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Dynamics of acyclic interval maps

Morris W. Hirsch

He who can digest a second or third fluxion . . . need not, methinks, be squeamish about any point in divinity.
—Bishop George Berkeley, “The Analyst,” 1734

Abstract. We investigate conditions under which a map f in a possibly non-compact interval is acyclic—the only periodic orbits are fixed points. Several earlier results are generalized to maps with multiple fixed points. The chief tools are convergence results due to Coppel and Sharkovski, and the Schwarzian derivative. Illustrative examples are given and open problems are suggested.

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1. Introduction

The population of a periodically reproducing species is traditionally modeled by a continuous map $f: I \rightarrow I$ where I is an interval (typically $[0, 1]$ or $[0, \infty)$). In this approach the size of the species (population, density, biomass, . . .) is represented by a number $x \in I$, and $f(x)$ denotes the size of the next generation.¹

The dynamical complexity of many maps given by simple formulas was emphasized in a 1976 article in *Nature* by R. May [12]. The prevalence of complex dynamics makes it desirable to have tools for determining, for a specific class of maps, whether every population converges to some fixed point (the value of which may depend on the initial state). We call such maps *convergent*.

The goal of this article is to give simple conditions implying that a map is convergent, or more generally *acyclic*, meaning only fixed points have finite orbits.

¹Such models can be traced back to the famous thesis of Reverend Malthus [11], that unchecked population will outrun food supply. His theory is still being debated after more than two centuries, despite its paucity of data and lack of predictions (except gloom and doom). Its success can be attributed to plausible hypotheses and clear conclusions, and above all to mathematics that is not only simple and rigorous, but also extremely robust.

According to Poincaré, the task of Dynamical Systems Theory is to describe the long-run behavior of trajectories $\{f^n(x)\}$, $n = 0, 1, \dots$, where f^n denotes the n th iterate $f \circ \dots \circ f$ of f . A basic topological tool is the *omega limit set* $\omega(x)$, comprising the limits in I of subsequences of the trajectory of x .² Ideally we should be able to answer the following questions for each point x (or “most” points): Is $\omega(x)$ nonempty and compact? If so, is it a *fixed point* $p = f(p)$, or a *k-cycle*—an orbit of finite cardinality $k > 1$ —or something else?

In the famous article “Period three implies chaos” [10], T. Li and J. Yorke proved in 1975 that a 3-cycle implies cycles of all periods.³ Subsequently a 1964 paper by O. Sharkovski⁴ [14] was belatedly discovered in a little-known Ukrainian journal by western scholars. This work proved an astonishing universal property of interval maps: There is a simple ordering $\prec$ of the positive integers, independent of f , such that if f has a j -cycle and $j \prec k$, then f also has a k -cycle. Since $3 \prec n$ for all n , the theorem of Li and Yorke is an immediate consequence. A corollary of this ordering is crucial for our results:

Maps without 2-cycles are acyclic.

We also rely on another, less well-known theorem of Sharkovski [15]:

For acyclic maps in compact intervals, every trajectory converges.

In fact both this and the previous result were proved much earlier by W. Coppel [1].⁵ In Theorem 2.3 we use this to show for acyclic maps in noncompact intervals that a trajectory converges if it lies in a compact subset of the domain.

This paper was inspired by earlier work of P. Cull [2, 3, 4], T. Huang [9] and H. Thieme [17]. Cull and Huang considered maps in $[0, \infty)$ that fixed the origin and exactly one other point. Thieme also considered maps in the open interval $]0, \infty[$ having a unique fixed point. We extend some of these results to maps with any number of fixed points by exploiting the convergence theorems of Coppel and Sharkovski. Following earlier work by D. Singer [16] and G. Enciso [8], we use the Schwarzian derivative to establish acyclicity (Theorems 4.13 and 4.18).

Except for Schwarzian theory and the results of Coppel and Sharkovski, this paper is self-contained. Several abstract examples illustrate the theory. The works cited above contain many applications to classical biological models.

Terminology

All maps are assumed continuous. If $-\infty \leq a \leq b \leq \infty$ then $]a, b[$ denotes the open interval $\{x: a < x < b\}$ (which is empty if $a = b$). When $-\infty < a \leq b < \infty$ then $[a, b]$ denotes the closed interval $[a, b]$ (which reduces to a point if $a = b$). I always

²This definition derives from G. D. Birkhoff. Poincaré used the concept of limit point, but considered notions such as “the set of all limit points of f ” to be impredicative—excessively self-referencing—and therefore logically impermissible.

³F. Dyson [6, p. 221] wrote: “The Li–Yorke paper is one of the immortal gems in the literature of mathematics. . . . To my mind, this theorem and its proof throw more light than a thousand beautiful pictures on the basic nature of chaos.”

⁴Also referred to as A. Sharkovski, Sarkovski, Šarkovs’kiĭ, etc.

⁵I thank the anonymous referee for this reference and several other improvements.

denotes an interval. The letters i, j, k, l, m, n stand for nonnegative integers. f, g, h denote real-valued functions on intervals. f is *increasing* if $x > y \Rightarrow f(x) > f(y)$, and *nondecreasing* if $x > y \Rightarrow f(x) \geq f(y)$. The definitions of *decreasing* and *nonincreasing* are analogous. x has *period* k for if $k \geq 1$ and $f^k(x) = x$. The orbit of x is a *cycle*, more precisely a *k-cycle*, if its smallest period is $k \geq 2$. When there are no cycles f is *acyclic*. We say x is *convergent* if $\omega(x)$ is a fixed point. When every point is convergent, f is called convergent. The Intermediate Value Theorem is referred to as IVT.

2. Acyclicity and convergence

We rely on the following corollary of Sharkovski's ordering:

Theorem 2.1. $f: I \rightarrow I$ is acyclic if and only if it has no 2-cycles.

In less well known works, W. Coppel [1] and Sharkovski [15] proved:

Theorem 2.2. If I is compact and $f: I \rightarrow I$ is acyclic, the following hold for all $x \in I$:

- (i) The trajectory of x converges.
- (ii) If $f(x) \geq x$ (respectively, $\leq x$) then $f^m(x) \geq x$ (respectively, $\leq x$) for all $m > 0$.

