Title
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Zero sets of Lie algebras of analytic vector fields on real and complex 2-dimensional manifolds

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Abstract

Let M denote a real or complex analytic manifold with empty boundary, having dimension 2 over the ground field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let Y, X denote analytic vector fields on M. Say that Y *tracks* X provided $[Y, X] = fX$ with $f\colon M \to \mathbb{F}$ continuous. Let K be a compact component of the zero set $\mathsf{Z}(X)$ whose Poincaré-Hopf index is nonzero.

Theorem. If Y tracks X then $\mathsf{Z}(Y) \cap K \neq \varnothing$.

Theorem. Let $\mathcal{G}$ be a Lie algebra of analytic vector fields that track X, with $\mathcal{G}$ finite-dimensional and supersolvable when M is real. Then $\bigcap_{Y\in\mathcal{G}} \mathsf{Z}(Y)$ meets K.

Contents

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1 Introduction

Let M denote a manifold with empty boundary ∂M. A fundamental issue in Dynamical Systems is deciding whether a vector field has a zero. When is M compact with Euler characteristic $\chi(M) \neq 0$, positive answer is given by the celebrated POINCARÉ-HOPF Theorem.

Determining whether two or more vector fields have a common zero is much more challenging. For two commuting analytic vector fields on a possibly noncompact manifold of dimension ≤ 4, Theorem 1.2, due to C. BONATTI, not only gives conditions sufficient for common zeros, it locates specific compact sets. Our results establish common zeros for other sets of vector fields. The basic results are Theorems 1.3 and 1.4. Theorem 1.5 is an application to fixed points of transformnation groups.

Denote the zero set of a vector field X on M by $\mathsf{Z}(X)$. A *block* of zeros for X, or an *X-block*, is a compact set $K \subset \mathsf{Z}(X)$ having an open neighborhood $U \subset M$ such that $\overline{U}$ is compact and $\mathsf{Z}(X) \cap \overline{U} = K$. Such a set U is *isolating* for X and for (X, K).

The *index* of the X-block K is the integer $\mathsf{i}_K(X) := \mathsf{i}(X, U)$ defined as the Poincaré-Hopf index of any sufficiently close approximation to X having only finitely many zeros in U (Definitions 3.3, 3.6).[1] This number is independent of U, and is stable under perturbations of X (Theorems 3.9, 3.13). When X is C^1 and generates the local flow ϕ, for sufficiently small $t > 0$ the index $\mathsf{i}(X, U)$ equals the fixed-point index $I(\phi_t|U)$ defined by DOLD [4].

An X-block K is *essential* if $\mathsf{i}_K(X) \neq 0$. This implies $\mathsf{Z}(X) \cap K \neq \varnothing$ because every isolating neighborhood of K meets $\mathsf{Z}(X)$.

Theorem 1.1 (POINCARÉ-HOPF)**.** *If M is compact $\mathsf{i}(X, M) = \chi(M)$ for all continuous vector fields X on M.*

For calculations of the index in more general settings see MORSE [18], PUGH [22], GOTTLIEB [6], JUBIN [13].

This paper was inspired by a remarkable result of C. Bonatti, which does not require compactness of M:[2]

[1] Equivalently: $\mathsf{i}(X, U)$ is the intersection number of $X|U$ with the zero section of the tangent bundle (BONATTI [2]).

[2] "The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X, as an analytic Y-invariant set." —P. MOLINO [17]

Theorem 1.2 (Bonatti [2])**.** *Assume M is a real manifold of dimension ≤ 4 and X, Y are analytic vector fields on M such that $[X, Y] = 0$. Then $\mathsf{Z}(Y)$ meets every essential X-block.*[3]

Statement of results

Theorems 1.3 and 1.4 extend Bonatti's Theorem to certain pairs of noncommuting vector fields, and Lie algebras of vector fields.

M always denotes a real or complex manifold without boundary, over the corresponding ground field $\mathbb{F} = \mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers). In the main results M has dimension $\mathsf{dim}_{\mathbb{F}} M = 2$ over $\mathbb{F}$.

$\mathcal{V}(M)$ is the vector space over $\mathbb{F}$ of continuous vector fields on M, with the compact-open topology. The subspace of C^r vector fields is $V^r(M)$, where r is a positive integer, ∞ (meaning C^k for all finite k), or ω (analytic over $\mathbb{F}$). When M is complex, $\mathcal{V}^r(M)$ and $\mathcal{V}^\omega(M)$ are identical as real vector spaces, and $\mathcal{V}^\omega(M)$ is a complex Lie algebra. A linear subspace of $\mathcal{V}^r(M)$ is called a Lie algebra if it is closed under Lie brackets. The set of common zeros of a set $\mathfrak{s} \subset \mathcal{V}(M)$ is $\mathsf{Z}(\mathfrak{s}) := \bigcap_{X \in \mathfrak{s}} \mathsf{Z}(X)$.

X and Y denote vector fields on M.

Y *tracks* X provided $Y, X \in \mathcal{V}^1(M)$ and $[Y, X] = fX$ with $f\colon M \to \mathbb{F}$ continuous. When $\mathbb{F} = \mathbb{R}$ this implies the local flow generated by Y locally permutes orbits of Φ^X (Proposition 2.4). We say that a set of vector fields tracks X when each of its elements tracks X.

Example (A). Suppose $\mathcal{G}$ is a Lie algebra of C^1 vector fields on M. If $X \in \mathcal{G}$ spans an ideal then $\mathcal{G}$ tracks X, and the converse holds provided $\mathcal{G}$ is finite dimensional.

Here are the main results. The manifold M is real or complex with $\mathsf{dim}_{\mathbb{F}} M = 2$.

