Combinatorial patterns in syzygies

by

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Abstract

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Affine semigroup rings are the coordinate rings of not necessarily normal toric varieties. They include the coordinate rings of the Segre-Veronese embeddings of projective spaces, and special projections of those. The study of affine semigroup rings lies in the intersection of commutative algebra, algebraic geometry and combinatorics. In this thesis, we study the syzygies for certain classes of affine semigroup rings.

The Betti numbers of affine semigroup rings can be computed as the dimensions of homology groups of certain simplicial complexes. Therefore, the study of the Betti numbers of affine semigroup rings can be translated into some combinatorial problems. The idea of using combinatorial topology to study syzygies originated from the work of Hochster, Reisner and Stanley in the seventies and eighties and since then have been an active area of research and proved to be useful in lots of cases.

In the first chapter, we introduce the problems concerned in our dissertation, and their relations to topology of simplicial complexes and representation theory of symmetric groups. We also include some background material from combinatorial commutative algebra, algebraic geometry, and representation theory.

In the second chapter, we use combinatorial and representation theoretic methods arising from work of Karaguerian, Reiner and Wachs [30] to reduce the study of the syzygies of Veronese varieties to the study of homology groups of matching complexes. In turn we use combinatorial methods to show the vanishings of certain homology groups of these matching complexes, giving a lower bound on the length of the linear part of the resolution of the Veronese varieties. In the case of third Veronese embeddings, we carry out the computation to prove the Ottaviani-Paoletti conjecture.

In the final chapter, we study a conjecture of Herzog and Srinivasan and a higher analog. The conjecture says that the Betti numbers of affine monomial curves under translations are eventually periodic. We prove the conjecture, and use it to study the analogous question for higher dimensional affine semigroup rings under translations and some other consequences.

To my family

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Chapter 1

Introduction

1.1 Overview

The study of affine semigroup rings is important in numerous field of recent research: combinatorial commutative algebra, toric geometry, geometric modeling and algebraic statistics [8], [11], [12], [34]. Of particular interest in these studies is the defining equations of affine semigroup rings and their syzygies.

Syzygies of affine semigroup rings are the main objects of study in this dissertation. Our research is motivated by the Ottaviani-Paoletti conjecture [35], and the Herzog-Srinivasan conjecture. In this dissertation, we give approaches to allow one to attack the first conjecture. We also settle the second conjecture and make further conjectures relating to this work.

The underlying theme for the connection to combinatorics is a result of Bruns and Herzog [9], (which was motivated from work of Campillo and Marijuan [10]) where Betti numbers of affine semigroup rings are computed as the dimensions of homology groups of squarefree divisor simplicial complexes. Let us introduce some notation.

Let K be an arbitrary field. All simplicial homology have coefficients in K. Let V be an additive semigroup generated by vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{N}^m$. The semigroup ring $K[V]$ generated by V is a subring of $K[t_1, \dots, t_m]$ generated by $\mathbf{t}^{\mathbf{v}_i} = t_1^{v_{i1}} \cdots t_m^{v_{im}}$. The polynomial ring $R = K[x_1, ..., x_n]$ maps surjectively onto $K[V]$ by sending x_i to $\mathbf{t}^{\mathbf{v}_i}$. Denote $I(V)$ be the defining ideal of $K[V]$ in R. Note that R is multi-graded with the grading given by $\deg x_i = \mathbf{v}_i$. Under this grading the Betti numbers of $I(V)$ are related to the homology of squarefree divisor simplicial complexes as follows.

Definition 1.1 (Squarefree divisor simplicial complex)**.** *For each $\mathbf{v} \in V$, let $\Delta_{\mathbf{v}}$ be the simplicial complex on the vertices $\{1, ..., n\}$ such that $F \subseteq \{1, ..., n\}$ is a face of $\Delta_{\mathbf{v}}$ if and only if*

$$\mathbf{v} - \sum_{i \in F} \mathbf{v}_i \in V.$$

Theorem 1.2 (Bruns-Herzog)**.** *For each i, and each element $\mathbf{v} \in V$,*

$$\beta_{i,\mathbf{v}}(I(V)) = \beta_{i+1,\mathbf{v}}(R/I(V)) = \dim_K \tilde{H}_i(\Delta_{\mathbf{v}}).$$

Proof. The Betti numbers of $I(V)$ are computed as homology of the complex obtained by tensoring $I(V)$ with the Koszul complex $\mathcal{K}$ which is the resolution of the residue field K over R. The $\mathbf{v}$-graded components of the complex $\mathcal{K} \otimes I(V)$ can be expressed as

$$(\mathcal{K} \otimes I(V))_{i\mathbf{v}} = \bigoplus_{F \in \Delta_{\mathbf{v}},\ |F|=i} k(-x^F).$$

The differentials of the complex $\mathcal{K} \otimes I(V)_{\mathbf{v}}$ are the differentials of the simplicial complex $\Delta_{\mathbf{v}}$. The conclusion follows. □

The rest of this chapter contains introduction to the two above-mentioned conjectures, and some background materials from algebraic geometry, combinatorics and representation theory of symmetric groups.

In Chapter 2, we present a new proof of a result of Athanasiadis [2] on the shellability of skeleta of matching complexes, as well as geometric proof of the Ottaviani-Paoletti conjecture in the case of fourth Veronese embedding of $\mathbb{P}^3$. Using computational results, we settle the Ottaviani-Paoletti conjecture in the case of third Veronese embeddings of projective spaces.

In Chapter 3, we present a proof of the Herzog-Srinivasan conjecture on the periodicity of Betti numbers of affine monomial curves under translation. To accomplish this goals, we prove the analogous result for projective monomial curves, and then prove that the total Betti numbers of affine and projective monomial curves are equal after a high enough translation.

1.2 N_p property of algebraic varieties

Let X be a smooth projective variety over an algebraically closed field k of characteristic 0. Let $\mathcal{L}$ be a very ample line bundle on X. Thus $\mathcal{L}$ defines an embedding

$$X \subseteq \mathbb{P}^r = \mathbb{P}H^0(X, \mathcal{L})$$

where $r = r(\mathcal{L}) = h^0(X, \mathcal{L}) - 1$. Let $S = \operatorname{Sym} H^0(X, \mathcal{L})$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and let $R = R(\mathcal{L}) = \oplus H^0(X, \mathcal{L}^k)$ be the homogeneous coordinate ring of X embedded by $\mathcal{L}$, viewed as an S-module. Since the influential papers by Green [17], [18], the study of syzygies of R as S-module have been extensively carried out. Green's idea was that the higher the degree of $\mathcal{L}$, the simpler are the syzygies of R (at least from the begining of the resolution). When X is a curve, this philosophy has proved to govern the shape of the resolution of high degree embeddings of X. Green proved that, when $\deg \mathcal{L} = 2g + 1 + p$, where g is the genus of the curve X, then the embedding of X defined by $\mathcal{L}$ satisfies property N_p. That is R is normally generated, and all syzygies up to homological degree p are linear. Following Green and Lazarsfeld [19], [20], we define:

Definition 1.3. *The Green-Lazarsfeld index of a very ample line bundle $\mathcal{L}$ on a smooth projective variety X is the largest integer p such that the embedding of X by $\mathcal{L}$ satisfies property N_p. We denote it by $p(X, \mathcal{L})$.*

When $\dim X \geq 2$, much less is known about the syzygies of embeddings of X, even in the simplest case when $X = \mathbb{P}^n$. Green proved that when $\mathcal{L} = \mathcal{O}(d)$, then R satisfies property N_d. Nevertheless, this is far from the actual Green-Lazarsfeld index of Veronese embeddings as conjectured in:

Conjecture 1.4 (Ottaviani-Paoletti)**.** *The dth Veronese embeddings of projective spaces satisfy property N_{3d-3}.*

For simplicity, for each p, q let

$$K_{p,q}(X, \mathcal{L}) = \{\text{minimal generators of } p\text{th syzygies of } R \text{ of degree } p+q\}.$$

Ottaviani and Paoletti in [35] showed that $K_{3d-2,2}(\mathbb{P}^n, \mathcal{O}(d)) \neq 0$ when $n \geq 2$ and $d \geq 3$. In other words, the conjecture is sharp.

The Ottaviani-Paoletti conjecture is known for $d = 2$ by the work of Jósefiak-Pragacz-Weyman [29] and also known for $\mathbb{P}^1$ and $\mathbb{P}^2$ by the work of Green [17] and Birkenhake [3]. The recent improvement of the conjecture by Bruns, Conca and Römer in [7] is that the dth Veronese embeddings of projective spaces satisfy property N_{d+1}. In a preparation work, we prove that they satisfy property N_{2d-2}.

Since we know more about the syzygies of embeddings of curves, we could try to take general hyperplane sections of the embeddings of $\mathbb{P}^n$ to bring it to the case of curves. Nevertheless, one immediately gets trouble as long as $n \geq 4$ or $d \geq 5$, as the curve obtained from the process would have the degree of the embedding less than the degree of its canonical divisor. Moreover, the curve would lie on a surface of general type, over which our knowledge of their syzygies are very limited.

The conjecture would be related to an extension of the Green-Lazarsfeld gonality conjecture if we could compute the Clifford index of complete intersection curves in projective spaces. Unfortunately, the computation of Clifford index of curves are a very delicate task which we do not know how to deal with even in the case of complete intersection curves in projective spaces.

Since the Green-Lazarsfeld index of Veronese embeddings have been determined for $\mathbb{P}^2$, we may assume that $n \geq 3$. When $n \geq 4$, almost nothing was known. When $n = 3$, in the same paper, Ottaviani and Paoletti showed that the third Veronese embedding of $\mathbb{P}^3$ satisfies property N_6. We will see that in the case of fourth Veronese embedding of $\mathbb{P}^3$, we are in the situation that we have a canonical curve lying on a K3 surface. Recently, Green's conjecture has been proved in this case by Aprodu and Farkas [1]. This reduces the determination of $p(\mathbb{P}^3, \mathcal{O}(4))$ to the computation of the Clifford index of the curve which is the complete intersection of two quartic surfaces in $\mathbb{P}^3$. We will define the Clifford index and compute it in our situation in section 2.1.

We will now introduce a different approach to Ottaviani-Paoletti conjecture using representation theory and combinatorics. Note that representation theory comes into play naturally in the land of syzygies of Segre-Veronese varieties. This fact has been exploited and successfully used in certain problems, for example see [29], [32], [38], [41]. The idea of using representation theory of general linear groups and symmetric groups to study syzygies has been an active area of research in recent years. For more of this direction, we refer to the book of Weyman [44], and the article of Raicu [37]. We will now switch the notation a little bit to move to the world of representation theory of general linear groups. For unexplained terminology, we refer to the book by Fulton and Harris [15].

Let k be a field of characteristic 0. Let V be a finite dimensional vector space over k of dimension $n+1$. The projective space $\mathbb{P}(V)$ has coordinate ring naturally isomorphic to $\operatorname{Sym} V$. For each natural number d, the d-th Veronese embedding of $\mathbb{P}(V)$, which is naturally embedded into the projective space $\mathbb{P}(\operatorname{Sym}^d V)$ has coordinate ring $\operatorname{Ver}(V,d) = \oplus_{k=0}^{\infty} \operatorname{Sym}^{kd} V$. For each set of integers p, q, b, let $K^d_{p,q}(V,b)$ be the associated Koszul cohomology group defined as the homology of the 3-term complex

$$\bigwedge^{p+1} \operatorname{Sym}^d V \otimes \operatorname{Sym}^{(q-1)d+b} V \to \bigwedge^{p} \operatorname{Sym}^d V \otimes \operatorname{Sym}^{qd+b} V$$
$$\to \bigwedge^{p-1} \operatorname{Sym}^d V \otimes \operatorname{Sym}^{(q+1)d+b} V.$$

Then $K^d_{p,q}(V,b)$ is the space of minimal p-th syzygies of degree $p+q$ of the $\operatorname{GL}(V)$-module $\operatorname{Ver}(V,d,b) = \oplus_{k=0}^{\infty} \operatorname{Sym}^{kd+b} V$. We write $K^d_{p,q}(b) : \text{Vect} \to \text{Vect}$ for the functor on finite dimensional k-vector spaces that assigns to a vector space V the corresponding syzygy module $K^d_{p,q}(V,b)$. In this notation, the Ottaviani-Paoletti conjecture is:

$$K^d_{p,q}(V,0) = 0 \text{ for } q \geq 2 \text{ and } p \leq 3d-3. \tag{1.1}$$

Moreover, the Veronese modules $\operatorname{Ver}(V,d,b)$ are Cohen-Macaulay, the equation (1.1) can be replaced by

$$K^d_{p,2}(V,0) = 0 \text{ for } p \leq 3d-3. \tag{1.2}$$

Though we are mainly interested in the vanishings of the Koszul homology groups $K^d_{p,2}(V,0)$, we will see later that, there is a long exact sequence which connects the Koszul homology groups $K^d_{p,q}(V,b)$; thus the understanding of the syzygies of the Veronese modules $\operatorname{Ver}(V,d,b)$ are very useful in analyzing the syzygies of the Veronese varieties.

From the definition, it is clear that $K^d_{p,q}(V,b)$ are $\operatorname{GL}(V)$-representations, in particular, the functors $K^d_{p,q}(b)$ are polynomial functors and decompose into irreducible polynomial functors, i.e. Schur functors. By a result of Karaguerian, Reiner and Wachs [30], these decompositions are closely related to decompositions of homology groups of matching complexes into irreducible representations as representations of symmetric groups.

Definition 1.5 (Matching Complexes)**.** *Let $d > 1$ be a positive integer and A a finite set. The matching complex C^d_A is the simplicial complex whose vertices are all the d-element subsets of A and whose faces are $\{A_1, \ldots, A_r\}$ so that $A_1, \ldots, A_r$ are mutually disjoint.*

The symmetric group S_A acts on C^d_A by permuting the elements of A making the homology groups of C^d_A representations of S_A. For each partition λ, we denote by V^λ the irreducible representation of $S_{|\lambda|}$ corresponding to the partition λ, and S_λ the Schur functor corresponding to the partition λ. For each vector space V, $S_\lambda(V)$ is an irreducible representation of $\mathrm{GL}(V)$. The relation between the syzygies of the Veronese embeddings and the homology groups of matching complexes is given by the following theorem of Karaguezian, Reiner and Wachs [30].

Theorem 1.6. *Let p, q be non-negative integers, let d be a positive integer and let b be a non-negative integer. Write $N = (p+q)d+b$. Consider a partition λ of N. Then the multiplicity of S_λ in $K^d_{p,q}(b)$ coincides with the multiplicity of the irreducible S_N representation V^λ in $\tilde{H}_{p-1}(C^d_N)$.*

Theorem 1.6 implies that the equation (1.2), and so the Ottaviani-Paoletti conjecture is equivalent to:

Conjecture 1.7. *The only non-zero homology groups of C^d_{kd} for $k = 1, \ldots, 3d-1$ is $\tilde{H}_{k-2}$.*

Besides the connection to the syzygies of Veronese embeddings, the study of connectivity of matching complexes is also of interest among the combinatorialists (see [4], [40]), and have connection to problems in group theory (see [31] and [39] and references therein). For interesting aspects and some open questions related to matching complexes we refer to the survey article by Wachs [43].

To compute the homology groups of the matching complexes inductively, the following equivariant long-exact sequence originated from Bouc [5] is useful. Let A be a finite set with $|A| \geq 2d$. Let $a \in A$ be an element of A. Let α be a d-element subset of A such that $a \in \alpha$. Let $\beta = \alpha \setminus \{a\}$, and let $C = A \setminus \alpha$, $B = A \setminus \{a\}$. Then we have the following long-exact sequence of representations of S_B.

$$\cdots \to \mathrm{Ind}(\tilde{H}_r(C^d_C) \otimes 1) \to \tilde{H}_r(C^d_B) \to \mathrm{Res}(\tilde{H}_r(C^d_A)) \to \mathrm{Ind}(\tilde{H}_{r-1}(C^d_C) \otimes 1) \to \cdots \quad (1.3)$$

where Ind is $\mathrm{Ind}^{S_B}_{S_C \times S_\beta}$, Res is $\mathrm{Res}^{S_A}_{S_B}$ and 1 is the trivial representation.