When (i) and (ii) hold we say that x and its trajectory are *neatly convergent*. If every trajectory is neatly convergent we say f is neatly convergent. This condition has an interesting biological interpretation: If generation $n + 1$ is larger (smaller) than generation n , all later generations will also be larger (smaller) than generation n .

We extend Theorem 2.2 to noncompact intervals I :

Theorem 2.3. If $f: I \rightarrow I$ is acyclic, every trajectory lying in a compact subset of I is neatly convergent.

As convergent trajectories lie in compact sets, an immediate consequence is:

Corollary 2.4. If $f: I \rightarrow I$ is convergent, every trajectory is neatly convergent. $\square$

Theorem 2.3 is a special case of a more general result:

Theorem 2.5. Let $g: I \rightarrow \mathbb{R}$ be an acyclic map and $\Lambda \subset I$ a compact set such that $g(\Lambda) \subset \Lambda$. Then every trajectory in Λ is S -convergent.

Notice that g need not map I into itself.

Proof. The definition of “acyclic” is extended to g in the obvious way. An x is a convergent point for g if there is a sequence $\{x_n\}$ converging to a limit in I with

$x_0 = x$ and $g(x_n) = x_{n+1}$. Denote the convex hull of Λ by $[a, b]$ and note that $a, b \in \Lambda \subset [a, b] \subset I$. Define the map

$$h: [a, b] \rightarrow [a, b], \quad h(x) = \begin{cases} a & \text{if } g(x) \leq a, \\ g(x) & \text{if } g(x) \in [a, b], \\ b & \text{if } g(x) \geq b. \end{cases}$$

Then h is acyclic and hence convergent by Theorem 2.1 and the following proposition:

$$h \circ h(y) = y \Rightarrow g \circ g(y) = y. \quad (1)$$

This implication holds when both $g(y)$ and $g \circ g(y)$ are in $[a, b]$. If $g(y) \notin [a, b]$ then $h(y) = a$ or b ; for definiteness assume $h(y) = a$. Then $g(a) = g \circ h(y) = h \circ g(a) = a$, whence $g(a) = a$, implying (1). Suppose finally that $g(y) \in [a, b]$ but $g \circ g(y) \notin [a, b]$. In this case $y = h \circ h(y) = h \circ g(y) = a$ or b , so $y \in \Gamma$ and (1) follows. Now apply Theorem 2.2 to h , and use the fact that each point in Λ has the same trajectory under h and g . $\square$

H. Thieme proved a convergence theorem (Corollary 2.7 below) that is remarkable for its simple hypotheses, the most stringent being existence of only one fixed point. Here we extend this result to allow any number of fixed points:

Theorem 2.6. *Assume $f:]0, \infty[\rightarrow]0, \infty[$ has no 2-cycles, and there are fixed points $p_* \leq p$ such that:*

- (a) f is bounded on $(0, p_*]$,
- (b) $x < p_* \Rightarrow f(x) > x$,
- (c) $x \geq p \Rightarrow f(x) < x$.

Then every trajectory is neatly convergent.

Hypothesis (a) holds in the common situation that f extends to a continuous map on $[0, \infty)$.

Proof. f is acyclic by the Coppel–Sharkovski Theorem 2.1. The hypotheses imply the existence of $a, b \in I$ such that

$$b = \sup\{f(x) : x \leq p\}, \quad p \leq b < \infty, \quad (2)$$

$$a = \inf\{f(x) : p_* \leq x \leq b\}, \quad 0 < a \leq p_*. \quad (3)$$

It follows that for all x ,

$$\limsup_{n \rightarrow \infty} f^n(x) \leq b. \quad (4)$$

Next we show that for all x ,

$$\liminf_{n \rightarrow \infty} f^n(x) \geq a. \quad (5)$$

Given any $\epsilon > 0$, choose $a_0 \in [a - \epsilon, a)$ and $b_0 \in [b, b + \epsilon)$ such that

$$x \in [p, b_0] \Rightarrow f(x) > a_0. \quad (6)$$

It suffices to prove

$$\liminf_{n \rightarrow \infty} f^n(x) \geq a_0, \quad (7)$$

and by (4) we can assume that for all n ,

$$f^n(x) \leq b_0. \quad (8)$$

If the orbit of x lies above a_0 then (7) is trivial, therefore we assume $x < a_0$. If the trajectory of x lies below p_* it increases to p_* by hypothesis (b), thereby implying (7). Therefore we can assume $x \geq p_*$. It now suffices to prove that for all k ,

$$f^k(x) \geq a_0. \quad (9)$$

This holds for $k = 0$ because $x \geq p_* > a_0$. Assume inductively that $l \geq 0$ and (9) holds for all $k \leq l$. If $f^l(x) \geq p_*$ then $f^l(x) \in [p_*, b_0]$ by (8), whence $f^{l+1}(x) \geq a_0$ by (6). If $f^l(x) < p_*$, hypothesis (b) implies $f^{l+1}(x) > f^l(x) \geq a_0$. This completes the proof of (7) and hence (5).

From (4) and (5) we see that $\omega(x)$ is a nonempty compact set in $[a, b]$. As $f(\omega(x)) \subset \omega(x)$, the conclusion of Theorem 2.6 follows from Theorem 2.5. $\square$

Theorem 2.6 was inspired by the special case due to H. Thieme [17, Theorem 9.6]:

Corollary 2.7. *Assume $f:]0, \infty[\rightarrow]0, \infty[$ has the unique fixed point p and is bounded on $(0, p]$, and there exist $x_1 < p < x_2$ with $f(x_1) > x_1$ and $f(x_2) < x_2$. If f has no 2-cycles then every trajectory converges to p .*

Proof. This follows from Theorem 2.6 with $p_* = p$, because hypotheses (b) and (c) are consequences of the existence of x_1 and x_2 , uniqueness of p , and the IVT. $\square$

More convergence theorems can be found in Thieme's book.

