal dominant in the sense that there is a $\delta > 0$ with*

$$\sum_{j:j\neq i} |M_{ij}| + \delta \leq M_{ii} = 1 \qquad \text{for all } i \in [N].$$

If the interaction decays fast enough, i.e. there are positive constants C and α such that

$$|M_{ij}| \leq C \frac{1}{|i-j|^{2+\alpha} + 1} \qquad \text{for all } i, j \in [N],$$

then the gce μ_N^σ satisfies a LSI(ϱ), where $\varrho > 0$ is independent of the system size N and the external fields σ, s.

The main result of this chapter is that the ce also satisfies on the one-dimensional lattice a uniform LSI for arbitrary strong, finite-range interaction.

Theorem 6.2.3. *The ce $\mu_{N,m}$ given by* (1.1.7) *satisfies a uniform LSI(ϱ), where $\varrho > 0$ is independent of the system size N, the external field s and the mean spin $m \in \mathbb{R}$.*

Remark 15 (From Glauber to Kawasaki). We want to point out that the ce $\mu_{N,m}$ is defined on the space $X_{N,m}$ given by (1.1.6). Because the space $X_{N,m}$ is endowed with the standard Euclidean structure inherited from $\mathbb{R}^N$, the bound on the right-hand side of the LSI (see (6.2.1)) is given in terms of the Glauber dynamics. By the discrete Poincaré inequality one can recover the bound for the Kawasaki dynamics (cf. [Cap03] or [GOVW09, Remark 15]) in the sense that one endows $X_{N,m}$ with the Euclidean structure coming from the discrete H^{-1}-norm. The so obtained diffusive scaling of LSI constant for the Kawasaki dynamics is known to be optimal (see [Yau96] and also Remark 2 in [Men11]).

We give the proof of Theorem 6.2.3 in Section 6.2.2. There are several basic criteria for the LSI (cf. Appendix 8.2), but none of them applies to the ce. The Tensorization Principle [Gro75] for LSI does not apply because of the restriction to the hyperplane $X_{N,m}$ and the presence of interaction i.e. $M \neq 0$. The criterion of Bakry & Émery [BE85] does not directly apply because the single-site potentials ψ_i are non-convex. A combination of the Bakry & Émery and the Holley & Stroock [HS87] would only yield a LSI with constant ϱ that is exponentially small in the system size N. The criterion of Otto & Reznikoff [OR07] does not apply because of the restriction to the hyper-plane $X_{N,m}$.

More advanced methods are needed for deducing a uniform LSI for the ce $\mu_{N,m}$. The most common approach to LSI for Kawasaki dynamics is the Lu & Yau martingale method [LY93, LPY02, Cha03]. Using this method Landim, Panizo & Yau [LPY02] proved Theorem 6.2.3 in the special case M = 0 for the Kawasaki bound. An adaptation of this approach by Chafai [Cha03] lead to the stronger bound for Glauber Dynamics. Providing a new technique, called the two-scale approach, Grunewald, Otto, Westdickenberg (former Reznikoff) & Villani [GOVW09] reproduced Theorem 6.2.3 for M = 0. In [Men11], the uniform LSI was deduced for weak interaction, i.e. $\|M\| \ll 1$, by perturbing the two-scale approach.

The drawback of the two-scale approach is that it elementarily takes advantage of having no interaction term in the Hamiltonian i.e. setting $M = 0$. The basic idea in the two-scale approach is to decompose the lattice $[N]$ into blocks. This yields a decomposition of the ce $\mu_{N,m}$ into a conditional measure, conditioned on the mean spins of the blocks, and a marginal measure for the mean spins. The task is then to deduce two LSIs: A microscopic LSI for the conditional measure and a macroscopic LSI for the marginal measure. After that, the two LSIs are combined into a single LSI for the ce $\mu_{N,m}$. If there is no interaction term, blocks do not interact for the conditional and marginal measure. This helps a lot when deducing the microscopic and the macroscopic LSI. If there is a small interaction term, i.e. $\|M\| \ll 1$, blocks only interact weakly. In [Men11], one took advantage of this observation by essentially perturbing the two-scale approach. If there is a large interaction term in the Hamiltonian H then blocks also interact strongly. It becomes very difficult to deduce the microscopic and macroscopic LSI in the original setting of the two-scale approach.

Here, we overcome this difficulty by using the following strategy. In [Zeg96], Zegarlinski deduced the uniform LSI for the gce μ_N^σ with strong finite range interaction on the one-dimensional lattice. We follow his approach and decompose the lattice $[N]$ into odd blocks Λ_1 and even blocks Λ_2 (see Figure 6.1). The difference to the two-scale approach is that one does not condition on the mean-spins of blocks but on the spin-values of every even block. The resulting conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ is also a canonical ensemble but with the advantage that spins in distinct odd blocks do not interact within the Hamiltonian of $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ due to the assumption of finite-range interaction. The next step in the Zegarlinski method is to deduce a uniform LSI for $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$. In our situation this is achieved via the two-scale approach described above. The main new ingredient is the local Cramér theorem (cf. Theorem 3.2.4). The last step in the Zegarlinski method [Zeg96] is to iteratively condition on the spin values in Λ_1 and Λ_2 deducing a LSI via a convergence argument. This is where we deviate from the Zegarlinski method. Instead of using an iterative argument we apply the two-scale criterion for LSI (cf. [OR07] or Theorem 8.2.5 in the appendix), which in the opinion of the author is a more direct argument.

In the two-scale criterion for the LSI one needs two ingredients: a uniform LSI for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ and a uniform LSI for the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$. Then, the criterion combines both LSIs into a LSI for the full measure $\mu_{N,m}$. Let us explain how we deduce the LSI for the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$ which is active on Λ_2 and integrates out Λ_1. For this, we use the Otto-Reznikoff criterion for LSI (cf. Theorem 1 in [OR07] or Theorem 8.2.4 in the appendix). The main observation needed is that blocks in Λ_2 only interact weakly, if the block size of Λ_1 is large enough. We deduce this subtle fact by a series of calculations, building up on the equivalence of ensembles, the decay of correlations, and the moment estimates for the gce. For details of the argument we refer the reader to Section 6.2.2.

6.2.2 Outline of the proof of main results

In the proof of Theorem 6.2.3 we use an idea of Zegarlinski and decompose the lattice $[N]$ into two parts. This idea was used to prove the uniform LSI for the gce μ_N^σ on the one-dimensional lattice (see [Zeg96]). However, instead of using an iterative argument as in [Zeg96] we make use of ideas outlined in [OR07] and [GOVW09].

We decompose the lattice into two types of blocks (see Figure 6.1):

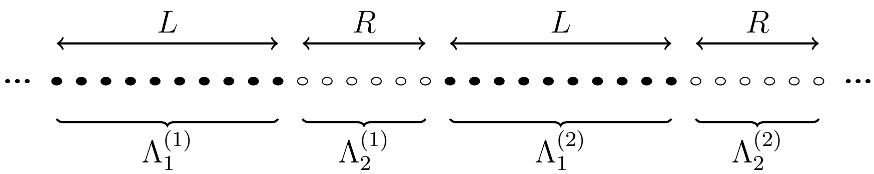

Figure 6.1: Arrangement in the cell $[1, 2(L + R)]$ for $L = 10$ and $R = 6$

$$\begin{aligned}
\Lambda &:= [N] = \{1, 2, \cdots, N\}, \\
\Lambda_1 &:= \bigcup_{n \in \mathbb{Z}} \Lambda \cap ([1, L] + (L + R)(n - 1)) = \bigcup_{n=1}^{M} \Lambda_1^{(n)}, \\
\Lambda_2 &:= \bigcup_{n \in \mathbb{Z}} \Lambda \cap ([L + 1, L + R] + (L + R)(n - 1)) = \bigcup_{n=1}^{M} \Lambda_2^{(n)},
\end{aligned}$$

where R is the range of interactions between particles (cf. (1.1.1)). The number L will be chosen later.

Recall the definition (1.1.7) of the ce $\mu_{N,m}$

$$\mu_{N,m}(dx) = \frac{1}{Z} \mathbb{1}_{\left\{\frac{1}{N}\sum_{k=1}^{N} x_k = m\right\}}(x) \exp\left(-H\left(x\right)\right) \mathcal{L}^{N-1}(dx).$$

The decomposition of $\Lambda = \Lambda_1 \cup \Lambda_2$ into odd blocks given by Λ_1 and even blocks given by Λ_2 yields a decomposition of the ce $\mu_{N,m}$ into conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ and a marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$. That is, for any test function f, it holds that

$$\int f(x) \mu_{N,m}(dx) = \int \left(\int f(x) \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \right) \bar{\mu}_m\left(dx^{\Lambda_2}\right). \tag{6.2.2}$$

Now, the strategy is to deduce uniform LSIs for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ and the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$. The uniform LSI for the full measure $\mu_{N,m}$ is then deduced via the two-scale criterion for the LSI (see eg. [OR07] or [GOVW09]).

Let us explain how the uniform LSI for the conditional measures $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ is deduced. By conditioning onto the even blocks Λ_2, the spins in one odd block (say $x_i \in \Lambda_1^{(n)}$) do not interact with the spins in another odd block (say $x_j \in \Lambda_2^{(l)}$, $l \neq n$) within the Hamiltonian of $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$. The only way they interact is through the constraint (1.1.8). With this observation one can modify the proof of [GOVW09, Theorem 14] (see also Theorem 8.2.5) to deduce a uniform LSI for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ via the two-scale approach (see Proposition 6.2.4 below).

Let us turn to the uniform LSI for the marginal measure $\bar{\mu}_m$. We observe that the marginal measure $\bar{\mu}_m$ is not constrained onto a hyperplane. Because $\bar{\mu}_m$ can be interpreted as a gce on a one-dimensional lattice, the marginal measure $\bar{\mu}_m$ should heuristically satisfy a uniform LSI. Rigorously, the uniform LSI for the marginal measure $\bar{\mu}_m$ is deduced via the the Otto-Reznikoff criterion for LSI (cf. Theorem 8.2.4). For this we need two ingredients: The first ingredient is to show that on each block $\Lambda_2^{(n)}$, the conditional measures $\bar{\mu}_m\left(dx^{\Lambda_2^{(n)}} \middle| x^{\Lambda_2^{(l)}}, l \neq n\right)$ satisfy a LSI with a constant that is uniform in the conditioned values $x^{\Lambda_2^{(l)}}, l \neq n$. This ingredient is derived via an adaptation of the argument of [Men14]. The main ingredients for this part are the local Cramér theorem (see Theorem 3.2.4), the Holley-Stroock Principle (see Theorem 8.2.2), the decay of correlations of the gce (Theorem 1.2.6) and a decomposition method for Hamiltonians introduced in [Men14]. The second ingredient is to show that, in the Hamiltonian of $\bar{\mu}_m$, the interactions between blocks $\Lambda_2^{(l)}$ and $\Lambda_2^{(n)}$, $l \neq n$, are sufficiently small. This is achieved by observing that the correlations of the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ decay due to Theorem 5.2.1 and choosing the parameter L sufficiently large. For more details, see Proposition 6.2.5 from below.

Now, let us turn to the detailed argument. We prove the following three propositions. The first one is the uniform LSI for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$.

Proposition 6.2.4. *The conditional measure $\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)$ satisfies a LSI with constant $\varrho_1 > 0$ that is uniform in m, s and the conditioned spins x^{Λ_2}. The constant $\varrho_1 > 0$ is uniform in $K = |\Lambda_1|$, x^{Λ_2}, s and m.*

The second proposition states the LSI for the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$.

Proposition 6.2.5. *There is a constant L_0 such that if $L \geq L_0$ the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$ satisfies a LSI with constant $\varrho_2 > 0$ that is uniform in m, s and L. The constant ϱ_2 only depends on the interaction range R.*

The last proposition is

Proposition 6.2.6. *Let $\mu_{N,m}$ be the ce defined by (1.1.7). Assume that*

- *The conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ satisfies LSI(ϱ_1). The constant ϱ_1 is uniform in m, s and the conditioned spins x^{Λ_2}.*
- *The marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$ satisfies LSI(ϱ_2). The constant ϱ_2 is uniform in m, s and L.*

Then the ce $\mu_{N,m}$ satisfies LSI(ϱ) with a constant $\varrho > 0$ that is uniform in N, m and s.

The main work of this chapter consists of deducing Proposition 6.2.4 and Proposition 6.2.5 which are done in Section 6.3.1 and Section 6.3.2 respectively. Proposition 6.2.6 is a slight modification of [GOVW09, Theorem 3]. For convenience of the readers, we give the proof of Proposition 6.2.6 in Section 6.3.6.

Proof of Theorem 6.2.3. Without loss of generality, it is sufficient to prove the uniform LSI for large enough system size $N \geq N_0$. For small system size $N \leq N_0$ the LSI can be deduced via a combination of Bakry-Émery criterion (cf. Theorem 8.2.3) and the Holley-Stroock perturbation principle (see Theorem 8.2.2). The uniform LSI for large systems $N \geq N_0$ is obtained by choosing L large but fixed and a combination of Proposition 6.2.4, Proposition 6.2.5 and Proposition 6.2.6. □

6.3 Proof of Propositions

6.3.1 Proof of Proposition 6.2.4

The goal is to deduce a uniform LSI for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$. As described in Section 6.2.2, the proof of Proposition 6.2.4 is an adaptation of [GOVW09, Theorem 14], where a uniform LSI was deduced for a ce with non-interacting Hamiltonian. As explained in Section 6.2.2 the main observation is that by conditioning on even blocks, odd blocks do not interact within the Hamiltonian and the setting of [GOVW09] applies with minor adaptation. More precisely, let us fix $x^{\Lambda_2} \in \mathbb{R}^{\Lambda_2}$ and denote $K := |\Lambda_1|$. We can rewrite the spin restriction (1.1.8) as

$$\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \frac{N}{K}m - \frac{1}{K}\sum_{i\in\Lambda_2} x_i =: \widetilde{m}. \tag{6.3.1}$$

Assume that we have M blocks in Λ_1. That is,

$$\Lambda_1 = \bigcup_{n=1}^{M} \Lambda_1^{(n)}.$$

Then the Hamiltonian H can be written as

$$H(x) = \sum_{n=1}^{M} H_n\left(x^{\Lambda_1^{(n)}}\right) + C\left(x^{\Lambda_2}\right), \tag{6.3.2}$$

where

$$H_n\left(x^{\Lambda_1^{(n)}}\right) = \sum_{i\in\Lambda_1^{(n)}}\left(\psi(x_i) + s_i x_i + \frac{1}{2}\sum_{j:\ 1\le|j-i|\le R} M_{ij}x_i x_j\right),$$

and $C\left(x^{\Lambda_2}\right)$ is a constant that only depends on x^{Λ_2}. We observe that for each n the potential $H_n\left(x^{\Lambda_1^{(n)}}\right)$ only depends on x^{Λ_1} through the spin values $x^{\Lambda_1^{(n)}}$. The potential $H_n\left(x^{\Lambda_1^{(n)}}\right)$ depends on the conditioned spin values x^{Λ_2} only via a linear term. Most importantly, there is no interaction between odd blocks $\left\{\Lambda_1^{(n)} : n = 1, \cdots, M\right\}$ within the Hamiltonian H.

Let us now explain how the argument from [GOVW09, Theorem 14] applies. We only point out the main differences and leave the details as an exercise. We start with observing that the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ is a ce wrt. the fixed mean spin $\widetilde{m}$ given by (6.3.1). Let L denote the size of a block in Λ_1 i.e. $L = |\Lambda_1^{(1)}|$. Let P be defined as the map

$$Px^{\Lambda_1} = \left(\frac{1}{L}\sum_{i\in\Lambda_1^{(n)}} x_i\right)_{n\in[M]} =: (y_n)_{n\in[M]} =: y$$

that associates to every block $\Lambda_1^{(n)}$ it's mean spin y_n. The mapping P yields a decomposition of the measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ into a conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$ and a marginal measure $\bar{\mu}_{N,m}\left(dy\middle|\ x^{\Lambda_2}\right)$ i.e.

$$\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right) = \mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)\ \bar{\mu}_{N,m}\left(dy\middle|\ x^{\Lambda_2}\right).$$

The core of the argument in [GOVW09] is to deduce a LSI for the measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$ and for $\bar{\mu}_{N,m}\left(dy\middle|\ x^{\Lambda_2}\right)$. Those two LSIs then combine into a LSI for the full measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$.

Let us consider the LSI for the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$. We observe that due to (6.3.2) and the conditioning onto x^{Λ_2} the measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$ is a product measure on the blocks. It follows that the measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$ satisfies a LSI via a combination of the Bakry-Émery Criterion (cf. Theorem 8.2.3), the Holley-Stroock Perturbation Principle (cf. Theorem 8.2.2) and the Tensorization Principle (cf. Theorem 8.2.1). Because the conditioning x^{Λ_2} only enters the Hamiltonian of $\mu_{N,m}\left(dx^{\Lambda_1}\middle|Px^{\Lambda_1}=y,\ x^{\Lambda_2}\right)$ as a linear term, the obtained LSI constant is uniform in the conditioned values x^{Λ_2}, the mean spin m, the linear term s and the overall system size N. The constant may depend on the size of the odd blocks. Let us turn to the LSI for the marginal measure $\bar{\mu}_{N,m}\left(dy\middle|\ x^{\Lambda_2}\right)$. We use the same strategy as in the proof of [GOVW09, Theorem 14]. We observe that the Hamiltonian $\bar{H}(y)$ of $\bar{\mu}_{N,m}\left(dy\middle|\ x^{\Lambda_2}\right)$ can be written after cancellation of constant terms as

$$\bar{H}(y) = L\sum_{n=1}^{M}\bar{\psi}_n(y_n),$$

where the function $\bar{\psi}_n$ is given by

$$\bar{\psi}_n(z) = -\frac{1}{L} \log \int_{\left\{x^{\Lambda_1^{(n)}} \in \mathbb{R}^{\Lambda_1^{(n)}} \ \mid \ \frac{1}{K}\sum_{i\in\Lambda_1^{(n)}} x_i = z\right\}} \exp(-H_n(x^{\Lambda_1^{(n)}}))\mathcal{L}(dx^{\Lambda_1^{(n)}}).$$

We observe that $H_n(x^{\Lambda_1^{(n)}})$ satisfies the same structural assumptions as the Hamiltonian H in Section 1.1.1. Hence, an application of Corollary 3.2.5 yields that the function $\bar{\psi}_n$ is uniformly strictly convex for large enough L. Hence, it follows that $\bar{H}(y)$ is uniformly strictly convex and therefore the marginal measure $\bar{\mu}_{N,m}\left(dy \middle|\, x^{\Lambda_2}\right)$ satisfies a uniform LSI by the Bakry-Émery criterion (cf. Theorem 8.2.3). Again, the LSI constant is uniform in the conditioned values x^{Λ_2}. It is left to combine both LSIs to a single LSI for the measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ which is done in a similar fashion as in [GOVW09].

6.3.2 Proof of Proposition 6.2.5

The goal is to deduce a uniform LSI for the marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ given by (6.2.2). We recall the decomposition of $\Lambda = \Lambda_1 \cup \Lambda_2$ into odd blocks given by Λ_1 and even blocks given by Λ_2 (cf. Figure 6.1). The marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ arises from $\mu_{N,m}$ by integrating out the spins located in the odd blocks Λ_1. We will deduce the uniform LSI for $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ by applying the Otto-Reznikoff criterion [OR07] (see Theorem 8.2.4 in the appendix). Applying this criterion needs two ingredients. For explaining the first ingredient let us consider the conditional measures $\bar{\mu}_m\left(dx^{\Lambda_2^{(n)}}\middle|x^{\Lambda_2^{(l)}}, l \neq n\right)$. This measure arises from the ce $\mu_{N,m}$ by conditioning on all spins in even blocks except of one and integrating out all spins in odd blocks. The first ingredient is a LSI for the measures $\bar{\mu}_m\left(dx^{\Lambda_2^{(n)}}\middle|x^{\Lambda_2^{(l)}}, l \neq n\right)$ that is uniform in the conditioned spins $x^{\Lambda_2^{(l)}}$, $l \neq n$ (see Lemma 6.3.1 below). The second ingredient is that the interactions in the measure $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ between even blocks are sufficiently small (see Lemma 6.3.2).

The first ingredient looks innocent on the first sight. By integrating out the odd blocks, the marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ is not restricted to a hyperplane anymore. Therefore a

LSI should hold because one considers a gce on a one-dimensional lattice. The difficult part is to show that the LSI constant is uniform in the conditioned spin values $x^{\Lambda_2^{(l)}}$, $l \neq n$. Those values enter the Hamiltonian of the measure $\bar{\mu}_{N,m}(dx^{\Lambda_2})$ via a subtle interaction term, namely the free energy of the ce $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2})$. This interaction term is non-quadratic and has infinite range and therefore is hard to control.

We derive the uniform LSI by extending a method of [Men14] for gces of one variable to ces of multiple variables (see (cf. [Men14, Lemma 3.1, Lemma 3.2])). In the first step of the argument we use the Holley-Stroock Perturbation Principle (see Theorem 8.2.2) to modify the interaction term in the Hamiltonian. By the equivalence of ensembles (see Chapter 3) this allows us to consider in the Hamiltonian an interaction term that is the free energy of the associated gce, which is easier to control than the free energy of the ce. Then we follow the approach of [Men14] and write the Hamiltonian H on a block $\Lambda_2^{(n)}$ as a bounded perturbation of a uniform strictly convex potential and deduce the uniform LSI via a combination of the Bakry-Émery criterion (see Theorem 8.2.3) and the Holley-Stroock Perturbation Principle (see Theorem 8.2.2).

In the next two lemmas we formulate the main ingredients for the proof of Proposition 6.2.5. They correspond to [Men14, Lemma 3.1, Lemma 3.2] with the difference that the lemmas in this chapter are more general in the sense that we consider multi-variate measures and that the interaction term is more complicated.

Lemma 6.3.1. *Assume $K = |\Lambda_1|$ is large enough. Then for each $n \in \{1, \cdots, M\}$, the block marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2^{(n)}}|x^{\Lambda_2^{(l)}}, l \neq n\right)$ satisfies the LSI(τ) for some $\tau > 0$ that depends on R but is independent of the mean spin m, the external field s and the conditioned spins $x^{\Lambda_2^{(l)}}, l \neq n$.*

Lemma 6.3.2. *Let $\bar{H}_{\Lambda_2}$ be the Hamiltonian of the marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2}\right)$, i.e.*

$$\begin{aligned}\bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right) &= -\log \int_{\frac{1}{N}\sum_{k\in\Lambda_1} x_k = m - \frac{1}{N}\sum_{i\in\Lambda_2} x_i} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1}) \quad (6.3.3)\\ &\overset{(6.3.1)}{=} -\log \int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1}).\end{aligned}$$

For any $n, l \in \{1, \cdots M\}$ with $n \neq l$, There is a constant $\tilde{C} > 0$ such that

$$\left\| \left(\nabla_i \nabla_j \bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)\right)_{i\in\Lambda_2^{(n)},\ j\in\Lambda_2^{(l)}} \right\| \leq \tilde{C}\frac{R}{K} + \tilde{C}R\exp\left(-CL|n-l|\right).$$

Here, $\|\cdot\|$ denotes the operator norm of a bilinear form and L denotes the size of a single odd block and K denotes the overall size of all combined odd blocks, i.e. $K = |\Lambda_1| = LM$.

The Lemma 6.3.1 and Lemma 6.3.2 are proved in Section 6.3.3 and Section 6.3.5, respectively.

Proof of Proposition 6.2.5. Recall that $K = LM$. By choosing L large enough, a combination of Lemma 6.3.1, Lemma 6.3.2 and the Otto-Reznikoff criterion (cf. Theorem 8.2.4) yields that the marginal measure $\bar{\mu}_m$ satisfies a LSI with the desired properties. □

6.3.3 Proof of Lemma 6.3.1

We note that the block Hamiltonian $\bar{H}_{\Lambda_2^{(n)}}$ of the block marginal measure $\bar{\mu}_m\left(dx^{\Lambda_2^{(n)}} \middle| x^{\Lambda_2^{(l)}}, l \neq n\right)$ is written as follows:

$$\begin{aligned}\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}} \middle| x^{\Lambda_2^{(l)}},\ l \neq n\right) &= -\log \int_{\frac{1}{N}\sum_{k\in\Lambda_1} x_k = m - \frac{1}{N}\sum_{i\in\Lambda_2} x_i} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})\\ &\overset{(6.3.1)}{=} -\log \int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1}).\end{aligned}$$

For notational convenience we abbreviate

$$\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right) = \bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}} \middle| x^{\Lambda_2^{(l)}},\ l \neq n\right).$$

Our aim is to decompose the block Hamiltonian $\bar{H}_{\Lambda_2^{(n)}}$ into a strictly convex function and a bounded function. Then the proposition will follow from the Bakry-Émery criterion (cf. Theorem 8.2.3) and the Holley-Stroock Perturbation Principle (cf. Theorem 8.2.2). More precisely, we want to find functions $\tilde{\psi}^c, \tilde{\psi}^b : \mathbb{R}^{\Lambda_2^{(n)}} \to \mathbb{R}$ such that

$$\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right) = \tilde{\psi}^c\left(x^{\Lambda_2^{(n)}}\right) + \tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right), \tag{6.3.4}$$

$$\operatorname{Hess}_{\mathbb{R}^{\Lambda_2^{(n)}}} \tilde{\psi}^c \geq c > 0 \qquad \text{and} \qquad |\tilde{\psi}^b|_\infty \leq C < \infty. \tag{6.3.5}$$

Let us introduce auxiliary set E_n and Hamiltonian $H_{\text{aux}} : \mathbb{R}^N \to \mathbb{R}$. These are

$$E_n := \{k \in \Lambda : \ \exists\, i \in \Lambda_2^{(n)} \text{ such that } |k-i| \leq S\}, \tag{6.3.6}$$

$$H_{\text{aux}}(x) := H(x) - \sum_{j \in E_n} \psi^b(x_j),$$

where S is a positive integer that will be chosen later. By definition, H_{aux} is strictly convex in the space restricted to the spins x_i with $i \in E_n$. The associated gce $\mu^\sigma_{\text{aux}}(dx^{\Lambda_1})$ and the ce $\mu_{\text{aux},\widetilde{m}}\left(dx^{\Lambda_1}\right)$ are

$$\mu^\sigma_{\text{aux}}\left(dx^{\Lambda_1}\right) = \mu^\sigma_{\text{aux}}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right) := \frac{1}{Z}\exp\left(\sigma\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1},$$

$$\begin{aligned}\mu_{\text{aux},\widetilde{m}}\left(dx^{\Lambda_1}\right) &= \mu_{\text{aux},\widetilde{m}}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)\\ &:= \frac{1}{Z}\mathbb{1}_{\left\{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}\right\}}\left(x^{\Lambda_1}\right)\exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\mathcal{L}^{K-1}(dx^{\Lambda_1}).\end{aligned} \tag{6.3.7}$$

As described in Section 2, we choose $\sigma = \sigma(\widetilde{m})$ such that (cf. Definition 2.1.6)

$$\widetilde{m} = \frac{1}{K}\sum_{k\in\Lambda_1}\int x_k \mu^\sigma_{\text{aux}}(dx^{\Lambda_1}) =: \frac{1}{K}\sum_{k\in\Lambda_1}\widetilde{m}_k. \tag{6.3.8}$$

Motivated by the approach of [Men14], the first attempt to decompose $\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right)$ according to (6.3.4) is to set

$$\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right) = -\log\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1})$$

$$- \log \frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}. \tag{6.3.9}$$

The motivation for decomposition is the following. Because H_{aux} is strictly convex on the spins in E_n, and log-integration conserves convexity, one expects that the first term on the right hand side of (6.3.9) is strictly convex. One expects that the second term on the right hand side of (6.3.9) is bounded because the Hamiltonian H and the auxiliary Hamiltonian H_{aux} are close up to bounded perturbations.

