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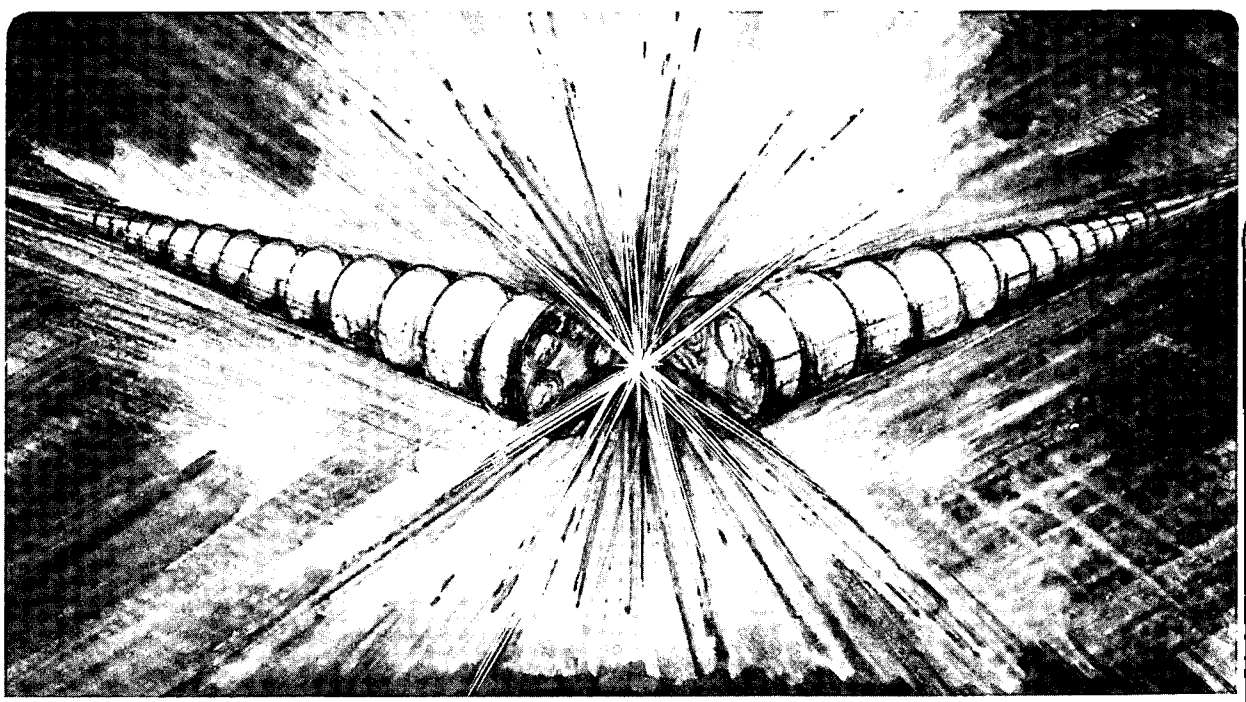

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Transverse Mode Coupling Instability in a Double RF System

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Abstract

The equations for transverse mode coupling in a storage ring with a double rf system are derived from a Hamiltonian formalism. The resulting integral equation is expanded into a set of orthogonal polynomials, and the expansion coefficients are then given by the solution of an infinite determinant. Truncation of this determinant permits solution of the problem on a computer, and a code has been written which finds the complex mode frequencies. The stability limits of LEP with a third harmonic are determined by equating the imaginary part of the solution to the radiation damping rate.

Geneva, Switzerland

5 February 1993

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1 Introduction

The bunch current in large electron (or positron) storage rings like LEP at CERN is limited by the transverse mode-coupling instability[1]. This instability - also called "fast head-tail effect" was first described as "transverse turbulence" [2], when it seemed to limit the current in the DESY storage ring PETRA. However, it was already experienced earlier at the storage ring SPEAR, and was clearly a limitation in PEP, both at the Stanford Linear Accelerator Center[3]. More detailed investigations were made at the next larger electron storage ring TRISTAN at KEK in Japan[4].

A number of methods have been proposed to overcome the resulting limitations of bunch current and hence luminosity. The most obvious one is the reduction of transverse impedance already during construction of the machine, by making the vacuum chamber as smooth as possible and careful shielding of all unavoidable cross-section variations such as bellows or flange-gaps. The major source of transverse impedance then remains the rf cavity system, which is usually designed to have a large impedance at the accelerating frequency in order to minimize the cost of rf power. This can be avoided by the use of superconducting cavities, for which the extremely high quality factor reduces the input power and a high shunt impedance is no longer required.

Nevertheless, in existing machines with copper cavities like the present version of LEP, the bunch current is severely limited. Another means to increase the threshold are lengthening of the bunches, e.g. by wigglers or changing the damping partition numbers. However, this method is limited by the concurrent increase of the energy spread which can not exceed the energy aperture of the machine. A way to increase the bunch length without increase of energy spread is the use of a second, "higher-harmonic" rf system. The increase of bunch length is largest when the phase and amplitude of the second rf system is adjusted such that the synchrotron frequency vanishes at the center of the bunch (i.e., the slope of the total rf voltage is zero there).

The effect on the TMC instability is then not so obvious. On one hand, the longer bunches should increase the threshold, but on the other hand, the smaller synchrotron frequency tends to reduce it. In order to study this situation in detail, it was necessary to develop a theoretical description which permits study of the equations of motion in a strongly deformed rf bucket. This is done by describing the system in a Hamiltonian formalism.

The result can be expressed as an eigenvalue equation, respectively the vanishing of the determinant, of an infinite matrix. Such a problem can be solved numerically by truncating the matrix to finite dimensions, given by the product of the number of radial and azimuthal

modes to be included. For not too long bunches, these dimensions can be kept to quite small values.

The eigenvalues yield the complex mode-frequencies - and hence the frequency shift and growth rates - as function of bunch current. Contrary to the problem for a single rf system, there is always a finite imaginary part, i.e. the growth rates are always non-zero. However, due to radiation damping (neglected in the Vlasov equation formulation), a finite threshold is found nevertheless. The computer code MOSDRF has been written and is available on the CERN-IBM system. It has been applied to a third harmonic system for LEP. It is found to be capable of increasing the threshold current at injection if it is adjusted correctly.

2 Double RF System

In this section, we define some quantities to describe the double rf system. The voltage seen by beam particles for the double rf system case is defined by

$$V(\phi) = V_p[\sin(\phi + \phi_s) + k\sin(n\phi + n\phi_n)], \tag{1}$$

where

V_p	=	peak voltage of the main rf system
kV_p	=	peak voltage of the higher-harmonic rf system
h	=	harmonic number of the main rf system
$n \cdot h$	=	harmonic number of the higher-harmonic rf system
ϕ_s	=	synchronous phase angle of the main rf system
$n \cdot \phi_n$	=	synchronous phase angle of the higher-harmonic rf system.

The Hamiltonian for the synchrotron motion is given by[6]

$$H_s = \frac{h^2\eta\omega_0}{2E_0}w^2 + \frac{eV_p}{2\pi h\omega_0}U(\phi), \tag{2}$$

where

$$w = \frac{\Delta E}{h\omega_0}, \tag{3}$$

and

$$U(\phi) = -\frac{1}{V_p}\int_0^\phi d\phi(V(\phi) - V(0)). \tag{4}$$

The variable w is the canonical momentum conjugate to ϕ. The other parameters are as follows:

$$\begin{aligned}
\omega_0 &= \text{angular revolution frequency} \\
\eta &= \alpha - 1/\gamma^2 = \text{phase slip factor} \\
\alpha &= \text{momentum compaction factor} \\
\gamma &= \text{Lorentz factor} \\
E &= \Delta + E_0 = \text{particle energy} \\
E_0 &= \text{energy of the synchronous particle} \\
e &= \text{elementary charge.}
\end{aligned}$$

The quantity $eV(0)$ represents the synchrotron radiation loss per turn that must be replenished by the rf system.