3. Further sufficient conditions for acyclicity

Definition 3.1. A *test interval* for $f: I \rightarrow I$ is a nondegenerate compact interval $[w, p] \subset I$ such that:

- (T1) p is the smallest fixed point in $[w, f(w)]$,
- (T2) $w \leq x < p \Rightarrow x < f(x) < f(w)$.

Proposition 3.2. *If $\{u, f(u)\}$ is a 2-cycle with $u < f(u)$, there exists a smallest fixed point p such that $u < p < f(u)$, and a test interval $[w, p] \subset [u, p]$ where w is the largest point in $[u, p]$ such that $f(w) = f(u)$.*

Proof. The existence of p follows from IVT applied to the function $f(x) - x$ in $[u, f(u)]$, and (T1) holds. Evidently there exists w as described in the proposition, and it is easy to see that $u \leq w < p$ and $f(p) = p < f(u)$. It follows that (T2) holds, as otherwise another application of IVT yields a fixed point between u and w , contradicting the definition of p . Thus $[w, p]$ is a test interval. $\square$

Consider the following condition on an interval $J \subset I$:

(N) If $[w, p] \subset J$ is any test interval then $f \circ f(w) > w$

Adapting Cull [2, Theorem 1], we have:

Theorem 3.3. *Suppose $f: I \rightarrow I$ and let $J \subset I$ be an interval.*

- (i) *Assume $J = [q, p]$, $J \cap \text{Fix}(f) = \{q, p\}$, and J meets no 2-cycle. Then (N) holds.*
- (ii) *If (N) holds, J contains no 2-cycle.*

Proof. (i) Suppose (N) is false and $[w, p] \subset [q, p]$ is a test interval with $f \circ f(w) \leq w$. Then w is not a fixed point because $f(w) > w$, so $f \circ f(w) < w$ because w does not belong to a 2-cycle. Then IVT shows that $f(x) > x$ if $q < x < p$ because this holds for $x = w$ and there are no fixed points in $]q, p[$. Therefore if $y \in]q, w[$ is close enough to q then $f \circ f(y) > y$. From IVT we get the contradiction that $]y, w[$ meets a 2-cycle.

(ii) Suppose *per contra* there is a 2-cycle $\{u, f(u)\} \subset J$. Let $[w, p] \subset [u, f(u)]$ be the test interval described in Proposition 3.2. Then $f \circ f(w) = f \circ f(u) = u \leq w$, in contradiction to (N). $\square$

General population models

The following definition broadens Cull's notion of "population model" (see Section 5):

Definition 3.4. Let $f: [0, \infty) \rightarrow [0, \infty)$. We say that f , or more precisely (f, p_*, m_0, p) , is a *general population model* provided $p_*, p \in \text{Fix}(f)$, $f(m_0)$ is the maximum value of f , and the following conditions are satisfied:

- (P1) p is the unique fixed point $> p_*$,
- (P2) $x > m_0 \Rightarrow f(x) < f(m_0)$,
- (P3) every point in $[0, p_*]$ is convergent,
- (P4) $p_* < x < m_0 \Rightarrow f(x) > x$.

Note that (P1), (P2) and (P4) imply:

(P5) $[m_0, p]$ is a test interval (see 3.1), and all test intervals lie in $[m_0, p]$.

Mild assumptions imply (P3), for instance: $p_* = 0$, or more generally, $f|_{[0, p_*]}$ is nondecreasing.

The next result extends Cull [2, Theorem 1] (see Theorem 5.1 below):

Theorem 3.5. *The following conditions are equivalent for a general population model (f, p_*, m_0, p) :*

- (a) *every trajectory converges,*
- (b) *there are no 2-cycles,*
- (c) $p_* < x < p \Rightarrow f \circ f(x) > x$,
- (d) $m_0 \leq x < p \Rightarrow f \circ f(x) > x$.

Proof. Clearly (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (b): It suffices by Theorem 3.3(ii) to show that $f \circ f(w) > w$ for every test interval $[w, p]$, and this inequality follows from (P5) and (d).

(b) $\Rightarrow$ (c): By (b) and (P4) there exists $\epsilon > 0$ such that $p - \epsilon < x < p \Rightarrow f \circ f(x) > x$. Therefore (c) holds—otherwise IVT implies existence of $z \in [p_*, p]$ with $f \circ f(x) = z$, making z a fixed point by (b) in contradiction to (P1).

(b) $\Rightarrow$ (a): Define the maps

$$\begin{aligned} g: [p_*, \infty) &\rightarrow [p_*, \infty), & g(x) &= \max(p_*, f(x)), \\ h: [0, \infty) &\rightarrow [0, \infty), & h(x) &= g(x + p_*) - p_*. \end{aligned}$$

Notice that $h(0) = 0$ and g is topologically conjugate to h . Assumption (b) implies h has no 2-cycles. Therefore h and hence g is convergent by Theorem 2.6 applied to the restriction of h to $]0, \infty[$. If $x \in [0, \infty)$, either the trajectory of x enters $[0, p_*]$, in which case it converges by (P3), or else the trajectory converges because it coincides with the trajectory of x under g . Thus (a) holds. $\square$

3.1. Envelopment

Adapting a useful notion of Cull, we say that $f: I \rightarrow I$ *envelops* $g: I \rightarrow I$ if the following conditions are satisfied:

- (E1) for each $x \in I$, $g(x)$ is a convex combination of $f(x)$ and x ,
- (E2) $\text{Fix}(g) = \text{Fix}(f)$.

The next result generalizes Theorem 3 of Cull [2]:

Theorem 3.6. *If (f, p_*, m_0, p) is a convergent general population model, every general population model enveloped by f is convergent.*

Proof. Any general population model enveloped by f has the form (g, p_*, m_1, p) . Suppose *per contra* that g is not convergent. By Theorem 3.5, g has a 2-cycle $\{u, g(u)\}$, $u < g(u)$, and $m_1 < u < p < g(u)$ by (P3), (P4) and Proposition 3.2.