However when doing so, it is not clear if the strict convexity of the first term on the right hand side of (6.3.9) is uniform in the conditioned spins $x^{\Lambda_2^{(l)}}$, $l \neq n$. We circumvent this obstacle by adding and subtracting an additional term, i.e.

$$\begin{aligned}
\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right) = &-\Bigg(\log \int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1}) \\
&\qquad + K\sigma(\widetilde{m})\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\sigma(\widetilde{m}) \sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\Bigg) \\
&+ \left(K\sigma(\widetilde{m})\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\sigma(\widetilde{m}) \sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \\
&- \log \frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}.
\end{aligned}$$

We define

$$\tilde{\psi}^c\left(x^{\Lambda_2^{(n)}}\right) := K\sigma(\widetilde{m})\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\sigma(\widetilde{m}) \sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1},$$

$$\begin{aligned}
\tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right) := &-\Bigg(\log \int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1}) \\
&+ K\sigma(\widetilde{m})\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\sigma(\widetilde{m}) \sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\Bigg) \qquad (6.3.10) \\
&- \log \frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1})}. \qquad (6.3.11)
\end{aligned}$$

Hence, we have the splitting

$$\bar{H}_{\Lambda_2^{(n)}}\left(x^{\Lambda_2^{(n)}}\right) = \tilde{\psi}^c\left(x^{\Lambda_2^{(n)}}\right) + \tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right).$$

Let us now explain, why the terms $\tilde{\psi}^c$ and $\tilde{\psi}^b$ heuristically satisfy (6.3.4) and (6.3.5). Let us start with the property (6.3.5) i.e. that $\tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right)$ is bounded. We observe that $\tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right)$ consists out of two terms, namely (6.3.10) and (6.3.11). The term (6.3.11) is unchanged from the first attempt and should be bounded by the same argument as above. The term (6.3.10) is representing the difference between the free energy of an canonical ensemble and the free energy of an associated modified grand-canonical ensemble. Hence, this term should be uniformly bounded by the equivalence of ensembles.
Let us explain why (6.3.4) is satisfied i.e. why the term $\tilde{\psi}^c$ is uniformly strictly convex. We observe that $\tilde{\psi}^c$ is a Legendre-transformation. By basic properties of the Legendre transform the convexity properties of $\tilde{\psi}^c$ are determined by the convexity properties of the log-partition function of the associated grand-canonical ensemble. In the first attempt we would have to analyze the convexity of the log-partition function of a canonical ensemble. Now, we only have to analyze the convexity of the log-partition function of a grand-canonical ensemble. This is easier to analyze and turns out to be more robust wrt. the conditioned spin-values.

This argument is made rigorous in the following two statements.

Lemma 6.3.3. *For both $K = |\Lambda_1|$ and S large enough (cf.* (6.3.6)*), the function $\tilde{\psi}^c$ is strictly convex in the sense that there is a positive constant $c > 0$ with*

$$\operatorname{Hess}_{\mathbb{R}^{\Lambda_2^{(n)}}} \tilde{\psi}^c \geq c \operatorname{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}},$$

where $\operatorname{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}}$ denotes the identity map on $\mathbb{R}^{\Lambda_2^{(n)}}$.

Lemma 6.3.4. *The function $\tilde{\psi}^b$ is uniformly bounded. More precisely, it holds that*

$$|\tilde{\psi}^b|_\infty \lesssim 1 + R + S.$$

We shall see how the statements from above yield Lemma 6.3.1.

Proof of Lemma 6.3.1. By choosing S large but fixed, the block Hamiltonian $\bar{H}_{\Lambda_2^{(n)}}$ is a sum of strictly convex function $\widetilde{\psi}^c$ and a bounded function $\widetilde{\psi}^b$. Then the Lemma 6.3.1 follows from a combination of Bakry-Émery criterion (see Theorem 8.2.3) and Holley-Stroock Perturbation Principle (see Theorem 8.2.2). □

It remains to prove Lemma 6.3.3 and Lemma 6.3.4. Let us begin with providing auxiliary statements which will be verified in Section 6.3.4. For the notational convenience, we simply write $\tilde{\sigma} = \sigma(\widetilde{m})$ (cf. see (6.3.8)).

For the proof of Lemma 6.3.3 we need to show that there exists a positive constant $c > 0$ such that the following holds for K and S large enough.

$$\mathrm{Hess}_{\mathbb{R}^{\Lambda_2^{(n)}}}\left(K\tilde{\sigma}\widetilde{m} - \log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\mathrm{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \geq c\,\mathrm{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}}, \quad (6.3.12)$$

where $\mathrm{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}}$ denotes the identity map on $\mathbb{R}^{\Lambda_2^{(n)}}$. As a first step, we calculate a formula for the left hand side of (6.3.12).

Lemma 6.3.5. *For each $i, j \in \Lambda_2^{(n)}$, it holds that*

$$\begin{aligned}
&\frac{d^2}{dx_i dx_j}\left(K\tilde{\sigma}\widetilde{m} - \log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{aux}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right)\\
&\qquad = \mathrm{var}_{\mu_{aux}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1} X_k\right)\frac{d\tilde{\sigma}}{dx_i}\frac{d\tilde{\sigma}}{dx_j} \qquad (6.3.13)\\
&\qquad\quad + \mathbb{E}_{\mu_{aux}^{\tilde{\sigma}}}\left[\frac{\partial^2}{\partial x_i \partial x_j} H_{aux}(X)\right] - \mathrm{cov}_{\mu_{aux}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i} H_{aux}(X), \frac{\partial}{\partial x_j} H_{aux}(X)\right). \qquad (6.3.14)
\end{aligned}$$

Next, we provide an auxiliary estimate of partial derivative of $\tilde{\sigma}$ with respect to x_i, where $i \in \Lambda_2^{(n)}$.

Lemma 6.3.6. *For each $i \in \Lambda_2^{(n)}$, it holds that*

$$\left|\frac{d\tilde{\sigma}}{dx_i}\right| \lesssim \frac{1}{K}.$$

The last step towards to the proof of Lemma 6.3.3 is the strict convexity of (6.3.14).

Lemma 6.3.7. *There is a positive constant $c > 0$ such that for both $K = |\Lambda_1|$ and S large enough (cf. (6.3.6)) the following holds.*

$$\left(\mathbb{E}_{\mu_{aux}^{\tilde{\sigma}}}\left[\frac{\partial^2}{\partial x_i \partial x_j} H_{aux}(X)\right] - \operatorname{cov}_{\mu_{aux}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i} H_{aux}(X), \frac{\partial}{\partial x_j} H_{aux}(X)\right)\right)_{i,j \in \Lambda_2^{(n)}} \geq c \, \mathrm{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}} . \tag{6.3.15}$$

The proof of Lemma 6.3.5, Lemma 6.3.6 and Lemma 6.3.7 are presented in Section 6.3.4. Let us see how these statements yield Lemma 6.3.3.

Proof of Lemma 6.3.3. Due to Lemma 6.3.6 it holds that for $i, j \in \Lambda_2^{(n)}$

$$\frac{d^2}{dx_i dx_j}\left(K\tilde{\sigma}\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k \in \Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) = T_{(6.3.13)} + T_{(6.3.14)}.$$

Hence, the desired statement follows from Lemma 6.3.7 and the estimate

$$\left|T_{(6.3.13)}\right| \lesssim \frac{1}{K}. \tag{6.3.16}$$

Indeed, (6.3.16) directly follows from a combination of Lemma 6.3.6 and the observation that (cf. Lemma 2.1.4)

$$\operatorname{var}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k \in \Lambda_1} X_k\right) \lesssim K.$$

□

Let us turn to the proof of Lemma 6.3.4. We need an auxiliary statement to begin with.

Lemma 6.3.8. *Let $X = (X_k)_{k \in \Lambda_1}$ be a random variable distributed according to the measure $\mu_{aux}^{\tilde{\sigma}}$. Denote g_{aux} by the density of the random variable $\frac{1}{\sqrt{K}} \sum_{k \in \Lambda_1} (X_k - \widetilde{m}_k)$. It holds that*

$$\begin{aligned}\log g_{aux}(0) = \Bigg(&\log \int_{\frac{1}{K}\sum_{k \in \Lambda_1} x_k = \widetilde{m}} \exp\left(-H_{aux}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}(dx^{\Lambda_1}) \\ &+ K\tilde{\sigma}\widetilde{m} - \log \int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k \in \Lambda_1} x_k - H_{aux}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\Bigg).\end{aligned}$$

The proof of Lemma 6.3.8 is given in Section 6.3.4. Let us now turn to the proof of Lemma 6.3.4.

Proof of Lemma 6.3.4. We observe that

$$\tilde{\psi}^b\left(x^{\Lambda_2^{(n)}}\right) = T_{(6.3.10)} + T_{(6.3.11)}.$$

Estimation of $T_{(6.3.10)}$: Because $\mu_{\text{aux}}^{\sigma}$ is a one-dimensional gce, the desired estimate

$$|T_{(6.3.10)}| \lesssim 1 \tag{6.3.17}$$

follows from a combination of Lemma 6.3.8 and Proposition 3.2.7.

Estimation of $T_{(6.3.11)}$: It holds that

$$\begin{aligned}\left|T_{(6.3.11)}\right| &= \left|\log \frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \tilde{m}} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1})}{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \tilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1})}\right| \\ &\overset{(6.3.7)}{=} \left|\log \int \exp\left(-\sum_{j\in E_n} \psi^b(x_j)\right)\mu_{\text{aux},\tilde{m}}\left(dx^{\Lambda_1}\right)\right| \\ &\leq \sum_{j\in E_n} \|\psi^b\|_\infty \lesssim R + S. \end{aligned} \tag{6.3.18}$$

Then a combination of (6.3.17) and (6.3.18) yields, as desired,

$$|\tilde{\psi}^b|_\infty \lesssim 1 + R + S.$$

□

6.3.4 Proof of Auxiliary Statements

In this section, we provide the proof of Lemma 6.3.5, Lemma 6.3.6, Lemma 6.3.7 and Lemma 6.3.8. Let us begin with providing the proof of Lemma 6.3.5.

Proof of Lemma 6.3.5. A direct calculation yields that

$$
\begin{aligned}
&\frac{d}{dx_j}\left(K\tilde{\sigma}\widetilde{m} - \log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \\
&\quad = K\left(\frac{d}{dx_j}\tilde{\sigma}\right)\widetilde{m} + K\tilde{\sigma}\frac{d\widetilde{m}}{dx_j} \\
&\qquad - \frac{\partial}{\partial\tilde{\sigma}}\left(\log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right)\frac{d\tilde{\sigma}}{dx_j} \\
&\qquad - \frac{\partial}{\partial x_j}\left(\log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \\
&\quad \overset{(6.3.8)}{=} K\tilde{\sigma}\frac{d\widetilde{m}}{dx_j} - \frac{\partial}{\partial x_j}\left(\log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \\
&\quad \overset{(6.3.1)}{=} -\tilde{\sigma} - \mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[-\frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right] \\
&\quad = -\tilde{\sigma} + \mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right].
\end{aligned}
$$

Taking further derivative, one gets

$$
\begin{aligned}
&\frac{d^2}{dx_i dx_j}\left(K\tilde{\sigma}\widetilde{m} - \log\int_{\mathbb{R}^{\Lambda_1}} \exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1} x_k - H_{\text{aux}}\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}\right) \\
&\quad = -\frac{d\tilde{\sigma}}{dx_i} + \frac{d}{dx_i}\mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right] \\
&\quad = -\frac{d\tilde{\sigma}}{dx_i} + \frac{\partial}{\partial\tilde{\sigma}}\mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right]\frac{d\tilde{\sigma}}{dx_i} + \frac{\partial}{\partial x_i}\mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right] \\
&\quad = \frac{d\tilde{\sigma}}{dx_i}\left(\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1} X_k, \frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right) - 1\right) \\
&\qquad + \mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial^2}{\partial x_i\partial x_j}H_{\text{aux}}(X)\right] - \operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i}H_{\text{aux}}(X), \frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right) \\
&\quad \overset{(6.3.19)}{=} \operatorname{var}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1} X_k\right)\frac{d\tilde{\sigma}}{dx_i}\frac{d\tilde{\sigma}}{dx_j} \\
&\qquad + \mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial^2}{\partial x_i\partial x_j}H_{\text{aux}}(X)\right] - \operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i}H_{\text{aux}}(X), \frac{\partial}{\partial x_j}H_{\text{aux}}(X)\right).
\end{aligned}
$$

□

Let us now turn to the proof of Lemma 6.3.6.

Proof of Lemma 6.3.6. A combination of the definition (6.3.1) of $\widetilde{m}$ and the equality (6.3.8) yields

$$\frac{N}{K}m - \frac{1}{K}\sum_{i\in\Lambda_2} x_i = \frac{1}{K}\sum_{k\in\Lambda_1} \widetilde{m}_k.$$

We recall that $\mu_{\text{aux}}^{\tilde{\sigma}}\left(dx^{\Lambda_1}\right) = \mu_{\text{aux}}^{\tilde{\sigma}}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ depends on $x_i, i \in \Lambda_2^{(n)}$ via the field $\tilde{\sigma}$ and the conditioned spin value x^{Λ_2}. Then differentiating both sides with respect to $x_i, i \in \Lambda_2^{(n)}$ gives

$$\begin{aligned}
-\frac{1}{K} &= \frac{1}{K}\frac{d}{dx_i}\left(\sum_{k\in\Lambda_1}\widetilde{m}_k\right) \\
&= \frac{1}{K}\frac{\partial}{\partial\tilde{\sigma}}\left(\sum_{k\in\Lambda_1}\widetilde{m}_k\right)\frac{d\tilde{\sigma}}{dx_i} + \frac{1}{K}\frac{\partial}{\partial x_i}\left(\sum_{k\in\Lambda_1}\widetilde{m}_k\right) \\
&= \frac{1}{K}\operatorname{var}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k\right)\frac{d\tilde{\sigma}}{dx_i} + \frac{1}{K}\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k, -\frac{\partial}{\partial x_i}H_{\text{aux}}(X)\right). \qquad (6.3.19)
\end{aligned}$$

Our strategy is to estimate every factor appearing in (6.3.19). Then rearranging (6.3.19) will yield the desired estimate. Note that the measure $\mu_{\text{aux}}^{\tilde{\sigma}}$ is only active on Λ_1 while $i \in \Lambda_2^{(n)}$. Since the covariance is invariant under adding constants, it holds that

$$\begin{aligned}
\left|\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k, \frac{\partial}{\partial x_i}H_{\text{aux}}(X)\right)\right| &= \left|\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k, X_i + s_i + \frac{1}{2}\sum_{l:1\le|l-i|\le R}M_{il}X_l\right)\right| \\
&= \frac{1}{2}\left|\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k, \sum_{\substack{l\in\Lambda_1\\ l:1\le|l-i|\le R}}M_{il}X_l\right)\right|. \qquad (6.3.20)
\end{aligned}$$

We note that the properties described in Chapter 2 hold for $\mu_{\text{aux}}^{\tilde{\sigma}}$ because it is a gce in the one-dimensional lattice. In particular $\mu_{\text{aux}}^{\tilde{\sigma}}$ has exponential decay of correlations as in Theorem 1.2.6. Therefore it holds that

$$T_{(6.3.20)} \le \frac{1}{2}\sum_{k\in\Lambda_1}\sum_{\substack{l\in\Lambda_1\\ l:1\le|l-i|\le R}}\left|\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(X_k, M_{il}X_l\right)\right| \lesssim R \lesssim 1. \qquad (6.3.21)$$

As a consequence, by rearranging (6.3.19) we get

$$\left|\frac{d\tilde{\sigma}}{dx_i}\right| = \frac{K}{\operatorname{var}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k\right)}\left|\frac{1}{K}\operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\sum_{k\in\Lambda_1}X_k, \frac{\partial}{\partial x_i}H_{\text{aux}}(X)\right) - \frac{1}{K}\right| \overset{\textit{Lemma } 2.1.4,\ (6.3.21)}{\lesssim} \frac{1}{K}.$$

$\square$

The proof of Lemma 6.3.7 consists of a lengthy calculations. We provide the proof of Lemma 6.3.7 modulo an auxiliary statement (see Lemma 6.3.9 from below).

Proof of Lemma 6.3.7. The argument is inspired by calculations done in [Men14, Lemma 3.1]. The main difference is that we consider the multi-variable case (in $\mathbb{R}^{\Lambda_2^{(n)}}$) while [Men14, Lemma 3.1] only considers the single variable case. Recall the definition (6.3.6) of the set E_n. Let us decompose the auxiliary measure $\mu_{\text{aux}}^{\tilde{\sigma}}$ into the conditional and the marginal measure as follows:

$$\mu_{\text{aux}}^{\tilde{\sigma}}\left(dx^{\Lambda_1}\right) = \mu_{\text{aux}}^{\tilde{\sigma}}\left((dx_k)_{k\in\Lambda_1, k\in E_n} \,\middle|\, (x_l)_{l\in\Lambda_1, l\notin E_n}\right) \bar{\mu}_{\text{aux}}^{\tilde{\sigma}}\left((dx_l)_{l\in\Lambda_1, l\notin E_n}\right).$$

We write $\mu_{\text{aux,c}}^{\tilde{\sigma}} = \mu_{\text{aux}}^{\tilde{\sigma}}\left((dx_k)_{k\in\Lambda_1, k\in E_n} \,\middle|\, (x_l)_{l\in\Lambda_1, l\notin E_n}\right)$ for notational convenience. Then it follows that

$$\begin{aligned}
&\mathbb{E}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left[\frac{\partial^2}{\partial x_i \partial x_j} H_{\text{aux}}(X)\right] - \operatorname{cov}_{\mu_{\text{aux}}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i} H_{\text{aux}}(X), \frac{\partial}{\partial x_j} H_{\text{aux}}(X)\right) \\
&\quad = \int \left(\int \frac{\partial^2}{\partial x_i \partial x_j} H_{\text{aux}}(x) \mu_{\text{aux,c}}^{\tilde{\sigma}} - \operatorname{cov}_{\mu_{\text{aux,c}}}\left(\frac{\partial}{\partial x_i} H_{\text{aux}}(x), \frac{\partial}{\partial x_j} H_{\text{aux}}(x)\right)\right) \bar{\mu}_{\text{aux}}^{\tilde{\sigma}} \\
&\qquad - \operatorname{cov}_{\bar{\mu}_{\text{aux}}^{\tilde{\sigma}}}\left(\int \frac{\partial}{\partial x_i} H_{\text{aux}}(x) \mu_{\text{aux,c}}^{\tilde{\sigma}}, \int \frac{\partial}{\partial x_j} H_{\text{aux}}(x) \mu_{\text{aux,c}}^{\tilde{\sigma}}\right). \qquad (6.3.22)
\end{aligned}$$

As noted before, if restricted to spins x_i with $i \in E_n$, the Hamiltonian H_{aux} is strictly convex. Then by Brascamp-Lieb inequality (cf. [BL76]), there is a constant $c > 0$ with

$$\left(\int \frac{\partial^2}{\partial x_i \partial x_j} H_{\text{aux}}(x) \mu_{\text{aux,c}}^{\tilde{\sigma}} - \operatorname{cov}_{\mu_{\text{aux,c}}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i} H_{\text{aux}}(x), \frac{\partial}{\partial x_j} H_{\text{aux}}(x)\right)\right)_{i,j} \geq c \operatorname{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}}.$$

and as a consequence,

$$\left(\int \left(\int \frac{\partial^2}{\partial x_i \partial x_j} H_{\text{aux}}(x) \mu_{\text{aux,c}}^{\tilde{\sigma}} - \operatorname{cov}_{\mu_{\text{aux,c}}^{\tilde{\sigma}}}\left(\frac{\partial}{\partial x_i} H_{\text{aux}}(x), \frac{\partial}{\partial x_i} H_{\text{aux}}(x)\right)\right) \bar{\mu}_{\text{aux}}^{\tilde{\sigma}}\right)_{i,j} \geq c \operatorname{Id}_{\mathbb{R}^{\Lambda_2^{(n)}}}. \qquad (6.3.23)$$

Then a combination of (6.3.22), (6.3.23) and Lemma 6.3.9 from below yields the desired property (6.3.15) by choosing S large enough. □

The following statement, Lemma 6.3.9, is an estimation of the second therm in the right hand side of (6.3.22).

Lemma 6.3.9. *Recall the definition* (6.3.6) *of the set* E_n*.*

$$E_n := \{k \in \Lambda : \ \exists\, i \in \Lambda_2^{(n)} \text{ such that } |k-i| \leq S\}.$$

For each $i, j \in \Lambda_2^{(n)}$*, it holds that*

$$\left|\operatorname{cov}_{\bar\mu^{\tilde\sigma}_{aux}}\left(\int \frac{\partial}{\partial x_i} H_{aux}(x)\mu^{\tilde\sigma}_{aux,c}, \int \frac{\partial}{\partial x_j} H_{aux}(x)\mu^{\tilde\sigma}_{aux,c}\right)\right| \lesssim S^2 \exp\left(-CS\right).$$

Proof of Lemma 6.3.9. We begin with a simple observation:

$$\frac{\partial}{\partial x_i} H_{\text{aux}}(x) = x_i + s_i + \frac{1}{2} \sum_{p: 1 \leq |p-i| \leq R} M_{ip} x_p.$$

As the measures $\mu^{\tilde\sigma}_{\text{aux,c}}$ and $\bar\mu^{\tilde\sigma}_{\text{aux}}$ are defined in the subspace of x^{Λ_1}, we may regard x_l's with $l \in \Lambda_2$ as constants. As a consequence, it holds that

$$\begin{aligned}
&\operatorname{cov}_{\bar\mu^{\tilde\sigma}_{\text{aux}}}\left(\int \frac{\partial}{\partial x_i} H_{\text{aux}}(x)\mu^{\tilde\sigma}_{\text{aux,c}}, \int \frac{\partial}{\partial x_j} H_{\text{aux}}(x)\mu^{\tilde\sigma}_{\text{aux,c}}\right) \\
&\quad = \frac{1}{4} \operatorname{cov}_{\bar\mu^{\tilde\sigma}_{\text{aux}}}\left(\int \sum_{\substack{p \in \Lambda_1, \\ 1 \leq |p-i| \leq R}} M_{ip} x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}, \int \sum_{\substack{q \in \Lambda_1, \\ 1 \leq |q-j| \leq R}} M_{jq} x_q \ \mu^{\tilde\sigma}_{\text{aux,c}}\right).
\end{aligned}$$

To estimate the covariance from above, let us double the variables to get

$$\begin{aligned}
&\operatorname{cov}_{\bar\mu^{\tilde\sigma}_{\text{aux}}}\left(\int \sum_{\substack{p \in \Lambda_1, \\ 1 \leq |p-i| \leq R}} M_{ip} x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}, \int \sum_{\substack{q \in \Lambda_1, \\ 1 \leq |q-j| \leq R}} M_{jq} x_q \ \mu^{\tilde\sigma}_{\text{aux,c}}\right) \\
&= \int\int \left(\int \sum_{\substack{p \in \Lambda_1, \\ 1 \leq |p-i| \leq R}} M_{ip} x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|y) - \int \sum_{\substack{p \in \Lambda_1, \\ 1 \leq |p-i| \leq R}} M_{ip} x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|z)\right)
\end{aligned}$$

$$\times \left(\int \sum_{\substack{q\in\Lambda_1,\\ 1\le|q-j|\le R}} M_{jq}x_q \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|y) - \int \sum_{\substack{q\in\Lambda_1,\\ 1\le|q-j|\le R}} M_{jq}x_q \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|z) \right) \bar\mu^{\tilde\sigma}_{\text{aux}}(dy)\bar\mu^{\tilde\sigma}_{\text{aux}}(dz). \tag{6.3.24}$$

Then it follows from the fundamental theorem of calculus that

$$\begin{aligned}
&\left| \int \sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} M_{ip}x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|y) - \int \sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} M_{ip}x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|z) \right| \\
&\quad = \left| \int_0^1 \left(\frac{d}{dt} \int \sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} M_{ip}x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|ty+(1-t)z) \right) dt \right| \\
&\quad = \left| \int_0^1 \operatorname{cov}_{\mu^{\tilde\sigma}_{\text{aux,c}}(dx|ty+(1-t)z)} \left(\sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} M_{ip}x_p, \sum_{\substack{r_1\in E_n\\ s_1\notin E_n\\ 1\le|r_1-s_1|\le R}} M_{r_1s_1}x_{r_1}(y_{s_1}-z_{s_1}) \right) dt \right| \\
&\quad \le \sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} \sum_{\substack{r_1\in E_n\\ s_1\notin E_n\\ 1\le|r_1-s_1|\le R}} |M_{ip}||M_{r_1s_1}||y_{s_1}-z_{s_1}| \sup_{0\le t\le 1} \left| \operatorname{cov}_{\mu^{\tilde\sigma}_{\text{aux,c}}(dx|ty+(1-t)z)} (x_p, x_{r_1}) \right|.
\end{aligned} \tag{6.3.25}$$

We note that $\mu^{\tilde\sigma}_{\text{aux}}$, $\mu^{\tilde\sigma}_{\text{aux,c}}$ are also gces on the one-dimensional lattice and they satisfy properties listed in Chapter 2. Therefore an application of Theorem 5.2.1 yields

$$T_{(6.3.25)} \lesssim \sum_{\substack{p\in\Lambda_1,\\ 1\le|p-i|\le R}} \sum_{\substack{r_1\in E_n\\ s_1\notin E_n\\ 1\le|r_1-s_1|\le R}} |M_{ip}||M_{r_1s_1}||y_{s_1}-z_{s_1}| \exp\left(-C|p-r_1|\right). \tag{6.3.26}$$

Similarly, one gets

$$\begin{aligned}
&\left| \int \sum_{\substack{q\in\Lambda_1,\\ |q-j|\le R}} M_{ip}x_p \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|y) - \int \sum_{\substack{q\in\Lambda_1,\\ |q-j|\le R}} M_{jq}x_q \ \mu^{\tilde\sigma}_{\text{aux,c}}(dx|z) \right| \\
&\qquad \lesssim \sum_{\substack{q\in\Lambda_1,\\ |q-j|\le R}} \sum_{\substack{r_2\in E_n\\ s_2\notin E_n\\ |r_2-s_2|\le R}} |M_{jq}||M_{r_2s_2}||y_{s_2}-z_{s_2}| \exp\left(-C|q-r_2|\right).
\end{aligned} \tag{6.3.27}$$

Note that

$$\int\int |y_s-z_s|\,|y_r-z_r|\,\bar\mu^{\tilde\sigma}_{\text{aux}}(dy)\bar\mu^{\tilde\sigma}_{\text{aux}}(dz) \le \frac{1}{2}\int\int (y_s-z_s)^2 + (y_r-z_r)^2\,\bar\mu^{\tilde\sigma}_{\text{aux}}(dy)\bar\mu^{\tilde\sigma}_{\text{aux}}(dz)$$

$$= \frac{1}{2}\left(\mathrm{var}_{\bar{\mu}^{\tilde{\sigma}}_{\mathrm{aux}}}(y_s) + \mathrm{var}_{\bar{\mu}^{\tilde{\sigma}}_{\mathrm{aux}}}(y_r)\right)$$
$$= \frac{1}{2}\left(\mathrm{var}_{\mu^{\tilde{\sigma}}_{\mathrm{aux}}}(y_s) + \mathrm{var}_{\mu^{\tilde{\sigma}}_{\mathrm{aux}}}(y_r)\right) \overset{\textit{Lemma } 2.1.4}{\lesssim} 1. \tag{6.3.28}$$

We also note that finite range interaction with strictly diagonal dominant condition

$$\sum_{1 \le |j-i| \le R} |M_{ij}| + \delta \le M_{ii} = 1 \qquad \text{for all } i \in [N]$$

imply there is a constant C such that for all $i, j \in [N]$

$$|M_{ij}| \le \exp\left(-C|i-j|\right). \tag{6.3.29}$$

Plugging the estimates (6.3.26), (6.3.27), (6.3.28) and (6.3.29) into (6.3.24) yields

$$\begin{aligned}
\left|T_{(6.3.24)}\right| &\lesssim \left(\sum_{\substack{p \in \Lambda_1, \\ |p-i| \le R}} \sum_{\substack{r_1 \in E_n \\ s_1 \notin E_n \\ |r_1 - s_1| \le R}} |M_{ip}||M_{r_1 s_1}| \exp\left(-C|p-r_1|\right) \right) \\
&\quad \times \left(\sum_{\substack{q \in \Lambda_1, \\ |q-j| \le R}} \sum_{\substack{r_2 \in E_n \\ s_2 \notin E_n \\ |r_2 - s_2| \le R}} |M_{jq}||M_{r_2 s_2}| \exp\left(-C|q-r_2|\right) \right) \\
&\lesssim \left(\sum_{\substack{p \in \Lambda_1, \\ |p-i| \le R}} \sum_{\substack{r_1 \in E_n \\ s_1 \notin E_n \\ |r_1 - s_1| \le R}} \exp\left(-C|i-p|\right) \exp\left(-C|r_1-s_1|\right) \exp\left(-C|p-r_1|\right) \right) \\
&\quad \times \left(\sum_{\substack{q \in \Lambda_1, \\ |q-j| \le R}} \sum_{\substack{r_2 \in E_n \\ s_2 \notin E_n \\ |r_2 - s_2| \le R}} \exp\left(-C|j-q|\right) \exp\left(-C|r_2-s_2|\right) \exp\left(-C|q-r_2|\right) \right) \\
&\lesssim R^2 (R+2S)^2 \exp\left(-CS\right) \lesssim S^2 \exp\left(-CS\right).
\end{aligned}$$

as desired. □

It remains to provide the proof of Lemma 6.3.8. This follows from a straightforward calculations.