When the higher-harmonic parameters k and ϕ_n are chosen so that the first and the second derivatives of $V(\phi)$ vanish at the bunch center, namely,

$$nk\cos n\phi_n = -\cos\phi_s, \qquad n^2 k \sin n\phi_n = -\sin\phi_s, \tag{5}$$

the rf potential function $U(\phi)$ becomes quartic with respect to ϕ in the vicinity of the bunch center:

$$U(\phi) = -\frac{n^2-1}{24}\cos\phi_s \cdot \phi^4. \tag{6}$$

Then, the phase motion can be expressed by the Jacobian elliptic function $\mathrm{cn}(u)$ of modulus $1/\sqrt{2}$ and the product of $\mathrm{sn}(u)$ and $\mathrm{dn}(u)$:

$$\begin{aligned}
\phi &= -\phi_{max}\mathrm{cn}(u), \\
w &= w_{max}\sqrt{2}\mathrm{sn}(u)\mathrm{dn}(u),
\end{aligned} \tag{7}$$

where ϕ_{max} and w_{max} are the ϕ and w coordinates where the particle trajectory intersects with the positive ϕ and w axes, respectively. The angle u in phase space is given by

$$u = \phi_{max}\sqrt{\frac{n^2-1}{6}}\nu_{s0}\omega_0 t, \tag{8}$$

where ν_{s0} is the synchrotron tune in the absence of the higher-harmonic system.

Using the action-angle variables I_s and Φ_s for the synchrotron motion, the above phase space motion can be described as

$$\phi_{max} = \left(\frac{2h^2\eta\omega_0}{E_0\nu_{s0}}\frac{3\pi\sqrt{2\pi}}{\Gamma(\frac{1}{4})^2}\sqrt{\frac{3}{n^2-1}}I_s\right)^{1/3}, \tag{9}$$

$$w_{max} = \left(\frac{E_0\nu_{s0}}{2\omega_0 h^2\eta}\sqrt{\frac{n^2-1}{3}}\right)^{1/3} \cdot \left(\frac{3\pi\sqrt{2\pi}}{\Gamma(\frac{1}{4})^2}I_s\right)^{2/3}, \tag{10}$$

and

$$u = \frac{\Gamma(\frac{1}{4})^2}{2\pi^{3/2}}\Phi_s, \tag{11}$$

where $\Gamma(x)$ is the Gamma Function. The Hamiltonian (2) can be expressed as function only of I_s:

$$H_s = (\frac{2h^2\eta\omega_0\nu_{s0}^2}{E_0}\frac{(n^2-1)}{3})^{1/3}\cdot(\frac{3\pi^{3/2}}{2\Gamma(\frac{1}{4})^2}I_s)^{4/3}. \tag{12}$$

3 Canonical Formulation

We start with the Vlasov eq. for the particle distribution function $\Lambda(I_x, \Phi_x, I_s, \Phi_s, \theta_l)$:[5]

$$\frac{\partial\Lambda}{\partial\theta_l} + \Phi'_x\frac{\partial\Lambda}{\partial\Phi_x} + I'_x\frac{\partial\Lambda}{\partial I_x} + \Phi'_s\frac{\partial\Lambda}{\partial\Phi_s} + I'_s\frac{\partial\Lambda}{\partial I_s} = 0, \tag{13}$$

where we take the angular position θ_l in a ring as an independent variable, and a prime denotes the derivative with respect to θ_l. The symbols I_x and Φ_x are the action-angle variables for the transverse motion. They are related to the transverse coordinate x and its canonical momentum p_x by

$$\begin{aligned} x &= (\frac{2cI_x}{E_0}\beta_x)^{1/2}\cos\Phi_x, \\ p_x &= -(\frac{2I_xE_0}{\beta_xc})^{1/2}(\alpha_x\cos\Phi_x + \sin\Phi_x), \end{aligned} \tag{14}$$

where α_x and β_x are the Twiss parameters, and c is the speed of light.

The unperturbed Hamiltonian for the transverse and synchrotron motion is given by

$$H_0 = (\nu_x + \xi\frac{\Delta E}{E_0})I_x + H_s, \tag{15}$$

where H_s is given by Eq. (12), ν_x is the incoherent betatron tune, and ξ is the chromaticity. The changes in the action-angle variables are calculated from H as

$$\Phi'_\alpha = \frac{\partial H}{\partial I_\alpha}, \quad I'_\alpha = -\frac{\partial H}{\partial\Phi_\alpha}, \tag{16}$$

where α is x or s.

The unperturbed particle distribution function is a function only of I_x and I_s:

$$\Lambda = f_0(I_x)g_0(I_s), \tag{17}$$

where they are normalized in such a way that

$$\int_0^\infty dI_x\int_0^{2\pi} d\Phi_x f_0(I_x) = 1, \tag{18}$$

and

$$\int_0^\infty dI_s \int_0^{2\pi} d\Phi_s g_0(I_s) = Ne, \tag{19}$$

where N is the total number of particles in the bunch.

Now, let us calculate the potential term $U_x(I_x, \Phi_x, I_s, \Phi_s)$ for the Hamiltonian due to the interaction with the environment[5]. For this purpose, we first define the perturbed part of the particle distribution function by

$$\Lambda = f_0(I_x)g_0(I_s) + \Lambda_1(I_x, \Phi_x, I_s, \Phi_s)\exp(-i\nu\theta_l + i\frac{\xi}{\alpha}\frac{\phi}{h}), \tag{20}$$

where ν is the tune of the coherent oscillation to be determined and $\exp(i\xi\phi/(\alpha h))$ is the head-tail phase factor. The transverse force F_x felt by a particle at rf phase angle ϕ at θ_l is given by

$$\begin{aligned} F_x(\phi, \theta_l) &= \frac{e}{2\pi R}\sum_{k=-\infty}^{\infty}\int_0^\infty dI_x' \int_0^{2\pi} d\Phi_x' \int_0^\infty dI_s' \int_0^{2\pi} d\Phi_s' x' \Lambda_1(I_x', \Phi_x', I_s', \Phi_s') \\ &\quad \times e^{-i\nu\theta_l + i\frac{\xi}{\alpha}\frac{\phi'}{h}} \cdot W(2\pi k + \frac{\phi'}{h} - \frac{\phi}{h})e^{2\pi i\nu}, \end{aligned} \tag{21}$$

where $W(\theta)$ is the transverse wake potential, and R is the average radius of the ring. The effect of the wake of all previous revolutions is expressed by the summation over k. Now, we define the dipole distribution function $D(I_s, \Phi_s)$ by

$$D(I_s, \Phi_s) = \int_0^\infty dI_x \int_0^{2\pi} d\Phi_x x\Lambda_1(I_x, \Phi_x, I_s, \Phi_s)/\sqrt{\beta_x}. \tag{22}$$