Theorem 1.3. *Assume $X, Y \in \mathcal{V}^\omega(M)$, Y tracks X, and $K \subset \mathsf{Z}(X)$ is an essential X-block. Then $\mathsf{Z}(Y) \cap K \neq \varnothing$.*

A Lie algebra $\mathcal{G}$ is *supersolvable* if it is real and faithfully represented by upper triangular matrices. If $\mathcal{G}$ is the Lie algebra of Lie group G we call G supersolvable.

Theorem 1.4. *Assume $X \in \mathcal{V}^\omega(M)$, K is an essential X-bloc, and $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a Lie algebra that tracks X. Let one of the following conditions hold:*

(a) *M is complex,*

[3]In [2] this is stated for $\mathsf{dim}(M) = 3$ or 4. If $\mathsf{dim}(M) = 2$ the same conclusion is obtained by applying the 3-dimensional case to the vector fields $X \times t\frac{\partial}{\partial t}$, $Y \times t\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$.

(b) *M is real and $\mathcal{G}$ is supersolvable.*

Then $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$.

Example (B). Let $M, \mathcal{G}, X$ be as in Theorem 1.4 and assume the local flow Φ^X has a compact global attractor. It can be shown that M is an isolating neighborhood for the X-block $\mathsf{Z}(X)$, M has finitely generated homology, and $\mathsf{i}(X, M) = \chi(M)$. Theorem 1.4 thus implies:

- *If* $\chi(M) \neq 0$ *then* $\mathsf{Z}(\mathcal{G}) \neq \varnothing$.

For instance:

- *Let $\mathcal{G}$ be a Lie algebra of holomorphic vector fields on $\mathbb{C}^2$. If $X \in \mathcal{G}$ spans an ideal and Φ^X has a global attractor, then* $\mathsf{Z}(\mathcal{G}) \neq \varnothing$.

Now let G denote a connected Lie group over the same ground field as M. An *analytic action* of G on M is a homomorphism $\alpha\colon\ g \mapsto g^\alpha$ from G to the group of analytic diffeomorphisms of M; this action is also denoted by (α, G, M). The action is *effective* if its kernel is trivial. Its *fixed point set* is

$$\mathsf{Fix}(\alpha) := \{p \in M\colon\ g^\alpha(p) = p, \quad (g \in G)\}.$$

Theorem 1.5. *Assume:*

- *M is a compact complex 2-manifold and* $\chi(M) \neq 0$,
- (α, G, M) *is an effective analytic action,*
- *G contains a 1-dimensional normal subgroup.*

Then $\mathsf{Fix}(\alpha) \neq \varnothing$.

Proof. This follows from Theorem 1.4(a) (see Section 4). ∎

The analogous result for analytic actions of supersolvable Lie groups on real surfaces is due to Hirsch & Weinstein [9]. But Lima [15] and Plante [20] have shown that every compact surface supports a continuous fixed-point free action by the 2-dimensional group whose Lie algebra has the structure $[Y, X] = X$. Whether X and Y can be smooth is unknown.

Terminology

The closure of subset Λ of a topological space S is denoted by $\overline{\Lambda}$, the frontier by $\mathsf{Fr}(\Lambda) := \overline{\Lambda} \cap \overline{S \setminus \Lambda}$, and the interior by $\mathsf{Int}(\Lambda)$.

Maps are continuous unless otherwise characterized. A map is *null homotopic* if it is homotopic to a constant map.

If is vector ξ in $\mathbb{R}^n$, or a tangent vector to a Riemannian manifold, its norm is $\|\xi\|$. The unit sphere in $\mathbb{R}^n$ is $\mathbf{S}^n$.

Let $\mathsf{dim}_{\mathbb{F}}(M) = n$. The tangent bundle $\tau(M)$ is an $\mathbb{F}^n$-bundle, meaning a fibre bundle over M with total space $T(M)$, projection $\pi_M\colon\ T(M) \to M$, standard fibre $\mathbb{F}^n$, and structure group $GL(n, \mathbb{F})$ (see Steenrod [25]). The fibre over $p \in M$ is tangent space to M at p is $T_p(M) := \pi_M^{-1}(p)$. When M is an open set in $\mathbb{F}^n$ we identify $T_p(M)$ with $\mathbb{F}^n$, $\tau(M)$ with the trivial vector bundle $M \times \mathbb{F}^n \to M$, and $X \in \mathcal{V}(M)$ with the map $M \to \mathbb{F}^n,\ p \mapsto X_p$.

Assume $X \in \mathcal{V}^r(M)$ and $\partial M = \varnothing$. The local flow on M whose trajectory through p is the X-trajectory of p is denoted by $\Phi^X := \left\{\Phi_t^X\right\}_{t\in\mathbb{R}}$, referred to informaly as the *X-flow*. The maps Φ_t^X are C^r diffeomorphisms between open subsets of M.

An *X-curve* is the image of an integral curve $t \mapsto y(t)$ of X. This is either a singleton and hence a zero of X, or an interval. The maximal X-curve through p is the *orbit* of p under X. A set $S \subset M$ is *X-invariant* if it contains the orbits under X of its points. When this holds for all X in a set $\mathcal{H} \subset \mathcal{V}^1(M)$ then S is *$\mathcal{H}$-invariant.*

If X, Y are vector fields on M, the alternating tensor field $X \wedge Y \in \Lambda^2(M)$ may be denoted by $X \wedge_{\mathbb{F}} Y$ in order to emphasize the ground field. $X \wedge_{\mathbb{F}} Y = 0$ means X_p and Y_p are linearly dependent over $\mathbb{F}$ at all $p \in M$.

2 Consequences of tracking

Throughout this section we assume:

- *M is a real or complex manifold, $\mathsf{dim}_{\mathbb{F}}(M) = n \geq 1$, $\partial M = \varnothing$,*
- *$X, Y \in \mathcal{V}^1(M)$,*

where $\mathbb{F}$ denotes the ground field $\mathbb{R}$ or $\mathbb{C}$.