Proof of Lemma 6.3.8. This follows from a direct computation. Indeed, it holds that

$$\begin{aligned}
&\Bigg(\log \int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k=\widetilde{m}} \exp\left(-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1}) \\
&\qquad +K\tilde{\sigma}\widetilde{m}-\log\int_{\mathbb{R}^{\Lambda_1}}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)dx^{\Lambda_1}\Bigg) \\
&\quad =\log\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k=\widetilde{m}}\exp\left(K\tilde{\sigma}\widetilde{m}-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1}) \\
&\qquad -\log\int_{\mathbb{R}^{\Lambda_1}}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)dx^{\Lambda_1} \\
&\quad =\log\frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k=\widetilde{m}}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1})}{\int_{\mathbb{R}^{\Lambda_1}}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)dx^{\Lambda_1}} \\
&\quad \overset{(6.3.8)}{=}\log\frac{\int_{\frac{1}{\sqrt{K}}\sum_{k\in\Lambda_1}(x_k-\widetilde{m}_k)=0}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dx^{\Lambda_1})}{\int_{\mathbb{R}^{\Lambda_1}}\exp\left(\tilde{\sigma}\sum_{k\in\Lambda_1}x_k-H_{\text{aux}}\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)dx^{\Lambda_1}}=\log g_{\text{aux}}(0).
\end{aligned}$$

□

6.3.5 Proof of Lemma 6.3.2

Let us begin with computing the second derivatives of $\bar{H}_{\Lambda_2}$.

Lemma 6.3.10. *Assume $i\in\Lambda_2^{(n)}$ and $j\in\Lambda_2^{(l)}$ with $n\neq l$. Then it holds that*

$$\begin{aligned}
\frac{d^2}{dx_i dx_j}\bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)=-\operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)}\Bigg(&\frac{\partial}{\partial x_i}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)-\frac{\partial}{\partial x_{i-R}}H\left(x^{\Lambda_1},x^{\Lambda_2}\right), \\
&\frac{\partial}{\partial x_j}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)-\frac{\partial}{\partial x_{j-R}}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)\Bigg).
\end{aligned}$$

Proof of Lemma 6.3.10. Recall the definition (6.3.3) of $\bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)$ given by

$$\bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)=-\log\int_{\frac{1}{N}\sum_{k\in\Lambda_1}x_k=m-\frac{1}{N}\sum_{i\in\Lambda_2}x_i}\exp\left(-H\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}^{K-1}(dx^{\Lambda_1}).$$

Fix $x^{\Lambda_2}\in\mathbb{R}^{\Lambda_2}$ and define a vector $z^{\Lambda_1}\in\mathbb{R}^{\Lambda_1}$ as

$$z_k:=\begin{cases}x_{k+R} & \text{if } k+R\in\Lambda_2,\\ 0 & \text{otherwise,}\end{cases}\qquad \text{for each } k\in\Lambda_1, \tag{6.3.30}$$

and let

$$y^{\Lambda_1} := x^{\Lambda_1} + z^{\Lambda_1}. \tag{6.3.31}$$

Note that $\sum_{k\in\Lambda_1} z_k = \sum_{i\in\Lambda_2} x_i$. In particular, it holds that

$$\sum_{k=1}^{N} x_k = \sum_{k\in\Lambda_1} x_k + \sum_{i\in\Lambda_2} x_i = \sum_{k\in\Lambda_1} x_k + \sum_{k\in\Lambda_1} z_k = \sum_{k\in\Lambda_1} y_k. \tag{6.3.32}$$

With this observation, it follows from change of variables that

$$\bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right) = -\log \int_{\frac{1}{N}\sum_{k\in\Lambda_1} y_k = m} \exp\left(-H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dy^{\Lambda_1}). \tag{6.3.33}$$

Note also that a direct calculation yields for $i \in \Lambda_2$,

$$\begin{aligned}
\frac{d}{dx_i} H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right) &= \frac{\partial}{\partial x_i} H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right) - \sum_{k\in\Lambda_1} \frac{\partial}{\partial x_k} H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right) \cdot \frac{dz_k}{dx_i} \\
&= \frac{\partial}{\partial x_i} H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right) \\
&= \frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right).
\end{aligned} \tag{6.3.34}$$

Then a combination of (6.3.32), (6.3.33) and (6.3.34) followed by change of variables yields

$$\begin{aligned}
&\frac{d}{dx_i} \bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right) \\
&= -\frac{\int_{\frac{1}{N}\sum_{k\in\Lambda_1} y_k = m} \frac{d}{dx_i}\left(-H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right)\right) \exp\left(-H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dy^{\Lambda_1})}{\int_{\frac{1}{N}\sum_{k\in\Lambda_1} y_k = m} \exp\left(-H\left(y^{\Lambda_1} - z^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dy^{\Lambda_1})} \\
&= -\frac{\int_{\frac{1}{N}\sum_{k=1}^{N} x_k = m} \left(-\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) + \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1})}{\int_{\frac{1}{N}\sum_{k=1}^{N} x_k = m} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1})} \\
&\overset{(6.3.1)}{=} -\frac{\int_{\frac{1}{K}\sum_{k\in\Lambda_1} x_k = \tilde{m}} \left(-\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) + \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1})}{\int_{\frac{1}{N}\sum_{k=1}^{N} x_k = m} \exp\left(-H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) \mathcal{L}^{K-1}(dx^{\Lambda_1})} \\
&= \mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right].
\end{aligned}$$

Let us turn to the computation of the second derivative of $\bar{H}$. Observe that for any distinct $k, l \in [N]$,

$$\frac{\partial^2}{\partial x_k \partial x_l} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) = M_{kl}.$$

In particular if $|k-l|>R$, we have

$$\frac{\partial^2}{\partial x_k \partial x_l} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) = M_{kl} = 0.$$

Suppose $i \in \Lambda_2^{(n)}$ and $j \in \Lambda_2^{(l)}$ with $n \neq l$. For $L > 2R$ (see Figure 6.1), it holds that $|i-j| \geq L > 2R$. This implies

$$\operatorname{dist}\left(\{i, i-R\}, \{j, j-R\}\right) > R.$$

With this observation, a similar computation from above yields the desired formula

$$\begin{aligned}\frac{d^2}{dx_i dx_j} \bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right) = -\operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)} &\left(\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right),\right. \\ &\left.\frac{\partial}{\partial x_j} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{j-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right).\end{aligned}$$

□

The main ingredient for proving Lemma 6.3.2 is the decay of correlations. We consider the conditional measure $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ which is a ce with mean spin $\widetilde{m}$ given by (6.3.1). We define the corresponding conditional gce $\mu_N^\tau\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ as

$$\mu_N^\tau\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right) := \frac{1}{Z} \exp\left(\tau \sum_{k\in\Lambda_1} x_k - H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right) dx^{\Lambda_1}, \tag{6.3.35}$$

where $\tau = \tau(\widetilde{m})$ is given by (cf. Definition 2.1.6)

$$\widetilde{m} = \frac{1}{K} \int \sum_{k\in\Lambda_1} x_k \mu_N^\tau\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right).$$

Note that $\mu_N^\tau\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ and $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ satisfy the same structural assumptions made in Chapter 1.1.1. As a consequence, Theorem 1.2.6 and Theorem 5.2.1, i.e. decay of correlations, also hold for $\mu_N^\tau\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ and $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$, respectively. Now we are ready to give a proof of Lemma 6.3.2.

Proof of Lemma 6.3.2. Assume $n \neq l$ and let $i \in \Lambda_2^{(n)}$, $j \in \Lambda_2^{(l)}$ be given. It is enough to prove that

$$\left|\frac{d^2}{dx_i dx_j} \bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)\right| \lesssim \frac{1}{K} + \exp\left(-CL|n-l|\right).$$

Note that for each $k \in [N]$

$$\frac{\partial}{\partial x_k} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) = x_k + \psi_b'(x_k) + s_k + \sum_{l:\ 1 \leq |l-k| \leq R} M_{kl} x_l.$$

In particular for L large enough we have

$$\begin{aligned} \operatorname{dist}\Bigg(&\operatorname{supp}\left(\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right), \\ &\operatorname{supp}\left(\frac{\partial}{\partial x_j} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{j-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\Bigg) \geq (L-2R)|n-l| \geq \frac{L}{2}|n-l|. \end{aligned} \tag{6.3.36}$$

Recall the definition (6.3.35) of μ_N^τ. Because it is the gce on the one-dimensional lattice, one has the uniform moment estimate as in Lemma 2.1.2, i.e. for each $n \in \mathbb{N}$, there is a constant $C = C(n)$ such that for each $i \in \Lambda_1$,

$$\mathbb{E}_{\mu_N^\tau}\left[\left|X_i - \mathbb{E}_{\mu_N^\tau}[X_i]\right|^n\right] \leq C(n). \tag{6.3.37}$$

Therefore a combination of Theorem 5.2.1, Lemma 6.3.10, the fact that covariances are invariant under the addition of constants, (6.3.36), and (6.3.37) yields the desired estimate

$$\left|\frac{d^2}{dx_i dx_j} \bar{H}_{\Lambda_2}\left(x^{\Lambda_2}\right)\right| \lesssim \frac{1}{K} + \exp\left(-CL|n-l|\right).$$

□

6.3.6 Proof of Proposition 6.2.6

The proof is a slight adaptation of the argument for the two-scale criterion for LSI (cf. Theorem 8.2.5, [GOVW09, Theorem 3]). For the convenience of the reader we give all details. Let $\Phi : \mathbb{R} \to \mathbb{R}$ denote the function $\Phi(x) = x \log x$. Observe that we can decompose the relative entropy into

$$\begin{aligned} &\int \Phi(f(x))\mu_{N,m}(dx) - \Phi\left(\int f(x)\mu_{N,m}(dx)\right) \\ &\quad = \int \left[\int \Phi\left(f(x)\right)\mu_{N,m}\left(dx^{\Lambda_1}|x^{\Lambda_2}\right) - \Phi\left(\int f(x)\mu_{N,m}\left(dx^{\Lambda_1}|x^{\Lambda_2}\right)\right)\right]\bar{\mu}_m\left(dx^{\Lambda_2}\right) \end{aligned} \tag{6.3.38}$$

$$+\int \Phi\left(\bar f\left(x^{\Lambda_2}\right)\right)\bar\mu_m\left(dx^{\Lambda_2}\right)-\Phi\left(\int \bar f\left(x^{\Lambda_2}\right)\bar\mu_m\left(dx^{\Lambda_2}\right)\right), \tag{6.3.39}$$

where $\bar f\left(x^{\Lambda_2}\right) := \int f(x)\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$. Let us begin with estimation of (6.3.38). As $\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)$ satisfies LSI(ρ_1), it follows that

$$\begin{aligned}
&\int\left[\int \Phi\left(f(x)\right)\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)-\Phi\left(\int f(x)\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)\right)\right]\bar\mu_m\left(dx^{\Lambda_2}\right)\\
&\quad\le \int\left[\frac{1}{2\rho_1}\int\frac{\left|\nabla_{\Lambda_1}f\right|^2}{f}\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)\right]\bar\mu_m\left(dx^{\Lambda_2}\right)\\
&\quad=\frac{1}{2\rho_1}\int\frac{\left|\nabla_{\Lambda_1}f\right|^2}{f}\mu_{N,m}\left(dx\right).
\end{aligned}$$

Let us turn to the estimation of (6.3.39). It holds by the assumption on $\bar\mu_m\left(dx^{\Lambda_2}\right)$ satisfying LSI(ρ_2) that

$$\int \Phi\left(\bar f\left(x^{\Lambda_2}\right)\right)\bar\mu_m\left(dx^{\Lambda_2}\right)-\Phi\left(\int \bar f\left(x^{\Lambda_2}\right)\bar\mu_m\left(dx^{\Lambda_2}\right)\right)\le\frac{1}{2\rho_2}\int\frac{\left|\nabla_{\Lambda_2}\bar f\right|^2}{\bar f}\bar\mu_m\left(dx^{\Lambda_2}\right). \tag{6.3.40}$$

Lemma 6.3.11 (Analogue of Lemma 21 in [GOVW09])**.** *It holds that*

$$\begin{aligned}
\nabla_{\Lambda_2}\bar f=&\left(\mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)}\left[\frac{\partial}{\partial x_i}f(x)-\frac{\partial}{\partial x_{i-R}}f(x)\right]\right)_{i\in\Lambda_2}\\
&-\operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)}\left(f(x),\left(\frac{\partial}{\partial x_i}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)-\frac{\partial}{\partial x_{i-R}}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right)_{i\in\Lambda_2}\right).
\end{aligned}$$

Proof of Lemma 6.3.11. Given $x^{\Lambda_2}\in\mathbb{R}^{\Lambda_2}$, recall the definition (6.3.30) and (6.3.31) of z^{Λ_1} and y^{Λ_1}, respectively. Similar calculations as in Lemma 6.3.10 implies

$$\begin{aligned}
\frac{\partial}{\partial x_i}\bar f\left(x^{\Lambda_2}\right)&=\frac{\partial}{\partial x_i}\frac{\int_{\frac{1}{N}\sum_{k\in\Lambda_1}y_k=m}f\left(y^{\Lambda_1}-z^{\Lambda_1},x^{\Lambda_2}\right)\exp\left(-H\left(y^{\Lambda_1}-z^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dy^{\Lambda_1})}{\int_{\frac{1}{N}\sum_{k\in\Lambda_1}y_k=m}\exp\left(-H\left(y^{\Lambda_1}-z^{\Lambda_1},x^{\Lambda_2}\right)\right)\mathcal{L}(dy^{\Lambda_1})}\\
&=\mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)}\left[\frac{\partial}{\partial x_i}f(x)-\frac{\partial}{\partial x_{i-R}}f(x)\right]\\
&\quad-\operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1}\middle|x^{\Lambda_2}\right)}\left(f(x),\frac{\partial}{\partial x_i}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)-\frac{\partial}{\partial x_{i-R}}H\left(x^{\Lambda_1},x^{\Lambda_2}\right)\right).
\end{aligned}$$

□

The next ingredient is the estimation of covariances.

Lemma 6.3.12 (Lemma 22 in [GOVW09]). *Let μ be a probability measure on an Euclidean space X. Suppose μ satisfies LSI(ρ) for some $\rho > 0$. Then for any two Lipschitz functions $f : X \to [0, +\infty)$ and $g : X \to \mathbb{R}$,*

$$|\mathrm{cov}_\mu (f, g)| \le \frac{\|\nabla g\|_{L^\infty(\mu)}}{\rho} \sqrt{\left(\int f d\mu\right)\left(\int \frac{|\nabla f|^2}{f} d\mu\right)}.$$

Now we estimate the integrand in the right hand side of (6.3.40) with the help of Lemma 6.3.11 and Lemma 6.3.12.

Lemma 6.3.13. *It holds that*

$$\frac{\left|\nabla_{\Lambda_2} \bar{f}\right|^2}{\bar{f}} \lesssim \frac{1}{\rho_1^2} \int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) + \int \frac{\left|\nabla_{\Lambda_2} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right).$$

Proof of Lemma 6.3.13. An application of Young's inequality yields

$$\begin{aligned}
\frac{\left|\nabla_{\Lambda_2} \bar{f}\right|^2}{\bar{f}} &\overset{Lemma\ 6.3.11}{=} \frac{1}{\bar{f}} \Bigg| \left(\mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\frac{\partial}{\partial x_i} f(x) - \frac{\partial}{\partial x_{i-R}} f(x)\right]\right)_{i \in \Lambda_2} \\
&\qquad - \mathrm{cov}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left(f(x), \left(\frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) - \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)\right)_{i \in \Lambda_2} \Bigg|^2 \\
&\le \frac{4}{\bar{f}} \left| \left(\mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\frac{\partial}{\partial x_i} f(x)\right]\right)_{i \in \Lambda_2} \right|^2 &(6.3.41) \\
&\quad + \frac{4}{\bar{f}} \left| \left(\mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\frac{\partial}{\partial x_{i-R}} f(x)\right]\right)_{i \in \Lambda_2} \right|^2 &(6.3.42) \\
&\quad + \frac{4}{\bar{f}} \left| \mathrm{cov}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left(f(x), \frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)_{i \in \Lambda_2} \right|^2 &(6.3.43) \\
&\quad + \frac{4}{\bar{f}} \left| \mathrm{cov}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left(f(x), \frac{\partial}{\partial x_{i-R}} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right)\right)_{i \in \Lambda_2} \right|^2. &(6.3.44)
\end{aligned}$$

Let us begin with estimation of (6.3.41). Cauchy's inequality implies that

$$\begin{aligned}
\left| \mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\nabla_{\Lambda_2} f(x)\right] \right|^2 &\le \int f(x) \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \int \frac{\left|\nabla_{\Lambda_2} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \\
&= \bar{f}\left(x^{\Lambda_2}\right) \int \frac{\left|\nabla_{\Lambda_2} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right),
\end{aligned}$$

and as a consequence

$$T_{(6.3.41)} \leq 4 \int \frac{\left|\nabla_{\Lambda_2} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right).$$

Let us turn to the estimation of (6.3.42). Note that $i - R \in \Lambda_1$ for each $i \in \Lambda_2$. Then a similar computation from above yields

$$T_{(6.3.42)} \leq \frac{4}{\bar{f}} \left| \mathbb{E}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left[\nabla_{\Lambda_1} f(x)\right] \right|^2 \leq 4 \int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right).$$

Let us turn to the estimation of (6.3.43).

$$\begin{aligned}
&\left| \operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left(f(x), \frac{\partial}{\partial x_i} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) \right)_{i \in \Lambda_2} \right|^2 \\
&\quad = \sup_{\substack{|y| \leq 1 \\ y \in \mathbb{R}^{\Lambda_2}}} \left[\operatorname{cov}_{\mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)} \left(f(x), \nabla_{\Lambda_2} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) \cdot y \right) \right]^2 \\
&\quad \overset{\textit{Proposition } 6.2.4, \textit{Lemma } 6.3.12}{\leq} \frac{1}{\rho_1^2} \left(\sup_{\substack{|y| \leq 1 \\ y \in \mathbb{R}^{\Lambda_2}}} \sup_{x^{\Lambda_1} \in \mathbb{R}^{\Lambda_1}} \left| \nabla_{\Lambda_1} \left(\nabla_{\Lambda_2} H\left(x^{\Lambda_1}, x^{\Lambda_2}\right) \cdot y \right) \right|^2 \right) \\
&\qquad \times \left(\int f(x) \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \right) \left(\int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \right) \\
&\quad \lesssim \frac{1}{\rho_1^2} \bar{f}\left(x^{\Lambda_2}\right) \left(\int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right) \right).
\end{aligned}$$

Therefore

$$T_{(6.3.43)} \lesssim \frac{1}{\rho_1^2} \int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right).$$

Similar computation also yields

$$T_{(6.3.44)} \lesssim \frac{1}{\rho_1^2} \int \frac{\left|\nabla_{\Lambda_1} f\right|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right).$$

□

Let us now give a proof of Proposition 6.2.6.

Proof of Proposition 6.2.6. A combination of (6.3.40) and Lemma 6.3.13 yields that

$$\begin{aligned}
T_{(6.3.39)} &\lesssim \frac{1}{2\rho_2} \max\left\{1, \frac{1}{\rho_1^2}\right\} \int \left(\int \frac{|\nabla f|^2}{f} \mu_{N,m}\left(dx^{\Lambda_1} \middle| x^{\Lambda_2}\right)\right) \bar{\mu}_m\left(dx^{\Lambda_2}\right) \\
&= \frac{1}{2\rho_2} \max\left\{1, \frac{1}{\rho_1^2}\right\} \int \frac{|\nabla f|^2}{f} \mu_{N,m}\left(dx\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int \Phi(f) d\mu - \Phi\left(\int f d\mu\right) &= T_{(6.3.38)} + T_{(6.3.39)} \\
&\lesssim \frac{1}{2\rho_1} \int \frac{|\nabla_{\Lambda_1} f|^2}{f} \mu_{N,m}\left(dx\right) + \frac{1}{2\rho_2} \max\left\{1, \frac{1}{\rho_1^2}\right\} \int \frac{|\nabla f|^2}{f} \mu_{N,m}\left(dx\right) \\
&\leq \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \max\left\{1, \frac{1}{\rho_1^2}\right\}\right) \int \frac{|\nabla f|^2}{f} \mu_{N,m}\left(dx\right).
\end{aligned}$$

This finishes the proof of Proposition 6.2.6. □

Chapter 7

Hydrodynamic limit of Kawasaki dynamics

We derive the hydrodynamic limit of the Kawasaki dynamics for the one-dimensional conservative system of unbounded real-valued spins with arbitrary strong, quadratic and finite-range interactions. This significantly extends prior results for bounded interaction by Rezakhanlou and complements results obtained by H.T. Yau. The result is obtained by adapting two scale approach of Grunewald, Otto, Villani and Westdickenberg combined with the results developed in the previous chapters.

7.1 Introduction

The broader scope of this chapter is the study of the continuum approximations of large discrete systems. Since the fundamental observation of Boltzmann that large particle systems in equilibrium are governed by Gibbs states, understanding the connection between discrete systems and their approximation to the continuum has been one of the main challenges in statistical physics. One of the most actively studied problem in this field is the hydrodynamic limit, which can be thought as a dynamical version of law of large numbers. This means that in a proper time and space macroscopic scales, the random evolution of microscopic system can be macroscopically described by a solution of deterministic partial differential equations.

Since 1980s, hydrodynamic limits were deduced in different settings. One interesting example is the Kawasaki dynamics, a stochastic dynamics which preserves the mean spin of the system (cf. [Fri87]). In an attempt to provide a more general strategy for deducing the hydrodynamic limits, several methods were presented by different authors. For example, in [GPV88a]. Guo, Papanicolaou and Varadhan presented the martingale method to prove the hydrodynamic limit of Kawasaki dynamics. In [Yau91], Yau introduced the entropy method based on Gronwall-type estimate for a relative entropy functional. More recently, Grunewald, Otto, Villani and Westdickenberg presented the two-scale method (cf. [GOVW09]). For more details, we refer to [GOVW09] or the classical reference on hydrodynamic limits [KL99].

Recalling that the hydrodynamic limit can be understood as a dynamical version of law of large numbers, it is obvious that the problem becomes relatively much easier if the underlying stationary distribution is a product measure; because this leads to independence of random variables. If the Hamiltonian H is non-interacting, we observe that the grand canonical ensemble is a product measure whereas the canonical ensemble is not due to the restriction to a hyperplane. This makes deducing hydrodynamic limit for the Kawasaki dynamics non-trivial even in the non-interactive Hamiltonian case. However, given the equivalence of ensembles (cf. Chapter 3 and Chapter 4) meaning that the canonical ensemble is equivalent to properly modified grand canonical ensemble, it is not surprising that one is able to deduce the hydrodynamic limit of Kawasaki dynamics in the case of a non-interactive Hamiltonian.

The problem becomes a lot more subtle if we consider interactions between spins within the Hamiltonian H. In this case, even the grand canonical ensemble is not a product measure, making the task of deducing hydrodynamic limit for the Kawasaki dynamics even more challenging. Another difficulty of studying Kawasaki dynamics is that the lack of understanding of properties of the canonical ensemble with interacting Hamiltonian. Therefore, even though the case of absent interactions was solved in 1980s (e.g. [Fri87], [GPV88a]), up to author' knowledge, there are only few known results for interacting lattice systems. In [Rez90], the hydrodynamic limit of Kawasaki dynamics was deduced in the case of translation-invariant, finite-range and bounded interactions. Also, in [Yau91], a super-quadratic single-site potential with quadratic, nearest neighbor interactions was considered. In this thesis we consider a perturbed quadratic single-site potential with strong finite-range quadratic interactions. Both arguments in [Rez90], [Yau91] cannot be applied to our situation because they heavily rely on the fact that the interaction is dominated by the single-site potentials. In particular, the strong dominance by single-site potentials plays a crucial role to prove a local ergodic theorem, which is a key ingredient to deduce a hydrodynamic limit in [Rez90], [Yau91]. Our result seems to be the first one for which the interaction is not dominated by the single-site potential but of the same order.

