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Some Results for the Ambient Obstruction Flow

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by

Christopher Lopez

Dissertation Committee:
Professor Jeffrey Streets, Chair
Professor Patrick Guidotti
Professor Xiangwen Zhang

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ABSTRACT OF THE DISSERTATION

Some Results for the Ambient Obstruction Flow

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2017

Professor Jeffrey Streets, Chair

We establish several results for the ambient obstruction flow (AOF), a parabolic flow of Riemannian metrics introduced by Bahuaud-Helliwell which is based on the Fefferman-Graham ambient obstruction tensor. The flow may be regarded as a higher order analogue of Ricci flow, and the critical metrics for this flow may be regarded as generalizations of Einstein metrics. First, we obtain local L2 smoothing estimates for the curvature tensor along AOF and use them to prove pointwise smoothing estimates for the curvature tensor. We use the pointwise smoothing estimates to show that the curvature must blow up for a finite time singular solution to AOF. We also use the pointwise smoothing estimates to prove a compactness theorem for a sequence of solutions to AOF with bounded C0 curvature norm and injectivity radius bounded from below at one point. The compactness theorem allows us to obtain a singularity model from a finite time singular solution to AOF and to characterize the behavior at infinity of a nonsingular solution to AOF. Our final result is a rigidity theorem, which states that under suitable conditions a metric that is critical for AOF and has small scale-invariant integral energy has vanishing Riemann curvature tensor.

Chapter 1

Introduction

1.1 Introduction

The uniformization theorem ensures that for a compact two dimensional Riemannian manifold (M, g), there is a metric $\tilde{g}$ conformal to g for which $(M, \tilde{g})$ has constant sectional curvature equal to K. Moreover, the sign of K can be determined via the Gauss-Bonnet theorem. In higher dimensions, curvature functionals have been used with great success to define and locate optimal metrics in higher dimensions; see [32]. One conformally invariant curvature functional for a 4-dimensional Riemannian manifold (M, g) is given by

$$\mathcal{F}_W^4(g) = \int_M |W_g|^2 \, dV_g,$$

where W_{ijkl} is the Weyl tensor. The negative gradient of $\mathcal{F}_W^4$ is the Bach tensor B_{ij} defined as

$$B_{ij} = -\nabla^k \nabla^l W_{kijl} - \tfrac{1}{2} R^{kl} W_{kijl}.$$

The study of critical metrics for $\mathcal{F}^4_W$, ie. Bach-flat metrics, has been fruitful. The class of Bach-flat metrics contains, as shown in [5], familiar metrics such as locally conformally Einstein metrics and scalar flat (anti) self-dual metrics.

Another conformally invariant functional for a 4-dimensional Riemannian manifold (M, g) is given by

$$\mathcal{F}^4_Q(g) = \int_M Q(g)\, dV_g,$$

where $Q(g)$ is a scalar quantity introduced by Branson in [7] called the Q curvature. Via the Chern-Gauss-Bonnet theorem, this functional is related to $\mathcal{F}^4_W$ by $\mathcal{F}^4_Q = 8\pi^2\chi(M) - \frac{1}{4}\mathcal{F}^4_W$. The Bach tensor is also the gradient of $\mathcal{F}^4_Q$. Unlike the Weyl tensor, the Q curvature is not pointwise conformally covariant.

One can generalize the Q curvature to a scalar quantity defined on n dimensional Riemannian manifolds (M, g), where n is even. Consider the functionals defined for n even by

$$\mathcal{F}^n_Q(g) = \int_M Q(g)\, dV_g.$$

These functionals are conformally invariant. The gradient of $\mathcal{F}^n_Q$ is a symmetric 2-tensor $\mathcal{O}$, introduced by Fefferman and Graham in [18], called the ambient obstruction tensor. This tensor arises in physics: for example, Anderson and Chruściel use $\mathcal{O}$ in [1] to construct global solutions of the vacuum Einstein equation in even dimensions. In dimension 4, $\mathcal{O}$ is just the Bach tensor. The ambient obstruction tensor is conformally covariant in n dimensions. This is in contrast to the n dimensional generalization of the Bach tensor, which is only conformally covariant in dimension 4. This fact follows from a result in Graham-Hirachi [21] stating that in even dimensions 6 and greater, the only conformally covariant tensors essentially are W and $\mathcal{O}$. Extending the 4-dimensional case, Fefferman and Graham showed in [19] that $\mathcal{O}$ vanishes for Einstein metrics for all even dimensions. However, there also exist non conformally Einstein metrics for which $\mathcal{O} = 0$, as shown by Gover and Leitner in [20]. The conformal covariance of $\mathcal{O}$ and the fact that obstruction flat metrics generalize conformally Einstein metrics suggest that studying the critical points of $\mathcal{F}_Q^n$ via its gradient flow may aid in the study of optimal metrics on M. Our main goal is to establish fundamental results for this gradient flow.

1.2 Main Results

1.2.1 Fundamental Results

We will continue the study of a variant of the gradient flow of $\mathcal{F}_Q^n$, that was introduced by Bahuaud and Helliwell in [3], establishing fundamental results. This flow, which we will refer to as the **ambient obstruction flow** (AOF), is defined for a family of metrics $g(t)$ on a smooth manifold M by

$$\begin{cases} \partial_t g = (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g \\ g(0) = h. \end{cases} \tag{1.1}$$

The conformal term involving the scalar curvature was added in order to counteract the invariance of $\mathcal{O}$ under the action of the conformal group on the space of metrics on M. In the papers [3, 4] they proved the short time existence and uniqueness, respectively, of solutions to AOF given by (1.1) when M is compact. Mantegazza and Martinazzi provided an existence proof for parabolic quasilinear PDE on compact manifolds in [35]. Kotschwar has given in [30] an alternate uniqueness proof via a classical energy argument without using the DeTurck trick.

Gradient flows have been studied extensively since Hamilton in [23, 25, 24] and Perelman in [37, 38, 39] (expositions are given in [9, 29, 36]) used the Ricci flow to study the geometry of 3-manifolds. In the past fifteen years, these have begun to include higher order flows. Mantegazza studied a family of higher order mean curvature flows in [34], Kuwert-Schätzle studied the gradient flow of the Willmore functional in [31], Streets studied the gradient flow of $\int_M |\mathrm{Rm}|^2$ in [42], Chen-He studied the Calabi flow in [11, 12], and Kişisel-Sarıoğlu-Tekin studied the Cotton flow in [28]. Bour studied the gradient flows of certain quadratic curvature functionals in [6], including some variants of $\int_M |W|^2$.

Our first result gives pointwise smoothing estimates for the C^0 norms of the derivatives of the curvature. Since the AOF PDE (1.1) is of order n, the maximum principle cannot be used to obtain these estimates. Instead, we first use interpolation inequalities derived by Kuwert and Schätzle in [31] in order to derive local integral Bernstein-Bando-Shi-type smoothing estimates. Then, we use a blowup argument adapted from Streets [44] in order to convert the integral smoothing estimates to pointwise smoothing estimates, as stated in the following theorem. During the proof, we use the local integral smoothing estimates to take a local subsequential limit of renormalized metrics.

Theorem 1.2.1. *Let $m \geq 0$ and $n \geq 4$. There exists a constant $C = C(m,n)$ so that if $(M^n, g(t))$ is a complete solution to AOF on $[0,T]$ satisfying*

$$\max\left(1, \sup_{M\times[0,T]} |\mathrm{Rm}|\right) \leq K,$$

then for all $t \in (0,T]$,

$$\sup_M |\nabla^m \mathrm{Rm}|_{g(t)} \leq C\left(K + t^{-\frac{2}{n}}\right)^{1+\frac{m}{2}}.$$

We obtain from the pointwise smoothing estimates two additional theorems. The first theorem gives an obstruction to the long time existence of the flow. Since the pointwise smoothing estimates do not require that the Sobolev constant be bounded on $[0, T)$, we rule out that the manifold collapses with bounded curvature.

Theorem 1.2.2. *Let $g(t)$ be a solution to the AOF on a compact manifold M that exists on a maximal time interval $[0, T)$ with $0 < T \leq \infty$. If $T < \infty$, then we must have*

$$\limsup_{t \uparrow T} \|\mathrm{Rm}\|_{C^0(g(t))} = \infty.$$

The second theorem allows us to extract convergent subsequences from a sequence of solutions to AOF with uniform C^0 curvature bound and uniform injectivity radius lower bound. We prove this in section 4 by using the Cheeger-Gromov compactness theorem to obtain subsequential convergence of solutions at one time. Then, after extending estimates on the covariant derivatives of the metrics from one time to the entire time interval, we obtain subsequential convergence over the entire time interval.

Theorem 1.2.3. *Let $\{(M_k^n, g_k(t), O_k)\}_{k \in \mathbb{N}}$ be a sequence of complete pointed solutions to AOF for $t \in (\alpha, \omega)$, with $t_0 \in (\alpha, \omega)$, such that*

1. $|\mathrm{Rm}(g_k)|_{g_k} \leq C_0$ *on $M_k \times (\alpha, \omega)$ for some constant $C_0 < \infty$ independent of k*

2. $\mathrm{inj}_{g_k(t_0)}(O_k) \geq \iota_0$ *for some constant $\iota_0 > 0$.*

Then there exists a subsequence $\{j_k\}_{k \in \mathbb{N}}$ such that $\{(M_{j_k}, g_{j_k}(t), O_{j_k})\}_{k \in \mathbb{N}}$ converges in the sense of families of pointed Riemannian manifolds to a complete pointed solution to AOF $(M_\infty^n, g_\infty(t), O_\infty)$ defined for $t \in (\alpha, \omega)$ as $k \to \infty$.

We use this compactness theorem to prove two corollaries. For a compact Riemannian manifold (M, g), let $C_S(M, g)$ denote the L^2 Sobolev constant of (M, g), defined as the smallest constant C_S such that

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq C_S\big(\|\nabla f\|^2_{L^2} + V^{-\frac{2}{n}}\|f\|^2_{L^2}\big),$$

where $V = \mathrm{vol}(M,g)$, for all $f \in C^1(M)$. The following result states that if the Sobolev constant and the integral of Q-curvature are bounded along the flow, there exists a sequence of renormalized solutions to AOF that converge to a singularity model.

Theorem 1.2.4. *Let $(M^n, g(t))$, $n \geq 4$, be a compact solution to AOF that exists on a maximal time interval $[0,T)$ with $T < \infty$. Suppose that $\sup\{C_S(M, g(t)) : t \in [0,T)\} < \infty$. Let $\{(x_i,t_i)\}_{i\in\mathbb{N}} \subset M \times [0,T)$ be a sequence of points satisfying $t_i \to T$, $|\mathrm{Rm}(x_i,t_i)| = \sup\{|\mathrm{Rm}(x,t)| : (x,t) \in M \times [0,t_i]\}$, and $\lambda_i \to \infty$, where $\lambda_i = |\mathrm{Rm}(x_i,t_i)|$. Then the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = \lambda_i g(t_i + \lambda_i^{-\frac{n}{2}} t), \quad t \in [-\lambda_i^{\frac{n}{2}} t_i, 0]$$

subsequentially converges in the sense of families of pointed Riemannian manifolds to a nonflat, noncompact complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty, 0]$. Moreover, if $n = 4$ or

$$\sup_{t\in[0,T)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $\mathcal{O}(g_\infty(t)) \equiv 0$ for all $t \in (-\infty, 0]$.

The next result states that if a nonsingular solution to AOF does not collapse at time ∞ and the integral of Q-curvature is bounded along the flow, there exists a sequence of times $t_i \to \infty$ for which $g(t_i)$ converges to an obstruction flat metric. We note that in cases (2) and (3), the boundedness of the integral of the Q curvature along the flow implies that $g_\infty(t)$ is obstruction flat. However, this does not imply that $\partial_t g_\infty = 0$. Rather,

$\partial_t g_\infty = (-1)^{n/2} C(n)(\Delta^{\frac{n}{2}-1} R) g_\infty$, i.e. the metric is still flowing by the conformal term of AOF within the conformal class of $g_\infty(0)$.

Theorem 1.2.5. *Let $(M, g(t))$ be a compact solution to AOF on $[0,\infty)$ such that*

$$\sup_{t\in[0,\infty)} \|\mathrm{Rm}\|_{C^0(g(t))} < \infty.$$

Then exactly one of the following is true:

1. *M collapses when $t = \infty$, i.e.*

$$\lim_{t\to\infty} \inf_{x\in M} \mathrm{inj}_{g(t)}(x) = 0.$$

2. *There exists a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0,\infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty)$$

subsequentially converges in the sense of pointed Riemannian manifolds to a complete noncompact finite volume pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty,\infty)$. If $n = 4$ or

$$\sup_{t\in[0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $g_\infty(t)$ is obstruction flat for all $t \in (-\infty,\infty)$.

3. *There exists a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0,\infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty)$$

subsequentially converges in the sense of pointed Riemannian manifolds to a compact pointed solution $(M_\infty, g_\infty(t), x_\infty)$ *to AOF defined for* $t \in (-\infty, \infty)$*, where* M_∞ *is diffeomorphic to* M*. If* $n = 4$ *or*

$$\sup_{t\in[0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $g_\infty(t)$ *is obstruction flat for all* $t \in (-\infty, \infty)$ *and there exists a family of metrics* $\hat{g}_\infty(t)$ *conformal to* $g_\infty(t)$ *for all* $t \in (-\infty, \infty)$*, with* $\hat{g}_\infty(t) = \hat{g}_\infty(0)$ *for all* $t \in (-\infty, \infty)$*, such that* $\hat{g}_\infty(0)$ *is obstruction flat and has constant scalar curvature.*

1.2.2 Rigidity Result

We can gain further insight into noncompact singularity models for AOF by studying the rigidity of such spaces. The rigidity of obstruction - flat metrics has been studied in the past twenty years, especially in dimension four, in which the ambient obstruction tensor and the Bach tensor coincide. Let (M, g) be a Riemannian manifold. The **Bach tensor** in dimension $n \geq 4$ is given by

$$B_{ij} = \tfrac{1}{n-3}\nabla^k\nabla^l W_{kijl} + \tfrac{1}{n-2}R^{kl}W_{kijl}.$$

We note that, in all dimensions greater than 3, the Bach tensor is (derivative) order 4 in g, whereas the ambient obstruction tensor is, in dimension n, order n in g. Let $Y = Y_{[g]}$ denote the **Yamabe constant** of the conformal class $[g]$:

$$Y_{[g]} = \inf_{\substack{u\in C_0^\infty(M)\\ u\neq 0}} \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)}Ru^2)\, dV}{\left(\int_M u^{\frac{2n}{n-2}}\, dV\right)^{\frac{n-2}{n}}}.$$

Here, all geometric objects are with respect to the conformal representative g. Kim proved in [27] that if (M,g) is a noncompact complete Bach - flat Riemannian 4-manifold with (the scalar curvature) $R=0$ and $Y>0$, then there exists $\epsilon_0>0$ such that if $\|\mathrm{Rm}\|_2^2<\epsilon_0$, then $\mathrm{Rm}=0$.

Let $\mathring{\mathrm{Rm}}$ denote the **traceless curvature tensor**:

$$(\mathring{\mathrm{Rm}})_{ijkl}=R_{ijkl}-\tfrac{R}{12}(g_{ik}g_{jl}-g_{il}g_{jk}).$$

Streets proved in [43] a similar result labeled Theorem 3, replacing Rm with $\mathring{\mathrm{Rm}}$ and the Yamabe constant with the Sobolev constant, as follows. If (M,g) is a noncompact, complete 4-manifold that is critical for the L^2 flow, and satisfies $R=0$ and $C_S<\infty$, then there exists $\epsilon_0>0$ such that if $\|\mathring{\mathrm{Rm}}\|_2^2<\epsilon_0$, then $\mathrm{Rm}=0$.

Chu generalized in [16] the result of Kim given above by showing that if (M,g) is a noncompact complete Bach - flat Riemannian 4-manifold with $R\geq 0$ and $Y>0$, then there exists $\epsilon_0>0$ such that if $\|\mathring{\mathrm{Rm}}\|_2^2<\epsilon_0$, then (M,g) has constant sectional curvature. Chu and Feng generalized further to n-dimensional Riemannian manifolds:

Theorem 1.2.6 (Chu and Feng [17], Theorem 1.1)**.** *Let (M^n,g) $(n\geq 4)$ be an n-dimensional complete noncompact Bach - flat Riemannian manifold with constant scalar curvature and positive Yamabe constant. Assume that $\int_{M^n}|\mathring{\mathrm{Rm}}|^2\,dV_g$ is finite when $n\geq 5$ and one of (i)-(iii) holds:*

(i) The scalar curvature $R>0$.

(ii) The scalar curvature $R=0$.

(iii) The scalar curvature $R<0$ and $n>9$.

Then there exists a small number c_1 such that if $\int_{M^n} |\mathring{\mathrm{Rm}}|^{n/2}\, dV_g < c_1$, then (M^n, g) is a space of constant curvature.

Streets used the rigidity results proved in [43] to rule out bubbles (curvature is concentrated in a small ball) along the L^2 flow in the proof of his convergence result, therein labeled Theorem 1. This theorem states that 4-manifolds with positive Yamabe constant and small traceless curvature tensor L^2 energy converge under the L^2 flow to a spherical space form. We believe that our rigidity results may be similarly useful in a proof of a sphere theorem for AOF.

The next theorem states that if noncompact singularity models possess sufficiently small energy, they are actually flat. Let (M^n, g) be a complete noncompact Riemannian manifold, where n is even and $n \geq 4$. Denote the open geodesic ball of radius ρ about $x \in M$ by $B(x, \rho, g)$. Also let $S(\rho) = \partial B(x, \rho)$ and $A(a, b) = B(x, b) \setminus \overline{B(x, a)}$. Define $\|\cdot\|_p$ for $1 \leq p \leq \infty$ by $\|\cdot\|_p = \|\cdot\|_{L^p(M,g)}$.

Definition 1.2.7. The L^2 **Sobolev constant** of a noncompact Riemannian manifold (M, g) is the smallest C_S such that $\|f\|^2_{\frac{2n}{n-2}} \leq C_S \|\nabla f\|_2^2$ for all $f \in C_c^1(M)$.

Definition 1.2.8. We say that (M, g) possesses a **volume growth upper bound** if there exist $x_0 \in M$ and $C_V > 0$ such that $\mathrm{vol}\, B(x_0, \rho, g) < C_V \cdot \rho^n$ for all $\rho > 0$.

Definition 1.2.9. Fix $x_0 \in M$. If $x, y \in M$, let $d_g(x, y)$ denote the distance between x and y with respect to the metric g. Define ρ by $\rho(x) = d_g(x_0, x)$ for $x \in M$. We say that (M, g) has **quadratic curvature decay** if there exists $C_Q > 0$ such that $|\mathrm{Rm}| \leq C_Q \cdot \rho^{-2}$ on M.

Theorem 1.2.10. *Let (M^n, g) be a complete noncompact Riemannian manifold. Suppose that n is even and $n \geq 6$. Assume that*

1. *(M, g) is obstruction-flat and has constant scalar curvature.*
2. *The L^2 Sobolev constant C_S of (M, g) is bounded.*

3. (M,g) *possesses a volume growth upper bound.*

4. There exists $K>0$ *such that* $\|\mathrm{Rm}\|_\infty \le K$.

5. (M,g) *has quadratic curvature decay.*

Then there exists $\epsilon_0 = \epsilon_0(n, C_S) > 0$ *such that if* $\|\mathrm{Rm}\|_{\frac{n}{2}} < \epsilon_0$*, then* (M,g) *is flat.*

In dimension 4, we can omit the volume growth upper bound, the L^∞ bound on Rm, and the quadratic curvature decay on Rm to obtain the following stronger result:

Theorem 1.2.11. *Let* (M^4, g) *be a complete noncompact Riemannian* 4 *- manifold. Assume that*

1. (M,g) *is obstruction-flat and has constant scalar curvature.*

2. The L^2 *Sobolev constant* C_S *of* (M,g) *is bounded.*

Then there exists $\epsilon_0 = \epsilon_0(C_S) > 0$ *such that if* $\|\mathrm{Rm}\|_2 < \epsilon_0$*, then* (M,g) *is flat.*

By assumption, we can choose $x_0 \in M$ for which there exists $C_V > 0$ such that $\mathrm{vol}\, B(x_0,\rho) < C_V \cdot \rho^n$ for all ρ. We also assume that at the same point x_0, there exists $C_Q > 0$ such that $|\mathrm{Rm}| \le C_Q \cdot \rho^{-2}$ on M. We define a cutoff function φ for each $R > 0$ by

(a) $\varphi \equiv 1$ on $B(x_0, R)$

(b) $\varphi \equiv 0$ on $M \setminus B(x_0, 2R)$

(c) $\|\nabla\varphi\|_\infty < \Lambda\rho^{-1}$ for all $\rho > 0$, where $\Lambda > 0$ is independent of ρ.

We prove our rigidity results via a Liouville theorem type argument, using integral estimates, in a manner similar to the method of proof in Streets [43]. First, we estimate $\|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|_2^2$

by $\langle \Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle$ and lower order terms. This estimate allows us to apply the fact that M is obstruction-flat and has constant scalar curvature to replace $\langle \Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle$ with a sum of lower order terms in Rm. The various lower order terms can be estimated via interpolation for $R \gg 1$ by

$$C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2}, \tag{1.2}$$

where $C_1 = C_1(n, s, C_S)$ and $C_2 = C_2(n, \Lambda, C_Q, C_V)$.

We explain the dependence of C_1, C_2 on the constants C_S, Λ, C_V, C_Q. We employ interpolation results that we derived from a Sobolev inequality and interpolation results (dependent on Λ) from Kuwert and Schätzle [31]. A bound on the Sobolev constant is needed to generalize the Sobolev inequality from $\mathbb{R}^n$ to complete noncompact manifolds. Since we only assumed a bound on $\|\mathrm{Rm}\|_{n/2}$, we used our volume growth upper bound to control $\|\mathrm{Rm}\|_\alpha$ for $\alpha < \frac{n}{2}$. We used these two types of estimates in the derivation of our interpolation results. We also needed to estimate via interpolation terms in Rm containing derivatives of orders at most $n-4$. While the Sobolev constant bound and volume growth upper bound were sufficient to estimate terms of derivative order at most $\frac{n}{2}-1$, we were not able to estimate in this fashion terms with orders between $\frac{n}{2}-1$ and $n-4$. Instead, we integrate by parts, which eventually yields a sum of integrals whose integrands are of order at most $\frac{n}{2}-1$. This process generates several error terms containing $\nabla\varphi$. The quadratic curvature decay bound induces pointwise decay bounds for all derivatives of Rm, which suffice to control these error terms.

We choose ϵ_0 small enough in inequality (1.2) to allow us to bound $\|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|_2^2$ by $C_2\epsilon_0 R^{-2}$; ϵ_0 only depends on n and C_S (we choose s sufficiently large so that the exponent of φ remains nonnegative throughout the proof). Our bound on $\|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|_2^2$ by $C_2\epsilon_0 R^{-2}$ implies that we can bound $\|\mathrm{Rm}\|_n^2$ by $C_2\epsilon_0 R^{-2}$ as well. We estimate $\|\mathrm{Rm}\|_n^2$ by $\|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|_2^2$ and $C_2\epsilon_0 R^{-2}$

via a Sobolev inequality. Then, our previous estimate of $\|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|_2^2$ by $C_2\epsilon_0 R^{-2}$ allows us to bound $\|\mathrm{Rm}\|_n^2$ by $C_2\epsilon_0 R^{-2}$. Letting $R \to \infty$, we conclude that $\|\mathrm{Rm}\|_n^2 = 0$ and M is flat.

1.3 Background

1.3.1 Q Curvature

Here we recall a description of Q curvature given by Chang et al. in [10]. The Q curvature was introduced in four dimensions by Riegert in [41] and Branson-Ørsted in [8] and in even dimensions by Branson in [7]. It is a scalar quantity defined on an even dimensional Riemannian manifold (M^n, g). If $n = 2$, we define Q to be $Q = -\frac{1}{2}R = -K$, where K is the Gaussian curvature of M. The Gauss-Bonnet theorem gives $\int Q\,dV = -2\pi\chi(M)$. The Q curvature of a metric $\tilde{g} = e^{2f}g$ is given by $e^{2f}\widetilde{Q} = Q + \mathscr{P}f$, where the Paneitz operator $\mathscr{P}$ introduced by Graham-Jenne-Mason-Sparling in [22] is given by $\mathscr{P}f = \Delta f$. If $n = 4$, we define Q to be

$$Q = -\tfrac{1}{6}\Delta R - \tfrac{1}{2}R^{ab}R_{ab} + \tfrac{1}{6}R^2.$$

The Chern-Gauss-Bonnet theorem gives

$$\int Q\,dV = 8\pi^2\chi(M) - \tfrac{1}{4}\int |W|^2\,dV.$$

In particular, if M is conformally flat, then $\int Q\,dV = 8\pi^2\chi(M)$. The Q curvature of a metric $\tilde{g} = e^{2f}g$ is given by $e^{4f}\widetilde{Q} = Q + \mathscr{P}f$, where the Paneitz operator $\mathscr{P}$ is given by

$$\mathscr{P}f = \nabla_a[\nabla^a\nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab}]\nabla_b f.$$

In general when n is even, we are only able to write down the highest order terms of Q and $\mathscr{P}$:

$$Q = -\tfrac{1}{2(n-1)}\Delta^{\frac{n}{2}-1}R + \text{lots}, \quad \mathscr{P}f = \Delta^{\frac{n}{2}}f + \text{lots}.$$

Nonetheless, Q still has nice conformal properties. Under a conformal change of metric $\tilde{g} = e^{2f}g$, we have $e^{nf}\widetilde{Q} = Q + \mathscr{P}f$. The integral of Q is conformally invariant. In particular, if M is locally conformally flat, we have an analogue of the Gauss-Bonnet theorem:

$$\int Q\,dV = (-1)^{\frac{n}{2}}(\tfrac{n}{2}-1)!\,2^{n-1}\pi^{\frac{n}{2}}\chi(M).$$

1.3.2 Ambient Obstruction Tensor

Fefferman and Graham proposed in [18] a method to determine the conformal invariants of a manifold from the pseudo-Riemannian invariants of an ambient space it is embedded into. They introduced the **ambient obstruction tensor** $\mathcal{O}$ as an obstruction to such an embedding. They subsequently provided a detailed description of the properties of $\mathcal{O}$ in their monograph [19].

We define several tensors that we will use to express $\mathcal{O}$. The Schouten tensor A, Cotton tensor C, and Bach tensor B are defined as

$$\mathsf{A}_{ij} = \tfrac{1}{n-2}\big(R_{ij} - \tfrac{1}{2(n-1)}Rg_{ij}\big), \quad C_{ijk} = \nabla_k\mathsf{A}_{ij} - \nabla_j\mathsf{A}_{ik}, \quad B_{ij} = \nabla^k C_{ijk} - \mathsf{A}^{kl}W_{kijl}.$$

We obtain via the identity $\nabla^l\nabla^k W_{kijl} = (3-n)\nabla^k C_{ijk}$ that

$$B_{ij} = \tfrac{1}{3-n}\nabla^l\nabla^k W_{kijl} + \tfrac{1}{2-n}R^{kl}W_{kijl}.$$

We define the notation $P_k^m(A)$ for a tensor A by

$$P_k^m(A) = \sum_{i_1+\cdots+i_k=m} \nabla^{i_1} A * \cdots * \nabla^{i_k} A.$$

The following result describes $\mathcal{O}$. The form of the lower order terms is implied by the proofs.

Theorem 1.3.1 (Fefferman-Graham [19], Theorem 3.8; Graham-Hirachi [21], Theorem 2.1)**.** *Let $n \geq 4$ be even. The obstruction tensor $\mathcal{O}_{ij}$ of g is independent of the choice of ambient metric $\tilde{g}$ and has the following properties:*

1. *$\mathcal{O}$ is a natural tensor invariant of the metric g; ie. in local coordinates the components of $\mathcal{O}$ are given by universal polynomials in the components of g, g^{-1}, and the curvature tensor of g and its covariant derivatives, and can be written just in terms of the Ricci curvature and its covariant derivatives. The expression for $\mathcal{O}_{ij}$ takes the form*

$$\begin{aligned} \mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}(\Delta \mathsf{A}_{ij} - \nabla_j \nabla_i \mathsf{A}_k{}^k) + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\ &= \frac{1}{3-n} \Delta^{\frac{n}{2}-2} \nabla^l \nabla^k W_{kijl} + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}), \end{aligned}$$

   where $\Delta = \nabla^i \nabla_i$ and lots *denotes quadratic and higher terms in curvature involving fewer derivatives.*

2. *One has $\mathcal{O}_i{}^i = 0$ and $\nabla^j \mathcal{O}_{ij} = 0$.*

3. *$\mathcal{O}_{ij}$ is conformally invariant of weight $2-n$; ie. if $0 < \Omega \in C^\infty(M)$ and $\hat{g}_{ij} = \Omega^2 g_{ij}$, then $\hat{\mathcal{O}}_{ij} = \Omega^{2-n} \mathcal{O}_{ij}$.*

4. *If g_{ij} is conformal to an Einstein metric then $\mathcal{O}_{ij} = 0$.*

C.R. Graham and K. Hirachi express the gradient of Q in terms of $\mathcal{O}$:

Theorem 1.3.2 ([21], Theorem 1.1). *If $g(t)$ is a 1-parameter family of metrics on a compact manifold M of even dimension $n \geq 4$ and $h = \partial_t|_{t=0}\, g(t)$, then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int_M Q(g(t))\, dV_{g(t)} = (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \big\langle \mathcal{O}(g(0)), h \big\rangle \, dV_{g(0)}.$$

Define the adjusted ambient obstruction tensor $\widehat{\mathcal{O}}$ to be

$$\widehat{\mathcal{O}} = (-1)^{\frac{n}{2}} \mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} (\Delta^{\frac{n}{2}-1} R) g. \tag{1.3}$$

We rewrite $\widehat{\mathcal{O}}$ in terms of the Ricci and scalar curvatures.

Proposition 1.3.3. *If (M, g) is a Riemannian manifold, then*

$$\mathcal{O} = \Delta^{\frac{n}{2}-1} \mathsf{A} - \frac{1}{2(n-1)} \Delta^{\frac{n}{2}-2} \nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \tag{1.4}$$

$$\widehat{\mathcal{O}} = \frac{(-1)^{\frac{n}{2}}}{n-2} \Delta^{\frac{n}{2}-1} \mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)} \Delta^{\frac{n}{2}-2} \nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

Proof. First, we re-express $\mathcal{O}$:

$$\begin{aligned} \mathsf{A}_k{}^k &= \tfrac{1}{n-2} \left[g^{jk} R_{kj} - \tfrac{1}{2(n-1)} R g^{jk} g_{kj} \right] \\ &= \tfrac{1}{n-2} \left[R - \tfrac{n}{2(n-1)} R \right] \\ &= \tfrac{1}{2(n-1)} R \end{aligned}$$

and

$$\mathcal{O}_{ij} = \Delta^{\frac{n}{2}-2} (\Delta \mathsf{A}_{ij} - \nabla_j \nabla_i \mathsf{A}_k{}^k) + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm})$$

$$= \Delta^{\frac{n}{2}-1}\mathsf{A}_{ij} - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla_j\nabla_i R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

Next, we re-express $\widehat{\mathcal{O}}$ using (1.4):

$$\begin{aligned}
\widehat{\mathcal{O}} &= (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g \\
&= (-1)^{\frac{n}{2}}\Delta^{\frac{n}{2}-1}\mathsf{A} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&\quad + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g \\
&= \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R \\
&\quad + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).
\end{aligned}$$

□

Chapter 2

Short Time Existence and Uniqueness

2.1 Preliminaries

We collect some facts about Riemannian manifolds that will be used to derive the evolution equations.

Lemma 2.1.1. *(Hamilton [23], Lemma 7.2) On any Riemannian manifold, the following identity holds:*

$$\Delta R_{jklm} = \nabla_j \nabla_m R_{lk} - \nabla_j \nabla_l R_{mk} + \nabla_k \nabla_l R_{mj} - \nabla_k \nabla_m R_{lj} + \mathrm{Rm}^{*2}.$$

We prove a proposition that allows us to move k covariant derivatives past l Laplacians.

