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A "Scandinavian Consensus" Solution for Efficient Income Distribution Among Nonmalevolent Consumers

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INTRODUCTION

In the familiar models of competitive equilibrium, [2, 7 and 10], it is assumed that the level of preference of each individual depends only on the bundle of commodities which he consumes himself. In competitive equilibrium each consumer maximizes his own preferences subject to the condition that the value at equilibrium prices of the bundle which he consumes does not exceed the value of the resources which he owns. It is well-known that if preferences of all consumers are individualistic and locally nonsatiated, any competitive equilibrium is Pareto optimal. If a consumer's level of preference depends in part on the consumption of others, not every competitive equilibrium is necessarily Pareto optimal. Where there are benevolent consumers, for example, it may be that transfers of purchasing power are required to reach a Pareto optimal allocation.

In an earlier paper [3], I discussed the possibility of expanding the notion of equilibrium to allow mutually voluntary bilateral gifts. It turns out that if the only interrelatedness of preferences is nonmalevolent (though somewhat "selfish") interaction between monogamous pairs of consumers, a competitive gift equilibrium exists and there is a correspondence between such equilibria and Pareto optima.

If there are networks of consumer interrelatedness more complex than simple pairwise relatedness, a competitive equilibrium where bilateral gifts are allowed ceases in general to be Pareto optimal. For example, if there is one good and three persons such that A and B are each benevolent toward C , but are not benevolent toward each other, it may be that A and B both wish to contribute to C . If A and B each contribute unilaterally to C , the total contribution to C will be less than the Pareto optimal amount since a contribution by A to C also benefits B , but A is not rewarded for this. In this example, a Pareto optimum could be reached

if A and B agreed to some kind of a matching grant arrangement. It turns out that this kind of arrangement can be generalized to provide a Pareto optimal solution for networks of interrelationships among many persons. An allocation mechanism, which is called a *distributive Lindahl mechanism* will be described informally here and more precisely in Section I.

An idea is used which is similar to the "Scandinavian consensus" solution of the public goods problem, discussed by Wicksell [15] and Lindahl [9]. Lindahl considered the case in which there are two consumers, one public good, and one private good. The problem is to determine the amount of the public good to be produced. Lindahl's suggestion is that if each individual were assigned a share of the cost of the public good which he must pay, he would vote for a certain level of expenditures on the public good. Shares are adjusted until both agree on the level of public expenditures. This quantity is the equilibrium quantity and the corresponding shares the equilibrium shares. Figure 1 illustrates equilibrium. The quantity OF is the equilibrium expenditure on public goods. SA and SB are the equilibrium shares.

FIGURE 1.

For an economy with interrelated preferences, a distributive Lindahl mechanism is described as follows. For any commodity price vector, each consumer has an initial wealth which is determined by the initial distribu-

tion of property rights. Shares, α_j^i , of the cost of Consumer j 's consumption to be borne by Consumer i are assigned to each $i \in I$ for each $j \in I$ in such a way that $\sum_{i \in I} \alpha_j^i = 1$ for each $j \in I$. Each consumer $i \in I$ then states the allocation (or allocations) which he likes best among those allocations which he can afford if for each $j \in I$ he must pay the fraction α_j^i of the cost of any bundle allocated to Consumer j . At an equilibrium set of prices and shares all consumers agree on the same allocation and there are no excess demands or supplies in any commodity market.

One could think of an adjustment mechanism which moves toward equilibrium in a manner similar to the tatonnement process. If an arbitrary set of shares is assigned, and if each individual i declares the value of the consumption bundle which he desires for j , the auctioneer could raise the shares, α_j^i , of those who desire the highest expenditure for j and lower the shares for those who desire lower expenditures for j until all are in agreement. This process would have to be repeated for many price vectors until there is equilibrium in the product markets as well as agreement on allocations.

This paper presents a set of assumptions which guarantee that a distributive Lindahl equilibrium exists and is Pareto optimal. It is shown that a large class of Pareto optima can be sustained as distributive Lindahl equilibria. For simplicity of exposition, attention is confined to pure exchange economies. The results can be extended to economies with convex production technologies in a straightforward way either by the method used by Debreu [7] or by Rader [11].

SECTION I

A. Nonmalevolent Preferences on Allocations

There are assumed to be n consumers and m goods. Let I be the index set of consumers. An allocation is a point $(x_1, x_2, \dots, x_n) \in \Omega_+^{mn}$ (the non-negative orthant in E^{mn}) where x_i represents the bundle of goods allocated to Consumer i . Each consumer $i \in I$ has a complete preordering R_i defined on the set of allocations in Ω_+^{mn} . The corresponding strict preference relation P_i is defined in the usual way. For $x \in \Omega_+^{mn}$, the upper contour set, $R_i(x)$, is $\{x' \mid x' R_i x\}$ and the lower contour set, $R_i^{-1}(x)$, is $\{x' \mid x R_i x'\}$.

Preferences are said to be *separable between individuals* if each consumer's preference between any two allocations which contain the same commodity bundles for everyone except some one consumer j is unaffected by *what* consumers other than j consume, so long as in each of the two allocations compared the amount consumed by the others is the same. This is the notion of separability familiar in consumption theory. Separation

bility rules out such Veblenesque effects as the desire to imitate the consumption of others or a desire for a commodity solely because of its scarcity but does allow persons to be concerned about the consumption of others.

Formally, *preferences of Consumer i are separable between individuals if for all $j \in I$ and all allocations x and y in Ω_+^{mn} such that $x_k = y_k$ for all $k \neq j$, if $x R_i y$ then $x' R_i y'$ for any allocations x' and y' such that $x'_j = x_j$, $y'_j = y_j$ and $x'_k = y'_k$ for $k \neq j$.¹*

If preferences are separable between individuals, one can define the notion of a *private preference ordering* derived from the individual's preferences on allocations. In particular, a bundle x_i is privately preferred by Consumer i to a bundle y_i if and only if for every pair of allocations u and v which differ only in that Consumer i receives x_i in allocation u and y_i in allocation v , Consumer i prefers allocation u to allocation v .