Definition. *Y tracks X* provided $Y, X \in \mathcal{V}^1(M)$ and $[Y, X] = fX$ with $f\colon\ M \to \mathbb{F}$ continuous.

Proposition 2.1. *Suppose $Z \in \mathcal{V}^1(M)$. If Y and Z track X and $[Y, Z]$ is C^1, then $[Y, Z]$ tracks X.*

Proof. Follows from the Jacobi identity. ∎

Definition 2.2. The *dependency set* of X and Y (over the ground field) is

$$\mathsf{Dep}_{\mathbb{F}}(X, Y) := \left\{p \in M\colon\ X_p \wedge_{\mathbb{F}} Y_p = 0\right\}.$$

Proposition 2.3. *If Y tracks X then $\mathsf{Z}(X)$ and $\mathcal{D}(X, Y)$ are X- and Y-invariant.*

Proof. As the statement is local, we assume M is an open set in $\mathbb{F}^n$.

Invariance of $\mathsf{Z}(X)$*:* Evidently $\mathsf{Z}(X)$ is X-invariant and $\mathsf{Z}(X) \cap \mathsf{Z}(Y)$ is Y-invariant. To show that $\mathsf{Z}(X) \setminus \mathsf{Z}(Y)$ is Y-invariant, fix $p \in \mathsf{Z}(X) \setminus \mathsf{Z}(Y)$.

Let $(y_1, \ldots, y_n)$ be flowbox coordinates in a neighborhood V_p of p, representing $Y|V_p$ as $\frac{\partial}{\partial y_1}$ in a convex open subset of $\mathbb{R}^n$, and the Y-trajectory of p as

$$t \mapsto y(t) := p + te_1$$

where $e_1, \ldots, e_n \in \mathbb{F}^n$ are the standard basis vectors.

Let $J_p \subset \mathbb{R}$ be an open interval around 0 such that

$$y(t) \in V_p, \qquad (t \in J_p).$$

Then

$$\frac{d}{dt}\big(T\Phi_t^Y(X_p)\big) = [Y, X]_{y(t)}, \qquad (t \in J_p). \tag{1}$$

Since Y tracks X, there is a continuous $\mathbb{F}$-valued function $t \mapsto g(t)$ such that in the flowbox coordinates for Y, the vector-valued function $t \mapsto X_{y(t)}$ satisfies the linear initial value problem

$$\frac{d}{dt}X_{y(t)} = g(t)X_{y(t)}, \qquad X_p = 0. \tag{2}$$

Therefore $X_{y(t)}$ vanishes identically in t.

Invariance of $\mathsf{D}(X, Y)$*:* We need to prove: for all $t \in J_p$,

$$X_p \wedge_{\mathbb{F}} Y_p = 0 \implies T\Phi_t^Y(X_p) \wedge_{\mathbb{F}} X_{\Phi_t^Y(p)} = 0. \tag{3}$$

Assume $Y_p \neq 0$ and fix flowbox coordinates for Y at p. It suffices to verify (3) for all $t \in J_p$. Equations (1) and (2) imply

$$\begin{aligned}\frac{d}{dt}\big(T\Phi_t^Y(X_p) \wedge_{\mathbb{F}} X_{y(t)}\big) &= \big([Y, X] \wedge_{\mathbb{F}} X\big)_{y(t)} + T\Phi_t^Y(X_p) \wedge_{\mathbb{F}} g(t)X_{y(t)} \\ &= 0 \ \text{ identically in } t.\end{aligned}$$

As (3) holds for $t = 0$, the proof is complete. ∎

The proof of the following result is similar and left to the reader:

Proposition 2.4. *If M is real and Y tracks X, each map Φ_t^Y sends orbits of $X|\mathcal{D}\Phi_t^Y$ to orbits of $X|\mathcal{R}\Phi_t^Y$.* ∎

When M is complex there is a simliar result for the holomorphic local actions of $\mathbb{C}$ on M generated by X and Y.

3 The index function

In this section M is a real surface of dimension $n \geq 1$ with empty boundary. Assume $X \in \mathcal{V}(M)$, K is an X-block, and the precompact open set $U \subset M$ is isolating for (X, K).

Definition 3.1. A *deformation* from X to X' is path in $\mathcal{V}(M)$ of the form

$$t \mapsto X^t, \quad X^0 := X, \quad X^1 = Y, \qquad (0 \le t \le 1).$$

The deformation is *nonsingular* in a set $S \subset M$ provided $\mathsf{Z}(X^t) \cap S = \varnothing$.

Proposition 3.2. *X has arbitrarily small convex open neighborhoods $\mathcal{B} \subset \mathcal{V}(M)$ such that for all $Y, Z \in \mathcal{B}$:*

(i) *U is isolating for Y,*

(ii) *the deformation $Y^t := (1-t)Y + tZ$, $(0 \le t \le 1)$ is nonsingular in $\mathsf{Fr}(U)$.*

These conditions imply:

(iii) *the set of $Y \in \mathcal{B}$ such that $\mathsf{Z}(Y) \cap U$ is finite contains a dense open subset*
of $\mathcal{B}$.