The function $D(I_s, \Phi_s)$ represents the distribution function of the dipole moment of Λ_1 in synchrotron phase space. We also define the Fourier transform of $D(I_s, \Phi_s)$ (projected onto ϕ axis) with respect to ϕ by

$$\tilde{D}(p) = \frac{1}{2\pi}\int_0^\infty dI_s \int_0^{2\pi} d\Phi_s D(I_s, \Phi_s)e^{-ip\frac{\phi}{h}}. \tag{23}$$

Substituting Eq. (22) into Eq. (21), and using $\tilde{D}$, the LHS of Eq. (22) can be transformed into the frequency domain . The result is

$$F_x(\phi, \theta_l) = \frac{ie\omega_0}{2\pi R}\sqrt{\beta_x}e^{-i\nu\theta_l}\sum_{p=-\infty}^{\infty}\tilde{D}(p'')Z_T(p')e^{ip\frac{\phi}{h}}, \tag{24}$$

where Z_T is the transverse impedance given by the Fourier transform of the wake potential:

$$Z_T(\frac{\omega}{\omega_0}) = \frac{1}{i}\int_{-\infty}^{\infty} d\theta W(\theta)e^{i\frac{\omega}{\omega_0}\theta}, \tag{25}$$

p is an integer, and $p' = p + \nu$ and $p'' = p + \nu - \frac{\xi}{\alpha}$.

The potential U_x can be calculated from F_x as

$$-\frac{\partial U_x}{\partial x} = \frac{F_x}{\omega_0}. \tag{26}$$

It follows

$$U_x = -ix\frac{e\sqrt{\beta_x}}{2\pi R}e^{-i\nu\theta_l}\sum_{p=-\infty}^{\infty}\tilde{D}(p'')Z_T(p')e^{ip'\frac{\phi}{h}}. \tag{27}$$

The total Hamiltonian is given by

$$H = H_0 + U_x. \tag{28}$$

Hamilton's equations are

$$\Phi_x' = \frac{\partial H}{\partial I_x} = \nu_x + \frac{\partial U_x}{\partial I_x}, \tag{29}$$

$$I_x' = -\frac{\partial H}{\partial \Phi_x} = -i\frac{e\beta_x}{2\pi R}(\frac{2cI_x}{E_0})^{1/2}e^{-i\nu\theta_l}\sum_p \tilde{D}(p'')Z_T(p')e^{ip'\frac{\phi}{h}}\sin\Phi_x, \tag{30}$$

$$\Phi_s' = \frac{\partial H}{\partial I_s} = \nu_s(I_s) + \frac{\partial U_x}{\partial I_s}, \tag{31}$$

$$I_s' = -\frac{\partial H}{\partial \Phi_s} = -x\frac{e\sqrt{\beta_x}}{2\pi Rh}e^{-i\nu\theta_l}\sum_p \tilde{D}(p'')p'Z_T(p')e^{ip'\frac{\phi}{h}}\frac{\partial\phi}{\partial\Phi_s}, \tag{32}$$

where

$$\nu_s(I_s) = \frac{2\pi^{3/2}\nu_{s0}}{\Gamma(\frac{1}{4})^2}\sqrt{\frac{n^2-1}{6}}\phi_{max} \tag{33}$$

is the synchrotron tune of particles at amplitude ϕ_{max}. We can see from the last non-zero I_s' term that a longitudinal force is also created by the transverse dipole oscillation. This effect is, however, normally negligible.

Inserting all these Eqs. (30-32) and Eq. (20) into Vlasov eq. (13), and linearizing it with respect to Λ_1, we obtain

$$-i\nu\Lambda_1 - \nu_s(I_s)\frac{\partial\Lambda_1}{\partial\Phi_s} + \nu_x\frac{\partial\Lambda_1}{\partial\Phi_x} =$$

$$\frac{e\beta_x}{4\pi R}(\frac{2cI_x}{E_0})^{1/2}\sum_{p=-\infty}^{\infty}\tilde{D}(p'')Z_T(p')e^{ip''\frac{\phi}{h}}(e^{i\Phi_x} - e^{-i\Phi_x})g_0(I_s)\frac{df_0(I_x)}{dI_x}, \tag{34}$$

where we have expressed $\sin\Phi_x$ by exponential. The perturbed distribution function Λ_1 can be Fourier expanded in Φ_x and Φ_s with period 2π due to its periodicity in transverse and synchrotron phase spaces as

$$\Lambda_1(I_x, \Phi_x, I_s, \Phi_s) = \sum_{q=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Lambda_{qm}(I_x, I_s)e^{iq\Phi_x}e^{-im\Phi_s}, \tag{35}$$

where q and m are integers. If we multiply Eq. (35) by $\exp(-iq\Phi_x)$ and integrate over Φ_x from 0 to 2π, we get

$$[-i\nu - \nu_s(I_s)\frac{\partial}{\partial\Phi_s} + iq\nu_x]\sum_{m=-\infty}^{\infty}\Lambda_{qm}(I_x,I_s)e^{-im\Phi_s}$$

$$= (\delta_{1q}-\delta_{-1q})\frac{e\beta_x}{4\pi R}(\frac{2cI_x}{E_0})^{1/2}\sum_{p=-\infty}^{\infty}\tilde{D}(p'')Z_T(p')e^{ip''\frac{\phi}{h}}g_0(I_s)\frac{df_0(I_x)}{dI_x}, \tag{36}$$

where δ_{kl} is Kronecker's delta. The RHS of the above equation has non-zero values only for the dipole mode terms, $q = 1$ and $q = -1$. In a similar way, by multiplying Eq. (36) by $\exp(im\Phi_s)$ and integrating it over Φ_s from 0 to 2π, we can extract $\Lambda_{qm}(I_x,I_s)$ component out of the summation. In doing so, since the Jacobian function $\mathrm{cn}(u)$ can be expanded as

$$\mathrm{cn}(u) = 0.95501\cos\Phi_s + 0.04305\cos 3\Phi_s + ..., \tag{37}$$

we can well approximate $\mathrm{cn}(u)$ in $\exp(ip''\frac{\phi}{h})$ on RHS by $\cos\Phi_s$. Then, we can use the formula

$$\int_0^{2\pi}d\Phi_s\exp(-i\frac{p''}{h}\phi_{max}\cos\Phi_s + m\Phi_s) = 2\pi i^{-m}\mathrm{J}_m(\frac{p''}{h}\phi_{max}). \tag{38}$$

In this approximation, Eq. (36) becomes

$$-i[\nu - m\nu_s(I_s) - q\nu_x]\Lambda_{qm}(I_x,I_s)$$

$$= (\delta_{1q}-\delta_{-1q})i^{-m}\frac{e\beta_x}{4\pi R}(\frac{2cI_x}{E_0})^{1/2}\sum_{p=-\infty}^{\infty}\tilde{D}(p'')Z_T(p')\mathrm{J}_m(\frac{p''}{h}\phi_{max})g_0(I_s)\frac{df_0(I_x)}{dI_x}. \tag{39}$$

The dipole distribution $D(I_s,\Phi_s)$ also can be expanded with respect to Φ_s as

$$D(I_s,\Phi_s) = \sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}D_{rn}(I_s)e^{-in\Phi_s}, \tag{40}$$

where $D_{rn}(I_s)$ can be expressed by $\Lambda_{rn}(I_x,I_s)$ as

$$D_{rn}(I_s) = \int_0^{\infty}dI_x\int_0^{2\pi}d\Phi_x x\Lambda_{rn}(I_x,I_s)e^{ir\Phi_x}/\sqrt{\beta_x}, \tag{41}$$

and r and n are integers. The Fourier transform $\tilde{D}(p'')$ can be also expanded as

$$\tilde{D}(p'') = \sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}i^n\int_0^{\infty}dI_s D_{rn}(I_s)\mathrm{J}_n(\frac{p''}{h}\phi_{max}), \tag{42}$$

where we have approximated $\mathrm{cn}(u)$ by $\cos\Phi_s$ and used the formula (38).