Let us also mention that in [Rez90], the translation invariance of the interaction was crucially used to deduce a large deviation for the empirical measure under the grand canonical ensemble. Our argument does not heavily rely on the assumption of translation-invariance. We only assume it for a convenience simplifying some calculations when analyzing the convergence of the free energies.

Another distinguishing characteristic between [Rez90], [Yau91] and this thesis is that we work with the canonical ensemble instead of the grand canonical ensemble. This is makes a crucial difference on assumptions of initial configuration of the Kawasaki dynamics. In [Rez90], the entropy bound of initial profile with respect to the grand canonical ensemble was assumed. In [Yau91], it is assumed that the initial state is close to the local Gibbs state. This excludes initial conditions with a fixed mean spin because a measure defined on a hyperplane is singular to the grand canonical ensemble and any local Gibbs state. In this manuscript, we assume the entropy bound of initial configuration with respect to the canonical ensemble. This allows initial configurations with fixed mean spin. Therefore, in contrast to the approaches of [Rez90] and [Yau91], it might even be possible to extend our method to allow deterministic initial conditions. The reason is that one expects that the relative entropy of the Kawasaki dynamics with respect to the canonical ensemble becomes of order N within finite time (see e.g. [Lu95], [Kos01]). This suggests that it is feasible to relax our assumption on the initial condition allowing deterministic state.

Our approach to the hydrodynamic limit is also different to [Rez90], [Yau91] in terms of the mode of convergence. In [Rez90], the dynamics is shown to converge in probability, and in [Yau91], the convergence of relative entropy with respect to local Gibbs state is established. Because we follow the framework of the two-scale approach, we show that the dyancmis converge wrt. Wasserstein distance associated to the H^{-1} norm, which is natural choice considering the underlying geometry of the Kawasaki dynamics.

In all mentioned methods, i.e., the martingale method (cf. [GPV88a]), the entropy method (cf. [Yau91]) and the two-scale approach (cf. [GOVW09]), a key ingredient to deduce the hydrodynamic limit is the control of the relative entropy. Here, the entropy method has a technical advantage over the two-scale approach. In the entropy method, one only needs to show that the normalized relative entropy stays bounded. The two-scale approach needs a tighter control on the relative entropy. It is controlled by a uniform logarithmic Sobolev inequality (LSI), which yields an exponential decay of the relative entropy.

Deducing the uniform LSI for the canonical ensemble even in the case of non-interacting Hamiltonian is a non-trivial problem. This is because there exist long-range interactions due to mean spin conservation. For instance, LSI for conservative systems with bounded discrete spin system was studied in [Yau96], where the LSI –scaling optimal in the systems size– was deduced under the Dobrushin-Shlosman mixing conditions. However, the problem becomes more subtle for the continuous unbounded spin systems due to the technical difficulties arising from lack of compactness. In [LPY02], the LSI for the conservative dynamics with non-interacting Hamiltonian was obtained. A major breakthrough in the case of interacting Hamiltonian was accomplished in [Men11]. There, the problem of deriving the uniform LSI was solved in the case of weakly interactive system.

While the preparatory study of the canonical ensemble in the case of weak interactions (cf. [Men11]) would have prepared the ground to derive the hydrodynamic limit in the weakly interacting system, the author chose to tackle the much harder problem of studying arbitrary strong interactions. The case of weak interactions does not face the problems we encounter in the study of arbitrary strong interactions, because one would expect that everything is close to the case of absent interaction. Indeed, the results were obtained by a perturbation argument, proving that weakly interactive system is close to a perturbed non-interactive system.

In previous chapters (Chapter 3, 4, 6,and 5), we provided a better understanding of the Kawasaki dynamics from the aspect of the canonical ensemble. The most important result is the uniform LSI for the canonical ensemble (Theorem 6.2.3). With this powerful control of relative entropy, it became possible to apply two-scale approach to deduce the hydrodynamic limit of the Kawasaki dynamics.

In this thesis, we follow the two-scale approach (cf. [GOVW09]) to deduce the hydrodynamic limit of Kawasaki dynamics. One reason is that it seems feasible to relax our assumptions on the initial state allowing deterministic configuration as we expect that the relative entropy with respect to the canonical ensemble becomes of order N in finite time (cf. [Lu95], [Kos01]). Another reason is, it is also possible to prove the quantitative hydrodynamic limit with two-scale approach. However, we do not prove the quantitative hydrodynamic limit of Kawasaki dynamics as this approach will result in sub-optimal scaling of convergence and unnecessarily complicate our argument. Nevertheless, the quantitative hydrodynamic limit would be an important ingredient when studying fluctuations not starting in equilibrium.

One possible way to improve the scaling of the the convergence would be to adapt two-scale approach with a more carefully chosen mesoscopic dynamics, as was done in [DMOW18a] for the non-interactive case. The main difference to [GOVW09] is that [DMOW18a] introduces a mesoscopic dynamics as the Galerkin approximation of the macroscopic dynamics, while [GOVW09] uses a projection onto piece-wise constant functions to define the mesoscopic scale. This approach using Galerkin approximation has an advantage of gaining regularity of the mesoscopic scale, resulting an optimal error estimate. It would be a challenging problem to extend this approach to the case of strongly interactive Hamiltonian.

Let us mention main challenges when applying the two-scale approach in the case of strong interactions. First of all, the convergence of the one-dimensional coarse-grained Hamiltonian should be handled. In case of non-interacting spin system, the local Cramèr theorem implies that the one-dimensional coarse-grained Hamiltonian converges to the Cramèr transform of a single-site potential. However, this is not true anymore under existence of strong interactions. Second, a uniform LSI should be extended from one block to multi blocks. That is, we consider the ensemble with conservation laws in each block and deduce the uniform LSI independent of block size, number of blocks, and the whole system size. Last, due to the strong finite-range interactions, the neighboring blocks are not independent anymore, resulting that the coarse-grained Hamiltonian is not a sum of one-dimensional coarse-grained Hamiltonians.

To overcome the first difficulty, we recall that the local Cramèr theorem implies the uniform convergence of one-dimensional coarse-grained Hamiltonian to a Legendre transform of the free energy of the grand canonical ensemble in the absence of interactions. Motivated by this, we first prove the convergence of the free energy of the grand canonical ensemble under the presence of strong interactions. In fact, we show that the sequence of (non-normalized) free energy is sub-additive up to moment bounds. The moments are then compared with Gaussian moments, resulting bounds uniform on system size and depend only on the mean spin m. Then we argue that the coarse-grained Hamiltonian converges to the Legendre transform of the limit of the free energy of the grand canonical ensemble. We refer to Section 7.2.2 for more details.

The second difficulty, when deducing the multi-block LSI, is handled by applying a combination of the two-scale approach (cf. [GOVW09]) and the Zegarlinski decomposition (cf. [Zeg96]). We decompose the lattice into two types of blocks Λ_1 and Λ_2 motivated by Zegarlinski's decomposition (cf. Figure 7.2). Then the measure is decomposed into a conditional distribution conditioned on Λ_2 and marginal distribution. By a careful choice of Λ_1 and Λ_2, the conditional distribution factorizes and thus the uniform LSI for the conditional distributions follows from a uniform LSI for the canonical ensemble (cf. Theorem 6.2.3) and the Tensorization Principle. For the marginal distribution, we apply Otto-Reznikoff Criterion (see [OR07]) where interactions between blocks are controlled via decay of correlations. Then a usual two-scale argument for LSI combines the LSIs for conditional and marginal distributions and the uniform LSI for the original measure is obtained. For more details, we refer to Section 7.4.

For the last difficulty, we artificially introduce an auxiliary Hamiltonian H_{aux} where we remove the interactions between neighboring blocks. Removing the interactions makes each block independent, and as a consequence, the corresponding coarse-grained Hamiltonian of M blocks is decomposed into a sum of M coarse-grained Hamiltonians of single blocks.

Because we assume finite range interactions, the number of interactions we remove is relatively small compared to the whole system size. Therefore, as expected, we prove that difference between the coarse-grained Hamiltonians arising from the formal Hamiltonian H and an auxiliary Hamiltonian H_{aux} goes to 0 as we increase the block size K. This is well explained in Section 7.6.

7.2 Main results and outline of the proof

In this chapter, we assume that $\Lambda = [N] := \{1, 2, \cdots, N\}$ and there are no external fields $s = (s_i), x^{\mathbb{R}\setminus\Lambda}$ in the definition (1.1.1) of Hamiltonian H. We also assume that interactions M_{ij} are spacial homogeneous. That is, there exists a function $h : \mathbb{Z} \to (-1, 1)$ such that

$$M_{ij} = h(|i - j|), \qquad \forall i, j \in \Lambda.$$

We follow the conventions made at the end of Section 1.1.1.

7.2.1 Main Results

A natural dynamics for the conservative system is the Kawasaki dynamics, which is defined as follows. Let A denote the second-order difference operator given by the $N \times N$ matrix

$$A_{ij} := N^2 \left(-\delta_{i,j-1} + 2\delta_{i,j} - \delta_{i,j+1}\right),$$

where we define $\delta_{i,0} = \delta_{i,N}$ and $\delta_{i,N+1} = \delta_{i,1}$. The Kawasaki dynamics is a stochastic process $X(t) \in \mathbb{R}^N$ satisfying the following stochastic differential equation:

$$dX(t) = -A\nabla H(X(t))dt + \sqrt{2A}dB(t),$$

where $B(t)$ denotes a standard Brownian motion on $\mathbb{R}^N$. The Kawasaki dynamics preserves its mean spins, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N} X_i(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(0) = m.$$

This implies that we can restrict the state space $\mathbb{R}^N$ to the hyperplane $X = X_{N,m}$ and consider the corresponding ce $\mu_{N,m}$ as an invariant measure. If the process X_t is distributed according to $f\mu_{N,m}$, then the time dependent probability density $f = f(t,x)$ satisfies

$$\frac{\partial}{\partial t}(f\mu_{N,m}) = \nabla \cdot (A\nabla f\mu_{N,m}) . \tag{7.2.1}$$

In order to define a continuous counterpart of the configuration space $X_{N,m}$, let us define the space $\bar{X} = \bar{X}_{N,m}$ of piecewise constant functions on $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$ with mean m by

$$\bar{X} := \left\{ \bar{x} : \mathbb{T}^1 \to \mathbb{R};\ \bar{x} \text{ is constant on } \left(\frac{j-1}{N}, \frac{j}{N}\right] \text{ for } j = 1, \cdots, N, \text{ and has mean } m \right\}.$$

We shall identify the space $X = X_{N,m}$ with $\bar{X}$ by the following relation:

- For each $x \in X$, the step function $\bar{x} \in \bar{X}$ associated to x is

$$\bar{x}(\theta) = x_j, \qquad \text{if } \theta \in \left(\frac{j-1}{N}, \frac{j}{N}\right].$$

- For each step function $\bar{x} \in \bar{X}$, the corresponding vector $x \in X$ is

$$x_j = \bar{x}\left(\frac{j}{N}\right), \qquad j = 1, \cdots, N.$$

We equip the space of locally integrable functions $f : \mathbb{T}^1 \to \mathbb{R}$ having a mean m with H^{-1} norm by

$$\|f\|^2_{H^{-1}} = \int_{\mathbb{T}^1} \omega^2(\theta) d\theta,$$

where ω is a function such that

$$\omega' = f, \qquad \int_{\mathbb{T}^1} \omega(\theta) d\theta = 0.$$

Now we are ready to formulate our main result, namely the hydrodynamic limit of the Kawasaki dynamics for the canonical ensemble (1.1.7) having strong interactions. We establish that the evolution along the Kawasaki dynamics gets close to the solution to a certain nonlinear parabolic equation as $N \to \infty$.

Theorem 7.2.1. *Let $f = f(t,x)$ be a solution of the Kawasaki dynamics (7.2.1) with initial condition $f(0,\cdot) = f_0(\cdot)$. Assume that there exists a constant $C > 0$ such that for any $N \geq 1$,*

$$\int f_0(x) \log f_0(x) \mu_{N,m}(dx) \leq CN. \tag{7.2.2}$$

Assume also that there is a $\zeta_0 \in L^2(\mathbb{T}^1)$ such that $\int \zeta_0 d\theta = m$ and

$$\lim_{N\to\infty} \int \|\bar{x} - \zeta_0\|_{H^{-1}}^2 f_0(x) \mu_{N,m}(dx) = 0.$$

Let $\zeta = \zeta(t,\theta)$ be the unique weak solution of the nonlinear parabolic equation

$$\begin{cases} \frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\zeta), \\ \zeta(0,\cdot) = \zeta_0, \end{cases} \tag{7.2.3}$$

where φ is defined as

$$\varphi(m) := \lim_{N\to\infty} -\frac{1}{N} \log \int_{\{\frac{1}{N}\sum_{i=1}^N x_i = m\}} \exp\left(-H(x)\right) \mathcal{L}^{N-1}(dx). \tag{7.2.4}$$

Then, for any $T > 0$,

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \int \|\bar{x} - \zeta(t,\cdot)\|_{H^{-1}}^2 f(t,x) \mu_{N,m}(dx) = 0.$$

Here, we say that $\zeta = \zeta(t,\theta)$ is a weak solution of (7.2.3) on $[0,T] \times \mathbb{T}^1$ if

$$\zeta \in L_t^\infty(L_\theta^2), \qquad \frac{\partial \zeta}{\partial t} \in L_t^2(H_\theta^{-1}), \qquad \varphi'(\zeta) \in L_t^2(L_\theta^2),$$

and

$$\left\langle \xi, \frac{\partial \zeta}{\partial t} \right\rangle_{H^{-1}} = -\int_{\mathbb{T}^1} \xi \varphi'(\zeta) d\theta, \qquad \forall \xi \in L^2, \text{ for almost every } t \in [0,T].$$

Remark 16. There are some issues in Theorem 7.2.1 to be resolved. First, one has to verify that the pointwise limit of (7.2.4) exists and is differentiable. This will be established in Section 7.2.2. In addition, the existence and uniqueness of a weak solution of (7.2.3) for convex φ follows from the standard argument in the nonlinear PDE theory (see for example [GOVW09, Lemma 38]).

Remark 17. Although we assumed the initial relative entropy bound (7.2.2) in Theorem 7.2.1, we expect that the condition (7.2.2) can be relaxed to cover more general class of initial configurations including deterministic states. In fact, it was proved in [Lu95] that for the Kawasaki dynamics with non-interacting Hamiltonians starting from deterministic configurations, the relative entropy with respect to the canonical ensemble instantaneously becomes of order N.

Remark 18. The quantity inside the limit of (7.2.4), denoted by $\bar{H}_N(m)$, represents the distribution $f_N(m)dm$ of the mean value $(x_1 + \cdots + x_N)/N$ under $\mu_{N,m}$:

$$f_N(m)dm = \frac{1}{Z_N} e^{-N\bar{H}_N(m)} dm.$$

In the case of ce without interactions, i.e. $M_{ij} = 0$, as a consequence of local Cramèr theorem (see [GOVW09, Proposition 31]), φ in (7.2.4) is a Legendre transform of the logarithmic generating function of the distribution $\frac{1}{Z} e^{-\psi(x)} dx$. On the other hand, in the presence of interactions, φ in (7.2.4) can also be expressed in terms of the Legendre transform of the thermodynamic free energy. This point will be discussed in Section 7.2.2.

7.2.2 Two-scale decomposition

In this section, we introduce a two-scale decomposition method, originally introduced in [GOVW09], which plays a crucial role to study the concentration properties of the ce and their hydrodynamic limit. Then, we state key results on the logarithmic Sobolev inequality and the strict convexity for the coarse-grained Hamiltonian, generalizing the previous results in Chapter 6 and Chapter 3, which are crucial to implement a two-scale approach to establish a hydrodynamic limit.

Let us divide N spins into M blocks with size K (see Figure 7.1), denoted by

$$B(l) := \{(l-1)K + 1, \cdots, lK\} \qquad \text{for each } l \in [M].$$

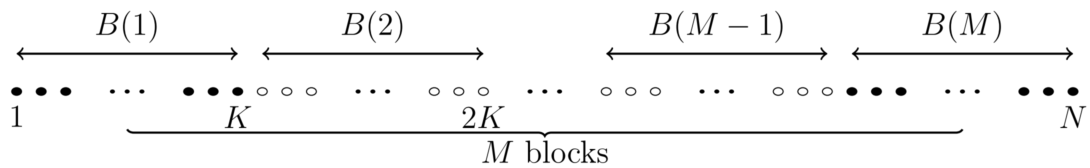

Figure 7.1: Block decomposition of lattice $[N]$.

We then define the mesoscopic space Y as

$$Y = Y_{M,m} := \left\{ (y_1, \cdots, y_M);\ \frac{1}{M}\sum_{l=1}^{M} y_l = m \right\}.$$

The L^2 inner product on Y is defined as follows:

$$\langle y, \tilde{y} \rangle_Y := \frac{1}{M}\sum_{l=1}^{M} y_l \tilde{y}_l.$$

The projection $P = P_{N,K} : X \to Y$ is defined via

$$P(x_1, \cdots, x_N) := (y_1, \cdots, y_M), \qquad y_l = \frac{1}{K}\sum_{i \in B(l)} x_i.$$

We observe that the adjoint operator $P^* : Y \to X$ given by

$$P^*(y_1, \cdots, y_M) = \frac{1}{N}(\underbrace{y_1, \cdots, y_1}_{K \text{ times}}, \cdots, \underbrace{y_M, \cdots, y_M}_{K \text{ times}})$$

satisfies the identity $PNP^* = \mathrm{Id}_Y$, where Id_Y is the identity operator on Y.

Remark 19. For notational simplicity, we assumed that all blocks $B(l)$ have equal size K so that $N = MK$. If N/K is not an integer, we decompose $[N]$ into M blocks with different sizes $K_1, \cdots, K_M$. More precisely, we define

$$Px = (y_1, \cdots, y_M),$$

where

$$y_l = \frac{1}{K_l}\sum_{i \in B(l)} x_i \qquad \text{for each } l \in [M].$$

The space Y is defined as

$$Y = \left\{ (y_1, \cdots, y_M); \ \frac{1}{M} \sum_{l=1}^{M} \alpha_l y_l = m, \ \text{where } \alpha_l = \frac{MK_l}{N} \right\}.$$

Here, we choose block sizes $\{K_l\}_{l=1}^{M}$ carefully so that $1 \le \alpha_l \le 2$ for all $l \in [M]$.

Finally, we disintegrate the ce $\mu_{N,m}$ into the conditional measure $\mu_{N,m}(dx|y) = \mu_{N,m}(dx|Px = y)$ and the marginal measure $\bar{\mu}_{N,m}(y)$ defined on Y. This means that for any test function ξ,

$$\int_X \xi d\mu_{N,m} = \int_Y \left(\int_{Px=y} \xi(x) \mu_{N,m}(dx|y) \right) \bar{\mu}_{N,m}(dy).$$

7.2.3 Outline of the proof of main results

In this section, we provide the proof of our main theorem, a hydrodynamic limit of Kawasaki dynamics (Theorem 7.2.1). As mentioned in the introduction, the main idea of proof is a two-scale approach (cf. [GOVW09, Theorem 8]). In [GOVW09], the hydrodynamic limit of the Kawasaki dynamics was deduced via two-scale approach where there is no-interactions within the Hamiltonian (see [GOVW09, Theorem 17]). The problem becomes a lot more subtle when we add strong finite-range interactions within the Hamiltonian. For example, because neighboring blocks interact with each other, the coarse-grained Hamiltonian $\bar{H}_Y$ (cf. (7.3.2)) cannot be decomposed into a sum of one-dimensional coarse-grained Hamiltonian of the form (7.3.3).

We overcome this difficulty by introducing an auxiliary Hamiltonian H_{aux} obtained by removing the interactions between different blocks in the Hamiltonian H (see (7.6.4) later). Due to the nature of finite range interactions of the Hamiltonian H, the amount of such removed interactions is negligible compared to the original Hamiltonian H. Therefore one can expect that H and H_{aux} are *close*, which will be quantified explicitly later. This allows us to take advantage of nice structure of H_{aux} such as block decomposition and tensorization property.

Recall the definition of the mesoscopic space Y in Section 7.2.2, and let us introduce a mesoscopic version of Kawasaki dynamics. First of all, define the coarse-grained operator $\bar{A} : Y \to Y$ by

$$(\bar{A})^{-1} = PA^{-1}NP^{*}.$$

For given $\eta_0 \in Y$, consider the mesoscopic analog of Kawasaki dynamics:

$$\begin{cases} \frac{d\eta}{dt} = -\bar{A}\nabla_Y \bar{H}_Y(\eta), \\ \eta(0) = \eta_0. \end{cases} \tag{7.2.5}$$

Recalling the identification of X and $\bar{X}$ (see Section 7.2), we identify Y with the space $\bar{Y}$ of piecewise constant functions on $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$ with mean m:

$$\bar{Y} := \left\{ \bar{y} : \mathbb{T}^1 \to \mathbb{R};\ \bar{y} \text{ is constant on } \left(\frac{l-1}{M}, \frac{l}{M} \right] \text{ for } l \in [M], \text{ and has mean } m \right\}.$$

The main idea of the two-scale approach is to prove the closeness of microscopic-mesoscopic solutions and mesoscopic-macroscopic solutions.

Consider a sequence $\{M_\nu, N_\nu\}_{\nu=1}^{\infty}$ such that

$$M_\nu \to \infty, \qquad N_\nu \to \infty, \qquad K_\nu = \frac{N_\nu}{M_\nu} \to \infty.$$

This means that the size of each block and the number of blocks are simultaneously increasing to the infinity.

Convention. Following the convention of [GOVW09], we write M, N, K for M_ν, N_ν, K_ν. We also denote $X = X_{N^\nu,m}$, $Y = Y_{M^\nu,m}$, and so on in the remaining sections.

For given ζ_0, choose a sequence of step functions $\{\bar{\eta}_0^\nu\}_{\nu=1}^{\infty}$ in $\bar{Y}$ that converges to ζ_0 in L^2:

$$\|\bar{\eta}_0^\nu - \zeta_0\|_{L^2} \to 0 \qquad \text{as } \nu \to \infty. \tag{7.2.6}$$

For each ν, let $\eta_0^\nu \in Y$ be the vector that corresponds to the step function $\bar{\eta}_0^\nu$, and denote η^ν by a solution of the mesoscopic parabolic equation (7.2.5) with the initial data $\eta(0) = \eta_0^\nu$.

The first main ingredient is the closeness of microscopic-mesoscopic solutions.

Proposition 7.2.2. *For any $T > 0$,*

$$\lim_{\nu\to\infty}\sup_{0\le t\le T}\int \|\bar{x}-\bar{\eta}^{\nu}(t,\cdot)\|_{H^{-1}} f(t,x)\mu_{N,m}(dx) = 0.$$

The second ingredient is the closeness of mesoscopic-macroscopic solutions.

Proposition 7.2.3. *The step functions $\bar{\eta}^{\nu}$ converge in $L^{\infty}(H^{-1})$ to the unique weak solution ζ of* (7.2.3)*. In particular, for any $T > 0$,*

$$\lim_{\nu\to\infty}\sup_{0\le t\le T}\|\bar{\eta}^{\nu}(t,\cdot)-\zeta(t,\cdot)\|_{H^{-1}}^2 = 0.$$

We provide the proof of Proposition 7.2.2 and 7.2.3 in Section 7.6 and 7.7, respectively. Assuming Proposition 7.2.2 and 7.2.3, one can conclude the proof of Theorem 7.2.1.

Proof of Theorem 7.2.1. Following the notations from above, Proposition 7.2.2 and 7.2.3 imply

$$\begin{aligned}
&\lim_{N\to\infty}\sup_{0\le t\le T}\int \|\bar{x}-\zeta(t,\cdot)\|_{H^{-1}}^2 f(t,x)\mu(dx)\\
&\quad\le 2\lim_{N\to\infty}\sup_{0\le t\le T}\left(\int \|\bar{x}-\bar{\eta}^{v}(t,\cdot)\|_{H^{-1}}^2 f(t,x)\mu(dx) + \int \|\bar{\eta}^{v}-\zeta(t,\cdot)\|_{H^{-1}}^2 f(t,x)\mu(dx)\right)\\
&\quad= 2\lim_{N\to\infty}\sup_{0\le t\le T}\left(\int \|\bar{x}-\bar{\eta}^{v}(t,\cdot)\|_{H^{-1}}^2 f(t,x)\mu(dx) + \|\bar{\eta}^{v}-\zeta(t,\cdot)\|_{H^{-1}}^2\right)\\
&\quad= 0.
\end{aligned}$$

□

7.3 Auxiliary Results

7.3.1 Logarithmic Sobolev inequality and coarse-grained Hamiltonian

The two-scale decomposition method has been successfully used to study the hydrodynamic limit of the Kawasaki dynamics of ce without interactions. Key ingredients to establish hydrodynamic limit are the uniform logarithmic Sobolev inequality for the conditional distributions and the strict convexity of the coarse-grained Hamiltonian. In this section, we state new results on the logarithmic Sobolev inequality and coarse-grained Hamiltonians in the context of ce with strong interactions.

There have been numerous works on studying LSI for the conservative spin systems. Important works include [LY93], where a martingale method was implemented, and [GOVW09], where a two-scale method was introduced. In Chapter 6, we proved that the ce $\mu_{N,m}$ with strong interactions satisfy the uniform LSI (Theorem 6.2.3. In particular, Theorem 6.2.3 says that for any Lipschitz density $f : X_{N,m} \to \mathbb{R}$,

$$\int_{X_{N,m}} f \log f d\mu_{N,m} - \int_{X_{N,m}} f d\mu_{N,m} \log \Big(\int_{X_{N,m}} f d\mu_{N,m} \Big) \leq \frac{1}{2\varrho} \int_{X_{N,m}} \frac{1}{f} \sum_{i=1}^{N} \Big(\frac{\partial f}{\partial x_i} \Big)^2 d\mu_{N,m},$$

where f is extended to be constant in a direction normal to the hypersurface $X_{N,m}$.

Recall that the measure $\mu_{N,m}$ conditions on the mean value m of the spins $x_1, \ldots, x_N$. We therefore call the LSI for the measure $\mu_{N,m}$ the *one-block LSI*.

Remark 20. Although Theorem 6.2.3 states the uniform LSI for the Dirichlet form associated with Glauber dynamics, one immediately obtains LSI for the Kawasaki dynamics. In fact, by discrete Poincarè inequality, for any f satisfying $\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} = 0$,

$$\sum_{i=1}^{N} \Big(\frac{\partial f}{\partial x_i} \Big)^2 \leq C N^2 \sum_{i=1}^{N} \Big(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \Big)^2. \tag{7.3.1}$$

Thus, by Theorem 6.2.3 and (7.3.1),

$$\begin{aligned} &\int_{X_{N,m}} f \log f d\mu_{N,m} - \int_{X_{N,m}} f d\mu_{N,m} \log \Big(\int_{X_{N,m}} f d\mu_{N,m} \Big) \\ &\qquad \leq C \frac{N^2}{2\varrho} \int_{X_{N,m}} \frac{1}{f} \sum_{i=1}^{N} \Big(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \Big)^2 d\mu_{N,m}. \end{aligned}$$

The uniform LSI implies a strong concentration property of the corresponding Gibbs measure and provides an exponential decay of entropy along the associated dynamics. More precisely, recalling the factor N^2 in the difference operator A, Theorem 6.2.3 and Remark 20 imply the exponential decay in the relative entropy along the Kawasaki dynamics:

$$\mathrm{Ent}(f_t \mu_{N,m} | \mu_{N,m}) \leq \exp(-Ct) \mathrm{Ent}(f_0 \mu_{N,m} | \mu_{N,m}),$$

where Ent denotes the relative entropy function. We refer to [Men11, Remark 3] for more details.

