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by

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Abstract

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This dissertation studies the problem of allocating heterogeneous indivisible goods to agents without the use of monetary compensation when each agent receives at most one good. The three central concerns in designing allocation mechanisms are incentives of the agents, efficiency, and fairness. There are inherent trade-offs between competing notions of the three desiderata in such assignment mechanisms. This dissertation further explores these tradeoffs and constructs novel mechanisms for such settings.

Chapter 2 focuses on random assignment mechanisms for the canonical house allocation problem. It shows that any strategy-proof and envy-free mechanism must sacrifice efficiency in a very weak form captured by the notion of *contention-free efficiency*. The chapter also characterizes a large family of strategy-proof and envy-free mechanisms called *Rank Exchange Mechanisms* thereby showing the existence of mechanisms apart from the equal division mechanism that satisfy these two properties.

Chapter 3 studies the allocation problem in the presence of arbitrary linear constraints when agents exhibit indifferences in their preferences. It proposes the *Constrained Serial Rule*, a mechanism that is a generalization of the well-known Probabilistic Serial mechanism, and shows that this mechanism satisfies constrained ordinal efficiency and envy-freeness among agents of the same type.

Chapter 4 introduces *Generalized Hierarchical Exchange* mechanisms and shows that they satisfy strategy-proofness, Pareto efficiency, and admit many bossy mechanisms. Using this generalization, the chapter proposes novel mechanisms called priority trading mechanisms. Finally, it also provides a characterization of an important sub-class of Generalized Hierarchical Exchange mechanisms.

To my parents and my husband.

Contents

List of Tables

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Chapter 1

Introduction

The house allocation problem is a fundamental resource allocation problem that deals with the assignment of indivisible objects to agents without the use of monetary transfers. Many real-world applications such as placement of students to public schools (Abdulkadiroğlu and Sönmez 2003), course allocation (Budish 2011), organ donation (Roth, Sönmez, and Ünver 2005), and on-campus housing allocation (Chen and Sönmez 2002) can be modeled as house allocation problems since monetary transfers are often undesirable in these settings. Policy makers or central planners (for instance, the campus housing office allocates dorms rooms to students) often use centralized procedures, or what we will refer to as *mechanisms*, to organize these markets and determine allocations. In many of these settings, agents report a preference ranking over the set of alternatives and the mechanism outputs an assignment of objects to agents. Incentive compatibility, market efficiency, and fairness are the three primary concerns in the design of allocation mechanisms. Unfortunately, a growing body of literature has shown that it is impossible to design mechanisms that satisfy all these three desiderata simultaneously in a large number of practical settings. The focus of this dissertation is to further explore the trade-offs between these properties. Specifically, this dissertation consists of three chapters in which we design mechanisms that try to achieve two of the three properties at a time.

In Chapter 2, we study random mechanisms in a canonical house allocation problem, where there are as many objects as agents and agents have strict preferences over these objects. Randomization is commonly used as a tool to achieve fairness in such discrete settings. In the presence of randomization, a natural notion of fairness is envy-freeness, which requires that every agent must prefer her allocation to anyone else's allocation. Incentives are measured through strategy-proofness; a mechanism is strategy-proof if truthful reporting of preferences is a weakly dominant strategy for every agent. Focusing on strategy-proof and envy-free mechanisms, we are motivated by the following question: Is there a strategy-proof and envy-free random assignment mechanism more efficient than equal division? In asnwering this question, we, first, further explore the incompatibility between efficiency, fairness, and strategy-proofness within random assignment mechanisms. We define a new notion of efficiency, called contention-free efficiency, that is weaker than ex-post efficiency and prove that any strategy-proof and envy-free mechanism must sacrifice efficiency even in this very weak sense. Next, we introduce a new family of mechanisms called *Rank Exchange Mechanisms* and show that they are strategy-proof and envy-free and stochastically dominate equal division. Further, we show that rank exchange mechanisms characterize the set of neutral, strategy-proof, and envy-free mechanisms that also satisfy a natural separability axiom that may be of independent interest.

In Chapter 3, we consider a more general model of the problem from Chapter 2, where agents are allowed to be indifferent between objects and we need to assign objects to agents in the presence of arbitrary linear constraints. Our main contribution is the generalization of the (Extended) Probabilistic Serial mechanism via a new mechanism called the *Constrained Serial Rule*. This mechanism is computationally efficient and maintains desirable efficiency and fairness properties namely constrained ordinal efficiency and envy-freeness among agents of the same type. Our mechanism is based on a linear programming approach that accounts for all constraints and provides a re-interpretation of the "bottleneck" set of agents that form a crucial part of the Extended Probabilistic Serial mechanism.

In Chapter 4, we take a different approach and focus our attention to deterministic mechanisms for the house allocation problem. Despite a significant amount of research interest, a complete characterization of strategy-proof and Pareto-efficient mechanisms remains unknown. Indeed, the class of hierarchical exchange rules (Pápai 2000) and trading cycles mechanisms (Pycia and Ünver 2017) that characterize all group-strategyproof and Pareto efficient mechanisms admit only non-bossy mechanisms and thus are unable to capture all strategy-proof and Pareto efficient mechanisms. Our main contribution in this chapter is to explore the set of strategy-proof and Pareto efficient mechanisms beyond those that are non-bossy as a stepping stone to a complete characterization. We present a new family called *Generalized Hierarchical Exchange Mechanisms* that admit all hierarchical exchange mechanisms as well as a number of novel bossy, strategy-proof and Pareto efficient mechanisms. We also characterize an important subset of this family through a notion of local dictatorship.

Chapter 2

Strategy-proof and Envy-free Mechanisms for House Allocation

2.1 Introduction

In this chapter, we study the problem of allocating heterogeneous indivisible objects to agents when there are as many objects as agents and agents have strict preferences over the objects. Since objects are indivisible, any assignment of objects to agents is bound to be perceived as unfair, ex-post. Randomization is, therefore, commonly used as a tool to restore fairness from an ex-ante perspective. As agents report ordinal rankings of objects, using the stochastic dominance relation allows for a way to extend agents' preferences over sure objects to lotteries of objects. Perhaps the most natural random allocation mechanism is the *Random Serial Dictatorship* (RSD) mechanism, also known as *Random Priority* mechanism. In this mechanism, agents are ordered uniformly at random, and each agent successively chooses her favorite object from the set of available objects according to that order. RSD is known to satisfy a number of attractive properties (Abdulkadiroğlu and Sönmez 1998; Bogomolnaia and Moulin 2001). It is strategy-proof, meaning that for any agent, revealing true preferences first-order stochastically dominates any other strategy. Or in other words, reporting true preferences is always a dominant strategy for every agent for any utility representation consistent with their preferences. It is also ex-post efficient, which implies that it always induces efficient eventual outcomes. However, it satisfies fairness only in the weak sense of equal treatment of equals (where agents with identical preferences face identical lotteries over objects).

A stronger notion of fairness is envy-freeness. Introduced by Foley (1967), the classic definition of envy-freeness requires that each agent should prefer her allocation to anyone else's allocation. This notion is often considered as the gold standard of fairness in many different settings such as resource allocation (Foley 1967), cake-cutting (Robertson and Webb 1998), and rent division (Edward Su 1999). In the context of random assignment mechanisms, Bo-

This work is joint with Manish Purohit.

gomolnaia and Moulin (2001) formulate this property using the first-order stochastic dominance relation. They proposed the *Probablistic Serial* (PS) mechanism that enjoys stronger efficiency and fairness properties than RSD. The mechanism allocates objects through an "eating" procedure as follows. All agents simultaneously start eating their most preferred object at an uniform rate. After an object is fully consumed, the agents consuming this object move on to eating their next best available object and this procedure continues until all objects are consumed. The random allocation an agent receives corresponds to the shares of the objects eaten by that agent. PS is ordinally efficient* and envy-free. However, this mechanism is not strategy-proof.

Unfortunately, a growing body of work (Bogomolnaia and Moulin 2001; Nesterov 2017; Zhou 1990; Martini 2016) has demonstrated the inherent incompatibility between efficiency, fairness, and strategy-proofness. Bogomolnaia and Moulin (2001) proved that no mechanism simultaneously satisfies strategy-proofness, ordinal efficiency and equal treatment of equals. More recently, Nesterov (2017) showed the incompatibility between strategy-proofness, envy-freeness and ex-post efficiency. Given these trade-offs, it is natural to consider mechanisms that can be designed when one would like to have two of the three properties of efficiency, fairness and strategy-proofness to be satisfied in their strongest notions, where the choice of the properties depends on the application, while trying to improve on the third dimension.

Since the primary motivation for randomization in house allocation mechanisms is to provide fairness guarantees, in this chapter, we focus our attention on strategy-proof random allocation mechanisms that satisfy envy-freeness. Indeed, there has been a resurgence of interest in fairness at the intersection of Economics and Computer Science in recent years. See, for instance, the recent survey by Moulin (2019) and the EC workshop† on fair resource allocation for an excellent overview. To the best of our knowledge, the equal division (ED) mechanism that allocates each object equally among all agents is the only known mechanism that is strategy-proof and envy-free. However, since this mechanism completely ignores agents' preferences, it is almost always inefficient. This raises the natural question: are there other strategy-proof and envy-free mechanisms that are more efficient than the equal division mechanism?

2.1.1 Our Contributions

We first show a strong impossibility result to demonstrate that strategy-proof and envy-free mechanisms must sacrifice efficiency even in a very weak sense. In Section 2.3, we define a notion of *contention-free efficiency* that is much weaker than ex-post efficiency and show that no strategy-proof mechanism that satisfies envy-freeness can be contention-free efficient. Our result thus strengthens and subsumes the hardness result by Bogomolnaia and Moulin (2001)

*A mechanism is ordinally efficient if its outcome is not first-order stochastically dominated by any other random assignment.

†Workshop on Fairness at ACM Conference on Economics and Computation, 2019: https://users.cs.duke.edu/~rupert/fair-division-ec19/index.html

and Nesterov (2017) regarding the incompatibility of ex-post efficiency, strategy-proofness and envy-freeness.

In Section 2.4, we introduce a family of mechanisms called Rank Exchange mechanisms that are strategy-proof and envy-free and stochastically dominate equal division. We characterize the Pareto frontier of such mechanisms in Section 2.4.2 and demonstrate that Rank Exchange mechanisms are, in general, incomparable with both RSD and PS (with respect to the stochastic dominance relation).

In Section 2.5, we introduce a new axiom called *separability* and show that all envy-free and separable mechanisms can be represented via a decomposition into pairwise transfers of fractional shares between agents. Further, we demonstrate that within this class of mechanisms, called pairwise exchange mechanisms, strategy-proofness is equivalent to envy-freeness. Therefore, we observe that all separable and envy-free mechanisms are also strategy-proof. Finally, in our main characterization result, we show that Rank Exchange mechanisms characterize the set of all neutral, separable, and envy-free mechanisms.

2.1.2 Other Related Work

This chapter adds to the broad literature on random assignment of indivisible objects that was pioneered by Hylland and Zeckhauser (1979). They adapt the competitive equilibrium from equal incomes (CEEI) solution to define a pseudo-market mechanism that elicits agents' von Neumann-Morgenstern preferences over individual objects and gives a solution that is efficient and fair with respect to the utility functions (i.e., ex-ante efficient and envy-free). However, the mechanism is not strategy-proof. Zhou (1990), in fact, shows (proving a conjecture by Gale (1987)) that there exists no strategy-proof mechanism that satisfies ex-ante efficiency and equal treatment of equals.

When agents' report only ordinal preferences over individual objects, the simplest and the most widely known strategy-proof mechanism is serial dictatorship (SD) (Svensson 1994; Satterthwaite and Sonnenschein 1981). Abdulkadiroğlu and Sönmez (1998) show that these are the only Pareto efficient matching mechanisms. However, while these are very unfair, using a random ordering of agents results in restoration of fairness in the sense of equal treatment of equals. The resulting mechanism, RSD, was also analyzed by Abdulkadiroğlu and Sönmez (1998) and shown to be equivalent to the *core from random endowments* mechanism, where each agent is initially endowed with an object that is chosen uniformly at random and the mechanism then uses Gale's *Top Trading Cycles* (TTC) algorithm (Shapley and Scarf 1974) to arrive at a random assignment.

Several papers have focused on characterizing (families of) mechanisms that satisfy certain desirable properties. For deterministic mechanisms, Svensson (1999) proves that serial dictatorships are the only group strategy-proof and neutral mechanisms. Pápai (2000) introduces a class of mechanisms called hierarchical exchange mechanisms and proves that these mechanisms characterize the class of group strategy-proof, Pareto efficient and reallocation-proof mechanisms. More recently, Pycia and Ünver (2017) have shown that their trading-cycles mechanisms characterize the full class of group strategy-proof and Pareto efficient mechanisms. Within random mechanisms, when there are three agents and three objects Bogomolnaia and Moulin (2001) show that RSD is the unique mechanism that satisfies strategy-proofness, ex-post efficiency and equal treatment of equals while PS is the unique mechanism that satisfies ordinal efficiency, envy-freeness and weak strategy-proofness. Bogomolnaia and Heo 2012; Heo 2014b; 2014a; Hashimoto et al. 2014; Heo and Yılmaz 2015 provide other axiomatic characterizations of PS. Kojima and Manea 2010 show that PS becomes asymptotically strategy-proof. In large markets, Liu and Pycia (2016) show that there is a unique mechanism that is ordinally efficient, envy-free and strategy-proof and all uniform randomizations over known deterministic mechanisms such as serial dictatorships, hierarchical exchange and trading-cycles mechanisms coincide with this unique mechanism. Chambers 2004 introduces a notion of probabilistic consistency and shows that ED is the only mechanism that satisfies probabilistic consistency and equal treatment of equals.

In other recent work, Budish et al. (2013) introduce constraints in the random assignment problem and generalize known mechanisms and results to these settings. Bade (2016) and Zhang (2019) show that there exists no group-strategyproof and ex-post efficient random assignment mechanism that satisfies equal treatment of equals. Harless (2019) proposes a new algorithm to construct ordinally efficient assignments. Kesten, Kurino, and Nesterov (2017) construct new mechanisms that stochastically improve upon RSD. Basteck (2018) proposes another novel envy-free (but not strategy-proof) solution to the random assignment problem, which is based on Walrasian equilibria. In subsequent ongoing work, Basteck and Ehlers (2021) independently obtain the incompatibility between strategy-proofness, envy-freeness and contention-free efficiency (Theorem 1), and construct a strategy-proof and envy-free mechanism called Random-Dictatorship-cum-Equal-Division.

2.2 Preliminaries

2.2.1 Model and Notation

Let N be the set of agents and O be the set of objects. Throughout this chapter, we assume that the sets N and O are fixed and finite with $|N| = |O| = n \geq 3$. Each agent $i \in N$ has a strict preference relation $\succ_i$ on O. The corresponding weak preference relation on O is denoted by $\succeq_i$. We denote the preference relation $a_1 \succ a_2 \succ \ldots \succ a_n$ as $\succ = \langle a_1, a_2, \ldots, a_n \rangle$. Let $\sigma(\succ, k)$ denote the kth most preferred object according to the preference relation $\succ$ and $rank(\succ, a)$ to denote the rank of object a in $\succ$.

A set of individual preference relations of all agents constitutes a preference profile $\boldsymbol{\succ} = (\succ_i)_{i \in N}$. Let $\boldsymbol{\succ}_{-i} = \boldsymbol{\succ} \setminus \{\succ_i\}$ denote the set of preferences of all agents other than agent i. We will use $\boldsymbol{\succ} = (\succ_i = \succ, \succ_j = \succ', \boldsymbol{\succ}_{-\{i,j\}})$ to denote a preference profile when agent i's preference is $\succ$, agent j's preference is $\succ'$ and all other agents' preferences are given by $\boldsymbol{\succ}_{-\{i,j\}}$. Let $\mathcal{R}$ be the set of all individual preferences and $\mathcal{R}^n$ be the set of all possible preference profiles.

A *deterministic* assignment is a bijection from N to O, where every agent receives one object and every object is assigned to exactly one agent. Let $\mathcal{D}$ be the set of all deterministic assignments. A *random* assignment $P = \left[P_{i,a}\right]_{i \in N, a \in O}$ is a doubly stochastic matrix of size $n \times n$, where each element $P_{i,a}$ of the matrix P represents the probability with which agent i is assigned object $a^{\ddagger}$. By the Birkhoff-von Neumann theorem (Birkhoff 1946; Neumann 1953), any random assignment can be implemented as a lottery over deterministic assignments. Let $\mathcal{L}(\mathcal{D})$ be a set of all possible random assignments. The ith row of the matrix P, P_i, represents the allocation received by agent i in the random assignment. This allocation is simply a probability distribution over the set of objects O.

In order to compare random assignments, we first extend agents' preferences over the set of objects to the set of random allocations. Agent i prefers a random allocation P_i to another random allocation Q_i (denoted as $P_i \geq^{(\succ_i)} Q_i$) if and only if P_i first-order stochastically dominates Q_i according to agent i's preference $\succ_i$. Formally, let $U(\succ, a) = \{o \in O \mid o \succeq a\}$ be the set of objects that are weakly preferred to object a according to the preference relation $\succ$. Then we have,

$$P_i \geq^{(\succ_i)} Q_i \Longleftrightarrow \sum_{o \in U(\succ_i, a)} P_{i,o} \geq \sum_{o \in U(\succ_i, a)} Q_{i,o} \ \forall a \in O$$

Under the expected utility framework, if P_i stochastically dominates Q_i at $\succ_i$, then for any von Neumann-Morgenstern utility index consistent with $\succ_i$, agent i prefers P_i to Q_i. We say that agent i strictly prefers P_i to Q_i, denoted as $P_i >^{(\succ_i)} Q_i$, if $P_i \geq^{(\succ_i)} Q_i$ and $P_i \neq Q_i$.

A random assignment P *weakly dominates* another assignment Q at preference profile $\succ$ if every agent prefers the random allocation that she receives in P to her random allocation in Q. That is, P *weakly dominates* Q at $\succ = (\succ_i)_{i \in N}$, if $\forall i \in N$, $P_i \geq^{(\succ_i)} Q_i$. Additionally, P *(strictly) dominates* Q if there exists an $i \in N$ such that $P_i >^{(\succ_i)} Q_i$.

A (randomized) *mechanism* is a mapping, $\varphi : \mathcal{R}^n \to \mathcal{L}(\mathcal{D})$, that associates each preference profile $\succ \in \mathcal{R}^n$ with some random assignment $P \in \mathcal{L}(\mathcal{D})$. A mechanism φ *(strictly) dominates* another mechanism φ' if $\varphi(\succ)$ weakly dominates $\varphi'(\succ)$ at every profile $\succ \in \mathcal{R}^n$, with strict dominance at at least one preference profile.

2.2.2 Properties of Mechanisms

We now define the different notions of efficiency, fairness, and incentive-compatibility that we address in this chapter.

Efficiency. A mechanism φ is said to be *ex-post efficient* if for any preference profile $\succ$, the random assignment $\varphi(\succ)$ is a distribution over Pareto-optimal deterministic assignments. An even stronger notion of efficiency was proposed by Bogomolnaia and Moulin (2001). A

‡For convenience in this chapter, we will often think of each object as an infinitely divisible good of one unit that will be distributed among n agents. Allocating a fractional unit x of an object a to agent i is interpreted as setting $P_{i,a} = x$.

mechanism φ is said to be *ordinally efficient* if at every profile $\succ$, $\varphi(\succ)$ is not stochastically dominated by another random assignment.

Incentives. A mechanism φ is said to be *strategy-proof* (SP) if reporting true preferences is a dominant strategy for every agent. Formally, a mechanism φ is strategy-proof if for every $i \in N$, for every $\succ \in \mathcal{R}^n$ and $\succ'_i \in \mathcal{R}$, $\varphi_i(\succ) \geq^{(\succ_i)} \varphi_i(\succ'_i, \succ_{-i})$.

As shown by Mennle and Seuken (2014a), 2014b, strategy-proofness is equivalent to the following three axioms: *swap-monotonicity, upper-invariance*, and *lower-invariance*. We define these axioms below. For any preference relation $\succ_i$, we define the *neighborhood* of $\succ_i$, $\Gamma(\succ_i)$, to be set of all preferences that can arise by swapping any two consecutively ranked objects in the preference $\succ_i$. For example, suppose $O = \{a, b, c, d\}$ and $\succ_i := \langle a, b, c, d\rangle$. Then $\Gamma(\succ_i) = \{\langle b, a, c, d\rangle, \langle a, c, b, d\rangle, \langle a, b, d, c\rangle\}$.

A mechanism φ is swap-monotonic, if for every agent $i \in N$, for every preference profile $\succ \in \mathcal{R}^n$ and every $\succ'_i \in \Gamma(\succ_i)$ with $a \succ_i b$ but $b \succ'_i a$ for some $a, b \in O$, then either $\varphi_i(\succ'_i, \succ_{-i}) = \varphi_i(\succ)$ or $\varphi_{i,b}(\succ'_i, \succ_{-i}) > \varphi_{i,b}(\succ)$. In other words, swap-monotonicity requires the mechanism φ to either disregard agent i's mis-report or to react to her mis-report by allocating her a higher probability of the object that's been brought up in the preference.

A mechanism is upper-invariant if no agent, by swapping some object a with a less preferred b in her preference, can get more allocation for any of the objects that are strictly preferred to object a. Formally, φ is upper-invariant if $\forall\ i \in N$, $\forall \succ \in \mathcal{R}^n$ and $\forall \succ'_i \in \Gamma(\succ_i)$ with $a \succ_i b$ but $b \succ'_i a$ for some $a, b \in O$, we have $\forall o \in U(\succ_i, a) \setminus \{a\}, \quad \varphi_{i,o}(\succ) = \varphi_{i,o}(\succ'_i, \succ_{-i})$.

Similarly, a mechanism is lower-invariant if $\forall\ i \in N$, $\forall \succ \in \mathcal{R}^n$ and $\forall \succ'_i \in \Gamma(\succ_i)$ with $a \succ_i b$ but $b \succ'_i a$ for some $a, b \in O$, we have $\forall o \notin U(\succ_i, b), \quad \varphi_{i,o}(\succ) = \varphi_{i,o}(\succ'_i, \succ_{-i})$.

Lemma 1 (Mennle and Seuken 2014a; 2014b)**.** *A mechanism φ is strategy proof if and only if it is swap-monotonic, upper-invariant, and lower-invariant.*

Fairness. Different notions of fairness have been considered in literature. *Equal treatment of equals* is a weak fairness criterion that requires two agents with the same reported preferences to get the same random allocations. On the other hand, *envy-freeness* is a well established strong fairness criterion that requires that no agent envies the allocation of any other agent. Formally a random assignment P is envy-free at preference profile $\succ = (\succ_k)_{k\in N}$, if we have for all $i, j \in N$, $P_i \geq^{(\succ_i)} P_j$. A mechanism φ is envy-free if it always produces envy-free assignments. It can be readily seen that envy-freeness implies equal treatment of equals under strict preferences.

Neutrality. In this chapter, we often restrict our attention to neutral mechanisms. Neutrality is a natural notion of symmetry that restricts the mechanism to treat all objects identically, i.e., the mechanism is invariant to any renaming of objects.

Formally, $\pi : O \to O$ be a permutation, i.e. a bijection from O to itself. For any preference relation $\succ = \langle a_1, a_2, \ldots, a_n\rangle$, let $\pi(\succ) = \langle \pi(a_1), \pi(a_2), \ldots, \pi(a_n)\rangle$ be obtained by applying the permutation to each object in order, and let $\pi(\succ) = (\pi(\succ_i))_{i \in N}$ be the preference profile obtained by relabeling all objects according to π. Then, a mechanism φ is *neutral* if for any preference profile $\succ \in \mathcal{R}^n$ and permutation $\pi : O \to O$, and every agent $i \in N$ and object $a \in O$, we have $\varphi_{i,a}(\succ) = \varphi_{i,\pi(a)}(\pi(\succ))$.

2.3 An Impossibility Result

In this section, we present our first main result regarding the inherent trade-offs between the competing notions of efficiency, fairness, and strategy-proofness. Our goal is to demonstrate that any strategy-proof and envy-free mechanism must sacrifice efficiency in even the weakest sense.

Before stating our theorem, we first introduce a very weak notion of efficiency, which we call *contention-free efficiency.* Intuitively, this notion captures the desideratum that if there is no competition for objects among the different agents, then an efficient mechanism must allocate to each agent her most preferred object[§]. We capture this intuition formally as follows.

Definition 1 (Contention-Free Profile). *A preference profile $\succ$ is called a* contention-free preference profile *if every agent prefers a distinct object as her top choice. Formally, a preference profile $\succ \in \mathcal{R}^n$ is* contention-free *if and only if $\forall i \neq j \in N$, $\sigma(\succ_i, 1) \neq \sigma(\succ_j, 1)$.*

Let $\mathcal{C} \subset \mathcal{R}^n$ be the set of all contention-free preference profiles.

Definition 2 (Contention-Free Efficiency). *A mechanism φ is defined to be* contention-free efficient *if and only if for every contention-free preference profile it allocates to every agent, her most preferred object fully. That is, φ is contention-free efficient $\Leftrightarrow \forall \succ \in \mathcal{C}$ and $\forall i \in N, \varphi_{i,\sigma(\succ_i,1)}(\succ) = 1$.*

Such a notion clearly imposes a very minimal efficiency requirement on a mechanism. Indeed, it places no restrictions at all at any profile that is not contention-free. Further, for any contention-free profile, the unique Pareto-optimal deterministic assignment is one that allocates to every agent her most preferred object. So any ex-post efficient mechanism must also be contention-free efficient, and indeed RSD and PS both satisfy this property. We, however, show that it is impossible for any mechanism to satisfy strategy-proofness, envy-freeness, and contention-free efficiency simultaneously.

Theorem 1. *For any $n \geq 3$, no strategy-proof and envy-free mechanism can be contention-free efficient.*

Proof. We first prove the theorem for the case when $n = 3$ and defer the extension for $n > 3$ to Appendix A.1.

[§]This property is analogous to the unanimity axiom defined in the social choice literature (Muller and Satterthwaite 1977).

Let $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$ denote the set of agents and objects respectively. Suppose for contradiction that there exists a mechanism φ that is strategy-proof, envy-free and contention-free efficient. For ease of exposition, for any profile $\succ^{(x)}$, we adopt the notation $P^{(x)} := \varphi(\succ^{(x)})$. We will proceed by considering six preference profiles, which are shown in Table 2.1. The corresponding random assignments given by the mechanism φ are shown in Table 2.2.

1	a	b	c
2	b	a	c
3	c	a	b

(a) Profile A

1	a	b	c
2	a	b	c
3	c	a	b

(b) Profile B

1	a	b	c
2	a	c	b
3	c	a	b

(c) Profile C

1	b	a	c
2	a	c	b
3	c	a	b

(d) Profile D

1	a	b	c
2	a	c	b
3	a	c	b

(e) Profile E

1	b	a	c
2	a	c	b
3	a	c	b

(f) Profile F

Table 2.1: Six preference profiles to demonstrate incompatibility of strategy-proofness, envy-freeness, and contention-free efficiency.

	a	b	c
1	1	0	0
2	0	1	0
3	0	0	1

(a) Profile A

	a	b	c
1	$1/2$	$1/2$	0
2	$1/2$	$1/2$	0
3	0	0	1

(b) Profile B

	a	b	c
1	$1/2$	$1 - 2y$	$2y - 1/2$
2	$1/2$	y	$1/2 - y$
3	0	y	$1 - y$

(c) Profile C

	a	b	c
1	0	1	0
2	1	0	0
3	0	0	1

(d) Profile D

	a	b	c
1	$1/3$	$1/2$	$1/6$
2	$1/3$	$1/4$	$5/12$
3	$1/3$	$1/4$	$5/12$

(e) Profile E

	a	b	c
1		1	$1/6$
2		0	
3		0	

(f) Profile F

Table 2.2: Assignments for the six preference profiles in Table 2.1.

Profile A ($\succ^{(A)}$) : Consider, first, the preference profile in Table 2.1a, where agent 1 prefers $a \succ_1^{(A)} b \succ_1^{(A)} c$, agent 2 prefers $b \succ_2^{(A)} a \succ_2^{(A)} c$ while agent 3 prefers $c \succ_3^{(A)} a \succ_3^{(A)} b$.

Since the mechanism φ is contention-free efficient, every agent must receive her top choice with probability 1. That is, $P^{(A)}_{1,a} = P^{(A)}_{2,b} = P^{(A)}_{3,c} = 1$ as shown in Table 2.2a.

Profile B ($\succ^{(B)}$) : Next, suppose agents 1 and 3 report their respective preferences as in profile A. But agent 2 swaps her first two choices and reports $a \succ^{(B)}_2 b \succ^{(B)}_2 c$. First, since strategy-proofness implies lower-invariance, we have $P^{(B)}_{2,c} = P^{(A)}_{2,c} = 0$. Second, envy-freeness implies equal treatment of equals. Since agents 1 and 2 have the same preferences, both must receive the same random allocation of objects. So, $P^{(B)}_{1,c} = P^{(B)}_{2,c} = 0$ and thus $P^{(B)}_{3,c} = 1$. This implies that $P^{(B)}_{1,a} = P^{(B)}_{2,a} = \frac{1}{2}$ and $P^{(B)}_{1,b} = P^{(B)}_{2,b} = \frac{1}{2}$.

Profile C ($\succ^{(C)}$) : If instead agent 2 reports $a \succ^{(C)}_2 c \succ^{(C)}_2 b$ while agents 1 and 3 report the same preferences as in profile B, upper invariance implies that she should receive an equal probability of being allocated her top object a. Therefore, $P^{(C)}_{2,a} = P^{(B)}_{2,a} = \frac{1}{2}$. For agent 1 to not envy agent 2, we must have $P^{(C)}_{1,a} = P^{(C)}_{2,a} = \frac{1}{2}$. Consequently, $P^{(C)}_{3,a} = 0$. To maintain envy-freeness between agents 2 and 3, we have $P^{(C)}_{2,a} + P^{(C)}_{2,c} = P^{(C)}_{3,a} + P^{(C)}_{3,c}$ and $P^{(C)}_{2,b} = P^{(C)}_{3,b}$. Let $P^{(C)}_{2,b} = P^{(C)}_{3,b} = y$. Note than $y \in [0, \frac{1}{2}]$. So we have, $P^{(C)}_{2,c} = \frac{1}{2} - y$ and $P^{(C)}_{3,c} = 1 - y$.

Profile D ($\succ^{(D)}$) : Consider the contention-free profile where agents 2 and 3 do not change their preferences from profile C but agent 1 reports $b \succ^{(D)}_1 a \succ^{(D)}_1 c$. By the assumption that the mechanism is contention-free efficient, each agent receives their top choice with probability 1. For the mechanism to be strategy-proof, agent 1 must receive the same probability of receiving object c in profiles C and D. That is, $P^{(C)}_{1,c} = P^{(D)}_{1,c} \implies 2y - \frac{1}{2} = 0 \implies y = \frac{1}{4}$.

Thus the resulting assignment for profile C in mechanism φ is shown in Table 2.3.

1	a	b	c
2	a	c	b
3	c	a	b

	a	b	c
1	1/2	1/2	0
2	1/2	1/4	1/4
3	0	1/4	3/4

Table 2.3: Profile C and its final assignment

Profile E ($\succ^{(E)}$) : Suppose, next, that agent 3 mimics agent 2 in profile C while agents 1 and 2 keep the same preferences as profile C. For all three agents to not envy one another, the mechanism must allocate object a equally to all agents. That is, $P^{(E)}_{1,a} = P^{(E)}_{2,a} = P^{(E)}_{3,a} = \frac{1}{3}$. By strategy-proofness, $P^{(E)}_{3,b} = P^{(C)}_{3,b} = \frac{1}{4}$. This implies that $P^{(E)}_{3,c} = \frac{5}{12}$. Since agent 2 and 3 have the same preferences, they must get the same allocation. So $P^{(E)}_{2,b} = \frac{1}{4}$ and $P^{(E)}_{2,c} = \frac{5}{12}$. The allocations for the two agents 2 and 3 completely determine the allocation that agent 1 must receive.

Profile F ($\succ^{(F)}$) : Finally, consider the profile in which agent 1 reports $b \succ_1^{(F)} a \succ_1^{(F)} c$ while agents 2 and 3 keep the same preferences as in profile E. Since the mechanism is strategy-proof, for agent 3 to report her preferences truthfully in profiles F and D, $P_{3,b}^{(F)} = P_{3,b}^{(D)} = 0$. For agents 2 and 3 to not envy each other, $P_{2,b}^{(F)} = 0$, which in turn implies $P_{1,b}^{(F)} = 1$. However, since the mechanism is strategy-proof, agent 1 must receive the same probability for object c in profiles E and F. That is, $P_{1,c}^{(F)} = P_{1,c}^{(E)} = \frac{1}{6}$, which contradicts feasibility of the allocation. □

When there are three agents and three objects, Bogomolnaia and Moulin (2001) mention that strategy-proofness and envy-freeness are incompatible with ex-post efficiency. This was proven formally for $n \geq 3$ by Nesterov (2017). Since ex-post efficiency implies contention-free efficiency, their results arise as an immediate corollary of Theorem 1.