By continuity of $f - g$ and compactness of $[u, p]$, there is a largest y such that $u \leq y < p$ and $f(y) = g(u)$. Because $f(p) = g(p) = p$, this implies $[y, p]$ is a test interval for f , and therefore $f(f(y)) > y$ by Theorem 3.5(c). Because $g(g(u)) = u < g(u)$, from (E1) we infer $f(g(u)) \leq g(g(u))$. Consequently, $u \leq y < f(f(y)) = f(g(u)) \leq g(g(u))$, giving the contradiction $u < g(g(u))$. $\square$

For further results on envelopment see Cull [3] and Cull *et al.* [4].

4. Differential conditions for acyclicity

The map $x \mapsto -x$ in $\mathbb{R}$ has 2-cycles but no cycles of higher period. This is the simplest instance of the following result:

Theorem 4.1. *Let $J \subset I$ be an interval in which $f: I \rightarrow I$ is absolutely continuous, and $f' \geq -1$ almost everywhere or $f' \leq -1$ almost everywhere. Then:*

- (a) *There are no n -cycles in J for $n > 2$.*
- (b) *If $\{p, q\}$ is a 2-cycle in J , then $f' \equiv -1$ almost everywhere in $]p, q[$.*

Proof. (a) We assume $f' \geq -1$ almost everywhere, the other case being similar. Suppose $n \geq 3$ and $f^i x \in J$ for $i \leq n$. By the telescoping identity and the fundamental theorem of calculus,

$$\begin{aligned} f^n(x) - x &= \sum_{j=0}^{n-1} [f(f^j(x)) - f^j(x)] = f(x) - x + \sum_{j=1}^{n-1} \int_{f^{j-1}(x)}^{f^j(x)} f'(t) dt \\ &\geq f(x) - x - \sum_{j=1}^{n-1} [f^j(x) - f^{j-1}(x)] \\ &= f^{n-1}(x) - f(x). \end{aligned}$$

Therefore if $f^n(x) = x$ then $f^n(x) = f(f^{n-1}(x)) = f^2(x)$, so $f^2(x) = x$ and n is not the minimal period of x .

(b) To fix ideas assume $p < f(p) = q$, $f' \geq -1$ almost everywhere, and $f' > 0$ in a positive measure subset of $[p, q]$. Then $f(q) - f(p) = \int_p^q f' > -1(q-p) = p-q$, hence $\{p, q\}$ is not a 2-cycle. $\square$

Corollary 4.2. *f is acyclic provided f' is continuous, and $f' > -1$ in a dense set or $f' < -1$ in a dense set.*

Example 4.3. Consider maps of the form $f_C: \mathbb{R} \rightarrow \mathbb{R}$, $f_C(x) = Ax^5 + Cx + D$, $A > 0$. The minimum value of $f'_C(x)$ is $f'_C(0) = C$, hence by Corollary 4.2:

- f_C is acyclic provided $C \geq -1$.

Example 4.4. The maps $f_b: [0, \infty) \rightarrow [0, \infty)$, $f_b(x) = xe^{b-x}$, $b > 0$, have often been used as biological models (see references in Cull [2]). The unique positive fixed point is at $x = b$. Using

$$f'_b(x) = (1-x)e^{b-x}, \quad f''_b(x) = (x-2)e^{b-x}, \quad f'''_b(x) = (3-x)e^{b-x}$$

we see that f'_b takes its minimum value only at the inflection point $x = 2$, and $f'_b(2) = e^{b-2}$. From Corollary 4.2, boundedness of orbits and Theorem 2.5, we recover a result due to Cull [2, Model 1]:

- f_b is convergent provided $b \leq 2$.

As b increases past 2 the positive fixed point loses stability and a 2-cycle bifurcates from $x = 2$ (see R. Devaney [5, Theorem 12.7] or S. Elaydi [7, Theorem 2.7]). This suggests:

Conjecture 4.5. *f_b has a 2-cycle if $b > 2$. If b is sufficiently large f_b has cycles of all periods.*

4.1. Stable acyclicity

If J is an interval and $k \geq 1$, let $\mathcal{M}^k(J)$ denote the space of maps $J \rightarrow J$ having k continuous derivatives, endowed with the topology of uniform convergence of f and its derivatives of orders $\leq k$ on compact subsets of J .

Proposition 4.6. *Assume $\lim_{n \rightarrow \infty} f_n = f$ in $\mathcal{M}^k(J)$ and f has no 2-cycles. For each n let $\{a_n, b_n\}$ be a 2-cycle for f_n and assume $a_n \rightarrow p \in J$ and $b_n \rightarrow q \in J$. Then $q = p \in \text{Fix}(f)$, and:*

- (i) $r \geq 1 \Rightarrow f'(p) = -1$,
- (ii) $r \geq 2 \Rightarrow f''(p) = 0$.

Proof. Clearly $f(p) = q$ and $f(q) = p$, hence $q = p \in \text{Fix}(f)$ because f has no 2-cycles.

(i) Follows from

$$-1 = \frac{f_n(b_n) - f_n(a_n)}{b_n - a_n} \rightarrow f'(p).$$

(ii) Assume $a_n < b_n$ for all $n \in \mathbb{N}_+$. There exists $p_n \in \text{Fix}(f_n)$ such that $a_n < p_n < b_n$. This implies $p_n \rightarrow p$, and a_n, p_n, b_n are fixed points of $f_n \circ f_n$. By IVT there exist s_n, t_n such that

$$a_n < s_n < p_n < t_n < b_n, \quad (f_n \circ f_n)'(s_n) = (f_n \circ f_n)'(t_n) = 1.$$

Another use of IVT yields points $z_n \in]s_n, t_n[$ such that as $n \rightarrow \infty$ we have

$$0 = (f_n \circ f_n)''(z_n) \rightarrow (f \circ f)''(p).$$

Now (ii) is proved by computing the Taylor polynomial of $f \circ f(x) - p$ from that of $f(x) - p$ and using (i). $\square$

Theorem 4.7. *Assume I is a compact interval and $f \in \mathcal{M}^k(I)$ is convergent. In the following cases f has a neighborhood of convergent maps in $\mathcal{M}^k(I)$:*

- (i) $k = 1$, and $p \in \text{Fix}(f) \Rightarrow f'(p) \neq -1$,
- (ii) $k = 2$, and $p \in \text{Fix}(f) \Rightarrow f'(p) \neq -1$ or $f''(p) \neq 0$.