The first main result of this section is an improvement of Lemma 6.2.3 to multi-blocks. More precisely, let us recall that the measure $\mu_{N,m}(dx|y)$ conditions on the mean spins $y_1, \dots y_M$ on each block. We show that this measure also satisfies the uniform LSI, which is called *multi-block LSI.* This is one of the key ingredients to implement the two-scale approach to establish the hydrodynamic limit. It is also is highly non-trivial because, due to the interactions between blocks, the measure $\mu_{N,m}(dx|y)$ does not tensorize.

Theorem 7.3.1. *The conditional measure $\mu_{N,m}(dx|y)$ satisfies a uniform LSI(ϱ), where $\varrho > 0$ is independent of the system size N, the mean spin m, and the macroscopic state y.*

Theorem 7.3.1 will be proved in Section 7.4.

Now, we define and study some properties about the coarse-grained Hamiltonian. Recall the disintegration

$$\mu_{N,m}(dy) = \mu_{N,m}(dx|y) \bar{\mu}_{N,m}(dy).$$

The coarse-grained Hamiltonian $\bar{H}_Y(y)$ is defined to be a Hamiltonian corresponding to $\bar{\mu}_{N,m}(dy)$:

$$\bar{\mu}_{N,m}(dy) = \exp(-N\bar{H}_Y(y))dy.$$

In other words, one can define a coarse-grained Hamiltonian $\bar{H}_Y : Y \to \mathbb{R}$ as follows:

$$\bar{H}_Y(y) := -\frac{1}{N} \log \int_{Px=y} \exp(-H(x)) \mathcal{L}^{N-M}(dx). \tag{7.3.2}$$

In particular, in the simple case $Y = \mathbb{R}$ and $Px = (1/N)\sum_{i=1}^N x_i$, we define $\bar{H}_N : \mathbb{R} \to \mathbb{R}$, called one-dimensional coarse-grained Hamiltonian, by

$$\bar{H}_N(m) := -\frac{1}{N} \log \int_{\{\frac{1}{N}\sum_{i=1}^N x_i = m\}} \exp\left(-H(x)\right) \mathcal{L}^{N-1}(dx). \tag{7.3.3}$$

The strict convexity of coarse-grained Hamiltonian of the ce plays a crucial role in [GOVW09] to establish a uniform LSI and the hydrodynamic limit without interactions. In the one-block setting, this has been verified under strong interactions.

Lemma 7.3.2 (Corollary 3.2.5)**.** *The coarse-grained Hamiltonian $\bar{H}_N : X \to \mathbb{R}$ is uniformly strictly convex. In other words, there exists a constant $C > 0$ such that for any $N \geq 1$ and $m \in \mathbb{R}$,*

$$\frac{1}{C} \leq \bar{H}_N''(m) \leq C.$$

The second main result of this section is an extension of Lemma 7.3.2 to the multi-block case:

Theorem 7.3.3. *The coarse-grained Hamiltonian $\bar{H}_Y$ is uniformly strictly convex. In other words, there exists a constant $\lambda > 0$ independent of the system size N and the mean spin m, such that for any $y \in Y$,*

$$\lambda \, \mathrm{Id}_Y \leq \mathrm{Hess}_Y \, \bar{H}_Y(y) \leq \frac{1}{\lambda} \, \mathrm{Id}_Y \, .$$

Theorem 7.3.3 will be proved in Section 7.5.

Remark 21. One should compare Theorem 7.3.1 and 7.3.3 with Theorem 6.2.3 and 7.3.2, respectively. In Theorem 6.2.3 and 7.3.2, we considered the case where Gibbs measure has only one constraint

$$\frac{1}{N}\sum_{i=1}^N x_i = m.$$

Theorem 7.3.1 and 7.3.3 are generalization of Theorem 6.2.3 and 7.3.2 in the sense that the measure $\mu_{N,m}(dx|y)$ has multiple constraints, having one conservation law for each block:

$$\frac{1}{K}\sum_{i\in B(l)} x_i = y_l, \qquad l = 1, \cdots, M.$$

That is, if we let $M = 1$, the statements of Theorem 7.3.1 and 7.3.3 reduce to that of Theorem 6.2.3 and 7.3.2, respectively.

Finally, we verify that the pointwise limit of the one-dimensional coarse grained Hamiltonian $\bar{H}_N(m)$ in (7.3.3) exists as the system size goes to infinity. It turns out that this limiting function, denoted by φ, appears in the nonlinear parabolic equation (7.2.3). The following lemma provides a quantitative convergence of $\bar{H}_N(m)$ as $N \to \infty$.

Proposition 7.3.4. *There exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for each $m \in \mathbb{R}$,*

$$\bar{H}_N(m) \to \varphi(m) \qquad \text{as } N \to \infty. \tag{7.3.4}$$

Moreover, there exist a constant $C > 0$ such that for any $N \geq 1$ and $m \in \mathbb{R}$,

$$\left|\bar{H}_N(m) - \varphi(m)\right| \leq C\frac{m^2+1}{N}.$$

Remark 22. In the case of ce without interactions, as mentioned in Remark 18, φ is a Legendre transform of the logarithmic generating function of the distribution $\frac{1}{Z}e^{-\psi(x)}dx$, and moreover the convergence in (7.3.4) holds in the C^2 topology. This is a consequence of the local Cramèr theorem obtained in [GOVW09, Proposition 31]. Whereas, under the presence of strong interactions, we find a candidate φ for Theorem 7.2.1 as a limit of one-dimensional coarse-grained Hamiltonians. It turns out that φ can also be represented by a Legendre transform of the thermodynamic free energy of the corresponding gce.

In the rest of section, we prove Proposition 7.3.4. As a key ingredient, we first establish a sharp moment estimates with respect to the ce $\mu_{N,m}$. Then, equipped with moment estimates, we establish Proposition 7.3.4 by showing that φ is a Legendre transform of the thermodynamic free energy of gce.

7.3.2 Proof of Proposition 7.3.4

In this section, we prove the quantitative convergence result for the coarse-grained Hamiltonian. Let us first define a (non-normalized) free energy of the gce

$$a_N(\sigma) := \log \int_{\mathbb{R}^N} \exp\left(\sigma \sum_{i=1}^{N} x_i - H_N(x)\right) dx.$$

First of all, we prove that for each $\sigma \in \mathbb{R}$, $a_N(\sigma)$ and $a'_N(\sigma)$ are sub-additive up to constants.

Lemma 7.3.5. *There exists a constant $C > 0$ such that for any $N_1, N_2 \in \mathbb{N}$ and $\sigma \in \mathbb{R}$,*

$$|a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma)| \leq C(\sigma^2 + 1), \tag{7.3.5}$$

$$\left|a'_{N_1+N_2}(\sigma) - a'_{N_1}(\sigma) - a'_{N_2}(\sigma)\right| \leq C(|\sigma| + 1). \tag{7.3.6}$$

Proof of (7.3.5) *in Lemma 7.3.5.* We write

$$\begin{aligned} &a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma) \\ &\quad= \log \frac{\int_{\mathbb{R}^{N_1+N_2}} \exp\left(\sigma \sum_{k=1}^{N_1+N_2} w_k - H_{N_1+N_2}(w)\right) dw}{\int_{\mathbb{R}^{N_1}} \exp\left(\sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u)\right) du \cdot \int_{\mathbb{R}^{N_2}} \exp\left(\sigma \sum_{j=1}^{N_2} v_i - H_{N_2}(v)\right) dv} \\ &\quad= \log \frac{\int_{\mathbb{R}^{N_1+N_2}} \exp\left(\sigma \sum_{k=1}^{N_1+N_2} w_k - H_{N_1+N_2}(w)\right) dw}{\int_{\mathbb{R}^{N_1+N_2}} \exp\left(\sigma \sum_{i=1}^{N_1} u_i + \sigma \sum_{j=1}^{N_2} v_j - H_{N_1}(u) - H_{N_2}(v)\right) dudv}. \end{aligned}$$

Let us define

$$I_{N_1,N_2} := \{(i,j) \mid i \in \{1, \cdots, N_1\}, j \in \{N_1 + 1, \cdots, N_1 + N_2\}, |i - j| \leq R\}. \tag{7.3.7}$$

Writing $w = (u, v) \in \mathbb{R}^N \times \mathbb{R}^M$, we have

$$H_{N_1+N_2}(w) - H_{N_1}(u) - H_{N_2}(v) = \sum_{(i,j)\in I_{N_1,N_2}} M_{ij} w_i w_j. \tag{7.3.8}$$

Thus we can write

$$a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma) = \log \left(\mathbb{E}_{\mu^\sigma_{N_1} \otimes \mu^\sigma_{N_2}} \left[\exp\left(- \sum_{(i,j)\in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right] \right). \tag{7.3.9}$$

We shall only prove that (7.3.9) is bounded from above as the proof of lower bound is almost identical to that of upper bound.

To begin with, by Young's inequality,

$$\begin{aligned} T_{(7.3.9)} &\leq \log\left(\mathbb{E}_{\mu^\sigma_{N_1}\otimes\mu^\sigma_{N_2}}\left[\exp\left(\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|\left(w_i^2+w_j^2\right)\right)\right]\right) \\ &= \log\left(\mathbb{E}_{\mu^\sigma_{N_1}}\left[\exp\left(\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2\right)\right]\right) \qquad (7.3.10) \\ &\quad+\log\left(\mathbb{E}_{\mu^\sigma_{N_2}}\left[\exp\left(\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|v_j^2\right)\right]\right). \qquad (7.3.11) \end{aligned}$$

We then apply a method of interpolation to obtain

$$\begin{aligned} T_{(7.3.10)} &= \log\left(\int_{\mathbb{R}^{N_1}}\exp\left(\sigma\sum_{i=1}^{N_1}u_i - H_{N_1}(u)+\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2\right)du\right) \\ &\quad-\log\left(\int_{\mathbb{R}_1^N}\exp\left(\sigma\sum_{i=1}^{N_1}u_i - H_{N_1}(u)\right)du\right) \\ &= \int_0^1\frac{d}{ds}\log\left(\int_{\mathbb{R}_1^N}\exp\left(\sigma\sum_{i=1}^{N_1}u_i - H_{N_1}(u)+s\cdot\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2\right)du\right)ds \\ &= \int_0^1\mathbb{E}_{\mu^\sigma_{N_1}(s)}\left[\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2\right]ds, \qquad (7.3.12) \end{aligned}$$

where $\mu^\sigma_{N_1}(s)$ is the probability distribution given by

$$\mu^\sigma_{N_1}(s)(du) := \frac{1}{Z}\exp\left(\sigma\sum_{i=1}^{N_1}u_i - H_{N_1}(u)+s\cdot\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2\right)du.$$

We observe that

$$\begin{aligned} &H_{N_1}(u) - s\cdot\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|u_i^2 \\ &\quad=\sum_{i=1}^{N_1-R}\left(\psi(u_i)+\frac{1}{2}\sum_{j:\ 1\leq|j-i|\leq R}M_{ij}u_iu_j\right) \end{aligned}$$

$$+\sum_{i=N_1-R+1}^{N_1}\left(\psi(u_i)-s\cdot\frac{1}{2}\sum_{\substack{j\in\{N_1+1,\cdots,N_1+N_2\}\\1\le|j-i|\le R}}|M_{ij}|u_i^2+\frac{1}{2}\sum_{\substack{j\in\{1,\cdots,N_1\}\\1\le|j-i|\le R}}M_{ij}u_iu_j\right)$$

Due to the strictly diagonal dominant assumption (1.1.3),

$$\frac{1}{2}-s\cdot\frac{1}{2}\sum_{\substack{j\in\{N_1+1,\cdots,N_1+N_2\}\\1\le|j-i|\le R}}|M_{ij}|\ge\frac{1}{2}\sum_{\substack{j\in\{1,\cdots,N_1\}\\1\le|j-i|\le R}}M_{ij}+\frac{1}{2}\delta.$$

This means that the interaction terms of $\mu_N^\sigma(s)$ also satisfy the strictly diagonal dominant assumption (1.1.3) and hence $\mu_N^\sigma(s)$ is a gce. Therefore an application of Corollary 2.2.2 implies

$$\begin{aligned}\left|T_{(7.3.10)}\right| &\overset{(7.3.12)}{\le}\int_0^1\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|\sup_{s\in[0,1]}\left(\mathbb{E}_{\mu_{N_1}^\sigma(s)}\left[u_i^2\right]\right)ds\\&\overset{Corollary\ 2.2.2}{\lesssim}\frac{1}{2}\sum_{(i,j)\in I_{N_1,N_2}}|M_{ij}|(\sigma^2+1)\sim\sigma^2+1.\end{aligned}$$

Similarly, one gets $|T_{(7.3.11)}|\lesssim\sigma^2+1$ and thus

$$T_{(7.3.9)}\lesssim\sigma^2+1.$$

Proof of (7.3.6) *in Lemma 7.3.5.* Following the notations used in the proof of (7.3.5) in Lemma 7.3.5, recalling $w=(u,v)\in\mathbb{R}^N\times\mathbb{R}^M$, we have

$$\begin{aligned}&a'_{N_1+N_2}(\sigma)-a'_{N_1}(\sigma)-a'_{N_2}(\sigma)\\&\quad=\mathbb{E}_{\mu_{N_1+N_2}^\sigma}\left[\sum_{k=1}^{N_1+N_2}w_k\right]-\mathbb{E}_{\mu_{N_1}^\sigma}\left[\sum_{i=1}^{N_1}u_i\right]-\mathbb{E}_{\mu_{N_2}^\sigma}\left[\sum_{j=1}^{N_2}v_j\right]\\&\quad=\mathbb{E}_{\mu_{N_1+N_2}^\sigma}\left[\sum_{k=1}^{N_1+N_2}w_k\right]-\mathbb{E}_{\mu_{N_1}^\sigma\otimes\mu_{N_2}^\sigma}\left[\sum_{k=1}^{N_1+N_2}w_k\right].\end{aligned}\tag{7.3.13}$$

Let us recall the definition (7.3.7) of I_{N_1,N_2}. For $s\in[0,1]$, define

$$\nu_{N_1+N_2}^\sigma(s)(dw):=\frac{1}{Z}\exp\left(\sigma\sum_{k=1}^{N_1+N_2}w_k-H_{N_1+N_2}(w)+s\sum_{(i,j)\in I_{N_1,N_2}}M_{ij}w_iw_j\right).$$

In particular, recalling (7.3.8), it holds that

$$\nu^\sigma_{N_1+N_2}(0) = \mu^\sigma_{N_1+N_2}, \qquad \nu^\sigma_{N_1+N_2}(1) = \mu^\sigma_{N_1} \otimes \mu^\sigma_{N_2}.$$

Therefore, by the method of interpolation,

$$\begin{aligned} \left|T_{(7.3.13)}\right| &= \left| - \int_0^1 \frac{d}{ds} \mathbb{E}_{\nu^\sigma_{N_1+N_2}(s)} \left[\sum_{k=1}^{N_1+N_2} w_k \right] dx \right| \\ &\leq \int_0^1 \left| \operatorname{cov}_{\nu^\sigma_{N_1+N_2}(s)} \left(\sum_{k=1}^{N_1+N_2} w_k, \sum_{(i,j)\in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| ds \\ &\leq \int_0^1 \sum_{k=1}^{N_1+N_2} \left| \operatorname{cov}_{\nu^\sigma_{N_1+N_2}(s)} \left(w_k, \sum_{(i,j)\in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| ds. \end{aligned} \tag{7.3.14}$$

Since $\nu^\sigma_{N_1+N_2}(s)$ is a gce and $|I_{N_1,N_2}| \leq 2R^2$, by the decay of correlation for gce (Proposition 1.2.6) and the second moment estimate (Corollary 2.2.2),

$$\begin{aligned} \left| \operatorname{cov}_{\nu^\sigma_{N_1+N_2}(s)} \left(w_k, \sum_{(i,j)\in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| &\leq C_s \left(2R^2(\sigma^2+1)\right)^{\frac{1}{2}} \exp\left(-C_s \operatorname{dist}(k, I_{N_1,N_2})\right) \\ &\leq C(|\sigma|+1) \exp\left(-C \operatorname{dist}(k, I_{N_1,N_2})\right) \end{aligned} \tag{7.3.15}$$

for some $C > 0$. Plugging (7.3.15) into (7.3.14) yields

$$\left|T_{(7.3.13)}\right| \leq C(|\sigma|+1) \sum_{k=1}^{N_1+N_2} \exp\left(-C \operatorname{dist}(k, I_{N_1,N_2})\right) = C\left(|\sigma|+1\right).$$

□

As an application of Fekete's Lemma to Lemma 7.3.5, we have the following corollary.

Corollary 7.3.6. *The normalized free energy*

$$A_N(\sigma) := \frac{1}{N} a_N(\sigma) = \frac{1}{N} \log \int_{\mathbb{R}^N} \exp\left(\sigma \sum_{i=1}^N x_i - H_N(x) \right) dx$$

and its derivative $A_N'(\sigma)$ *converge pointwisely to some functions* $A(\sigma), B(\sigma) : \mathbb{R} \to \mathbb{R}$ *as* $N \to \infty$*, respectively.*

Next, we provide quantitative bounds of the convergences of A_N and A'_N as $N \to \infty$.

Lemma 7.3.7. *There exists a constant $C > 0$ such that*

$$|A_N(\sigma) - A(\sigma)| \le C\frac{\sigma^2+1}{N}, \qquad \forall \sigma \in \mathbb{R}, \tag{7.3.16}$$

$$|A'_N(\sigma) - B(\sigma)| \le C\frac{|\sigma|+1}{N}, \qquad \forall \sigma \in \mathbb{R}. \tag{7.3.17}$$

Proof of Lemma 7.3.7. We shall only provide the proof of (7.3.16) as there is only a cosmetic difference between the proof of (7.3.16) and that of (7.3.17).

Let us fix $N \in \mathbb{N}$. We fisrt claim that for each $k \in \mathbb{N}$,

$$|A_{kN}(\sigma) - A_N(\sigma)| \le C\frac{k-1}{k} \cdot \frac{\sigma^2+1}{N}, \tag{7.3.18}$$

where $C > 0$ is a constant from Lemma 7.3.5.

First of all, (7.3.18) is obviously true for $k = 1$. The case $k = 2$ also holds by putting $M = N$ in (7.3.5) and dividing it by $2N$. Let us assume that (7.3.18) holds for some $k = p \in \mathbb{N}$. That is,

$$|A_{pN}(\sigma) - A_N(\sigma)| \le C\frac{p-1}{p} \cdot \frac{\sigma^2+1}{N}. \tag{7.3.19}$$

Then

$$\begin{aligned}
\left|A_{(p+1)N}(\sigma) - A_N(\sigma)\right| &= \left|\frac{a_{(p+1)N}(\sigma)}{(p+1)N} - \frac{a_N(\sigma)}{N}\right| \\
&\le \left|\frac{a_{(p+1)N}(\sigma) - a_{pN}(\sigma) - a_N(\sigma)}{(p+1)N}\right| + \left|\frac{a_{pN}(\sigma) - pa_N(\sigma)}{(p+1)N}\right| \\
&\overset{(7.3.5)}{\le} C\frac{1}{p+1} \cdot \frac{\sigma^2+1}{N} + \frac{p}{p+1}\,|A_{pN}(\sigma) - A_N(\sigma)| \\
&\overset{(7.3.19)}{\le} C\frac{1}{p+1} \cdot \frac{\sigma^2+1}{N} + C\frac{p-1}{p+1} \cdot \frac{\sigma^2+1}{N} = C\frac{p}{p+1} \cdot \frac{\sigma^2+1}{N}.
\end{aligned}$$

Therefore (7.3.18) holds for $k = p + 1$ as well and thus it holds for all $k \in \mathbb{N}$.

We now take $k \to \infty$ in (7.3.18) to conclude that

$$|A(\sigma) - A_N(\sigma)| \leq C \frac{\sigma^2 + 1}{N}.$$

□

Lemma 7.3.7 implies that A_N and A'_N uniformly converge to A and B on each bounded interval $[a, b]$, respectively. Since B is continuous, we have the following statement:

Corollary 7.3.8. *A is a C^1 function, and $A' = B$. In other words,*

$$A'(\sigma) = \lim_{N \to \infty} A'_N(\sigma), \qquad \forall \sigma \in \mathbb{R}.$$

Let $\mathcal{H}_N$ and φ be the Legendre transforms of A_N and A, respectively:

$$\begin{aligned} \mathcal{H}_N(m) &:= \sup_{\sigma \in \mathbb{R}} \left(\sigma m - A_N(\sigma) \right), \\ \varphi(m) &:= \sup_{\sigma \in \mathbb{R}} \left(\sigma m - A(\sigma) \right). \end{aligned} \tag{7.3.20}$$

We recall that

$$A'_N(\sigma) = \frac{1}{N} \mathbb{E}_{\mu_N^\sigma} \left[\sum_{i=1}^N X_i \right] \qquad \text{and} \qquad A''_N(\sigma) = \frac{1}{N} \operatorname{var}_{\mu_N^\sigma} \left(\sum_{i=1}^N X_i \right).$$

Then Lemma 2.2.1 and Lemma 2.1.4 imply that there exists a constant $C > 0$ such that

$$-C \leq A'_N(0) \leq C \qquad \text{and} \qquad \frac{1}{C} \leq A''_N(\sigma) \leq C. \tag{7.3.21}$$

Since the bounds (7.3.21) are uniform in N, it also holds that

$$-C \leq A'(0) \leq C \tag{7.3.22}$$

and A is strictly convex in the sense that

$$\frac{1}{C}|x - y| \leq |A'(x) - A'(y)| \leq C|x - y|. \tag{7.3.23}$$

The strict convexity of A_N implies that for each N, there exists a unique real number $\sigma_N \in \mathbb{R}$ such that

$$\mathcal{H}_N(m) = \sup_{\sigma \in \mathbb{R}} (\sigma m - A_N(\sigma)) = \sigma_N m - A_N(\sigma_N). \tag{7.3.24}$$

We also denote σ_∞ by a unique real number satisfying

$$\varphi(m) = \sup_{\sigma \in \mathbb{R}} (\sigma m - A(\sigma)) = \sigma_\infty m - A(\sigma_\infty). \tag{7.3.25}$$

Next, we prove that $\mathcal{H}_N$ converges pointwisely to φ as $N \to \infty$.

Lemma 7.3.9. *There exists a constant $C > 0$ such that*

$$|\mathcal{H}_N(m) - \varphi(m)| \le C \frac{m^2 + 1}{N}, \qquad \forall m \in \mathbb{R}.$$

Proof of Lemma 7.3.9. By definition (7.3.24) and (7.3.25) of σ_N and σ_∞, it holds that

$$A'_N(\sigma_N) = A'(\sigma_\infty) = m. \tag{7.3.26}$$

By (7.3.21), (7.3.22), (7.3.23) and (7.3.26), there exist constants $\gamma_1, \gamma_2 > 0$ such that for any $N \ge 1$,

$$|\sigma_N|, |\sigma_\infty| \le \gamma_1 |m| + \gamma_2. \tag{7.3.27}$$

Note that by (7.3.24) and (7.3.25),

$$|\mathcal{H}_N(m) - \varphi(m)| \le |\sigma_\infty - \sigma_N| \, |m| + |A_N(\sigma_N) - A(\sigma_\infty)| \, . \tag{7.3.28}$$

Let us begin with the estimation of the first term in the right hand side of (7.3.28). By (7.3.23),

$$\frac{1}{C} |\sigma_\infty - \sigma_N| \le |A'(\sigma_\infty) - A'(\sigma_N)| \le C \, |\sigma_\infty - \sigma_N| \, .$$

Therefore, we have

$$|\sigma_\infty - \sigma_N| \le C \, |A'(\sigma_\infty) - A'(\sigma_N)| \overset{(7.3.26)}{=} C \, |A'_N(\sigma_N) - A'(\sigma_N)|$$

$$\stackrel{(7.3.17)}{\leq} C\frac{|\sigma_N|+1}{N} \stackrel{(7.3.27)}{\leq} C\frac{|m|+1}{N}. \tag{7.3.29}$$

Let us turn to the estimation of the second term in the right hand side of (7.3.28). It holds that

$$\begin{aligned} |A_N(\sigma_N) - A(\sigma_\infty)| &\leq |A_N(\sigma_N) - A(\sigma_N)| + |A(\sigma_N) - A(\sigma_\infty)| \\ &\stackrel{(7.3.16)}{\leq} C\frac{\sigma_N^2+1}{N} + |A(\sigma_N) - A(\sigma_\infty)| \\ &= C\frac{\sigma_N^2+1}{N} + |A'(\sigma_N^*)|\,|\sigma_N - \sigma_\infty|\,, \end{aligned} \tag{7.3.30}$$

where σ_N^* is some number between σ_N and σ_∞. Hence, applying (7.3.22), (7.3.23), (7.3.27) and (7.3.29) to (7.3.30),

$$|A_N(\sigma_N) - A(\sigma_\infty)| \leq C\frac{m^2+1}{N}. \tag{7.3.31}$$

Plugging the estimates (7.3.29) and (7.3.31) into (7.3.28) gives the desired estimate

$$|\mathcal{H}_N(m) - \varphi(m)| \leq C\frac{m^2+1}{N}.$$

□

We can now conclude the proof of Proposition 7.3.4.

Proof of Proposition 7.3.4. Let φ be the function defined by (7.3.20). A combination of Lemma 7.3.9 and Theorem 3.2.4 implies that

$$\begin{aligned} \left|\bar{H}_N(m) - \varphi(m)\right| &\leq \left|\bar{H}_N(m) - \mathcal{H}_N(m)\right| + |\mathcal{H}_N(m) - \varphi(m)| \\ &\leq C\frac{1}{N} + C\frac{m^2+1}{N} = C\frac{m^2+1}{N}. \end{aligned}$$

The differentiability of φ is obvious. In fact, the Legendre transform of a C^1 strictly convex function with superlinear growth is also differentiable.

□

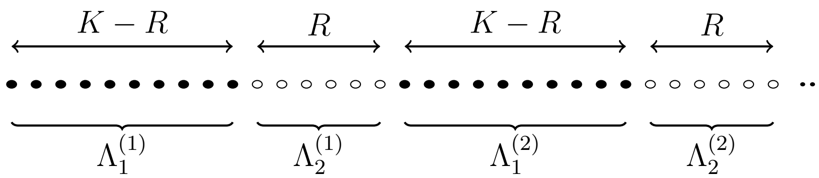

Figure 7.2: Arrangement in the cell $[1, 2K]$ for $K = 16$ and $R = 6$

7.4 Proof of Theorem 7.3.1

The proof of Theorem 7.3.1 is motivated by Zegarlinski's decomposition which was used to prove the uniform LSI for the gce μ_N (cf. [Zeg96]). In Chapter 6, we used this idea combined with the two-scale approach (cf. [GOVW09]) to prove that the ce $\mu_{N,m}$ satisfies a uniform LSI on the one-dimensional lattice. In this proof, we adapt this idea using Zegarlinski's decomposition and two-scale approach to deduce the uniform LSI for the measure $\mu_{N,m}(dx|y)$.

