Title
K\"ahler Geometry of non-Compact Toric Manifolds

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Publication Date
2021

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

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Professor John Lott
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Spring 2021

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Abstract

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Toric manifolds and toric varieties have played an important and particularly illuminating role in algebraic, symplectic, and Kähler geometry going back at least to the 1970's. The high degree of symmetry in many cases allows one to reduce the complexity of a given geometric question, giving the impression that we can really "see" the structure. A notable example of this in the Kähler setting is Donaldson's classic paper [27].

Until relatively recently, this has been mostly confined to the setting of compact manifolds (complete varieties). Both the algebraic and the symplectic perspectives have come to be fairly well understood individually in the non-compact case, but there has been little application of the intersection of the two theories, which naturally comprises toric Kähler geometry. The notable exception to this rule is the case of toric Kähler cones [56, 55, 39, 17, 16]. These non-compact manifolds inherit a great deal of structure from their compact *link*, which are interesting spaces in their own right. However, these manifolds are (typically) highly singular at precisely one point.

In this dissertation, we describe a procedure for the study of Kähler geometry on smooth non-compact toric manifolds. This involves tying together the algebraic and symplectic perspectives. We study a class of toric manifolds and show that here we can apply a non-compact version of the classical Delzant classification together with its application to Kähler geometry [4, 41, 24, 42]. We apply this to non-compact shrinking Kähler-Ricci solitons, where we propose a useful class of complete Kähler metrics, and ultimately prove a general uniqueness theorem for shrinking Kähler-Ricci solitons on non-compact toric manifolds.

This dissertation is dedicated to Ketchup and Mustard.

Contents

List of Figures

Acknowledgments

Most of all I would like to thank my advisor, Song Sun. He has helped me in countless ways throughout my time in graduate school. Through his constant support, encouragement, guidance, dedicated mentorship, and genuine enthusiasm for my learning and mathematical development, his efforts made all the difference for me, and I will always appreciate it.

I would also like to thank Richard Bamler, Claude LeBrun, Simon Donaldson, Blaine Lawson, John Lott, Christian Schnell, and Dror Varolin for stimulating conversations, advice, and lectures. I am grateful for Claude LeBrun, Alexander Kirillov, Matt Szczesny, and Dror Varolin specifically for helping me make the decision to pursue graduate school at Stony Brook, and for Richard Bamler and John Lott for making me feel welcome at Berkeley. I also want to thank Ronan Conlon for helping me learn how to turn a jumble of ideas into a paper.

I also want to thank Michael Albanese, Alexander Appleton, Jean-François Arbour, Frederick Bernirschke, Jack Burkart, Paula Burkhardt-Guim, Nic Brody, Xuemiao Chen, Paul Frigge, Silvia Ghinassi, Marlon de Oliveira Gomes, Lisandra Hernandez-Vasquez, David Hu, Yi Lai, Matt Lam, Hongyi Liu, Holly Mandel, Lisa Marquand, Aleksandar Milivojevic, Tim Ryan, Meredith Shea, John Sheridan, Melissa Sherman-Bennett, Tobias Shin, David Stapleton, Hong Suh, Ben Wu, Xiaohan Yan, Mengyuan Zhang, Junsheng Zhang, and Mu Zhao for learning math with me, goofing around, and being my friends. I'm somewhat annoyed that David already took the idea of calling you all by funny names.

Most importantly I want to thank all the people who've kept me sane(ish). My mom, the Cifchez household, Joe, my Boston friends, and most of all Ellis. Your love and support has meant the world to me.

Chapter 1

Introduction

The terms *toric manifold* or *toric variety* typically refer to one of two types of geometric objects:

1. A complete algebraic variety $\mathcal{M}$ which is an equivariant compactification of the algebraic torus $\mathbb{T}_{\mathbb{C}} = (\mathbb{C}^*)^n$. This means that $\mathcal{M}$ contains $\mathbb{T}_{\mathbb{C}}$ as a Zariski-open subset, and that the natural action of $\mathbb{T}_{\mathbb{C}}$ on itself extends to an algebraic action of $\mathbb{T}_{\mathbb{C}}$ on $\mathcal{M}$.

2. A compact $2n$ real-dimensional symplectic manifold (M, ω) together with an effective Hamiltonian action of the *real* torus $\mathbb{T}_{\mathbb{R}} = T^n$. This means that $\mathbb{T}_{\mathbb{R}}$ leaves the symplectic form ω invariant and that each vector field on M corresponding to an element of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}_{\mathbb{R}}$ admits a Hamiltonian function.

The significant interest in such objects from the perspective of Kähler geometry dates back to the work of Atiyah, Guillemin-Sternberg, and Delzant [4, 41, 24], which collectively says that these two notions are equivalent. Somewhat more specifically, given a compact toric manifold in the symplectic sense, there is a *canonically* associated algebraic variety satisfying the equivariant compactification condition above. Moreover given a toric variety in the algebraic sense, we can always associate a manifold which is toric in the symplectic sense, and this association can be made canonical if we further choose a *polarization*. We will explain this point in detail and explore the role of polarizations in Chapter 3. We refer to the equivalence of the two definitions in this sense as the *Delzant classification*.

The mechanism for this correspondence, perhaps unsurprisingly, passes through Kähler geometry. The key insight of Delzant is that given *either* a (M, ω) *or* $\mathcal{M}$, there exists a Kähler manifold $(M, J, \tilde{\omega})$ such that

1. $(M, \tilde{\omega})$ is $\mathbb{T}_{\mathbb{R}}$-equivariantly symplectomorphic to (M, ω) *and*

2. (M, J) admits a $\mathbb{T}_{\mathbb{C}}$-action, and is $T_{\mathbb{C}}$-equivariantly biholomorphic to $\mathcal{M}$.

Thus, starting with a symplectic toric manifold (M, ω), we obtain an algebraic toric variety $\mathcal{M}$ by passing through $(M, J, \tilde{\omega})$, and vice-versa. What's more, is that the $\mathbb{T}_{\mathbb{C}}$-action on

(M, J) *extends* the action of $\mathbb{T}_\mathbb{R}$ on M with respect to the natural inclusion $\mathbb{T}_\mathbb{R} \subset \mathbb{T}_\mathbb{C}$. We call an extension of a real torus action of this form a *complexification*. Toric manifolds have subsequently played an important role in the study of special metrics on compact Kähler manifolds, starting with the work of Abreu [2, 1] and Donaldson [27, 26] on constant scalar curvature metrics.

As soon as one begins to explore these ideas and how they behave in the non-compact setting, we very quickly encounter examples that make it clear that a Delzant classification cannot hold in all possible situations. We will see several such examples, but the basic one goes like this.

Example 1.1. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Then we have a natural S^1-action by rotation which preserves the usual Euclidean symplectic form $\omega_E = \frac{i}{2} dz \wedge d\bar{z}$. One readily checks that the function $f = x^2 + y^2$ is the relevant Hamiltonian function, making $(\mathbb{D}, \omega_E)$ into a toric manifold in the symplectic sense above. However, there is no corresponding variety. One perspective on how to see this is that if the S^1-action were to admit a complexification, then the radial vector field $r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ would have to be a complete vector field, which it clearly is not.

We refer to Chapter 3 for the details on this and other examples. This leads to the following natural question:

Question 1. *To what extent do the results above hold true for non-compact manifolds, and how can we use this to study Kähler geometry?*

One of the primary purposes of this dissertation is to provide a general framework for the study of Kähler metrics on smooth non-compact toric manifolds. We propose a class of toric manifolds, the *AK-toric* manifolds, and show that it is necessary and sufficient for a manifold to admit this structure in order to obtain a Delzant classification. The main results of Chapter 3 can be summarized as follows:

Theorem 1.2. *The Delzant classification holds for AK-toric manifolds.*

Chapter 2 contains a summary of some of the well-known basic theory for toric manifolds, with an emphasis on the new non-compact perspective that we'll need in Chapter 3, culminating in the definition of an AK-toric manifold. In order to get there, we introduce yet another notion of what it means for a manifold to be "toric," and study *complex toric manifolds*. These are essentially complex manifolds (M, J) which admit an effective and holomorphic $(\mathbb{C}^*)^n$-action. In the compact setting the Delzant classification renders a definition of this form basically useless, we may as well just view each of these in their corresponding symplectic or algebraic categories. However in the non-compact case we will see this as a useful intermediate, encapsulating one of the main issues with Example 1.1: the question of whether a $\mathbb{T}_\mathbb{R}$-action admits a complexification.

We achieve the classification in Chapter 3 by studying GIT constructions and relating this back to existing structure theorems in the symplectic/complex analytic literature [11, 46, 61, 64]. This chapter focuses on extending the well-known approach of Guillemin [42] and Abreu [2] (c.f. [3]) for studying Kähler metrics on toric manifolds to the AK-toric case.

In Chapter 4, we illustrate the application of this framework in a study of Kähler-Ricci solitons on non-compact toric manifolds. A *shrinking Ricci soliton* is a Riemannian metric g together with a vector field X on M which satisfy

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_X g = \frac{1}{2}g. \tag{1.1}$$

They arise originally as blowup limits of Type I singularities of the Ricci flow [33, 58]. A Ricci soliton is Kähler if X is real holomorphic and g is Kähler, and we say that it is gradient if $X = \nabla^g f$ for a smooth function $f \in C^\infty(M)$. Kähler-Ricci solitons are similarly related to the Kähler-Ricci flow. The tendency of the Kähler-Ricci flow to converge (in some sense) to a shrinking Kähler-Ricci soliton, together with the relative simplicity of equation (1.1), makes shrinking Kähler-Ricci solitons a natural generalization of a Kähler-Einstein metric on a Fano manifold. The reason this is important is that shrinking Kähler-Ricci solitons are known to exist [70] in many situations when there are known obstructions to the existence of a Kähler-Einstein metric with positive scalar curvature. Indeed, any compact Fano manifold admitting a shrinking Kähler-Ricci soliton with nontrivial vector field X cannot admit a Kähler-Einstein metric [69, 68].

Toric geometry really entered the scene for the study of canonical metrics in Kähler geometry with the work of Batyrev-Salivanova [6], who showed that any compact toric Fano manifold with vanishing Futaki invariant admits a Kähler-Einstein metric. This result was generalized by Wang-Zhu [70], who showed using Tian-Zhu's modified Futaki invariant [68] that every toric Fano manifold admits a shrinking gradient Kähler-Ricci soliton. This was followed by a good deal of related work, including an interesting formula of Li [51] relating toric combinatorics with the existence of Kähler-Einstein metrics, Berman-Berndtsson's generalization to the singular setting [8], and the afforementioned papers of Abreu [2, 1] and Donaldson [27, 26].

The non-compact setting is considerably less well-known. Feldman-Ilmanen-Knopf [34] found shrinking gradient Kähler-Ricci solitons with even larger symmetry $T^n \subset U(n)$ on the total space of line bundles $\mathcal{O}(-k) \to \mathbb{P}^{n-1}$ for $k = 1, \ldots, n$. More recently, there were examples of Fuatki [38], based on work of Futaki-Wang [40] which generalize the FIK examples. In their setting M is the total space of any root of the canonical bundle over a compact toric Fano manifold N, i.e. M is the total space of the negative line bundle $L \to N$ with the property that $L^p = K_N$ for $1 < p < n$. Other than this there is not much currently known about existence. In this dissertation we focus on two problems related to understanding shrinking Kähler-Ricci solitons on toric manifolds:

1. Using the Delzant classification for AK-toric manifolds to study shrinking Kähler-Ricci solitons on AK-toric manifolds in general.

2. Apply this specifically to understand the uniqueness of these metrics.

Uniqueness already is an interesting question. There is interesting work of Kotschwar-Wang on asymptotically conical [49] and asymptotically cylindrical [48] manifolds, but even here in each case we can only deduce uniqueness among all metrics which are asymptotic to a *fixed* model metric at infinity. Conlon-Deruell-Sun were able to get around this by restricting attention: they show that the FIK examples are the unique shrinking gradient Kähler-Ricci solitons with bounded Ricci curvature in the special case that $M = O(-k)_{\mathbb{P}^{n-1}}$ for $n = 1, \ldots, n-1$ [19, Theorem E] . Our main uniqueness result follows in this direction. We restrict attention to those complex manifolds which admit an effective and holomorphic $\mathbb{T}_{\mathbb{R}}$-action and obtain a general uniqueness theorem:

Theorem 1.3. *Suppose (M, J) is an n-dimensional complex manifold admitting an effective and holomorphic $\mathbb{T}_{\mathbb{R}} = T^n$-action and suppose that g is a shrinking gradient Kähler-Ricci soliton on M with bounded Ricci curvature with respect to the solition vector field X with $JX \in \mathfrak{t}$. Then (M, J) is quasiprojective, and g is unique up to biholomorphism.*

Consequently we recover [19, Theorem E], and moreover we see that the Futaki examples are the only such metrics with bounded Ricci curvature in the case that M is the total space of a root of the canonical bundle K_N of a compact toric Fano manifold. This is the main result of Chapter 4. Along the way, we will explore various ways that the theory developed in Chapter 3 can be applied to study Kähler-Ricci solitons. In particular, we introduce a space of Kähler metrics which admits an interesting space $\mathcal{P}$ of potentials. We introduce an analoge of the Futaki invariant, considered originally in [19] for non-compact solitons, and study its properties in the toric setting. We introduce an analog of the Ding functional $\mathcal{D}$ [25] and again interpret its properties for AK-toric manifolds. The uniqueness result comes as a consequence of the convexity of $\mathcal{D}$.

Every real torus $\mathbb{T}_{\mathbb{R}}$ has a canonically defined lattice $\Gamma \cong \mathbb{Z}^n \subset \mathfrak{t}$ in its Lie algebra, defined by all the closed one-parameter subgroups $\tau : S^1 \hookrightarrow \mathbb{T}_{\mathbb{R}}$. If we fix a $\mathbb{Z}$-basis $(\Xi_1, \ldots, \Xi_n)$ for Γ, this in particular determines a isomorphisms

$$\mathbb{T}_{\mathbb{R}} \to T^n = S^1 \times \cdots \times S^1$$

and

$$\mathbb{T}_{\mathbb{C}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}^* \to (\mathbb{C}^*)^n = \mathbb{C}^* \times \cdots \times \mathbb{C}^*.$$

It will be convenient throughout the dissertation to fix some such basis $(\Xi_1, \ldots, \Xi_n)$. Correspondingly we will refer to our algebraic torus $\mathbb{T}_{\mathbb{C}}$ and real torus $\mathbb{T}_{\mathbb{R}}$ as $(\mathbb{C}^*)^n$ and T^n directly indicating this choice.

Disclaimer. Parts of all Chapters 2, 3, 4 are based on our paper [15].

Chapter 2

Toric manifolds

In this chapter we will recall some of the basic definitions and properties of toric manifolds and algebraic toric varieties that we will use throughout the dissertation. Our focus is to introduce the framework in which the rest of the text takes place. This chapter has some new definitions based on our paper [15], but most of the material here is the result of many authors in various fields throughout the last half-century.

The theme of this chapter is to meet each of our three main distinct different geometric structures which we will refer to as “toric.” When working on compact manifolds (or, in the algebraic setting, complete varieties), these notions all turn out to be equivalent in the appropriate sense. This equivalence turns out to break down in the non-compact setting, and recovering what we can of this will be a large focus of the material in Chapter 3. As we will see moving forward, a key aspect of the study of toric manifolds is the rich interplay between the algebraic, symplecitc, combinatorial, and complex perspectives.

2.1 Algebraic toric varieties

Here we will introduce most of the algebro-geometric preliminaries that we will use throughout the text. Almost everything that we will mention here is contained in the comprehensive textbook [21] (c.f. [36]). We do not attempt in any way to provide a comprehensive treatment of the algebraic geometry of toric varieties, we present only what we explicitly use in the later sections. We begin with the basic definition.

Definition 2.1 (Toric variety)**.** A *toric variety* M is an equivariant compactification of the algebraic torus $(\mathbb{C}^*)^n$. More precisely, this means that M is an algebraic variety which contains the torus $(\mathbb{C}^*)^n$ as a Zariski-open subset, and that the natural action of $(\mathbb{C}^*)^n$ on itself extends to a morphism of algebraic varieties $\tau : (\mathbb{C}^*)^n \times M \to M$.

We will use the term toric *variety* exclusively in this sense. Note that there is an associated complex manifold (M, J) to any toric variety which comes naturally equipped with an effective and holomorphic $(\mathbb{C}^*)^n$-action. We will return to objects of this form and their relationship with toric varieties in subsequent sections.

2.1.1 Combinatorics: fans, polytopes, polyhedra

Fix a real n-dimensional vector space V and an additive subgroup $\Gamma \subset V$ such that $\Gamma \cong \mathbb{Z}^n$. As a matter of terminology, we will refer to $\Gamma \subset V$ as the *lattice* and the points $\nu \in \Gamma$ as *lattice points*. We will use V^* to denote the real dual vector space and $\Gamma^* \subset V^*$ the corresponding dual lattice. We will use $\langle \cdot, \cdot \rangle$ to denote the dual pairing $V \times V^* \to \mathbb{R}$.

Definition 2.2. A *rational polyhedral cone* $\sigma \subset V$ is a convex subset of the form

$$\sigma = \left\{ \sum \lambda_i \nu_i \mid \lambda_i \in \mathbb{R}_+ \right\},$$

where $\nu_1, \ldots, \nu_k \in \Gamma$ is a fixed finite collection of lattice points.

In this dissertation we will always tacitly assume that rational polyhedral cones are *strongly convex*, meaning that they do not contain any (nontrivial) linear subspace of V. The topological boundary of any rational polyhedral cone is itself a finite union of rational polyhedral cones. We call these cones the *faces* of σ.

Definition 2.3. A *fan* Σ in V is a finite set consisting of strongly convex rational polyhedral cones $\sigma \subset V$ satisfying

1. For every $\sigma \in \Sigma$, each face of σ also lies in Σ.
2. For every pair $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each.
3. There exists at least one $\sigma \in \Sigma$ whose interior is open in V.

We highlight a few properties that a fan may or may not have, which will turn out to be in correspondence with some natural geometric properties of toric varieties.

Definition 2.4. Let Σ be a fan in V.

1. A rational polyhedral cone σ is *smooth* if there are exactly n one-dimensional edges (or *rays*) $\{v_1, \ldots, v_n\}$, and if one can find a $\mathbb{Z}$-basis $\{\nu_1, \ldots, \nu_n\}$ for the lattice $\Gamma \subset V$ such that $\nu_i \in v_i$. We say Σ is *smooth* if each n-dimensional cone $\sigma \in \Sigma$ is smooth.
2. We say that Σ is *complete* if the union $\bigcup_{\sigma \in \Sigma} \sigma \subseteq V$ is the whole of V.

Definition 2.5. A *polyhedron* $P \subset V^*$ is any finite intersection of affine half spaces $H_{\nu,a} = \{x \in V^* \mid \langle \nu, x \rangle \geq a\}$ with $\nu \in V, a \in \mathbb{R}$. A *polytope* is a bounded polyhedron.

The reason for considering polyhedra in V^* as opposed to V is mainly for consistence of notation with the following sections. In most cases we will not make a careful notational distinction between a polyhedron P and its interior, but where confusion may arise we will denote by $\overline{P}$ the closed object and P the interior. Again, the topological boundary of a a polyhedron P is a union of polyhedra $\{F_\nu\}$ of one less dimension which we refer to as the *facets* of P. Here the index ν is such that F_ν lies in the plane $\langle \nu, x\rangle = a$. The intersections of any number of the F_ν's are again lower-dimensional polyhedra and together form the collection of *faces* of P.

Definition 2.6. Let P be a polyhedron given by the intersection of the half spaces H_{ν_i,a_i}. We define the *recession cone* (or asymptotic cone) C of P by

$$C = \{x \in V^* \mid \langle \nu_i, x\rangle \geq 0\}\,.$$

Given any convex cone $C \subset V$, the *dual cone* $C^* \subset V^*$ is defined by

$$C^* = \{\xi \in V \mid \langle \xi, x\rangle > 0 \text{ for all } x \in C\}. \tag{2.1}$$

Note that C^* is necessarily an open cone in V, even when C is not full-dimensional.

Definition 2.7. A polyhedron P is *rational* if its vertices lie in the dual lattice $\Gamma^* \subset V^*$.

There is a natural fan Σ_P in V associated to any full-dimensional rational polyhedron $P \subset V^*$. If we identify $V \cong \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ with the natural Euclidean inner product, we can think of Σ_P as encapsulating all the information about the directions *normal* to the faces of P. More precisely, we produce a finite set Σ_P as follows. Fix a vertex $v \in P$ and consider only those facets $\{F_{\nu_i}\}_{i=1}^k$ which contain v. Recall that each facet lies on a linear subspace of the form $\langle \nu_i, x\rangle \geq -a_i$ for $\nu_i \in V$. Set $C_v = \{x \in V^* \mid \langle \nu_i, x\rangle \geq 0\}$, and let $\sigma_v = C_v^*$. Finally, we let Σ_P be the collection of all such σ_v running through each vertex of P, together with each of their faces. In particular, note that the one-dimensional cones in Σ_P are precisely the rays generated by the "inner normals" $\nu_i \in V$.

Proposition 2.8 (c.f. [21, Theorem 2.3.2])**.** *The set Σ_P is a fan.*

2.1.2 Construction and structure of toric varieties

The goal of this section is to describe a general mechanism for constructing toric varieties from combinatorial data as in the previous section. We fix a real n-dimensional vector space V and a lattice $\Gamma \subset V$ as above. The starting point for the discussion is the observation that there is a natural process by which one can associate an affine variety to any strongly convex rational polyhedral cone $\sigma \subset V$. To see this, fix such a σ, and let σ^* denote the dual cone. The set of lattice points $S_\sigma = \sigma^* \cap \Gamma^*$ in σ^* inherits the structure of a semigroup under addition in V^*. The semigroup ring $\mathbb{C}[S_\sigma]$ is defined as a set by

$$\mathbb{C}[S_\sigma] = \left\{ \sum \lambda_s s \mid s \in S_\sigma \right\},$$

and the ring structure is then defined on monomials by $\lambda_{s_1} s_1 \cdot \lambda_{s_2} s_2 = (\lambda_{s_1}\lambda_{s_2})(s_1 + s_2)$ and extended in the natural way. The affine variety is then defined to be

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

A careful study of the ring structure of $\mathbb{C}[S_\sigma]$ shows that U_σ comes endowed with a canonical $(\mathbb{C}^*)^n$-action and an open subvariety $U \subset U_\sigma$ which is equivariantly isomorphic to $(\mathbb{C}^*)^n$ itself. In other words, we have:

Proposition 2.9 ([21, Theorem 1.2.18])**.** *Let $\sigma \subset V$ be a strongly convex rational polyhedral cone. Then the associated affine variety U_σ is a toric variety in the sense of Definition 2.1.*

Example 2.10. Let $V = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$, and $\sigma = \mathbb{R}^n_+ \subset \mathbb{R}^n$ be the positive orthant. Then $\sigma^* = \mathbb{R}^n_+$ and $S_\sigma \cong \mathbb{N}^n$ as a semigroup. The semigroup algebra $\mathbb{C}[S_\sigma]$ is then naturally identified with $\mathbb{C}[z_1, \ldots, z_n]$, and correspondingly we see that the affine toric variety U_σ constructed is isomorphic to the affine space $\mathbb{C}^n$ with the standard $(\mathbb{C}^*)^n$-action.

Remark 2.11*.* One can see directly from Definition 2.4 that if $\sigma \subset V$ is any smooth strongly convex rational polyhedral cone, then there exists a basis of Γ which determines an isomorphisms $V \cong \mathbb{R}^n$, $\Gamma \cong \mathbb{Z}^n$, and $\sigma \cong \mathbb{R}^n_+$. Consequently we see immediately that any smooth affine toric variety is equivariantly isomorphic to $\mathbb{C}^n$.

The combinatorial tools that allows one to generalize these concepts to produce toric varieties which are not necessarily affine are fans. The basic idea is that a collection of (n-dimensional) strongly convex rational polyhedral cones $\{\sigma\}$ in V produces a collection of affine toric varieties $\{U_\sigma\}$, which are then glued together in the appropriate way. The place where the full information in the data of a fan (Definition 2.3) is used is precisely to determine this gluing. We will not need to make any use of the details of this construction, but the core idea is as follows. Suppose that $\sigma_1, \sigma_2 \in \Sigma$ are full-dimensional and set $\tau = \sigma_1 \cap \sigma_2$. Then τ is again a strongly convex rational polyhedral cone, and as such there is a corresponding affine toric variety U_τ. In fact, U_τ is further equipped with open $(\mathbb{C}^*)^n$-equivariant embeddings $\phi_1 : U_\tau \hookrightarrow U_{\sigma_1}$ and $\phi_2 : U_\tau \hookrightarrow U_{\sigma_2}$. Thus the map $\phi_1 \circ \phi_2^{-1}$ can be used to glue U_{σ_2} to U_{σ_1} along U_τ in a way which is compatible with the $(\mathbb{C}^*)^n$-actions on each. Running through each such pair of maximal-dimensional cones in Σ, we conclude that the fan Σ in V gives rise naturally to a toric variety which we refer to as $\mathcal{M}_\Sigma$. The existence and basic properties of $\mathcal{M}_\Sigma$ are summarized as follows:

Proposition 2.12 ([21, Theorem 3.1.5, Theorem 3.1.19])**.** *Let Σ be a fan in V. Then there exists a normal and separated variety $\mathcal{M}_\Sigma$ constructed as above, which is toric in the sense of Definition 2.1. Moreover we have that*

1. *$\mathcal{M}_\Sigma$ is smooth as a variety if and only if Σ is smooth as a fan.*

2. *$\mathcal{M}_\Sigma$ is complete as a variety (or compact, in the analytic topology) if and only if Σ is complete as a fan.*

Remark 2.13. From a geometric perspective, one can understand the meaning of this process as follows. We can write the torus iteslf by identifying $V \oplus V^* \cong \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n$, and then setting $(\mathbb{C}^*)^n \cong (V \oplus V^*)/\Gamma$, where Γ acts only on the first factor. Then the k-dimensional cones in Σ tell us, through the dual pairing, in what directions we should add in a $(n-k)$-dimensional torus $(\mathbb{C}^*)^{n-k}$ to construct $\mathcal{M}_\Sigma$ from $(\mathbb{C}^*)^n$.

Example 2.14. Let $\Sigma = \{\sigma_1 = [0, \infty), \sigma_2 = (-\infty, 0], \tau = \{0\}\}$ be a fan in $V = \mathbb{R}$. As we saw in Example 2.10, $U_{\sigma_1} \cong U_{\sigma_2} \cong \mathbb{C}$. Considering $\tau = \{0\}$, we see that τ^* is the semigroup generated by $\{1\}$ and $\{-1\}$. The semigroup algebra is isomorphic to the Laurent polynomial ring $S_\tau \cong \mathbb{C}[z_1, z_1^{-1}]$, and $U_\tau = \mathrm{Spec}(S_\tau) \cong \mathbb{C}^*$. The maps $\phi_1 : \mathbb{C}^* \to \mathbb{C}$ and $\phi_2 : \mathbb{C}^* \to \mathbb{C}$ are given by $\phi_1(z) = z$ and $\phi_2(z) = z^{-1}$, and so we see that the variety we have constructed is $\mathcal{M}_\Sigma \cong \mathbb{P}^1$.

So far we have given a description of one possible way that toric varieties can be constructed.It turns out that, in fact, all toric varieties can be obtained in this way. The following result is well-known and goes back to the work of Sumihiro [65]:

Theorem 2.15 ([21, Corollary 3.1.8])**.** *Let $\mathcal{M}$ be a toric variety. Then there exists a fan Σ such that $\mathcal{M} = \mathcal{M}_\Sigma$.*

Proposition 2.16 (Orbit-Cone correspondence, [21, Theorem 3.2.6])**.** *Let Σ be a fan in V and $\mathcal{M}_\Sigma$ be the associated toric variety. The k-dimensional cones $\sigma \in \Sigma$ are in natural one-to-one correspondence with the $(n-k)$-dimensional orbits O_σ of the $(\mathbb{C}^*)^n$-action on $\mathcal{M}_\Sigma$. Moreover, there is a natural identification $V \cong \mathfrak{t}$ of V with the Lie algebra $\mathfrak{t}$ of the standard real torus $T^n \subset (\mathbb{C}^*)^n$. Under this identification, given a k-dimensional cone $\sigma \in \Sigma$ and a corresponding orbit $O_\sigma \subset \mathcal{M}_\Sigma$, we have that σ lies as an open subset of the Lie algebra $\mathfrak{t}_\sigma$ of the k-dimensional real subtorus $T_\sigma \subset T^n$ that stabilizes the points on O_σ.*

In this way the data of the fan encodes the information of all torus-invariant subvarieties. Of special interest are the two extremes, i.e. $k = n$ and $k = 1$. We can see immediately that the fixed point set of the $(\mathbb{C}^*)^n$-action on $\mathcal{M}_\Sigma$ corresponds to the set of n-dimensional cones and is therefore always finite for any toric variety. On the other hand when $k = 1$, we see that each ray $\sigma \in \Sigma$ determines a unique torus-invariant divisor D_σ. This latter case will be of particular importance throughout this dissertation. For our purposes, the key property is that a torus-invariant Weil divisor D on $\mathcal{M}_\Sigma$ naturally determines a polyhedron $P_D \subset V^*$. To see this, first decompose D uniquely as $D = \sum_{i=1}^N a_i D_{\sigma_i}$, where $\sigma_i \in \Sigma$, $i = 1, \ldots, N$ is the collection of rays. By assumption, there exists a unique minimal $\nu_i \in \sigma_i \cap \Gamma$. Then set

$$P_D = \{x \in V^* \mid \langle \nu_i, x \rangle \geq -a_i \text{ for all } i = 1, \ldots, N\}. \tag{2.2}$$

In fact, this procedure is essentially reversable. That is, given a polyhedron $P \subset V^*$, we can associate a polarized variety $(\mathcal{M}_P, D)$ such that polyhedron determined by the divisor D

on M through (2.2) is precisely P. In view of the goals of the subsequent sections, we introduce a new definition, which is simply a natural extension of the (essentially local) concept studied by Delzant [24] and many authors thereafter in the compact setting. Essentially, we allow for polyhedra which represent $\mathbb{R}$-divisors on toric varieties. Practically, this amounts to studying polyhedra P of the form 2.2 where $\nu_i \in \Gamma$ but now the coefficients a_i may vary continuously (in contrast, note that $a_i \in \mathbb{Z}$ in 2.2). More precisely we have the following. Here we state something slightly more general than what was stated in [15].

Definition 2.17. [15, Definition 7] Let $P \subset V^*$ be a (possibly irrational) polyhedron. Let Σ_P be the set of cones in V constructed as in Proposition 2.8. This may *a priori* not be a fan if P is not rational.