By a result of Athanasiadis [2], which we will give a different proof in section 2.2, for $d \leq 4$ and $N \leq d(3d-1)$ the matching complexes C^d_N has at most two non-zero homology groups. In particular, the Euler characteristic of the matching complex C^d_N is equal to

$$\chi(C^d_N) = (-1)^i \tilde{H}_i + (-1)^{i+1} \tilde{H}_{i+1}$$

for $i = \lfloor \frac{N}{d} \rfloor - 3$. Morever, it can be computed as the alternating sum of the space of i-faces of C^d_N. The space $F_{i-1}(C^d_N)$ of $i-1$-faces of C^d_N as representation of S_N has character given as follows.

Proposition 1.8 (Shareshian-Wachs [39]). *The character of $F_{i-1}(C_N^d)$ is given by:*

$$\mathrm{ch}\, F_{i-1}(C_N^d) = e_i[h_d]h_{n-id},$$

where e (respectively h) denotes the elementary (resp. homogeneous) symmetric functions.

Note that for representations of the symmetric groups, the character maps the group of virtual representations of S_N isomorphically to the group of symmetric functions on N-letters. This map is also a map of algebras when the group of virtual representations are endowed with a multiplication induced by tensor product. The transition to the characters makes the computation and notation easier.

Since the Veronese embeddings of projective spaces are the very natural embeddings, it is generally believed that the spaces of minimal syzygies behave naturally in the following sense. For each p, q, d, b, there is no common subrepresentation between $K^d_{p,q}(V, b)$ and $K^d_{p+1,q-1}(V, b)$. Equivalently, for each i, d, n, there is no common subrepresentation between $\tilde{H}_i C^d_n$ and $\tilde{H}_{i+1} C^d_n$. Note that if this naturality property holds, then proving Ottaviani-Paoletti conjecture would be easier, as by induction and the long exact sequence (1.3), we know that there is at most two non-zero homology groups for C^d_{kd} when k is in the range of our interest. Given the naturality, it suffices to show that the Euler characteristic of C^d_{kd} corresponds to either an honest representation or negative of such an honest one. By the character formula of Shareshian and Wachs, the Pieri's rule [15], and the plethysm formula of Doran [25], it seems to be an achievable goal. The naturality would also imply a conjecture of Shareshian and Wachs [40] on the top dimensional homology group of matching complexes. When $d \leq 4$, the computation of the virtual character of the Euler characteristic of C^d_N in our desired range is made possible by Sage [42]. These computational experiments support the naturality property.

When $d = 3$, we prove the Ottaviani-Paoletti conjecture in section 2.3 basically by proving that the naturality property holds in the range of our interest.

Theorem 1.9. *The third Veronese embeddings of projective spaces satisfy property N_6.*

When $d = 4$, we actually prove that the naturality property fails making the Ottaviani-Paoletti conjecture more appealing. Using geometric method, we prove in section 2.1 that the fourth Veronese embedding of $\mathbb{P}^3$ satisfies property N_9.

1.3 Betti numbers of monomial curves

Let K denote an arbitrary field. Let R be the polynomial ring $K[x_1, ..., x_n]$. Let $\mathbf{a} = (a_1 < \cdots < a_n)$ be a sequence of positive integers. The sequence $\mathbf{a}$ gives rise to a monomial curve $C(\mathbf{a})$ whose parametrization is given by $x_1 = t^{a_1}, ..., x_n = t^{a_n}$. Let $I(\mathbf{a})$ be the defining ideal of $C(\mathbf{a})$. For each positive integer j, let $\mathbf{a} + j$ be the sequence $(a_1 + j, ..., a_n + j)$. In chapter 3, we consider the behaviour of the Betti numbers of the defining ideals $I(\mathbf{a} + j)$ and their homogenizations $\bar{I}(\mathbf{a} + j)$ for positive integers j.

For each finitely generated R-module M and each integer i, let

$$\beta_i(M) = \dim_K \operatorname{Tor}_i^R(M, K)$$

be the i-th total Betti number of M. The following conjecture was communicated to us by Herzog and Srinivasan.

Conjecture 1.10 (Herzog-Srinivasan)**.** *The Betti numbers of $I(\mathbf{a}+j)$ are eventually periodic in j with period $a_n - a_1$.*

In general, the problem of finding defining equations of monomial curves is difficult. For example, in [6], Bresinsky gave an example of a family of monomial curves in $\mathbb{A}^4$ whose numbers of minimal generators of the defining ideals are unbounded. Recently, in the case $n \leq 3$, Conjecture 1.10 was proven by Jayanthan and Srinivasan in [26]. In the case when $\mathbf{a}$ is an arithmetic sequence, Conjecture 1.10 was proven by Gimenez, Sengupta and Srinivasan in [16]. In chapter 3, we prove the conjecture in full generality:

Theorem 1.11. *The Betti numbers of $I(\mathbf{a} + j)$ are eventually periodic in j with period $a_n - a_1$.*

To prove Theorem 1.11 we first prove the eventual periodicity in j for total Betti numbers of the homogenization $\bar{I}(\mathbf{a} + j)$, and then prove the equalities for total Betti numbers of $I(\mathbf{a} + j)$ and $\bar{I}(\mathbf{a} + j)$ when $j \gg 0$.

To simplify notation, for each i, $1 \leq i \leq n$, let $b_i = a_n - a_i$. Note that if f is homogeneous then $f \in I(\mathbf{a})$ if and only if $f \in I(\mathbf{a} + j)$ for all j. Denote by $J(\mathbf{a})$ the ideal generated by homogeneous elements of $I(\mathbf{a})$. In general, for each finitely generated graded R-module M, $\operatorname{Tor}_i^R(M, K)$ is a finitely generated graded module for each i. Let

$$\beta_{ij}(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$$

be the i-th graded Betti number of M in degree j. Moreover, let

$$\operatorname{reg} M = \sup_{i,j}\{j - i : \beta_{ij} \neq 0\}$$

be the Castelnuovo-Mumford regularity of M.

Let x_0 be a homogenizing variable. In Proposition 3.3, we prove that when $j > b_1(n + \operatorname{reg} J(\mathbf{a}))$, each binomial in $\bar{I}(\mathbf{a} + j)$ involving x_0 has degree greater than $n + \operatorname{reg} J(\mathbf{a})$. Thus the Betti table of $\bar{I}(\mathbf{a} + j)$ separates into two parts. One part is the Betti table of $J(\mathbf{a})$ which lies in degree at most $n + \operatorname{reg} J(\mathbf{a})$. The other part lies in degree larger than $n + \operatorname{reg} J(\mathbf{a})$. We call the part of Betti table of $\bar{I}(\mathbf{a} + j)$ lying in degree larger than $n + \operatorname{reg} J(\mathbf{a})$ the high degree part. We will prove that when $j \gg 0$, the Betti table of $\bar{I}(\mathbf{a} + j + b_1)$ is obtained from the Betti table of $\bar{I}(\mathbf{a} + j)$ by shifting the high degree part of $\bar{I}(\mathbf{a} + j)$ (see Theorem 3.12).

Example 1.12. *Let $\mathbf{a} = (1, 2, 3, 7, 10)$. A computation in Macaulay2 shows that the Betti tables of $\bar{I}(\mathbf{a} + 49)$ and $\bar{I}(\mathbf{a} + 58)$ are as follows:*

	0	1	2	3
2	1	–	–	–
3	6	8	1	–
4	–	2	4	1
5	–	–	–	–
6	–	–	–	–
7	2	1	–	–
8	1	11	13	3
9	–	–	–	1

	0	1	2	3
2	1	–	–	–
3	6	8	1	–
4	–	2	4	1
5	–	–	–	–
6	–	–	–	–
7	–	–	–	–
8	2	1	–	–
9	1	11	13	3
10	–	–	–	1

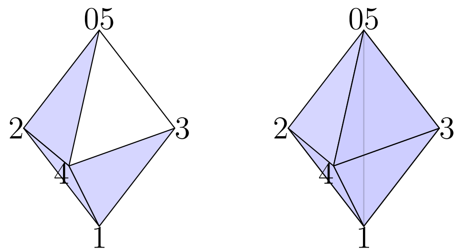
where the entry in the column-index i and row-index j of each table represents the Betti number $\beta_{i,i+j}$ of the corresponding ideals. A dash represents 0*.*

The example shows that the separation of Betti tables of $\bar{I}(\mathbf{a}+j)$ might occur much earlier, i.e., $j < b_1(n+\operatorname{reg} J(\mathbf{a})$. Also, it is natural to expect that the periodicity of the Betti table of $\bar{I}(\mathbf{a}+j)$ begins as early as the separation of the Betti table occurs. Computational experiments suggest that this is correct.

To prove the shifting behaviour of the Betti tables of $\bar{I}(\mathbf{a}+j)$ as well as the equalities of total Betti numbers of $I(\mathbf{a}+j)$ and $\bar{I}(\mathbf{a}+j)$, we note that $\bar{I}(\mathbf{a}+j)$ and $I(\mathbf{a}+j)$ are defining ideals of certain semigroup rings. Moreover, by Theorem 1.2, Betti numbers of a semigroup ring can be given in term of homology groups of certain simplicial complexes associated to elements of the semigroup. Thus we reduce the problem to proving equalities among homology groups of these simplicial complexes.

In our situation, Betti numbers of $\bar{I}(\mathbf{a}+j)$ can be expressed in terms of homology groups of squarefree divisor simplicial complexes $\Delta_{l,r}(j)$ (defined in section 3.1). In section 3.1, we prove that if $l > n+\operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups, then $\Delta_{l,r}(j)$ is a double cone. By a double cone, we mean the union of two cones. From that, we derive the equalities of homology groups among these squarefree divisor simplicial complexes. The following example illustrates the double cone structure on $\Delta_{l,r}(j)$.

Example 1.13. *Let* $\mathbf{a} = (1,2,3,7,10)$*. We consider the Betti numbers of* $\bar{I}(\mathbf{a}+49)$*. The Betti table in Example 1.12 shows that* $\beta_{1,9}(\bar{I}(\mathbf{a}+49))$ *and* $\beta_{2,10}(\bar{I}(\mathbf{a}+49))$ *are nonzero. A more precise computation in Macaulay2 with multi-grading shows that* $\Delta_{9,73}$ *and* $\Delta_{10,83}$ *contribute to the Betti numbers* $\beta_{1,9}$ *and* $\beta_{2,10}$ *of* $\bar{I}(\mathbf{a}+49)$ *respectively. The complex* $\Delta_{9,73}$ *is the simplicial complex on the vertices* $\{0,...,5\}$ *with facets* $0524, 053, 124, 134$*, which is the double cone* $\{05\} * \{3,24\} \cup \{1\} * \{24,34\}$*. Also,* $\Delta_{10,83}$ *is the simplicial complex on the vertices* $\{0,...,5\}$ *with facets* $0514, 05123, 0534, 124, 134$*, which is the double cone* $\{05\} * \{14,123,34\} \cup \{1\} * \{24,34\}$*. The picture for these simplicial complexes are given in the following where we have identified* 0 *and* 5*.*



In the following, we give a bound for the place when the periodicity of the Betti numbers of $I(\mathbf{a}+j)$ happens.

Let $d = \gcd(b_1, ..., b_{n-1})$ be the greastest common divisor of $b_1, ..., b_{n-1}$. Let c be the conductor of the numerical semigroup generated by $b_1/d, ..., b_{n-1}/d$. Let $B = \sum_{i=1}^{n} b_i + n + d$. Let

$$N = \max\left\{ b_1(n + \operatorname{reg} J(\mathbf{a})), b_1 b_2 \left(\frac{dc + b_1}{b_{n-1}} + B \right) \right\}. \tag{1.4}$$

Fix $j > N$. Let $k = a_n + j$. Let $e = d/\gcd(d, k)$. In the case $l > n + \operatorname{reg} J(\mathbf{a})$, we prove that, for each pair (l, r) whose $\Delta_{l,r}(j)$ has non-trivial homology groups then $\Delta_{l,r}(j) = \Delta_{l+e,r+eb_1}(j + b_1)$ proving that the Betti numbers of $\bar{I}(\mathbf{a}+j)$ and $\bar{I}(\mathbf{a}+j+b_1)$ are equal (see section 3.2 for more details).

Denote by $(\mathbf{a}+j)$ the semigroup generated by $a_1 + j, ..., a_n + j$. For each pair (l, r) corresponding to an element of $\overline{\mathbf{a}+j}$ (defined in section 3.1), $m = lk - r$ is an element of $(\mathbf{a}+j)$. In the case $l > n + \operatorname{reg} J(\mathbf{a})$, we prove that if $\Delta_{l,r}(j)$ has non-trivial homology groups then Δ_m is obtained from $\Delta_{l,r}(j)$ by the deletion of the vertex 0. The double cone structures on $\Delta_{l,r}(j)$ and Δ_m show that they have the same homology groups, proving the equalities of Betti numbers of $I(\mathbf{a}+j)$ and $\bar{I}(\mathbf{a}+j)$ (see section 3.2 for more details). Consequently, we prove that the Betti numbers of $I(\mathbf{a}+j)$ are periodic in j with period b_1 when $j > N$. The condition that $j > N$ where N is technically defined in (1.4) will naturally arise in the proofs throughout chapter 3.

In the case where $\mathbf{a}$ is a Bresinsky's sequence, we prove in Proposition 3.27 that the period b_1 is exact.

Finally, we consider the analogous question for higher dimensional affine semigroup rings.

Chapter 2

Syzygies of Veronese varieties

2.1 Clifford index and canonical curves

In this section, we briefly introduce the notion of Clifford index and Green's conjecture and their connection to the Ottaviani-Paoletti conjecture. For more information, we refer to [14] and [27].

Throughout this section, we assume that k is an algebraically closed field of characteristic 0. Let C be a smooth projective curve of genus $g \geq 4$. A line bundle $\mathcal{L}$ on X is called special if $h^1(\mathcal{L}) \neq 0$. The Clifford index of $\mathcal{L}$ is defined to be

$$\operatorname{Cliff} \mathcal{L} = \deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) = g + 1 - h^0(\mathcal{L}) - h^1(\mathcal{L}).$$

The second equality follows from the Riemann-Roch Theorem [22].