Let us begin with decomposing the lattice with two types of blocks (cf. Figure 7.2):

$$\begin{aligned}
\Lambda &:= [N] = \{1, 2, \cdots, N\}, \\
\Lambda_1 &:= \bigcup_{l=1}^{M} \Lambda \cap ([1, K-R] + (l-1)K) = \bigcup_{l=1}^{M} \Lambda_1^{(l)}, \\
\Lambda_2 &:= \bigcup_{l=1}^{M} \Lambda \cap ([K-R+1, K] + (l-1)K) = \bigcup_{l=1}^{M} \Lambda_2^{(l)},
\end{aligned}$$

where R is the interaction range of the particles. We disintegrate the measure $\mu_{N,m}(dx|y)$ as follows:

$$\mu_{N,m}(dx|y) = \mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)\bar{\mu}_{N,m}(dx^{\Lambda_2}|y).$$

In other words, for any test function ξ,

$$\int \xi(x)\mu_{N,m}(dx|y) = \int \left(\int \xi(x^{\Lambda_1}, x^{\Lambda_2})\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y) \right) \bar{\mu}_{N,m}(dx^{\Lambda_2}|y).$$

We prove the uniform LSI for the conditional measure $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)$ and the marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ separately. Then uniform LSI for the full measure $\mu_{N,m}$ is deduced via the two-scale criterion for the LSI (cf. [GOVW09, Theorem 3] or Chapter 6). More precisely, we have

Lemma 7.4.1. *The conditional measure $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)$ satisfies LSI(ϱ_1), where $\varrho_1 > 0$ is a constant independent of the system size $|\Lambda_1|$, the mean spin m, conditioned spins x^{Λ_2}, and the macroscopic state y.*

Lemma 7.4.2. *The marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ satisfies LSI(ρ_2), where $\rho_2 > 0$ is a constant independent of the mean spin m and the macroscopic state y.*

Lemma 7.4.3. *Assume that*

- *The conditional measure $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)$ satisfies LSI(ϱ_1), where $\varrho_1 > 0$ is a constant independent of $|\Lambda_1|$, m, x^{Λ_2} and y.*
- *The marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ satisfies LSI(ρ_2), where $\rho_2 > 0$ is a constant independent of m and y.*

Then the ce $\mu_{N,m}$ satisfies LSI(ρ), where $\rho > 0$ is a constant independent of the system size N and the mean spin m.

Let us briefly summarize the main ideas to prove those lemmas:

- Lemma 7.4.1 is a consequence of Theorem 6.2.3 and tensorization principle (cf. Theorem 8.2.1). Indeed, by conditioning on the spins x^{Λ_2}, the blocks $\Lambda_1^{(l)}$ do not interact within the Hamiltonian. Therefore the conditional measure $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)$ tensorizes on $\bigotimes_{l=1}^{M} X_{K-R,\tilde{y}_l}$, where

$$\tilde{y}_l := \frac{K y_l - \sum_{j \in \Lambda_2^{(l)}} x_j}{K - R}. \qquad (7.4.1)$$

  Because each tensorized measure on $X_{K-R,\tilde{y}_l}$ has the same structure as one dimensional ce $\mu_{N,m}$, it satisfies a uniform LSI by Theorem 6.2.3. Then an application of Theorem 8.2.1 yields Lemma 7.4.1; and we omit the details of the argument.

- The proof of Lemma 7.4.2 utilizes the Otto-Reznikoff Criterion (cf. Theorem 8.2.4). The details are provided in Section 7.4.1.

- The proof of Lemma 7.4.3 is almost identical to that of Proposition 6.2.6. We refer to Chapter 6 for more details.

With the help of the lemmas above, we establish Theorem 7.3.1:

Proof of Theorem 7.3.1. A combination of Lemma 7.4.1, 7.4.2 and 7.4.3 immediately yields the proof. □

7.4.1 Proof of Lemma 7.4.2

The main idea of the proof of Lemma 7.4.2 is to apply the Otto-Reznikoff Criterion (Theorem 8.2.4).

For each $l \in [M]$, we define (with a slight abuse of notation) a conditional version of Hamiltonian

$$H_l(x^{B(l)}|\bar{x}^{B(l)}) := \sum_{i \in B(l)} \psi(x_i) + \frac{1}{2} \sum_{i,j \in B(l)} M_{ij} x_i x_j + \sum_{\substack{i \in B(l) \\ j \notin B(l)}} M_{ij} x_i x_j \tag{7.4.2}$$

and

$$\begin{aligned} H_{\bar{l}}(\bar{x}^{B(l)}) &:= H(x) - H_l(x^{B(l)}|\bar{x}^{B(l)}) \\ &= \sum_{i \notin B(l)} \psi(x_i) + \frac{1}{2} \sum_{i,j \notin B(l)} M_{ij} x_i x_j. \end{aligned}$$

Note that the Hamiltonian Q associated to the marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ is

$$Q(x^{\Lambda_2}|y) = -\log \int_{\substack{\frac{1}{K-R} \sum_{i \in \Lambda_1^{(l)}} x_i = \tilde{y}_l \\ \forall l \in [M]}} \exp\left(-H(x)\right) \mathcal{L}(dx^{\Lambda_1}),$$

where $\tilde{y}_l$ is given by (7.4.1). In particular, a rearrangement of the integral gives

$$Q(x^{\Lambda_2}|y) = -\log \int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i = \tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right) \exp\left(-Q_l(x^{\Lambda_2^{(l)}}|\bar{x}^{B(l)})\right) \mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}}),$$

where Q_l is the block Hamiltonian defined by

$$Q_l(x^{\Lambda_2^{(l)}}|\bar{x}^{B(l)}) := -\log \int_{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(l)}} x_i = \tilde{y}_l} \exp\left(-H(x^{B(l)}|\bar{x}^{B(l)})\right) \mathcal{L}(dx^{\Lambda_1^{(l)}}).$$

In Chapter 6, we deduced that the block Hamiltonian Q_l can be decomposed into a sum of strictly convex function $\tilde{\Psi}_l^c$ and bounded perturbation $\tilde{\Psi}_l^b$ (cf. (6.3.4) and (6.3.5)).

Lemma 7.4.4. *There exist functions $\tilde{\Psi}_l^c$ and $\tilde{\Psi}_l^b$ such that*

- $Q_l = \tilde{\Psi}_l^c + \tilde{\Psi}_l^b$.
- *For block size K large enough, $\tilde{\Psi}_l^c$ is strictly convex.*
- *$\tilde{\Psi}_l^b$ is uniformly bounded.*

Moreover, the strict convexity of $\tilde{\Psi}_l^c$ and boundedness of $\tilde{\Psi}_l^b$ is independent of the conditioned spins $\bar{x}^{B(l)}$.

The crucial step towards the proof of Lemma 7.4.2 is to prove that each block marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ satisfies a uniform LSI.

Lemma 7.4.5. *The block marginal measure $\bar{\mu}_{N,m}(dx^{\Lambda_2}|y)$ satisfies a uniform LSI.*

Proof of Lemma 7.4.5. Let us fix $l \in [M]$ and decompose a block Hamiltonian $Q_l(x^{\Lambda_2^{(l)}}|\bar{x}^{B(l)})$ into the strictly convex part $\tilde{\Psi}_l^c$ and bounded perturbation part $\tilde{\Psi}_l^b$ using Lemma 7.4.4. Our aim is to decompose the Hamiltonian $Q(x^{\Lambda_2}|y)$ into two parts $\tilde{\Phi}_l^c$ and $\tilde{\Phi}_l^b$ so that when restricted to the spins $x^{\Lambda_2^{(1)}}$, $\tilde{\Phi}_l^c$ is strictly convex and $\tilde{\Phi}_l^b$ is bounded. Then the desired statement follows from the application of Bakry-Émery criterion (Theorem 8.2.3) and Holley-Stroock Perturbation Principle (Theorem 8.2.2).

To see this, let us decompose $Q(x^{\Lambda_2}|y)$ as follows:

$$
\begin{aligned}
Q(x^{\Lambda_2}|y) &= -\log \int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c-\tilde{\Psi}_l^b\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}}) \\
&= -\log \int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}}) \\
&\quad + \Bigg(\log \int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}}) \\
&\quad\quad - \log \int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c-\tilde{\Psi}_l^b\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}})\Bigg) \\
&=: \tilde{\Phi}_l^c + \tilde{\Phi}_l^b.
\end{aligned}
$$

Step 1. Strict convexity of $\tilde{\Phi}_l^c$, restricted to the spins $x^{\Lambda_2^{(l)}}$.

We note that $H(\bar{x}^{B(l)})$ is independent of the spins $x^{\Lambda_2^{(l)}}$ and thus

$$
\frac{d}{dx_i}\tilde{\Psi}_l^c = \frac{d}{dx_i}\left(H(\bar{x}^{B(l)}) + \tilde{\Psi}_l^c\right) \qquad \text{for } i \in \Lambda_2^{(l)}.
$$

In particular, $\tilde{\Psi}_l^c$ is strictly convex, if restricted to the spins $x^{\Lambda_2^{(l)}}$. Therefore by Brascamp-Lieb inequality (cf. [BL76], [DMOW18b]), $\tilde{\Phi}_l^c$ is also uniformly strictly convex on $\mathbb{R}^{\Lambda_2^{(l)}}$.

Step 2. Boundedness of $\tilde{\Phi}_l^b$, restricted to the spins $x^{\Lambda_2^{(l)}}$.

We write

$$
\begin{aligned}
\tilde{\Phi}_l^b &= -\log \frac{\int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c-\tilde{\Psi}_l^b\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}})}{\int_{\substack{\frac{1}{K-R}\sum_{i\in\Lambda_1^{(k)}} x_i=\tilde{y}_k \\ \forall k\in[M]\setminus\{l\}}} \exp\left(-H(\bar{x}^{B(l)})\right)\exp\left(-\tilde{\Psi}_l^c\right)\mathcal{L}(d\bar{x}^{\Lambda_1^{(l)}})} \\
&= -\log \mathbb{E}_{\tilde{\mu}_l}\left[\exp\left(-\tilde{\Psi}_l^b\right)\right].
\end{aligned}
$$

Therefore the boundedness of $\tilde{\Phi}_l^b$ follows from the boundedness of $\tilde{\Psi}_l^b$.

□

Next, we shall prove that the strength of interactions between different blocks $\Lambda_2^{(l)}$ and $\Lambda_2^{(n)}$ become arbitrary small for large enough block size K. We first introduce a formula for the second derivative of the Hamiltonian $Q(x^{\Lambda_2}|y)$.

Lemma 7.4.6 (Lemma 6.3.10)**.** *For any $i \in \Lambda_2^{(n)}$ and $j \in \Lambda_2^{(l)}$ with $n \neq l$,*

$$\frac{d^2}{dx_i dx_j} Q(x^{\Lambda_2}|y) = -\operatorname{cov}_{\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2},y)} \left(\frac{\partial}{\partial x_i} H(x) - \frac{\partial}{\partial x_{i-R}} H(x), \frac{\partial}{\partial x_j} H(x) - \frac{\partial}{\partial x_{j-R}} H(x) \right).$$

The following statement is the second main ingredient for proving Lemma 7.4.2.

Lemma 7.4.7. *For any $n, l \in [M]$ with $n \neq l$, it holds that*

$$\left| \frac{d^2}{dx_i dx_j} Q(x^{\Lambda_2}|y) \right| \lesssim \frac{R}{K} + R \exp\left(-CK|n-l|\right) \qquad \text{for all } i \in \Lambda_2^{(n)}, j \in \Lambda_2^{(l)}.$$

Proof of Lemma 7.4.7. Let f and g be functions supported on $\Lambda_1^{(l)}$ and $\Lambda_1^{(n)}$, respectively. Since the measure $\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2}, y)$ tensorizes on $\bigotimes_{l=1}^{M} X_{K-R,\tilde{y}_l}$, for $l \neq n$,

$$\operatorname{cov}_{\mu_{N,m}(dx^{\Lambda_1}|x^{\Lambda_2},y)}(f, g) = 0. \tag{7.4.3}$$

Then a combination of Lemma 7.4.6, (7.4.3) and Proposition 5.2.1 yields

$$\left| \frac{d^2}{dx_i dx_j} Q(x^{\Lambda_2}|y) \right| \lesssim \frac{R}{K} + R \exp\left(-CK|n-l|\right).$$

□

Now we are ready to present the proof of Lemma 7.4.2.

Proof of Lemma 7.4.2. By choosing the block size K large enough, the Otto-Reznikoff Criterion (Theorem 8.2.4) applied with Lemma 7.4.5 and 7.4.7 concludes the proof of Lemma 7.4.2.

□

7.5 Proof of Theorem 7.3.3

In this section, we establish the strict convexity of coarse-grained Hamiltonian, Theorem 7.3.3. The proof consists of two ingredients. The first one is a uniform estimate on the diagonal elements of $\operatorname{Hess}_Y \bar{H}(y)$ and the second one is a control on off-diagonal elements of $\operatorname{Hess}_Y \bar{H}(y)$. First, we state the uniform positivity of the diagonal terms of $\operatorname{Hess}_Y \bar{H}(y)$, obtained by the strict convexity of the one-dimensional coarse-grained Hamiltonian.

Lemma 7.5.1. *There is a constant $\tau > 0$ such that for sufficiently large K and each $l \in [M]$,*

$$\tau \leq \left(\operatorname{Hess}_Y \bar{H}(y)\right)_{ll} \leq \frac{1}{\tau}.$$

Next, using the decay of correlations result (cf. Proposition 5.2.1), we establish the smallness of off-diagonal terms of $\operatorname{Hess}_Y \bar{H}(y)$.

Lemma 7.5.2. *For each $l \neq n \in [M]$,*

$$\left|\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{ln}\right| \lesssim \frac{1}{K}.$$

The proof of Lemma 7.5.1 and 7.5.2 are given in Section 7.5.1 and 7.5.2, respectively. Theorem 7.3.3 is a direct consequence of these two lemmas.

Proof of Theorem 7.3.3. It follows directly from Lemma 7.5.1 and 7.5.2 by choosing K large enough. □

7.5.1 Proof of Lemma 7.5.1

Recall the definition (7.4.2) of the conditional Hamiltonian $H(x^{B(l)}|\bar{x}^{B(l)})$. The coarse-grained Hamiltonian associated to $H(x^{B(l)}|\bar{x}^{B(l)})$ is given as follows: for $y_l \in \mathbb{R}$,

$$\bar{H}(y_l|\bar{x}^{B(l)}) := -\frac{1}{K} \log \int_{\frac{1}{K}\sum_{i \in B(l)} x_i = y_l} \exp\left(-H(x^{B(l)}|\bar{x}^{B(l)})\right) \mathcal{L}^{K-1}(dx^{B(l)}).$$

In addition, we disintegrate the measure $\mu_{N,m}(dx|y)$ into $\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y)$ and $\bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y)$.

Observing that the Hamiltonian $H(x^{B(l)}|\bar{x}^{B(l)})$ has the same structure as the Hamiltonian given by (1.1.1), a straightforward calculation yields the following statement:

Lemma 7.5.3 ((34) in [Men11]). *For any $l \in [M]$,*

$$\begin{aligned}
&\left(\mathrm{Hess}_Y \bar{H}(y)\right)_{ll} \\
&= \int \frac{d^2}{dy_l^2} \bar{H}(y_l|\bar{x}^{B(l)}) \bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y) \\
&\quad - \frac{1}{K} \mathrm{var}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right). \quad (7.5.1)
\end{aligned}$$

The next ingredient is that one-dimensional coarse-grained Hamiltonian is uniformly strictly convex.

Lemma 7.5.4 (Corollary 3.2.5). *There is a constant $\lambda > 0$ independent of the system size N, mean spin m, block $B(l)$ and conditioning $\bar{x}^{B(l)}$ such that*

$$\lambda \leq \frac{d^2}{dy_l^2} \bar{H}(y_l|\bar{x}^{B(l)}) \leq \frac{1}{\lambda}, \qquad \forall y \in Y.$$

The next statement implies that the right hand side of (7.5.1) can be arbitrary small for small enough K.

Lemma 7.5.5. *It holds that*

$$\frac{1}{K} \mathrm{var}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right) \lesssim \frac{1}{K}.$$

Proof of Lemma 7.5.5. Note that the conditional measure $\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y)$ is given by

$$\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) = \frac{1}{Z} \mathbb{1}_{\left\{\frac{1}{K}\sum_{i \in B(l)} x_i = y_l\right\}}(x^{B(l)}) \exp\left(-H(x^{B(l)}|\bar{x}^{B(l)})\right) \mathcal{L}^{K-1}(dx^{B(l)}).$$

For each $l \in [M]$, define $E_l := \{k \notin B(l) : \exists i \in B(l) \text{ such that } |i - k| \leq R\}$. Note that

$$\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \qquad (7.5.2)$$

depends only on the spins x_k with $k \in E_l$. In particular, (7.5.2) is a function of $\bar{x}^{B(l)}$. Thus,

$$\operatorname{var}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right)$$
$$= \operatorname{var}_{\mu_{N,m}(dx|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right).$$

Then an application of Poincaré inequality for $\mu_{N,m}(dx|y)$ (cf. Theorem 7.3.1) yields

$$\operatorname{var}_{\mu_{N,m}(dx|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right)$$
$$\lesssim \int \left| \nabla \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right) \right|^2 \mu_{N,m}(dx|y).$$

Because (7.5.2) depends only on the spins x_k, $k \in E_l$, it holds that

$$\left| \nabla \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right) \right|$$
$$= \sum_{k \in E_l} \left(\frac{\partial}{\partial x_k} \int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right)^2$$
$$= \sum_{k \in E_l} \left(\sum_{j \in B(l)} M_{kj} - \operatorname{cov}_{\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y)} \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} x_i \right) + \psi_b'(x_j), \sum_{\substack{i \in B(l) \\ |k-i| \le R}} M_{ik} x_i \right) \right)^2. \tag{7.5.3}$$

Lastly, an application of Proposition 5.2.1 combined with the fact that $|E_l| \le 2R$ implies

$$T_{(7.5.3)} \lesssim 1.$$

This finishes the proof of Lemma 7.5.5. □

Proof of Lemma 7.5.1. Lemma 7.5.1 is a direct consequence of Lemma 7.5.3, 7.5.4 and 7.5.5. Indeed, by choosing K large enough, we can find a constant $\tau > 0$ such that

$$\tau \le \left(\operatorname{Hess}_Y \bar{H}(y) \right)_{ll} \le \frac{1}{\tau}.$$

□

7.5.2 Proof of Lemma 7.5.2.

The main ingredient for the proof of Lemma 7.5.2 is the following representation of the Hessian of $\bar{H}$ from Lemma 8.3.3.

Lemma 7.5.6. *For any $l \neq n \in [M]$,*

$$\begin{aligned}
&\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{ln} \\
&= \frac{1}{K} \sum_{i \in B(l), j \in B(n)} M_{ij} && (7.5.4)\\
&\quad - \frac{1}{K} \operatorname{cov}_{\mu_{N,m}(dx|y)} \left(\sum_{j \in B(l)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j), \sum_{j \in B(n)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right). && (7.5.5)
\end{aligned}$$

Proof of Lemma 7.5.2. We observe that for $l \neq n$, there are at most R^2 many pairs (i, j) with $i \in B(l)$, $j \in B(n)$ and $|i - j| \leq R$. For such (i, j), we know $|M_{ij}|$ is uniformly bounded by 1 and hence

$$\left| T_{(7.5.4)} \right| \lesssim R^2 \cdot \frac{1}{K} \lesssim \frac{1}{K}.$$

Let us turn to the estimation of (7.5.5). The law of total variance yields

$$\begin{aligned}
&\operatorname{cov}_{\mu_{N,m}(dx|y)} \left(\sum_{j \in B(l)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j), \sum_{j \in B(n)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \\
&= \operatorname{cov}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y)} \left(\int \left(\sum_{j \in B(l)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y), \right. \\
&\qquad\qquad \left. \int \left(\sum_{j \in B(n)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right) \\
&\quad + \int \operatorname{cov}_{\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y)} \left(\sum_{j \in B(l)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j), \right. \\
&\qquad\qquad \left. \sum_{j \in B(n)} \left(\sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \bar{\mu}_{N,m}(d\bar{x}^{B(l)}|y).
\end{aligned}$$

Then an application of Proposition 5.2.1 as in Lemma 7.5.1 yields the desired estimate

$$\left| T_{(7.5.5)} \right| \lesssim \frac{1}{K}.$$

Hence we conclude

$$\left|\left(\mathrm{Hess}_Y \bar{H}(y)\right)_{ln}\right| \leq \left|T_{(7.5.4)}\right| + \left|T_{(7.5.5)}\right| \lesssim \frac{1}{K}.$$

□

7.6 Proof of Proposition 7.2.2

The key ingredient of the proof of Proposition 7.2.2 is a two-scale criterion for the hydrodynamic limit which was originally obtained in [GOVW09, Theorem 8]. This was successfully used to establish a hydrodynamic limit of Kawasaki dynamics where there is no-interactions within the Hamiltonian. We first introduce a general two-scale criterion for the hydrodynamic limit developed in [GOVW09], and then apply this to the general ce using the results established so far.

Theorem 7.6.1 (Two-Scale Criterion for the hydrodynamic limit [GOVW09])**.** *Let*

$$\mu(dx) = \frac{1}{Z} \exp\left(-H(x)\right) dx$$

be a probability measure on X. Assume a linear operator $P : X \to Y$ satisfies $PNP^* = \mathrm{Id}_Y$ for some large $N \in \mathbb{N}$. Assume further the following:

(i) It holds that

$$\kappa := \max\left\{\langle \mathrm{Hess}\, H(x) \cdot u, v\rangle : \ u \in \mathrm{Ran}(NP^*P), v \in \mathrm{Ran}(\mathrm{Id}_X - NP^*P), |u| = |v| = 1\right\} < \infty;$$

(ii) There is $\rho > 0$ such that $\mu(dx|y)$ satisfies LSI(ρ) for all y;

(iii) There is $\lambda > 0$ such that $\langle \tilde{y}, \mathrm{Hess}\, \bar{H}(y)\tilde{y}\rangle_Y \geq \lambda\langle \tilde{y}, \tilde{y}\rangle_Y$;

(iv) There is $\alpha > 0$ such that $\int |x|^2 \mu(dx) \leq \alpha N$;

(v) There is $\beta > 0$ such that $\inf_{y \in Y} \bar{H}(y) \geq -\beta$;

Define $M := \dim Y$ and let $A : X \to X$ be a symmetric linear operator such that:

(vi) There is $\gamma > 0$ such that for all $x \in X$, $|(\mathrm{Id}_X - NP^*P)x|^2 \leq \gamma M^{-2}\langle x, Ax\rangle_X$.

Let $f(t,x)$ and $\eta(t)$ be the solution of (7.2.1) and

$$\frac{d\eta}{dt} = -\bar{A}\nabla_Y \bar{H}(\eta),$$

with initial data $f(0,\cdot)$ and η_0 respectively, where $(\bar{A})^{-1} = PA^{-1}NP^*$. Assume

(vii) $\int f(0,x)\log f(0,x)\mu(dx) \leq C_1 N,\ \bar{H}(\eta_0) \leq C_2.$

Define

$$\Theta(t) := \frac{1}{2N}\int \left\langle (x - NP^*\eta(t)), A^{-1}(x - NP^*\eta(t))\right\rangle f(t,x)\mu(dx).$$

Then for any $T > 0$,

$$\max\left\{\sup_{0<t\leq T}\Theta(t), \frac{\lambda}{2}\int_0^T\left(\int_Y |y-\eta(t)|_Y^2 \bar{f}(t,y)\bar{\mu}(dy)\right)dt\right\}$$
$$\leq \Theta(0) + T\left(\frac{M}{N}\right) + \left(\frac{C_1\gamma\kappa^2}{2\lambda\rho^2}\frac{1}{M^2}\right) + \left[\sqrt{2T\gamma}\left(\alpha + \frac{2C_1}{\hat{\rho}}\right)^{\frac{1}{2}}\left(C_1^{\frac{1}{2}} + (C_2+\beta)^{\frac{1}{2}}\right)\right]\frac{1}{M}.$$

As a corollary, we have both microscopic and mesoscopic closeness between the microscopic Kawasaki dynamics and the evolution (7.2.5).

Corollary 7.6.2 (Propagation of hydrodynamic behavior [GOVW09])**.** *Consider a sequence $\{X_\nu, Y_\nu, P_\nu, A_\nu, \mu_\nu, f_{0,\nu}, \eta_{0,\nu}\}_{\nu=1}^{\infty}$ of data satisfying the assumptions of Theorem 7.6.1 for every ν with uniform constants $\lambda, \rho, \kappa, \alpha, \beta, \gamma, C_1, C_2$. Suppose that*

$$M_\nu \to \infty, \qquad N_\nu \to \infty, \qquad \frac{N_\nu}{M_\nu} \to \infty,$$

and the initial data satisfies

$$\lim_{\nu\to\infty}\frac{1}{N_\nu}\int (x - N_\nu P_\nu^t \eta_{0,\nu})\cdot A_\nu^{-1}(x - N_\nu P_\nu^t\eta_{0,\nu}) f_{0,\nu}(x)\mu_\nu(dx) = 0. \tag{7.6.1}$$

Then for any $T > 0$,

$$\lim_{\nu\to\infty}\sup_{0\leq t\leq T}\frac{1}{N_\nu}\int (x - N_\nu P_\nu^t\eta)\cdot A_\nu^{-1}(x - N_\nu P_\nu^t\eta) f(t,x)\mu(dx) = 0, \tag{7.6.2}$$

and

$$\lim_{v\to\infty}\int_0^T\int_Y |y-\eta(t)|_Y^2\bar{f}(y)\bar{\mu}(dy)dt = 0. \tag{7.6.3}$$

The convergence (7.6.2) implies the closeness of microscopic variables in the weak norm induced by A^{-1}, and the convergence (7.6.3) implies the closeness of macroscopic variables in the strong $L^2(Y)$ norm.

7.6.1 Auxiliary Hamiltonian

As mentioned at the beginning of Section 7.2.3, we introduce the auxiliary Hamiltonian and related notions. First of all, define the auxiliary Hamiltonian $H_{N,\text{aux}} = H_{\text{aux}}$ as

$$H_{\text{aux}}(x) := H(x) - \frac{1}{2} \sum_{\substack{n,l \in [M] \\ n \neq l}} \sum_{\substack{i \in B(l) \\ j \in B(n)}} M_{ij} x_i x_j. \tag{7.6.4}$$

Because the interactions between different blocks are removed, H_{aux} is decomposed as follows:

$$\begin{aligned} H_{\text{aux}}(x) &= \sum_{l=1}^{M} \left(\sum_{i \in B(l)} \left(\psi(x_i) + \frac{1}{2} \sum_{\substack{j \in B(l), \\ 1 \leq |j-i| \leq R}} M_{ij} x_i x_j \right) \right) \\ &=: \sum_{l=1}^{M} H_K(x^{B(l)}). \end{aligned}$$

Here, we note that there are at most $2R^2M$ many pairs of (i,j,l,n) such that

- $l, n \in [M]$ and $l \neq n$,
- $i \in B(l)$, $j \in B(n)$ and $|i-j| \leq R$.