Proposition 2.1.2. *If A is a tensor on a Riemannian manifold and $k, l \geq 1$, then*

$$\nabla^k \Delta^l A = \Delta^l \nabla^k A + \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i} \mathrm{Rm} * \nabla^i A.$$

Proof. First we claim that $\nabla\Delta^l A = \Delta^l \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i}\mathrm{Rm} * \nabla^i A$. For any tensor A,

$$\begin{aligned}
\nabla\Delta A &= \nabla_i \nabla^j \nabla_j A \\
&= \nabla^j \nabla_i \nabla_j A + \mathrm{Rm} * \nabla A \\
&= \nabla^j \nabla_j \nabla_i A + \nabla\mathrm{Rm} * A + \mathrm{Rm} * \nabla A \\
&= \Delta\nabla A + \nabla\mathrm{Rm} * A + \mathrm{Rm} * \nabla A.
\end{aligned}$$

Suppose the claim is true for $l-1$. Then

$$\begin{aligned}
\nabla^2 \sum_{i=0}^{2l-3} \nabla^{2l-3-i}\mathrm{Rm} * \nabla^i A) &= \nabla \sum_{i=0}^{2l-3} (\nabla^{2l-2-i}\mathrm{Rm} * \nabla^i A + \nabla^{2l-3-i}\mathrm{Rm} * \nabla^{i+1} A) \\
&= \nabla \left(\sum_{i=0}^{2l-3} \nabla^{2l-2-i}\mathrm{Rm} * \nabla^i A + \mathrm{Rm} * \nabla^{2l-2} A \right) \\
&= \sum_{i=0}^{2l-3} (\nabla^{2l-1-i}\mathrm{Rm} * \nabla^i A + \nabla^{2l-2-i}\mathrm{Rm} * \nabla^{i+1} A) \\
&\qquad + \nabla\mathrm{Rm} * \nabla^{2l-2} A + \mathrm{Rm} * \nabla^{2l-1} A \\
&= \sum_{i=0}^{2l-3} \nabla^{2l-1-i}\mathrm{Rm} * \nabla^i A + \nabla\mathrm{Rm} * \nabla^{2l-2} A + \mathrm{Rm} * \nabla^{2l-1} A \\
&= \sum_{i=0}^{2l-1} \nabla^{2l-1-i}\mathrm{Rm} * \nabla^i A.
\end{aligned}$$

Next,

$$\begin{aligned}
\nabla\Delta^l A &= \nabla\Delta\Delta^{l-1} A \\
&= \Delta\nabla\Delta^{l-1} A + \nabla\mathrm{Rm} * \nabla^{2l-2} A + \mathrm{Rm} * \nabla^{2l-1} A \\
&= \Delta \left(\Delta^{l-1}\nabla A + \sum_{i=0}^{2l-3} \nabla^{2l-3-i}\mathrm{Rm} * \nabla^i A \right) + \nabla\mathrm{Rm} * \nabla^{2l-2} A + \mathrm{Rm} * \nabla^{2l-1} A \\
&= \Delta^l \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i}\mathrm{Rm} * \nabla^i A + \nabla\mathrm{Rm} * \nabla^{2l-2} A + \mathrm{Rm} * \nabla^{2l-1} A
\end{aligned}$$

$$= \Delta^l \nabla A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \mathrm{Rm} * \nabla^i A.$$

We have proved the claim. Assume the proposition holds for $k-1$. Then

$$\begin{aligned}
\nabla \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \mathrm{Rm} * \nabla^i A &= \sum_{i=0}^{2l+k-3} \nabla^{2l+k-2-i} \mathrm{Rm} * \nabla^i A + \mathrm{Rm} * \nabla^{2l+k-2} A \\
&= \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i} \mathrm{Rm} * \nabla^i A.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\nabla^k \Delta^l A &= \nabla \nabla^{k-1} \Delta^l A \\
&= \nabla \left(\Delta^l \nabla^{k-1} A + \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \mathrm{Rm} * \nabla^i A \right) \\
&= \Delta^l \nabla \nabla^{k-1} A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \mathrm{Rm} * \nabla^i \nabla^{k-1} A + \nabla \sum_{i=0}^{2l+k-3} \nabla^{2l+k-3-i} \mathrm{Rm} * \nabla^i A \\
&= \Delta^l \nabla^k A + \sum_{i=0}^{2l-1} \nabla^{2l-1-i} \mathrm{Rm} * \nabla^{i+k-1} A + \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i} \mathrm{Rm} * \nabla^i A \\
&= \Delta^l \nabla^k A + \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i} \mathrm{Rm} * \nabla^i A.
\end{aligned}$$

□

We prove a proposition that allows us to move k covariant derivatives past l covariant derivatives.

Lemma 2.1.3. *Let $l \geq 1$. Then, for any tensor A,*

$$\nabla \nabla^l A = \nabla^l \nabla A + \sum_{i=0}^{l-1} \nabla^{l-i-1} \mathrm{Rm} * \nabla^i A.$$

Proof. The lemma is true for $l = 1$ since

$$\nabla_p \nabla_q A = \nabla_q \nabla_p A + \mathrm{Rm} * A.$$

Assume that the lemma is true for all integers at most $l-1$. The inductive hypothesis yields

$$\begin{aligned}
\nabla_p \nabla^l A &= \nabla_p \nabla_q \nabla^{l-1} A \\
&= \nabla_q \nabla_p \nabla^{l-1} A + \mathrm{Rm} * \nabla^{l-1} A \\
&= \mathrm{Rm} * \nabla^{l-1} A + \nabla_q \left[\nabla^{l-1} \nabla_p A + \sum_{i=0}^{l-2} \nabla^{l-2-i} \mathrm{Rm} * \nabla^i A \right] \\
&= \mathrm{Rm} * \nabla^{l-1} A + \nabla^l \nabla_p A + \sum_{i=0}^{l-2} (\nabla^{l-1-i} \mathrm{Rm} * \nabla^i A + \nabla^{l-2-i} \mathrm{Rm} + \nabla^{i+1} A) \\
&= \nabla^l \nabla_p A + \mathrm{Rm} * \nabla^{l-1} A + \sum_{i=0}^{l-2} \nabla^{l-1-i} \mathrm{Rm} * \nabla^i A + \sum_{i=1}^{l-1} \nabla^{l-1-i} \mathrm{Rm} * \nabla^i A \\
&= \nabla^l \nabla_p A + \sum_{i=0}^{l-1} \nabla^{l-1-i} \mathrm{Rm} * \nabla^i A.
\end{aligned}$$

We have obtained the desired equation. □

Proposition 2.1.4. *Let $k, l \geq 1$. Then, for any tensor A,*

$$\nabla^k \nabla^l A = \nabla^l \nabla^k A + \sum_{i=0}^{k+l-2} \nabla^{k+l-2-i} \mathrm{Rm} * \nabla^i A.$$

Proof. We apply the inductive hypothesis:

$$\begin{aligned}
\nabla^k \nabla^l A &= \nabla(\nabla^{k-1} \nabla^l A) \\
&= \nabla \left[\nabla^l \nabla^{k-1} A + \sum_{i=0}^{k+l-3} \nabla^{k+l-3-i} \mathrm{Rm} * \nabla^i A \right] \\
&= \nabla(\nabla^l \nabla^{k-1} A) + \nabla \left[\sum_{i=0}^{k+l-3} \nabla^{k+l-3-i} \mathrm{Rm} * \nabla^i A \right].
\end{aligned}$$

We expand the two terms on the right side of the last equality. Using Lemma 2.1.3, we expand the first term:

$$\begin{aligned}\nabla(\nabla^l\nabla^{k-1}A) &= \nabla\nabla^l(\nabla^{k-1}A)\\ &= \nabla^l\nabla\nabla^{k-1}A + \sum_{i=0}^{l-1}\nabla^{l-1-i}\mathrm{Rm} * \nabla^{i+k-1}A\\ &= \nabla^l\nabla^k A + \sum_{i=k-1}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A.\end{aligned}$$

We expand the second term:

$$\begin{aligned}\nabla\left[\sum_{i=0}^{k+l-3}\nabla^{k+l-3-i}\mathrm{Rm} * \nabla^i A\right] &= \sum_{i=0}^{k+l-3}(\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A + \nabla^{k+l-3-i}\mathrm{Rm} * \nabla^{i+1}A)\\ &= \sum_{i=0}^{k+l-3}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A\\ &\quad + \sum_{i=1}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A\\ &= \sum_{i=0}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A.\end{aligned}$$

Finally,

$$\begin{aligned}\nabla^k\nabla^l A &= \nabla(\nabla^l\nabla^{k-1}A) + \nabla\left[\sum_{i=0}^{k+l-3}\nabla^{k+l-3-i}\mathrm{Rm} * \nabla^i A\right]\\ &= \nabla^l\nabla^k A + \sum_{i=k-1}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A + \sum_{i=0}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A\\ &= \nabla^l\nabla^k A + \sum_{i=0}^{k+l-2}\nabla^{k+l-2-i}\mathrm{Rm} * \nabla^i A.\end{aligned}$$

We have obtained the desired equation. □

Proposition 2.1.5. *Let M be a manifold and $g(t)$ be a one-parameter family of metrics on M. If A is a tensor on M and $k \geq 1$, then*

$$\partial_t \nabla^k A = \nabla^k \partial_t A + \sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g * \nabla^{k-1-j} A).$$

Proof. First

$$\begin{aligned}
\partial_t \nabla A = {} & \partial_i \partial_t A^{k_1 \cdots k_s}_{j_1 \cdots j_r} - \sum_{m=1}^{r} \left[\partial_t \Gamma^l_{i j_m} A^{k_1 \cdots k_s}_{j_1 \cdots j_{m-1} l j_{m+1} \cdots j_r} + \Gamma^l_{i j_m} \partial_t A^{k_1 \cdots k_s}_{j_1 \cdots j_{m-1} l j_{m+1} \cdots j_r} \right] \\
& + \sum_{p=1}^{s} \left[\partial_t \Gamma^{k_p}_{iq} A^{k_1 \cdots k_{p-1} q k_{p+1} \cdots k_s}_{j_1 \cdots j_r} + \Gamma^{k_p}_{iq} \partial_t A^{k_1 \cdots k_{p-1} q k_{p+1} \cdots k_s}_{j_1 \cdots j_r} \right] \\
= {} & \nabla \partial_t A + \partial_t \Gamma * A \\
= {} & \nabla \partial_t A + \nabla \partial_t g * A,
\end{aligned}$$

so the proposition is true when $k = 1$. Assume the proposition holds for $k - 1$. Then

$$\begin{aligned}
\partial_t \nabla^k A &= \partial_t \nabla \nabla^{k-1} A \\
&= \nabla \partial_t \nabla^{k-1} A + \nabla \partial_t g * \nabla^{k-1} A \\
&= \nabla^k \partial_t A + \nabla \partial_t g * \nabla^{k-1} A + \nabla \sum_{j=0}^{k-2} \nabla^j (\nabla \partial_t g * \nabla^{k-2-j} A) \\
&= \nabla^k \partial_t A + \nabla \partial_t g * \nabla^{k-1} A + \nabla \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g * \nabla^{k-2-i} A \\
&= \nabla^k \partial_t A + \nabla \partial_t g * \nabla^{k-1} A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} (\nabla^{i+2} \partial_t g * \nabla^{k-2-i} A \\
&\quad + \nabla^{i+1} \partial_t g * \nabla^{k-1-i} A) \\
&= \nabla^k \partial_t A + \sum_{j=0}^{k-2} \sum_{i=0}^{j} \nabla^{i+1} \partial_t g * \nabla^{k-1-i} A + \nabla \partial_t g * \nabla^{k-1} A
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=0}^{k-2} \nabla^{j+2}\partial_t g * \nabla^{k-2-j} A \\
&= \nabla^k \partial_t A + \sum_{j=0}^{k-2}\sum_{i=0}^{j} \nabla^{i+1}\partial_t g * \nabla^{k-1-i} A + \nabla\partial_t g * \nabla^{k-1} A \\
&+ \sum_{j=1}^{k-1} \nabla^{j+1}\partial_t g * \nabla^{k-1-j} A \\
&= \nabla^k \partial_t A + \sum_{j=0}^{k-2}\sum_{i=0}^{j} \nabla^{i+1}\partial_t g * \nabla^{k-1-i} A + \sum_{j=0}^{k-1} \nabla^{j+1}\partial_t g * \nabla^{k-1-j} A \\
&= \nabla^k \partial_t A + \sum_{j=0}^{k-2} \nabla^j (\nabla\partial_t g * \nabla^{k-1-j} A) + \nabla^{k-1}(\nabla\partial_t g * A) \\
&= \nabla^k \partial_t A + \sum_{j=0}^{k-1} \nabla^j (\nabla\partial_t g * \nabla^{k-1-j} A).
\end{aligned}
$$

□

2.2 Evolution Equations

We derive the equations for $\partial_t \nabla^k \mathrm{Rm}$ for every $k \geq 0$.

Proposition 2.2.1. *If $(M, g(t))$ is a solution to AOF, then*

$$\partial_t \mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \mathrm{Rm} + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\mathrm{Rm}).$$

Proof. Let $\hat{g}(t)$ be a one-parameter family of metrics on M and $h = \partial_t \hat{g}$. The evolution of Rm is given by ([23], Theorem 7.1)

$$\partial_t R_{ijkl} = \tfrac{1}{2}[\nabla_i\nabla_k h_{jl} + \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il}] + \mathrm{Rm} * h.$$

If $h = \Delta^{\frac{n}{2}-1}\mathrm{Rc}$ then, using Proposition 2.1.2 in the second line and Lemma 2.1.1 in the third line,

$$
\begin{aligned}
\partial_t R_{ijkl} &= \tfrac{1}{2}[\nabla_i \nabla_k \Delta^{\frac{n}{2}-1} R_{jl} + \nabla_j \nabla_l \Delta^{\frac{n}{2}-1} R_{ik} - \nabla_i \nabla_l \Delta^{\frac{n}{2}-1} R_{jk} - \nabla_j \nabla_k \Delta^{\frac{n}{2}-1} R_{il}] \\
&\quad + \mathrm{Rm} * \Delta^{\frac{n}{2}-1}\mathrm{Rc} \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-1}[\nabla_i \nabla_k R_{jl} + \nabla_j \nabla_l R_{ik} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il}] \\
&\quad + \sum_{i=0}^{n-2} \nabla^{n-2-i}\mathrm{Rm} * \nabla^i \mathrm{Rc} + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-1}[-\Delta R_{ijkl} + \mathrm{Rm}^{*2}] + P_2^{n-2}(\mathrm{Rm}) \\
&= -\tfrac{1}{2}\Delta^{\frac{n}{2}} R_{ijkl} + P_2^{n-2}(\mathrm{Rm}).
\end{aligned}
$$

If $h = \Delta^{\frac{n}{2}-2}\nabla^2 R$ then, using Proposition 2.1.2 in the second and fourth lines,

$$
\begin{aligned}
\partial_t R_{ijkl} &= \tfrac{1}{2}[\nabla_i \nabla_k \Delta^{\frac{n}{2}-2} \nabla_j \nabla_l R + \nabla_j \nabla_l \Delta^{\frac{n}{2}-2} \nabla_i \nabla_k R - \nabla_i \nabla_l \Delta^{\frac{n}{2}-2} \nabla_j \nabla_k R \\
&\quad - \nabla_j \nabla_k \Delta^{\frac{n}{2}-2} \nabla_i \nabla_l R] + \mathrm{Rm} * \Delta^{\frac{n}{2}-2}\nabla^2 R \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i \nabla_k \nabla_j \nabla_l R + \nabla_j \nabla_l \nabla_i \nabla_k R - \nabla_i \nabla_l \nabla_j \nabla_k R - \nabla_j \nabla_k \nabla_i \nabla_l R] \\
&\quad + \sum_{i=0}^{n-2} \nabla^{n-2-i}\mathrm{Rm} * \nabla^i \nabla^2 R + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i \nabla_k \nabla_j \nabla_l R + \nabla_j \nabla_l \nabla_i \nabla_k R - \nabla_i \nabla_l \nabla_j \nabla_k R - \nabla_j \nabla_k \nabla_i \nabla_l R] \\
&\quad + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i \nabla_k \nabla_j \nabla_l R + \nabla_j \nabla_l \nabla_i \nabla_k R - \nabla_i \nabla_k \nabla_j \nabla_l R - \nabla_j \nabla_l \nabla_i \nabla_k R \\
&\quad + \nabla \mathrm{Rm} * \nabla R + \mathrm{Rm} * \nabla^2 R] + P_2^{n-2}(\mathrm{Rm}) \\
&= P_2^{n-2}(\mathrm{Rm}).
\end{aligned}
$$

If $h = \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm})$, then

$$\begin{aligned}
\partial_t \mathrm{Rm} &= \nabla^2 \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) + \mathrm{Rm} * \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&= \sum_{j=2}^{n/2} P_j^{n-2j+2}(\mathrm{Rm}) + \sum_{j=2}^{n/2} P_{j+1}^{n-2j}(\mathrm{Rm}).
\end{aligned}$$

Combining these results, we conclude that if $h = \widehat{\mathcal{O}}$ then

$$\begin{aligned}
\partial_t \mathrm{Rm} &= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \mathrm{Rm} + P_2^{n-2}(\mathrm{Rm}) + P_2^{n-2}(\mathrm{Rm}) \\
&\quad + \sum_{j=2}^{n/2} P_j^{n-2j+2}(\mathrm{Rm}) + \sum_{j=2}^{n/2} P_{j+1}^{n-2j}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \mathrm{Rm} + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\mathrm{Rm}).
\end{aligned}$$

□

Proposition 2.2.2. *If $(M, g(t))$ is a solution to AOF, then*

$$\partial_t \nabla^k \mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \nabla^k \mathrm{Rm} + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(\mathrm{Rm}).$$

Proof. We compute:

$$\begin{aligned}
\sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g * \nabla^{k-1-j} \mathrm{Rm}) &= \sum_{j=0}^{k-1} \nabla^j \left(\sum_{l=2}^{n/2} P_l^{n-2l+1}(\mathrm{Rm}) * \nabla^{k-1-j} \mathrm{Rm} \right) \\
&= \sum_{j=0}^{k-1} \nabla^j \sum_{l=2}^{n/2} P_{l+1}^{n-2l+k-j}(\mathrm{Rm}) \\
&= \sum_{j=0}^{k-1} \sum_{l=2}^{n/2} P_{l+1}^{n-2l+k}(\mathrm{Rm})
\end{aligned}$$

$$= \sum_{l=2}^{n/2} P_{l+1}^{n-2l+k}(\mathrm{Rm})$$
$$= \sum_{l=3}^{n/2+1} P_{l}^{n-2l+k+2}(\mathrm{Rm}).$$

Then, using Proposition 2.1.5 in the first line, Proposition 2.2.1 in the second line, and Proposition 2.1.2 in the third line, we get

$$\begin{aligned}
\partial_t \nabla^k \mathrm{Rm} &= \nabla^k \partial_t \mathrm{Rm} + \sum_{j=0}^{k-1} \nabla^j (\nabla \partial_t g * \nabla^{k-1-j} \mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \nabla^k \Delta^{\frac{n}{2}} R_{ijkl} + \nabla^k \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\mathrm{Rm}) \\
&\quad + \sum_{l=3}^{n/2+1} P_l^{n-2l+k+2}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \nabla^k R_{ijkl} + P_2^{n+k-2}(\mathrm{Rm}) + \sum_{j=2}^{n/2+1} P_j^{n-2j+k+2}(\mathrm{Rm}) \\
&\quad + \sum_{l=3}^{n/2+1} P_l^{n-2l+k+2}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)} \Delta^{\frac{n}{2}} \nabla^k R_{ijkl} + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(\mathrm{Rm}).
\end{aligned}$$

□

2.3 Short Time Existence and Uniqueness

The ambient obstruction flow is a quasilinear flow of order n in the metric g. E. Bahuaud and D. Helliwell have shown the following existence and uniqueness result for AOF:

Theorem 2.3.1 ([3] Theorem C, [4] Theorem C). *Let h be a smooth metric on a compact manifold M of even dimension $n \geq 4$. Then there is a unique smooth short time solution to the following flow:*

$$\begin{cases} \partial_t g = \widehat{\mathcal{O}} = (-1)^{\frac{n}{2}} \mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} (\Delta^{\frac{n}{2}-1} R) g \\ g(0) = h, \end{cases} \tag{2.1}$$

where $\mathcal{O}$ is the ambient obstruction tensor on M and R is the scalar curvature of M.

We briefly illustrate that applying the DeTurck trick to the system (2.1) results in a strongly parabolic system. Due to the diffeomorphism invariance of M, the system (2.1) is not strongly parabolic. We define the following vector fields:

$$\begin{aligned} V^k &= g^{ij}(\Gamma^k_{ij} - \Gamma(h)^k_{ij}) \\ X &= \frac{(-1)^{\frac{n}{2}-1}}{2(n-2)} \Delta^{\frac{n}{2}-1} V \\ Y &= \frac{(-1)^{\frac{n}{2}}}{4(n-1)} (\nabla \Delta^{\frac{n}{2}-2} R)^{\sharp} \\ W &= X + Y. \end{aligned}$$

We show that the following system is strongly parabolic:

$$\begin{cases} \partial_t g = \widehat{\mathcal{O}} + \mathcal{L}_W g \\ g(0) = h. \end{cases} \tag{2.2}$$

We show this by computing the principal symbol σ of the linearization of $\widehat{\mathcal{O}} + \mathcal{L}_W g$ at h. We know from Proposition 1.3.3 that

$$\widehat{\mathcal{O}} = \frac{(-1)^{\frac{n}{2}}}{n-2} \Delta^{\frac{n}{2}-1} \mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)} \Delta^{\frac{n}{2}-2} \nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

We then rewrite the system (2.2) as follows:

$$\partial_t g = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \mathcal{L}_X g + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \mathcal{L}_Y g + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}). \qquad (2.3)$$

Let $\zeta \in T^*M$. The principal symbol of the first two terms of (2.3) is given by

$$\begin{aligned}
&\sigma\left[D\left(\tfrac{(-1)^{n/2}}{n-2}\Delta^{n/2-1}\mathrm{Rc} + \mathcal{L}_X g\right)\right](\zeta)(h)\\
&\quad = \tfrac{(-1)^{n/2-1}}{2(n-2)}\sigma[D(\Delta^{n/2-1})](\zeta)\cdot\sigma[D(-2\mathrm{Rc} + \mathcal{L}_V g)](\zeta)(h)\\
&\quad = \tfrac{(-1)^{n/2-1}}{2(n-2)}|\zeta|^n h.
\end{aligned}$$

We used the fact that the Ricci-DeTurck flow is strongly parabolic (Chow-Knopf [14], Theorem 3.13). The highest order terms of the next two terms of (2.3) cancel each other out, and the remaining terms are of lower order. Therefore the principal symbol of the system (2.2) is $\frac{(-1)^{n/2-1}}{2(n-2)}|\zeta|^n h$, implying that this system is strongly parabolic.

We show that we can pull back the short time solution of (2.2) to give a solution of (2.1) that exists for $t \in [0, \epsilon)$. It follows from the parabolicity shown above that there exists $\epsilon > 0$ for which the solution to (2.2) exists for $t \in [0, \epsilon)$ via parabolic PDE theory. Next, there exists a family $\varphi_t : M \to M$ of diffeomorphisms satisfying

$$\begin{cases} \frac{\partial \varphi_t}{\partial t} = -W(\varphi_t, t) \\ \varphi_0 = \mathrm{id}_M. \end{cases}$$

for $t \in [0, \epsilon)$. The existence of the φ_t follows from the existence and uniqueness theorem for nonautonomous ODE on manifolds, and the uniform ϵ follows from bounds on W that result from the compactness of M. We now show that $\varphi_t^* \partial_t g$ satisfies (2.1):

$$\partial_t(\varphi_t^* g) = \partial_s|_{s=0}(\varphi_{s+t}^* g(s+t))$$

$$
\begin{aligned}
&= \lim_{s\to 0} \frac{\varphi^*_{s+t} g(s+t) - \varphi^*_t g(t)}{s} \\
&= \lim_{s\to 0} \frac{\varphi^*_{s+t} g(s+t) - \varphi^*_{s+t} g(t)}{s} + \lim_{s\to 0} \frac{\varphi^*_{s+t} g(t) - \varphi^*_t g(t)}{s} \\
&= \varphi^*_t \partial_t g + \partial_s|_{s=0} (\varphi^*_{t+s} g(t)) \\
&= \varphi^*_t [(-1)^{\frac{n}{2}} \mathcal{O}(g) + \tfrac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} [\Delta^{\frac{n}{2}-1} R(g(t))] g(t) + \mathcal{L}_W g(t)] \\
&\quad + \partial_s|_{s=0} [(\varphi_t^{-1} \circ \varphi_{t+s})^* \varphi^*_t g(t)] \\
&= (-1)^{\frac{n}{2}} \mathcal{O}(\varphi^*_t g(t)) + \tfrac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} [\Delta^{\frac{n}{2}-1} R(\varphi^*_t g(t))] \varphi^*_t g(t) + \varphi^*_t (\mathcal{L}_W g(t)) \\
&\quad - \mathcal{L}_{[(\varphi_t^{-1})_* W(t)]} (\varphi^*_t g(t)) \\
&= (-1)^{\frac{n}{2}} \mathcal{O}(\varphi^*_t g(t)) + \tfrac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)} [\Delta^{\frac{n}{2}-1} R(\varphi^*_t g(t))] \varphi^*_t g(t).
\end{aligned}
$$

Since $\varphi^*_0 g(0) = g(0) = h$, $\varphi^*_t g(t)$ satisfies (2.1). Therefore these diffeomorphisms pull back the short time solution of (2.2) to give a solution of (2.1) that exists for $t \in [0, \epsilon)$.

Chapter 3

Smoothing Estimates and Long Time Existence

3.1 Local Integral Estimates

In this section, let (M^n, g) be a Riemannian manifold that is a solution to the AOF on a time interval $[0, T)$. We give local L^2 estimates for $\nabla^k \mathrm{Rm}$ for all $k \in \mathbb{N}$. We need to use local L^2 estimates since we can only convert L^2 estimates to pointwise estimates locally. These local pointwise estimates are used in the proof of the pointwise smoothing estimates given in Theorem 1.2.1. Specify the Laplace operator by $\Delta = -\nabla^* \nabla$. Let $\varphi \in C_c^\infty(M)$ be a cutoff function with constants $\Lambda, \Lambda_1 > 0$ such that

$$\sup_{t \in [0,T)} |\nabla \varphi| \leq \Lambda_1, \qquad \max_{0 \leq i \leq \frac{n}{2}} \sup_{t \in [0,T)} |\nabla^i \varphi| \leq \Lambda.$$

Lemma 3.1.1. *Suppose M, φ satisfy the above hypotheses. Let A be any tensor and $p \geq 1, q \geq 2$. Then*

$$\int_M \varphi^p \langle \Delta^q A, A\rangle = (-1)^q \int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A.$$

Proof. We first claim that if $q \geq 2$, then

$$\Delta^q A = (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A.$$

If $q = 2$, we get, using Proposition 2.1.2, that

$$\begin{aligned}
\Delta^2 A &= -\nabla^*\nabla\Delta A \\
&= -\nabla^*\Delta\nabla A + \nabla^*[\nabla\mathrm{Rm} * A + \mathrm{Rm} * \nabla A] \\
&= (\nabla^*)^2\nabla^2 A + \nabla^2\mathrm{Rm} * A + \nabla\mathrm{Rm} * \nabla A + \mathrm{Rm} * \nabla^2 A,
\end{aligned}$$

which agrees with the claim. Suppose the claim is true for every integer less than q. First,

$$\begin{aligned}
\Delta^q A &= -\nabla^*\nabla\Delta^{q-1} A \\
&= -\nabla^* \left[\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-3-i}\mathrm{Rm} * \nabla^i A\right] \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \left[\nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A + \nabla^{2q-3-i}\mathrm{Rm} * \nabla^{i+1} A\right] \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A + \sum_{i=1}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A.
\end{aligned}$$

Applying the last equation above and then the inductive hypothesis,

$$
\begin{aligned}
\Delta^q A &= -\nabla^* \Delta^{q-1} \nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A \\
&= -\nabla^* \left[(-1)^{q-1} (\nabla^*)^{q-1} \nabla^{q-1} \nabla A + \sum_{i=0}^{2q-4} \nabla^{2q-4-i} \mathrm{Rm} * \nabla^i \nabla A \right] \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A \\
&= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-4} \nabla^{2q-3-i} \mathrm{Rm} * \nabla^{i+1} A + \sum_{i=0}^{2q-4} \nabla^{2q-4-i} \mathrm{Rm} * \nabla^{i+2} A \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A \\
&= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=1}^{2q-3} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A + \sum_{i=2}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A \\
&= (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i} \mathrm{Rm} * \nabla^i A.
\end{aligned}
$$

This proves the claim. We compute

$$
\begin{aligned}
(-1)^{q+1} \int_M \nabla^q A * \nabla^q (\varphi^p A) &= (-1)^{q+1} \int_M \nabla^q A * \sum_{i=0}^{q} \nabla^{q-i} (\varphi^p) * \nabla^i A \\
&= (-1)^{q+1} \int_{[\varphi>0]} \sum_{i=0}^{q} \sum_{|\alpha|=q-i} \nabla^i A * \nabla^q A * \prod_{j=1}^{p} \nabla^{\alpha_j} \varphi_j \\
&= (-1)^{q+1} \int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A.
\end{aligned}
$$

Finally, applying the claim,

$$\begin{aligned}\int_M \varphi^p \langle \Delta^q A, A\rangle &= \int_M \varphi^p \left\langle (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A, A \right\rangle \\ &= (-1)^q \int_M \nabla^q A * \nabla^q(\varphi^p A) + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A \\ &= (-1)^q \int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A \\ &\quad + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A.\end{aligned}$$

□

Proposition 3.1.2. *Suppose M, φ satisfy the above hypotheses. If $p \geq 1, k \geq 0$, then*

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 = -\frac{1}{n-2}\int_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 + \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) \\ + \int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm}. \quad (3.1)\end{aligned}$$

Proof. First, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 \, dV_g = 2\int_M \varphi^p \left\langle \frac{\partial}{\partial t}\nabla^k \mathrm{Rm}, \nabla^k \mathrm{Rm}\right\rangle dV_g \\ + \int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 \frac{\partial g}{\partial t}\, dV_g.\end{aligned}$$

We can expand the first integral by substituting Proposition 2.2.2, which states that for our flow,

$$\frac{\partial}{\partial t}\nabla^k \mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\nabla^k \mathrm{Rm} + \sum_{i=2}^{\frac{n}{2}+1} P_i^{n-2i+k+2}(\mathrm{Rm}).$$

Applying Lemma 3.1.1 to the first term of $\frac{\partial}{\partial t}\nabla^k \mathrm{Rm}$ gives that

$$\begin{aligned}
\frac{(-1)^{\frac{n}{2}+1}}{n-2}\int\limits_M \varphi^p \langle \Delta^{\frac{n}{2}}\nabla^k \mathrm{Rm}, \nabla^k \mathrm{Rm}\rangle &= \frac{(-1)^{n+1}}{n-2}\int\limits_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}} [P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} \\
&\quad * \nabla^{k+\frac{n}{2}}\mathrm{Rm}] \\
&\quad + \int\limits_M \sum_{i=0}^{n-2} \varphi^p \nabla^{n-2-i}\mathrm{Rm} * \nabla^{k+i}\mathrm{Rm} * \nabla^k \mathrm{Rm} \\
&= -\frac{1}{n-2}\int\limits_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\
&\quad + \int\limits_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm} \\
&\quad + \int\limits_M \varphi^p P_3^{n+2k-2}(\mathrm{Rm}).
\end{aligned}$$

Substituting the second term of $\frac{\partial}{\partial t}\nabla^k \mathrm{Rm}$ into the inner product gives that

$$\begin{aligned}
\int\limits_M \varphi^p \left\langle \nabla^k \mathrm{Rm}, \sum_{i=2}^{\frac{n}{2}+1} P_i^{n-2i+k+2}(\mathrm{Rm})\right\rangle &= \int\limits_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+2k+4}(\mathrm{Rm}) \\
&= \int\limits_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \Delta^{\frac{n}{2}-1}\mathrm{Rc} + \Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm}) \\
&= \nabla^{n-2}\mathrm{Rm} + \nabla^{n-4+2}\mathrm{Rm} + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm}) \\
&= \sum_{i=1}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm}),
\end{aligned}$$

we have

$$
\begin{aligned}
\int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 \frac{\partial g}{\partial t} &= \int_M \varphi^p (\nabla^k \mathrm{Rm})^{*2} \sum_{i=1}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm}) \\
&= \int_M \varphi^p \sum_{i=1}^{\frac{n}{2}} P_{i+2}^{n-2i+2k}(\mathrm{Rm}) \\
&= \int_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+2k+4}(\mathrm{Rm}) \\
&= \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-l+2}^{2l}(\mathrm{Rm}).
\end{aligned}
$$

Combining all of these results yields

$$
\begin{aligned}
\frac{\partial}{\partial t} \int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 = &-\frac{1}{n-2} \int_M \varphi^p |\nabla^{\frac{n}{2}+k} \mathrm{Rm}|^2 \\
&+ \int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i} \mathrm{Rm} * \nabla^{k+\frac{n}{2}} \mathrm{Rm} \\
&+ \int_M \varphi^p P_3^{n+2k-2}(\mathrm{Rm}) + \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-l+2}^{2l}(\mathrm{Rm}) \\
&+ \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-l+2}^{2l}(\mathrm{Rm}) \\
= &-\frac{1}{n-2} \int_M \varphi^p |\nabla^{\frac{n}{2}+k} \mathrm{Rm}|^2 \\
&+ \int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i} \mathrm{Rm} * \nabla^{k+\frac{n}{2}} \mathrm{Rm} \\
&+ \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P_{\frac{n}{2}+k-l+2}^{2l}(\mathrm{Rm}).
\end{aligned}
$$

□

We estimate the last two terms of (3.1). First, we recall two corollaries from the paper [31] of E. Kuwert and R. Schätzle.