Definition 2.1. *The Clifford index of a smooth projective curve C of genus $g \geq 4$ is:*

$$\operatorname{Cliff} C = \min\{\operatorname{Cliff} \mathcal{L} | h^0\mathcal{L} \geq 2 \text{ and } h^1\mathcal{L} \geq 2\}.$$

A line bundle $\mathcal{L}$ is said to compute the Clifford index of C if $\operatorname{Cliff} \mathcal{L} = \operatorname{Cliff} C$. Green's conjecture says that the Clifford index of C determines the Green-Lazarsfeld index of the canonical line bundle of C:

Conjecture 2.2 (Green)**.** *Let C be a smooth nonhyperelliptic curve over a field of characteristic 0. Then*

$$p(C, \omega_C) = \operatorname{Cliff} C - 1.$$

Let us come back to the Ottaviani-Paoletti conjecture in the case of $\mathbb{P}^3$ and $\mathcal{L} = \mathcal{O}(4)$. Let $S \in |\mathcal{L}|$ be a general hyperplane section of the embedding of $\mathbb{P}^3$ by $\mathcal{L}$, then by the adjunction formula [22], we see that $K_S = 0$. In particular, S is a K3 surface. Let $C \in |\mathcal{O}_S(1)|$ be a general hyperplane section of S, then by adjunction formula again, we see that we have a canonical curve lying on a K3 surface. Throughout the process, the Green-Lazarsfeld index stay the same, as all the embeddings are Cohen-Macaulay. Moreover, we have

Theorem 2.3 (Aprodu-Farkas [1]). *Green's conjecture holds for every smooth curve lying on an arbitrary K3 surface.*

Therefore, $p(\mathbb{P}^3, \mathcal{O}(4)) = \operatorname{Cliff} C - 1$. Thus it suffices to determine the Clifford index of a curve which is a complete intersection of two general quartic hypersurfaces in $\mathbb{P}^3$. Moreover, we have:

Theorem 2.4 (Green-Lazarsfeld [21]). *Let $\mathcal{L}$ be a base point free line bundle on a K3 surface S with $\mathcal{L}^2 > 0$. Then $\operatorname{Cliff} C$ is constant for all smooth irreducible $C \in |\mathcal{L}|$, and if $\operatorname{Cliff} C < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $M_C = M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |\mathcal{L}|$.*

Now let us determine the genus of C. Since the embedding of C is obtained by taking two general hyperplane sections of the fourth Veronese embedding of $\mathbb{P}^3$, it is embedded into $\mathbb{P}^{32}$. Moreover, the embedding of C is the canonical embedding, so $g = 33$. (This can also be found by using adjunction formula). Moreover, by Ottaviani and Paoletti, $\operatorname{Cliff} C \leq 10$, therefore we are in the second situation of Theorem 2.4. In other words, there exists a line bundle M on S such that $M_C = M \otimes \mathcal{O}_C$ computes the Clifford index of C. Note that S is a general quartic hypersurface, in this case, we have

Theorem 2.5 (Noether-Lefschetz [33]). *If $S \subset \mathbb{P}^3$ is a general surface of degree $d \geq 4$, then the restriction map $\operatorname{Pic} \mathbb{P}^3 \to \operatorname{Pic} S$ is an isomorphism.*

We are now ready for a proof of the Ottaviani-Paoletti conjecture in the case of $\mathbb{P}^3$ and $\mathcal{O}(4)$:

Theorem 2.6. *The fourth Veronese embedding of $\mathbb{P}^3$ satisfies property N_9.*

Proof. By Noether-Lefschetz Theorem $\operatorname{Pic} S \cong \mathbb{Z}\ell$ where ℓ is the class of the hyperplane section of $\mathbb{P}^3$. By Green-Lazarsfeld Theorem and the fact that $\operatorname{Cliff} C < \lfloor \frac{g-1}{2} \rfloor$, we have $\ell_C = \ell \otimes \mathcal{O}_C$ computes the Clifford index of C. Moreover ℓ_C corresponds to a g^3_{16} on C. Therefore,

$$\operatorname{Cliff} C = \operatorname{Cliff} \ell_C = 16 - 2 \cdot 3 = 10.$$

The theorem follows from Aprodu-Farkas Theorem. □

2.2 Shellability of skeleta of matching complexes

In this section, we prove the vanishings of certain homology groups of matching complexes by showing that certain skeleta of matching complexes are shellable. Let us first recall the notion of shellability of simplicial complexes.

Definition 2.7. *Let Δ be a pure simplicial complex with the set of facets $\mathcal{F}$. A total ordering $>$ on $\mathcal{F}$ is said to be a shelling of Δ if for every facet F which is not smallest with respect* *to $>$, we have $F \cap \mathcal{F}_{<F}$ is a pure simplicial complex of codimension 1 in F, where $\mathcal{F}_{<F}$ is the simplicial complex whose facets are the facets of Δ which is smaller than F with respect to $>$. When Δ has a shelling, we say that Δ is shellable.*

The shellability of a simplicial complex has a strong consequence on the homology groups of the simplicial complex itself.

Proposition 2.8. *If Δ is a shellable simplicial complex of pure dimension n, then $\tilde{H}_i(\Delta) = 0$ for all $i \neq n$.*

From the definition of the matching complex, it is easy to see that C_N^d is a pure simplicial complex of dimension $\lfloor N/d \rfloor - 1$. By Proposition 2.8, the shellability of k-skeleton of C_N^d implies that $\tilde{H}_i(C_N^d) = 0$ for all $i < k$. In other words, to show that C_N^d has trivial homology groups $\tilde{H}_i$ for all $i < k$, one can try to prove that k-skeleton of C_N^d is shellable. That is the main goal of this section.

The following notation will be used throughout the section.

For a finite ordered set S, we denote $\min S$ the smallest element in S. When $A = \{1, ..., n\}$, for any t, we order the t-element subsets of A lexicographically. Suppose $B = \{\beta_1, \ldots, \beta_t\}$ is an ordered set. When we wish to indicate that the elements of B are in the order $\beta_1 < \cdots < \beta_t$, then we write $B = \beta_1 \cdots \beta_t$ without commas and parenthesis. For example, if $F = \{a_1, \ldots, a_{k+1}\}$ is a k-face of C_A^d, then writing $F = a_1 \cdots a_{k+1}$ signifies that $a_1 < a_2 < \cdots < a_{k+1}$ in lexicographic ordering.

Let $F = a_1 \cdots a_{k+1}$ and $G = b_1 \cdots b_{k+1}$ be two k-faces of C_A^d. We say that F is larger than G in lexicographic ordering, denoted by $F > G$, if $a_i > b_i$ for the first index i where $a_i \neq b_i$.

Finally, assume that $F = a_1 a_2 \cdots a_{k+1}$ is a face of C_A^d. Let x be the smallest element in the complement of $a_1 \cup \cdots \cup a_{k+1}$. We set $c(F) = a_1 \cup \{x\}$.

For example, set $n = 5$ and $d = 2$. Then $134 > 123$ as 3-element subsets of $A = \{1, 2, \ldots, 5\}$. In the matching complex C_A^2, the 1-face 13 24 is larger than the 1-face 12 45 in the lexicographic ordering. And finally, when $F = 13\ 24$, we have $c(F) = 123$.

Note that a graph (pure 1-dimensional simplicial complex) is shellable if and only if it is connected. Therefore, the 1-skeleton of the matching complex C_n^d is shellable when $n \geq 2d + 1$.

Lemma 2.9. *Assume that $k \geq 2$ and that the $(k-1)$-skeleton of C_n^d is shellable. The k-skeleton of C_{n+d+1}^d is shellable.*

Proof. Let $A = \{1, \ldots, n+d+1\}$. Fix a shelling $\succ_1$ of $(k-1)$-skeleton of C_n^d as in the assumption. For any n-element subset B of A, we denote $\succ_B$ the shelling of $(k-1)$-skeleton of C_B^d coming from $\succ_1$ corresponding to the order-preserving bijection between B and $\{1, \ldots, n\}$.

We define an ordering $\succ$ of the k-faces of C_A^d as follows. Let $F = a_1 \cdots a_{k+1}$ and $G = b_1 \cdots b_{k+1}$ be two k-faces of C_A^d. Then $F \succ G$ if and only if

1. $1 \notin a_1$, and $F > G$ in the lexicographic ordering; or

2. $1 \in a_1$, $1 \in b_1$, and $c(F) > c(G)$; or
3. $1 \in a_1$, $1 \in b_1$, $c(F) = c(G)$ and $a_1 > b_1$; or
4. $1 \in a_1$, $1 \in b_1$, $c(F) = c(G)$, $a_1 = b_1$, and $a_2 \cdots a_{k+1} \succ_B b_2 \cdots b_{k+1}$, where B is the complement of $c(F)$.

We will now show that this is a shelling of the k-skeleton of C_A^d. Let $F = a_1 \cdots a_{k+1}$ be a k-face of C_A^d. Furthermore, assume that F is not smallest with respect to the $\succ$-order. Let $\mathcal{F}$ be the simplicial complex whose facets are all the k-faces of C_A^d that are less than F in the $\succ$-order. Let $H = F \cap G$ be a facet of $F \cap \mathcal{F}$.We need to show that H has codimension 1 in F.

Moreover assume that $G = b_1 \cdots b_{k+1}$.

Case 1: $1 \notin a_1$. Assume that $a_i \notin H$ for some i. Let b_1 be any d-element subset containing 1 of the complement of $a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_{k+1}$. Let $F' = b_1 a_1 \cdots a_{i-1} a_{i+1} \cdots a_{k+1}$, then $F \succ F'$, and $F \cap F' = F \setminus \{a_i\}$. Since H is a facet of $F \cap \mathcal{F}$, we have $H = F \setminus \{a_i\}$ having codimension 1 in F.

Case 2: $1 \in a_1$ and $c(F) > c(G)$. Assume that $c(F) = a_1 \cup \{x\}$ and $c(G) = b_1 \cup \{y\}$. There are two subcases:

Subcase 2a: $a_1 = b_1$. Since $c(F) > c(G)$, we have $x > y$, this implies that $y \in a_i$ for some $i > 1$. Since $y \in c(G)$, this implies that $a_i \neq b_j$ for any j. In other words, $a_i \notin H$. Let $a_i' = a_i \setminus \{y\} \cup \{x\}$. Let $F' = \{a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_{k+1}\}$. Then $c(F') = a_1 \cup \{y\} < c(F)$. Moreover $F \cap F' = F \setminus \{a_i\}$ which is of codimension 1. Since $a_i \notin H$, and H is a facet of $F \cap \mathcal{F}$, $H = F \cap F'$ is of codimension 1 in F.

Subcase 2b: $a_1 \neq b_1$. In this case, as $1 \in a_1$ and $1 \notin b_i$ for any $i > 1$, we have $a_1 \notin H$. If a_1 is not the smallest d-element subset of $c(F)$, let a_1' be the smallest such element. Let $F' = a_1' a_2 \cdots a_{k+1}$. We have $F \succ F'$ and $F \cap F' = a_2 \cdots a_{k+1}$ which is of codimension 1 in F. Therefore, we may assume that a_1 is the smallest d-element subset of $c(F)$. This implies that x is larger than any element in a_1. Let z be the smallest element in $b_1 \setminus a_1$. Since $c(F) > c(G)$, this implies that $z < x$. Therefore, z must belong to a_i for some $i \geq 2$, and $a_i \notin H$. Again, let $a_i' = a_i \setminus \{z\} \cup \{x\}$. Let $F' = \{a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_{k+1}\}$, then we have $c(F') = a_1 \cup \{z\} < c(F)$, and $F \cap F' = F \setminus \{a_i\} \supsetneq H$, which is a contradiction.

Case 3: $1 \in a_1$, $c(F) = c(G)$ and $a_1 > b_1$. In this case $a_1 \notin H$. Since $c(F) = c(G)$, this implies that if we let $F' = b_1 a_2 \cdots a_{k+1}$, then $F \succ F'$ and $F \cap F' = a_2 \cdots a_{k+1}$ which is of codimension 1. Therefore, H is of codimension 1 in F.

Case 4: $1 \in a_1$, $c(F) = c(G)$ and $a_1 = b_1$. In this case, by the shelling on the complement of $c(F)$, H also has codimension 1 in F. □

As an application, we give another proof of a theorem of Athanasiadis [2].

Theorem 2.10. *The k-skeleton of C_n^d is shellable when $k \leq \frac{n+1}{d+1} - 1$.*

Proof. For any $n \geq 2d + 1$, the 1-skeleton of C_n^d is shellable. By Lemma 2.9, for any $n \geq kd + k - 1$, the $k-1$-skeleton of C_n^d is shellable. This finishes the proof of the theorem. □

2.3 Third Veronese embeddings of projective spaces

In this section, using the long exact sequence (1.3), Theorem 2.10, and the computation of the Euler characteristic of the matching complex C_N^3 for $N \leq 24$, we establish the Ottaviani-Paoletti conjecture for third Veronese embeddings of projective spaces.

The following notation will be used throughout this section and the forth coming section. We denote the partition λ with row lengths $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ by the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and we use the same notation for the representation V^λ. To simplify notation, we omit the subscript and superscript when we use the operators Ind and Res. It is clear from the context and the equivariant long exact sequence what the induction and restriction are. From section 1.2, we see that in the range of our interest

$$\chi(C_N^d) = (-1)^i \tilde{H}_i + (-1)^{i+1} \tilde{H}_{i+1}$$

for some i. Writing $\chi(C_N^d) = A - B$ where A and B are representations of S_N, we have when i is even, $\tilde{H}_i = A + C$ and $\tilde{H}_{i+1} = B + C$; when i is odd, $\tilde{H}_i = B + C$ and $\tilde{H}_{i+1} = A + C$ for some representation C. We say that A and B are the expected values of $\tilde{H}_i$ and $\tilde{H}_{i+1}$ of C_N^d. Note that A or B might be zero.

Throughout this section $d = 3$. For simplicity, we sometimes write C_N for C_N^3. In this case, the long exact sequence (1.3) is:

$$\cdots \to \operatorname{Ind} \tilde{H}_i C_{n-3}^3 \to \tilde{H}_i C_{n-1}^3 \to \operatorname{Res} \tilde{H}_i C_n^3 \to \operatorname{Ind} \tilde{H}_{i-1} C_{n-3}^3 \to \cdots \tag{2.1}$$

From the definition of the matching complexes, and the long exact sequence (2.1), it is not hard to see the following.

Proposition 2.11. *The homology groups of C_n^3 with $n \leq 15$ are the expected ones.*

Proposition 2.12. *The homology groups of C_{16}^3 are the expected ones.*

Proof. By Theorem 2.10, we have $\tilde{H}_i C_{16}^3 = 0$ for $i \neq 3, 4$. Applying the equivariant long exact sequence (2.1) with $n = 16$ and Proposition 2.11, we have an exact sequence

$$0 \to \operatorname{Res} \tilde{H}_4 C_{16}^3 \to \operatorname{Ind} \tilde{H}_3 C_{13}^3 \to \tilde{H}_3 C_{15}^3 \to \operatorname{Res} \tilde{H}_3 C_{16}^3 \to \operatorname{Ind} \tilde{H}_2 C_{13}^3 \to 0. \tag{2.2}$$

From the computation of the Euler characteristic of C_{16}^3, we get the expected values of $\tilde{H}_3 C_{16}$ and $\tilde{H}_4 C_{16}$, called A and B. We have $\tilde{H}_4 C_{16} = B + C$ and $\tilde{H}_3 C_{16} = A + C$ for some representation C. By Proposition 2.11, the exact sequence (2.2) becomes

$$0 \to \operatorname{Res} C \to X \to X \to \operatorname{Res} C \to 0$$

where

$$\begin{aligned} X = \operatorname{Ind} \tilde{H}_3 C_{13} - \operatorname{Res} B &= (4,3,3,3,2) \oplus (5,3,3,3,1) \oplus (5,4,3,2,1) \oplus (5,5,3,1,1) \\ &\oplus (6,3,3,2,1) \oplus (6,4,3,1,1) \oplus (6,5,2,1,1) \oplus (7,3,3,1,1) \oplus (7,4,2,1,1) \\ &\oplus (7,5,1,1,1) \oplus (8,3,2,1,1) \oplus (8,4,1,1,1) \oplus (9,3,1,1,1) \oplus (10,2,1,1,1). \end{aligned}$$

Since there is no representation of S_{16} whose restriction can be mapped injectively into X, we see that $C = 0$. The proposition follows. □

Proposition 2.13. *The homology groups of C^3_{17} are the expected ones.*

Proof. By Theorem 2.10, we have $\tilde{H}_i C^3_{17} = 0$ for $i \neq 3, 4$. It remains to prove that $\tilde{H}_3 C_{17} = 0$. Applying the equivariant long exact sequence (2.1) with $n = 17$ we have $\tilde{H}_3 C_{16}$ maps surjectively onto $\operatorname{Res} \tilde{H}_3 C_{17}$. Moreover, by Proposition 2.12, we have

$$\begin{aligned}\tilde{H}_3 C_{16} = {} & (5,5,3,3) \oplus (5,5,5,1) \oplus (6,5,3,2) \oplus (6,5,4,1) \oplus (6,6,2,2) \oplus (6,6,4) \oplus (7,3,3,3) \\ & \oplus (7,4,3,2) \oplus 2(7,5,3,1) \oplus (7,5,4) \oplus (7,6,2,1) \oplus (7,6,3) \oplus (7,7,1,1) \oplus (8,4,3,1) \\ & \oplus (8,5,2,1) \oplus (8,5,3) \oplus (8,6,2) \oplus (9,3,3,1) \oplus (9,5,1,1).\end{aligned}$$