Proof. (i) and (ii) follow from the definition of the compact-open topology on $\mathcal{V}(M)$. Standard approximation theory gives (iii). ∎

Definition 3.3. When K is finite, the *Poincaré-Hopf index* of X at K, and in U, is the integer $i^{PH}_K(X) = \mathsf{i}^{PH}(X, U)$ defined as follows. For each $p \in K$ choose an open set $W \subset U$ meeting K only at p, such that W is the domain of a C^1 chart

$$\phi\colon\ W \approx W' \subset \mathbb{R}^n, \quad \phi(p) = p'.$$

The transform of X by ϕ is

$$X' := T\phi \circ X \circ \phi^{-1} \in \mathcal{V}(W').$$

There is a unique map of pairs

$$F_p\colon\ (W', 0) \to \mathbb{R}^n, 0)$$

that expresses X' by the formula

$$X'_x = \big(x, F_p(x)\big) \in \{x\} \times \mathbb{R}^n, \qquad (x \in W').$$

Noting that $F^{-1}(0) = p$, we define $\mathsf{i}^{PH}_p(X) \in \mathbb{Z}$ as the degree of the map defined for any sufficiently small $\epsilon > 0$ as

$$\mathbf{S}^{n-1} \to \mathbf{S}^{n-1}, \quad u \mapsto \frac{F_p(\epsilon u)}{\|F_p(\epsilon u)\|}\,.$$

This degree is independent of ϵ and the chart ϕ, by standard properties of the degree function. Therefore the integer

$$\mathsf{i}^{PH}_K(X) = i^{PH}(X, U) := \begin{cases} \sum_{p \in K} i^{PH}_p(X) & \text{if } K \ne \varnothing, \\ 0 & \text{if } K = \varnothing. \end{cases}$$

is well defined and depends only on X and K.

Proposition 3.4. *Let $\{X^t\}$ be a deformation that is nonsingular in $\mathsf{Fr}(U)$. If both $\mathsf{Z}(X^0) \cap U$ and $\mathsf{Z}(X^1) \cap U$ are finite, then*

$$i^{PH}(X, U) = i^{PH}(X^1, U).$$

Proof. The proof is similar to that of a standard result on homotopy invariance of intersection numbers in oriented manifolds (compare HIRSCH [8, Theorem 5.2.1]). ∎

Definition 3.5. The *support* of a deformation $\{X^t\}$ is the closed set

$$\mathsf{supp}\{X^t\} := \{p \in M \colon \; X^t_p = X^0_p, \quad 0 \le t \le 1\}.$$

The deformation is *compactly supported in S* provided $\mathsf{supp}\{X^t\}$ is a compact subset of S.

Definition 3.6. The *index of X in U* is

$$\mathsf{i}(X, U) := \mathsf{i}^{PH}(X', U) \tag{4}$$

where X' is any vector field on M such that $\mathsf{Z}(X') \cap U$ is finite and there is a deformation from X to X' compactly supported in $\mathsf{Int}(U)$. This integer is well defined because the right hand side of Equation (4) depends only on X and U, by Proposition 3.4. The notation $\mathsf{i}(X, U)$ tacitly assumes U is isolating for X.

Lemma 3.7. *If U and U_1 are isolating for (X, K) then $\mathsf{i}(X, U) = \mathsf{i}(X, U_1)$.*

Proof. Let W be isolating for (X, K), with $\overline{W} \subset U_1 \cap U$. It suffices to show that $\mathsf{i}(X, U) = \mathsf{i}(X, W)$, for this also implies $\mathsf{i}(X, U_1) = \mathsf{i}(X, W$. By definition, $\mathsf{i}(X, W) = \mathsf{i}^{PH}(X', W)$ provided X and X' are homotopic by deformation with compact support in W and $\mathsf{Z}(Y^1) \cap W$ is finite. Let $\{Y^t\}$ be the deformation defined by

$$Y^t_p = \begin{cases} X^t_p & \text{if } p \notin W, \\ Y^t_p & \text{if } p \in \overline{W}. \end{cases}$$

Therefore $\mathsf{i}(X, U) = \mathsf{i}(X, W)$, because this deformation is compactly supported in U and $\mathsf{Z}(Y^1) \cap U$ is finite. ∎

It follows that $\mathsf{i}(X, U)$ depends only on X and K. The *index of X at K* is

$$\mathsf{i}_K(X) := \mathsf{i}(X, U).$$

It is easy to see that the index function enjoys the following additivity:

Proposition 3.8. *Let K_1, K_2 be disjoint X-blocks, with isolating neighborhoods U_1, U_2 respectively. Then*

$$\mathsf{i}_{K_1 \cup K_2}(X) = \mathsf{i}_{K_1}(X) + \mathsf{i}_{K_2}(X),$$
$$\mathsf{i}(X, U_1 \cup U_2) = \mathsf{i}(X, U_1) + \mathsf{i}(X, U_2).$$

The following property is crucial:

Theorem 3.9 (Stability)**.** *Let $U \subset M$ be isolating for X.*

(a) *If $\mathsf{i}(X,U) \neq 0$ then $\mathsf{Z}(X) \cap U \neq \varnothing$.*

(b) *If Y is sufficiently close to X then U is isolating for Y and $\mathsf{i}(Y,U) = \mathsf{i}(X,U)$.*

(c) *Let $\{X^t\}$ be a deformation of X that is nonsingular in $\mathsf{Fr}(U)$. Then*

$$\mathsf{i}(X^t,U) = \mathsf{i}(X,U), \qquad (0 \le t \le 1).$$

Proof. If $\mathsf{i}(X,U) \neq 0$, Definition 3.6 shows that X is the limit of a convergent sequence $\{X^n\}$ in $\mathcal{V}(M)$ such that $\mathsf{Z}(X^n) \cap U \neq \varnothing$. Passing to a subsequence and using compactness of $\overline{U}$ shows that $\mathsf{Z}(X) \cap \overline{U} \neq \varnothing$, and (a) follows because $\mathsf{Z}(X) \cap \mathsf{Fr}(U) = \varnothing$. Parts (b) and (c) are implied by Propositions 3.2 and 3.4. ∎

Proposition 3.10. *Assume $X, Y \in \mathcal{V}(M)$ and $U \subset M$ is isolating for both X and Y. For each component U' of U that meets $\mathsf{Z}(X) \cup \mathsf{Z}(Y)$, let one of the following conditions hold:*

(a) $X_p \neq \lambda Y_p, \quad (p \in \mathsf{Fr}(U'),\ \lambda < 0),$

or

(b) $X_p \neq \lambda Y_p, \quad (p \in \mathsf{Fr}(U'),\ \lambda > 0).$

Then $\mathsf{i}(X,U) = \mathsf{i}(Y,U)$.