Proposition 3.1.3 ([31], Corollary 5.2)**.** *Suppose M, φ satisfy the above hypotheses. Let A be a tensor. If $2 \leq p < \infty$ and $s \geq p$, then for every $\epsilon > 0$,*

$$\left(\int_M |\nabla A|^p \varphi^s\right)^{\frac{1}{p}} \leq \epsilon \left(\int_M |\nabla^2 A|^p \varphi^{s+p}\right)^{\frac{1}{p}} + \frac{c}{\epsilon}\left(\int_{[\varphi>0]} |A|^p \varphi^{s-p}\right)^{\frac{1}{p}},$$

where $c = c(n, p, s, \Lambda_1)$.

Proposition 3.1.4 ([31], Corollary 5.5)**.** *Suppose M, φ satisfy the above hypotheses. Let A be a tensor. Let $0 \leq i_1, \ldots, i_r \leq k$, $i_1 + \cdots + i_r = 2k$, and $s \geq 2k$. Then*

$$\left|\int_M \varphi^s \nabla^{i_1} A * \cdots * \nabla^{i_r} A\right| \leq c\|A\|_\infty^{r-2}\left(\int_M \varphi^s |\nabla^k A|^2 \, dV + \|A\|^2_{2,[\varphi>0]}\right),$$

where $c = c(k, n, r, s, \Lambda_1)$.

We estimate the last term of (3.1).

Lemma 3.1.5. *Suppose M, φ satisfy the above hypotheses. If $l \geq 1, q \geq 0$, then for every $\epsilon > 0$,*

$$\int_M \varphi^{2l+q} |\nabla^l \mathrm{Rm}|^2 \leq \epsilon \int_M \varphi^{2l+q+2} |\nabla^{l+1}\mathrm{Rm}|^2 + \frac{C}{\epsilon^l}\int_{[\varphi>0]} \varphi^q |\mathrm{Rm}|^2. \tag{3.2}$$

where $C = C(n, l, \Lambda_1, q)$.

Proof. We prove the inequality (3.2) by induction on l. If $l = 1$, the inequality (3.2) follows immediately from Proposition 3.1.3. Assume that $l \geq 2$ and (3.2) is true for all integers at most l. Then, applying Proposition 3.1.3 in the first line and the inductive hypothesis in the second line,

$$\begin{aligned}\int_M \varphi^{2l+2+q}|\nabla^{l+1}\mathrm{Rm}|^2 &\leq \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{C}{\epsilon}\int_M \varphi^{2l+q}|\nabla^{l}\mathrm{Rm}|^2 \\ &\leq \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{C}{\epsilon}\frac{\epsilon}{2C}\int_M \varphi^{2l+q+2}|\nabla^{l+1}\mathrm{Rm}|^2 \\ &\quad + \frac{C}{\epsilon}\frac{C}{\epsilon^l}\int_{[\varphi>0]} \varphi^{q}|\mathrm{Rm}|^2 \\ &= \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{1}{2}\int_M \varphi^{2l+q+2}|\nabla^{l+1}\mathrm{Rm}|^2 \\ &\quad + \frac{C}{\epsilon^{l+1}}\int_{[\varphi>0]} \varphi^{q}|\mathrm{Rm}|^2.\end{aligned}$$

Collecting terms, we see that (3.2) is also true for $l+1$. □

Lemma 3.1.6. *Suppose M, φ satisfy the above hypotheses. If $q \geq 0$ and $0 \leq l \leq q$, then for every $\epsilon > 0$,*

$$\int_M \varphi^{2l+r}|\nabla^{l}\mathrm{Rm}|^2 \leq \epsilon^{q-l}\int_M \varphi^{2q+r}|\nabla^{q}\mathrm{Rm}|^2 + C\epsilon^{-l}\int_{[\varphi>0]} \varphi^{r}|\mathrm{Rm}|^2.$$

where $C = C(n, l, \Lambda_1, r)$.

Proof. Let $m = q - l$. The desired inequality is equivalent to

$$\int_M \varphi^{2q-2m+r}|\nabla^{q-m}\mathrm{Rm}|^2 \leq \epsilon^{m}\int_M \varphi^{2q+r}|\nabla^{q}\mathrm{Rm}|^2 + C\epsilon^{m-q}\int_{[\varphi>0]} \varphi^{r}|\mathrm{Rm}|^2. \tag{3.3}$$

We prove this inequality by induction on m. If $m = 0$ the inequality is true:

$$\int_M \varphi^{2q+r}|\nabla^{q}\mathrm{Rm}|^2 \leq \int_M \varphi^{2q+r}|\nabla^{q}\mathrm{Rm}|^2 + C\epsilon^{-q}\int_{[\varphi>0]} \varphi^{r}|\mathrm{Rm}|^2.$$

Assume the inequality (3.3) is true for every integer less than m. Then

$$\begin{aligned}
\int_M \varphi^{2q-2m+r}|\nabla^{q-m}\mathrm{Rm}|^2 &\le \epsilon \int_M \varphi^{2q-2m+r+2}|\nabla^{q-m+1}\mathrm{Rm}|^2 \\
&\quad + C\epsilon^{m-q}\int_{[\varphi>0]} \varphi^r|\mathrm{Rm}|^2 \\
&\le \epsilon\epsilon^{m-1}\int_M \varphi^{2q+r}|\nabla^q\mathrm{Rm}|^2 + \epsilon C\epsilon^{m-q-1}\int_{[\varphi>0]}\varphi^r|\mathrm{Rm}|^2 \\
&\quad + C\epsilon^{m-q}\int_{[\varphi>0]} \varphi^r|\mathrm{Rm}|^2 \\
&= \epsilon^m \int_M \varphi^{2q+r}|\nabla^q\mathrm{Rm}|^2 + C\epsilon^{m-q}\int_{[\varphi>0]}\varphi^r|\mathrm{Rm}|^2.
\end{aligned}$$

We applied Lemma 3.1.5 in the first line and the inductive hypothesis in the second line. □

Lemma 3.1.7. *Suppose M, φ satisfy the above hypotheses. Let $0 \le i \le \frac{n}{2}-1$ and $p \ge n+2k$. Then for every $\delta > 0$,*

$$\begin{aligned}
\int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k}\mathrm{Rm} * \nabla^{\frac{n}{2}+k}\mathrm{Rm} &\le C\delta \int_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\
&\quad + C\delta^{\frac{-n-2i-4k}{n-2i}} \int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2,
\end{aligned}$$

where $C = C(n, k, p, \Lambda, i)$.

Proof. We apply the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k}\mathrm{Rm} * \nabla^{\frac{n}{2}+k}\mathrm{Rm} &\le C(\Lambda)\int_M |\varphi^{p-(\frac{n}{2}-i)} * \nabla^{i+k}\mathrm{Rm} * \nabla^{\frac{n}{2}+k}\mathrm{Rm}| \\
&\le C\epsilon^{\beta}\int_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\
&\quad + C\epsilon^{-\beta}\int_{[\varphi>0]} \varphi^{p-n+2i}|\nabla^{i+k}\mathrm{Rm}|^2.
\end{aligned}$$

The second term can be estimated using Lemma 3.1.6:

$$\int_{[\varphi>0]} \varphi^{p-n+2i}|\nabla^{i+k}\mathrm{Rm}|^2 = \int_{[\varphi>0]} \varphi^{2(i+k)+(p-n-2k)}|\nabla^{i+k}\mathrm{Rm}|^2$$
$$\leq \epsilon^{\frac{n}{2}-i}\int_M |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 + C\epsilon^{-i-k}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2.$$

If $\beta = \frac{n}{2} - i - \beta$, then $\beta = \frac{n-2i}{4}$. If we set $\delta = \epsilon^{\frac{n-2i}{4}}$, then $\epsilon = \delta^{\frac{4}{n-2i}}$ and

$$\epsilon^{-\beta-i-k} = \delta^{\frac{4}{n-2i}\left(\frac{2i-n}{4}-i-k\right)} = \delta^{\frac{-n-2i-4k}{n-2i}}.$$

Therefore

$$\int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k}\mathrm{Rm} * \nabla^{\frac{n}{2}+k}\mathrm{Rm} \leq C\epsilon^{\beta}\int_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2$$
$$+ C\epsilon^{-\beta+\frac{n}{2}-i}\int_M |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2$$
$$+ C\epsilon^{-\beta-i-k}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2$$
$$\leq C\delta\int_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2$$
$$+ C\delta^{\frac{-n-2i-4k}{n-2i}}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2.$$

□

We estimate the penultimate term of (3.1).

Lemma 3.1.8. *Suppose M, φ satisfy the above hypotheses. Let $K = \max\{1, \|\mathrm{Rm}\|_\infty\}$. If $p \geq n + 2k$ and $k \leq l \leq \frac{n}{2} + k - l$, then for every δ satisfying $0 < \delta \leq 1$,*

$$\int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) \leq C\delta\int_M \varphi^{p+n+2k-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2$$
$$+ CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\|\mathrm{Rm}\|^2_{2,[\varphi>0]},$$

where $C = C(n, k, p, \Lambda_1, l)$.

Proof. Since $p \geq n + 2k \geq n + 2k - 2 = 2(\frac{n}{2} + k - 1)$, Proposition 3.1.4 implies

$$\int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) \leq C\|\mathrm{Rm}\|_\infty^{\frac{n}{2}+k-l}\left(\int_M \varphi^p |\nabla^l \mathrm{Rm}|^2 + \|\mathrm{Rm}\|^2_{2,[\varphi>0]}\right)$$

Let $\epsilon = K^{-1}\delta^{\frac{2}{n+2k-2l}}$. We have $p - 2l \geq n + 2k - (n + 2k - 1) = 1$. Via Lemma 3.1.6,

$$\begin{aligned} C\|\mathrm{Rm}\|_\infty^{\frac{n}{2}+k-l}\int_M \varphi^p|\nabla^l \mathrm{Rm}|^2 &\leq CK^{\frac{n}{2}+k-l}\epsilon^{\frac{n}{2}+k-l}\int_M \varphi^{n+2k+p-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k-l}\epsilon^{-l}\int_{[\varphi>0]}\varphi^{p-2l}|\mathrm{Rm}|^2 \\ &= C\delta\int_M \varphi^{n+2k+p-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\int_{[\varphi>0]}\varphi^{p-2l}|\mathrm{Rm}|^2. \end{aligned}$$

Since $k \leq l \leq \frac{n}{2} + k - l$ and $0 < \delta \leq 1$, we get $\delta^{\frac{2l}{2l-n-2k}} \geq \delta^{-\frac{2k}{n}} \geq 1$ and $K^{\frac{n}{2}+k-l} \leq K^{\frac{n}{2}}$. Therefore

$$\begin{aligned} \int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) &\leq C\delta\int_M \varphi^{n+2k+p-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\int_{[\varphi>0]}\varphi^{p-2l}|\mathrm{Rm}|^2 \\ &\quad + K^{\frac{n}{2}+k-l}\|\mathrm{Rm}\|^2_{2,[\varphi>0]} \\ &\leq C\delta\int_M \varphi^{p+n+2k-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\|\mathrm{Rm}\|^2_{2,[\varphi>0]}. \end{aligned}$$

□

Proposition 3.1.9. *Suppose M, φ satisfy the above hypotheses. Let*

$$K = \max\{1, \|\mathrm{Rm}\|_\infty\}.$$

If $p \geq n + 2k$, then for every δ satisfying $0 < \delta \leq 1$,

$$\partial_t \|\varphi^{\frac{p}{2}} \nabla^k \mathrm{Rm}\|_2^2 \leq -\tfrac{1}{2(n-2)} \|\varphi^{\frac{p}{2}} \nabla^{\frac{n}{2}+k} \mathrm{Rm}\|_2^2 + CK^{\frac{n}{2}+k} \|\mathrm{Rm}\|_{2,[\varphi>0]}^2$$

where $C = C(n, k, p, \Lambda)$.

Proof. Applying the estimates from Lemmas 3.1.8 and 3.1.7 to the equation (3.1) in Proposition 3.1.2, we obtain

$$\begin{aligned}
\partial_t \|\varphi^{\frac{p}{2}} \nabla^k \mathrm{Rm}\|_2^2 &\leq -\tfrac{1}{n-2} \|\varphi^{\frac{p}{2}} \nabla^{\frac{n}{2}+k} \mathrm{Rm}\|_2^2 \\
&+ \sum_{l=k}^{\frac{n}{2}+k-1} \left[C_1 \delta \|\varphi^{\frac{p}{2}+\frac{n}{2}+k-l} \nabla^{\frac{n}{2}+k} \mathrm{Rm}|_2^2 + C_1 K^{\frac{n}{2}+k} \delta^{\frac{2l}{2l-n-2k}} \|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \right] \\
&\qquad + \sum_{i=0}^{\frac{n}{2}-1} \left[C_2 \delta \|\varphi^{\frac{p}{2}} \nabla^{\frac{n}{2}+k} \mathrm{Rm}\|_2^2 + C_2 \delta^{\frac{-n-2i-4k}{n-2i}} \|\varphi^{\frac{p}{2}-\frac{n}{2}-k} \mathrm{Rm}\|_{2,[\varphi>0]}^2 \right],
\end{aligned}$$

where $C_1 = C_1(n, k, p, \Lambda, l)$ and $C_2 = C_2(n, k, p, \Lambda_1, i)$. From the inequalities

$$1 - n - 2k \leq 1 - \frac{2n+4k}{n-2i} \leq -\frac{n+4k}{n}, \qquad \frac{2-n-2k}{2} \leq 1 + \frac{n+2k}{2l-n-2k} \leq -\frac{2k}{n}$$

we conclude

$$\max\left(\{\delta^{\frac{2l}{2l-n-2k}} : k \leq l \leq \tfrac{n}{2} + k - 1\} \cup \{\delta^{\frac{-n-2i-4k}{n-2i}} : 0 \leq i \leq \tfrac{n}{2} - 1\} \right) = \delta^{1-n-2k}.$$

Therefore

$$\begin{aligned}\partial_t\|\varphi^{\frac{p}{2}}\nabla^k\mathrm{Rm}\|_2^2 &\le -\tfrac{1}{n-2}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2+\widetilde{C}\delta\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2\\ &\quad+\widetilde{C}K^{\frac{n}{2}+k}\delta^{1-n-2k}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\\ &\le -\tfrac{1}{2(n-2)}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2+CK^{\frac{n}{2}+k}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2,\end{aligned}$$

where

$$\widetilde{C}\equiv\sum_{l=k}^{\frac{n}{2}+k-1}C_1+\sum_{i=0}^{\frac{n}{2}-1}C_2,\quad \delta\equiv\min\{\tfrac{1}{2(n-2)}\widetilde{C}^{-1},1\}.$$

□

Proposition 3.1.10. *Suppose M,φ satisfy the above hypotheses. Suppose*

$$\max\{\|\mathrm{Rm}\|_\infty,1\}\le K$$

for all $t\in[0,\alpha K^{-\frac{n}{2}}]$. Then

$$\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2\le Ct^{-\frac{m}{2}}\sup_{t\in[0,\alpha K^{-\frac{n}{2}}]}\|\mathrm{Rm}\|_{L^2(t),[\varphi>0]},$$

where $C=C(m,n,\alpha,\Lambda)$, for all $t\in(0,\alpha K^{-\frac{n}{2}}]$.

Proof. Let β_k for $0\le k\le m$ denote constants given by $\beta_k=(2n-4)^{m-k}m!/k!$. Define

$$G(t)\equiv t^m\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2+\sum_{k=0}^{m-1}\beta_k t^k\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}k}\mathrm{Rm}\|_{2,[\varphi>0]}^2.$$

Using Proposition 3.1.9,

$$\begin{aligned}
\frac{dG}{dt} \le\ & mt^{m-1}\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 \\
& + t^m\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}(m+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
& + \sum_{k=1}^{m-1}\beta_k k t^{k-1}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}k}\mathrm{Rm}\|_2^2 \\
& + \sum_{k=0}^{m-1}\beta_k t^k\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
\le\ & mt^{m-1}\|\varphi^{\frac{n}{2}m}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 + t^m\Big(C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
& + \sum_{k=0}^{m-2}\beta_{k+1}(k+1)t^k\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 \\
& + \sum_{k=0}^{m-1}\beta_k t^k\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big).
\end{aligned}$$

Choose $t_0 \in [0, \alpha K^{-\frac{n}{2}}]$ such that

$$\|\mathrm{Rm}\|_{L^2(t_0),[\varphi>0]} = \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]}\|\mathrm{Rm}\|_{L^2(t),[\varphi>0]}.$$

Our choice of the constants β_k yields

$$\begin{aligned}
\frac{dG}{dt} &\le \alpha^m K^{-\frac{n}{2}m}C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&\quad + \sum_{k=0}^{m-1}\beta_k\alpha^k K^{-\frac{n}{2}k}C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&= \sum_{k=0}^{m}\beta_k C_{\frac{n}{2}k}\alpha^k K^{\frac{n}{2}}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&= CK^{\frac{n}{2}}\|\mathrm{Rm}\|_{L^2(t_0),[\varphi>0]}^2.
\end{aligned}$$

Therefore

$$
\begin{aligned}
t^m \|\varphi^{\frac{n}{2}(m+1)} \nabla^{\frac{n}{2}m} \mathrm{Rm}\|_2^2 \le G &\le \beta_0 \|\mathrm{Rm}\|^2_{L^2(0),[\varphi>0]} + C K^{\frac{n}{2}} \|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]} t \\
&\le (\beta_0 + \alpha C) \|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]} \\
&= C \|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]},
\end{aligned}
$$

proving the proposition. □

Proposition 3.1.11. *Let $(M^n, g(t))$ be a solution to the AOF for $t \in [0,T)$. Let $\varphi \in C_c^\infty(M)$ be a cutoff function such that*

$$
\max_{0 \le i \le \frac{n}{2}} \sup_{t \in [0,T)} \|\nabla^i \varphi\|_{C^0(M,g(t))} \le \Lambda.
$$

Suppose $\max\{\|\mathrm{Rm}\|_{C^0(M,g(t))}, 1\} \le K$ *for all $t \in [0, \alpha K^{-\frac{n}{2}}]$. Then, for every $l \ge 0$ and all $t \in (0, \alpha K^{-\frac{n}{2}}]$,*

$$
\|\varphi^{l+\frac{n}{2}} \nabla^l \mathrm{Rm}\|_{L^2(M,g(t))} \le C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0, \alpha K^{-\frac{n}{2}}]} \|\mathrm{Rm}\|_{L^2(\mathrm{supp}(\varphi), g(t))},
$$

where $C = C(l, n, \alpha, \Lambda)$.

Proof. Let $l = \frac{n}{2}m + r, 1 \le r \le \frac{n}{2}$. Then, applying Lemma 3.1.6 and Proposition 3.1.10, we get

$$
\begin{aligned}
\int_M \varphi^{n(m+1)+2r} |\nabla^{\frac{n}{2}m+r} \mathrm{Rm}|^2 &\le \int_M \varphi^{n(m+2)} |\nabla^{\frac{n}{2}(m+1)} \mathrm{Rm}|^2 + C' \int_{[\varphi>0]} \varphi^n |\mathrm{Rm}|^2 \\
&\le t^{-(m+1)} C \Theta^2 + C' \Theta^2 \\
\|\varphi^{l+\frac{n}{2}} \nabla^l \mathrm{Rm}\|_{L^2(t)} &\le \Theta(C t^{-\frac{m+1}{2}} + C'),
\end{aligned}
$$

where

$$\Theta = \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]} \|\mathrm{Rm}\|_{L^2(t),[\varphi>0]}.$$

□

3.2 Pointwise Smoothing Estimates

Let $(M, g(t))$ be a solution to AOF and let φ be a cutoff function on M. We give estimates of $|\nabla^i\varphi|_{g(t)}$ for $1 \leq i \leq \frac{n}{2}$ that depend on spacetime derivatives of the metric and $|\nabla^i\varphi|_{g(0)}$ for $0 \leq i \leq \frac{n}{2}$. We then give a proof of the pointwise smoothing estimates given in Theorem 1.2.1.

Lemma 3.2.1. *Let M be a manifold and $g(t)$ be a one-parameter family of metrics on M. For a function $\varphi \in C^i(M)$ and $i \geq 2$,*

$$\partial_t \nabla^i \varphi = \sum_{j=1}^{i-1} \nabla^{i-j}\partial_t g * \nabla^j \varphi.$$

Proof. Apply Proposition 2.1.5 with $k = i - 1$ and $A = \nabla\varphi$. □

Proposition 3.2.2. *Let M be a manifold and $g(t)$ be a one-parameter family of metrics on M. For a function $\varphi \in C^i(M)$ and $i \geq 1$,*

$$\partial_t |\nabla^i \varphi|^2_{g(t)} = \sum_{j=1}^{i} \nabla^{i-j}\partial_t g * \nabla^j \varphi * \nabla^i \varphi.$$

Proof. We compute, using the preceding Lemma 3.2.1 in the second line:

$$\begin{aligned}\partial_t|\nabla^i\varphi|^2_{g(t)} &= \partial_t g * \nabla^i\varphi^{*2} + \partial_t\nabla^i\varphi * \nabla^i\varphi \\ &= \partial_t g * \nabla^i\varphi^{*2} + \sum_{j=1}^{i-1}\nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi \\ &= \sum_{j=1}^{i}\nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi.\end{aligned}$$

□

Proposition 3.2.3. *Let M be a Riemannian manifold with a one-parameter family of metrics $\{g(t)\}_{t\in[0,T]}$ and $\varphi \in C_c^\infty(M)$. Fix $i \geq 1$. Suppose that, for each j satisfying $0 \leq j \leq i-1$, there exists $K_j > 0$ such that $|\nabla^j\partial_t g(x,t)|_{g(t)} \leq K_j$ on $\operatorname{supp}\varphi \times [0,T]$ and, for each j satisfying $1 \leq j \leq i$, there exists $C'_j > 0$ such that $|\nabla^j\varphi|_{g(0)} \leq C'_j$ on $\operatorname{supp}\varphi$. Then there exists a constant C_i such that, for every $t \in [0,T]$,*

$$|\nabla^i\varphi|^2_{g(t)} \leq C_i = C_i(K_0,\ldots,K_{i-1},C'_1,\ldots,C'_i,T).$$

Proof. Let $i = 1$. Then Proposition 3.2.2 gives

$$\partial_t|\nabla\varphi|^2_{g(t)} = \partial_t g * \nabla\varphi^{*2} \leq CK_0|\nabla\varphi|^2_{g(t)}.$$

Solving the differential inequality, we get

$$|\nabla\varphi|^2_{g(t)} \leq |\nabla\varphi|^2_{g(0)}e^{CK_0T} \equiv C_1^2$$

which proves the proposition for $i = 1$.

Fix $i \geq 2$ and suppose that the proposition is true for every j satisfying $1 \leq j \leq i-1$. Let $f(t) = |\nabla^i \varphi|^2_{g(t)}$. Then, via Proposition 3.2.2,

$$\begin{aligned}
\frac{df}{dt} &\leq \sum_{j=1}^{i} \nabla^{i-j}\partial_t g * \nabla^j \varphi * \nabla^i \varphi \\
&\leq \sum_{j=1}^{i-1} |\nabla^{i-j}\partial_t g||\nabla^j \varphi||\nabla^i \varphi| + |\partial_t g||\nabla^i \varphi|^2 \\
&\leq \sum_{j=1}^{i-1} C K_{i-j} C_j f^{\frac{1}{2}} + C K_0 f \\
&\leq \widetilde{C}(K_0, \ldots, K_{i-1}, C_1, \ldots, C_{i-1})(1+f) \\
&= \widetilde{C}(K_0, \ldots, K_{i-1}, C_1', \ldots, C_{i-1}', T)(1+f).
\end{aligned}$$

Solving the differential inequality, we get

$$\begin{aligned}
1 + f(t) &\leq (1 + f(0))e^{\widetilde{C}T} \\
|\nabla^i \varphi|^2_{g(t)} &\leq (1 + |\nabla^i \varphi|^2_{g(0)})e^{\widetilde{C}T} \\
&\leq (1 + (C_i')^2)e^{\widetilde{C}T} \equiv C_i^2.
\end{aligned}$$

□

Proposition 3.2.4. *Let $(M^n, g(t))$ solve AOF on $[0,T]$, where $n \geq 4$. Fix $r > 0$. Suppose there exist $x \in M$, $r > 0$, and $K > 0$ such that*

$$\max\left[1, \sup_{[0,T]} \|\mathrm{Rm}\|_{C^0(B_{g(T)}(x,2r),g(t))}\right] + \sum_{j=1}^{3n/2-3} \sup_{[0,T]} \|\nabla^j \mathrm{Rm}\|^{\frac{2}{j+2}}_{C^0(B_{g(T)}(x,2r),g(t))} < K. \tag{3.4}$$

Then for all $l \geq 0$ *and* $t \in (0, T]$,

$$\|\nabla^l \mathrm{Rm}\|_{L^2(B_{g(T)}(x,r),g(t))} \leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0,T]} \|\mathrm{Rm}\|_{L^2(B_{g(T)}(x,2r),g(t))}, \tag{3.5}$$

where $C = C(n, l, K, T, r)$.

Proof. Let φ be a cutoff function that is equal to 1 on $B_{g(T)}(x, r)$ and supported on $B_{g(T)}(x, 2r)$. The inequality (3.4) provides C^0 bounds for the first $\frac{n}{2} - 1$ covariant derivatives of Rm, so that

$$\max_{0 \leq j \leq \frac{n}{2}} \|\nabla^j \varphi\|_{C^0(M,g(T))} \leq C'(n, K, r). \tag{3.6}$$

The inequality (3.6) provides bounds for the first $\frac{n}{2}$ covariant derivatives of φ at time T, and the inequality (3.4) induces bounds on the first $\frac{n}{2} - 1$ covariant derivatives of $\widehat{\mathcal{O}}$. We therefore are able to, for each $t \in [0, T]$ and j satisfying $0 \leq j \leq \frac{n}{2}$, to obtain via Proposition 3.2.3 bounds given by

$$\|\nabla^j \varphi\|_{C^0(M,g(t))} \leq \widetilde{C}_j(n, K, r, T).$$

Therefore, via Proposition 3.1.11,

$$\begin{aligned}
\|\nabla^l \mathrm{Rm}\|_{L^2(B_{g(T)}(x,r),g(t))} &\leq \|\varphi^{l+\frac{n}{2}} \nabla^l \mathrm{Rm}\|_{L^2(M,g(t))} \\
&\leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0,T]} \|\mathrm{Rm}\|_{L^2(\mathrm{supp}(\varphi),g(t))} \\
&= C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t \in [0,T]} \|\mathrm{Rm}\|_{L^2(B_{g(T)}(x,2r),g(t))}
\end{aligned}$$

where $C = C(n, l, K, T, r)$. □

We are now able to prove the pointwise smoothing estimates given in Theorem 1.2.1.

Proof of Theorem 1.2.1. We adapt the proof of Theorem 1.3 in Streets [44]. We will show that if this inequality fails, we can construct a blowup limit that is flat and has nonzero curvature. Consider the function given by

$$f_m(x,t,g) = \sum_{j=1}^{m} |\nabla^j \mathrm{Rm}(g(x,t))|_{g(t)}^{\frac{2}{j+2}}.$$

It suffices to show that

$$f_m(x,t,g) \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right) \tag{3.7}$$

since for every l satisfying $1 \leq l \leq m$,

$$|\nabla^l \mathrm{Rm}(g(x,t))|_{g(t)}^{\frac{2}{l+2}} \leq \sum_{j=1}^{m} |\nabla^j \mathrm{Rm}(g(x,t))|_{g(t)}^{\frac{2}{j+2}} = f_m(x,t,g) \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)$$

and

$$|\nabla^l \mathrm{Rm}(g(x,t))|_{g(t)} \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)^{\frac{l+2}{2}} \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)^{\frac{m+2}{2}}.$$

Suppose that the inequality (3.7) fails. It suffices to take $m \geq \frac{3n}{2} - 3$. Without loss of generality, for each $i \in \mathbb{N}$ there exists a solution to AOF $(M_i^n, g_i(t))$ and $(x_i, t_i) \in M_i \times (0,T]$ such that

$$i < \frac{f_m(x_i,t_i,g_i)}{K + t_i^{-\frac{2}{n}}} = \sup_{M_i \times (0,T]} \frac{f_m(x,t,g_i)}{K + t^{-\frac{2}{n}}} < \infty.$$

and define a new sequence of blown up metrics by

$$\widetilde{g}_i(t) = \lambda_i g_i(t_i + \lambda_i^{-\frac{n}{2}} t),$$

where $\lambda_i = f_m(x_i, t_i, g_i)$. We will show in the proof of Theorem 1.2.4 that these metrics also solve AOF. These metrics, which are defined for $t \in [-\lambda_i^{\frac{n}{2}} t_i, 0]$, are eventually defined on $[-1, 0]$ since as $i \to \infty$,

$$t_i^{\frac{2}{n}} \lambda_i = \frac{f_m(x_i, t_i, g_i)}{t_i^{-\frac{2}{n}}} \geq \frac{f_m(x_i, t_i, g_i)}{K + t_i^{-\frac{2}{n}}} \to \infty.$$

Replace the sequence of AOF solutions $\{(M_i, \widetilde{g}_i(t))\}_{i \in \mathbb{N}}$ with the tail subsequence for which $\lambda_i^{\frac{n}{2}} t_i > 1$. The curvatures of these manifolds converge to 0 since as $i \to \infty$,

$$|\mathrm{Rm}(\widetilde{g}_i)|_{\widetilde{g}_i} \leq \frac{K}{\lambda_i} = \frac{K}{f_m(x_i, t_i, g_i)} \leq \frac{K + t_i^{-\frac{2}{n}}}{f_m(x_i, t_i, g_i)} \to 0. \tag{3.8}$$

Furthermore, there is a uniform C^m estimate on the curvature given by

$$\begin{aligned} f_m(x, t, \widetilde{g}_i) &= \frac{f_m(x, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{\lambda_i} \\ &= \frac{f_m(x, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{f_m(x_i, t_i, g_i)} \\ &\leq \frac{K + (t_i + t\lambda_i^{-\frac{n}{2}})^{-\frac{2}{n}}}{K + t_i^{-\frac{2}{n}}} \\ &\leq \frac{K + t_i^{-\frac{2}{n}}(1 + \frac{t}{2})^{-\frac{2}{n}}}{K + t_i^{-\frac{2}{n}}} \\ &\leq 2^{\frac{2}{n}} \end{aligned} \tag{3.9}$$

for all $i \in \mathbb{N}$ and $(x, t) \in M_i \times [-1, 0]$.

Let $B(0, 1)$ be the open Euclidean ball in $\mathbb{R}^n$ centered at 0 with radius 1, $\varphi_i : B(0, 1) \to M_i$ be given by $\exp_{x_i}$ with respect to $g_i(0)$ for each $i \in \mathbb{N}$, and $h_i(t) \equiv \varphi_i^* g_i(t)$. The uniform C^0 bound on $\mathrm{Rm}(\widetilde{g}_i(t))$ given by (3.9) induces a uniform bound on $(\varphi_i)_*$ (see Petersen [40]) which permits the uniform C^m estimate (3.9) on $\mathrm{Rm}(\widetilde{g}_i(t))$ to lift to a uniform C^m estimate on $\mathrm{Rm}(h_i(t))$. Furthermore, $h_i(t)$ solves AOF for all i since φ_i does not depend on t.