Formally, *the private preference ordering, $\succsim_i$, of Consumer i is defined as follows: If $x_i, y_i \in \Omega_+^m$, then $x_i \succsim_i y_i$ if and only if $u R_i v$ whenever $u_i = x_i$, $v_i = y_i$ and $u_j = v_j$ for all $j \in I, j \neq i$. The corresponding strict preference relation $\succ_i$ is defined in the natural way.*

It is readily verified that $\succsim_i$ is a complete preordering on Ω_+^m if preferences of i are separable between individuals.

Consumer i is said to be nonmalevolently related to Consumer j if and only if for any two allocations x and y which contain the same bundles for everyone except j and such that j privately prefers his bundle in x to his bundle in y , consumer i respects j 's private preference to the extent that he either prefers x to y or is indifferent between them. If under these circumstances, i always prefers x to y , Consumer i is said to be benevolently related to j . Nonmalevolence rules out the possibility that i disagrees with j about what kinds of goods j should consume. A special case of nonmalevolence is the traditional individualistic assumption.

Formally, *Consumer i is nonmalevolently (benevolently) related to j if preferences of i and j are separable between individuals and for any x and y in Ω_+^{mn} such that $x_k = y_k$ for $k \neq j$ and $x_j \succsim_j y_j$ ($x_j \succ_j y_j$) it is true that $x R_i y$ ($x P_i y$).²*

¹ If preferences of Consumer i are representable by a continuous utility function, then they are separable between individuals if and only if they can be represented by a utility function of the form $F_i(g_1(x_1), g_2(x_2), \dots, g_n(x_n))$, where $g_j(\cdot)$ is a continuous real-valued function and x_j is the consumption bundle received by Consumer j .

² If preferences of Consumer i are representable by a continuous utility function, then Consumer i is nonmalevolently related to all consumers if and only if preferences of i can be represented by a utility function of the form $F_i(u_1(x_1), u_2(x_2), \dots, u_n(x_n))$, where for each $j \in I$, $u_j(\cdot)$ is a continuous function which represents the private preferences of Consumer j .

B. Distributive Lindahl Equilibrium

Let w be the vector of resources available to an exchange economy. An initial wealth distribution W is a function $W(p) = (W_1(p), \dots, W_n(p))$ whose value depends on prices and the initial distribution of property rights and such that for any price vector $p \in E^m$, $\sum_{i \in I} W_i(p) = pw$.

The set of admissible Lindahl shares is the set of $n \times n$ matrices $A \equiv \{(\alpha_j^i) \mid \alpha_j^i \geq 0 \text{ for all } i, j \in I \text{ and } \sum_{i \in I} \alpha_j^i = 1 \text{ for all } j \in I\}$.

Where $\alpha \in A$ and $p \in E^m$, the budget set $B_i(p, \alpha)$ of the i -th consumer is the set $\{(x_1, \dots, x_n) \in \Omega_+^{mn} \mid \sum_{j \in I} \alpha_j^i(p x_j) \leq W_i(p)\}$.

A distributive Lindahl equilibrium for an exchange economy in which the initial wealth distribution is $W(p)$, is a point $(\bar{p}, \bar{\alpha}, \bar{x}) \in (E^m, A, \Omega_+^{mn})$ such that

- (1) For all $i \in I$, $\bar{x}$ maximizes R_i on $B_i(\bar{p}, \bar{\alpha})$.
- (2) $\sum_{i \in I} \bar{x}_i = w$.

If $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium, then there is unanimous agreement about what the consumption bundle of each consumer should be, given that each i must pay the fraction, $\bar{\alpha}_j^i$ of the cost of the bundle consumed by each j .

C. An Example of Distributive Lindahl Equilibrium for Consumers with Cobb–Douglas Utility Functions

Consideration of the following special case may help to illuminate the nature of a distributive Lindahl equilibrium. Suppose there is just one good and there are n consumers. Let W_i be the quantity of the good initially held by Consumer i . Suppose the preferences of Consumer i are represented by the function, $U_i(x) = \sum_{j=1}^n a_{ij} \log x_j$, where x_j is the amount consumed by j and where $a_{ij} \geq 0$ for all i and j . Since the representation is unique up to a monotonic transformation, there is no loss of generality in assuming that $\sum_{j \in I} a_{ij} = 1$. It is not hard to show that when i chooses a utility maximizing allocation subject to the budget $\sum_{j \in I} \alpha_j^i x_j \leq W_i$, he will desire the quantity $x_j^i(\alpha_j^i) = (a_{ij}/\alpha_j^i) W_i$ for consumer j . If we choose the shares $\bar{\alpha}_j^i = a_{ij} W_i / \sum_{i \in I} a_{ij} W_i$, then for each i , $x_j^i(\bar{\alpha}_j^i) = \sum_{i \in I} a_{ij} W_i$. Therefore with shares $\bar{\alpha}_j^i$ each consumer desires $\bar{x}_j = \sum_{i \in I} a_{ij} W_i$ for consumer j . It is easily verified that $\sum_{i \in I} \bar{\alpha}_j^i = 1$. Also, $\sum_{j \in I} \bar{x}_j = \sum_{j \in I} \sum_{i \in I} a_{ij} W_i = \sum_{i \in I} W_i$ since $\sum_{j \in I} a_{ij} = 1$. Therefore the shares $\bar{\alpha}_j^i$ and the consumptions $\bar{x}_j = \sum_{i \in I} a_{ij} W_i$ for all j constitutes a Lindahl equilibrium.

If for all i and j in I , Consumer i is required to pay the share $\bar{\alpha}_j^i$ of the cost, $\bar{p} \bar{x}_j$, of Consumer j 's consumption and if the total payments made by Consumer i are not allowed to exceed W_i , then all consumers will agree on the allocation $(\bar{x}_1, \dots, \bar{x}_n)$.

SECTION II

A. *The Existence of a Distributive Lindahl Equilibrium for Nonmalevolent Consumers*

Before the traditional proofs of the existence of competitive equilibrium can be modified to prove the existence of a distributive Lindahl equilibrium, some attention must be paid to the problem of whether the usual assumptions of convexity, local nonsatiation, and continuity of preferences seem plausible when preferences are defined on the entire space, Ω_+^{mn} of allocations.