Therefore $\tilde{H}_3 C_{17} = 0$, the proposition follows. □

Proposition 2.14. *The homology groups of C^3_{18} are the expected ones.*

Proof. By Theorem 2.10, we have $\tilde{H}_i C_{18} = 0$ for $i \neq 3, 4$. It suffices to prove that $\tilde{H}_3 C_{18} = 0$. By Proposition 2.13 and the long exact sequence (2.1) applied to $n = 18$, we have $\tilde{H}_3 C_{17}$ maps surjectively onto $\operatorname{Res} \tilde{H}_3 C_{18}$. Since $\tilde{H}_3 C_{17} = 0$, the proposition follows. □

Proposition 2.15. *The homology groups of C^3_{19} are the expected ones.*

Proof. By Theorem 2.10, we have $\tilde{H}_i C_{19} = 0$ for $i \neq 4, 5$. Applying the equivariant long exact sequence (2.1) with $n = 19$, Proposition 2.12 and Proposition 2.14, we have an exact sequence

$$0 \to \operatorname{Res} \tilde{H}_5 C^3_{19} \to \operatorname{Ind} \tilde{H}_4 C^3_{16} \to \tilde{H}_4 C^3_{18} \to \operatorname{Res} \tilde{H}_4 C^3_{19} \to \operatorname{Ind} \tilde{H}_3 C^3_{16} \to 0. \tag{2.3}$$

From the computation of the Euler characteristic of C_{19}, we get the expected values of $\tilde{H}_4 C_{19}$ and $\tilde{H}_5 C_{19}$, called A and B. We have $\tilde{H}_5 C_{19} = B + C$ and $\tilde{H}_4 C_{19} = A + C$ for some representation C, and an exact sequence

$$0 \to \operatorname{Res} C \to X \to X \to \operatorname{Res} C \to 0$$

where

$$\begin{aligned}X = {} & (4,3,3,3,3,2) \oplus (5,3,3,3,3,1) \oplus (5,4,3,3,2,1) \oplus (5,5,3,3,1,1) \oplus (5,5,4,2,1,1) \\ & \oplus (6,3,3,3,2,1) \oplus (6,4,3,3,1,1) \oplus (6,5,3,2,1,1) \oplus (6,5,4,1,1,1) \oplus (7,3,3,3,1,1) \\ & \oplus (7,4,3,2,1,1) \oplus 2(7,5,3,1,1,1) \oplus (7,6,2,1,1,1) \oplus (8,3,3,2,1,1) \oplus (8,4,3,1,1,1) \\ & \oplus (8,5,2,1,1,1) \oplus (8,6,1,1,1,1) \oplus (9,3,3,1,1,1) \oplus (9,4,2,1,1,1) \oplus (9,5,1,1,1,1) \\ & \oplus (10,3,2,1,1,1) \oplus (10,4,1,1,1,1) \oplus (11,3,1,1,1,1) \oplus (12,2,1,1,1,1).\end{aligned}$$

Since there is no representation of S_{19} whose restriction can be mapped injectively into X, we see that $C = 0$. The proposition follows. □

Proposition 2.16. *The homology groups of C^3_{20} are the expected ones.*

Proof. Applying the equivariant long exact sequence (2.1) with $n = 20$, Proposition 2.15 and Proposition 2.14, we have exact sequences

$$0 \to \operatorname{Res} \tilde{H}_i C^3_{20} \to 0$$

for $i \neq 4, 5$ and an exact sequence

$$0 \to \tilde{H}_5 C^3_{19} \to \operatorname{Res} \tilde{H}_5 C^3_{20} \to \operatorname{Ind} \tilde{H}_4 C^3_{17} \to \tilde{H}_4 C^3_{19} \to \operatorname{Res} \tilde{H}_4 C^3_{20} \to 0. \tag{2.4}$$

Therefore, $\tilde{H}_i C^3_{20} = 0$ for $i \neq 4, 5$. From the computation of the Euler characteristics of C_{20}, we get the expected value of $\tilde{H}_4 C_{20}$ is $A = (8, 8, 4) \oplus (8, 6, 6)$. Let $M = S_\lambda(V) \oplus S_\mu(V)$ where $\lambda = (8, 8, 4)$ and $\mu = (8, 6, 6)$. By Theorem 1.6, $K_{4,2}(2) = 0$ and $K_{5,1}(V, 2) = M + N$ for some representation N. Moreover, using Macaulay2 to compute the dimensions of minimal linear syzygies of the module $\oplus_{k=0}^{\infty} \operatorname{Sym}^{3k+2}(V)$ with $\dim V = 4$, we get $\dim K_{6,0}(V, 2) = 14003$. Since $\oplus_{k=0}^{\infty} \operatorname{Sym}^{3k+2}(V)$ is a Cohen-Macaulay module of codimension 16 with h-vector $(10, 16, 1)$,

$$\dim K_{5,1}(V, 2) = 14003 - 10 \cdot \binom{16}{6} + 16 \cdot \binom{16}{5} - \binom{16}{4} = 1991.$$

Since $\dim M = 1991$, we have $K_{5,1}(V, 2) \cong M$ and so $N = 0$. By Theorem 1.6, $\tilde{H}_4 C^3_{20} = A$. The proposition follows. □

Proposition 2.17. *The homology groups of C^3_{21} are the expected ones.*

Proof. By Theorem 2.10, $\tilde{H}_i C^3_{21} = 0$ for $i \neq 4, 5$. It suffices to show that $\tilde{H}_4 C_{21} = 0$. Applying the long exact sequence (2.1) with $n = 21$ we have $\tilde{H}_4 C_{20}$ maps surjectively onto $\operatorname{Res} \tilde{H}_4 C_{21}$. The proposition follows from Proposition 2.16. □

Proposition 2.18. *The homology groups of C^3_{22} are the expected ones.*

Proof. Applying the equivariant long exact sequence (2.1) with $n = 22$ we have exact sequences

$$0 \to \operatorname{Res} \tilde{H}_i C^3_{22} \to 0$$

for $i \neq 5, 6$ and an exact sequence

$$0 \to \operatorname{Res} H_6 C^3_{22} \to \operatorname{Ind} H_5 C^3_{19} \to \tilde{H}_5 C^3_{21} \to \operatorname{Res} H_5 C^3_{22} \to \operatorname{Ind} \tilde{H}_4 C^3_{19} \to 0. \tag{2.5}$$

Therefore, $\tilde{H}_i C^3_{22} = 0$ for $i \neq 5, 6$. From the computation of the Euler characteristic of C_{22}, we get the expected values of $\tilde{H}_5 C_{22}$ and $\tilde{H}_6 C_{22}$, called A and B. In other words, $\tilde{H}_6 C_{22} = B + C$ and $\tilde{H}_5 C_{22} = A + C$ for some representation C. By Proposition 2.15 and Proposition 2.17, the exact sequence (2.5) becomes

$$0 \to \operatorname{Res} C \to X \to X \to \operatorname{Res} C \to 0$$

where

$$
\begin{aligned}
X = \operatorname{Ind} H_5C_{19} - \operatorname{Res} B &= (4,3,3,3,3,3,2) \oplus (5,3,3,3,3,3,1) \oplus (5,4,3,3,3,2,1) \\
&\oplus (5,5,3,3,3,1,1) \oplus (5,5,4,3,2,1,1) \oplus (5,5,5,3,1,1,1) \oplus (6,3,3,3,3,2,1) \\
&\oplus (6,4,3,3,3,1,1) \oplus (6,5,3,3,2,1,1) \oplus (6,5,4,3,1,1,1) \oplus (6,5,5,2,1,1,1) \\
&\oplus (7,3,3,3,3,1,1) \oplus (7,4,3,3,2,1,1) \oplus 2(7,5,3,3,1,1,1) \oplus (7,5,4,2,1,1,1) \\
&\oplus (7,5,5,1,1,1,1) \oplus (7,6,3,2,1,1,1) \oplus (7,6,4,1,1,1,1) \oplus (7,7,3,1,1,1,1) \\
&\oplus (8,3,3,3,2,1,1) \oplus (8,4,3,3,1,1,1) \oplus (8,5,3,2,1,1,1) \oplus (8,5,4,1,1,1,1) \\
&\oplus (8,6,3,1,1,1,1) \oplus (8,7,2,1,1,1,1) \oplus (9,3,3,3,1,1,1) \oplus (9,4,3,2,1,1,1) \\
&\oplus 2(9,5,3,1,1,1,1) \oplus (9,6,2,1,1,1,1) \oplus (9,7,1,1,1,1,1) \oplus (10,3,3,2,1,1,1) \\
&\oplus (10,4,3,1,1,1,1) \oplus (10,5,2,1,1,1,1) \oplus (10,6,1,1,1,1,1) \oplus (11,3,3,1,1,1,1) \\
&\oplus (11,4,2,1,1,1,1) \oplus (11,5,1,1,1,1,1) \oplus (12,3,2,1,1,1,1) \oplus (12,4,1,1,1,1,1) \\
&\oplus (13,3,1,1,1,1,1) \oplus (14,2,1,1,1,1,1).
\end{aligned}
$$

Since there is no representation of S_{22} whose restriction can be mapped injectively into X, we see that $C = 0$. The proposition follows. □

Proposition 2.19. *The homology groups of C^3_{23} are the expected ones.*

Proof. Applying the equivariant long exact sequence (2.1) with $n = 23$ we have exact sequences

$$0 \to \operatorname{Res} \tilde{H}_i C^3_{23} \to 0$$

for $i \neq 5, 6$, and an exact sequence

$$0 \to H_6C^3_{22} \to \operatorname{Res} H_6C^3_{23} \to \operatorname{Ind} \tilde{H}_5C^3_{20} \to \tilde{H}_5C^3_{22} \to \operatorname{Res} \tilde{H}_5C^3_{23} \to \operatorname{Ind} \tilde{H}_4C^3_{20} \to 0. \quad (2.6)$$

Therefore, $\tilde{H}_iC^3_{23} = 0$ for $i \neq 5, 6$. From the computation of the Euler characteristic of C_{23}, the expected value of $\tilde{H}_5C_{23}$ is

$$
\begin{aligned}
A = (8,6,6,3) &\oplus (8,7,6,2) \oplus (8,8,4,3) \oplus (8,8,5,2) \oplus (8,8,6,1) \oplus (9,6,6,2) \\
&\oplus (9,7,6,1) \oplus (9,8,4,2) \oplus (9,8,5,1) \oplus (9,8,6) \oplus (10,6,6,1) \oplus (10,8,4,1)
\end{aligned}
$$

Let M be the corresponding representation of $\mathrm{GL}(V)$ with $\dim V = 4$. By Theorem 1.6, $K_{6,1} \cong M \oplus N$ for some representation N. Moreover, using Macaulay2 to compute the dimensions of minimal linear syzygies of the module $\oplus_{k=0}^{\infty} \operatorname{Sym}^{3k+2}(V)$ with $\dim V = 4$, we get $\dim K_{7,0}(V,2) = 5400$. Since $\oplus_{k=0}^{\infty} \operatorname{Sym}^{3k+2}(V)$ is a Cohen-Macaulay module of codimension 16 with h-vector $(10,16,1)$ and $\tilde{H}_4C^3_{23} = 0$, we have

$$\dim K_{6,1}(V,2) = 10 \cdot \binom{16}{7} - 16 \cdot \binom{16}{6} + \binom{16}{5} - 5400 = 14760.$$

Finally, note that $\dim M = 14760$, thus $N = 0$. The proposition follows. □

We are ready for the proof of Theorem 1.9. We restate it here for convenience.

Theorem 2.20. *The third Veronese embeddings of projective spaces satisfy property N_6.*

Proof. It remains to prove that the only non-zero homology groups of the matching complex C^3_{24} is $\tilde{H}_6 C^3_{24}$. Applying the equivariant long exact sequence (2.1) with $n = 24$ we have exact sequences

$$0 \to \operatorname{Res} \tilde{H}_i C^3_{24} \to 0$$

for $i \neq 5, 6$, and an exact sequence

$$0 \to \tilde{H}_6 C^3_{23} \to \operatorname{Res} \tilde{H}_6 C^3_{24} \to \operatorname{Ind} \tilde{H}_5 C^3_{21} \to \tilde{H}_5 C^3_{23} \to \operatorname{Res} \tilde{H}_5 C^3_{24} \to 0.$$

Therefore, $\tilde{H}_i C^3_{24} = 0$ for $i \neq 5, 6$ and $\tilde{H}_5 C^3_{23}$ maps surjectively onto $\operatorname{Res} \tilde{H}_5 C^3_{24}$. Moreover, by a result of Ottaviani and Paoletti [35], the third Veronese embedding of $\mathbb{P}^3$ satisfies property N_6. By Theorem 1.6, $\tilde{H}_5 C^3_{24}$ does not contain any irreducible representations whose corresponding partitions have at most 4 rows. By Proposition 2.19, $\tilde{H}_5 C^3_{23}$ contains only irreducible representations whose corresponding partitions have at most 4 rows, thus $\tilde{H}_5 C^3_{24}$ is zero. □

Note that if we use the following set of equivariant long exact sequences we can prove Theorem 1.9 without knowing the homology groups of C^3_{23}. The idea for constructing such long exact sequences also come from Bouc [5] (see also [28]). Assume that $d \geq 3$, and $n \geq 3d$ (the sequences will be different in the case $d = 2$). Let a, b be two fixed element of $[n]$, for example $a = 1$ and $b = 2$. We have the following filtration of C^d_n.

$$C^d_{n-2} \subset \Delta_0 \subset \Delta_1 \subset C^d_n$$

where

$$\Delta_1 = C^d_n \setminus \{f : ax, by \in f \text{ for some x, y}\}.$$

and

$$\Delta_0 = C^d_n \setminus \{f : ax \text{ or } by \in f \text{ for some x, y}\}.$$

Since $C^d_n/\Delta_1 \cong \{axbyg\}$, $\Delta_1/\Delta_0 \cong \{axg \text{ or } byg\}$ and $\Delta_0/C^d_{n-2} = \{abzg\}$, we have the following equivariant long exact sequences of representations of S_{n-2}

$$\cdots \to \tilde{H}_i(\Delta_1) \to \operatorname{Res} \tilde{H}_i(C^d_n) \to \operatorname{Ind} \operatorname{Ind} \tilde{H}_{i-2} C^d_{n-2d} \to \tilde{H}_{i-1}(\Delta_1) \to \cdots \quad (2.7)$$

$$\cdots \to \tilde{H}_i(\Delta_0) \to \tilde{H}_i(\Delta_1) \to 2 \operatorname{Ind} \tilde{H}_{i-1} C^d_{n-d-1} \to \tilde{H}_{i-1}(\Delta_0) \to \cdots \quad (2.8)$$

$$\cdots \to \tilde{H}_i(C_{n-2}) \to \tilde{H}_i(\Delta_0) \to \operatorname{Ind} \tilde{H}_{i-1}(C_{n-d}) \to \tilde{H}_{i-1} C_{n-2} \to \cdots \quad (2.9)$$

where for simplicity we have omitted the subscripts in the operators Ind and Res. It is clear from the sequences what the inductions and restrictions are.

2.4 Fourth Veronese embeddings of projective spaces

In this section, using the long exact sequence (1.3) we prove that the naturality property fails for general matching complexes. Using an idea similar to section 2.3, one might hope to be able to settle the Ottaviani-Paoletti conjecture by induction, and computation of the Euler characteristic of matching complexes C_N^4 for $N \leq 44$, which is still doable by our machine. Nevertheless, one should also note that, in Proposition 2.16 and Proposition 2.19, we need to compute certain Betti numbers to determine the exact values of the corresponding homology groups. The computation of the Betti numbers is however out of reach of computer very quickly. This fact prevents us from determining the precise values of certain homology groups of C_N^4 by merely these machineries. Also, the failure of the naturality property for general matching complexes make the situation more complicated. We also want to note that the series of exact sequences (2.7)-(2.9) introduced in the last paragraph of the previous section does not help very much in understanding the homology of C_N^4.