1. We say that P is *algebraic* if Σ_P is a fan.
2. We say that P is *Delzant* if Σ_P is a smooth fan.

It is a straightforward consequence of the definitions that the normal fan of an algebraic polyhedron P is complete if and only if P is a polytope. We then have the following corollary of Proposition 2.12:

Proposition 2.18. *Given any algebraic polyhedron P with normal fan Σ_P, we can define an associated normal and separated toric variety $\mathcal{M}_P = \mathcal{M}_{\Sigma_P}$. Moreover, we have*

1. *$\mathcal{M}_P$ is smooth as a variety if and only if P is Delzant.*
2. *$\mathcal{M}_P$ is complete as a variety (or compact, in the analytic topology) if and only if P is a polytope.*

Proposition 2.19 ([21, Theorem 7.1.10])**.** *Let P be a full-dimensional Delzant polyhedron in V^*. Then the variety $\mathcal{M}_P$ constructed above is quasiprojective.*

We saw in (2.2) how the Orbit-Cone correspondence allows us to associate a rational polyhedron P_D to any torus-invariant divisor D on a toric manifold M. Suppose now that P is a *rational* polyhedron, so that P admits a representation

$$P = \{x \in V^* \mid \langle \nu_i, x\rangle \geq -a_i\}$$

for $\nu_i \in \Gamma$, $a_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Let $\mathcal{M}_P$ be the associated toric variety. By construction, the one-dimensional cones in the normal fan Σ_P are spanned by the inner normals $\nu_1, \ldots, \nu_N \in \Gamma$ of the facets of P. By the Orbit-Cone correspondence, there is a unique torus-invariant Weil divisor D_{ν_i} on $\mathcal{M}_P$ associated to each ν_i. If we then set

$$D = \sum_{i=1}^{N} a_i D_{\nu_i}, \tag{2.3}$$

then it is clear that the polyhedron P_D of 2.2 is equal to P. In sum, we have the following.

Lemma 2.20. *Let $P \subset V^*$ be a full-dimensional rational polyhedron. Then there is a divisor D on $\mathcal{M}_P$, given explicitly by* (2.3)*, whose associated polyhedron is equal to P.*

2.2 Complex and symplectic toric manifolds

The primary purpose of this section is to extend the previous discussion to the analytic setting, ultimately with the goal of studying Kähler metrics on toric manifolds. In this direction, there are two distinct viewpoints. From the complex-geometric point of view, we study complex manifolds (M, J) of complex dimension n with *holomorphic* $(\mathbb{C}^*)^n$-actions. The alternative is to work in the symplectic setting, in which one considers real $2n$-dimensional symplectic manifolds (M, ω) together with an action of the real n-dimensional torus T^n. In this latter case one usually restricts attention to those T^n-actions which admit a *moment map*. The key feature of both is that the dimension of the torus is "as large as possible."

Of course Kähler geometry concerns both viewpoints simultaneously. As we will see, in the Kähler setting, on a compact manifold M the two perspectives are essentially equivalent, and the Delzant classification is the piece providing the precise sense in which this is true. For non-compact manifolds this will no longer hold, and the extent to which one can recover a Delzant classification will be the subject of Chapter 3.

2.2.1 Complex toric manifolds

We begin this section by defining what it means for a complex manifold (M, J) to be toric.

Definition 2.21 (Complex toric manifold)**.** A *complex toric manifold* is a complex manifold (M, J) together with an effective and holomorphic action of the complex torus $(\mathbb{C}^*)^n$ such that

1. (M, J) is of Kähler type, meaning it admits at least one Kähler metric.

2. The fixed-point set of the $(\mathbb{C}^*)^n$-action is non-empty.

Notation 2.22. Let $\mathrm{Aut}(M, J)$ denote the holomorphic automorphism group of (M, J). We will occasionally refer to the inclusion $\tau_{\mathbb{C}} : (\mathbb{C}^*)^n \hookrightarrow \mathrm{Aut}(M, J)$ explicitly, in which case we will say that $(M, J, \tau_{\mathbb{C}})$ is complex toric.

Remark 2.23. The condition 1 above is purely technical, and is not necessary for any of the results of this chapter. In fact, we suspect that it is not necessary at all. Since the goal of this dissertation is to study Kähler metrics, we do not dwell on the issue here, and in any case it will be trivially satisfied in all of our applications. Note that the second condition implies that the neither torus $(\mathbb{C}^*)^n$ itself nor any product $N \times (\mathbb{C}^*)^k$ of a lower-dimensional toric manifold satisfies Definition 2.21. In the algebraic literature, these are sometimes referred to as "without flat factors."

A fundamental property of complex toric manifolds is that they, like toric varieties, are always partial equivariant compactifications of the torus $(\mathbb{C}^*)^n$ itself. When no confusion is likely to arise, we will sometimes suppress the notation and simply write "a complex toric manifold M" to mean the complex manifold (M, J) together with the data of the $(\mathbb{C}^*)^n$-action.

Lemma 2.24. *A complex toric manifold M always admits an open and dense subset $U \subset M$ which is equivariantly biholomorphic to $(\mathbb{C}^*)^n$ with its standard action on itself.*

Proof. Choose a basis $(\Xi_1, \ldots, \Xi_n)$ for the canonical lattice $\Gamma \subset \mathfrak{t}$. By differentiating the T^n-action on M, we can view each Ξ_i as a holomorphic vector field on M. In particular, each Ξ_i vanishes along an analytic subvariety $V_i \subset M$. Set $D = \bigcup_{i=1}^n V_i$ and $U = M - D$. Then no element of $\mathfrak{t}$ (viewed again as a holomorphic vector field on M) vanishes at any point of U. Since the action is effective, the set $\{\Xi_1, J\Xi_1, \ldots, \Xi_n, J\Xi_n\}$ viewed as real vector fields form an $\mathbb{R}$-basis for T_pM for each $p \in U$. They clearly commute since $(\mathbb{C}^*)^n$ is an abelian Lie group. Moreover they are complete: the integral curves consist of real one-parameter subgroups of $(\mathbb{C}^*)^n$ and as such exist for all time. Thus, they can be integrated starting from any basepoint $p \in U$ to determine a $(\mathbb{C}^*)^n$-equivariant holomorphic injection $(\mathbb{C}^*)^n \hookrightarrow U$. Since none of the vector fields Ξ_i vanish on U, this map is surjective as soon as we know that U is connected, which is immediate since $\operatorname{codim}_{\mathbb{R}}(D) \geq 2$. □

In fact, the real codimension of D is *exactly* 2, at least generically.

Lemma 2.25. *The analytic set D is a divisor, and moreover $\mathcal{O}(D) \cong -K_M$.*

Proof. Since none of the holomorphic vector fields Ξ_i vanish at any point in U, it follows that the wedge product

$$s = \Xi_1 \wedge \cdots \wedge \Xi_n \in H^0(M, -K_M)$$

is not identically zero on M. Clearly the zero set $Z(s)$ is a divisor with the property that $O(Z(s)) \cong -K_M$. Moreover, s vanishes whenever any one Ξ_i vanishes, which means that $Z(s) = D$. □

Clearly Definition 2.1 implies that the underlying complex manifold to any algebraic toric variety is a complex toric manifold. The converse turns out not to be true in general. An example of a (non-compact) complex toric manifold which is not a toric variety can be found in [46]. We will treat this example in a systematic study of the various equivalences of definition in Chapter 3.

Example 2.26. The standard $(\mathbb{C}^*)^n$-action on $\mathbb{C}^n$ is given by

$$(\lambda_1, \ldots, \lambda_n) \cdot (z_1, \ldots, z_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n),$$

which is clearly effective and holomorphic. The orbit of $p = (1, \ldots, 1)$ is simply the inclusion of $(\mathbb{C}^*)^n \hookrightarrow \mathbb{C}^n$. We can compactify this example by adding a $\mathbb{P}^{n-1}$ at infinity. Indeed, let

$(\mathbb{C}^*)^n$ act on $\mathbb{P}^n$ by

$$(\lambda_1, \cdots, \lambda_n) \cdot [Z_0 : Z_1 : \cdots : Z_n] = [Z_0 : \lambda_1 Z_1 : \cdots : \lambda_n Z_n] .$$

Then, in the homogeneous coordinate system $z_j = \frac{Z_j}{Z_0}$, we are precisely in the situation above. In this way, we can view $\mathbb{C}^n$ as an equivariant compactification of $(\mathbb{C}^*)^n$ by adding a hyperplane whenever any of the coordinates $z_j \to 0$. Similarly, the usual picture of $\mathbb{P}^n$ as the union of a (projective) hyperplane with $\mathbb{C}^n$ is equivariant with respect to these $(\mathbb{C}^*)^n$-actions.

2.2.2 Symplectic toric manifolds

Here we encounter another geometric object which is classically referred to as a "toric manifold." We begin with a general definition concerning smooth group actions on symplectic manifolds.

Definition 2.27. Let G be a connected real Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts smoothly on a symplectic manifold (M, ω) in a way which preserves the symplectic form ω. This action is called *Hamiltonian* if it admits a *moment map*. This, by definition, is a smooth map $\mu : M \to \mathfrak{g}^*$ satisfying

$$d\langle \mu, V \rangle = -i_V \omega. \tag{2.4}$$

Here we view $V \in \mathfrak{g}$ on the right-hand side as a vector field on M by differentiating the action, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$.

In other words, the moment map is a smoothly-varying consistent choice of Hamiltonian function for each vector field on M coming from the G-action. We will sometimes refer to the smooth function $\langle \mu, v \rangle : M \to \mathbb{R}$ as a *Hamiltonian potential* for $V \in \mathfrak{g}$. Note that this all is *a priori* dependent on the symplectic form ω.

In this dissertation we will only really be concerned with the case when G is a compact abelian group of dimension equal to half the real dimension of M. This brings us to the analog of Definitions 2.1 and 2.21. As before, we let T^n always denotes the real n-dimensional torus and $\mathfrak{t}$ its Lie algebra.

Definition 2.28 (Symplectic toric manifold)**.** A *symplectic toric manifold* is a real $2n$-dimensional symplectic manifold (M, ω) together with an effective and Hamiltonian action of the real torus T^n.

Remark 2.29. There can be no action of a torus of higher dimension on M which is both effective and Hamiltonian. Indeed, these together imply that there exists a point $p \in M$ such that the derivative $d\mu_p : T_pM \to \mathfrak{t}^*$ is surjective. Since the action preserves ω, however, it is clear from (2.4) that the Lie algebra $\mathfrak{t} \subset T_pM$ is contained in $\ker(d\mu_p)$. It follows immediately that the dimension of the torus cannot exceed half the dimension of M.

Notation 2.30. Let $\mathrm{Ham}(M, \omega)$ denote the Hamiltonian symplectomorphism group of (M, ω). This is, by definition, the group of symplectomorphisms of (M, ω) whose Lie algebra consists only of Hamiltonian vector fields. Similarly to 2.22, we will occasionally refer to the inclusion $\tau : T^n \hookrightarrow \mathrm{Ham}(M, \omega)$ explicitly, in which case we will say that (M, ω, τ) is symplectic toric.

Many of the examples we have already encountered are symplectic toric.

Example 2.31. Consider $\mathbb{C}^n$ with the Euclidean symplectic form $\omega_E = \frac{i}{2}\sum_{j=1}^n dz^j \wedge d\bar{z}^j$. The real n-torus T^n acts in the usual way through $U(n)$:

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n).$$

Identifying $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}^n$, it is straightforward to check that the map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ given by

$$\mu(z_1, \ldots, z_n) = \left(|z_1|^2, \ldots, |z_n|^2\right)$$

satisfies equation (2.4).

Example 2.32. Consider $\mathbb{P}^n$ with the T^n-action given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot [Z_0 : Z_1 : \cdots : Z_n] = \left[Z_0 : e^{i\theta_1} Z_1 : \cdots : e^{i\theta_n} Z_n\right].$$

Choose homogeneous coordinates of the form $z_j = \frac{Z_j}{Z_0}$, in which setting one can write the Fubini-Study symplectic form as $\omega_{FS} = i\partial\bar{\partial}\log(1 + |z|^2)$. This action is Hamiltonian with respect to ω_{FS}, with moment map $\mu_{FS} : \mathbb{P}^n \to \mathbb{R}^n$ given by

$$\mu_{FS}\left([Z_0 : Z_1 : \cdots : Z_n]\right) = \left(\frac{|Z_1|^2}{|Z|^2}, \ldots, \frac{|Z_n|^2}{|Z|^2}\right).$$

In homogeneous coordinates this takes the form

$$\mu_{FS}(z_1, \ldots, z_n) = \left(\frac{|z_1|^2}{1 + |z|^2}, \ldots, \frac{|z_n|^2}{1 + |z|^2}\right),$$

in which setting one can check immediately that it satisfies equation (2.4).

We will also be interested in studying symplectic toric structures on complex manifolds. To this end we make the following definition.

Definition 2.33 (Compatible complex structure)**.** An integrable complex structure J on a symplectic toric manifold (M, ω) is *compatible* if the following conditions are met:

1. J preserves ω, i.e. $\omega(J-, J-) = \omega(-, -)$.
2. The symmetric tensor $g_{J,\omega}(-, -) = \omega(-, J-)$ is positive-definite (i.e. is a Riemannian metric)

3. The T^n-action leaves J invariant (i.e. it is a holomorphic action with respect to J)

In fact, examples 2.31 and 2.32 fit neatly into a general picture consisting of our key examples of Definition 2.33. Let $T^n \subset (\mathbb{C}^*)^n$ denote the real n-torus, whose inclusion in $(\mathbb{C}^*)^n$ as a real Lie subgroup is induced by the standard inclusion $S^1 \subset \mathbb{C}^*$. We let $\mathfrak{t}_\mathbb{C}$ be the Lie algebra of $(\mathbb{C}^*)^n$ and $\mathfrak{t}$ the Lie algebra of T^n, so that $\mathfrak{t}_\mathbb{C} \cong \mathfrak{t} \oplus iJ\mathfrak{t}$, where J is the standard complex structure on $(\mathbb{C}^*)^n$. If (M, J) is any complex toric manifold, then clearly the inclusion above induces an effective and holomorphic action of the *real* torus T^n on (M, J).

Definition 2.34. We say that a Kähler manifold (M, J, ω) is a *AK-toric* (or algebraic Kähler toric) if the following are satisfied:

1. (M, J) is complex toric with $(\mathbb{C}^*)^n$-action $\tau_\mathbb{C} : (\mathbb{C}^*)^n \hookrightarrow \mathrm{Aut}(M, J)$,
2. The fixed point set of $\tau_\mathbb{C}$, and thus the fixed point set of the underlying real torus action $\tau : T^n \hookrightarrow \mathrm{Aut}(M, J)$, is finite and nonempty,
3. (M, ω) is symplectic toric with respect to τ,
4. There exists an element $b \in \mathfrak{t}$ whose Hamiltonian potential $\langle \mu, b \rangle : M \to \mathbb{R}$ is proper and bounded from below.

The reason for "algebraic" in the terminology will be made clear in Chapter 3. In all such cases, we see that J is compatible with (M, ω) in the sense of Definition 2.33. Given a symplectic toric manifold (M, ω), however, there may well be compatible complex structures J which do not give rise to an AK-toric manifold. Indeed, we will see examples of this in Chapter 3. The essential problem is that (M, J) may not admit an action of the larger complex group $(\mathbb{C}^*)^n$ extending the original T^n-action (i.e. the T^n-action may not have a "complexification").

If M is compact, we will see in Chapter 3 (Theorem 3.23, Proposition 3.24) that this definition is vacuous, in the sense that any symplectic toric manifold (M, ω), complex toric manifold (M, J), or toric variety $\mathcal{M}$ can be given the structure of an AK-toric manifold. For non-compact manifolds (varieties which are not complete), we will see several examples where some but not all of these conditions are met.

Chapter 3

Kähler geometry

Perhaps the single most interesting and important feature of Kähler geometry is that seemingly purely geometric phenomena and structures often turn out to have seemingly purely complex algebraic counterparts or explanations (and indeed vice versa). The very definition of a Kähler metric induces and naturally intertwines analytic, geometric, and symplectic structures, yielding complex and interesting relationships. From the analytic perspective, one of the key consequences is *simplification*.

In studying geometry, we very often ask questions of the form "If I start with a manifold M, can I find a Riemannian metric g on M with nice properties?" One of the most basic examples of the "simplification" phenomenon goes as follows. If we restrict our attention to metrics which are Kähler with respect to a given complex structure on M, we can choose a background metric g with Kähler form ω, and only search within those metrics whose Kähler form is related to ω by $\omega' = \omega + i\partial\bar{\partial}\varphi$, with $\varphi \in C^\infty(M)$. If M is compact, any other metric with the property that $[\omega'] = [\omega]$ satisfies this property. In this way we have reduced the question of finding a Riemannian metric $g \in C^\infty(\mathrm{Sym}^2(TM))$ (an analytic question which *a priori* depends on $\frac{(n-1)(n-2)}{2}$ parameters) to a question of finding a *single* smooth function φ. This is not to say that analytic simplification is the *only* feature which sets the study of Kähler metrics apart, but it is certainly a powerful tool which helps us see the picture, often leads to a clearer and deeper understanding of the underlying structures that make things work.

The goal of this chapter is to provide a general framework for studying Kähler geometry on non-compact toric manifolds. This is an extension of the well-understood situation for compact manifolds, which goes back to the work of Atiyah, Abreu, Delzant, Guillemin, and Sternberg [4, 2, 24, 42, 41] (we will explain this in detail in the following sections, for a reference see the excellent survey [29]). The result in essence is that the extra symmetry imposed by a good torus action allows us to even further reduce many questions about Kähler metrics on M to questions about convex functions on Euclidean space. This latter subject is extremely well-understood, with a history going back at least to Archimedes. The entanglement of the complex analytic, symplectic, algebraic, and geometric structures involved will, as expected, play a central role.

Throughout this Chapter our primary interest will be AK-toric manifolds (M, J, ω) (Definition 2.34). Recall that, as a consequence of Lemma 2.24, there exists in this setting an orbit $U = (\mathbb{C}^*)^n \cdot p \subset M$ which is open and dense in M. We will begin by trying to understand more explicitly the local geometry of T^n-invariant Kähler metrics on the dense orbit $U \subset M$. With this in mind, in Section 3.1 we study T^n-invariant Kähler metrics on the torus $(\mathbb{C}^*)^n$ itself. We will ultimately treat the question of when these Kähler metrics extend to metrics on M. To this end, we study in Section 3.2 how to classify the image of the moment map associated to (M, ω, τ). The situation in the compact case is well-understood, going back to the work of Atiyah, Guillemin-Sternberg, and Delzant [4, 41, 24]. As we will see, there are many subtleties that arise in the non-compact setting, and we will see clearly through explicit examples that one cannot hope for the situation to be as neat as in the compact case. Through explicit examples and by piecing together some existing results in the literature [46, 61, 64], we show that the assumptions in Definition 2.34 are necessary and sufficient for the full Delzant classification to go through, in a sense we will make precise in Section 3.2. In Section 3.3 we bring this discussion together with the local theory and treat the extension problem of Kähler metrics on the dense orbit of an AK-toric manifold. In Section 3.4, we briefly discuss an interesting class of AK-toric manifolds, the asymptotically conical toric manifolds, and prove that they are indeed AK-toric.

3.1 Kähler metrics on toric manifolds part one: Local theory

Here we will review some of the theory of torus-invariant Kähler metrics on $(\mathbb{C}^*)^n$. Everything in this section is well-known, and we review it only because it is essential for the rest of the dissertation. The main references for this section are [29, 66]. We begin by pointing out that there is a canonical coordinate system $(z_1, \ldots, z_n)$ on $(\mathbb{C}^*)^n$, and throughout this section we will use this notation to refer to these fixed background coordinates (this in fact is the primary reason for considering actions of $(\mathbb{C}^*)^n$ with a given coordinate system as opposed to actions of a more abstract algebraic torus). If we have a complex toric manifold $(M, J, \tau_{\mathbb{C}})$, once we choose a basepoint $p \in U$ the dense orbit $U \cong (\mathbb{C}^*)^n \cdot p \subset M$ then gives rise to a local holomorphic coordinate system on M. One of the main analytic properties that we exploit is that these "local" coordinates in fact cover almost all of M. As we see later on, this makes the study of integral and other measure-theoretic quantities particularly useful on toric manfiolds. Notice also that this justifies our convention of denoting by "J" both the complex structure on M and the one on $(\mathbb{C}^*)^n$ itself. Similarly, we will not make a notational distinction between a Kähler metric ω and its restriction to the dense orbit, simply denoting $\omega|_U = \omega$. In general, the strategy will be to reduce questions about T^n-invariant Kähler geometry on M to questions about T^n-invariant Kähler geometry on $(\mathbb{C}^*)^n$.

Although our primary interest of course is the situation above where we have a Kähler metric ω on $(\mathbb{C}^*)^n$ which is induced the restriction of a T^n-invariant Kähler metric on an ambient toric manifold M, throughout the rest of this section we will treat the problem abstractly and study arbitrary T^n-invariant metrics on $(\mathbb{C}^*)^n$. In light of the picture of a complex toric manifold as an equivariant compactification of $(\mathbb{C}^*)^n$, one can ask when this process can be reversed, i.e. when a Kähler metric on $(\mathbb{C}^*)^n$ induces such a metric on a given ambient toric manifold. As far as the author is aware, this question was first studied by Abreu [2]. We will return to this in detail in the later sections of this chapter.

3.1.1 Logarithmic coordinates

We begin by fixing some notation that we will use throughout the rest of the dissertation.

Notation 3.1. Fix a basis $\beta = (\Xi_1, \ldots, \Xi_n)$ for $\Gamma \subset \mathfrak{t}$ which determines a natural bi-invariant metric g_β on T^n with respect to which it is orthonormal. This, in particular, induces a real coordinate system $(\xi_1, \ldots, \xi_n)$ on $\mathfrak{t}$. The metric g_β induces an identification $\mathfrak{t} \cong \mathfrak{t}^*$, and thereby a (real) coordinate system $(x_1, \ldots, x_n)$ on $\mathfrak{t}^*$. Moreover, with these conventions, the dual pairing $\langle \cdot, \cdot \rangle : \mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ is naturally identified with the Euclidean dot product on $\mathbb{R}^n$.

As a smooth manifold, the algebraic torus $(\mathbb{C}^*)^n$ can be identified with the tangent bundle of the real torus T^n. In fact, the perhaps most natural identification turns out to translate much of the T^n-invariant complex geometry on $(\mathbb{C}^*)^n$ into *convex* geometry on $\mathfrak{t}$. To see this let $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathfrak{t} \times T^n$ denote that map

$$\mathrm{Log}(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) = (r_1, \ldots, r_n, \theta_1, \ldots, \theta_n). \tag{3.1}$$

The next lemma is immediate, but we highlight it anyway because it is essential to our approach that follows.

Lemma 3.2. *The map* (3.1) *induces a one-to-one correspondence between T^n-invariant continuous functions on $(\mathbb{C}^*)^n$ and continuous functions on $\mathfrak{t}$.*

If we let $C^0_T((\mathbb{C}^*)^n)$ be those real-valued continuous functions on $(\mathbb{C}^*)^n$ which are T^n-invariant, then the above Lemma says that there is a natural identification of $C^0_T((\mathbb{C}^*)^n)$ with $C^0(\mathfrak{t})$. The key to the relationship with complex analysis is the following simple observation. For simplicity of notation we state the result for smooth functions only, but it will be clear from the proof that there are corresponding statements with less regularity.

Lemma 3.3. *Under the correspondence given by Lemma 3.2, the operators*

$$2\bar{z}_j \frac{\partial}{\partial \bar{z}_j} : C^\infty_T((\mathbb{C}^*)^n) \to C^\infty_T((\mathbb{C}^*)^n)$$

and

$$\frac{\partial}{\partial \xi_j} : C^\infty(\mathfrak{t}) \to C^\infty(\mathfrak{t})$$

are equal for any $j = 1, \ldots, n$. In particular, any T^n-invariant holomorphic function on $(\mathbb{C}^*)^n$ is constant.

Proof. Fix any branch of the *holomorphic* logarithm function and let $w_j = \log(z_j)$ be the coordinates on $(\mathbb{C}^*)^n$ defined away from the corresponding branch cuts. On this branch cut, notice that the *real* coordinates induced by (3.1) (ξ, θ) are well-defined and that

$$w_j = \xi_j + i\theta_j. \tag{3.2}$$

Let $h \in C^\infty_T((\mathbb{C}^*)^n)$ be smooth and T^n-invariant. Clearly, this implies that h written in the real coordinates (ξ, θ) is independent of θ. Writing now h in the local holomorphic coordinates w_j, we can see that

$$\bar{z}_j \frac{\partial h}{\partial \bar{z}_j}(z) = \frac{\partial h}{\partial \bar{w}_j}(w).$$

Finally, using that $\frac{\partial}{\partial \bar{w}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \xi^j} + i\frac{\partial}{\partial \theta^j}\right)$ from (3.2), we see that

$$\frac{\partial h}{\partial \bar{w}_j}(w) = \frac{1}{2}\frac{\partial h}{\partial \xi_j}(\xi, \theta) = \frac{1}{2}\frac{\partial h}{\partial \xi_j}(\xi).$$

Since this description has no dependence on θ, it follows that it is in fact independent of the branch of $w = \log(z)$ that we chose, which completes the proof of the first statement. Now suppose that $h \in C^\infty_{T,\mathbb{C}}((\mathbb{C}^*)^n)$ is a T^n-invariant *complex-valued* function. Write $h = h(w)$ in the local coordinates 3.2 and then decompose $h(w) = h_1(w) + ih_2(w)$, where $h_i \in C^\infty_T((\mathbb{C}^*)^n)$ are smooth T^n-invariant real-valued functions. If h is holomorphic, then for any j we have that $\frac{\partial h}{\partial \bar{w}_j} = \frac{\partial h_1}{\partial \bar{w}_j} + i\frac{\partial h_2}{\partial \bar{w}_j} = 0$. By Lemma 3.2, the functions h_k are induced from smooth real-valued functions on $\mathfrak{t}$, and we continue to abuse notation and refer to these functions as h_k. Therefore

$$\frac{\partial h_k}{\partial \bar{w}_j}(w) = \frac{\partial h_k}{\partial \xi^j}(\xi)$$

is real-valued, and hence it follows that $\frac{\partial h_k}{\partial \xi^j} = 0$ for any $k = 1, 2$, $j = 1, \ldots, n$, so that h is constant. □

Remark 3.4. As a consequence, we see that the restriction of the $\bar{\partial}$-operator to the real-valued functions

$$\bar{\partial} : C^\infty_T((\mathbb{C}^*)^n) \to \Lambda^{1,0}_T((\mathbb{C}^*)^n)$$

is identified with the usual exterior derivative

$$d : C^\infty(\mathfrak{t}) \to \Lambda^1(\mathfrak{t}).$$

Remark 3.5. The local holomorphic coordinates induced by $w = \log(z)$ (3.2) introduced in the proof above are a useful computational tool that we will return to in later sections. The proof exhibits a general principle, that T^n-invariant complex data can be understood "invariantly" in the coordinates (3.2), even though they themselves depend *a priori* on a choice of branch of the logarithm.

Another important elementary quality of this coordinate system is the following.

Lemma 3.6. *Let ψ be any T^n-invariant C^2 function on $(\mathbb{C}^*)^n$ with the property that*

$$i\partial\bar{\partial}\psi = 0.$$

Then, as a function on $\mathfrak{t}$, ψ is affine-linear.

Proof. Choose a branch of log, and then write in the coordinates (3.2):

$$0 = i\partial\bar{\partial}\psi = i\frac{\partial^2\psi}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j = \frac{1}{2}\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j} d\xi^i \wedge d\theta^j.$$

Since second partial derivatives of $\psi : \mathfrak{t} \to \mathbb{R}$ vanish, it follows that ψ is affine-linear. □

3.1.2 Invariant metrics

As we have seen, the introduction of logarithmic coordinates allows us to understand properties functions on $(\mathbb{C}^*)^n$ in terms of properties of functions on $\mathbb{R}^n$. The next step is to exploit this in the study of Kähler metrics. The starting point for this discussion is an observation due to Gullemin [42]:

Proposition 3.7 ([42, Theorem 4.1])**.** *Let ω be any T^n-invariant Kähler form on the algebraic torus $(\mathbb{C}^*)^n$. Then the action is Hamiltonian with respect to ω if and only if there exists a T^n-invariant potential ϕ such that $\omega = 2i\partial\bar{\partial}\phi$.*

Suppose then that we have some Kähler metric ω on $(\mathbb{C}^*)^n$ with respect to which the natural T^n-action is Hamiltonian. Proposition 3.7 furnishes for us a Kähler potential $\omega = 2i\partial\bar{\partial}\phi$ which we see immediately must be T^n-invariant. We can therefore write ω in the local holomorphic coordinate system (3.2):

$$\omega = 2i\frac{\partial^2\phi}{\partial w_j \partial \bar{w}_k} dw_j \wedge d\bar{w}_k.$$

Just as in the proof of Lemma 3.3, since the T^n-action in these coordinates is given simply by translation in the imaginary direction, we see that $\phi(w)$ is independent of θ. Abusing notation as above, we write $\phi = \phi(\xi)$ for the smooth function on $\mathfrak{t}$ determined through Lemma 3.2. Thus, the complex hessian of ϕ is just a multiple of its real hessian:

$$\frac{\partial^2\phi}{\partial w_j \partial \bar{w}_k} = \frac{1}{4}\frac{\partial^2\phi}{\partial\xi^j\partial\xi^k}.$$

Expressing dw_j and $d\bar{w}_k$ in terms of ξ and θ via (3.2), we see all in all that we have:

$$\omega = \frac{i}{2}\frac{\partial^2\phi}{\partial\xi^j\partial\xi^k} dw_j \wedge d\bar{w}_k = \frac{\partial^2\phi}{\partial\xi^j\partial\xi^k} d\xi^j \wedge d\theta^k. \tag{3.3}$$

Since the coefficients of ω define a metric on $(\mathbb{C}^*)^n$ by assumption, we have by the middle equality that if we view ϕ as a smooth function on $\mathfrak{t}$, its (real, Euclidean) hessian is strictly positive definite! In other words, we have

Lemma 3.8. *The Kähler potential ϕ for a T^n-invariant Kähler metric ω on $(\mathbb{C}^*)^n$ is determined by strictly convex function on $\mathfrak{t}$, which we also denote by ϕ.*

The associated Riemannian metric $g(-,-) = \omega(-, J-)$ can also be understood explicitly in these terms. That $(w_j = \xi^j + i\theta^j)$ are holomorphically compatible with the standard complex structure J on $(\mathbb{C}^*)^n$ is equivalent to the fact that $J\frac{\partial}{\partial \xi^j} = \frac{\partial}{\partial \theta^j}$ and $J\frac{\partial}{\partial \theta^j} = -\frac{\partial}{\partial \xi^j}$ for all $j = 1, \ldots, n$. From this we can readily calculate:

$$g = \frac{\partial^2 \phi}{\partial \xi^j \partial \xi^k} d\xi^j \otimes d\xi^k + \frac{\partial^2 \phi}{\partial \xi^j \partial \xi^k} d\theta^j \otimes d\theta^k. \tag{3.4}$$

As a final remark, we set the stage for the later sections of this chapter. Once we know that the Kähler form is $\partial\bar{\partial}$-exact, we have an explicit representation for the moment map $\mu : (\mathbb{C}^*)^n \to \mathfrak{t}^*$. Indeed, we define μ directly in logarithmic coordinates:

$$\mu(\xi, \theta) = \nabla \phi(\xi), \tag{3.5}$$

where $\nabla\phi$ denotes the Euclidean gradient

$$\nabla \phi : \mathfrak{t} \to \mathfrak{t}^* \tag{3.6}$$

of ϕ, computed in terms of our fixed background coordinates $(\xi^1, \ldots, \xi^n)$ on $\mathfrak{t}$. In terms of the dual pairing, this says that for any $b \in \mathfrak{t}$, the moment map μ is defined by the relation

$$\langle \mu(\xi, \theta), b \rangle = \langle \nabla \phi(\xi), b \rangle.$$

This can be checked directly. Fix $b = (b^1, \ldots, b^n) \in \mathfrak{t}$, so that the corresponding holomorphic vector field X_b on $(\mathbb{C}^*)^n$ is given by

$$X_b = \sum_{j=1}^n b^j \frac{\partial}{\partial \theta^j}.$$

Then we have

$$-i_{X_b}\omega = b^j \frac{\partial^2 \phi}{\partial \xi^j \partial \xi^k} d\xi^k = d\left(b^j \frac{\partial \phi}{\partial \xi^j} \right) = d\langle \mu, b \rangle.$$

Observe that, by the strict convexity of $\phi : \mathfrak{t} \to \mathbb{R}$, the gradient $\nabla\phi : \mathfrak{t} \to \mathfrak{t}^*$ is a diffeomorphism onto its image $\Omega = \nabla\phi(\mathfrak{t}) \subset \mathfrak{t}^*$. In this way the geometry begins to emerge explicitly. The moment map $\mu = \nabla\phi$ writes the torus $(\mathbb{C}^*)^n$ as a specific choice of fibration by real tori:

$$\begin{array}{ccc} T^n & \hookrightarrow & (\mathbb{C}^*)^n \\ & & \big\downarrow \nabla\phi \\ & & \Omega \end{array}$$

The metric on each fiber $\mu^{-1}(\xi^*)$ is the flat metric with constant coefficients given by the hessian $\mathrm{Hess}_{\xi^*}\phi$. Moreover, if we denote $\phi_{ij}(\xi) = \mathrm{Hess}_{\xi}\phi = \frac{\partial^2\phi}{\partial\xi^i\partial\xi^j}(\xi)$ and then equip the Lie algebra $\mathfrak{t}$ with the Riemannian metric

$$g_{\mathfrak{t}} = \phi_{ij}(\xi)d\xi^i \otimes d\xi^j, \tag{3.7}$$

then for any $\theta^* \in T^n$ the natural inclusion $\mathfrak{t} \hookrightarrow (\mathbb{C}^*)^n \cong \mathfrak{t} \times T^n$ is an isometric immersion onto the totally geodesic submanifold $\mathfrak{t} \times \{\theta^*\}$.