Proof. Since I is compact, a map in I is convergent if and only if it has no 2-cycles (Theorems 2.1 and 2.2). Therefore the conclusions follow from Proposition 4.6. $\square$

4.2. Schwarzian derivatives

Recall that the *Schwarzian* of f at x is

$$\text{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

provided f is thrice differentiable at x and $f'(x) \neq 0$. These conditions are tacitly assumed in all statements referring to $\text{S}f(x)$. The set in which they hold is the domain of a function denoted by $\text{S}f$, or by $\text{S}[\psi]$ where ψ denotes a formula for f . For the use of the Schwarzian in geometry the book of V. Ovsienko & S. Tabachnikov [13] can be consulted.

Proposition 4.8. *Assume $Sf < 0$ at every point of an open interval J . Then the following hold:*

- (i) *f' does not have a positive local minimum value or a negative local maximum value in J .*
- (ii) *If $Sh \leq 0$ at every point of $f(J)$, then $S[h \circ f] < 0$.*
- (iii) *Suppose $y \in J$ is such that $f^k(y) \in J$ for $k = 1, \dots, n-1$. Then $S[f^n](y) < 0$ (where f^n denotes the n th iterate of f).*
- (iv) *Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a real polynomial of degree $n \geq 2$ having n distinct real roots. Then $S[\Phi + c] < 0$ for all real constants c .*

Proof. See Propositions 11.2 and 11.3 in R. Devaney's book [5]. □

Many maps with negative Schwarzians can be constructed using (ii), (iii), (iv) of Proposition 4.8 together with the following facts:

Proposition 4.9.

- (a) *If $f(x) = \frac{ax+b}{cx+d}$ then $Sf = 0$.*
- (b) *Assume $f(x) := g(x) + \lambda x$, $\lambda > 0$. Then $Sf < 0$ provided $Sg < 0$ and $f' < 0$.*

Proof. (a) To prove this classical result make the variable change $y = cx + d$, $x = c^{-1}(y - d)$, set $h(y) := f(x)$, and note that $Sh(y) = Sf(x)$. As h has the form $h(y) = A + By^{-1}$, it suffices to prove $S[By^{-1}] = 0$, which is left to the reader.

(b) This follows from the identity

$$(f')^2 \cdot Sf = \lambda g''' + (g')^2 \cdot Sg$$

at points where $g''' \leq 0$, and from the identity

$$(f')^2 \cdot Sf = f'g''' - \frac{3}{2}(g''')^2$$

wherever $g''' > 0$. □

Next we exhibit a connection between Schwarzians and local dynamics at a fixed point:

Lemma 4.10. *Assume*

$$f(p) = p, \quad 0 < |f'(p)| \leq 1, \quad Sf(p) < 0$$

and set $g = f \circ f$. There exists $\epsilon > 0$ such that

$$x \in [p - \epsilon, p) \Rightarrow x < g(x) < p, \quad x \in (p, p + \epsilon] \Rightarrow p < g(x) < x, \quad (10)$$

hence p is an asymptotically stable fixed point for g and f .

Proof. After translating the coordinate we assume $p = 0$. In a suitably small neighborhood of 0 the function g satisfies $Sg < 0$ by Proposition 4.8. A calculation yields

$$g(0) = 0, \quad g'(0) = A, \quad 0 < A \leq 1, \quad g''(0) = 0.$$

Therefore $Sg(0) \times g'(0) = g'''(0)$, whence $g'''(0) = -B < 0$. A Taylor expansion shows that

$$g(x) = x \left(A - \frac{B}{2} x^2 \right) + o(x^3)$$

As $0 < A - \frac{B}{2} x^2 < 1$ when $|x|$ is sufficiently small, this proves the existence of ϵ satisfying (10). Therefore $|x - p| < \epsilon \Rightarrow g^n(x) \rightarrow p$. This shows that p is asymptotically stable for $f \circ f$, hence also for f . $\square$

The key to several acyclicity criteria are the following assumptions on the behavior of $f: I \rightarrow I$ in open intervals $J \subset I$:

Hypothesis 4.11. If $\text{Fix}(f) \cap J \neq \emptyset$, then:

(H1) $x \in J \Rightarrow f'(x) < 0$ and $Sf(x) < 0$,

(H2) $p \in \text{Fix}(f) \cap J \Rightarrow f'(p) \geq -1$.

Note that (H2) implies p is asymptotically stable, by Lemma 4.10.

Proposition 4.12. Given $f: I \rightarrow I$, assume $m \in I$ is such that the open interval $]m, f(m)[$ satisfies Hypothesis 4.11. Then $[m, f(m)]$ does not meet any 2-cycle.

Similar result have been found by D. Singer [16] and G. Enciso [8].

Proof. Assume *per contra* that $[m, f(m)]$ meets a 2-cycle $\{u, v\}$, $u < v$. As $f' < 0$ in $]m, f(m)[$ we have $v < f(m)$. Thus $[u, v] \subset J$, and f maps $[u, v]$ homeomorphically onto itself. By Proposition 3.2 there is a fixed point $p \in]u, v[$, and p is asymptotically stable by (H1) and Lemma 4.10. The connected component of p in its basin of attraction is an interval L that excludes u and v , hence L is an open interval $]a, b[\subset]u, v[$. It follows that f maps L homeomorphically onto itself, and $\{a, b\}$ is a 2-cycle.

The iterate $f \circ f$ restricts to a homeomorphism $g: [a, b] \rightarrow [a, b]$ such that

$$x \in [a, b] \Rightarrow g'(x) > 0 \text{ and } Sg(x) < 0. \quad (11)$$

As $\text{Fix}(g) = \{a, p, b\}$, by IVT there exist s, t such that

$$a < s < p, \quad g'(s) = 1, \quad p < t < b, \quad g'(t) = 1.$$

Therefore g' has a local minimum $w \in]s, t[$ and $g'(w) > 0$ by (11). But this is contradicted by Proposition 4.8(i) applied to g . $\square$

The following result extends a theorem of Y. Huang [9] (see Theorem 5.3 below):

Theorem 4.13. Let (f, p_*, m_0, p) be a general population model such that the interval $]m_0, f(m_0)[$ satisfies Hypothesis 4.11. Then f is convergent.