Proposition 2.21. *The naturality property fails for general matching complexes C_N^4.*

Proof. If the naturality fails for C_n^4 for some $n \leq 25$, then we are done. Assume that the naturality holds for all C_n^4 for $n \leq 25$. Applying the equivariant long exact sequence (1.3) with $n = 26$, we have an exact sequence

$$0 \to \tilde{H}_5 C_{25} \to \operatorname{Res} \tilde{H}_5 C_{26} \to \operatorname{Ind} \tilde{H}_4 C_{22} \to \tilde{H}_4 C_{25} \to \operatorname{Res} \tilde{H}_4 C_{26} \to \operatorname{Ind} \tilde{H}_3 C_{22} \to 0.$$

In particular, $\operatorname{Res} \tilde{H}_4 C_{26}$ maps surjectively onto $\operatorname{Ind} \tilde{H}_3 C_{22}$. Moreover, from the computation of the Euler characteristic of C_{26}^4, we get the expected value A of $\tilde{H}_4 C_{26}$. Since $\operatorname{Res} A$ does not map surjectively onto $\operatorname{Ind} \tilde{H}_3 C_{22}$, we must have $\tilde{H}_4 C_{26} = A + C$ for some non-zero representation C. In other words, C is a common representation of $\tilde{H}_4 C_{26}$ and $\tilde{H}_5 C_{26}$. $\square$

Chapter 3

Shifted families of affine semigroup rings

3.1 Double cone structure on simplicial complexes $\Delta_{l,r}(j)$

In this section, we prove the double cone structure of the squarefree divisor simplicial complexes $\Delta_{l,r}(j)$ defined below.

The following notation from section 1.3 and facts will be used throughout the chapter.

- For each $i = 1, ..., n$, let $b_i = a_n - a_i$. In particular, $b_n = 0$.
- Let d be the greatest common divisor of $b_1, ..., b_{n-1}$.
- Let c be the conductor of the semigroup generated by $b_1/d, ..., b_{n-1}/d$.
- Let $B = \sum_{i=1}^{n} b_i + n + d$.

Let

$$N = \max\left\{b_1(n + \operatorname{reg} J(\mathbf{a})), b_1 b_2 \left(\frac{dc + b_1}{b_{n-1}} + B\right)\right\}. \tag{3.1}$$

Fix $j > N$. Let $k = a_n + j$. Let $e = d/\gcd(d, k)$. Note that $d|b_1$, so $e = d/\gcd(d, k + b_1)$. Since $k > N$, it follows that k satisfies

$$k/b_1 > n + \operatorname{reg} J(\mathbf{a}), \tag{3.2}$$

and

$$\frac{k}{b_1 b_2} > \frac{dc + b_1}{b_{n-1}} + B. \tag{3.3}$$

We will use the following notation when dealing with faces of simplicial complexes on the vertices $\{0, ..., n\}$. If $F \subseteq \{0, ..., n\}$, let $|F|$ be the cardinality of F. Moreover, for $i \in \{0, ..., n\}$, let

$$\delta_{iF} = \begin{cases} 0 & \text{if } i \notin F, \\ 1 & \text{if } i \in F. \end{cases}$$

Finally, the following representation of a natural number will be used frequently in the paper. Let u be a natural number such that u is divisible by d and $u \geq dc$. We can write $u = tb_1 + v$ for some non-negative integer numbers t and v such that $dc \leq v < dc + b_1$. Because $d|b_1$, it follows that $d|v$. Since $v/d \geq c$, the conductor of the numerical semigroup generated by $b_1/d, ..., b_{n-1}/d$, we can write

$$v/d = w_1(b_1/d) + ... + w_{n-1}(b_{n-1}/d),$$

for non-negative integers $w_1, ..., w_{n-1}$. If we denote by $\mathbf{b} = (b_1, ..., b_{n-1})^t$ and $\mathbf{w} = (w_1, ..., w_{n-1})^t$ the column vectors with coordinates $b_1, ..., b_{n-1}$ and $w_1, ..., w_{n-1}$ respectively, then we have the following representation of $u = tb_1 + v$ as

$$u = tb_1 + \mathbf{w} \cdot \mathbf{b}, \tag{3.4}$$

where $\mathbf{w} \cdot \mathbf{b} = \sum_{i=1}^{n-1} w_i b_i$ is the usual dot product of these two vectors. With this representation, if we denote by $|\mathbf{w}| = \sum_{i=1}^{n-1} w_i$ then

$$t < \frac{u}{b_1} \text{ and } |\mathbf{w}| < \frac{dc + b_1}{b_{n-1}}. \tag{3.5}$$

Note that $\bar{I}(\mathbf{a} + j)$ is the defining ideal of the semigroup ring $K[\overline{\mathbf{a} + j}]$ where $\overline{\mathbf{a} + j}$ is the additive semigroup generated by vectors $\mathbf{w}_0 = (k, 0), \mathbf{w}_1 = (b_1, k - b_1), ..., \mathbf{w}_n = (0, k)$. Note that $|\mathbf{w}_i| = k$ for all $i = 0, ..., n$. Thus each element $\mathbf{v}$ of the semigroup $\overline{\mathbf{a} + j}$ corresponds uniquely to a pair (l, r) where $l = |\mathbf{v}|/k$ and r is the first coordinate of $\mathbf{v}$. The definition of squarefree divisor simplicial complex in Definition 1.1 translates to

Definition 3.1. *For each pair of natural numbers (l, r), let $\Delta_{l,r}(j)$ be the simplicial complex on the vertices $\{0, ..., n\}$ such that $F \subseteq \{0, ..., n\}$ is a face of $\Delta_{l,r}(j)$ if and only if the equation*

$$y_0 k + \sum_{i=1}^{n} y_i b_i = r$$

has a non-negative integer solution $y = (y_0, ..., y_n)$ such that $|y| = \sum_{i=0}^{n} y_i = l$ and $F \subseteq$ $\operatorname{supp} y = \{i : y_i > 0\}$.

By Theorem 1.2, if we consider $S = K[x_0, \ldots, x_n]$ with standard grading $\deg x_i = 1$ then the graded Betti numbers of $\bar{I}(\mathbf{a} + j)$ are given by

Proposition 3.2. *For each i and each l, we have*

$$\beta_{il}(\bar{I}(\mathbf{a}+j)) = \dim_K \operatorname{Tor}_i^S(\bar{I}(\mathbf{a}+j), K)_l = \sum_{r\geq 0} \dim_K \tilde{H}_i(\Delta_{l,r}(j)).$$

An easy consequence of inequality (3.2) is the separation of the Betti table of $\bar{I}(\mathbf{a}+j)$ when $j > N$.

Proposition 3.3. *Assume that $j > N$. Any minimal binomial generator of $\bar{I}(\mathbf{a}+j)$ involving x_0 has degree greater than $n + \operatorname{reg} J(\mathbf{a})$. In particular, any syzygy of $\bar{I}(\mathbf{a}+j)$ of degree at most $n + \operatorname{reg} J(\mathbf{a})$ is a syzygy of $J(\mathbf{a})$.*

Proof. Assume that $f = x_0^{u_0} x_1^{u_1} ... x_n^{u_n} - x_1^{v_1} ... x_n^{v_n}$ is a minimal binomial generator of $\bar{I}(\mathbf{a}+j)$. We have

$$u_0 k + u_1 b_1 + ... + u_n b_n = v_1 b_1 + ... + v_n b_n.$$

Since $u_0 > 0$, it follows that $v_1 b_1 + ... + v_n b_n \geq k$. By inequality (3.2),

$$\deg f = \sum_{i=1}^{n} v_i \geq k/b_1 > n + \operatorname{reg} J(\mathbf{a}).$$

The second part follows immediately. □

In the remaining of the section, we fix $j > N$, and simply denote by $\Delta_{l,r}$ the simplicial complex $\Delta_{l,r}(j)$. We will prove that in the case $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}$ has non-trivial homology groups, $\Delta_{l,r}$ is a double cone, the union of a cone over the vertices $\{0, n\}$ and another cone over the vertex 1.

Our first technical lemma says that the range for r so that $\Delta_{l,r}$ has non-trivial homology groups is quite small.

Lemma 3.4. *Assume that $j > N$. For any $l > n + \operatorname{reg} J(\mathbf{a})$, if $\Delta_{l,r}(j)$ has non-trivial homology groups then $ek \leq r < ek + dc + B$ and $l \geq r/b_1$. In particular, any solution $y = (y_0, ..., y_n)$ of the equation $y_0 k + y_1 b_1 + ... + y_n b_n = r$ with $y_0 > 0$ satisfies $y_0 = e$.*

Proof. Since $l > n + \operatorname{reg} J(\mathbf{a})$, if $\Delta_{l,r}$ has non-trivial homology groups then it supports a nonzero syzygies of $\bar{I}(\mathbf{a}+j)$ in degree larger than $n + \operatorname{reg} J(\mathbf{a})$. By Proposition 3.3, $\Delta_{l,r}$ must have at least a facet containing 0. By Definition 3.1, the equation

$$y_0 k + y_1 b_1 + ... + y_n b_n = r \tag{3.6}$$

has a non-negative integer solution $y = (y_0, ..., y_n)$ such that $y_0 > 0$. Moreover, for $\Delta_{l,r}$ to have non-trivial homology groups, it must have at least a facet that does not contain $\{0\}$. Again, by Definition 3.1, the equation (3.6) has a solution $z = (z_0, ..., z_n)$ such that $z_0 = 0$ and $\sum_{i=1}^{n} z_i = l$. This implies that $d|r$ and $l \geq r/b_1$. Therefore, we must have $d|y_0 k$ or $e|y_0$. Thus, $r \geq y_0 k \geq ek$.

Now assume that $r \geq ek + dc + B$. Since $r - ek - \sum_{i=1}^{n-1} b_i > dc$, as in (3.4), we can write

$$r = ek + \sum_{i=1}^{n-1} b_i + tb_1 + \mathbf{w} \cdot \mathbf{b} \tag{3.7}$$

with

$$|\mathbf{w}| < \frac{dc + b_1}{b_{n-1}} \text{ and } t < \frac{r - ek}{b_1}. \tag{3.8}$$

By inequality (3.3), it follows that

$$\frac{dc + b_1}{b_{n-1}} + B + \frac{r - ek}{b_1} < \frac{k}{b_1 b_2} + \frac{r - ek}{b_1} \leq \frac{r}{b_1} \leq l.$$

Together with (3.8), this implies

$$|\mathbf{w}| + e + n + t < l.$$

Therefore, the equation (3.6) has a solution $u = (u_0, ..., u_n)$ such that $u_0 = e$, $u_1 = t + 1 + w_1$, $u_i = w_i + 1$ for $i = 2, ..., n-1$ and $u_n = 1 + l - (t + e + n + |\mathbf{w}|)$. In particular, $u_i > 0$ for all i; consequently, by Definition 3.1, $\Delta_{l,r}$ is the simplex $\{0, ..., n\}$ which has trivial homology groups. This is a contradiction.

Finally, any solution $y = (y_0, ..., y_n)$ of the equation (3.6) with $y_0 > 0$ satisfies

$$y_0 \leq \frac{r}{k} < \frac{ek + dc + B}{k} < e + 1.$$

Since $e | y_0$, it follows that $y_0 = e$. □

The following theorem is the main technical result of the chapter where we prove that if $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups then $\Delta_{l,r}(j)$ has a structure of a double cone.

Theorem 3.5 (Double cone structure)**.** *Assume that $j > N$, $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups. Given a facet F of $\Delta_{l,r}(j)$, one either has $0 \notin F$ and $1 \in F$, or $0 \in F$ and $n \in F$.*

Proof. Assume that there exists a facet F of $\Delta_{l,r}$ such that $0, 1 \notin F$. By Definition 3.1, the equation

$$y_1 b_1 + ... + y_n b_n = r \tag{3.9}$$

has a solution $y = (y_1, ..., y_n)$ such that $\operatorname{supp} y = F$, $y_1 = 0$ and $\sum_{i=1}^{n} y_i = l$. Thus $l \geq \frac{r}{b_2}$. As in (3.4), we can write

$$r = \sum_{i \in F} b_i + tb_1 + \mathbf{w} \cdot \mathbf{b}$$

with
$$|\mathbf{w}| < \frac{dc + b_1}{b_{n-1}} \text{ and } t < \frac{r}{b_1}.$$
Together with inequality (3.3), this implies
$$|\mathbf{w}| + n + t < \frac{dc + b_1}{b_{n-1}} + B + \frac{r}{b_1} < \frac{k}{b_1 b_2} + \frac{r}{b_1} \leq \frac{r}{b_1 b_2} + \frac{r}{b_1} \leq \frac{r}{b_2} \leq l.$$
Note that $t > 0$, since $b_1 + ... + b_{n-1} + dc + b_1 < k \leq r$. Therefore, there is a solution $z = (z_1, ..., z_n)$ of the equation (3.9) such that $z_1 = t + w_1, z_i = \delta_{iF} + w_i$ for $i = 2, ..., n-1$, and $z_n = l - (|\mathbf{w}| + t + |F|)$. In particular, $\operatorname{supp} z \supsetneq \operatorname{supp} y$ which is a contradiction.

Now assume that F is a facet of $\Delta_{l,r}$ such that $0 \in F$ and $n \notin F$. By Definition 3.1 and Lemma 3.4, the equation
$$ek + y_1 b_1 + ... + y_n b_n = r$$
has a solution $y = (y_1, ..., y_n)$ such that $y_n = 0$ and $\sum_{i=1}^{n-1} y_i = l - e$. Moreover, by Lemma 3.4, we have $l \geq \frac{r}{b_1}$. By inequality (3.3), it follows that
$$\left(\frac{r}{b_1} - e\right) b_{n-1} \geq \left(\frac{k}{b_1} - e\right) b_{n-1} > \left(\left(\frac{dc + b_1}{b_{n-1}} + B\right) b_2 - e\right) b_{n-1} \geq dc + B.$$
Therefore,
$$r = ek + \sum_{i=1}^{n} y_i b_i \geq ek + \left(\frac{r}{b_1} - e\right) b_{n-1} > ek + dc + B,$$
which is a contradiction to Lemma 3.4. □

One of the surprising consequences of the double cone structure is the following characterization of minimal inhomogeneous generators of $I(\mathbf{a} + j)$ when $j \gg 0$.

Corollary 3.6. *Assume that $j > N$. Let $e = d/\gcd(d, a_n + j)$. Any minimal binomial inhomogeneous generator of $I(\mathbf{a} + j)$ is of the form*
$$x_1^u f - g x_n^v$$
where $u, v > 0$, f and g are monomials in the variables $x_2, ..., x_{n-1}$, and moreover, $u + \deg f = v + \deg g + e$.

Proof. By Lemma 3.4 and the double cone structure, any minimal binomial homogeneous generator of $\bar{I}(\mathbf{a} + j)$ involving an x_0 has the form
$$x_1^u f - x_0^e g x_n^v$$
where $u, v > 0$ and f, g are monomials in $x_2, ..., x_{n-1}$. The corollary follows since any minimal binomial inhomogeneous generator of $I(\mathbf{a} + j)$ is obtained from dehomogenization of a binomial homogeneous generator of $\bar{I}(\mathbf{a} + j)$ involving an x_0. □

3.2 Periodicity of Betti numbers of monomial curves

In this section we prove the Herzog-Srinivasan conjecture. It is accomplished by Theorem 3.12 where we prove that when $j > N$, the Betti table of $\bar{I}(\mathbf{a}+j+b_1)$ is obtained from the Betti table of $\bar{I}(\mathbf{a}+j)$ by shifting the high degree part by e rows and Theorem 3.21 where we prove that total Betti numbers of $I(\mathbf{a}+j)$ and those of $\bar{I}(\mathbf{a}+j)$ are equal. As in section 3.1, we fix $j > N$ and denote by $\Delta_{l,r}(j)$ the squarefree divisor simplicial complexes associated to elements of the semigroup $\overline{\mathbf{a}+j}$ and $\Delta_{l,r}(j+b_1)$ the squarefree divisor simplicial complexes associated to elements of the semigroup $\overline{\mathbf{a}+j+b_1}$.

As an appliciation of the double cone structure, we will prove that if $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}$ has non-trivial homology groups, then $\Delta_{l,r}(j) = \Delta_{l+e,r+eb_1}(j+b_1)$ as simplicial complexes. First we prove that if $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups then l is controlled in a small range by r.