Corollary 1 (Bogomolnaia and Moulin 2001; Nesterov 2017)**.** *For $n \geq 3$, there does not exist a mechanism that is strategy-proof, envy-free, and ex-post efficient.*

Theorem 1 demonstrates that in our search for envy-free and strategy-proof mechanisms we must necessarily deviate from (variants of) efficient mechanisms. For example, considering that the RSD mechanism is strategy-proof and satisfies equal treatment of equals, one might try to obtain a strategy-proof and envy-free mechanism by starting from RSD assignments and then successively attempting to remove envy from each assignment. For instance, a similar strategy has been exploited by Harless and Phan (2019) to show that a variant of the RSD mechanism where randomizing only over the three adjacent agent positions (instead of all orderings) helps recover ordinal efficiency without sacrificing strategy-proofness. But such a strategy is unlikely to yield envy-free and strategy-proof mechanisms considering all natural variants of known efficient mechanisms also maintain contention-free efficiency.

2.4 Rank Exchange Mechanisms

While both strategy-proofness and envy-freeness are highly desirable properties for any random assignment mechanism, the only strategy-proof and envy-free mechanism known prior to this work is the *Equal Division (ED)* mechanism that simply allocates all objects equally to all agents and totally disregards the agents' preferences. In some sense, the equal division mechanism can be considered as the "most inefficient" strategy-proof and envy-free mechanism. In view of the strong impossibility result presented in Theorem 1, it is natural to wonder whether there exist other strategy-proof and envy-free mechanisms that are more efficient than equal division. In this section, we answer this question in the positive and describe a large family of mechanisms that are strategy-proof, envy-free and stochastically dominate the equal division mechanism.

2.4.1 Mechanism

Definition 3 (Rank Exchange Mechanism)**.** *A mechanism φ is said to be a* Rank Exchange mechanism *if there exists a vector $v = (v_1, v_2, \ldots, v_n) \in [0, \frac{1}{n(n-1)}]^n$, where $\forall k \in \{1, 2, \ldots, n-1\}$, $v_k \geq v_{k+1}$ and $v_n = 0$, such that, $\forall \succ \in \mathcal{R}^n, \forall\ i \in N$, and $\forall\ a \in O$*

$$\varphi_{i,a}(\succ) = \frac{1}{n} + \sum_{j \in N \setminus \{i\}} \Big(v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)} \Big)$$

For any such vector v, we will denote the corresponding Rank Exchange mechanism by φ^v.

A Rank Exchange mechanism can be equivalently described as follows. All agents are initially endowed with the equal division random assignment. Every agent now splits her endowment into $(n-1)$ equal partial allocations, one part for each of the other $n-1$ agents. Every pair of agents then mutually exchange fractional shares of some objects in a way that is dictated by single vector $v \in [0, \frac{1}{n(n-1)}]^n$. Formally, for any pair of agents i and j and any object a, we interpret that $(v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)})$ amount of the object is transferred from agent j to agent i. Note that this amount can be negative, which is interpreted as the transfer occurs in the other direction. Through such a pairwise exchange, each pair of agents swap fractional shares of all the objects: an agent always receives shares of objects that she prefers more than the other agent and donates shares of objects that she prefers less. The net probability shares transferred across all objects between any pair of agents is zero and we maintain a feasible random assignment.

To understand the mechanism concretely, let us consider the following example. Let $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$. Suppose the exchange vector is $v = (\frac{1}{6}, \frac{1}{6}, 0)$. Consider the preference profile where $\succ_1 = \langle b, c, a \rangle$, $\succ_2 = \langle a, c, b \rangle$ and $\succ_3 = \langle a, b, c \rangle$. Let us, first, look at the transfers that agents 1 and 2 engage in. Since agent 1 ranks object a third and agent 2 ranks object a first in their respective preferences, agent 1 transfers $v_1 - v_3 = \frac{1}{6}$ shares of object a to agent 2. In return, agent 2 transfers $v_1 - v_3 = \frac{1}{6}$ shares of object b to agent 1. Agents 1 and 2 both rank object c as their second choice. Therefore, they do not exchange any shares of object c with each other. Between the pair of agents 1 and agent 3, agent 1 transfers $v_1 - v_3 = \frac{1}{6}$ shares of object a to agent 3 and receives $v_2 - v_3 = \frac{1}{6}$ shares of object c. Since $v_1 = v_2$, agents 1 and 3 do not transfer any shares of object b. Lastly, if we consider the transfers between agents 2 and 3, since both rank a as their top choice, they do not exchange any shares of object a. However, since the ranks of objects b and c are flipped in their preferences, they trade $v_2 - v_3 = \frac{1}{6}$ shares of both these objects. The assignment that results from these simultaneous pairwise exchanges in shown in Table 2.4.

We now state our main result of this section on the fairness and incentives of these class of mechanisms. In particular, we show that every Rank Exchange mechanism is strategy-proof and envy-free. We cover the proof of this theorem in Appendix A.2.

Theorem 2. *Every Rank Exchange mechanism φ^v is strategy-proof and envy-free.*

1	b	c	a
2	a	c	b
3	a	b	c

	a	b	c
1	0	1/2	1/2
2	1/2	0	1/2
3	1/2	1/2	0

Table 2.4: Profile and assignment with exchange vector $v = (\frac{1}{6}, \frac{1}{6}, 0)$

The argument in the proof relies on the constraints imposed on the vector v that defines the mechanism. The non-increasing coordinates of v are sufficient to ensure that the total allocation that an agent i receives for the top t objects in her preference is at least as much as the total allocation that any other agent i' receives for the same objects. The same condition also guarantees that misreporting her preference would yield a weakly lower total allocation for the top t objects in her true preference.

At this stage, we would like to contrast our Rank Exchange mechanisms from the *Top-Trading Cycles from Equal Division (TTCED)* mechanism proposed by Kesten (2009). The TTCED mechanism first starts by allocating an equal amount of each object to every agent. Then, any pair of agents who have received each other's top objects are allowed to exchange their shares sequentially. Once an agent can no longer trade her top object, her allocation of that object is finalized and the trading continues with the next object in the preference. While superficially similar in the sense that both mechanisms start with equal division and allow pairwise trading between agents, the TTCED and Rank Exchange mechanisms differ in one key aspect - a Rank Exchange mechanism requires all the trades to occur in parallel independently from each other, while the agent pairs trade sequentially in the TTCED mechanism. Indeed, as shown by Kesten (2009), the TTCED mechanism is equivalent to the the PS mechanism of Bogomolnaia and Moulin (2001) and hence is not strategy-proof. In contrast, we demonstrate that Rank Exchange mechanisms are strategy-proof and envy-free.

2.4.2 Efficiency Discussion

We now discuss the efficiency considerations of our class of mechanisms. It can be readily seen that if the first coordinate of the vector v is strictly greater than zero, then the corresponding Rank Exchange mechanism strictly stochastically dominates ED. Since the coordinates of v are sorted, at every preference profile the total allocation that any agent receives for her top t objects is always weakly greater than $\frac{t}{n}$. However, with $v_1 > 0$, there exist preference profiles where at least one agent is guaranteed to receive strictly more than $\frac{1}{n}$ for her top ranked object. For instance, consider the preference profile where agent i ranks object a as her top choice while the other agents rank this object as their last choice. In this case, agent i receives $v_1 - v_n = v_1 > 0$ additional shares of object a from all the other agents leading to an allocation of $\frac{1}{n} + (n-1)v_1 > \frac{1}{n}$ for her top choice. We state this proposition below.

Proposition 1. *Consider a vector* $v \in [0, \frac{1}{n(n-1)}]^n$ *such that* $\forall k \in \{1, 2, \ldots, n-1\}$, $v_k \geq v_{k+1}$ *and* $v_n = 0$. *If* $v_1 > 0$, *then* φ^v *dominates ED.*

From the impossibility result in Section 2.3, we know that, in any strategy-proof and envy-free mechanism, agents cannot receive their top choice with probability one at all contention-free preference profiles. In subsequent work, Basteck and Ehlers (2021) show that, at all such profiles, agents cannot each receive more than $\frac{2}{n}$ of their top ranked object. Indeed, in any Rank Exchange mechanism, every agent receives at most $\frac{2}{n}$ of her top choice at all contention-free preference profiles. This follows since $v_i \leq \frac{1}{n(n-1)}$ for any $i \in \{1, 2, \ldots, n\}$.

In fact, consider the specific mechanism in this class with $v_1 = \frac{1}{n(n-1)}$ and $v_k = 0$ for all other $k \in \{2, 3, \ldots, n\}$. This mechanism achieves the upper bound of $\frac{2}{n}$ at all contention-free profiles. To see this, fix any contention-free profile $\succ \in \mathcal{C}$, and any agent $i \in N$. Suppose $\sigma(\succ_i, 1) = a$. Then, the probability that agent receives object a is given by,

$$\begin{aligned}
\varphi_{i,a}(\succ) &= \frac{1}{n} + \sum_{j \in N \setminus \{i\}} \Big(v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)} \Big) \\
&= \frac{1}{n} + \sum_{j \in N \setminus \{i\}} \Big(v_1 - 0 \Big) = \frac{1}{n} + \sum_{j \in N \setminus \{i\}} \frac{1}{n(n-1)} = \frac{2}{n}
\end{aligned}$$

The second equality follows since $\succ$ is a contention-free profile, which implies that if $rank(\succ_i, a) = 1$, then $rank(\succ_j, a) \neq 1$ for any $j \neq i$.

Comparison with RSD and PS

In this section, we show that Rank Exchange mechanisms are not dominated by RSD and PS and can in fact lead to incomparable assignments. In contrast, the "Random-Dictatorship-cum-Equal-Division" mechanism proposed by Basteck and Ehlers (2021) is indeed dominated by RSD.

As an example, consider the Rank Exchange mechanism with vector $v = (\frac{1}{6}, \frac{1}{6}, 0)$ and the profile that we have considered before in Table 2.4. Table 2.5 shows the profile and assignments obtained by both the Rank Exchange mechanism and RSD at this profile. It can be readily seen that although agent 1 prefers her allocation in RSD, agent 3 strictly prefers her allocation in the Rank Exchange mechanism. Therefore, neither the RSD assignment nor the assignment in the Rank Exchange mechanism stochastically dominate the other at this preference profile.

In a similar vein, Table 2.6 shows the assignments of the same Rank Exchange mechanism and PS at a different profile. Here, we can see that agents 1 and 2 both strictly prefer their allocations in the Rank Exchange mechanism over their allocations under PS whereas agent 3 prefers her allocation in PS.

1	b	c	a
2	a	c	b
3	a	b	c

(a) Profile

	a	b	c
1	0	1/2	1/2
2	1/2	0	1/2
3	1/2	1/2	0

(b) $\varphi^{(\frac{1}{6},\frac{1}{6},0)}$ assignment

	a	b	c
1	0	5/6	1/6
2	1/2	0	1/2
3	1/2	1/6	1/3

(c) RSD assignment

Table 2.5: Comparison of RSD with Rank Exchange mechanism with vector $v = (\frac{1}{6}, \frac{1}{6}, 0)$

1	a	b	c
2	a	b	c
3	b	c	a

(a) Profile

	a	b	c
1	1/2	1/3	1/6
2	1/2	1/3	1/6
3	0	1/3	2/3

(b) $\varphi^{(\frac{1}{6},\frac{1}{6},0)}$ assignment

	a	b	c
1	1/2	1/6	1/3
2	1/2	1/6	1/3
3	0	2/3	1/3

(c) PS assignment

Table 2.6: Comparison of PS with Rank Exchange mechanism with vector $v = (\frac{1}{6}, \frac{1}{6}, 0)$

Efficient Frontier of Rank Exchange Mechanisms

In our last result of this section, we explore the Pareto efficient frontier of this class of mechanisms. Specifically, we characterize the set of Rank Exchange mechanisms that are not dominated by any other mechanism within this class.

Proposition 2. *A Rank Exchange mechanism φ^v is not dominated by another Rank Exchange mechanism if and only if the vector v satisfies $v_1 = \frac{1}{n(n-1)}$.*

Proof. Proof can be found in Appendix A.2 □

2.5 Characterization

In this section, we show that Rank Exchange mechanisms characterize the set of all strategy-proof, envy-free mechanisms that satisfy a certain separability axiom. As discussed earlier, Rank Exchange mechanisms can be described as first starting from the equal division allocation and then allowing each pair of agents to exchange their respective fractional shares of objects. This view of decomposing the mechanism into multiple pairwise interactions leads to a natural and appealing notion of separability in object allocations. Informally, we call a mechanism φ separable if the difference in the allocation received by an agent i in two preference profiles, where agent i's preference is unchanged, can be attributed to each of the other agents independently changing her preference. Formally, we define the following property.

Definition 4 (Separability). *A mechanism φ is separable if for all $\succ = (\succ_k)_{k\in N} \in \mathcal{R}^n$, for every agent $i \in N$, for each $\succ'_{-i} = (\succ'_k)_{k\in N\setminus\{i\}} \in \mathcal{R}^{n-1}$, and for every object $a \in O$, we have*

$$\varphi_{i,a}((\succ_i, \succ'_{-i})) - \varphi_{i,a}(\succ) = \sum_{j\in N\setminus\{i\}} \Big[\varphi_{i,a}((\succ'_j, \succ_{-j})) - \varphi_{i,a}(\succ)\Big]$$

Our main characterization result is presented below.

Theorem 3. *A mechanism φ is neutral, envy-free, and separable if and only if it is a Rank Exchange mechanism. Further, all envy-free and separable mechanisms are also strategy-proof.*

In the rest of this section, we focus on proving that Rank Exchange mechanisms are the only ones that satisfy our criteria. The necessity direction of the proof is structured as follows. In Section 2.5.1, we show that any envy-free and separable mechanism can be represented using a decomposition into pairwise transfers of fractional shares between agents. In Section 2.5.2, we show that envy-freeness and separability together also imply strategy-proofness. Finally, in Section 2.5.3, we show that the introduction of neutrality and strategy-proofness allows us to arrive at a representation as a Rank Exchange mechanism completing the necessity direction of Theorem 3.

2.5.1 Pairwise Transfers

We first show that envy-freeness and separability imply that any random mechanism can be alternatively represented as follows. The mechanism starts off by allocating all objects equally among the agents and then shares are transferred from one agent to another, where the amount transferred is dictated by a single function f. We will often refer to this function as a *transfer function* following this intuition. The domain of function f is defined on pairs of individual preferences and an object. We call such mechanisms pairwise exchange mechanisms.

Definition 5. *A mechanism φ is said to be a* pairwise exchange mechanism *if there exists a function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$ such that $\forall \succ = (\succ_k)_{k\in N} \in \mathcal{R}^n, \forall i \in N,$ and $\forall a \in O$*

$$\varphi_{i,a}(\succ) = \frac{1}{n} + \sum_{j\in N\setminus\{i\}} f(\succ_i, \succ_j, a)$$

Given such a function f, we denote the corresponding mechanism by φ^f.

The key restriction that envy-freeness and separability impose is that the transfer function f is only a function of the preferences of the two agents involved in the transfer and in particular is independent of the agent identities and the preferences of the other agents. The following key lemma allows us to focus on deriving a functional form for the function f in subsequent steps. The proof for this lemma can be found in Appendix A.3.1.

Lemma 2. *If a mechanism φ is envy-free and separable, then it is a pairwise exchange mechanism.*

To ensure feasibility of the random assignment, the transfer function f must satisfy certain properties that we tabulate in the following lemma. These conditions capture some natural notions and restrictions that one would expect in a bilateral exchange. For instance, the no transfers between equals property states that if a pair of agents have identical preferences over objects, then they would never engage in an exchange of fractional shares for any object. The second property, balanced transfers, ensures the total amount of probabilistic shares that an agent transfers to another agent must equal the amount that she gets back from that same agent.

Lemma 3. *Consider a pairwise exchange mechanism φ^f and its corresponding function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$. Then f satisfies the following properties -*

1. No transfers between equals: $\forall \succ \in \mathcal{R}$ *and* $\forall a \in O$, $f(\succ, \succ, a) = 0$.
2. Balanced transfers: *For any pair of preferences* $\succ, \succ' \in \mathcal{R}$, $\sum_{a \in O} f(\succ, \succ', a) = 0$.
3. Anti-symmetry: *For any pair of preferences* $\succ, \succ' \in \mathcal{R}$ *and* $\forall a \in O$, $f(\succ, \succ', a) = -f(\succ', \succ, a)$.
4. Bounded range: *For any pair of preferences* $\succ, \succ' \in \mathcal{R}$ *and* $\forall a \in O$, $f(\succ, \succ', a) \in [-\frac{1}{n(n-1)}, \frac{1}{n(n-1)}]$.

We often refer back to these properties to prove our primary characterization result. The lemma follows directly from the definition of feasible mechanisms and we include the proof in Appendix A.3.1 for completeness.

2.5.2 Fairness and Incentives

In the next step in the proof, we show that if a mechanism is envy-free and separable, then it is also strategy-proof. This result arises as a corollary of a surprising observation that within pairwise exchange mechanisms, incentive compatibility and fairness go hand in hand. In particular, we demonstrate the equivalence of strategy-proofness and envy-freeness within this class. To the best of our knowledge, this is the only family of random assignment mechanisms for finite markets where such an equivalence between the two normally competing notions of fairness and strategy-proofness has been established[¶].

Theorem 4. *For any pairwise exchange mechanism φ^f, φ^f is strategy-proof if and only if φ^f is envy-free.*

[¶]Such an equivalence has been observed in large economies (for instance, see Jackson and Kremer (2007), Che and Kojima (2010), Liu and Pycia (2016), Noda (2018), and Azevedo and Budish (2019)).

Strategy-proofness and envy-freeness both necessitate that the transfer function satisfies a natural, but informal, notion of monotonicity - that the transfer between two agents is well-aligned with their corresponding preferences. We formalize this notion through a series of lemmas.

In Lemma 11 we show that in a pairwise exchange mechanism that is strategy-proof, the transfers of probabilistic shares received by an agent reporting $\succ$ from any other agent stochastically dominate (with respect to $\succ$) the transfers that she receives when reporting any other $\succ'$. Analogous to Lemma 1, in Lemma 12, we show that strategy-proofness in pairwise exchange mechanisms is equivalent to analogous properties on the function f. Finally, Lemmas 13 and 14 show that the transfer function for any envy-free mechanism must be "selfish". In other words, for any agent, the total amount of objects exchanged with any other agent is compatible with the agent's preferences. Lemma 13 shows that if two agents have identical preferences for their top t objects, then they do not exchange any of those t objects. Similarly, Lemma 14 shows that the total transfers received by any agent from any other agent stochastically dominates zero transfers.

These four lemmas hint towards strategy-proofness and envy-freeness being dependent on slightly different notions of "monotonicity" in the transfer functions. Surprisingly, we can in fact show that the two concepts are much more closely tied together. Appendix A.3.2 provides details of the proof. The following corollary follows from Lemma 2 and Theorem 4.

Corollary 2. *If a mechanism φ is envy-free and separable, then φ is strategy-proof.*

2.5.3 Characterizing Neutral and Strategy-proof Pairwise Exchange Mechanisms

Corollary 2 shows that we get strategy-proofness for free with envy-freeness and separability. Therefore, in our last step to prove the necessity direction, we introduce strategy-proofness and neutrality and use these properties to derive a functional form for the function f.

The following lemma shows that neutrality and strategy-proofness impose stronger restrictions on the function f. In particular, if an object is ranked at the same position in two agents' preferences, which may otherwise be different, then the amount of this object transferred by both these agents to any other agent must be the same.

Lemma 4. *If pairwise-exchange mechanism φ^f is neutral and strategy-proof, then for any object $a \in O$, and preferences $\succ, \succ', \succ'' \in \mathcal{R}$, we have*

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

Lemma 4 is non-trivial to prove and forms the keystone to arrive at a representation for the function f. We prove the result using induction on n. Surprisingly, proving the base case also requires a subtle use of the properties of f from Lemmas 12 and 15. Appendix A.3.3 provides the details.

Lemma 4 already implies that the function f admits a simpler form, where the transfer of any object a between two agents i and j only depends on the ranks of the object in their corresponding preference orderings. This insight in conjunction with the balanced transfers and the no transfers between equals property in Lemma 3 allow us to effectively define the transfer function f in terms of a single vector v of dimension n, which we formalize in Lemmas 17 and 18 in Appendix A.3.3. We are now formally ready to prove Theorem 3

Proof of Theorem 3. We cover the necessity and sufficiency directions below.

Necessity. Consider a mechanism φ which is neutral, envy-free, and separable. By Lemma 2, φ is a pairwise exchange mechanism. Let f be any corresponding function that represents it. From Corollary 2, we know that, φ is also strategy-proof.

Lemmas 3, 4, 17, and 18 together imply the existence of a vector $v \in \mathbb{R}^n$ such that $f(\succ, \succ', a) = v_{rank(\succ,a)} - v_{rank(\succ',a)}$. The additional restrictions on the vector v stem from property (4) in Lemma 3 and the strategy-proofness of the mechanism φ. If the vector v is not sorted, it is easy to find a profile that violates the strategy-proofness of the mechanism. The restriction of $v \in [0, \frac{1}{n(n-1)}]^n$ is needed to ensure that $f(\succ, \succ', a) \in [-\frac{1}{n(n-1)}, \frac{1}{n(n-1)}]$.

Sufficiency. Suppose φ^v is a Rank Exchange mechanism with the corresponding exchange vector v. From Theorem 2, we know that φ^v is strategy-proof and envy-free. We now show that φ^v satisfies the remaining two properties.

- Separability: To see separability, consider any agent $i \in N$, any two preference profiles $\succ = (\succ_k)_{k\in N} \in \mathcal{R}^n$ and $\succ' = (\succ'_k)_{k\in N\setminus\{i\}} \in \mathcal{R}^{n-1}$. From the definition of a Rank Exchange mechanism, we know that

$$\begin{aligned}
\sum_{j\in N\setminus\{i\}} \Big[\varphi_{i,a}((\succ'_j, \succ_{-j})) - \varphi_{i,a}(\succ)\Big] &= \sum_{j\in N\setminus\{i\}} \Bigg[\Bigg(\Bigg(\frac{1}{n} + \sum_{k\in N\setminus\{i,j\}} \Big(v_{rank(\succ_i,a)} - v_{rank(\succ_k,a)}\Big) \\
&\quad + \Big(v_{rank(\succ_i,a)} - v_{rank(\succ'_j,a)}\Big)\Bigg) \\
&\quad - \Bigg(\frac{1}{n} + \sum_{k\in N\setminus\{i\}} \Big(v_{rank(\succ_i,a)} - v_{rank(\succ_k,a)}\Big)\Bigg)\Bigg] \\
&= \sum_{j\in N\setminus\{i\}} \Big[v_{rank(\succ_i,a)} - v_{rank(\succ'_j,a)}\Big] \\
&\quad - \sum_{j\in N\setminus\{i\}} \Big[v_{rank(\succ_i,a)} - v_{rank(\succ_j,a)}\Big]
\end{aligned}$$

$$= \left(\frac{1}{n} + \sum_{j \in N \setminus \{i\}} \left[v_{rank(\succ_i, a)} - v_{rank(\succ'_j, a)}\right]\right)$$
$$- \left(\frac{1}{n} + \sum_{j \in N \setminus \{i\}} \left[v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)}\right]\right)$$
$$= \varphi_{i,a}((\succ_i, \succ'_{-i})) - \varphi_{i,a}(\succ)$$

- Neutrality: This property directly follow from definition of φ^v since the vector v is independent of the identity of the objects. □

In Appendices A.4 and A.5, we discuss independence of axioms for the characterization result. In Appendix A.4, we give an example of a mechanism which is neutral and envy-free but violates separability to demonstrate that separability is independent of the other properties. Similarly, Appendix A.5 provides a mechanism that is envy-free and separable, but violates neutrality. Finally, to see that envy-freeness is independent of neutrality and separability, consider any Rank Exchange mechanism φ^v, except that the coordinates of the vector v are now sorted in an increasing order, with the first coordinate being the smallest. The mechanism resulting from this shuffled vector still continues to be neutral and separable. However, the mechanism is no longer envy-free.

2.6 Conclusion

In this work, we focused on strategy-proof and envy-free random assignment mechanisms for the house allocation problem. We first defined a very weak notion of efficiency, called contention-free efficiency, and showed that strategy-proofness and envy-freeness must sacrifice efficiency even in this weak sense.

We then designed a new family of mechanisms called Rank Exchange mechanisms that are strategy-proof and envy-free and stochastically dominate the equal division mechanism. We also provide an axiomatic characterization of this class and show that all neutral, strategy-proof, envy-free, and separable mechanisms are Rank Exchange mechanisms.

Given the trade-offs between incentives, fairness and efficiency, characterizing the Pareto efficient frontier within the class of strategy-proof and envy-free mechanisms is an important question. Indeed, by imposing *separability* and thereby restricting the transfer of fractional shares of objects to depend only on the preferences of the agents involved in the exchange, the mechanisms we characterize do not allow us to reach this frontier. For instance, the mechanism in Appendix A.4 is an example of a mechanism that is strategy-proof and envy-free and stochastically dominates the closest Rank Exchange mechanism. We consider our work as only the first step towards characterizing the Pareto efficient frontier of strategy-proof and envy-free mechanisms. We expect studying more general transfer functions that take into account preferences of other agents not involved in the exchange to yield insight into more efficient strategy-proof and envy-free mechanisms.

Chapter 3

Constrained Serial Rule on the Full Preference Domain

3.1 Introduction

In this chapter, we focus on allocating a number of indivisible objects among a set of agents in a *fair* and *efficient* manner. Classical models for object allocation without monetary transfers such as the house allocation model (Hylland and Zeckhauser 1979) are now well understood and mechanisms that guarantee any maximal subset of attainable properties along the dimensions of *fairness*, *efficiency*, and *incentive compatibility* are well known. However, many applications in practice impose additional constraints on the set of allowed allocations and designing allocation mechanisms in such constrained settings remains an ongoing challenge. In this work, we present a novel *Constrained Serial Rule* mechanism that always obtains efficient and fair outcomes for object allocation under a large class of general constraints.

In the context of house allocation, Bogomolnaia and Moulin (2001) introduced the notion of *ordinal efficiency*. A random assignment is said to be ordinally efficient if there exists no other assignment that stochastically dominates it. Indeed, they showed that the well-studied *Random Priority* mechanism that orders agents in a uniformly random order and then allocates each agent her most preferred object from the set of remaining objects is not ordinally efficient and only satisfies a weaker notion of ex-post efficiency. The notion of *envy-freeness* that requires each agent to prefer her own allocation to anyone else's allocation is often considered as the gold standard of fairness in many different settings such as resource allocation (Foley 1967), cake-cutting (Robertson and Webb 1998), and rent division (Edward Su 1999). In the context of random assignment mechanisms, this property is formulated using the first-order stochastic dominance relation. In their seminal work, Bogomolnaia and Moulin 2001 introduced the *Probabilistic Serial* mechanism that always produces ordinally efficient and envy-free outcomes. The probabilistic serial mechanism can be described as follows. At time zero, each agent begins "eating" her most preferred object. An object becomes unavailable once the total time spend by agents eating it equals one. Once an object becomes unavailable, all agents that were eating it, switch to eating their most preferred object among the ones still available. Finally, the probability that an agent receives some object is the time spent by that agent to eat the object.

Unfortunately, the probabilistic serial mechanism assumes that agents have strict preferences over objects, which is a fairly restrictive assumption in practice. Indeed, as discussed by Katta and Sethuraman (2006) and Erdil and Ergin (2017), ties in preferences are widespread in many practical applications. For example, agents may treat some objects as identical. Even when objects are all distinct, evaluating and ranking all objects may be computationally prohibitive, and agents may only reveal coarse rankings with indifferences. In their influential paper, Katta and Sethuraman (2006) present the *Extended Probabilistic Serial* mechanism that generalizes the probabilistic serial mechanism to the full preference domain and retains the desirable properties of ordinal efficiency and envy-freeness.

Widespread applicability of these mechanisms is hindered by the fact that these mechanisms assume that every random assignment is feasible. In a large number of practical applications, legal and policy requirements necessitate studying mechanisms where the set of feasible assignments is constrained in some way. For example, in the course allocation problem, there are often requirements on the minimum (and maximum) number of students assigned to a course. Similarly, in school choice applications, it is required to find assignments that maintain a minimum level of diversity (Ehlers et al. 2014). In resident matching, it is often necessary for the allocation of doctors to hospitals to satisfy geographic constraints (Kamada and Kojima 2015). In the refugee resettlement problem, objects represent settlement facilities and a feasible assignment must be such that the total demands of all agents assigned to a facility must be met by the total supply of resources at that facility (Delacrétaz, Kominers, and Teytelboym 2016). Similarly, in kidney matching applications (Roth, Sönmez, and Ünver 2005), blood-type compatibility imposes constraints on feasible matchings. In this chapter, we study the object allocation problem with arbitrary linear constraints on the set of feasible probabilistic assignments. Following the work of Balbuzanov (2019), this formalization supports an arbitrary set of constraints on ex-post allocations.

3.1.1 Our Contributions

Our primary contribution is to generalize the probabilistic serial mechanism to the full preference domain and support arbitrary linear constraints on the set of feasible random assignments via a new mechanism called the *Constrained Serial Rule*. Our mechanism is computationally efficient and only requires a running time that is polynomial in the number of constraints, agents, and objects. For the classical unconstrained house allocation setup on the full preference domain, our mechanism coincides with the extended probabilistic serial mechanism of Katta and Sethuraman (2006). Various generalizations of the probabilistic serial mechanism have been proposed for multi-unit demand (Kojima 2009), specific type of constraints such as bi-hierarchical constraints (Budish et al. 2013), type-dependent distributional constraints (Ashlagi, Saberi, and Shameli 2020), combinatorial demand (Nguyen, Peivandi, and Vohra 2016), property rights with individual rationality (Yılmaz 2010), and even arbitrary constraints on the ex-post allocations (Balbuzanov 2019). Our *constrained serial rule* unifies this literature and provides a common generalization of all these mechanisms and also provides an extension to the full preference domain.

We show that the *constrained serial rule* maintains the desirable efficiency and fairness properties of the probabilistic serial mechanism even in our general constrained setting. In particular, our mechanism is constrained ordinally efficient. While it is easy to observe that arbitrary constraints rule out the existence of envy-free mechanisms, we show that the *constrained serial rule* maintains a compelling notion of fairness. Intuitively, we say agents i and j are of the same *type* if the constraint structure does not distinguish between the two agents. We show that the constrained serial rule mechanism guarantees envy-freeness among any pair of agents of the same type. However, our mechanism is not strategyproof or even weak-strategyproof. This is unsurprising since even in the unconstrained setting, weak-strategyproofness is incompatible with ordinal efficiency and envy-freeness on the full preference domain (Katta and Sethuraman 2006).

3.1.2 Other Related Work

There is a growing body of literature on assignment and matching mechanisms subject to constraints. Several studies have considered floor and ceiling constraints in the context of controlled school choice, college admissions, and affirmative action (Kojima 2012; Kominers and Sönmez 2013; Hafalir, Yenmez, and Yildirim 2013; Hamada, Iwama, and Miyazaki 2016; Fragiadakis and Troyan 2017; Fleiner and Kamiyama 2016; Goto et al. 2015; Westkamp 2013; Ehlers et al. 2014; Echenique and Yenmez 2015; Biró et al. 2010; Ashlagi, Saberi, and Shameli 2020). Echenique, Miralles, and Zhang 2019 consider arbitrary ex-post constraints as in Balbuzanov 2019 and provide a pseudo-market equilibrium solution that is constrained ex-ante efficient and fair. In independent and ongoing work, Aziz and Brandl (2020) also generalize the probabilistic serial mechanism to allow for arbitrary constraints. However, by restricting ourselves to linear constraints, we are able to provide stronger fairness guarantees.

Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2014) have studied the implementability of random matching mechanisms. Budish et al. (2013) identify bi-hierarchical constraint structures as a necessary and sufficient condition for implementing a random assignment using lottery of feasible assignments. They also provide a generalization of the (extended) probabilistic serial mechanism in the case when there are no floor constraints. Indeed, our mechanism is able to accommodate bi-hierarchical constraint inequalities in the presence of non-zero floor constraints. Akbarpour and Nikzad (2014) consider more general constraints beyond bi-hierarchical structures and show how feasible random assignments can be implemented approximately. Pycia and Ünver (2015) provide sufficient conditions on the properties of random mechanisms that continue to be satisfied on the deterministic mechanisms when random mechanisms are decomposed as a lottery over these deterministic mechanisms. While the focus of our chapter is not on implementability, we provide a small discussion of this in Section 3.3.4.

