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# L-functions of Exponential Sums over Finite Fields

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for the degree of

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in Mathematics

by

Chao Chen

Dissertation Committee:
Professor Daqing Wan, Chair
Professor Nathan Kaplan
Professor Vladimir Baranovsky

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# VITA

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# ABSTRACT OF THE DISSERTATION

L-functions of Exponential Sums over Finite Fields

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2021

Professor Daqing Wan, Chair

Let $\mathbb{F}_q$ be a finite field with characteristic $p$. A fundamental problem in number theory is to estimate the reciprocal zeros and poles of L-functions of exponential sums over $\mathbb{F}_q$. In this dissertation, we focus on two classical families of exponential sums which have been widely used in the literature. For the type I family, we compute the weights and $q$-adic slopes of the associated L-functions. One consequence of our main result is a sharp estimate of these exponential sums. Another consequence is to obtain an explicit counterexample of Adolphson-Sperber's conjecture on the weights of toric exponential sums [3]. For the type II family, the associated L-functions has pure weights. We study the $q$-adic slopes of the reciprocal roots and extend Zhang and Feng's results [47] of Hasse polynomials. Our main tools include Adolphson-Sperber's work [1] on toric exponential sums and Wan's [44] decomposition theorems.

# Chapter 1

# Introduction

Let $\mathbb{F}_q$ be the finite field of $q$ elements with characteristic $p$. For each positive integer $k$, let $\mathbb{F}_{q^k}$ denote the degree $k$ finite extension of $\mathbb{F}_q$ and $\mathbb{F}_{q^k}^*$ denote the set of non-zero elements in $\mathbb{F}_{q^k}$. Assume that $\zeta_p$ is a fixed primitive $p$-th root of unity in $\mathbb{C}$. For any Laurent polynomial $f(x_1, \ldots, x_{n+1}) \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$, the $k$-th exponential sum associated to $f$ is defined by,

$$S_k^*(f) = \sum_{x_1,\ldots,x_{n+1} \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr_k} f(x_1,\ldots,x_{n+1})}, \quad k = 1, 2, 3, \ldots \tag{1.1}$$

where $\mathrm{Tr_k} : \mathbb{F}_{q^k} \mapsto \mathbb{F}_p$ is the trace map.

In analytic number theory, it's a classical problem to give good estimates for archimedean and non-archimedean sizes of $S_k^*(f)(1 \leq k < \infty)$. To understand the sequence of exponential sums, one naturally forms the generating L-function

$$L^*(f, T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f) \frac{T^k}{k}\right) \in \mathbb{Q}(\zeta_p)[[T]] \tag{1.2}$$

which contains all information of the given sequence.

The study of $L^*(f,T)$ is closely related to the fundamental problem of counting rational points over various finite extension fields of an algebraic variety defined over a finite field, whose resulting generating function is the zeta function introduced by Weil [46]. Consider the toric affine hypersurfaces defined by $f$:

$$U_f(\mathbb{F}_{q^k}) = \{x_1, \dots, x_{n+1} \in \mathbb{F}_{q^k}^* | f(x_1, \dots, x_{n+1}) = 0\}, \quad k = 1, 2, 3, \dots \tag{1.3}$$

The corresponding zeta function is defined as:

$$Z(U_f, T) = \exp\left(\sum_{k=1}^{\infty} \#U_f(\mathbb{F}_{q^k}) \frac{T^k}{k}\right) \tag{1.4}$$

where $\#U_f(\mathbb{F}_{q^k})$ is the number of solutions of $f$ over the torus $\mathbb{G}_m^{n+1}$. It's easy to check that

$$L^*(x_0 f, T) = \frac{Z(U_f, qT)}{Z(\mathbb{G}_m^{n+1}, T)}. \tag{1.5}$$

In the celebrated Weil conjectures [46], the rationality and the evaluation of such zeta functions have been fully discussed. Dwork's [13] rationality proof pioneered his $p$-adic theory of zeta functions and the refinement of his theory [15, 16, 17, 18] is to introduce commuting differential operators which relates zeta functions to GKZ hypergeometric differential equations. Much of the relevant theory also works for the L-functions.

Furthermore, the toric L-function $L^*(f,T)$ is a trivial case of Dwork's unit root L-function [18], identified as the L-function attached to the Galois representation arising from the étale cohomology of an algebraic variety defined over a finite field of characteristic $p$. In Chapter 5, we will discuss a typical non-trivial unit root L-function constructed in terms of the unit zero of $L^*(f,T)^{(-1)^n}$. The meromorphic continuation property of unit root L-functions has potential significance in the general case. For arithmetic applications, it implies the existence of an exact $p$-adic formula [17, 18] for geometric $p$-adic character sums in terms of the $p$-adically bounded solutions of certain differential equations. It also indicates the existence of a general $p$-adic equi-distribution theorem [37] for the roots of zeta functions, which states that the zeros with a given slope of zeta function of a variety over a finite field are equi-distributed in a suitable $p$-adic sense when the variety moves through an algebraic family. In addition, the p-adic evaluation of zeros or poles of the unit root L-function can be viewed as an extension of Gouvéa-Mazur conjecture or Ramanujan-Peterson conjecture on the zeros of the Hecke polynomial [10, 27, 32]. This connects the L-functions to the $p$-adic automorphic forms. For instance, the unit root L-function of the Kloosterman family should be related to arithmetic of Maass forms.

## 1.1 History and motivation

By well-known theorems of Dwork-Bombieri-Grothendieck [13, 14, 6, 29], the generating L-function is a rational function,

$$L^*(f,T) = \frac{\prod_{i=1}^{d_1}(1-\alpha_i T)}{\prod_{j=1}^{d_2}(1-\beta_j T)}, \tag{1.6}$$

where all the reciprocal poles and zeros are non-zero algebraic integers. After taking logarithmic derivatives, we have the formula,

$$S_k^*(f) = \sum_{j=1}^{d_2} \beta_j^k - \sum_{i=1}^{d_1} \alpha_i^k, \quad k = 1, 2, 3, \ldots \tag{1.7}$$

The reciprocal zeros and poles of the L-function contain critical information of the exponential sums. Thus, the fundamental problem of estimating the exponential sums reduces to two parts: (a) determine the number of all reciprocal zeros and poles; (b) estimate the absolute values of all reciprocal zeros and poles.

For part (a), without any condition on $f$, there's an effective (but not efficient) algorithm to compute the L-function. Hence, there's an effective (but not efficient) algorithm to compute the number of zeros and poles. Bombieri [7] gives good upper bounds on the degree and total degree of the L-function associated to an exponential sum. These bounds are expressed in terms of the degree and number of variables of the polynomial $f$ appearing in the exponential sum. Adolphson and Sperber [1] sharpen Bombieri's bounds using the Newton polyhedron of $f$. In addition, they [2] find that $L^*(f,T)^{(-1)^n}$ is a polynomial of fixed degree if the Laurent polynomial $f$ is non-degenerate (a suitable smooth condition on the leading form of $f$). This study is continued by Katz [38] who applies $\ell$-adic method and optimizes Adolphson and Sperber's and Bombieri's bounds on the degree and total degree of L-function. These results are all based on Dwork's $p$-adic theory which will be introduced Chapter 2.

For part (b), Deligne's theorem [11] on the analog of the Riemann hypothesis for algebraic varieties over a finite field provides general information about the valuations of the zeros and poles. The complex absolute values of reciprocal zeros and poles are bounded as follows,

$$|\alpha_i| = q^{u_i/2}, |\beta_j| = q^{v_j/2}, u_i \in \mathbb{Z} \cap [0, 2(n+1)], v_j \in \mathbb{Z} \cap [0, 2(n+1)]. \tag{1.8}$$

For non-archimedean values, $|\alpha_i|_\ell = |\beta_j|_\ell = 1$ when $\ell$ is a prime and $\ell \neq p$. For $p$-adic absolute values,

$$|\alpha_i|_p = q^{-r_i}, |\beta_j|_p = q^{-s_j}, r_i \in \mathbb{Q} \cap [0, n+1], s_j \in \mathbb{Q} \cap [0, n+1].$$

The integer $u_i$ (resp. $v_j$) is called the *weight* of $\alpha_i$ (resp. $\beta_j$) and the rational number $r_i$ (resp. $s_j$) is called the *slope* of $\alpha_i$ (resp. $\beta_j$). In the past few decades, it has been tremendous interest in determining the weights and slopes of the generating L-functions, which is known as a typical version of Riemann hypothesis for the L-functions. This problem turns out to be extremely complicated in general and there's no exact theorem or formula that can completely determine all the weights and slopes without any condition on $f$.

Restricting to non-degenerate Laurent polynomials, the problem becomes solvable since the generating L-function is a polynomial or the inverse of a polynomial. For complex absolute values, Adolphson and Sperber [2] provide a sharper estimate of the reciprocal zeros: $|\alpha_i| \leq q^{(n+1)/2}$. These complex absolute values can be determined explicitly by a theorem of Denef and Loeser [12] proved using $\ell$-adic methods. This formula is derived from intersection cohomology and becomes combinatorially complicated to use when the dimension $n \geq 5$ and the Newton polyhedron of $f$ is not simple at the origin. For $q$-adic slopes, the problem is equivalent to compute the $q$-adic Newton polygon of the polynomial $L^*(f, T)^{(-1)^n}$. A standard lower bound of the Newton polygon is provided by Adolphson and Sperber [2] and one can determine the slopes if the Newton polygon coincides with its lower bound. However, there does not exist a clean general answer that works for arbitrary non-degenerate $f$ and prime $p$. It's still unknown if there exists a polynomial time algorithm to determine the $q$-adic slopes with a given $f$ and $p$. So we may have to restrict to certain classes of $f$ and $p$ to get a precise result.

## 1.2 Main objects: two types of exponential sums

In this dissertation, we study the weights and slopes of the generating L-functions of two widely used families of exponential sums arising from analytic number theory. The type I exponential sums were first systematically studied by Katz [36] from $\ell$-adic point of view and the type II exponential sums were first systematically studied by Zhang and Feng [47] from $p$-adic point of view. Our main tools include Adolphson-Sperber's [1] work on toric exponential sums and Wan's decomposition theorems [44]. All the basic definitions and useful technical theorems will be introduced in Chapter 2.