Although it is useful to allow a consumer to contemplate all allocations in Ω_+^{mn} , it does not seem reasonable to assume that each consumer can physically survive if he receives any nonnegative commodity bundle.

For each consumer, $i \in I$, it will be assumed that there is a closed, convex subset C_i of Ω_+^m called the *survival set* of i such that for any allocation x in which Consumer i receives the bundle $x_i \in C_i$ the allocation x permits Consumer i to survive.

If an allocation does not provide a consumer or any of his friends with enough resources to survive, there may be an entire neighborhood about that allocation such that he is indifferent among all allocations in that neighborhood. For this reason, a consumer will be assumed to be locally nonsatiated only at allocations such that the bundle which is allocated to him is in his survival set. Similar considerations make the assumption of ordinary convexity implausible. The theorem presented below uses only weak convexity.

The following example illustrates a seemingly reasonable case which lacks full continuity of preferences (closedness of all upper and lower contour sets). Suppose that Consumer i is benevolent toward Consumer j and that Consumer j is privately indifferent between any two bundles in E^m which are not in C_j , while he privately prefers any bundle in C_j to any bundle not in C_j . Then if C_j is closed and if there are two points x_j and y_j on the boundary of C_j such that $x_j \succ_j y_j$, it is easily seen that the lower contour set $R_i^{-1}(y)$ is not closed if y is an allocation in which Consumer j receives the bundle, y_j . Fortunately, existence can be proved with a weaker continuity assumption. In particular, it will be assumed that for all $i \in I$ and all $x \in \Omega_+^{mn}$ the set $R_i(x)$ is closed and that for all $i \in I$, preferences are continuous along straight lines in the survival sets.³

In the usual proofs of the existence of competitive equilibrium, it

³ The formal statement of this assumption is Assumption 7(b) of Theorem 1. This continuity assumption also allows one to employ Rader's "Principle of Equivalence" [11] to extend the results of Theorem 1 to economies with convex production sets.

must be guaranteed that under the relevant price systems, each consumer can afford some bundle which is in his survival set but which is not the minimal cost bundle in this set. In the existence proof offered here assumptions are made which ensure not only that he can *afford* a non cost-minimizing bundle in his survival set but also that he will *choose* an allocation in which he receives such a bundle.

An important role in the existence proof will be played by a cone of vectors in E^m with the property that if the resources in any vector of the cone are added to the total resources used in any allocation in $\prod_{i \in I} C_i$, the new aggregate vector of resources can be divided in such a way as to achieve an allocation in Ω_+^{mn} which each consumer likes at least as well as the original allocation. *This cone is defined as* $S \equiv \{s \mid \text{if } x = (x_1, \dots, x_n) \in \prod_{i \in I} C_i, \text{ there exists a } y = (y_1, \dots, y_n) \in \Omega_+^{mn} \text{ such that } \sum_{i \in I} x_i + s = \sum_{i \in I} y_i \text{ and } y R_i x \text{ for all } i \in I\}$.

It is a straightforward exercise to show that weak convexity of preferences implies that S is a convex cone. Closedness of upper contour sets implies that S is closed. An easy consequence of the definition of S is that S is contained in the nonnegative orthant.

It will be assumed that S has a nonempty interior. The set of price vectors which will be considered is the set $P = \{p \mid px \geq 0 \text{ for all } x \in S\} \cap \{p \mid p\dot{x} = 1\}$ where $\dot{x}$ is a point chosen from the interior of S . It can readily be verified that P is a nonempty, closed, bounded, convex set. (See Debreu [5]). Clearly, if $p \in P$, then $p\dot{s} > 0$ for all $\dot{s} \in \text{int } S$.

THEOREM 1. *There exists a distributive Lindahl equilibrium for an exchange economy which satisfies the following assumptions:*

(1) *There is an initial allocation of resource ownership, $(w_1, \dots, w_n) \in \Omega_+^{mn}$ such that $\sum_{i \in I} w_i = w$ and $W_i(p) = pw_i$.*

(2) *For all $i \in I$, the survival set, $C_i \subset E^m$, is nonempty, convex, and closed in E^m .*

(3) *For all $i \in I$, R_i is a complete preordering on Ω_+^{mn} .*

(4) *(Weak convexity) For all $i \in I$, and all $x \in \Omega_+^{mn}$, the set $R_i(x)$ is convex.*

(5) *(Nonmalevolence) For all $i \in I$ and all $j \in I$, Consumer i is non-malevolently related to Consumer j .*

(6) *(Nonsatiation) For all $i \in I$, if $x_i \in C_i$, then in every open neighborhood of x_i in C_i there is an $x_i' \in C_i$ such that $x_i' >_i x_i$.*

(7) *(Continuity) For all $i \in I$,*

(a) *If $x \in \Omega_+^{mn}$, then $R_i(x)$ is closed in Ω_+^{mn} .*

(b) If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are allocations such that $x P_i y$ and if $x_j \in C_j$ where $j \in I$, then for every $z_j \in C_j$ there exists a λ such that $0 < \lambda < 1$, and $(x_1, \dots, x_{j-1}, \lambda z_j + (1 - \lambda) x_j, x_{j+1}, \dots, x_n) P_i y$.

(8) For all $i \in I$, w_i is contained in the interior of S .⁴

(9) For all $i \in I$, if $\hat{w}^i = (0, \dots, 0, w_i, 0, \dots, 0)$ and if $(x_1, \dots, x_n) R_i \hat{w}^i$, then $x_i \in C_i$ and $x_i - \hat{s} \in C_i$ for some $\hat{s}$ in the interior of S .

The theorem will first be proved for choice restricted to allocations in a closed, bounded cube in Ω_+^{mn} . The artificial boundary can later be removed by means of the technique used by Debreu [6].

Let Z_i be a closed, bounded cube in Ω_+^m such that the set $\{x_i \mid x_i \in \Omega_+^m \text{ and } x_i \leq \sum_{i \in I} w_i\}$ is contained in the interior of Z_i . Let $Z = \prod_{i \in I} Z_i$.