Since $m \geq \frac{3n}{2} - 3$, we have uniform C^0 bounds on $\nabla^j \widehat{\mathcal{O}}(h(t))$ for $0 \leq j \leq \frac{n}{2} - 1$. Via Proposition 3.2.4, we obtain uniform bounds on the $L^2(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all covariant derivatives of $\mathrm{Rm}(h_i(0))$. Since the metrics $h_i(0)$ are uniformly equivalent to the Euclidean metric, the Sobolev constant of $B_{h_i(0)}(0, \frac{1}{2})$ is uniformly bounded for all i. Via the Kondrakov compactness theorem, we thus obtain uniform bounds on the $C^0(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all covariant derivatives of $\mathrm{Rm}(h_i(0))$. The Taylor expansion for h_i in terms of geodesic coordinates about 0 with curvature coefficients can then be used to obtain uniform bounds on the $C^0(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all partial derivatives of $h_i(0)$. Finally, by the Arzelà-Ascoli theorem, after taking a subsequence, still named $\{h_i(0)\}_{i\in\mathbb{N}}$, we get $h_i(0) \to h_\infty$ in $C^\infty(B(0, \frac{1}{2}))$ for some Riemannian metric h_∞. We have already shown with inequality (3.8) that $(B(0, \frac{1}{2}), h_\infty)$ is flat. However, for all $i \in \mathbb{N}$,

$$\begin{aligned}
f_m(x_i, 0, \tilde{g}_i) &= \sum_{j=1}^{m} |\nabla^j_{\widetilde{g}_i} \mathrm{Rm}(\widetilde{g}_i)(x_i, 0)|^{\frac{2}{2+j}}_{\widetilde{g}_i(0)} \\
&= \sum_{j=1}^{m} \left(\lambda_i^{-\frac{j+2}{2}} |\nabla^j \mathrm{Rm}(x_i, t_i)|_{g(t_i)} \right)^{\frac{2}{2+j}} \\
&= \sum_{j=1}^{m} \lambda_i^{-1} |\nabla^j \mathrm{Rm}(x_i, t_i)|^{\frac{2}{2+j}}_{g(t_i)} \\
&= \lambda_i^{-1} \lambda_i = 1.
\end{aligned}$$

Also, $f_m(0, 0, h_i) = 1$ for all i since $(\varphi_i)_*$ is the identity map at $0 = \varphi_i^{-1}(x_i)$. Therefore $f_m(0, 0, h_\infty) = 1$. This is a contradiction, thereby proving the inequality (3.7). □

3.3 Long Time Existence

In this section, we prove that if a solution $(M, g(t))$ to the AOF only exists for a finite time T, then $\|\mathrm{Rm}\|_{C^0(g(t))}$ becomes unbounded along a sequence $\{(x_n, t_n)\}_{n=1}^{\infty} \subset M \times [0, T)$ with $t_n \uparrow T$. We will prove this theorem by showing that if actually

$$\sup_{t \in [0,T)} \|\mathrm{Rm}\|_{C^0(g(t))} = K < \infty, \tag{3.10}$$

then the solution $g(t)$ exists past the time T. In order to show this, we show that (3.10) and the pointwise smoothing estimates on $|\nabla^k \mathrm{Rm}|_{g(t)}$ induce bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ with respect to some fixed background metric $\bar{g}$ and connection $\bar{\nabla}$. We also show that (3.10) implies uniform convergence of $g(t)$ to some continuous metric $g(T)$. The bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ imply that $g(T)$ is smooth, so that we can extend the solution $g(t)$ past the time T via the short time existence Theorem 2.3.1.

We first show that if (3.10) holds, the metrics $g(t)$ converge uniformly as $t \uparrow T$ to a continuous metric $g(T)$ equivalent to each $g(t)$. The following lemma is from Chow-Knopf [14]:

Lemma 3.3.1. *Let M be a closed manifold. For $0 \leq t < T \leq \infty$, let $g(t)$ be a one-parameter family of metrics on M depending smoothly on both space and time. If there exists a constant $C < \infty$ such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(t)} dt \leq C$$

for all $x \in M$, then

$$e^{-C} g(x,0) \leq g(x,t) \leq e^{C} g(x,0)$$

for all $x \in M$ and $t \in [0,T)$. Furthermore, as $t \uparrow T$, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that for all $x \in M$,

$$e^{-C} g(x,0) \leq g(x,T) \leq e^{C} g(x,0).$$

Lemma 3.3.2. *Let M be a compact manifold and let $(M, g(t))$ be a solution to AOF on $[0,T)$ such that*

$$\sup_{t\in[0,T)} \|\mathrm{Rm}\|_{C^0(g(t))} = K < \infty.$$

Then $g(t)$ converges uniformly as $t \uparrow T$ to a continuous metric $g(T)$ that is uniformly equivalent to $g(t)$ for every $t \in [0,T]$.

Proof. Since Proposition 1.3.3 states that

$$\frac{\partial g}{\partial t} = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}),$$

in order to apply the preceding Lemma 3.3.1 it suffices to show that $|\nabla^k \mathrm{Rm}|_{g(t)}$ is bounded on $M \times [0,T)$ for all k satisfying $0 \leq k \leq n-2$. Using the smoothing estimate provided in Theorem 1.2.1, we get

$$\max_{0\leq k\leq n-2}\ \sup_{M\times[0,T)} |\nabla^k \mathrm{Rm}|_{g(t)} \leq \max_{0\leq k\leq n-2}\ \sup_{M\times[0,\frac{T}{2}]} |\nabla^k \mathrm{Rm}|_{g(t)} + C\big(\widetilde{K} + (\tfrac{T}{2})^{-\frac{2}{n}}\big)^{\frac{n}{2}},$$

where $C = C(n)$ and $\widetilde{K} = \max\{K, 1\}$.

So $\frac{\partial g}{\partial t}$ is bounded on $M \times [0,T)$ and the metrics $g(t)$ converge uniformly as $t \uparrow T$ to a continuous metric $g(T)$ uniformly equivalent to each $g(t)$. ∎

Since M is a compact manifold, we can obtain bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ by taking the maximum of bounds taken on finitely many coordinate patches. On such a coordinate patch, we can assume that the fixed metric is just the Euclidean one. Thus we will only need to bound the partial derivatives of g and $\widehat{\mathcal{O}}$.

Lemma 3.3.3. *Let M be a compact manifold and let $(M, g(t))$ be a solution to AOF on $[0, T)$. Let U be a coordinate patch on M. Fix $m \geq 0$. Suppose that for $0 \leq i \leq m+n-1$, there exist constants C_i such that $|\nabla^i_{g(t)} \mathrm{Rm}(g(t))|_{g(t)} < C_i$ on $M \times [0, T)$. Then for all $(x, t) \in U \times [0, T)$,*

$$|\partial^m g(x,t)|_{g(t)} < \widetilde{C}_1(g(0), C_0, \ldots, C_{m+n-1})$$
$$|\partial^m \widehat{\mathcal{O}}(x,t)|_{g(t)} < \widetilde{C}_2(g(0), C_0, \ldots, C_{m+n-1}).$$

Proof. We prove this by induction. First we bound ∂g. We have

$$\partial_t \partial g = \partial \partial_t g = (\nabla + \Gamma) * \partial_t g = \nabla \widehat{\mathcal{O}} + \Gamma * \widehat{\mathcal{O}}.$$

From the definition of $\widehat{\mathcal{O}}$, we obtain the bound $|\nabla \widehat{\mathcal{O}}| < C(C_0, \ldots, C_{n-1})$. Then, since $\partial_t \Gamma = \nabla \partial_t g = \nabla \widehat{\mathcal{O}}$, Γ can be bounded in terms of the initial metric and $\nabla \widehat{\mathcal{O}}$ after integrating. So $\partial \widehat{\mathcal{O}} = \partial_t \partial g$ is uniformly bounded by $C(g(0), C_0, \ldots, C_{n-1})$, and so is ∂g after integrating.

Assume that $m \geq 2$ and

$$|\partial^i g| < C(g(0), C_0, \ldots, C_{i+n-1}) \text{ for } 0 \leq i \leq m-1,$$
$$|\partial^i \widehat{\mathcal{O}}| < C(g(0), C_0, \ldots, C_{i+n-1}) \text{ for } 0 \leq i \leq m-1,$$
$$|\partial^i \Gamma| < C(g(0), C_0, \ldots, C_{i+n-1}) \text{ for } 0 \leq i \leq m-2.$$

We wish to bound $\partial^m g$. It suffices to bound $\partial^m \widehat{\mathcal{O}}$ since $\partial_t \partial^m g = \partial^m \partial_t g = \partial^m \widehat{\mathcal{O}}$. We define $\mathsf{P}^m_k(\Gamma)$ to be a polynomial in $\Gamma, \ldots, \partial^{k-1}\Gamma$ where each term contains m partial derivatives of g:

$$\mathsf{P}_k^m(\Gamma) = \sum_{\substack{l+i_1+\cdots+i_l=m \\ 1\le l\le k}} \partial^{i_1}\Gamma * \cdots * \partial^{i_l}\Gamma.$$

We can express $\partial^m\widehat{\mathcal{O}}$ as

$$\partial^m\widehat{\mathcal{O}} = \nabla^m\widehat{\mathcal{O}} + \sum_{i=0}^{m-1} \partial^i\widehat{\mathcal{O}} * \mathsf{P}_i^{m-i}(\Gamma). \tag{3.11}$$

We prove equation (3.11) by induction. First, the equation holds when $m = 1$: $\partial\widehat{\mathcal{O}} = (\nabla + \Gamma) * \widehat{\mathcal{O}} = \nabla\widehat{\mathcal{O}} + \Gamma * \widehat{\mathcal{O}}$. Assume the equation (3.11) holds for $0 \le i \le m$. Then

$$\begin{aligned}
\nabla^{m+1}\widehat{\mathcal{O}} &= (\partial + \Gamma)\nabla^m\widehat{\mathcal{O}} \\
&= \partial^{m+1}\widehat{\mathcal{O}} + \partial^m\widehat{\mathcal{O}} * \Gamma + \sum_{i=0}^{m-1}\left[\partial^{i+1}\widehat{\mathcal{O}} * \mathsf{P}_i^{m-i}(\Gamma) + \partial^i\widehat{\mathcal{O}} * \mathsf{P}_{i+1}^{m+1-i}(\Gamma)\right] \\
&= \partial^{m+1}\widehat{\mathcal{O}} + \partial^m\widehat{\mathcal{O}} * \Gamma + \sum_{i=1}^{m} \partial^i\widehat{\mathcal{O}} * \mathsf{P}_i^{m+1-i}(\Gamma) + \sum_{i=0}^{m-1} \partial^i\widehat{\mathcal{O}} * \mathsf{P}_{i+1}^{m+1-i}(\Gamma) \\
&= \partial^{m+1}\widehat{\mathcal{O}} + \sum_{i=0}^{m} \partial^i\widehat{\mathcal{O}} * \mathsf{P}_{i+1}^{m+1-i}(\Gamma).
\end{aligned}$$

From the equation (3.11), we see that in order to bound $\partial^m\widehat{\mathcal{O}}$, we only need to bound $\partial^{m-1}\Gamma$. We have

$$\begin{aligned}
|\partial_t\partial^{m-1}\Gamma| &= |\partial^{m-1}\partial_t\Gamma| \\
&= |\partial^{m-1}(g^{-1} * \nabla\widehat{\mathcal{O}})| \\
&= \sum_{i=0}^{m-1} |\partial^i\nabla\widehat{\mathcal{O}} * \partial^{m-1-i}g| \\
&\le C(g(0), C_0, \ldots, C_{m+n-2}) \sum_{i=0}^{m-1} |\partial^i\nabla\widehat{\mathcal{O}}|.
\end{aligned} \tag{3.12}$$

We bound $\partial^i \nabla \widehat{\mathcal{O}}$ via the equation

$$\partial^i \nabla \widehat{\mathcal{O}} = \nabla^{i+1} \widehat{\mathcal{O}} + \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+1}^{i-j+1}(\Gamma). \tag{3.13}$$

In order to verify this via induction, we have that for $i = 1$, $\partial \nabla \widehat{\mathcal{O}} = \nabla^2 \widehat{\mathcal{O}} + \Gamma * \nabla \widehat{\mathcal{O}}$. If equation (3.13) holds for the ith partial derivative,

$$\begin{aligned}
\partial^{i+1} \nabla \widehat{\mathcal{O}} &= (\nabla + \Gamma) \nabla^{i+1} \widehat{\mathcal{O}} + \partial \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+1}^{i-j+1}(\Gamma) \\
&= \nabla^{i+2} \widehat{\mathcal{O}} + \nabla^{i+1} \widehat{\mathcal{O}} * \Gamma + \sum_{j=1}^{i} (\nabla + \Gamma) \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+1}^{i-j+1}(\Gamma) + \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+1}^{i-j+2}(\Gamma) \\
&= \nabla^{i+2} \widehat{\mathcal{O}} + \nabla^{i+1} \widehat{\mathcal{O}} * \Gamma + \sum_{j=2}^{i+1} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+2}^{i-j+2}(\Gamma) + \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+2}^{i-j+2}(\Gamma) \\
&\quad + \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+1}^{i-j+2}(\Gamma) \\
&= \nabla^{i+2} \widehat{\mathcal{O}} + \sum_{j=1}^{i+1} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}_{i-j+2}^{i-j+2}(\Gamma).
\end{aligned}$$

If $0 \leq i \leq m-1$, then the highest partial derivative of Γ that appears in equation (3.13) is of order at most $m-2$, so $\partial^i \nabla \widehat{\mathcal{O}}$ is bounded in terms of covariant derivatives of $\widehat{\mathcal{O}}$ and previously bounded partial derivatives of Γ. Therefore, via equation (3.12), $\partial^{m-1}\Gamma$ and $\partial^m \widehat{\mathcal{O}}$ are bounded. □

Proof of Theorem 1.2.2. Suppose that equation (3.10) holds. By Lemma 3.3.2, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ as $t \uparrow T$. We show that $g(T)$ is C^∞ on M. It suffices to show for each $k \in \mathbb{N}$ that $g(T)$ is C^k on any coordinate patch since we can take a maximum over finitely many of them to show that $g(T)$ is C^k on M. We have

$$g(t) = g(0) + \int_0^t \widehat{\mathcal{O}}(\tau)\, d\tau.$$

Taking limits as $t \uparrow T$, we get

$$g(T) = g(0) + \int_0^T \widehat{\mathcal{O}}(\tau)\, d\tau.$$

This permits us to take the kth partial derivative:

$$\partial^k g(T) = \partial^k g(0) + \int_0^T \partial^k \widehat{\mathcal{O}}(\tau)\, d\tau.$$

The bounds on $\partial^k g$ and $\partial^k \widehat{\mathcal{O}}$ from Lemma 3.3.3 therefore imply a bound on $\partial^k g(T)$. So $g(T)$ is C^∞ on M. Furthermore, since

$$|\partial^k g(T) - \partial^k g(t)| \leq \int_t^T |\partial^k \widehat{\mathcal{O}}(\tau)|\, d\tau \leq C_k(T - t),$$

the metrics $g(t)$ converge in C^∞ to $g(T)$. So $g(t)$ is a C^∞ solution to AOF on $[0, T]$. Then the short time existence Theorem 2.3.1 applied to $g(t)$ with initial metric $g(T)$ allows us to extend $g(t)$ past T. This contradicts the assumption that T was the maximal time for the solution $(M, g(t))$. $\square$

Chapter 4

Compactness and Convergence

4.1 Compactness

In this section, we give compactness results for AOF similar to Hamilton's compactness theorem for solutions of the Ricci flow. We first prove a proposition that states that for a sequence of metrics, uniform bounds on the spacetime derivatives of curvature and the derivatives of the metric at one time extend to uniform bounds on the spacetime derivatives of the metric. This is used to prove the compactness Theorem 1.2.3 for a sequence of complete pointed solutions of AOF. We then give the proofs of Theorem 1.2.4, which allows us to obtain a singularity model from a singular solution, and Theorem 1.2.5, which describes the behavior at time ∞ of a nonsingular solution.

The type of convergence of manifolds we will consider is pointed C^∞ Cheeger-Gromov convergence.

Definition 4.1.1 (C^∞ Cheeger-Gromov convergence ([13] Definition 3.5))**.** A sequence $\{(M_k^n, g_k, O_k)\}_{k\in\mathbb{N}}$ of complete pointed Riemannian manifolds **converges** (in the Cheeger-Gromov topology) to a complete pointed Riemannian manifold $(M_\infty^n, g_\infty, O_\infty)$ if there exist

1. an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$,

2. a sequence of diffeomorphisms $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$, such that $\left(U_k, \Phi_k^*\left[g_k|_{V_k}\right]\right)$ converges in C^∞ to (M_∞, g_∞) uniformly on compact sets in M_∞.

The following compactness result of Hamilton allows us to extract a convergent subsequence of manifolds at a fixed time.

Theorem 4.1.2 (Cheeger-Gromov compactness theorem ([25] Theorem 2.3))**.** *Let* $\{(M_k^n, g_k, O_k)\}_{k\in\mathbb{N}}$ *be a sequence of complete pointed Riemannian manifolds that satisfy*

$$|\nabla_k^p \mathrm{Rm}_k|_k \le C_p \text{ on } M_k$$

for all $p \ge 0$ *and* k *where* $C_p < \infty$ *is a sequence of constants independent of* k *and*

$$\mathrm{inj}_{g_k}(O_k) \ge \iota_0$$

for some constant $\iota_0 > 0$*. Then there exists a subsequence* $\{j_k\}_{k\in\mathbb{N}}$ *such that* $\{M_{j_k}, g_{j_k}, O_{j_k})\}_{k\in\mathbb{N}}$ *converges to a complete pointed Riemannian manifold*

$$(M_\infty^n, g_\infty, O_\infty)$$

as $k \to \infty$*.*

The following proposition allows us to extend bounds on the derivatives of a sequence of metrics at one time to bounds that are uniform over an interval.

Proposition 4.1.3. *Let* (M, g) *be a Riemannian manifold and* L *be a compact subset of* M*. Let* $\{g_i\}_{i\in\mathbb{N}}$ *be a collection of Riemannian metrics that are solutions of AOF on neighborhoods containing* $L \times [\beta, \psi]$*. Let* $t_0 \in [\beta, \psi]$ *and fix* $k \ge n-2$*. Let unmarked objects such as* ∇ *and*

$|\cdot|$ be taken with respect to g, and let objects such as ∇_k and $|\cdot|_k$ be taken with respect to g_k. Suppose that:

1. *The metrics $g_i(t_0)$ are uniformly equivalent to g for every $i \in \mathbb{N}$: for some $B_0 > 0$, $B_0^{-1} g \leq g_i(t_0) \leq B_0 g$.*
2. *For each $1 \leq p \leq k$, there exists a uniform bound C_p on L independent of i such that $|\nabla^p g_i(t_0)| \leq C_p$.*
3. *For each $0 \leq p+q \leq k+n-2$, there exists a uniform bound $C'_{p,q}$ on $L \times [\beta, \psi]$ independent of i such that $|\partial_t^q \nabla_{g_i}^p \mathrm{Rm}(g_i)|_{g_i} \leq C'_{p,q}$.*

Then:

1. *The metrics $g_k(t)$ are uniformly equivalent to g for every $i \in \mathbb{N}$ and $t \in [\beta, \varphi]$: for some $B = B(t, t_0) > 0$, $B^{-1} g \leq g_i(t) \leq Bg$.*
2. *For every p, q satisfying $0 \leq p+q \leq k$, there is a uniform bound $\widetilde{C}_{p,q}$ on $L \times [\beta, \psi]$ independent of i such that $|\partial_t^q \nabla^p g_i(t)| \leq \widetilde{C}_{p,q}$.*

Proof. We adapt the proof of Lemma 3.11 in Chow et. al. [13]. The uniform equivalence of the g_k and g on $L \times [\beta, \varphi]$ follow from the given bounds for $|\nabla_{g_i}^p \mathrm{Rm}(g_i)|_{g_i}$ on $L \times [\beta, \psi]$. Define the bounds $\overline{C}_j$ for j satisfying $0 \leq j \leq j - n + 2$ by

$$|\nabla_k^j \widehat{\mathcal{O}}_k| \leq \sum_{p=j}^{n-2+j} a_p C C'_{p,0} \equiv \overline{C}_j.$$

Suppose that $(p, q) = (1, 0)$. Hamilton showed in Theorem 7.1 of [23] that $\partial_t \Gamma = g^{-1} * \nabla \partial_t g$. Then

$$|\partial_t (\Gamma_k - \Gamma)|_k \leq C |\nabla_k \widehat{\mathcal{O}}_k|_k \leq C \overline{C}_1.$$

It follows that

$$
\begin{aligned}
|\nabla g_k(t)| &\le B(t,t_0)^{3/2}|\nabla g_k(t)|_k \\
&\le B(\psi,\beta)^{3/2} 2|\Gamma_k(t)-\Gamma|_k \\
&\le B(\psi,\beta)^{3/2}(C\overline{C}_1|\psi-\beta| + 3B_0^{3/2}C_1) \equiv \tilde{C}_{1,0}.
\end{aligned}
$$

Next, we prove the lemma for p satisfying $p \le k$ when $q = 0$. We will show that for $p \ge 1$,

$$
|\nabla^p \partial_t g_k| \le C_p''|\nabla^p g_k| + C_p''', \quad |\nabla^p g_k| \le \tilde{C}_{p,0}. \tag{4.1}
$$

If $p = 1$, then

$$
\begin{aligned}
|\nabla \partial_t g_k(t)| &\le B(t,t_0)^{3/2}|(\nabla - \nabla_k)\partial_t g_k + \nabla_k \partial_t g_k|_k \\
&\le B(t,t_0)^{3/2} C|\Gamma - \Gamma_k|_k |\partial_t g_k|_k + |\nabla_k \partial_t g_k|_k \\
&\le B(t,t_0)^{3/2} C|\nabla g_k|\overline{C}_0 + \overline{C}_1
\end{aligned}
$$

and we have already shown that $|\nabla g_k| \le \tilde{C}_{1,0}$.

Let $N \ge 2$ and assume that (4.1) is true for $0 \le p \le N-1$. The telescoping identity

$$
\nabla^N A - \nabla_k^N A = \sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1} A
$$

results in the following inequality:

$$
|\nabla^N \partial_t g_k| \le |\nabla^{N-1}(\nabla - \nabla_k)\partial_t g_k| + \sum_{i=2}^{N} |\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\partial_t g_k| + |\nabla_k^N \partial_t g_k|. \tag{4.2}
$$

Using the induction hypothesis and the given estimates for $|\nabla^p_{g_i}\mathrm{Rm}(g_i)|_{g_i}$, we estimate the terms of the preceding inequality (4.2). Collecting terms yields

$$|\nabla^N \partial_t g_k| \leq C''_N |\nabla^N g_k| + C'''_N.$$

Applying the preceding inequality, we get

$$\begin{aligned}\partial_t |\nabla^N g_k|^2 &= 2\langle \partial_t \nabla^N g_k, \nabla^N g_k\rangle \\ &\leq |\partial_t \nabla^N g_k|^2 + |\nabla^N g_k|^2 \\ &\leq (1 + 2(C''_N)^2)|\nabla^N g_k|^2 + 2(C'''_N)^2.\end{aligned}$$

After solving an ODE, we obtain

$$|\nabla^N g_k|^2(t) \leq e^{(1+2(C''_N)^2)(\psi - t_0)} \left[C_N + \frac{2(C'''_N)^2}{1+2(C''_N)^2}\left(1 - e^{(1+2(C''_N)^2)(t_0-\beta)}\right)\right] \equiv \tilde{C}^2_{N,0}.$$

This completes the inductive proof of (4.1) and the proof of the proposition when $q = 0$. Since $\partial_t^q \nabla^p g_k = \nabla^p \partial_t^q g_k$, a similar procedure may be used to prove the proposition when $q > 0$. □

We are now able to prove the compactness Theorem 1.2.3 for solutions of the AOF via a modification of the proof given by Hamilton in [25] of the compactness theorem for Ricci flow. We need the following lemma.

Proposition 4.1.4. *(Chow et al. [13] Corollary 3.15) Let (M^n, g) be a Riemannian manifold and let $L \subset M^n$ be compact. Furthermore, let p be a nonnegative integer. If $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of Riemannian metrics on L such that*

$$\sup_{0\leq|\alpha|\leq p+1}\ \sup_{x\in L} |\nabla^\alpha g_k| \leq C < \infty$$

and if there exists $\delta > 0$ such that $g_k(V,V) \geq \delta g(V,V)$ for all $V \in TM$,then there exists a subsequence $\{g_k\}$ and a Riemannian metric g_∞ on L such that g_k converges in C^p to g_∞ as $k \to \infty$.

Proof of Theorem 1.2.3. Since we are given a uniform bound on $|\mathrm{Rm}(g_k)|_{g_k}$, the pointwise smoothing estimates given by Theorem 1.2.1 furnish uniform bounds on
$\|\nabla^m_{g_k(t_0)}\mathrm{Rm}(g_k(t_0))\|_{C^0(g_k(t_0))}$ for all $m \in \mathbb{N}$. Therefore, since the (M_k, g_k) are complete, the Cheeger-Gromov compactness Theorem 4.1.2 yields a subsequence of $\{(M_k, g_k(t), O_k)\}_{k\in\mathbb{N}}$, also called $\{(M_k, g_k(t), O_k)\}_{k\in\mathbb{N}}$, for which $\{(M_k, g_k(t_0), O_k)\}_{k\in\mathbb{N}}$ converges to a complete pointed Riemannian manifold
$(M^n_\infty, h, O_\infty)$.

Fix a compact subset L of M_∞ and a closed interval $[\beta,\psi]$, with $t_0 \in (\beta,\psi)$ of (α,ω). Since $\{(M_k, g_k(t_0), O_k)\}_{k\in\mathbb{N}}$ converges to (M_∞, h, O_∞), by definition there exists an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$ and a sequence of diffeomorphisms $\Phi_k : U_k \to V_k \equiv \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$, such that if $h_k \equiv \Phi_k^*\left[g_k(t_0)|_{V_k}\right]$, then (U_k, h_k) converges in C^∞ to (M_∞, h) on compact sets in M_∞. Since the U_k exhaust M_∞, $L \subset U_k$ for some k. So the metrics h_k are uniformly equivalent to h on L. We also obtain from the C^∞ convergence that for each $p \geq 1$, there exists a C_p independent of $x \in L$ and k such that $|\nabla^p_h h_k|_h \leq C_p$.

Let $G_k(t) \equiv \Phi_k^*\left[g_k(t)|_{V_k}\right]$; then $h_k = G_k(t_0)$. From the pointwise smoothing estimates given by Theorem 1.2.1, for each p we obtain a bound $C'_{p,0}$ uniform on $L \times [\beta,\psi]$ independent of k such that $|\nabla^p_{G_k}\mathrm{Rm}(G_k)|_{G_k} \leq C'_{p,0}$ on $L \times [\beta,\psi]$. Using the expression of $\partial_t\nabla^p_{G_k}\mathrm{Rm}(G_k)$ in terms of covariant derivatives of $\mathrm{Rm}(G_k)$ given by Proposition 2.2.2, for each (p,q) we obtain a bound $C'_{p,q}$ uniform on $L \times [\beta,\psi]$ independent of k such that $|\partial^q_t\nabla^p_{G_k}\mathrm{Rm}(G_k)|_{G_k} \leq C'_{p,q}$ on $L \times [\beta,\psi]$. We then conclude via Proposition 4.1.3 that the metrics G_k are uniformly equivalent to h on $L \times [\beta, \psi]$ and that for every $p, q \geq 0$, there is a constant $\tilde{C}_{p,q}$ independent of k such that $|\partial_t^q \nabla_h^p G_k|_h \leq \tilde{C}_{p,q}$ on $L \times [\beta, \psi]$.

The uniform equivalence of the G_k to h and the uniform bounds $|\partial_t^q \nabla_h^p G_k|_h \leq \tilde{C}_{p,q}$ allow us to apply an Arzelà-Ascoli type Proposition 4.1.4 to the metrics $G_k(t) + dt^2$ on $L \times [\beta, \psi]$ and obtain a subsequence that converges in $C^\infty(L \times [\beta, \psi], h + dt^2)$ to a metric $g_\infty(t) + dt^2$ such that $g_\infty(0) = h$; we relabel the convergent subsequence as $\{G_k(t) + dt^2\}_{k \in \mathbb{N}}$. It follows that $g_\infty(t) + dt^2$ is uniformly equivalent to $G_1(t) + dt^2$ on $L \times [\beta, \psi]$. Then $g_\infty(t) + dt^2$ is uniformly equivalent to $h + dt^2$ on $L \times [\beta, \psi]$ since $G_1(t) + dt^2$ is uniformly equivalent to $h + dt^2$ on $L \times [\beta, \psi]$. Since (M, h) is complete, $(M \times (\alpha, \omega), h + dt^2)$ is also complete. The uniform equivalence of $g_\infty(t) + dt^2$ to $h + dt^2$ on compact subsets of $M \times (\alpha, \omega)$ and the Hopf-Rinow theorem imply that $(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$ is complete.

Since $(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$ is complete, compact sets are equivalent to closed, bounded ones. A compact set in $M_\infty \times (\alpha, \omega)$ is contained in the compact set that is the product of a closed geodesic ball in M_∞ and a closed interval in (α, ω). So the metrics $G_k(t) + dt^2$ subsequentially converge in $C^\infty(M_\infty \times (\alpha, \omega), h + dt^2)$. Let $\{G_k(t) + dt^2\}_{k \in \mathbb{N}}$ be the convergent subsequence. Then $\{(M_k, g_k(t), O_k)\}_{k \in \mathbb{N}}$ converges to $(M_\infty, g_\infty(t), O_\infty)$. It follows that for each p, q, $\partial_t^q \nabla_{g_\infty}^p G_k \to \partial_t^q \nabla_{g_\infty}^p g_\infty$ and $\widehat{\mathcal{O}}(G_k) \to \widehat{\mathcal{O}}(g_\infty)$ in $C(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2)$. Therefore $(M_\infty, g_\infty, O_\infty)$ is a complete pointed solution to AOF for $t \in (\alpha, \omega)$. □

4.2 Convergence to Singularity Models

As our first corollary of the compactness theorem 1.2.3, we show that under suitable conditions, we can obtain a singularity model for the ambient obstruction flow.

Proof of Theorem 1.2.4. We first show that the g_i are also solutions to AOF by showing that if $\tilde{g} = \lambda g$ and g satisfies AOF, given up to constants by

$$\partial_t g = \Delta^{\frac{n}{2}-1}\mathrm{Rc} + \Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}),$$

then $\tilde{g}$ satisfies

$$\partial_t \tilde{g} = \widetilde{\Delta}^{\frac{n}{2}-1}\widetilde{\mathrm{Rc}} + \widetilde{\Delta}^{\frac{n}{2}-2}\widetilde{\nabla}^2 \tilde{R} + \sum_{j=2}^{n/2} P_j^{n-2j}(\widetilde{\mathrm{Rm}}). \tag{4.3}$$

We evaluate the first term of the right side of (4.3):

$$\widetilde{\Delta}^{\frac{n}{2}-1}\widetilde{\mathrm{Rc}} = \left(\lambda^{-1} g^{-1} \nabla^2\right)^{\frac{n}{2}-1} \mathrm{Rc} = \lambda^{1-\frac{n}{2}} \Delta^{\frac{n}{2}-1}\mathrm{Rc}.$$

Similarly, the second term is equal to $\lambda^{1-\frac{n}{2}}\Delta^{\frac{n}{2}-1}\mathrm{Rc}$. The remaining terms are contractions of terms of the form

$$\widetilde{\nabla}^{i_1}\widetilde{\mathrm{Rm}} \otimes \cdots \otimes \widetilde{\nabla}^{i_j}\widetilde{\mathrm{Rm}}$$

with $2 \leq j \leq \frac{n}{2}$ and $i_1 + \cdots + i_j = n - 2j$. In order to contract on all but two indices of the above term, we need to contract $\frac{1}{2}(i_1 + \cdots + i_j + 3j - j - 2) = \frac{n}{2} - 1$ pairs of indices. This implies that $P_j^{n-2j}(\widetilde{\mathrm{Rm}}) = \lambda^{1-\frac{n}{2}} P_j^{n-2j}(\mathrm{Rm})$. The left side of (4.3) is equal to $\lambda^{1-\frac{n}{2}}\partial_t g$. So $\tilde{g}$ satisfies (4.3).

We have $|\mathrm{Rm}(g_i)|_{g_i} \leq 1$ on $M \times [-\lambda_i^{n/2} t_i, 0]$ for each i since the definition of the λ_i implies

$$|\mathrm{Rm}(g_i)|_{g_i}^2 = \lambda_i^{-2}|\mathrm{Rm}|^2 \leq \lambda_i^{-2}\lambda_i^2 \leq 1.$$

Let $k \in \mathbb{N}$. There exists i_k such that if $i \geq i_k$, then $\lambda_i^{n/2} t_i > k$. Then $\{g_i\}_{i \geq i_k}$ is a sequence of complete pointed solutions to AOF on $(-k, 0]$. Since the Sobolev constant is scaling invariant, the uniform bound of $C_S(M,g)$ on $[0,T)$ implies a uniform bound independent of i of $C_S(M,g_i)$ on $[0,T)$. We conclude from Lemma 3.2 of Hebey [26] that there exists a uniform lower bound independent of i for $\inf_{x\in M}\operatorname{vol}(B_{g_i}(x,1))$. This and the bound $|\mathrm{Rm}(g_i)|_{g_i}\le 1$ on $M\times[-\lambda_i^{n/2}t_i,0]$ for all i give a uniform lower bound independent of i for $\mathrm{inj}_{g_i(0)}(x_i)$ via the Cheeger-Gromov-Taylor theorem.

