Title

Beta-Ray Spectrometer with Reduced Spherical Aberration

Author

Sessler, Andrew M

Publication Date

1963-02-01

UCRL-10668

University of California

Ernest O. Lawrence Radiation Laboratory

BETA-RAY SPECTROMETER WITH REDUCED SPHERICAL ABERRATION

DISCLAIMER

Submitted to Nuclear Insts. and Methods

UCRL-10668

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

BETA-RAY SPECTROMETER WITH REDUCED SPHERICAL ABERRATION

Andrew M. Sessler

February 4, 1963

BETA-RAY SPECTROMETER WITH REDUCED SPHERICAL ABERRATION†

Andrew M. Sessler

Lawrence Radiation Laboratory
University of California
Berkeley, California

February 4, 1963

Modern β-ray spectrometers are based upon the concept, first introduced by Svartholm and Siegbahn in 1946, of focusing in both the radial and vertical directions[1]. The theory of axially symmetric devices has been carefully studied by a large number of workers[2], culminating in the analysis, in 1956, of Lee-Whiting and Taylor[3]. These last authors calculate aberrations through the sixth order and show that by appropriate choice of the magnetic field a spectrometer can be designed with a relatively large transmission and a high resolution. The acceptable transmission is remarkable because the second-order "spherical" aberration in the median plane of the image cannot be made to vanish identically, and consequently the design is forced to a tall thin aperture (or a slightly less advantageous short wide aperture) which a priori would seem to imply a low transmission.

It is the purpose of this communication to show that if the arbitrary restriction to axially symmetric fields is removed, then both the radial and the vertical contributions to the "spherical" aberration can be made to vanish in second order. That azimuthally varying fields (AVF) afford the freedom to accomplish this end may well be suspected in view of the technological revolution that the concept has brought to other particle-handling devices.

A general treatment of optics in arbitrary static electric and magnetic fields has been given by Sturrock[4]), and our calculations can be considered a special case of his work. We give a self-contained analysis for an AVF spectrometer, and obtain explicit formulas for the "spherical" aberration through second order, and then demonstrate, by example, that the aberration can be made to vanish.

Restricting attention to devices with median-plane symmetry and a central trajectory which lies on a circle, one can show that the most general field in the (source-free) region of the optic circle has cylindrical components

$$\begin{aligned} \frac{B_z}{B_0} &= 1 + a_1(\theta)x + a_2(\theta)x^2 - \left[\frac{a_1(\theta)}{2} + a_2(\theta)\right] y^2 + \cdots , \\ \frac{B_r}{B_0} &= a_1(\theta)y + 2a_2(\theta)xy + \cdots , \\ \frac{B_\theta}{B_0} &= \frac{da_1(\theta)}{d\theta}\, xy + \cdots , \end{aligned} \tag{1}$$

where r_0 is the optic circle radius,

$$\begin{aligned} x &= \frac{r - r_0}{r_0} , \\ y &= \frac{z}{r_0} , \end{aligned} \tag{2}$$

and $a_1(\theta)$ and $a_2(\theta)$ are two arbitrary functions. The path of an electron of momentum p_0, where

$$p_0 = -\frac{er_0B_0}{c} , \tag{3}$$

is described in the neighborhood of the optic circle by the equations (valid through second order):

$$x'' + \left[1 + a_1(\theta)\right] x = -\left[1 + 2a_1(\theta) + a_2(\theta)\right] x^2 + \frac{x'^2}{2} - \frac{y'^2}{2} + \left[\frac{a_1(\theta)}{2} + a_2(\theta)\right] y^2 , \tag{4}$$

$$y'' - a_1(\theta)y = 2\left[a_1(\theta) + a_2(\theta)\right] xy + x'y' ,$$

where the primes denote differentiation with respect to θ . An electron of slightly different momentum

$$p = p_0(1 + \epsilon) , \tag{5}$$

has radial motion which in first order is determined by

$$x'' + [1 + a_1(\theta)]x = \epsilon . \tag{6}$$

Explicit formulas for the second-order aberration coefficients can be given in terms of the independent solutions of the linearized approximations to eqs. (4). A convenient set of such solutions, x_{L1} and x_{L2} , may be specified by the boundary conditions

$$\begin{aligned} x_{L1}(0) &= 0 , & x'_{L1}(0) &= 1 ; \\ x_{L2}(0) &= 1 , & x'_{L2}(0) &= 0 ; \end{aligned} \tag{7}$$

and similarly for the y-equation. Solutions of eqs. (4) which are correct through second order and satisfy the boundary conditions