Define the correspondence $E_i : P \times A \rightarrow Z$ so that $E_i(p, \alpha) = \{x \mid x = (x_1, \dots, x_n), \text{ where}$

- (i) $x \in B_i(p, \alpha) \cap Z$.⁵
- (ii) $x R_i \hat{w}^i$.
- (iii) If $x' \in Z$ and $\sum_{j \in I} \alpha_j^i p x_j' < p w_i$, then $x R_i x'$.
- (iv) For all $j \in I$, if $x_j' \in Z_j$ and $p x_j' < p x_j$, then $x_j \succ_j x_j'$.

Observe that if the inequality in (iii) of the definition of $E_i(p, \alpha)$ were $\leq$ rather than the strict inequality, then $E_i(p, \alpha)$ would be the set of allocations which maximize R_i on $B_i(p, \alpha) \cap Z$. The reason this "smoothed" correspondence is used rather than simply the set of preference maximizing allocations on $B_i(p, \alpha) \cap Z$ is that the latter correspondence fails in general to be upper semicontinuous (u.s.c.) where $\alpha_i^i = 0$.

Assumption 8 implies that $p w_i > 0$ for all $p \in P$. Therefore $\hat{w}_i \in B_i(p, \alpha) \cap Z$ for all $(p, \alpha) \in P \times A$. Since $B_i(p, \alpha) \cap Z$ is compact and nonempty and since upper contour sets are closed, there is at least one allocation $(\hat{x}_1, \dots, \hat{x}_n)$ which maximizes R_i on $B_i(p, \alpha) \cap Z$. Similar reasoning shows that there is an allocation $(x_1', \dots, x_n')$ such that for all $j \in I$, x_j' maximizes $\succ_j$ on the set $\{x_j \mid p x_j \leq p \hat{x}_j\} \cap Z_j$.

It is easily shown, using the assumption of nonmalevolence, that $(x_1', \dots, x_n') \in E_i(p, \alpha)$. Therefore the set, $E_i(p, \alpha)$, is nonempty for all $(p, \alpha) \in P \times A$. Weak convexity of preferences implies that $E_i(p, \alpha)$ is a convex set. It follows easily from the closedness of upper contour sets that E_i is u.s.c.

⁴ If the model is generalized to include production, then the proof of Theorem 1 will hold if in Assumptions (8) and (9) the set S' is substituted for S where S' consists of all vectors which can be transformed, through a process in the asymptotic cone of the aggregate production possibility set, into some vector in S . The substitution of the larger set S' for S increases the plausibility of Assumptions (8) and (9).

⁵ $B_i(p, \alpha)$ is the budget set defined in Section I(B).

Define the correspondence $E : P \times A \rightarrow \prod_{i \in I} Z$ so that $E = \prod_{i \in I} E_i$. A typical element $(x_1^i, \dots, x_n^i)$ of $E_i(p, \alpha) \subset \Omega_+^{mn}$ will be denoted x^i . A typical element $(x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2, \dots, x_1^n, \dots, x_n^n)$ of $E(p, \alpha) \subset \Omega_+^{mn^2}$ will be denoted by $(x^1, \dots, x^n)$.

Since for each $i \in I$, E_i is u.s.c. and has nonempty convex image sets, the same is true of E .

The following lemma, which will be used to prove Theorem 1, is an adaptation of a theorem by Debreu [5].

LEMMA 1. *If the assumptions of Theorem 1 are satisfied, then for some $(\bar{p}, \bar{\alpha}) \in (P, A)$ and some $(\bar{x}^1, \dots, \bar{x}^n) \in E(\bar{p}, \bar{\alpha})$,*

- (a) *If for all $j \in I$, $\tilde{x}_j = \sum_{i \in I} \bar{\alpha}_j^i \bar{x}_j^i$, then $\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j \in -S$.*
- (b) *For all i and j in I , either $\bar{p} \bar{x}_j^i = \max_{k \in I} \bar{p} x_j^k$ or $\bar{\alpha}_j^i = 0$.*

Proof. Define the correspondence $M_\alpha : P \times \prod_{i \in I} Z \rightarrow A$, so that if $p \in P$ and $x = (x^1, \dots, x^n) \in \prod_{i \in I} Z$, then $M_\alpha(p, x) = \{\alpha \mid \alpha \text{ maximizes } \sum_{j \in I} \sum_{i \in I} \alpha_j^i p x_j^i \text{ in } A\}$.

Define the correspondence $M_p : A \times \prod_{i \in I} Z \rightarrow P$ so that $M_p(\alpha, x) = \{p \mid p \text{ maximizes } p(\sum_{j \in I} \sum_{i \in I} \alpha_j^i x_j^i - \sum_{i \in I} w_i) \text{ in } P\}$.

It is easily verified that the correspondences M_α and M_p are u.s.c. and have nonempty convex image sets.

Consider the mapping F of the set $P \times A \times \prod_{i \in I} Z$ into itself which is defined so that $F(p, \alpha, z) = M_p(\alpha, z) \times M_\alpha(p, z) \times E(p, \alpha)$. The correspondence F is u.s.c. and has nonempty convex image sets since it is the product of mappings with these properties.

According to Kakutani's fixed point theorem, there is a point $(\bar{p}, \bar{\alpha}, \bar{x}) \in P \times A \times \prod_{i \in I} Z$ such that $(\bar{p}, \bar{\alpha}, \bar{x}) \in F(\bar{p}, \bar{\alpha}, \bar{x})$. Therefore $\bar{x} \in E(\bar{p}, \bar{\alpha})$, $\bar{\alpha} \in M_\alpha(\bar{p}, \bar{x})$, and $\bar{p} \in M_p(\bar{\alpha}, \bar{x})$. Since $\bar{p} \in M_p(\bar{\alpha}, \bar{x})$ it must be that for all $p \in P$, $p(\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j) \leq \bar{p}(\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j)$. Since $\bar{x} \in E(\bar{p}, \bar{\alpha})$, it follows that $\bar{p}(\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j) \leq 0$. Therefore $\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j$ belongs to the set $\{x \mid px \leq 0 \text{ for all } p \in P\}$. From the duality theorem for closed convex cones it follows that $\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j \in -S$.