Proof. $U \cap \big(\mathsf{Z}(X) \cup \mathsf{Z}(Y)\big)$ is the compact set $\overline{U} \cap \big(\mathsf{Z}(X) \cup \mathsf{Z}(Y)\big)$. This implies only finitely many components of U meet $\mathsf{Z}(X) \cup \mathsf{Z}(Y)$. The union U_1 of these components is isolating for X and Y. The index function is additive over disjoint unions, and both X and Y have index zero in the open set $U \backslash U_1$, which is disjoint from $\mathsf{Z}(X) \cup \mathsf{Z}(Y)$. Therefore

$$\begin{aligned} \mathsf{i}(X,U) &= \mathsf{i}(X,U_1), \\ \mathsf{i}(Y,U) &= \mathsf{i}(Y,U_1). \end{aligned}$$

Replacing U by U_1, we assume U has only finitely many components. As it suffices to prove X and Y have the same index in each component of U, we also assume U is connected.

Let $p \in \mathsf{Fr}(U)$ be arbitrary. If (a) holds, consider the deformation

$$X^t := (1-t)X + tY, \quad (0 \le t \le 1).$$

If $t = 0$ or 1 then $X^t_p \neq 0$ because U is isolating for X and Y, while if $0 < t < 1$ then $X^t_p \neq 0$ by (a). Therefore the conclusion follows from the Stability Theorem 3.9. If (b) holds the same argument works for the deformation $(1-t)X - tY$. ∎

Proposition 3.11. *Assume U is an isolating neighborhood for both X and Y, whose closure N is a C^1 submanifold such that*

$$X_p \wedge_{\mathbb{R}} Y_p = 0, \quad (p \in \partial N) \tag{5}$$

and one of the following conditions holds:

(a) *M is even-dimensional*

(b) *M is odd-dimensional and X and Y are tangent to ∂N.*

Then $\mathsf{i}(X, U) = \mathsf{i}(Y, U)$.

Proof. By the Stability Theorem 3.9 it suffices to find a deformation from X to Y that is nonsingular on ∂N. As ∂N is a subcomplex of a smooth triangulation of M (Whitehead [27], Munkres [19]), The Homotopy Extension Theorem (Steenrod [25, Th. 34.9]) shows that this deformation exists provided $X|\partial N$ and $Y|\partial N$ are connected by a homotopy of nonsingular sections of $T_{\partial N}M$. Such a homotopy exists in case (a) because the antipodal map and identity maps of $\mathbb{R}^n \setminus \{0\}$ are homotopic, and in case (b) because these maps in $\mathbb{R}^{n-1} \setminus \{0\}$ are homotopic. ∎

Fix U and $N = \overline{U}$ as in Proposition 3.11, so that $\mathsf{i}(X, U) = \mathsf{i}(X, N \backslash \partial N)$. An orientation of N corresponds to a generator

$$\nu_N \in H_n(N, \partial N) \cong \mathbb{Z}.$$

Let $\nu^N \in H^n(N, \partial N)$ be the dual generator.

Evaluating cocyles on cycles defines the canonical dual pairing (the Kronecker Index):

$$H^n(N, \partial N) \times H_n(N, \partial N) \to \mathbb{Z}, \qquad (c, \lambda) \mapsto c \cdot \lambda.$$

Let $c_{X,N} \in H^n(N, \partial N)$ be the obstruction to extending $X|\partial N$ to a nonsingular vector field on N. Unwinding definitions proves:

Proposition 3.12. *If N is oriented, $\mathsf{i}(X, U) = c_{X,N} \cdot \nu_N$.* ∎

A similar result holds for nonorientable manifolds, using homology with coefficients twisted by the orientation sheaf.

Theorem 3.13. *If X can be approximated by vector fields X' with no zeros in $\overline{U}$ then $\mathsf{i}(X, U) = 0$, and the converse holds provided U is connected.*

Proof. If the approximation is possible then the index vanishes by the Stability Theorem 3.9. To prove the converse fix a Riemann metric on M and $\epsilon > 0$. There exists an isolating neighborhood U' of K whose closure is a compact submanifold $N \subset U$ and

$$\|X_p\| < \epsilon, \quad (p \in N).$$

Define

$$E_\epsilon := \{x \in \ (N)\colon\ 0 < \|x\| < \epsilon, \quad (p \in N)\}.$$

This is the total space of a fibre bundle η over N that is fibre homotopically equivalent to the sphere bundle associated to the tangent bundle of N.

$X|\partial N$ extends to a section $X''\colon N \to E_\epsilon$ of η, by Proposition 3.12. Let $X' \in \mathcal{V}(M)$ be the extension of X'' that agrees with X outside N. Then X' is an ϵ-approximation to X with no zeros in $\overline{U}$. ∎

Examination of the proof, together with standard approximation theory, yields the following addendum to Theorem 3.13:

Corollary 3.14. *Assume* $\mathsf{i}(X, U) = 0$.

(i) *If X is analytic, the approximations in* Theorem 3.13 *can be chosen to be analytic.*

(ii) *If X is C^r and $0 \le r \le \infty$, the approximations can be chosen to be C^r and to agree with X in $M \backslash U$.* ∎

Definition 3.15. Let η denote a real or complex vector bundle with total space E and n-dimensional fibres. A *trivialization* of η is a map $\psi\colon\ E \to \mathbb{F}^n$ that restricts to linear isomorphisms on fibres.