If we multiply Eq. (39) by $x\exp(iq\Phi_x)/\sqrt{\beta_x}$, and integrate it over I_x and Φ_x, we obtain

$$
\begin{aligned}
&[\nu - m\nu_s(I_s) - q\nu_x]D_{qm}(I_s) \\
&\quad = -i(\delta_{1q} - \delta_{-1q})i^{-m}\frac{e\beta_x}{2TE_0/e}\cdot g_0(I_s)\sum_{p=-\infty}^{\infty} Z_T(p')\mathrm{J}_m(\frac{p''}{h}\phi_{max}) \\
&\quad \times \sum_{n=-\infty}^{\infty} i^n \int_0^{\infty} dI_s' D_{rn}(I_s')\mathrm{J}_n(\frac{p''}{h}\phi_{max}'),
\end{aligned} \tag{43}
$$

where we have used Eqs. (41) and (42) and T is the revolution period.

We solve Eq. (43) by expanding $D_{qm}(I_s)$ using a complete set of orthogonal functions as

$$
D_{qm}(I_s) = W(I_s)\sum_{k=\infty}^{\infty} a_{qk}^{(m)} f_k^{(|m|)}(I_s). \tag{44}
$$

Here, the weight function $W(I_s)$ is defined by

$$
W(I_s) = Cg_0(I_s) \tag{45}
$$

where C is a normalization constant to be chosen. The orthogonal functions $f_k^{(|m|)}(I_s)$ are determined so as to satisfy the following orthogonal relationship

$$
\int_0^{\infty} dI_s W(I_s) f_k^{(|m|)}(I_s) f_l^{(|m|)}(I_s) = \delta_{kl}. \tag{46}
$$

Dividing Eq. (43) by $[\nu - m\nu_s(I_s) - q\nu_x]$, inserting Eq. (44) into it, multiplying by $f_k^{(|m|)}(I_s)$ and integrating over I_s, we have a matrix equation for the coefficients $a_{qk}^{(m)}$:

$$
\begin{aligned}
a_{qk}^{(m)} \;=\; & -i(\delta_{1q} - \delta_{-1q})i^{-m}\frac{\beta_x}{2TE_0/e}\frac{1}{C}\sum_{j=0}^{\infty}\int_0^{\infty} dI_s \frac{W(I_s)f_k^{(|m|)}(I_s)f_j^{(|m|)}(I_s)}{\nu - m\nu_s(I_s) - q\nu_x} \\
& \cdot \sum_{p=-\infty}^{\infty} Z_T(p')C_{mj}(p'')\sum_{n=-\infty}^{\infty} i^n \sum_{l=0}^{\infty} a_{ql}^{(n)} C_{nl}(p''),
\end{aligned} \tag{47}
$$

where

$$
C_{mj}(p'') = \int_0^{\infty} dI_s \mathrm{J}_m(\frac{p''}{h}\phi_{max})W(I_s)f_j^{(|m|)}(I_s). \tag{48}
$$

If we define new coefficients $b_k^{(m)}$ by

$$
b_k^{(m)} = \sum_{q=-\infty}^{\infty} a_{qk}^{(m)}, \tag{49}
$$

Eq. (47) can be rewritten as a matrix equation for $b_k^{(m)}$:

$$
b_k^{(m)} = \sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty} N_{nl}^{mk} b_l^{(n)}, \tag{50}
$$

where

$$N_{nl}^{mk} = \sum_{j=0}^{\infty} F_{kj}^{(m)} M_{nl}^{mj}, \tag{51}$$

where

$$F_{kj}^{(m)} = \sum_{q=-\infty}^{\infty} (\delta_{1q} - \delta_{-1q}) \int_0^{\infty} dI_s \frac{W(I_s) f_k^{(|m|)}(I_s) f_j^{(|m|)}(I_s)}{\nu - m\nu_s(I_s) - q\nu_x}, \tag{52}$$

$$M_{nl}^{mj} = -iKi^{n-m} \sum_{p=-\infty}^{\infty} Z_T(p') C_{mj}(p'') C_{nl}(p''), \tag{53}$$

and

$$K = \frac{\beta_x}{2TE_0/e} \frac{1}{C}. \tag{54}$$

The nontrivial solution of Eq. (51) requires that

$$det(\delta_{mn}\delta_{kl} - N_{nl}^{mk}) = 0. \tag{55}$$

This dispersion relation gives a coherent tune ν.

Now, let us find appropriate orthogonal functions $f_k^{(|m|)}(I_s)$ and derive more explicit expressions of matrixes $F_{kj}^{(m)}$ and M_{nl}^{mj}. The normalized unperturbed longitudinal distribution function $g_0(I_s)$ is given by

$$g_0(I_s) = \frac{Neh\omega_0 8^{1/4}}{\Gamma(\frac{1}{4})^2 \sigma_\phi \frac{\sigma_E}{E_0} E_0} \exp(-\lambda(\frac{\phi_{max}}{\sigma_\phi})^4), \tag{56}$$

where

$$\sigma_\phi = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{4})}(\frac{3}{n^2-1})^{1/4}(\frac{h\alpha\sigma_E/E_0}{\nu_{s0}})^{1/2} \approx \frac{1.28678}{(n^2-1)^{1/4}}(\frac{h\alpha\sigma_E/E_0}{\nu_{s0}})^{1/2} \tag{57}$$

is the rms bunch length in units of rf phase angle, σ_E/E_0 is the relative rms energy spread, and the parameter λ is defined by

$$\lambda = \frac{2\pi^2}{\Gamma(\frac{1}{4})^4} \approx 0.11419. \tag{58}$$

To obtain a simple expression of the orthogonal functions $f_k^{(|m|)}(I_s)$, we chose the normalization factor C to be

$$C = \frac{Neh\omega_0 8^{1/4}}{\Gamma(\frac{1}{4})^2 \sigma_\phi \frac{\sigma_E}{E_0} E_0}. \tag{59}$$

It is convenient to change the variable from I_s to $z \doteq \phi_{max}/\sigma_\phi$. The action variable I_s is proportional to cubic of ϕ_{max} (see Eq. (9)):

$$I_s = A\phi_{max}^3, \tag{60}$$

where

$$A = \frac{1}{\frac{2h^2\eta\omega_0}{E_0\nu_{s0}}\frac{3\pi\sqrt{2\pi}}{\Gamma(\frac{1}{4})^2}\sqrt{\frac{3}{n^2-1}}} \tag{61}$$

We factorize the functions $f_k^{(|m|)}(I_s)$ as

$$f_k^{(|m|)}(I_s) = \frac{1}{\sqrt{3A\sigma_\phi^3}} z^{|m|} e_k^{(|m|)}(z). \tag{62}$$

If we substitute Eqs. (59) and (62) into Eq. (46), we find that the functions $e_k^{(|m|)}(z)$ satisfy the following orthogonal relationship:

$$\int_0^\infty dz \exp(-\lambda z^4) z^{2|m|+2} e_k^{(|m|)}(z) e_l^{(|m|)}(z) = \delta_{kl}. \tag{63}$$