Since $\bar{\alpha} \in M_\alpha(\bar{p}, \bar{x})$, it follows easily from the definitions of M_α and A that $\bar{\alpha}_j^i > 0$ only if $\bar{p} \bar{x}_j^i \geq \bar{p} x_j^k$ for all $k \in I$. Q.E.D.

It will be demonstrated that at prices $\bar{p}$ and Lindahl shares $\bar{\alpha}$ which satisfy Lemma 1, there is an allocation, $\bar{x}$, which satisfies the requirements for a Lindahl equilibrium when the allocation space is restricted to Z . Lemma 2 presents results which are useful to this end.

LEMMA 2. *Suppose that $(\bar{p}, \bar{\alpha}, \bar{x}^1, \dots, \bar{x}^n)$ satisfies Lemma 1. Let $\tilde{x}_j = \sum_{i \in I} \bar{\alpha}_j^i \bar{x}_j^i$ and let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$. Then for all $i \in I$:*

- (a) If $\bar{\alpha}_j^i > 0$, then $\bar{x}_j^i \succsim_j \bar{x}_j^j$
 (b) $\tilde{x}$ maximizes R_i on $B_i(\bar{p}, \bar{\alpha}) \cap Z$.
 (c) $\bar{\alpha}_i^i > 0$.

Proof. Since for all $j \in I$, $\bar{x}^j \in E_j(\bar{p}, \bar{\alpha})$, it must be that $\bar{x}^j R_j \bar{w}^j$. Assumption 9 together with the fact that $S \subset \Omega_+^m$ implies that $\bar{x}_j^j \in C_j$ and $\bar{x}_j^j - s \in C_j \cap Z_j$ for some $s \in \text{int } S$. Suppose that $\bar{\alpha}_j^i > 0$ and $\bar{x}_j^j \succ_j \bar{x}_j^i$. Then, by Assumption 7(b), there is a $\lambda > 0$ such that $\bar{x}_j^j - \lambda s \succ_j \bar{x}_j^i$. Since $s \in \text{int } S$, $\bar{p}(\bar{x}_j^j - \lambda s) < \bar{p}\bar{x}_j^j \leq \bar{p}\bar{x}_j^i$. But $\bar{x}^i \in E_i(\bar{p}, \bar{\alpha})$. This contradicts condition (iv) of the definition of $E_i(p, \alpha)$. Hence Part (a) of the lemma must be true.

Since $\bar{x}_j^i \succsim_j \bar{x}_j^j$ if $\bar{\alpha}_j^i > 0$, Assumption 9 can be used to show that where $\bar{\alpha}_j^i > 0$, if $x_j \succsim_j \bar{x}_j^i$ then $x_j \in C_j$ and $x_j - s \in C_j$ for some $s \in \text{int } S$. Again applying Assumption 7(b) one can show that if $\bar{\alpha}_j^i > 0$, then $\bar{x}_j^i \succsim_j x_j$ for all $x_j \in Z_j$ such that $\bar{p}x_j \leq \bar{p}\bar{x}_j^i$. But if $\bar{\alpha}_j^i > 0$, then $\bar{p}\bar{x}_j^k \leq \bar{p}\bar{x}_j^i$ for all $k \in I$. It is an immediate consequence of weak convexity of preferences that $\tilde{x}_j \succsim_j \bar{x}_j^k$ for all $k \in I$. Since preferences are nonmalevolent, $\tilde{x} R_i \bar{x}^i$ for all $i \in I$. It follows that if $x \in Z$ and $\sum_{j \in I} \bar{\alpha}_j^i(\bar{p}x_j) < \bar{p}w$, then $\tilde{x} R_i \bar{x}^i$. It is also readily verified that $\tilde{x} \in B_i(\bar{p}, \bar{\alpha}) \cap Z$. All that remains to be shown is that $\tilde{x} R_i x$, where x satisfies the budget constraint with equality.

Since $\sum_{j \in I} \tilde{x}_j - \sum_{j \in I} w_j \in -S$ and since the set S is contained in the nonnegative orthant, it must be that $\tilde{x} \in \text{int } Z$. Suppose that $\bar{\alpha}_j^i = 0$ for all $j \in I$. Since $w_i \in \text{int } S$, $\bar{p}w_i > 0$. It follows immediately that $\bar{x}^i$ and $\tilde{x}$ maximize R_i on the set, Z . But local nonsatiation then implies that $\tilde{x} \notin \text{int } Z$. This is a contradiction. Therefore $\bar{\alpha}_j^i > 0$ for some $j \in I$. Suppose that there is an allocation x in Z such that $x P_i \tilde{x}$ and $\sum_{j \in I} \bar{\alpha}_j^i \bar{p}x_j = \bar{p}w_i$. Using assumption 7(b), and observing that $\tilde{x}_j$ is a non-cost-minimizing bundle in C_j , one can show that there is an allocation which is preferred by i to $\tilde{x}$ and which satisfies the budget constraint with strict inequality. This contradicts the result of the preceding paragraph. Therefore $\tilde{x}$ maximizes R_i on $B_i(\bar{p}, \bar{\alpha}) \cap Z$ for all $i \in I$.

If $\bar{\alpha}_i^i = 0$, then it follows from Part (b) of the lemma that $\tilde{x}_i$ maximizes $\succsim_i$ on Z_i . But if this is the case, then $\tilde{x}$ cannot be contained in the interior of Z . Therefore $\bar{\alpha}_i^i > 0$ for all $i \in I$. Q.E.D.

Since $\sum_{i \in I} \tilde{x}_i - \sum_{i \in I} w_i = -s$ where $s \in S$, there is an allocation $\bar{x}$ in Z such that $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} \tilde{x}_i + s = \sum_{i \in I} w_i$ and such that $\bar{x} R_i \tilde{x}$ for all $i \in I$. As was observed above, $\tilde{x} \in \text{int } Z$ and $\bar{\alpha}_i^i > 0$ for all $i \in I$. It follows from the nonsatiation assumption that $\bar{p} \sum_{j \in I} \bar{\alpha}_j^i \bar{x}_j - \bar{p}w_i \geq 0$ for all $i \in I$. But $\sum_{i \in I} (\bar{p} \sum_{j \in I} \bar{\alpha}_j^i \bar{x}_j - \bar{p}w_i) = \bar{p} \sum_{j \in I} \bar{x}_j - \bar{p} \sum_{j \in I} w_j = 0$. Therefore $\bar{p} \sum_{j \in I} \bar{\alpha}_j^i \bar{x}_j = \bar{p}w_i$ for all $i \in I$. Hence $\bar{x} \in B_i(\bar{p}, \bar{\alpha})$ for all $i \in I$. But $\bar{x} R_i \tilde{x}$. Therefore $\bar{x}$ maximizes R_i on $B_i(\bar{p}, \bar{\alpha}) \cap Z$. It follows that $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium on the restricted allocation space Z .