Proposition 3.16. *Assume $N \subset U$ is a compact, connected real n-manifold whose interior is isolating for (X, K). Let ψ be a trivialization of $\tau_{\partial N}(M)$. Then $\mathsf{i}(X, U)$ equals the degree $\mathsf{deg}(F_X)$ of the map*

$$F_X\colon\ \partial N \to \mathbf{S}^{n-1}, \quad p \mapsto \frac{\psi(X_p)}{\|\psi(X_p)\|}.$$

Proof. Follows from Proposition 3.12, because $\mathsf{deg}(F_X) = c_{X,N} \cdot \nu_N$ by obstruction theory. ∎

This result will be used in the proofs Theorem 1.3:

Proposition 3.17. *Let $W \subset M$ be a connected isolating neighbrohood for (X, K). Assume the following data:*

- $\Phi\colon\ T(W \backslash K) \to \mathbb{R}^n$ *is a trivialization of* $\tau(W \backslash K)$,
- $E \subset \mathbb{R}^{n \times n}$ *is a linear space of matrices,* $\mathsf{dim}(E) < n$,
- $A\colon\ W \setminus K \to E$ *is a map such that*

$$\Phi(X_q) = A(q) \cdot \Phi(Y_q), \qquad (q \in W \setminus K). \tag{6}$$

If W is isolating for $Y \in \mathcal{V}(M)$, then $\mathsf{i}(Y, W) = 0 \implies \mathsf{i}(X, W) = 0$.

Proof. Consider the maps

$$F_X\colon\ \partial N \to \mathbf{S}^{n-1}, \quad p \mapsto \frac{\Phi(X_p)}{\|\Phi(X_p)\|},$$
$$F_Y\colon\ \partial N \to \mathbf{S}^{n-1}, \quad p \mapsto \frac{\Phi(Y_p)}{\|\Phi(Y_p)\|}.$$

Corollary 3.16 implies $\mathsf{deg}(F_Y) = 0$, hence F_Y is null homotopic, and it suffices to prove F_X null homotopic. Degree theory shows that F_Y is homotopic to a constant map

$$\tilde{F}_Y\colon\ \partial N \to \mathbf{S}^{n-1}, \quad \tilde{F}_Y(p) = c \in \mathbf{S}^{n-1}.$$

By Equation (6) there exists $\lambda\colon\ \partial N \to \mathbb{R}$ such that

$$F_X(p) = \lambda(p)A(p)F_Y(p), \quad \lambda(p) > 0.$$

Consequently F_X is homotopic to

$$\tilde{F}_X\colon\ \partial N \to \mathbf{S}^{n-1}, \qquad \tilde{F}_X(p) = \lambda(p)A(p)c.$$

The map

$$H\colon\ E\backslash\{0\} \to \mathbf{S}^{n-1}, \quad B \mapsto \frac{B(c)}{\|B(c)\|}$$

satisfies:

$$\tilde{F}_X(\partial N) \subset H(E\backslash\{0\}) \subset \mathbf{S}^{n-1}. \tag{7}$$

Since the unit sphere $\Sigma \subset E\backslash\{0\}$ is a deformation retract of $E\backslash\{0\}$, Equation (7) shows that $\tilde{F}_X$ is homotopic to a map

$$G\colon\ \partial N \to H(\Sigma) \subset \mathbf{S}^{n-1}.$$

Now $\mathsf{dim}(\Sigma) = \mathsf{dim}(E) - 1 \le n-2$. As H is Lipschitz, $\mathsf{dim}(H(\Sigma)) \le n-2$. Therefore $H(\Sigma)$ is a proper subset of Σ^{n-1} containing $G(\partial N)$, implying G is null homotopic. The conclusion follows because the homotopic maps

$$F_X, \tilde{F}_X, G\colon\ \partial N \to \mathbf{S}^{n-1}$$

have the same degree. ∎

Example (C). Let $\mathcal{A}$ denote a finite dimensional algebra over $\mathbb{R}$ with multiplication $(a,b) \mapsto a \bullet b$. Let X, Y be vector fields on a connected open set $U \subset \mathcal{A}$, whose respective zero sets K, L are compact. Assume there is a map $A\colon\ U \to \mathcal{A}$ such that

$$X_p = A(p) \bullet Y_p, \qquad (p \in U).$$

Then $\mathsf{i}_Y(U) = 0 \implies \mathsf{i}_X(U) = 0$, by Proposition 3.17.

4 Proofs of Theorems 1.3, 1.4, 1.5

Henceforth M denotes a connected real or complex 2-manifold with $\partial M = \varnothing$.

Proof of Theorem 1.3

The hypotheses are:

- X and Y are analytic vector fields on M,
- Y tracks X,
- K is an essential X-block,

The conclusion is that $\mathsf{Z}(Y) \cap K \neq \varnothing$. It suffices to prove:

$\mathsf{Z}(Y)$ *meets every neighborhood of* K.

Many sets $S \subset M$ associated to analytic vector fields, including zero sets and dependency sets, are *analytic spaces*: Each point of S has an open neighborhood $V \subset M$ such that $S \cap V$ is the zero set of an analytic map $V \to \mathbb{F}^k$. This implies S is covered by a locally finite family of disjoint analytic submanifolds.

The local topology of analytic spaces of is rather simple, owing to the theorem of Łojasiewicz [16]:

Theorem 4.1 (Triangulation). *If S is a locally finite collection of closed analytic spaces in M, there is a triangulation of M such that each element of S a subcomplex.*

We justify three simplifying assumptions by showing that if any one of them is violated the conclusion of Theorem 1.3 holds:

(A1) *K is connected, and $\chi(K) = 0$.*

K is compact and triangulable and hence has only finitely many components. As K is an essential X-block, so is some component (Proposition 3.8), and we can assume K is that component. If $\chi(K) \neq 0$, the flow induced by Y on the triangulable space K fixes a point $p \in \mathsf{Z}(Y) \cap K$ by Lefschetz's Fixed Point Theorem (Lefschetz [14], Spanier [24], Dold [4]). This justifies (A1).

Note that (A1) implies K has arbitrarily small connected neighborhoods U that are isolating for (X, K).