### 1.2.1 Type I exponential sums

The type I exponential sums will be discussed in Chapter 3. This study is based on the unpublished forthcoming joint work with Xin Lin. For part of the details, see [9]. Let $n = \sum_{i=1}^{r} n_i$ be a partition of a positive integer $n$. Consider $n$ variables

$$x_{ij}, \quad \text{for } 1 \leq i \leq r,\ 1 \leq j \leq n_i,$$

and $n$ positive integral constants

$$b_{ij}, \quad \text{for } 1 \leq i \leq r,\ 1 \leq j \leq n_i.$$

Let $\psi : \mathbb{F}_p \to \mathbb{C}^*$ be a fixed nontrivial additive character over $\mathbb{F}_p$ and $\mathrm{Tr}_k : \mathbb{F}_{q^k} \to \mathbb{F}_p$ be the trace map ($k \in \mathbb{Z}_{>0}$). The type I exponential sum is defined as: for $a_i \in \mathbb{F}_q^*$ and $b_{ij} \in \mathbb{Z}_{>0}$,

$$S_k(\vec{a}) = \sum_{\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} = 1} \psi\left(\mathrm{Tr}_k\left(\sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij}\right)\right),$$

where the sum is over all $x_{ij} \in \mathbb{F}_{q^k}^*$. The associated L-function is given by

$$L(\vec{a}, T) = \exp\left(\sum_{k=1}^{\infty} S_k(\vec{a}) \frac{T^k}{k}\right).$$

Our main results include a formula for the lower bound of a $q$-adic Newton polygon of $L(\vec{a}, T)^{(-1)^n}$ and a congruence condition on $p$, under which the Newton polygon coincides with its lower bound. In addition, we determine part of the weights and find some trivial factors of $L(\vec{a}, T)^{(-1)^n}$. Since the detailed theorems for the whole family is very complicated, here we take a special case as an example.

Assume $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \leq i \leq 2, 1 \leq j \leq 2)$. Then the exponential sum has the following simpler form: for $a_1, a_2 \in \mathbb{F}_q^*$,

$$S_k(\vec{a}) = S_k(a_1, a_2) = \sum_{\substack{\frac{a_1}{x_1x_2} + \frac{a_2}{x_3x_4} = 1 \\ x_1,\ldots,x_4 \in \mathbb{F}_{q^k}^*}} \psi(\mathrm{Tr}_k(x_1 + x_2 + x_3 + x_4)).$$

This family of exponential sums has long been objects of study and has been valuable in analytic number theoretic applications. Based on Deligne's main theorem [11], Birch and Bombieri [5] prove that if $p > c_0$, then $|S_k(a_1, a_2)| \leq c_1 q^{3k/2}$ for some absolute constants $c_0$,$c_1$. Their estimate contributes to Friedlander and Iwaniec's [21] work on the estimate of the certain averages of incomplete Kloosterman sums in application to the divisor problem of $d_3(n)$. Following the previous results of $S_k(a_1, a_2)$, Zhang [48] gains a boundary of the error terms in his work on twin prime conjecture. The depth of this special case should give an indication about the potential significance of the estimate of type I exponential sums in analytic number theory.

Applying the systematic theory for toric L-functions, we can completely determine the slopes and weights of $L(\vec{a}, T)$ in this special case and get the following decomposition.

**Theorem 1.1.** *The generating L-function of $S_k(a_1, a_2)$ is a polynomial of degree 8, i.e.,*

$$L(\vec{a}, T) = (1 - T)(1 - qT) \prod_{i=1}^{6} (1 - \alpha_i T),$$

*where $\alpha_i \in \overline{\mathbb{Q}}$ with the complex absolute value $|\alpha_i| = q^{3/2}$. If we view the reciprocal roots $\alpha_i$ as p-adic numbers and enumerate them with respect to the q-adic slopes, their p-adic norms* *are given by*

$$|\alpha_i|_p = \begin{cases} 1 & \textit{if} \quad i = 1. \\ q^{-1} & \textit{if} \quad i = 2, 3. \\ q^{-2} & \textit{if} \quad i = 4, 5. \\ q^{-3} & \textit{if} \quad i = 6. \end{cases}$$

A consequence of our theorem is a sharp estimate for the complex value of $S_k(\vec{a})$.

**Corollary 1.1.** *Notations as above. For all* $p, k$ *and* $\vec{a}$*,*

$$|S_k(a_1, a_2)| \leq 6q^{3k/2} + q^k + 1.$$

This special case also provides the first explicit counterexample of Adolphson-Sperber's conjecture [3] on weights of toric exponential sums.

### 1.2.2 Type II exponential sums

In Chapter 4, we study the $p$-adic estimate of exponential sums

$$S_k^*(f) = \sum_{x_1,\ldots,x_{n+1} \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr}_k f(x_1,\ldots,x_{n+1})}, \quad k = 1, 2, 3, \ldots$$

associated to a family of Laurent polynomials $f(x_1, x_2 \ldots, x_{n+1})$ over $\mathbb{F}_q$, denoted by $\mathcal{F}$:

$$f(x_1, x_2 \ldots, x_{n+1}) = \sum_{i=1}^{n} a_i x_{n+1} \left( x_i + \frac{1}{x_i} \right) + a_{n+1} x_{n+1} + \frac{1}{x_{n+1}}$$

where $a_i \in \mathbb{F}_q^*$, $i = 1, 2, \ldots, n+1$. The study of type II exponential sums stated in Chapter 4 comes from my own published paper [8].

The estimate of type II exponential sums applies to many critical problems in analytic number theory. For instance, in Iwaniec's work [33] on small eigenvalues of Laplacian for $\Gamma_0(p)$, he improves the lower bound of eigenvalues conjectured by Selberg via the estimate for the specific exponential sum $S_1^*(f)$ where $f \in \mathcal{F}$ is in 3 variables.

For this family, the reciprocal roots of the associated L-function have the same weight and the lower bound of the Newton polygon has been computed by Zhang and Feng [47]. Zhang and Feng also find that this family is generically ordinary for all prime $p$, i.e., there exists a non-zero polynomial depending on the coefficients of $f$ that characterizes when the Newton polygon reaches its lower bound. The non-zero polynomial, called *Hasse polynomial*, is the main object of study in Chapter 4.

Let $\vec{a} = (a_1, \ldots, a_{n+1})$ denote the coefficients of a non-degenerate Laurent polynomial $f \in \mathcal{F}$ and $\Delta_n = \Delta(f)$ denote the Newton polyhedron of $f$ (defined in Section 2.1). Wan [44] provides a general method to directly calculate the Hasse polynomials but the method becomes insufficient for higher dimensional Newton polyhedrons. Based on Wan's method, Zhang and Feng [47] obtain an explicit formula of Hasse polynomials in low dimensions, i.e., $n \leq 3$. Assume $h_p(\Delta_n)$ is the Hasse polynomial of $\Delta_n$ satisfying: if $h_p(\Delta_n) \neq 0$, the $q$-adic Newton polygon of the associated L-function coincides with its lower bound. Let $h_p(\Delta_n, \leq 1)$ be a factor of $h_p(\Delta_n)$ satisfying: if $h_p(\Delta_n, \leq 1) \neq 0$, Newton polygon coincides with the lower bound for all sides of slope $\leq 1$. Note that $h_p(\Delta_n)$ is the full Hasse polynomial, while $h_p(\Delta_n, \leq 1)$ is the partial Hasse polynomial which is a factor of the full Hasse polynomial. This indicates that the full Hasse polynomial is harder than the partial Hasse polynomial. Extending Zhang and Feng's result, we give an explicit formula for the partial Hasse polynomial $h_p(\Delta_n, \leq 1)$ ($n \in \mathbb{Z}_{\geq 0}$) in general case.

**Theorem 1.2.** *Let $f(x_1, \ldots, x_{n+1}) \in \mathcal{F}$ be a non-degenerate Laurent polynomial with $\Delta_n = \Delta(f)$. When $n \geq 2$, a Hasse polynomial of slope at most one side can be taken to be,*

$$h_p(\Delta_n, \leq 1)(\vec{a}) = \sum_{\substack{0 \leq v_1 + \ldots + v_n \leq \frac{p-1}{2} \\ v_1, \ldots, v_n \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)}}{(v_1! v_2! \ldots v_n!)^2 (p-1-2(\sum_{i=1}^n v_i))!}.$$

Furthermore, Zhang-Feng's formula for the full Hasse polynomial $h_p(\Delta_3)$ when $n = 3$ is very complicated that involves the determinant of a $4 \times 4$ matrix whose entries are all polynomials. Here we provide a much simpler formula for the $n = 3$ case based on Denef-Loeser's theorem [12] and the symmetric property of NP$(f)$, which answers an open question of Zhang and Feng [47].

**Theorem 1.3.** *For $n = 3$, let $f(x_1, \ldots, x_4) \in \mathcal{F}$ be a non-degenerate Laurent polynomial with $\Delta(f) = \Delta_3$. A Hasse polynomial of $\Delta_3$ can be written as,*

$$h_p(\Delta_3)(\vec{a}) = h_p(\Delta_3, \leq 1)(\vec{a}) = \sum_{\substack{0 \leq v_1 + v_2 + v_3 \leq \frac{p-1}{2} \\ v_1, v_2, v_3 \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} a_3^{2v_3} a_4^{p-1-2(\sum_{i=1}^3 v_i)}}{(v_1! v_2! v_3!)^2 (p-1-2(\sum_{i=1}^3 v_i))!}.$$

# Chapter 2

# Prelimaries

## 2.1 Non-degenerate Laurent polynomials

Consider a Laurent polynomial $f(x_1, \ldots, x_{n+1}) \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ given by

$$f(x_1, \ldots, x_{n+1}) = \sum_{j=1}^{J} a_j x^{V_j}$$

where $a_j \in \mathbb{F}_q^*$ and $V_j = (v_{1j}, \ldots, v_{n+1j}) \in \mathbb{Z}^{n+1}$ $(1 \leq j \leq J)$. The *Newton polyhedron* of $f$, $\Delta(f)$, is defined to be the convex closure in $\mathbb{R}^{n+1}$ generated by the origin and the lattice points $V_j$ $(1 \leq j \leq J)$. For $\delta \subset \Delta(f)$, let the Laurent polynomial

$$f^{\delta} = \sum_{V_j \in \delta} a_j x^{V_j}$$

be the restriction of $f$ to $\delta$.