All that remains is to remove the artificially bounding cones. Following the procedure of Debreu [6], one can consider an increasing sequence of closed bounded cubes $\{Z(n)\}$ such that the cube Z discussed above is included in each of them and such that $\lim_{n \rightarrow \infty} Z(n) \supset \Omega_+^{mn}$. The results above show that for each n , there is a distributive Lindahl equilibrium, $(p(n), \alpha(n), x(n))$ on the allocation space $Z(n)$. Since for all n , $\sum_{i \in I} x_i(n) = \sum_{i \in I} w_i$, the sequence $\{x(n)\}$ is contained in the compact set Z . Also, $\{\alpha(n)\}$ and $\{p(n)\}$ are contained in the compact sets, A and P . One can therefore extract a subsequence which converges to $(p^*, \alpha^*, x^*) \in P \times A \times Z$. It is not hard to verify that (p^*, α^*, x^*) is a distributive Lindahl equilibrium. This completes the proof of Theorem 1.

B. Distributive Lindahl Equilibrium is Pareto Optimal

THEOREM 2. *If $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium⁶ and if each consumer is locally nonsatiated at $\bar{x}$, then $\bar{x}$ is Pareto optimal.*

Following the lines of the traditional proof that a competitive equilibrium is Pareto optimal, one can show that if $x R_i \bar{x}$ then $\sum_{j \in I} \bar{\alpha}_j^i(\bar{p}x_j) \geq W_i(\bar{p})$ and if $x P_i \bar{x}$ then $\sum_{j \in I} \bar{\alpha}_j^i(\bar{p}x_j) > W_i(\bar{p})$. Therefore if x is Pareto superior to $\bar{x}$, $\sum_{i \in I} \sum_{j \in I} \bar{\alpha}_j^i(\bar{p}x_j) > \sum_{i \in I} W_i(\bar{p}) = \bar{p}w$. But since $\sum_{i \in I} \bar{\alpha}_j^i = 1$ for all $j \in I$, this implies that $\bar{p} \sum_{j \in I} x_j > \bar{p}w$. This cannot be if x is feasible. Therefore $\bar{x}$ is Pareto optimal. Q.E.D.

C. Pareto Optima Can Be Sustained as Lindahl Equilibria

If $\bar{x}$ is a Pareto optimal allocation for an exchange economy with an aggregate vector of resources, $w \in \Omega_+^m$ and if

- (1) *Assumptions 2–7 of Theorem 1 are true,*
- (2) *For each $i \in I$, $\bar{x}_i \in C_i$ and $\bar{x}_i - \hat{s}_i \in C_i$ for some $\hat{s}_i \in \text{int } S$,*
- (3) *For each $k \in I$, there is an allocation x , such that $\sum_{i \in I} x_i = \sum_{i \in I} \bar{x}_i$ and $x P_i \bar{x}$ for all $i \in I$, where $i \neq k$, then there is an initial wealth distribution, $(\bar{W}_1, \dots, \bar{W}_n)$ such that $\sum_{i \in I} \bar{W}_i = \bar{p}w$ and for some $\bar{\alpha} \in A$, $\bar{p} \in E^m$, $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium when the initial wealth distribution is $(\bar{W}_1, \dots, \bar{W}_n)$.*

Assumption 2 restricts the class of Pareto optimal allocations to those in which each consumer would be able to survive even if some vector of goods, which could be allocated in such a way as to make everyone better off, were taken away from him. Assumption 3 restricts the class of alloca-

⁶ The theorem is stated for an exchange economy. The extension to production economies can be made in the same way as in the familiar proof that competitive equilibrium is Pareto optimal (see Arrow [1]).

tions to those in which there is sufficient conflict of interest so that although there is no feasible allocation which is preferred by all consumers to $\bar{x}$, there is, for any consumer k , some allocation which is preferred to $\bar{x}$ by all consumers other than k .⁷

The proof of the theorem proceeds as follows. Lemma 3 proves the existence of a nonzero commodity price vector which separates the aggregate wealth vector w from the (convex) set of aggregate outputs which can be distributed in such a way as to produce an allocation Pareto superior to $\bar{x}$. Lemma 4 is an extension theorem for linear functionals, closely related to the Hahn-Barach theorem. Lemma 5 uses Lemma 4 to extend the functional $\bar{p}$ in such a way as to find an $\bar{\alpha} \in A$, where for all $i \in I$, if $xR_i\bar{x}$ then $\sum_{j \in I} \bar{\alpha}_j^i(\bar{p}x_j) \geq \sum_{j \in I} \bar{\alpha}_j^i(\bar{p}\bar{x}_j)$.⁸ Assumption 3 of Theorem 3 is then used to show that for each $i \in I$, the vector $(\bar{\alpha}_1^i, \dots, \bar{\alpha}_n^i)$ is nontrivial. Finally, Assumption 2 of Theorem 3 and the continuity Assumption 7(b) of Theorem 1 are both used to show that $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium when the initial wealth distribution is $(\bar{W}_1, \dots, \bar{W}_n) = (\sum_{j \in I} \bar{\alpha}_j^1(\bar{p}\bar{x}_j), \dots, \sum_{j \in I} \bar{\alpha}_j^n(\bar{p}\bar{x}_j))$.