Solutions of the above equation are given by

$$e_k^{(|m|)}(z) = 2\lambda^{\frac{|m|}{4}+\frac{3}{8}} \left(\frac{k!}{\Gamma(\frac{|m|}{2}+k+\frac{3}{4})}\right)^{1/2} L_k^{(\frac{|m|}{2}-\frac{1}{4})}(\lambda z^4), \tag{64}$$

where $L_k^{(\alpha)}$ are the generalized Laguerre polynomials. By inserting the above equation into Eq. (48), the function $C_{mj}(p'')$ becomes

$$C_{mj}(p'') = I_{mj}(\frac{p''}{h}\sigma_\phi) \cdot \sqrt{3A\sigma_\phi^3}, \tag{65}$$

where I_{mj} is given by

$$I_{mj}(\frac{p''}{h}\sigma_\phi) = \int_0^\infty dz \mathrm{J}_m(\frac{p''}{h}\sigma_\phi z) \exp(-\lambda z^4) z^{|m|+2} e_j^{(|m|)}(z). \tag{66}$$

There is no analytical expression for the above integral in terms of known special functions. One must carry out integration numerically. In a similar way, the function $F_{kj}^{(m)}$ can be expressed explicitly as

$$F_{kj}^{(m)} = \sum_{q=-\infty}^{\infty} (\delta_{1q} - \delta_{-1q}) \int_0^\infty dz \frac{\exp(-\lambda z^4) z^{2|m|+2} e_k^{(|m|)}(z) e_j^{(|m|)}(z)}{\nu - m\nu_s(z) - q\nu_x}, \tag{67}$$

where the amplitude dependent synchrotron tune is given by

$$\nu_s(z) = \frac{2\pi^{3/2}\nu_{s0}}{\Gamma(\frac{1}{4})^2}\sqrt{\frac{n^2-1}{6}}\sigma_\phi z \approx 0.34587 \cdot \nu_{s0}\sqrt{n^2-1}\sigma_\phi z. \tag{68}$$

The matrix M_{nl}^{mj} can be also rewritten using I_{mj} as

$$M_{nl}^{mj} = -iK' i^{n-m} \sum_{p=-\infty}^{\infty} Z_T(p') I_{mj}(\frac{p''}{h}\sigma_\phi) I_{nl}(\frac{p''}{h}\sigma_\phi), \tag{69}$$

where the constant K' is given by

$$K' = \frac{4\sqrt{2}8^{1/4}}{\sqrt{\pi}\Gamma(\frac{1}{4})^2} \cdot \frac{Ne\beta_x}{2TE_0/e} \approx 0.20417 \cdot \frac{I_b\beta_x}{E_0/e}, \tag{70}$$

where I_b is the bunch current.

4 Numerical Examples and Discussions

Let us apply the present formalism to LEP to see how the mode-coupling instability will be affected by installation of higher-harmonic cavities into the current single rf system. For this purpose, a computer code MOSDRF has been written. The main LEP parameters used are summarized in Table 1. In the following calculations, we included three azimuthal modes ($m = -1, 0, 1$) and two radial modes ($k = 0, 1$). Figures 1 and 2 show the real and imaginary parts of the coherent tune shift in the single rf system in units of the incoherent synchrotron tune ν_{s0}, respectively. One can see the clear onset of the mode-coupling instability at the bunch current of about 0.75 mA. When higher-harmonic cavities are installed (double rf system), they are changed as shown by the broken curves in Figs. 3 and 4. The single curve starting from the origin in Fig. 3 represents actually the two eigenmodes. In Figs. 3 and 4, only the eigenmodes which have non-zero imaginary parts of coherent tunes are plotted. Other eigenmodes have singularities at the denominators of the integration $F_{kj}^{(m)}$ given by Eq. (67), and thus cannot be obtained by the present method. The sharpe rise of growth rate at the onset of mode-coupling instability is replaced by a steady increase which starts from the zero bunch current. However, the magnitude of growth rate is far smaller in the latter case (notice the difference of the vertical scale in Figs. 2 and 4.). This behavior change can be explained as follows. Since the incoherent synchrotron tune is diminished at the zero synchrotron amplitude in the double rf system, the $m = 0$ and $m = -1$ modes can couple to each other even at the zero bunch current to yield an instability. However, a strong Landau damping due to a large synchrotron tune spread drags down the growth rate to a low level. The threshold current will be now defined by the current for which the growth rate is balanced by the radiation damping rate. In LEP, the numerical result shows the threshold current will increase from 0.75 mA to 1.16 mA.

Next, let us examine the effect of the bunch length in the double rf system. Figures 5 and 6 show the real and imaginary parts of the coherent tune shift in units of ν_{s0} in the double rf system, when the bunch length is halved by halving the energy spread and the momentum compaction factor. The growth rate gets enlarged by two orders of magnitude. In Fig. 7, we plot the growth rate at $I_b = 0.75$mA as a function of the bunch length σ_ϕ in units of $\sigma_{\phi 0}$, the original bunch length in units of rf phase angle, when we shorten the bunch by reducing the energy spread and the moment compaction factor proportionally. The bunch length in the real space corresponding to $\sigma_{\phi 0}$ is given by $\sigma_{\phi 0} R/h = 13.12$cm. As can be seen, the growth rate rises up very rapidly as the bunch shortens. The growth rate exceeds the radiation damping rate at about $\sigma_\phi/\sigma_{\phi 0} = 0.82$, or $\sigma_\phi R/h = 10.8$cm. Figure 8 shows the threshold

current against the bunch length σ_ϕ in units of $\sigma_{\phi 0}$. The threshold current breaks below the present threshold of 0.75 mA of the single rf system again at $\sigma_\phi/\sigma_{\phi 0} = 0.82$. These results show that if a double rf system does not produce a long enough bunch, the threshold current could become lower than that of the single rf system. This may imply that the LEP case might be a rather fortunate one.

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Table 1. The main LEP parameters used for the calculations.

Beam energy, E_0 (GeV)	20.0
Revolution frequency, f_0 (kHz)	11.2455
Average machine radius, R (km)	4.2429
Momentum compaction factor, α	0.0019
Harmonic number of the single rf system, h	31320
Rms bunch length in the single rf system, σ_z (cm)	2.0
Rms relative energy spread in the single rf system, σ_E/E_0	0.0021
Radiation damping time, τ_r (sec)	0.3
Synchrotron tune of the single rf system, ν_{s0}	0.078
Ratio of the higher-harmonic frequency to the main frequency, n	3
Rms bunch length in the double rf system, $\sigma_{\phi 0}R/h$ (cm)	13.12
Beta function at the impedance, β_x (m)	40
Resonant frequency of the broadband impedance, f_r (GHz)	2.0
Peak value of the broadband impedance, R_r (MΩ/m)	1.5
Q-factor of the broadband impedance, Q	1.0