The proof of the compactness theorem 1.2.3 is unchanged if we replace (α,ω) with $(-k,0]$. Thus, by theorem 1.2.3, we obtain subsequential convergence of

$$\{(M,g_i(t),x_i)\}_{i\ge i_k}$$

to a complete pointed solution $(M_\infty,g_\infty(t),x_\infty)$ to AOF for $t\in(-k,0]$. By taking a further diagonal subsequence over the k, we get that $\{(M,g_i(t),x_i)\}_{i\ge 1}$ subsequentially converges to a complete pointed solution $(M_\infty,g_\infty(t),x_\infty)$ to AOF for $t\in(-\infty,0]$. The limit $(M_\infty,g_\infty(t))$ is not flat since

$$|\mathrm{Rm}(g_\infty(0))(x_\infty)|_{g_\infty(0)}=1$$

by the definition of the $g_i(t)$.

We show that M_∞ is noncompact. Lemma 3.9 of Chow-Knopf [14] states that for a one parameter family of Riemannian manifolds $(M,g(t))$, the volume element evolves by $\partial_t dV_g = \frac{1}{2}g^{ij}\partial_t g_{ij}$. Applying the fact that $\mathcal{O}$ is traceless and the divergence theorem,

$$\begin{aligned}
\frac{\partial}{\partial t}\operatorname{vol}(M,g(t)) &= \frac{1}{2}\int_M g^{ij}\frac{\partial g_{ij}}{\partial t}\,dV_{g(t)}\\
&= \frac{1}{2}\int_M [(-1)^{\frac{n}{2}}g^{ij}\mathcal{O}_{ij}+C(n)(\Delta^{\frac{n}{2}-1}R)g^{ij}g_{ij}]\,dV_{g(t)}\\
&= C(n)\int_M \Delta^{\frac{n}{2}-1}R\,dV_{g(t)}\\
&= 0.
\end{aligned}$$

Therefore the volume of $(M, g(t))$ is preserved along the flow. Since $\lambda_i \to \infty$,

$$\operatorname{vol}(M_\infty, g_\infty(t)) = \lim_{i\to\infty} \operatorname{vol}(M, g_i(t)) = \lim_{i\to\infty} \lambda_i^{n/2} \operatorname{vol}(M, g(t_i + \lambda_i^{\frac{n}{2}} t)) = \infty$$

for all $t \in (-\infty, 0]$. So the volume of $(M, g_\infty(t))$ is infinite for all $t \in (-\infty, 0]$. The uniform volume lower bound for the (M, g_i) passes in the limit to a uniform volume lower bound for (M, g_∞). Therefore M_∞ is noncompact by Lemma 8.1 of Bour [6].

Next, we show that the integral of the Q-curvature is nondecreasing along the flow on M. Along the flow, the derivative of $\int_M Q$ is given by

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M Q &= (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \langle \mathcal{O}, \partial_t g \rangle \\
&= (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M (-1)^{\frac{n}{2}} |\mathcal{O}|^2 + C(n) \int_M \langle \mathcal{O}, (\Delta^{\frac{n}{2}-1} R) g \rangle \\
&= \frac{n-2}{2} \int_M |\mathcal{O}|^2,
\end{aligned}$$

where the third line holds since $\mathcal{O}$ is traceless. So the integral of the Q-curvature does not decrease along the flow.

Suppose that

$$\sup_{t\in[0,T)} \int_M Q(g(t))\, dV_{g(t)} < \infty.$$

This is always true when $n = 4$ since the Chern-Gauss-Bonnet theorem gives that for all $t \in [0, T)$,

$$\int_M Q = 8\pi^2 \chi(M) - \tfrac{1}{4} \int_M |W|^2 \leq 8\pi^2 \chi(M).$$

So if the integral of the Q curvature is bounded along the flow,

$$\begin{aligned}\int_0^T \int_M |\mathcal{O}|^2 &= \int_0^T \frac{\partial}{\partial t} \int_M Q \\ &= \lim_{t \uparrow T} \int_M Q(g(t)) - \int_M Q(g(0)) \\ &< \infty.\end{aligned}$$

Let $\{(M, g_i(t), x_i)\}_{i \geq 1}$ be the convergent subsequence previously found in the proof. Fix $k \in \mathbb{N}$. Since $t_i \to T$ and $\lambda_i \to \infty$, we can choose a subsequence of times $\{t_{i_j}\}_{j \in \mathbb{N}}$ as follows:

$$i_1 = \inf \left\{ i : t_i \geq \tfrac{T}{2}, \lambda_i \geq \left(\tfrac{2k}{T}\right)^{\frac{2}{n}} \right\}, \quad i_j = \inf \left\{ i : t_i \geq \tfrac{1}{2}(T + t_{i_{j-1}}), \lambda_i \geq \left(\tfrac{2k}{T - t_{i_{j-1}}}\right)^{\frac{2}{n}} \right\}$$

for $j \geq 2$. We relabel $\{t_{i_j}\}_{j \in \mathbb{N}}$ as$\{t_i\}_{i \in \mathbb{N}}$. Then

$$\sum_{i=1}^{\infty} \int_{t_i - k\lambda_i^{-\frac{n}{2}}}^{t_i} \int_M |\mathcal{O}|^2 < \int_0^T \int_M |\mathcal{O}|^2 < \infty,$$

implying that, using the scaling law $\mathcal{O}(\lambda g) = \lambda^{\frac{2-n}{2}} \mathcal{O}(g)$,

$$\begin{aligned}0 &= \lim_{i \to \infty} \int_{t_i - k\lambda_i^{-\frac{n}{2}}}^{t_i} \int_M |\mathcal{O}(g)|_g^2 \, dV_g \, dt \\ &= \lim_{i \to \infty} \int_{-k}^{0} \int_M \lambda_i^n |\mathcal{O}(g_i)|_{g_i}^2 \lambda_i^{-\frac{n}{2}} \lambda_i^{-\frac{n}{2}} \, dV_{g_i} \, dt \\ &= \lim_{i \to \infty} \int_{-k}^{0} \int_M |\mathcal{O}(g_i)|_{g_i}^2 \, dV_{g_i} \, dt.\end{aligned}$$

Since $\mathcal{O}(g_i) \to \mathcal{O}(g_\infty)$ in C^∞ on compact subsets, this implies that $\mathcal{O}(g_\infty) \equiv 0$ on $[-k, 0]$. So for each $k \in \mathbb{N}$, there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on $[-k, 0]$. By taking a further diagonal subsequence over the k, we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on $(-\infty, 0]$. $\square$

Finally, we provide a corollary of the compactness theorem 1.2.3 characterizing limits of nonsingular solutions to AOF.

Proof of Theorem 1.2.5. Suppose M does not collapse at ∞. Then there exists a sequence $\{(x_i, t_i)\}_{i \in \mathbb{N}} \subset M \times [0, \infty)$ such that $\inf_i \operatorname{inj}_{g(t_i)}(x_i) > 0$. Let $g_i(t) = g(t + t_i)$ for $t \in [-t_i, \infty)$. Let $k \in \mathbb{N}$. Then there exists $i_k \in \mathbb{N}$ such that $t_i > k$ for all $i \geq i_k$. Since $\sup_{t \in [0,\infty)} \|\mathrm{Rm}\|_\infty < \infty$ and $\inf_i \operatorname{inj}_{g_i(0)}(x_i) > 0$, we apply Theorem 1.2.3 to obtain subsequential convergence in the sense of families of pointed Riemannian manifolds of $\{(M, g_i(t), x_i)\}_{i \geq i_k}$ to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF on $(-k, \infty)$. By taking a further diagonal subsequence over the k, we get that $\{(M, g_i(t), x_i)\}_{i \geq 1}$ subsequentially converges to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF on $(-\infty, \infty)$.

If M_∞ is compact, then by the definition of convergence of complete pointed Riemannian manifolds, M_∞ is diffeomorphic to M. Just as in the proof of Theorem 1.2.4, the volume of $(M, g(t))$ is preserved along the flow. So for all $t \in (-\infty, \infty)$,

$$\mathrm{vol}(M_\infty, g_\infty(t)) = \lim_{i \to \infty} \mathrm{vol}(M, g_i(t)) = \lim_{i \to \infty} \mathrm{vol}(M, g(t_i + t)) < \infty.$$

Suppose that

$$\sup_{t \in [0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty.$$

This is always true when $n = 4$ by the Chern-Gauss-Bonnet theorem. Using the same argument as in the proof of Theorem 1.2.4, we obtain

$$\int_0^\infty \int_M |\mathcal{O}|^2 < \infty.$$

Let $\{(M, g_i(t), x_i)\}_{i\geq 1}$ be the convergent subsequence previously found in the proof. Since $t_i \to \infty$, we can choose a subsequence of times $\{t_{i_j}\}_{j\in\mathbb{N}}$ as follows:

$$i_1 = \inf\{i : t_i \geq k\}, \quad i_j = \inf\{i : t_i \geq t_{i_{j-1}} + 2k\}$$

for $j \geq 2$. We relabel $\{t_{i_j}\}_{j\in\mathbb{N}}$ as$\{t_i\}_{i\in\mathbb{N}}$. Then

$$\sum_{i=1}^{\infty} \int_{t_i-k}^{t_i+k} \int_M |\mathcal{O}|^2 < \int_0^{\infty} \int_M |\mathcal{O}|^2 < \infty$$

implies that

$$0 = \lim_{i\to\infty} \int_{t_i-k}^{t_i+k} \int_M |\mathcal{O}(g)|_g^2 \, dV_g \, dt = \lim_{i\to\infty} \int_{-k}^{k} \int_M |\mathcal{O}(g_i)|_{g_i}^2 \, dV_{g_i} \, dt.$$

Since $\mathcal{O}(g_i) \to \mathcal{O}(g_\infty)$ in C^∞ on compact subsets, this implies that $\mathcal{O}(g_\infty) \equiv 0$ on $[-k, k]$. So for each $k \in \mathbb{N}$, there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on $[-k, k]$. By taking a further diagonal subsequence over the k, we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on $(-\infty, \infty)$. Since g_∞ solves the conformal flow $\partial_t g_\infty = (-1)^{n/2} C(n)(\Delta^{\frac{n}{2}-1} R) g$, we see that $g_\infty(t)$ is in the conformal class of $g_\infty(0)$ for all $t \in (-\infty, \infty)$. If M_∞ is compact, we can solve the Yamabe problem for $(M_\infty, [g_\infty(0)])$; the Yamabe problem was solved by Aubin, Trudinger, and Schoen (see [2, 33]). Due to the conformal covariance of $\mathcal{O}$, we obtain a obstruction flat, constant scalar curvature complete pointed solution $(M_\infty, \hat{g}_\infty(t))$ to AOF with $\hat{g}_\infty(t) = \hat{g}_\infty(0)$ for all $t \in (-\infty, \infty)$. □

Chapter 5

Rigidity in Dimension Four and Initial Sobolev Estimate

5.1 Proof of Theorem 1.2.11

The arguments in the proof of the gap theorem for 4-manifolds given by Theorem 1.2.11 are a special case of the arguments used for n-manifolds with $n \geq 6$, and do not require the machinery we develop in subsequent sections. In particular, we need neither a volume growth upper bound nor a quadratic curvature decay bound. Therefore, we will present the proof of Theorem 1.2.11 in this section. All manifolds in this section are 4-dimensional.

Proposition 5.1.1. *Suppose M, φ satisfy the above hypotheses. Suppose $s \geq 1$. For every $\delta > 0$, there exists C such that*

$$\int_M \varphi^{2s-1} \nabla\varphi * \nabla \mathrm{Rm} * \mathrm{Rm} \leq \delta \int_M \varphi^{2s} |\nabla \mathrm{Rm}|^2 + C\Lambda^2 \epsilon_0^2 \delta^{-1} R^{-2}.$$

In particular, there exists C such that

$$\int_M \varphi^{2s-1}\nabla\varphi * \nabla \mathrm{Rm} * \mathrm{Rm} \leq \epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C\Lambda^2\epsilon_0 R^{-2}.$$

Proof. The Cauchy-Schwarz inequality gives

$$\int_M \varphi^{2s-1}\nabla\varphi * \nabla \mathrm{Rm} * \mathrm{Rm} \leq \delta \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + \frac{C\Lambda^2}{\delta R^2}\int_M \varphi^{2s-2}|\mathrm{Rm}|^2.$$

Set $\delta = \epsilon_0$ to obtain the second inequality. □

Proposition 5.1.2. *Suppose M, φ satisfy the above hypotheses. Suppose $s \geq 1$. Then there exists C such that*

$$\Big(\int_M \varphi^{4s}|\mathrm{Rm}|^4\Big)^{\frac{1}{2}} \leq CC_S \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + CC_S\Lambda^2\epsilon_0^2 R^{-2}.$$

Proof. We estimate:

$$\begin{aligned}
\|\varphi^s|\mathrm{Rm}|\|_4^2 &\leq C_S\|\nabla[\varphi^s|\mathrm{Rm}|]\|_2^2 \\
&\leq CC_S\|\varphi^s\nabla|\mathrm{Rm}|\|_2^2 + CC_S\|\varphi^{s-1}|\mathrm{Rm}|\nabla\varphi\|_2^2 \\
&\leq CC_S\int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + CC_S\int_M \varphi^{2s-2}|\mathrm{Rm}|^2|\nabla\varphi|^2 \\
&\leq CC_S\int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + CC_S\Lambda^2\epsilon_0^2 R^{-2}.
\end{aligned}$$

We used the Sobolev inequality in the first line and the Kato inequality in the third line. □

Proposition 5.1.3. *Suppose M, φ satisfy the above hypotheses. Suppose $s \geq 1$. Then there exists C such that*

$$\int_M \varphi^{2s}\mathrm{Rm}^{*3} \leq CC_S\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + CC_S\Lambda^2\epsilon_0^3 R^{-2}.$$

Proof. We apply Hölder's inequality, followed by Proposition 5.1.2:

$$\begin{aligned}\int_M \varphi^{2s}\mathrm{Rm}^{*3} &\leq C\Big(\int_M |\mathrm{Rm}|^2\Big)^{1/2}\Big(\int_M \varphi^{4s}|\mathrm{Rm}|^4\Big)^{1/2}\\ &\leq C\epsilon_0\Big(CC_S\int_M \varphi^{2s}|\nabla\mathrm{Rm}|^2 + CC_S\Lambda^2\epsilon_0^2R^{-2}\Big)\\ &= CC_S\epsilon_0\int_M \varphi^{2s}|\nabla\mathrm{Rm}|^2 + CC_S\Lambda^2\epsilon_0^3R^{-2}.\end{aligned}$$

Our choice of the exponent of φ was sufficiently large to apply Proposition 5.1.2 since $s \geq 1$. □

Proposition 5.1.4. *Suppose M, φ satisfy the above hypotheses. Suppose $s \geq 1$. Then there exist $C_1 = C_1(C_S)$ and $C_2 = C_2(C_S, \Lambda)$ such that*

$$-\int_M \varphi^{2s}\langle\Delta\mathrm{Rm}, \mathrm{Rm}\rangle \leq 2\int_M \varphi^{2s}|\nabla\mathrm{Rc}|^2 + C_1\epsilon_0\int_M \varphi^{2s}|\nabla\mathrm{Rm}|^2 + C_2\epsilon_0R^{-2}.$$

Proof. Let I denote

$$I = -\int_M \varphi^{2s}\langle\Delta\mathrm{Rm}, \mathrm{Rm}\rangle.$$

First,

$$\begin{aligned}I &= -\int_M \varphi^{2s}g^{ja}g^{kb}g^{lc}g^{md}R_{adcb}(\nabla_j\nabla_mR_{lk} - \nabla_j\nabla_lR_{mk} + \nabla_k\nabla_lR_{mj} - \nabla_k\nabla_mR_{lj})\\ &\quad + \int_M \varphi^{2s}\mathrm{Rm}^{*3}\\ &= -2\int_M \varphi^{2s}g^{ja}g^{kb}g^{lc}g^{md}R_{adcb}(\nabla_j\nabla_mR_{lk} - \nabla_j\nabla_lR_{mk}) + \int_M \varphi^{2s}\mathrm{Rm}^{*3}\\ &= -2\int_M \varphi^{2s}g^{ja}g^{kb}g^{lc}g^{md}R_{adcb}\nabla_j\nabla_mR_{lk} + 2\int_M \varphi^{2s}g^{ja}g^{kb}g^{lc}g^{md}R_{adcb}\nabla_j\nabla_lR_{mk}\\ &\quad + \int_M \varphi^{2s}\mathrm{Rm}^{*3}\\ &= 2I_1 + 2I_2 + \int_M \varphi^{2s}\mathrm{Rm}^{*3}. \qquad (5.1)\end{aligned}$$

The first line follows from an identity for $\Delta\mathrm{Rm}$. Then

$$
\begin{aligned}
I_1 &= \int_M \varphi^{2s} g^{kb} g^{lc} g^{md} \nabla_m R_{lk} g^{ja} \nabla_j R_{adcb} + \int_M \varphi^{2s-1} \nabla\varphi * \nabla\mathrm{Rm} * \mathrm{Rm} \\
&\le \int_M \varphi^{2s} g^{kb} g^{lc} g^{md} \nabla_m R_{lk} g^{ja} \nabla_j R_{adcb} + \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C\Lambda^2 \epsilon_0 R^{-2} \\
&= \int_M \varphi^{2s} g^{kb} g^{lc} g^{md} \nabla_m R_{lk} g^{ja} (-1)(\nabla_c R_{adbj} + \nabla_b R_{adjc}) + \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 \\
&\quad + C\Lambda^2 \epsilon_0 R^{-2} \\
&= -\int_M \varphi^{2s} g^{kb} g^{lc} g^{md} \nabla_m R_{lk} (\nabla_c R_{db} - \nabla_b R_{dc}) + \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C\epsilon_0 \Lambda^2 R^{-2} \\
&= \int_M \varphi^{2s} g^{kb} g^{lc} R_{lk} g^{md} (\nabla_m \nabla_c R_{db} - \nabla_m \nabla_b R_{dc}) + \int_M \varphi^{2s-1} \nabla\varphi * \nabla\mathrm{Rm} * \mathrm{Rm} \\
&\quad + \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C\Lambda^2 \epsilon_0 R^{-2} \\
&\le \int_M \varphi^{2s} g^{kb} g^{lc} R_{lk} g^{md} (\nabla_m \nabla_c R_{db} - \nabla_m \nabla_b R_{dc}) + 2\epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C\Lambda^2 \epsilon_0 R^{-2} \\
&= \int_M \varphi^{2s} g^{kb} g^{lc} R_{lk} (\nabla_c \nabla^d R_{db} - \nabla_b \nabla^d R_{dc}) + \int_M \varphi^{2s} \mathrm{Rm}^{*3} + 2\epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 \\
&\quad + C\Lambda^2 \epsilon_0 R^{-2} \\
&= \int_M \varphi^{2s} \mathrm{Rm}^{*3} + 2\epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C\Lambda^2 \epsilon_0 R^{-2} \\
&\le C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C_2 \epsilon_0 R^{-2},
\end{aligned}
$$

where $C_1 = C_1(C_S)$ and $C_2 = C_2(C_S, \Lambda)$. We estimated the gradient term in the second and sixth lines via Proposition 5.1.1. The first line follows from integration by parts. The third line follows from a Bianchi identity. The fifth line follows from integration by parts. We obtained the seventh line by commuting derivatives. The eighth line follows from the assumption that (M, g) has constant scalar curvature and a Bianchi identity. The ninth line follows from Proposition 5.1.3.

Similarly,

$$
I_2 \le \int_M \varphi^{2s} g^{kb} g^{lc} g^{md} \nabla_l R_{mk} (\nabla_c R_{db} - \nabla_b R_{dc}) + C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla\mathrm{Rm}|^2 + C_2 \epsilon_0^2 R^{-2}
$$

$$
\begin{aligned}
&= \int_M \varphi^{2s}|\nabla \mathrm{Rc}|^2 + \int_M \varphi^{2s}\langle \nabla \operatorname{div} \mathrm{Rc}, \mathrm{Rc}\rangle + \int_M \varphi^{2s}\mathrm{Rm}^{*3} + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 \\
&\quad + C_2\epsilon_0^2 R^{-2} \\
&\leq \int_M \varphi^{2s}|\nabla \mathrm{Rc}|^2 + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2},
\end{aligned}
$$

where $C_1 = C_1(C_S)$ and $C_2 = C_2(C_S, \Lambda)$. The second line follows from integration by parts and commuting derivatives. The third line follows from the assumption that (M, g) has constant scalar curvature and a Bianchi identity as well as Proposition 5.1.3.

We complete the proof of the proposition by applying the estimates for I_1 and I_2 and applying Proposition 5.1.3 to the initial estimate for I given by the inequality (5.1). □

Proposition 5.1.5. *Suppose M, φ satisfy the above hypotheses. Suppose $s \geq 1$. Then there exist $\epsilon_0 = \epsilon_0(C_S)$ and $C_2 = C_2(C_S, \Lambda)$ such that, for all $R > 0$,*

$$
\int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 \leq C_2\epsilon_0 R^{-2}.
$$

Proof. We apply the previous estimates obtained in this section to obtain that there exist $C_1 = C_1(C_S)$ and $C_2 = C_2(C_S, \Lambda)$ such that

$$
\begin{aligned}
\int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 &= -\int_M \varphi^{2s}\langle \Delta \mathrm{Rm}, \mathrm{Rm}\rangle + \int_M \varphi^{2s-1}\nabla\varphi * \nabla \mathrm{Rm} * \mathrm{Rm} \\
&\leq -\int_M \varphi^{2s}\langle \Delta \mathrm{Rm}, \mathrm{Rm}\rangle + \epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C\epsilon_0 R^{-2} \\
&\leq 2\int_M \varphi^{2s}|\nabla \mathrm{Rc}|^2 + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2} \\
&= -2\int_M \varphi^{2s}\langle \Delta \mathrm{Rc}, \mathrm{Rc}\rangle + \int_M \varphi^{2s-1}\nabla\varphi * \nabla \mathrm{Rm} * \mathrm{Rm} \\
&\quad + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2} \\
&\leq -2\int_M \varphi^{2s}\langle \Delta \mathrm{Rc}, \mathrm{Rc}\rangle + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2} \\
&\leq \int_M \varphi^{2s}\mathrm{Rm}^{*3} + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2}
\end{aligned}
$$

$$\leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2}.$$

We obtain the first and fourth lines by integrating by parts, and the second and fifth lines via Proposition 5.1.1. We obtain the third line via Proposition 5.1.4. We obtain the seventh line via Proposition 5.1.3. We justify the sixth line as follows. Since we have assumed that (M, g) is obstruction - flat and has constant scalar curvature, we have, with $n = 4$,

$$0 = \mathcal{O} = \tfrac{(-1)^{n/2}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) = \tfrac{1}{2}\Delta \mathrm{Rc} + \mathrm{Rm}^{*2}.$$

So the sixth line follows from substituting $\Delta \mathrm{Rc} = \mathrm{Rm}^{*2}$.

Now, since $C_1 = C_1(C_S)$, we can choose $\epsilon_0 = \epsilon_0(C_S)$ so that $C_1\epsilon_0 = 1$. Using this choice of ϵ_0, the previous estimate yields

$$\begin{aligned} 2\int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 &\leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2} \\ &\leq \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 + C_2\epsilon_0 R^{-2} \\ \int_M \varphi^{2s}|\nabla \mathrm{Rm}|^2 &\leq C_2\epsilon_0 R^{-2} \end{aligned}$$

as desired. □

We are now able to complete the proof of Theorem 1.2.11.

Proof of Theorem 1.2.11. We apply the Sobolev inequality given by Proposition 5.1.2 with $s = 1$, followed by the energy estimate given by Proposition 5.1.5, to obtain that there exists $\epsilon_0 = \epsilon_0(C_S)$ for which

$$\|\varphi \mathrm{Rm}\|_4^2 \leq C_1\|\varphi\nabla \mathrm{Rm}\|_2^2 + C_2\epsilon_0^2 R^{-2} \leq C_2\epsilon_0 R^{-2},$$

where $C_1 = C_1(C_S)$ and $C_2 = C_2(C_S, \Lambda)$. Letting $R \to \infty$, we get $\|\varphi \mathrm{Rm}\|_4^2 = 0$, which implies that (M, g) is flat.

□

5.2 Initial Sobolev Estimate

We prove the following.

Proposition 5.2.1. *Suppose M, φ satisfy the above hypotheses. For $p \geq 1$ and $1 \leq k \leq n-1$, there exist $C_1 = C(n, k, p, C_S)$ and $C_2 = C(n, k, p, C_S, \Lambda, C_V)$ such that*

$$\|\varphi^p \mathrm{Rm}\|_n^2 \leq C_1 \|\varphi^p \nabla^k \mathrm{Rm}\|_{\frac{n}{k+1}}^2 + C_2 \epsilon_0^2 R^{-2}.$$

In order to control lower order terms arising from repeated applications of the Sobolev inequality, we use the following estimate.

Proposition 5.2.2. *Suppose M, φ satisfy the above hypotheses. Assume that $2 \leq \alpha \leq \frac{n}{2}$ and $s \geq \alpha l$. For $l \geq 0$ and $\delta > 0$, there exists $C = C(n, \alpha, s, l, \Lambda, C_V)$ such that*

$$\frac{1}{R^\beta} \int_M \varphi^s |\nabla^l \mathrm{Rm}|^\alpha \leq \frac{\delta}{R^{\beta-\gamma}} \int_M \varphi^{s+\alpha} |\nabla^{l+1} \mathrm{Rm}|^\alpha + \frac{C}{\delta^l} \epsilon_0^\alpha R^{n-2\alpha-\beta-l\gamma}.$$

Lemma 5.2.3. *Suppose M, φ satisfy the above hypotheses. Then for $0 \leq \alpha \leq \frac{n}{2}$, there exists $C = C(n, \alpha, C_V)$ such that*

$$\int_{[\varphi>0]} |\mathrm{Rm}|^\alpha \leq C \epsilon_0^\alpha R^{n-2\alpha}.$$

Proof. Using Hölder's inequality in the first line and the volume growth upper bound in the second line, we get (per its definition, φ is nonzero on $B(x,2R)$)

$$\begin{aligned}\int_{[\varphi>0]} |\mathrm{Rm}|^\alpha &\leq \left(\int_{[\varphi>0]} |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{2\alpha}{n}} \left(\int_{[\varphi>0]} 1\right)^{\frac{n-2\alpha}{n}} \\ &\leq C\epsilon_0^\alpha R^{\frac{n(n-2\alpha)}{n}} \\ &= C\epsilon_0^\alpha R^{n-2\alpha}.\end{aligned}$$

□

Proof of Proposition 5.2.2. We prove this by induction on l. If $l=0$ then, by Lemma 5.2.3, there exists $C = C(n,\alpha,C_V)$ such that

$$\begin{aligned}\frac{1}{R^\beta}\int_M \varphi^s|\mathrm{Rm}|^\alpha &\leq C\epsilon_0^\alpha R^{n-2\alpha-\beta} \\ &\leq \frac{\delta}{R^{\beta-\gamma}}\int_M \varphi^{s+\alpha}|\nabla \mathrm{Rm}|^\alpha + C\epsilon_0^\alpha R^{n-2\alpha-\beta}.\end{aligned}$$

We assume the proposition is true for all integers at most l, where $l \geq 0$, and show that the proposition holds for $l+1$. In particular, this means we assume that $s \geq \alpha(l+1)$. Then, using Proposition 3.1.3 in the first line and setting $\epsilon = \delta R^\gamma$,

$$\begin{aligned}\frac{1}{R^\beta}\int_M \varphi^s|\nabla^{l+1}\mathrm{Rm}|^\alpha &\leq \frac{\epsilon}{R^\beta}\int_M \varphi^{s+\alpha}|\nabla^{l+2}\mathrm{Rm}|^\alpha + \frac{C}{\epsilon R^\beta}\int_{[\varphi>0]} \varphi^{s-\alpha}|\nabla^l \mathrm{Rm}|^\alpha \\ &\leq \frac{\delta}{R^{\beta-\gamma}}\int_M \varphi^{s+\alpha}|\nabla^{l+2}\mathrm{Rm}|^\alpha + \frac{C}{\delta R^{\beta+\gamma}}\int_{[\varphi>0]} \varphi^{s-\alpha}|\nabla^l \mathrm{Rm}|^\alpha,\end{aligned}$$

where $C = C(n,\alpha,s,\Lambda)$. Note that $s \geq \alpha(l+1)$ implies $s-\alpha \geq \alpha l$, which is required to apply the inductive hypothesis to estimate the second term on the right hand side. We use the inductive hypothesis, replacing δ with $\delta/(2C)$, β with $\beta+\gamma$, and s with $s-\alpha$, to estimate the second term on the right hand side:

$$\begin{aligned}\frac{C}{\delta R^{\beta+\gamma}}\int_{[\varphi>0]}\varphi^{s-\alpha}|\nabla^l \mathrm{Rm}|^\alpha &\le \frac{C}{\delta}\Big[\frac{\delta}{2CR^{\beta+\gamma-\gamma}}\int_M \varphi^{s-\alpha+\alpha}|\nabla^{l+1}\mathrm{Rm}|^\alpha \\ &\quad + \frac{C}{\delta^l}\epsilon_0^\alpha R^{n-2\alpha-(\beta+\gamma)-l\gamma}\Big] \\ &= \frac{1}{2}R^\beta\int_M \varphi^s|\nabla^{l+1}\mathrm{Rm}|^\alpha + \frac{C}{\delta^{l+1}}\epsilon_0^\alpha R^{n-2\alpha-\beta-(l+1)\gamma},\end{aligned}$$

where $C = C(n,\alpha,s,l,\Lambda,C_V)$. Collecting terms, we conclude that the proposition is true for $l+1$ as well. □

Proposition 5.2.4. *Suppose M,φ satisfy the above hypotheses. For $0\le q\le n-2$, $p\ge q+1$, and $1\le k\le n-q-1$, there exist $C_1 = C_1(n,k,p,q,C_S)$ and $C_2 = C_2(n,k,p,q,C_S,\Lambda,C_V)$ such that*

$$\|\varphi^p\nabla^q\mathrm{Rm}\|_{\frac{n}{q+1}} \le C_1\|\varphi^p\nabla^{q+k}\mathrm{Rm}\|_{\frac{n}{q+k+1}} + C_2\epsilon_0 R^{-1}.$$

Proof. We prove this by induction on k. Suppose $k=1$. Then the proposition states

$$\|\varphi^p\nabla^q\mathrm{Rm}\|_{\frac{n}{q+1}} \le C\|\varphi^p\nabla^{q+1}\mathrm{Rm}\|_{\frac{n}{q+2}} + C\epsilon_0 R^{-1}. \tag{5.2}$$

Using the Sobolev inequality in the first line and the Kato inequality in the third line,

$$\begin{aligned}\|\varphi^p\nabla^q\mathrm{Rm}\|_{\frac{n}{q+1}} &\le C_S\big\|\nabla\left[\varphi^p|\nabla^q\mathrm{Rm}|\right]\big\|_{\frac{n}{q+2}} \\ &\le C\|\varphi^p\nabla|\nabla^q\mathrm{Rm}|\|_{\frac{n}{q+2}} + C\|\varphi^{p-1}|\nabla^q\mathrm{Rm}|\nabla\varphi\|_{\frac{n}{q+2}} \\ &\le C\|\varphi^p\nabla^{q+1}\mathrm{Rm}\|_{\frac{n}{q+2}} + C\|\varphi^{p-1}|\nabla^q\mathrm{Rm}|\nabla\varphi\|_{\frac{n}{q+2}},\end{aligned} \tag{5.3}$$

where $C = C(n,p,q,C_S)$. We estimate the second term of the right side of the inequality (5.3). Using Proposition 5.2.2 with $\alpha=\beta=\gamma=\frac{n}{q+2}$, $s=\frac{n}{q+2}(p-1)$, $l=q$, and

$\delta = \Lambda^{-n/(q+2)}$, we get

$$\begin{aligned}
\int_M \varphi^{\frac{n}{q+2}(p-1)} |\nabla \varphi|^{\frac{n}{q+2}} |\nabla^q \mathrm{Rm}|^{\frac{n}{q+2}} &\leq \frac{C\Lambda^{\frac{n}{q+2}}}{R^{\frac{n}{q+2}}} \int_M \varphi^{\frac{n}{q+2}(p-1)} |\nabla^q \mathrm{Rm}|^{\frac{n}{q+2}} \\
&\leq C_1 \int_M \varphi^{\frac{n}{q+2}p} |\nabla^{q+1} \mathrm{Rm}|^{\frac{n}{q+2}} + C_2 \epsilon_0^{\frac{n}{q+2}} R^{-\frac{n}{q+2}} \\
\|\varphi^{p-1} |\nabla^q \mathrm{Rm}| \nabla \varphi\|_{\frac{n}{q+2}} &\leq C_1 \|\varphi^p \nabla^{q+1} \mathrm{Rm}\|_{\frac{n}{q+2}} + C_2 \epsilon_0 R^{-1}
\end{aligned}$$

with $C_1 = C_1(n, p, q)$ and $C_2 = C_2(n, p, q, \Lambda, C_V)$. Our choice of the exponent of φ was sufficiently large to apply the proposition since $p \geq q + 1$ implies

$$\frac{n}{q+2}(p-1) \geq \frac{n}{q+2} \cdot q.$$

Applying the above estimate to (5.3) and collecting terms, we conclude that the proposition is true for $k = 1$.