The next step is to classify the moment image $\Omega \subset \mathfrak{t}^*$, which will ultimately bring us back to our discussion of how to understand when the local picture we have just discussed compactifies with respect to a given toric manifold M with $(\mathbb{C}^*)^n \subset M$.

3.2 The Delzant classification

The main goal of this section is to classify the image of the moment map μ associated to a symplectic toric manifold (M, ω, τ).

3.2.1 GIT construction of toric manifolds

In Section 2.1, we discussed an algebraic procedure which produced a toric *variety* $\mathcal{M}_P$ from the combinatorial data of an algebraic polyhedron P. We present here a different construction which produces an AK-toric manifold (M_P, J_P, ω_P) (recall from Definition 2.34 that this means there is a $(\mathbb{C}^*)^n$-action $\tau_{\mathbb{C}}$ such that $(M_P, J_P, \tau_{\mathbb{C}})$ is complex toric and (M_P, ω_P, τ) is symplectic toric). The construction is an example of the general procedure known as *Kähler reduction* [47], which exploits the compatibility of the complex analog of Mumford's geometric invariant theory [57] with the Marsten-Weinstein symplectic quotient [54]. The advantage to this perspective is that it simultaneously addresses the symplectic and complex perspectives. For toric manifolds the picture in its most general form is due to Burns-Guillemin-Lerman [11], but goes back to the work of Delzant, Kirwan and others [5, 3, 24, 47]. We will address the question of the precise sense and setting in which this construction coincides with the algebraic one in the later sections of this chapter.

As always $\mathfrak{t}$ will denote the Lie algebra of the real n-dimensional torus T^n, and we fix an integral lattice $\Gamma \cong \mathbb{Z}^n \subset \mathfrak{t}$. In this way, we work in the context of Section 2.1 with our real vector space $V = \mathfrak{t}$. Let $P \subset \mathfrak{t}^*$ be an algebraic polyhedron (see Definition 2.17). The algebraic condition guarantees the existence of elements $\nu_1, \ldots, \nu_N \in \Gamma$ and numbers $a_1, \ldots, a_N \in \mathbb{R}$ such that

$$P = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x \rangle \geq -a_i\} \tag{3.8}$$

for all $i = 1, \dots, N$. We will always assume that the collection $\{\nu_i\}$ is minimal, in the sense that the deletion of one element of the collection would necessarily change the set P.

Definition 3.9. Let P be an algebraic polyhedron as above with a minimal collection of $\nu_1, \dots, \nu_N \in \Gamma$ such that (3.8) holds. We say that ν_i are the *inner normals* of P if further each ν_i is the minimal generator in Γ of the ray $\mathbb{R} \cdot \nu_i \subset \mathfrak{t}$.

We define a map $\Psi : \mathbb{Z}^N \to \Gamma$ by

$$\Psi(y_1, \dots, y_N) = \sum_{i=1}^{N} y_i \nu_i,$$

which is necessarily surjective as a consequence of Definition 2.3. There is a natural $\mathbb{R}$-linear extension $\Psi : \mathbb{R}^N \to \mathfrak{t}$ which is also surjective, and thereby induces the short exact sequences

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^N \xrightarrow{\Psi} \mathfrak{t} \to 0 \tag{3.9}$$

and

$$0 \to K \xrightarrow{\iota} T^N \xrightarrow{\Psi} T^n \to 0 \tag{3.10}$$

Clearly, we can view $K \subset T^N$ as a subtorus with Lie algebra $\mathfrak{k}$. Moreover, we can complexify the picture and obtain

$$0 \to \mathfrak{k}_{\mathbb{C}} \xrightarrow{\iota} \mathbb{C}^N \xrightarrow{\Psi} \mathfrak{t}_{\mathbb{C}} \to 0 \tag{3.11}$$

and

$$0 \to K_{\mathbb{C}} \xrightarrow{\iota} (\mathbb{C}^*)^N \xrightarrow{\Psi} (\mathbb{C}^*)^n \to 0 \tag{3.12}$$

Heuristically speaking, the complex manifold (M_P, J_P) is obtained by removing a union V of some linear subspaces in $\mathbb{C}^N$ and taking the quotient $\mathbb{C}^N / K_{\mathbb{C}}$. If P is *Delzant*, then this is literally true, the essential reason being that the $K_{\mathbb{C}}$-action is free on $\mathbb{C}^N - V$. In general, the space M_P will need to be singular, and the picture becomes more complicated. Since the primary interest of this dissertation is the smooth case we will restrict ourselves to the case where P is Delzant. With the appropriate definitions the same procedure can be made to work in the analytic category, and we will describe briefly what these changes are below. The situation is particularly well-understood in the algebraic category, see [21, Chapter 14] for an overview.

We assume then for the moment that P is Delzant. In this case, the set V is described as follows. By definition, set of cones Σ_P in $\mathfrak{t}$ constructed by 2.8 is a fan. For each cone $\sigma \in \Sigma_P$, set

$$V_\sigma = \{y \in \mathbb{C}^N \mid y_i = 0 \text{ if } \nu_i \notin \Sigma\},$$

and then

$$V = \bigcup_{\sigma \in \Sigma_P} V_\sigma.$$

Notice that if $\sigma' \subset \sigma$, that $V_{\sigma'} \subset V_\sigma$, which implies the union above may as well be taken over only the maximal cones in Σ_P. In any case, we have the following. For a detailed proof, see [21, Theorem 5.1.11].

Proposition 3.10 ([5, 21]). *The $K_{\mathbb{C}}$-action on $\mathbb{C}_N - V$ is free, and therefore there is a well-defined quotient $(M_P, J_P) = (C_N - V)/K_{\mathbb{C}}$. The quotient $(\mathbb{C}^*)^N/K_{\mathbb{C}}$, which is naturally identified with $(\mathbb{C}^*)^n$ through Ψ, acts effectively and holomorphically on (M_P, J_P).*

We will sometimes denote $(M_P, J_P) = C^N /\!\!/ K_{\mathbb{C}}$. Note that the set V, and thus the complex structure J_P, depends only on the normal fan Σ_P of P. This is highly suggestive of the fact that (M_P, J_P) should be related to the toric variety $\mathcal{M}_{\Sigma_P}$ constructed in Section 2.1. This is true; in fact the GIT description for toric varieties is due originally in the algebraic setting to Cox [20].

Proposition 3.11. *The toric manifold (M_P, J_P) is equivariantly biholomorphic to the underlying complex manifold of $\mathcal{M}_{\Sigma_P}$, where Σ_P is the normal fan of P.*

Proof. Once we know that we can find a *rational* polyhedron P' such that $\Sigma_{P'} = \Sigma_P$, this will be a straightforward consequence of the fact that toric *varieties* are all isomorphic to GIT quotients of a large enough affine space. The latter result is well-known, see [20, 21]. The existence of such a P' is also straightforward. Given a Delzant polyhedron (or indeed any algebraic polyhedron), we can find polyhedra P_ε arbitrarily close to P but whose coefficients a_i as in (3.8) are rational. By the inherent discreteness of the normal fan, we see, at least when ε is small enough, that the normal fan $\Sigma_{P_\varepsilon} = \Sigma_P$. So make some such choice of P_ε with $a_i \in \mathbb{Q}$. We then clear denominators: if $k \in \mathbb{Z}$ is such that $ka_i \in \mathbb{Z}$ for all $i = 1, \ldots, N$, then the polyhedron $P' = kP_\varepsilon$ is rational and satisfies $\Sigma_{P'} = \Sigma_{P_\varepsilon} = \Sigma_P$. $\square$

To obtain the corresponding symplectic form ω_P we first dualize (3.9):

$$0 \to \mathfrak{t} \xrightarrow{\Psi^*} (\mathbb{R}^N)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0. \tag{3.13}$$

As we saw in Example 2.31, the action of T^N on C^N is Hamiltonian with respect to the Euclidean symplectic form ω_E, with moment map $\mu_E : \mathbb{C}^N \to (\mathbb{R}^N)^*$ given by

$$\mu_E(z_1, \ldots, z_N) = \sum_{i=1}^N |z_i|^2 e_i^*,$$

where $e_i^* \in (\mathbb{Z}^N)^*$ are the dual basis associated to the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$. The subgroup $K \subset T^N$ of course then also acts on $\mathbb{C}^N$, and it is not hard to see that this action admits a moment map $\mu_K : \mathbb{C}^N \to \mathfrak{k}^*$ with respect to ω_E by

$$\mu_K = \iota^* \circ \mu_E.$$

Let $a_i \in \mathbb{R}$ be as in (3.8), and define an element $\lambda_P \in (\mathbb{R}^N)^*$ by

$$\lambda_P = \sum_{i=1}^{N} a_i e_i^*. \tag{3.14}$$

Theorem 3.12 ([5, 11, 24, 47]). *The K-action on $\mu_K^{-1}(\lambda_P)$ is free, thus the quotient $M_P = \mu_K^{-1}(\lambda_P)/K$ is a smooth manifold. The restriction of the Euclidean symplectic form ω_E to $\mu_K^{-1}(\lambda_P)$ determines a unique symplectic form ω_P on M_P. As in the complex case the quotient $T^n \cong T^N/K$ acts effectively on M_P, preserving ω_P. This action is Hamiltonian, and the moment map $\mu_P : M_P \to (\mathbb{R}^N)^*/\mathfrak{k}^* \cong \mathfrak{t}^*$ is determined by the restriction of ω_E to $\mu_K^{-1}(\lambda_P)$, and the image $\mu_P(M_P) \subset \mathfrak{t}^*$ is precisely P. Moreover, the natural map $(\mathbb{C}^N - V)/K_{\mathbb{C}} \to \mu^{-1}(\lambda_P)/K$ indentifying orbits is a T^n-equivariant diffeomorphism. Under this identification, the symplectic form ω_P inherits the property of being Kähler with respect to J_P from the original Kähler triple $(\mathbb{C}^N, J_E, \omega_E)$.*

Similarly to the complex quotient construction, we will occasionally use the notation $(M_P, \omega_P) = \mathbb{C}^N /\!\!/_{\lambda_P} K$ for the symplectic quotient construction above. Putting together Theorem 3.12 with Proposition 3.10, we obtain from the data of a Delzant polyhedron a Kähler manifold (M_P, J_P, ω_P) together with a $(\mathbb{C}^*)^n$-action $\tau_{\mathbb{C}}$ with the property that:

1. $\tau_{\mathbb{C}}$ is J_P-holomorphic,

2. The underlying real torus action τ is Hamiltonian with respect to ω_P,

3. The moment map $\mu_P : M_P \to \mathfrak{t}^*$ is has image equal precisely to P.

Remark 3.13. One of the key points in [11] is that this situation can be made to hold in an appropriate sense when P is a more general algebraic polyhedron. As we already mentioned, the manifold M_P will necessarily be singular. The result is that the same description as above holds on the *interior* $\mathring{F}$ of each k-dimensional face F of P for $k = 0, \ldots, n$ (the $k = n$ case being the dense orbit, which is fibered over the interior of P itself). In this way M_P is stratified by smooth Kähler spaces $(M_{\mathring{F}}, J_{\mathring{F}}, \omega_{\mathring{F}})$, which each have the properties described above.

Suppose now further that P is *rational*. This forces the coefficients a_i of (3.8) to be integers, which in turn means that the element λ_P of (3.14) lies in the dual lattice $(\mathbb{Z}^N)^* \subset (\mathbb{R}^N)^*$. As a consequence, there is a well-defined *character* $\chi^{\lambda_P} : (\mathbb{C}^*)^N \to \mathbb{C}^*$,

$$\chi^{\lambda_P}(\lambda_1, \ldots, \lambda_N) = \lambda_1^{a_1} \ldots \lambda_N^{a_N}. \tag{3.15}$$

In particular, we can define an action of $(\mathbb{C}^*)^N$ on the total space of the trivial bundle $\mathcal{O}_{\mathbb{C}^N} \cong \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ by

$$\lambda \cdot (z_1, \ldots, z_N, \zeta) = (\lambda_1 z_1, \ldots, \lambda_N z_N, \chi^{\lambda_P}(\lambda)\zeta), \tag{3.16}$$

for $\lambda = (\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N$. In particular, there is a free action of $K_{\mathbb{C}}$ on the total space of $\mathcal{O}_{\mathbb{C}^N - V} \cong (\mathbb{C}^N - V) \times \mathbb{C}$, and therefore a well-defined quotient line bundle $\pi : L_P = \mathcal{O}_{\mathbb{C}^N - V}/K_{\mathbb{C}} \to M_P$. Moreover, we have:

Proposition 3.14 ([21, Theorem 14.2.13]). *Using Proposition 3.11 to identify $\mathcal{M}_P$ and (M_P, J_P), the line bundle L_P is isomorphic to $\mathcal{O}_{M_P}(D)$, where D is the divisor* (2.3) *on $\mathcal{M}_P$ associated to P.*

We omit the proof here, but it is essentially a consequence of the (algebraic) GIT construction of $\mathcal{M}_P$ mentioned above. The result is that $\mathcal{M}_P$ is isomorphic to $\mathrm{Proj}(R_{\chi^{\lambda_P}})$, where $R_{\chi^{\lambda_P}}$ is the ring of $K_{\mathbb{C}}$-invariant sections $\mathcal{O}_{\mathbb{C}^N}$ under the action (3.16). From this perspective, it is not hard to see that the divisor aH_i given by the vanishing of z_i^a on $\mathbb{C}^N$ passes via the quotient to the divisor aD_{ν_i} (recall that $\Phi : \mathbb{Z}^N \to \Gamma$ is defined by sending the standard basis $\{e_1, \ldots, e_N\}$ to the inner normals $\{\nu_1, \ldots, \nu_N\}$). Finally, observe that we have a natural $K_{\mathbb{C}}$-invariant section s_P of $\mathcal{O}_{C^N}$:

$$s_P(z_1, \ldots, z_N) = (z_1, \ldots, z_N, z_1^{a_1} \ldots z_N^{a_N}). \tag{3.17}$$

Clearly, the zero divisor of s_P is $Z(s_P) = \sum a_i H_i$. Moreover, s_P descends by construction to a section of L_P, whose zero divisor is then $\sum a_i D_{\nu_i}$.

The symplectic perspective, as it turns out, also fits in neatly with the picture.

Corollary 3.15. *Using Theorem 3.18 to view (M_P, J_P, ω_P) as a Kähler manifold, we have that $\omega_P \in 2\pi c_1(L_P)$.*

This is a consequence of Guillemin's explicit formula for ω_P (see [11] for the non-compact case), which we will see in more detail in Section 3.3.

3.2.2 The compact setting

We next begin to address the question of the relationship between the three definitions (2.1, 2.21, 2.28) of a toric space that we have mentioned here. Although it seems tempting to phrase these relationships in categorical language, we will avoid doing so here. Essentially, the a careful study of the morphisms in each category is sufficiently cumbersome that it detracts from our main purpose, the study of Kähler metrics on an individual toric manifold. We will nevertheless attempt to state the correspondences as clearly as possible. The story begins with the pioneering work of Atiyah [4] and Guillemin-Sternberg [41]:

Theorem 3.16 ([4, 41]). *Let (M, ω) be a compact symplectic toric manifold with moment map $\mu : M \to \mathfrak{t}^*$. Then the image of the moment map $\mu(M) \subset \mathfrak{t}^*$ is a polytope P, equal to the convex hull of the image under μ of the fixed point set of T^n on M.*

The connection with our general picture is due to Delzant:

Theorem 3.17 ([24]). *Let (M, ω) be a symplectic toric manifold, which by the previous theorem we know has the property that its moment image $\mu(M) \subset \mathfrak{t}^*$ is a polytope P. Then there exists a T^n-equivariant symplectomorphism $G : (M, \omega) \to (M_P, \omega_P)$, unique up to symplectic automorphisms of (M, ω).*

We note some immediate consequences of this result. First of all, this immediately imples that the image P of any moment map μ with respect to ω is *unique up to translations in* $\mathfrak{t}^*$. Moreover, by Theorem 3.12, we can use G to pull back the complex structure of (M_P, ω_P, J_P) and obtain a compatible complex structure $J = G^* J_P$ on (M, ω) with respect to which ω is Kähler. But in fact we have more: we can also use G to pull back the $(\mathbb{C}^*)^n$-action on (M_P, J_P) to obtain a J-holomorphic $(\mathbb{C}^*)^n$-action $\tau_{\mathbb{C}}$ on (M, J). Since G is equivariant with respect to the T^n-actions, the underlying real torus action τ of $\tau_{\mathbb{C}}$ coincides with the original T^n-action on (M, ω). In other words, we can associate to (M, ω, τ) a complex toric manifold $(M, J, \tau_{\mathbb{C}})$. More precisely, we have:

Corollary 3.18. *Given any compact symplectic toric manifold (M, ω), there exists a compatible complex structure J making (M, J, ω) into an AK-toric manifold (Definition 2.34). Moreover, if J' is any other such complex structure on M, then there exists a T^n-equivariant symplectic automorphism g of (M, ω) such that $J' = g^* J$.*

In fact we have a converse as well:

Lemma 3.19. *Let $(M, J, \tau_{\mathbb{C}})$ be a complex toric manifold (not necessarily compact). Then there exists a symplectic toric structure with respect to which the real torus action τ underlying $\tau_{\mathbb{C}}$ is Hamiltonian.*

Proof. Choose a Kähler metric ω on (M, J) which, by averaging, we can assume to be invariant under the T^n-action τ. Since the fixed point set of τ is finite, a result of Frankel [35, Lemma 2] says that the T^n-action is Hamiltonian with respect to ω. □

Using Lemma 3.19 together with Delzant's theorem, we obtain a complex analog of Corollary 3.18 when M is compact. The proof we give here comes from [15], and is based on an idea of Abreu [2]. Our main tool is a structure theorem for Hamiltonian group actions which admit holomorphic complexifications due to Sjamaar [64]:

Theorem 3.20 (Holomorphic Slice Theorem [64, Theorem 1.12, Theorem 1.23]). *Let M be a complex manifold and $G^{\mathbb{C}}$ be a connected complex reductive Lie group which acts holomorphically on M. A* slice *at $x \in M$ for the $G^{\mathbb{C}}$-action is an analytic subvariety $S \subset M$ such that*

1. *$x \in S$,*
2. *the set $G^{\mathbb{C}} S$ of all orbits intersecting S (the "saturation") is open in M,*

3. *the stabilizer* $(G^{\mathbb{C}})_x$ *preserves* S*,*

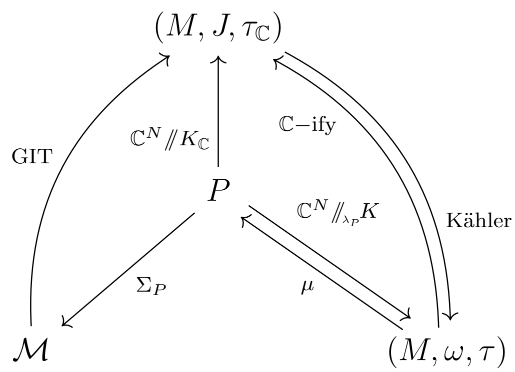
4. *the map* $H_x^{-1} : S \times_{(G^{\mathbb{C}})_x} G^{\mathbb{C}} \to G^{\mathbb{C}} S$ *defined on equivalence classes by* $H_x^{-1}(p, g) = g \cdot p$ *is a biholomorphism.*

Suppose there exists a Kähler metric ω *on* M *with respect to which the action of the compact real form* $G \subset G^{\mathbb{C}}$ *is Hamiltonian. Then there exists slices through any point* $x \in M$ *such that the* G*-orbit is* ω*-isotropic. If* G *is a real torus (and thus* $G^{\mathbb{C}} \cong (\mathbb{C}^*)^n$*), then all orbits are isotropic, and consequently there exist slices through all points of* M*.*

Proposition 3.21 ([15, Lemma 2.14],[2, Proposition A.1])**.** *Every compact complex toric manifold* (M, J) *is equivariantly isomorphic to the underlying complex manifold of some toric variety* $\mathcal{M}$*. More specifically, there exits a* $(\mathbb{C}^*)^n$*-equivariant biholomorphism* $H : M \to M_P$*, where* P *is the image of the moment map of any* T^n*-invariant compatible Kähler metric.*

Proof. Let ω be a T^n-invariant compatible Kähler metric on (M, J) with respect to which the action is Hamiltonian, as guaranteed by Lemma 3.19. By Theorem 3.17, we know that the image of the corresponding moment map $\mu : M \to \mathfrak{t}^*$ is a polytope P. For each k-dimensional face F_i of P, choose a point p_i in the interior. By [44, Theorem 4.1, part (v)] (c.f. [41, 60]), each point $q \in \mu^{-1}(F_i)$ is stabilized by a common torus $T^{n-k}_{F_i} \subset T^n$ with Lie algebra $\mathfrak{t}_i$, and moreover F_i lies as an open subset of the dual k-plane $\mathfrak{t}^{\perp}_{F_i} \subset \mathfrak{t}^*$. By the Holomorphic Slice Theorem 3.20, there exists a $(\mathbb{C}^*)^n$-invariant open neighborhood $U_i \subset M$ of the orbit $(\mathbb{C}^*)^n \cdot p_i \subset M$ and an equivariant biholomorphism $H_i : U_i \to (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$, with the standard $(\mathbb{C}^*)^n$-action such that $H_i(p_i) = (1, 0)$ and $H_i(\mu^{-1}(F_i) \cap U_i) = (\mathbb{C}^*)^k \times \{0\}$. We see that the stabilizer $T^{n-k}_{F_i}$ acts in the coordinates induced by H_i by the standard action on $\mathbb{C}^{n-k}$. In this way, we produce an equivariant holomorphic coordinate covering of M by running through each p_i. Suppose now that F_1, F_2 are two k-dimensional faces that which lie on the boundary of a higher-dimensional face E of P, and let $H_{F_1} : U_{F_1} \to (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}, H_{F_2} : U_{F_2} \to (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}, H_E : U_E \to (\mathbb{C}^*)^l \times \mathbb{C}^{n-l}$ denote the corresponding maps as above. By equivariance, the transition map $H_{F_2} \circ H_{F_1}^{-1}$ is uniquely determined by the inclusions of $(\mathbb{C}^*)^l \times \mathbb{C}^{n-l} \subset (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$ given by H_E. These in turn are determined uniquely by the inclusions of the stabilizer algebra $\mathfrak{t}_E \subset \mathfrak{t}_{F_1}, \mathfrak{t}_{F_2}$. As we have seen, the stabilizer algebras $\mathfrak{t}_E, \mathfrak{t}_{F_1}, \mathfrak{t}_{F_2}$ comprise the normal directions to the faces E, F_1, F_2 in $\mathfrak{t}^*$, respectively. In particular, the transition data of this covering is determined uniquely by the normal fan Σ_P of P. Now let $(W_i, \tilde{H}_i)$ be a cover of M_P constructed in the same way. For each face F_i of P, we have maps $\tilde{H}_i^{-1} \circ H_i : U_i \to W_i$. Since the transition data for each covering is uniquely determined by Σ_P, we see that these local maps patch together to form a well-defined biholomorphism $M \to M_P$. □

Loosely speaking, we can summarize the correspondences so far by the following diagram:



Our next goal is to understand how the correspondences above behave with respect to polarizations. Polarizations appear often in algebraic/Kähler geometry, and usually refer to a piece of cohomological or intersection-theoretic data that we would like to fix before studying a given problem. We will need the following fact from algebraic (toric) geometry [21, Proposition 2.4.2, Proposition 6.1.10, Theorem 6.1.15]:

Lemma 3.22. *Let $\mathcal{M}$ be a smooth complete toric variety with fan Σ in $\mathfrak{t}$. A torus-invariant divisor D on $\mathcal{M}$ is ample (in fact, very ample) if and only if $P_D \subset \mathfrak{t}^*$ is full-dimensional and $\Sigma_{P_D} = \Sigma$.*

Putting this together with the previous results this section and of Section 2.1, we see that we have in fact encountered two examples of this:

1. A polarization $(\mathcal{M}, D)$ on a toric variety $\mathcal{M}$ is a choice of ample torus-invariant Cartier divisor D. We have already seen that associated to any polarization D on a smooth toric variety $\mathcal{M}$ there is a corresponding polytope P_D (2.2). Moreover, we can associate a *polarized* toric variety $(\mathcal{M}_P, D_P)$ to any polytope $P \subset \mathfrak{t}^*$.

2. A polarization (M, L) on a complex toric manifold is a choice of positive holomorphic line bundle L. Choosing any T^n-invariant Kähler metric $\omega \in c_1(L)$, we obtain by Lemma 3.19 a polytope P defined as the image of the moment map μ corresponding to ω. Conversely, starting with a polytope $P \subset \mathfrak{t}^*$ we have constructed a complex toric manifold (M_P, ω_P) and a holomorphic line bundle L_P, which is positive by Corollary 3.15.

3. A polarization on a symplectic toric manifold (M, ω) can be thought of as simply the cohomology class $2\pi[\omega]$. In this way, symplectic manifolds are all trivially polarized. We've seen that the Atiyah, Guillemin-Sternberg Theorem allows us to associate a polytope P to any compact symplectic toric manifold (M, ω), and that conversely we have the symplectic quotient (M_P, ω_P) to any polytope $P \subset \mathfrak{t}^*$.

In sum, we have:

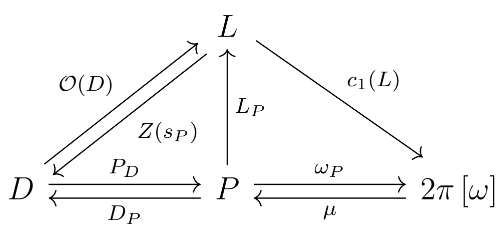
Theorem 3.23. *Let (M, J, ω) be a compact AK-toric manifold. Suppose that $2\pi[\omega]$ is an integral class, so that there exists a holomorphic line bundle $L \to M$ with the property that $\omega \in c_1(L)$. Clearly then L is a polarization for (M, J) and $2\pi[\omega]$ is a polarization for (M, ω). We can also associate the structure of a polarized toric variety $(\mathcal{M}, D)$, and each type of polarization is determined by a polytope $P \subset \mathfrak{t}^*$, unique up to translations in $\mathfrak{t}^*$, in the following way:*

1. *Let $P = \mu(M) \subset \mathfrak{t}^*$ be the image of any moment map with respect to ω associated to the T^n-action. Then P is a rational polytope, and moreover $(M, \omega) \cong (M_P, \omega_P)$.*

2. *$(M, J, L) \cong (M_P, J_P, L_P)$ as polarized complex toric varieties.*

3. *By Proposition 3.21, we can view (M, J) as an algebraic toric variety $\mathcal{M}$. Let $s_P \in H^0(M, L_P)$ be the section defined by (3.17). Then $D = Z(s_P)$ defines a polarization on $\mathcal{M}$.*

The polarizations are related by



Let (M, J, ω) be an arbitrary compact AK-toric manifold. Using Proposition 3.21, we can identify (M, J) with (M_P, J_P). Pulling back ω by this identification, we can view M_P with two *different* symplectic structures (M_P, J_P, ω_P) and (M_P, J_P, ω), both compatible with J_P. By the Delzant Theorem 3.17 the cohomology classes of ω and ω_P coincide. Moreover, Proposition 3.21 tells us that M admits a compatible algebraic structure. It is clear from the proof of Proposition 3.21 that the polytope P still determines a divisor D_P on M if we allow for real coefficients.

Proposition 3.24. *If (M, J, ω) is an arbitrary compact AK-toric manifold (i.e. $2\pi[\omega]$ need not be integral), then Theorem 3.23 still holds in the sense of $\mathbb{R}$-divisors, $\mathbb{R}$-line bundles, and Delzant polytopes. Moreover, given M which is either*

1. *a symplectic toric manifold (M, ω),*

2. *a complex toric manifold (M, J),*

3. *the underlying complex manifold of a smooth projective toric variety $\mathcal{M}$,*

then M admits all three such structures, compatible with the original given one. All three structures admit polarizations, and they too are compatible in the sense of Theorem 3.23.

3.2.3 Smooth non-compact toric manifolds

Next, we begin to attack the problem of extending these classic results to the non-compact setting. As we will see, there are real complications in the non-compact case and we cannot hope for the Delzant classification to hold in full generality. We start by presenting some examples, first of the "ideal" behavior which mirrors the structure present in the compact setting, and then of some of the new problems that arise in the non-compact setting.