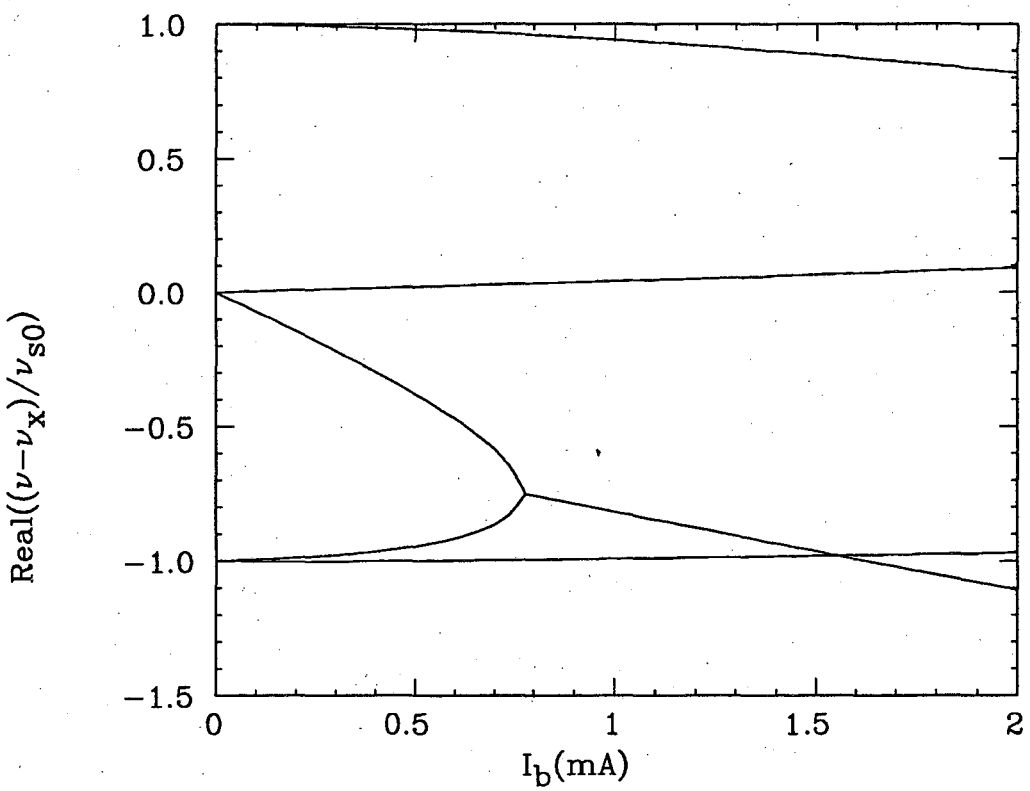

Figure 1: Real parts of $(\nu - \nu_x)/\nu_{s0}$ in the single rf system.

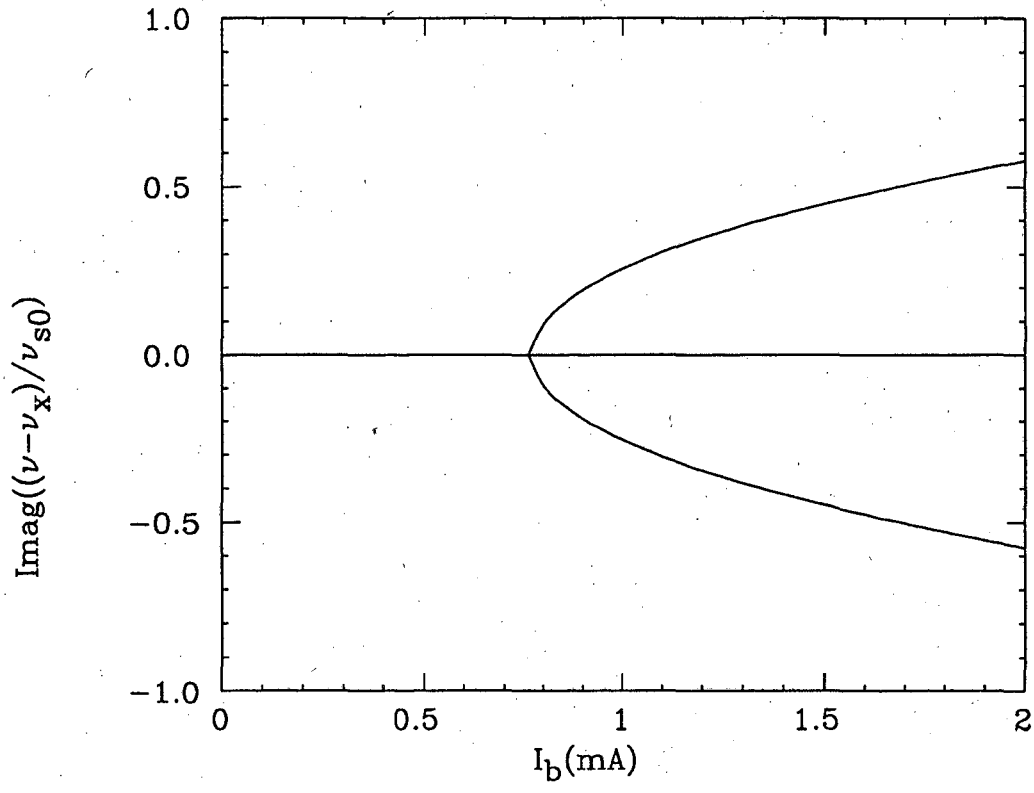

Figure 2: Imaginary parts of $(\nu - \nu_x)/\nu_{s0}$ in the single rf system.

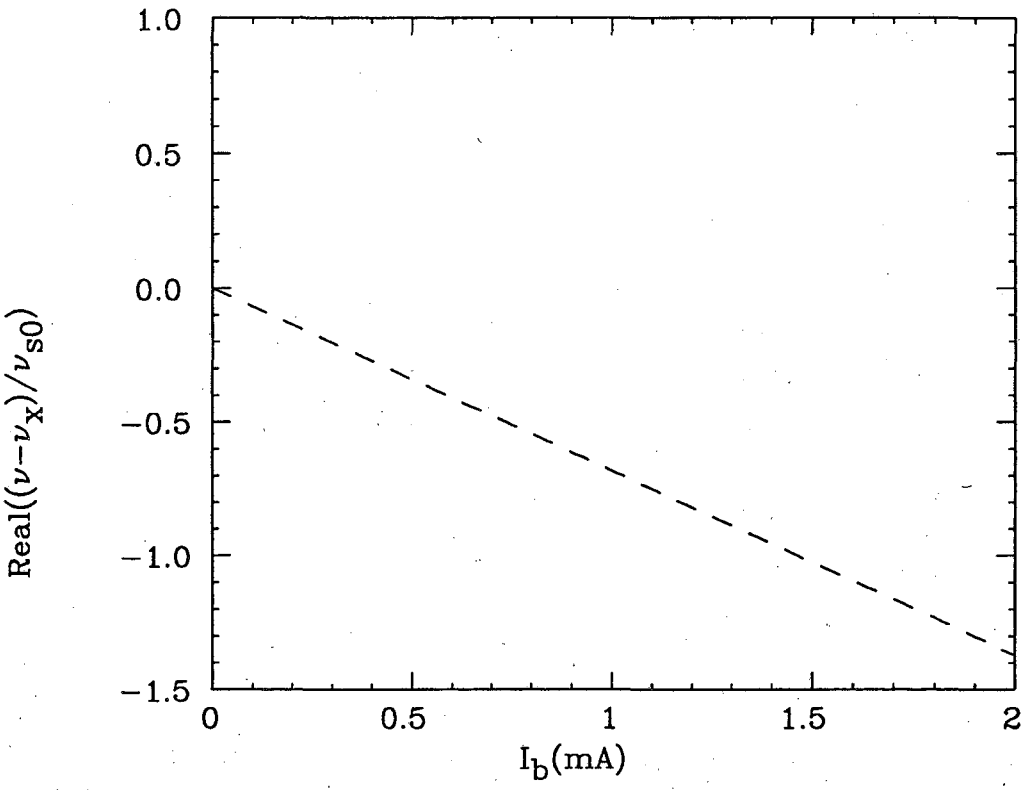

Figure 3: Real parts of $(\nu - \nu_x)/\nu_{s0}$ in the double rf system. $\sigma_{\phi 0} R/h = 13.12$cm.