Lemma 3.7. *Assume that $j > N$. If $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups, then*

$$l < \frac{r}{b_1} + \frac{dc+b_1}{b_{n-1}} + n.$$

Proof. In order for $\Delta_{l,r}(j)$ to have non-trivial homology groups, there must exist at least a facet F of $\Delta_{l,r}(j)$ such that $n \notin F$. By Theorem 3.5, this implies that $0 \notin F$. Therefore, by Definition 3.1, the equation

$$y_1 b_1 + ... + y_n b_n = r \tag{3.10}$$

has a solution $y = (y_1, ..., y_n)$ such that $\operatorname{supp} y = F$, $y_n = 0$, $\sum_{i=1}^{n-1} y_i = l$. As in (3.4), we can write

$$r = \sum_{i \in F} b_i + tb_1 + \mathbf{w} \cdot \mathbf{b},$$

with

$$|\mathbf{w}| < \frac{dc+b_1}{b_{n-1}} \text{ and } t < \frac{r}{b_1}.$$

Therefore, if $l \geq \frac{r}{b_1} + \frac{dc+b_1}{b_{n-1}} + n$ then $t + |\mathbf{w}| + n < l$. In particular, the equation (3.10) has a solution $z_1 = t + w_1$, $z_i = \delta_{iF} + w_i$ for $2 \leq i \leq n-1$ and $z_n = l - t - |\mathbf{w}| - |F| > 0$. By Definition 3.1, $F \cup \{n\} \subseteq \operatorname{supp} z$ is a face of $\Delta_{l,r}(j)$, which is a contradiction. □

The following two lemmas are the first indication of a relation between $\Delta_{l,r}(j)$ and $\Delta_{l+e,r+eb_1}(j+b_1)$.

Lemma 3.8. *Assume that $j > N$. If $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups, then*

$$\Delta_{l,r}(j) \subseteq \Delta_{l+e,r+eb_1}(j+b_1).$$

Proof. Let F be a facet of $\Delta_{l,r}(j)$. We need to prove that $F \in \Delta_{l+e,r+eb_1}(j+b_1)$. If $0 \notin F$ then by Definition 3.1, the equation

$$y_1b_1 + ... + y_nb_n = r$$

has a solution $y = (y_1, ..., y_n)$ such that $\operatorname{supp} y = F$ and $\sum_{i=1}^{n} y_i = l$. Therefore, the equation

$$y_1b_1 + ... + y_nb_n = r + eb_1$$

has a solution $z = (y_1 + e, y_2, ..., y_n)$ such that $\operatorname{supp} z \supseteq F$ and $\sum_{i=1}^{n} z_i = l + e$. By Definition 3.1, $F \in \Delta_{l+e,r+eb_1}(j+b_1)$.

If $0 \in F$, then by Definition 3.1 and Lemma 3.4, the equation

$$y_0k + y_1b_1 + ... + y_nb_n = r$$

has a solution $y = (e, y_1, ..., y_n)$ such that $\operatorname{supp} y = F$ and $\sum_{i=0}^{n} y_i = l$. Thus the equation

$$y_0(k + b_1) + y_1b_1 + ... + y_nb_n = r + eb_1$$

has a solution $z = (e, y_1, ..., y_{n-1}, y_n + e)$ such that $\operatorname{supp} z \supseteq F$ and $\sum_{i=0}^{n} z_i = l + e$. By Definition 3.1, $F \in \Delta_{l+e,r+eb_1}(j+b_1)$. □

Similarly, we have

Lemma 3.9. *Assume that $j > N$. If $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j + b_1)$ has non-trivial homology groups, then*

$$\Delta_{l-e,r-eb_1}(j) \subseteq \Delta_{l,r}(j + b_1).$$

Proof. Let F be a facet of $\Delta_{l-e,r-eb_1}(j)$. We need to prove that $F \in \Delta_{l,r}(j+b_1)$. If $0 \notin F$ then by Definition 3.1, the equation

$$y_1b_1 + ... + y_nb_n = r - eb_1$$

has a solution $y = (y_1, ..., y_n)$ such that $\operatorname{supp} y = F$ and $\sum_{i=1}^{n} y_i = l - e$. Therefore, the equation

$$y_1b_1 + ... + y_nb_n = r$$

has a solution $z = (y_1 + e, y_2, ..., y_n)$ such that $\operatorname{supp} z \supseteq F$ and $\sum_{i=1}^{n} z_i = l$. By Definition 3.1, $F \in \Delta_{l,r}(j + b_1)$.

If $0 \in F$ then by Definition 3.1, the equation

$$y_0k + y_1b_1 + ... + y_nb_n = r - eb_1$$

has a solution $y = (y_0, ..., y_n)$ such that $\operatorname{supp} y = F$ and $\sum_{i=0}^{n} y_i = l - e$. Since $\Delta_{l,r}(j + b_1)$ has non-trivial homology group, the proof of Lemma 3.4 shows that $d|r$, thus $e|y_0$. Also, by Lemma 3.4, $r < e(k + b_1) + dc + B$, thus $y_0k \leq r - eb_1 < ek + dc + B$. This implies that $y_0 = e$. Therefore, the equation

$$y_0(k + b_1) + y_1b_1 + ... + y_nb_n = r$$

has a solution $z = (e, y_1, ..., y_{n-1}, y_n + e)$ such that $\operatorname{supp} z \supseteq F$ and $\sum_{i=0}^{n} z_i = l$. By Definition 3.1, $F \in \Delta_{l,r}(j + b_1)$. □

Proposition 3.10. *Assume that $j > N$. If $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups, then*

$$\Delta_{l,r}(j) = \Delta_{l+e,r+eb_1}(j + b_1).$$

Proof. By Lemma 3.8, it suffices to show that for any facet F of $\Delta_{l+e,r+eb_1}(j + b_1)$, we have $F \in \Delta_{l,r}(j)$.

If $0 \notin F$, then by Definition 3.1, the equation

$$y_1b_1 + ... + y_nb_n = r + eb_1$$

has a solution $y = (y_1, ..., y_n)$ such that $\operatorname{supp} y = F$ and $\sum_{i=1}^n y_i = l+e$. Assume that $y_1 \leq e$, then

$$r = \sum_{i=1}^n y_ib_i - eb_1 \leq \sum_{i=2}^n y_ib_i < (l + e)b_2.$$

By Lemma 3.7 and inequality (3.3), it follows that

$$r < (l + e)b_2 < \left(\frac{r}{b_1} + \frac{dc + b_1}{b_{n-1}} + n + e\right) b_2 < \left(\frac{r}{b_1} + \frac{r}{b_1b_2}\right) b_2 \leq r.$$

This is a contradiction. Therefore, we must have $y_1 > e$. This implies that the equation

$$y_1b_1 + ... + y_nb_n = r$$

has the solution $z = (y_1 - e, ..., y_n)$ such that $\operatorname{supp} z = \operatorname{supp} y = F$ and $\sum_{i=1}^n z_i = l$. By Definition 3.1, $F \in \Delta_{l,r}(j)$.

If $0 \in F$, then by Definition 3.1 and Lemma 3.4, the equation

$$e(k + b_1) + y_1b_1 + ... + y_nb_n = r + eb_1$$

has a solution $y = (y_1, ..., y_n)$ such that $\{0\} \cup \operatorname{supp} y = F$ and $e + \sum_{i=1}^n y_i = l + e$. Assume that $y_n \leq e$. This implies that $\sum_{i=1}^{n-1} y_i \geq l - e$. By Lemma 3.4, and inequality (3.3), it follows that

$$\begin{aligned} r - ek = \sum_{i=1}^{n-1} y_ib_i &\geq (l - e)b_{n-1} \geq \left(\frac{r}{b_1} - e\right) b_{n-1} \\ &> b_2(dc + b_1 + Bb_{n-1}) - eb_{n-1} > dc + B. \end{aligned}$$

In other words, $r > ek + dc + B$. By Lemma 3.4, $\Delta_{l,r}(j)$ has trivial homology, which is a contradiction. Thus, $y_n > e$. In particular, the equation

$$ek + y_1b_1 + ... + y_nb_n = r$$

has the solution $z = (y_1, ..., y_{n-1}, y_n - e)$ such that $\operatorname{supp} z = \operatorname{supp} y = F$ and $e + \sum_{i=1}^n z_i = l$. Therefore, by Definition 3.1, $F \in \Delta_{l,r}(j)$. □

Proposition 3.11. *Assume that $j > N$. If $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j + b_1)$ has non-trivial homology groups, then*

$$\Delta_{l,r}(j + b_1) = \Delta_{l-e,r-eb_1}(j).$$

Proof. The proof is similar to that of Proposition 3.10. □

The equality of the simplicial complexes $\Delta_{l,r}(j)$ and $\Delta_{l+e,r+eb_1}(j + b_1)$ shows that the Betti table of $\bar{I}(\mathbf{a} + j + b_1)$ is obtained from the Betti table of $\bar{I}(\mathbf{a} + j)$ by shifing the high degree part by e rows as follows.

Theorem 3.12. *Assume that $j > N$. If $l \leq n + \operatorname{reg} J(\mathbf{a})$, then*

$$\beta_{i,l}(\bar{I}(\mathbf{a} + j) = \beta_{i,l}(\bar{I}(\mathbf{a} + j + b_1).$$

If $l > n + \operatorname{reg} J(\mathbf{a})$, then

$$\beta_{i,l}(\bar{I}(\mathbf{a} + j)) = \beta_{i,l+e}(\bar{I}(\mathbf{a} + j + b_1)).$$

Proof. First part follows since syzygies of $\bar{I}(\mathbf{a} + j)$ and $\bar{I}(\mathbf{a} + j + b_1)$ of degrees at most $n + \operatorname{reg} J(\mathbf{a})$ are the syzygies of $J(\mathbf{a})$ by Proposition 3.3.

Assume that $l > n + \operatorname{reg} J(\mathbf{a})$. By Proposition 3.10, if $\Delta_{l,r}(j)$ has non-trivial homology groups, then $\Delta_{l,r}(j) = \Delta_{l+e,r+eb_1}(j + b_1)$. Thus, by Proposition 3.2, it follows that

$$\begin{aligned} \beta_{i,l}(\bar{I}(\mathbf{a} + j)) &= \sum_{r \geq 0} \dim_K \tilde{H}_i(\Delta_{l,r}(j)) \\ &\leq \sum_{r \geq 0} \dim_K \tilde{H}_i(\Delta_{l+e,r+eb_1}(j + b_1)) \\ &\leq \beta_{i,l+e}(\bar{I}(\mathbf{a} + j + b_1)). \end{aligned}$$

Moreover, by Proposition 3.11, if $\Delta_{l+e,r}(j + b_1)$ has non-trivial homology groups, then $\Delta_{l+e,r}(j + b_1) = \Delta_{l,r-eb_1}(j)$. By Lemma 3.4, if $\Delta_{l+e,r}(j + b_1)$ has non-trivial homology groups then $r \geq ek + eb_1$. Thus, by Proposition 3.2, it follows that

$$\begin{aligned} \beta_{i,l+e}(\bar{I}(\mathbf{a} + j + b_1)) &= \sum_{r \geq ek+eb_1} \dim_K \tilde{H}_i(\Delta_{l+e,r}(j + b_1)) \\ &\leq \sum_{r \geq ek+eb_1} \dim_K \tilde{H}_i(\Delta_{l,r-eb_1}(j)) \\ &\leq \beta_{i,l}(\bar{I}(\mathbf{a} + j)). \end{aligned}$$

Therefore, $\beta_{i,l}(\bar{I}(\mathbf{a} + j)) = \beta_{i,l+e}(\bar{I}(\mathbf{a} + j + b_1))$. □

Remark 3.13. *Note that to establish results in this section and section 3.1, we only require that inequality (3.2) and inequality (3.3) hold true for k. Since these inequalities are still valid when we replace k by $k - a_n$, the periodicity of Betti numbers of $\bar{I}(\mathbf{a} + j)$ happens when $j > N - a_n$.*

As a corollary, we have

Corollary 3.14. *If $j > N$, then*

$$\operatorname{reg} \bar{I}(\mathbf{a} + j + b_1) = \operatorname{reg} \bar{I}(\mathbf{a} + j) + e.$$

In particular, $\operatorname{reg} \bar{I}(\mathbf{a} + j)$ *is quasi-linear in* j *when* $j > N$.

Proof. By Theorem 3.12, it suffices to show that there is at least one minimal binomial generator of $\bar{I}(\mathbf{a} + j)$ involving x_0. This is always the case, since $I(\mathbf{a} + j)$ always contains at least one inhomogeneous minimal generator. □

In the remaining of the section, we establish the equality of total Betti numbers of $\bar{I}(\mathbf{a}+j)$ and $I(\mathbf{a}+j)$. We simply denote $\Delta_{l,r}(j)$ by $\Delta_{l,r}$. Denote by $(\mathbf{a}+j)$ the semigroup generated by $k - b_1, ..., k - b_{n-1}, k$. The ideal $I(\mathbf{a}+j)$ is the defining ideal of the semigroup ring $K[(\mathbf{a}+j)]$. In this setting, Definition 1.1 gives

Definition 3.15. *For each* $m \in (\mathbf{a} + j)$, *let* Δ_m *be the simplicial complex on the vertices* $\{1, ..., n\}$ *such that* $F \subseteq \{1, ..., n\}$ *is a face of* Δ_m *if and only if the equation*

$$\sum_{i=1}^{n} y_i(k - b_i) = m \tag{3.11}$$

has a non-negative integer solution $y = (y_1, ..., y_n)$ *such that* $F \subseteq \operatorname{supp} y = \{i : y_i > 0\}$.

In considering the Betti numbers of $I(\mathbf{a} + j)$, we will use the following grading coming from the semigroup $(\mathbf{a} + j)$.

Definition 3.16 ($(\mathbf{a} + j)$-grading)**.** *The* $(\mathbf{a} + j)$*-grading on* $R = K[x_1, ..., x_n]$ *is given by* $\deg x_i = a_i + j = k - b_i$.

The ideal $I(\mathbf{a} + j)$ is homogeneous when R is endowed with $(\mathbf{a} + j)$-grading, Theorem 1.2 gives

Proposition 3.17. *For each* i *and each* m, *we have*

$$\beta_{im}(I(\mathbf{a} + j)) = \dim_K \operatorname{Tor}_i^R(I(\mathbf{a} + j), K)_m = \dim_K \tilde{H}_i(\Delta_m),$$

where $\operatorname{Tor}_i^R(I(\mathbf{a} + j), K)_m$ *is the* $(\mathbf{a} + j)$*-degree* m *part of* $\operatorname{Tor}_i^R(I(\mathbf{a} + j), K)$.

Note that for each pair (l, r) corresponding to an element of $\overline{\mathbf{a} + j}$, $lk - r$ is an element of $(\mathbf{a} + j)$. To prove the equality of total Betti numbers of $I(\mathbf{a}+j)$ and of $\bar{I}(\mathbf{a}+j)$, we prove the equality of homology groups of Δ_{lk-r} and of $\Delta_{l,r}(j)$. More precisely, applying the double cone structure theorem, we will prove that for each $l > n + \operatorname{reg} J(\mathbf{a})$, if $\Delta_{l,r}(j)$ has non-trivial homology groups then Δ_{lk-r} is obtained from $\Delta_{l,r}(j)$ by deleting the vertex 0. The double cone structure again applies to prove that $\Delta_{l,r}(j)$ and Δ_m have the same homology groups.

Similar to Proposition 3.3, we first establish the separation in the Betti table of $I(\mathbf{a}+j)$.