**Definition 2.1.** *A Laurent polynomial $f$ is called non-degenerate if for each closed face $\delta$ of*

*$\Delta(f)$ of arbitrary dimension which doesn't contain the origin, the $n+1$ partial derivatives*

$$\left\{\frac{\partial f^{\delta}}{\partial x_1}, \ldots, \frac{\partial f^{\delta}}{\partial x_{n+1}}\right\}$$

*have no common zeros with $x_1 \ldots x_{n+1} \neq 0$ over the algebraic closure of $\mathbb{F}_q$.*

Based on Dwork's $p$-adic theory, Adolphson and Sperber [2] extend Dwork's cohomology theory from smooth projective hypersurfaces [14] to a general class of exponential sums defined by non-degenerate Laurent polynomials. Through calculating the dimensions of corresponding cohomology spaces, they find that the associated L-function $L^*(f,T)^{(-1)^n}$, defined by Equation (1.2), is a polynomial whose degree is expressed in terms of the Newton polyhedron of $f$. By Grothendieck's Lefschetz trace formula, the L-function can be viewed as an alternalting product of characteristic polynomials of associated $\ell$-adic cohomology groups. So to obtain archimedean estimates, Adolphson and Sperber compute the cohomology groups using Deligne's fundamental theorem [11] and get the following sharper result.

**Theorem 2.1** (Adolphson and Sperber, [2])**.** *For any non-degenerate $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ with $(n+1)$-dimensional Newton polyhedron $\Delta(f)$, the associated L-function $L^*(f,T)^{(-1)^n}$ is of the following form,*

$$L^*(f,T)^{(-1)^n} = \prod_{i=1}^{(n+1)!\,\mathrm{Vol}(\Delta(f))} (1-\alpha_i T),$$

*where $|\alpha_i| = q^{\omega_i/2}$, $\omega_i \in \mathbb{Z} \cap [0, n+1]$ and $i = 1, 2, \ldots, (n+1)!\,\mathrm{Vol}(\Delta)$.*

Deligne's integrality theorem also implies that the $p$-adic absolute values of reciprocal roots are given by $|\alpha_i|_p = q^{-r_i}$ where $r_i \in \mathbb{Q} \cap [0, n+1]$. For simplicity, we normalize $p$-adic absolute value to be $|q|_p = q^{-1}$. To obtain the $q$-adic slopes of its reciprocal roots, it's equivalent to compute the $q$-adic Newton polygon of $L^*(f,T)^{(-1)^n}$ since its shape can determine all the $q$-adic norms of the reciprocal roots. Note that although the Newton polygon is independent of the base field $\mathbb{F}_q$ of $f$, it varies a lot as $f$ and prime $p$ change. So there's no simple general formula that can precisely determine all such Newton polygons for arbitrary $f$ and $p$.

By a theorem of Adolphson and Sperber [2], the Newton polygon of the associated L-function of a non-degenerate Laurent polynomial lies above a certain topological or combinatorial lower bound called *Hodge polygon* which only depends in a simple way on the Newton polyhedron $\Delta$ and is easier to compute with a combinatorial formula. In contrast, the Newton polynomial depends not only on the combinatorial property of $\Delta$, but also on the arithmetic property of $\Delta$ and on the actual coefficients of $f$ which make the situation much more complicated. So generally, the slope problem concerning with a non-degenerate Laurent polynomial is reduced to the computation of the Hodge polygon and determination of the condition when the Newton polygon coincides with its corresponding Hodge polygon. The basic definitions of the Newton polygon and Hodge polygon will be discussed in section 2.2. The coincidence between a Newton polygon and its Hodge polygon will be discussed in section 2.3.

## 2.2 Newton polygon and Hodge polygon

### 2.2.1 Newton polygon

**Definition 2.2** (Newton polygon). *Let* $L(T) = \sum_{i=0}^{n} a_i T^i \in 1 + T\overline{\mathbb{Q}}_p[T]$*, where* $\overline{\mathbb{Q}}_p$ *is the algebraic closure of* $\mathbb{Q}_p$*. The* $q$*-adic Newton polygon of* $L(T)$ *is defined to be the lower convex closure of the set of points* $\{(k, \operatorname{ord}_q(a_k)) \,|k = 0, 1, \ldots, n\}$ *in* $\mathbb{R}^2$*.*

Here $\operatorname{ord}_q$ denotes the standard $q$-adic ordinal on $\overline{\mathbb{Q}}_p$ where the valuation is normalized by assuming $\operatorname{ord}_q(q) = 1$. The following lemma relates the $q$-adic valuation of reciprocal roots to the shape of the corresponding $q$-adic Newton polygon.

**Lemma 2.1.** *In the above notation, let* $L(T) = (1 - \alpha_1 T) \dots (1 - \alpha_n T)$ *be the factorization of* $L(T)$ *in terms of reciprocal roots* $\alpha_i \in \overline{\mathbb{Q}}_p$. *Let* $\lambda_i = \operatorname{ord}_q \alpha_i$. *If* $\lambda$ *is the slope of a* $q$*-adic Newton polygon with horizontal length* $l$, *then precisely* $l$ *of the* $\lambda_i$ *are equal to* $\lambda$.

Assume $f$ is a non-degenerate Laurent polynomial in $n + 1$ variables and consequently $L^*(f, T)^{(-1)^n}$ is a polynomial. The $q$-adic Newton polygon of $L^*(f, T)^{(-1)^n}$ is denoted by $\mathrm{NP}(f)$. Adolphson and Sperber bound below $\mathrm{NP}(f)$ by a polygon constructed involving Hodge numbers. These Hodge numbers are identified with the dimensions of the graded pieces of the cohomology space and can be determined by a combinatorial formula involving the number of lattice points contained in $\Delta(f)$.

### 2.2.2 Hodge number

Let $\Delta$ be an $(n + 1)$-dimensional integral polytope containing the origin in $\mathbb{R}^{n+1}$. Define $C(\Delta)$ to be the cone generated by $\Delta$ in $\mathbb{R}^{n+1}$.

**Definition 2.3.** *For any point* $u \in \mathbb{R}^{n+1}$, *the weight function of* $\Delta$*:*

$$w(u) := \begin{cases} \text{the smallest } c \in \mathbb{R}_{\geq 0} \text{ such that } u \in c\Delta = \{cx | x \in \Delta\}, & \text{if } u \in C(\Delta), \\ \infty & \text{if } u \notin C(\Delta). \end{cases} \tag{2.1}$$

*If* $\Delta$ *is generated by vertices* $V_1, \dots, V_m$, *then*

$$w(u) = \inf_{\vec{c}} \left\{ \sum_{l=1}^{m} c_l \;\middle|\; \sum_{l=1}^{m} c_l V_l = u, \vec{c} = (c_1, \dots, c_m) \in \mathbb{R}^m_{\geq 0} \right\}.$$

From the description of the weight function, one can easily get the following proposition.

**Proposition 2.1.** *Notations as above.*

1. *Let $u \in \mathbb{R}^{n+1}$. If $k$ is a non-negative integer, then $w(ku) = kw(u)$.*

2. *If $u, u' \in C(\Delta)$, then $u+u' \in C(\Delta)$. Furthermore, $w(u+u') \leq w(u)+w(u') < \infty$ where the equation holds if and only if $u$ and $u'$ are cofacial, i.e. , $w(u)^{-1}u$ and $w(u')^{-1}u'$ lie on the same closed face of $\Delta$.*

Assume $\delta$ is a co-dimension 1 face of $\Delta$ not containing the origin. Let $D(\delta)$ be the least common multiple of the denominators of the coefficients in the implicit equation of $\delta$, normalized to have constant term 1. We define the *denominator* of $\Delta$ to be the least common multiple of all such $D(\delta)$:

$$D = D(\Delta) = \mathrm{lcm}_{\delta} D(\delta) \tag{2.2}$$

where $\delta$ runs over all the co-dimension 1 faces of $\Delta$ that don't contain the origin. It's easy to check

$$w(\mathbb{Z}^{n+1}) \subseteq \frac{1}{D(\Delta)}\mathbb{Z}_{\geq 0} \cup \{+\infty\}.$$

For $k \in \mathbb{Z}_{\geq 0}$, let

$$W_{\Delta}(k) = \#\left\{u \in \mathbb{Z}^{n+1} | w(u) = \frac{k}{D}\right\} \tag{2.3}$$

be the number of lattice points in $\mathbb{Z}^{n+1}$ with weight $k/D$.

**Definition 2.4** (Hodge number)**.** *Let $\Delta$ be an $(n+1)$-dimensional integral polytope containing the origin in $\mathbb{R}^{n+1}$. For a non-negative integer $k$, the $k$-th Hodge number of $\Delta$ is defined to be*

$$H_{\Delta}(k) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} W_{\Delta}(k - iD). \tag{2.4}$$

Here $H_\Delta(k)$ represents the number of lattice points of weight $k/D$ in a certain fundamental domain corresponding to a basis of the $p$-adic cohomology space used to compute the L-function. Thus, for $k \in \mathbb{Z}_{\geq 0}$, $H_\Delta(k)$ is a natural number. Adolphson and Sperber[2] prove that $H_\Delta(k)$ coincides with the usual Hodge number in the toric hypersurface case in which case $D = 1$. So similar to the more traditional Hodge number,

$$\sum_{k=0}^{(n+1)D} H_\Delta(k) = (n+1)!\mathrm{Vol}(\Delta),$$

and

$$H_\Delta(k) = 0, \quad \text{if } k > (n+1)D.$$

### 2.2.3 Hodge polygon

Based on the Hodge numbers, we define the Hodge polygon of a given polyhedron $\Delta \in \mathbb{R}^{n+1}$.

**Definition 2.5** (Hodge polygon)**.** *The Hodge polygon HP*$(\Delta)$ *of* $\Delta$ *is the lower convex polygon in* $\mathbb{R}^2$ *with vertices* $(0,0)$ *and*

$$Q_k = \left( \sum_{m=0}^{k} H_\Delta(m), \frac{1}{D} \sum_{m=0}^{k} m H_\Delta(m) \right), \quad k = 0, 1, \ldots, (n+1)D,$$

*where* $H_\Delta(k)$ *is the* $k$*-th Hodge number of* $\Delta$*,* $k = 0, 1, \ldots, (n+1)D$*.*

*That is, HP*$(\Delta)$ *is a polygon starting from the origin* $(0,0)$ *with a slope* $k/D$ *side of horizontal length* $H_\Delta(k)$ *for* $k = 0, 1, \ldots, (n+1)D$*. The vertex* $Q_k$ *is called a break point if* $H_\Delta(k+1) \neq 0$ *where* $k = 1, 2, \ldots, (n+1)D - 1$*.*

In the direction of Katz's conjecture [34], Adolphson and Sperber construct the combinitorial Hodge polygon and prove that it's a lower bound of the corresponding Newton polygon.