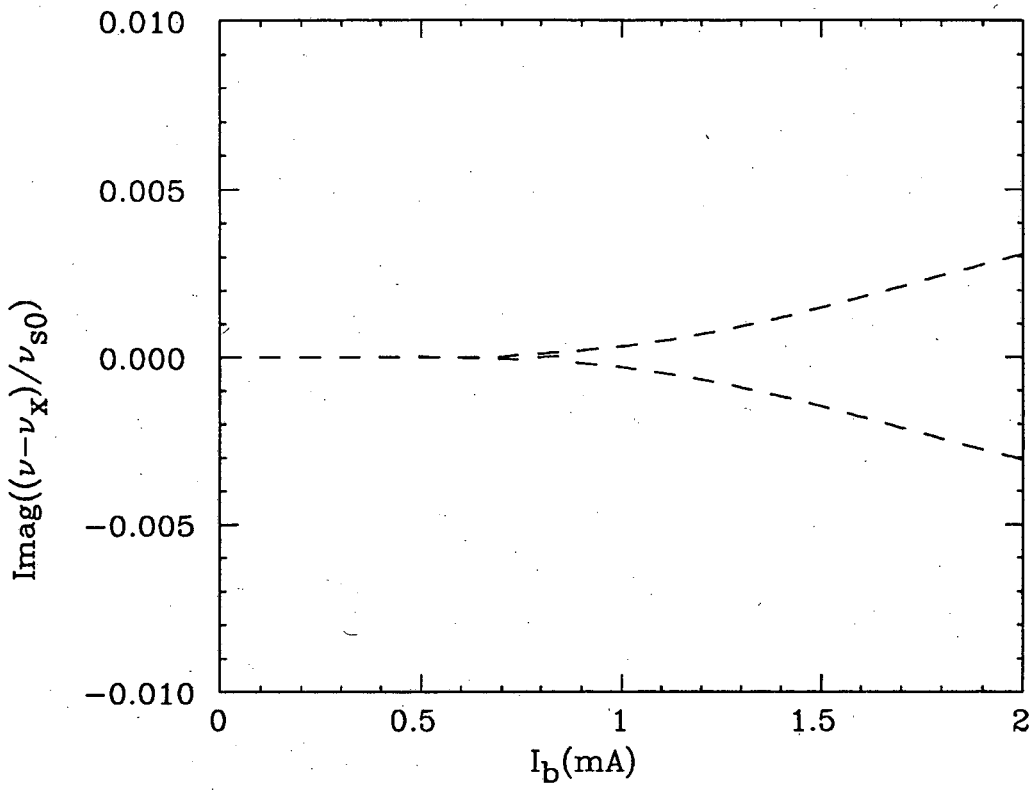

Figure 4: Imaginary parts of $(\nu - \nu_x)/\nu_{s0}$ in the double rf system. $\sigma_{\phi 0} R/h = 13.12$cm.

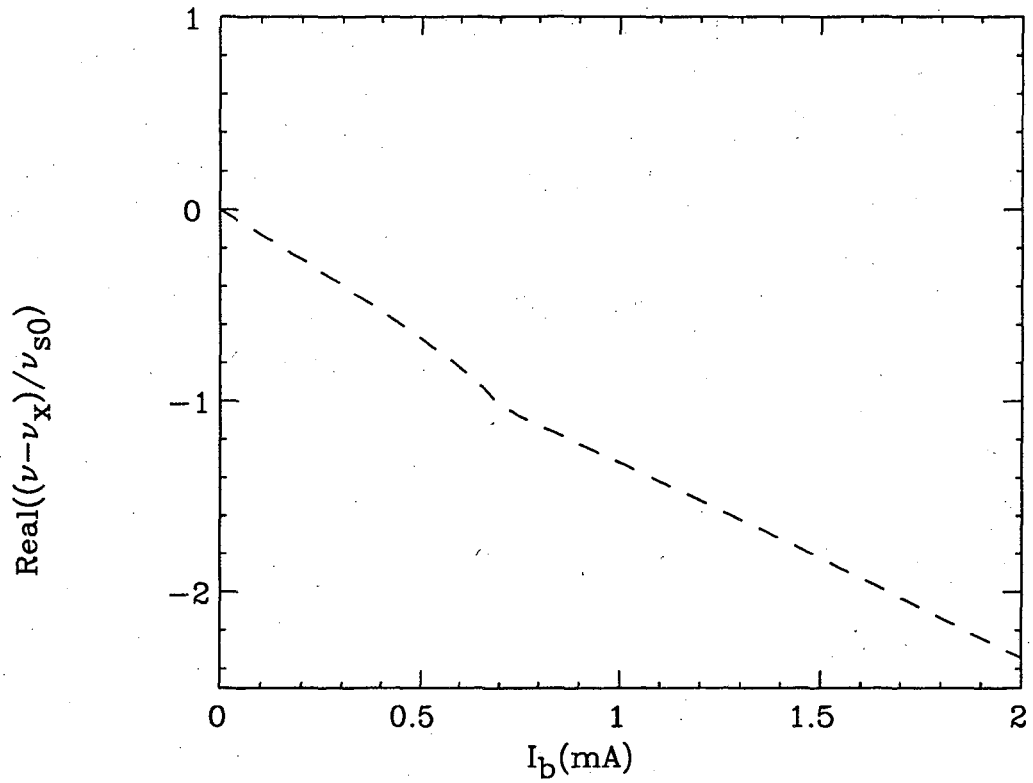

Figure 5: Real parts of $(\nu - \nu_x)/\nu_{s0}$ in the double rf system. $\sigma_\phi R/h = 10.8$cm.