(A2) $\mathsf{dim}_{\mathbb{F}}(K) = 1$ *and K is an analytic submanifold.*

If $\mathsf{dim}_{\mathbb{F}}(K) = 0$ then K is a singleton by (A1) and the conclusion of the theorem is obvious.

If $\mathsf{dim}_{\mathbb{F}}(K) = 2$ then $K = M$ because X is analytic, M is connected, and both are 2-dimensional. Therefore $\chi(M) \neq 0$ and the Poincaré-Hopf Theorem 1.1 implies $\mathsf{Z}(Y) \cap K = \mathsf{Z}(Y) \neq \varnothing$.

Let $\mathsf{dim}(K) = 1$. Suppose K is not an analytic submanifold. Its singular set is nonempty, finite and Y-invariant, and thus contained in $\mathsf{Z}(Y) \cap K$.

(A3) *U is a connected isolating neighborhood for (X, K) and $\mathsf{Z}(Y) \cap \overline{U} \subset K$.*

If no such U exists, (A1) implies there is a nested sequence $\{U_j\}$ of connected isolating neighborhoods for (X, K) whose intersection is K, and each U_j contains a $p_j \in \mathsf{Z}(Y) \cap \overline{U_j}$. A subsequence of $\{p_j\}$ tends to a point of $\mathsf{Z}(Y) \cap K$ by compactness of K.

Note that (A3) implies U is isolating for Y.

Henceforth we assume (A1), (A2) and (A3).

It suffices to prove

$$\mathsf{i}(Y, U) \neq 0, \tag{8}$$

because then (A3) implies $\mathsf{Z}(Y) \cap U$ meets K.

Both K and the dependency set $\mathsf{D} := \mathsf{Dep}_{\mathbb{F}}(X, Y)$ (Definition 2.2) are Y-invariant analytic spaces (Proposition 2.3), and (A2) implies $\mathsf{dim}_{\mathbb{F}}(\mathsf{D}) = 1$ or 2. Because $\mathsf{dim}_{\mathbb{F}}(K) = 1$ by (A3), one of the following conditions is satisfied:

(B1) *K is a component of D,*

(B2) $\mathsf{dim}_{\mathbb{F}}(\mathsf{D}) = 1$ *and K is not a component of D,*

(B3) $\mathsf{dim}_{\mathbb{F}}(\mathsf{D}) = 2$.

Assume (B1): Choose the isolating neighborhood U so small that $\mathsf{Fr}(U) \cap \mathsf{D} = \varnothing$. Proposition 3.10 implies $i(Y, U) = \mathsf{i}(X, U) \neq 0$, yielding (8).

Assume (B2): Because D and K are 1-dimensional and $K \subset \mathsf{D}$, the frontier in K of $K \cap \left(\overline{\mathsf{D} \setminus K}\right)$ is Y-invariant and 0-dimensional, hence a nonempty subset of $\mathsf{Z}(Y) \cap K$.

Assume (B3): In this case $\mathsf{D} = M$ because X and Y are analytic and M is connected, hence $X \wedge_{\mathbb{F}} Y = 0$.

If M is real, Proposition 3.11(a) implies

$$\mathsf{i}(Y, U) = \mathsf{i}(X, U) \neq 0,$$

whence $\mathsf{Z}(Y) \cap U \neq \varnothing$ and $\mathsf{Z}(Y) \cap K \neq \varnothing$ by (A3).

This completes the proof of Theorem 1.3 for real M.

Henceforth we assume M is complex. Therefore (A1) and (A2) imply

(C1) *K is a compact connected Riemann surface of genus 1, holomorphically embedded in U,*

(C2) *The tangent bundle $\tau(K)$ is a holomorphically trivial complex line bundle,*

Note that (B3) implies X_p and Y_p are linearly dependent over $\mathbb{C}$ at all $p \in M$, because X and Y are analytic and M is connected. Together with (A3) this implies:

(C3) *There is an open neighborhood $W \subset U$ of K and a holomorphic map $f\colon W \to \mathbb{C}$ satisfying:*

$$p \in W \implies X_p = f(p)Y_p, \qquad f^{-1}(0) = K.$$

Since K is a compact, connected, complex submanifold of M having codimension 1, it can be viewed as a divisor of the complex analytic variety M (Griffiths & Harris [7]). This divisor determines a holomorphic line bundle $[K]$ over M, canonically associated to the pair (M, K).

Proposition 4.2.

(i) *The restriction of $[K]$ to the submanifold K is holomorphically isomorphic to the algebraic normal bundle $\nu(K, M) := \tau_K(M)/\tau(K)$ of K.*

(ii) *$[K]$ is holomorphically trivial.*

(iii) *$\nu(K, M)$ is holomorphically trivial.*

Proof. Working through the definition of $[K]$ in [7] demonstrates (i). Part (ii) follows from (C3) and the italicized statement on [7, page 134]), and (ii) implies (iii).[4] ∎

From (C1), (C2) and Proposition 4.2(ii) with $V := K$ we see that the complex vector bundle $\tau_K(M) \cong \tau(K) \oplus \nu(K, M)$ is holomorphically trivial. As K is triangulable we can choose W in (C3) so that it admits K as a deformation retract. Therefore:

(C4) *$\tau(W)$ is a trivial complex vector bundle*

by the Homotopy Extension Theorem (Steenrod [25, Thm. 34.9], Hirsch [8, Chap. 4, Thm. 1.5]).

Define

$$\theta\colon \mathbb{C} \to \mathbb{R}^{2\times 2}, \quad a + b\sqrt{-1} \mapsto \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right], \qquad (a, b \in \mathbb{R}).$$

Let

$$\Theta\colon \mathbb{C}^{2\times 2} \to \mathbb{R}^{4\times 4}$$

[4]An elegant explanation was kindly supplied by D. Eisenbud [5]: The ideal defining $[K]$ is the dual $\mathcal{K}^*$ of the sheaf $\mathcal{K}$ of ideals of the analytic space K. Because $\mathcal{K}$ is generated by the single function f it is a product sheaf, and so also is $\mathcal{K}^*$. Therefore $[K]$ is a holomorphically trivial line bundle, as is its restriction to K, which is $\nu(K, M)$.

be the $\mathbb{R}$-linear isomorphism that replaces each matrix entry z by the 2×2 block $\theta(z)$.