3.2 Model and Preliminaries

We consider a finite set N of agents and a finite set O of objects. Let $n = |N|$ be the number of distinct agents and $\rho = |O|$ be the number of distinct objects. Every agent has a unit* demand and each object $o \in O$ is supplied in $q_o \in \mathbb{N}$ copies. When objects are scarce, we can include the null object, $\emptyset$, in the set O, which is supplied in a quantity sufficient to meet the demand of all agents. That is, $q_\emptyset \geq |N|$. We can, therefore, without loss of generality, assume that $\sum_{o\in O} q_o \geq n$. Each agent $i \in N$ has a preference relation $\succeq_i$ on the set of objects in O. The preference $\succeq_i$ is assumed to be complete and transitive. In particular, we allow agents to be indifferent between any pair of objects in O. Let $E(\succeq)$ be the number of indifference classes within the preference $\succeq$. For any $\ell \leq E(\succeq_i)$, let $T_i(\ell)$ be the set of objects in the first ℓ indifference classes of the preference $\succeq_i$. A set of individual preferences of all agents constitutes a preference profile $\succeq = (\succeq_i)_{i\in N}$. Let $\mathcal{R}$ denote the set of all complete and transitive relations on O and $\mathcal{R}^n$ be the set of all possible preference profiles.

A *random assignment* of objects to agents is given by a vector $\mathbf{x} = (x_{i,o})_{i\in N, o\in O} \in [0,1]^{n\rho}$ such that

$$\sum_{o\in O} x_{i,o} = 1 \qquad \forall i \in N$$
$$\sum_{i\in N} x_{i,o} \leq q_o \quad \forall o \in O$$

In assignment $\mathbf{x}$, every agent i's *allocation* is given by the sub-vector $\mathbf{x}_i = (x_{i,o})_{o\in O}$, where the quantity $x_{i,o}$ is interpreted to be the probability with which object o is assigned to agent i. Let $x_i(S) = \sum_{o\in S} x_{i,o}$ be agent i's cumulative allocation for the set of objects in set S. An assignment is *deterministic* whenever $x_{i,o} \in \{0,1\}$, i.e, every agent is assigned a single object with probability 1. Let $\mathcal{D}$ denote the set of all deterministic assignments and $\Delta\mathcal{D}$ denote the set of all random assignments. A random assignment *mechanism* is a mapping, $\varphi : \mathcal{R}^n \to \Delta\mathcal{D}$, that associates each preference profile $\succeq \in \mathcal{R}^n$ with some random assignment $\mathbf{x} \in \Delta\mathcal{D}$.

We extend agents' preferences from the set of objects to the set of random allocations using the stochastic dominance relation. Given two random assignments $\mathbf{x}$ and $\mathbf{y}$, allocation $\mathbf{x}_i$ *stochastically dominates* allocation $\mathbf{y}_i$ with respect to $\succeq_i$, denoted by $\mathbf{x}_i sd(\succeq_i)\mathbf{y}_i$, if and only if $\sum_{o'\succeq_i o} x_{i,o'} \geq \sum_{o'\succeq_i o} y_{i,o'}$ for all $o \in O$. If the inequality is strict for some $o \in O$, then $\mathbf{x}_i$ *strictly* stochastically dominates $\mathbf{y}_i$, in which case we denote it by $\mathbf{x}_i sd(\succ_i)\mathbf{y}_i$.

We now introduce a general class of constraints into our model. At any given preference profile, we assume that the set of feasible random assignments can be described as a convex polytope. Formally, at preference profile $\succeq$, the set of feasible random assignments $\Delta\mathcal{C}(\succeq)$ is parameterized by a matrix $A = [a^c_{i,o}]_{1\leq c\leq m, \{i,o\}\in N\times O} \in \mathbb{R}^{m\times n\rho}$ and a vector $\boldsymbol{b} = [b^c]_{1\leq c\leq m} \in \mathbb{R}^m$, where c is a generic constraint and m is the number of constraints, and is defined as:

$$\Delta\mathcal{C}(\succeq) = \{\mathbf{x} \in \Delta\mathcal{D} \mid A\mathbf{x} \leq \boldsymbol{b}\}$$

*Our results also generalize to the case when all agents demand $d \geq 1$ objects.

We assume that at each preference profile $\succeq$, the set of feasible random assignments is non-empty. That is, $\Delta\mathcal{C}(\succeq) \neq \emptyset$. Such a formulation of the constraints enables us to apply our model to many different specific applications that we describe in Section 3.4. Let $\Delta\mathcal{C} = \{\Delta C(\succeq)\}_{\succeq \in \mathcal{R}^n}$ be the collection of constraint polytopes for all preference profiles. Given a collection of constraints $\Delta\mathcal{C}$, a mechanism is *feasible* if at every preference profile $\succeq$, $\varphi(\succeq) \in \Delta\mathcal{C}(\succeq)$.

We next define the normative properties of efficiency and fairness in the presence of constraints.

Definition 6 (Constrained Ordinal Efficiency). *A random assignment* $\mathbf{x}$ *is* constrained ordinally efficient *at a preference profile* $\succeq$ *and constraint set* $\Delta\mathcal{C}(\succeq)$ *if there does not exist another random assignment* $\mathbf{x}' \in \Delta\mathcal{C}(\succeq)$ *such that* $\mathbf{x}'_i sd(\succeq_i)\mathbf{x}_i$ *for all* $i \in N$, *with* $\mathbf{x}'_i sd(\succ_i)\mathbf{x}_i$ *for at least one* $i \in N$. *A mechanism* φ *is* constrained ordinally efficient *if for every preference profile* $\succeq$, $\varphi(\succeq)$ *is constrained ordinally efficient.*

The classic notion of fairness requires that no agent should envy the allocation received by some other agent. When faced with arbitrary feasibility constraints, it is easy to see that one cannot guarantee the existence of envy-free assignments. Therefore, we restrict ourselves to fairness comparisons between agents that belong to the same type. For a given constraint matrix A, we say agents i and j belong to the same type if for every object o, the variables $x_{i,o}$ and $x_{j,o}$ have the same coefficients in every constraint in A.

Definition 7 (Agent Type). *Let* $\Delta\mathcal{C}(\succeq) = \{\mathbf{x} \in \Delta\mathcal{D} \mid A\mathbf{x} \leq \boldsymbol{b}\}$ *where* $A = [a^c_{i,o}]_{1 \leq c \leq m, \{i,o\} \in N \times O}$ *denote the constraint set a preference profile* $\succeq$. *Two agents* i *and* j *are said to be of the* same type *at this profile, if for every constraint* $1 \leq c \leq m$, *and for every object* $o \in O$, *we have* $a^c_{i,o} = a^c_{j,o}$.

Definition 8 (Envy-freeness among agents of the same type). *A random assignment* $\mathbf{x}$ *is* envy-free among agents of the same type *if for every pair of agents* $i, j \in N$ *of the same type,* $\mathbf{x}_i sd(\succeq_i)\mathbf{x}_j$. *A mechanism* φ *is* envy-free for agents of the same type *if for every preference profile* $\succeq$, $\varphi(\succeq)$ *is envy-free among agents of the same type.*

3.3 The Constrained Serial Rule

We first give a brief intuitive description of the classical probabilistic serial mechanism (Bogomolnaia and Moulin 2001) in the simple house allocation model†. The mechanism is best described as a continuous time procedure for $t \in [0, 1]$: at each infinitesimal time interval $[t, t + dt)$, each agent i consumes dt amount of her most preferred object among the set of objects currently available. When this procedure terminates, the probability that an agent is assigned an object is given by the fraction of the object consumed by the agent.

†Agents have strict preferences over objects and there are no additional constraints on the assignment.

While attempting to extend the probabilistic serial mechanism to our general model on the full preference domain and with arbitrary constraints on the eventual random assignment, one faces two key challenges. First, when agents have strict preferences, each agent at any point in time has a unique most preferred object and hence the mechanism can simply allocate that object to the agent. On the other hand, when agents are indifferent between two or more objects, the mechanism can no longer uniquely identify an object to assign. Katta and Sethuraman (2006) deal with this difficulty by constructing a flow graph where every agent points to her set of most preferred objects and then using a parametric flow formulation to find a set of *bottleneck* agents and objects. Intuitively, the set of bottleneck agents are those that compete the most among themselves and the bottleneck objects are those desired by bottleneck agents. Once the bottleneck agents have been identified, the extended probabilistic serial mechanism allocates all bottleneck objects among these agents uniformly. As in the classic probabilistic serial rule, the mechanism then proceeds by each bottleneck agent simply pointing to her next most preferred object. A key observation is that the (extended) probabilistic serial mechanism attempts to assign each agent her most preferred object for as long as possible. In fact, as observed by Bogomolnaia (2015), one can provide a welfarist interpretation of the probabilistic serial rule as follows: for any agent i, let $x_i(\ell)$ be the total probability share of objects that agent i receives for her top ℓ indifference classes; then the probabilistic serial mechanism leximin maximizes the vector of all such shares $(x_i(\ell))_{i\in N,k\leq E(\succeq_i)}$. We crucially use this observation in our *constrained serial rule* mechanism.

The second challenge arises due to the presence of arbitrary constraints on the space of feasible random assignments. We observe that at any step of the mechanism, the (extended) probabilistic serial mechanism always assigns each agent her most preferred object (at that time). In the presence of constraints, however, it is essential to not allow agents to obtain their most preferred object if such an allocation leads to infeasibility. More precisely, the mechanism needs to *look ahead* in time as it builds up a partial allocation to ensure that there is at least one way to extend the partial allocation to a feasible assignment. Our *constrained serial rule* mechanism uses a linear program that explicitly accounts for all constraints and at every step maintains a feasible solution.

We now describe the linear program that is used crucially by the mechanism. Let $S \subseteq N$ denote any subset of agents and let $\ell_i \in \{1, 2, \ldots, E(\succeq_i)\}$ for each agent $i \in N$ denote an indifference threshold. Let F be a set of triples that denotes prior promised assignments. Formally, a triple $(i, \ell, \gamma) \in F$ indicates that agent i must receive a total probability share of at least γ from her top ℓ indifference classes. The linear program $LP(S, F, (\ell_i)_{i\in N})$ specified in Figure 3.1 finds a random assignment $\mathbf{x} \in \mathbb{R}_+^{n\rho}$ that satisfies all constraints specified by F in addition to the imposed feasibility constraints and maximizes the total probability share that each agent $i \in S$ receives from her top ℓ_i indifference classes. The variables h_i for each $i \in N$ represent the total probability share received by agent i for objects in her top ℓ_i indifference classes. Constraints (3.1) and (3.2) enforce the requirement that the linear program maximizes $\min_{i\in S} h_i$. Constraints (3.3) and (3.4) enforce that the obtained random assignment is feasible, and finally constraint (3.5) requires the assignment to be consistent

$$
\begin{aligned}
LP(S, F, (\ell_i)_{i \in S}) = \quad & \underset{\mathbf{x},\mathbf{h},\lambda}{\text{maximize}} & \lambda & & \\
& \text{s.t.} & h_i \geq \lambda & \quad \forall i \in S & (3.1)\\
& & \sum_{o \in T_i(\ell_i)} x_{i,o} \geq h_i & \quad \forall i \in N & (3.2)\\
& & A\mathbf{x} \leq \boldsymbol{b} & & (3.3)\\
& & \mathbf{x} \in \Delta\mathcal{D} & & (3.4)\\
& & \sum_{o \in T_i(\ell)} x_{i,o} \geq \gamma & \quad \forall (i, \ell, \gamma) \in F & (3.5)\\
& & \mathbf{h}, \lambda \geq 0 & &
\end{aligned}
$$

Figure 3.1: Linear Program used by the Constrained Serial Rule

with the requirements specified by the triples in F.

3.3.1 Mechanism

We are now ready to describe the constrained serial rule mechanism formally. The mechanism proceeds in multiple rounds. We initialize $F^1 = \emptyset$ and for each agent $i \in N$, we initialize $\ell_i^1 = 1$. Intuitively, ℓ_i^t denotes the threshold indifference class for agent i in round t. In other words, in round t, we consider the total probability share of objects in $T_i(\ell_i^t)$ assigned to agent i. Let h_i^t denote the total probability share of objects in the top ℓ_i^t indifference classes assigned to agent i. We use the linear program described in Figure 3.1 to find a feasible random assignment such that $\min_i h_i^t$ is maximized. The mechanism then identifies a set B^t of *bottleneck* agents. Intuitively, these are the set of agents who are responsible for the LP objective in this round to be only λ^t. Since we are dealing with arbitrary linear constraints, our definition of *bottleneck* agents needs to be more subtle than that of Katta and Sethuraman (2006). We define B_t to be a *minimal* set of agents such that solving the linear program while only attempting to maximize the utility of agents in that set also yields the same objective value of λ^t. Our definition of bottleneck agents is central to the validity of the mechanism as well as its efficiency and fairness properties. Finally, once the bottleneck set of agents has been identified, we update F^t to guarantee that in future rounds each agent $i \in B^t$ obtains at least the promised λ^t probability share from her top ℓ_i^t indifference classes and then increment the threshold ℓ_i^t for all such agents. The mechanism then proceeds to the next round and the process continues until every agent receives a total probability share of 1. Algorithm 1 provides a complete formal description of the algorithm.

We provide a simple example run of Algorithm 1 on a constrained allocation problem to illustrate our constrained serial rule.

Algorithm 1: The Constrained Serial Rule

Initialize:
$t \leftarrow 1$;
$\ell_i^t \leftarrow 1, \quad \forall i \in N$;
$F^t \leftarrow \emptyset$;
for $t = 1, 2, \ldots$ **do**
  $(\mathbf{x}^t, \mathbf{h}^t, \lambda^t) \leftarrow LP(N, F^t, (\ell_i^t)_{i \in N})$;
  if $\lambda^t = 1$ **then**
    $\mathbf{x} \leftarrow \mathbf{x}^t$;
    terminate;
  else
    Find a minimal set B^t such that $LP(B^t, F^t, (\ell_i^t)_{i \in N})$ has objective value λ^t;
    Update $F^{t+1} = F^t \cup \{(i, \ell_i^t, \lambda^t) \mid i \in B^t\}$;
    Update $\ell_i^{t+1} = \begin{cases} \ell_i^t + 1 & \forall i \in B^t \\ \ell_i^t & \text{otherwise} \end{cases}$
  end
end

Example 1. *Consider an allocation problem with three agents $N = \{1, 2, 3\}$ and three objects $O = \{a, b, c\}$. Our goal is to obtain an assignment that satisfies the usual bistochastic constraints, i.e. $\sum_{o \in O} x_{i,a} = 1 \ \forall i \in N$ and $\sum_{i \in N} x_{i,a} = 1 \ \forall o \in O$. In addition, there are two additional constraints as follows: $x_{1,a} + x_{2,a} \leq 0.5$ and $x_{1,c} + x_{2,c} \geq 0.5$. The agents' preferences are given as follows.*

$$\begin{array}{llll} \succ_1: & a & b & c \\ \succ_2: & \{a, b\} & & c \\ \succ_3: & c & b & a \end{array}$$

We illustrate how our mechanism works through this example. In the first round, we initialize $\ell_1^1 = \ell_2^1 = \ell_3^1 = 1$ and solve the linear program to find a feasible solution that maximizes $\lambda^1 = \min\{x_{1,a}, x_{2,a} + x_{2,b}, x_{3,c}\}$. With the given constraints, one potential optimum solution assigns $x_{1,a} = x_{2,b} = x_{3,c} = 0.5$ to obtain $\lambda^1 = 0.5$. We then proceed to find the set of bottleneck agents. In this example, either of the singleton sets with agents 1 or 3 could be the bottleneck set. This is because the constraint $x_{1,a} + x_{2,a} \leq 0.5$ prevents agent 1 from receiving a larger probability share of object a. Similarly, the constraint $x_{1,c} + x_{2,c} \geq 0.5$ prevents agent 3 from receiving a larger probability share of object c. Suppose we select agent 1 to be the bottleneck agent. Then, we increment $\ell_1^2 = 2$ and maintain $\ell_2^2 = \ell_3^2 = 1$. We also set $F^2 = (1, 1, 0.5)$ to signify that agent 1 must continue to obtain 0.5 amount of her top choice.

In the second round, we again solve the linear program to find a feasible solution that now maximizes $\lambda^2 = \min\{x_{1,a} + x_{1,b}, x_{2,a} + x_{2,b}, x_{3,c}\}$. As described earlier, agent 3 is unable *to receive more than* 0.5 *amount of object c in any feasible solution and hence we obtain* $\lambda^2 = 0.5$. *In this case, agent* 3 *is the unique bottleneck agent and as earlier we increment her indifference threshold and add the triple* $(3, 1, 0.5)$ *to* F^3. *Similarly, in the third round, we solve the linear program to find a feasible solution that maximizes* $\lambda^3 = \min\{x_{1,a} + x_{1,b}, x_{2,a} + x_{2,b}, x_{3,c} + x_{3,b}\}$. *However, the constraints* $x_{1,a} + x_{2,a} \leq 0.5$ *and* $x_{1,a} + x_{2,a} + x_{3,a} = 1$ *together imply that* $x_{3,a} \geq 0.5$ *and thus* $x_{3,c} + x_{3,b} \leq 0.5$. *So yet again, we have* $\lambda^3 = 0.5$ *and agent* 3 *is the bottleneck agent and we increment her indifference threshold and add the triple* $(3, 2, 0.5)$ *to* F^4.

In the fourth round, the linear program attempts to find a feasible solution that respects all the constraints in F^4 *and maximizes* $\lambda^4 = \min\{x_{1,a} + x_{1,b}, x_{2,a} + x_{2,b}, x_{3,c} + x_{3,b} + x_{3,a}\}$. *In this case, one potential optimum solution assigns* $x_{1,a} = 0.5, x_{1,b} = 0.25, x_{2,b} = 0.75, x_{3,c} = 0.5, x_{3,a} = 0.25$ *to obtain* $\lambda^4 = 0.75$. *Since the constraint* $x_{1,a} + x_{2,a} = 0.5$ *is already tight and object b has been fully allocated, both agents* 1 *and* 2 *are in the bottleneck set in this round. We increment the indifference threshold of both the agents to obtain* $\ell_1^5 = 3$ *and* $\ell_2^5 = 2$ *and add the triples* $(1, 2, 0.75)$ *and* $(2, 1, 0.75)$ *to* F^5.

Finally, in the fifth round, we again solve the linear program to find a feasible solution that respects all the constraints in F^5 *and maximizes* $\lambda^5 = \min_{i \in N}\{x_{i,a} + x_{i,b} + x_{i,c}\}$. *Since any feasible solution satisfies* $x_{i,a} + x_{i,b} + x_{i,c} = 1$, *we obtain* $\lambda^5 = 1$ *and the mechanism terminates. The outcome of the mechanism is any feasible solution that satisfies all constraints in* F^5. *For this example, the unique such solution is given by the following random assignment.*

	a	b	c
1	0.5	0.25	0.25
2	0	0.75	0.25
3	0.5	0	0.5

3.3.2 Properties

We first show that the constrained serial rule presented in Algorithm 1 is well-defined and always produces a feasible random assignment.

Proposition 3. *Algorithm 1 terminates and always produces a feasible random assignment.*

Proof. We first observe that in any step t, the LP always has a feasible solution. This is because, for any $t > 1$, the solution $(\mathbf{x}^{t-1}, \mathbf{h}^{t-1}, \lambda^{t-1})$ from the previous iteration continues to be a feasible solution; whereas for $t = 1$, the existence of a feasible solution is guaranteed since the assignment constraints are assumed to be satisfiable.

In any step t, it is easy to see that the bottleneck set of agents B^t is guaranteed to exist by observing that the set of all agents N satisfies the required constraint by definition. Further, we observe that any agent $i^* \in N$ such that $\ell_{i^*}^t = E(\succeq_{i^*})$ does not appear in the bottleneck set B^t. This is because, for any such agent $i^* \in N$, the linear programs $LP(B^t, F^t, (\ell_i^t)_{i \in N})$

and $LP(B^t \setminus \{i^*\}, F^t, (\ell_i^t)_{i\in N})$ are actually identical and by definition B^t is the *minimal* set that satisfies the condition. Thus in any step t, we increment the indifference threshold of at least one agent. Finally, by definition of feasible assignment, when $\ell_i^t = E(\succeq_i)$, $\forall i \in N$, we may set $h_i^t = \sum_{o\in O} x_{i,o}^t = 1$ for all $i \in N$ and hence obtain $\lambda^t = 1$ and the algorithm terminates. Since any agent $i \in N$ has $E(\succeq_i) \leq \rho$, the mechanism terminates in at most $n\rho$ rounds.

Finally, since the linear program in Figure 3.1 explicitly maintains the constraints $A\mathbf{x} \leq \boldsymbol{b}$ and $\mathbf{x} \in \Delta\mathcal{D}$, the outcome is guaranteed to be feasible. □

We then proceed to show two technical lemmas. The first lemma shows that even though we allow arbitrary linear constraints on the random assignment, the projection of the feasibility polytope on the $\mathbf{h}$ and λ variables satisfies a desirable monotonicity property. Specifically, the constraints on the $\mathbf{h}$ variables can be expressed as a set of linear upper-bound constraints with only non-negative coefficients.

Lemma 5. *Fix any step t of the algorithm. Let P be the polytope defined by the constraints in $LP(B^t, F^t, (\ell_i^t)_{i\in N})$ and let $Q = \{(\mathbf{h}, \lambda) \mid \exists(\mathbf{x}, \mathbf{h}, \lambda) \in P\}$ be its projection. Then there exists a non-negative matrix $\tilde{A}$ and a non-negative vector $\tilde{b}$ such that*

$$Q = \begin{cases} h_i \geq \lambda, \quad \forall i \in B^t \\ \tilde{A}\mathbf{h} \leq \tilde{\boldsymbol{b}} \\ \mathbf{h}, \lambda \geq 0 \end{cases}$$

Proof. By the Fourier-Motzkin theorem, we know that Q is also a polytope that can be obtained from P using Fourier-Motzkin elimination. Since the only constraints involving λ in P do not contain any $\mathbf{x}$ variables, those constraints remain unchanged in polytope Q. Let $\tilde{A} = [\tilde{a}_i^c]$ and $\tilde{b} = [\tilde{b}^c]$ denote the minimal linear constraints on the variables $\mathbf{h}$ obtained after Fourier-Motzkin elimination. We now need to show that $\tilde{A}$ and $\tilde{b}$ are non-negative.

We first observe that the polytope Q is downward closed on the $\mathbf{h}$ variables, i.e. for any $\mathbf{h}' \leq \mathbf{h}$, if $(\mathbf{h}, \lambda) \in Q$ then there exists a $\lambda' \leq \lambda$ such that $(\mathbf{h}', \lambda') \in Q$. This is because, by definition, if $(\mathbf{h}, \lambda) \in Q$ then there exists an $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{h}, \lambda) \in P$. Further, observing the polytope P (refer to Figure 3.1), it is clear that $(\mathbf{x}, \mathbf{h}', \lambda')$ also belongs to P where $\lambda' = \min_{i\in B^t} h_i'$, and thus $(\mathbf{h}', \lambda') \in Q$.

Now, suppose for contradiction that there exists a constraint c such that the entry $\tilde{a}_j^c < 0$ for some $j \in N$. Since this constraint is not redundant, there exists a vector $\tilde{\mathbf{h}} \in \mathbb{R}_+^n$ such that the cth constraint is the only binding constraint, i.e., $\sum_i \tilde{a}_i^c \tilde{h}_i = \tilde{b}^c$ and further $\tilde{h}_i > 0$ for all $i \in N$. Define $\mathbf{h}' \in \mathbb{R}_+^n$ as $h_i' = \tilde{h}_i$, $\forall i \in N \setminus \{j\}$ and $h_j' = 0$. By definition $\mathbf{h}' < \tilde{\mathbf{h}}$, and yet $\sum_i \tilde{a}_i^c h_i' > \tilde{b}^c$ and thus $\mathbf{h}' \notin Q$ which is a contradiction.

Finally if the matrix $\tilde{A}$ is non-negative and the vector $\tilde{b}$ has a negative entry, then the polytope Q must be empty. Since Q is guaranteed to be non-empty, we must have that $\tilde{b}$ is non-negative. □

The following lemma demonstrates the importance of our definition of *bottleneck* agents. Informally, it states that if B^t is the set of bottleneck agents at some step t of the algorithm, then no agent in B^t can obtain a higher total allocation for her top ℓ_i^t indifference classes without hurting some other agent in B^t. This lemma is crucial to proving the constrained serial rule produces constrained ordinally efficient outcomes, and also to demonstrate its fairness properties.

Lemma 6. *Fix any step t of the algorithm and let $(\mathbf{x}^t, \mathbf{h}^t, \lambda^t)$ denote an optimal solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. Then $\nexists\ \mathbf{y} = (y_{i,o})_{i\in N, o\in O} \in \mathbb{R}_+^{n\rho}$ such that*

1. *$A\mathbf{y} \le b$, $\mathbf{y} \in \Delta\mathcal{D}$*
2. *$y_i(T_i(\ell)) \ge \gamma,\ \forall (i,\ell,\gamma) \in F^t$*
3. *$y_i(T_i(\ell_i^t)) \ge \lambda^t,\ \forall i \in B^t$ with atleast one strict inequality.*

Proof. Let P be the polytope defined by the constraints in $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. Let $Q = \{(\mathbf{h}, \lambda) \mid \exists (\mathbf{x}, \mathbf{h}, \lambda) \in P)\}$ be the projection of P on the $\mathbf{h}$ and λ variables. By Lemma 5, there exists a non-negative matrix $\tilde{A} = [\tilde{a}_i^c]$ and a non-negative vector $\tilde{b}$ such that

$$Q = \begin{cases} h_i \ge \lambda, & \forall i \in B^t \\ \tilde{A}\mathbf{h} \le \tilde{\boldsymbol{b}} \\ \mathbf{h}, \lambda \ge 0 \end{cases}$$

By definition, $(\mathbf{x}^t, \mathbf{h}^t, \lambda^t)$ is an optimal solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. Let $\tilde{\mathbf{h}}$ be defined as $\tilde{h}_i = \lambda^t$ for all $i \in B^t$ and $\tilde{h}_i = 0$ for all $i \notin B^t$. We now have $(\mathbf{x}^t, \tilde{\mathbf{h}}, \lambda^t)$ is also an optimal solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. Thus, $(\tilde{\mathbf{h}}, \lambda^t)$ does not lie in the interior of the polytope Q. Thus at least one of the constraints $\tilde{A}\mathbf{h} \le \tilde{b}$ must be tight at this point, i.e., there exists a constraint c such that $\sum_{i\in B^t} \tilde{a}_i^c \tilde{h}_i = \tilde{b}^c$. We now consider two cases.

Case 1: $\tilde{a}_i^c > 0$ for all $i \in B^t$. Suppose for contradiction that there exists a $\mathbf{y}$ that satisfies the premises of the lemma. For any agent $i \in N$, let $h_i' = y_i(T_i(\ell_i^t))$, so we have $h_i' \ge \lambda^t$ for all $i \in B^t$ with at least one strict inequality. Thus since $(\mathbf{y}, \mathbf{h}', \lambda^t)$ is a feasible solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$, we have $(\mathbf{y}, \mathbf{h}', \lambda^t) \in P$. By definition of Q, we have $(\mathbf{h}', \lambda^t) \in Q$. However, we have

$$\sum_{i\in B^t} \tilde{a}_i^c h_i' > \sum_{i\in B^t} \tilde{a}_i^c \lambda^t = \sum_{i\in B^t} a_i^c \tilde{h}_i = \tilde{b}^c$$

which is a contradiction to the statement that $(\mathbf{h}', \lambda^t) \in Q$.

Case 2: $\tilde{a}_j^c = 0$ for some $j \in B^t$. Consider the set $B' = B^t \setminus \{j\}$. We now have $\sum_{i\in B'} \tilde{a}_i^c \tilde{h}_i = \tilde{b}^c$. Let P' be the polytope defined by the constraints in $LP(B', F^t, (\ell_i^t)_{i\in N})$ and Q' be its projection. Since B^t is the minimal bottleneck set, there exists a $(\mathbf{y}', h', \lambda') \in P'$ such that $\lambda' > \lambda^t$ and $h_i' = \lambda',\ \forall i \in B'$. Hence, we have

$$\sum_{i\in B'} \tilde{a}_i^c h_i' > \sum_{i\in B'} \tilde{a}_i^c \lambda^t = \sum_{i\in B'} a_i^c \tilde{h}_i = \tilde{b}^c$$

and thus $(h', \lambda') \notin Q'$. But, this contradicts the fact that $(\mathbf{y}', h', \lambda') \in P'$. $\square$

We are now ready to prove that the *constrained serial rule* algorithm finds a constrained ordinally efficient assignment.

Theorem 5. *For any preference profile $\succeq$ and constraint set $\Delta\mathcal{C}(\succeq)$, the outcome of Algorithm 1 is constrained ordinally efficient.*

Proof. Let $\mathbf{x}$ be the solution of the algorithm for any preference profile $\succeq$ and constraint set $\Delta\mathcal{C}(\succeq)$. In order to prove that $\mathbf{x}$ is constrained ordinally efficient it suffices to show that if there exists $\mathbf{x}' \in \Delta\mathcal{C}(\succeq)$ such that $\mathbf{x}'_i sd(\succeq_i)\mathbf{x}_i$ for all $i \in N$, then $x'_i(T_i(\ell)) = x_i(T_i(\ell))$, for all $i \in N$, for all $\ell \in \{1, 2, ..., E(\succeq_i)\}$. We prove this using contradiction.

For any round t, let $(\mathbf{x}^t, \mathbf{h}^t, \lambda^t)$ denote the optimal solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. For any agent i in the bottleneck set B^t, the mechanism fixes the cumulative allocation received by agent i for her top ℓ_i^t indifference classes. Hence we have $x_i(T_i(\ell_i^t)) = x_i^t(T_i(\ell_i^t)) = \lambda^t$. Towards a contradiction, let t be the first step in the algorithm such that $x'_j(T_j(\ell_j^t)) \neq x_j^t(T_j(\ell_j^t))$ for some agent $j \in B^t$. Since $\mathbf{x}'_i sd(\succeq_i)\mathbf{x}_i$ for all $i \in N$, it must be that $x'_j(T_j(\ell_j^t)) > x_j(T_j(\ell_j^t)) = x_j^t(T_j(\ell_j^t)) = \lambda^t$. Also, we have $x'_i(T_i(\ell_i^t)) \geq x_i(T_i(\ell_i^t)) = \lambda^t$ for all agents $i \in B^t$ with $i \neq j$. Further, since $\mathbf{x}'$ is a feasible random assignment, we have $\mathbf{x}' \in \Delta\mathcal{D}$ and $A\mathbf{x}' \leq \boldsymbol{b}$. Lastly, since round t is the first time $\mathbf{x}'$ differs from $\mathbf{x}$, $x'_i(T_i(\ell)) = x_i(T_i(\ell)) \geq \gamma$, $\forall (i, \ell, \gamma) \in F^t$. This is in direct contradiction to Lemma 6. Therefore, $\mathbf{x}$ is constrained ordinally efficient. $\square$

In the presence of arbitrary constraints, the existence of envy maybe inevitable between any pair of agents if one agent is more constrained than the other. However, as the next theorem shows we can guarantee envy-freeness among any pair of agents of identical type.

Theorem 6. *At any preference profile $\succeq$, the constrained serial rule guarantees envy-freeness among agents of the same type.*

Proof. Let $\mathbf{x}$ denote the outcome of the algorithm. Consider any pair of agents i and j that are of the same type. Suppose for contradiction that agent i envies j, i.e. there exists an indifference class $\ell < E(\succeq_i)$ such that $x_i(T_i(\ell)) < x_j(T_i(\ell))$. Let t denote the step of the algorithm when $\ell_i^t = \ell$ and agent i is in the bottleneck set B^t. Let $(\mathbf{x}^t, \mathbf{h}^t, \lambda^t)$ denote the optimal solution to $LP(B^t, F^t, (\ell_i^t)_{i\in N})$. Since agent i is in the bottleneck set B^t, the mechanism fixes the cumulative allocation received by agent i for her top ℓ indifference classes. Hence we have $x_i(T_i(\ell)) = x_i^t(T_i(\ell)) = \lambda^t$.

Let's consider two cases:

Case 1: Suppose agent $j \in B^t$. We have $x_j(T_j(\ell_j^t)) = x_j^t(T_j(\ell_j^t)) = \lambda^t$. However, as $x_j(T_i(\ell)) > \lambda^t$, there must exist some object $o \in T_i(\ell) \setminus T_j(\ell_j^t)$ such that $x_{j,o} > 0$.

Case 2: Suppose agent $j \notin B^t$. By construction, for any triple $(j, \ell', \gamma) \in F^t$, we have $\ell' \leq \ell_j^{t-1}$ and $x_j(T_j(\ell')) = \gamma \leq \lambda^t$. Thus, there must exist some object $o \in T_i(\ell) \setminus T_j(\ell_j^{t-1})$ such that $x_{j,o} > 0$.