$$x(0) = x_0 \ , \qquad x'(0) = x'_0 \ ;$$
$$y(0) = y_0 \ , \qquad y'(0) = y'_0 \ ; \tag{8}$$

can be expressed in the form

$$x(\theta) = x_0 x_{L2}(\theta) + x'_0 x_{L1}(\theta) + A_{xx}(\theta)x_0^2 + A_{yy}(\theta)y_0^2 + A_{x'x'}(\theta)x'^2_0 + A_{x'x}(\theta)x_0 x'_0 + A_{y'y'}(\theta)y'^2_0 + A_{y'y}(\theta)y_0 y'_0 \ ; \tag{9}$$

and similarly for $y(\theta)$. The condition for simultaneous vertical and radial linear focusing is that for some angle θ_f ,

$$x_{L1}(\theta_f) = y_{L1}(\theta_f) = 0 \ . \tag{10}$$

For a sufficiently small source the resolution of the spectrometer is limited by the "spherical" aberration in the radial dimension of the image, namely by the magnitude of the coefficients $A_{x'x'}(\theta_f)$ and $A_{y'y'}(\theta_f)$, which are given explicitly by

$$A_{x'x'}(\theta_f) = -x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta) \left\{ -[1 + 2a_1(\theta) + a_2(\theta)][x_{L1}(\theta)]^2 + \tfrac{1}{2}[x'_{L1}(\theta)]^2 \right\} d\theta \ ,$$
$$A_{y'y'}(\theta_f) = -x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta) \left\{ \left[\frac{a_1(\theta)}{2} + a_2(\theta) \right] [y_{L1}(\theta)]^2 - \tfrac{1}{2}[y'_{L1}(\theta)]^2 \right\} d\theta \ . \tag{11}$$

The dispersion of the spectrometer is proportional to the displacement D of the image of an electron with $x(0) = x'(0) = 0$ and momentum fractionally differing from p_0 by ϵ [as in eq. (5)]. From eq. (6) one finds the displacement $D(\epsilon)$, given to lowest order by

$$D(\epsilon) = -\epsilon\, x_{L2}(\theta_f) \int_0^{\theta_f} x_{L1}(\theta)d\theta \quad . \tag{12}$$

For spectrometers in which $a_1(\theta)$ is independent of azimuth, eq. (10) can only be satisfied with $\theta_f < 2\pi$ by choosing $a_1(\theta) = -1/2$, as in the spectrometer of Svartholm and Siegbahn. In this case $\theta_f = \sqrt{2}\,\pi$, and the aberration coefficients of eq. (11) are:

$$\begin{aligned} A_{x'x'}(\theta_f) &= \frac{2}{3} - 4 \int_0^{\pi} \sin^3\psi\, a_2(\sqrt{2}\ \psi)d\psi \ , \\ A_{y'y'}(\theta_f) &= -2 + 4 \int_0^{\pi} \sin^3\psi\, a_2(\sqrt{2}\ \psi)d\psi \ . \end{aligned} \tag{13}$$

It is thus clear that <u>no choice</u> of $a_2(\theta)$ will allow the removal of the second-order "spherical" aberration[5]. To accomplish this end we must, in general, choose a_1 to vary with θ. The physical reason is clear: Maxwell's equations [see eq. (1)] relate B_z to B_r and thus relate the effect of $a_2(\theta)$ on particles with a certain x-amplitude to the effect on particles with an equal y-amplitude. This relation, as may be seen in eq. (13), does not allow $A_{x'x'}$ and $A_{y'y'}$ to vanish simultaneously. In order to affect particles with x displacements differently from particles with y displacements we must break the cylindrical symmetry about the optic circle; i.e., choose $a_1(\theta)$ different from $-\frac{1}{2}$, and consequently, in general, varying[6] with θ.

As an idealized example, consider a spectrometer in which

$$a_1(\theta) = -1 , \qquad a_2(\theta) = a_{20} ; \qquad 0 \leqslant \theta \leqslant \theta_f/2$$
$$a_1(\theta) = 0 , \qquad a_2(\theta) = a_{21} ; \qquad \theta_f/2 \leqslant \theta \leqslant \theta_f . \tag{14}$$