Suppose the proposition is true for all integers at most k, where $k \geq 1$. Using the inductive hypothesis in the first line and the inequality (5.2) in the second line,

$$\begin{aligned}
\|\varphi^p \nabla^q \mathrm{Rm}\|_{\frac{n}{q+1}} &\leq C_1 \|\varphi^p \nabla^{q+k} \mathrm{Rm}\|_{\frac{n}{q+k+1}} + C_2 \epsilon_0 R^{-1} \\
&\leq C_1 \|\varphi^p \nabla^{q+k+1} \mathrm{Rm}\|_{\frac{n}{q+k+2}} + C_2 \epsilon_0 R^{-1}.
\end{aligned}$$

Thus the proposition is true for $k + 1$ as well. □

Proof of Proposition 5.2.1. Apply Proposition 5.2.4 with $q = 0$. □

Chapter 6

Estimates for Certain Lower Order Terms

Applying Proposition 5.2.1 with $1 \leq k \leq n-1$, we obtain

$$\left(\int_M \varphi^{ns}|\mathrm{Rm}|^n\right)^{\frac{2}{n}} \leq \left(\int_M \varphi^{\frac{ns}{k+1}}|\nabla^k \mathrm{Rm}|^{\frac{n}{k+1}}\right)^{\frac{2k+2}{n}} + C\epsilon_0^2 R^{-2}$$

where $C = C(n, s, C_S, \Lambda, C_V)$. It follows that if $\|\nabla^k \mathrm{Rm}\|_\infty = 0$ for some k satisfying $1 \leq k \leq n-1$, then $\|\mathrm{Rm}\|_\infty$=0 as well. From now on, we assume that

$$\|\nabla^{n-1}\mathrm{Rm}\|_\infty > 0.$$

In particular, when $k = \frac{n}{2} - 1$, we obtain

$$\left(\int_M \varphi^{ns}|\mathrm{Rm}|^n\right)^{\frac{2}{n}} \leq \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C\epsilon_0^2 R^{-2}$$

where $C = C(n, s, C_S, \Lambda, C_V)$. We wish to obtain an analogous estimate with Rm replaced by Rc. To facilitate this, in this section we obtain estimates for several lower order terms.

First, we obtain estimates for

$$\int_M \varphi^{2s} \nabla^{\frac{n}{2}-1}\mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-3-k}\mathrm{Rm}$$

for $0 \leq k \leq \frac{n}{2} - 3$ and

$$\int_M \varphi^{2s} \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-2-k}\mathrm{Rm}$$

for $0 \leq k \leq \frac{n}{2} - 2$. We then obtain estimates for gradient terms that arise when we integrate by parts.

6.1 The Cutoff = 1 Case

As an illustrative special case, we begin by deriving estimates for the case where $\varphi \equiv 1$. We will use the following proposition.

Proposition 6.1.1 ([31], Lemma 5.1)**.** *Suppose M, φ satisfy the above hypotheses. Let A be a tensor on M. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 \leq p, q, r \leq \infty$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$. For $s \geq \max\{\alpha q, \beta p\}$ and $-\frac{1}{p} \leq t \leq \frac{1}{q}$ there exists $C = C(n, r)$ such that*

$$\left(\int_M \varphi^s |\nabla A|^{2r}\right)^{\frac{1}{r}} \leq C \left(\int_{[\varphi>0]} \varphi^{s(1-tq)}|A|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)}|\nabla^2 A|^p\right)^{\frac{1}{p}} + C\Lambda s R^{-1} \left(\int_{[\varphi>0]} \varphi^{s-\alpha q}|A|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s-\beta p}|\nabla A|^p\right)^{\frac{1}{p}}.$$

We note that if $y \in A(R, 2R)$, then $R < \rho(y) < 2R$, so that $|\nabla\varphi| < \Lambda\rho^{-1} < \Lambda R^{-1}$.

Lemma 6.1.2. *Suppose M satisfies the above hypotheses. Let A be a tensor on M. Then*

$$\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{1}{r}} \le C \left(\int_M |A|^q\right)^{\frac{2}{(k+1)q}} \left(\int_M |\nabla^{k+1} A|^p\right)^{\frac{2k}{(k+1)p}},$$

where $\frac{1}{r} = \frac{2}{(k+1)q} + \frac{2k}{(k+1)p}$, $1 \le p, q, r \le \infty$, $k \ge 1$, and $C = C(n, p, q, r, k)$.

Proof. When $k = 1$ the lemma reduces to Proposition 6.1.1 with $\varphi \equiv 1$:

$$\left(\int_M |\nabla A|^{2r}\right)^{\frac{1}{r}} \le C \left(\int_M |A|^q\right)^{\frac{1}{q}} \left(\int_M |\nabla^2 A|^p\right)^{\frac{1}{p}}, \tag{6.1}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Assume the lemma is true for all integers at most $k - 1$. Applying inequality (6.1) and the inductive hypothesis, we obtain

$$\begin{aligned}\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{1}{r}} &\le C \left(\int_M |\nabla^{k-1} A|^{2u}\right)^{\frac{1}{2u}} \left(\int_M |\nabla^{k+1} A|^p\right)^{\frac{1}{p}} \\ &\le C \left(\int_M |A|^q\right)^{\frac{1}{kq}} \left(\int_M |\nabla^k A|^{2r}\right)^{\frac{k-1}{2kr}} \left(\int_M |\nabla^{k+1} A|^p\right)^{\frac{1}{p}},\end{aligned}$$

with

$$\frac{1}{r} = \frac{1}{2u} + \frac{1}{p}, \quad \frac{1}{u} = \frac{2}{kq} + \frac{k-1}{kr}.$$

We solve for $\frac{1}{r}$:

$$\begin{aligned}\frac{1}{r} &= \frac{1}{kq} + \frac{k-1}{2kr} + \frac{1}{p} \\ \frac{k+1}{2kr} &= \frac{1}{kq} + \frac{1}{p} \\ \frac{1}{r} &= \frac{2}{(k+1)q} + \frac{2k}{(k+1)p}.\end{aligned}$$

Since

$$\frac{1}{r} - \frac{k-1}{2kr} = \frac{k+1}{2kr}, \quad \frac{1}{r} = \frac{k+1}{2kr} \cdot \frac{2k}{k+1},$$

we obtain

$$\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{k+1}{2kr}} \leq C \left(\int_M |A|^q\right)^{\frac{1}{kq}} \left(\int_M |\nabla^{k+1} A|^p\right)^{\frac{1}{p}}$$
$$\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{1}{r}} \leq C \left(\int_M |A|^q\right)^{\frac{2}{(k+1)q}} \left(\int_M |\nabla^{k+1} A|^p\right)^{\frac{2k}{(k+1)p}}.$$

□

Proposition 6.1.3. *Suppose M satisfies the above hypotheses. Let A be a tensor on M. Then*

$$\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{1}{r}} \leq C \left(\int_M |A|^q\right)^{\frac{2j}{(k+j)q}} \left(\int_M |\nabla^{k+j} A|^p\right)^{\frac{2k}{(k+j)p}},$$

where $\frac{1}{r} = \frac{2j}{(k+j)q} + \frac{2k}{(k+j)p}$, $1 \leq p, q, r \leq \infty$, $j, k \geq 1$, and $C = C(n, p, q, r, j, k)$.

Proof. We prove the proposition by induction on j. When $j = 1$ the proposition reduces to the preceding Lemma 6.1.2. Assume the proposition is true for all integers at most $j - 1$. We have

$$\frac{j-1}{k+j-1} + \frac{k}{(k+j-1)(k+j)} = \frac{j}{k+j}.$$

Applying the inductive hypothesis in the first line and the preceding Lemma 6.1.2 in the second line, we obtain

$$\left(\int_M |\nabla^k A|^{2r}\right)^{\frac{1}{r}} \leq C \left(\int_M |A|^q\right)^{\frac{2(j-1)}{(k+j-1)q}} \left(\int_M |\nabla^{k+j-1} A|^{2v}\right)^{\frac{k}{(k+j-1)v}}$$

$$
\begin{aligned}
&\leq C\left(\int_M |A|^q\right)^{\frac{2(j-1)}{(k+j-1)q}} \\
&\quad \times \left[\left(\int_M |A|^q\right)^{\frac{2}{(k+j)q}} \left(\int_M |\nabla^{k+j}A|^p\right)^{\frac{2(k+j-1)}{(k+j)p}}\right]^{\frac{k}{k+j-1}} \\
&= C\left(\int_M |A|^q\right)^{\frac{2(j-1)}{(k+j-1)q}} \left(\int_M |A|^q\right)^{\frac{2k}{(k+j-1)(k+j)q}} \\
&\quad \times \left(\int_M |\nabla^{k+j}A|^p\right)^{\frac{2k}{(k+j)p}} \\
&= C\left(\int_M |A|^q\right)^{\frac{2j}{(k+j)q}} \left(\int_M |\nabla^{k+j}A|^p\right)^{\frac{2k}{(k+j)p}},
\end{aligned}
$$

where $\frac{1}{r} = \frac{2(j-1)}{(k+j-1)q} + \frac{k}{(k+j-1)v}$, $\frac{1}{v} = \frac{2}{(k+j)q} + \frac{2(k+j-1)}{(k+j)p}$. We compute $\frac{1}{r}$:

$$
\begin{aligned}
\frac{1}{r} &= \frac{2(j-1)}{(k+j-1)q} + \frac{2k}{(k+j-1)(k+j)q} + \frac{2k}{(k+j)p} \\
&= \frac{2j}{(k+j)q} + \frac{2k}{(k+j)p}.
\end{aligned}
$$

□

Proposition 6.1.4. *Suppose M satisfies the above hypotheses. Then for i satisfying $0 \leq i \leq \frac{n}{2} - 2$,*

$$
\int_M \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \nabla^{\frac{n}{2}-2-i}\mathrm{Rm} * \nabla^i \mathrm{Rm} \leq C\epsilon_0 \int_M |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2,
$$

where $C = C(n, i, C_S)$.

Proof. Define r_1, r_2 by

$$
\frac{1}{r_1} = \frac{2(\frac{n}{2}-2-i)}{(\frac{n}{2}-2)\frac{n}{2}} + \frac{2i}{(\frac{n}{2}-2)\frac{2n}{n-2}}, \quad \frac{1}{r_2} = \frac{2i}{(\frac{n}{2}-2)\frac{n}{2}} + \frac{2(\frac{n}{2}-2-i)}{(\frac{n}{2}-2)\frac{2n}{n-2}}.
$$

Proposition 6.1.3 gives

$$\left(\int_M |\nabla^i \mathrm{Rm}|^{2r_1}\right)^{\frac{1}{r_1}} \le C\left(\int_M |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{2(n/2-2-i)}{(n/2-2)n/2}} \times \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{2i}{(n/2-2)[2n/(n-2)]}}$$

$$\left(\int_M |\nabla^{\frac{n}{2}-2-i} \mathrm{Rm}|^{2r_2}\right)^{\frac{1}{r_2}} \le C\left(\int_M |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{2i}{(n/2-2)n/2}} \times \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{2(n/2-2-i)}{(n/2-2)[2n/(n-2)]}},$$

where $C = C(n,i)$. Define p, q by

$$\frac{1}{q} = \frac{n}{n+2}\cdot\frac{1}{r_1}, \quad \frac{1}{p} = \frac{n}{n+2}\cdot\frac{1}{r_2}.$$

Then

$$\begin{aligned}\frac{1}{p}+\frac{1}{q} &= \frac{n}{n+2}\left[\frac{1}{r_2}+\frac{1}{r_1}\right]\\ &= \frac{n}{n+2}\left[\frac{2(\frac{n}{2}-2)}{(\frac{n}{2}-2)\frac{n}{2}} + \frac{2(\frac{n}{2}-2)}{(\frac{n}{2}-2)\frac{2n}{n-2}}\right]\\ &= 1.\end{aligned}$$

Since

$$\frac{1}{2}\left[\frac{2(\frac{n}{2}-2-i)}{(\frac{n}{2}-2)\frac{n}{2}} + \frac{2i}{(\frac{n}{2}-2)\frac{n}{2}}\right] = \frac{2}{n}$$

$$\frac{1}{2}\left[\frac{2i}{(\frac{n}{2}-2)\frac{2n}{n-2}} + \frac{2(\frac{n}{2}-2-i)}{(\frac{n}{2}-2)\frac{2n}{n-2}}\right] = \frac{n-2}{2n},$$

we have

$$\left(\int_M |\nabla^{\frac{n}{2}-2-i}\mathrm{Rm}|^{\frac{2np}{n+2}}\right)^{\frac{n+2}{2np}} \left(\int_M |\nabla^i \mathrm{Rm}|^{\frac{2nq}{n+2}}\right)^{\frac{n+2}{2nq}} \le \left(\int_M |\nabla^{\frac{n}{2}-2-i}\mathrm{Rm}|^{2r_2}\right)^{\frac{1}{2r_2}}$$

$$\times \left(\int_M |\nabla^i \mathrm{Rm}|^{2r_1}\right)^{\frac{1}{2r_1}}$$
$$\leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}$$
$$\times \left(\int_M |\nabla^{\frac{n}{2}-2} \mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}}. \tag{6.2}$$

Let I denote

$$I = \int_M \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \nabla^{\frac{n}{2}-2-i}\mathrm{Rm} * \nabla^i \mathrm{Rm}.$$

Applying Hölder's inequality in the first and second lines, inequality (6.2) in the third line, and the Sobolev inequality in the fifth line,

$$\begin{aligned}
I &\leq C \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_M |\nabla^{\frac{n}{2}-2-i}\mathrm{Rm} * \nabla^i \mathrm{Rm}|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \\
&\leq C \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_M |\nabla^{\frac{n}{2}-2-i}\mathrm{Rm}|^{\frac{2np}{n+2}}\right)^{\frac{n+2}{2np}} \left(\int_M |\nabla^i \mathrm{Rm}|^{\frac{2nq}{n+2}}\right)^{\frac{n+2}{2nq}} \\
&\leq C \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_M |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \\
&= C\epsilon_0 \|\nabla^{\frac{n}{2}-2}\mathrm{Rm}\|^2_{\frac{2n}{n-2}} \\
&\leq C\epsilon_0 \|\nabla^{\frac{n}{2}-1}\mathrm{Rm}\|^2_2,
\end{aligned}$$

where $C = C(n, i, C_S)$. □

6.2 General Case

We generalize the argument from the previous subsection. We will be able to avoid many error terms since we have assumed that $\|\nabla^{n-1}\mathrm{Rm}\|_\infty > 0$.

A small modification of the proof of Proposition 6.1.1 gives

Proposition 6.2.1. *Suppose M, φ satisfy the above hypotheses. Let A be a tensor on M. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 \le p, q, r \le \infty$ and $\alpha + \beta = 1$, $\alpha, \beta \ge 0$. For $s \ge \max\{\alpha q, \beta p\}$ and $-\frac{1}{p} \le t \le \frac{1}{q}$ there exists $C = C(n, r)$ such that*

$$\left(\int_M \varphi^s |\nabla A|^{2r}\right)^{\frac{1}{r}} \le C \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |A|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)} |\nabla^2 A|^p\right)^{\frac{1}{p}}$$
$$+ C\Lambda s R^{-1} \left(\int_{[\varphi>0]} \varphi^{s(1-tq)-\alpha q} |A|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)-\beta p} |\nabla A|^p\right)^{\frac{1}{p}}.$$

Lemma 6.2.2. *Suppose M, φ satisfy the above hypotheses. In addition, suppose that $n \ge 8$, $1 \le p, q, r \le \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\frac{n}{4} < p \le \frac{n}{2}$, $-\frac{1}{p} \le t \le \frac{1}{q}$, $s \ge \max\{p, \frac{2p}{1+tp}\}$. Then there exists R_0 independent of ϵ_0 such that for all $R \ge R_0$,*

$$\left(\int_M \varphi^s |\nabla \mathrm{Rm}|^{2r}\right)^{\frac{1}{r}} \le C(1+s) \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |\mathrm{Rm}|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p\right)^{\frac{1}{p}},$$

where $C = C(n, r)$.

Proof. We can apply Proposition 5.2.2 since $n \ge 8$ and $2 \le p \le \frac{n}{2}$, taking $\delta = \Lambda^{-1}$, to show that there exists R_0 such that for all $R \ge R_0$,

$$R^{-p} \int_M \varphi^{s(1+tp)-p} |\nabla \mathrm{Rm}|^p \le \Lambda^{-1} \int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p + C_1 \epsilon_0 R^{n-4p}$$
$$\le \Lambda^{-1} \int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p,$$

where $C_1 = C_1(n, r, s, \Lambda, C_V)$. We justify the second inequality as follows. The term $C_1\epsilon_0 R^{n-4p} \to 0$ as $R \to \infty$ since $n - 4p < 0$. Therefore, since we assume $\|\nabla^2 \mathrm{Rm}\|_\infty > 0$, there exists R_0 independent of ϵ_0 such that $C_1\epsilon_0 R^{n-4p}$ is absorbed by the integral term for all $R \geq R_0$. Applying Proposition 6.2.1, we obtain that there exist $C = C(n, r)$ and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}\left(\int_M \varphi^s |\nabla \mathrm{Rm}|^{2r}\right)^{\frac{1}{r}} &\leq C \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |\mathrm{Rm}|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p\right)^{\frac{1}{p}} \\ &\quad + C\Lambda R^{-1} s \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |\mathrm{Rm}|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)-p} |\nabla \mathrm{Rm}|^p\right)^{\frac{1}{p}} \\ &\leq C(1 + \Lambda\Lambda^{-1}s) \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |\mathrm{Rm}|^q\right)^{\frac{1}{q}} \\ &\quad \times \left(\int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p\right)^{\frac{1}{p}} \\ &= C(1+s) \left(\int_{[\varphi>0]} \varphi^{s(1-tq)} |\mathrm{Rm}|^q\right)^{\frac{1}{q}} \left(\int_M \varphi^{s(1+tp)} |\nabla^2 \mathrm{Rm}|^p\right)^{\frac{1}{p}}.\end{aligned}$$

□

For the rest of this paper, when there is a radius R_0 for which an inequality holds for all $R \geq R_0$, the radius R_0 will be assumed to be independent of ϵ_0.

Proposition 6.2.3. *Suppose M, φ satisfy the above hypotheses. Suppose that $n \geq 8$, $1 \leq k \leq \frac{n}{2} - 3$, $\frac{n}{2(k+2)} < r_k \leq \frac{n}{4}$, and $s \geq \frac{4kr_k}{k+1}$. Let $q_1 = \frac{n}{2}$, $t_k = \frac{1}{kq_1}$. Define p_k by*

$$\frac{1}{r_k} = \frac{2}{(k+1)q_1} + \frac{2k}{(k+1)p_k}. \tag{6.3}$$

Then there exists R_0 such that for all $R \geq R_0$,

$$\left(\int_M \varphi^s |\nabla^k \mathrm{Rm}|^{2r_k}\right)^{\frac{1}{r_k}} \leq C\epsilon_0^{\frac{2}{k+1}} \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k}\right)^{\frac{2k}{(k+1)p_k}},$$

where $C = C(n, k, r_k, s)$.

Proof. We will use induction on k. If $k = 1$, the proposition reduces to Lemma 6.2.2. For a fixed k satisfying $2 \le k \le \frac{n}{2} - 3$, assume that the proposition is true for all integers at most $k - 1$. We verify the hypotheses of Proposition 5.2.2. First, we obtain from equation (6.3) and the inequalities $\frac{n}{2(k+2)} < r_k \le \frac{n}{4}$ the following:

$$\frac{1}{p_k} = \frac{k+1}{2kr_k} - \frac{1}{kq_1}$$
$$\frac{1}{p_k} \ge \frac{k+1}{2k} \cdot \frac{4}{n} - \frac{2}{kn} = \frac{2}{n}$$
$$\frac{1}{p_k} < \frac{k+1}{2k} \cdot \frac{2(k+2)}{n} - \frac{2}{kn} = \frac{k+3}{n}.$$

Also, $\frac{n}{k+3} \ge 2$ since $k \le \frac{n}{2} - 3$. Therefore

$$2 \le \frac{n}{k+3} < p_k \le \frac{n}{2}. \tag{6.4}$$

Finally, since $s \ge \frac{4kr_k}{k+1}$ and $\frac{1}{p_k} = \frac{k+1}{2kr_k} - \frac{1}{kq_1}$,

$$\frac{2p_k}{1 + t_k p_k} = \frac{2}{\frac{1}{p_k} + t_k} = \frac{4kr_k}{k+1},$$

implying $s \ge \frac{2p_k}{1+t_k p_k}$ and $s(1 + t_k p_k) - p_k \ge p_k$. Therefore, we can apply Proposition 5.2.2, taking $\delta = \Lambda^{-1}$, to conclude that there exists R_0 such that, for all $R \ge R_0$,

$$\begin{aligned} R^{-p_k} \int_M \varphi^{s(1+t_k p_k)-p_k} |\nabla^k \mathrm{Rm}|^{p_k} &\le \Lambda^{-1} \int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k} \\ &\quad + C_1 \epsilon_0^{p_k} R^{n-(k+3)p_k} \\ &\le \Lambda^{-1} \int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k}, \end{aligned} \tag{6.5}$$

where $C_1 = C_1(n, k, r_k, s, \Lambda, C_V)$. We justify the second inequality as follows. The term $C_1\epsilon_0 R^{n-(k+3)p_k} \to 0$ as $R \to \infty$ since $\frac{n}{k+3} < p_k$. Therefore, since $k+1 \leq \frac{n}{2} - 2$ and we assume $\|\nabla^{n-1}\mathrm{Rm}\|_\infty > 0$, there exists R_0 such that $C_1\epsilon_0 R^{n-(k+3)p_k}$ is absorbed by the integral term for all $R \geq R_0$.

We verify the hypotheses of Proposition 6.2.1. Define q_k by

$$\frac{1}{q_k} = \frac{2}{kq_1} + \frac{k-1}{kr_k}. \tag{6.6}$$

This implies

$$\frac{1}{2q_k} \geq \frac{1}{kq_1} = t_k.$$

From equation (6.6) and the inequalities $\frac{n}{2(k+2)} < r_k \leq \frac{n}{4}$, we obtain

$$\begin{aligned} \frac{1}{q_k} &< \frac{4}{kn} + \frac{k-1}{k} \cdot \frac{2(k+2)}{n} = \frac{2(k+1)}{n} \\ \frac{1}{q_k} &\geq \frac{4}{kn} + \frac{k-1}{k} \cdot \frac{4}{n} = \frac{4}{n}, \end{aligned}$$

so that $\frac{n}{2(k+1)} < q_k \leq \frac{n}{4}$. We required that $r_k > \frac{n}{2(k+2)}$. So $2q_k, r_k \geq 1$. From (6.4), we have $p_k \geq 1$. From $p_k \leq \frac{n}{2}$ in (6.4) and $t_k = \frac{2}{kn}$ we get $\frac{2p_k}{1+t_k p_k} \geq p_k$. So $s \geq p_k$ since $s \geq \frac{2p_k}{1+t_k p_k}$. From equations (6.3) and (6.6), we obtain

$$\frac{1}{r_k} = \frac{1}{2q_k} + \frac{1}{p_k}. \tag{6.7}$$

Let I denote

$$I = \int_M \varphi^s |\nabla^k \mathrm{Rm}|^{2r_k}.$$

We apply Proposition 6.2.1 and inequality (6.5) to obtain that there exists R_0 such that, for all $R \geq R_0$,

$$\begin{aligned}
I^{\frac{1}{r_k}} &\leq C \left(\int_{[\varphi>0]} \varphi^{s(1-2t_k q_k)} |\nabla^{k-1}\mathrm{Rm}|^{2q_k} \right)^{\frac{1}{2q_k}} \\
&\quad \times \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1}\mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\
&\quad + C\Lambda R^{-1} s \left(\int_{[\varphi>0]} \varphi^{s(1-2t_k q_k)} |\nabla^{k-1}\mathrm{Rm}|^{2q_k} \right)^{\frac{1}{2q_k}} \\
&\quad \times \left(\int_M \varphi^{s(1+t_k p_k)-q_k} |\nabla^{k}\mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\
&\leq C(1+\Lambda\Lambda^{-1}s) \left(\int_{[\varphi>0]} \varphi^{s(1-2t_k q_k)} |\nabla^{k-1}\mathrm{Rm}|^{2q_k} \right)^{\frac{1}{2q_k}} \\
&\quad \times \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1}\mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\
&\leq C(1+s) \left(\int_{[\varphi>0]} \varphi^{s(1-2t_k q_k)} |\nabla^{k-1}\mathrm{Rm}|^{2q_k} \right)^{\frac{1}{2q_k}} \\
&\quad \times \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1}\mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}}, \qquad (6.8)
\end{aligned}$$

where $C = C(n, r_k)$. We wish to apply the inductive hypothesis to the integral containing $\nabla^{k-1}\mathrm{Rm}$. We have shown that $\frac{n}{2(k+1)} < q_k \leq \frac{n}{4}$. Using equation (6.6), $t_k = \frac{1}{kq_1}$, and $s \geq \frac{4kr_k}{k+1}$, we obtain that there exists R_0 such that, for all $R \geq R_0$,

$$\begin{aligned}
t_k &= \frac{1}{2q_k} - \frac{k-1}{2r_k} \\
s(1-2t_k q_k) &\geq \frac{4kr_k}{k+1} \cdot \frac{(k-1)q_k}{r_k} \\
&= \frac{4k(k-1)q_k}{k+1} \\
&\geq \frac{4(k-1)q_k}{k},
\end{aligned}$$

since $k \geq 2$. We obtain from equation (6.6)

$$\begin{aligned} 1 - 2t_k q_k &= \frac{(k-1)q_1}{2r_k + (k-1)q_1} \\ 1 + 2t_{k-1} r_k &= \frac{(k-1)q_1 + 2r_k}{(k-1)q_1} \end{aligned}$$

$$(1 - 2t_k q_k)(1 + 2t_{k-1} r_k) = 1. \tag{6.9}$$

As a result, we can apply the inductive hypothesis to inequality (6.8) with $r_{k-1} = q_k$ and $p_{k-1} = 2r_k$: there exists R_0 such that, for all $R \geq R_0$,

$$\begin{aligned} I^{\frac{1}{r_k}} &\leq C\epsilon_0^{\frac{1}{k}} \left(\int_M \varphi^{s(1-2t_k q_k)(1+2t_{k-1}r_k)} |\nabla^k \mathrm{Rm}|^{2r_k} \right)^{\frac{k-1}{2kr_k}} \\ &\quad \times \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\ &= C\epsilon_0^{\frac{1}{k}} \left(\int_M \varphi^{s} |\nabla^k \mathrm{Rm}|^{2r_k} \right)^{\frac{k-1}{2kr_k}} \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\ I^{\frac{k+1}{2kr_k}} &\leq C\epsilon_0^{\frac{1}{k}} \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k} \right)^{\frac{1}{p_k}} \\ I^{\frac{1}{r_k}} &\leq C\epsilon_0^{\frac{2}{k+1}} \left(\int_M \varphi^{s(1+t_k p_k)} |\nabla^{k+1} \mathrm{Rm}|^{p_k} \right)^{\frac{2k}{(k+1)p_k}}, \end{aligned}$$

where $C = C(n, k, r_k, s)$. In the second line, we apply equation (6.9). We have obtained the desired inequality. □

Proposition 6.2.4. *Let M, φ satisfy the above hypotheses. Suppose that $n \geq 8$ and the following hold:*

(a) $k \geq 1$, $j \geq 1$, $2 \leq k + j \leq \frac{n}{2} - 2$

(b) $\frac{n}{2(k+2)} < r_k \leq \frac{n}{4}$, $s \geq \frac{4jkr_k}{k+1}$.

Let $q_1 = \frac{n}{2}$. *For each* $i \geq 1$, *let* $t_i = \frac{1}{iq_1}$. *Define* p_{k+j-1} *by*

$$\frac{1}{r_k} = \frac{2j}{(k+j)q_1} + \frac{2k}{(k+j)p_{k+j-1}} \tag{6.10}$$

and define v_i *for* $i \geq k$ *by*

$$\frac{1}{r_k} = \frac{2(i+1-k)}{(i+1)q_1} + \frac{k}{(i+1)v_i}. \tag{6.11}$$

Then there exists R_0 *such that for all* $R \geq R_0$,

$$\left(\int_M \varphi^s |\nabla^k \mathrm{Rm}|^{2r_k}\right)^{\frac{1}{r_k}} \leq C \left(\int_M |\mathrm{Rm}|^{q_1}\right)^{\frac{2j}{(k+j)q_1}} \times \left(\int_M \varphi^{sE_{k+j-1}} |\nabla^{k+j} \mathrm{Rm}|^{p_{k+j-1}}\right)^{\frac{2k}{(k+j)p_{k+j-1}}},$$

where $E_k = 1 + t_k p_k$,

$$E_{k+j-1} = (1 + t_{k+j-1} p_{k+j-1}) \prod_{i=k}^{k+j-2} (1 + 2t_i v_i) \quad \text{if } j \geq 2,$$

and $C = C(n, j, k, r_k, s)$.

Proof. We use induction on j. If $j = 1$, the proposition reduces to Proposition 6.2.3. For a fixed $j \geq 2$, assume the proposition is true for all integers at most $j-1$. We prepare to apply Proposition 6.2.3. We show that $\frac{n}{2(k+j+1)} < v_{k+j-2} \leq \frac{n}{4}$. From equation (6.11), we obtain

$$\frac{1}{r_k} = \frac{2(j-1)}{(k+j-1)q_1} + \frac{k}{(k+j-1)v_{k+j-2}}$$
$$\frac{1}{v_{k+j-2}} = \frac{k+j-1}{kr_k} - \frac{2(j-1)}{kq_1}. \tag{6.12}$$

Since $\frac{n}{2(k+2)} < r_k \leq \frac{n}{4}$,

$$\begin{aligned}
\frac{1}{v_{k+j-2}} &< \frac{k+j-1}{k} \cdot \frac{2(k+2)}{n} - \frac{4(j-1)}{kn} \\
&= \frac{2(k+j-1)}{n} \\
\frac{1}{v_{k+j-2}} &\geq \frac{k+j-1}{k} \cdot \frac{4}{n} - \frac{4(j-1)}{kn} \\
&= \frac{4}{n}.
\end{aligned}$$

Therefore $\frac{n}{2(k+j+1)} < v_{k+j-2} \leq \frac{n}{4}$.

Next, we show that

$$s(1 + 2t_{k+j-2}v_{k+j-2}) \geq \frac{4(k+j-1)v_{k+j-2}}{k+j}. \tag{6.13}$$

First, we obtain from equation (6.12)

$$\begin{aligned}
\frac{1}{v_{k+j-2}} &= \frac{k+j-1}{kr_k} - \frac{2(j-1)(k+j-2)2t_{k+j-2}}{k} \\
2t_{k+j-2} &= \frac{k+j-1}{(j-1)(k+j-2)r_k} - \frac{k}{(j-1)(k+j-2)v_{k+j-2}}.
\end{aligned}$$

This yields

$$\begin{aligned}
1 + 2t_{k+j-2}v_{k+j-2} &= 1 + v_{k+j-2}\left(\frac{k+j-1}{(j-1)(k+j-2)r_k} - \frac{k}{(j-1)(k+j-2)v_{k+j-2}}\right) \\
&= 1 + \frac{(k+j-1)v_{k+j-2}}{(j-1)(k+j-2)r_k} - \frac{k}{(j-1)(k+j-2)} \\
&\geq 1 + \frac{(k+j-1)v_{k+j-2}}{(j-1)(k+j-2)r_k} - 1 \\
&= \frac{(k+j-1)v_{k+j-2}}{(j-1)(k+j-2)r_k}.
\end{aligned}$$

Since $s \geq \frac{4jkr_k}{k+1}$,

$$s(1 + 2t_{k+j-2}v_{k+j-2}) \geq \frac{4jkr_k}{k+1} \cdot \frac{(k+j-1)v_{k+j-2}}{(j-1)(k+j-2)r_k}$$
$$= \frac{jk(k+j)}{(k+1)(j-1)(k+j-2)} \cdot \frac{4(k+j-1)v_{k+j-2}}{k+j}.$$

We will have proved inequality (6.13) if we show that

$$\frac{jk(k+j)}{(k+1)(j-1)(k+j-2)} \geq 1. \tag{6.14}$$

We have

$$jk(k+j) - (j-1)(k+1)(k+j-2) = k^2 + (2j-1)k - (j-2)(j-1).$$

We estimate the value of the larger root k_0 of the above quadratic in k:

$$\begin{aligned} k_0 &= \tfrac{1}{2}\big(-(2j-1) + \sqrt{(2j-1)^2 + 4(j-2)(j-1)}\big) \\ &\leq \tfrac{1}{2}\big(-(2j-1) + \sqrt{(2j-4)^2}\big) \\ &= \tfrac{1}{2}(-(2j-1) + 2j - 4) \\ &= -\tfrac{3}{2}. \end{aligned}$$

So $k^2 + (2j-1)k - (j-2)(j-1) \geq 0$ for all j, k since we require that $k \geq 1$. This implies that inequality (6.14) is true. Therefore inequality (6.13) holds.