Proof. By Theorem 3.3(ii) it suffices to prove $f \circ f(x) > x$ for all $x \in [m_0, p)$. Suppose the contrary: assume $v \in [m_0, p)$, $f \circ f(v) \leq v$. Since f is decreasing in $[m_0, p]$ by (H1), $[m_0, p]$ is a test interval, hence $f \circ f(v) < v$. As p is asymptotically stable (Lemma 4.10), there exists $w \in]v, p[$ such that $f \circ f(w) > w$. By IVT

there exists $q = f \circ f(q) \in]v, w[$. As $q \neq f(q)$ by (PM1), $\{q, f(q)\}$ is a 2-cycle, and it meets $[m_0, p]$ because $q \in [v, w] \subset [m_0, p]$. But this contradicts Proposition 4.12. $\square$

Example 4.14. Consider a unimodal map

$$f: [0, \infty) \rightarrow [0, \infty), \quad x \mapsto Cx^2e^{-x}, \quad C > 0.$$

Denote the maximum fixed point by $p \geq 0$ and let p_* be the largest fixed point in $[0, p)$. The unique critical point is the global maximum point at $x = 2$. It is easy to see that $(f, p_*, 2, p)$ is a general population model (Definition 3.4).

We use Theorem 4.13 to prove:

- f is convergent provided $p \leq 3$.

It suffices to verify Hypothesis 4.11 for the open interval $J :=]2, \infty[= \{x: f'(x) < 0\}$. Condition (H1) of 4.11 requires that $Sf(x) < 0$ in J . This inequality is a consequence of Proposition 4.8(ii): Express f as the composition $f = h \circ \phi$, $\phi(x) = C^{1/2}xe^{-x/2}$, $h(x) = x^2$, and use the fact that $x > 0 \Rightarrow Sh(x) < 0$, $S\phi(x) < 0$. For H(2) we need $f'(p) \geq -1$, which holds because $f'(x) = f(x)(2/x - 1)$ for $x > 0$, $f(p) = p$, and $0 < p \leq 3$.

Example 4.15. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a cubic polynomial with positive leading coefficient.

- f is acyclic provided there exists $p \in \text{Fix}(f)$ such that either $-1 \leq f'(p) < 0$, or else $f'(p) = 1$ and $f''(p) \neq 0$.

For the first case there are exactly three fixed points $a < p < b$ and $f'(x) < 0 \Leftrightarrow a < x < b$. Propositions 4.8(iv) and 4.9(b) show that $Sf(x) < 0$ if $a < x < b$. Hypothesis 4.17 is satisfied, and acyclicity follows from Theorem 4.18. In the second case there are just two fixed points, but the same argument proves acyclicity. f is not convergent, because $|f(x)| > |x|$ if x is outside the convex hull H of the fixed point set. It can be shown that every point of H is convergent.

On the other hand, Theorem 6.4 shows that acyclicity fails for any cubic having three fixed points and negative leading coefficient.

Example 4.16. One can construct many acyclic general population models that are far from satisfying Hypothesis 4.11, by combining Theorem 4.13 with Theorem 3.6 on enveloping. Let f be any acyclic general population model. Define $g(x) = \phi(x)f(x) + (1 - \phi(x))x$, where $\phi: [0, \infty) \rightarrow [0, 1]$ is a continuous map such that $0 < \phi(x) < 1$ if $f(x) \neq x$. Then g is a general population model enveloped by f . Theorem 4.13 implies f is acyclic, hence g is acyclic by Theorem 3.6. One can choose ϕ to make g highly oscillatory between fixed points, with arbitrarily large values of $|g'|$. At fixed points of g , however, g' is restricted to convex combinations of f' and 1.

To obtain acyclicity for maps more complicated than general population models we introduce the following conditions:

Hypothesis 4.17. I is a closed interval and $f: I \rightarrow I$ has the following properties:

(G0) If $p \in \text{Fix}(f)$, $a < p < b$, and $f(a) > f(b)$, there exists $y_0 < p$ such that $f(y_0) < f(a)$.

This holds in case $I = [0, \infty)$ and $f(0) = 0$.

(G1) If m_1, m_2 are local maxima and $m_1 < m_2 \leq f(m_1)$, then $f(m_1) \leq f(m_2)$.

(G2) If f is nonincreasing in an open interval J and $p \in J \cap \text{Fix}(f)$, then

- (a) $f'(x) < 0$ and $Sf(x) < 0$ for all $x \in J$,
- (b) $f'(p) \geq -1$.

Theorem 4.18. Any map f satisfying Hypothesis 4.17 is acyclic.

Proof. Assume *per contra* there is a 2-cycle $\{a, b\}$, $a < b$. By IVT there is a smallest fixed point $p \in]a, b[$. Choose $y_0 < p$ as in (G0), and let m be the largest point in $[y_0, p]$ at which $f|_{[y_0, p]}$ takes its maximum value. Then

$$m < p, \quad f(m) \geq b.$$

Let r be the largest point in $[a, p)$ such that $f(r) = b$, and let s be the smallest point in $(p, b]$ such that $f(s) = a$. Then $f([r, s]) = [a, b]$. Observe that $f|_{[r, s]}$ is nonincreasing: otherwise f would have a local maximum point $z \in]r, s[$ such that $m < z$ and $f(m) \geq b > f(m)$, contradicting (G1).