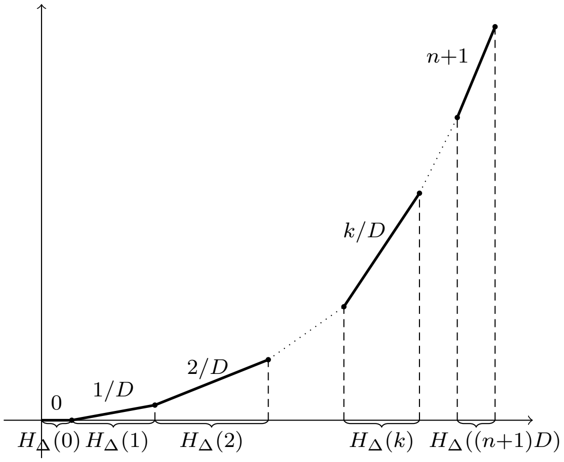

Figure 2.1: Hodge Polygon

**Theorem 2.2** (Adolphson and Sperber, [2]). *For every prime $p$ and non-degenerate Laurent polynomial $f$ with $\Delta(f) = \Delta \subset \mathbb{R}^{n+1}$, we have*

$$NP(f) \geq HP(\Delta),$$

*where $NP(f)$ is the $q$-adic Newton polygon of $L^*(f,T)^{(-1)^n}$. Furthermore, the endpoints of $NP(f)$ and $NP(\Delta)$ coincide.*

## 2.3 Ordinary and generically ordinary families

For a fixed $(n+1)$-dimensional integral polytope $\Delta \subset \mathbb{R}^{n+1}$ containing origin, let $\mathcal{N}_p(\Delta)$ be the parameter space of Laurent polynomials $g$ defined over $\overline{\mathbb{F}}_p$ with $\Delta(g) = \Delta$. Obviously, $\mathcal{N}_p(\Delta)$ is a smooth affine variety over $\mathbb{F}_p$.

Let $\mathcal{M}_p(\Delta)$ be the subset of $\mathcal{N}_p(\Delta)$ consisting of all non-degenerate Laurent polynomials. By the definition of non-degeneracy, $\mathcal{M}_p(\Delta)$ is a Zariski open smooth affine subset of $\mathcal{N}_p(\Delta)$ which is the complement of a certain discriminant locus [24]. Also $\mathcal{M}_p(\Delta)$ is non-empty for $p$ sufficiently large, i,e, $p > (n+1)!\mathrm{Vol}(\Delta)$. Then it's natural to consider how Newton polygon

NP($f$) varies as $f$ varies in $\mathcal{M}_p(\Delta)$. Although there's no simple formula for the precise determination of NP($f$), Grothendieck's specialization theorem provides a general structure of how the Newton polygon varies for arbitrary $f$ in a certain algebraic family [35, 43]. By Grothendieck's theorem, Newton polygons "go up" under specialization: no point of the Newton polygon of a closed fiber in the family is below the Newton polygon of the generic fiber. This implies that there is a lowest polygon for the set $\{\mathrm{NP}(f) \mid f \in \mathcal{M}_p(\Delta)\}$ and the lowest polygon is attained for $f$ in some Zariski open dense subset of $\mathcal{M}_p(\Delta)$. The lowest possible Newton polygon is called the generic Newton polygon, denoted as,

$$\mathrm{GNP}(\Delta, p) := \inf_{f \in \mathcal{M}_p} \mathrm{NP}(f).$$

Combining with Adolphson and Sperber's theorem [2], we have the following inequalities.

**Proposition 2.2.** *For every prime $p$ and $f \in \mathcal{M}_p(\Delta)$, we have*

$$NP(f) \geq GNP(\Delta, p) \geq HP(\Delta).$$

*Furthermore, the endpoints of $NP(f)$ and $NP(\Delta)$ coincide.*

**Definition 2.6.** *A Laurent polynomial $f$ is called ordinary if $NP(f) = HP(\Delta)$.*

**Definition 2.7.** *The family $\mathcal{M}_p(\Delta)$ is called generically ordinary if $GNP(\Delta, p)$=$HP(\Delta)$.*

Let $\mathcal{H}_p(\Delta) = \{g \in \mathcal{M}_p(\Delta) | \mathrm{NP}(g) = \mathrm{HP}(\Delta)\}$ be the subset of $\mathcal{M}_p(\Delta)$ containing non-degenerate ordinary Laurent polynomials, which is called the *Hasse domain* in Dwork's terminology. Similarly, let $\mathcal{H}_p(\Delta, \leq k) = \{g \in \mathcal{M}_p(\Delta) |$ NP($g$) = HP($\Delta$) for all sides of slopes $\leq k/D\}$ and $\mathcal{H}_p(\Delta, k) = \{g \in \mathcal{M}_p(\Delta) | \mathrm{NP}(g) = \mathrm{HP}(\Delta)$ for the slope $k/D$ side$\}$. It's easy to check that $\mathcal{H}_p(\Delta)$, $\mathcal{H}_p(\Delta, \leq k)$ and $\mathcal{H}_p(\Delta, k)$ are Zariski-open subsets of $\mathcal{M}_p(\Delta)$ (possibly empty).

A basic question is whether $\mathcal{H}_p(\Delta)$ is empty or not. If $\mathcal{H}_p(\Delta) = \varnothing$, then $\mathrm{GNP}(\Delta, p) > \mathrm{HP}(\Delta)$ which implies $\mathcal{M}_p(\Delta)$ is not generically ordinary. In this case, every Laurent polynomial in $\mathcal{M}_p(\Delta)$ is not ordinary. Let $\vec{a}$ denote the coefficients of $f \in \mathcal{M}_p(\Delta)$. If $\mathcal{H}_p(\Delta)$ is not empty, there exists a non-zero polynomial $h_p(\Delta)(\vec{a}) \in \mathbb{F}_p[\vec{a}]$ such that if the coefficients of $f \in \mathcal{M}_p(\Delta)$ satisfying $h_p(\Delta)(\vec{a}) \neq 0$, then $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$.

**Definition 2.8** (Hasse polynomial)**.** *If* $\mathcal{H}_p(\Delta)$ *is not empty, then* $\mathcal{H}_p(\Delta)$ *is Zariski dense in* $\mathcal{M}_p(\Delta)$ *and the complement of* $\mathcal{H}_p(\Delta)$ *is a hypersurface determined by a non-zero polynomial* $h_p(\Delta)$ *over* $\mathbb{F}_p$ *which is called a Hasse polynomial with respect to* $\Delta$*.*

Similarly, let $h_p(\Delta, k)$ be the partial Hasse polynomial of slope $k/D$ side and let $h_p(\Delta, \leq k)$ be the partial Hasse polynomial of all sides with slope $\leq k/D$. Note that $h_p(\Delta)$ is the full Hasse polynomial, while $h_p(\Delta, \leq k)$ or $h_p(\Delta, k)$ is a partial Hasse polynomial which is a factor of the full Hasse polynomial. For a non-degenerate Laurent polynomial $f$ with $\Delta(f) = \Delta$, Hasse polynomials determine the amount of coincidence between $\mathrm{NP}(f)$ and $\mathrm{HP}(\Delta)$.

## 2.4 Newton polygons of Fredholm determinants

Based on Dwork's trace formula, the associated L-function can be expressed in terms of the Fredholm determinant of a certain infinite Frobenius matrix. To fully understand the $p$-adic variation of L-functions, it's sufficient to compute the Newton polygon of the Fredholm determinant. This section will introduce a natural lower bound for the Newton polygon of Fredholm determinant and reduce the determination of an ordinary family to a simpler case only consisting of the determinant of a computable Frobenius matrix. At the end of this section, we also provide a standard way to compute a Hasse polynomial which represents the ordinary property of a given non-degenerate Laurent polynomial.

### 2.4.1 Basic settings

Let $p$ be a prime and $q = p^a$ for some positive integer $a$. Let $\mathbb{Q}_p$ be the field of $p$-adic numbers and $\Omega$ be the completion of $\overline{\mathbb{Q}}_p$. Pick a fixed primitive $p$-th root of unity in $\Omega$, denoted by $\zeta_p$. In $\mathbb{Q}_p(\zeta_p)$, choose a fixed element $\pi$ satisfying

$$\sum_{m=0}^{\infty} \frac{\pi^{p^m}}{p^m} = 0 \quad \text{and} \quad \mathrm{ord}_p \pi = \frac{1}{p-1}.$$

By Krasner's lemma, it's easy to check $\mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_p)$. The fixed $\pi$ is the uniformizer of the totally ramified extension $\mathbb{Q}_p(\zeta_p)$ over $\mathbb{Q}_p$. Let $K$ be an unramified extension of $\mathbb{Q}_p$ of degree $a$ and $\Omega_a$ be the compositum of $\mathbb{Q}_p(\zeta_p)$ and $K$.

$$\begin{tikzcd}
 & \Omega_a \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash, "a"'] & & \mathbb{Q}_p(\pi) \arrow[dl, dash, "p-1"] \\
 & \mathbb{Q}_p &
\end{tikzcd}$$

To define a Frobenius generator $\tau$ of $\mathrm{Gal}(\Omega_a/\mathbb{Q}_p(\pi))$, we lift the Frobenius automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ to a generator $\tau$ of $\mathrm{Gal}(K/\mathbb{Q}_p)$ and extend it to $\Omega_a$ with the restriction that $\tau(\pi) = \pi$. The action of $\tau$ on $\Omega_a$ can be described by $\tau(\zeta_{q-1}) = \zeta_{q-1}^p$, where $\zeta_{q-1}$ is the primitive $(q-1)$-th root of unity.