Simultaneous linear focusing may be obtained if

$$\tan \theta_f/2 = -\theta_f/2 , \tag{15}$$

which implies $\theta_f = 4.0576$ (corresponding to about 232°). The linear x-solutions are easily seen to be

$$x_{L1}(\theta) = \begin{cases} \theta , & 0 \leqslant \theta \leqslant \theta_f/2 \\ \dfrac{\sin(\theta - \theta_f)}{\cos \theta_f/2} , & \theta_f/2 \leqslant \theta \leqslant \theta_f \end{cases} ;$$
$$x_{L2}(\theta) = \begin{cases} 1 , & 0 \leqslant \theta \leqslant \theta_f/2 \\ \cos(\theta - \theta_f/2) , & \theta_f/2 \leqslant \theta \leqslant \theta_f \end{cases} ; \tag{16}$$

with quite similar solutions of the y-equation. From eq. (12), the dispersion is

$$\frac{D(\epsilon)}{\epsilon} = 1 + \frac{2}{S}(1 + \theta_f^2/8) , \tag{17}$$

where

$$S = (4 + \theta_f^2)^{1/2} . \tag{18}$$

Numerical evaluation yields[7] $D(\epsilon)/\epsilon = 2.35$. The aberration coefficients are

$$A_{x'x'}(\theta_f) = \frac{1}{S}\left[\frac{\theta_f^4}{32} - \frac{\theta_f^2}{8}(3+S) - \frac{S}{2} - 1\right] - \frac{\theta_f^4}{32S}\, a_{20} - \frac{a_{21}}{S}\left[\frac{\theta_f^2}{2}\left(\frac{S}{3}+1\right) + \frac{2}{3}S + \frac{4}{3}\right],$$

$$A_{y'y'}(\theta_f) = \frac{-1}{S^3}\left[\frac{\theta_f^4}{8} + \frac{\theta_f^2}{2} + 4 + 2S\right] + \frac{a_{20}\theta_f^2}{S^3}\left[2 + \frac{\theta_f^2}{8}\right] - \frac{4a_{21}}{S^3}\left[2S + 4 - \frac{3}{2}\theta_f^2\right], \tag{19}$$

which can both be made zero by choosing $a_{20} = 1.46$, and $a_{21} = -0.91$.

A design study is now being undertaken at this laboratory with the aim of extending these concepts to higher orders as well as of realizing them in an iron-free configuration. In this way it is hoped to obtain a high-precision spectrometer with good resolution and a large transmission.

The author wishes to thank Dr. R. L. Graham, Dr. J. M. Hollander, Dr. D. L. Judd, and Dr. P. Sturrock for informative discussions. He is indebted to Dr. J. Young for a critical reading of the manuscript.

FOOTNOTES AND REFERENCES

† Work done under the auspices of the U. S. Atomic Energy Commission.

1. N. Svartholm and K. Siegbahn, Arkiv. f. Mat. Astr. Fys. 33A: No. 21 (1946).

2. N. Svartholm, Arkiv. f. Mat. Astr. Fys. 33A: No. 24 (1946).

F. B. Shull and D. M. Dennison, Phys. Rev. 71, 681; 72, 256 (1947).

P. H. Stoker, Ong Ping Hok, E. F. de Haan, and G. J. Sizoo, Physica 20, 337 (1954).

N. F. Verster, Physica 16, 815 (1950).

P. P. Pavinski, Izv. Akad. Nauk SSSR Ser. Fiz. 18, 175 (1954).

V. R. Saulit, Izv. Akad. Nauk SSSR Ser. Fiz. 18, 227 (1954).

E. Huster, G. Lehr, and W. Walcher, Z. Naturforsch. 10A, 83 (1955).

3. G. E. Lee-Whiting and E. A. Taylor, Can. J. Phys. 35, 1 (1957) and Atomic Energy of Canada Limited, Chalk River Project, Report CRT-668 (1956).

4. P. A. Sturrock, Phil. Trans. Roy. Soc. London A245, 155 (1952).

5. If $a_2(\theta)$ is taken independent of θ then eqs. (13) become

$$A_{x'x'} = \frac{2}{3}(1 - 8a_2) \ ,$$

$$A_{y'y'} = \frac{2}{3}(-3 + 8a_2) \ .$$

The choices $a_2 = \frac{1}{8}$ and $a_2 = \frac{3}{8}$, correspond respectively to the "wide aperture" and "high aperture" spectrometers of Lee-Whiting and Taylor.

6. If a_1 is taken independent of θ but chosen so that the x and y frequencies are unequal (although commensurate), then suitable choice of $a_2(\theta)$ will make $A_{x'x'}$ and $A_{y'y'}$ simultaneously vanish. However, θ_f will necessarily be larger than 2π.

7. For this particular example the value of $D(\epsilon)/\epsilon$ is approximately half of that associated with the axially symmetric spectrometer of Svartholm and Siegbahn. However, it can be seen from eq. (12) that there is no reason, in general, to expect the dispersion to be poorer in an AVF spectrometer than in an axially symmetric spectrometer.