Next, define the corresponding canonical ensemble $\mu_{N,m,\text{aux}}$ by

$$\mu_{N,m,\text{aux}}(dx) := \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\}}(x) \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-1}(dx).$$

Now, let us decompose the ce $\mu_{N,m,\text{aux}}$ into the conditional measure $\mu_{N,m,\text{aux}}(dx|y)$ and the marginal measure $\bar{\mu}_{N,m,\text{aux}}(y)$, and then define the corresponding coarse-grained Hamiltonian $\bar{H}_{Y,\text{aux}}$ by

$$\begin{aligned} \bar{H}_{Y,\text{aux}}(y) &:= -\frac{1}{N} \log \int_{Px=y} \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-M}(dx) \\ &= \frac{1}{M} \sum_{l=1}^{M} \left(-\frac{1}{K} \log \int_{\left\{ \frac{1}{K} \sum_{i \in B(l)} x_i = y_l \right\}} \exp\left(-H_K(x^{B(l)}) \mathcal{L}^{K-1}(dx^{B(l)})\right) \right) \end{aligned}$$

$$= \frac{1}{M} \sum_{l=1}^{M} \bar{H}_K(y_l). \tag{7.6.5}$$

Convention. In Section 7.6 and 7.7, we write $\mu = \mu_{N,m}$, $\mu_{\text{aux}} = \mu_{N,m,\text{aux}}$, $\bar{H} = \bar{H}_Y$, and $\bar{H}_{\text{aux}} = \bar{H}_{Y,\text{aux}}$ to reduce our notational burden.

7.6.2 Auxiliary Lemmas

In this section, we provide auxiliary statements that are needed in the proof of Proposition 7.2.2. We first obtain the quantitative closeness between $\bar{H}$ and $\bar{H}_{\text{aux}}$.

Lemma 7.6.3. *There exists a constant $C > 0$ such that for any $y \in Y$,*

$$\left|\bar{H}(y) - \bar{H}_{\text{aux}}(y)\right| \leq \frac{C}{K}\left(1 + \|y\|^2_{L^2(Y)}\right).$$

Proof of Lemma 7.6.3. As there is only cosmetic difference between the proof of Lemma 7.6.3 and Lemma 7.3.5, we briefly outline the proof.

Recalling the definition (7.6.4) of H_{aux}, we have

$$\begin{aligned}
\bar{H}_{\text{aux}}(y) - \bar{H}(y) &= \frac{1}{N} \log \frac{\int_{Px=y} \exp\left(-H(x)\right) \mathcal{L}^{N-M}(dx)}{\int_{Px=y} \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-M}(dx)} \\
&= \frac{1}{N} \log \left(\mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[\exp\left(-\frac{1}{2} \sum_{\substack{n,l \in [M] \\ n \neq l}} \sum_{\substack{i \in B(l) \\ j \in B(n)}} M_{ij} x_i x_j \right) \right] \right) \\
&\leq \frac{1}{N} \log \left(\mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[\exp\left(\frac{1}{4} \sum_{\substack{n,l \in [M] \\ n \neq l}} \sum_{\substack{i \in B(l) \\ j \in B(n)}} |M_{ij}|(x_i^2 + x_j^2) \right) \right] \right) \\
&= \frac{1}{N} \log \left(\mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[\exp\left(\frac{1}{2} \sum_{\substack{n,l \in [M] \\ n \neq l}} \sum_{\substack{i \in B(l) \\ j \in B(n)}} |M_{ij}| x_i^2 \right) \right] \right).
\end{aligned} \tag{7.6.6}$$

Because the conditional measure $\mu_{\text{aux}}(dx|y)$ tensorizes, i.e.,

$$\mu_{\text{aux}}(dx|y) = \bigotimes_{l=1}^{M} \frac{1}{Z} \mathbb{1}_{\left\{\frac{1}{K}\sum_{i \in B(l)} x_i = y_l\right\}} \left(x^{B(l)}\right) \exp\left(-H_K(x^{B(l)})\right) \mathcal{L}^{K-1}(dx^{B(l)})$$

$$=: \bigotimes_{l=1}^{M} \mu_K(dx^{B(l)}|y_l),$$

we have

$$T_{(7.6.6)} = \frac{1}{N}\sum_{l=1}^{M} \log\left(\mathbb{E}_{\mu_K(dx^{B(l)}|y_l)}\left[\exp\left(\frac{1}{2}\sum_{\substack{n,l\in[M]\\ n\neq l}}\sum_{\substack{i\in B(l)\\ j\in B(n)}} |M_{ij}|x_i^2\right)\right]\right).$$

Since $\mu_K(dx^{B(l)}|y_l)$ is a canonical ensemble with a single constraint for each $l \in [M]$, a similar argument from the proof of Lemma 7.3.5 using Corollary 2.2.4 yields

$$T_{(7.6.6)} \lesssim \frac{1}{N}\sum_{l=1}^{M}\left(1+y_l^2\right) = \frac{1}{K}\left(1+\|y\|^2_{L^2(Y)}\right).$$

The lower bound of $T_{(7.6.6)}$ is similarly deduced.

□

Next, we bound the coarse-grained Hamiltonian $\bar{H}$ by quadratic functions.

Lemma 7.6.4. *There exists a constant $C > 0$ such that*

$$-C + \frac{1}{C}\|y\|^2_{L^2(Y)} \le \bar{H}(y) \le C\left(1+\|y\|^2_{L^2(Y)}\right).$$

Proof of Lemma 7.6.4. By Lemma 7.6.3, it suffices to prove

$$-C + \frac{1}{C}\|y\|^2_{L^2(Y)} \le \bar{H}_{\text{aux}}(y) \le C\left(1+\|y\|^2_{L^2(Y)}\right). \tag{7.6.7}$$

First of all, by Proposition 7.3.4,

$$\bar{H}_{\text{aux}}(\mathbf{0}) \overset{(7.6.5)}{=} \frac{1}{M}\sum_{l=1}^{M} \bar{H}_K(0) = \bar{H}_K(0) \to \varphi(0) \qquad \text{as } \nu \to \infty.$$

Thus $\bar{H}_{\text{aux}}(\mathbf{0})$ is uniformly bounded.

Next, Lemma 8.3.2 implies

$$\left|\frac{\partial}{\partial y_l}\bar{H}_{\text{aux}}(\mathbf{0})\right| = \left|\frac{1}{N}\mathbb{E}_{\mu_{\text{aux}}(dx|0)}\left[\sum_{i,j\in B(l)} M_{ij}X_i + \sum_{i\in B(l)} \psi_b'(X_i)\right]\right|$$

$$\overset{\text{Lemma 2.2.3}}{\lesssim} \frac{1}{N}\left(2KR^2+K\right) \sim \frac{1}{M}.$$

In particular the partial derivatives of $\bar{H}_{\text{aux}}(\mathbf{0})$ are also bounded.

Since $\bar{H}_{\text{aux}}$ is uniformly strictly convex (Theorem 7.3.3), by Taylor's theorem, we obtain (7.6.7).

□

As a special case of Lemma 7.6.4, we have the following corollary.

Corollary 7.6.5. *The one-dimensional coarse-grained Hamiltonian $\bar{H}_K$ and its limit φ are strictly convex. In particular, there exists a constant $C > 0$ such that*

$$-C + \frac{1}{C}m^2 \leq \bar{H}_K(m), \varphi(m) \leq C(1+m^2).$$

We now state lemmas which are additional ingredients for deducing Proposition 7.2.2.

Lemma 7.6.6. *Define κ by*

$$\kappa := \max\left\{\langle \operatorname{Hess} H(x) \cdot u, v\rangle : \; u \in \operatorname{Ran}(NP^*P), v \in \operatorname{Ran}(\operatorname{Id}_X - NP^*P), |u| = |v| = 1\right\}.$$

Then, we have $\kappa < \infty$.

Note that κ measures the strength of interactions between the microscopic and macroscopic scales.

Proof of Lemma 7.6.6. Since $\operatorname{Hess} H(x)$ is symmetric, for any $|u| = |v| = 1$,

$$|\langle \operatorname{Hess} H(x) \cdot u, v\rangle| \leq \|\operatorname{Hess} H(x)\|_{L^2} \leq \|\operatorname{Hess} H(x)\|_{L^1}.$$

Due to the conditions (1.1.2) and (1.1.3), we have $\max_x \|\operatorname{Hess} H(x)\|_{L^1} < \infty$. This concludes the proof.

□

Lemma 7.6.7 ((92), (93) in [GOVW09]). *There exists a constant $C > 0$ such that*

$$\frac{1}{C}\langle \bar{x}, \bar{x}\rangle_{H^{-1}} \leq \frac{1}{N}\langle x, A^{-1}x\rangle_X \leq C\langle \bar{x}, \bar{x}\rangle_{H^{-1}}. \tag{7.6.8}$$

If $\bar{x}$ is bounded in L^2, then

$$\left|\langle \bar{x}, \bar{x}\rangle_{H^{-1}} - \frac{1}{N}\langle x, A^{-1}x\rangle_X\right| \leq \frac{C}{N}.$$

Since the statement of Lemma 7.6.7 is same as (92) and (93) in [GOVW09], we omit the proof.

7.6.3 Proof of Proposition 7.2.2

In this section, we prove Proposition 7.2.2 using Corollary 7.6.2.

Proof of Proposition 7.2.2. Let us begin with verifying the assumptions of Theorem 7.6.1.

- (i) is the same as Lemma 7.6.6.
- (ii) is a consequence of Theorem 7.3.1.
- (iii) is a consequence of Theorem 7.3.3.
- (iv) follows from Lemma 2.2.4.
- (v) is a consequence of Lemma 7.6.4.
- (vi) is the same as (7.6.8) in Lemma 7.6.7.

Lastly, we verify the condition (vii). The first assumption of (vii) is the same as (7.2.2). To verify the second condition, let us recall (7.2.6). Because $\zeta_0 \in L^2(\mathbb{T}^1)$, there is a positive constant C such that

$$\|\bar{\eta}_0^\nu\|_{L^2} \leq C. \tag{7.6.9}$$

Thus Lemma 7.6.4 implies the second condition of (vii) as follows:

$$\bar{H}(\eta_0^\nu) \leq C(1 + \|\eta_0^\nu\|_{L^2(Y)}^2) = C(1 + \|\bar{\eta}_0^\nu\|_{L^2}^2) \leq C.$$

Therefore, all assumptions in Theorem 7.6.1 is are satisfied and hence one can apply Corollary 7.6.2, once the closeness of initial data is verified. In fact,

$$
\begin{aligned}
0 &\overset{(7.6.8)}{\leq} \frac{1}{N}\int (x - NP^t\eta_0^\nu)\cdot A^{-1}(x - NP^t\eta_0^\nu) f_0(x)\mu(dx) \\
&\overset{(7.6.8)}{\leq} \int C\|\bar{x} - \bar{\eta}_0^\nu\|_{H^{-1}}^2 f_0(x)\mu(dx) \\
&\leq \int 2C\left(\|\bar{x} - \zeta_0\|_{H^{-1}}^2 + \|\zeta_0 - \bar{\eta}_0^\nu\|_{H^{-1}}^2\right) f_0(x)\mu(dx) \overset{(7.2.2),(7.2.6)}{\longrightarrow} 0 \qquad \text{as } \nu \to \infty.
\end{aligned}
$$

Therefore, by (7.6.1) in Corollary 7.6.2, we obtain

$$\lim_{\nu\to\infty}\sup_{0\leq t\leq T}\frac{1}{N}\int (x - NP^t\eta)\cdot A^{-1}(x - NP^t\eta) f(t,x)\mu(dx) = 0.$$

In particular, (7.6.8) implies

$$\lim_{\nu\to\infty}\sup_{0\leq t\leq T}\int \|\bar{x} - \bar{\eta}^\nu(t,\cdot)\|_{H^{-1}} f(t,x)\mu_{N,m}(dx) = 0.$$

□

7.7 Proof of Proposition 7.2.3

7.7.1 Auxiliary Lemmas

In this Section, we provide auxiliary statements that will be needed in the proof of Proposition 7.2.3. The statements are extensions of [GOVW09], where the Hamiltonian with no-interactions is considered, to the general ce with strong interactions. Recall that $\{\eta^\nu\}_{\nu=1}^\infty$ are solutions to the mesoscopic parabolic equation (7.2.5).

Lemma 7.7.1 (Analogue of Lemma 34 in [GOVW09])**.** *Let $\{\eta^\nu\}_{\nu=1}^\infty$ be solutions to the mesoscopic parabolic equation* (7.2.5) *satisfying* (7.6.9)*. Then, there is a constant $C > 0$ such that*

$$\sup_{0\leq t\leq T}\langle \eta^\nu(t), \eta^\nu(t)\rangle_Y \leq C, \tag{7.7.1}$$

$$\int_0^T \left\langle \frac{d\eta^\nu}{dt}(t), (\bar{A})^{-1}\frac{d\eta^\nu}{dt}(t)\right\rangle_Y dt \leq C.$$

In particular, (7.7.1) implies, up to a subsequence, the associated step functions $\bar{\eta}^\nu$ converges to η_* weak-$*$ in $L_t^\infty(L_\theta^2) = (L_t^1(L_\theta^2))^*$. Next lemma provides some properties of the function η_*.

Lemma 7.7.2 (Analogue of Lemma 35 in [GOVW09]). *Let $\{\eta^\nu\}_{\nu=1}^\infty$ be solutions to the mesoscopic parabolic equation (7.2.5) satisfying (7.6.9). Assume that the associated step functions of any subsequence of $\{\eta^\nu\}_{\nu=1}^\infty$ weak* converges to η_* in $L_t^\infty(L_\theta^2) = (L_t^1(L_\theta^2))^*$. Then η_* satisfies*

$$\eta_* \in L_t^\infty(L_\theta^2), \qquad \frac{\partial \eta_*}{\partial t} \in L_t^2(H_\theta^{-1}), \qquad \varphi'(\eta_*) \in L_t^2(L_\theta^2).$$

The following lemma provides a integral criteria to ensure a function to be a weak solution to the nonlinear parabolic equation.

Lemma 7.7.3 (Analogue of Lemma 36 in [GOVW09]). *Assume $\bar{H}$ is convex. Then η satisfies*

$$\frac{d\eta^\nu}{dt} = -\bar{A}\nabla_Y \bar{H}(\eta^\nu)$$

if and only if for all $\xi \in Y$ and smooth $\beta : [0,T] \to [0,\infty)$,

$$\int_0^T \bar{H}(\eta)\beta(t)dt \le \int_0^T \bar{H}(\eta+\xi)\beta(t)dt - \int_0^T \langle \xi, (\bar{A})^{-1}\eta\rangle_Y \dot{\beta}(t)dt.$$

Similarly, if φ is convex, then ζ is a weak solution of

$$\frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2}\varphi'(\zeta)$$

if and only if for all $\xi \in L^2(\mathbb{T}^1)$ and smooth $\beta : [0,T] \to [0,\infty)$,

$$\int_0^T \int_{\mathbb{T}^1} \varphi(\zeta(t,\theta))\beta(t)d\theta dt \le \int_0^T \int_{\mathbb{T}^1} \varphi(\zeta(t,\theta)+\xi(\theta))\beta(t)d\theta dt - \int_0^T \langle \xi(\cdot), \zeta(t,\cdot)\rangle_{H^{-1}}\dot{\beta}(t)dt.$$

In addition, thanks to the convexity of φ (see (7.3.20)), we have a uniqueness of the weak solution to the nonlinear parabolic equation (7.2.3).

Lemma 7.7.4 (Analogue Lemma 38 in [GOVW09]). *There is at most one weak solution to the equation* (7.2.3).

We shall not provide the proof of Lemma 7.7.1, 7.7.2, 7.7.3 and 7.7.4 as there are only cosmetic differences to that of Lemma 34, 35, 36 and 38 in [GOVW09], respectively. The last ingredient of the proof of Proposition 7.2.3 is the following lemma.

Lemma 7.7.5 (Lemma 37 in [GOVW09]). *Let* $\{\eta^\nu\}_{\nu=1}^\infty$ *and* η_* *as in Lemma 7.7.2. Define* $\xi^\nu := \pi(\xi + \eta_*) - \eta^\nu$, *where* π *is the* L^2 *projection onto* Y. *Then it holds that*

$$\liminf_{\nu\to\infty} \int_0^T \bar{H}(\eta^\nu(t))\beta(t)dt \geq \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*(t,\theta))\beta(t)d\theta dt, \tag{7.7.2}$$

$$\lim_{\nu\to\infty} \int_0^T \bar{H}(\eta^\nu(t)+\xi^\nu(t))\beta(t)dt = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*(t,\theta)+\xi(\theta))\beta(t)d\theta dt, \tag{7.7.3}$$

$$\lim_{\nu\to\infty} \int_0^T \left\langle \xi^\nu(t), (\bar{A})^{-1}\eta^\nu(t)\right\rangle_Y \dot{\beta}(t)dt = \int_0^T \langle \xi(\theta), \eta_*(t,\theta)\rangle_{H^{-1}}\dot{\beta}(t)dt. \tag{7.7.4}$$

Since the proof of a statement (108) in [GOVW09] can be adapted to prove (7.7.4) without any changes, we only present the proof of (7.7.2) and (7.7.3). Compared to [GOVW09], the main difficulty one encounters when deducing (7.7.2) and (7.7.3) is the lack of uniform convergence of the coarse grained Hamiltonian $\bar{H}_K$ towards φ. The key ingredient to solve this problem is Proposition 7.3.4, which gives a quantitative convergence of $\bar{H}_K$ towards φ.

Proof of (7.7.2) *in Lemma 7.7.5.* Let us write

$$\int_0^T \bar{H}(\eta^\nu)\beta(t)dt = \int_0^T \left(\bar{H}(\eta^\nu) - \bar{H}_{\mathrm{aux}}(\eta^\nu)\right)\beta(t)dt \tag{7.7.5}$$

$$+ \int_0^T \bar{H}_{\mathrm{aux}}(\eta^\nu)\beta(t)dt. \tag{7.7.6}$$

To begin with, by Lemma 7.6.3 and (7.7.1),

$$\left|T_{(7.7.5)}\right| \leq \frac{C}{K}\int_0^T (1+\|\eta^\nu\|_{L^2}^2)\beta(t)dt \leq \frac{C}{K}\int_0^T \beta(t)dt. \tag{7.7.7}$$

Next, we have

$$T_{(7.7.6)} \overset{(7.6.5)}{=} \int_0^T \int_{\mathbb{T}^1} \bar{H}_K(\bar{\eta}^\nu)\beta(t)d\theta dt$$
$$= \int_0^T \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu)\beta(t)d\theta dt + \int_0^T \int_{\mathbb{T}^1} \left(\bar{H}_K(\bar{\eta}^\nu) - \varphi(\bar{\eta}^\nu)\right)\beta(t)d\theta dt.$$

Since φ is convex, the functional $f \mapsto \int\int \varphi(f)\beta(t)d\theta dt$ is weakly lower semicontinuous with respect to the weak-$*$ $L_t^\infty(L_\theta^2)$ topology. Thus, we have

$$\liminf_{\nu\to\infty} \int_0^T \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu)\beta(t)d\theta dt \geq \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*)\beta(t)d\theta dt. \tag{7.7.8}$$

Next, by Proposition 7.3.4,

$$\left|\int_0^T \int_{\mathbb{T}^1} \left(\bar{H}_K(\bar{\eta}^\nu) - \varphi(\bar{\eta}^\nu)\right)\beta(t)d\theta dt\right| \leq \frac{C}{K}\int_0^T (1 + \|\eta^\nu\|_{L^2}^2)\beta(t)dt \overset{(7.7.1)}{\leq} \frac{C}{K}\int_0^T \beta(t)dt. \tag{7.7.9}$$

Therefore, plugging the estimations (7.7.7), (7.7.8) and (7.7.9) into (7.7.5), (7.7.6) and then taking $\nu \to \infty$, we conclude the proof.

□

Proof of (7.7.3) *in Lemma 7.7.5.* Let us write

$$\int_0^T \bar{H}(\eta^\nu + \xi^\nu)\beta(t)dt - \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi)\beta(t)d\theta dt$$
$$\overset{(7.6.5)}{=} \int_0^T \int_{\mathbb{T}^1} \bar{H}_K(\bar{\eta}^\nu + \bar{\xi}^\nu)\beta(t)d\theta dt - \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi)\beta(t)d\theta dt$$
$$= \int_0^T \int_{\mathbb{T}^1} \left(\bar{H}_K(\bar{\eta}^\nu + \bar{\xi}^\nu) - \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu)\right)\beta(t)d\theta dt \tag{7.7.10}$$
$$+ \int_0^T \int_{\mathbb{T}^1} \left(\varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) - \varphi(\eta_* + \xi)\right)\beta(t)d\theta dt. \tag{7.7.11}$$

Let us begin with the estimation of (7.7.10). By Proposition 7.3.4,

$$\left|T_{(7.7.10)}\right| \leq \frac{C}{K}\int_0^T \int_{\mathbb{T}^1} \left(1 + \|\bar{\eta}^\nu + \bar{\xi}^\nu\|_{L^2}^2\right)\beta(t)d\theta dt$$
$$\leq \frac{C}{K}\int_0^T \int_{\mathbb{T}^1} (1 + \|\eta_* + \xi\|^2)\beta(t)d\theta dt$$
$$\leq \frac{C}{K}\int_0^T \beta(t)dt. \tag{7.7.12}$$

Let us turn to the estimation of (7.7.11). Recalling that ξ^ν is defined by $\eta^\nu + \xi^\nu = \pi(\xi + \eta_*)$, we have

$$\bar{\eta}^\nu + \bar{\xi}^\nu \to \eta_* + \xi \qquad \text{in } L^2 \text{ for a.e. } t. \tag{7.7.13}$$

A combination of Corollary 7.6.5 and (7.7.13) yields

$$\int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) d\theta \to \int_{\mathbb{T}^1} \varphi(\eta_* + \xi) d\theta \qquad \text{for a.e. } t. \tag{7.7.14}$$

In addition, we have

$$\left| \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) d\theta \right| \le \int_{\mathbb{T}^1} C \left(1 + |\bar{\eta}^\nu + \bar{\xi}^\nu|^2\right) d\theta \le \int_{\mathbb{T}^1} C \left(1 + |\eta_* + \xi|^2\right) d\theta \le C. \tag{7.7.15}$$

Thus the Dominated Convergence theorem applied with (7.7.14) and (7.7.15) gives

$$\lim_{\nu\to\infty} T_{(7.7.11)} = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi)\beta(t) d\theta dt. \tag{7.7.16}$$

Now letting $\nu \to \infty$ in (7.7.12) and plugging this with (7.7.16) into (7.7.10) and (7.7.11) yields

$$\lim_{\nu\to\infty} \int_0^T \bar{H}(\eta^\nu(t) + \xi^\nu(t))\beta(t) dt = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*(t,\theta) + \xi(\theta))\beta(t) d\theta dt.$$

□

7.7.2 Proof of Proposition 7.2.3

In this Section, we prove Proposition 7.2.3.