Let $E_p(t)$ be the Artin-Hasse exponential series:

$$E_p(t) = \exp\left(\sum_{m=0}^{\infty} \frac{t^{p^m}}{p^m}\right) = \sum_{m=0}^{\infty} \lambda_m t^m.$$

Based on Dwork's lemma, it's easy to check that the coefficients of $E_p(t)$ are $p$-adic integers, i.e., $E_p(t) \in \mathbb{Z}_p[[x]]$. After simple calculation, we have

$$\lambda_m = \begin{cases} \frac{1}{m!}, & \text{if } 0 \leq m \leq p-1, \\ \frac{1}{m!} + \frac{1}{p(m-p)!}, & \text{if } p \leq m \leq 2p-1. \end{cases}$$

In Dwork's terminology, a splitting function $\theta(t)$ is defined to be,

$$\theta(t) = E(\pi t) = \sum_{m=0}^{\infty} \lambda_m \pi^m t^m.$$

Note that $\theta(1)$ is a primitive $p$-th root of unity that can be identified with $\zeta_p$ in $\Omega$.

### 2.4.2 Dwork's trace formula

Consider a Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ given by

$$f = \sum_{j=1}^{J} \bar{a}_j x^{V_j}$$

where $V_j \in \mathbb{Z}^{n+1}$ and $\bar{a}_j \in \mathbb{F}_q^*$. Let $a_j$ be the Teichmüller lifting of $\bar{a}_j$ in $\Omega$ satisfying $a_j^q = a_j$.

We are interested in the behaviour of $L^*(f,T)^{(-1)^n}$ as a function in $\Omega_a$. By Dwork's trace formula, we can describe the associated L-function in terms of the Fredholm determinant of a completely continuous $\Omega_a$-linear operator on a $p$-adic infinite-dimensional Banach space. This Banach space is a set of $p$-adic Laurent series satisfying certain growth conditions related to weight functions. After choosing a suitable orthonormal basis of the Banach space, the action of the $\Omega_a$-linear operator is determined by a computable infinite matrix.

The operator is constructed based on the following function:

$$F(f,x)=\prod_{j=1}^{J}\theta(a_j x^{V_j})=\sum_{r\in\mathbb{Z}^n}F_r(f)x^r$$

with coefficients

$$F_r(f)=\sum_{u}(\prod_{j=1}^{J}\lambda_{u_j}a_j^{u_j})\pi^{u_1+\ldots+u_J} \tag{2.5}$$

where the sum is over all the solutions of the following linear system

$$\sum_{j=1}^{J}u_jV_j=r,\ u_j\in\mathbb{Z}_{\geq 0}$$

and $\lambda_m$ is $m$-th coefficient of the Artin-Hasse exponential series $E_p(t)$.

Let $\Delta$ be the Newton polyhedron of $f$ and $L(\Delta)=\mathbb{Z}^{n+1}\cap C(\Delta)$ be the set of lattice points in the closed cone generated by origin and $\Delta$. Recall that for a given point $r\in\mathbb{R}^{n+1}$, the weight function is defined to be

$$w(r):=\inf_{\vec{u}}\left\{\sum_{j=1}^{J}u_j|\sum_{j=1}^{J}u_jV_j=r,\ u_j\in\mathbb{R}_{\geq 0}\right\}.$$

**Definition 2.9.** *Notations as above. In Dwork's terminology,*

1. *The infinite semilinear Frobenius matrix $A_1(f)$ is an infinite matrix whose rows and columns are indexed by the lattice points in $L(\Delta)$ with respect to the weights:*

   $$A_1(f)=(a_{r,s}(f))=(F_{ps-r}(f)\pi^{w(r)-w(s)}) \tag{2.6}$$

   *where $r,s\in L(\Delta)$.*

*2. The infinite linear Frobenius matrix* $A_a(f)$ *is defined to be*

$$A_a(f) = A_1(f) A_1^{\tau}(f) \cdots A_1^{\tau^{a-1}}(f).$$

**Theorem 2.3** ([44]). *Let* $f \in \mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$. *The associated L-function can be expressed in the following form:*

$$L^*(f,T)^{(-1)^n} = \prod_{i=0}^{n+1} \det(I - Tq^i A_a(f))^{(-1)^i \binom{n+1}{i}}. \tag{2.7}$$

*Equivalently, we have,*

$$\det(I - TA_a(f)) = \prod_{i=0}^{\infty} \left( L^*(f, q^i T)^{(-1)^n} \right)^{\binom{n+i}{i}}. \tag{2.8}$$

Based on the fact that $\mathrm{ord}_p F_r(f) \geq \frac{w(r)}{p-1}$ and Proposition 2.1, we have the estimate

$$\mathrm{ord}_p(a_{r,s}(f)) \geq \frac{w(ps-r) + w(r) - w(s)}{p-1} \geq w(s).$$

Let $\xi$ be an element in $\Omega$ satisfying $\xi^D = \pi^{p-1}$. Then $A_1(f)$ can be written in the following block form,

$$A_1(f) = \begin{pmatrix} A_{00} & \xi A_{01} & \cdots & \xi^i A_{0i} & \cdots \\ A_{10} & \xi A_{11} & \cdots & \xi^i A_{1i} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ A_{i0} & \xi A_{i1} & \cdots & \xi^i A_{ii} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}$$

where the block $A_{ii}$ is a $p$-adic integral $W_{\Delta}(i) \times W_{\Delta}(i)$ matrix and $W_{\Delta}(i) = \#\{u \in \mathbb{Z}^{n+1} | w(u) = \frac{i}{D}\}$. Thus, the $p$-adic Newton polygon of $\det(I - TA_1(f))$ has a natural lower bound which can be identified with the chain level version of the Hodge polygon.

**Definition 2.10** (chain level Hodge polygon). *Let $P(\Delta)$ be a polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and*

$$\left(\sum_{i=0}^{k} W_\Delta(i), \frac{1}{D}\sum_{i=0}^{k} iW_\Delta(i)\right),\ k = 0, 1, 2, \ldots$$

Based on the block form of $A_1(f)$, we have the following result.

**Proposition 2.3** ([44]). *Let $f$ be a Laurent polynomial with $\Delta(f) = \Delta$, then*

1. *The $p$-adic Newton polygon of* $\det(I - TA_1(f))$ *lies above $P(\Delta)$.*
2. *The $q$-adic Newton polygon of* $\det(I - TA_a(f))$ *lies above $P(\Delta)$.*

Furthermore, the ordinary property of $f$ is related to the $p$-adic Newton polygon of $\det(I - TA_1(f))$ as follows.

**Theorem 2.4** ([44]). *Let $f \in \mathbb{F}_q[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_{n+1}^{\pm1}]$ with $\Delta = \Delta(f)$. Assume that the L-function $L^*(f,T)^{(-1)^n}$ is a polynomial. Then*

1. *NP$(f)$ = HP$(\Delta)$ if and only if the $q$-adic Newton polygon of* $\det(I - TA_a(f))$ *coincides with its lower bound $P(\Delta)$ if and only if the $p$-adic Newton polygon of* $\det(I - TA_1(f))$ *coincides with its lower bound $P(\Delta)$.*
2. *NP$(f)$ coincides with HP$(\Delta)$ for all sides with slopes $\leq k/D$ if and only if the $p$-adic Newton polygon of* $\det(I - TA_1(f))$ *computed with respect to $p$ coincides with $P(\Delta)$ for the sides with slopes $\leq k/D$.*

### 2.4.3 General method for computing Hasse polynomials

Recall that the Hasse domain $\mathcal{H}_p(\Delta) = \{g \in \mathcal{M}_p(\Delta) | \mathrm{NP}(g) = \mathrm{HP}(\Delta)\}$ is a Zariski open subset of $\mathcal{M}_p(\Delta)$. If $\mathcal{H}_p(\Delta)$ is not empty, its complement is a hypersurface determined by Hasse polynomial $h_p(\Delta)$. Similarly, $h_p(\Delta,k)$ defines the complement of $\mathcal{H}_p(\Delta,k) = \{g \in \mathcal{M}_p(\Delta) | \mathrm{NP}(g) = \mathrm{HP}(\Delta) \text{ for slope-}k/D \text{ side}\}$. Let $\det(I - A_1(f)) = \sum_{j=0}^{\infty} c_j T^j$. The Newton polygon of $\det(I - A_1(f))$ computed with respect to $p$ is the lower convex closure of $\{(j, \mathrm{ord}_p(c_j)) \mid k = 0, 1, \ldots\}$. Let $(j, P(\Delta, j))$ be a point on $P(\Delta)$ for $j \in \mathbb{Z}_{\geq 0}$. By Proposition 2.3, we have $\mathrm{ord}_p(c_j) \geq P(\Delta, j)$. Let $j' = \sum_{i=0}^{k} W_\Delta(i)$. From the block form of $A_1(f)$, it's easy to check that

$$c_{j'} = \xi^{\sum_{i=0}^{k} iW_\Delta(i)} \det \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0k} \\ A_{10} & A_{11} & \cdots & A_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k0} & A_{k1} & \cdots & A_{kk} \end{pmatrix} + \xi^{1+\sum_{i=0}^{k} iW_\Delta(i)} \cdot u_{j'}$$

where $u_{j'}$ is a p-adic integer and $\xi^D = -p$. So $\mathrm{ord}_p(c_{j'}) \geq \frac{1}{D} \sum_{i=0}^{k} iW_\Delta(i) = P(\Delta, j')$ and $\mathrm{ord}_p(c_{j'}) = P(\Delta, j')$ if and only if

$$h_p(\Delta, k) := \det \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0k} \\ A_{10} & A_{11} & \cdots & A_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k0} & A_{k1} & \cdots & A_{kk} \end{pmatrix} \not\equiv 0 \mod p. \tag{2.9}$$

Consequently, we have the following formula for computing the Hasse polynomial,

$$h_p(\Delta) = \prod_{k=0}^{(n+1)D} h_p(\Delta, k). \tag{2.10}$$

## 2.5 Local diagonal theory

In the simplest case when $f$ is a monomial in one variable, the toric exponential sum can be expressed in terms of standard Gauss sums. So the rationality of the L-function follows from the Hasse-Davenport relation on Gauss sums and each reciprocal zero of the L-function is a radical of a Gauss sum whose $p$-adic absolute value can be determined by Stickelberger's theorem. In addition, there's a precise $p$-adic formula for the exponential sums in terms of the $p$-adic $\Gamma$-function [28]. Generally, if $f$ is a higher dimensional diagonal Laurent polynomial whose Newton polyhedron is a simplex, the L-function can also be computed explicitly using Gauss sum and the Hasse Davenport relation [44]. In this section, we give some non-degenerate and ordinary criteria for the diagonal Laurent polynomials.