LEMMA 3. *If $\bar{x}$ is a Pareto optimal allocation satisfying Assumptions 2–7 of Theorem 1, then there exists a nonzero $\bar{p} \in E^m$ such that if $xR_i\bar{x}$ for all $i \in I$, then $\bar{p} \sum_{i \in I} x_i \geq \bar{p} \sum_{i \in I} \bar{x}_i$.*

The proof of Lemma 3 is a simple adaptation of the traditional proof that a Pareto optimum is a competitive equilibrium.

LEMMA 4. *Let K be a cone in a real linear space E , and let F be a linear subspace which intersects the radial kernel of K . Then each linear functional f on F which is nonnegative on $K \cap F$ can be extended to a linear functional on E which is nonnegative on K .*

A proof of Lemma 4 can be found in Kelley and Namioka [8, p. 20].

⁷ To see that an assumption like (3) is required, consider the following example. There are two consumers and one commodity. The total supply of the commodity available is 1 unit. Consumer 1 has the utility function $u_1 = x_1^{1/2}x_2^{1/2}$, where x_1 and x_2 are the quantities held by consumers 1 and 2, respectively. Consumer 2's utility function is $u_2 = x_2$. It is easily seen that any allocation such that $\frac{1}{2} \leq x_2 \leq 1$ and such that $x_1 = 1 - x_2$ is Pareto optimal. Consider the Pareto optimal allocation $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$. This allocation can satisfy Assumptions (1) and (2) of Theorem 3. Assumption 3, however, is violated since there is no feasible allocation which is preferred by Consumer 2 to $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$. If the functional $\alpha_1^1x_1 + \alpha_2^1x_2$ supports $R_1(\bar{x}_1, \bar{x}_2)$ then it must be that $\alpha_1^1 = \alpha_2^1$. If $\alpha_1^2x_1 + \alpha_2^2x_2$ supports $R_2(\bar{x}_1, \bar{x}_2)$, then $\alpha_1^2 = 0$ and $\alpha_2^2 > 0$. Thus, if (α_1^1, α_2^1) and (α_1^2, α_2^2) are both nontrivial, it cannot be that $\alpha_1^1 + \alpha_1^2 = 1$ and $\alpha_2^1 + \alpha_2^2 = 1$.

⁸ Lemma 4 is used for similar purposes by Starrett [14].

LEMMA 5. *If Assumptions (1) and (3) of Theorem 3 are true then there is an $\bar{\alpha} \in A$ such that if $\bar{p}$ satisfies Lemma 3, then for all $i \in I$, $\sum_{j \in I} \bar{\alpha}_j^i \bar{p} x_j \geq \sum_{j \in I} \bar{\alpha}_j^i \bar{p} \bar{x}_j$ if $x R_i \bar{x}$. Furthermore for all $i \in I$, $(\bar{\alpha}_1^i, \dots, \bar{\alpha}_n^i) \neq 0$.*

Proof. Let $K_i = \{\lambda z \mid \lambda \geq 0 \text{ and } z = (\bar{p}(x_1 - \bar{x}_1), \dots, \bar{p}(x_n - \bar{x}_n))\}$, where $(x_1, \dots, x_n) R_i(\bar{x}_1, \dots, \bar{x}_n)$. Let $K = \prod_{i \in I} K_i$. Since preferences are weakly convex and nonmalevolent, K is a convex cone containing $\Omega_+^{n^2}$.

Let $F = \{z \mid z = (z_1^1, \dots, z_n^1, z_1^2, \dots, z_n^2, \dots, z_1^n, \dots, z_n^n) \in E^{n^2} \text{ and for all } j \in I, z_j^1 = z_j^2 = \dots = z_j^n = z_j^j\}$. Clearly, F is a linear subspace of E . Since K contains the entire nonnegative orthant in E^{n^2} , F intersects the radial kernel of K . If $z \in K \cap F$, then $1/n \sum_{i \in I} \sum_{j \in I} z_j^i = \sum_{j \in I} z_j = \sum_{j \in I} \bar{p}(x_j - \bar{x}_j)$ for some $(x_1, \dots, x_n)$ such that $(x_1, \dots, x_n) R_i(\bar{x}_1, \dots, \bar{x}_n)$ for all $i \in I$. According to Lemma 3, $1/n \sum_{i \in I} \sum_{j \in I} z_j^i \geq 0$ for all $z \in K \cap F$. Lemma 4 implies that the linear functional f (such that $f(z) = 1/n \sum_{i \in I} \sum_{j \in I} z_j^i$, where $z \in F$) can be extended to a linear function g on E^{n^2} which is nonnegative on K . In particular, let $g(z) = \sum_{i \in I} \sum_{j \in I} \bar{\alpha}_j^i z_j^i$. If $z \in F$, then $g(z) = \sum_{j \in I} \bar{z}_j$. Therefore $\sum_{i \in I} \bar{\alpha}_j^i = 1$ for all $j \in I$. Since K contains the nonnegative orthant in E^{n^2} , $\bar{\alpha}_j^i \geq 0$ for all $i, j \in I$. Hence $\bar{\alpha} \in A$. It is easily verified that if $x R_i \bar{x}$, then $\sum_{j \in I} \bar{\alpha}_j^i \bar{p}(x_j - \bar{x}_j) \geq 0$.

It will be shown that $(\bar{\alpha}_1^k, \dots, \bar{\alpha}_n^k)$ is nontrivial for each $k \in I$. According to Assumption 3, there is an allocation x such that $x P_i \bar{x}$ for all $i \neq k$ and such that $\sum_{j \in I} x_j = \sum_{j \in I} \bar{x}_j$. Clearly, $\sum_{j \in I} \bar{p}(x_j - \bar{x}_j) = 0$, Assumption 2, with the continuity assumption of Theorem 1, implies that there is an allocation x' such that $\sum_{j \in I} x'_j = \sum_{j \in I} x_j - s$ for some $s \in \text{int } S$. But $\bar{p} \sum_{j \in I} (x'_j - \bar{x}_j) = -\bar{p}s < 0$. Since if $x R_i \bar{x}$ for all $i \neq k$, $\bar{p} \sum_{i \in I; i \neq k} \sum_{j \in I} \bar{\alpha}_j^i (x_j - \bar{x}_j) \geq 0$, it cannot be that $\sum_{i \in I; i \neq k} \bar{\alpha}_j^i = 1$ for all $j \in I$. But $(\bar{\alpha}_1^k, \dots, \bar{\alpha}_n^k) = (1 - \sum_{i \in I; i \neq k} \bar{\alpha}_1^i, \dots, 1 - \sum_{i \in I; i \neq k} \bar{\alpha}_n^i)$. Therefore, $(\bar{\alpha}_1^k, \dots, \bar{\alpha}_n^k) \neq 0$. This completes the proof of Lemma 5.