Proposition 3.18. *Assume that $j > N$. Any minimal binomial inhogeneous generator of $I(\mathbf{a}+j)$ has $(\mathbf{a}+j)$-degree larger than $k(n+\operatorname{reg} J(\mathbf{a}))$. In particular, any syzygy of $I(\mathbf{a}+j)$ of $(\mathbf{a}+j)$-degree at most $k(n+\operatorname{reg} J(\mathbf{a}))$ is a syzygy of $J(\mathbf{a})$.*

Proof. By Theorem 3.12 and Remark 3.13, each inhomogeneous generator of $I(\mathbf{a}+j)$ has degree at least $n+\operatorname{reg} J(\mathbf{a})+e$. Thus its $(\mathbf{a}+j)$-degree is at least

$$(n+e+\operatorname{reg} J(\mathbf{a}))(k-b_1) > k(n+\operatorname{reg} J(\mathbf{a}))$$

since $k = a_n + j > b_1 + N$, and by the choice of N in (3.1),

$$e(k-b_1) \geq eN > b_1(n+\operatorname{reg} J(\mathbf{a})).$$

The second statement follows immediately. □

The following technical lemma says that for each pair (l, r) for which $l > n+\operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non trivial homology groups, the corresponding simplicial complex Δ_{lk-r} is obtained from $\Delta_{l,r}(j)$ by the deletion of the vertex 0. From Theorem 3.5, we see that $\Delta_{l,r}(j)$ and Δ_{lk-r} have the same homology groups.

Lemma 3.19. *Assume that $j > N$, $l > n+\operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}(j)$ has non-trivial homology groups. Let $m = lk-r$. If $(y_1, ..., y_n)$ is a non-negative integer solution of* (3.11) *then $\sum_{i=1}^n y_i = l-e$ or $\sum_{i=1}^n y_i = l$. In particular, Δ_m is obtained from $\Delta_{l,r}(j)$ by the deletion of the vertex* 0.

Proof. Note that $\sum_{i=1}^n y_i(k-b_i) = lk-r$ is equivalent to

$$\left(\sum_{i=1}^n y_i - l\right) k = \sum_{i=1}^n y_i b_i - r.$$

Since the right hand side is divisible by d, the left hand side is divisible by d. Therefore $\sum_{i=1}^n y_i - l$ is divisible by e. Thus it suffices to show that $\sum_{i=1}^n y_i \geq l-e$ and $\sum_{i=1}^n y_i < l+e$.

By inequality (3.3), it follows that

$$\sum_{i=1}^n y_i \geq \frac{lk-r}{k} \geq \frac{lk-ek-B-dc}{k} > l-e-1.$$

Moreover, from the equation $\sum_{i=1}^n y_i(k-b_i) = lk-r$, it follows that

$$\sum_{i=1}^n y_i \leq \frac{lk-r}{k-b_1}.$$

To prove that $\frac{lk-r}{k-b_1} < l+e$ is equivalent to prove that

$$lb_1 + eb_1 < r + ek.$$

This follows from Lemma 3.7, and inequality (3.3) since

$$lb_1 + eb_1 < r + \left(\frac{dc + b_1}{b_{n-1}} + n + e\right) b_1 < r + \frac{k}{b_1 b_2} b_1 \leq r + ek.$$

Let Δ be the simplicial complex obtained by deleting the vertex 0 of the simplicial complex $\Delta_{l,r}$. From Definition 3.1 and Definition 3.15, $\Delta \subseteq \Delta_m$. It suffices to show that $\Delta_m \subseteq \Delta$. Let F be any facet of Δ_m. By Definition 3.15, there exists a solution $(y_1, ..., y_n)$ of the equation $\sum_{i=1}^n y_i(k - b_i) = m$ such that $\operatorname{supp} y = F$. We have two cases:

If $\sum_{i=1}^n y_i = l$, then y is also a solution of the equation $\sum_{i=1}^n y_i b_i = r$. By Definition 3.1, F is a face of $\Delta_{l,r}$ which is also a face of Δ.

If $\sum_{i=1}^n y_i = l - e$, then $z = (e, y_1, ..., y_n)$ is a solution of the equation $y_0 k + \sum_{i=1}^n y_i b_i = r$ such that $\sum_{i=0}^n z_i = l$. By Definition 3.1, $F \cup \{0\}$ is a face of $\Delta_{l,r}$, thus F is a face of Δ. □

Lemma 3.20. *Assume that $j > N$, $l_1, l_2 > n + \operatorname{reg} J(\mathbf{a})$ and Δ_{l_1,r_1}, Δ_{l_2,r_2} have non-trivial homology groups. If $(l_1, r_1) \neq (l_2, r_2)$, then $l_1 k - r_1 \neq l_2 k - r_2$.*

Proof. Assume that $l_1 k - r_1 = l_2 k - r_2$. It follows that $k | r_1 - r_2$. Moreover, by Lemma 3.4, we have

$$ek \leq r_1, r_2 \leq ek + dc + B.$$

Together with inequality (3.3) this implies

$$|r_1 - r_2| \leq dc + B < k.$$

Thus $r_1 = r_2$ and $l_1 = l_2$. □

Theorem 3.21. *Assume that $j > N$. For each i, $\beta_i(\bar{I}(\mathbf{a} + j)) = \beta_i(I(\mathbf{a} + j))$.*

Proof. Since $I(\mathbf{a} + j)$ is the dehomogenization of $\bar{I}(\mathbf{a} + j)$, $\beta_i(\bar{I}(\mathbf{a} + j)) \geq \beta_i(I(\mathbf{a} + j))$.

Moreover, by Theorem 3.5 and Lemma 3.19, we have if $l > n + \operatorname{reg} J(\mathbf{a})$ and $\Delta_{l,r}$ has non-trivial homology groups then $\Delta_{l,r}$ and Δ_{lk-r} have isomorphic homology groups. Together with Proposition 3.17 and Proposition 3.2, we have for each i,

$$\begin{aligned}
\beta_i(I(\mathbf{a} + j)) &= \sum_{m \geq 0} \dim_K \tilde{H}_i(\Delta_m) \\
&= \sum_{m \leq k(n + \operatorname{reg} J(\mathbf{a}))} \dim_K \tilde{H}_i(\Delta_m) + \sum_{m > k(n + \operatorname{reg} J(\mathbf{a}))} \dim_K \tilde{H}_i(\Delta_m) \\
&\geq \sum_{l \leq n + \operatorname{reg} J(\mathbf{a})} \dim_K \tilde{H}_i(\Delta_{l,r}) + \sum_{l > n + \operatorname{reg} J(\mathbf{a})} \dim_K \tilde{H}_i(\Delta_{l,r}) \\
&= \sum_{l,r} \dim_K \tilde{H}_i(\Delta_{l,r}) = \beta_i(\bar{I}(\mathbf{a} + j))
\end{aligned}$$

Therefore, $\beta_i(\bar{I}(\mathbf{a} + j)) = \beta_i(I(\mathbf{a} + j))$. □

We are now ready for a proof of the Herzog-Srinivasan conjecture. For convenience we restate it here.

Theorem 3.22. *The Betti numbers of $I(\mathbf{a}+j)$ are eventually periodic in j with period $a_n - a_1$.*

Proof. Fix $j > N$. By Theorem 3.21, for each i, $\beta_i(I(\mathbf{a}+j)) = \beta_i(\bar{I}(\mathbf{a}+j))$. By Theorem 3.12, $\beta_i(\bar{I}(\mathbf{a}+j)) = \beta_i(\bar{I}(\mathbf{a}+j+b_1))$. Thus the Betti numbers of $I(\mathbf{a}+j)$ are equal to the corresponding Betti numbers of $I(\mathbf{a}+j+b_1)$. □

As a corollary of Theorem 3.22, the Betti numbers of any monomial curve $I(\mathbf{a})$ is bounded by a function of n and $b_1 = a_n - a_1$.

Corollary 3.23. *There exists a function $B(n,b)$ such that for any monomial curve $I(\mathbf{a})$ whose corresponding sequence $\mathbf{a} = (a_1, ..., a_n)$ satisfies $a_n - a_1 \leq b$, we have*

$$\beta_i(I(\mathbf{a})) \leq B(n,b)$$

for any i.

Proof. For each n and b there are only finitely many sequences of differences $(b_1, ..., b_{n-1})$ such that $b_{n-1} < \cdots < b_1 \leq b$. By Theorem 3.22, the Betti numbers of $I(\mathbf{a})$ is bounded by the maximum of the Betti numbers of $I((j, j+b_{n-1}, ..., j+b_1))$ for $1 \leq j \leq N$. The corollary follows. □

3.3 Periodicity in the case of Bresinsky's sequences

Applying the results in previous sections, in this section we will analyze the number of minimal generators of $I(\mathbf{a}+j)$ where $\mathbf{a}$ is a Bresinsky's sequence. The main result of this section is Proposition 3.27 where we prove that the period b_1 of the eventual periodicity of $I(\mathbf{a}+j)$ is exact in the case of Bresinsky's sequences.

It is worth noting that, the period of the periodicity of all the Betti numbers coincide with that of the number of minimal generators. From the description of the minimal generators of monomial curves [23], [36], the period b_1 are also exact in these cases.

Recall from [6] the following definition of sequences of integers. For each h, let $\mathbf{a}^h = ((2h-1)2h, (2h-1)(2h+1), 2h(2h+1), 2h(2h+1)+2h-1)$ be a Bresinsky sequence. Since the minimal homogeneous generators of $I(\mathbf{a}^h + j)$ are the same when $j \gg 0$, it suffices to consider the number of minimal inhomogeneous generators of $I(\mathbf{a}^h + j)$ when $j \gg 0$. For an ideal I, we denote by $\mu'(I)$ the number of minimal inhomogeneous generators of I.

Fix $h \geq 2$. We simply denote $\mathbf{a}^h$ by $\mathbf{a}$. In this case, we have $b_1 = 6h-1$, $b_2 = 4h$, $b_3 = 2h-1$ and $B = 12h+3$. Note that $R = K[x_1, x_2, x_3, x_4]$.

Lemma 3.24. *If $j \geq 4b_1 b_2(b_2+1)$ then $\mu'(I(\mathbf{a}+j)) = \mu'(I(\mathbf{a}+j+b_1)) \leq 6h+1$.*

Proof. We first compute the number N in equation (3.1) in this situation. Note that $x_2x_3 - x_1x_4 \in J(\mathbf{a})$, therefore, any minimal binomial generator of $J(\mathbf{a})$ must be of the form $x_1^{\alpha}x_3^{\beta} - x_2^{\gamma}x_4^{\delta}$, where the non-negative integers β, γ, δ satisfy

$$\beta b_2 = \gamma b_1 + \delta b_3.$$

Equivalently,

$$(\beta - \delta)4h = (\gamma + \delta)(2h - 1).$$

Since $(4h, 2h - 1) = 1$, one deduces that there exists l such that

$$\begin{aligned}\beta &= (2h-1)l + \delta\\ \gamma &= 4hl - \delta\\ \alpha &= (2h+1)l - \delta.\end{aligned}$$

From this, one deduces easily that

$$J(\mathbf{a}) = (x_2x_3 - x_1x_4, x_3^{4h} - x_2^{2h-1}x_4^{2h+1}, x_1x_3^{4h-1} - x_2^{2h}x_4^{2h}, ..., x_1^{2h+1}x_3^{2h-1} - x_2^{4h}).$$

By Buchberger's algorithm, [13], these elements form a Gröbner basis for $J(\mathbf{a})$ with respect to grevlex order. Thus the initial ideal of $J(\mathbf{a})$ is

$$\text{in}(J(\mathbf{a})) = (x_2x_3, x_3^{4h}, x_1x_3^{4h-1}, ..., x_1^{2h}x_3^{2h}, x_2^{4h}).$$

Since

$$(x_2x_3) : x_3^{4h} = (x_2),$$
$$(x_2x_3, x_3^{4h}, ..., x_1^{i}x_3^{4h-i}) : x_1^{i+1}x_3^{4h-i-1} = (x_2, x_3)$$

for all $i = 0, ..., 2h - 1$, and

$$(x_2x_3, x_3^{4h}, ..., x_1^{2h}x_3^{2h}) : x_2^{4h} = (x_3),$$

$\text{in}(J(\mathbf{a}))$ has linear quotient. By [24], $\operatorname{reg}\text{in}(J(\mathbf{a})) = 4h$. Moreover, by [13],

$$\operatorname{reg} J(\mathbf{a}) \le \operatorname{reg}\text{in}(J(\mathbf{a})) = 4h;$$

therefore, $\operatorname{reg} J(\mathbf{a}) = 4h$.

Note that the conductor of the numerical semigroup generated by b_1, b_2, b_3 is $c = 4h(2h - 1) - 4h - (2h - 1) + 1$. Therefore,

$$N = \max\{b_1(4 + \operatorname{reg} J(\mathbf{a})), b_1b_2(\frac{c + b_1}{b_3} + B)\} < 4b_1b_2(b_2 + 1).$$

By Theorem 3.22 and the fact that minimal homogeneous generators of $I(\mathbf{a} + j)$ and $I(\mathbf{a} + j + b_1)$ are the same, $\mu'(I(\mathbf{a} + j)) = \mu'(\mathbf{a} + j + b_1)$ when $j \ge 4b_1b_2(b_2 + 1)$.

Fix $j \geq 4b_1b_2(b_2+1)$. We simply denote by $\Delta_{l,r}$ the simplicial complexes associated to elements of the semigroup $\overline{\mathbf{a}+j}$. By Corollary 3.6, and the fact that $x_1x_4 - x_2x_3 \in J(\mathbf{a})$, any minimal binomial inhomogeneous generator of $I(\mathbf{a}+j)$ is of the form

$$x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_1+u_2-1-u_3}, \text{ or } x_1^{v_1}x_3^{v_3} - x_2^{v_2}x_4^{v_1+v_3-1-v_2}.$$

Moreover, for each u_3, and each v_2, there can be at most one minimal binomial generator of $I(\mathbf{a}+j)$ of the two forms above. By Theorem 3.21 and Definition 3.1, these minimal binomial inhomgeneous generators correpond to $\Delta_{l,r}$ where r is of the form $a_4 + j + 4hv_2$ or $a_4 + j + (2h-1)u_3$.

Assume that either $v_2 \geq 2h$ or $u_3 \geq 4h+1$, then $r - (a_4 + j + 4h + 2h - 1) \geq c$. Using representation as in (3.4), we can write

$$r = a_4 + j + tb_1 + b_2 + b_3 + w_2b_2 + w_3b_3 \tag{3.12}$$

for some non-negative integer w_2, w_3 such that $c \leq w_2b_2 + w_3b_3 < c + b_1$. From equation (3.12), it follows that

$$3 + t + w_2 + w_3 \leq \frac{r - a_4 - j}{b_1} + \frac{c + b_1}{b_3} + 3 < \frac{r}{b_1} \leq l$$

since $(c+b_1)/b_3 < b_2 + 4$ and $a_4 + j > b_1(b_2+7)$. By Definition 3.1, $\{0,2,3,4\}$ is a face of $\Delta_{l,r}$. Moreover, $b_1 = b_2 + b_3$, thus equation (3.12) can be rewritten as

$$r = a_4 + j + tb_1 + b_1 + w_2b_2 + w_3b_3.$$

By Definition 3.1, $\{0,1,4\}$ is a face of $\Delta_{l,r}$. Thus $\Delta_{l,r}$ is connected, so $\Delta_{l,r}$ does not support any minimal generator of $\bar{I}(\mathbf{a}+j)$. Thus $u_3 \leq 4h$ and $v_2 \leq 2h-1$. □

We keep notation as in the proof of Lemma 3.24. For each u_3 and each v_2, the following lemma gives the explicit form of minimal inhomogeneous generators of $I(\mathbf{a}+j)$.