In either case, since $x_i(T_i(\ell)) = \lambda^t < 1$, there exists some object $p \notin T_i(\ell)$ where $x_{i,p} > 0$. Let $0 < \varepsilon < \min\{x_{j,o}, x_{i,p}\}$ be some fixed constant. We can now define a new outcome $\mathbf{y}$ as follows. Let $y_{i',o'} = x_{i',o'}$ for all objects $o' \in O$ and agents $i' \notin \{i, j\}$. For agents $i' \in \{i, j\}$, let $y_{i',o'} = x_{i',o'}$ for all objects $o' \notin \{o, p\}$. Let $y_{i,o} = x_{i,o} + \varepsilon$, $y_{i,p} = x_{i,p} - \varepsilon$, and $y_{j,o} = x_{j,o} - \varepsilon$, $y_{j,p} = x_{j,p} + \varepsilon$. Since agents i and j are of the same type and we have $y_{i,o'} + y_{j,o'} = x_{i,o'} + x_{j,o'}$ for all objects $o' \in O$, we must have $A\mathbf{y} = A\mathbf{x} \leq \mathbf{b}$. Further, by construction we have $\mathbf{y} \in \Delta\mathcal{D}$, i.e. $\mathbf{y}$ is a feasible outcome. In addition, by our choice of objects o and p, we have $y_k(T_k(\ell)) \geq x_k(T_k(\ell)) \geq \gamma$ for all $(k, \ell, \gamma) \in \mathcal{F}^t$ and also $y_k(T_k(\ell_k^t)) \geq x_k(T_k(\ell_k^t))$ for all $k \in B^t$. However, since $y_i(T_i(\ell_i^t)) > x_i(T_i(\ell_i^t)) \geq \lambda^t$, this contradicts Lemma 6. □

3.3.3 Computational Complexity

As shown in Proposition 3, the mechanism terminates in at most $n\rho$ rounds where n and ρ denote the number of distinct agents and objects respectively. In each round t, the algorithm solves one instance of the linear program to compute the value of λ^t. Further, the algorithm needs to find a set of bottleneck agents B^t. We now show that B^t can be found in polynomial time by solving a sequence of at most n linear programs.

We recall that B^t is defined as any minimal set of agents such that the objective value of $LP(B^t, F^t, (\ell_i^t)_{i \in N})$ equals λ^t. Algorithm 2 provides a simple iterative procedure to find such a minimal set. We first initialize B^t to be the set of all agents. In each step, the algorithm considers removing an agent i from B^t. If removing such an agent allows the linear program to obtain a higher objective value, then clearly agent i must belong to the bottleneck set. On the other hand, if the linear program obtains an objective value of only λ^t, then agent i can be safely removed from consideration since by definition $B^t \setminus \{i\}$ is a smaller candidate set.

Algorithm 2: Procedure to find the bottleneck set

```
Input: F^t, (ℓ_i^t)_{i∈N}, λ^t as defined in Algorithm 1
Result: B^t: Set of bottleneck agents
B^t ← N;
for i = 1, 2, ..., n do
    λ ← objective value of LP(B^t \ {i}, F^t, (ℓ_i^t)_{i∈N});
    if λ == λ^t then
        B^t ← B^t \ {i}
    end
end
Return B^t
```

Algorithm 2 terminates in at most n iterations and thus in total each round of the constrained serial rule requires solving at most $(n + 1)$ linear programs. Algorithm 1 can thus be executed in time that is polynomial in the size of the constraints, number of agents and objects.

3.3.4 Implementability

Randomization in object allocation mechanisms is often used as a tool to incorporate fairness from an ex-ante perspective. The outcome of the random assignment mechanism is treated as a probability distribution over deterministic outcomes and an outcome drawn from this distribution is what gets implemented in practice. By the Birkhoff-von Neumann theorem, it is well known that every bistochastic random assignment can be implemented efficiently as a lottery over feasible deterministic assignments, i.e., a deterministic assignment where every agent is assigned one object and each object is assigned to one agent. However, in the presence of arbitrary constraints on the random assignment, such a decomposition into a lottery over deterministic assignments that satisfy those constraints may not exist. The following example illustrates such a situation.

Example 2. *Consider a simple example with one agent, $N = \{1\}$, and three objects $O = \{a, b, c\}$. In the presence of constraints $x_{1,a} + x_{1,b} \leq 2/3$, $x_{1,b} + x_{1,c} \leq 2/3$, and $x_{1,a} + x_{1,c} \leq 2/3$, a potential feasible solution to the random assignment problem is to set $x_{1,a} = x_{1,b} = x_{1,c} = 1/3$. However, it can be readily seen that there exists no deterministic assignment X that satisfies all three constraints and still obtains $X_{1,a} + X_{1,b} + X_{1,c} = 1$.*

In many practical applications of constrained object allocation, one can show that any random assignment satisfying these constraints can be represented as a lottery over deterministic assignments that are approximately feasible. For example, for the combinatorial assignment problem with limited complementarities, Nguyen, Peivandi, and Vohra (2016) show that one can always decompose a feasible random assignment into a lottery over deterministic assignments where the capacity of each object is violated by at most additive k where k denotes the size of the largest bundle. We discuss such implementation details where applicable in the specific applications in Section 3.4.

On the other hand, our model of imposing constraints on the random assignment solution generalizes the approach of imposing constraints on the ex-post deterministic outcomes. As shown by Balbuzanov (2019), any set of arbitary constraints on the ex-post outcomes can be represented by a set of linear inequalities on the random assignment. While such a reduction always exists, we note that it may not be computationally efficient.

3.4 Applications

In this section, we discuss how the constrained serial rule can be applied for several concrete applications of constrained object allocation.

3.4.1 Unconstrained Object Allocation

We can apply our model to the unconstrained object assignment problem by simply setting $\Delta\mathcal{C}(\succeq) = \Delta\mathcal{D}$ for all $\succeq \in \mathbb{R}^n$. In this case, the random assignment output by the *constrained* *serial rule* coincides with that given by the extended probabilistic serial algorithm of Katta and Sethuraman (2006).

We first briefly discuss the extended probabilistic serial algorithm. At every round t, the extended probabilistic serial algorithm constructs a flow network where agents point to their most preferred objects among the set of objects available at that round. Using the parametric max-flow algorithm, the algorithm then identifies a bottleneck set of agents X to be those that satisfy:

$$X = \operatorname*{argmin}_{Y \subseteq N} \frac{|\Gamma(Y)|}{|Y|}$$

where $\Gamma(Y)$ denotes the set of objects that are most preferred by atleast one agent in Y. We note that this set of agents X is precisely the most constrained set in this round, i.e., an agent $i \in X$ can get exactly $\frac{|\Gamma(X)|}{|X|}$ (and not more) total probability share from her preferred objects. In the constrained serial rule algorithm, since there are not other constraints restricting the random assignments, this same set X of agents will be chosen as the bottleneck set B^t.

3.4.2 Bi-hierarchical Constraints

Budish et al. (2013) considers the object allocation problem with general quota constraints and identified a large class of constraints called 'bi-hierarchical constraints' that are universally implementable, i.e., any random assignment satisfying these constraints can be implemented as a lottery over deterministic assignments that satisfy the same constraints.

A constraint in their setup is of the form $\underline{q}_S \leq \sum_{(i,o)\in S} \overline{x}_{i,o} \leq \overline{q}_S$, where $S \subseteq N \times O$ is a set of agent-object pairs, $\overline{\mathbf{x}}$ is a deterministic assignment, and $\underline{q}_S, \overline{q}_S$ are both integers. A constraint structure $\mathcal{H} = (S, \boldsymbol{q}_S)$ comprises of collection of such constraints. An additional requirement on $\mathcal{H}$ is that it must include all singleton sets. A constraint structure $\mathcal{H}$ is a hierarchy if for every $S, S' \in \mathcal{H}$, either $S \subseteq S'$ or $S' \subseteq S$ or $S \cap S' = \emptyset$. Finally, $\mathcal{H}$ is a bi-hierarchy if there exist hierarchies $\mathcal{H}_1, \mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$. Budish et al. (2013) proposed the *Generalized Probabilistic Serial* mechanism for object allocation with bi-hierarchical quota constraints. However, their mechanism only works for upper-bound quotas and assumes that all lower bounds $\underline{q}_S = 0$.

In contrast, we can directly use the inequalities in the constraint structure $\mathcal{H}$ in our *constrained serial rule* algorithm (even in the presence of indifferences in the preference relations) by defining:

$$\Delta\mathcal{C}(\succeq) = \{\mathbf{x} \in \Delta\mathcal{D} \mid \underline{q}_S \leq \sum_{(i,o)\in S} x_{i,o} \leq \overline{q}_S \text{ for all } (S, \boldsymbol{q}_S) \in \mathcal{H}\} \text{ for all } \succeq \in \mathbb{R}^n$$

Thus our algorithm generalizes the approach of Budish et al. even for lower bound quota constraints. The key technical innovation that allows us to do so lies in the fact that our algorithm *looks ahead* in time to ensure that the partial solution obtained at any time leads to a feasible random assignment.

3.4.3 Type-Dependent Distributional Constraints

Ashlagi, Saberi, and Shameli (2020) study type dependent distributional constraints that do not conform to a bi-hierarchical structure. In this setup, every agent $i \in N$ is associated with a type $t_i \in T$ where T denotes a finite set of types. Let $R \subseteq T$ denote an arbitrary set of agent types and let $o \in O$ denote an arbitrary object. A single constraint is of the form $\underline{q}_{R,o} \leq \sum_{i \in N | t_i \in R} x_{i,o} \leq \overline{q}_{R,o}$, i.e., the mechanism imposes floor and ceiling quotas on the total allocation of all agents belonging to a specific set of agent types at a given object.

As earlier, it can be readily seen that such distributional constraints can be easily represented in our framework. Since the constraints do not conform to a bi-hierarchical structure, the outcome of our mechanism cannot always be implemented as a lottery over feasible deterministic assignments. However, as shown by Ashlagi, Saberi, and Shameli (2020), any random assignment satisfying these constraints can be decomposed into a distribution over almost feasible deterministic outcomes where every floor and ceiling constraint is violated by at most $|T|$.

3.4.4 Explicit Ex-post Constraints

Balbuzanov (2019) considers the problem of random object assignment when we are given an explicit list of ex-post feasible allocations and the random assignment must be implemented as a lottery over these allocations. For a preference profile $\succeq$, let $C(\succeq)$ be the set of all permissible deterministic assignments and $\Delta\mathcal{C}(\succeq)$ be the convex hull of the set $\mathcal{C}$. He shows that for every $C(\succeq)$, there exists a minimal set of constraints parameterized by the matrix A, with $a^c_{i,o} \geq 0$ for all $(i, o) \in N \times O$ and constraint c, and the vector $\boldsymbol{b} \geq \mathbf{0}$ such that $\Delta\mathcal{C}(\succeq) = \{\mathbf{x} \in \Delta\mathcal{D} \mid A\mathbf{x} \leq \boldsymbol{b}\}$. He generalizes the probabilistic serial mechanism to incorporate these inequalities for the case when agents have strict preferences. The *constrained serial rule* algorithm generalizes his mechanism to the full preference domain.

3.4.5 Combinatorial Assignment

Another class of problems where our mechanism can be applied to is the problem of allocating bundles of indivisible objects to agents when preferences exhibit complementarities (Budish 2011; Budish and Cantillon 2012; Nguyen, Peivandi, and Vohra 2016). Formally, let G be an underlying set of objects, where each object $g \in G$ is supplied in q_g copies. A bundle of objects can be represented by a vector in $\mathbb{N}_+^{|G|}$, where the tth co-ordinate of this vector corresponds to the number of copies of the object t and $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. Let $O = \{o \in \mathbb{N}_+^{|G|} \mid \sum_{g \in G} o_g \leq k\}$ now be the set of all bundles of size at most k. We assume that each bundle is available in a single copy. Each agent $i \in N$ is interested in consuming one bundle from the set O and has a complete and transitive preference $\succeq_i$ on the set O. A common application that fits in this class of problems is the course allocation problem. Every student is to be assigned a schedule of at most k courses, where each course g has a finite number of seats q_g.

The set of feasible random allocations can be described by the following set of constraints that enforce that the total amount allocated of any object $g \in G$ is at most its supply q_g. At every preference profile $\succeq$, we have:

$$\Delta\mathcal{C}(\succeq) = \Delta\mathcal{C} = \{x \in \Delta\mathcal{D} \mid \sum_{i \in N} \sum_{o \in O} o_g \cdot x_{i,o} \leq q_g, \quad \forall g \in G\}$$

From Definition 7, it is easy to see that all agents under these feasibility constraints are of the same type. Therefore, the *constrained serial rule* guarantees constrained ordinally efficient and envy-free outcomes. Further, as discussed by Nguyen, Peivandi, and Vohra (2016), any outcome of the mechanism can be implemented as a lottery over deterministic assignments that violate the supply constraints by at most $k - 1$.

Chapter 4

Generalized Hierarchical Exchange Mechanisms

4.1 Introduction

In this chapter, we consider the same problem setting as in Chapter 2 and look at the canonical house allocation problem. We, however, now focus on deterministic mechanisms and study the incentive and efficiency considerations of such mechanisms. A complete characterization of the class of strategy-proof and Pareto efficient mechanisms is as yet unknown. Pápai (2000) proposed the class of hierarchical exchange rules which included existing mechanisms – the serial dictatorship mechanism (Svensson 1994; 1999) and the top trading cycles (TTC) mechanism attributed to Gale by Shapley and Scarf (1974) as special cases. A hierarchical exchange mechanism can be thought of as imitating a market through an iterative procedure in which individuals exchange objects from hierarchically determined endowment sets that closely resembles the TTC mechanism developed in the context of the housing market*. She showed that hierarchical exchange mechanisms are strategy-proof, non-bossy, and Pareto-efficient. Recently, Pycia and Ünver (2017) provided a complete characterization of strategy-proof, non-bossy and Pareto efficient mechanisms. They showed that this class can be implemented as trading cycles mechanisms.

Non-bossiness, a criterion introduced by Satterthwaite and Sonnenschein (1981), specifies that an agent cannot change the allocation of other agents without changing her own allocation†. Indeed if a mechanism is bossy, it may provide agents with incentives to bribe. Strategy-proofness and non-bossiness were also shown to be equivalent to group-strategyproofness, a property that states that no coalition of agents can jointly misreport their preferences such that all of them weakly benefit while at least one agent in the group strictly benefits. Yet there are some applications where non-bossiness is at odds with individ-

*The housing market problem is a variant of the house allocation problem where each agent *owns* an object.

†See Thomson (2016) for a discussion.

ual incentives and efficiency. For example, in the housing market problem, when agents' preferences may exhibit indifferences, there exists no mechanism that is strategy-proof, Pareto efficient, individually rational, and non-bossy[‡] (Bogomolnaia, Deb, and Ehlers 2005; Jaramillo and Manjunath 2012). Therefore, bossiness may confer some degree of flexibility in designing desirable mechanisms.

4.1.1 Our Contributions

Our main contribution in this chapter is to explore the set of strategy-proof, Pareto efficient mechanisms beyond those that are non-bossy as a stepping stone to a complete characterization. We generalize the hierarchical exchange mechanisms to construct a large class of mechanisms that admit many strategy-proof, Pareto efficient and bossy mechanisms. Through this generalization, we also encounter many examples of novel mechanisms.

Pycia and Ünver (2017) redefined Pápai (2000)'s hierarchical exchange mechanisms as TTC mechanisms implemented with a structure of ownership rights. We follow their reformulation and generalize this structure in two ways. First, we introduce flexibility by allowing ownerships of un-assigned objects to depend on not only the assignments of agents but also the prevailing preferences of these agents. Second, we equip the mechanism with *resolution rules* that determine which agents should be assigned objects in a particular round of the mechanism. We formulate consistency conditions on the ownership structure and resolution rules to ensure that the resulting mechanism is strategy-proof.

An important sub-class within this family are mechanisms equipped with *identity* resolution rules. We introduce a concept of *local dictatorship* and show that this sub-class characterizes precisely those mechanisms that are strategy-proof, Pareto efficient, and locally dictatorial.

4.1.2 Related Work

Our work builds on a rich literature on the housing market problem introduced by Shapley and Scarf (1974) and the housing allocation problem formalized by Hylland and Zeckhauser (1979). A fundamental class of mechanisms for these problems, serial dictatorships, were analyzed by Svensson (1994) and Svensson (1999). Svensson (1999) characterize serial dictatorships as the set of all strategy-proof, non-bossy, and neutral mechanisms. Roth (1982) show that David Gale's TTC mechanism (reported by Shapley and Scarf 1974) is strategy-proof. Ma (1994) characterize TTC mechanism as the unique strategy-proof and Pareto efficient mechanism that satisfies individual rationality. Abdulkadiroğlu and Sönmez (1999) construct a larger class of strategy-proof, non-bossy, and Pareto efficient mechanisms. Abdulkadiroğlu and Sönmez (2003) extend the TTC mechanism to priority-based allocation problems. TTC mechanisms have been also been analyzed by Morrill (2013), 2015a. While

[‡]Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) generalize the TTC algorithm to accommodate indfferences and show that the mechanisms are strategy-proof, Pareto efficiency, and individually rational.

all of these mechanisms are non-bossy, several other papers Ergin (2000), Ehlers, Klaus, and Pápai (2002), Ehlers and Klaus (2007), and Kesten et al. (2012) study mechanisms without the non-bossiness property. However, in Ehlers, Klaus, and Pápai (2002) and Ehlers and Klaus (2007), non-bossiness is implied by other assumptions. Morrill (2015b) construct a strategy-proof and bossy variation of the TTC mechanism.

4.2 Model

Let $I = \{i_1, i_2, \ldots, i_n\}$ be the set of agents and let $O = \{o_1, o_2, \ldots, o_k\}$ be the set of objects. We will often refer to the objects as houses. The sets of agents and objects are assumed to be fixed and finite. Every agent demands at most one object. Each agent $i \in I$ has a strict preference relation $\succ_i$ over O. We denote the preference relation $o_1 \succ_i o_2 \succ_i \ldots \succ_i o_n$ as $\succ_i = \langle o_1, o_2, \ldots, o_n \rangle$. Let $\succeq_i$ be the weak preference relation associated with $\succ_i$. Let $\mathcal{P}_i$ denote the set of all strict preference relations of agent i. A set of individual preference relations of all agents constitutes a preference profile $\succ = (\succ_i)_{i \in I}$. Let $\mathcal{P}$ be the set of all preferences profiles. Let $\mathcal{P}_S$ be the set of all preference profiles of agents in $S \subseteq I$. For any fixed profile $\succ$, the preferences of a group of agents $G \subset I$ is denoted by $\succ_G$. Similarly, the preferences of agents who are not in G are denoted by $\succ_{-G}$. Thus, $\succ = (\succ_G, \succ_{-G})$. Throughout the chapter, we assume that there are at least as many objects as agents[§].

A house allocation problem is defined by the triple $(I, O, \mathcal{P})$. A submatching σ matches a subset of agents $I_\sigma \subset I$ to a subset of objects $O_\sigma \subset O$, i.e., it is a one-to-one function $\sigma : I_\sigma \to O_\sigma$. For any $i \in I_\sigma$, $\sigma(i)$ is the object that agent i is matched with. Similarly, for any $o \in O_\sigma$, $\sigma^{-1}(o)$ is the agent that object o is matched to. Let $\emptyset$ denote the empty submatching in which no agent is matched. Let $\mathcal{S}$ be the set of all submatchings. For any submatching $\sigma \in \mathcal{S}$, $\bar{I}_\sigma := I \setminus I_\sigma$ and $\bar{O}_\sigma := O \setminus O_\sigma$ are the set of agents and objects not matched by σ. A matching is a submatching σ where $I_\sigma = I$ holds. Let $\mathcal{M} \subset \mathcal{S}$ be the set of all matchings. For convenience, we often refer to a submatching $\sigma \in \mathcal{S}$ with the set of agent - object pairs $\{(i, o) : \sigma(i) = o\}$. A mechanism is a mapping $\varphi : \mathcal{P} \to \mathcal{M}$ that selects a matching $\sigma \in \mathcal{M}$ for every $\succ \in \mathcal{P}$. Let $\varphi(\succ)(i)$ denote the assignment that agent i receives under the mechanism φ at the preference profile $\succ$.

A matching σ is Pareto efficient at $\succ$, if there exists no other matching that makes all agents weakly better off and at least one agent strictly better off. Formally, a matching $\sigma \in \mathcal{M}$ is Pareto efficient at $\succ$ if there does not exist another $\mu \in \mathcal{M}$ such that $\forall i \in I, \mu(i) \succeq_i \sigma(i)$, and for some $i \in I$, $\mu(i) \succ_i \sigma(i)$. A mechanism φ is Pareto efficient if $\forall \succ \in \mathcal{P}$, $\varphi(\succ)$ is Pareto efficient. A mechanism φ is strategy-proof if for all $i \in I$, for every $\succ \in \mathcal{P}$, and for every $\succ' \in \mathcal{P}_i$, $\varphi(\succ)(i) \succeq_i \varphi(\succ', \succ_{-i})(i)$. That is, truth-telling is a weakly dominant strategy for every agent. A mechanism is bossy if there exist $\succ \in \mathcal{P}$, $i, j \in I$,

[§] When there are fewer objects than agents, we can assume that O contains a null object, $\emptyset$, which has unlimited copies, and matching with $\emptyset$ represents being unmatched. Each agent $i \in I$ has a strict preference relation $\succ_i$ over $O \cup \{\emptyset\}$ such that she always prefers any object in O to being unmatched. That is, for all $i \in G$ and $o \in O$, $o \succ_i \{\emptyset\}$.

and $\succ \in \mathcal{P}_i$ such that $\varphi(\succ)(i) = \varphi(\succ', \succ_{-i})(i)$ and $\varphi(\succ)(j) \neq \varphi(\succ', \succ_{-i})(j)$. A mechanism is non-bossy if it not bossy. That is, a mechanism is non-bossy if an agent can change the matching of other agents without affecting her own assignment.

4.3 Preliminaries

4.3.1 Hierarchical Exchange Mechanisms

The hierarchical exchange mechanisms proposed by Pápai (2000) built upon Gale's TTC algorithm (Shapley and Scarf 1974) by allowing agents to own multiple objects and by defining inheritance rules. As agents with ownership rights get matched with objects in the TTC algorithm and leave the market, the ownership rights to the remaining objects are passed on to the other agents according to the inheritance rule. These inheritance rules were reformulated by Pycia and Ünver (2017) as a collection of ownership mappings. We describe the idea behind the hierarchical exchange mechanisms following the formulation presented by Pycia and Ünver (2017).

Formally, a hierarchical exchange mechanism is described through a collection of mappings $\{c_\sigma : \bar{O}_\sigma \rightarrow \bar{I}_\sigma\}_{\sigma \in \mathcal{S}}$. Once a submatching σ is fixed, $c_\sigma(o)$ specifies the unmatched agent in $\bar{I}_\sigma$ that gains ownership rights to the unmatched object $o \in \bar{O}_\sigma$. We have the following consistency restrictions on the collection of mappings: For all $\sigma \subseteq \sigma' \in \mathcal{S}$, for any $i \in \bar{I}_{\sigma'}$, and for any $o \in \bar{O}_{\sigma'}$, if $c_\sigma(o) = i$, then $c_{\sigma'}(o) = i$[¶]. That is, if an agent owns an object at a certain submatching, then that agent continues to own that object at a larger submatching, as long as both the agent and that object are unmatched.

With these ownership structures defined, the mechanism uses the TTC algorithm to find the allocation. The TTC algorithm is an iterative procedure that allocates objects to agents in a sequence of rounds. At the beginning of each round, each unmatched object is owned by some unmatched agent. The ownership mapping for the submatching that was fixed at the end of the previous round is used to determine who owns every unmatched object. Each remaining agent now points to her most preferred object among the remaining objects, and each unmatched object points to the agent who owns it. This gives rise to a directed graph where at least one cycle exists. The algorithm clears each cycle by assigning every agent in the cycle the object that she was pointing to. At the end of the round, the algorithm creates a larger submatching that consists of the submatching that was fixed at the beginning of the round and the matches determined by clearing the cycles. When all agents are matched, the algorithm terminates. Since each round matches at least one agent to an object, the algorithm terminates in finite number of rounds.

Pápai (2000) showed that all hierarchical exchange mechanisms are strategy-proof, non-bossy and Pareto efficient.

[¶]The subset relationship over submatchings is defined in the natural way; $\sigma \subseteq \sigma' \Leftrightarrow I_\sigma \subseteq I_{\sigma'}$ and further $\forall i \in I_\sigma, \sigma(i) = \sigma'(i)$.

4.3.2 Motivating Examples

We now present two examples that are strategy-proof, Pareto efficient, and bossy. Through these examples, we introduce ways to generalize the hierarchical exchange mechanisms to incorporate bossy mechanisms.

In the first example, we consider a simple generalization of sequential dictatorships. Sequential dictatorships are a special class of hierarchical exchange mechanisms (Pápai 2000; Pycia and Ünver 2016). These mechanisms work similar to serial dictatorships in that the choice of the first dictator is fixed. However, the choice of other dictators that follow can depend on the assignments of the previous dictators.

Example 1. Suppose there are three agents and three objects. Let $I = \{i_1, i_2, i_3\}$ and let $O = \{o_1, o_2, o_3\}$. Let agent i_1 be the first dictator. If agent i_1's preference is $o_1 \succ_{i_1} o_2 \succ_{i_1} o_3$, then the second dictator to follow is agent i_2. But if her preference is $o_1 \succ_{i_1} o_3 \succ_{i_1} o_2$, then agent i_3 is the next dictator. For all of agent i_1's other preferences, the second dictator to follow is agent i_2.

It is easy to see that this mechanism is strategy-proof and Pareto-efficient. However, this mechanism is bossy. Consider the preference profile where $\succ_{i_1} = \succ_{i_2} = \succ_{i_3} = \langle o_1, o_2, o_3 \rangle$. In this case, agent i_1 gets her most preferred object, o_1. Since agent i_2 comes second in the dictatorship sequence, she gets object o_2 while agent i_3 gets object o_3. However, if agent i_1 were to change her preference to $\succ'_{i_1} = \langle o_1, o_3, o_2 \rangle$, she will continue to get object o_1. But, agent i_3 now gets object o_2 instead of agent i_2. Thus, agent i_1 is bossy.

Notice that in any hierarchical exchange mechanism, the ownership rights of unassigned objects are defined at every submatching. At the submatching (i_1, o_1), the ownership of other objects o_2 and o_3 are not allowed to differ based on the prevailing preference of the matched agent i_1. Indeed, by allowing the ownership at the submatching (i_1, o_1) to depend on the preference of agent i_1, we can continue to get a strategy-proof and Pareto efficient mechanism. □

Recall that in every round of its iterative procedure, the TTC algorithm clears all cycles that are formed‖. Intuitively, this means that once an agent obtains ownership rights of some object at a submatching, she retains these rights until the end of the mechanism. In our second example, we relax this requirement and give a mechanism that is bossy, and yet remains strategy-proof and Pareto efficient.

Example 2. Consider three agents and three objects. Let $I = \{i_1, i_2, i_3\}$ and let $O = \{o_1, o_2, o_3\}$. Suppose we have an initial ownership of object o_ℓ to agent i_ℓ for $\ell = 1, 2, 3$. At the submatching (i_1, o_1), agent i_3 owns objects o_2 and o_3. At the submatchings (i_2, o_2) and (i_3, o_3), we maintain the consistency restriction as in a hierarchical exchange mechanism,

‖Even if only one cycle is resolved per round of the algorithm, the consistency restriction on the ownership structure ensures that the cycles that were not resolved in the previous rounds continue to form in every subsequent round until they are resolved.

where every unmatched agent i_ℓ continues to own the unmatched object o_ℓ that they previously owned. In addition, let us suppose that there is a priority ordering over the agents such that $i_1 > i_2 > i_3$. Given this structure**, we run the TTC algorithm with one key modification. In every round, as before, objects point to their owners. However, each agent, according to the priority ordering defined, points to their most favorite object one at a time. So with the priority ordering $i_1 > i_2 > i_3$, first, agent i_1 points to her top choice, then i_2 points to her top choice and finally agent i_3. At the first occurrence of a cycle, the other remaining agent(s) in the priority ordering are not allowed to point to any object. The unique cycle in the resulting graph is cleared and the round ends.

This mechanism is strategy-proof and Pareto efficient. Pareto efficiency holds since we continue to use the TTC algorithm and resolve cycles in which agents get their top choice. To see strategy-proofness, observe that we maintain the consistency restriction on the ownerships for agents i_1 and i_3. Thus, the set of objects that each of these agents could get either by submitting their true preferences or by changing their preferences, for a fixed profile of other agents' preferences, weakly increases with each round. To see that agent i_2 cannot do better by mis-reporting her preference, notice that in any profile where agent i_1 prefers object o_1 as her top choice, the cycle where i_1 points to o_1 and o_1 points to i_1 is the only cycle resolved in the first round. Thereafter, agent i_3 owns both objects o_2 and o_3. So agent i_2 always gets the object that agent i_3 does not prefer. In all other preference profiles, the consistency restriction continues to hold for the ownership rights of agent i_2. Thus, agent i_2 cannot do better by mis-reporting her preferences.

The mechanism is different from any hierarchical exchange mechanism in that the cycle where i_2 points to o_2 and o_2 points to i_2 is never resolved whenever agent i_1 is involved in a cycle. This relaxation of the requirement of clearing all cycles allows us to introduce bossiness into the mechanism. Consider the preference profile where $\succ_{i_1} = \langle o_2, o_1, o_3 \rangle$, $\succ_{i_2} = \langle o_2, o_1, o_3 \rangle$ and $\succ_{i_3} = \langle o_2, o_1, o_3 \rangle$. In this case, the mechanism allocates o_1 to agent i_1, o_2 to agent i_2, and o_3 to agent i_3. However, when agent i_1 changes her preference to $\succ'_{i_1} = \langle o_1, o_2, o_3 \rangle$, she is bossy towards agents i_2 and i_3. After agent i_1 gets object o_1, she transfers the ownership rights for object o_2 to i_3. This leads to the assignment where agent i_2 gets object o_3 and agent i_3 gets object o_2. □

4.4 Generalized Hierarchical Exchange Mechanisms

The examples described in the previous section suggest two ways in which we can generalize the hierarchical exchange mechanisms to accommodate bossy, strategy-proof, and Pareto efficient mechanisms. In this section, we formally introduce a large class of mechanisms called Generalized Hierarchical Exchange Mechanisms that incorporate both these features and prove that these mechanisms are strategy-proof and Pareto efficient.

**The ownerships at other submatchings are irrelevant, since either, those submatchings never arise given the initial ownership, or there is only one unmatched agent and one unmatched object remaining.

Notation. Before we proceed, let us define some extended notation. For any submatching $\sigma \in \mathcal{S}$ and any $\succ \in \mathcal{P}$, let $\succ_\sigma := \succ_{I_\sigma}$ denote the preferences of the agents matched in σ and let $\mathcal{P}_\sigma$ be the set of all preference profiles of agents in I_σ. From here on, we will use term "submatching" to refer to either a $\sigma \in \mathcal{S}$ or a pair $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$. The use of the specific term will be clear from context. For any pair of submatchings $(\sigma, \succ_\sigma), (\sigma', \tilde{\succ}_{\sigma'})$, we say that $(\sigma, \succ_\sigma) \subseteq (\sigma', \tilde{\succ}_{\sigma'}) \iff \sigma \subseteq \sigma'$ and $\forall i \in I_\sigma$, we have $\succ_i = \tilde{\succ}_i$. For any $\sigma \in \mathcal{S}$, let $\Omega_\sigma = \{\omega \mid \omega : \bar{O}_\sigma \to \bar{I}_\sigma\}$ be the set of all functions that map the unmatched objects in $\bar{O}_\sigma$ to unmatched agents in $\bar{I}_\sigma$ and $H_\sigma = \{h \mid h : \bar{I}_\sigma \to \bar{O}_\sigma\}$ be the set of all functions from unmatched agents in $\bar{I}_\sigma$ to unmatched objects in $\bar{O}_\sigma$.

At a submatching $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$, for any $\omega \in \Omega_\sigma$ and $h \in H_\sigma$, let $G_\sigma(\omega, h)$ be a directed graph with vertex set $\bar{I}_\sigma \cup \bar{O}_\sigma$, where every agent $i \in \bar{I}_\sigma$ points to $h(i)$ and every object $o \in \bar{O}_\sigma$ points to $\omega(o)$. Let $\mathcal{C}(G)$ be the set of all cycles in graph G. For any cycle C, let $\mathcal{I}(C)$ be the set of agents in the cycle and let $\mathcal{O}(C)$ be the set of objects in the cycle. For any submatching $(\sigma, \succ_\sigma)$ and any $\omega \in \Omega_\sigma$ and $h \in H_\sigma$ and cycle $C \in \mathcal{C}(G_\sigma(\omega, h))$, *resolving* or *clearing* the cycle C means that every agent $i \in \mathcal{I}(C)$ is matched with the object $h(i)$. We overload notation and let $\sigma \cup C$ denote the submatching obtained by resolving cycle C. For a given submatching $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$ and $\omega \in \Omega_\sigma$, let $\mathcal{C}_\omega = \bigcup_{h \in H_\sigma} \mathcal{C}(G_\sigma(\omega, h)$ be the set of cycles that can form in all of the graphs, where unmatched objects at σ point to unmatched agents according to the function ω. Finally, let $2^{\mathcal{C}_\omega}_{dis}$ be the set of all subsets of cycles such that the cycles within each subset are disjoint.