Example 3.25 (Ideal behavior)**.** As we saw in Examples 2.10, 2.26, 2.31, Euclidean space $(\mathbb{C}^n, J_E, \omega_E)$ is AK toric. The corresponding moment map $\mu_E : \mathbb{C}^n \to \mathbb{R}^n$ is a polyhedron $P = \mathbb{R}^n_+$, which has defining equations

$$P = \{x \in \mathbb{R}^n \mid \langle x, e_i \rangle \geq 0,\ i = 1, \dots, n\},$$

where $e_i \in \mathbb{Z}^n$ are the standard basis elements. Notice that there are exactly n inner normals $e_1, \dots, e_n$, and that as a consequence the Kähler quotient construction has $K_{\mathbb{C}} = \{0\}$ acting on $(\mathbb{C}^n, J_E, \omega_E)$ itself. We have already seen that $\mathbb{C}^n \cong \mathcal{M}_P$. The divisor D_P is the sum of the coordinate hyperplanes $H_i = Z(z_i)$:

$$D_P = \sum_{i=1}^{n} H_i.$$

Of course the line bundle L_P is trivial, but specifically we can see that the character λ_P 3.14 itself acts trivially on $\mathcal{O}_{\mathbb{C}^n}$.

As we can see, with this simple example the correspondences of Theorem 3.23 still hold. As we'll see next, this is not the general expectation.

Example 3.26 (Image of the moment map need not be convex)**.** Let $\Delta \subset \mathbb{C}^n$ be the standard unit polydisc and set $M = \mathbb{C}^n - \Delta$. The T^n-action is still Hamiltonian with respect to the restriction of the Euclidean metric ω_E to M, the moment map $\mu : M \to \mathbb{R}^n$ being simply the restriction of μ_E. As such, however, the image of μ inside of $\mathbb{R}^n$ is equal to the image of $\mu_E(\mathbb{C}^n) - \mu_E(\Delta)$, which is equal to the positive orthant without the "unit cube," which is not convex. For example, the two-dimensional case $n = 2$ is given by:

In this example (M, ω_E) is symplectic toric, and J_E is compatible with ω_E, but (M, J_E, ω_E) fails to be AK-toric. The reason is that the T^n-action does not admit a complexification. This can be seen in at least two ways. Firstly, if there was such a complexification, the corresponding holomorphic vector field would have to coincide with the Euler vector field:

$$X = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i},$$

which is incomplete on M. From another perspective, if such a complexification existed, analytic continuation would furnish a biholomorphism $M \to (\mathbb{C}^*)^n$. Note that in this example, there is no algebraic variety to talk about, since the polyhedral set $\mu(M)$ is not a polyhedron.

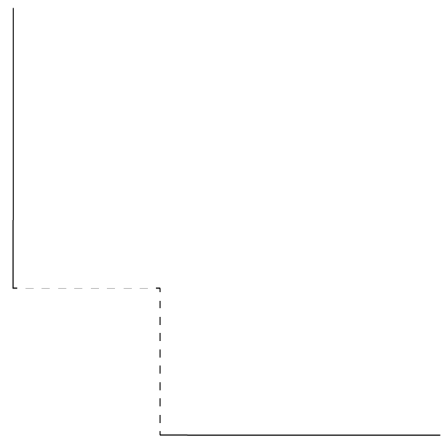


Figure 3.1: A symplectic toric manifold whose moment image is not convex.

Even if the image of the moment map is convex, there are still issues with the Delzant correspondence.

Example 3.27 (Another incomplete Kähler symplectic toric manifold which is not complex toric)**.** Now let $M = \Delta$ be the unit polydisc of the previous example. By the exact same reasoning as abive, the restriction of the Euclidean metric gives (M, ω_E) the structure of a symplectic toric manifold with respect to which J_E is compatible. The image of the moment map $\mu(M) \subset \mathbb{R}^n$ is the half-open unit cube

$$\mu(M) = [0,1) \times \cdots \times [0,1),$$

which is *convex*, but not a polyhedron. Once again, the T^n-action does not admit a complexification, as such a $(\mathbb{C}^*)^n$-action would give rise to a biholomorphism $(M, J_E) \to \mathbb{C}^n$. Again there is no algebraic structure to speak of, since $\mu(M)$ is not a polyhedron.

In fact, there can still be issues even if the image of the moment map *is* a polyhedron.

Example 3.28 (A complete Kähler symplectic toric manifold which is not complex toric)**.** Consider the unit disc $\Delta = \mathbb{D} \subset \mathbb{C}$, but instead with the hyperbolic Poincaré metric

$$\omega_H = \frac{2i}{(1-|z|^2)^2} dz \wedge d\bar{z}.$$

This metric is *complete*, and the S^1-action on $\mathbb{D}$ is Hamiltonian with respect to ω_H, with moment map $\mu : \mathbb{D} \to \mathbb{R}$ given by

$$\mu(z) = \frac{2|z|^2}{1 - |z|^2}.$$

The image of μ is a polyhedron $P = [0, \infty)$, but as we have already seen, the S^1 action here does not complexify. Although there exists a toric variety $\mathcal{M}_P \cong \mathbb{C}$ associated to the polyhedron P, we have for the same reason that $(\mathbb{D}, J_E)$ is *not* biholomorphic to $\mathcal{M}_P$.

Example 3.29 (Incomplete Kähler manifolds with $(\mathbb{C}^*)^n$-actions can have bad moment maps)**.** Let P be a polytope in $\mathbb{R}^n$, $M = \mathcal{M}_P$ be the associated projective toric variety, and ω_P be the Kähler metric on M determined by Proposition 3.21. By the Orbit-Cone correspondence, each facet F of P determines a $(\mathbb{C}^*)^n$-invariant divisor D_F on M. Choose one such F and for simplicity of notation let $D = D_F$. It follows then that there is a well-defined restriction of the $(\mathbb{C}^*)^n$-action to the quasiprojective variety $N = M - D$ which is still effective and holomorphic. The restriction of ω_P to N is Hamiltonian, and the moment map $\mu : N \to \mathfrak{t}^*$ is just the restriction of $\mu_P : M \to \mathfrak{t}^*$. Thus $\mu(N) = \mu_P(M) - \mu_P(D)$. By the same reasoning as in the proof of Proposition 3.21, we can see that $\mu_P(D) = F$. Thus $\mu(N) = P - F$ is not a polyhedron. See Figure 3.29 below for an example when $M \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D = \{0\} \times \mathbb{P}^1$.

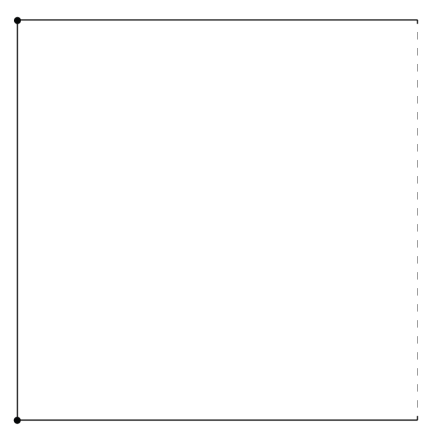


Figure 3.2: A quasiprojective toric variety whose moment image does not describe the algebraic structure.

These examples seem to suggest that the well-behavedness of the moment map is entirely dependent on the completeness of the metric. The next example shows that even this is too much to ask.

Theorem/Example 3.30. *There exists a complete S^1-invarint Kähler metric ω on $\mathbb{C}$ whose moment image $\mu(\mathbb{C}) \subset \mathbb{R}$ is bounded.*

Proof. We will choose ω of the form

$$\omega = \frac{i}{2}F(|z|)dz \wedge d\bar{z},$$

where $F : [0, \infty) \to (0, \infty)$ is a smooth function. The corresponding Riemannian metric is then

$$g = F(|z|)\left(dx \otimes dx + dy \otimes dy\right).$$

From this we can see that the antiholomorphic diffeomorphism $z \mapsto \bar{z}$ is an isometry with respect to g. This, together with S^1-invariance, implies that radial lines emanating from 0 are the images of geodesics. Thus, there are geodesics $\gamma : [a, \infty) \to \mathbb{C}$ whose images lie in the positive reals $\mathbb{R}_+ \subset \mathbb{C}$, and so we may think of $\gamma(t)$ simply as a positive real number. In this way, write $\gamma = (\gamma(t), 0)$ in the real coordinates (x, y). Since $\dot{\gamma}$ is a multiple of $\frac{\partial}{\partial x}$, the geodesic equation restricted to $\mathbb{R}_+$ reduces to

$$\ddot{\gamma} + \Gamma^1_{11}(\dot{\gamma})^2 = 0.$$

Since the metric is diagonal, the Christoffel symbol Γ^1_{11}, again restricted to the positive reals, is given simply by

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x} = \frac{F'(x)}{2F(x)}.$$

The geodesic equation is then separable:

$$\frac{\partial \dot{\gamma}}{\partial t}\frac{1}{\dot{\gamma}} = -\frac{1}{2}\frac{\partial}{\partial t}\log(F(\gamma(t))),$$

so that

$$\dot{\gamma} = \frac{1}{\sqrt{F(\gamma)}}. \tag{3.18}$$

We now let $F : [0, \infty) \to (0, \infty)$ be any smooth function with the property that

$$F(r) = \begin{cases} \left(\frac{1}{r\log(r)}\right)^2 & r \geq 2 \\ 1 & r \leq 1 \end{cases}.$$

We first claim that with this choice the metric ω is complete. Indeed, this will be true if and only if γ is defined for all time. Choose a such that $\gamma(a) = 2$. We solve (3.18) explicitly. Along γ, we have that $F = \frac{1}{(x\log(x))^2}$, so that

$$\dot{\gamma} = \gamma \log(\gamma).$$

In this case we can solve for $\gamma(t) = e^{e^{t+c}}$. By modifying the choice of a, we can assume that $c = 0$. In any case γ is clearly defined for all time, and so ω is complete.

Now let $f : \mathbb{C} \to \mathbb{R}$ be the Hamiltonian potential for the tangent vector X of the S^1-action, so that $\nabla^g f = JX = r\frac{\partial}{\partial r}$. Let $\eta_t : [0, \infty) \to \mathbb{C}$ be the flow line of JX starting at $2 \in \mathbb{C}$, so that $\eta(s) = e^s \in \mathbb{R}_+ \subset \mathbb{C}$ for $s \in [\log(2), \infty)$. The energy of of η is given by

$$\begin{aligned} E(\eta_t) &= \int_{\log(2)}^{t} g(\dot{\eta}, \dot{\eta}) ds = \int_{\log(2)}^{t} g(\nabla^g f, \dot{\eta}) ds \\ &= \int_{\log(2)}^{t} df(\dot{\eta}) ds = f(\eta(t)) - f(\eta(\log(2))) = f(e^t) - f(2). \end{aligned}$$

But now we see that

$$\begin{aligned} \int_{\log(2)}^{t} g(\dot{\eta}, \dot{\eta}) ds = \int_{\log(2)}^{t} F(e^s) e^{2s} ds = \int_{2}^{e^t} xF(x) dx \\ = \int_{2}^{e^t} \frac{1}{x(\log(x))^2} dx = \frac{1}{\log(2)} - e^{-t}. \end{aligned}$$

Letting $t \to \infty$, we see that f is bounded on the positive reals. Since f is S^1-invariant, $f(\mathbb{C})$ is bounded.

□

In particular, even though $\mathbb{C}$ is certainly an affine toric variety, even the completeness of ω is not enough to guarantee that there exists a variety $\mathcal{M}_{\mu(\mathbb{C})}$ such that $\mathbb{C} \cong \mathcal{M}_{\mu(\mathbb{C})}$. We will see in a moment (Lemma 3.32) that the existence of a Hamiltonian potential which is proper will be enough to guarantee that the image of the moment map is a Delzant polyhedron P. If there is also an action of the complexified group, then (Proposition 3.33) we can also show that $M \cong \mathcal{M}_P$. Thus, the AK-toric case is the correct setting in which to expect a Delzant correspondence for non-compat manifolds. Indeed, as Example 3.30 shows, the assumption of a proper Hamiltonian potential is non-trivial, and need not be satisfied even for a complete T^n-invariant Kähler metric on a complex toric manifold. In general, one must impose extra geometric assumptions on the metric for such a potential to exist. This will always be the case if ω is a shrinking gradient Kähler-Ricci soliton, for example, a situation which we will study in detail in Chapter 4.

We turn now to an example first described, to the author's knowledge, by Karshon-Lerman [46]. Here we see yet another new complication in the non-compact case, namely infinite topology.

Example 3.31 (A complex toric manifold which is not a toric variety) [46, Example 6.9])**.** We begin by defining a polyhedral set $P \subset \mathbb{R}^2$ as the convex hull of the set of points $v_k = (k(k-1)/2, k)$ for $k = 0, 1, 2, \ldots$.

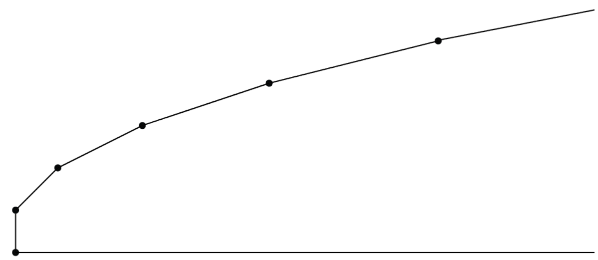


Figure 3.3: A toric manifold M with infinite topology. In particular, M is not a Kähler quotient of $\mathbb{C}^N$.

Note that this is *not* a polyhedron as defined by Definition 2.5. However, [46, Theorem 1.3] implies that there exists a symplectic toric manifold (M, ω) with moment map $\mu : M \to \mathbb{R}^2$ whose image is equal to P. In fact, M also admits a compatible complex structure J and an effective and holomorphic $(\mathbb{C}^*)^2$-action; the complex manifold (M, J) can be obtained from $\mathbb{P}^1 \times \mathbb{C}$ as a limit by sequentially blowing up a fixed point of the $(\mathbb{C}^*)^2$-action. The fixed point set is the preimage under μ of the set of vertices $\{v_k\}$, which is isolated but not finite. Once again there is no algebraic structure, in this case because the normal fan of P is infinite.

So, clearly some extra care is required in the non-compact case. From the symplectic perspective, Prato-Wu [61] observed that one can obtain a version of the Atiyah, Guillemin-Sternberg theorem if we make the *a priori* assumption on the existence of a certain well-behaved Hamiltonian potential. From this stage, the general theory developed by Karshon-Lerman [46] for non-compact symplectic toric manifolds applies, and from this we can obtain a version of the Delzant theorem suitable for our purposes.

Lemma 3.32. *Let (M, ω) be a symplectic toric manifold with finite, nonempty fixed point set. Suppose that there exists $b \in \mathfrak{t}$ such that the function $\langle \mu, b \rangle : M \to \mathbb{R}$ is proper and bounded from below. Then the image of the moment map μ is a Delzant polyhedron P, and moreover (M, ω) is equivariantly symplectomorphic to (M_P, ω_P).*

Proof. Since the fixed point set of the T^n-action is finite, it follows from [61, Proposition 1.4] and the preceeding remarks that the existence of such a $b \in \mathfrak{t}$ is sufficient to show that the image of the moment map μ is a polyhedral set in $\mathfrak{t}^*$. This means by definition that $\mu(M)$ is equal to the intersection of finitely many half spaces. It then follows immediately from [46, Proposition 1.1] that P is a Delzant (unimodular) polyhedron. Finally, [46, Theorem 1.3, c.f. Theorem 6.7] furnishes the desired equivariant symplectomorphism.

□

We recall that an AK-toric manifold admits such a Hamiltonian potential $\langle \mu, b \rangle : M \to \mathbb{R}$ by definition. Thus, the image $\mu(M) \subset \mathfrak{t}^*$ of the moment map is always a Delzant polyhedron. This lets us put together an analog of Proposition 3.21 in the AK-toric setting:

Proposition 3.33 ([15, Lemma 2.14]). *An AK-toric manifold (M, J, ω) equivariantly biholomorphic to (M_P, J_P), where P is the moment polyhedron. Consequently, (M, J) is in particular quasiprojective.*

Proof. To prove the proposition, we proceed as in Proposition 3.21. The crux of that proof was the Holomorphic Slice Theorem 3.20, whose key components were:

1. a Kähler manifold (M, J, ω),
2. a Hamiltonian group action of a compact group G, and
3. a holomorphic action of the complexified group $G^{\mathbb{C}}$, extending the original action.

These are guaranteed for an AK-toric manifold by definition with $G = T^n$, $G^{\mathbb{C}} = (\mathbb{C}^*)^n$. Since we know that the image of the moment map is a Delzant polyhedron, the proof of Proposition 3.21 then carries through.

□

This, together with Lemma 3.19, treats the first part of the Delzant correspondence for non-compact AK-toric manifolds. The next step is to understand the polarizations.

Proposition 3.34 ([15, Proposition 2.15]). *If (M, J, ω) is AK-toric, then the moment polyhedron P is determined up to translation by the cohomology class $[\omega]$.*

Proof. The polyhedron P determines a torus-invariant divisor D_ω on (M, J) as follows. Since (M, J) is biholomorphic to (M_P, J_P), we use this biholomorphism and assume without loss of generality that $(M, J, \omega) = (M_P, J_P, \omega)$ with ω not necessarily equal to ω_P. Recall that (M_P, J_P) naturally carries the structure of the algebraic toric variety $\mathcal{M}_P$. Thus, we can identify the normal fan Σ of P with the fan corresponding to $\mathcal{M}_P$. Let ν_i be the minimal generator in Γ of the ray $\sigma_i \in \Sigma$ corresponding to the direction normal to each facet F_i of P. Then each F_i of P has the local defining equation $\ell_i(x) + a_i = 0$, where $\ell_i(x) = \langle \nu_i, x \rangle$ for some $a_i \in \mathbb{R}$. Recall that σ_i defines via the Orbit-Cone correspondence an irreducible Weil divisor D_i. The divisor D_ω is then given by

$$D_\omega = \sum a_i D_i. \tag{3.19}$$

We can assume without loss of generality that the irreducible component D_1 of D_ω is compact. If there is no such D_1, then it follows that there is a $b \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n, \mathbb{Z})$ such that the affine transformation $Ax + b$ takes P to the positive orthant $\mathbb{R}^n_+$, and so $M \cong \mathbb{C}^n$. Note that the entire construction behaves well with respect to restriction, so that $D_1 = \mathcal{M}_{F_1}$. Since P is Delzant, so is F_1, and so it follows that D_1 is a nonsingular projective variety. If we restrict ω to D_1, we obtain a moment map for the T^{n-1}-action $\mu_1 : D_1 \to \mathfrak{t}_1$, where $\mathfrak{t}_1 \subset \mathfrak{t}$ is the orthogonal complement of the stabilizer algebra of D_1. Then the image of μ_1 is the face F_1 of P corresponding to D_1. After potentially acting by an element of $\mathrm{GL}(n, \mathbb{Z})$, we can assume that $\langle \nu_1, x \rangle = x_1$, so that $\mathfrak{t}_1$ can be identified with the subspace $x_1 = 0$. Inside of $\mathfrak{t}_1$, F_1 is then defined by $\langle \eta_i, (x_2, \ldots, x_n) \rangle \geq -\alpha_i$ for some η_i in the lattice and $\alpha_i \in \mathbb{R}$. Thus, the Delzant polytope F_1 determines a divisor $\Delta = \sum \alpha_i \Delta_i$ on D_1, where Δ_i are the torus-invariant divisors on D_1 corresponding to η_i through the Orbit-Cone correspondence.

Since $(D_1, \omega|_{D_1})$ is itself a compact symplectic toric manifold, we can now appeal to the well-established theory of the previous section. Specifically, it follows from Proposition 3.24 that the cohomology class of the symplectic form $\omega|_{D_1}$ is given by

$$[\omega|_{D_1}] = \sum \alpha_i [\Delta_i].$$

The coefficients α_i, by definition, fix the defining equations of F_1 inside $\mathfrak{t}_1$. Thus, we see that the facet F_1 is uniquely determined by $[\omega]$ up to translation in $\mathfrak{t}_1$. By the Orbit-Cone correspondence, the subspace $\mathfrak{t}_1$ on which F_1 lies is uniquely determined by the fixed fan Σ, up to translation in its normal direction. We see then that the set of vertices $\{v_1, \ldots, v_k\}$ of F_1, which is the image under μ of the set of fixed points T^n-action that lie in $\mu^{-1}(F_1)$, is determined uniquely up to a translation in $\mathfrak{t}^*$ by $[\omega]$. Now each vertex of P lies on at least one compact facet, otherwise again we would be in the situation where $P = \mathbb{R}^n_+$. Hence, we can repeat this process for each compact torus-invariant divisor to see that the set of *all* vertices $\{v_1, \ldots, v_K\}$ of P is determined up to translation in $\mathfrak{t}^*$ by $[\omega]$. It is clear then that the same is true of P.

□

Corollary 3.35 ([15, Corollary 2.16]). *Let M be AK-toric with moment polyhedron $P = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x \rangle \geq -a_i$ for all $i = 1, \ldots, N\}$, and suppose that ω represents $c_1(L)$. Then $L \cong \mathcal{O}(D_\omega)$ is the line bundle associated to the divisor $D_\omega = \sum a_i D_i$.*

Proof. Recall that $(M, J) \cong (M_P, J_P) \cong \mathcal{M}_P$. Let Σ be the normal fan of P so that $\mathcal{M}_P = \mathcal{M}_\Sigma$. Since M is smooth we have by [21, Proposition 4.6.2] that $L \cong \mathcal{O}(D)$ for some torus-invariant divisor $D = \sum \beta_i D_i$ with $\beta_i \in \mathbb{Z}$. Recall that $P_D \subset \mathfrak{t}^*$ denotes the polyhedron associated to D given by

$$P_D = \{x \in \mathfrak{t}^* \mid \langle x, \nu_i \rangle \geq -\beta_i, \text{ for all } i = 1, \ldots, N\}, \tag{3.20}$$

where $\nu_1, \ldots, \nu_N$ are the minimal generators of the rays $\sigma_i \in \Sigma$. If D and D' are any two torus-invariant divisors on M with integer coefficients, we define an equivalence relation by declaring that $D \sim D'$ if and only if there exists some $\nu \in \Gamma^*$ such that $P_{D'} = P_D + \nu$, where P_D and $P_{D'}$ are the polyhedra (3.20) defined above. By [21, Theorem 4.1.3], $D \sim D'$ if and only if $\mathcal{O}(D) \cong \mathcal{O}(D')$. Suppose that D_1 is a compact torus-invariant Weil divisor in M. As before, such a D_1 must exist unless $M \cong \mathbb{C}^n$ and $P = \mathbb{R}^n_+$. Perhaps by modifying D by the equivalence relation, we can assume that the coefficient β_1 corresponding to D_1 is zero. In other words, there is a section s_1 of L which does not vanish identically on D_1. Let $F_1 \subset \overline{P}$ be the facet corresponding to D_1. As before, the Delzant polyhedron F_1 determines a unique torus-invariant Weil divisor $\Delta = \sum \alpha_i \Delta_i$ on D_1. The restriction of s_1 to D_1 is a section of $L|_{D_1}$ which vanishes along $\Delta_i = D_1 \cap D_i$ to order α_i. In particular, we see that the coefficients α_i of Δ_i are equal to those β_i such that $D_i \cap D_1 \neq \emptyset$. Let $D_\omega = \sum a_i D_i$ be the divisor on M associated to ω as in (3.19) of the previous proof. We claim that $D_\omega \sim D$. As before, we can act by $\mathrm{GL}(n, \mathbb{Z})$ so that $\langle \nu_1, x \rangle = x_1$. Write $P_1 = P_\omega + \nu_1$ so that the face $F_1 + \nu_1$ corresponding to D_1 now lies on the hyperplane $x_1 = 0$, and in general P_1 is defined by $\langle x, \nu_i \rangle \geq \langle \nu_1, \nu_i \rangle - a_i = -\tilde{a}_i$. Then it is straightforward to compute that the coefficients α_i are equal to those $\tilde{a}_i$ such that $D_i \cap D_1 \neq \emptyset$. Running across all compact divisors of M, we see that the coefficients a_i in the defining equations for P_ω are uniquely determined by β_i up to equivalence. In particular, $D_\omega \sim D$.

□

In sum, we have established:

Theorem 3.36. *Theorem 3.23 and Proposition 3.24 hold for AK-toric manifolds in the sense of* $\mathbb{R}$*-divisors,* $\mathbb{R}$*-line bundles, and Delzant polyhedra.*

3.3 Kähler metrics on toric manifolds part two: Moment maps and extension

We return now to the direct study of Kähler metrics as in Section 3.1.

3.3.1 The Legendre transform

Let (M, J) be a complex toric manifold and ω be a compatible Kähler metric with respect to which the T^n-action is Hamiltonian. As we saw in Section 3.1, there exists a point $p \in M$ such that the orbit $U = (\mathbb{C}^*)^n \cdot p \subset M$ is open and dense, and moreover there is a smooth T^n-invariant function $\phi \in C^\infty_T((\mathbb{C}^*)^n)$ such that $\omega|_U = 2i\partial\bar{\partial}\phi$. We will abuse notation and denote the restriction $\omega|_U$ simply by ω. The function ϕ can be understood (Lemma 3.2) as a strictly convex function on $\mathfrak{t}$, and the moment map $\mu|_U : U \to \mathfrak{t}^*$ is just the Euclidean gradient $\nabla\phi : \mathfrak{t} \to \mathfrak{t}^*$. Since $\nabla\phi$ is a diffeomorphism, we can use this to define an alternate angular coordinate system on $(\mathbb{C}^*)^n$:

$$(\mathbb{C}^*)^n \xrightarrow{\mathrm{Log}} \mathfrak{t} \times T^n \xrightarrow{\nabla\phi \times \mathrm{id}} \Omega \times T^n.$$

Recall our background coordinate system $(x^1, \ldots, x^n)$ on $\mathfrak{t}^*$. By construction, when we change coordinates by the transformation above, the symplectic form ω is just the standard Euclidean form

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\theta^i. \tag{3.21}$$

Since the coordinate change was not *holomorphic*, we cannot expect the coefficients of ω to determine those of the corresponding Riemannian metric. To understand the metric structure, we need some basic machinery from convex function theory. That we are changing coordinates by means of the gradient of a strictly convex function is highly suggestive of the following general procedure.

Lemma 3.37. *Let V be a real vector space and ϕ be a smooth and strictly convex function on a convex domain $\Omega' \subset V$. Then there is a unique function $L(\phi) = u$ defined on $\Omega = \nabla\phi(\Omega') \subset V^*$ by:*

$$\phi(\xi) + u(x) = \langle \xi, x \rangle \tag{3.22}$$

for $x = \nabla\phi(\xi)$. The function u is smooth and strictly convex on Ω. Moreover, L has the following properties:

1. *$L(L(\phi)) = \phi$,*
2. *$\nabla\phi : \Omega' \to \Omega$ and $\nabla u : \Omega \to \Omega'$ are inverse to each other,*
3. *$\phi_{ij}(\xi) = u^{ij}(\nabla\phi(\xi))$, where $\phi_{ij} = \frac{\partial^2\phi}{\partial\xi^i\partial\xi^j}$ and $u^{ij} = \left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right)^{-1}_{ij}$,*
4. *$L((1-t)\phi + t\phi') \leq (1-t)L(\phi) + tL(\phi')$.*

For a comprehensive reference on the Legendre transform and its basic properties, see Rockafellar's book [62].

Remark 3.38. The first two items above together with the symmetry of equation (3.22) imply that the Legendre transform is symmetric in u and ϕ.

In the next lemma, we collect some useful properties of the behavior of the Legendre transform under certain linear operations, all immediate consequences of Lemma 3.37.

Lemma 3.39. *Let ϕ be a C^1 strictly convex function on V and $u = L(\phi)$ be its Legendre transform. Let Ω denote the image of the gradient $\nabla\phi : V \to V^*$. Suppose that we are given a lattice $\Gamma \subset V$, and identify $GL(n, \mathbb{Z})$ with the maximal subgroup of $GL(V)$ which preserves Γ.*

1. *For $B \in GL(n, \mathbb{Z})$, set $\phi_B(\xi) = \phi(B\xi)$. Then $L(\phi_B)(x) = u((B^T)^{-1}x)$, and the image of $\nabla\phi_B : V \to V^*$ is equal to $B^T(\Omega)$.*
2. *For $b_1 \in V$, set $\phi_{b_1}(\xi) = \phi(\xi - b_1)$. Then $L(\phi_{b_1})(x) = u(x) + \langle b_1, x\rangle$. Clearly, the image of $\nabla\phi_{b_1}$ is also equal to Ω.*

3. *Symmetrically, for* $b_2 \in V^*$*, set* $\phi^{b_2}(\xi) = \phi(\xi) + \langle b_2, \xi\rangle$*. Then* $L(\phi^{b_2})(x) = u(x - b_2)$ *and the image of* $\nabla\phi^{b_2}$ *is equal to* $\Omega + b_2$*.*

4. *If* $a \in \mathbb{R}$*, then* $L(\phi + a) = u - a$*.*

Sketch of the proof. As we mentioned above, these are all elementary consequences of 3.37, and so we will only prove the second property to give the flavor of how proofs of this sort go. Let $u = L(\phi)$. Applying (3.22) to ϕ, we have

$$\phi(\xi - b_1) + u(\nabla\phi(\xi - b_1)) = \langle(\xi - b_1), \nabla\phi(\xi - b_1)\rangle.$$

Since $\nabla\phi_{b_1}(\xi) = (\nabla\phi)(\xi - b_1)$, we see that

$$\phi_{b_1}(\xi) + u(\nabla\phi_{b_1}(\xi)) + \langle b_1, \nabla\phi_{b_1}(\xi)\rangle = \langle\xi, \nabla\phi_{b_1}(\xi)\rangle,$$

from which it follows immediately from (3.22) that $L(\phi_{b_1})(x) = u(x) + \langle b_1, x\rangle$. □

3.3.2 Symplectic coordinates

Returning to the scenario at the beginning of this section, let $\omega = 2i\partial\bar{\partial}\phi$ on $U \cong (\mathbb{C}^*)^n \subset M$ as before. Let $u(x) = L(\phi) \in C^\infty(\Omega)$ be the Legendre transform of $\phi \in \mathbb{C}^\infty(\mathfrak{t})$. Combining (3.4) with Lemma 3.37, we see that after changing coordinates by $\nabla\phi$, the metric g on $\Omega \times T^n$ is given by

$$g = u_{ij}dx^i \otimes dx^j + u^{ij}d\theta^i \otimes d\theta^j, \tag{3.23}$$

where as before u_{ij} is the Euclidean Hessian of u and u^{ij} its inverse. From this we see immediately that the complex structure J is given by

$$J = \begin{pmatrix} 0 & -u^{ij} \\ u_{ij} & 0 \end{pmatrix}. \tag{3.24}$$

Now the picture becomes more clear. From the complex perspective, the complex structure J is kept standard and the Kähler structure (g, ω) is determined by the convex function ϕ on $\mathfrak{t}$, whereas from this new perspective the symplectic form ω is standard and the Kähler structure (g, J) is determined by the convex function u on $\Omega \subset \mathfrak{t}^*$.

Definition 3.40. The convex function u defined as above by the Legendre transform of the Kähler potential ϕ is called a *symplectic potential* for the Kähler structure on M.