Finally, we obtain from equations (6.12) and (6.10)

$$\begin{aligned} \frac{1}{v_{k+j-2}} &= \frac{k+j-1}{k}\left(\frac{2j}{(k+j)q_1} + \frac{2k}{(k+j)p_{k+j-1}}\right) - \frac{2(j-1)}{kq_1} \\ &= \frac{2}{(k+j)q_1} + \frac{2(k+j-1)}{(k+j)p_{k+j-1}}. \end{aligned}$$

We are ready to apply Proposition 6.2.3. Define

$$I = \left(\int_M \varphi^s |\nabla^k \mathrm{Rm}|^{2r_k} \right)^{\frac{1}{r_k}}$$

and V by $V(k-1) = 1$, $V(l) = \prod_{i=k}^{l}(1+2t_i v_i)$ for $l \geq k$. Then there exist $C = C(n, j, k, r_k, s)$ and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}
I &\leq C\epsilon_0^{\frac{2(j-1)}{k+j-1}} \left(\int_M \varphi^{sV(k+j-3)(1+2t_{k+j-2}v_{k+j-2})} |\nabla^{k+j-1}\mathrm{Rm}|^{2v_{k+j-2}} \right)^{\frac{k}{(k+j-1)v_{k+j-2}}} \\
&\leq C\epsilon_0^{\frac{2(j-1)}{k+j-1}} \\
&\quad \times \left[C\epsilon_0^{\frac{2}{k+j}} \left(\int_M \varphi^{sV(k+j-2)(1+t_{k+j-1}p_{k+j-1})} \right. \right. \\
&\quad \left. \left. \cdot\ |\nabla^{k+j}\mathrm{Rm}|^{p_{k+j-1}} \right)^{\frac{2(k+j-1)}{(k+j)p_{k+j-1}}} \right]^{\frac{k}{k+j-1}} \\
&= C\epsilon_0^{\frac{2(j-1)}{k+j-1}} \epsilon_0^{\frac{2k}{(k+j-1)(k+j)}} \\
&\quad \times \left(\int_M \varphi^{sV(k+j-2)(1+t_{k+j-1}p_{k+j-1})} |\nabla^{k+j}\mathrm{Rm}|^{p_{k+j-1}} \right)^{\frac{2k}{(k+j)p_{k+j-1}}} \\
&= C\epsilon_0^{\frac{2j}{k+j}} \left(\int_M \varphi^{sV(k+j-2)(1+t_{k+j-1}p_{k+j-1})} |\nabla^{k+j}\mathrm{Rm}|^{p_{k+j-1}} \right)^{\frac{2k}{(k+j)p_{k+j-1}}}.
\end{aligned}$$

We used the inductive hypothesis in the first line and Proposition 6.2.3 in the second line. We have obtained the desired inequality. □

6.3 Estimates for Certain Integrals

We apply Proposition 6.2.4 to estimate the integrals from the beginning of the section.

Lemma 6.3.1. *Let M, φ satisfy the above hypotheses, with $n \geq 6$, and let k satisfy $0 \leq k \leq \frac{n}{2} - 3$. Suppose $s \geq \frac{1}{4}n^2$. Then there exist $C = C(n, k, s, C_S)$ and R_0 such that, for all $R \geq R_0$,*

$$\int_M \varphi^{2s}|\nabla^k \mathrm{Rm}|^2|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^2 \leq C\epsilon_0^2 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.$$

Proof. If $k = 0$ or $k = \frac{n}{2} - 3$, then the Hölder inequality and Proposition 5.2.4 imply that there exist $C = C(n, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}\int_M \varphi^{2s}|\mathrm{Rm}|^2|\nabla^{\frac{n}{2}-3}\mathrm{Rm}|^2 &\leq C\left(\int_M |\mathrm{Rm}|^{\frac{n}{2}}\right)^{\frac{4}{n}}\left(\int_M \varphi^{\frac{2ns}{n-4}}|\nabla^{\frac{n}{2}-3}\mathrm{Rm}|^{\frac{2n}{n-4}}\right)^{\frac{n-4}{n}} \\ &\leq C\epsilon_0^2 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.\end{aligned}$$

The exponent of φ was sufficiently large to apply Proposition 5.2.4 since, if $s \geq \frac{1}{4}n^2$, then

$$\begin{aligned}\frac{2ns}{n-4} &\geq \frac{n^3}{2(n-4)} \geq \frac{n(n-4)^2}{2(n-4)} = \frac{n(n-4)}{2} \\ &\geq \frac{n-4}{2} = \frac{n}{2} - 3 + 1.\end{aligned}$$

Suppose $1 \leq k \leq \frac{n}{2} - 4$. In this case, $n \geq 10$. The Hölder inequality gives

$$\begin{aligned}&\int_M \varphi^{2s}|\nabla^k \mathrm{Rm}|^2|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^2 \\ &\qquad \leq \left(\int_M \varphi^{\alpha_k ps}|\nabla^k \mathrm{Rm}|^{2p}\right)^{\frac{1}{p}}\left(\int_M \varphi^{\beta_k qs}|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^{2q}\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\alpha_k = \frac{4k}{n-6}, \quad \beta_k = \frac{2(n-6-2k)}{n-6}.$$

For $1 \leq k \leq \frac{n}{2} - 4$, define r_k by

$$r_k = \frac{n(n-6)}{4(n-6) + 2(n-8)k} = \frac{n(n-6)}{2(k+2)n - 8(2k+3)}.$$

Set $p = r_k$ and $q = r_{n/2-3-k}$. Then $\frac{1}{p} + \frac{1}{q} = 1$. Using equation (6.10), we calculate that $p_{n/2-4} = \frac{2n}{n-4}$. Using equation (6.11), we calculate that

$$1 + 2t_i v_i = \frac{i+1}{i} \cdot \frac{(i+2)n - 4(2i+3)}{(i+3)n - 4(2i+5)}.$$

This enables us to compute the following telescoping product:

$$\prod_{i=k}^{l} (1 + 2t_i v_i) = \frac{l+1}{k} \cdot \frac{(k+2)n - 4(2k+3)}{(l+3)n - 4(2l+5)}.$$

We estimate the integral containing $\nabla^k \mathrm{Rm}$. First, if $1 \leq k \leq \frac{n}{2} - 5$ then

$$\prod_{i=k}^{n/2-5} (1 + 2t_i v_i) = \frac{n-8}{k} \cdot \frac{(k+2)n - 4(2k+3)}{n^2 - 12n + 40}.$$

Applying Proposition 6.2.4, we obtain that there exists R_0 such that for all $R \geq R_0$,

$$\left(\int_M \varphi^{\alpha_k r_k s} |\nabla^k \mathrm{Rm}|^{2r_k} \right)^{\frac{1}{r_k}} \leq C \epsilon_0^{\frac{2(n/2-3-k)}{n/2-3}} \left(\int_M \varphi^{sE_k} |\nabla^{\frac{n}{2}-3} \mathrm{Rm}|^{\frac{2n}{n-4}} \right)^{\frac{2k}{(\frac{n}{2}-3)\frac{2n}{n-4}}}, \tag{6.15}$$

where $C = C(n, k, s)$,

$$E_{n/2-4} = \alpha_{n/2-4} r_{n/2-4} (1 + t_{n/2-4} p_{n/2-4}) = \frac{2n}{n-4}$$

and for $1 \leq k \leq \frac{n}{2} - 5$,

$$E_k = \alpha_k r_k (1 + t_{n/2-4} p_{n/2-4}) \prod_{i=k}^{n/2-5} (1 + 2t_i v_i) = \frac{2n}{n-4}.$$

We show that the exponent of φ was sufficiently large to apply Proposition 6.2.4. Since $1 \leq k \leq \frac{n}{2} - 4$,

$$\frac{(n-6)(\frac{n}{2} - 3 - k)}{k+1} \leq \frac{(n-6)(\frac{n}{2} - 4)}{2} = \frac{(n-6)(n-8)}{4} \leq \frac{n^2}{4}.$$

So, the exponent of φ was sufficiently large to apply Proposition 6.2.4 since, if $s \geq \frac{1}{4}n^2$,

$$\begin{aligned} \alpha_k r_k s &= \frac{4k r_k s}{n-6} \\ &\geq \frac{4k r_k}{n-6} \cdot \frac{(n-6)(\frac{n}{2} - 3 - k)}{k+1} \\ &= \frac{4(\frac{n}{2} - 3 - k) k r_k}{k+1}. \end{aligned}$$

We estimate the integral containing $\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}$. First,

$$r_{n/2-3-k} = \frac{n(n-6)}{4n - 24 + (n-8)(n-6-2k)} = \frac{n(n-6)}{n^2 - 2(k+5)n + 8(2k+3)}.$$

For $2 \leq k \leq \frac{n}{2} - 4$,

$$\prod_{i=n/2-3-k}^{n/2-5} (1 + 2t_i v_i) = \frac{n-8}{n-6-2k} \cdot \frac{n^2 - 2(k+5)n + 8(2k+3)}{n^2 - 12n + 40}.$$

Applying Proposition 6.2.4, we obtain that there exists R_0 such that for all $R \geq R_0$,

$$\left(\int_M \varphi^{\beta_k r_{n/2-3-k} s} |\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^{2 r_{n/2-3-k}} \right)^{\frac{1}{r_{n/2-3-k}}}$$

$$\leq C\epsilon_0^{\frac{2k}{n/2-3}} \left(\int_M \varphi^{sF_k} |\nabla^{\frac{n}{2}-3}\mathrm{Rm}|^{\frac{2n}{n-4}} \right)^{\frac{2(\frac{n}{2}-3-k)}{(\frac{n}{2}-3)\frac{2n}{n-4}}}, \quad (6.16)$$

where $C = C(n, k, s)$,

$$F_1 = \beta_1 r_{n/2-4}(1 + t_{n/2-4}p_{n/2-4}) = \frac{2n}{n-4}$$

and for $2 \leq k \leq \frac{n}{2} - 4$,

$$F_k = \beta_k r_{n/2-3-k}(1 + t_{n/2-4}p_{n/2-4}) \prod_{i=n/2-3-k}^{n/2-5} (1 + 2t_i v_i) = \frac{2n}{n-4}.$$

We show that the exponent of φ was sufficiently large to apply Proposition 6.2.4. Since $1 \leq k \leq \frac{n}{2} - 4$,

$$\frac{(n-6)k}{\frac{n}{2} - 3 - k + 1} \leq \frac{(n-6)(\frac{n}{2}-4)}{\frac{n}{2} - 2 - (\frac{n}{2} - 4)} \leq \frac{n(\frac{n}{2})}{2} = \frac{n^2}{4}.$$

So, the exponent of φ was sufficiently large to apply Proposition 6.2.4 since, if $s \geq \frac{1}{4}n^2$,

$$\begin{aligned}
\beta_k r_{\frac{n}{2}-3-k} s &= \frac{4(\frac{n}{2} - 3 - k) r_{\frac{n}{2}-3-k} s}{n-6} \\
&\geq \frac{4(\frac{n}{2} - 3 - k) r_{\frac{n}{2}-3-k}}{n-6} \cdot \frac{(n-6)k}{\frac{n}{2} - 3 - k + 1} \\
&= \frac{4k(\frac{n}{2} - 3 - k) r_{\frac{n}{2}-3-k}}{\frac{n}{2} - 3 - k + 1}.
\end{aligned}$$

Let I denote

$$I = \int_M \varphi^{2s} |\nabla^k \mathrm{Rm}|^2 |\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^2.$$

Therefore there exist $C = C(n, k, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$
\begin{aligned}
I &\leq \left(\int_M \varphi^{\alpha_k ps}|\nabla^k \mathrm{Rm}|^{2p}\right)^{\frac{1}{p}} \left(\int_M \varphi^{\beta_k qs}|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^{2q}\right)^{\frac{1}{q}} \\
&= \left(\int_M \varphi^{s\alpha_k r_k}|\nabla^k \mathrm{Rm}|^{2r_k}\right)^{\frac{1}{r_k}} \left(\int_M \varphi^{s\beta_k r_{n/2-3-k}}|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^{2r_{n/2-3-k}}\right)^{\frac{1}{r_{n/2-3-k}}} \\
&\leq C\epsilon_0^2 \left(\int_M \varphi^{\frac{2ns}{n-4}}|\nabla^{\frac{n}{2}-3}\mathrm{Rm}|^{\frac{2n}{n-4}}\right)^{\frac{n-4}{n}} \\
&\leq C\epsilon_0^2 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.
\end{aligned}
$$

We obtained the third line by combining inequalities (6.15) and (6.16). We obtained the fourth line by applying Proposition 5.2.4. □

Proposition 6.3.2. *Let M, φ satisfy the above hypotheses, with $n \geq 6$, and let k satisfy $0 \leq k \leq \frac{n}{2} - 3$. Suppose $s \geq \frac{1}{4}n^2$. Then there exist $C = C(n, k, s, C_S)$ and R_0 such that, for all $R \geq R_0$,*

$$
\int_M \varphi^{2s}\nabla^{\frac{n}{2}-1}\mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-3-k}\mathrm{Rm} \leq C\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.
$$

Proof. Applying the Hölder inequality and Lemma 6.3.1 , we obtain

$$
\begin{aligned}
\int_M \varphi^{2s}\nabla^{\frac{n}{2}-1}\mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-3-k}\mathrm{Rm} &\leq C\left(\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_M \varphi^{2s}|\nabla^k \mathrm{Rm}|^2|\nabla^{\frac{n}{2}-3-k}\mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&\leq C\left(\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&\quad \times C\epsilon_0\left(\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&= C\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.
\end{aligned}
$$

□

Proposition 6.3.3. *Let M, φ satisfy the above hypotheses, with $n \geq 4$, and let k satisfy $0 \leq k \leq \frac{n}{2} - 2$. Suppose that $s \geq \frac{1}{4}n^2$. Then there exist $C = C(n, k, s, C_S)$ and R_0 such that, for all $R \geq R_0$,*

$$\int_M \varphi^{2s} \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-2-k}\mathrm{Rm} \leq C\epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.$$

Proof. We combine the proofs of Lemma 6.3.1 and Proposition 6.3.2. If $k = 0$ or $k = \frac{n}{2} - 2$, then the Hölder inequality and Proposition 5.2.4 give that there exist $C = C(n, s, C_S)$ and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned}\int_M \varphi^{2s} \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \mathrm{Rm} * \nabla^{\frac{n}{2}-2}\mathrm{Rm} &\leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \\ &\quad \times \left(\int_M \varphi^{\frac{2ns}{n-2}} |\nabla^{\frac{n}{2}-2}\mathrm{Rm}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ &\leq C\epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.\end{aligned}$$

The exponent of φ was sufficiently large to apply Proposition 5.2.4 since, if $s \geq \frac{1}{4}n^2$, then

$$\begin{aligned}\frac{2ns}{n-2} &\geq \frac{n^3}{2(n-2)} \geq \frac{n(n-2)^2}{2(n-2)} = \frac{n(n-2)}{2} \\ &\geq \frac{n-2}{2} = \frac{n}{2} - 2 + 1.\end{aligned}$$

Suppose $1 \leq k \leq \frac{n}{2} - 3$. In this case, $n \geq 8$. First, let I_1 denote

$$I_1 = \left(\int_M \varphi^{\frac{2ns}{n+2}} |\nabla^k \mathrm{Rm}|^{\frac{2n}{n+2}} |\nabla^{\frac{n}{2}-2-k}\mathrm{Rm}|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}}.$$

We obtain an interpolation estimate for I_1 by arguing as in the proof of Lemma 6.3.1. Define α_k, β_k by

$$\alpha_k = \frac{4k}{n-4}, \quad \beta_k = \frac{2(n-2k-4)}{n-4}.$$

Define r_k by

$$r_k = \frac{n(n-4)}{4(n-4)+2(n-6)k} = \frac{n(n-4)}{2(k+2)n-4(3k+4)}.$$

Let $p = \left(\frac{n+2}{n}\right) r_k$, $q = \left(\frac{n+2}{n}\right) r_{n/2-2-k}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Applying the Hölder inequality and imitating the application of Proposition 6.2.4 in the proof of Lemma 6.3.1, we obtain that there exist $C = C(n,k,s,C_S)$ and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned} I_1 &\leq \left(\int_M \varphi^{\frac{\alpha_k nps}{n+2}} |\nabla^k \mathrm{Rm}|^{\frac{2np}{n+2}}\right)^{\frac{n+2}{2np}} \left(\int_M \varphi^{\frac{\beta_k nqs}{n+2}} |\nabla^{\frac{n}{2}-2-k} \mathrm{Rm}|^{\frac{2nq}{n+2}}\right)^{\frac{n+2}{2nq}} \\ &= \left(\int_M \varphi^{s\alpha_k r_k} |\nabla^k \mathrm{Rm}|^{2r_k}\right)^{\frac{1}{2r_k}} \\ &\quad \times \left(\int_M \varphi^{s\beta_k r_{n/2-2-k}} |\nabla^{\frac{n}{2}-2-k} \mathrm{Rm}|^{2r_{n/2-2-k}}\right)^{\frac{1}{2r_{n/2-2-k}}} \\ &\leq C\epsilon_0 \left(\int_M \varphi^{\frac{2ns}{n-2}} |\nabla^{\frac{n}{2}-2} \mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}}. \end{aligned}$$

Let I_2 denote

$$I_2 = \int_M \varphi^{2s} \nabla^{\frac{n}{2}-2} \mathrm{Rm} * \nabla^k \mathrm{Rm} * \nabla^{\frac{n}{2}-2-k} \mathrm{Rm}.$$

Therefore

$$I_2 \leq C \left(\int_M \varphi^{\frac{2ns}{n-2}} |\nabla^{\frac{n}{2}-2} \mathrm{Rm}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_M \varphi^{\frac{\alpha_k nps}{n+2}} |\nabla^k \mathrm{Rm}|^{\frac{2np}{n+2}}\right)^{\frac{n+2}{2np}}$$

$$
\begin{aligned}
&\quad \times \left(\int_M \varphi^{\frac{\beta_k nqs}{n+2}} |\nabla^{\frac{n}{2}-2-k} \mathrm{Rm}|^{\frac{2nq}{n+2}} \right)^{\frac{n+2}{2nq}} \\
&= C \left(\int_M \varphi^{\frac{2ns}{n-2}} |\nabla^{\frac{n}{2}-2} \mathrm{Rm}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} I_1 \\
&\le C\epsilon_0 \left(\int_M \varphi^{\frac{2ns}{n-2}} |\nabla^{\frac{n}{2}-2} \mathrm{Rm}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\
&\le C\epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2.
\end{aligned}
$$

We obtained the first line by applying the Hölder inequality. We obtained the third line by applying the above bound on I_1. We obtained the fourth line by applying Proposition 5.2.4. □

6.4 Estimates for Gradient Terms

In this subsection, we obtain estimates for lower order terms containing the 1-form $\nabla\varphi$, which we refer to as a gradient.

Let (M, g) be a complete obstruction-flat constant scalar curvature Riemannian manifold. We recall the following definitions from section 1.2.2. Fix $x_0 \in M$. If $x, y \in M$, let $d_g(x, y)$ denote the distance between x and y with respect to the metric h. Define ρ by $\rho(x) = d_g(x_0, x)$ for $x \in M$. The manifold M has quadratic curvature decay if there exists $C_Q > 0$ such that $|\mathrm{Rm}| \le C_Q \cdot \rho^{-2}$ on M. We prove the following proposition.

Proposition 6.4.1. *Suppose that there exists $K \ge 1$ such that $\|\mathrm{Rm}\|_\infty \le K$. Suppose that M has quadratic curvature decay. Then for all $j \ge 1$, there exists $C = C(n, j, C_Q)$ such that $|\nabla^j \mathrm{Rm}| \le C\rho^{-(j+2)}$.*

Proof. We use a blowup argument to prove the proposition by contradiction. Let $m \geq \frac{3n}{2} - 3$. For a point $x \in M$ and metric h on M, define $f_m(x,h)$ as

$$f_m(x,h) = \sum_{j=1}^{m} |\nabla^j \mathrm{Rm}(h)|_h^{\frac{2}{j+2}}.$$

Let x_0 be the point at which the cutoff function is centered. We have assumed that quadratic curvature decay holds with respect to x_0.

Suppose that the conclusion is false. Then there exists a sequence $\{y_i\}_{i\in\mathbb{N}}$ such that

$$\rho(y_i)^2 f_m(y_i, g) > i, \quad \lim_{i\to\infty} \rho(y_i) = \infty.$$

In particular, $\limsup_{\rho(x)\to\infty} f_m(x,g)\rho(x)^2 = \infty$. We are then able to apply Theorem 8.44 in [15], with f_m replacing the scalar curvature R. This theorem provides a sequence of points $(x_i)_{i=1}^{\infty}$, a sequence of radii $(r_i)_{i=1}^{\infty}$, and a sequence $(\epsilon_i)_{i=1}^{\infty}$ such that

(a) the balls $B(x_i, r_i, g)$ are disjoint

(b) $\lim_{i\to\infty} \rho_i / r_i = \infty$

(c) $\sup\{f_m(x,g) : x \in B(x_i, r_i, g)\} \leq (1+\epsilon_i) f_m(x_i, g)$,

where $\rho_i = \rho(x_i)$. We can obtain more precise information from the proof of Theorem 8.44 in [15]. Let A_i and δ_i be as in the proof we are referencing. The proof sets $\epsilon_i = (1-\delta_i)^{-3} - 1$. If we set $A_i = \rho_i^4$ and $\delta_i = \rho_i^{-1}$, we obtain from the proof that

$$f_m(x_i, g) r_i^2 = (1-\delta_i) A_i \delta_i^2 = (1-\rho_i)\rho_i^4 \rho_i^{-2} = (1-\rho_i)\rho_i^2$$
$$1 + \epsilon_i = (1-\delta_i)^{-3} = \frac{\rho_i^3}{(\rho_i - 1)^3} \leq \frac{\rho_i^3}{(\rho_i/2)^3} = 8.$$

Define a sequence of metrics g_i by $g_i = \lambda_i g$, where $\lambda_i = f_m(x_i, g)$. We show that

$$\lim_{i\to\infty} |\mathrm{Rm}(g_i)|_{g_i} = 0$$

on M. We have

$$\begin{aligned}\frac{1}{f_m(x_i,g)} &= \frac{r_i^2}{r_i^2 f_m(x_i,g)}\\ &\leq \frac{r_i^2}{(1-\rho_i^{-1})\rho_i^2} = \frac{r_i^2}{(\rho_i - 1)\rho_i}\\ &\leq \frac{r_i}{\rho_i}\cdot\frac{r_i}{\rho_i/2}.\end{aligned}$$

Since $\|\mathrm{Rm}(g)\|_\infty \leq K$ and (b) from above states that $\lim_{i\to\infty} \rho_i/r_i = \infty$, we get

$$|\mathrm{Rm}(g_i)|_{g_i} \leq \frac{K}{\lambda_i} \leq \frac{r_i}{\rho_i}\cdot\frac{r_i}{\rho_i/2} \to 0$$

as $i \to \infty$. We also have a uniform C^m estimate for $\mathrm{Rm}(g_i)$ given by

$$\begin{aligned}f_m(x, g_i) &\leq \frac{f_m(x,g)}{\lambda_i} = \frac{f_m(x,g)}{f_m(x_i,g)}\\ &\leq \frac{(1+\epsilon_i)f_m(x_i,g)}{f_m(x_i,g)}\\ &\leq 8,\end{aligned}$$

for each $i \in \mathbb{N}$ and $x \in B(x_i, r_i, g)$.

Let g_e denote the Euclidean metric on $\mathbb{R}^n$. Define exponential maps for all i by

$$\varphi_i : B(0,1,g_e) \to M, \quad v \mapsto \exp_{x_i}(v).$$

For all i, let $h_i = \varphi_i^* g_i$. Since g is obstruction-flat and scalar-flat, we know that (M, g) is a stationary point for AOF. Arguing as in the blowup argument that comprises the proof of the pointwise smoothing estimates for AOF, we are able to extract a subsequential limit metric h_∞ on $B(0,1/2,g_e)$. Since

$$f_m(0,h_i) = f_m(0,g_i) = \lambda_i^{-1}\lambda_i = 1$$

for all i, we get that $f_m(0,h_\infty) = 1$ as well. We have already shown that $\lim_{i\to\infty} |\mathrm{Rm}(g_i)|_{g_i} = 0$ on M, which implies that $f_m(0,h_\infty) = 0$. This is a contradiction, from which the proposition follows. □

We use the estimates in Proposition 6.4.1 to prove the following proposition, which allows us to estimate lower order terms that arise when we integrate by parts.

Proposition 6.4.2. *Let $(M,g),\varphi$ satisfy the above hypotheses. Suppose that $s \geq \frac{1}{2}$. Let dV denote the volume form of g. Then, for $j \geq -1$, there exists $C = C(n,j,\Lambda,C_Q,C_V)$ such that for all $R > 0$,*

$$\int_M \varphi^{2s-1}\nabla\varphi * P_{j+1}^{n-2j-1}(\mathrm{Rm})\, dV \leq CR^{-2}.$$

Proof. Let I denote

$$I = \int_M \varphi^{2s-1}\nabla\varphi * P_{j+1}^{n-2j-1}(\mathrm{Rm})\, dV.$$

Let x_0 be the point at which the cutoff function is centered. We have assumed that the volume growth upper bound assumption holds for balls centered at x_0. We have also assumed that quadratic curvature decay holds with respect to x_0. Let $dS(\rho)$ denote the volume form of $S(\rho)$, recalling that $S(\rho) = \partial B(x_0,\rho)$ and $A(a,b) = B(x_0,b) \setminus \overline{B(x_0,a)}$. Then

$$\begin{aligned} I &= \int_{A(R,2R)} \varphi^{2s-1}\nabla\varphi * P_{j+1}^{n-2j-1}(\mathrm{Rm})\, dV \\ &\leq \Lambda C(C_Q) \int_R^{2R} \Big[\int_{S(\rho)} \rho^{-1}\rho^{-(n-2j-1)}\rho^{-2(j+1)}\, dS(\rho)\Big]\, d\rho \end{aligned}$$

$$
\begin{aligned}
&= C \int_R^{2R} \rho^{-(n+2)} \Big[\int_{S(\rho)} dS(\rho) \Big] \, d\rho \\
&\leq C R^{-(n+2)} \int_R^{2R} \Big[\int_{S(\rho)} dS(\rho) \Big] \, d\rho \\
&\leq C R^{-(n+2)} \cdot C_V R^n \\
&= C R^{-2}.
\end{aligned}
$$

We obtained the second line by applying Proposition 6.4.1 and our decay bound for φ. We obtained the penultimate line by applying the volume growth upper bound. □

Chapter 7

Elliptic Estimates and Rigidity in Even Dimensions

7.1 Estimate of Full Riemann Tensor by Ricci Tensor

In this section, let n be even and let $n \geq 6$. Applying Proposition 5.2.1 with $q = \frac{n}{2} - 1$, we obtain for $R \gg 1$

$$\left(\int_M \varphi^{ns}|\mathrm{Rm}|^n\right)^{\frac{2}{n}} \leq C \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2,$$

where $C = C(n, s, \Lambda, C_S)$. We wish to obtain L^2 estimates of $\nabla^{\frac{n}{2}-1}\mathrm{Rm}$ by $\nabla^{\frac{n}{2}-3}\Delta\mathrm{Rm}$ and, subsequently, $\nabla^{\frac{n}{2}-1}\mathrm{Rc}$ for $R \gg 1$.

Proposition 7.1.1. *Let M, φ satisfy the above hypotheses. Suppose $s \geq \frac{1}{4}n^2$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that, for all $R \geq R_0$,*

$$\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 \leq \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-3}\Delta\mathrm{Rm}|^2 + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}.$$

Proof. Let I denote

$$I = \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2.$$

We estimate I:

$$\begin{aligned}
I &= -\int_M \langle \varphi^{2s} \Delta \nabla^{\frac{n}{2}-2} \mathrm{Rm}, \nabla^{\frac{n}{2}-2} \mathrm{Rm} \rangle + \int_M \varphi^{2s-1} \nabla \varphi * \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^{\frac{n}{2}-2} \mathrm{Rm} \\
&\leq -\int_M \langle \varphi^{2s} \Delta \nabla^{\frac{n}{2}-2} \mathrm{Rm}, \nabla^{\frac{n}{2}-2} \mathrm{Rm} \rangle + C_2 R^{-2} \\
&= -\int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-2} \Delta \mathrm{Rm}, \nabla^{\frac{n}{2}-2} \mathrm{Rm} \rangle + \sum_{k=0}^{n/2-2} \int_M \varphi^{2s} \nabla^{\frac{n}{2}-2} \mathrm{Rm} * \nabla^{\frac{n}{2}-2-k} \mathrm{Rm} * \nabla^k \mathrm{Rm} \\
&\quad + C_2 R^{-2} \\
&\leq -\int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-2} \Delta \mathrm{Rm}, \nabla^{\frac{n}{2}-2} \mathrm{Rm} \rangle + C_1 \epsilon_0 I + C_2 R^{-2} \\
&= \int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm}, \Delta \nabla^{\frac{n}{2}-3} \mathrm{Rm} \rangle + \int_M \varphi^{2s-1} \nabla \varphi * \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^{\frac{n}{2}-2} \mathrm{Rm} \\
&\quad + C_1 \epsilon_0 I + C_2 R^{-2} \\
&\leq \int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm}, \Delta \nabla^{\frac{n}{2}-3} \mathrm{Rm} \rangle + C_1 \epsilon_0 I + C_2 R^{-2} \\
&= \int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm}, \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm} \rangle + \sum_{k=0}^{n/2-3} \int_M \varphi^{2s} \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^{\frac{n}{2}-3-k} \mathrm{Rm} * \nabla^k \mathrm{Rm} \\
&\quad + C_1 \epsilon_0 I + C_2 R^{-2} \\
&\leq \int_M \langle \varphi^{2s} \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm}, \nabla^{\frac{n}{2}-3} \Delta \mathrm{Rm} \rangle + C_1 \epsilon_0 I + C_2 R^{-2}.
\end{aligned}$$

We obtained the first and fifth lines via integration by parts. We obtained the second and sixth lines by applying Lemma 6.4.2. We obtained the third and seventh lines by commuting derivatives via Proposition 2.1.2. We obtained the fourth and eighth lines by applying Propositions 6.3.3 and 6.3.2, respectively. □

Proposition 7.1.2. *Let M, φ satisfy the above hypotheses. Suppose $s \geq \frac{1}{4}n^2$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that, for all $R \geq R_0$,*

$$\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 \leq \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}.$$

Proof. Let I_1 denote

$$I_1 = \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.$$

Then there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned} I_1 &\leq \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-3}\Delta\mathrm{Rm}|^2 + C\epsilon_0 I_1 + CR^{-2} \\ &= \int_M \varphi^{2s}\left|\nabla^{\frac{n}{2}-3}\left[\nabla_j\nabla_m R_{lk} - \nabla_j\nabla_l R_{mk} + \nabla_k\nabla_l R_{mj} - \nabla_k\nabla_m R_{lj} + \mathrm{Rm}^{*2}\right]\right|^2 \\ &\quad + C_1\epsilon_0 I_1 + C_2 R^{-2}. \end{aligned} \tag{7.1}$$

We obtained the first line by applying Proposition 7.1.1. We obtained the second line by applying a result of Hamilton.