Mechanism Description. A generalized hierarchical exchange mechanism utilizes the TTC algorithm, where agents and objects are matched in trading cycles over a sequence of rounds. We, however, equip the algorithm with some flexibility. Instead of clearing all cycles that are formed in a given round, the set of cycles that are resolved are dictated by a cycle resolution rule.

Definition 9. *For a given submatching $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$ and $\omega \in \Omega_\sigma$, a* resolution rule *is a choice correspondence $f_{(\sigma,\succ_\sigma,\omega)} : 2^{\mathcal{C}_\omega}_{dis} \setminus \{\emptyset\} \rightrightarrows \mathcal{C}_\omega$ such that for all $\mathcal{A} \in 2^{\mathcal{C}_\omega}_{dis} \setminus \{\emptyset\}$, $f_{(\sigma,\succ_\sigma,\omega)}(\mathcal{A}) \subseteq \mathcal{A}$ and $f_{(\sigma,\succ_\sigma,\omega)}(\mathcal{A}) \neq \{\emptyset\}$. For brevity, we drop ω from the subscript when it is clear from context and let $f_{(\sigma,\succ_\sigma)} = f_{(\sigma,\succ_\sigma,\omega_{(\sigma,\succ_\sigma)})}$.*

The assignment produced by the algorithm also depends on the structure of ownership rights. We now allow the ownership rights to depend on the submatching as well as the prevailing preferences of the agents in the submatching. That is, the ownership rights are defined for every pair $(\sigma, \succ_\sigma)$.

Definition 10. *A* structure of ownership-resolution rights *is a collection of mappings* $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)}) | \omega_{(\sigma,\succ_\sigma)} \in \Omega_\sigma$ *and* $f_{(\sigma,\succ_\sigma)}$ *is a resolution rule*$\}_{(\sigma,\succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma}$[††].

Intuitively, the ownership-resolution rights structure specifies the ownership rights for each unmatched object at every submatching and further specifies a resolution rule at that

[††]We only need to define this structure for all $\sigma \in \mathcal{S} \setminus \mathcal{M}$.

submatching to decide which of the cycles formed at that submatching need to be cleared[‡‡]. For any ownership rights $\omega_{(\sigma,\succ_\sigma)}$, we think of sets in $2_{dis}^{\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}} \setminus \{\emptyset\}$ as a menu of sets of cycles that could potentially form when the agents point to their top choice among the remaining object and the ownership rights are defined by the function $\omega_{(\sigma,\succ_\sigma)}$. Then a particular set $\mathcal{A} \in 2_{dis}^{\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}} \setminus \{\emptyset\}$ is interpreted as a set of available cycles that form in a particular preference profile, and the of these cycles the algorithm chooses to clear the cycles dictated by the resolution rule $f(\mathcal{A})$.

TTC Algorithm. Fix a preference profile $\succ \in \mathcal{P}$. The TTC algorithm induced by a structure of ownership-resolution rights $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$ proceeds as follows:

Initialization: *We start with* $\sigma^0 = \emptyset$. *The algorithm recursively constructs submatchings* σ^k *in rounds* $k = 1, 2, \ldots$.
Round k: *Each remaining object in* $\bar{O}_{\sigma^{k-1}}$ *points to its owner defined in* $\omega_{(\sigma^{k-1},\succ_{\sigma^{k-1}})}$ *and each remaining agent in* $\bar{I}_{\sigma^{k-1}}$ *points to her top choice object among the remaining objects in* $\bar{O}_{\sigma^{k-1}}$. *In the directed graph that is obtained, there exists at least one cycle. Let* $\mathcal{C}^k$ *be the set of cycles formed. We clear all cycles in* $f_{(\sigma^{k-1},\succ_{\sigma}^{k-1})}(\mathcal{C}^k)$ *by assigning each agent in the cycle the object she is pointing to. Let* $\sigma^k = \sigma^{k-1} \cup f_{(\sigma^{k-1},\succ_{\sigma}^{k-1})}(\mathcal{C}^k)$ *be formed by the union of* σ^{k-1} *and the set of newly determined matches. If* σ^k *matches all the agents, i.e., if it is a matching, then the algorithm terminates and* σ^k *is the outcome of the algorithm. Otherwise, we continue to round* $k+1$.

Since the number of agents and objects are finite, every agent points to a unique object and every object points to a unique owner in a round, there always exists at least one cycle in the directed graph constructed in any round. Moreover, the cycles formed will always be disjoint. Because we always resolve at least one cycle in every round, the algorithm will terminate in a finite number of rounds. Observe that in every round, matched agents always leave the market with their most preferred object among the set of all remaining objects and thus the resulting assignment is always Pareto efficient. Therefore, given any structure of ownership-resolution rights, a mechanism that uses the TTC algorithm described above is Pareto efficient.

To ensure that such a mechanism is strategy-proof, we need to impose additional constraints on the structure of the ownership-resolution rights. There are three primary reasons that ensure that a hierarchical exchange mechanism is strategy-proof. First, when an agent is part of a cycle that gets resolved in any round, she gets her top choice among all the objects that remain. Second, an object that is "available" to an agent in some round, meaning that the agent can leave the market with that object in that round by forming a cycle, continues to be "available" in subsequent rounds as long as that agent is unmatched. Lastly, an agent can only affect the ownership rights of the unmatched objects after she leaves the

[‡‡]Recall that in a hierarchical exchange mechanism Pápai (2000), all cycles that are formed at a submatching must eventually be resolved. But we relax this requirement here.

market with her most preferred object under her reported preference. These three points ensure that, for a fixed preference profile of the other agents, an agent cannot alter the set of "available" objects to by mis-reporting her preferences.

Before we translate these ideas into constraints, let us introduce some additional terminology. Fix a structure of ownership-resolution rights $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$.

Definition 11 (1-Step Reachability)**.** *For any pair of submatchings* $(\sigma,\succ_\sigma),(\sigma',\tilde{\succ}_{\sigma'})$*, we say that* $(\sigma',\tilde{\succ}_{\sigma'})$ *is 1-step reachable from* $(\sigma,\succ_\sigma)$ *if* $(\sigma,\succ_\sigma)\subseteq(\sigma',\tilde{\succ}_{\sigma'})$ *and there exists a* $h\in H_\sigma$ *such that* $\sigma'=\sigma\cup\mathcal{C}(G(\omega_{(\sigma,\succ_\sigma)},h))$*,* $f_{(\sigma,\succ_\sigma)}(\mathcal{C}(G(\omega_{(\sigma,\succ_\sigma)},h)))=\mathcal{C}(G(\omega_{(\sigma,\succ_\sigma)},h))$*, and* $\forall i\in I_{\sigma'}\setminus I_\sigma, \tilde{\succ}_i$ *ranks* $h(i)$ *higher than all objects in* $\bar{O}_\sigma$*.*

If a submatching $(\sigma',\tilde{\succ}_{\sigma'})$ *is 1-step reachable from* $(\sigma,\succ_\sigma)$*, we denote the corresponding set of cycles by* $\mathcal{C}_{(\sigma,\succ_\sigma)\to(\sigma',\tilde{\succ}_{\sigma'})}$*.*

Note that since the ownership function $\omega_{(\sigma,\succ_\sigma)}$ determines the edges that point from objects in $\bar{O}_\sigma$ to agents $\bar{I}_\sigma$, there is be a unique set of cycles $\mathcal{C}(G(\omega_{(\sigma,\succ_\sigma)},h))$ that will lead to 1-step reachability between $(\sigma',\tilde{\succ}_{\sigma'})$ and $(\sigma,\succ_\sigma)$.

Definition 12 (Reachability)**.** *We say that* $(\sigma',\tilde{\succ}_{\sigma'})$ *is* reachable *from* $(\sigma,\succ_\sigma)$ *if either* $(\sigma',\tilde{\succ}_{\sigma'})=(\sigma,\succ_\sigma)$ *or if there exists as sequence of submatchings* $(\sigma^{(1)},\succ^{(1)}_{\sigma^{(1)}}),(\sigma^{(2)},\succ^{(2)}_{\sigma^{(2)}}),\ldots,(\sigma^{(k)},\succ^{(k)}_{\sigma^{(k)}})$*, where* $(\sigma,\succ_\sigma)=(\sigma^{(1)},\succ^{(1)}_{\sigma^{(1)}})$ *and* $(\sigma',\tilde{\succ}_{\sigma'})=(\sigma^{(k)},\succ^{(k)}_{\sigma^{(k)}})$*, such that* $(\sigma^{(\ell+1)},\succ^{(\ell+1)}_{\sigma^{(\ell+1)}})$ *is 1-step reachable from* $(\sigma^{(\ell)},\succ^{(\ell)}_{\sigma^{(\ell)}})$*.*

Let $\mathcal{R}_\emptyset$ be the set of all submatchings in $\mathcal{S}$ that are reachable from the empty submatching $\emptyset$. We now impose the following consistency requirements on the structure of ownership-resolution rights.

Consistency requirements:

1. Consider submatchings $(\sigma,\succ_\sigma),(\sigma',\tilde{\succ}_{\sigma'})\in\mathcal{R}_\emptyset$ such that $(\sigma',\tilde{\succ}_{\sigma'})$ is 1-step reachable from $(\sigma,\succ_\sigma)$ and let $\mathcal{D}\in 2^{\mathcal{C}_{\omega(\sigma,\succ_\sigma)}}$ be any set of cycles such that $f_{(\sigma,\succ_\sigma)}(\mathcal{D})=\mathcal{C}_{(\sigma,\succ_\sigma)\to(\sigma',\tilde{\succ}_{\sigma'})}$.
   Let $C\in\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}$ be such that C is disjoint from any cycle in $\mathcal{D}$ and $C\in f_{(\sigma,\succ_\sigma)}(C\cup\mathcal{D})$. Then for any object $o\in\mathcal{O}(C)$ and agent $i=\omega_{(\sigma,\succ_\sigma)}(o)$, the following holds: At all submatchings $(\tilde{\sigma},\succ^*_{\tilde{\sigma}})$ that are reachable from $(\sigma',\tilde{\succ}_{\sigma'})$ such that $\mathcal{I}(C)\subseteq\bar{I}_{\tilde{\sigma}}$ and $\mathcal{O}(C)\subseteq\bar{O}_{\tilde{\sigma}}$, we have $\omega_{(\tilde{\sigma},\succ^*_{\tilde{\sigma}})}(o)=i$.

2. For any set of cycles $\mathcal{A}\in 2^{\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}}_{dis}\setminus\{\emptyset\}$ and any $C\in\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}$ such that C is disjoint from every cycle in $\mathcal{A}$, either $f_{(\sigma,\succ_\sigma)}(\mathcal{A}\cup C)=f_{(\sigma,\succ_\sigma)}(\mathcal{A})$ or $C\in f_{(\sigma,\succ_\sigma)}(\mathcal{A}\cup C)$.

By resolving cycles, the TTC algorithm ensures that when an agent leaves the market she always get matched to her best choice among all the remaining objects. The consistency property (1) postulates that in a particular round of the TTC algorithm, if an agent can match to some object and leave the market by changing her preference, then that object must remain available to the agent in subsequent rounds as long as the agent is unmatched. In other words, if an agent can form a cycle in some round and that cycle will be chosen by the resolution rule in that round, then the ownerships within that cycle must persist in subsequent rounds. The consistency requirement (2) places restrictions on the resolution rule. It states that an agent can only influence the ownership-resolution rights structure in subsequent rounds of the algorithm if that agent can match to an object and leave the market in this round.

We are now ready to define a Generalized Hierarchical Exchange Mechanism.

Definition 13. *A mechanism is a Generalized Hierarchical Exchange Mechanism if its outcomes are determined by running the TTC algorithm with a structure of ownership-resolution rights that satisfy consistency requirements (1) and (2).*

We now present two lemmas that are useful in proving the strategy-proofness of our class of mechanisms. The following lemma states that for a fixed preference profile of agents, the submatchings that are formed in any two consecutive rounds of the TTC algorithm satisfy our notion of 1-step reachability. As a consequence, larger submatchings formed in the subsequent rounds of the algorithm are reachable from the smaller submatchings formed in the previous rounds.

Lemma 7. *Fix a preference profile $\succ$. For any two consecutive rounds k and $k+1$ of the TTC algorithm, $(\sigma^{k+1}, \succ_{\sigma^{k+1}})$ is 1-step reachable from $(\sigma^k, \succ_{\sigma^k})$.*

Proof. Let $\mathcal{C}^{k+1} = \{C_1, C_2, \ldots, C_m\}$ be the set of cycles formed in round $k+1$ of the algorithm and suppose the algorithm resolves cycles $\tilde{\mathcal{C}}^{k+1} := f_{(\sigma^k, \succ_{\sigma^k})}(\mathcal{C}^{k+1}) \subseteq \mathcal{C}^{k+1}$ in round $k+1$. By definition, we have that $\sigma^{k+1} = \sigma^k \cup \tilde{\mathcal{C}}^{k+1}$.

Without loss of generality, let $\tilde{\mathcal{C}}^{k+1} = \{C_1, C_2, \ldots, C_n\}$ for some $n \leq m$. We first prove the following claim.

Claim 1. $f_{(\sigma^k, \succ_{\sigma^k})}(\tilde{\mathcal{C}}^{k+1}) = \tilde{\mathcal{C}}^{k+1}$

Proof. We use the consistency requirement (2) to prove this claim. If $n = m$, then $\tilde{\mathcal{C}}^{k+1} = \mathcal{C}^{k+1} \implies f_{(\sigma^k, \succ^k_\sigma)}(\tilde{\mathcal{C}}^{k+1}) = \tilde{\mathcal{C}}^{k+1}$.

Suppose $n < m$. Since $C_{n+1} \notin f_{(\sigma^k, \succ_{\sigma^k})}(\mathcal{C}^{k+1})$, by the consistency requirement (2), we know that $f_{(\sigma^k, \succ^k_\sigma)}(\mathcal{C}^{k+1} \setminus \{C_{n+1}\}) = f_{(\sigma^k, \succ^k_\sigma)}(\mathcal{C}^{k+1}) = \tilde{\mathcal{C}}^{k+1}$. Similarly, since $C_{n+2}, \ldots, C_m \notin f_{\sigma^k, \succ_{\sigma^k}}(\mathcal{C}^{k+1})$, we can iteratively remove all these cycles to show $f_{\sigma^k, \succ_{\sigma^k}}(\tilde{C}^{k+1}) = f_{\sigma^k, \succ_{\sigma^k}}(C^{k+1} \setminus \{C_{n+1}, \ldots, C_m\}) = f_{\sigma^k, \succ_{\sigma^k}}(\tilde{C}^{k+1}) = \tilde{C}^{k+1}$. □

We can define a function $h : \bar{I}_{\sigma^k} \to \bar{O}_{\sigma^k}$ where every agent in $\mathcal{I}(\tilde{\mathcal{C}}_{k+1})$ continues to point to the object that she was pointing to in round $k+1$ of the algorithm while all the other agents in $\bar{I}_{\sigma^k} \setminus \mathcal{I}(\tilde{\mathcal{C}}_{k+1})$ point to some object in $\mathcal{O}(\tilde{\mathcal{C}}_{k+1})$. In this case, the set of cycles $\mathcal{C}(G(\omega_{(\sigma^k, \succ^k_\sigma)}, h))$ that form are exactly equal to $\tilde{\mathcal{C}}^{k+1}$. Since $\sigma^{k+1} = \sigma^k \cup \tilde{\mathcal{C}}^{k+1}$, $\mathcal{I}(\tilde{\mathcal{C}}_{k+1}) = I_{\sigma^{k+1}} \setminus I_{\sigma^k}$. So, $\forall i \in \mathcal{I}(\tilde{\mathcal{C}}_{k+1})$, $h(i)$ continues to point to i's top choice among all remaining objects in $\bar{O}_{\sigma^k}$.

Additionally we also know that, $(\sigma^k, \succ^k_\sigma) \subseteq (\sigma^{k+1}, \succ^{k+1}_\sigma)$. Hence, the claim follows.

□

The next lemma is key in proving strategy-proofness. It states that as long as an agent is unmatched, she cannot influence the ownerships of the objects by choosing which preferences to report. For any $\succ \in \mathcal{P}$, let $TTC^k(\succ)$ be the submatching that the algorithm constructs in round k under the preference profile $\succ$.

Lemma 8. *Suppose an agent $i \in I$ is unmatched at end of some round k of the TTC algorithm under preference profiles $\succ = (\succ_i, \succ_{-i})$ and $\succ' = (\succ'_i, \succ_{-i})$. Then $TTC^k(\succ) = TTC^k(\succ')$.*

Proof. We can prove this lemma by induction on the round k. Let $\mathcal{C}_k, \mathcal{C}'_k$ be the set of cycles that are formed in round k of the algorithm under $\succ$ and $\succ'$ respectively.

Base Step: At the beginning of the TTC algorithm, since we have $TTC^0(\succ) = TTC^0(\succ') = \emptyset$, the same initial ownerships are used under both the preference profiles. Since agent i is unmatched at the end of round 1, we have the following three cases:

1. Agent i does not form a cycle in round 1 under both $\succ$ and $\succ'$.

   In this case, $\mathcal{C}_1 = \mathcal{C}'_1 \implies f_\emptyset(\mathcal{C}_1) = f_\emptyset(\mathcal{C}'_1)$. Thus, $TTC^1(\succ) = TTC^0(\succ) \cup f_\emptyset(\mathcal{C}_1) = TTC^0(\succ') \cup f_\emptyset(\mathcal{C}'_1) = TTC^1(\succ')$.

2. Agent i forms a cycle in round 1 under $\succ$ but does not form a cycle under $\succ'$.

   Let C be the cycle that agent i forms under $\succ$. We have $\mathcal{C}_1 = C \cup \mathcal{C}'_1$ since the cycles that form under $\succ'$ continue to form under $\succ$. By the consistency requirement (2), since agent 1 is unmatched, $C \notin f_\emptyset(\mathcal{C}_1) \implies f_\emptyset(\mathcal{C}'_1) = f_\emptyset(C \cup \mathcal{C}'_1) = f_\emptyset(\mathcal{C}_1)$. Thus, $TTC^1(\succ) = TTC^1(\succ')$.

3. Agent i forms a cycle in round 1 under both $\succ$ and $\succ'$.

Let C, C' be the cycle that agent i forms under $\succ$ and $\succ'$ respectively. We have $\mathcal{C}_1 \setminus C = \mathcal{C}'_1 \setminus C'$. Again by consistency requirement (2), since $C \notin f_\emptyset(\mathcal{C}_1)$ and $C' \notin f_\emptyset(\mathcal{C}'_1) \implies f_\emptyset(\mathcal{C}_1) = f_\emptyset(\mathcal{C}_1 \setminus C) = f_\emptyset(\mathcal{C}'_1 \setminus C') = f_\emptyset(\mathcal{C}'_1)$. Therefore, $TTC^1(\succ) = TTC^1(\succ')$.

Induction Step: Suppose the lemma is true for round $k-1$. We need to prove that the lemma is true for round k. At the end of round $r-1$, we have $TTC^{k-1}(\succ) = TTC^{k-1}(\succ') = \sigma^{k-1}$. Since the same agents are matched at the end of round $r-1$ under both profiles and agent 1 is not matched, the same submatching $(\sigma^{k-1}, \succ_{\sigma^{k-1}})$ forms under the profiles $\succ$ and $\succ'$. Hence the ownership rights function $\omega_{(\sigma^{k-1}, \succ_{\sigma^{k-1}})}$ and resolution rule $f_{(\sigma^{k-1}, \succ_{\sigma^{k-1}})}$ that are used in round k are the same under $\succ$ and $\succ'$. We can use the same three cases as in the base step and show that $TTC^k(\succ) = TTC^k(\succ')$.

□

We now present our main result of this section.

Theorem 7. *Every Generalized Hierarchical Exchange Mechanism is strategy-proof and Pareto efficient.*

Proof. Consider a Generalized Hierarchical Exchange Mechanism φ with a structure of ownership - resolution rights $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$. We first show that φ is Pareto efficient. We can use a recursive argument to prove Pareto efficiency. In the first round of the TTC algorithm, all agents that leave the market are matched with their most preferred object. In any round $t > 1$, any agent i matched in that round leaves the market with her most preferred object among all the remaining objects. The only way to assign an object that she strictly prefers to her current assignment is to match her with an object that has already been matched to some agent j in a prior round $t' < t$ and would thus make agent j worse off.

To show strategy-proofness, fix an agent $i \in I$. Consider a preference profile $\succ = (\succ_i, \succ_{-i}) \in \mathcal{P}$. We need to show that i cannot benefit by submitting preference $\succ_i' \neq \succ_i$ while others continue to report $\succ_{-i}$. Let $\succ' = (\succ_i', \succ_{-i})$.

Suppose agent i is matched in rounds t and t' of the TTC algorithm when she reports $\succ_i$ and $\succ_i'$ respectively. By Lemma 8, at the end of the round just before round $\min\{t,t'\}$, we obtain the same submatching. Let that submatching be $(\sigma^{\min\{t,t'\}-1}, \succ_{\sigma^{\min\{t,t'\}-1}})$. Therefore, the same set of objects and agents remain in the market and the ownership rights and the resolution rule are the same under $\succ$ and $\succ'$. We consider two cases:

1. $t' < t$:

   In this case $\min\{t,t'\} = t'$. Let $\varphi(\succ')(i) = o$. Since agent i is matched in round t' under profile $\succ'$, let $C' := (i_1 \to o_1 \to i_2 \ldots \to o_n \to i)$ where $i_1 = i$ and $o_1 = o$ be the cycle formed at time t' that contains agent i. Since agent i is not matched in round t' when she reports her true preference $\succ_i$, there exist other cycles that are formed in round t' of the algorithm when agent i reports $\succ_i'$. Let $\mathcal{C}^{t'}$ be the set of other cycles that are formed in round t' of the algorithm under $\succ'$.

   Note that, these cycles continue to form in round t' of the algorithm when agent i reports $\succ_i$. Let $(\sigma^{t'}, \succ_{\sigma^{t'}})$ be the submatching that is formed at the end of round t' when agent i reports her true preference $\succ_i$. Since agent i is not matched in round t', either agent i does not form a cycle in that round or she forms some cycle that is not resolved. In any case, we have $\sigma^{t'} = \sigma^{t'-1} \cup f_{(\sigma^{t'-1},\succ_{\sigma^{t'-1}})}(\mathcal{C}^{t'})$.

   By Lemma 7, $(\sigma^{t'}, \succ_{\sigma^{t'}})$ and all submatchings formed by the algorithm until round $t-1$ under profile $\succ$ are reachable from $(\sigma^{t'}, \succ_{\sigma^{t'}})$. Further since $C' \in f_{(\sigma^{t'-1},\succ_{\sigma^{t'-1}})}(C' \cup \mathcal{C}^{t'})$, by the consistency requirement (1), at the submatching $(\sigma^{t'}, \succ_{\sigma^{t'}})$ we have that $\forall j = 1 \ldots n-1$, $\omega_{(\sigma^{t'},\succ_{\sigma^{t'}})}(o_j) = i_{j+1}$ and $\omega_{(\sigma^{t'},\succ_{\sigma^{t'}})}(o_n) = i_1 = i$. Also note that $\forall j = 2 \ldots n$, agent i_j has o_j as her most preferred remaining object. Thus, in the directed graph constructed by the TTC algorithm in round $t'+1$, the unique path originating from object $o = o_1$ must include agent i. Consequently objects $\{o_1, \ldots, o_n\}$ and agents $\{i_2, \ldots, i_n\}$ cannot get matched in round $t'+1$ unless agent i is also matched. A similar argument shows that objects $\{o_1, \ldots, o_n\}$ and agents $\{i_2, \ldots, i_n\}$ remain unmatched until the beginning of round t. Since agent i leaves the market with her top choice among all remaining objects in round t, we have $\varphi(\succ)(i) \succeq_i o$.

2. $t \leq t'$:

   In round t, submitting her true preference $\succ_i$ gives agent i her top choice among all the remaining objects. Therefore, she cannot be better off by reporting $\succ_i'$.

Thus, the mechanism is strategy-proof.

□

4.5 Examples

In Section 4.5.1, we first present some examples of cycle resolution rules that satisfy the consistency requirement (2). Later, in Section 4.5.2, we show how these resolution rules along with appropriate ownership rights structure can be used to describe the hierarchical exchange mechanisms (Pápai 2000; Pycia and Ünver 2017), generalizations of sequential dictatorships, as well as a novel class of strategy-proof and Pareto efficient mechanisms called "Priority Trading Mechanisms".

4.5.1 Resolution Rules

Recall from Definition 9, that a resolution rule for a set of cycles $\mathcal{C}$ is simply a choice correspondence $f : 2^{\mathcal{C}} \setminus \{\emptyset\} \rightrightarrows \mathcal{C}$ such that for all $\mathcal{A} \in 2^{\mathcal{C}} \setminus \{\emptyset\}$, $f(\mathcal{A}) \subseteq \mathcal{A}$ and $f(\mathcal{A}) \neq \emptyset$. The consistency requirement (2) states that $\forall \mathcal{A} \in 2^{\mathcal{C}} \setminus \{\emptyset\}$ and $C \in \mathcal{C}$, either $f(\mathcal{A} \cup \{C\}) = f(\mathcal{A})$ or $C \in f(\mathcal{A} \cup \{C\})$. In what follows, we give two examples of resolution rules that satisfy this property.

Example 1 (Identity). For all $\mathcal{A} \in 2^{\mathcal{C}} \setminus \{\emptyset\}$, $f(\mathcal{A}) = \mathcal{A}$. That is, all cycles that are formed in any round of the algorithm are always resolved. In this case, the consistency requirement (2) is trivially satisfied.

Example 2 (Rational Choice Functions). It is well-known in the choice theory literature that Sen's α and Sen's β are necessary and sufficient conditions for rationalizability of choice functions Sen's α, also known as, Independence of Irrelevant Alternatives, states that for any $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{C}} \setminus \{\emptyset\}$, if $X \in \mathcal{A} \subseteq \mathcal{B}$ and $X \in f(\mathcal{B})$, then $X \in f(\mathcal{A})$. Sen's β condition states that if $X, Y \in f(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{B}$, and $Y \in f(\mathcal{B})$, then $x \in f(\mathcal{A})$. Interestingly, in the following proposition we show that if a resolution rule satisfies these two properties then the resolution rule satisfies our consistency requirement (2). In other words, if there is a complete preference relation $\succcurlyeq$ on the set of cycles such that $f(\mathcal{A}) = \{X \in \mathcal{A} \mid X \succcurlyeq Y \; \forall Y \in \mathcal{A}\}$, then the induced resolution rule satisfies consistency requirement (2).

Proposition 4. *For any submatching $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$, if the resolution rule $f_{(\sigma,\succ_\sigma)}$ satisfies Sen's α and Sen's β, then $f_{(\sigma,\succ_\sigma)}$ satisfies consistency property (2).*

Proof. Suppose $C \notin f(\mathcal{A} \cup \{C\})$. We want to show that $f(\mathcal{A} \cup \{C\}) = f(\mathcal{A})$.

Consider $X \in f(\mathcal{A} \cup \{C\})$. We know that $X \in \mathcal{A}$. By Sen's α, we have $X \in \mathcal{A} \subseteq \mathcal{A} \cup \{C\}$ and $X \in f(\mathcal{A} \cup \{C\}) \implies X \in f(\mathcal{A})$. This implies that $f(\mathcal{A} \cup \{C\}) \subseteq f(\mathcal{A})$. If $|f(\mathcal{A})| = 1$, then $f(\mathcal{A} \cup \{C\}) = f(\mathcal{A})$ since $f(\mathcal{A} \cup \{C\}) \subseteq f(\mathcal{A})$ and $|f(\mathcal{A} \cup \{C\})| \neq 0$.

Suppose $|f(\mathcal{A})| > 1$. Let $X \in \mathcal{A}$ be such that $X \in f(\mathcal{A} \cup \{C\}) \subseteq f(A)$. Consider any other $Y \in f(A)$. By Sen's β, we must have $Y \in f(\mathcal{A} \cup \{C\})$ and thus we have $f(\mathcal{A}) \subseteq f(\mathcal{A} \cup \{C\})$ and the proposition follows. □

4.5.2 Example Mechanisms

We now present examples of three classes of mechanisms that are captured by Generalized Hierarchical Exchange Mechanisms.

Hierarchical Exchange Mechanisms. It can be readily seen that any Hierarchical Exchange Mechanism can be expressed as a Generalized Hierarchical Exchange mechanism. Indeed, in this case, the resolution rule at all submatchings is simply the identity function as all cycles that are formed in a round get cleared. Further, in a hierarchical exchange mechanism, at any submatching σ, if an agent i owns object o, then that agent continues to own o in all submatchings $\sigma' \supseteq \sigma$. Let $\{c_\sigma\}_{\sigma \in \mathcal{S}}$ denote the ownership rights that define a particular hierarchical exchange mechanism. Then it can be verified easily that the ownership rights structure defined by $\forall \sigma \in \mathcal{S}, \forall \succ_\sigma \in \mathcal{P}_\sigma,\ \omega_{(\sigma,\succ_\sigma)} := c_\sigma$ satisfies the consistency requirement (1).

Generalized Sequential Dictatorships. Following Example 1 from Section 4.3, we can generalize sequential dictatorships to incorporate a large class of bossy, strategy-proof, and Pareto efficient mechanisms. In a generalized sequential dictatorship, we first choose an initial dictator $i \in I$ who chooses her top choice from all the objects. Then, depending on her assignment and reported preference $\succ_i$, the second dictator $j \neq i \in I$ is chosen. The choice of the third dictator $k \in I \setminus \{i, j\}$ depends on the assignment of agents i and j and their reported preferences $\succ_i$ and $\succ_j$. This process continues until all agents are matched to an object.

Any generalized sequential dictatorship can be cast as a Generalized Hierarchical Exchange Mechanism in the following way: $\forall (\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma,\ \forall o, o' \in \bar{O}_\sigma,\ \omega_{(\sigma,\succ_\sigma)}(o) = \omega_{(\sigma,\succ_\sigma)}(o')$. That is, at any submatching, a particular unmatched agent (the dictator) owns all unmatched objects. With this structure of ownerships, at any submatching $(\sigma, \succ_\sigma)$,

$2_{dis}^{\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}}$ only includes singleton subsets of cycles. Therefore, the consistency requirements (1) and (2) are trivially satisfied.

Priority Trading Mechanisms. We extend Example 2 from Section 4.3 to define a new class of mechanisms called Priority Trading Mechanisms. A Priority Trading Mechanism is defined by $(\pi, \{\omega_{(\sigma,\succ_\sigma)}\}_{(\sigma,\succ_\sigma \in \mathcal{S}\times\mathcal{P}_\sigma)})$, where π is a strict priority ordering over the agents, with $\pi(i) = 1$ if agent i has the highest priority and $\pi(i) = n$ if agent has the lowest priority. At every submatching $(\sigma, \succ_\sigma)$, we define an ownership function $\omega_{(\sigma,\succ_\sigma)}$ of unmatched houses to agents. The mechanism uses a modified TTC algotithm and proceeds as follows: At every round, unmatched objects point to their owners specified by the ownership function. Each agent, following the priority order, points to her most favorite object, one agent at a time. At the first occurrence of a cycle, no other agent remaining according to the priority order is allowed to point to any objects. The cycle is resolved and the round ends. At the next round, given the new submatching, the ownership structure is determined and the round proceeds as before. The consistency requirement for ownerships across rounds is defined for for any pair of $(\sigma, \succ_\sigma) \subseteq (\sigma', \tilde{\succ}'_\sigma)$ as follows: Let agent j be the last agent in π such that $j \in \sigma'$. For all $i \in \bar{I}_{\sigma'}$ such that $\pi(i) \leq \pi(j)$, we have if i owns an object in $(\sigma, \succ_\sigma)$ then i owns that object in $(\sigma', \tilde{\succ}'_\sigma)$.

A Priority Trading Mechanism with structure $(\pi, \{\omega_{(\sigma,\succ_\sigma)}\}_{(\sigma,\succ_\sigma \in \mathcal{S}\times\mathcal{P}_\sigma)})$ can be expressed as a Generalized Hierarchical Exchange Mechanism. At every submatching $(\sigma, \succ_\sigma)$, we can define the same ownership function as in the priority trading mechanism. Since the agents point to their most preferred object one at a time according to the priority order, the resolution rule now depends this priority order π as follows: Given a set of disjoint cycles $\mathcal{C}$, for any $\mathcal{A} \in 2^{\mathcal{C}} \setminus \{\emptyset\}$, $f(\mathcal{A}) := \text{argmin}_{C\in\mathcal{A}} \max_{i\in\mathcal{I}(C)} \pi(i))$. Note that the same resolution rule is utilized at all submatchings. To see that the rule satisfies consistency requirement (2), observe that the resolution rule always selects a single cycle from a set of cycles. Thus, Sen's β is trivially satisfied. Additionally, we can always verify that f satisfies Sen's α. Note that in a priority trading mechanism, the ownerships of the agents, who get a chance to point to their top choice in any round, persist in subsequent rounds and thus it can be verified that consistency requirement (1) is also satisfied.