As we have noted, the moment map $\mu : M \to \mathfrak{t}^*$ is defined only up to translation. Moreover, we have defined our choice of moment map in terms of a choice of Kähler potential ϕ for ω. By Lemma 3.6, the choice of ϕ is unique up to the additon of an affine function on $\mathfrak{t}$. Let $a(\xi) = \langle b_a, \xi\rangle + c_a$ be affine, and set $\phi_a = \phi + a$. Then we have

$$\nabla\phi_a = \nabla\phi + b_a,$$

which clearly acts to translate the image of the moment map. By Lemma 3.39, we have

Lemma 3.41. *The gauge group $\mathfrak{t}^* \times \mathbb{R}$ acts feely and transitively on the set of Kähler potentials ϕ for a given Kähler form ω on $(\mathbb{C}^*)^n$ by the addition of the affine-linear functions. The action of the subgroup $\mathfrak{t}^* \times \{0\}$ is naturally identified with the action of translating the image of the moment map, with the effect on the symplectic potential being simply the pullback $u(x) \mapsto u(x - b_a)$.*

There are in fact other gauge symmetries at play here. Recall that we have fixed a basis $(\Xi_1, \ldots, \Xi_n)$ for $\Gamma \subset \mathfrak{t}$ from the outset, and as a consequence we have our preferred coordinate system $(\xi^1, \ldots, \xi^n)$ on $\mathfrak{t}$. This is equivalent to viewing $(\mathbb{C}^*)^n$ as a torus with *coordinates* $(z_1, \ldots, z_n)$, rather than simply an abstract algebraic torus. The $(\mathbb{C}^*)^n$-equivariant automorphisms of $(\mathbb{C}^*)^n$ are precisely $\mathrm{GL}(n, \mathbb{Z})$, and they act on $\mathfrak{t}$ by the usual matrix multiplication. Appealing to Lemma 3.39, we have

Lemma 3.42. *The equivariant automorphism group $GL(n, \mathbb{Z})$ of $(\mathbb{C}^*)^n$ acts on our choice of coordinates on $\mathfrak{t}$, and therefore our coordinate representation of the image $\Omega \subset \mathfrak{t}^*$ of the moment map. The action is by usual matrix multiplication $\Omega \mapsto B^T(\Omega)$, with the action on Kähler potentials given by $\phi(\xi) \mapsto \phi(B\xi)$ and on symplectic potentials given by $u(x) \mapsto u((B^T)^{-1}x)$.*

Finally, there is the action of $(\mathbb{C}^*)^n$ on M to account for. Recall that in order to introduce coordinates on our dense orbit $U \subset M$, we had to choose a base point $p \in U$. Of course $(\mathbb{C}^*)^n$ acts transitively on U by construction, and the ambiguity of this choice is reflected as follows:

Lemma 3.43. *In logarithmic coordinates, the action of $(\mathbb{C}^*)^n$ on itself is given by*

$$\lambda \cdot (\xi^1, \ldots, \xi^n, \theta^1, \ldots, \theta^n) = (\xi^1 + \log(|\lambda_1|), \ldots, \xi^n + \log(|\lambda_n|), \theta^1 + \arg(\lambda_1), \ldots, \theta^n + \arg(\lambda_1)).$$

If $|\lambda|^2 \neq 1$, then multiplication by $\lambda : M \to M$ will not preserve a T^n-invariant Kähler form ω on (M, J). For $b_1 \in \mathfrak{t}$, the pullback of $\omega = 2i\partial\bar{\partial}\phi$ by the element $e^{-b_1} \in (\mathbb{C}^*)^n$ has Kähler potential equal to $\phi_{b_1}(\xi) = \phi(\xi - b_1)$, and consequently symplectic potential $u^{b_1}(x) = u(x) + \langle b_1, x\rangle$.

For the last point, we have once again used Lemma 3.39.

3.3.3 The canonical potential

We now turn to one of the most useful and interesting reasons to study toric manifolds in symplectic coordinates. Recall that, given a Delzant polyhedron P, we have a canonically associated AK-toric manifold (M_P, J_P, ω_P) obtained as a Kähler quotient of $\mathbb{C}^N$. Suppose, as always, that $P \subset \mathfrak{t}^*$ has defining equations

$$P = \{x \in \mathfrak{t}^* \mid \langle x, \nu_i\rangle \geq -a_i\} \tag{3.25}$$

for $\nu_1, \ldots, \nu_N \in \Gamma \subset \mathfrak{t}$ and $a_1, \ldots, a_N \in \mathbb{R}$.

Notation 3.44. Let P be a Delzant polyhedron described as in (3.25). When no confusion is likely to arise, we will sometimes denote by $\ell_i(x)$ the linear function $\langle x, \nu_i \rangle$ determined by the inner normal ν_i.

As above, we can write $\omega_P = 2i\partial\bar{\partial}\phi_P$ for $\phi_P \in C^\infty(\mathfrak{t})$ strictly convex. By Lemma 3.41, we can choose ϕ_P uniquely up to the addition of a constant such that the image of $\nabla\phi_P : \mathfrak{t} \to \mathfrak{t}^*$ is equal to P. With such a choice, let $u_P = L(\phi_P) \in \mathbb{C}^\infty(P)$ be the corresponding symplectic potential. It turns out that u_P has a simple explicit formula in terms of the data of the polyhedron P, due originally to Guillemin [42] in the smooth compact case and more recently to Burns-Guillemin-Lerman [11] in the much more general setting of Remark 3.13.

Proposition 3.45 ([11])**.** *The smooth convex function u_P is given explicitly on P by*

$$u_P(x) = \frac{1}{2}\sum_{i=1}^{N} (\ell_i(x) + a_i) \log (\ell_i(x) + a_i) . \tag{3.26}$$

We can finally now treat the question of to what extent these local descriptions can be related to the global picture. To this end, suppose that we have an AK-toric manifold (M, J, ω) with moment polyhedron $P \subset \mathfrak{t}^*$. An important consequence of the existence of an equivariant symplectomorphism $(M, \omega) \cong (M_P, J_P)$, which is guaranteed by Lemma 3.32, is the following.

Proposition 3.46. *Let (M, J, ω) be an AK-toric manifold with moment polyhedron P. In particular, there is a symplectic potential $u \in C^\infty(P)$ associated by the moment map. The function u takes the special form*

$$u = u_P + v, \tag{3.27}$$

where $v \in C^\infty(\overline{P})$ extends smoothly to the boundary of P.

Proof. By Lemma 3.32 (see also [61]), the moment map $\mu : M \to \mathfrak{t}^*$ is proper. The result now follows immediately from [3, Proposition 1]. Alternatively, according to [3, Lemma 2, c.f. Remark 3], u admits such a description as soon as we know that (M, ω) is T^n-equivariantly symplectomorphic to (M_P, ω_P). This follows from Lemma 3.32. □

If we now consider the situation of Section 3.1 once again, we can ask the converse: When does a T^n-invariant Kähler metric $\omega = 2i\partial\bar{\partial}\phi$ on $(\mathbb{C}^*)^n$ extend to a Kähler metric on some ambient toric manifold? The Delzant correspondence of Section 3.2 makes it clear that this must depend at least on the image of the corresponding moment map μ, which in this situation is given by the Euclidean gradient $\nabla\phi$. We will only answer this question when the image of $\nabla\phi$ is a polyhedron; this will be more than sufficient for our purposes. For a proof, see [3, Lemma 2].

Lemma 3.47 ([2, 1, 3])**.** *Let $\omega = 2i\partial\bar{\partial}\phi$ be a T^n-invariant Kähler metric on $(\mathbb{C}^*)^n$, and suppose that the image of $\mu = \nabla\phi : (\mathbb{C}^*)^n \to \mathfrak{t}^*$ is equal to the interior of a Delzant polyhedron*

P. Let $u = L(\phi) \in C^\infty(P)$ be the symplectic potential associated to ϕ via the Legendre transform. Then ω extends to a smooth Kähler metric on (M_P, J_P) if the function

$$v = u - u_P$$

extends smoothly past the boundary of P.

3.4 Asymptotically conical metrics

We conclude this chapter with a brief foray into an interesting class of examples of this theory: Kähler cones and especially asymptotically conical metrics. Toric geometry has been instrumental in the study of Sasaki-Einstein metrics on Kähler cones. These metrics are Ricci-flat, and thus form a natural component of the study of the AdS/CFT correspondence in string theory. Toric Sasaki-Einstein geometry was first studied by Martelli-Sparks-Yau [56, 55] from this perspective. This served as a starting point for great deal of recent work in the purely mathematical realm on K-stability and Sasaki-Einstein geometry (see [18] and the survey [52] for results in this direction). Sasaki-Einstein metrics were discovered to exist in great generality on certain toric Kähler cones by Futaki-Ono-Wang [39], thus providing a large class of non-compact Ricci flat Kähler metrics. These in turn were used to construct special *complete* metrics on resolutions of these cones, for example Ricci-flat metrics [17] and Kähler-Ricci solitons [40, 38].

In this section we give a short overview of the necessary background before explaining how to fit this in to the general theory of this chapter.

3.4.1 Kähler cones and affine toric varieties

Here we give some backround Riemannian cones and their relation to toric geometry. The main references for this section are Martelli-Sparks-Yau [56, 55] and VanCoevering [17, 16]. Let S be a compact smooth manifold of odd dimension $\dim_{\mathbb{R}} S = 2m + 1$, and consider the product $M = \mathbb{R}_+ \times S$. A *cone metric* g on M is a Riemannian metric of the form

$$g = dr^2 + r^2 g_S, \tag{3.28}$$

where r is the "radial" coordinate on $\mathbb{R}_+ = (0, \infty)$ and g_S is a Riemannian metric on S. Note that a cone metric is always incomplete as $r \to 0$. If M is a complex manifold equipped with a complex structure J, we say that g is a Kähler cone if g is Kähler with respect to J in the usual sense. Let $n = 2m+2$ be the complex dimension of M. In case that g is Kähler, the radial vector field $r\frac{\partial}{\partial r}$ is holomorphic and thus there is a natural real holomorphic vector field

$$K = Jr\frac{\partial}{\partial r}, \tag{3.29}$$

usually called the *Reeb vector field.* Let ω be the Kähler form associated to (g, J). The Reeb vector field K is Hamiltonian, with potential given by $r^2/2$:

$$i_K\omega = -\frac{1}{2}dr^2. \tag{3.30}$$

In fact ω is always $\partial\bar{\partial}$-exact:

$$\omega = i\partial\bar{\partial}r^2. \tag{3.31}$$

We now suppose that (M, ω) is symplectic toric, which we recall means that there is an effective action of the real n-dimensional torus T^n on M which is Hamiltonian with respect to ω. The basic result is that *the Delzant correspondence holds for cones* [11, 50, 56, 17, 16]. What this means, in sum, is as follows. The moment map $\mu : M \to \mathfrak{t}^*$ always has image equal to a rational polyhedral cone C. Recall from Section 2.1.2 that associated to any rational polyhedral cone $C \subset \mathfrak{t}^*$ is a unique affine toric variety $\mathcal{M}_C$. By the same process as in Section 3.2.1, we have the symplectic and holomorphic structures (M, ω) and (M, J) are (independently) isomorphic to that of a unique Kähler quotient (M_C, J_C, ω_C) of $\mathbb{C}^N$ [11, 50, 17]. That is, we have $(M, \omega) \cong (M_C, \omega_C)$ and $(M, J) \cong (M_C, J_C) \cong \mathcal{M}_C$.

In sum, we know that, up to biholomorphism, we may view g as a Kähler metric on the affine toric variety $\mathcal{M}_C$. In particular, the T^n-action always complexifies. It is not difficult to compte [56] that whenever (M, ω) is symplectic toric, moreover, the Reeb vector field K necessarily lies in $\mathfrak{t}$. One of the key results of [56] is:

Lemma 3.48 ([56]). *The Reeb vector field $K \in \mathfrak{t}$ is determined by an element of the dual cone $C^* \subset \mathfrak{t}$ of $C \subset \mathfrak{t}^*$.*

3.4.2 Asymptotically conical toric manifolds

Let $P \subset \mathfrak{t}^*$ be a Delzant polyhedron with recession cone C, and suppose that the $C \subset \mathfrak{t}^*$ has full dimension, so that we have toric varieties $\mathcal{M}_P$ and $\mathcal{M}_C$ corresponding to the normal fans Σ_P and Σ_C. Now clearly every cone $\sigma \in \Sigma_C$ also lies in Σ_P, and this induces a morphism of varieties $\pi : \Sigma_P \to \Sigma_C$ which restricts to the identity on the dense orbit of $\mathcal{M}_P$. In fact, if we let E be the union of all compact torus-invariant divisors in $\mathcal{M}_P$, then $\pi : \mathcal{M}_P - E \to \mathcal{M}_C - \{o\}$ is an isomorphism and hence π is a $(\mathbb{C}^*)^n$-equivariant resolution of singularities [21, Chapter 11]. In particular, if we remove the unique fixed point o of the $(\mathbb{C}^*)^n$-action on $\mathcal{M}_C$, then $\mathcal{M}_C - \{o\}$ is smooth [1].

The smooth complex manifold (M_C, J_C) corresponding to $\mathcal{M}_C - \{o\}$ is then diffeomorphic to $\mathbb{R} \times S$ for some compact smooth manifold S. This decomposition is by no means unique, but in any case we can equip M_C with various Riemannian cone metrics. Let $(M, J) = (M_P, J_P)$ be the complex toric manifold associated to $\mathcal{M}_P$. We introduce a class of Kähler metrics on M_P which are asymptotic to some such Riemannian cone metric on M_C.

[1] It is perhaps not hard to see that this is an intrinsic combinatorial property of the cone $\mathbb{C} \subset \mathfrak{t}^*$ which it inherits from the Delzant condition on P. In the literature, a cone C of this type is referred to as a *good cone* [50, 56]

Definition 3.49. Fix a real holomorphic vector field X on (M, J) such that $JX \in \mathfrak{t}$. Let $\pi : M \to M_C$ be the $(\mathbb{C}^*)^n$-equivariant resolution of singularities above. We say that a T^n-invariant Kähler metric ω with corresponding metric g on (M, J) lies in the class $\mathcal{A}_{P,X}$ of *asymptotically conical* metrics if

1. $\omega \in [\omega_P] = 2\pi c_1(L_P)$,
2. the T^n-action is Hamiltonian,
3. $\pi_* X = r\frac{\partial}{\partial r}$, and
4. there exists a Riemannian cone metric g_0 on M_C and compact subsets $K_1 \subset M$, $K_2 \subset M_C$ such that $\pi|_{M-K_1} : M - K_1 \to M_C - K_2$ is a biholomorphism and
$$|(\nabla^{g_0})^k(\pi_* g - g_0)|_{g_0} < C_k r^{-k-1-\varepsilon}, \tag{3.32}$$
for $k = 0, 1$ and some $\varepsilon > 0$ on all of $M_C - K_2$.

We do not claim that this rate is optimal, we simply make a convenient choice for this basic discussion. It is straightforward to verify that g is T^n-invariant if and only if g_0 is T^n-invariant, and that any such g_0 is necessarily Kähler with respect to J_C.

Lemma 3.50. *Let $\pi : M \to M_C$ be an equivariant resolution of a toric Kähler cone M_C. Suppose that we have a T^n-invariant Kähler metric g on M, a Riemannian metric g_0 on M_C, and $\varepsilon > 0$ such that*
$$|(\nabla^{g_0})^k(\pi_* g - g_0)|_{g_0} \le C_k r^{-\varepsilon-k},$$
on $M - K_1 \cong M_C - K_2$, where r is the radial coordinate on M_C, and $k = 0, 1$. If there is a holomorphic vector field X on M which commutes with the T^n action and such that $\pi_* X = r\frac{\partial}{\partial r}$, then g_0 is a toric metric on M_C, i.e. it is Kähler with respect to J_C and invariant under the T^n action.

Proof. The lemma essentially follows from the fact that there exists is a one parameter family of holomorphic automorphisms $r_\lambda : M \to M$, commuting with the T^n action, which descend via π to the scaling of the radial coordinate of M_C by λ. The asymptotically conical condition for $k = 0$ immediately implies that pointwise
$$\pi^* g_0 = \lim_{\lambda\to\infty} \lambda^{-2} r_\lambda^* g,$$
since $\lambda^{-2} r_\lambda^* g_0 = g_0$. Therefore if $\tau \in T^n$,
$$\tau^*\pi^* g_0 = \lim_{\lambda\to\infty} \lambda^{-2}\tau^* r_\lambda^* g = \lim_{\lambda\to\infty} \lambda^{-2} r_\lambda^* g = \pi^* g_0.$$

Since $\pi : M - E \to M_C - \{o\}$ is an equivariant biholomorphism, it follows that $\tau^* g_0 = g_0$. The same line of reasoning shows that J_C is g_0-orthogonal. Once we know this, the same argument repeated again, this time with $k = 1$ shows that J_C is g_0-parallel.

□

In particular, there is a Kähler form ω_0 and Reeb vector field $K = J_C r\frac{\partial}{\partial r}$ associated to g_0 on M_C, and by construction $\pi_* JX = K$. In what follows we will drop the dependence on π as a matter of notation, and simply consider $JX = K$, $J = J_C$, g, and g_0 on $M_C - K_2$ directly.

Lemma 3.51. *Let f be the Hamiltonian potential for JX on M. Then there exists a constant C such that*

$$\sup_{p \in M_C - K_2} \left| f - \frac{r^2}{2} \right| < C. \tag{3.33}$$

Proof. First we show that $|i_{JX}(\omega - \omega_0)|_{g_0} \leq Cr^{-\varepsilon}$. Note that $|JX|^2_{g_0} = |K|^2_{g_C} = \left|r\frac{\partial}{\partial r}\right|^2_{g_C} = r^2$. This implies

$$|i_{JX}(\omega - \omega_0)|^2_{g_0} = |i_X(g - g_0)|^2_{g_0} \leq |X|^2_{g_0}|g - g_0|^2_{g_0} \leq Cr^{-2-\varepsilon}.$$

Since f and $r^2/2$ are Hamiltonian potentials for JX with respect to ω and ω_0 respectively, we see that

$$\begin{aligned} \left|\frac{\partial}{\partial r}\left(f - \frac{r^2}{2}\right)\right| &= \left|d\left(f - \frac{r^2}{2}\right)\left(\frac{\partial}{\partial r}\right)\right| = \left|i_{JX}\left(\omega - \omega_0\right)\left(\frac{\partial}{\partial r}\right)\right| \\ &= \left|\langle i_{JX}(\omega - \omega_0), dr\rangle_{g_0}\right| \leq |i_{JX}(\omega - \omega_0)|^2_{g_0} \leq Cr^{-2-\varepsilon}, \end{aligned}$$

since dr is the dual one-form to $\frac{\partial}{\partial r}$ with respect to g_0 and $|dr|^2_{g_0} = \left|\frac{\partial}{\partial r}\right|^2_{g_0} = 1$. Choosing a large enough so that $K_2 \subset \{r < a\} \subset M_C$, we obtain

$$\begin{aligned} \left|f(r) - \frac{r^2}{2}\right| &\leq \left|\int_a^r \frac{\partial}{\partial s}\left(f(s) - \frac{s^2}{2}\right) ds\right| + C \leq \int_a^r \left|\frac{\partial}{\partial s}\left(f(s) - \frac{s^2}{2}\right)\right| ds + C \\ &\leq C\int_a^r s^{-2-\varepsilon} ds + C \leq C_1 - C_2 r^{-1-\varepsilon} \leq C \end{aligned}$$

□

In particular, the Hamiltonian potential f is proper and bounded from below. Therefore we automatically have

Corollary 3.52. *If $\omega \in \mathcal{A}_{P,X}$, then (M, J, ω) is AK-toric.*

Chapter 4

Kähler-Ricci solitons

We begin this chapter with an introduction to the study of Kähler-Ricci solitons on toric manifolds. We show that our main uniqueness result Theorem 1.3 can be reduced to a similar uniqueness question of independent interest on AK-toric manifolds. We give a description of solitons in terms of toric data, and reduce the equation (4.2) to two real equivalent real Monge-Ampère equations, depending on whether we take the complex or the symplectic viewpoint. As an important consequence, we prove that the real Monge-Ampère equation completely determines the image of the moment map for a Kähler-Ricci soliton on an AK-toric manifold.

Section 4.2 is dedicated to the Futaki invariant. Originally defined in [37], provides an obstruction to the existence of Kähler-Einstein metrics on compact Fano manifolds. This was generalized by Tian-Zhu [68] to the case of shrinking Kähler-Ricci solitons. In principle this version still provides an obstruction, but in this setting it has the added benefit of allowing us to characterize the soliton vector field X. This was further generalized to the non-compact setting, originally by Martelli-Sparks-Yau [56, 55] (c.f. [18]) for Sasaki-Einstein metrics and then Conlon-Deruelle-Sun [19] for Kähler-Ricci solitons. We show, independently of [19], that there is a reasonable sense in which the Futaki invariant can be defined and use this to prove uniqueness of the soliton vector field X, thus recovering their work in the toric case. Moreover, we show that a holomorphic vector field with respect to which the Futaki invariant vanishes always exists.

We move on to discuss a functional $\mathcal{D}$, introduced by Ding in [25]. The Ding functional has played an increasingly important role in the study of special metrics in Kähler geometry, particularly since the work of Berman [7] making the connection with algebro-geometric stability. It was shown by Berndtsson [9, 10] that $\mathcal{D}$ is convex along geodesics in the space of Kähler metrics on a compact Fano manifold. In the toric setting, these goedesics take a particularly simple form when we pass to the symplectic viewpoint. The properness of the Ding functional was first established by Tian [67] for compact Fano manifolds not admitting any holomorphic vector fields (see also [59]). The Ding functional was extended to treat solitons with non-trivial vector field by Tian-Zhu [68], and the properness of $\mathcal{D}$ was eventually shown to be equivalent to the existence of a soliton by Cao-Tian-Zhu [12]. We begin Section

4.3 by introducing a suitable space $\mathcal{P}$ of Kähler metrics on which much of the picture goes through. We show that the Ding functional can be defined on $\mathcal{P}$, and we prove convexity and properness statements for $\mathcal{D}$ along the lines of [8, 43] from which we can deduce one of our main theorems 1.3

4.1 Kähler-Ricci solitons on a toric manifold

4.1.1 Background

Let (M, J) be a complex manifold.

Definition 4.1. A pair (ω, X) is a *shrinking Kähler-Ricci soliton* if ω is a Kähler metric, X is a real holomorphic vector field, and

$$\mathrm{Ric}_\omega + \frac{1}{2}\mathcal{L}_X\omega = \omega. \tag{4.1}$$

A Kähler-Ricci soliton is *gradient* if $X = \nabla^g f$ for a smooth function $f \in C^\infty(M)$.

Lemma 4.2. *A shrinking gradient Kähler-Ricci* (ω, X) *soliton satisfies*

$$Ric_\omega + i\partial\bar{\partial} f = \omega. \tag{4.2}$$

Proof. This is just the statement that, if f is any smooth function, and if $X = \nabla^g f$, then $\frac{1}{2}\mathcal{L}_X\omega = i\partial\bar{\partial}f$. We compute

$$\begin{aligned} Jdf(V) &= df(JV) = -i_{JX}\omega(JV) \\ &= -\omega(JX, JV) = -\omega(X, V) = -i_X\omega(V), \end{aligned}$$

so that

$$\frac{1}{2}\mathcal{L}_X\omega = \frac{1}{2}di_X\omega = -\frac{1}{2}dJdf = id\frac{1}{2}\left(df + iJdf\right) = i\partial\bar{\partial}f.$$

□

Remark 4.3. A shrinking Kähler-Ricci soliton being gradient is actually equivalent to the equation

$$\mathrm{Ric}_\omega + \frac{1}{2}\mathcal{L}_{X^{1,0}}\omega = \omega.$$

This is the definition used in Tian-Zhu [69, 68], for example.

A Kähler-Ricci soliton is *complete* if both the metric ω and the vector field X are complete. However, we have from [72] that the soliton vector field X is complete whenever the metric is, so this is really only a condition on ω.

4.1.2 Reduction to the AK-toric case

This section will be occupied with the proof of the following theorem, which we originally proved in [15].

Theorem 4.4 ([15]). *Let (M, J) be a complex manifold with an effective and holomorphic action of the real torus T^n with finite non-empty fixed point set. Suppose that $JX \in \mathfrak{t}$, and that (ω, X) is a complete shrinking gradient Kähler-Ricci soliton with bounded Ricci curvature. Then there exists an automorphism $\alpha \in Aut(M, J)$ of (M, J) such that $(M, J, \alpha^*\omega)$ is AK-toric. In particular, (M, J) is quasiprojective.*

Throughout this section, (M, J, ω, X) will be as in Theorem 4.4, and g will be the corresponding Riemannian metric. The proof consists of several steps. We first show that the T^n-action automatically complexifies. To do this, we introduce some notation from [19].

Notation 4.5. We let $\mathfrak{aut}^X$ denote the space of holomorphic vector fields on M which commute with the soliton vector field X, and $\mathfrak{g}^X \subset \mathfrak{aut}^X$ be the subspace of real holomorphic Killing fields commuting with X.

Then we have the following general structure theorem from [19]:

Theorem 4.6 ([19, Theorem 5.1]). *For (M, J, ω, X) as in Theorem 4.4, we have*

$$\mathfrak{aut}^X = \mathfrak{g}^X \oplus J\mathfrak{g}^X. \tag{4.3}$$

Furthermore, $\mathfrak{aut}^X$ and $\mathfrak{g}^X$ are the Lie algebras of finite-dimensional Lie groups Aut^X and G^X corresponding to holomorphic automorphisms and holomorphic isometries commuting with the flow of X. Moreover, the identity component G_0^X of G^X is maximal compact in Aut^X

Lemma 4.7. *There exists a complexification of the T^n-action on M, i.e. an action of $(\mathbb{C}^*)^n$ whose underlying real torus corresponds to the original T^n-action.*

Proof. As always, fix a basis $(\Xi_1, \ldots, \Xi_n)$ for $\Gamma \subset \mathfrak{t}$. Since $JX \in \mathfrak{t}$, it is clear that $[X, \Xi_i] = [X, J\Xi_i] = 0$ for any i. In particular, $\mathfrak{t} \subset \mathfrak{aut}^X$. Since the scalar curvature of g is bounded by assumption, we have by [19, Lemma 2.26] that the zero set of X is compact. Therefore by [19, Lemma 2.34], it follows that for each i, Ξ_i and $J\Xi_i$ are complete. In particular, the flow of $(\Xi_i, J\Xi_i)$ determines a unique effective and holomorphic action of $\mathbb{C}^*$. Repeating this process for each i, we can integrate out the system $(\Xi_1, J\Xi_1, \ldots, \Xi_n, J\Xi_n)$ to obtain an effective and holomorphic $(\mathbb{C}^*)^n$-action, whose corresponding action of the underlying real torus coincides with the original one by construction. □

Lemma 4.8. *There exists an automorphism α of (M, J) such that $\alpha^*X = X$ and α^*g is T^n-invariant.*

Proof. By construction, the $(\mathbb{C}^*)^n$-action on M of Lemma 4.7 satisfies $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus J\mathfrak{t} \subset \mathfrak{aut}^X$. It follows that the $(\mathbb{C}^*)^n$-action $\tau_{\mathbb{C}}$ on M embeds $(\mathbb{C}^*)^n \subset Aut^X$, and so the real torus $T^n \subset (\mathbb{C}^*)^n$ lies in some maximal compact subgroup G of Aut^X. Since any two maximal compact subgroups of a reductive group are conjugate by Iwasawa's theorem [45], it follows such that there exists an automorphism α sending G to G_0^X by conjugation. Consequently, $T^n \subset G$ preserves the metric $\alpha^* g$. The fact that $\alpha^* X = X$ follows because α is the time-one flow of a holomorphic vector field which commutes with X. □

Lemma 4.9. *The T^n-action is Hamiltonian with respect to $\alpha^*\omega$.*

Proof. By [71, Theorem 1.1] it follows that any manifold which admits a complete shrinking Ricci soliton must satisfy $H^1(M) = 0$. If as before $\Xi_1, \ldots, \Xi_n$ is our basis for $\mathfrak{t}$, let $\theta_j \in C^\infty(M)$ be smooth functions satisfying $-i_{\Xi_j}\omega = d\theta_j$. Then one defines a moment map explicitly by the formula $\mu(x) = (\theta_1, \ldots, \theta_n)$. □

Remark 4.10. There is of course an ambiguity in the choice of each θ_j of the addition of a constant. Put together, this corresponds to a translation of the image $\mu(M) \subset \mathfrak{t}^*$. As we saw in Section 3.3, this can be controlled for at the level of the Kähler potential ϕ, and we will see in later sections that there is a natural choice of such normalization.

By definition, we have that $X = \nabla^g f$, and so we see that $X = \nabla^{\alpha^* g}\alpha^* f$. It follows then that

$$d\alpha^* f = i_X \alpha^* g = -i_{JX}\alpha^*\omega, \tag{4.4}$$

so that $\alpha^* f$ is a Hamiltonian potential for the flow of JX with respect to $\alpha^*\omega$. If we let $JX = b_X \in \mathfrak{t}$, it follows that, up to a constant

$$\alpha^* f = \langle \mu, b_X \rangle,$$

where $\mu : M \to \mathfrak{t}^*$ is the moment map for the T^n-action with respect to $\alpha^*\omega$ of Lemma 4.9. The final piece we need to prove Theorem 4.4 is the following.

Proposition 4.11 ([13, Theorem 1.1])**.** *Let (M, g, f) be any non-compact complete shrinking gradient Ricci soliton. The soliton potential f grows quadratically with respect to the distance function d_g defined by g, so there is a constant c_f such that*

$$\frac{1}{4}(d_p - c_f)^2 \leq f \leq \frac{1}{4}(d_p + c_f)^2.$$

Proof of Theorem 4.4. Lemmas 4.8 and 4.9 establish the existence of a holomorphic automorphism $\alpha : (M, J) \to (M, J)$ such that $\alpha^* X = X$ and (M, ω) is symplectic toric. By Lemma 4.7, there is an effective holomorphic $(\mathbb{C}^*)^n$-action extending the original T^n-action. Proposition 4.11 guarantees that the soliton potential $\alpha^* f$ corresponding to $\alpha^* g$ is proper and bounded from below. Thus $(M, J, \alpha^*\omega)$ is AK-toric, and so by Proposition 3.33 (M, J) must be quasiprojective. □

Remark 4.12. Recall that, by Lemma 3.32, we know that the image of the moment map $\mu : M \to \mathfrak{t}^*$ is a Delzant polyhedron P, and that Proposition 3.33 actually guarantees that (M, J) is biholomorphic to the toric variety $\mathcal{M}_P$ associated to P. By Theorem 3.36, there is an effective holomorphic line bundle $L_P \cong \mathcal{O}(D) \to M$ such that some multiple of ω lies in $c_1(L_P)$.