**Definition 2.11.** *A Laurent polynomial* $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ *is called diagonal if* $f$ *has exactly* $n+1$ *non-constant terms and* $\Delta(f)$ *is an* $(n+1)$*-dimensional simplex in* $\mathbb{R}^{n+1}$.

Let $f$ be a Laurent polynomial over $\mathbb{F}_q$,

$$f(x_1, x_2, \ldots x_{n+1}) = \sum_{j=1}^{n+1} a_j x^{V_j},$$

where $a_j \in \mathbb{F}_q^*$ and $V_j = (v_{1j}, \ldots, v_{n+1j}) \in \mathbb{Z}^{n+1}, \quad j = 1, 2, \ldots, n+1$. Let $\Delta = \Delta(f)$. Then the vertex matrix of $\Delta$ is defined to be

$$M(\Delta) = (V_1, \ldots, V_{n+1}),$$

where the $i$-th column is the $i$-th exponent of $f$. If $f$ is diagonal, $M(\Delta)$ is invertible over $\mathbb{Q}$.

**Proposition 2.4.** *Suppose* $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ *is diagonal with* $\Delta = \Delta(f)$*. Then* $f$ *is non-degenerate if and only if* $p$ *is relatively prime to* $\det(M(\Delta))$.

Let $S(\Delta)$ be the solution set of the following linear system

$$M(\Delta)\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n+1} \end{pmatrix} \equiv 0(\mathrm{mod} 1), \quad r_i \in \mathbb{Q} \cap [0,1).$$

It's easy to prove that $S(\Delta)$ is an abelian group and its order is given by

$$|\det M(\Delta)| = (n+1)! \operatorname{Vol}(\Delta). \tag{2.11}$$

By the fundamental structure theorem of finite abelian group, we decompose $S(\Delta)$ into a direct product of invariant factors,

$$S(\Delta) = \bigoplus_{i=0}^{n+1} \mathbb{Z}/d_i\mathbb{Z},$$

where $d_i | d_{i+1}$ for $i = 1, 2, \ldots, n$.

For such diagonal Laurent polynomial, each reciprocal zero of the L-function is a radical of a product of Gauss sums. From the Stickelberger theorem for Gauss sums, one can derive the following ordinary criterion for a non-degenerate Laurent polynomial [44].

**Proposition 2.5.** *Suppose* $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ *is a non-degenerate diagonal Laurent polynomial with* $\Delta = \Delta(f)$. *Let* $d_{n+1}$ *be the largest invariant factor of* $S(\Delta)$. *If* $p \equiv 1(\bmod d_{n+1})$, *then* $f$ *is ordinary at* $p$.

In practice, non-trivial arithmetic and combinatorial problems ofter arise in the actual calculation of the Newton polygon even in the diagonal case due to the diversity of the polyhedron and prime $p$.

## 2.6 Wan's decomposition theorems

For non-diagonal Laurent polynomials, the Newton polygon problem become significantly harder and there's no simple explicit formula as the diagonal ones. In that case, we need to apply Wan's decomposition theorems, reducing the problem from global to local in some sense. Apparently, the ordinary property of a Laurent polynomial depends on its Newton polyhedron $\Delta$. Wan's decomposition theorems [41] enable us to decompose an arbitrary polytope $\Delta$ into small pieces in different ways where these small pieces may be much easier to deal with due to their better combinatorial and topological properties. So the determination of the ordinary property can be further simplified using these decomposition theorems.

### 2.6.1 Facial decomposition theorem

Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$. Assume $\Delta = \Delta(f)$ is an $(n+1)$-dimensional polytope containing the origin and $\delta_1, \ldots, \delta_h$ are all the co-dimension 1 faces of $\Delta$ which don't contain the origin.

**Definition 2.12** (Facial Decomposition)**.** *Let $\Delta_i$ $(1 \leq i \leq h)$ be the $(n+1)$-dimensional polytope generated by $\delta_i$ and the origin. The facial decomposition of $\Delta$ is defined by*

$$\Delta = \bigcup_{i=1}^{h} \Delta_i.$$

Let $f^{\delta_i}$ denote the restriction of $f$ to $\delta_i$ $(1 \leq i \leq h)$. By the definition of a non-degenerate Laurent polynomial, it's easy to check that $f$ is non-degenerate if and only if $f^{\delta_i}$ for $1 \leq i \leq h$.

**Theorem 2.5** (Facial decomposition theorem, [41])**.** *Assume $f$ is non-degenerate. Then $f$ is ordinary if and only if $f^{\delta_i}$ is ordinary for $1 \leq i \leq h$.*

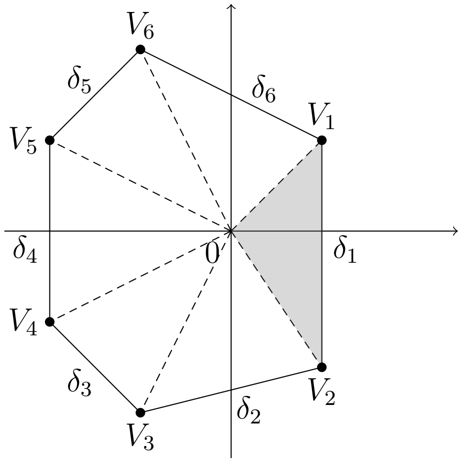

Figure 2.2: Facial Decomposition

### 2.6.2 Boundary decomposition theorem

Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ with $\Delta = \Delta(f)$, where $\Delta$ is an $(n+1)$-dimensional integral convex polyhedron in $\mathbb{R}^{n+1}$ containing the origin. Let $C(\Delta)$ be the cone generated by $\Delta$ in $\mathbb{R}^{n+1}$.

**Definition 2.13.** *The boundary decomposition*

$$B(\Delta) = \{ \text{ the interior of a closed face in } C(\Delta) \text{ containing the origin}\}$$

*is the unique interior decomposition of $C(\Delta)$ into a disjoint union of relatively open cones.*

If the origin is a vertex of $\Delta$, then it is the unique 0-dimensional open cone in $B(\Delta)$. By Definition 2.9, $A_1(f) = (a_{r,s}(f))$ is the infinite semilinear Frobenius matrix whose rows and columns are indexed by the lattice points in $L(\Delta)$. For $\Sigma \in B(\Delta)$, we define $A_1(\Sigma, f)$ to be the submatrix of $A_1(f)$ with $r, s \in \Sigma$. Let $f^{\overline{\Sigma}}$ be the restriction of $f$ to the closure of $\Sigma$. Then $A_1(\Sigma, f^{\overline{\Sigma}})$ denotes the submatrix of $A_1(f^{\overline{\Sigma}})$ with $r, s \in \Sigma$.

Let $B(\Delta) = \{\Sigma_0, \ldots, \Sigma_h\}$ such that $\dim(\Sigma_i) \leq \dim(\Sigma_{i+1})$, $i = 0, \ldots, h-1$. Define $B_{ij} = (a_{r,s}(f))$ with $r \in \Sigma_i$ and $s \in \Sigma_j$ $(0 \leq i, j \leq h)$. After permutation, the infinite semilinear

Frobenius matrix can be written as

$$A_1(f) = \begin{pmatrix} B_{00} & B_{01} & \cdots & B_{0h} \\ B_{10} & B_{11} & \cdots & B_{1h} \\ \vdots & \vdots & \ddots & \vdots \\ B_{h0} & B_{h1} & \cdots & B_{hh} \end{pmatrix}, \tag{2.12}$$

where $B_{ij} = 0$ for $i > j$. Then $\det(I - TA_1(f)) = \prod_{i=0}^{h} \det(I - TB_{ii})$ and we have the boundary decomposition theorem.

**Theorem 2.6** (Boundary decomposition, [41])**.** *Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ with $\Delta = \Delta(f)$. Then we have the following factorization*

$$\det(I - TA_1(f)) = \prod_{\Sigma \in B(\Delta)} \det\left(I - TA_1(\Sigma, f^{\overline{\Sigma}})\right).$$

**Corollary 2.1.** *Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ be a non-degenerate Laurent polynomial with an $(n+1)$-dimensional Newton polyhedron $\Delta$. If the origin is a vertex of $\Delta$, then the associated L-function*

$$L^*(f, T)^{(-1)^n} = (1 - \psi(\mathrm{Tr}(c))T) \prod_{i=1}^{(n+1)!\,\mathrm{Vol}(\Delta(f))-1} (1 - \alpha_i T),$$

*where $c$ is the constant term of $f$, $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map and $|\alpha_i| \leq q^{(n+1)/2}$.*

# Chapter 3

# Type I exponential sums

This chapter is based on the unpublished forthcoming joint work with Xin Lin. For part of the details, see [9]. Let $n$ be an integer with the following partition as the sum of $r$ integers

$$n = \sum_{i=1}^{r} n_i.$$

where $r$ and $n_i$ $(1 \leq i \leq r)$ are positive integers. Depending on this partition, we introduce $n$ variables

$$x_{ij}, \quad \text{for } 1 \leq i \leq r,\ 1 \leq j \leq n_i,$$

and $n$ positive integral constants

$$b_{ij}, \quad \text{for } 1 \leq i \leq r,\ 1 \leq j \leq n_i.$$

Let $V_{\vec{a}}$ be the subvariety of the torus $\mathbb{G}_m^n$ over $\overline{\mathbb{F}}_q$ (with coordinates $x_{ij}$) defined by the vanishing of the following equation

$$\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} = 1$$

where $\vec{a} = (a_1, \ldots, a_r) \in (\mathbb{F}_q^*)^r$ and the left hand side is a sum of inverse monomials in disjoint sets of variables, i.e., each variable occurs in precisely one monomial.