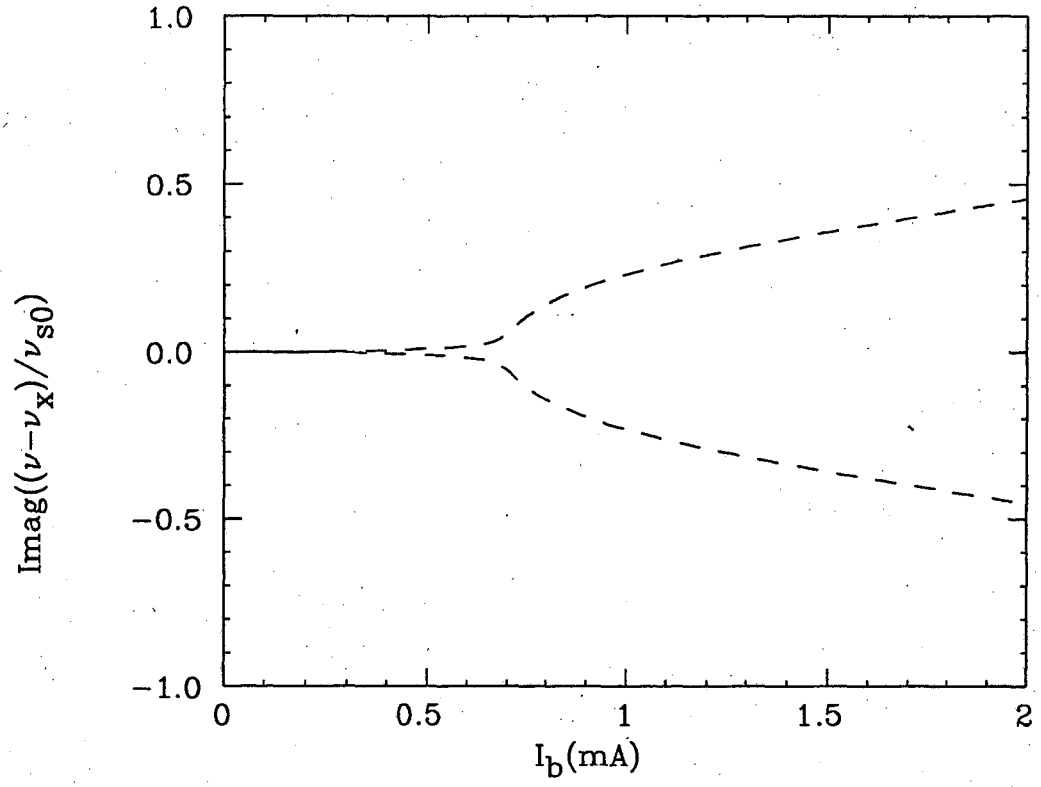

Figure 6: Imaginary parts of $(\nu - \nu_x)/\nu_{s0}$ in the double rf system. $\sigma_\phi R/h = 10.8$cm.

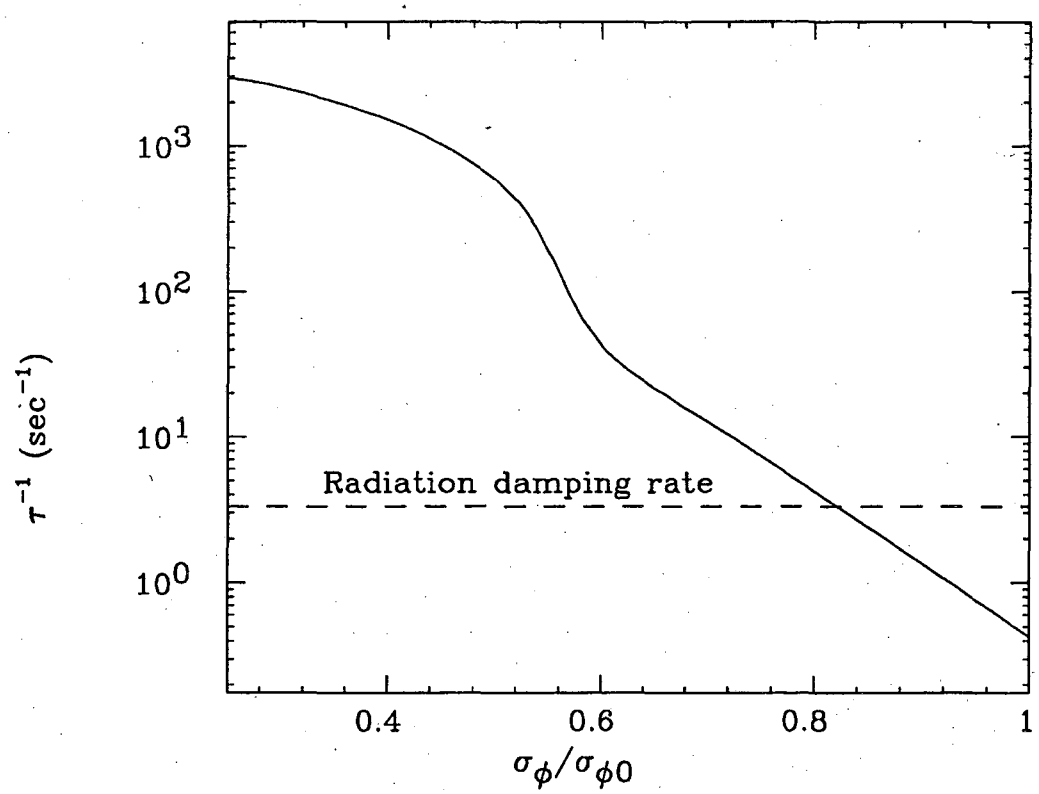

Figure 7: Growth rate τ^{-1} at $I_b = 0.75$mA as a function of the bunch length σ_ϕ in units of $\sigma_{\phi 0}$ where $\sigma_{\phi 0} R/h = 13.12$cm is the original bunch length.

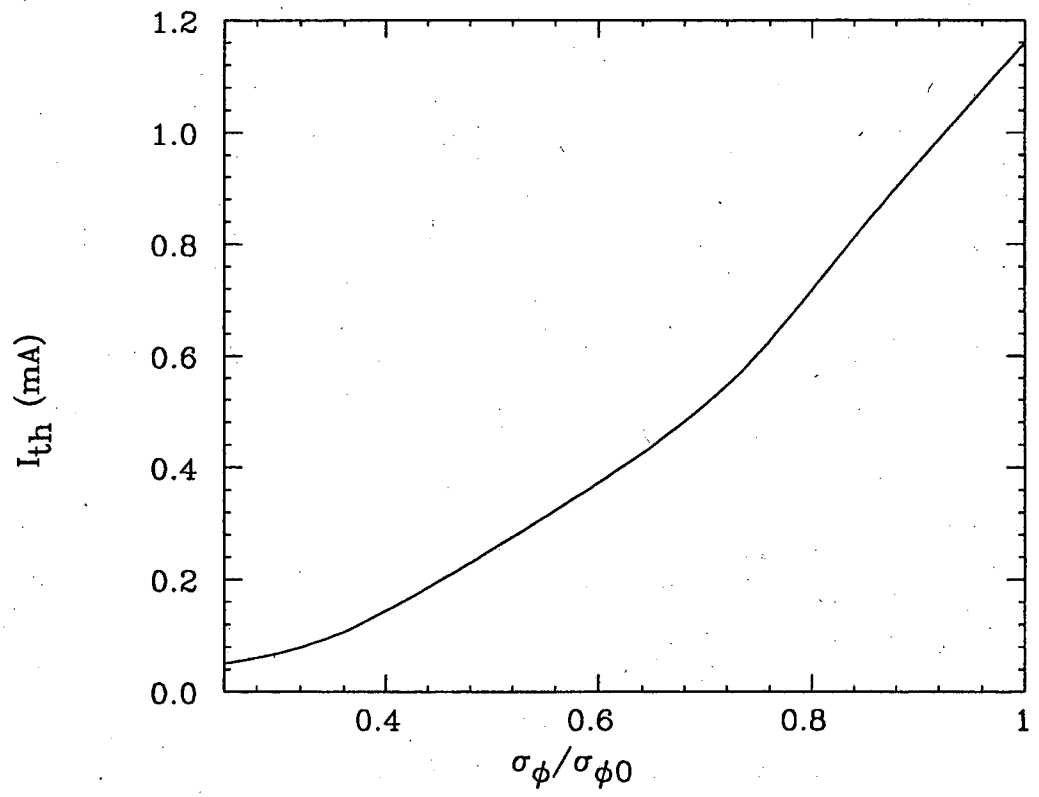

Figure 8: Threshold bunch current I_{th} as a function of the bunch length σ_ϕ in units of $\sigma_{\phi 0}$.

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