We will derive estimates for the integrals that arise from expanding the expression

$$\nabla^{\frac{n}{2}-3}\left[\nabla_j\nabla_m R_{lk} - \nabla_j\nabla_l R_{mk} + \nabla_k\nabla_l R_{mj} - \nabla_k\nabla_m R_{lj} + \mathrm{Rm}^{*2}\right]$$

contained in the above estimate of I_1. First, we estimate the terms whose integrals are of the form $(\nabla^{\frac{n}{2}-3}\nabla^2\mathrm{Rc})^{*2}$. We estimate the diagonal terms:

$$\int_M \varphi^{2s}\langle\nabla^{\frac{n}{2}-3}\nabla_j\nabla_m R_{lk}, \nabla^{\frac{n}{2}-3}\nabla_j\nabla_m R_{lk}\rangle = \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2.$$

We estimate the off-diagonal terms. Let I_2 denote

$$I_2 = \int_M \varphi^{2s} \langle \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m R_{lk}, \nabla^{\frac{n}{2}-3} \nabla_j \nabla_l R_{mk} \rangle.$$

Then there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned}
I_2 &= \int_M \varphi^{2s} \langle \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m R_{lk}, \nabla_l \nabla^{\frac{n}{2}-3} \nabla_j R_{mk} \rangle \\
&\quad + \sum_{i=0}^{n/2-3} \int_M \varphi^{2s} \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^{\frac{n}{2}-3-i} \mathrm{Rm} * \nabla^i \mathrm{Rm} \\
&\leq \int_M \varphi^{2s} \langle \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m R_{lk}, \nabla_l \nabla^{\frac{n}{2}-3} \nabla_j R_{mk} \rangle + C_1 \epsilon_0 I_1 \\
&= -\int_M \varphi^{2s} \langle \nabla^l \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m R_{lk}, \nabla^{\frac{n}{2}-3} \nabla_j R_{mk} \rangle \\
&\quad + \int_M \varphi^{2s-1} \nabla \varphi * \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^{\frac{n}{2}-2} \mathrm{Rm} + C_1 \epsilon_0 I_1 \\
&\leq -\int_M \varphi^{2s} \langle \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m \nabla^l R_{lk}, \nabla^{\frac{n}{2}-3} \nabla_j R_{mk} \rangle \\
&\quad + \sum_{i=0}^{n/2-2} \int_M \varphi^{2s} \nabla^{\frac{n}{2}-2} \mathrm{Rm} * \nabla^{\frac{n}{2}-2-i} \mathrm{Rm} * \nabla^i \mathrm{Rm} + C_1 \epsilon_0 I_1 + C_2 R^{-2} \\
&\leq C_1 \epsilon_0 I_1 + C_2 R^{-2}.
\end{aligned}$$

We obtained the first line by commuting derivatives via Proposition 2.1.4. We obtained the second line by applying Proposition 6.3.2. We obtained the third line via integration by parts. We obtained the fourth line by applying Proposition 6.4.2 and commuting derivatives via Proposition 2.1.4. The first term in the fourth line vanishes due to a Bianchi identity and the assumption that (M, g) has constant scalar curvature. We estimated the second term in the fourth line using Proposition 6.3.3.

Next, we estimate terms of the form $(\nabla^{\frac{n}{2}-3}(\mathrm{Rm}^{*2}))^{*2}$. Let I_3 denote

$$I_3 = int_M \varphi^{2s} \nabla^i \mathrm{Rm} * \nabla^{\frac{n}{2}-3-i} \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^{\frac{n}{2}-3-j} \mathrm{Rm}.$$

We estimate using the Hölder inequality and Lemma 6.3.1. There exist $C_1 = C_1(n, s, C_S)$ and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned} I_3 &\leq C_1 \left(\int_M \varphi^{2s} |\nabla^i \mathrm{Rm}|^2 |\nabla^{\frac{n}{2}-3-i} \mathrm{Rm}|^2 \right)^{\frac{1}{2}} \left(\int_M \varphi^{2s} |\nabla^j \mathrm{Rm}|^2 |\nabla^{\frac{n}{2}-3-j} \mathrm{Rm}|^2 \right)^{\frac{1}{2}} \\ &\leq C_1 \epsilon_0^2 I_1. \end{aligned}$$

Finally, we use Proposition 6.3.2 to estimate terms of the form $\nabla^{\frac{n}{2}-3} \nabla^2 \mathrm{Rc} * \nabla^{\frac{n}{2}-3}(\mathrm{Rm}^{*2})$. Let I_4 denote

$$I_4 = \int_M \varphi^{2s} \nabla^{\frac{n}{2}-3} \nabla_j \nabla_m R_{lk} * \nabla^i \mathrm{Rm} * \nabla^{\frac{n}{2}-3-i} \mathrm{Rm}.$$

There exist $C_1 = C_1(n, s, C_S)$ and R_0 such that, for all $R \geq R_0$,

$$I_4 = \int_M \varphi^{2s} \nabla^{\frac{n}{2}-1} \mathrm{Rm} * \nabla^i \mathrm{Rm} * \nabla^{\frac{n}{2}-3-i} \mathrm{Rm} \leq C_1 \epsilon_0 I_1.$$

We collect the preceding estimates in order to conclude that there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that, for all $R \geq R_0$,

$$\begin{aligned} \int_M \varphi^{2s} \big| \nabla^{\frac{n}{2}-3} \big[\nabla_j \nabla_m R_{lk} - \nabla_j \nabla_l R_{mk} + \nabla_k \nabla_l R_{mj} - \nabla_k \nabla_m R_{lj} + \mathrm{Rm}^{*2} \big] \big|^2 \\ \leq \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rc}|^2 + C_1 \epsilon_0 I_1 + C_2 R^{-2}. \end{aligned}$$

The desired inequality now follows from the inequality (7.1). □

7.2 General Interpolation Estimates

In this section, let n be even and let $n \geq 6$. We prove an interpolation inequality and use it to provide energy estimates for various lower order terms. This will enable us to use the fact that (M, g) is obstruction-flat to complete the proof of the main theorem.

The following proposition provides estimates for terms of the form

$$\mathrm{Rm}^{*l_0} * (\nabla \mathrm{Rm})^{*l_1} * \cdots * (\nabla^{\frac{n}{2}-2} \mathrm{Rm})^{*l_{n/2-2}}$$

containing j factors that possess $n - 2j$ derivatives in total.

Proposition 7.2.1. *Let (M, φ) satisfy the above hypotheses. Suppose that*

(a) n is even, $n \geq 4$, $s \geq n^2$, $2 \leq j \leq \frac{n}{2}$, $l_0 \geq -1$, $l_k \geq 0$ for $1 \leq k \leq \frac{n}{2} - 2$

(b) $\sum_{k=0}^{n/2-2} l_k = j$, $\sum_{k=1}^{n/2-2} k l_k = n - 2j$.

Let $\alpha = \max\{k : l_k > 0\}$. Then there exist $C = C(n, j, \alpha, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$\int_M \varphi^{2s} |\mathrm{Rm}|^{l_0+1} \prod_{k=1}^{n/2-2} |\nabla^k \mathrm{Rm}|^{l_k} \leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(j-1)} \left(\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2 \right). \tag{7.2}$$

Proof. We apply Hölder's inequality, followed by the interpolation inequality given by Proposition 6.2.4. Let

$$m = \begin{cases} n - 2j - \alpha, & \alpha l_\alpha > \frac{1}{2}(n-2j) \\ \frac{1}{2}(n-2j), & \alpha l_\alpha \leq \frac{1}{2}(n-2j). \end{cases}$$

Define c, v, w by the following equations:

$$\sum_{k=1}^{c-1} kl_k < m \leq \sum_{k=1}^{c} kl_k, \quad \sum_{k=1}^{c-1} kl_k + cv = m, \quad v + w = l_c.$$

We also define α_k for $1 \leq k \leq c$ and β_k for $c \leq k \leq \frac{n}{2} - 3$ as follows:

$$\frac{1}{\alpha_k} = \frac{m-k}{mn/2} + \frac{k}{mn/(m+1)} = \frac{(k+2)m - k}{nm}$$
$$\frac{1}{\beta_k} = \frac{(k+2)(n-2j-m) - k}{n(n-2j-m)}.$$

We note that

$$1 = \frac{2}{n}(l_0 + 1) + \sum_{k=1}^{c-1} \frac{l_k}{\alpha_k} + \frac{v}{\alpha_c} + \frac{w}{\beta_c} + \sum_{k=c+1}^{n/2-2} \frac{l_k}{\beta_k} \tag{7.3}$$

and

$$j - 1 = 1 + l_0 + \sum_{k=1}^{c-1} \frac{(m-k)l_k}{m} + \frac{(m-c)v}{m} + \frac{(n-2j-m-c)w}{n-2j-m} + \sum_{k=c+1}^{n/2-2} \frac{(n-2j-m-k)l_k}{n-2j-m}. \tag{7.4}$$

Define I by

$$I = \int_M \varphi^{2s} |\mathrm{Rm}|^{l_0+1} \prod_{k=1}^{n/2-2} |\nabla^k \mathrm{Rm}|^{l_k}.$$

Then there exist $C = C(n, j, \alpha, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$I = \left(\int_M |\mathrm{Rm}|^{l_0+1}\right) \left[\prod_{k=1}^{c-1} \varphi^{\frac{kl_k s}{m}} |\nabla^k \mathrm{Rm}|^{l_k}\right] \left(\varphi^{\frac{clv}{m}} |\nabla^c \mathrm{Rm}|^v\right)$$

$$
\begin{aligned}
&\times \left([\varphi^{\frac{cw}{n-2j-m}} |\nabla^c \mathrm{Rm}|^w \right) \left[\prod_{k=c+1}^{n/2-2} \varphi^{\frac{kl_k s}{n-2j-m}} |\nabla^k \mathrm{Rm}|^{l_k} \right] \\
&\leq \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(l_0+1)} \left[\prod_{k=1}^{c-1} \left(\int_M \varphi^{\frac{k\alpha_k s}{m}} |\nabla^k \mathrm{Rm}|^{\alpha_k} \right)^{\frac{l_k}{\alpha_k}} \right] \left(\int_M \varphi^{\frac{c\alpha_c s}{m}} |\nabla^c \mathrm{Rm}|^{\alpha_c} \right)^{\frac{v}{\alpha_c}} \\
&\times \left(\int_M \varphi^{\frac{c\beta_c s}{n-2j-m}} |\nabla^c \mathrm{Rm}|^{\beta_c} \right)^{\frac{w}{\beta_c}} \left[\prod_{k=c+1}^{n/2-2} \left(\int_M \varphi^{\frac{k\beta_k s}{n-2j-m}} |\nabla^k \mathrm{Rm}|^{\beta_k} \right)^{\frac{l_k}{\beta_k}} \right] \\
&\leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(j-1)} \left[\prod_{k=1}^{c-1} \left(\int_M \varphi^{\frac{ns}{m+1}} |\nabla^m \mathrm{Rm}|^{\frac{n}{m+1}} \right)^{\frac{k(m+1)l_k}{mn}} \right] \\
&\times \left(\int_M \varphi^{\frac{ns}{m+1}} |\nabla^m \mathrm{Rm}|^{\frac{n}{m+1}} \right)^{\frac{c(m+1)v}{mn}} \\
&\times \left(\int_M \varphi^{\frac{ns}{n-2j-m+1}} |\nabla^{n-2j-m} \mathrm{Rm}|^{\frac{n}{n-2j-m+1}} \right)^{\frac{c(n-2j-m+1)w}{(n-2j-m)n}} \\
&\times \left[\prod_{k=c+1}^{n/2-2} \left(\int_M \varphi^{\frac{ns}{n-2j-m+1}} |\nabla^{n-2j-m} \mathrm{Rm}|^{\frac{n}{n-2j-m+1}} \right)^{\frac{k(n-2j-m+1)l_k}{(n-2j-m)n}} \right] \\
&= C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(j-1)} \left(\int_M \varphi^{\frac{ns}{m+1}} |\nabla^m \mathrm{Rm}|^{\frac{n}{m+1}} \right)^{\frac{m+1}{n}} \\
&\times \left(\int_M \varphi^{\frac{ns}{n-2j-m+1}} |\nabla^{n-2j-m} \mathrm{Rm}|^{\frac{n}{n-2j-m+1}} \right)^{\frac{n-2j-m+1}{n}} \\
&\leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(j-1)} \left(\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2 \right)^{\frac{1}{2}} \left(\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\int_M |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}(j-1)} \left(\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2 \right).
\end{aligned}
$$

We obtained the second line via Hölder's inequality and equation (7.3). The definition of m and equation (7.4) allow us to apply the interpolation inequality Proposition 6.2.4, which yields the third line. We obtained the fifth line via the Sobolev inequality Proposition 5.2.4.

It follows from the inequalities

$$0 \leq m \leq n - 2j \leq n - 4, \quad s \geq n^2,$$

and from our definitions of α_k for $1 \leq k \leq c$ and β_k for $c \leq k \leq \frac{n}{2} - 3$, that the exponents of the cutoff function φ are sufficiently large to allow the application of Propositions 6.2.4 and 5.2.4. $\square$

In the proof of the following proposition, we integrate by parts to reduce the orders of the various terms to less than $\frac{n}{2} - 1$ so that we can apply the interpolation estimate from the previous proposition. Since we only integrate by parts when the order of the terms is at least $\frac{n}{2} - 1$, we avoid obtaining, after integrating by parts, terms $(\mathrm{Rm})^{1+l_0} \prod_{k=1}^{n-4} (\nabla^k \mathrm{Rm})^{l_k}$ of the same type, i.e. the sequences $(l_k)_{k=1}^{n-4}$ are equal for both of the terms (AB denotes $A * B$).

Proposition 7.2.2. *Let $n \geq 8$ and $2 \leq j \leq \frac{n}{2}$. Suppose $s \geq n^2$. Then there exist $C_1 = C_1(n, j, s, C_S)$, $C_2 = C_2(n, j, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,*

$$\int_M \varphi^{2s} \mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm}) \leq C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1} \mathrm{Rm}|^2 + C_2 R^{-2}.$$

Proof. Let $A \in \mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm})$. Then A can be expressed as

$$A = (\mathrm{Rm})^{1+l_0} \prod_{k=1}^{d(A)} (\nabla^k \mathrm{Rm})^{l_k}$$

(AB denotes $A * B$), where $d = d(A) = \max\{k : l_k > 0\}$ and

$$\sum_{k=0}^{d(A)} l_k = j, \quad \sum_{j=0}^{d(A)} k l_k = n - 2j.$$

Let I denote

$$I = \int_M \varphi^{2s} A.$$

Suppose that $d(A) \leq \frac{n}{2} - 2$. Then Proposition 7.2.1 provides the desired estimate: there exist $C = C(n, j, d, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$I \leq C\epsilon_0^{j-1} \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.$$

It follows that, for all A such that $d(A) \leq \frac{n}{2} - 2$, there exists a uniform $C_1 = C_1(n, j, s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$I \leq C\epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.$$

We estimate the terms $A \in \mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm})$ for which $d(A) \geq \frac{n}{2} - 2$ by induction. We have already estimated the terms $A \in \mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm})$ for which $d(A) \leq \frac{n}{2} - 2$. Now suppose that $d(A) > \frac{n}{2} - 2$ and that we have already estimated the terms $B \in \mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm})$ for which $d(B) < d(A)$. We have $d(A) \geq \frac{n}{2} - 1$ and

$$n - 4 \geq n - 2j = \sum_{i=0}^{d(A)} i l_i.$$

Since $d = d(A) = \max\{k : l_k > 0\}$, the above equation implies that $l_d = 1$ and $l_{d-1} = 0$. We integrate by parts:

$$\begin{aligned} I = &\int_M \varphi^{2s-1} \nabla\varphi (\mathrm{Rm})^{1+l_0} \prod_{k=1}^{d-2} (\nabla^k \mathrm{Rm})^{l_k} \\ &+ \sum_{\substack{1 \leq i \leq d-2 \\ l_i > 0}} \int_M \left[\varphi^{2s} (\nabla^i \mathrm{Rm})^{-1+l_i} (\nabla^{i+1}\mathrm{Rm})^{1+l_{i+1}} (\mathrm{Rm}) \prod_{\substack{0 \leq k \leq d-2 \\ k \neq i, i+1}} (\nabla^k \mathrm{Rm})^{l_k} \right] \end{aligned}$$

$$+\int_M \varphi^{2s}(\mathrm{Rm})^{l_0}(\nabla\mathrm{Rm})^{1+l_1}\prod_{2\leq k\leq d-2}(\nabla^k\mathrm{Rm})^{l_k}. \tag{7.5}$$

The fact that $l_{d-1}=0$ ensures that none of the terms on the right hand side of the above equation (7.5) matches the integral on the left hand side of the equation. We estimate the term on the right hand side containing $\nabla\varphi$ via Proposition 6.4.2: there exists $C_2 = C_2(n,j,\Lambda,C_Q,C_V)$ such that

$$\int_M \varphi^{2s-1}\nabla\varphi(\mathrm{Rm})^{1+l_0}\prod_{k=1}^{d-2}(\nabla^k\mathrm{Rm})^{l_k}\leq CR^{-2}.$$

The remaining terms on the right hand side are integrals containing integrands $B\in\mathrm{Rm}*P_j^{n-2j}(\mathrm{Rm})$ for which $d(B)\leq d-1$. As a result, these terms have already been estimated via the induction hypothesis. Collecting terms, we obtain that there exist $C_1 = C_1(n,j,s,C_S)$, $C_2 = C_2(n,j,\Lambda,C_Q,C_V)$, and R_0 such that for all $R\geq R_0$,

$$I\leq C_1\epsilon_0\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2}.$$

Collecting the estimates for all $A\in\mathrm{Rm}*P_j^{n-2j}(\mathrm{Rm})$ yields the desired estimate. □

7.3 Conclusion of the Proof of Theorem 1.2.10

In this section, let n be even and let $n\geq 6$. We use the assumptions that (M,g) is obstruction - flat and has constant scalar curvature to obtain estimates that allow us to complete the proof of Theorem 1.2.10.

Proposition 7.3.1. *Let M, φ satisfy the above hypotheses. Suppose $s \geq n^2$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,*

$$\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 \leq \int_M \varphi^{2s} \langle (-1)^{\frac{n}{2}-1} \Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle + C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}.$$

Proof. Let I denote

$$I = \int_M \varphi^{2s} \langle (-1)^i \nabla^{\frac{n}{2}-1-i} \Delta^i \mathrm{Rc}, \nabla^{\frac{n}{2}-1-i}\mathrm{Rc}\rangle.$$

We show that, for all i satisfying $0 \leq i \leq \frac{n}{2} - 1$, there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$\int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 \leq I + C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}. \tag{7.6}$$

We prove this by induction on i. Suppose this estimate holds for all nonnegative integers at most i, where i satisfies $0 \leq i \leq \frac{n}{2} - 1$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}
I &= (-1)^{i+1} \int_M \varphi^{2s} \langle \Delta \nabla^{\frac{n}{2}-i-2} \Delta^i \mathrm{Rc}, \nabla^{\frac{n}{2}-i-2}\mathrm{Rc}\rangle \\
&\quad + \int_M \varphi^{2s-1} \nabla\varphi * \nabla^{\frac{n}{2}+i-1}\mathrm{Rm} * \nabla^{\frac{n}{2}-i-2}\mathrm{Rm} \\
&\leq (-1)^{i+1} \int_M \varphi^{2s} \langle \Delta \nabla^{\frac{n}{2}-i-2} \Delta^i \mathrm{Rc}, \nabla^{\frac{n}{2}-i-2}\mathrm{Rc}\rangle + C_2 R^{-2} \\
&= (-1)^{i+1} \int_M \varphi^{2s} \langle \nabla^{\frac{n}{2}-i-2} \Delta^{i+1} \mathrm{Rc}, \nabla^{\frac{n}{2}-i-2}\mathrm{Rc}\rangle \\
&\quad + \sum_{j=0}^{n/2-i-2} \int_M \varphi^{2s} \nabla^{\frac{n}{2}-i-2}\mathrm{Rm} * \nabla^{\frac{n}{2}-i-2-j}\mathrm{Rm} * \nabla^{j+2i}\mathrm{Rm} + C_2 R^{-2} \\
&\leq \int_M \varphi^{2s} \langle (-1)^{i+1} \nabla^{\frac{n}{2}-i-2} \Delta^{i+1} \mathrm{Rc}, \nabla^{\frac{n}{2}-i-2}\mathrm{Rc}\rangle + C_1 \epsilon_0 \int_M \varphi^{2s} |\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}.
\end{aligned}$$

We obtained the first line via integration by parts. We obtained the second line via Proposition 6.4.2. We obtained the third line by commuting derivatives via Proposition 2.1.2. We obtained the fourth line via Proposition 7.2.2, since each lower order term is of the form $P_3^{n-4}(\mathrm{Rm})$. Therefore, there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 &\leq I + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2} \\ &\leq \int_M \varphi^{2s}\langle(-1)^{i+1}\nabla^{\frac{n}{2}-i-2}\Delta^{i+1}\mathrm{Rc}, \nabla^{\frac{n}{2}-i-2}\mathrm{Rc}\rangle \\ &\quad + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2},\end{aligned}$$

so that inequality (7.6) also holds for $i + 1$. We conclude inequality (7.6) holds for all i satisfying $0 \leq i \leq \frac{n}{2} - 1$.

We obtain the desired inequality by setting $i = \frac{n}{2} - 1$ in inequality (7.6). □

Proposition 7.3.2. *Let M, φ satisfy the above hypotheses. Suppose $s \geq n^2$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,*

$$\int_M \varphi^{2s}\langle(-1)^{\frac{n}{2}-1}\Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle \leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2}.$$

Proof. Let I denote

$$I = \int_M \varphi^{2s}\langle(-1)^{\frac{n}{2}-1}\Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle.$$

Since we have assumed that (M, g) is obstruction - flat and has constant scalar curvature, we have

$$0 = \mathcal{O} = \tfrac{(-1)^{n/2}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

We use this equation to express $\Delta^{\frac{n}{2}-1}\mathrm{Rc}$ as a sum of lower order terms. If $n \geq 8$, then we can apply Proposition 7.2.2 to obtain that, for each j satisfying $2 \leq j \leq \frac{n}{2}$, there exist $C_1 = C_1(n, j, s, C_S)$, $C_2 = C_2(n, j, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$\int_M \varphi^{2s}\mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm}) \leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2}.$$

Summing over the j yields that there exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$I = \sum_{j=2}^{n/2}\int_M \varphi^{2s}\mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm}) \leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2}.$$

We give a separate argument for $n = 6$. We have

$$I = \sum_{j=2}^{n/2}\int_M \varphi^{2s}\mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm}) = \int_M \varphi^{2s}[\nabla^2\mathrm{Rm} * \mathrm{Rm}^{*2} + \nabla\mathrm{Rm}^{*2} * \mathrm{Rm} + \mathrm{Rm}^{*4}].$$

Since

$$\nabla^2\mathrm{Rm} * \mathrm{Rm}^{*2} = \nabla^{\frac{n}{2}-1}\mathrm{Rm} * \nabla^k\mathrm{Rm} * \nabla^{\frac{n}{2}-3-k}\mathrm{Rm}$$

when $n = 6, k = 0$ and

$$\nabla\mathrm{Rm}^{*2} * \mathrm{Rm} = \nabla^{\frac{n}{2}-2}\mathrm{Rm} * \nabla^k\mathrm{Rm} * \nabla^{\frac{n}{2}-2-k}\mathrm{Rm}$$

when $n = 6, k = 1$, we can apply Propositions 6.3.2 and 6.3.3, respectively, to obtain that there exist $C = C(s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$\int_M \varphi^{2s}[\nabla^2\mathrm{Rm} * \mathrm{Rm}^{*2} + \nabla\mathrm{Rm}^{*2} * \mathrm{Rm}] \leq C\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + CR^{-2}. \tag{7.7}$$

We estimate the remaining term by applying the Hölder inequality and Proposition 5.2.4: there exist $C = C(s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}
\int_M \varphi^{2s}\mathrm{Rm}^{*4} &\leq C\int_M (\varphi^{2s}|\mathrm{Rm}|^2)|\mathrm{Rm}|^2 \\
&\leq C\Big(\int_M \varphi^{6s}|\mathrm{Rm}|^6\Big)^{\frac{1}{3}}\Big(\int_M |\mathrm{Rm}|^3\Big)^{\frac{2}{3}} \\
&= C\epsilon_0^2\|\varphi^s\mathrm{Rm}\|_6^2 \\
&\leq C\epsilon_0^2\int_M \varphi^{2s}|\nabla^2\mathrm{Rm}|^2. \qquad (7.8)
\end{aligned}$$

Combining the estimates (7.7) and (7.8) implies that there exist $C = C(s, C_S)$ and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}
I &= \sum_{j=2}^{n/2}\int_M \varphi^{2s}\mathrm{Rm} * P_j^{n-2j}(\mathrm{Rm}) \\
&= \int_M \varphi^{2s}[\nabla^2\mathrm{Rm} * \mathrm{Rm}^{*2} + \nabla\mathrm{Rm}^{*2} * \mathrm{Rm} + \mathrm{Rm}^{*4}] \\
&\leq C\epsilon_0\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2.
\end{aligned}$$

So, we have obtained the desired inequality for $n \geq 6$. □

Proposition 7.3.3. *Let M, φ satisfy the above hypotheses. Suppose $s \geq n^2$. There exist $C_1 = C_1(n, s, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,*

$$\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 \leq C_1\epsilon_0\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2R^{-2}.$$

Proof. We apply Propositions 7.3.1 and 7.3.2:

$$\begin{aligned}
\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 &\leq \int_M \varphi^{2s}\langle(-1)^{\frac{n}{2}-1}\Delta^{\frac{n}{2}-1}\mathrm{Rc}, \mathrm{Rc}\rangle + C\epsilon_0\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + CR^{-2} \\
&\leq C\epsilon_0\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + CR^{-2}.
\end{aligned}$$

□

We are now able to complete the proof of Theorem 1.2.10.

Proof of Theorem 1.2.10. Let $s = n^2$. Propositions 7.1.2 and 7.3.3 imply that there exist $C_1 = C_1(n, C_S)$, $C_2 = C_2(n, \Lambda, C_Q, C_V)$, and R_0 such that for all $R \geq R_0$,

$$\begin{aligned}\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 &\leq \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rc}|^2 + C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2} \\ &\leq C_1\epsilon_0 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 + C_2 R^{-2}.\end{aligned}$$

If we let $\epsilon_0 = \frac{1}{2C_1}$ we obtain

$$\int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 \leq C_2 R^{-2}.$$

Applying the above inequality and Proposition 5.2.1 yields

$$\left(\int_M \varphi^{ns}|\mathrm{Rm}|^n\right)^{\frac{2}{n}} \leq C_1 \int_M \varphi^{2s}|\nabla^{\frac{n}{2}-1}\mathrm{Rm}|^2 \leq C_2 R^{-2}.$$

Letting $R \to \infty$, we conclude that $\mathrm{Rm} = 0$, so that (M, g) is flat. We note that, since $C_1 = C_1(n, C_S)$, we have shown that we can choose ϵ_0 to depend only on n and C_S. □

Bibliography

[1] M. T. Anderson and P. T. Chruściel, Asymptotically simple solutions of the vacuum Einstein equations in even dimensions, Comm. Math. Phys. **260** (2005), no. 3, 557–577. MR2183957 (2007b:58041)

[2] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296. MR0431287 (55 #4288)

[3] E. Bahuaud and D. Helliwell, Short-time existence for some higher-order geometric flows, Comm. Partial Differential Equations **36** (2011), no. 12, 2189–2207. MR2852074 (2012k:53125)

[4] ———, Uniqueness for some higher-order geometric flows, Bull. Lond. Math. Soc. **47** (2015), no. 6, 980–995. MR3431578

[5] A. L. Besse, *Einstein manifolds*, reprint of the 1987 edition, Classics in Mathematics, Springer, Berlin, 2008. MR2371700 (2008k:53084)

[6] V. Bour, Fourth Order Curvature Flows and Geometric Applications, arXiv:1012.0342v1

[7] T. P. Branson, *The functional determinant*, Lecture Notes Series, 4, Seoul Nat. Univ., Seoul, 1993. MR1325463 (96g:58203)

[8] T. P. Branson and B. Ørsted, Explicit functional determinants in four dimensions, Proc. Amer. Math. Soc. **113** (1991), no. 3, 669–682. MR1050018 (92b:58238)

[9] H.-D. Cao and X.-P. Zhu, A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow, Asian J. Math. **10** (2006), no. 2, 165–492. MR2233789 (2008d:53090)

[10] S.-Y. A. Chang et al., What is Q-curvature?, Acta Appl. Math. **102** (2008), no. 2-3, 119–125. MR2407525 (2009f:53049)

[11] X. X. Chen and W. Y. He, On the Calabi flow, Amer. J. Math. **130** (2008), no. 2, 539–570. MR2405167 (2009c:53086)

[12] ———, The Calabi flow on Kähler surfaces with bounded Sobolev constant (I), Math. Ann. **354** (2012), no. 1, 227–261. MR2957626

[13] B. Chow et al., *The Ricci flow: techniques and applications. Part I*, Mathematical Surveys and Monographs, 135, Amer. Math. Soc., Providence, RI, 2007. MR2302600 (2008f:53088)

[14] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, 110, Amer. Math. Soc., Providence, RI, 2004. MR2061425 (2005e:53101)

[15] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, 77, American Mathematical Society, Providence, RI, 2006. MR2274812

[16] Y. Chu, A rigidity theorem for complete noncompact Bach-flat manifolds, J. Geom. Phys. **61** (2011), no. 2, 516–521. MR2746134

[17] Y. Chu and P. Feng, Rigidity of complete noncompact Bach-flat n-manifolds, J. Geom. Phys. **62** (2012), no. 11, 2227–2233. MR2964656

[18] C. Fefferman and C. R. Graham, Conformal invariants, Astérisque **1985**, Numéro Hors Série, 95–116. MR0837196 (87g:53060)

[19] ———, *The ambient metric*, Annals of Mathematics Studies, 178, Princeton Univ. Press, Princeton, NJ, 2012. MR2858236

[20] A. R. Gover and F. Leitner, A sub-product construction of Poincaré-Einstein metrics, Internat. J. Math. **20** (2009), no. 10, 1263–1287. MR2574315 (2010m:53064)

[21] C. R. Graham and K. Hirachi, The ambient obstruction tensor and Q-curvature, in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, 59–71, IRMA Lect. Math. Theor. Phys., 8, Eur. Math. Soc., Zürich. MR2160867 (2006k:53045)

[22] C. R. Graham et al., Conformally invariant powers of the Laplacian. I. Existence, J. London Math. Soc. (2) **46** (1992), no. 3, 557–565. MR1190438 (94c:58226)

[23] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. **17** (1982), no. 2, 255–306. MR0664497 (84a:53050)

[24] ———, The formation of singularities in the Ricci flow, in *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, 7–136, Int. Press, Cambridge, MA. MR1375255 (97e:53075)

[25] ———, A compactness property for solutions of the Ricci flow, Amer. J. Math. **117** (1995), no. 3, 545–572. MR1333936 (96c:53056)

[26] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics, 1635, Springer, Berlin, 1996. MR1481970 (98k:46049)

[27] S. Kim, Rigidity of noncompact complete Bach-flat manifolds, J. Geom. Phys. **60** (2010), no. 4, 637–642. MR2602377

[28] A. U. Ö. Kişisel, Ö. Sarıoğlu and B. Tekin, Cotton flow, Classical Quantum Gravity **25** (2008), no. 16, 165019, 15 pp. MR2429736 (2009k:53170)

[29] B. Kleiner and J. Lott, Notes on Perelman's papers, Geom. Topol. **12** (2008), no. 5, 2587–2855. MR2460872 (2010h:53098)

[30] B. Kotschwar, An energy approach to uniqueness for higher-order geometric flows, J. Geom. Anal. **26** (2016), no. 4, 3344–3368. MR3544962

[31] E. Kuwert and R. Schätzle, Gradient flow for the Willmore functional, Comm. Anal. Geom. **10** (2002), no. 2, 307–339. MR1900754 (2003c:53099)

[32] C. LeBrun, Curvature functionals, optimal metrics, and the differential topology of 4-manifolds, in *Different faces of geometry*, 199–256, Int. Math. Ser. (N. Y.), 3, Kluwer/Plenum, New York. MR2102997 (2005h:53055)

[33] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91. MR0888880 (88f:53001)

[34] C. Mantegazza, Smooth geometric evolutions of hypersurfaces, Geom. Funct. Anal. **12** (2002), no. 1, 138–182. MR1904561 (2003f:53118)

[35] C. Mantegazza and L. Martinazzi, A note on quasilinear parabolic equations on manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), no. 4, 857–874. MR3060703

[36] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, 3, Amer. Math. Soc., Providence, RI, 2007. MR2334563 (2008d:57020)

[37] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159

[38] ———, Ricci flow with surgery on three-manifolds, arXiv:math/0303109

[39] ———, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math/0307245

[40] P. Petersen, *Riemannian geometry*, second edition, Graduate Texts in Mathematics, 171, Springer, New York, 2006. MR2243772 (2007a:53001)

[41] R. J. Riegert, A nonlocal action for the trace anomaly, Phys. Lett. B **134** (1984), no. 1-2, 56–60. MR0729802 (85g:81160)

[42] J. D. Streets, The gradient flow of $\int_M |\mathrm{Rm}|^2$, J. Geom. Anal. **18** (2008), no. 1, 249–271. MR2365674 (2008i:53095)

[43] ———, The gradient flow of the L^2 curvature energy near the round sphere, Adv. Math. **231** (2012), no. 1, 328–356. MR2935391

[44] ———, The long time behavior of fourth order curvature flows, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 39–54. MR301650