4.6 Characterization

In this section, we characterize Generalized Hierarchical Exchange Mechanisms with identity resolution rules. Indeed, as shown in Section 4.5, many known mechanisms such as Papai's hierarchical exchange mechanisms, serial dictatorships, Gale's TTC mechanisms belong to this sub-class of mechanisms. Recall that identity resolution rules for a Generalized Hierarchical Mechanism are those resolution rules that satisfy: for all submatchings $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}$, for all $\mathcal{A} \in 2_{dis}^{\mathcal{C}_{\omega_{(\sigma,\succ_\sigma)}}} \setminus \{\emptyset\}$, $f_{(\sigma,\succ_\sigma)}(\mathcal{A}) = \mathcal{A}$.

We now define a new notion of local dictatorship to help us with the characterization. Given a mechanism φ, we say that agent j *envies* agent i at $\succ \in \mathcal{P}$ if $\varphi(\succ)(i) \succ_j \varphi(\succ)(j)$. In other words, an agent j envies another agent i at preference profile $\succ$ if she prefers i's assignment to her own assignment. We say that agent j *affects* agent i at $\succ \in \mathcal{P}$, if there exists some $\succ'_j \in \mathcal{P}_j$ such that $\varphi(\succ)(i) \neq \varphi(\succ'_j, \succ_{-j})(i)$.

Definition 14 (locally dictatorial)**.** *A mechanism φ is* locally dictatorial *if for all $\succ \in \mathcal{P}$, for any agent $i \in N$, if there exists another agent $j \in N$ such that j envies i at $\succ$, then j does not affect i at $\succ$.*

Intuitively, a locally dictatorial mechanism is one in which at every profile $\succ \in \mathcal{P}$, there is a priority ordering over agents such that every agent receives their most preferred remaining object in this priority order. In addition, this priority ordering forms a "dictatorship" in the sense that if some agent j envies an agent i that arrives earlier in the priority ordering, then j cannot affect the allocation received by agent i. We remark that Pápai (2000) shows that all strategy-proof, Pareto efficient, reallocation-proof and non-bossy mechanisms are locally dictatorial. Since she showed that all mechanisms with these properties are hierarchical exchange mechanisms, the following lemma is a strict generalization of Lemma 7 from Pápai (2000).

Lemma 9. *Every Generalized Hierarchical Exchange Mechanism with identity resolution rules is locally dictatorial.*

Proof. Consider a Generalized Hierarchical Exchange Mechanism $\varphi := \varphi^{\omega,f}$ with structure of ownership-resolution rights $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$, where at each submatching $(\sigma, \succ_\sigma$ $)$, $f_{(\sigma,\succ_\sigma)}$ is a identity resolution rule.

Consider a preference profile $\succ \in \mathcal{P}$ where there exist $i, j \in I$ such that j envies i. Suppose i is matched in round t of the TTC algorithm and $\varphi(\succ)(i) = o$. Since j envies i it must be that j is matched in round $t' > t$. Consider any preference $\succ'_j$ of agent j. Let $\succ' = (\succ'_j, \succ_{-j})$. We need to show that $\varphi(\succ')(i) = o$.

If in the profile $\succ'$, agent j is matched in round $t'' \geq t$, then by Lemma 8, we know that the same submatchings form under $\succ$ and $\succ'$ in round t of the TTC algorithm. Therefore, $\varphi(\succ')(i) = \varphi(\succ)(i) = o$ and agent j does not affect agent i.

On the other hand, suppose that in profile $\succ'$, agent j is matched in round $t'' < t$. Let $(\sigma_r, \succ_{\sigma_r})$ and $(\sigma'_r, \succ'_{\sigma'_r})$ be the submatchings that are formed in round r of the TTC algorithm for profiles $\succ$ and $\succ'$ respectively. Let G_r and G'_r denote the directed graphs formed in round r by the TTC algorithm for preferences $\succ$ and $\succ'$ respectively and recall that the submatchings $(\sigma_r, \succ_{\sigma_r})$ and $(\sigma'_r, \succ'_{\sigma'_r})$ are obtained by resolving the cycles formed in graphs G_r and G'_r respectively. Finally, for any agent k, let $out(G_r, k)$ denote the set of all nodes u in graph G_r such that there is a directed path from k to u in the graph (we define $k \in out(G_r, k)$). We observe that, by construction, the set of nodes in $out(G_r, k)$ always form a single directed path that terminates in a cycle in G_r.

We first observe that consistency requirement 1 and the identity resolution rules imply that for any pair of nodes $u, v \in G_{r+1}$ such that edge $u \to v$ exists in graph G_r, we also have that $u \to v$ exists in graph G_{r+1} (and similarly for G'_r and G'_{r+1}). To see this, note that the consistency rule implies that for any object a that is unmatched at round $r+1$, if a is owned by agent k at round r, then a must continue to be owned by agent k at round $r+1$. Also agents always point to their most preferred object and the set of remaining objects only decreases between rounds. We now show the following claim.

Claim 2. *In any round r of the TTC algorithm such that $t'' \leq r \leq t$, we have the following:*

1. *Submatching $(\sigma'_{r-1}, \succ'_{\sigma'_{r-1}})$ is reachable from $(\sigma_{r-1}, \succ_{\sigma_{r-1}})$.*

2. *For any agent $k \in \bar{I}_{\sigma_{r-1}}$, either $out(G_r, k) = out(G'_r, k)$ or $j \in out(G_r, k)$.*

Proof. We prove this claim by induction on round r.

Base Step: Round $r = t''$.

Since agent j is matched in round t'' of the algorithm under $\succ'$, we know by Lemma 8 that the same submatchings are formed at the beginning of round t'' under $\succ$ and $\succ'$. That is, $(\sigma_{t''-1}, \succ_{\sigma_{t''-1}}) = (\sigma'_{t''-1}, \succ_{\sigma'_{t''-1}})$ and hence the first part of the claim is trivially true. Since only agent j changes her preference between $\succ$ and $\succ'$, the graphs $G_{t''}$ and $G'_{t''}$ are identical with the exception of the edge leaving from agent j. Thus we have the second part of the claim.

Induction Step: Round $r+1$.

Suppose the claim is true for round $r < t$. We need to show that the claim continues to hold true for round $r+1$. Let $\mathcal{C}_r$ and $\mathcal{C}'_r$ denote the set of cycles in graphs G_r and G'_r respectively. We note that since agent j is not matched in round r under the profile $\succ$, we know that j is not a part of any cycle in $\mathcal{C}_r$. Thus, by the induction hypothesis, any cycle $C \in \mathcal{C}_r$ is also formed in the graph G'_r, and thus we have $(\sigma_r, \succ_{\sigma_r}) \subseteq (\sigma'_r, \succ'_{\sigma'_r})$ and the first part of the claim follows.

Consider any node u in graph G_r that is not a part of any cycle in $\mathcal{C}_r$. We observe that consistency requirement 1 implies that any such node u continues to have the same out-neighbor in graph G_{r+1} unless its out-neighbor is part of a cycle in $\mathcal{C}_r$ (and similarly for G'_r and G'_{r+1}). Consequently we have that for any node $k \in \bar{I}_{\sigma_r}$, $out(G_r, k) \setminus \mathcal{C}_r \subseteq out(G_{r+1}, k)$. Hence for any node $k \in \bar{I}_{\sigma_r}$, $j \in out(G_r, k)$ implies that $j \in out(G_r, k) \setminus \mathcal{C}_r$ (since j is not part of any cycle in $\mathcal{C}_r$) which in turn implies that $j \in out(G_{r+1}, k)$.

Consider any node $k \in \bar{I}_{\sigma_r}$ such that $j \notin out(G_r, k)$. Since $out(G_r, k)$ always forms a directed path that ends in some cycle, let $u \to v$ denote the edge on this path such that v is in the cycle but u is not. Suppose that the out-neighbor of u in graph G_{r+1} is some vertex w. We have the following two cases. (i) If $j \in out(G_r, w)$, then $j \in out(G_{r+1}, k)$ and we are done. (ii) If $j \notin out(G_r, w)$, then since $(\sigma'_r, \succ'_{\sigma'_r})$ is reachable from $(\sigma_r, \succ_{\sigma_r})$, we know that the edge $u \to w$ exists in both G_{r+1} and G'_{r+1}. Further by the induction hypothesis, we have $out(G_r, w) = out(G'_r, w)$. If $w \in out(G_r, k)$, then we have $out(G_{r+1}, k) = out(G'_{r+1}, k)$

as desired. Otherwise, $out(G_r, w)$ must again end in some cycle. In that case, repeating the same argument again, we obtain the desired claim. □

Note that since j envies i and is matched in round $t' > t$, we have that $j \notin out(G_r, i)$ in any round $r \leq t$. Therefore, by Claim 2, $out(G_r, i) = out(G'_r, i)$ for any round $r \leq t$. In particular, $out(G_t, i) = out(G'_t, i)$ and therefore $\varphi(\succ)(i) = \varphi(\succ')(i)$. That is, j does not affect i.

□

We now present our main characterization result of this section.

Theorem 8. *A mechanism is strategy-proof, Pareto efficient, and locally dictatorial if and only if it is a Generalized Hierarchical Exchange Mechanism with identity resolution rules.*

Proof. We first prove the sufficiency direction of the theorem.

Sufficiency. Consider a Generalized Hierarchical Exchange Mechanism $\varphi^{\omega,f}$ with identity resolution rules. From Theorem 7, we know that $\varphi^{\omega,f}$ is strategy-proof and Pareto efficient. In addition, Lemma 9 shows that $\varphi^{\omega,f}$ is locally dictatorial.

Necessity. Let φ be a strategy-proof, Pareto efficient, and locally dictatorial mechanism. We will show that φ is a Generalized Hierarchical Exchange Mechanism with identity resolution rules in three steps.

Step 1: Construction of $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$

For any agent i, let $\mathcal{P}_i^o \subseteq \mathcal{P}_i$ be the set of preference relations where object o is agent i's most preferred object. For any submatching $(\sigma, \succ_\sigma)$ and any object $o \in \bar{O}_\sigma$, let $\mathcal{P}_{\bar{\sigma}}^o = \{\succ := (\succ_i)_{i\in\bar{I}_\sigma} \mid \succ_i \in \mathcal{P}_i^o\}$ be the set of all preference profiles of unmatched agents such that all agents have object o as most preferred. We define the ownership function $\omega_{(\sigma,\succ_\sigma)} : \bar{O}_\sigma \to \bar{I}_\sigma$ as follows: For any $o \in \bar{O}_\sigma$, let $\omega_{(\sigma,\succ_\sigma)}(o) = i$ if for all $\succ_{\bar{\sigma}} \in \mathcal{P}_{\bar{\sigma}}^o$, we have $\varphi(\succ_\sigma, \succ_{\bar{\sigma}})(i) = o$. Otherwise, let $\omega_{(\sigma,\succ_\sigma)}(o) = k$ where k is an arbitrary agent in $\bar{I}_\sigma$. For any $(\sigma, \succ_\sigma) \in \mathcal{S} \times \mathcal{P}_\sigma$, let $f_{(\sigma,\succ_\sigma)}$ be the identity resolution rule.

Step 2: Generalized Hierarchical Exchange Mechanism with $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$ *equals* φ

Let $\varphi^{(\omega,f)}$ be the Generalized Hierarchical Exchange Mechanism with the structure of ownership-resolution rights $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$ defined in Step 1. In this step, we will show that for all $\succ \in \mathcal{P}$, $\varphi(\succ) = \varphi^{(\omega,f)}(\succ)$.

Fix a preference profile $\succ \in \mathcal{P}$. Let T be the last round when the TTC algorithm described in Section 4.4 terminates. In mechanism $\varphi^{(\omega,f)}$, the TTC algorithm starts with $\sigma^0 = \emptyset$ and iteratively constructs submatchings σ^t in rounds $t = 1, 2, \ldots, T$. Using induction, we will show that for all rounds $t \leq T$, for all $i \in I_{\sigma^t}$, $\varphi(\succ)(i) = \varphi^{(\omega,f)}(\succ)(i)$. Further, at the end of any round $t < T$, the ownership function $\omega_{(\sigma^t,\succ_{\sigma^t})}$ is well-defined in the sense that it never assigns any object $o \in \bar{O}_{\sigma^t}$ to an arbitrary agent (called k in Step 1).

Base Step: Round 0.

The statement is vacuously true since we start with $\sigma^0 = \emptyset$ which implies that $I_{\sigma^0} = \emptyset$. We know that $\bar{I}_{\sigma^0} = I$ and $\bar{O}_{\sigma^0} = O$. Consider any $o \in O$. Let $\succ \in \mathcal{P}^o_{\bar{\sigma}^0}$ be an arbitrary profile where all agents have o as their most preferred object. By Pareto efficiency, there must exist an agent $i \in I$ such that $\varphi(\succ)(i) = o$. We now argue that $\forall \succ' \in \mathcal{P}^o_{\bar{\sigma}^0}, \varphi(\succ')(i) = o$.

Let $\succ' \in \mathcal{P}^o_{\bar{\sigma}^0}$ be an arbitrary profile. Indeed, by strategyproofness, for the profile $(\succ'_i, \succ_{-i}) \in \mathcal{P}^o_{\bar{\sigma}^0}$, we must have $\varphi(\succ'_i, \succ_{-i})(i) = o$. Consider any agent $j \neq i$. Since j *envies* i at $\succ$ and φ is locally dictatorial, agent j cannot affect the assignment for agent i, and thus $\varphi(\succ'_i, \succ'_j, \succ_{I\setminus\{i,j\}})(i) = o$. Continuing this way and changing the preferences of each agent $j \neq i$ one by one, we obtain $\varphi(\succ')(i) = o$ as desired.

Induction Step: Round $t + 1$.

Fix $t < T$. Suppose that for all $i \in I_{\sigma^t}$, $\varphi(\succ)(i) = \varphi^{(\omega,f)}(\succ)(i)$ and that at the end of round t, the ownership function at submatching $(\sigma^t, \succ_{\sigma^t})$ is well-defined.

Let $\mathcal{P}_{\bar{\sigma}}$ be the set of all preferences profiles for agents in $\bar{I}_\sigma$. First, we prove the following claim.

Claim 3. *For all $i \in \bar{I}_{\sigma^t}$, for all $o \in \bar{O}_{\sigma^t}$ such that $\omega_{(\sigma^t, \succ_{\sigma^t})}(o) = i$, for all $\succ' \in \mathcal{P}_{\bar{\sigma}^t}$, $\varphi(\succ_{\sigma^t}, \succ')(i) \succeq'_i o$.*

Proof. Fix $i \in \bar{I}_{\sigma^t}$, $o \in \bar{O}_{\sigma^t}$ such that $\omega_{(\sigma^t, \succ_{\sigma^t})}(o) = i$. By the induction hypothesis, we know that the construction of $\omega_{(\sigma^t, \succ_{\sigma^t})}$ is well-defined. Thus for any $\bar{\succ} \in \mathcal{P}^o_{\bar{\sigma}^t}$, we have $\varphi(\succ_\sigma, \bar{\succ})(i) = o$. Consider any agent $j \in \bar{I}_{\sigma^t} \setminus \{i\}$. Since j envies i at $(\succ_\sigma, \bar{\succ})$, j cannot affect i and hence we have $\varphi(\succ_\sigma, \succ'_j, \bar{\succ}_{-j})(i) = o$. Again, continuing this way and changing the preferences of each agent $j \neq i$ one by one and finally changing the preference of agent i, we obtain the claim. □

Since we have assumed that for all $i \in I_{\sigma^t}$, $\varphi(\succ)(i) = \varphi^{(\omega,f)}(\succ)(i)$, it is enough to show that for all $i \in I_{\sigma^{t+1}} \setminus I_{\sigma^t}$, $\varphi(\succ)(i) = \varphi^{(\omega,f)}(\succ)(i)$. With the ownership function $\omega_{(\sigma^t, \succ_{\sigma^t})}$, let $\mathcal{C}^{t+1}$ be the set of cycles formed in round $t+1$ of the TTC. By construction, $f_{(\sigma^t, \succ_{\sigma^t})}$ is an identity resolution rule. Therefore, $f_{(\sigma^t, \succ_{\sigma^t})}(\mathcal{C}^{t+1}) = \mathcal{C}^{t+1}$. Thus, $\mathcal{I}(\mathcal{C}^{t+1}) = I_{\sigma^{t+1}} \setminus I_{\sigma^t}$.

Suppose there exists $C \in \mathcal{C}^{t+1}$ such that $|\mathcal{I}(C)| = 1$. Let $\mathcal{I}(C) = i$ and $\mathcal{O}(C) = o$. We know that $\varphi^{(\omega,f)}(\succ)(i) = o$. Since $\omega_{(\sigma^t, \succ_{\sigma^t})}(o) = i$, o is agent i's most preferred object among the the set of objects in $\bar{O}_{\sigma^t}$ in $\succ_i$, and σ^t matches all objects in $O \setminus \bar{O}_{\sigma^t}$, by Claim 3, we must have $\varphi(\succ)(i) = o = \varphi^{(\omega,f)}(\succ)(i)$.

Suppose $C \in \mathcal{C}^{t+1}$ is such that $|\mathcal{I}(C)| > 1$. Let $C := (i_1 \to o_1 \to i_2 \ldots \to o_n \to i_{n+1})$ where $i_{n+1} = i_1$ be the cycle formed. We know that o_1 is $i_1 (= i_{n+1})$'s most preferred object in $\succ_{i_1}$, o_2 is i_2's most preferred object in $\succ_{i_2}$, and so on. We also know that $\varphi^{(\omega,f)}(\succ)(i_1) = o_1$. We now argue that $\varphi(\succ)(i_1) = o_1$. Since i_1 is an arbitrary agent in the cycle C, this is sufficient to show the induction step.

Let $\succ'_{i_1}$ rank o_1 first and o_n second. For any $k = 2, \ldots, n$, let $\succ'_{i_k}$ rank o_k first, and o_{k-1} second. Suppose $\varphi(\succ)(i_1) \neq o_1$. Then by strategyproofness, $\varphi(\succ'_{i_1}, \succ_{-i_1})(i_1) \neq o_1$. Since $\omega_{(\sigma^t, \succ_{\sigma^t})}(o_n) = i_{n+1} = i_1$, by Claim 3, $\varphi(\succ'_{i_1}, \succ_{-i_1})(i_1) = o_n$. Here, i_n envies $i_{n+1} = i_1$. Since φ is locally dictatorial, $\varphi(\succ'_{i_{n+1}}, \succ'_n, \succ_{I\setminus\{i_n, i_{n+1}\}})(i_{n+1}) = o_n$. By Claim 3, $\varphi(\succ'_{i_{n+1}}$

$, \succ'_n, \succ_{I\setminus\{i_n,i_{n+1}\}})(i_n) = o_{n-1}$. Here, i_{n-1} envies i_n. Therefore, by a similar argument, we have $\varphi(\succ'_{i_{n+1}}, \succ'_n, \succ'_{n-1}, \succ_{I\setminus\{i_{n-1},i_n,i_{n+1}\}})(i_{n-1}) = o_{n-2}$. We can iteratively apply the same argument to get $\varphi(\succ'_{\mathcal{I}(C)}, \succ_{I\setminus\mathcal{I}(C)})(i_2) = o_1$. Once again, by repeatedly applying Claim 3, we obtain $\varphi(\succ'_{\mathcal{I}(C)}, \succ_{I\setminus\mathcal{I}(C)})(i_1) = o_n$ and for all $k \in \{2, \ldots, n\}$, $\varphi(\succ'_{\mathcal{I}(C)}, \succ_{I\setminus\mathcal{I}(C)})(i_k) = o_{k-1}$. However, this violates Pareto efficiency.

Step 3: Generalized Hierarchical Exchange Mechanism with $\{(\omega_{(\sigma,\succ_\sigma)}, f_{(\sigma,\succ_\sigma)})\}_{(\sigma,\succ_\sigma)\in\mathcal{S}\times\mathcal{P}_\sigma}$ *satisfies consistency requirements*

We first show that our constructed structure of ownership-resolution rights in Step 1 satisfy consistency requirement 1. Let $\mathcal{R}_\emptyset$ be the set of all reachable submatchings from the empty submatching. From the definition of 1-step reachability, it is easy to see that for any submatching $(\sigma, \succ_\sigma) \in \mathcal{R}_\emptyset$, there exists a preference profile $\succ$ such that $(\sigma, \succ_\sigma)$ is obtained in some round t of the TTC algorithm. Also, by the induction step in Step 2, we know that the ownership functions at all submatchings in $\mathcal{R}_\emptyset$ are well-defined.

Consider any two submatchings in $(\sigma, \succ_\sigma), (\sigma', \succ_{\sigma'}) \in \mathcal{R}_\emptyset$ such that $(\sigma', \succ_{\sigma'})$ is reachable from $(\sigma, \succ_\sigma)$. For any $i \in \bar{I}_\sigma$ and any $o \in \bar{O}_\sigma$ such that $\omega_{(\sigma,\succ_\sigma)}(o) = i$, from Step 2 we know that agent i is allocated object o in any profile where i has o as most-preferred, i.e., for any $\succ_i \in \mathcal{P}_i^o$ and any preferences of the other unmatched agents $\succ_{\bar{\sigma}\setminus\{i\}} \in \mathcal{P}_{\bar{I}_\sigma\setminus\{i\}}$, we have $\varphi(\succ_\sigma, \succ_i, \succ_{\bar{\sigma}\setminus\{i\}})(i) = o$. In particular, consider a preference profile where agents in $I_{\sigma'}$ report preferences $\succ_{\sigma'}$. Thus, for any preference profile $\succ' \in \mathcal{P}^o_{\bar{\sigma}'}$, we have $\varphi(\succ_{\sigma'}, \succ')(i) = o$; and thus by definition we have $\omega_{(\sigma',\succ_{\sigma'})}(o) = i$.

Since we have $\omega_{(\sigma',\succ_{\sigma'})}(o) = \omega_{(\sigma,\succ_\sigma)}(o)$ for all objects $o \in \bar{O}_{\sigma'}$, the consistency property 1 (that only requires such a condition for some objects) is satisfied. □

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Appendix A

Chapter 2: Supplementary Material

A.1 Proof from Section 2.3

Theorem 1. *For any $n \geq 3$, no strategy-proof and envy-free mechanism can be contention-free efficient.*

Proof. We show the impossibility result from the case when $n > 3$.

Let $N = \{1, 2, 3, \ldots n\}$ and $O = \{a_1, a_2, a_3, \ldots a_n\}$ denote the set of agents and objects respectively. Let us again suppose for contradiction that there exists a mechanism φ that is strategy-proof, envy-free and contention-free efficient. We consider a family of preference profiles $\mathcal{F}^n$, where agents 1, 2 and 3 prefer objects a_1, a_2 and a_3, in any order, to $\langle a_4, a_5, \ldots a_n\rangle$, while every agent $i > 3$ has the preference relation $\succ_i = \langle a_i, \ldots a_n, a_1, a_2, \ldots, a_{i-1}\rangle$. Table A.1 illustrates this family of profiles. The $\star$ in the table refers to any ordering of objects a_1, a_2, and a_3.

1				a_4	a_5	$\ldots$	$\ldots$	$\ldots$	a_n
2		$\star$		a_4	a_5	$\ldots$	$\ldots$	$\ldots$	a_n
3				a_4	a_5	$\ldots$	$\ldots$	$\ldots$	a_n
4	a_4	a_5	a_6	$\ldots$	$\ldots$	$\ldots$	a_1	a_2	a_3
5	a_5	a_6	a_7	$\ldots$	$\ldots$	$\ldots$	a_2	a_3	a_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	a_n	a_1	a_2	$\ldots$	$\ldots$	$\ldots$	a_{n-3}	a_{n-2}	a_{n-1}

Table A.1: Family of preference profiles $\mathcal{F}^n$

We first prove the following lemma to show that if the mechanism φ satisfies all the three premises of the theorem, then in every preference profile in $\mathcal{F}^n$ agents 1, 2, and 3 are assigned a zero probability of receiving any object from $\{a_4, a_5, \ldots, a_n\}$.

Lemma 10. *If mechanism φ is strategy-proof, envy-free and contention-free efficient, then for every $\succ \in \mathcal{F}^n$, for $i \in \{1,2,3\}$ and for $j \in \{4,\dots n\}$, $\varphi_{i,a_j}(\succ) = 0$.*

Proof. Consider any profile $\succ = (\succ_1, \succ_2, \succ_3, \succ_{N\setminus\{1,2,3\}}) \in \mathcal{F}^n$. Consider also a preference profile $\succ^{(0)} = (\succ_1^{(0)}, \succ_2^{(0)}, \succ_3^{(0)}, \succ_{N\setminus\{1,2,3\}}) \in \mathcal{F}^n$ where agents $1,2$, and 3 prefer a_1, a_2, and a_3 as their top choice respectively. Notice that $\succ^{(0)}$ is a contention-free preference profile. For convenience, we adopt the notation that $P^{(x)} := \varphi(\succ^{(x)})$ for any profile $\succ^{(x)}$. Since φ is contention-free efficient, $P^{(0)}_{1,a_1} = P^{(0)}_{2,a_2} = P^{(0)}_{3,a_3} = 1$, which implies that $P^{(0)}_{i,a_j} = 0$ for $i \in \{1,2,3\}$ and $j \in \{4,5,\dots n\}$.

We now consider a sequence of preference profiles $\{\succ^{(1)}, \succ^{(2)}, \succ^{(3)} = \succ\}$ where agents $1,2$, and 3 successively report their preference as in $\succ$. That is, $\succ^{(1)} = (\succ_1, \succ_2^{(0)}, \succ_3^{(0)}, \succ_{N\setminus\{1,2,3\}})$, $\succ^{(2)} = (\succ_1, \succ_2, \succ_3^{(0)}, \succ_{N\setminus\{1,2,3\}})$, $\succ^{(3)} = \succ = (\succ_1, \succ_2, \succ_3, \succ_{N\setminus\{1,2,3\}})$. Observe that between $\succ^{(0)}$ and $\succ^{(1)}$, only agent 1's preferences differ. But in both cases, agent 1 prefers objects $\{a_1, a_2, a_3\}$ over all the other objects. Since φ is strategy-proof, it must be that

$$\begin{aligned} P^{(1)}_{1,a_1} + P^{(1)}_{1,a_2} + P^{(1)}_{1,a_3} = P^{(0)}_{1,a_1} + P^{(0)}_{1,a_2} + P^{(0)}_{1,a_3} = 1 \\ \implies P^{(1)}_{1,a_j} = 0 \text{ for } j \in \{4,5,\dots n\} \end{aligned}$$

Now, since φ is also envy-free, for agents $1,2$, and 3 to not envy each other, we must have,

$$\begin{aligned} P^{(1)}_{2,a_1} + P^{(1)}_{2,a_2} + P^{(1)}_{2,a_3} = P^{(1)}_{3,a_1} + P^{(1)}_{3,a_2} + P^{(1)}_{3,a_3} = P^{(1)}_{1,a_1} + P^{(1)}_{1,a_2} + P^{(1)}_{1,a_3} = 1 \\ \implies P^{(1)}_{2,a_j} = P^{(1)}_{3,a_j} = 0 \text{ for } j \in \{4,5,\dots n\} \end{aligned}$$

We can apply the same arguments as we move from $\succ^{(1)}$ to $\succ^{(2)}$ where only agent 2 changes her report and from $\succ^{(2)}$ to $\succ^{(3)}$ where only agent 3 reports different preferences. □

Lemma 10 implies that the assignment problem for $n > 3$ can be reduced to a problem on the first three agents. Since φ satisfies the premises of the theorem, we can define a new mechanism φ' for $n = 3$ by restricting to the domain of $\mathcal{F}^n$. Formally, let $\succ' = (\succ'_1, \succ'_2, \succ'_3) \in \mathcal{R}^3$ be a preference profile for the first three agents in the reduced problem and let $\succ = (\succ_1, \succ_2, \succ_3, \succ_{N\setminus\{1,2,3\}}) \in \mathcal{F}^n$ be a preference profile for n agents such that each agent $i \in \{1,2,3\}$ has the preference relation with objects a_1, a_2, and a_3 preferred according to $\succ'_i$ followed by $\langle a_4, a_5, \dots a_n\rangle$. In the random assignment $P = \varphi(\succ)$, Lemma 10 implies that $P_{i,a_j} = 0$ for $i \in \{1,2,3\}$ and $j \in \{4,5,\dots n\}$. Therefore, we can define $P' = \varphi(\succ')$ as $P' = [P_i]_{i\in\{1,2,3\}}$, which is a valid random assignment. However, since φ is contention-free, envy-free, and strategy-proof, φ' also satisfies all the three properties and thus we have arrived at a contradiction. This completes the proof of Theorem 1. □

A.2 Proofs from Section 2.4

Theorem 2. *Every Rank Exchange mechanism φ^v is strategy-proof and envy-free.*

Proof. Consider any vector $v \in [0, \frac{1}{n(n-1)}]^n$ with $v_k \geq v_{k+1}\ \forall k \in \{1, 2, \ldots, n-1\}$ and $v_n = 0$ and let φ^v be the corresponding Rank Exchange mechanism.

Feasibility. We first demonstrate that φ^v is a feasible random assignment mechanism. By definition, at any profile $\succ \in \mathcal{R}^n$, we have

$$\varphi^v_{i,a}(\succ) = \frac{1}{n} + \sum_{j \neq i} (v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)})$$

Since for any ranks ℓ and k, we have $v_\ell - v_k \in [\frac{-1}{n(n-1)}, \frac{1}{n(n-1)}]$, we have that $\varphi^v_{i,a} \in [0, 1]$ as desired. The total allocation over all objects for any agent i is given by

$$\begin{aligned}\sum_{a \in O} \varphi^v_{i,a}(\succ) &= \sum_{a \in O} \left(\frac{1}{n} + \sum_{j \neq i} (v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)}) \right) \\ &= 1 + \sum_{j \neq i} (\sum_{a \in O} v_{rank(\succ_i, a)} - \sum_{a \in O} v_{rank(\succ_j, a)}) = 1\end{aligned}$$

Similarly, the total allocation of an object $a \in O$ over all agents is

$$\begin{aligned}\sum_{i \in N} \varphi^v_{i,a}(\succ) &= \sum_{i \in N} \left(\frac{1}{n} + \sum_{j \neq i} (v_{rank(\succ_i, a)} - v_{rank(\succ_j, a)}) \right) \\ &= 1 + \sum_{i \in N} \left((n-1) \cdot v_{rank(\succ_i, a)} - \sum_{j \neq i} v_{rank(\succ_j, a)} \right) \\ &= 1 + (n-1) \cdot \sum_{i \in N} v_{rank(\succ_i, a)} - \sum_{i \in N} \sum_{j \neq i} v_{rank(\succ_j, a)} \\ &= 1 + (n-1) \cdot \sum_{i \in N} v_{rank(\succ_i, a)} - (n-1) \cdot \sum_{i \in N} v_{rank(\succ_i, a)} = 1\end{aligned}$$

Thus, the assignment obtained is doubly stochastic and hence the mechanism is feasible.

Envy-freeness. Consider any two agents $i, i' \in N$ and any preference profile $\succ$. Without loss of generality, let $\succ_i = \langle a_1, \ldots, a_n \rangle$. For any $t \in \{1, 2, \ldots, n\}$, let us consider the total allocation obtained by agents i and i' for the top t objects in agent i's preference. We have the following.

$$\begin{aligned}\sum_{k \leq t} \varphi^v_{i,a_k}(\succ) &= \frac{t}{n} + \sum_{j \in N \setminus \{i\}} \sum_{k \leq t} (v_k - v_{rank(\succ_j, a_k)}) \\ &= \frac{t}{n} + \sum_{j \in N \setminus \{i, i'\}} \sum_{k \leq t} (v_k - v_{rank(\succ_j, a_k)}) + \sum_{k \leq t} (v_k - v_{rank(\succ_{i'}, a_k)})\end{aligned}$$

However, for any agent i', $\sum_{k\leq t} v_k \geq \sum_{k\leq t} v_{rank(\succ_{i'},a_k)}$ since the vector v is sorted, and hence we have

$$\geq \frac{t}{n} + \sum_{j\in N\setminus\{i,i'\}} \sum_{k\leq t} (v_{rank(\succ_{i'},a_k)} - v_{rank(\succ_j,a_k)}) + \sum_{k\leq t} (v_{rank(\succ_{i'},a_k)} - v_k)$$
$$= \sum_{k\leq t} \varphi^v_{i',a_k}(\succ)$$

and hence the mechanism φ^v is envy-free.