Lemma 3.25. *Let $j = (6h-1)m + s$ for some m and s such that $0 \leq s \leq 6h-2$. Let $s = (2h-1)a - 4hb$ be the unique representation of s in term of $2h-1$ and $4h$ such that $0 \leq a \leq 4h$ and $0 \leq b \leq 2h-1$ and $(a,b) \neq (4h, 2h-1)$. If $j \geq 4b_1b_2(b_2+1)$ then the minimal inhomogeneous generators of $I(\mathbf{a}+j)$ are among the following forms*

$$\begin{aligned}
f_{v_2}^2 &= x_1^{v_2+m+1-b}x_3^{a+b+2h-v_2} - x_2^{v_2}x_4^{m+2h+a-v_2}\\
g_{v_2}^2 &= x_1^{v_2+m+2h-b}x_3^{a+b+1-4h-v_2} - x_2^{v_2}x_4^{m-2h+a-v_2}\\
f_{u_3}^3 &= x_1^{m-6h+a+1+u_3}x_2^{10h-2-a-b-u_3} - x_3^{u_3}x_4^{m+4h-2-b-u_3}\\
g_{u_3}^3 &= x_1^{m-2h+1+a+u_3}x_2^{4h-1-a-b-u_3} - x_3^{u_3}x_4^{m+2h-1-b-u_3}
\end{aligned}$$

where $0 \leq v_2 \leq 2h-1$, and $0 \leq u_3 \leq 4h$.

Proof. Assume that $f = x_1^{v_1}x_3^{v_3} - x_2^{v_2}x_4^{v_1+v_3-1-v_2}$ is a minimal generator of $I(\mathbf{a}+j)$. By Theorem 3.21 and Definition 3.15,

$$(6h-1)m + s + 2h(2h+1) + 2h - 1 + 4hv_2 = (6h-1)v_1 + (2h-1)v_3.$$

Equivalently,

$$s = (2h-1)(v_1 + v_3 - (m + 2h + 1)) + 4h(v_1 - v_2 - m - 1). \tag{3.13}$$

If f is a minimal generator of $I(\mathbf{a}+j)$ then $v_1 + v_3$ is as small as possible so that the equation (3.13) has non-negative integer solutions in v_1 and v_3. Moreover, by Lemma 3.4, $v_1 + v_3 - (m+2h+1) > -(2h+1)$. Therefore, either

$$v_1 + v_3 - (m + 2h + 1) = a, \text{ and } v_1 - v_2 - m - 1 = -b$$

or

$$v_1 + v_3 - (m + 2h + 1) = -(2h - 1 - b), \text{ and } v_1 - v_2 - m - 1 = 4h - a.$$

The first case gives the family $f_{v_2}^2$, while the second case gives the family $g_{v_2}^2$.

Assume that $g = x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_1+u_2-1-u_3}$ is a minimal generator of $I(\mathbf{a}+j)$. By Theorem 3.21 and Definition 3.15,

$$(6h-1)m + s + 2h(2h+1) + 2h - 1 + (2h-1)u_3 = (6h-1)u_1 + 4hu_2.$$

Equivalently,

$$s = (2h-1)(u_1 - u_3 - (m - 2h + 1)) + 4h(u_1 + u_2 - m - 2h). \tag{3.14}$$

If g is a minimal generator of $I(\mathbf{a}+j)$ then $u_1 + u_2$ is as small as possible so that the equation (3.14) has non-negative integer solutions in u_1 and u_2. Moreover, by Lemma 3.4, $u_1 + u_2 - (m+2h) > -2h$. Therefore, either

$$u_1 - u_3 - (m - 2h + 1) = a, \text{ and } u_1 + u_2 - m - 2h = -b$$

or

$$u_1 - u_3 - (m - 2h + 1) = -(2h - 1 - b), \text{ and } u_1 + u_2 - m - 2h = 4h - a.$$

The first case gives the family $g_{u_3}^3$, while the second case gives the family $f_{u_3}^3$. □

Lemma 3.26. *If $j \geq 4b_1b_2(b_2+1)$ and $j = 4h \mod 6h-1$ then $\mu'(I(\mathbf{a}+j)) = 6h-1$.*

Proof. We keep the notation as in Lemma 3.25. When $s = 4h$, then $a = 4h$ and $b = 2h-2$. In this case, $g_i^3 \notin R$ for any $i \geq 0$ and $f_{v_2}^2 - x_4^{4h}g_{v_2}^2 \in J(\mathbf{a})$, thus $f_{v_2}^2$ are not minimal for all $v_2 \geq 0$. Moreover,

$$g_0^2 = x_1^{m+2}x_3^{2h-1} - x_4^{m+2h}, \text{ and } f_0^3 = x_1^{m-2h+1}x_2^{4h} - x_4^{m+2h}$$

thus at most one of them can be in a minimal set of generators of $I(\mathbf{a}+j)$. Similarly,

$$g^2_{2h-1} = x_1^{m+2h+1} - x_2^{2h-1}x_4^{m+1}, \text{ and } f^3_{4h} = x_1^{m+2h+1} - x_3^{4h}x_4^{m-2h}$$

thus at most one of them can be in a minimal set of generators of $I(\mathbf{a}+j)$.

Therefore, the set of minimal inhomogeneous generator of $I(\mathbf{a}+j)$ can be chosen from $f^3_{u_3}$ for $u_3 = 1, ..., 4h$ and $g^2_{v_2}$ for $v_2 = 0, ..., 2h-2$. This implies that $\mu'(I(\mathbf{a}+j)) \leq 6h-1$. Moreover, for each r of the form $r = a_4 + j + 4hv_2$ or $r = a_4 + j + (2h-1)u_3$ where $0 \leq v_2 \leq 2h-2$ and $1 \leq u_3 \leq 4h$, we will prove that $\Delta_{m+2h+1,r}$ is disconnected. The lemma then follows from the Proposition 3.2 and Theorem 3.21.

In the following we consider $\Delta_{m+2h+1,r}$ for $r = a_4 + j + 4hv_2$ or $r = a_4 + j + (2h-1)u_3$. For simplicity, in each case, we simply denote $\Delta_{m+2h+1,r}$ by Δ. Since $b_2 + b_3 = b_1$, if $\{2,3\}$ is a face of Δ, then $\{1,4\}$ is also a face of Δ. Moreover, if $r \neq a_4 + j + b_2b_3$, then b_1 does not divide r, therefore $\{1,4\}$ is not a facet of Δ.

If $v_2 = 0$, then

$$r = a_4 + j = (m+2)b_1 + (2h-1)b_3 = (m-2h+1)b_1 + 4hb_2.$$

By Definition 3.1, $\{0,4\}$, $\{1,2\}$ and $\{1,3\}$ are faces of Δ. Also, $\{0,4\}$ is the only facet of Δ contaning 0. It suffices to show that if $y = (y_1, y_2)$ is a non-negative integer solution of

$$y_1b_1 + y_2b_2 = (m-2h+1)b_1 + 4hb_2 \tag{3.15}$$

or $y = (y_1, y_3)$ is a non-negative integer solution of

$$y_1b_1 + y_3b_3 = (m+2)b_1 + (2h-1)b_3 \tag{3.16}$$

then $y_1 + y_2 \geq m + 2h + 1$ and $y_1 + y_3 \geq m + 2h + 1$ respectively. Moreover, if (y_1, y_2) is a non-negative integer solution of (3.15), then $y_1 = m-2h+1-b_2t$ and $y_2 = 4h+b_1t$ for some integer t. Thus $t \geq 0$, therefore $y_1 + y_2 \geq m + 2h + 1$. Similarly, if (y_1, y_3) is a non-negative integer solution of (3.16), then $y_1 = m + 2 - b_3t$ and $y_3 = 2h - 1 + b_1t$ for some integer t. Thus $t \geq 0$, therefore $y_1 + y_3 \geq m + 2h + 1$.

If $1 \leq v_2 \leq 2h-2$, then

$$r = a_4 + j + 4hv_2 = (m + v_2 + 2)b_1 + (2h - 1 - v_2)b_3.$$

By Definition 3.1, $\{0,4,2\}$ and $\{1,3\}$ are faces of Δ. Also, $\{0,4,2\}$ is the only facet of Δ containing 0. It suffices to show that $\{1,2\}$ is not a face of Δ. If

$$y_1b_1 + y_2b_2 = r = (m+2h+1)b_1 - (2h-1-v_2)b_2 \tag{3.17}$$

then $y_1 = m+2h+1-b_2t$ and $y_2 = -(2h-1-v_2)+b_1t$ for some integer t. Thus for any non-negative integer solution $y = (y_1, y_2)$ of equation (3.18), $t \geq 1$, thus $y_1+y_2 \geq m+2h+1+b_3$. Therefore $\{1,2\}$ is not a face of Δ.

If $1 \leq u_3 \leq 4h-1$, then

$$r = a_4 + j + (2h-1)u_3 = (m + u_3 + 1 - 2h)b_1 + (4h - u_3)b_2.$$

By Definition 3.1, $\{0,4,3\}$ and $\{1,2\}$ are faces of Δ. Also, $\{0,4,3\}$ is the only facet of Δ containing 0. It suffices to show that $\{1,3\}$ is not a face of Δ. If

$$y_1b_1 + y_3b_3 = r = (m + 2h + 1)b_1 - (4h - u_3)b_3 \tag{3.18}$$

then $y_1 = m + 2h + 1 - b_3t$ and $y_2 = -(4h - u_3) + b_1t$ for some integer t. Thus for any non-negative integer solution $y = (y_1, y_2)$ of equation (3.18), $t \geq 1$, thus $y_1 + y_2 \geq m + 2h + 1 + b_2$. Therefore $\{1,3\}$ is not a face of Δ.

Finally, if $u_3 = 4h$, then

$$r = a_4 + j + (2h-1)b_2 = (m + 2h + 1)b_1 = a_4 + j + 4hb_3.$$

By Definition 3.1, $\{0,2,4\}$, $\{0,3,4\}$ and $\{1\}$ are faces of Δ. Also, $\{0,2,4\}$ and $\{0,3,4\}$ are the only facets of Δ contaning 0. Also, if

$$y_1b_1 + y_2b_2 + y_3b_3 = (m + 2h + 1)b_1 \tag{3.19}$$

then $y_1 + y_2 + y_3 \geq m + 2h + 1$ and equality happens if and only if $y_2 = y_3 = 0$. Therefore, $\{1\}$ is a facet of Δ. □

Proposition 3.27. *If $j \geq 4b_1b_2(b_2+1)$ then $\mu'(I(\mathbf{a}^h + j)) \leq 6h - 1$. Moreover, equality happens if and only if $j = 4h \mod 6h-1$. In particular, the period of the periodicity of the Betti numbers of $I(\mathbf{a}^h + j)$ in j is exactly $6h - 1$.*

Proof. We keep the notation as in Lemma 3.25. We have the following cases.

If $a + b < 4h - 1$, then $g_i^2 \notin R$ for any $i \geq 0$. Moreover, $f_0^2 - x_4^{a+b+1}g_0^3 \in J(\mathbf{a})$, thus f_0^2 is not minimal. Also, $f_{2h-1}^2 - x_3^{a+b+1}g_{4h-1-a-b}^3 \in J(\mathbf{a})$, thus f_{2h-1}^2 is not minimal. Finally, note that $g_i^3 \notin R$ for $i > 4h - 1 - a - b$, and $f_{4h-a-b}^3 - x_3x_4^{2h-2}g_{4h-1-a-b}^3 \in J(\mathbf{a})$ which is not minimal. Thus $\mu'(I(\mathbf{a} + j)) \leq 6h - 2$.

If $a+b = 4h-1$, then $g_0^2 = g_0^3$ is the only element in the family g^2 belongs to $I(\mathbf{a}+j)$ and $f_{v_2}^2 - x_2^{v_2}x_4^{4h-v_2}g_0^2 \in J(\mathbf{a})$, thus no element in the family f^2 are minimal. Thus $\mu'(I(\mathbf{a}+j)) \leq 4h+1$.

If $4h-1 < a+b \leq 6h-3$, then $g_i^3 \notin R$ for any $i \geq 0$. Moreover, $f_0^3 - x_4^{6h-2-(a+b)}g_0^2 \in J(\mathbf{a})$, thus f_0^3 is not minimal. Also, $f_{4h}^3 - x_2^{6h-2-(a+b)}g_{a+b+1-4h}^2 \in J(\mathbf{a})$, thus f_{4h}^3 is not minimal. Finally, note that $g_i^2 \notin R$ for $i > a + b + 1 - 4h$, and $f_{a+b+2-4h}^2 - x_2x_4^{4h-1}g_{a+b+1-4h} \in J(\mathbf{a})$ which is not minimal. Thus $\mu'(I(\mathbf{a} + j)) \leq 6h - 2$.

Finally, if $a+b = 6h-2$, then $a = 4h$ and $b = 2h-2$ and then $s = 4h$. By Lemma 3.26, $\mu'(I(\mathbf{a} + j)) = 6h - 1$. □

3.4 Higher dimensional affine semigroup rings

Finally, we consider an analogous question for higher dimensional semigroup rings. Let $V = \mathbf{v}_1, ..., \mathbf{v}_n$ be a collection of vectors in $\mathbb{N}^k$, and $\mathbf{v} \in \mathbb{N}^k$ is a fixed directional vectors. For each collection V, let $k[V]$ be the semigroup ring generated by $\mathbf{v}_1, ..., \mathbf{v}_n$. Let $I(V)$ be the defining ideal of $k[V]$. For each integer t, let $V + t\mathbf{v}$ be the collection of vectors $\mathbf{v}_1 + t\mathbf{v}, ..., \mathbf{v}_n + t\mathbf{v}$. From the one dimensional case, it is natural to ask if the number of minimal generators of $I(V + t\mathbf{v})$ are eventually periodic in t. In Example 3.29, we give a collection of vectors V in $\mathbb{N}^2$ and a directional vector $\mathbf{v} \in \mathbb{N}^2$ with the property $\mu(I(V + t\mathbf{v}) = 12t + 2$; in particular, they are not eventually periodic. Nevertheless, the behaviour of $I(V + t\mathbf{v})$ for $t \gg 0$ are quite structured. Lots of calculations in Macaulay2 suggest us to formulate the following:

Conjecture 3.28. *For each collection of vectors V, and each directional vector $\mathbf{v}$, there exists a constant k such that for each i, there exist constants a_i, b_i such that*

$$\beta_i(I(V + j\mathbf{v} + tk\mathbf{v})) = a_i t + b_i$$

for $j \gg 0$.

Example 3.29. *Let $S = \{(3, 4), (4, 5), (5, 7), (6, 8)\}$. Let $\mathbf{v} = (1, 1)$. Then we have*

$$\mu(I(S + 12t\mathbf{v})) = 12t + 2$$

for $t \geq 1$.

Proof. Let

$$A = \begin{pmatrix} 3 + 12t & 4 + 12t & 5 + 12t & 6 + 12t \\ 4 + 12t & 5 + 12t & 7 + 12t & 8 + 12t \end{pmatrix}$$

Then a minimal generator of $I(S + 12t\mathbf{v})$ is a binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ such that $A\mathbf{u}_+ = A\mathbf{u}_-$. Moreover, $\ker A$ can be described by parameters z_1 and z_2 as follows

$$\begin{pmatrix} -z_1 - (1 + 3t)z_2 \\ z_1 + (3t - 1)z_2 \\ z_1 - (9t + 1)z_2 \\ -z_1 + (9t + 2)z_2 \end{pmatrix}$$

Assume that our $\mathbb{Z}$ basis for the solution set always have x_1 nonnegative. Chosing $z_2 = 0$, we have solution $(1, -1, -1, 1)$. Chosing $z_2 = -1$, then z_1 can take values from $(3t - 1)$ to $-9t - 1$. We then have $12t + 2$ solutions. We will show that these solutions span the solution set, and that they are minimal. Note that except the first generator, $x_1x_4 - x_2x_3$, all other generators is of the form $f_i = x_1^i x_3^{12k+2-i} - x_2^{i-2} x_4^{12k+3-i}$ for $i = 2, ..., 12k + 2$, they all have degree $12k + 2$, and the positive parts of f_i, i.e. $x_2^i x_4^{12k+1-i}$ cover all possible choices. This implies that they are minimal.

Assume that a generetor f corresponds to some z_1, z_2. It suffices to consider the case $z_2 > 0$ and $z_2 < -1$.

If $z_2 < -1$ then $\deg f \geq 2(12k+2)$, thus we can find an f_i whose positive part is less than that of f. In other words, f is not minimal.

If $z_2 > 0$, then the positive part of f is larger than x_4^{12t+3}. In other words, f can be factored through $x_4^{12t+1} - x_1^2 x_3^{12k}$, thus f is not minimal. □

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