4.1.3 The equation in logarithmic coordinates

By Theorem 4.4, it suffices to work in the AK-toric setting for the purposes of studying shrinking gradient Kähler-Ricci solitons, at least if they have bounded Ricci curvature. In this section, we work with a given AK-toric manifold (M, J, ω) and a holomorphic vector field X such that (ω, X) define a shrinking Kähler-Ricci soliton. We make no assumption on the curvature at this stage. As always, we let $U \cong (\mathbb{C}^*)^n \subset M$ be the dense orbit in M with coordinates $(z_1, \dots, z_n)$. Recall that, since the T^n-action is Hamiltonian, we can write

$$\omega = 2i\partial\bar{\partial}\phi = \frac{\partial^2\phi}{\partial\xi^i\partial\xi^j} d\xi^i \wedge d\theta^j, \tag{4.5}$$

where $\phi \in C^\infty((\mathbb{C}^*)^n)$ comes from a smooth strictly convex function $\phi : \mathfrak{t} \to \mathbb{R}$ (see Lemma 3.8). We begin with the following

Lemma 4.13. *If ω is a T^n-invariant shrinking gradient Kähler-Ricci soliton on an AK-toric manifold with soliton vector field X, then $JX \in \mathfrak{t}$.*

Proof. Let $f : M \to \mathbb{R}$ be a smooth function so that $X = \nabla^g f$. Since ω and hence Ric_ω are T^n-invariant, we know from (4.2) that $i\partial\bar{\partial}f$ is also T^n-invariant. By averaging f over the torus, we can assume that f, and thereby $X = \nabla^g f$, is also T^n-invariant. So we are in a good setting to interpret this data in logarithmic coordinates. Let $w = \log(z)$ be local holomorphic logarithmic coordinates on $(\mathbb{C}^*)^n$ (3.2), so that $\xi = \mathrm{Re}(w)$ and $\theta = \mathrm{Im}(w)$ are our real logarithmic coordinates on $(\mathbb{C}^*)^n \cong \mathfrak{t} \times T^n$. Recall that the metric g is given by (3.4):

$$g = \phi_{ij} d\xi^i \otimes d\xi^j + \phi_{ij} d\theta^i \otimes d\theta^j,$$

where $\phi_{ij}(\xi) = \frac{\partial^2\phi}{\partial\xi^i\partial\xi^j}$. Since f satisfies $\frac{\partial f}{\partial\theta^j} = 0$ for all $j = 1, \dots, n$, it follows that $f = f(\xi)$ is independent of θ (c.f. Lemma 3.2). Therefore

$$X = \nabla^g f = \phi^{ij} \frac{\partial f}{\partial\xi^j} \frac{\partial}{\partial\xi^i} \tag{4.6}$$

Moreover, if we write

$$X^{1,0} = X^i(w) \frac{\partial}{\partial w_i},$$

it follows that

$$X^i(w) = \phi^{ij} \frac{\partial f}{\partial\xi^j}$$

is also independent of θ and *real-valued*. Thus, since X is holomorphic,

$$0 = 2\frac{\partial}{\partial \bar{w}_k} X^i = \frac{\partial}{\partial \xi^k}\left(\phi^{ij}\frac{\partial f}{\partial \xi^j}\right).$$

We see therefore that the coefficients $X^i(w) = \phi^{ij}\frac{\partial f}{\partial \xi^j} = b^i_X \in \mathbb{R}$ are constant. From (4.6) we see that

$$X = b^i_X \frac{\partial}{\partial \xi^i}, \tag{4.7}$$

so that

$$JX = b^i_X \frac{\partial}{\partial \theta^i} \in \mathfrak{t}.$$

□

Notation 4.14. We will let $b_X = (b^1_X, \ldots, b^n_X) \in \mathfrak{t}$ denote the coefficients of X.

We can use this to rewrite the soliton equation (4.1) on the dense orbit $U \cong (\mathbb{C}^*)^n \subset M$ in logarithmic coordinates.

Proposition 4.15. *Suppose that (M, J, ω) is an AK-toric manifold and that X is a holomorphic vector field on M such that (ω, X) is a shrinking gradient Kähler-Ricci soliton. Then there exists a unique affine function $a(\xi)$ on $\mathfrak{t}$ such that $\phi_a = \phi - a$ satisfies the real Monge-Ampère equation*

$$\det(\phi_a)_{ij} = e^{-2\phi_a + \langle \nabla \phi_a, b_X\rangle}. \tag{4.8}$$

Proof. Using (4.5) and (3.3), we see that

$$\mathrm{Ric}_\omega = -i\partial\bar{\partial}\log\det\left(\frac{\partial^2\phi}{\partial w_i \partial \bar{w}_j}\right) = -\frac{1}{2}\frac{\partial^2}{\partial \xi^i \partial \xi^j}\left(\log\det\left(\frac{\partial^2 \phi}{\partial \xi^i \partial \xi^j}\right)\right) d\xi^i \wedge d\theta^j.$$

Moreover, from (4.7), we have

$$\mathcal{L}_X \omega = 2i\partial\bar{\partial}\mathcal{L}_X\phi = 2i\partial\bar{\partial} X\cdot\phi = \frac{\partial^2}{\partial\xi^i\partial\xi^j}\langle\nabla\phi, b_X\rangle d\xi^i \wedge d\theta^j.$$

Combining all this with (4.1)

$$\omega - \mathrm{Ric}_\omega - \frac{1}{2}\mathcal{L}_X\omega = 0$$

we get

$$\begin{aligned} 0 &= i\partial\bar{\partial}\left(2\phi + \log\det(\phi_{ij}) - \langle\nabla\phi, b_X\rangle\right) \\ &= \frac{1}{2}\frac{\partial^2}{\partial\xi^i\partial\xi^j}\left(2\phi + \log\det(\phi_{ij}) - \langle\nabla\phi, b_X\rangle\right) d\xi^i \wedge d\theta^j, \end{aligned}$$

and so the function $2\phi + \log\det(\phi_{ij}) - \langle \nabla\phi, b_X\rangle$ on $\mathfrak{t}$ has vanishing Hessian, and is therefore equal to an affine function $a(\xi)$. Define

$$\phi_a(\xi) = \phi(\xi) - 2a(\xi)$$

and let c be the constant $c = 2\langle \nabla a, b_X\rangle$. Then it is clear that

$$2\phi_a + \log\det(\phi_{a,ij}) - \langle \nabla\phi_a, b_X\rangle = c.$$

Thus, by modifying a by the addition of a further constant, we have that ϕ_a satisfies (4.8). □

4.1.4 The equation in symplectic coordinates

Since (M, J, ω) is AK-toric, we know from Section 3.2 that the image of the moment map $\mu : M \to \mathfrak{t}^*$ is a Delzant polyhedron P. We can therefore, as in Section 3.3, use the moment map $\mu = \nabla\phi$ to change coordinates on $(\mathbb{C}^*)^n \cong \mathfrak{t} \times T^n \cong P \times T^n$. We would next like to understand how the soliton equation (4.8) behaves under this change. Applying Lemma 3.37 directly, one obtains

Proposition 4.16. *Let (M, J, ω) be AK-toric with moment polyhedron P, and suppose that (ω, X) is a shrinking gradient Kähler-Ricci soliton. Let $\phi \in C^\infty(\mathfrak{t})$ be the Kähler potential on the dense orbit which satisfies* (4.8)*. Then the corresponding symplectic potential $u = L(\phi) \in C^\infty(P)$ satisfies*

$$2(u_i x^i - u(x)) - \log\det(u_{ij}) = \langle x, b_X\rangle, \tag{4.9}$$

where $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$.

Recall that the moment polyhedron $P \subset \mathfrak{t}^*$ is defined uniquely only up to translation in $\mathfrak{t}^*$ from the original data (M, J, ω). By Proposition 4.15, however, using the fact that ω satisfies (4.2), we have a natural choice of normalization by demanding that the Kähler potential ϕ satisfies (4.8). This fixes our moment map $\mu = \nabla\phi$ uniquely.

In fact, we can describe the image P under this normalization explicitly. Recall from Remark 4.12 that M is biholomorphic to $\mathcal{M}_\Sigma$ where $\Sigma = \Sigma_P$ is the normal fan of the polyhedron P. Let $\nu_1, \dots, \nu_N$ be the corresponding inner normals (see Definition 3.9). Let D_i be the divisor associated to ν_i as guaranteed by the Orbit-Cone correspondence. From Lemma 2.25, we have

Lemma 4.17. *The divisor $D = \sum_{i=1}^N D_i$ represents the anticanonical class, i.e.*

$$\mathcal{O}\left(\textstyle\sum_{i=1}^N D_i\right) \cong -K_M. \tag{4.10}$$

This brings us to one of our main results of great practical significance. The observation is originally due to Donaldson [29], and by the results in Chapter 3 can be made to hold in the non-compact setting as well.

Theorem 4.18. *Let (M, J, ω) be AK-toric with moment polyhedron P, X be a holmorphic vector field such that (ω, X) is a shrinking gradient Kähler-Ricci soliton, and let $\phi \in C^\infty(\mathfrak{t})$ be the unique Kähler potential on the dense orbit which satisfies* (4.8)*. Then the image of $\nabla\phi : \mathfrak{t} \to \mathfrak{t}^*$ is precisely the polyhedron P_{-K_M} associated to the anticanonical divisor $-K_M = \sum_{i=1}^N D_i$, i.e.*

$$P_{-K_M} = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x\rangle \geq -1,\ i = 1, \dots, N\}\,. \tag{4.11}$$

Proof. We know that the image of $\nabla\phi$ is equal to some Delzant polyhedron P. We first claim that P is at least some translate in $\mathfrak{t}^*$ of P_{-K_M}. Since the metric satisfies (4.2), we can see that ω is the curvature form of the T^n-invariant hermitian metric h_X on $-K_M$ given by

$$h_X = e^{-f}\,(\omega^n)^{-1}\,.$$

The fact that P is a translate of P_{-K_M} now follows from Corollary 3.35. Next we nail down P explicitly. In our background coordinate system $(x_1, \dots, x_n)$ on $\mathfrak{t}^*$ we can, by definition, write

$$P = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x\rangle \geq -a_i\}$$

for some $a_i \in \mathbb{R}$, where $\nu_i \in \Gamma$, $i = 1, \dots, N$ is the set of inner normals. By Proposition 4.15, we know that the symplectic potential $u = L(\phi) \in C^\infty(P)$ satisfies

$$2(u_i x^i - u(x)) - \log\det(u_{ij}) = \langle x, b_X\rangle,$$

where $b_X \in \mathfrak{t}$ is the point corresponding to the real holomorphic vector field JX. For simplicity of notation, let $\rho(x) \in C^\infty(P)$ denote the auxiliary function

$$\rho(x) = 2(u_i x^i - u(x)) - \log\det(u_{ij}), \tag{4.12}$$

so that the equation 4.9 takes the form $\rho = \langle x, b_X\rangle$. Recall that the canonical symplectic potential $u_P(x)$ is given by

$$u_P(x) = \frac{1}{2}\sum(\ell_i(x) + a_i)\log(\ell_i(x) + a_i).$$

By Proposition 3.46, the fact that the Kähler metric $\omega = 2i\partial\bar{\partial}\phi$ on $(\mathbb{C}^*)^n$ extends to metric on $M \cong M_P$ implies that there exists a smooth function $v \in C^\infty(\overline{P})$ such that

$$u = u_P + v.$$

Fix any facet F of P'. We may assume that F is given by $\ell_1(x) = -a_1$. Up to a change of basis in $\mathfrak{t}^*$ (which notably does *not* affect the coefficients a_i, see Lemma 3.39), we may also assume by the Delzant condition that $\ell_1(x) = x_1$. Choose a point p in the interior of F. Near p, u_P can therefore be written

$$u_P(x) = \frac{1}{2}(x_1 + a_1)\log(x_1 + a_1) + v_1,$$

where v_1 extends smoothly across F. The symplectic potential u itself therefore is given by

$$u = u_P + v = \frac{1}{2}(x_1 + a_1)\log(x_1 + a_1) + v_1 + v.$$

It then follows that in a small half ball B in the interior of P' containing p, ρ_u can be expressed as

$$\rho_u(x) = x_1 \log(x_1 + a_1) - (x_1 + a_1)\log(x_1 + a_1) + \log(x_1 + a_1) + v_2,$$

where v_2 again extends smoothly across F in B. Since $\rho_u = \langle x, b_X\rangle$, and the linear function $\langle x, b_X\rangle$ clearly extends past the boundary of P, it follows that $a_1 = 1$. □

In accordance with the work of Section 3.2 (see Theorem 3.23, Proposition 3.24, Theorem 3.36), we define:

Definition 4.19. The pair $(\mathcal{M}, -K_{\mathcal{M}})$ for $\mathcal{M}$ a toric variety is called *anticanonically polarized* if $\mathcal{M} \cong \mathcal{M}_P$ where $P = P_{-K_{\mathcal{M}}}$.

Thus, Theorem 4.18 says that any toric manifold which admits complete shrinking gradient Kähler-Ricci soliton is an anticanonically polarized toric variety. Notice that the polyhedron P_{-K_M} always contains the origin. As it turns out, this is an important property, especially in the non-compact setting. To see this, let $P \subset \mathfrak{t}^*$ be any Delzant polyhedron containing zero in its interior. The function $\rho(x)$ of the previous proof turns out to have some interesting and useful properties that will help us study convex functions on P, and thereby Kähler metrics on M_P.

Notation 4.20. Given a function $u \in C^\infty(P)$ we will denote by ρ_u the function

$$\rho_u(x) = 2(\langle x, \nabla u\rangle - u(x)) - \log\det(u_{ij}). \tag{4.13}$$

Lemma 4.21. *Let u be a smooth and strictly convex function on P, and let $\phi = L(u)$ be the Legendre transform. Then*

$$\int_P e^{-\rho_u} dx = \int_{\nabla u(P)} e^{-2\phi} d\xi,$$

and consequently

$$\int_P e^{-\rho_u} dx < \infty.$$

To prove this, we begin with the following useful elementary lemma.

Lemma 4.22. *Let ϕ be a strictly convex function on an open convex domain $\Omega' \subset \mathbb{R}^n$. Let u be its Legendre transform defined on Ω. If $0 \in \Omega$, then there exists a $C > 0$ such that*

$$\phi(\xi) \geq C^{-1}|\xi| - C. \tag{4.14}$$

In particular, ϕ is proper.

For smooth functions, this is fairly straightforward. If $0 \in \Omega$, then there is a point $\xi^* \in \mathbb{R}^n$ such that $\nabla\phi(\xi^*) = 0$. Since ϕ is strictly convex, ϕ obtains its unique minimum at ξ^*. Considering then the values of ϕ over a small ball centered at ξ^*, using convexity again one easily establishes (4.14). We will not need this result for more general convex functions in this dissertation, but the same basic idea holds true, see for example [8] for a proof.

Proof of Lemma 4.21. We apply the change of coordinates $x = \nabla\phi_u(\xi)$, where ξ denotes coordinates on the domain $\Omega \subset \mathbb{R}^n$ of ϕ. Then from Lemma 3.37 we have

$$\det(u_{ij})dx = d\xi,$$

and

$$u - \langle \nabla u, x\rangle = -\phi(\xi).$$

Therefore

$$\int_P e^{-\rho_u} dx = \int_\Omega e^{-2\phi} d\xi.$$

Then from Lemma 4.22 we know that $e^{-\phi}$ is integrable on Ω.

□

Remark 4.23. In the situation of Lemma 4.21, even if $\Omega \neq \mathbb{R}^n$, since ϕ satisfies (4.14) we can extend the function $e^{-2\phi}$ from Ω to all of $\mathbb{R}^n$ by zero continuously. As such, whenever we write

$$\int_{\mathbb{R}^n} e^{-2\phi} d\xi,$$

we are referring to this choice of extension, which clearly does not affect the integral.

Corollary 4.24. *Let $P \subset \mathbb{R}^n$ be a Delzant polyhedron, $M = M_P$, and suppose that ω is a T^n-invariant Kähler metric with P as its moment polyhedron. Let v be the holomorphic vector field on M determined by $b_v \in \mathfrak{t}$ and θ_v be a Hamiltonian potential for Jv. Then*

$$\int_M e^{-\theta_v}\omega^n < \infty$$

if and only if b_v lies in the dual recession cone C^*.

Proof. We work on the dense orbit in symplectic coordinates $(\mathbb{C}^*)^n \cong P \times T^n$ in which case ω is given simply by $\omega = \sum dx^i \wedge d\theta^i$. The integral then becomes

$$\int_{(\mathbb{C}^*)^n} e^{-\theta_v}\omega^n = \int_{P\times T^n} e^{-\langle x, b_v\rangle} dx d\theta = (2\pi)^n \int_P e^{-\langle x, b_v\rangle} dx.$$

We prove that this integral is finite this is finite precisely when $b_v \in C^*$. Let $C \subset \mathfrak{t}^*$ be the recession cone of P with dual cone C^*. It follows immediately from the definition 2.6, 2.1

that the interior of C^* is characterized by those $b \in \mathfrak{t}$ such that the linear function $\langle x, b\rangle$ on P is positive outside of a compact set. Indeed, for each $b \in \mathfrak{t}$, set

$$H_b = \{x \in \mathbb{R}^n \mid \langle x, b\rangle \leq 0\},$$

and

$$Q_b = H_b \cap \overline{P}.$$

Then Q_b is compact if and only if $\langle b, \nu_i\rangle > 0$ for each inner normal ν_i of P. Since C is spanned by the inner normals $\{\nu_i\}$, it follows that Q_b is compact if and only if $b \in C^*$.

□

Corollary 4.25. *Let P be a Delzant polyhedron containing zero in its interior, and suppose that there exists a solution $u \in C^\infty(P)$ to (4.9). Then the element $b_X \in \mathfrak{t}$ determining JX lies in C^*.*

Proof. Since P contains zero in its interior, we have by Lemma 4.21 that

$$\int_P e^{-\rho_u} dx < \infty.$$

Since u satisfies $\rho_u = \langle x, b_X\rangle$, we have

$$\int_P e^{-\langle x, b_X\rangle} dx < \infty.$$

Since the restriction of the Hamiltonian potential θ_X for JX to $P \times T^n$ is given by $\theta_X|_{P\times T^n} = \langle x, b_X\rangle$, it follows from Corollary 4.24 that $b_X \in C^*$.

□

4.2 The Futaki invariant

The results of the previous section can be summarized, loosely, by the statement

Motto 4.26. Shrinking Kähler-Ricci solitons on a toric manifold, compact or otherwise, can be studied entirely in terms of convex functions on a *fixed* polyhedron in $\mathbb{R}^n$.

Namely, given any shrinking gradient Kähler-Ricci soliton on a toric manifold M, there is a corresponding symplectic potential u on the polyhedron P_{-K_M} which completely determines the metric on M. Moreover, u satisfies the real Monge-Ampère equation (4.9). In the sequel we will see how in many cases it is advantageous to study real Monge-Ampère equations with the same form as (4.9) on the polyhedra directly, even when our primary interest is global.

4.2.1 The weighted volume functional

Suppose that (N, ω) be a Fano manifold with a given Kähler metric $\omega \in 2\pi c_1(N)$. Given a holomorphic vector field v, we can always find a *complex-valued* function θ_v such that $\bar{\partial}\theta_v = i_{v^{1,0}}\omega$, which exists by the Hodge theorem since compact manifolds with $c_1 > 0$ always satisfy $H^1(N) = 0$. We can then define a functional on the space of holomorphic vector fields on N by

$$F(v) = \int_N e^{-\theta_v}\omega^n.$$

In order for this to be well-defined of course we must have some normalization for θ_v. With an appropriate choice, it turns out that $F(v)$ is independent of choice of the metric ω in its cohomology class [68]. The modified Futaki invariant of [68] is then defined as the derivative $F_X : \mathfrak{h} \to \mathbb{C}$ of F at a given holomorphic vector field X. Then F_X is independent of the choice of reference metric, and in [68] it is shown that F_X must therefore vanish identically if X is the vector field corresponding to a Kähler-Ricci soliton on N. A necessary condition therefore for X to occur as the vector field of a shrinking gradient Kähler-Ricci soliton on N is that $F_X \equiv 0$.

It is shown in [19] that these ideas can be generalized to the non-compact setting in the presence of a complete shrinking gradient Kähler-Ricci soliton with bounded Ricci curvature. As in [19], we refer to F as the *weighted volume functional*. Suppose that a real torus T^k acts on M holomorphically and effectively with Lie algebra $\mathfrak{t}$, and that the soliton vector field X satisfies $JX \in \mathfrak{t}$. In this case, there is still a smooth function θ_v associated to each v with $Jv \in \mathfrak{t}$. In fact θ_v is real-valued, and hence is a Hamiltonian potential for the T^k-action. By the Duistermaat-Heckman theorem [32, 31, 61], there is an open cone $\Lambda \subset \mathfrak{t}$ where the weighted volume functional F, and thereby the Futaki invariant, can be defined. Moreover, the domain Λ can be naturally identified with the dual asymptotic cone of $\mu(M) \subset \mathfrak{t}^*$ (see [61, Definition A.2, Definition A.6]). Just as in [61], we shall see that Λ is in natural bijection with the space of Hamiltonian potentials which are proper and bounded below on M. In this setting, the soliton vector field X has the property that $JX \in \Lambda$ and is the unique critical point of F [19, Lemma 5.17]. This is analogous to the volume minimization principle of [55] for the Reeb vector field of a Sasaki-Einstein metric.

We begin with an elementary proposition:

Proposition 4.27. *Let $P \subset \mathfrak{t}^*$ be a Delzant polyhedron containing zero in its interior. Then there exists a unique linear function $\ell_P(x)$ determined by P such that*

$$\int_P \ell(x)e^{-\ell_P(x)}dx = 0 \tag{4.15}$$

for any linear function ℓ on P.

Proof. By Corollary 4.24, the function $e^{-\langle b,x\rangle}$ is integrable on P if and only if b lies in the interior of the dual recession cone C^*. Hence there is a well-defined function $F : C^* \to \mathbb{R}$ given by

$$F(b) = \int_P e^{-\langle b,x\rangle} dx.$$

Then

$$\frac{\partial}{\partial b^j} F = -\left(\int_P x^j e^{-\langle b,x\rangle} dx\right).$$

The function F is clearly convex which immediately gives uniqueness. To show existence, it suffices to show that F is proper. That is, given a sequence b_j in the interior of C^* such that either $|b_j| \to \infty$ or the sequence $\{b_j\}$ approaches a point on the boundary, we need to show that $F(b_j) \to \infty$. Consider the former case first. Using the natural inner product on $\mathfrak{t}$, we can view the dual recession cone C^* as sitting inside of $\mathfrak{t}^*$. Since $0 \in P$, the intersection $Q = -C^* \cap P$ has positive measure in $\mathbb{R}^n$. Now suppose that $\{b_j\}$ is any sequence in C^* such that $|b_j| \to \infty$. Let $y \in Q$ be a fixed point in the interior and choose ε sufficiently small so that $B_\varepsilon(y) \subset Q$ has strictly positive Euclidean distance to the boundary ∂Q. In particular, we then have that $\inf_{v \in S^{n-1} \cap \overline{C}^*} \langle v, -y\rangle > 0$. We choose ε sufficiently small so that $\delta = \inf_{v \in S^{n-1} \cap \overline{C}^*} \langle v, -y\rangle - \varepsilon > 0$. For any $x \in B_\varepsilon(y)$, write $x = y + rw$ for $r \in [0, \varepsilon)$ and $w \in S^{n-1}$. Then we have, for any $(b, x) \in C^* \times B_\varepsilon(y)$,

$$-\langle b, x\rangle \geq \langle b, -y\rangle - r|b||v| \geq \left(\left\langle \frac{b}{|b|}, -y\right\rangle - \varepsilon\right)|b| \geq \delta|b|.$$

Therefore, we see immediately that

$$F(b_j) = \int_P e^{-\langle b_j,x\rangle} dx \geq \int_{B_\varepsilon(y)} e^{-\langle b_j,x\rangle} dx \geq \int_{B_\varepsilon(y)} e^{\delta|b_j|} dx.$$

Since $|b_j| \to \infty$, we have then that $F(b_j) \to \infty$.

Consider now the latter case. The key point is that ∂C^* is defined by those $\bar{b} \in \mathbb{R}^n$ such that there exists at least one $\bar{c} \in \overline{C}$ with $\langle \bar{b}, \bar{c}\rangle = 0$. Choose $\bar{b} \in \partial C^*$. The result essentially follows from the fact that the polyhedron $Q_{\bar{b}}$ defined above is unbounded. More explicitly, if $\bar{c}$ is a point with $\langle \bar{b}, \bar{c}\rangle = 0$, then for any $x_0 \in Q_{\bar{b}}$ we have that $x_0 + \lambda c \in Q_{\bar{b}}$ for any $\lambda \geq 0$. If we then fix a small $(n-1)$-disc $D_\varepsilon(x_0) \subset Q_{\bar{b}}$ perpendicular to c, consider the tubes $T_\lambda = \{x + rc \mid x \in D_\varepsilon(x_0), r \in (0, \lambda)\} \subset Q_{\bar{b}}$. Take a sequence of points $b_i \to \bar{b}$ with b_j in the interior of C^*, and define Q_{b_j} and H_{b_j} as above. Recall that each Q_{b_j} is bounded. Choosing ε small enough, and perhaps after removing finitely many terms from $\{b_j\}$, we can assume that $D_\varepsilon(x_0)$ is contained in Q_{b_1}. Let λ_j be the largest positive number such that $T_{\lambda_j} \subset Q_{b_j}$. Since $Q_{b_j} \to Q_{\bar{b}}$, we see that $\lambda_j \to \infty$. Then we have

$$F(b_j) = \int_P e^{-\langle b_j,x\rangle} dx \geq \int_{T_{\lambda_j}} e^{-\langle b_j,x\rangle} dx = \lambda_j \int_{D_\varepsilon(x_0)} e^{-\langle b_j,y\rangle} dy,$$

where y are the coordinates on $D_\varepsilon(x_0)$. Clearly $F(b_j) \to \infty$.

□

The function $F : C^* \to \mathbb{R}$ appearing in the proof in fact has the following geometric significance:

Definition 4.28. Let (M, J) be a complex toric manifold, and let ω be any symplectic form giving (M, J, ω) the structure of an AK-toric manifold with moment map $\mu : M \to \mathfrak{t}^*$ whose image is a Delzant polyhedron P. Let $C^* \subset \mathfrak{t}$ be the dual recession cone of P, and define the *P-weighted volume functional* $F_P : C^* \to \mathbb{R}$ by

$$F_P(b) = \frac{1}{(2\pi)^n} \int_M e^{-\langle \mu, b\rangle} \omega^n. \tag{4.16}$$

Lemma 4.29. *The P-weighted volume functional $F_P : C^* \to \mathbb{R}$ is well-defined, and independent of the choice of ω with moment image P.*

Remark 4.30. This can be viewed as a non-compact analog of the classic result that the Futaki invariant on a compact manifold is independent of the Kähler metric used to define it within a given cohomology class. In the non-compact setting, we need the extra assumption of a proper Hamiltonian potential, as there may well be metrics in a given cohomology class whose moment image is not a Delzant polyhedron, see Example 3.30.

Proof. We compute directly

$$\begin{aligned} F_P(b) &= \frac{1}{(2\pi)^n} \int_M e^{-\langle \mu, b\rangle} \omega^n = \frac{1}{(2\pi)^n} \int_{(\mathbb{C}^*)^n} e^{-\langle \mu, b\rangle} \omega^n \\ &= \frac{1}{(2\pi)^n} \int_{\mathfrak{t} \times T^n} e^{-\langle \nabla\phi, b\rangle} \det(\phi_{ij}) d\xi d\theta \\ &= \int_P e^{-\langle x, b\rangle} dx, \end{aligned}$$

where in the last line we have used the moment map $\mu = \nabla\phi$ to change coordinates $\nabla\phi : \mathfrak{t} \to P$. As we saw in the proof of Proposition 4.27, the integral is finite if and only if b lies in the interior of the dual recession cone C^*. Moreover, the right hand side is then completely independent of ω so long as the image of $\nabla\phi$ is equal to P. □

Note that F_P *does* depend *a priori* on the choice of Kähler potential $\omega = 2i\partial\bar{\partial}\phi$. Of course, one can always force this to be intrinsic to the polarization that P determines on M by making a specific artificial choice of P up to translation and then normalizing ϕ accordingly. However, as we have seen for Kähler-Ricci solitons there is a canonical choice such that ϕ satisfies the real Monge-Ampère equation (4.8), in which case the image of $\nabla\phi$ is equal to P_{-K_M}. The weighted volume functional of [19] is associated intrinsically to any complex manifold admitting a complete shrinking gradient Kähler-Ricci soliton. We recover their weighted volume funcitonal F (up to the factor of $(2\pi)^n$) as follows.

Definition 4.31. Let $(M, -K_M)$ be any anticanonically polarized AK-toric manifold. The *weighted volume functional* $F : C^* \to \mathbb{R}$ is defined by

$$F(b) = F_{P_{-K_M}}(b) = \frac{1}{(2\pi)^n} \int_M e^{-\langle \mu, b\rangle} \omega^n, \tag{4.17}$$

for any Kähler metric ω with moment map $\mu : M \to P_{-K_M}$.

4.2.2 The Futaki invariant and uniqueness of the soliton vector field

Definition 4.32. Let $(M, -K_M)$ be an anticanonically polarized AK-toric manifold. Given a real holomorphic vector field v with $Jv \in \mathfrak{t}$, let b_v denote the coefficients of Jv. The *Futaki invariant* $F_v : \mathfrak{t} \to \mathbb{R}$ is defined to be the total derivative of the weighted volume functional F at the point $b_v \in \mathfrak{t}$. That is,

$$\begin{aligned} F_v(b) &= -\left.\frac{d}{dt}\right|_{t=0} F(b_v + tb) = -\frac{1}{(2\pi)^n} \int_M \langle \mu, b\rangle e^{-\langle \mu, b_v\rangle} \omega^n \\ &= -\int_{P_{-K_M}} \langle x, b\rangle e^{\langle x, b_v\rangle} dx, \end{aligned} \tag{4.18}$$

where $\theta_v = \langle \mu, b_v\rangle$ is the Hamiltonian potential $d\theta_{Jv} = -i_{Jv}\omega$ determined by μ.

Remark 4.33. By extending F_v complex-linearly to $\mathfrak{t}_{\mathbb{C}} \cong \mathfrak{t} \oplus i\mathfrak{t}$, we can recover the more familiar complex-valued Futaki invariant $F_v : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$.

The goal of this section will be to prove the following theorem. This was originally proved in [19], see [19, Theorem D, Lemma 5.17] for general shrinking gradient Kähler-Ricci solitons when $\mathfrak{t}$ is the Lie algebra of the T^k-action generated by JX. We give an independent proof in the toric setting, which boils down to the elementary existence result 4.27.

Theorem 4.34 ([15])**.** *Let (M, J, ω) be an anticanonically polarized AK-toric manifold. Then there exists a unique vector field X with $JX \in \mathfrak{t}$ that could be the vector field associated to a shrinking gradient Kähler-Ricci soliton. If such a soliton (ω, X) exists, then*

1. *$JX \in C^* \subset \mathfrak{t}$, and*
2. *JX is the unique holomorphic vector field in $\mathfrak{t}$ such that $F_X \equiv 0$.*

Lemma 4.35. *Let P be a Delzant polyhedron containing zero in its interior, and suppose that there exists a solution $u \in C^\infty(P)$ to (4.9). Then the linear function $\langle x, b_X\rangle$ on P satisfies*

$$\int_P \ell(x) e^{-\langle x, b_X\rangle} dx = 0$$

for any linear function $\ell(x)$ on P. Thus, $\langle x, b_X\rangle$ is equal to the linear function ℓ_P of Proposition 4.35.