Proof of Proposition 7.2.3. Since $\zeta_0 \in L^2(\mathbb{T}^1)$ and $\bar{\eta}_0^\nu$ converges to ζ_0 in L^2, $\|\bar{\eta}_0^\nu\|_{L^2}$ is uniformly bounded. Thus by Lemma 7.7.1, up to a subsequence,

$$\bar{\eta}^\nu \rightharpoonup \eta_* \qquad \text{weak} * \text{ in } L_t^\infty(L_\theta^2) = (L_t^1(L_\theta^2))^*, \quad \text{strongly in } L_t^\infty(H_\theta^{-1}).$$

Lemma 7.7.2 implies that η_* satisfies

$$\eta_* \in L_t^\infty(L_\theta^2), \qquad \frac{\partial \eta_*}{\partial t} \in L_t^2(H_\theta^{-1}), \qquad \varphi'(\eta_*) \in L_t^2(L_\theta^2).$$

Next, by Lemma 7.7.3, for any smooth $\beta : [0,T] \to [0,\infty)$,

$$\int_0^T \bar{H}(\eta^\nu)\beta(t)dt \le \int_0^T \bar{H}(\eta^\nu + \xi^\nu)\beta(t)dt - \int_0^T \langle \xi^\nu, (\bar{A})^{-1}\eta^\nu\rangle_Y \dot{\beta}(t)dt, \tag{7.7.17}$$

where $\xi^\nu := \pi(\xi + \eta_*) - \eta^\nu$. By taking the limit in (7.7.17) and applying Lemma 7.7.5, one gets

$$\int_0^T \int_{\mathbb{T}^1} \varphi(\zeta(t,\theta))\beta(t)d\theta dt \le \int_0^T \int_{\mathbb{T}^1} \varphi(\zeta(t,\theta) + \xi(\theta))\beta(t)d\theta dt - \int_0^T \langle \xi(\cdot), \zeta(t,\cdot)\rangle_{H^{-1}} \dot{\beta}(t)dt.$$

Therefore Lemma 7.7.3 implies that η_* is a weak solution of

$$\begin{cases} \frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2}\varphi'(\zeta), \\ \zeta(0,\cdot) = \zeta_0, \end{cases}$$

and by uniqueness (Lemma 7.7.4), one can conclude that the sequence $\{\bar{\eta}^\nu\}_{\nu=1}^\infty$ converges to η_* in $L_t^\infty(H_\theta^{-1})$. $\square$

Chapter 8

Appendix

8.1 Proof of Lemma 4.2.4

Let $\zeta, \eta : \mathbb{R}^N \to \mathbb{R}$ be given. Let us fix $\sigma = \sigma(m)$ and $m = m(\sigma)$ as in Definition 2.1.6. We introduce auxiliary external fields with ζ, η in the definition of gce and ce. More precisely, let us denote for $\alpha, \beta \in \mathbb{R}$,

$$\mu_N^{\sigma,\alpha,\beta}(dx) := \frac{1}{Z} \exp\left(\sigma \sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x) \right) dx,$$
$$\mu_{N,m}^{\alpha,\beta}(dx) := \frac{1}{Z} \mathbb{1}_{\left\{\frac{1}{N}\sum_{i=1}^N x_i = m\right\}} \exp\left(\sigma \sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x) \right) \mathcal{L}^{N-1}(dx).$$

In particular, one should observe that

$$\mu_N^{\sigma,\alpha,\beta}(dx)\Big|_{\alpha,\beta=0} = \mu_N^{\sigma}(dx), \qquad \mu_{N,m}^{\alpha,\beta}(dx)\Big|_{\alpha,\beta=0} = \mu_{N,m}(dx). \tag{8.1.1}$$

Let us consider associated free energies $A_{gce}^{\zeta,\eta}$, $A_{ce}^{\zeta,\eta}$ defined by

$$A_{gce}^{\zeta,\eta}(\alpha,\beta) := \frac{1}{N} \log \int \exp\left(\sigma \sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x) \right) dx,$$
$$A_{ce}^{\zeta,\eta}(\alpha,\beta) := \frac{1}{N} \log \int_{\left\{\frac{1}{N}\sum_{i=1}^N x_i = m\right\}} \exp\left(\sigma \sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x) \right) \mathcal{L}^{N-1}(dx).$$

First, a straightforward calculation yields

$$\frac{d}{d\alpha} A_{gce}^{\zeta,\eta}\Big|_{\alpha,\beta=0} = \frac{1}{N} \mathbb{E}_{\mu_N^{\sigma}}\left[\zeta(X)\right], \qquad \frac{d}{d\beta} A_{gce}^{\zeta,\eta}\Big|_{\alpha,\beta=0} = \frac{1}{N} \mathbb{E}_{\mu_N^{\sigma}}\left[\eta(X)\right],$$
$$\frac{d}{d\alpha} A_{ce}^{\zeta,\eta}\Big|_{\alpha,\beta=0} = \frac{1}{N} \mathbb{E}_{\mu_{N,m}}\left[\zeta(X)\right], \qquad \frac{d}{d\beta} A_{ce}^{\zeta,\eta}\Big|_{\alpha,\beta=0} = \frac{1}{N} \mathbb{E}_{\mu_{N,m}}\left[\eta(X)\right],$$

$$\left.\frac{d^2}{d\alpha d\beta}A_{gce}^{\zeta,\eta}\right|_{\alpha,\beta=0} = \frac{1}{N}\operatorname{cov}_{\mu_N^\sigma}\left(\zeta(X),\eta(X)\right), \quad \left.\frac{d^2}{d\alpha d\beta}A_{ce}^{\zeta,\eta}\right|_{\alpha,\beta=0} = \frac{1}{N}\operatorname{cov}_{\mu_{N,m}}\left(\zeta(X),\eta(X)\right). \tag{8.1.2}$$

Next, the Cramer's representation yields

$$\begin{aligned} & A_{ce}^{\zeta,\eta}(\alpha,\beta) - A_{gce}^{\zeta,\eta}(\alpha,\beta) \\ &\quad = \frac{1}{N}\log\frac{\int_{\left\{\frac{1}{N}\sum_{i=1}^N x_i = m\right\}}\exp\left(\sigma\sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x)\right)\mathcal{L}^{N-1}(dx)}{\int_{\mathbb{R}}\exp\left(\sigma\sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x)\right)dx} \\ &\quad = \frac{1}{N}\log\frac{\int_{\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^N (x_i - m) = 0\right\}}\exp\left(\sigma\sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x)\right)\mathcal{L}^{N-1}(dx)}{\int_{\mathbb{R}}\exp\left(\sigma\sum_{i=1}^N x_i + \alpha\zeta(x) + \beta\eta(x) - H(x)\right)dx}. \end{aligned} \tag{8.1.3}$$

Let $W = (W_1, \cdots, W_N)$ be a real-valued random vector distributed according to $\mu_N^{\sigma,\alpha,\beta}$ and $g_{\alpha,\beta}$ be the density of random variable

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N (W_i - m).$$

Then in (8.1.3), it holds that

$$A_{ce}^{\zeta,\eta}(\alpha,\beta) - A_{gce}^{\zeta\eta}(\alpha,\beta) = \frac{1}{N}\log g_{\alpha,\beta}(0). \tag{8.1.4}$$

Note also that an application of inverse Fourier transformation yields

$$2\pi g_{\alpha,\beta}(0) = \int_{\mathbb{R}}\mathbb{E}_{\mu_N^{\sigma,\alpha,\beta}}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (W_i - m)\,\xi\right)\right]d\xi.$$

Again, a direct calculation with a help of identity (8.1.1) implies

$$\begin{aligned} 2\pi g_{\alpha,\beta}(0)|_{\alpha,\beta=0} &= \int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\left[\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m)\,\xi\right)\right]dx, \\ 2\pi\left.\frac{d}{d\alpha}g_{\alpha,\beta}(0)\right|_{\alpha,\beta=0} &= \int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\left[\left(\zeta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right]\right)\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m)\,\xi\right)\right]dx, \\ 2\pi\left.\frac{d}{d\beta}g_{\alpha,\beta}(0)\right|_{\alpha,\beta=0} &= \int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\left[\left(\eta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\eta(X)\right]\right)\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m)\,\xi\right)\right]dx, \\ 2\pi\left.\frac{d^2}{d\alpha d\beta}g_{\alpha,\beta}(0)\right|_{\alpha,\beta=0} &= \int_{\mathbb{R}}\mathbb{E}_{\mu_N^\sigma}\Bigg[\left(\zeta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\zeta(X)\right]\right)\left(\eta(X) - \mathbb{E}_{\mu_N^\sigma}\left[\eta(X)\right]\right) \\ &\qquad \times\exp\left(i\frac{1}{\sqrt{N}}\sum_{i=1}^N (X_i - m)\,\xi\right)\Bigg]dx. \end{aligned} \tag{8.1.5}$$

Then a combination of (8.1.2), (8.1.4) and (8.1.5) yields the desired result. □

8.2 Criteria for the logarithmic Sobolev inequality

In this section we state several standard criteria for deducing a LSI. For proofs we refer to the literature. For a general introduction and more comments on the LSI we refer the reader to [Led01b, Led01a, Roy99, BGL14].

Theorem 8.2.1 (Tensorization Principle [Gro75]). *Let μ_1 and μ_2 be probability measures on Euclidean spaces X_1 and X_2 respectively. Suppose that μ_1 and μ_2 satisfy LSI(ρ_1) and LSI(ρ_2) respectively. Then the product measure $\mu_1 \bigotimes \mu_2$ satisfies LSI(ρ), where $\rho = \min\{\rho_1, \rho_2\}$.*

Theorem 8.2.2 (Holley-Stroock Perturbation Principle [HS87]). *Let μ_1 be a probability measure on Euclidean space X and $\delta\psi : X \to \mathbb{R}$ be a bounded function. Define a probability measure μ_2 on X by*

$$\mu_2(dx) := \frac{1}{Z} \exp\left(-\delta\psi(x)\right) \mu_1(dx).$$

Suppose that μ_1 satisfies LSI(ρ_1). Then μ_2 also satisfies LSI with constant

$$\rho_2 = \rho_1 \exp\left(-osc\ \delta\psi\right),$$

where $osc\ \delta\psi := \sup \delta\psi - \inf \delta\psi$.

Theorem 8.2.3 (Bakry-Émery criterion [BE85]). *Let X be a N-dimensional Euclidean space and $H \in C^2(X)$. Define a probability measure μ on X by*

$$\mu(dx) := \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

Suppose there is a constant $\rho > 0$ such that $\operatorname{Hess} H \geq \rho$. More precisely, for all $u, v \in X$,

$$\langle v, \operatorname{Hess} H(u) v\rangle \geq \rho |v|^2.$$

Then μ satisfies LSI(ρ).

Theorem 8.2.4 (Otto-Reznikoff Criterion [OR07]). *Let $X = X_1 \times \cdots \times X_N$ be a direct product of Euclidean spaces and $H \in C^2(X)$. Define a probability measure μ on X by*

$$\mu(dx) := \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

Assume that

- *For each $i \in \{1, \cdots, N\}$, the conditional measures $\mu(dx_i|\bar{x}_i)$ satisfy LSI(ρ_i).*
- *For each $1 \leq i \neq j \leq N$ there is a constant $\kappa_{ij} \in (0, \infty)$ with*

$$|\nabla_i \nabla_j H(x)| \leq \kappa_{ij}. \qquad \text{for all } x \in X.$$

Here, $|\cdot|$ denotes the operator norm of a bilinear form.

- *Define a symmetric matrix* $A = (A_{ij})_{1\le i,j\le N}$ *by*

$$A_{ij} = \begin{cases} \rho_i, & \text{if } i = j \\ -\kappa_{ij}, & \text{if } i \neq j \end{cases}.$$

Assume that there is a constant $\rho \in (0,\infty)$ *with*

$$A \geq \rho \, \mathrm{Id},$$

in the sense of quadratic forms.

Then μ *satisfies LSI*(ρ).

Theorem 8.2.5 (Two-Scale Criterion [GOVW09])**.** *Let* X *and* Y *be Euclidean spaces. Consider a probability measure* μ *on* X *defined by*

$$\mu(dx) := \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

Let $P : X \to Y$ *be a linear operator such that for some* $N \in \mathbb{N}$,

$$PNP^t = \mathrm{Id}_Y .$$

Define

$$\kappa := \max\left\{ \langle \mathrm{Hess}\, H(x) \cdot u, v\rangle : \ u \in Ran(NP^tP), v \in Ran(\mathrm{Id}_X - NP^tP), |u| = |v| = 1 \right\}.$$

Assume that

- $\kappa < \infty$
- *There is* $\rho_1 \in (0,\infty)$ *such that the conditional measure* $\mu(dx|Px = y)$ *satisfies LSI*(ρ_1) *for all* $y \in Y$.
- *There is* $\rho_2 \in (0,\infty)$ *such that the marginal measure* $\bar{\mu} = P_\# \mu$ *satisfies LSI*($\rho_2 N$).

Then μ *satisfies LSI*(ρ)*, where*

$$\rho := \frac{1}{2}\left(\rho_1 + \rho_2 + \frac{\kappa^2}{\rho_1} - \sqrt{\left(\rho_1 + \rho_2 + \frac{\kappa^2}{\rho_1}\right)^2 - 4\rho_1\rho_2} \right) > 0.$$

8.3 Derivatives of coarse-grained Hamiltonian

In this section, we provide the first and second derivative formula for the coarse-grained Hamiltonian. First, we state an explicit formula for $\bar{H}$, obtained in [Men11].

Lemma 8.3.1 (Lemma 1 in [Men11])**.** *For* $z \in X$ *with* $Pz = 0$ *and* $y \in Y$*, define* $H_M(z, y)$ *by*

$$H_M(z,y) := \frac{1}{2}\langle z, (\mathrm{Id} + M)z\rangle + \langle z, MNP^* y\rangle + \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i), \tag{8.3.1}$$

where $M = (M_{ij})_{1\le i,j\le n}$ *is an interaction matrix in the Hamiltonian (cf.* (1.1.1)*). Then*

$$\bar{H}(y) = \frac{1}{2}\langle y, (\mathrm{Id} + PMNP^*)y\rangle_Y - \frac{1}{N}\log\int_{Px=0} \exp\left(-H_M(x,y)\right) \mathcal{L}(dx).$$

Proof of Lemma 8.3.1. For $x \in X$ with $Px = y$ and $y \in Y$, let $z = x - NP^*y$. Recalling the identity $PNP^* = Id_Y$, we have $Pz = 0$. We then write the Hamiltonian H as

$$\begin{aligned}
H(x) &= \sum_{i=1}^{N} \left(\psi(x_i) + \frac{1}{2} \sum_{j:\ 1 \le |j-i| \le R} M_{ij} x_i x_j \right) \\
&= \frac{1}{2} \langle x, (\mathrm{Id} + Mx \rangle + \sum_{i=1}^{N} \psi_b(x_i) \\
&= \frac{1}{2} \langle z + NP^*y, (\mathrm{Id} + M)(z + NP^*y) \rangle + \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i) \\
&= \frac{1}{2} \langle z, (\mathrm{Id} + M) z \rangle + \langle z, NP^*y \rangle + \langle z, MNP^*y \rangle + \frac{1}{2} \langle NP^*y, (\mathrm{Id} + M) NP^*y \rangle \\
&\quad + \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i). \qquad (8.3.2)
\end{aligned}$$

Because $Pz = 0$, we have

$$\langle z, NP^*y \rangle = N \langle Pz, y \rangle_Y = 0. \qquad (8.3.3)$$

It also holds by $PNP^* = \mathrm{Id}_Y$ that

$$\begin{aligned}
\frac{1}{2} \langle NP^*y, (\mathrm{Id} + M) NP^*y \rangle &= \frac{N}{2} \langle y, PNP^*y + PMNP^*y \rangle_Y \\
&= \frac{N}{2} \langle y, (\mathrm{Id} + PMNP^*) y \rangle_Y. \qquad (8.3.4)
\end{aligned}$$

Plugging (8.3.3) and (8.3.4) into (8.3.2) yields

$$H(x) = \frac{N}{2} \langle y, (\mathrm{Id} + PMNP^*) y \rangle_Y + H_M(z, y),$$

and hence

$$\begin{aligned}
\bar{H}(y) &= -\frac{1}{N} \log \int_{Px=y} \exp\left(-H(x)\right) \mathcal{L}(dx) \\
&= \frac{1}{2} \langle y, (\mathrm{Id} + PMNP^*) y \rangle_Y - \frac{1}{N} \log \int_{Pz=0} \exp\left(-H_M(z, y)\right) \mathcal{L}(dz).
\end{aligned}$$

□

Next, we compute the first derivative of the coarse-grained Hamiltonian $\bar{H}$ using Lemma 8.3.1.

Lemma 8.3.2. *For each $l \in [M]$, it holds that*

$$\frac{\partial}{\partial y_l} \bar{H}(y) = \frac{1}{M} y_l + \frac{1}{N} \mathbb{E}_{\mu_{N,m}(dx|y)} \left[\sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} X_i + \sum_{i \in B(l)} \psi_b'(X_i) \right].$$

Proof of Lemma 8.3.2. First of all, noting that

$$\langle y, PMNP^* y\rangle_Y = \frac{1}{N}\sum_{l,n=1}^{M}\sum_{i\in B(l),j\in B(n)} M_{ij} y_l y_n,$$

we have

$$\frac{\partial}{\partial y_l}\left(\frac{1}{2}\langle y, (\mathrm{Id} + PMNP^*)y\rangle_Y\right) = \frac{1}{M}y_l + \frac{1}{N}\sum_{n=1}^{M}\sum_{i\in B(l),j\in B(n)} M_{ij} y_n. \tag{8.3.5}$$

In addition, differentiating (8.3.1) yields

$$\begin{aligned}
&\frac{\partial}{\partial y_l}\left(H_M(x,y)\right)\\
&\quad= \frac{\partial}{\partial y_l}\left(\langle x, MNP^* y\rangle\right) + \frac{\partial}{\partial y_l}\left(\sum_{i=1}^{N}\psi_b(x_i + (NP^*y)_i)\right)\\
&\quad= \sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}x_i + \sum_{i\in B(l)}\psi_b'(x_i + (NP^*y)_i)\\
&\quad= \sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}(x_i + (NP^*y)_i) - \sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}(NP^*y)_i + \sum_{i\in B(l)}\psi_b'(x_i + (NP^*y)_i)\\
&\quad= \sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}(x_i + (NP^*y)_i) + \sum_{i\in B(l)}\psi_b'(x_i + (NP^*y)_i) - \sum_{n=1}^{M}\sum_{i\in B(n),j\in B(l)} M_{ij}y_n.
\end{aligned}$$

As a consequence we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y_l}\left(-\frac{1}{N}\log\int_{Px=0}\exp\left(-H_M(x,y)\right)\mathcal{L}(dx)\right)\\
&\quad= \frac{1}{N}\frac{\int_{Px=0}\frac{\partial}{\partial y_l}\left(H_M(x,y)\right)\exp\left(-H_M(x,y)\right)\mathcal{L}(dx)}{\int_{Px=0}\exp\left(-H_M(x,y)\right)\mathcal{L}(dx)}\\
&\quad= \frac{1}{N}\mathbb{E}_{\mu_{N,m}(dx|y)}\left[\sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}X_i + \sum_{i\in B(l)}\psi_b'(X_i) - \frac{1}{N}\sum_{n=1}^{M}\sum_{i\in B(n),j\in B(l)} M_{ij}y_n\right].
\end{aligned} \tag{8.3.6}$$

Combining (8.3.5) and (8.3.6) with symmetry of M_{ij}, i.e., $M_{ij} = M_{ji}$, we have

$$\frac{\partial}{\partial y_l}\bar{H}(y) = \frac{1}{M}y_l + \frac{1}{N}\mathbb{E}_{\mu_{N,m}(dx|y)}\left[\sum_{i=1}^{N}\sum_{j\in B(l)} M_{ij}X_i + \sum_{i\in B(l)}\psi_b'(X_i)\right].$$

□

The next statement is the second derivatives of the coarse-grained Hamiltonian $\bar{H}$.

Lemma 8.3.3 (Lemma 2 in [Men11]). *For $l, n \in [M]$, we have*

$$\left(\mathrm{Hess}_Y \bar{H}(y)\right)_{ln} = \delta_{ln} + \delta_{ln} \frac{1}{K} \int \sum_{i \in B(l)} \psi_b''(x_i) \mu_{N,m}(dx|y) + \frac{1}{K} \sum_{i \in B(l), j \in B(n)} M_{ij}$$
$$- \frac{1}{K} \operatorname{cov}_{\mu_{N,m}(dx|y)} \left(\sum_{j \in B(l)} \left(\sum_{i=1}^{N} M_{ij} X_i + \psi_b'(X_j) \right), \right.$$
$$\left. \sum_{j \in B(n)} \left(\sum_{i=1}^{N} M_{ij} X_i + \psi_b'(X_j) \right) \right).$$

Lemma 8.3.3 follows from a similar calculations done in the proof of Lemma 8.3.2.

Bibliography

[BE85] Dominique Bakry and M. Emery. Diffusions hypercontractives. *Lecture Notes in Math. Berlin. Mathematical Reviews*, 84, 01 1985.

[Ben73] Edward Bender. Central and local limit theorems applied to asymptotic enumeration. *Journal of Combinatorial Theory, Series A*, 15:91–111, 07 1973.

[BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[BHK82] Jean Bellissard and Raphael Høegh-Krohn. Compactness and the maximal Gibbs state for random gibbs fields on a lattice. *Communications in Mathematical Physics*, 84, 09 1982.

[BL76] Herm Jan Brascamp and Elliott H Lieb. Best constants in Young's inequality, its converse, and its generalization to more than three functions. *Advances in Mathematics*, 20(2):151 – 173, 1976.

[Cap03] Pietro Caputo. Uniform Poincaré inequalities for unbounded conservative spin systems: the non-interacting case. *Stochastic Process. Appl.*, 106(2):223–244, 2003.

[Cha03] D. Chafaï. Glauber versus Kawasaki for spectral gap and logarithmic Sobolev inequalities of some unbounded conservative spin systems. *Markov Process. Related Fields*, 9(3):341–362, 2003.

[CM00] Nicoletta Cancrini and Fabio Martinelli. Comparison of finite volume canonical and grand canonical Gibbs measures under a mixing condition. *Markov Process. Related Fields*, 6(1):23–72, 2000.

[CMR02] N. Cancrini, F. Martinelli, and C. Roberto. The logarithmic Sobolev constant of Kawasaki dynamics under a mixing condition revisited. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(4):385–436, 2002.

[CO17] Nicoletta Cancrini and Stefano Olla. Ensemble dependence of fluctuations: Canonical microcanonical equivalence of ensembles. *Journal of Statistical Physics*, 168(4):707–730, Aug 2017.

[DF16] M. H. Duong and M. Fathi. The two-scale approach to hydrodynamic limits for non-reversible dynamics. *Markov Process. Related Fields*, 22(1):1–36, 2016.

[DM16] Stephen DeSalvo and Georg Menz. A robust quantitative local central limit theorem with applications to enumerative combinatorics and random combinatorial structures. *arXiv:1610.07664*, 2016.

[DMOW18a] Dizdar Deniz, Georg Menz, Felix Otto, and Tianqi Wu. The quantitative hydrodynamic limit of the Kawasaki dynamics. *arXiv:1807.09850*, 2018.

[DMOW18b] Deniz Dizdar, Georg Menz, Felix Otto, and Tianqi Wu. Toward a quantitative theory of the hydrodynamic limit. *arXiv:1807.09857*, 2018.

[Dob68] R. L. Dobrushin. Description of a random field by means of conditional probabilities and conditions for its regularity. *Teor. Verojatnost. i Primenen*, 13:201–229, 1968.

[Dob74] R. L. Dobrushin. Conditions for the absence of phase transitions in one-dimensional classical systems. *Mat. Sb. (N.S.)*, 93(135):29–49, 151, 1974.

[DS85] R. L. Dobrushin and S. B. Shlosman. *Constructive criterion for the uniqueness of Gibbs field*, pages 347–370. Birkhäuser Boston, Boston, MA, 1985.

[DS87] R. L. Dobrushin and S. B. Shlosman. Completely analytical interactions: Constructive description. *Journal of Statistical Physics*, 46(5):983–1014, Mar 1987.

[DT77] R. L. Dobrushin and Brunello Tirozzi. The central limit theorem and the problem of equivalence of ensembles. *Comm. Math. Phys*, 54(2):173–192, Jun 1977.

[Dur10] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[DW90] R. L. Dobrushin and V. Warstat. Completely analytic interactions with infinite values. *Probability Theory and Related Fields*, 84(3):335–359, Sep 1990.

[ER60] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[ER61] P. Erdős and A. Rényi. On the evolution of random graphs. *Bull. Inst. Internat. Statist.*, 38:343–347, 1961.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[FFH07] P. Fronczak, A. Fronczak, and J. A. Hołyst. Phase transitions in social networks. *The European Physical Journal B*, 59(1):133–139, Sep 2007.

[Fri87] József Fritz. On the hydrodynamic limit of a one-dimensional Ginzburg-Landau lattice model. The a priori bounds. *Journal of Statistical Physics*, 47:551–572, 1987.

[Geo95] Hans-Otto Georgii. The equivalence of ensembles for classical systems of particles. *Journal of Statistical Physics*, 80(5):1341–1378, Sep 1995.

[GOVW09] Natalie Grunewald, Felix Otto, Cédric Villani, and Maria G. Westdickenberg. A two-scale approach to logarithmic Sobolev inequalities and the hydrodynamic limit. *Ann. Inst. H. Poincaré Probab. Statist.*, 45(2):302–351, 2009.

[GPV88a] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Comm. Math. Phys.*, 118(1):31–59, 1988.

[GPV88b] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Comm. Math. Phys.*, 118(1):31–59, 1988.

[Gro75] Leonard Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97:1061–1083, 1975.

[HM16] Christopher Henderson and Georg Menz. Equivalence of a mixing condition and the LSI in spin systems with infinite range interaction. *Stochastic Process. Appl.*, 126(10):2877–2912, 2016.

[HS87] Richard Holley and Daniel Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *Journal of Statistical Physics*, 46(5-6):1159–1194, 1987.

[Isi25] Ernst Ising. Contribution to the theory of ferromagnetism. *Z. Phys.*, 31:253–258, 1925.

[KL99] Claude Kipnis and Claudio Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 1999.

[KLM21] Younghak Kwon, Jaehun Lee, and Georg Menz. Equivalence of the grand canonical ensemble and the canonical ensemble of 1d-lattice systems. *Markov Process. Related Fields*, 1(27):63–109, 2021.

[KM18] Younghak Kwon and Georg Menz. Strict convexity of the free energy of the canonical ensemble under decay of correlations. *Journal of Statistical Physics*, 172, 08 2018.

[KM19] Younghak Kwon and Georg Menz. Decay of correlations and uniqueness of the infinite-volume gibbs measure of the canonical ensemble of 1d-lattice systems. *Journal of Statistical Physics*, 176, 08 2019.

[KM20] Younghak Kwon and Georg Menz. Uniform LSI for the canonical ensemble on the 1d-lattice. *ESAIM: PS*, 24:341–373, 2020.

[KMN20] Younghak Kwon, Georg Menz, and Kyeongsik Nam. Hydrodynamic limit of the Kawasaki dynamics on the 1d-lattice with strong, finite-range interaction. *arXiv:2010.01266*, 2020.

[Kos01] Elena Kosygina. The behavior of the specific entropy in the hydrodynamic scaling limit. *Ann. Probab.*, 29(3):1086–1110, 07 2001.

[KS13] Michael Krivelevich and Benny Sudakov. The phase transition in random graphs: a simple proof. *Random Structures Algorithms*, 43(2):131–138, 2013.

[Led01a] M. Ledoux. The concentration of measure phenomenon. *AMS Surveys and Monographs*, 89, 01 2001.

[Led01b] Michel Ledoux. Logarithmic Sobolev inequalities for unbounded spin systems revisted. *Sem. Probab. XXXV, Lecture Notes in Math, Springer*, 1755:167–194, 2001.

[LPY02] C. Landim, G. Panizo, and Horng Tzer Yau. Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(5):739–777, 2002.

[Lu95] Shenglin Lu. Hydrodynamic scaling limits with deterministic initial configurations. *Ann. Probab.*, 23(4):1831–1852, 10 1995.

[LY93] Sheng Lin Lu and Horng Tzer Yau. Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.*, 156(2):399–433, 1993.

[Mar99] Fabio Martinelli. *Lectures on Glauber dynamics for discrete spin models*, pages 93–191. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[Men11] Georg Menz. LSI for Kawasaki dynamics with weak interaction. *Comm. Math. Phys.*, 307(3):817–860, 2011.

[Men14] Georg Menz. The approach of Otto-Reznikoff revisited. *Electron. J. Probab.*, 19:no. 107, 27, 2014.

[MN14] Georg Menz and Robin Nittka. Decay of correlations in 1D lattice systems of continuous spins and long-range interaction. *Journal of Statistical Physics*, 156(2):239–267, 2014.

[MO13] Georg Menz and Felix Otto. Uniform logarithmic Sobolev inequalities for conservative spin systems with super-quadratic single-site potential. *Ann. Probab.*, 41(3B):2182–2224, 2013.

[OR07] Felix Otto and Maria G. Reznikoff. A new criterion for the logarithmic Sobolev inequality and two applications. *Journal of Functional Analysis*, 243(1):121 – 157, 2007.

[Pei36] R. Peierls. On Ising's model of ferromagnetism. *Mathematical Proceedings of the Cambridge Philosophical Society*, 32(3):477–481, 1936.

[Rai03] Martin Raič. Normal approximation by Stein's method. *Proceedings of the Seventh Young Statisticians Meeting*, 21:71–97, 01 2003.

[Rez90] Fraydoun Rezakhanlou. Hydrodynamic limit for a system with finite range interactions. *Comm. Math. Phys.*, 129(3):445–480, 1990.

[Roy99] Gilles Royer. Une initiation aux inégalités de Sobolev logarithmiques. *Cours Spéc., Soc. Math. France*, 1999.

[Rue68] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Comm. Math. Phys.*, 9:267–278, 1968.

[Sel16] Jonathan V. Selinger. *Introduction to the theory of soft matter: from ideal gases to liquid crystals*. Springer, 2016.

[WW90] Mei Wang and Michael Woodroofe. A local limit theorem for sums of dependent random variables. *Statist. Probab. Lett.*, 9(3):207–213, 1990.

[Yau91] Horng Tzer Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Lett. Math. Phys.*, 22(1):63–80, 1991.

[Yau96] Horng Tzer Yau. Logarithmic Sobolev inequality for lattice gases with mixing conditions. *Comm. Math. Phys.*, 181(2):367–408, 1996.

[Yos01] Nobuo Yoshida. The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on the lattice. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(2):223–243, 2001.

[Yos03] N. Yoshida. Phase transition from the viewpoint of relaxation phenomena. *Reviews in Mathematical Physics*, 15(07):765–788, 2003.

[Zeg96] Boguslaw Zegarlinski. The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice. *Comm. Math. Phys.*, 175(2):401–432, 1996.