Strategy-proofness. Consider a profile $\succ = (\succ_i, \succ_{-i})$ and another profile $\succ' = (\succ'_i, \succ_{-i})$ where agent i misreports her preference. Without loss of generality, let $\succ_i = \langle a_1, \ldots, a_n\rangle$. For any $t \in \{1, 2, \ldots, n\}$, we have the following.

$$\sum_{k\leq t} \varphi^v_{i,a_k}(\succ) = \frac{t}{n} + \sum_{j\in N\setminus\{i\}} \sum_{k\leq t} (v_k - v_{rank(\succ_j,a_k)})$$

Again, since the vector v is sorted, we have $\sum_{k\leq t} v_k \geq \sum_{k\leq t} v_{rank(\succ'_i,a_k)}$ for any $\succ'_i$, and hence we have

$$\geq \frac{t}{n} + \sum_{j\in N\setminus\{i\}} \sum_{k\leq t} (v_{rank(\succ'_i,a_k)} - v_{rank(\succ_j,a_k)}) = \sum_{k\leq t} \varphi^v_{i,a_k}(\succ')$$

and thus truth-telling is a dominant strategy for any agent i and the mechanism is strategy-proof. □

Proposition 2. *A Rank Exchange mechanism φ^v is not dominated by another Rank Exchange mechanism if and only if the vector v satisfies $v_1 = \frac{1}{n(n-1)}$.*

Proof. We first show that the condition $v_1 = \frac{1}{n(n-1)}$ is sufficient to guarantee that φ^v is not stochastically dominated by another Rank Exchange mechanism. Suppose for contradiction that there exists a vector $u \in [0, \frac{1}{n(n-1)}]^n$ such that the mechanism φ^u dominates φ^v. Since φ^u strictly dominates φ^v at some preference profile, it must be that $u \neq v$.

We first claim that any such vector u must satisfy that $\forall k \in \{1, 2, \ldots, n\}, u_k \leq v_k$. Indeed, since $u_1 \leq \dfrac{1}{n(n-1)} = v_1$, the claim is trivially true for $k = 1$. For any $k > 1$, consider a profile $\succ$ where the agent 1 has the preference $\succ_1 = \langle a_1, a_2, \ldots, a_n\rangle$, while all other agents have object a_1 in the kth position in their preference, i.e. $\forall j \neq i,\ rank(\succ_j, a_1) = k$. We have,

$$\varphi^u_{1,a_1}(\succ) = \frac{1}{n} + \sum_{j\neq i}(u_1 - u_k) = \frac{1}{n} + (n-1)(u_1 - u_k)$$
$$\leq \frac{1}{n} + (n-1)(v_1 - u_k) = \frac{1}{n} + (n-1)(v_1 - v_k + v_k - u_k)$$
$$= \varphi^v_{1,a_1}(\succ) + (n-1)(v_k - u_k)$$

However, our assumption that φ^u dominates φ^v implies that $\varphi^u_{1,a_1}(\succ) \geq \varphi^v_{1,a_1}(\succ)$, and hence we must have

$$v_k \geq u_k,\ \forall k \in \{1, 2, \dots, n\} \tag{A.1}$$

On the other hand, we can also similarly show that $\forall k \in \{1, 2, \dots, n\}, u_k \geq v_k$. Since $v_n = 0$, this is trivially true for $k = n$. For any $k < n$, consider a profile $\succ$ where agent 1 has the preference $\succ_1 = \langle a_1, a_2, \dots, a_n\rangle$, while all other agents have object a_n in the kth position in their preference, i.e. $\forall j \neq i,\ rank(\succ_j, a_n) = k$.

We have,

$$\begin{aligned}
\varphi^u_{1,a_n}(\succ) &= \frac{1}{n} + \sum_{j\neq i}(u_n - u_k) = \frac{1}{n} + (n-1)(u_n - u_k) \\
&\geq \frac{1}{n} + (n-1)(v_n - u_k) = \frac{1}{n} + (n-1)(v_n - v_k + v_k - u_k) \\
&= \varphi^v_{1,a_n}(\succ) + (n-1)(v_k - u_k)
\end{aligned}$$

But our assumption that φ^u dominates φ^v implies that $\varphi^u_{1,a_n}(\succ) \leq \varphi^v_{1,a_n}(\succ)$, and hence we have

$$v_k \leq u_k,\ \forall k \in \{1, 2, \dots, n\} \tag{A.2}$$

Inequalities (A.1) and (A.2) together imply that $\forall k \in \{1, 2, \dots, n\}, u_k = v_k$ which is a contradiction since we $u \neq v$.

We next prove that the condition $v_1 = \frac{1}{n(n-1)}$ is necessary for φ^v to not be stochastically dominated. We prove the contrapositive of the statement. Suppose $v_1 < \frac{1}{n(n-1)}$. Let $u \in [0, \dfrac{1}{n(n-1)}]^n$ be such that $u_1 = \frac{1}{n(n-1)}$ and for $k \in \{2, 3, \dots, n\}$, $u_k = v_k$. Consider any agent i and any profile $\succ \in \mathcal{R}^n$. Without loss of generality, let $\succ_i = \langle a_1, \dots, a_n\rangle$. For any $t \in \{1, 2, \dots, n\}$, let us consider the total allocation obtained by agent i for her top t objects under φ^u and φ^v.

$$\sum_{\ell\leq t} \varphi^v_{i,a_\ell}(\succ) = \frac{t}{n} + \sum_{j\in N\setminus\{i\}} \sum_{\ell\leq t}(v_\ell - v_{rank(\succ_j,a_\ell)})$$

Notice that if for an agent j and some $\ell \leq t$, $rank(\succ_j, a_\ell) = 1$, then $\sum_{\ell\leq t}(v_\ell - v_{rank(\succ_j,a_\ell)}) = \sum_{\ell\leq t}(u_\ell - u_{rank(\succ_j,a_\ell)})$. On the other hand, if for any agent j and all $\ell \leq t$, $rank(\succ_j, a_\ell) \neq 1$, then $\sum_{\ell\leq t}(v_\ell - (v_{rank(\succ_j,a_\ell)}) < \sum_{\ell\leq t}(u_\ell - u_{rank(\succ_j,a_\ell)})$. Therefore, we have

$$\sum_{\ell\leq t} \varphi^v_{i,a_\ell}(\succ) \leq \frac{t}{n} + \sum_{j\in N\setminus\{i\}} \sum_{m\leq t}(u_m - u_{rank(\succ_j,a_m)}) = \sum_{\ell\leq t} \varphi^u_{i,a_\ell}(\succ)$$

Further, the above inequality is strict at any profile $\succ$ where there exists an agent j with $\sigma(\succ_j, 1) \neq \sigma(\succ_i, 1)$ and $t = 1$. Therefore, φ^u dominates φ^v. □

A.3 Proofs from Section 2.5

A.3.1 Proofs from Section 2.5.1

Lemma 2. *If a mechanism φ is envy-free and separable, then it is a pairwise exchange mechanism.*

Proof. Suppose φ is envy-free and separable.

Let $\succ$ and $\succ'$ be two arbitrary preference relations. Let $\succ_S$ denote a preference profile where agents in the set $S \subseteq N$ have preference $\succ'$ and the remaining agents in $N \setminus S$ have preference $\succ$. We first show the following technical claim.

Claim 4. *$\forall i, j \in N$ and $\forall a \in O$, $\varphi_{i,a}(\succ_{\{i\}}) = \varphi_{j,a}(\succ_{\{j\}})$.*

Proof. For convenience, $\forall i \in N$ and $\forall a \in O$, let $x_{i,a} := \varphi_{i,a}(\succ_{\{i\}})$. Then feasibility of the random assignment and envy-freeness imply that

$$\forall i \in N, \forall j \in N \setminus \{i\}, \text{ and } \forall a \in O, \quad \varphi_{j,a}(\succ_{\{i\}}) = \frac{1 - x_{i,a}}{n-1} \tag{A.3}$$

Consider any two agents $i, j \in N$, and some $k \in N \setminus \{i, j\}$, and $a \in O$. By separability, we know that,

$$\varphi_{k,a}(\succ_{\{i,j\}}) - \varphi_{k,a}(\succ_{\emptyset}) = \varphi_{k,a}(\succ_{\{i\}}) - \varphi_{k,a}(\succ_{\emptyset}) + \varphi_{k,a}(\succ_{\{j\}}) - \varphi_{k,a}(\succ_{\emptyset})$$

Therefore,

$$\varphi_{k,a}(\succ_{\{i,j\}}) + \varphi_{k,a}(\succ_{\emptyset}) = \varphi_{k,a}(\succ_{\{i\}}) + \varphi_{k,a}(\succ_{\{j\}})$$

Since φ is envy-free, $\varphi_{k,a}(\succ_{\emptyset}) = \frac{1}{n}$. Thus substituting Eq (A.3), we have,

$$\begin{aligned}\varphi_{k,a}(\succ_{\{i,j\}}) &= \frac{1 - x_{i,a}}{n-1} + \frac{1 - x_{j,a}}{n-1} - \frac{1}{n} \\ &= \frac{n+1}{n(n-1)} - \frac{1}{n-1}(x_{i,a} + x_{j,a})\end{aligned}$$

By envy-freeness, we have,

$$\begin{aligned}\varphi_{i,a}(\succ_{\{i,j\}}) = \varphi_{j,a}(\succ_{\{i,j\}}) &= \frac{1}{2}\Big[1 - (n-2)\varphi_{k,a}(\succ_{\{i,j\}})\Big] \\ &= \frac{1}{2} - \frac{(n-2)(n+1)}{2n(n-1)} + \frac{n-2}{2(n-1)}(x_{i,a} + x_{j,a}) \\ &= \frac{1}{n(n-1)} + \frac{n-2}{2(n-1)}(x_{i,a} + x_{j,a}) \qquad \text{(A.4)}\end{aligned}$$

Finally, consider any $i \in N$. Since φ is separable,

$$\varphi_{i,a}(\succ_N) - \varphi_{i,a}(\succ_{\{i\}}) = \sum_{j \in N \setminus \{i\}} \Big[\varphi_{i,a}(\succ_{\{i,j\}}) - \varphi_{i,a}(\succ_{\{i\}})\Big]$$

Substituting Eq (A.4), we get,

$$\frac{1}{n} - x_{i,a} = \sum_{j \in N \setminus \{i\}} \Big[\frac{1}{n(n-1)} + \frac{n-2}{2(n-1)}(x_{i,a} + x_{j,a}) - x_{i,a}\Big]$$

Simplifying, we get that

$$x_{i,a} = \frac{\sum_{j \in N} x_{j,a}}{n}$$

Since the above equality holds true for every $i \in N$ and $a \in O$, it follows that for any $i, j \in N$ and $a \in O$, $x_{i,a} = x_{j,a}$. □

For any $\succ, \succ' \in \mathcal{R}$ and $\forall a \in O$, define the function f as $f(\succ, \succ', a) := \varphi_{k,a}(\succ_{\{i\}}) - \frac{1}{n}$ for any $i \in N$ and any $k \in N \setminus \{i\}$. By the claim above and due to envy-freeness, the function f is well-defined since the choice of i and k does not matter. Additionally, since φ is envy-free, we also have $0 \leq \varphi_{k,a}(\succ_{\{i\}}) \leq \frac{1}{n-1} \implies -\frac{1}{n} \leq f(\succ, \succ', a) \leq \frac{1}{n}$.

Consider any profile $\succ = (\succ_k)_{k \in N}$, any agent i, and any object a. Let $\succ^{\text{id}} = (\succ_i, \succ_i, \ldots, \succ_i)$ be the preference profile where all agents report the same preference as agent i in $\succ$. Due to envy-freeness, we know that $\varphi_{i,a}(\succ^{\text{id}}) = \frac{1}{n}$. By separability we have,

$$\begin{aligned}\varphi_{i,a}(\succ) - \varphi_{i,a}(\succ^{\text{id}}) &= \sum_{j \in N \setminus \{i\}} \Big[\varphi_{i,a}(\succ_j, \succ^{\text{id}}_{-j}) - \varphi_{i,a}(\succ^{\text{id}})\Big] = \sum_{j \in N \setminus \{i\}} \Big[\varphi_{i,a}(\succ_j, \succ^{\text{id}}_{-j}) - \frac{1}{n}\Big] \\ &= \sum_{j \in N \setminus \{i\}} f(\succ_i, \succ_j, a)\end{aligned}$$

which follows from our construction of the function f. Therefore we have our desired representation,

$$\varphi_{i,a}(\succ) = \frac{1}{n} + \sum_{j \in N \setminus \{i\}} f(\succ_i, \succ_j, a)$$

□

Lemma 3. *Consider a pairwise exchange mechanism φ^f and its corresponding function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$. Then f satisfies the following properties -*

1. No transfers between equals: *$\forall \succ \in \mathcal{R}$ and $\forall a \in O$, $f(\succ, \succ, a) = 0$.*
2. Balanced transfers: *For any pair of preferences $\succ, \succ' \in \mathcal{R}$, $\sum_{a \in O} f(\succ, \succ', a) = 0$.*

3. Anti-symmetry: *For any pair of preferences* $\succ, \succ' \in \mathcal{R}$ *and* $\forall a \in O$, $f(\succ, \succ', a) = -f(\succ', \succ, a)$.

4. Bounded range: *For any pair of preferences* $\succ, \succ' \in \mathcal{R}$ *and* $\forall a \in O$, $f(\succ, \succ', a) \in [-\frac{1}{n(n-1)}, \frac{1}{n(n-1)}]$.

Proof. We prove that the if the pairwise exchange mechanism φ^f induced by the transfer function is feasible, then f must satisfy the following properties.

1 *No transfers between equals:* $\forall \succ \in \mathcal{R}$ and $\forall a \in O$, $f(\succ, \succ, a) = 0$.

Suppose for contradiction that there exists $\succ \in \mathcal{R}$ and $a \in O$ such that $f(\succ, \succ, a) \neq 0$. Consider a preference profile $\boldsymbol{\succ} = (\succ_i)_{i\in N}$ such that for all $i \in N$, $\succ_i = \succ$. However, by definition

$$\begin{aligned}\sum_{i\in N} \varphi^f_{i,a}(\boldsymbol{\succ}) &= \sum_{i\in N} \frac{1}{n} + \sum_{i\in N}\sum_{j\neq i} f(\succ_i, \succ_j, a) \\ &= 1 + n \cdot (n-1) \cdot f(\succ, \succ, a) \\ &\neq 1\end{aligned}$$

This contradicts with the feasibility of the random assignment $\varphi^f(\boldsymbol{\succ})$.

2 *Balanced transfers:* For any pair of preferences $\succ, \succ' \in \mathcal{R}$, $\sum_{a\in O} f(\succ, \succ', a) = 0$.

We prove this by contradiction. Suppose that there exist a pair of preferences $\succ, \succ' \in \mathcal{R}$ such that $\sum_{a\in O} f(\succ, \succ', a) \neq 0$. Consider a preference profile $\boldsymbol{\succ} = (\succ_i, \boldsymbol{\succ}_{-i})$ such that for some $i \in N$, $\succ_i = \succ$ and for any $j \neq i$, $\succ_j = \succ'$. The total allocation for agent i is given by

$$\begin{aligned}\sum_{a\in O} \varphi^f_{i,a}(\boldsymbol{\succ}) &= \sum_{a\in O} \frac{1}{n} + \sum_{a\in O}\sum_{j\neq i} f(\succ_i, \succ_j, a) \\ &= 1 + \sum_{a\in O}\sum_{j\neq i} f(\succ, \succ', a) \\ &= 1 + (n-1) \cdot \sum_{a\in O} f(\succ, \succ', a) \\ &\neq 1\end{aligned}$$

which contradicts feasibility of $\varphi^f(\boldsymbol{\succ})$.

3 *Anti-symmetry:* For any pair of preferences $\succ, \succ' \in \mathcal{R}$ and $\forall a \in O$, $f(\succ, \succ', a) = -f(\succ', \succ, a)$.

We again proceed by contradiction. Suppose $\succ, \succ' \in \mathcal{R}$ and $a \in O$ are such that $f(\succ, \succ', a) \neq -f(\succ', \succ, a)$. Let us again consider the preference profile $\boldsymbol{\succ} = (\succ_k, \boldsymbol{\succ}_{-k})$ such that for some $k \in N$, $\succ_k = \succ$ and for any $j \neq k$, $\succ_j = \succ'$. If we sum the probability of being assigned object a over all agents

$$\begin{aligned}
\sum_{i\in N} \varphi^f_{i,a}(\boldsymbol{\succ}) &= \sum_{i\in N} \frac{1}{n} + \sum_{i\in N}\sum_{j\neq i} f(\succ_i, \succ_j, a) \\
&= 1 + \sum_{j\neq k} f(\succ_k, \succ_j, a) + \sum_{i\neq k}\sum_{j\neq i} f(\succ_i, \succ_j, a) \\
&= 1 + \sum_{j\neq k} f(\succ, \succ', a) + \sum_{i\neq k}\sum_{j\in N\setminus\{i,k\}} f(\succ_i, \succ_j, a) + \sum_{i\neq k} f(\succ_i, \succ_k, a) \\
&= 1 + \sum_{j\neq k} f(\succ, \succ', a) + \sum_{i\neq k}\sum_{j\in N\setminus\{i,k\}} f(\succ', \succ', a) + \sum_{i\neq k} f(\succ', \succ, a)
\end{aligned}$$

Since f satisfies the property that there are no transfers between identical preferences (Property (1))

$$\begin{aligned}
\sum_{i\in N} \varphi^f_{i,a}(\boldsymbol{\succ}) &= 1 + \sum_{j\neq k} f(\succ, \succ', a) + \sum_{i\neq k} f(\succ', \succ, a) \\
&\neq 1
\end{aligned}$$

which leads to a contradiction

4 *Bounded range:* For any $\succ, \succ' \in \mathcal{R}$ and $\forall a \in O$, $f(\succ, \succ', a) \in [-\frac{1}{n(n-1)}, \frac{1}{n(n-1)}]$.

Consider a preference profile $\boldsymbol{\succ} = (\succ_i, \boldsymbol{\succ}_{-i})$ such that for some $i \in N$, $\succ_i = \succ$ and for all $j \neq i$, $\succ_j = \succ'$. Since $\varphi^f_{i,a}(\boldsymbol{\succ}) \geq 0$, we have

$$\begin{aligned}
\frac{1}{n} + \sum_{j\neq i} f(\succ_i, \succ_j, a) &\geq 0 \\
\frac{1}{n} + (n-1) f(\succ, \succ', a) \geq 0 \implies f(\succ, \succ', a) &\geq \frac{-1}{n(n-1)}
\end{aligned}$$

Exchanging the roles of $\succ$ and $\succ'$, we similarly obtain $f(\succ', \succ, a) \geq \frac{-1}{n(n-1)}$. Thus, by the *anti-symmetry* property defined above, we have $f(\succ, \succ', a) = -f(\succ', \succ, a) \leq \frac{1}{n(n-1)}$.

□

A.3.2 Proofs from Section 2.5.2

To prove that that strategy-proofness is equivalent to envy-freeness for any pairwise exchange mechanism φ^f, we need the following four lemmas.

Lemma 11. *Suppose* $\succ = \langle a_1, a_2, \ldots, a_n \rangle$*. If* φ^f *is strategy-proof, then* $\forall \succ', \succ'' \in \mathcal{R}$ *and* $\forall t \in \{1, 2, \ldots, n\}$*,* $\sum_{k=1}^{t} f(\succ, \succ'', a_k) \geq \sum_{k=1}^{t} f(\succ', \succ'', a_k)$*.*

Proof. We prove the contrapositive. Suppose there exists $\succ', \succ'' \in \mathcal{R}$ and some $t \in \{1, 2, \ldots, n\}$, such that $\sum_{k=1}^{t} f(\succ, \succ'', a_k) < \sum_{k=1}^{t} f(\succ', \succ'', a_k)$. Consider a profile $\boldsymbol{\succ}$ such that for some $i \in N$, $\succ_i = \succ$ and for all $j \neq i$, $\succ_j = \succ''$. Let $\boldsymbol{\succ}'$ be a preference profile where agent i lies and reports $\succ_i' = \succ'$ while all other agents continue to report $\succ''$ as in $\boldsymbol{\succ}$.

We have that,

$$\begin{aligned}
\sum_{k=1}^{t} \varphi^f_{i,a_k}(\boldsymbol{\succ}) &= \frac{t}{n} + \sum_{j \neq i} \sum_{k=1}^{t} f(\succ_i, \succ_j, a_k) = \frac{t}{n} + \sum_{j \neq i} \sum_{k=1}^{t} f(\succ, \succ'', a_k) \\
&= \frac{t}{n} + (n-1) \cdot \sum_{k=1}^{t} f(\succ, \succ'', a_k) \\
&< \frac{t}{n} + (n-1) \cdot \sum_{k=1}^{t} f(\succ', \succ'', a_k) = \frac{t}{n} + \sum_{j \neq i} \sum_{k=1}^{t} f(\succ_i', \succ_j, a_k) \\
&= \sum_{k=1}^{t} \varphi^f_{i,a_k}(\boldsymbol{\succ}')
\end{aligned}$$

Thus φ^f is not strategy-proof. □

Lemma 12. *A mechanism* φ^f *is strategy-proof if and only if the function* f *satisfies the following two properties -*

1. Sender-invariance*: If* $\succ' = \langle a_1, a_2, \ldots, a_n \rangle$ *and* $\succ''$ *is such that* $\forall k \leq t$*,* $\sigma(\succ'', k) = \sigma(\succ', k) = a_k$ *, then* $\forall \succ \in \mathcal{R}$*,* $f(\succ, \succ', a_t) = f(\succ, \succ'', a_t)$*. Similarly, if* $\forall k \geq t$*,* $\sigma(\succ'', k) = \sigma(\succ', k) = a_k$ *, then* $f(\succ, \succ', a_t) = f(\succ, \succ'', a_t)$*.*

2. Swap-monotonicity: *For all* $\succ, \succ' \in \mathcal{R}$ *and* $\succ'' \in \Gamma(\succ')$ *with* $a \succ' b$ *but* $b \succ'' a$ *for some* $a, b \in O$*, we have* $f(\succ', \succ, a) \geq f(\succ'', \succ, a)$*.*

Proof. That swap-monotonicity and sender-invariance of the function f implies strategy-proofness of φ^f follows from Lemma 1 and definitions.

For the necessity part of the lemma, we first show that strategy-proofness implies sender-invariance. Let $\succ' = \langle a_1, a_2, \ldots, a_n \rangle$, $\succ''$ be such that $\forall k \leq t$, $\sigma(\succ', k) = \sigma(\succ'', k)$. Consider any $\succ \in \mathcal{R}$. At a preference profile $\boldsymbol{\succ}$, where for some agent $i \in N$, $\succ_i = \succ'$ and for all other agents $j \in N \setminus \{i\}$, $\succ_j = \succ$, and a mis-report $\succ_i' = \succ''$ for agent i, strategy-proofness implies that agent i must receive the same allocation for her top $(t-1)$ objects as well her top t objects under $\succ_i$ and $\succ_i'$ since $\forall k \leq t$, $\sigma(\succ', k) = \sigma(\succ'', k)$. Let $P^{\boldsymbol{\succ}} := \varphi^f(\boldsymbol{\succ})$ and

$P^{(\succ_i', \boldsymbol{\succ}_{-i})} := \varphi^f(\succ_i', \boldsymbol{\succ}_{-i})$. We have,

$$\sum_{k=1}^{t-1} P^{\boldsymbol{\succ}}_{i,a_k} = \sum_{k=1}^{t-1} P^{(\succ_i', \boldsymbol{\succ}_{-i})}_{i,a_k}$$

$$\implies \frac{t-1}{n} + (n-1)\sum_{k=1}^{t-1} f(\succ', \succ, a_k) = \frac{t-1}{n} + (n-1)\sum_{k=1}^{t-1} f(\succ'', \succ, a_k) \qquad \text{(A.5)}$$

and

$$\sum_{k=1}^{t} P^{\boldsymbol{\succ}}_{i,a_k} = \sum_{k=1}^{t} P^{(\succ_i', \boldsymbol{\succ}_{-i})}_{i,a_k}$$

$$\implies \frac{t}{n} + (n-1)\sum_{k=1}^{t} f(\succ', \succ, a_k) = \frac{t}{n} + (n-1)\sum_{k=1}^{t} f(\succ'', \succ, a_k) \qquad \text{(A.6)}$$

Equations (A.5) and (A.6) together imply that $f(\succ', \succ, a_t) = f(\succ'', \succ, a_t)$. Since f satisfies anti-symmetry, we have our desired result that $f(\succ, \succ', a_t) = f(\succ, \succ'', a_t)$. The proof for the case $\forall k \geq t$ follows similarly.

Next, if the transfer function f is not swap-monotonic, then there exists $\succ' \in \mathcal{R}$, which we can without loss of generality assume to be $\succ' = \langle a_1, a_2, \ldots, a_n \rangle$, $\succ'' \in \Gamma(\succ')$ such that $a_t \succ' a_{t+1}$ but $a_{t+1} \succ'' a_t$ for some $t \in \{1, 2, \ldots, n\}$ and $\succ \in \mathcal{R}$ such that $f(\succ', \succ, a_t) < f(\succ'', \succ, a_t)$. At the preference profile $\boldsymbol{\succ}$, where for some agent $i \in N$, $\succ_i = \succ'$ and for all other agents $j \in N \setminus \{i\}$, $\succ_j = \succ$, if agent i chooses to deviate and report $\succ_i' = \succ''$, we have

$$P^{\boldsymbol{\succ}}_{i,a_t} = \frac{1}{n} + (n-1)f(\succ', \succ, a_t) < \frac{1}{n} + (n-1)f(\succ'', \succ, a_t) = P^{(\succ_i', \boldsymbol{\succ}_{-i})}_{i,a_t}$$

which implies that the mechanism φ^f is not swap-monotonic and hence is not strategy-proof. □

Lemma 13. *Let $\succ = \langle a_1, a_2, \ldots, a_n \rangle$ be an arbitrary preference relation, and $\succ'$ be such that $\forall k \leq t$, $\sigma(\succ', k) = \sigma(\succ, k) = a_k$. If φ^f is envy-free, then $f(\succ, \succ', a_t) = 0$. Similarly, if $\succ'$ is such that $\forall k \geq t$, $\sigma(\succ', k) = \sigma(\succ, k) = a_k$, then $f(\succ, \succ', a_t) = 0$.*

Proof. Let $\succ = \langle a_1, a_2, \ldots, a_n \rangle$ and $\succ'$ be such that $\forall k \leq t$, $\sigma(\succ, k) = \sigma(\succ', k)$. Consider a preference profile $\boldsymbol{\succ}$ where for some agent $i \in N$, $\succ_i = \succ$ and for all other agents $j \in N \setminus \{i\}$, $\succ_j = \succ'$. For agent i to not envy any other agent $j \in N \setminus \{i\}$, they must get the same allocation for agent i's top $t-1$ and top t objects, since $\forall k \leq t$, $\sigma(\succ_i, k) = \sigma(\succ_j, k)$. That is,

$$\sum_{k=1}^{t-1} \varphi^f_{i,a_k}(\boldsymbol{\succ}) = \sum_{k=1}^{t-1} \varphi^f_{j,a_k}(\boldsymbol{\succ}) \text{ where } j \in N \setminus \{i\}$$

$$\implies \frac{t-1}{n} + (n-1)\sum_{k=1}^{t-1} f(\succ, \succ', a_k) = \frac{t-1}{n} + \sum_{k=1}^{t-1} f(\succ', \succ, a_k) + (n-2)\sum_{k=1}^{t-1} f(\succ', \succ', a_k)$$

By the no transfer among equals property from Lemma 3, $f(\succ', \succ', o) = 0$ for all $o \in O$, which in conjunction with the anti-symmetry property of f implies that

$$\sum_{k=1}^{t-1} f(\succ, \succ', a_k) = 0$$

Similarly we get,

$$\sum_{k=1}^{t} f(\succ, \succ', a_k) = 0 \implies f(\succ, \succ', a_t) = 0$$

The argument for the second part of the lemma is symmetric to the one made above. □

Lemma 14. *Let $\succ = \langle a_1, a_2, \ldots, a_n \rangle$ be an arbitrary preference relation. If φ^f is envy-free, then for all $\succ' \in \mathcal{R}$, for all $t \in \{1, 2, \ldots, n\}$, $\sum_{k=1}^{t} f(\succ, \succ', a_t) \geq 0$.*

Proof. Consider a preference profile $\boldsymbol{\succ}$ such that for some agent $i \in N$, $\succ_i = \succ$ and for all agents $j \in N \setminus \{i\}$, $\succ_j = \succ'$. By envy-freeness, the allocation that agent i receives for her top t objects is at least as much as the allocation any other agent j receives. That is,

$$\begin{aligned} \sum_{k=1}^{t} \varphi^f_{i,a_k}(\boldsymbol{\succ}) &\geq \sum_{k=1}^{t} \varphi^f_{j,a_k}(\boldsymbol{\succ}) \\ \implies \frac{t}{n} + (n-1) \cdot \sum_{k=1}^{t} f(\succ, \succ', a_k) &\geq \frac{t}{n} + \sum_{k=1}^{t} f(\succ', \succ, a_k) + (n-2) \cdot \sum_{k=1}^{t} f(\succ', \succ', a_k) \\ \implies \sum_{k=1}^{t} f(\succ, \succ', a_k) &\geq 0 \end{aligned}$$

where the last inequality follows from the no transfer and anti-symmetry properties. □

We are now ready to prove Theorem 4.

Theorem 4. *For any pairwise exchange mechanism φ^f, φ^f is strategy-proof if and only if φ^f is envy-free.*

Proof. We first show that if a pairwise exchange mechanism φ^f is strategy-proof, then it is envy-free. Suppose, for contradiction, that φ^f is strategy-proof but not envy-free. Therefore, there exists some $\boldsymbol{\succ} \in \mathcal{R}$, some agent $i \in N$, another agent $i' \in N \setminus \{i\}$, and some $t \in \{1, 2, \ldots, n\}$ such that the total allocation that agent i gets for her top t objects is strictly smaller than what agent i' receives. Without loss of generality, let $\succ_i = \langle a_1, a_2, \ldots, a_n \rangle$. Since agent i envies i' we have,

$$\sum_{k=1}^{t} \varphi^f_{i',a_k}(\boldsymbol{\succ}) > \sum_{k=1}^{t} \varphi^f_{i,a_k}(\boldsymbol{\succ})$$

However, since φ^f is strategy-proof, at preference profile $\succ$, agent i does not gain if she chooses to deviate and report her preference as $\succ_i' = \succ_{i'}$. Let $\succ' = (\succ_i' = \succ_{i'}, \succ_{-i})$ be the resulting preference profile, when agent i mis-reports her preference. Then we have $\sum_{k=1}^{t} \varphi^f_{i,a_k}(\succ) \geq \sum_{k=1}^{t} \varphi^f_{i,a_k}(\succ')$. Substituting back into the above inequality, we have

$$\sum_{k=1}^{t} \varphi^f_{i',a_k}(\succ) > \sum_{k=1}^{t} \varphi^f_{i,a_k}(\succ')$$

Substituting the allocations by their transfer function representation, we get

$$\sum_{k=1}^{t} f(\succ_{i'}, \succ_i, a_k) + \sum_{j \in N \setminus \{i,i'\}} \sum_{k=1}^{t} f(\succ_{i'}, \succ_j, a_k) > \sum_{k=1}^{t} f(\succ_i', \succ_{i'}, a_k) + \sum_{j \in N \setminus \{i,i'\}} \sum_{k=1}^{t} f(\succ_i', \succ_j, a_k)$$

$$= \sum_{k=1}^{t} f(\succ_{i'}, \succ_{i'}, a_k) + \sum_{j \in N \setminus \{i,i'\}} \sum_{k=1}^{t} f(\succ_{i'}, \succ_j, a_k)$$

By the no transfers between equals property in Lemma 3, we have that $f(\succ_{i'}, \succ_{i'}, a_k) = 0, \forall k$, and hence we obtain

$$\sum_{k=1}^{t} f(\succ_{i'}, \succ_i, a_k) > 0 = \sum_{k=1}^{t} f(\succ_i, \succ_i, a_k)$$

But this contradicts Lemma 11 when we set $\succ = \succ_i, \succ' = \succ_{i'}$, and $\succ'' = \succ_i$, and hence if φ^f is strategy-proof then it must also be envy-free.

We now prove that envy-freeness implies sender-invariance and swap-monotonicity, and consequently implies strategy-proofness. We prove by contradiction. Suppose that φ^f is envy-free, but the transfer function f is not sender-invariant. Then, there exists $\succ', \succ'' \in \mathcal{R}$ such that either $\forall k \leq t$, $\sigma(\succ', k) = \sigma(\succ'', k)$ or $\forall k \geq t$, $\sigma(\succ', k) = \sigma(\succ'', k)$ and a preference $\succ \in \mathcal{R}$ such that $f(\succ, \succ', \sigma(\succ', t)) \neq f(\succ, \succ'', \sigma(\succ'', t))$. Suppose $\succ'$ is such that $\forall k \leq t$, $\sigma(\succ', k) = \sigma(\succ'', k)$ (the argument for the other case is symmetric). Let $\succ' = \langle a_1, a_2, \ldots, a_n \rangle$. Let t be the smallest such index at which $f(\succ, \succ', a_t) \neq f(\succ, \succ'', a_t)$. Therefore, we have for all $k < t, f(\succ, \succ', a_k) = f(\succ, \succ'', a_k)$. Now, consider a preference profile $\succ$, where for some agent $i \in N$, $\succ_i = \succ'$, for another agent $i' \in N \setminus \{i\}$, $\succ_{i'} = \succ''$, and for all other agents $j \in N \setminus \{i, i'\}$, $\succ_j = \succ$. The total allocation that agent i gets for her top t objects is given by,

$$\begin{aligned}
\sum_{k=1}^{t} \varphi^f_{i,a_k}(\succ) &= \frac{t}{n} + \sum_{k=1}^{t} f(\succ', \succ'', a_k) + (n-2)\sum_{k=1}^{t} f(\succ', \succ, a_k) \\
&= \frac{t}{n} + (n-2)\sum_{k=1}^{t} f(\succ', \succ, a_k) \\
&\neq \frac{t}{n} + (n-2)\sum_{k=1}^{t} f(\succ'', \succ, a_k) \\
&= \frac{t}{n} + \sum_{k=1}^{t} f(\succ'', \succ', a_k) + (n-2)\sum_{k=1}^{t} f(\succ'', \succ, a_k) \\
&= \sum_{k=1}^{t} \varphi^f_{i',a_k}(\succ)
\end{aligned}$$

The second and fourth equality above follow from Lemma 13. Therefore, we find that one of agent i or i' would envy the other, a contradiction.