In this chapter, we are concerned with the following class of exponential sums: for $a_i \in \mathbb{F}_q^*$ and $b_{ij} \in \mathbb{Z}_{>0}$,

$$S_k(\vec{a}) = \sum_{V_{\vec{a}}(\mathbb{F}_{q^k}^*)} \psi\left(\mathrm{Tr}_k\left(\sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij}\right)\right),$$

where the sum is over all $\mathbb{F}_{q^k}^*$ rational points of the specified variety, $\psi : \mathbb{F}_p \to \mathbb{C}^*$ is a fixed nontrivial additive character over $\mathbb{F}_p$ and $\mathrm{Tr}_k : \mathbb{F}_{q^k} \to \mathbb{F}_p$ is the trace map. By the fact that $\mathrm{Tr}_k(x^p) = \mathrm{Tr}_k(x)$ for $x \in \mathbb{F}_{q^k}^*$, the case of $b_{ij}$ divisible by $p$ can be reduced to $p \nmid b_{ij}$ through a change of variables. So without loss of generality, we assume $p \nmid b_{ij}(1 \le i \le r,\ 1 \le j \le n_i)$.

In the past few decades, there have been extensive study and applications of the exponential sum $S_k(\vec{a})$ when $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \le i \le 2, 1 \le j \le 2)$. In this special case, the exponential sum will be in the following simpler form

$$S_k(\vec{a}) = S_k(a_1, a_2) = \sum_{\substack{\frac{a_1}{x_1x_2}+\frac{a_2}{x_3x_4}=1 \\ x_1,\ldots,x_4 \in \mathbb{F}_{q^k}^*}} \psi(\mathrm{Tr}_k(x_1 + x_2 + x_3 + x_4)).$$

Using Deligne's theory [11] on the weights of zeta functions, Birch and Bombieri [5] prove that if $p > c_0$, then $|S_k(a_1, a_2)| \le c_1 q^{3k/2}$ for some absolute constants $c_0$,$c_1$. Their estimate is a crucial ingredient in Heath Brown's work[31] on the divisor function $d_3(n)$ in arithmetic progressions. Birch and Bombieri's result also contributes to Friedlander and Iwaniec's work

[21] on estimating certain averages of incomplete Kloosterman sums in application to the divisor problem of $d_3(n)$. Relying on Birch and Bombieri's estimate of $S_k(\vec{a})$ and Friedlander-Iwaniec's result, Zhang [48] gains a boundary of the error terms in his work on twin prime conjecture. The depth of this special case should give an indication about the potential significance of the estimate of type I exponential sums in analytic number theory.

For the general form of exponential sum $S_k(\vec{a})$, Katz [36] obtains an optimal upper bound for the complex absolute value:

$$|S_k(\vec{a})| \leq c_1 q^{(n-1)k/2},$$

where $c_1 = \prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} b_{ij}\right) - 1$.

In this chapter, we study the reciprocal zeros and poles of the generating L-function of the exponential sum $S_k(\vec{a})$ given by

$$L(\vec{a}, T) = \exp\left(\sum_{k=1}^{\infty} S_k(\vec{a}) \frac{T^k}{k}\right). \tag{3.1}$$

Recall that $L(\vec{a}, T)$ is a rational function by the theorem of Dwork-Bombieri-Grothendiek [13, 14, 6, 29]. So studying the number and absolute values of the reciprocal zeros and poles could directly lead to the valuation of the sequence of exponential sums.

Our approach is to apply the systematic results available for toric exponential sums. However, type I exponential sums are different from the classical ones defined simply by a Laurent polynomial introduced in the previous chapters. To transfer these exponential sums into the classical forms, we construct a specified Laurent polynomial in $n+1$ variables,

$$f = \sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij} + x_{n+1}\left(\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} - 1\right),$$

where the corresponding generating functions are related in the following way

$$L(\vec{a},T)^{(-1)^n} = \frac{1}{1-T/q} L^*\left(f,T/q\right)^{(-1)^n}.$$

In that case, the problem can be reduced to studying the classical toric L-function $L^*(f,T)$ and we can apply the useful theorems in Chapter 2. It's easy to check $L^*(f,T)^{(-1)^n}$ is a polynomial by Theorem 2.1 (Adolphson and Sperber's theorem). In this chapter, we compute the $q$-adic Newton polygon of $L^*(f,T)^{(-1)^n}$, which indicates the $q$-adic slope information of the L-function. The denominator of the Newton polyhedron $D = \mathrm{lcm}\{b_{ij}\}$. Following from Wan's facial decomposition theorem and local diagonal theory, the Laurent polynomial $f$ is ordinary if $p \equiv 1 (\mathrm{mod} D)$. For the special case when $b_{ij} = 1$ $(1 \le i \le r, 1 \le j \le n_i)$, the condition works for arbitrary prime $p$ and so $f$ is ordinary at any prime. On this particular occasion, Wan's boundary decomposition theorem together with the slope information we obtained leads to an explicit expression of $L^*(f,T)^{(-1)^n}$. Furthermore, this expression provides a sharper estimate of the exponential sums compared to the previous results.

## 3.1 Main results

In this chapter, we obtain $q$-adic slopes and weights for the above general class of exponential sums. Note that our main results are consistent with Katz's complex estimate.

**Theorem 3.1.** *Let* $D = \mathrm{lcm}\{b_{ij} | 1 \le i \le r, 1 \le j \le n_i\}$ *and* $d = \prod_{i=1}^{r}\left(1+\sum_{j=1}^{n_i} b_{ij}\right)$. *Then we have the following results.*

1. *The associated L-function is a polynomial given by*

$$L(\vec{a},T)^{(-1)^n} = (1-T)^{r-1}\prod_{i=1}^{d-r}(1-\alpha_i T).$$

2. *If* $p \equiv 1(\mathrm{mod} D)$*, for each* $m \in \{0,1,\ldots,nD\}$*, the number of reciprocal zeros of* $L(\vec{a},T)^{(-1)^n}$ *with q-adic slope* $m/D$ *is the coefficient of* $x^{m+D}$ *in the following generating function*

$$G(x) = \left(1-x^D\right)^{n-r}\prod_{i=1}^{r}\frac{1-x^{(1+\sum_{j=1}^{n_i}\frac{1}{b_{ij}})D}}{\prod_{j=1}^{n_i}\left(1-x^{D/b_{ij}}\right)},$$

   *and for any rational number* $m \notin \{0,1,\ldots,nD\}$*, there is no reciprocal zero of* $L(\vec{a},T)^{(-1)^n}$ *with q-adic slope* $m/D$*.*

3. *For all* $p$*,* $k$ *and* $\vec{a}$*,*

$$|S_k(\vec{a})| \leq (d-r)q^{(n-1)k/2} + r - 1.$$

**Corollary 3.1.** *Assume* $b_{ij} = 1$ $(1 \leq i \leq r, 1 \leq j \leq n_i)$*. Let* $d = \prod_{i=1}^{r}(1+n_i)$*.*

1. *The polynomial* $L(\vec{a},T)^{(-1)^n}$ *has degree* $d-1$*.*

2. *For each* $m \in \{0,1,2,\ldots,n-1\}$*, the number of reciprocal zeros of* $L(\vec{a},T)^{(-1)^n}$ *with q-adic slope* $m$ *is the coefficient of* $x^{m+1}$ *in the following generating function*

$$G_1(x) = \prod_{i=1}^{r}\left(1+x+\cdots+x^{n_i}\right),$$

   *and for any rational number* $m \notin \{0,1,\ldots,n-1\}$*, there is no reciprocal zero of* $L(\vec{a},T)^{(-1)^n}$ *with q-adic slope* $m$*.*

3. *If $n_i$ is even for $1 \le i \le r$, we have*

$$L(\vec{a}, T)^{(-1)^n} = (1-T)^{r-1}(1-\gamma_0 T)(1-qT)^{\frac{r^2-r}{2}} \prod_{i=1}^{r}(1-\gamma_i T) \prod_{j=1}^{d-\frac{r^2+3r+2}{2}} (1-\alpha_j T),$$

   *where* $\mathrm{ord}_q\, \gamma_0 = 0$*,* $\mathrm{ord}_q\, \gamma_i = 1$ $(1 \le i \le r)$*,* $|\gamma_i| = q^{\frac{n-1}{2}}$ $(0 \le i \le r)$*,* $\mathrm{ord}_q\, \alpha_j > 1$ *and* $|\alpha_j| \le q^{\frac{n-1}{2}}$.

4. *If $n_i$ is even for $1 \le i \le r$, then for all $p, k$ and $\vec{a}$,*

$$|S_k(\vec{a})| \le \left(d - \frac{r^2+r}{2}\right) q^{(n-1)k/2} + \left(\frac{r^2-r}{2}\right) q^k + r - 1.$$

For the special case when $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \le i \le 2, 1 \le j \le 2)$, our main result is a complete determination of the weights and slopes of the L-function and a sharp estimate of the exponential sums.

**Corollary 3.2.** *Assume $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \le i \le 2, 1 \le j \le 2)$.*

1. *The generating L-function of $S_k(\vec{a})$ is a polynomial of degree 8, i.e.,*

$$L(\vec{a}, T) = (1-T)(1-qT) \prod_{i=1}^{6}(1-\alpha_i T),$$

   *where $\alpha_i \in \overline{\mathbb{Q}}$ with the complex absolute value $|\alpha_i| = q^{3/2}$. If we view the reciprocal roots $\alpha_i$ as p-adic numbers and enumerate them with respect to the q-adic slopes, their* *p-adic norms are given by*

$$|\alpha_i|_p = \begin{cases} 1 & \textit{if} \quad i = 1. \\ q^{-1} & \textit{if} \quad i = 2, 3. \\ q^{-2} & \textit{if} \quad i = 4, 5. \\ q^{-3} & \textit{if} \quad i = 6. \end{cases}$$

2. *For all* $p, k$ *and* $\vec{a}$,

$$|S_k(\vec{a})| \leq 6q^{3k/2} + q^k + 1.$$

For the weight computation, one can also apply Denef-Loeser's weight formula [12] obtained using intersection cohomology. There is a much simpler weight formula conjectured by Adolphson-Sperber [3] which is true in many interesting cases including all low dimensional cases $n \leq 4$. This conjectural formula however was disproved by Denef and Loeser [12] who showed the existence (no construction) of a counterexample in dimension 5. We also test this example of $L^*(f, T)$ ($n = 4$, $n_i = 2$ and $b_{ij} = 1$) using the Adolphson-Sperber formula and find it disagrees with our main theorem. This means that our 5-dimensional example provides the first explicit construction of a counterexample to the Adophson-Sperber conjecture.