Proof. First, we claim that any function $u \in C^\infty(P)$ which is the Legendre transform $u = L(\phi)$ of a smooth convex function ϕ on $\mathbb{R}^n$ satisfies

$$\int_P \ell(x) e^{-\rho_u} dx = 0$$

for any linear function $\ell(x)$ on P. Indeed, for each $j = 1, \dots, n$, compute

$$\int_P x^j e^{-\rho_u} dx = \int_{\mathbb{R}^n} \phi_j e^{-\phi} d\xi = -\int_{\mathbb{R}^n} \left(e^{-\phi}\right)_j d\xi.$$

By Lemma 4.22, we know that $e^{-\phi}$ decays at least exponentially in $|x|$. Thus, integration by parts yields that the term on the right-hand side is zero. Then if u satisfies $\rho_u = \langle x, b_X \rangle$, it follows that

$$\int_P x^j e^{-\langle x, b_X \rangle} dx = \int_{\mathbb{R}^n} \left(e^{-\phi}\right)_j d\xi = 0$$

for each j.

□

Proof of Theorem 4.34. Let ω_1 and ω_2 be two T^n-invariant Kähler metrics on M satisfying (4.2) on M with vector fields X_1 and X_2. By Theorem 4.18, we know that each moment map μ_s, $s = 1, 2$, has image equal to $P = P_{-K_M}$. Moreover, by Lemma 4.16, we know that ω_s is uniquely determined by a symplectic potential u_s on the fixed polyhedron P which satisfies the real Monge-Ampère equation $\rho_{u_s} = \langle x, b_s \rangle$. By Lemma 4.35, the function $\langle x, b_s \rangle$ satisfies

$$\int_P \ell(x) e^{-\langle x, b_s \rangle} dx = 0$$

for each linear function $\ell(x)$ on P. In particular, $\langle x, b_s \rangle$ is equal to the fixed linear function ℓ_P determined in Proposition 4.27. Clearly, there is a unique $b_P \in \mathfrak{t}$ such that $\ell_P(x) = \langle x, b_P \rangle$. Let X_P denote the holomorphic vector field with $JX \in \mathfrak{t}$ on M such that

$$X_P^{1,0} = \sum_{j=1}^n b_P^j z_j \frac{\partial}{\partial z_j}$$

on the dense orbit. (In other words, $X = \sum_{j=1}^n b_P^j \frac{\partial}{\partial \xi^j}$ and $JX = \sum_{j=1}^n b_P^j \frac{\partial}{\partial \theta^j}$). We have in particular that $\mathcal{L}_{X_P} \omega_s = \mathcal{L}_{X_s} \omega_s$. Since ω_s is T^n-invariant and $JX_P, JX_1, JX_2 \in \mathfrak{t}$, this immediately implies that $X_1 = X_2 = X_P$. □

4.3 The space of Kähler metrics

Throughout this section we work with the following assumptions. We have a fixed complex anticanonically polarized toric manifold $(M, J, -K_M)$. By Theorem 4.34, there exists a unique real holomorphic vector field X with $JX \in \mathfrak{t}$ being the unique critical point of the weighted volume functional F. We suppose that there exists a background Kähler metric $\omega' \in 2\pi c_1(M)$ such that

1. (M, J, ω') is AK-toric,
2. there exists $H \in C^\infty(M)$ such that $\omega' - \mathrm{Ric}_{\omega'} - \frac{1}{2}\mathcal{L}_X\omega' = i\partial\bar{\partial}H$, and
3. $||H||_{L^\infty(M)} < \infty$.

We denote the anticanonical polyhedron $P_{-K_M} \subset \mathfrak{t}^*$ of (4.11) simply by P. The soliton vector field X corresponds to the linear function $\langle x, b_X\rangle$ on P of Proposition 4.27, and we define the constant V_X by

$$V_X = \int_P e^{-\langle x, b_X\rangle} dx. \tag{4.19}$$

On any Kähler manifold (N, J, ω), compact or otherwise, we let $\mathcal{H}$ denote the space of all Kähler metrics $\omega_\varphi = \omega + 2i\partial\bar{\partial}\varphi$ such that the difference $\omega - \omega_\varphi$ is $\partial\bar{\partial}$-exact.

4.3.1 The space of symplectic potentials

By Propositions 3.33, 3.34 and Corollary 3.35, the image of the moment map $\mu : M \to \mathfrak{t}^*$ associated to *any* Kähler metric ω such that (M, J, ω) is AK-toric and $[\omega] = [\omega'] = 2\pi c_1(M)$ is a translate of the anticanonical polyhedron $P = P_{-K_M}$. If we normalize so that the image is precisely equal to P, then by Proposition 3.46 any such ω is uniquely determined by a symplectic potential $u = u_P + v \in C^\infty(P)$. We therefore consider the following space of symplectic potentials.

$$\mathcal{P} = \left\{ u = u_P + v \;\middle|\; \int_P u e^{-\langle x, b_X\rangle} dx < \infty\ ,\ (u)_{ij} > 0, v \in C^\infty(\overline{P}) \right\}. \tag{4.20}$$

Remark 4.36. Since $u_P = O(|x|\log|x|)$, we clearly have that $\int_P u_P e^{-\langle x, b_X\rangle} dx < \infty$. Thus for any $u = u_P + v \in \mathcal{P}$, we also have that $\int_P v e^{-\langle x, b_X\rangle} dx < \infty$.

Lemma 4.37. *The background metric ω' admits a symplectic potential $u' \in C^\infty(P)$ which lies in $\mathcal{P}$.*

In order to prove this, we need the following useful lemma.

Lemma 4.38 (c.f. [29, Lemma 1])**.** *Let P be a polyhedron containing zero in the interior and $u \in C^\infty(P)$ be any strictly convex function such that the gradient ∇u maps P diffeomorphically onto $\mathbb{R}^n$. Then*

$$\int_P u e^{\rho_u} dx < \infty.$$

Proof. Let $\phi(\xi) = L(u)$. Recall that by Lemma 4.22, ϕ grows at least linearly in $|\xi|$, and in particular is necessarily bounded from below. Assume that ϕ is minimized at the origin. Then

$$\int_P ue^\rho dx = \int_{\mathbb{R}^n} \left(\langle\nabla\phi, \xi\rangle - \phi\right) e^{-\phi} d\xi \leq \int_{\mathbb{R}^n} \left(\langle\nabla\phi, \xi\rangle + C\right) e^{-\phi} d\xi.$$

The second term $\int Ce^{-\phi}d\xi$ is bounded again by Lemma 4.22, so that

$$\int_P ue^\rho dx \leq \int_{\mathbb{R}^n} \langle\nabla\phi, \xi\rangle e^{-\phi} d\xi + C.$$

In polar coordinates we have

$$\int_{\mathbb{R}^n} \langle\nabla\phi, \xi\rangle e^{-\phi} d\xi = \int_{S^{n-1}} \int_0^\infty r^n \frac{\partial\phi}{\partial r} e^{-\phi} dr d\Theta.$$

Integrating by parts, we obtain

$$\int_{S^{n-1}} \int_0^\infty r^n \frac{\partial\phi}{\partial r} e^{-\phi} dr d\Theta = n \int_{S^{n-1}} \int_0^\infty r^{n-1} e^{-\phi} dr d\Theta = n \int_{\mathbb{R}^n} e^{-\phi} d\xi.$$

Note that the boundary term converges since $\phi = O(r)$ as $r \to \infty$. Thus

$$\int_P ue^\rho dx \leq \int_{\mathbb{R}^n} \langle\nabla\phi, \xi\rangle e^{-\phi} d\xi + C = n \int_{\mathbb{R}^n} e^{-\phi} d\xi + C < \infty.$$

If ϕ is not minimized at the origin, since $\nabla\phi = P$ and $0 \in P$ it follows that there is a point ξ^* such that $\nabla\phi(\xi^*) = 0$. Repeating the proof replacing ϕ with $\tilde{\phi}(\xi) = \phi(\xi - \xi^*)$ achieves the desired conclusion. □

Proof of Lemma 4.37. The metric ω' admits a symplectic potential $u \in C^\infty(P)$ by assumption, since

$$[\omega'] = [\mathrm{Ric}_{\omega'} + \frac{1}{2}\mathcal{L}_X\omega' + i\partial\bar{\partial}H] = 2\pi c_1(M).$$

Now, let $\phi' \in C^\infty(\mathfrak{t})$ be a Kähler potential for ω' on the dense orbit so that $\nabla\phi(\mathfrak{t}) = P$. It follows from Lemma 3.6 that there exists an affine function $a(\xi)$ such that

$$H + a(\xi) = 2\phi' + \log\det(\phi'_{ij}) - \langle\nabla\phi', b_X\rangle.$$

We claim that a is constant. Indeed, passing to the Legendre transform $u' = L(\phi')$, we see that

$$H(\nabla u') + \langle\nabla u', b_a\rangle + c_a = \rho_{u'} - \langle x, b_X\rangle.$$

Now H comes from a smooth function on all of M, so in particular $H(\nabla u') : P \to \mathbb{R}$ extends continuously to the boundary. As we saw in the proof of Theorem 4.18, since $P = P_{-K_M}$ it follows that $\rho_{u'}$ extends continuously to the boundary of P. This proves the claim, noting that $\nabla u' = \nabla u_P + \nabla v = O(\log(\mathrm{dist}(x, \partial P)))$ does *not* extend continuously to ∂P unless $b_a = 0$. Therefore we see that

$$\sup_{x \in P} |\rho_{u'} - \langle x, b_X \rangle| = \sup_{p \in M} |H| < C, \tag{4.21}$$

and hence

$$\int_P u' e^{-\langle x, b_X \rangle} dx < \infty$$

by Lemma 4.38. □

Proposition 4.39. *Any symplectic potential $u \in \mathcal{P}$ determines a unique $\mathcal{H}_T$ of T^n-invariant Kähler metric ω in the cohomology class $2\pi c_1(M)$.*

Proof. First we note that, since $\nabla u_P(P) = \mathbb{R}^n$, v extends smoothly past the boundary of P, and u is strictly convex, the gradient ∇u maps P diffeomorphically onto all of $\mathbb{R}^n \cong \mathfrak{t}$. By Lemma 3.47, if $\phi = L(u) \in C^\infty(\mathfrak{t})$, the Kähler metric $\omega = 2i\partial\bar{\partial}\phi$ extends smoothly to all of M. The extension is clearly Hamiltonian with gradient image equal to $P = P_{-K_M}$, and hence $[\omega] = 2\pi c_1(M)$. □

As such, we will sometimes say that a Kähler metric $\omega \in \mathcal{P}$ as a shorthand to mean that ω admits a symplectic potential u lying in $\mathcal{P}$.

Remark 4.40. As we saw above, any $u \in \mathcal{P}$ satisfies $\nabla u(P) = \mathbb{R}^n$. As a consequence of Lemma 4.22, therefore, any such u can only be negative outside of a compact set. It follows that any $u \in \mathcal{P}$ is L^1 with respect to the measure $e^{-\langle x, b_X \rangle} dx$ on P.

4.3.2 Geodesics

Lemma 4.41. *The space $\mathcal{P}$, as a subset of, $C^\infty(P)$ is convex.*

Proof. This is immediate. A convex combination $u_t = tu_1 + (1-t)u_0$ of two strictly convex functions u_0 and u_1 is strictly convex. If $u_i = u_P + v_i$, for $v_i \in C^\infty(\overline{P})$, then $u_t = u_P + v_t$ for $v_t = tv_1 + (1-t)v_0$. Finally,

$$\int_P u_t e^{-\langle x, b_X \rangle} dx = t \int_P u_1 e^{-\langle x, b_X \rangle} dx + (1-t) \int_P u_0 e^{-\langle x, b_X \rangle} dx < \infty.$$

□

Definition 4.42. Let $u_0, u_1 \in \mathcal{P}$ be any two symplectic potentials. The convex combination $u_t = tu_1 + (1-t)u_0$, $t \in [0, 1]$ is called a *geodesic*.

The terminology comes from a now standard picture originally due to Donaldson, Mabuchi, and Semmes [28, 53, 63]. The idea is that the space of Kähler metrics $\mathcal{H}$ on a compact manifold (M, J, ω) comes naturally equipped with a Riemannian metric (sometimes called the

"L^2-metric"), from which perspective one can understand $\mathcal{H}$ as a formal "non-compact dual" symmetric space to the symplectomorphism group of (M, ω). With respect to this metric, a path of Kähler metrics $\omega_t = \omega + 2i\partial\bar{\partial}\varphi$ is a geodesic if and only if

$$\ddot{\varphi}_t - \frac{1}{2}|\nabla^{g_t}\dot{\varphi}_t|^2_{g_t} = 0. \tag{4.22}$$

The existence of geodesics, even on a compact Kähler manifold, is an interesting problem. It was conjectured by Donaldson [28] that $\mathcal{H}$ is a metric space with respect to the L^2 metric distance, and this was verified by Chen [14] by proving the existence of paths which solve (4.22) in a weak sense. In fact, it was proved by Darvas [22] that on any compact Kähler manifold (except for the outlying case that $M = \mathbb{CP}^1$) that one can always find a pair of Kähler metrics which cannot be connected by a smooth geodesic. The familiar fact from the compact setting is that this obstruction disappears when we consider T^n-invariant metrics on toric manifolds, and smooth geodesics become readily available.

Proposition 4.43. *A path ω_t of the form $\omega_t = \omega + i2\partial\bar{\partial}\varphi_t$ in $\mathcal{P}$ satisfies* (4.22) *if and only if the corresponding path of symplectic potentials $u_t \in C^\infty(P)$ is a geodesic in the sense of Definition 4.42.*

The only proofs that we could find in the literature make use of the energy functional associated to $\mathcal{H}$, which is not obviously always finite if M is non-compact. It is natural enough to only work with paths of finite energy, considering this is how the equation (4.22) is established in the first place. However, (4.22) makes perfect sense as a pointwise equation. We give a local and entirely elementary proof of 4.43, thereby bypassing the need to restrict attention to only finite-energy paths. We begin with a lemma that is useful in its own right.

Lemma 4.44. *Let ω_t be* any *path in $\mathcal{P}$, and let ϕ_t be corresponding path of Kähler potentials on the dense orbit, normalized so that $\nabla\phi_t(\mathfrak{t}) = P$. If $u_t = L(\phi_t)$ is the path of symplectic potentials, then the time derivatives satisfy*

$$\dot{\phi}_t = -\dot{u}_t.$$

Proof. We have

$$\begin{aligned}
\dot{u}_t = \frac{\partial}{\partial t}u_t(x) &= \frac{\partial}{\partial t}\left(\langle \nabla u_t, x\rangle - \phi_t(\nabla u_t)\right) \\
&= \left\langle \frac{\partial}{\partial t}\nabla u_t, x\right\rangle - \dot{\phi}(\nabla u_t) - \left\langle \nabla\phi_t, \frac{\partial}{\partial t}\nabla u_t\right\rangle \\
&= -\dot{\phi}.
\end{aligned}$$

□

Proof of Proposition 4.43. Let $\omega_t = \omega + 2i\partial\bar{\partial}\varphi_t$ be a path satisfying (4.22). On the dense orbit, we can write $\omega = 2i\partial\bar{\partial}\phi$ for $\phi \in C^\infty(\mathfrak{t})$ and indeed $\omega_t = 2i\partial\bar{\partial}\phi_t$ for $\phi_t \in C^\infty(\mathfrak{t})$,

$t \in [0,1]$. In particular, $\omega_t = 2i\partial\bar{\partial}(\phi + \varphi_t)$. Since φ_t extends smoothly to all of M, the gradient $\nabla(\phi + \varphi_t)$ has the same image of that of $\nabla\phi$. Therefore, if we normalize that $\nabla\phi(\mathfrak{t}) = \nabla\phi_t(\mathfrak{t}) = P$, it follows that $\phi_t = \phi + \varphi_t$, and in particular $\dot{\phi}_t = \dot{\varphi}_t$. Thus, the Kähler potentials themselves satisfy

$$\ddot{\phi}_t - \frac{1}{2}|\nabla^{g_t}\dot{\phi}_t|^2_{g_t} = 0.$$

We now compute directly. The chain rule yields

$$\frac{\partial}{\partial \xi^j} = \frac{\partial x^i}{\partial t^j}\frac{\partial}{\partial t^i} = \frac{\partial^2 \phi_t}{\partial \xi^i \partial \xi^j}\frac{\partial}{\partial x^i} = u^{ij}\frac{\partial}{\partial x^i}.$$

It follows that

$$\frac{\partial \dot{\phi}_t}{\partial \xi^j} = -u_t^{ij}\frac{\partial \dot{u}_t}{\partial x^i}. \tag{4.23}$$

Now compute

$$\begin{aligned}
\ddot{u}_t &= -\frac{\partial}{\partial t}\dot{\phi}_t(\nabla^x u_t) = -\ddot{\phi}_t(\nabla^x u_t) - \sum_m \frac{\partial \dot{\phi}_t}{\partial \xi^m}\frac{\partial \dot{u}_t}{\partial x^m} \\
&= -\ddot{\phi}_t(\nabla^x u_t) + u_t^{lm}\frac{\partial \dot{u}_t}{\partial x^l}\frac{\partial \dot{u}_t}{\partial x^m}
\end{aligned},$$

so that

$$\begin{aligned}
\ddot{\phi}_t - \frac{1}{2}|\nabla_t\dot{\phi}_t|^2_t &= \ddot{\phi}_t - \phi_t^{ij}\frac{\partial \dot{\phi}_t}{\partial \xi^i}\frac{\partial \dot{\phi}_t}{\partial \xi^j} \\
&= -\ddot{u}_t + u_t^{lm}\frac{\partial \dot{u}_t}{\partial x^l}\frac{\partial \dot{u}_t}{\partial x^m} - (u_t)_{ij}u_t^{il}u_t^{mj}\frac{\partial \dot{u}_t}{\partial x^l}\frac{\partial \dot{u}_t}{\partial x^m} \\
&= -\ddot{u}_t = 0.
\end{aligned}$$

□

4.3.3 The Ding functional

Let $f' \in C^\infty(M)$ be the Hamiltonian function for JX with respect to our background metric ω', normalized as always so that the image of the moment map is equal to P_{-K_M}. In the compact setting, one can define the $\hat{F}$*-functional* on $\mathcal{H}$ by

$$\hat{F}(\varphi) = -\frac{1}{V}\int_0^1 \int_M \dot{\varphi}_s \left(e^{-f' - X(\varphi_s)}\omega^n_{\varphi_s}\right) \wedge ds, \tag{4.24}$$

where $\omega_{\varphi_s} = \omega + 2i\partial\bar{\partial}\varphi_s$ is any path in $\mathcal{H}$, and $V = \int_M e^{-f'}\omega'^n$. The *Ding functional* is then defined by

$$\mathcal{D}(\varphi) = \hat{F}(\varphi) - \frac{1}{2}\log\left(\int_M e^{f'-H-2\varphi}\omega'^n\right). \tag{4.25}$$

The upshot is that one readily calculates the first variation

$$\begin{aligned}\partial_\varphi\mathcal{D}(\psi) &= \partial_\varphi\hat{F}(\psi) + \frac{1}{\int_M e^{f'-H-\varphi}\omega'^n}\int_M \psi e^{H-f'-2\varphi}\omega'^n \\ &= \int_M \psi\left(\frac{e^{f'-H-2\varphi}\omega'^n}{\int_M e^{f'-H-2\varphi}\omega'^n} - \frac{e^{-f'-X(\varphi)}\omega_\varphi^n}{V}\right),\end{aligned}$$

so that the critical points of $\mathcal{D}$ satisfy

$$\begin{aligned}0 &= i\partial\bar{\partial}\left(f' - H - 2\varphi + f' + X(\varphi) - \log\left(\frac{\omega'^n}{\omega_\varphi^n}\right)\right) \\ &= \mathrm{Ric}_\omega - \omega' - 2i\partial\bar{\partial}\varphi + \frac{1}{2}\mathcal{L}_X(\omega' + 2i\partial\bar{\partial}\varphi) - \mathrm{Ric}_{\omega'} + \mathrm{Ric}_{\omega_\varphi} \\ &= \mathrm{Ric}_{\omega_\varphi} - \omega_\varphi + \frac{1}{2}\mathcal{L}_X\omega_\varphi,\end{aligned}$$

and thus define a shrinking Kähler-Ricci soliton with respect to X. When M is non-compact, we cannot expect all of these integrals to be well-defined. However, we prove that this picture does make sense when restricted to $\mathcal{P}$.

Lemma 4.45. *The $\hat{F}$-functional is well-defined on $\mathcal{P}$.*

Proof. Let ω be any element of $\mathcal{P}$. Suppose that there exists a path $\varphi_t \in C^\infty(M)$ of T^n-invariant smooth functions such that $\omega_t = \omega' + i\partial\bar{\partial}\varphi_t > 0 \in \mathcal{P}$, $\varphi_0 = 0$, and $\omega_1 = \omega$. Let ϕ_t be the Kähler potential on the dense orbit for ω_t normalized so that $\nabla\phi_t = P$, so that $\dot{\varphi}_t = \dot{\phi}_t$. Let $u_t = L(\phi_t)$ be the corresponding path of symplectic potentials. Then, the weighted volume

$$\begin{aligned}V = \int_M e^{-f'}\omega'^n &= \int_{\mathfrak{t}\times T^n} e^{-\langle\nabla\phi', b_X\rangle}\det(\phi'_{ij})d\xi d\theta \\ &= (2\pi)^n\int_P e^{-\langle x, b_X\rangle}dx = (2\pi)^n V_X.\end{aligned}$$

Then

$$\begin{aligned}\hat{F}(\omega) &= -\frac{1}{V}\int_0^1\int_M \dot{\varphi}_s\left(e^{-f'-X(\varphi_s)}\omega^n_{\varphi_s}\right)\wedge ds\\ &= -\frac{1}{(2\pi)^n V_X}\int_0^1\int_{\mathfrak{t}\times T^n}\dot{\phi}_s e^{-\langle\nabla\phi_s,b_X\rangle}\det(\phi_{s,ij})d\xi d\theta\\ &= \frac{1}{V_X}\int_0^1 \dot{u}_t e^{-\langle x,b_X\rangle}dx = \frac{1}{V_X}\int_P (u_1-u_0)\,e^{-\langle x,b_X\rangle}dx.\end{aligned}$$

Since $\int_P u_i e^{-\langle x,b_X\rangle}dx<\infty$ by assumption, it is clear that $\hat{F}(\omega)$ is finite. Moreover, noting also by assumptoin that $u_0=u'$ is the symplectic potential on P with respect to the background metric ω', we see that this is independent of the path φ_t chosen. Moreover, if no such path φ_t exists, we can still choose any path $\omega_t\in\mathcal{P}$ such that $\omega_0=\omega'$ and $\omega_1=\omega$ and define $\hat{F}$ directly by

$$\begin{aligned}\hat{F}(\phi_1) &= -\frac{1}{V}\int_0^1\int_{\mathfrak{t}\times T^n}\dot{\phi}_s e^{-\langle\nabla\phi_s,b_X\rangle}\det(\phi_{s,ij})d\xi d\theta\\ &= \frac{1}{V_X}\int_P (u_1-u')\,e^{-\langle x,b_X\rangle}dx,\end{aligned} \tag{4.26}$$

which again is clearly finite and independent of path. Such a path clearly exists, simply take ω_t determined by the geodesic $u_t=tu_1+(1-t)u'$. □

Lemma 4.46. *The Ding functional $\mathcal{D}$ is well-defined on $\mathcal{P}$.*

Proof. As above, let ω be any element of $\mathcal{P}$ and suppose that there exists a $\varphi\in C^\infty(M)$ such that $\omega=\omega_0+2i\partial\bar{\partial}\varphi$, so that $\omega=2i\partial\bar{\partial}\phi=2i\partial\bar{\partial}(\phi'+\varphi)$ on the dense orbit $U\cong(\mathbb{C}^*)^n\subset M$. As we saw in the proof of Lemma 4.37, when we restrict the function H to U, we get that up to a constant

$$H=2\phi'+\log\det\phi'_{ij}-\langle\nabla\phi',b_X\rangle,$$

so that

$$\begin{aligned}e^{f'-H-2\varphi}\omega'^n &= e^{-2(\phi'+\varphi)-\log\det(\phi'_{ij})}\det\phi'_{ij}d\xi d\theta\\ &= e^{-2\phi}d\xi d\theta\end{aligned}$$

Therefore, if we change coordinates by $\nabla\phi=\nabla(\phi'+\varphi):\mathfrak{t}\to P$, we get as in Lemma 4.21 that

$$\int_M e^{f'-H-2\varphi}\omega'^n=(2\pi)^n\int_{\mathfrak{t}}e^{-2\phi}d\xi=(2\pi)^n\int_P e^{-\rho_u}dx,$$

where $u=L(\phi)$ is the Legendre transform. The Ding functional is then given by

$$\begin{aligned}\mathcal{D}(\omega) &= \hat{F}(\omega)-\frac{1}{2}\log\left(\int_M e^{f'-H-2\varphi}\omega'^n\right)\\ &= \frac{1}{V_X}\int_P (u_1-u')\,e^{-\langle x,b_X\rangle}dx-\frac{1}{2}\log\left((2\pi)^n\int_P e^{-\rho_u}dx\right),\end{aligned} \tag{4.27}$$

which is well-defined by Lemma 4.21. As in the proof of Lemma 4.45, the formula (4.27) makes perfect sense for all $\omega \in \mathcal{P}$, regardless of whether $\omega - \omega'$ is $\partial\bar{\partial}$-exact. □

Remark 4.47. For convenience of notation, we will drop the additive factor of $n\log(2\pi)$, and simply denote

$$\mathcal{D}(\omega) = \frac{1}{V_X}\int_P (u_1 - u')\, e^{-\langle x, b_X\rangle} dx - \frac{1}{2}\log\left(\int_P e^{-\rho_u} dx\right).$$

Proposition 4.48. *The critical points of the functional $\mathcal{D}$ on $\mathcal{P}$ are precisely the solutions to the real Monge-Ampère equation* (4.9).

Proof. Let $u = u_P + v \in \mathcal{P}$ be a symplectic potential, and suppose that $w \in C^\infty(\overline{P})$ is compactly supported on $\mathbb{R}^n$, so that $u_t = u + tw \in \mathcal{P}$ for sufficiently small t. Set $\rho_t = \rho_{u_t}$. We first compute that

$$\begin{aligned}\frac{\partial}{\partial t}\int_P e^{-\rho_t} dx\bigg|_{t=0} &= \frac{\partial}{\partial t}\int_{\mathfrak{t}} e^{-2\phi_t} d\xi = -2\int_{\mathfrak{t}} \dot{\phi} e^{-2\phi} d\xi \\ &= 2\int_P \dot{u} e^{-\rho_u} dx = 2\int_P w e^{-\rho_u} dx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{D}(u_t)\bigg|_{t=0} &= \frac{\partial}{\partial t}\frac{1}{V_X}\int_P (u_t - u')\, e^{-\langle x, b_X\rangle} dx - \frac{1}{2}\frac{\frac{\partial}{\partial t}\int_P e^{-\rho_t} dx}{\int_P e^{-\rho_u} dx} \\ &= \frac{\partial}{\partial t}\frac{1}{V_X}\int_P (tw + (u - u'))\, e^{-\langle x, b_X\rangle} dx - \frac{\int_P w e^{-\rho_u} dx}{\int_P e^{-\rho_u} dx} \\ &= \int_P w\left(\frac{e^{-\langle x, b_X\rangle}}{V_X} - \frac{e^{-\rho_u}}{\int_P e^{-\rho_u} dx}\right) dx,\end{aligned}$$

so that $u \in \mathcal{P}$ is a critical point of $\mathcal{D}$ if and only if $\rho_u = \langle x, b_X\rangle$. □

Remark 4.49. Strictly speaking, $\mathcal{D}$ and $\hat{F}$ here are defined on potentials ϕ, or equivalently symplectic potentials u. Nonetheless, we have written $\mathcal{D}(\omega)$ or $\hat{F}(\omega)$ in some instances in order to illustrate the analogy with the compact setting. We will see in Proposition 4.50 below that in fact the Ding functional $\mathcal{D}$ is well-defined at the metric level. The $\hat{F}$ functional is truly a functional on potentials, for example notice that $\hat{F}(\phi + a) = \hat{F}(\phi) - a$.

4.3.4 Convexity

As we have seen, there assignment $u \mapsto \omega$ of a Kähler metric on (M, J) associated to a symplectic potential u is not injective. Namely, if $a(x)$ is any affine-linear function on P, then $u + a$ defines the same Kähler metric as u. Indeed, addition of the affine functions induces a free action of $\mathbb{R}^n \times \mathbb{R}$ on $\mathcal{P}$. We prove a strong convexity statement for $\mathcal{D}$ on the space $\mathcal{P}$ from [15], which adapts some of the ideas of Berman-Berndtsson [8] to the AK-toric setting.

Proposition 4.50 (c.f. [8, Proposition 2.15])**.** *The Ding functional $\mathcal{D}$ is convex on $\mathcal{P}$. It is invariant under the action of $\mathbb{R}^n \times \mathbb{R}$ given by addition of affine-linear functions, and it is strictly convex modulo this action. That is, if $u_0, u_1 \in \mathcal{P}$ and $\mathcal{D}(u_t)$ is affine, i.e. $\mathcal{D}(tu_1 + (1-t)u_0) = t\mathcal{D}(u_1) + (1-t)\mathcal{D}(u_0)$, then there exists an affine-linear function $a(x)$ on P such that $u_1 = u_0 + a$.*

Proof. If $u_0, u_1 \in \mathcal{P}$ satisfy $u_1 = u_0 + \langle x, b\rangle + c$ with $c \in \mathbb{R}$, it is straightforward to see that $\rho_{u_1} = \rho_{u_0} - 2c$. Using the fact that $\int_P \langle x, b\rangle e^{-\langle x, b_X\rangle} dx = 0$, we see

$$\begin{aligned}
\mathcal{D}(u_1) &= \frac{1}{V_X}\int_P (u_0 + \langle x, b\rangle + c - u')\, e^{-\langle x, b_X\rangle} dx - \frac{1}{2}\log\left(\int_P e^{-\rho_{u_0}+2c} dx\right) \\
&= \frac{1}{V_X}\int_P (u_0 - u')\, e^{-\langle x, b_X\rangle} dx + c - \frac{1}{2}\left(2c + \log\left(\int_P e^{-\rho_{u_0}} dx\right)\right) \\
&= \mathcal{D}(u_0).
\end{aligned}$$

We prove convexity directly, and show that

$$\mathcal{D}(u_t) \le t\mathcal{D}(u_1) + (1-t)\mathcal{D}(u_0),$$

where $u_t = tu_1 + (1-t)u_0$ for any $u_0, u_1 \in \mathcal{P}$. Let $\rho_t = \rho_{u_t}$ and $\phi_t = L(u_t)$ be the Legendre transform of u_t. To see the inequality, first notice that the functional $u \mapsto \int_P u e^{-\langle x, b_X\rangle} dx$ is clearly affine on $\mathcal{P}$ and in fact vanishes identically on $\mathcal{P}_0$. Therefore it suffices to show that the function

$$t \mapsto -\log \int_P e^{-\rho_t} dx$$

is convex in t. This follows from the fact that the Legendre transform $u \mapsto L(u)$ itself is, i.e.

$$\phi_t(\xi) \le t\phi_1(\xi) + (1-t)\phi_0(\xi), \tag{4.28}$$

which follows from Lemma 3.37. Fixing t, we see by changing coordinates that

$$-\log \int_P e^{-\rho_t} dx = -\log \int_{\mathbb{R}^n} e^{-\phi_t} d\xi.$$

It then follows immediately from the Prekopa-Leindler inequality [30] that this is convex in t. This says precisely that any family ϕ_t of convex functions satisfying (4.28) has the property that the function of one variable $\int_{\mathbb{R}^n} e^{-\phi_t} d\xi$ is log-concave (i.e. $t \mapsto -\log \int_{\mathbb{R}^n} e^{-\phi_t} d\xi$ is convex). The strict convexity follows from the equality case of the Prekopa-Leindler inequality, which was also studied in [30]. If the function $\int_{\mathbb{R}^n} e^{-\phi_t} d\xi$ is affine in t, then by [30, Theorem 12] there exists $m \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that

$$\phi_1(\xi) = \phi_0(m\xi + a) - n\log(m) - \log\left(\frac{\int_{\mathbb{R}^n} e^{-\phi_1} d\xi}{\int_{\mathbb{R}^n} e^{-\phi_0} d\xi}\right).$$

Firstly, we see that m must be equal to 1 since $u_0, u_1 \in \mathcal{P}$. Indeed $L(\phi_0(m\xi)) = u_0(m^{-1}x)$. If $u_0 \in \mathcal{P}$, then $u_0(m^{-1}x) - u_P(x) \in C^\infty(\overline{P})$ if and only if $m = 1$. Then we have that $\phi_1(\xi) = \phi_0(\xi + a) - C$ for some C. Again passing to the Legendre transform, we have that

$$u_1(x) = L(\phi_1(\xi)) = L(\phi_0(\xi + a) - C) = u_0(x) + \ell_a(x) + C.$$

□

As an immediate consequence, we have uniqueness for the real Monge-Ampère equation (4.9) on P.