Hypothesis 4.17 and Lemma 4.10 show that p is asymptotically stable. The component of p in its basin of attraction is an invariant open interval $K \subset]r, s[$. The map $f|_K$ is injective by hypothesis (G2)(b), therefore f maps the boundary $\{u, v\}$ of K onto itself. As neither u nor v is a fixed point, $\{u, v\}$ is a 2-cycle. Hypothesis (G2) shows that the interval $J :=]u, v[$ satisfies Hypothesis 4.11. Proposition 4.12 gives the contradiction that $[u, v]$ does not contain any 2-cycle. $\square$

Example 4.19. We show for a certain range of parameters that the maps

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x + \alpha \sin x + C, \quad \alpha \neq 0, C \in \mathbb{R}, \quad (12)$$

satisfy Hypothesis 4.17 and are therefore acyclic, but some orbits go to infinity. To ensure fixed points we assume

$$|\alpha| \geq |C|. \quad (13)$$

Condition (G0) of Hypothesis 4.17 is easily verified, using the fact that the function $f(x) - x$ is 2π -periodic. To check (G1), observe that local maximum points m_1 and $m_2 > m_1$ are zeroes of $\cos x + \alpha^{-1}$ at which $\sin x$ is positive. Hence they are separated by integer multiples of 2π , entailing $f(m_2) - f(m_1) = m_2 - m_1 > 0$.

Let J be an interval in which f is nonincreasing, as in (G2), so that $f' \leq 0$ in J . The function $g(x) := f(x) - x$ has the property that $g'(x) \neq 0 \Rightarrow Sg(x) < 0$. Therefore Lemma 4.9(b) gives

$$x \in J \Rightarrow Sf(x) < 0,$$

which validates (G2)(a). For (G2)(b) we need

$$p \in J \cap \text{Fix}(f) \Rightarrow f'(p) \geq -1,$$

or equivalently, $\alpha \sin p = -C \Rightarrow \alpha \cos p \geq -2$. From $\cos^2 p + \sin^2 p = 1$ we obtain the equivalent condition $\alpha^2 \leq 2 + C^2$. Combining this with (13) we postulate

$$C^2 \leq \alpha^2 \leq 2 + C^2. \quad (14)$$

This implies Hypothesis 4.17. Therefore by Theorem 4.18:

- f is acyclic and has fixed points if (14) is satisfied.

In this case there are many asymptotically stable fixed points by Lemma 4.10, hence many convergent points.

When f is acyclic there may be nonempty open sets of points y whose trajectories go to ∞ or $-\infty$. For example, this holds when

$$\alpha \sin y + C = 2k\pi, \quad k \text{ a nonzero integer}, \quad (15)$$

because induction on n shows that

$$f^n(y) - y = 2nk\pi \quad (n \in \mathbb{N}). \quad (16)$$

Equation (15) has a solution y if and only if $\alpha \geq |2k\pi - C|$. Given $k \in \mathbb{Z} \setminus 0$, both (15) and (14) can be satisfied for some choice of α and y if and only if

$$k > 0 \text{ and } C \geq k\pi - (2k\pi)^{-1}, \text{ or } k < 0 \text{ and } C \leq k\pi - (2k\pi)^{-1} \quad (17)$$

When $|k|$ is a sufficiently large odd number and both (15) and (17) hold, it can be shown that $|f'(y)| < 1$, and this implies that the trajectory of every point sufficiently near y goes to ∞ or $-\infty$.

In Section 6 Schwarzians are used in a different way, to prove existence of 2-cycles.

5. Comparison with results of Cull and Huang

Cull [2] defines a *population model* as a map f having the following properties:

- $f: [0, \infty) \rightarrow [0, \infty)$ and $f(0) = 0$,
- there is a unique positive fixed point p ,
- $f(x) > x$ if $0 < x < p$, and $f(x) < x$ if $x > p$,
- there is a unique local and global maximum point $m_1 > 0$; if $p < m_1 < x$ and $f(x) > 0$, then $f(x) > f(y)$ for all $y > x$.

These conditions have plausible biological interpretations as models of population growth: Populations are nonnegative and only positive populations reproduce, a very small population that does not die out rises to a unique maximum value and converges to a unique positive equilibrium, and so forth.

When $m_1 < p$, Cull's conditions correspond to a special kind of general population model (f, p_*, m_0, p) (Definition 3.4), with $m_0 = m_1$. When $m_1 \geq p$ they imply every nonzero orbit increases or decreases to p , or reaches 0 in finite time. In any case, every trajectory approaches the set $[0, p]$ and is therefore bounded, hence f is convergent provided it is acyclic (Theorem 2.3).

Cull calls f *globally stable* if

$$f(x) > 0 \Rightarrow \lim_{n \rightarrow \infty} f^n(x) = p.$$

His Theorem 1, a model for several results herein, states:

Theorem 5.1 (Cull). *A population model is globally stable if and only if it has no 2-cycles.*

This is implied by Theorem 3.5, and also by Corollary 2.7.

Cull also found a subtle differential condition for global stability:

Theorem 5.2 (Cull). *A population model with global maximum point m_1 and positive fixed point p is globally stable provided the following conditions hold:*

- (a) $f''(x) < 0$ if $m_1 \leq x < p$,
- (b) $f'''(x) \geq 0$ if $f''(x) < 0$,
- (c) f'' has at most one sign change,
- (d) $|f'(p)| \leq 1$.

This result and Theorem 4.13 above are very similar, but neither theorem directly implies the other.

Huang [9] proved a related result:

Theorem 5.3 (Huang). *A population model having global maximum point m_1 and positive fixed point p is globally stable if it has the following properties:*

- (a) f has an inflection point $x^* > p$ such that

$$x \in [m_1, x^*) \Rightarrow f''(x) < 0, \quad x \in (x^*, f(m_1)] \Rightarrow f''(x) > 0,$$

- (b) $x \in (m_1, f(x^*)) \Rightarrow f'(x) < 0 \leq f'''(x)$,
- (c) either $f'(p) = -1$, or $f'(p) > -1$ and $x^* \geq p$.

This result is a consequence of Theorem 4.13 because the assumptions imply Hypothesis 4.17.

6. Existence of 2-cycles

Most maps are not convergent and therefore have 2-cycles. In this section we give simple conditions on f implying a 2-cycle.