Define

$$H\colon \mathbb{C} \to \mathbb{R}^{4\times 4}, \quad a + b\sqrt{-1} \mapsto \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

Note that $E := H(\mathbb{C})$ is a 2-dimensional linear subspace of $\mathbb{R}^{4\times 4}$.

Let

$$\Psi\colon T(W) \to \mathbb{C}^2$$

be a trivialization of the complex vector bundle $\tau(W)$ (Definition 3.15). The real vector bundle $\tau(W^{\mathbb{R}})$ has the trivialization

$$\Phi := \Theta \circ \Psi\colon T(W) \to \mathbb{R}^4.$$

Let $f\colon W \to \mathbb{C}$ be as in (C3) and set $A := H \circ f\colon W \to E$. Then

$$\Phi(X_q) = A(q) \cdot \Phi(Y_q), \qquad (q \in W).$$

This implies $\mathsf{i}(Y, W) = \mathsf{i}(X, W) \neq 0$ by Proposition 3.17, because $\mathsf{i}(X, W) \neq 0$, Therefore (8) holds, completing the proof of Theorem 1.3.

Proof of Theorem 1.4

Recall the hypotheses:

- M is a connected real or complex 2-manifold with empty boundary
- $\mathcal{G} \subset \mathcal{V}^{\omega}(M)$ is a Lie algebra over ground field $\mathbb{F}$ that tracks $X \in \mathcal{V}^{\omega}(M)$
- If M is real, $\mathcal{G}$ is supersolvable.
- K is an essential X-block

To be proved: $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$.

Consider first the case that M is complex. We can assume:

(A1′) *K is connected and $\chi(K) = 0$.*

For otherwise the argument used above to justify (A1) shows that there is point of K fixed by the local flow of every $Y \in \mathcal{V}^1(M)$ that tracks X.

(A2′) $\mathsf{dim}_{\mathbb{F}}(K) = 1$*, and K is an analytic submanifold.*

If $\mathsf{dim}_{\mathbb{F}}(K) = 0$ then K is a singleton by (A1′) and the conclusion of the theorem is obvious. If $\mathsf{dim}_{\mathbb{F}}(K) = 2$ then $K = M$ because X is analytic and M is connected. But then $X = 0$, contradicting $\mathsf{i}_K(X) \neq \varnothing$. Thus we can assume $\mathsf{dim}(K) = 1$. If K is not an analytic submanifold, its singular set is a nonempty and $\mathcal{G}$-invariant; being 0-dimensional, it is finite hence contained in $\mathsf{Z}(\mathcal{G})$.

(A1′) and [(A2′) imply:

(C1′) *K is a compact connected Riemann surface of genus 1, holomorphically embedded in M.*

As every $Y \in \mathcal{G}$ tracks X, K is Y-invariant (Proposition 2.3. Therefore $Y|K$ is a holomorphic vector field on K, and $\mathsf{Z}(Y) \cap K \neq \varnothing$ (Theorem 1.3). Since a nontrivial holomorphic vector field on a compact Riemann surface has no zeros, $K \subset \mathsf{Z}(Y)$ for all $Y \in \mathcal{G}$. This completes the proof of Theorem 1.4 for complex manifolds.

Now assume: *M is real and $\mathcal{G}$ is supersolvable.*

Let $\mathsf{dim}\, G = d$, $1 \leq d < \infty$. Finite dimensionality and the assumption that $\mathcal{G}$ tracks X implies X spans an ideal $\mathcal{H}_1$

The conclusion of the theorem is trivial if $d = 1$. If $d = 2$ then $\mathcal{G}$ has a basis $\{X, Y\}$ and Theorem 1.3 implies $\mathsf{Z}(Y) \cap K \neq \varnothing$, whence $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$.

Proceeding inductively we assume $d > 2$ and that the conclusion holds for smaller values of d. Supersolvability implies there is a chain of ideals

$$\mathcal{H}_1 \subset \cdots \subset H_d = \mathcal{G}$$

such that $\mathsf{dim}, \mathcal{H}_k = k$. Note that

$$\mathsf{Z}(\mathcal{H}_1) = K.$$

The inductive hypothesis implies

$$L := \mathsf{Z}(\mathcal{H}_{d-1}) \cap K \neq \varnothing,$$

and L is $\mathcal{G}$-invariant because the zero sets of the ideals $\mathcal{H}_{d-1}$ and $\mathcal{H}_1$ are $\mathcal{G}$-invariant.

If $\mathsf{dim}\, L = 0$ it is a nonempty finite $\mathcal{G}$ invariant set, hence contained in $\mathcal{G} \cap K$. Suppose $\mathsf{dim}\, L = 1$. If $L = K$ there is nothing more to prove, and if $L \neq K$ its frontier in K is a nonempty finite $\mathcal{G}$-invariant set. The proof of Theorem 1.3 is complete. ∎

Proof of Theorem 1.5

The effective analytic action of G on M induces an isomorphism from the Lie algebra of G onto a Lie algebra $\mathcal{G} \subset \mathcal{V}^\omega(M)$. Let $X \in \mathcal{G}$ span the Lie algebra of a 1-dimensional ideal. Because $\chi(M) \neq 0$, the set $K := \mathsf{Z}(X)$ is an essential X-block by the Poincaré-Hopf Theorem 1.1. Theorem 1.4 shows that $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$. Connectedness of G implies $\mathsf{Z}(\mathcal{G}) = \mathsf{Fix}(\alpha)$, implying the conclusion. ∎

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