Theorem 4.51. *If u_0 and u_1 both solve* (4.9)*, then there exists an affine-linear function $a(x)$ such that $u_1 = u_0 + a$.*

Proof. Suppose u_0 and u_1 are two such solutions and set $u_t = tu_1 + (1-t)u_0$. Since $t \mapsto \mathcal{D}(u_t)$ is convex and $\frac{\partial}{\partial t}\mathcal{D}(u_t)\big|_{t=0} = \frac{\partial}{\partial t}\mathcal{D}(u_t)\big|_{t=1} = 0$, it follows that $\mathcal{D}(u_t)$ is constant, and in particular affine. Proposition 4.50 immediately yields the existence of such an $a(x)$. □

Using this, we can prove one of our main theorems.

Theorem 4.52 ([15, Theorem A])**.** *Let (M, J) be a complex toric manifold with finite fixed point set. Up to automorphisms, there is at most one complete T^n-invariant shrinking gradient Kähler-Ricci soliton on (M, J).*

Proof. We begin by recalling where we are so far. Suppose that ω is the Kähler form of a complete T^n-invariant shrinking gradient Kähler-Ricci soliton on (M, J). By Proposition 4.11, the triple (M, J, ω) is automatically AK-toric, and since $\omega \in 2\pi c_1(M)$ the image of the moment map is a translate of the anticanonical polyhedron $P = P_{-K_M}$. Then by Proposition 4.18, we know that there exists a Kähler potential $\omega = 2i\partial\bar{\partial}\phi$ on the dense orbit such that ϕ satisfies the real Monge-Ampère equation (4.8), in which case the image of the moment map $\mu = \nabla\phi$ is precisely equal to P. The corresponding symplectic potential $u = L(\phi)$ then satisfies

$$\rho_u = \langle x, b_X \rangle. \tag{4.29}$$

By Theorem 4.34, the soliton vector field JX and thereby the linear function $\langle x, b_X \rangle$ is the unique critical point of the weighted volume functional F. Therefore, suppose we have two such metrics ω_0 and ω_1. Then, repeating the description above, we have two symplectic potentials u_0, u_1 for ω_0, ω_1 respectively which both satisfy the *same* equation (4.29). By Theorem 4.51, there exists an affine-linear function $a(x) = \langle x, b_a \rangle + c_a$ such that $u_1 = u_0 + a$. Let $\phi_s = L(u_s) \in C^\infty(\mathfrak{t})$ be the Legendre transform, so that $\omega_0 = 2i\partial\bar{\partial}\phi_0$ and $\omega_1 = 2i\partial\bar{\partial}\phi_1$ on the dense orbit. As we saw in Chapter 3 (see Lemma 3.39, Lemma 3.43), the Kähler potentials are related by $\phi_1(\xi) = \phi_0(\xi - b_a) - c_a$, and so $2i\partial\bar{\partial}\phi_2(\xi) = 2i\partial\bar{\partial}\phi_1(\xi - b_a)$. Let $\alpha : M \to M$ denote the automorphism determined by the action of $e^{-b_a} \in (\mathbb{C}^*)^n$. Then it is clear that $\phi_0(\xi - b_a) = \phi_0 \circ \alpha(\xi)$, and therefore that $\omega_1 = \alpha^*\omega_0$. □

Together with Theorem 4.4, this also completes the proof of our other main uniqueness theorem 1.3.

Theorem 4.53 ([15, Theorem B]). *Let (M, J) be a n-dimensional complex manifold admitting an effective, holomorphic action of the real torus T^n. Up to automorphisms, there is at most one shrinking gradient Kähler-Ricci soliton (ω, X) on (M, J) with bounded Ricci curvature and $JX \in \mathfrak{t}$.*

4.3.5 Coercivity

The strict convexity of the Ding functional $\mathcal{D}$ suggests that there might be some absolute sense in which "$\mathcal{D}$ goes to infinity as u goes to infinity." As such, there is need for an infinite-dimensional version of the Euclidean norm in (4.14). This fits into a broad series of conjectures of Tian (see for example[23]), especially in the compact setting, relating to the existence of special metrics (in this case Kähler-Ricci solitons). Following Berman-Berndtsson [8], we show that the Ding functional is proper, using the $\hat{F}$ functional to measure distance in $\mathcal{P}$. In order to make sense of this, we need to work with the potentials $u \in \mathcal{P}$ directly rather than the equivalence classes under the $\mathbb{R}^n \times \mathbb{R}$-action. To this end, we make the following normalization.

Definition 4.54. The space $\mathcal{P}_1 \subset \mathcal{P}$ of C^1-normalized symplectic potentials is defined to be the space of all $u \in \mathcal{P}$ such that $\nabla u(0) = 0$, $u(0) = 0$.

Since the symplectic potentials $u \in \mathcal{P}$ are strictly convex, any $u \in \mathcal{P}_1$ attains the minimum value of 0 at the origin in P. In particular, any $u \in \mathcal{P}_1$ is nonnegative. Since the $\hat{F}$ functional (4.26) on $\mathcal{P}$ is given essentially by the weighted integral $\int_P ue^{-\langle x, b_X \rangle} dx$, this becomes a reasonable notion of "distance" in $\mathcal{P}$. Now any $u \in \mathcal{P}$ clearly admits a unique representative in $\mathcal{P}_1$ via the addition of a unique affine-linear function $a(x)$ with the property that $\nabla a = -\nabla u(0)$ and $a(0) = -u(0)$. In this way, we have simultaneously normalized for the constants as well as fixed gauge for the action of $(\mathbb{C}^*)^n$ on M.

Lemma 4.55. *There exists a constant C such that for any $u \in \mathcal{P}_1$ we have*

$$\log \left(\int_P e^{-\rho_u} dx \right) \leq C \int_P ue^{-\langle x, b_X \rangle} dx - C \tag{4.30}$$

Proof. Recall from (4.21) that our reference metric ω' admits a symplectic potential $u' \in \mathcal{P}$ with

$$|\rho_{u'} - \langle x, b_X \rangle| < C. \tag{4.31}$$

Now there exists a unique affine function $a'(x)$ such that $u' + a' \in \mathcal{P}_1$. Since adding an affine function only modifies ρ by a constant, it suffices to assume that $u' \in \mathcal{P}_1$. For simplicity of notation, set $\rho' = \rho_{u'}$. The bound (4.31) gives in particular

$$e^{-\rho'} \leq Ce^{-\langle x, b_X \rangle}. \tag{4.32}$$

Denote by C_0 the constant such that

$$C_0 V_X = \int_P e^{-\rho'} dx$$

Now let $u \in \mathcal{P}_1$ be any potential and $u_t = u' + tv$ be the geodesic connecting u' and u, so that $v = u - u'$. Differentiating,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{D}(u_t)\bigg|_{t=0} &= \frac{1}{V_X}\int_P v e^{-\langle x, b_X\rangle} dx - \frac{1}{C_0 V_X}\int_P v e^{-\rho'} dx \\
&= \frac{1}{V_X}\int_P u e^{-\langle x, b_X\rangle} dx - \frac{1}{C_0 V_X}\int_P u e^{-\rho'} dx + C_1 \\
&\geq \frac{1}{V_X}\left(1 - \frac{C}{C_0}\right)\int_P u e^{-\langle x, b_X\rangle} dx + C_1
\end{aligned}$$

Where the last step follows since $u \geq 0$. Explicitly $C_1 = \int_P u'\left(\frac{1}{C_0 V_X} e^{-\rho'} - \frac{1}{V_X} e^{-\langle x, b_X\rangle}\right) dx$. Now by the convexity of $\mathcal{D}(u_t)$ and the above that

$$\begin{aligned}
\mathcal{D}(u) &\geq \frac{\partial}{\partial t}\mathcal{D}(u_t)\bigg|_{t=0} + \mathcal{D}(u') \\
&\geq -\frac{C(u')}{V_X}\int_P u e^{-\langle x, b_X\rangle} dx + C(u'),
\end{aligned}$$

and therefore

$$\log\left(\int_P e^{-\rho_u} dx\right) \leq (1 + C)\int_P u e^{-\langle x, b_X\rangle} dx - C,$$

where $C > 0$ is a constant depending only on u'. □

Theorem 4.56 ([8, Theorem 2.16])**.** *For any $\delta < 1$, there exists a $C_\delta > 0$ such that*

$$\mathcal{D}(\omega) \geq (1 - \delta)\hat{F}(u) - C_\delta \tag{4.33}$$

For all $\omega \in \mathcal{P}_1$.

Proof. Let $u \in \mathcal{P}_1$ and let $\phi = L(u)$ be the Legendre transform. From the definition of the Legendre transform (3.22) directly it follows that ϕ too attains the minimum value of 0 at the origin, and is therefore positive. For $s \in (0, 1)$, denote by $u_s = su$ and let $\phi_s = L(u_s)$. Since $\phi_s(\xi) = s\phi\left(\frac{\xi}{s}\right)$, we have that both u_s and ϕ_s are still positive proper convex functions on P which are minimized at zero. Let $\rho_s = \rho_{u_s}$. It is also not hard to see that the inequality (4.55) still holds for u_s. Computing directly we see

$$\rho_s = -\log\det(u_{ij}) + 2s\left(u_i x^i - u\right) + n\log s,$$

and therefore

$$\begin{aligned} e^{-\rho_s}dx &= s^n e^{-2s(u_i x^i - u)}(\det(u_{ij}))dx \\ &= s^n e^{-2s\phi} d\xi. \end{aligned}$$

Since $\phi \geq 0$ we have that $e^{-2\phi} \leq e^{-2s\phi}$, and therefore

$$\begin{aligned} \int_P e^{-\rho_u} dx = \int_{\mathbb{R}^n} e^{-\phi} d\xi &\leq \int_{\mathbb{R}^n} e^{-s\phi} d\xi \\ &= s^{-n} \int_P e^{-\rho_s} dx. \end{aligned}$$

Then we have by (4.55)

$$\begin{aligned} \log\left(\int e^{-\rho} dx\right) &\leq \log\left(\int e^{-\rho_s} dx\right) + n\log(s^{-1}) \\ &\leq C \int_P u_s e^{-\langle x, b_X\rangle} dx + n\log(s^{-1}) - C \\ &= sC \int_P u e^{-\langle x, b_X\rangle} dx + n\log(s^{-1}) - C, \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{D}(u) &= \frac{1}{V_X}\int_P (u - u')e^{-\langle x, b_X\rangle} dx - \frac{1}{2}\log\left(\int_P e^{-\rho_u} dx\right) \\ &\geq \left(1 - \frac{sC}{2}\right)\frac{1}{V_X}\int_P (u - u')e^{-\langle x, b_X\rangle} dx - n\log(s^{-1}) + C_1, \end{aligned}$$

for some constant C_1 depending only on u'. This finishes the proof, taking $s = 2\delta/C$ and $C_\delta = n\log(s^{-1}) - C$. □

Bibliography

[1] Miguel Abreu. "Kähler geometry of toric manifolds in symplectic coordinates". In: *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*. Vol. 35. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2003, pp. 1–24.

[2] Miguel Abreu. "Kähler geometry of toric varieties and extremal metrics". In: *Internat. J. Math.* 9.6 (1998), pp. 641–651. ISSN: 0129-167X. DOI: 10.1142/S0129167X98000282. URL: https://doi.org/10.1142/S0129167X98000282.

[3] Vestislav Apostolov et al. "Hamiltonian 2-forms in Kähler geometry. II. Global classification". In: *J. Differential Geom.* 68.2 (2004), pp. 277–345. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1115669513.

[4] M. F. Atiyah. "Convexity and commuting Hamiltonians". In: *Bull. London Math. Soc.* 14.1 (1982), pp. 1–15. ISSN: 0024-6093. DOI: 10.1112/blms/14.1.1. URL: https://doi.org/10.1112/blms/14.1.1.

[5] Michèle Audin. *The topology of torus actions on symplectic manifolds.* Vol. 93. Progress in Mathematics. Translated from the French by the author. Birkhäuser Verlag, Basel, 1991, p. 181. ISBN: 3-7643-2602-6. DOI: 10.1007/978-3-0348-7221-8. URL: https://doi.org/10.1007/978-3-0348-7221-8.

[6] Victor V. Batyrev and Elena N. Selivanova. "Einstein-Kähler metrics on symmetric toric Fano manifolds". In: *J. Reine Angew. Math.* 512 (1999), pp. 225–236. ISSN: 0075-4102. DOI: 10.1515/crll.1999.054. URL: https://doi.org/10.1515/crll.1999.054.

[7] Robert J. Berman. "K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler- Einstein metrics". In: *Invent. Math.* 203.3 (2016), pp. 973–1025. ISSN: 0020-9910. DOI: 10.1007/s00222-015-0607-7. URL: https://doi.org/10.1007/s00222-015-0607-7.

[8] Robert J. Berman and Bo Berndtsson. "Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties". In: *Ann. Fac. Sci. Toulouse Math. (6)* 22.4 (2013), pp. 649–711. ISSN: 0240-2963. DOI: 10.5802/afst.1386. URL: https://doi.org/10.5802/afst.1386.

[9] Bo Berndtsson. "Curvature of vector bundles associated to holomorphic fibrations". In: *Ann. of Math. (2)* 169.2 (2009), pp. 531–560. ISSN: 0003-486X. DOI: 10.4007/annals.2009.169.531. URL: https://doi.org/10.4007/annals.2009.169.531.

[10] Bo Berndtsson. "Strict and nonstrict positivity of direct image bundles". In: *Math. Z.* 269.3-4 (2011), pp. 1201–1218. ISSN: 0025-5874. DOI: 10.1007/s00209-010-0783-5. URL: https://doi.org/10.1007/s00209-010-0783-5.

[11] Dan Burns, Victor Guillemin, and Eugene Lerman. "Kähler metrics on singular toric varieties". In: *Pacific J. Math.* 238.1 (2008), pp. 27–40. ISSN: 0030-8730. DOI: 10.2140/pjm.2008.238.27. URL: https://doi.org/10.2140/pjm.2008.238.27.

[12] Huai-Dong Cao, Gang Tian, and Xiaohua Zhu. "Kähler-Ricci solitons on compact complex manifolds with $C_1(M) > 0$". In: *Geom. Funct. Anal.* 15.3 (2005), pp. 697–719. ISSN: 1016-443X. DOI: 10.1007/s00039-005-0522-y. URL: https://doi.org/10.1007/s00039-005-0522-y.

[13] Huai-Dong Cao and Detang Zhou. "On complete gradient shrinking Ricci solitons". In: *J. Differential Geom.* 85.2 (2010), pp. 175–185. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1287580963.

[14] Xiuxiong Chen. "The space of Kähler metrics". In: *J. Differential Geom.* 56.2 (2000), pp. 189–234. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1090347643.

[15] Charles Cifarelli. "Uniqueness of shrinking gradient Kähler-Ricci solitons on non-compact toric manifolds". In: arXiv:2010.00166 (2020). arXiv: 2010.00166 [math.DG].

[16] Craig van Coevering. "Examples of asymptotically conical Ricci-flat Kähler manifolds". In: *Math. Z.* 267.1-2 (2011), pp. 465–496. ISSN: 0025-5874. DOI: 10.1007/s00209-009-0631-7. URL: https://doi.org/10.1007/s00209-009-0631-7.

[17] Craig van Coevering. "Ricci-flat Kähler metrics on crepant resolutions of Kähler cones". In: *Math. Ann.* 347.3 (2010), pp. 581–611. ISSN: 0025-5831. DOI: 10.1007/s00208-009-0446-1. URL: https://doi.org/10.1007/s00208-009-0446-1.

[18] Tristan C. Collins and Gábor Székelyhidi. "K-semistability for irregular Sasakian manifolds". In: *J. Differential Geom.* 109.1 (2018), pp. 81–109. ISSN: 0022-040X. DOI: 10.4310/jdg/1525399217. URL: https://doi.org/10.4310/jdg/1525399217.

[19] Ronan J. Conlon, Alix Deruelle, and Song Sun. "Classification results for expanding and shrinking gradient Kähler-Ricci solitons". In: arXiv:1904.00147 (2019).

[20] David A. Cox. "The homogeneous coordinate ring of a toric variety". In: *J. Algebraic Geom.* 4.1 (1995), pp. 17–50. ISSN: 1056-3911.

[21] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties.* Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7. DOI: 10.1090/gsm/124. URL: https://doi.org/10.1090/gsm/124.

[22] Tamás Darvas. "Morse theory and geodesics in the space of Kähler metrics". In: *Proc. Amer. Math. Soc.* 142.8 (2014), pp. 2775–2782. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2014-12105-8. URL: https://doi.org/10.1090/S0002-9939-2014-12105-8.

[23] Tamás Darvas and Yanir A. Rubinstein. "Tian's properness conjectures and Finsler geometry of the space of Kähler metrics". In: *J. Amer. Math. Soc.* 30.2 (2017), pp. 347–387. ISSN: 0894-0347. DOI: 10.1090/jams/873. URL: https://doi.org/10.1090/jams/873.

[24] Thomas Delzant. "Hamiltoniens périodiques et images convexes de l'application moment". In: *Bull. Soc. Math. France* 116.3 (1988), pp. 315–339. ISSN: 0037-9484. URL: http://www.numdam.org/item?id=BSMF_1988__116_3_315_0.

[25] Wei Yue Ding. "Remarks on the existence problem of positive Kähler-Einstein metrics". In: *Math. Ann.* 282.3 (1988), pp. 463–471. ISSN: 0025-5831. DOI: 10.1007/BF01460045. URL: https://doi.org/10.1007/BF01460045.

[26] S. K. Donaldson. "Interior estimates for solutions of Abreu's equation". In: *Collect. Math.* 56.2 (2005), pp. 103–142. ISSN: 0010-0757.

[27] S. K. Donaldson. "Scalar curvature and stability of toric varieties". In: *J. Differential Geom.* 62.2 (2002), pp. 289–349. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1090950195.

[28] S. K. Donaldson. "Symmetric spaces, Kähler geometry and Hamiltonian dynamics". In: *Northern California Symplectic Geometry Seminar*. Vol. 196. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 1999, pp. 13–33. DOI: 10.1090/trans2/196/02. URL: https://doi.org/10.1090/trans2/196/02.

[29] Simon K. Donaldson. "Kähler geometry on toric manifolds, and some other manifolds with large symmetry". In: *Handbook of geometric analysis. No. 1*. Vol. 7. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2008, pp. 29–75.

[30] Serge Dubuc. "Critères de convexité et inégalités intégrales". In: *Ann. Inst. Fourier (Grenoble)* 27.1 (1977), pp. x, 135–165. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1977__27_1_135_0.

[31] J. J. Duistermaat and G. J. Heckman. "Addendum to: "On the variation in the cohomology of the symplectic form of the reduced phase space"". In: *Invent. Math.* 72.1 (1983), pp. 153–158. ISSN: 0020-9910. DOI: 10.1007/BF01389132. URL: https://doi.org/10.1007/BF01389132.

[32] J. J. Duistermaat and G. J. Heckman. "On the variation in the cohomology of the symplectic form of the reduced phase space". In: *Invent. Math.* 69.2 (1982), pp. 259–268. ISSN: 0020-9910. DOI: 10.1007/BF01399506. URL: https://doi.org/10.1007/BF01399506.

[33] Joerg Enders, Reto Müller, and Peter M. Topping. "On type-I singularities in Ricci flow". In: *Comm. Anal. Geom.* 19.5 (2011), pp. 905–922. ISSN: 1019-8385. DOI: 10.4310/CAG.2011.v19.n5.a4. URL: https://doi.org/10.4310/CAG.2011.v19.n5.a4.

[34] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. "Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons". In: *J. Differential Geom.* 65.2 (2003), pp. 169–209. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1090511686.

[35] Theodore Frankel. "Fixed points and torsion on Kähler manifolds". In: *Ann. of Math. (2)* 70 (1959), pp. 1–8. ISSN: 0003-486X. DOI: 10.2307/1969889. URL: https://doi.org/10.2307/1969889.

[36] William Fulton. *Introduction to toric varieties*. Vol. 131. Annals of Mathematics Studies. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993, pp. xii+157. ISBN: 0-691-00049-2. DOI: 10.1515/9781400882526. URL: https://doi.org/10.1515/9781400882526.

[37] A. Futaki. "An obstruction to the existence of Einstein Kähler metrics". In: *Invent. Math.* 73.3 (1983), pp. 437–443. ISSN: 0020-9910. DOI: 10.1007/BF01388438. URL: https://doi.org/10.1007/BF01388438.

[38] Akito Futaki. "Irregular Eguchi-Hanson metrics and their soliton analogues". In: (2020). arXiv: 2008.07351 [math.DG].

[39] Akito Futaki, Hajime Ono, and Guofang Wang. "Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds". In: *J. Differential Geom.* 83.3 (2009), pp. 585–635. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1264601036.

[40] Akito Futaki and Mu-Tao Wang. "Constructing Kähler-Ricci solitons from Sasaki-Einstein manifolds". In: *Asian J. Math.* 15.1 (2011), pp. 33–52. ISSN: 1093-6106. DOI: 10.4310/AJM.2011.v15.n1.a3. URL: https://doi.org/10.4310/AJM.2011.v15.n1.a3.

[41] V. Guillemin and S. Sternberg. "Convexity properties of the moment mapping". In: *Invent. Math.* 67.3 (1982), pp. 491–513. ISSN: 0020-9910. DOI: 10.1007/BF01398933. URL: https://doi.org/10.1007/BF01398933.

[42] Victor Guillemin. "Kaehler structures on toric varieties". In: *J. Differential Geom.* 40.2 (1994), pp. 285–309. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214455538.

[43] Jiyuan Han and Chi Li. *On the Yau-Tian-Donaldson conjecture for generalized Kähler-Ricci soliton equations*. 2020. arXiv: 2006.00903 [math.DG].

[44] Joachim Hilgert, Karl-Hermann Neeb, and Werner Plank. "Symplectic convexity theorems and coadjoint orbits". In: *Compositio Math.* 94.2 (1994), pp. 129–180. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1994__94_2_129_0.

[45] Kenkichi Iwasawa. "On some types of topological groups". In: *Ann. of Math. (2)* 50 (1949), pp. 507–558. ISSN: 0003-486X. DOI: 10.2307/1969548. URL: https://doi.org/10.2307/1969548.

[46] Yael Karshon and Eugene Lerman. “Non-compact symplectic toric manifolds”. In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 11 (2015), Paper 055, 37. DOI: 10.3842/SIGMA.2015.055. URL: https://doi.org/10.3842/SIGMA.2015.055.

[47] Frances Clare Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Vol. 31. Mathematical Notes. Princeton University Press, Princeton, NJ, 1984, pp. i+211. ISBN: 0-691-08370-3. DOI: 10.2307/j.ctv10vm2m8. URL: https://doi.org/10.2307/j.ctv10vm2m8.

[48] Brett Kotschwar and Lu Wang. *A uniqueness theorem for asymptotically cylindrical shrinking Ricci solitons.* 2020. arXiv: 1712.03185 [math.DG].

[49] Brett Kotschwar and Lu Wang. “Rigidity of asymptotically conical shrinking gradient Ricci solitons”. In: *J. Differential Geom.* 100.1 (2015), pp. 55–108. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1427202764.

[50] Eugene Lerman. “Contact toric manifolds”. In: *J. Symplectic Geom.* 1.4 (2003), pp. 785–828. ISSN: 1527-5256. URL: http://projecteuclid.org/euclid.jsg/1092749569.

[51] Chi Li. “Greatest lower bounds on Ricci curvature for toric Fano manifolds”. In: *Adv. Math.* 226.6 (2011), pp. 4921–4932. ISSN: 0001-8708. DOI: 10.1016/j.aim.2010.12.023. URL: https://doi.org/10.1016/j.aim.2010.12.023.

[52] Chi Li, Yuchen Liu, and Chenyang Xu. *A Guided Tour to Normalized Volume.* 2019. arXiv: 1806.07112 [math.AG].

[53] Toshiki Mabuchi. “Some symplectic geometry on compact Kähler manifolds. I”. In: *Osaka J. Math.* 24.2 (1987), pp. 227–252. ISSN: 0030-6126. URL: http://projecteuclid.org/euclid.ojm/1200780161.

[54] Jerrold Marsden and Alan Weinstein. “Reduction of symplectic manifolds with symmetry”. In: *Rep. Mathematical Phys.* 5.1 (1974), pp. 121–130. ISSN: 0034-4877. DOI: 10.1016/0034-4877(74)90021-4. URL: https://doi.org/10.1016/0034-4877(74)90021-4.

[55] Dario Martelli, James Sparks, and Shing-Tung Yau. “Sasaki-Einstein manifolds and volume minimisation”. In: *Comm. Math. Phys.* 280.3 (2008), pp. 611–673. ISSN: 0010-3616. DOI: 10.1007/s00220-008-0479-4. URL: https://doi.org/10.1007/s00220-008-0479-4.

[56] Dario Martelli, James Sparks, and Shing-Tung Yau. “The geometric dual of a- maximisation for toric Sasaki-Einstein manifolds”. In: *Comm. Math. Phys.* 268.1 (2006), pp. 39–65. ISSN: 0010-3616. DOI: 10.1007/s00220-006-0087-0. URL: https://doi.org/10.1007/s00220-006-0087-0.

[57] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, pp. xiv+292. ISBN: 3-540-56963-4. DOI: 10.1007/978-3-642-57916-5. URL: https://doi.org/10.1007/978-3-642-57916-5.

[58] Aaron Naber. "Noncompact shrinking four solitons with nonnegative curvature". In: *J. Reine Angew. Math.* 645 (2010), pp. 125–153. ISSN: 0075-4102. DOI: 10.1515/CRELLE.2010.062. URL: https://doi.org/10.1515/CRELLE.2010.062.

[59] D. H. Phong et al. "The Moser-Trudinger inequality on Kähler-Einstein manifolds". In: *Amer. J. Math.* 130.4 (2008), pp. 1067–1085. ISSN: 0002-9327. DOI: 10.1353/ajm.0.0013. URL: https://doi.org/10.1353/ajm.0.0013.

[60] Elisa Prato. "Convexity properties of the moment map for certain non-compact manifolds". In: *Comm. Anal. Geom.* 2.2 (1994), pp. 267–278. ISSN: 1019-8385. DOI: 10.4310/CAG.1994.v2.n2.a5. URL: https://doi.org/10.4310/CAG.1994.v2.n2.a5.

[61] Elisa Prato and Siye Wu. "Duistermaat-Heckman measures in a non-compact setting". In: *Compositio Math.* 94.2 (1994), pp. 113–128. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1994__94_2_113_0.

[62] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Reprint of the 1970 original, Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997, pp. xviii+451. ISBN: 0-691-01586-4.

[63] Stephen Semmes. "Complex Monge-Ampère and symplectic manifolds". In: *Amer. J. Math.* 114.3 (1992), pp. 495–550. ISSN: 0002-9327. DOI: 10.2307/2374768. URL: https://doi.org/10.2307/2374768.

[64] Reyer Sjamaar. "Holomorphic slices, symplectic reduction and multiplicities of representations". In: *Ann. of Math. (2)* 141.1 (1995), pp. 87–129. ISSN: 0003-486X. DOI: 10.2307/2118628. URL: https://doi.org/10.2307/2118628.

[65] Hideyasu Sumihiro. "Equivariant completion". In: *J. Math. Kyoto Univ.* 14 (1974), pp. 1–28. ISSN: 0023-608X. DOI: 10.1215/kjm/1250523277. URL: https://doi.org/10.1215/kjm/1250523277.

[66] Gábor Székelyhidi. *An introduction to extremal Kähler metrics*. Vol. 152. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2014, pp. xvi+192. ISBN: 978-1-4704-1047-6. DOI: 10.1090/gsm/152. URL: https://doi.org/10.1090/gsm/152.

[67] Gang Tian. "Kähler-Einstein metrics with positive scalar curvature". In: *Invent. Math.* 130.1 (1997), pp. 1–37. ISSN: 0020-9910. DOI: 10.1007/s002220050176. URL: https://doi.org/10.1007/s002220050176.

[68] Gang Tian and Xiaohua Zhu. "A new holomorphic invariant and uniqueness of Kähler-Ricci solitons". In: *Comment. Math. Helv.* 77.2 (2002), pp. 297–325. ISSN: 0010-2571. DOI: 10.1007/s00014-002-8341-3. URL: https://doi.org/10.1007/s00014-002-8341-3.

[69] Gang Tian and Xiaohua Zhu. "Uniqueness of Kähler-Ricci solitons". In: *Acta Math.* 184.2 (2000), pp. 271–305. ISSN: 0001-5962. DOI: 10.1007/BF02392630. URL: https://doi.org/10.1007/BF02392630.

[70] Xu-Jia Wang and Xiaohua Zhu. "Kähler-Ricci solitons on toric manifolds with positive first Chern class". In: *Adv. Math.* 188.1 (2004), pp. 87–103. ISSN: 0001-8708. DOI: 10.1016/j.aim.2003.09.009. URL: https://doi.org/10.1016/j.aim.2003.09.009.

[71] William Wylie. "Complete shrinking Ricci solitons have finite fundamental group". In: *Proc. Amer. Math. Soc.* 136.5 (2008), pp. 1803–1806. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-07-09174-5. URL: https://doi.org/10.1090/S0002-9939-07-09174-5.

[72] Zhu-Hong Zhang. "On the completeness of gradient Ricci solitons". In: *Proc. Amer. Math. Soc.* 137.8 (2009), pp. 2755–2759. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-09-09866-9. URL: https://doi.org/10.1090/S0002-9939-09-09866-9.