Proposition 6.1. *Assume $f: I \rightarrow I$ is differentiable at $p \in \text{Fix}(f)$ and $|f'(p)| > 1$. Suppose $a \in I$, $a \neq p$, and no fixed point lies between a and p . If $a < p$ and $a < f \circ f(a)$ then $(a, p]$ meets a 2-cycle, while if $a > p$ and $a > f \circ f(a)$ then $(p, a]$ meets a 2-cycle.*

Proof. The first assertion follows from IVT: $f \circ f(x) > x$ for all $x \in (a, a + \epsilon]$, while a Taylor estimate shows that $f \circ f(x) < x$ if $0 < |x - p| < \epsilon$. The second assertion is similar, and in fact follows by applying the first to the map $x \mapsto -f(-x)$. $\square$

Theorem 6.2. *$f: \mathbb{R} \rightarrow \mathbb{R}$ has a 2-cycle provided:*

- (a) *there exists $K > 0$ such that $|x| > K \Rightarrow f(x)/x < -1$,*
- (b) *there is a maximal fixed point p , and $f'(p) < 0$, $Sf(p) < 0$.*

Proof. The topologically conjugate map $p + f(x - p)$ satisfies the same hypothesis and has maximal fixed point 0, hence we can assume $p = 0$. Hypothesis (a) implies

$$|x| > K \Rightarrow |f(x)| > |x|, \quad (18)$$

$$x < -K \Rightarrow f(x) > |x|, \quad (19)$$

and also

$$\lim_{k \rightarrow \infty} \frac{f^k(x)}{x} = -\infty, \quad (20)$$

$$\lim_{k \rightarrow \infty} f^{2k}(x) = \infty. \quad (21)$$

Now we proceed by contradiction, assuming there are no 2-cycles. The main task is to show that this implies $f'(0) = -1$. Note that $f \circ f$ satisfies

$$x > K \Rightarrow f \circ f(x) > x,$$

as otherwise IVT yields a positive fixed point for $f \circ f$ and hence a 2-cycle for f , contrary to our current assumption. In fact,

$$x > 0 \Rightarrow f \circ f(x) > x, \quad (22)$$

for otherwise IVT shows that $f \circ f(z) = z$ for some $z \in]0, K[$, and $\{z, f(z)\}$ has to be a 2-cycle for f . From (19) and Proposition 6.1 we have $-1 \leq f'(0) < 0$. If $-1 < f'(0)$, a Taylor approximation shows that $x \in (0, \epsilon] \Rightarrow f \circ f(x) < x$, contradicting (22). Therefore $f'(0) = -1$.

Set

$$a := \frac{f''(0)}{2!}, \quad b := \frac{f'''(0)}{3!}. \quad (23)$$

The third order Taylor expansions of f and $f \circ f$ are

$$f(x) = -x + ax^2 + bx^3 + o(x^3) \quad (24)$$

and

$$f \circ f(x) = x - (2a^2 + b)x^3 + o(x^3). \quad (25)$$

Applying (22) and (25) to sufficiently small $x > 0$ reveals that

$$2a^2 + b = 0. \quad (26)$$

From (23), (24) and (26) we have

$$Sf(0) = \frac{6b}{-1} - \frac{3}{2} \left[\frac{2a}{-1} \right]^2 = -6(b + a^2) = -6(-a^2) > 0.$$

But this contradicts assumption (b). $\square$

6.1. 2-cycles for odd degree polynomials

We start with a simple result:

Theorem 6.3. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial having odd degree $n \geq 3$, negative leading coefficient, and unique fixed point p . If $f'(p) < -1$, there is a 2-cycle.*

Proof. We can set $p = 0$. To fix ideas assume $\sigma = -1$, so that $f'(0) < -1$. Suppose there are no 2-cycles. Then $f(x) < x$ for all $x > 0$, hence the hypotheses imply $-1 < f'(0) \leq 0$. Then $g(x) := f \circ f(x) - x$ satisfies $g(x) > 0$ for very large $|x|$, hence for all $x \neq 0$. This implies $g'(0) \geq 0$. Thus $f'(0)^2 \geq 1$, which is inconsistent with the previous estimate. $\square$

Theorem 6.4. *Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial map having odd degree $n \geq 3$, negative leading coefficient, and n fixed points. Then f has a 2-cycle.*

Proof. After changing coordinates by a translation we assume the largest fixed point is 0. Assuming *per contra* that there are no 2-cycles, we will reach a contradiction by verifying hypotheses (a), (b) of Theorem 6.2. The algebraic conditions on f validate (a).

The first part of the proof of 6.2 shows that (18) through (22) hold for f . From (22) and a Taylor approximation we deduce $(f \circ f)'(0) \geq 1$ and therefore

$$|f'(0)| \geq 1. \quad (27)$$

The n th degree polynomial $g(x) := f(x) - x$ has n distinct roots, namely, the fixed points of f . Therefore Proposition 4.8(ii) shows that

$$g'(x) \neq 0 \Rightarrow Sg(x) < 0. \quad (28)$$

Because g has negative leading coefficient and negative roots, all roots of $g'(x)$ and g'' are negative, which implies $g'(0) < 0$ and $f'(0) < 1$. Therefore $f'(0) < 0$ by (27). By (28) and Proposition 4.9(c) we have $Sf(0) < 0$, which validates hypothesis (b) of Theorem 6.2. That result now gives the contradiction that f has a 2-cycle. $\square$

7. Suggestions for research

Progress on the following tasks would be of interest:

1. Find a set of semialgebraic conditions on the coefficients of polynomial maps that is equivalent to acyclicity. Or prove that there are no such conditions.
2. Find conditions on higher derivatives of f in an interval J , analogous to Hypothesis 4.17, that imply J does not meet any 2-cycles.

3. For some generic family $\{f_\mu\}_{\mu \in \mathbb{R}}$ of maps in I , describe the set of pairs $(\mu, x) \in \mathbb{R} \times I$ such that x belongs to a 2-cycle for f_μ .
4. Find conditions on acyclic maps f, g that make $f \circ g$ acyclic.
5. Investigate acyclic circle maps.
6. Estimate the complexity of deciding acyclicity for piecewise linear maps.
7. For a convenient Riemannian metric on the space of degree n polynomials, estimate the volume of the set of acyclic maps in a given ball of radius r .
8. Find a map that is acyclic if and only if the Riemann Hypothesis is true.
9. Show there is a map that is acyclic, but cannot be proved acyclic in ZFC set theory if the latter is consistent.

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