Consider the initial wealth distribution, $(\bar{W}_1, \dots, \bar{W}_n)$ such that for $i \in I$, $W_i = \sum_{j \in I} \bar{\alpha}_j^i (\bar{p} \bar{x}_j)$. Clearly, $\sum_{i \in I} \bar{W}_i = \bar{p} \sum_{j \in I} \bar{x}_j = \bar{p}w$. Suppose that there is an allocation x such that for some $i \in I$, $x P_i \bar{x}$ and $\sum_{j \in I} \bar{\alpha}_j^i (\bar{p} x_j) \leq \bar{W}_i$. Then since for each $j \in I$, $x_j \in C_j$ and $\bar{x}_j - s \in C_j$ for some $s \in \text{int } S$, and since $(\bar{\alpha}_1^i, \dots, \bar{\alpha}_n^i) \neq 0$, the continuity assumption implies that there is an allocation x' such that $x' P_i \bar{x}$ and $\sum_{j \in I} \bar{\alpha}_j^i \bar{p} x'_j < \bar{W}_i$. This contradicts Lemma 5. It follows $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium where the initial wealth distribution is $(\bar{W}_1, \dots, \bar{W}_n)$. Theorem 3 is now proved.

The following theorem is also of some interest.

THEOREM 4. *If $\bar{x}$ is a Pareto optimal allocation satisfying Assumptions (1) and (2) of Theorem 3, then there is a commodity price vector $\bar{p} \in E^m$ such that for all $i \in I$, if $x_i >_i \bar{x}_i$, then $\bar{p} x_i > \bar{p} \bar{x}_i$.*

Let $\bar{p}$ satisfy Lemma 3 above. An easy consequence of nonmalevolence

is that if $x_i \succeq_i \bar{x}_i$ then $\bar{p}x_i \geq \bar{p}\bar{x}_i$. Suppose that $x_i >_i \bar{x}_i$ and $\bar{p}x_i \leq \bar{p}\bar{x}_i$. Assumption (2) and the continuity assumption both imply that for some $\delta \in \text{int } S$, $x_i - \delta P_i \bar{x}_i$. But $\bar{p}(x_i - \delta) < \bar{p}\bar{x}_i$.

This is a contradiction. Theorem 4 must therefore be true. Q.E.D.

A similar theorem is proved by Winter [17]. One interpretation of Theorem 4 is that there is a price system which sustains a Pareto optimum if preferences are nonmalevolent and if all consumers are forbidden to make gifts.

CONCLUSION

The principal result of this paper is that an allocation mechanism is proposed which yields a Pareto optimal distribution of wealth in the presence of benevolence. Furthermore, this mechanism can be used to sustain *most* Pareto optimal allocations. Several related problems are suggested by the analysis. Some will be mentioned here.

A difficulty which is inherent in a Lindahl solution in the so-called "free-rider problem" [4, p. 78]. That is, if an individual's stated preference for allocations influences the distributional shares which he is assigned, it will generally be in his interest to misrepresent his preferences.

It therefore seems unlikely that the information required to reach a Lindahl equilibrium can be extracted by a simple process in which shares are adjusted in response to individual voting behaviour as is suggested in Section I(C). This does not necessarily mean that such information cannot be gained in other ways such as sampling procedures, jury proceedings, or representative parliaments. It is also possible that there are some kinds of group ethic which would result in honest voting. It might be, e.g., that if a man's peers were informed of the way in which he voted, he would tend to vote honestly to retain their "good opinion."

If $(\bar{p}, \bar{\alpha}, \bar{x})$ is a distributive Lindahl equilibrium when the initial wealth distribution is $W(\bar{p})$, then a central clearing house could make transfer payments of value, $T_i = \bar{p}\bar{x}_i - W_i(\bar{p})$ to each $i \in I$. Since $\sum_{i \in I} W_i(\bar{p}) = \bar{p}w = \bar{p} \sum_{i \in I} \bar{x}_i$, $\sum_{i \in I} T_i = 0$. The equilibrium allocation $\bar{x}$, could then be attained if each consumer maximizes his *private* preferences subject to the constraint that his personal expenditures do not exceed $W_i(\bar{p}) + T_i = \bar{p}\bar{x}_i$. The fact that the equilibrium allocation $\bar{x}$ can be maintained simply by transfers of income while commodities are priced competitively depends, of course, in a crucial way on the assumption that all consumers are nonmalevolent.

If there are more complicated interrelations in consumer preferences, a mechanism for which equilibrium is Pareto-optimal will in general

require that for each person there be different shares (some possibly negative) of the cost of *each good* for each other consumer. The informational requirements for the operation of such a mechanism would be truly formidable. Some of the "anti-libertarian" results of the application of the Pareto criterion in these cases would raise the question of whether the Pareto criterion is the most appropriate tool for evaluating the equilibrium states of a social allocation mechanism.

Professor A. K. Sen has recently dramatized this difficulty in his thought-provoking article, "The Impossibility of a Paretian Liberal" [13]. Sen's result can be roughly paraphrased to say that unless some restrictions are placed on the kinds of interrelatedness of consumer preference, there is no social decision mechanism which will always choose a Pareto optimal alternative and which will allow any individual complete control of the social decision regarding any aspect of his "private life." The traditional individualistic assumption of consumer theory is one example of a restriction on the range of possible preferences which allows a "Paretian" to be a "liberal" in Sen's sense. In this case the mechanism of competition would choose Pareto optimal points and allow any individual to determine the social choice between any pair of allocations which differ only in the commodity bundle he himself receives and which exhaust the same vector of resources as does the equilibrium allocation. If the assumption of nonmalevolence is made, a distributive Lindahl mechanism satisfies "Paretian liberality" in exactly the same way.

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