Next, suppose that φ^f is envy-free, but the transfer function f is not swap-monotonic. Then, there exists $\succ' \in \mathcal{R}$, which we without loss assume to be $\succ' = \langle a_1, a_2, \ldots, a_n \rangle$, $\succ'' \in \Gamma(\succ')$ with $a_t \succ' a_{t+1}$ but $a_{t+1} \succ'' a_t$ for some $t \in \{1, 2, \ldots, n\}$, and $\succ \in \mathcal{R}$ such that $f(\succ', \succ, a_t) < f(\succ'', \succ, a_t)$. By sender-invariance, we have $\sum_{k=1}^{t-1} f(\succ', \succ, a_k) = \sum_{k=1}^{t-1} f(\succ'', \succ, a_k)$ and $\sum_{k=t+2}^{n} f(\succ', \succ, a_k) = \sum_{k=t+2}^{n} f(\succ'', \succ, a_k)$. Since the transfers are balanced (Property (2) of Lemma 3), it must be that $f(\succ', \succ, a_{t+1}) > f(\succ'', \succ, a_{t+1})$.

Without loss, let $rank(\succ, a_t) < rank(\succ, a_{t+1})$, i.e. $a_t \succ a_{t+1}$ (for the other case, we can just interchange the roles of $\succ'$ and $\succ''$). We can now construct a preference $\tilde{\succ} \in \mathcal{R}$ such that $rank(\tilde{\succ}, a_{t+1}) = n$ and for all $s \in \{1, 2, \ldots, rank(\succ, a_t)\}, \sigma(\tilde{\succ}, s) = \sigma(\succ, s)$. Using sender-invariance we have, $f(\succ', \tilde{\succ}, a_t) = f(\succ', \succ, a_t) < f(\succ'', \succ, a_t) = f(\succ'', \tilde{\succ}, a_t)$. The same property also implies that $\sum_{k=1}^{t-1} f(\succ', \tilde{\succ}, a_k) = \sum_{k=1}^{t-1} f(\succ'', \tilde{\succ}, a_k)$ and $\sum_{k=t+2}^{n} f(\succ', \tilde{\succ}, a_k) = \sum_{k=t+2}^{n} f(\succ'', \tilde{\succ}, a_k)$, which in conjunction with the balancedness of f (Property (2) of Lemma 3) results in $f(\succ', \tilde{\succ}, a_{t+1}) > f(\succ'', \tilde{\succ}, a_{t+1})$.

Finally, let us construct a preference $\hat{\succ} \in \mathcal{R}$ such that $rank(\hat{\succ}, a_{t+1}) = n$ and for all $s \in \{1, 2, \ldots, t\}$, $\sigma(\hat{\succ}, s) = \sigma(\succ', s) = a_s$. For such a preference, sender invariance implies that $f(\succ', \hat{\succ}, a_{t+1}) = f(\succ', \tilde{\succ}, a_{t+1}) > f(\succ'', \tilde{\succ}, a_{t+1}) = f(\succ'', \hat{\succ}, a_{t+1})$. We again have that $\sum_{k=1}^{t-1} f(\succ', \hat{\succ}, a_k) = \sum_{k=1}^{t-1} f(\succ'', \hat{\succ}, a_k)$ and $\sum_{k=t+2}^{n} f(\succ', \hat{\succ}, a_k) = \sum_{k=t+2}^{n} f(\succ'', \hat{\succ}, a_k)$. Therefore, $f(\succ', \hat{\succ}, a_t) < f(\succ'', \hat{\succ}, a_t)$. But, $f(\succ', \hat{\succ}, a_t) = 0$ from Lemma 13, which implies that $f(\succ'', \hat{\succ}, a_t) > 0$. Consequently, $f(\hat{\succ}, \succ'', a_t) < 0$. But Lemma 13 implies that $f(\hat{\succ}, \succ'', a_k) = 0$ for all $k \leq (t-1)$. Therefore, we get $\sum_{k=1}^{t} f(\hat{\succ}, \succ'', a_k) < 0$, which contradicts Lemma 14. □

A.3.3 Proofs from Section 2.5.3

To prove Lemma 4, we need the following two lemmas in addition to Lemma 12. The following simple lemma shows that the transfer functions for a neutral and strategy-proof pairwise exchange mechanisms satisfy two additional properties, namely neutrality and receiver-invariance, that will be useful in arriving at characterization for this class of mechanisms. The proof follows directly from the definitions of the mechanism and the corresponding properties. Intuitively, if either of these two properties is not satisfied by the transfer function, then we can construct corresponding preference profiles that demonstrate that the mechanism cannot be strategy-proof.

Lemma 15. *If the mechanism φ^f is strategy-proof and neutral, then the function f satisfies the following properties -*

1. Neutrality: *The function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$ is neutral, i.e.* $f(\succ, \succ', a_i) = f(\pi(\succ), \pi(\succ'), \pi(a_i))$ *for any permutation* $\pi : O \to O$.
2. Receiver-invariance: *If* $\succ = \langle a_1, a_2, \dots, a_n \rangle$ *and* $\succ'$ *and* $\succ''$ *are such that* $rank(\succ', a_k) = rank(\succ'', a_k)$ $\forall k \leq t$, *then* $f(\succ, \succ', a_t) = f(\succ, \succ'', a_t)$. *Similarly, if* $rank(\succ', a_k) = rank(\succ'', a_k)$ $\forall k \geq t$, *then* $f(\succ, \succ', a_t) = f(\succ, \succ'', a_t)$.

Proof. From the definition of pairwise exchange mechanisms, it follows that mechanism φ_f is neutral if and only if the function f is neutral, i.e., it satisfies property (1).

We now proceed to show that the receiver-invariance property (2) is necessary. Suppose for contradiction that we have $\succ = \langle a_1, a_2, \dots, a_n \rangle$ and $\succ'$ and $\succ''$ such that $\forall k \leq t$, $rank(\succ', a_k) = rank(\succ'', a_k)$, but $f(\succ, \succ', a_t) \neq f(\succ, \succ'', a_t)$. Without loss of generality, let $f(\succ, \succ', a_t) < f(\succ, \succ'', a_t)$. Fix an agent $i \in N$. Now consider a preference profile $\boldsymbol{\succ} = (\succ_i, \boldsymbol{\succ}_{-i})$ where for some agent $i \in N$, $\succ_i = \succ$ and $\forall j \neq i$, $\succ_j = \succ'$. Let $\pi : O \to O$ be a permutation such that $\pi(\succ'') = \succ'$. Observe that since $rank(\succ', a_k) = rank(\succ'', a_k)$ $\forall k \leq t$, we have $\forall k \leq t$, $\pi(a_k) = a_k$. Let $\succ'_i = \pi(\succ_i)$ be the preference that results from re-labeling objects in $\succ_i$ using π. Note that in both $\succ_i$ and $\succ'_i$, we will have $\forall k \leq t$, $\sigma(\succ_i, k) = \sigma(\succ'_i, k)$. Since f satisfies neutrality, $f(\succ_i, \succ'', a_t) = f(\pi(\succ_i), \pi(\succ''), \pi(a_t)) = f(\succ'_i, \succ', a_t)$, which in conjunction with our assumption $f(\succ, \succ'', a_t) > f(\succ, \succ', a_t) = f(\succ_i, \succ', a_t)$ implies that $f(\succ'_i, \succ', a_t) > f(\succ_i, \succ', a_t)$. Now, starting from $\boldsymbol{\succ}$, suppose agent i chooses to instead report her preference as $\succ'_i$. Let $\boldsymbol{\succ}' = (\succ'_i, \boldsymbol{\succ}_{-i})$ be the resulting profile. Since the mechanism is strategy-proof, $\sum_{k=1}^{t-1} \varphi^f_{i,a_k}(\boldsymbol{\succ}) = \sum_{k=1}^{t-1} \varphi^f_{i,a_k}(\boldsymbol{\succ}')$. The probability that agent i is assigned the object a_t when she reports her true preferences is

$$\varphi^f_{i,a_t}(\boldsymbol{\succ}) = \frac{1}{n} + \sum_{j \neq i} f(\succ_i, \succ_j, a_t) = \frac{1}{n} + (n-1) \cdot f(\succ_i, \succ', a_t)$$

The second equality above comes from the construction of our profile, where $\succ_j = \succ' \forall j \neq i$. If agent i reports $\succ_i'$ we have

$$\begin{aligned}\varphi^f_{i,a_t}(\succ') &= \frac{1}{n} + \sum_{j\neq i} f(\succ_i', \succ_j, a_t)\\ &= \frac{1}{n} + (n-1)\cdot f(\succ_i', \succ', a_t)\\ &> \frac{1}{n} + (n-1)\cdot f(\succ_i, \succ', a_t)\\ &= \varphi^f_{i,a_t}(\succ)\end{aligned}$$

which leads to a contradiction. The proof for the second part of the receiver-invariance property follows symmetrically. □

Lemma 16. *If a transfer function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$ satisfies the properties of neutrality, sender invariance, and receiver invariance, as well as the following:*

$\sum_{a\in O} f(\succ, \succ', a) = c$ *for any pair of preferences* $\succ, \succ' \in \mathcal{R}$*, where* $c \in \mathbb{R}$ *is some constant.*

then for any object $a \in O$, and preferences $\succ, \succ', \succ'' \in \mathcal{R}$, we have

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

Proof. We prove this claim by induction on n.

Base Step. For the base case, let $n = 3$ and $O = \{a_1, a_2, a_3\}$. Since f is neutral, we will let $\succ = \langle a_1, a_2, a_3\rangle$. There are six preferences in $\mathcal{R}$. We list these preferences and the transfer functions for $\succ$ with each of them in Table A.2.

By the receiver invariance property, we know that for object a_1 and $\forall \succ', \succ'' \in \mathcal{R}$

$$rank(\succ', a_1) = rank(\succ'', a_1) \implies f(\succ, \succ', a_1) = f(\succ, \succ'', a_1)$$

Let $x_i = f(\succ, \succ', a_1)$ be the transfers when $rank(\succ', a_1) = i$. By the same property, we also have that for object a_3 and $\forall \succ', \succ'' \in \mathcal{R}$

$$rank(\succ', a_3) = rank(\succ'', a_3) \implies f(\succ, \succ', a_3) = f(\succ, \succ'', a_3)$$

Let $y_i = f(\succ, \succ', a_3)$ be the transfers when $rank(\succ', a_3) = i$.

By the sender invariance property for object a_2, if $rank(\succ', a_2) = rank(\succ'', a_2) = 1$ or $rank(\succ', a_2) = rank(\succ'', a_2) = 3$, then $f(\succ, \succ', a_2) = f(\succ, \succ'', a_2)$. Let $w_i = f(\succ, \succ', a_2)$ when $rank(\succ', a_2) = i$ for $i \in \{1, 3\}$. It remains to be shown that the claim is true when $rank(\succ', a_2) = rank(\succ'', a_2) = 2$.

Since $\forall \succ' \in \mathcal{R}, \sum_{a\in O} f(\succ, \succ', a) = c$, we have

$$\sum_{a\in O} f(\succ, \langle a_1, a_3, a_2\rangle, a) + \sum_{a\in O} f(\succ, \langle a_2, a_1, a_3\rangle, a) = \sum_{a\in O} f(\succ, \langle a_2, a_3, a_1\rangle, a) + \sum_{a\in O} f(\succ, \langle a_3, a_1, a_2\rangle, a)$$
$$x_1 + w_3 + y_2 + x_2 + w_1 + y_3 = x_3 + w_1 + y_2 + x_2 + w_3 + y_1$$
$$x_1 + y_3 = x_3 + y_1$$

This implies that when $rank(\succ', a_2) = rank(\succ'', a_2) = 2$, $f(\succ, \succ', a_2) = f(\succ, \succ'', a_2)$ as desired.

	$\langle a_1, a_2, a_3\rangle$	$\langle a_1, a_3, a_2\rangle$	$\langle a_2, a_1, a_3\rangle$	$\langle a_2, a_3, a_1\rangle$	$\langle a_3, a_1, a_2\rangle$	$\langle a_3, a_2, a_1\rangle$
a_1	x_1	x_1	x_2	x_3	x_2	x_3
a_2		w_3	w_1	w_1	w_3	
a_3	y_3	y_2	y_3	y_2	y_1	y_1

Table A.2: Transfer function f for $n = 3$

Induction Step. For the induction step, suppose the claim is true for $n-1 \geq 3$. We need to show that the claim continues to hold for n. Let $O = \{a_1, a_2, \dots a_n\}$. For ease of exposition, let for any $k \in \mathbb{N}, [k] := \{1, 2, \dots, k\}$. Without loss of generality, let $\succ = \langle a_1, a_2, \dots a_n\rangle$. For any $p, q \in [n]$, we define $\mathcal{R}_{p,q} = \{\succ' \in \mathcal{R} \mid rank(\succ', a_1) = p \text{ and } rank(\succ', a_n) = q\}$ to be the set of all preferences where the ranks of object a_1 and a_n are p and q respectively. Let $\mathcal{R}_{p,*} = \bigcup_{q\in[n]} \mathcal{R}_{p,q}$ and $\mathcal{R}_{*,q} = \bigcup_{p\in[n]} \mathcal{R}_{p,q}$.

Claim 5. *For all* $1 \leq p \leq n$, $\forall \succ', \succ'' \in \mathcal{R}_{p,*}$, *and* $\forall a \in O$,

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

Proof. By the definition of $\mathcal{R}_{p,*}$, we have for any two preferences $\succ', \succ'' \in \mathcal{R}_{p,*}$, $rank(\succ', a_1) = rank(\succ'', a_1) = p$. Thus, by receiver invariance, we must have $f(\succ, \succ', a_1) = f(\succ, \succ'', a_1)$.

Let $O^{(n-1)} = \{a_2, \dots, a_n\}$ denote the set of $n-1$ objects apart from a_1, and let $\mathcal{R}^{(n-1)}$ denote the set of all strict preferences over O^{n-1}. We can now define an auxiliary function $f' : \mathcal{R}^{(n-1)} \times \mathcal{R}^{(n-1)} \times O^{(n-1)} \to [-\frac{1}{n-1}, \frac{1}{n-1}]$ as follows

$$f'(\succ, \succ', a) = f(\tilde{\succ}, \tilde{\succ}', a) \ \forall \succ, \succ' \in \mathcal{R}^{(n-1)}, \forall a \in O^{(n-1)}$$

where $\tilde{\succ}$ is obtained by appending a_1 to the beginning of preference relation $\succ$, and $\tilde{\succ}'$ is obtained by inserting a_1 at the pth position in $\succ'$. Now, by receiver-invariance and neutrality, $f(\tilde{\succ}, \tilde{\succ}', a_1)$ is equal for any such preferences, and hence we maintain for all $\succ, \succ' \in \mathcal{R}^{[n-1]}$, $\sum_{a\in O^{[n-1]}} f'(\succ, \succ', a) = c - f(\tilde{\succ}, \tilde{\succ}', a_1) = c'$. Further if $\succ = \langle a_2, \dots, a_n\rangle$, and $\succ'$ and $\succ''$ satisfy that for all $2 \leq k \leq t$, $rank(\succ', a_k) = rank(\succ'', a_k)$, then we have $f'(\succ, \succ', a_t) = f(\tilde{\succ}, \tilde{\succ}', a_t) = f(\tilde{\succ}, \tilde{\succ}'', a_t) = f'(\succ, \succ'', a_t)$ where the second equality follows from the receiver invariance of function f. Thus, the constructed function f' also satisfies receiver invariance. We can similarly verify that the function f' also satisfies neutrality and sender invariance. Hence we can use the induction hypothesis to complete the proof of the claim. □

We can use a symmetric argument on object a_n to prove the next claim.

Claim 6. *For all* $1 \leq q \leq n$, $\forall \succ', \succ'' \in \mathcal{R}_{*,q}$, *and* $\forall a \in O$,

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

In order to complete the proof of the lemma, we now need to argue that the claim is true for any $\succ' \in \mathcal{R}_{p,q}$ and $\succ'' \in \mathcal{R}_{s,t}$ where $p \neq s$ and $q \neq t$.

Claim 7. *For all* $\succ' \in \mathcal{R}_{p,q}, \succ'' \in \mathcal{R}_{s,t}$ *such that* $p \neq s$ *and* $q \neq t$, *and* $\forall a \in O$,

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

Proof. Fix $p \neq s$ and $q \neq t$, $\succ' \in \mathcal{R}_{p,q}, \succ'' \in \mathcal{R}_{s,t}$, and some object $a \in O$ such that $rank(\succ', a) = rank(\succ'', a)$. Note that $a \neq a_n$ and $a \neq a_1$ by definition of the sets $\mathcal{R}_{p,q}$ and $\mathcal{R}_{s,t}$. We consider two cases on the object a.

Case 1: $a \neq a_{n-1}$

Let $\tilde{\succ}' \in \mathcal{R}_{*,q}$ and $\tilde{\succ}'' \in \mathcal{R}_{*,t}$ be arbitrary preferences such that

1. $rank(\tilde{\succ}', a) = rank(\succ', a) = rank(\tilde{\succ}'', a) = rank(\succ'', a)$
2. $rank(\tilde{\succ}', a_n) = q$ and $rank(\tilde{\succ}'', a_n) = t$
3. $\forall i \notin \{n-1, n\}$, $rank(\tilde{\succ}', a_i) = rank(\tilde{\succ}'', a_i)$

Note that since $a \neq a_{n-1}$, we are guaranteed to be able to find such preferences, by setting $rank(\tilde{\succ}', a_{n-1}) = rank(\succ'', a_n) = t$ and $rank(\tilde{\succ}'', a_{n-1}) = rank(\succ', a_n) = q$. Now, by Claim 6, we must have $f(\succ, \succ', a) = f(\succ, \tilde{\succ}', a)$ and $f(\succ, \succ'', a) = f(\succ, \tilde{\succ}'', a)$. But since f satisfies the receiver invariance property, we must have $f(\succ, \tilde{\succ}', a) = f(\succ, \tilde{\succ}'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$ as desired.

Case 2: $a = a_{n-1}$

Just as in the case above, our goal is to construct preferences $\tilde{\succ}'$ and $\tilde{\succ}''$ such that $f(\succ, \succ', a) = f(\succ, \tilde{\succ}', a)$ and $f(\succ, \succ'', a) = f(\succ, \tilde{\succ}'', a)$, and then argue that these two transfers must be equal by properties of $\tilde{\succ}'$ and $\tilde{\succ}''$.

Let $\tilde{\succ}' \in \mathcal{R}_{p,*}$ and $\tilde{\succ}'' \in \mathcal{R}_{s,*}$ be arbitrary preferences such that

1. $rank(\succ', a) = rank(\tilde{\succ}', a) = rank(\succ'', a) = rank(\tilde{\succ}'', a)$
2. $rank(\tilde{\succ}', a_1) = p$ and $rank(\tilde{\succ}'', a_1) = s$
3. $\forall i \notin \{1, 2\}$, $rank(\tilde{\succ}', a_i) = rank(\tilde{\succ}'', a_i)$

Again, we are guaranteed to find such preferences. Since $n \geq 4$, we have $a_{n-1} \neq a_2$ and thus we can set $rank(\tilde{\succ}', a_2) = rank(\succ'', a_1) = s$ and $rank(\tilde{\succ}'', a_2) = rank(\succ', a_1) = p$. Now, by Claim 5, $f(\succ, \succ', a) = f(\succ, \tilde{\succ}', a)$ and $f(\succ, \succ'', a) = f(\succ, \tilde{\succ}'', a)$. Since f satisfies the receiver invariance property, we must have $f(\succ, \tilde{\succ}', a) = f(\succ, \tilde{\succ}'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$ as desired. This concludes the proof of the claim. □

Lemma 16 follows as a direct consequence of Claims 5, 6 and 7. □

Lemma 4. *If pairwise-exchange mechanism φ^f is neutral and strategy-proof, then for any object $a \in O$, and preferences $\succ, \succ', \succ'' \in \mathcal{R}$, we have*

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

Proof. The lemma is an immediate consequence of Lemma 3 and Lemma 16. □

For the remainder of this section, recall that for any $k \in \mathbb{N}$, $[k] := \{1, 2, \ldots, k\}$. To prove Theorem 3, we additionally require the following two lemmas.

Lemma 17. *For any function $f : \mathcal{R} \times \mathcal{R} \times O \to [-\frac{1}{n}, \frac{1}{n}]$, if f is neutral and $\forall\ a \in O$, and $\forall\ \succ, \succ', \succ'' \in \mathcal{R}$,*

$$rank(\succ', a) = rank(\succ'', a) \implies f(\succ, \succ', a) = f(\succ, \succ'', a)$$

then there exists a function $g : [n] \times [n] \to [-\frac{1}{n}, \frac{1}{n}]$ such that for all $\succ, \succ' \in \mathcal{R}$ and $\forall a \in O$, we have $f(\succ, \succ', a) = g(rank(\succ, a), rank(\succ', a))$

Proof. Let $\succ^*$ be an arbitrary preference relation and let $\succ^* = \langle a_1, a_2, \ldots, a_n \rangle$ without loss of generality. We define a function $g : [n] \times [n] \to [-\frac{1}{n}, \frac{1}{n}]$ as follows.

$$g(r, s) = f(\succ^*, \hat{\succ}, a_r) \text{ where } \hat{\succ} \text{ is an arbitrary preference such that } rank(\hat{\succ}, a_r) = s.$$

Note that this function is well defined since we have $f(\succ^*, \hat{\succ}, a_r) = f(\succ^*, \hat{\succ}', a_r)$ for any other preference relation $\hat{\succ}'$ that has $rank(\hat{\succ}', a_r) = s$. Now consider any two preferences $\succ$ and $\succ'$ and an object $a \in O$. Suppose $rank(\succ, a) = \ell$. Let $\pi : O \to O$ be a permutation such that $\succ^* = \pi(\succ)$, and define $\tilde{\succ} = \pi(\succ')$. By definition, we have $rank(\tilde{\succ}, a_\ell) = rank(\succ', a)$. Since the function f is neutral, we must have

$$f(\succ, \succ', a) = f(\succ^*, \tilde{\succ}, a_\ell) = g(\ell, rank(\tilde{\succ}, a_\ell)) = g(rank(\succ, a), rank(\succ', a))$$ □

Lemma 18. *Any function $g : [n] \times [n] \to \mathbb{R}$ satisfies $g(i, i) = 0$, $\forall i \in [n]$ and $\sum_{i=1}^{n} g(i, \pi(i)) = 0$ for all permutations $\pi : [n] \to [n]$ if and only if there exists a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ with $v_n = 0$ such that $g(i, j) = v_i - v_j$.*

Proof. Let us prove the easy direction first. Suppose $\exists\, v \in \mathbb{R}^n$ s.t. $g(i,j) = v_i - v_j$. First, for any $i \in [n]$, $g(i,i) = v_i - v_i = 0$. Second, for any permutation π,

$$\sum_{i=1}^{n} g(i,\pi(i)) = \sum_{i=1}^{n} \left(v_i - v_{\pi(i)}\right) = \sum_{i=1}^{n} v_i - \sum_{i=1}^{n} v_{\pi(i)} = 0$$

since π is a permutation and hence a bijection on $[n]$.

For the other direction of the claim, we define the vector $v \in \mathbb{R}^n$ as,

$$v_i = g(i,n)\ \forall i \in [n]$$

Let π be any permutation and let π' be such that it is identical to π except for a swap on one pair of indices. That is, π, π' are such that for all $i \in [n] \setminus \{m,k\}$, $\pi(i) = \pi'(i)$ and for some $m,k \in [n]$ $\pi(m) = \pi'(k)$ and $\pi(k) = \pi'(m)$. Since the function g satisfies $\sum_{i=1}^{n} g(i,\pi(i)) = 0$ for all permutations π, we have:

$$\sum_{i=1}^{n} g(i,\pi(i)) = \sum_{i=1}^{n} g(i,\pi'(i))$$

$$\text{i.e. } \sum_{i\in[n]\setminus\{m,k\}} g(i,\pi(i)) + g(m,\pi(m)) + g(k,\pi(k)) = \sum_{i\in[n]\setminus\{m,k\}} g(i,\pi'(i)) + g(m,\pi'(m)) + g(k,\pi'(k))$$

$$\text{i.e. } g(m,\pi(m)) + g(k,\pi(k)) = g(m,\pi(k)) + g(k,\pi(m))$$

Thus for any $m,k \in [n]$, if we consider a particular permutation π such that $\pi(m) = k$ and $\pi(k) = n$, the above equality reduces to

$$g(m,k) + g(k,n) = g(m,n) + g(k,k)$$

But since $g(k,k) = 0$, this yields

$$g(m,k) = g(m,n) - g(k,n) = v_m - v_k$$

as desired. □

A.4 Neutral and Envy-free Mechanism that is not Separable

Below we give an example of a mechanism for $n = 3$ agents and $n = 3$ objects that is neutral and envy-free but violates separability. First, note that up to relabeling the objects and / or the agents, there are exactly ten different preference profiles. These profiles and their corresponding assignments are shown in Table A.3 and A.4 respectively.

It is easy to verify that this mechanism satisfies neutrality and envy-freeness. However, it violates separability. The mechanism fails to satisfy separability at exactly those contention-free preference profiles where it differs from the Rank Exchange mechanism with vector

1	a	b	c
2	a	b	c
3	a	b	c

(a) Profile A

1	a	c	b
2	a	b	c
3	a	b	c

(b) Profile B

1	b	a	c
2	a	b	c
3	a	b	c

(c) Profile C

1	b	c	a
2	a	b	c
3	a	b	c

(d) Profile D

1	c	a	b
2	a	b	c
3	a	b	c

(e) Profile E

1	c	b	a
2	a	b	c
3	a	b	c

(f) Profile F

1	b	a	c
2	a	c	b
3	a	b	c

(g) Profile G

1	b	c	a
2	a	c	b
3	a	b	c

(h) Profile H

1	c	a	b
2	b	a	c
3	a	b	c

(i) Profile I

1	c	a	b
2	b	c	a
3	a	b	c

(j) Profile J

Table A.3: Ten preference profiles

	a	b	c
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) Profile A

	a	b	c
1	$\frac{1}{3}$	0	$\frac{2}{3}$
2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(b) Profile B

	a	b	c
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(c) Profile C

	a	b	c
1	0	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

(d) Profile D

	a	b	c
1	$\frac{1}{3}$	0	$\frac{2}{3}$
2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(e) Profile E

	a	b	c
1	0	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

(f) Profile F

	a	b	c
1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
2	$\frac{1}{3}$	0	$\frac{2}{3}$
3	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(g) Profile G

	a	b	c
1	0	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	$\frac{1}{2}$	$\frac{1}{2}$	0

(h) Profile H

	a	b	c
1	$\frac{1}{3}$	0	$\frac{2}{3}$
2	0	$\frac{5}{6}$	$\frac{1}{6}$
3	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(i) Profile I

	a	b	c
1	$\frac{1}{6}$	0	$\frac{5}{6}$
2	0	$\frac{5}{6}$	$\frac{1}{6}$
3	$\frac{5}{6}$	$\frac{1}{6}$	0

(j) Profile J

Table A.4: Assignments at the ten preference profiles in Table A.3

$v = (\frac{1}{6}, \frac{1}{6}, 0)$. To see this, let us look at the amount of object b that agent 2 gets at Profile I. In what follows, for any profile $\succ^{(x)}$, let $P^{(x)} := \varphi(\succ^{(x)})$.

We know that $P^{(I)}_{2,b} = \frac{5}{6}$. Let $\succ^K = (\langle b, a, c\rangle, \langle b, a, c\rangle, \langle b, a, c\rangle)$. By neutrality, $P^{(K)}_{2,b} = P^{(A)}_{2,a} = \frac{1}{3}$. Consider $\succ^L = (\langle c, a, b\rangle, \langle b, a, c\rangle, \langle b, a, c\rangle)$ and $\succ^M = (\langle b, a, c\rangle, \langle b, a, c\rangle, \langle a, b, c\rangle)$. After working through appropriate agent and object rotations, we have, $P^{(L)}_{2,b} = P^{(F)}_{2,a} = \frac{1}{2}$

and $P^{(M)}_{2,b} = P^{(C)}_{2,a} = \frac{1}{3}$. However,

$$(P^{(L)}_{2,b} - P^{(K)}_{2,b}) + (P^{(M)}_{2,b} - P^{(K)}_{2,b}) = (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{3}) = \frac{1}{6} \neq \frac{1}{2} = (\frac{5}{6} - \frac{1}{3}) = P^{(I)}_{2,b} - P^{(K)}_{2,b}$$

leading to a violation of separability.

It can be readily seen that this mechanism cannot be represented as a Rank Exchange mechanism. At eight out of the ten preference profiles, the mechanism outputs the same assignments as those in a Rank Exchange mechanism with vector $v = (\frac{1}{6}, \frac{1}{6}, 0)$. However, the mechanism provides different assignments at the contention-free preference profiles (Profiles I and J).

A.5 Envy-free and Separable Mechanism that is not Neutral

We present an example of a mechanism for $n = 3$ agents and $n = 3$ objects that is envy-free and separable but violates neutrality. From Lemma 2, we know that an envy-free and separable mechanism can be represented as a pairwise exchange mechanism. Table A.5 defines a transfer function $f : \mathcal{R} \times \mathcal{R} \times O \rightarrow [\frac{-1}{6}, \frac{1}{6}]$.

It is easy to verify that this function f represents a valid pairwise exchange mechanism φ^f and is separable. It satisfies all the properties in Lemma 3. Further, one can also verify that f satisfies sender-invariance and swap-monotonicity. These two properties are sufficient to show that φ^f is strategy-proof (Lemma 12). Consequently φ^f is also envy-free (Theorem 4). However, φ^f violates neutrality. This is easy to see since $f(\langle a, b, c\rangle, \langle c, b, a\rangle, b) \neq f(\langle a, c, b\rangle, \langle b, c, a\rangle, c)$. Thus, function f violates neutrality (Lemma 15). Note that this mechanism cannot be represented as a Rank Exchange mechanism since all Rank Exchange mechanisms are neutral by definition.

	$\succ'$																	
$f(\succ, \succ', o)$	$\langle a,b,c\rangle$			$\langle a,c,b\rangle$			$\langle b,a,c\rangle$			$\langle b,c,a\rangle$			$\langle c,a,b\rangle$			$\langle c,b,a\rangle$		
$\succ$	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
$\langle a,b,c\rangle$	0	0	0	0	$\frac{1}{12}$	$\frac{-1}{12}$	$\frac{1}{12}$	$\frac{-1}{12}$	0	$\frac{1}{12}$	$\frac{-1}{12}$	0	0	$\frac{1}{12}$	$\frac{-1}{12}$	$\frac{1}{12}$	0	$\frac{-1}{12}$
$\langle a,c,b\rangle$	0	$\frac{-1}{12}$	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{-1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{-1}{6}$	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{-1}{12}$	0
$\langle b,a,c\rangle$	$\frac{-1}{12}$	$\frac{1}{12}$	0	$\frac{-1}{12}$	$\frac{1}{6}$	$\frac{-1}{12}$	0	0	0	0	0	0	$\frac{-1}{12}$	$\frac{1}{6}$	$\frac{-1}{12}$	0	$\frac{1}{12}$	$\frac{-1}{12}$
$\langle b,c,a\rangle$	$\frac{-1}{12}$	$\frac{1}{12}$	0	$\frac{-1}{12}$	$\frac{1}{6}$	$\frac{-1}{12}$	0	0	0	0	0	0	$\frac{-1}{12}$	$\frac{1}{6}$	$\frac{-1}{12}$	0	$\frac{1}{12}$	$\frac{-1}{12}$
$\langle c,a,b\rangle$	0	$\frac{-1}{12}$	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{-1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{-1}{6}$	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{-1}{12}$	0
$\langle c,b,a\rangle$	$\frac{-1}{12}$	0	$\frac{1}{12}$	$\frac{-1}{12}$	$\frac{1}{12}$	0	0	$\frac{-1}{12}$	$\frac{1}{12}$	0	$\frac{-1}{12}$	$\frac{1}{12}$	$\frac{-1}{12}$	$\frac{1}{12}$	0	0	0	0

Table A.5: Transfer function $f(\succ, \succ', o)$