## 3.2 Proof of the main results

In this section, we provide the proof of the main results. Recall that $b_{ij} \in \mathbb{Z}_{>0}$ and $p \nmid b_{ij}$ $(1 \le i \le r,\ 1 \le j \le n_i)$. For $a_i \in \mathbb{F}_q^*$,

$$S_k(\vec{a}) = \sum_{\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} = 1} \psi\left(\mathrm{Tr}_k\left(\sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij}\right)\right),$$

where the sum is over $x_{ij} \in \mathbb{F}_{q^k}^*$, $\psi : \mathbb{F}_p \to \mathbb{C}^*$ is a fixed nontrivial additive character over $\mathbb{F}_p$ and $\mathrm{Tr}_k : \mathbb{F}_{q^k} \to \mathbb{F}_p$ is the trace map. For such kind of exponential sums, the key structural features include the equation defining the variety of summation,

$$\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} = 1$$

and the sum of all variables inside the trace map

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij}.$$

Relying on these two components, we construct the following polynomial

$$f = \sum_{i=1}^{r}\sum_{j=1}^{n_i} x_{ij} + x_{n+1}\left(\sum_{i=1}^{r} \frac{a_i}{\prod_{j=1}^{n_i} x_{ij}^{b_{ij}}} - 1\right)$$

where $x_{n+1}$ is a new variable. For simplicity, we relabel $x_{ij}$ and $b_{ij}$ with one single index. Let $N_i = \sum_{l=1}^{i} n_l$ for $1 \le i \le r$ and $N_0 = 0$. Let $x_{ij} = x_{N_{i-1}+j}$ and $b_{ij} = b_{N_{i-1}+j}$ $(1 \le i \le r,$ $1 \le j \le n_i)$. In this way, the Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ is expressed in the following form

$$f(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n} x_i + x_{n+1}\left(\sum_{i=1}^{r} \frac{a_i}{\prod_{l=N_{i-1}+1}^{N_i} x_l^{b_l}} - 1\right), \tag{3.2}$$

where $a_i \in \mathbb{F}_q^*$ and $b_l \in \mathbb{Z}_{>0}$. For any positive integer $k$, let

$$S_k^*(f) = \sum_{x_1,\ldots,x_{n+1} \in \mathbb{F}_{q^k}^*} \psi(\mathrm{Tr}_k(f))$$

be the associated exponential sum. Its generating L-function is defined to be

$$L^*(f,T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f)\frac{T^k}{k}\right).$$

The following equation describes the relationship between $S_k(\vec{a})$ and $S_k^*(f)$,

$$S_k(\vec{a}) = \frac{(-1)^n}{q^k} + \frac{1}{q^k}S_k^*(f).$$

Based on the relationship, it's easy to check that

$$L(\vec{a},T)^{(-1)^n} = \frac{1}{1-T/q}L^*\left(f, T/q\right)^{(-1)^n}. \tag{3.3}$$

To determine slopes and find an explicit expression of $L(\vec{a},T)^{(-1)^n}$, it suffices to evaluate the reciprocal roots or poles of $L^*(f,T)^{(-1)^n}$.

### 3.2.1 Newton polyhedron of $f$

Let $\Delta = \Delta(f)$ be the Newton polyhedron of $f$. From Section 2.1 and Equation (3.2), we know $\Delta$ is an $(n+1)$-dimensional polytope in $\mathbb{R}^{n+1}$ with the following $n+r+2$ vertices,

$$V_0 = (0,\cdots,0)$$

$$V_1 = (1,0,\cdots,0),$$

$$\vdots$$

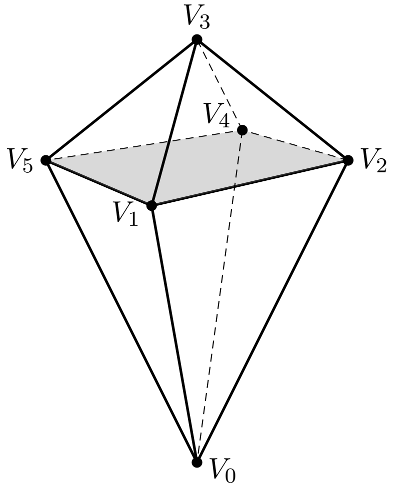

Figure 3.1: Polytope $\Delta$ for $n = 2$ and $r = 2$

$$\begin{aligned}
&V_{n+1} = (0, \cdots, 0, 1),\\
&V_{n+2} = (\underbrace{-b_1, \cdots, -b_{n_1}}_{n_1}, 0, \cdots, 0, 1),\\
&\vdots\\
&V_{n+i+1} = (0, \cdots, 0, \underbrace{-b_{N_{i-1}+1}, \cdots, -b_{N_i}}_{n_i}, 0, \cdots, 0, 1),\\
&\vdots\\
&V_{n+r+1} = (0, \cdots, 0, \underbrace{-b_{N_{r-1}+1}, \cdots, -b_{N_r}}_{n_r}, 1).
\end{aligned}$$

The polytope $\Delta$ has $\prod_{i=1}^{r}(1 + n_i)$ co-dimension 1 faces not containing the origin. Let $\delta_\ell$ ($1 \leq \ell \leq \prod_{i=1}^{r}(1 + n_i)$) be a co-dimension 1 face of $\Delta$ that doesn't contain the origin and let $\mathcal{V} = \{V_1, V_2, \cdots, V_n\}$ be a set consisting of the first $n$ vertices. On each face $\delta_\ell$, there are $n + 1$ vertices which should be chosen in the following way:

$$V_{n+1}, \quad \underbrace{\mathcal{V} - \{V_{i_1}, \cdots, V_{i_k}\}}_{n-k \text{ vertices}} \quad \text{and} \quad \underbrace{\{V_{n+1+j_1}, \ldots, V_{n+1+j_k}\}}_{k \text{ vertices}}, \tag{3.4}$$

where $N_{j_l - 1} + 1 \leq i_l \leq N_{j_l}$ and the $j_l$s are distinct ($1 \leq l \leq k$, $0 \leq k \leq r$). Note that there is a one-to-one correspondence between vertices $V_{i_l}$ and $V_{n+1+j_l}$ $(1 \leq l \leq k)$ whose $i_l$-th coordinates are nonzero. Relying on the selection of vertices, we can get the equation of $\delta_\ell$:

$$\sum_{i=1}^{n+1} c_i x_i = 1,$$

where

$$c_i = \begin{cases} 1 & \text{if } \ i \neq i_l, \\ -\dfrac{1}{b_{i_l}} \left( \displaystyle\sum_{m=N_{j_l-1}+1}^{N_{j_l}} b_m \right) + 1 & \text{if } \ i = i_l. \end{cases} \tag{3.5}$$

Here $b_{i_l}$ is the nonzero $i_l$-th coordinate of $V_{n+1+j_l}$. Let $\Delta_\ell$ be a polytope generated by the origin and face $\delta_\ell$ $(0 \leq \ell \leq \prod_{i=1}^{r}(1+n_i))$. Obviously, $\Delta_\ell$ is a simplex in $\mathbb{R}^{n+1}$ and so the restriction $f^{\delta_\ell}$ is diagonal. Recall that the vertex matrix of $\Delta_\ell$ is a square matrix whose columns are the $n+1$ coordinates of the generating vertices. By Formula (3.4) and (2.11), we have

$$(n+1)!\,\mathrm{Vol}(\Delta_\ell) = |\det M(\Delta_\ell)| = \prod_{l=1}^{k} b_{i_l} \tag{3.6}$$

where $b_{i_l}$ is the $i_l$-th coordinate of the vertex $V_{n+1+j_l}$ on $\Delta_\ell$ for $1 \leq l \leq k$. From the detailed information of these co-dimension 1 faces, we have the following proposition.

**Proposition 3.1.** *Notations as above.*

1. *The denominator* $D = \mathrm{lcm}\{b_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n_i\}$.
2. *The volume of the Newton polyhedron* $\Delta$ *is given by*

$$\mathrm{Vol}(\Delta) = \frac{1}{(n+1)!} \prod_{i=1}^{r} \left(1 + \sum_{j=1}^{n_i} b_{ij}\right).$$

*Proof.* Based on Formula (2.2) , we know the denominator of $\Delta$ can be deduced immediately from Equation (3.5), the equation of $\delta_\ell$. By Definition 2.12 (Wan's facial decomposition),

$$\mathrm{Vol}(\Delta) = \sum_{\ell=1}^{\prod_{i=1}^{r}(1+n_i)} \mathrm{Vol}(\Delta_\ell).$$

Combining with Formula (3.6), we obtain the second conclusion. □

### 3.2.2 Newton polygons of $L^*(f,T)^{(-1)^n}$ and $L(\vec{a},T)^{(-1)^n}$

To study the Newton polygon of $L^*(f,T)^{(-1)^n}$, we need to first determine when $L^*(f,T)^{(-1)^n}$ is a polynomial. So we begins this subsection with the following non-degenerate property.

**Proposition 3.2.** *The Laurent polynomial $f$ defined in Equation* (3.2) *is non-degenerate. Consequently, the associated L-function $L^*(f,T)^{(-1)^n}$ is a polynomial.*

*Proof.* By Definition 2.1, $f$ is non-degenerate if and only if $f^{\delta_\ell}$ is non-degenerate for all $0 \le \ell \le \prod_{i=1}^{r}(1+n_i)$. Since $f^{\delta_\ell}$ is a diagonal Laurent polynomial, one can apply the useful criterion, Proposition 2.4, which indicates the fact that $f^{\delta_\ell}$ is non-degenerate if and only if $p \nmid |\det M(\Delta_\ell)|$. The proposition then follows from Equation (3.6), Proposition 3.1 and the assumption that $p \nmid b_{ij}$ $(1 \le i \le r,\ 1 \le j \le n_i)$. □

**Proposition 3.3.** *Notations as above. If $p \equiv 1(\mathrm{mod} D)$, then $f$ is ordinary at prime $p$.*

*Proof.* By Theorem 2.5 (Wan's facial decomposition theorem), $f$ is ordinary if and only if $f^{\delta_\ell}$ is ordinary for $0 \le \ell \le \prod_{i=1}^{r}(1+n_i)$. As stated in Proposition 2.5, $f^{\delta_\ell}$ is ordinary at $p$ if $p \equiv 1(\mathrm{mod}\ d_\ell)$ where $d_\ell$ be the largest invariant factor of $\mathbb{Z}^{n+1}/(M(\Delta_\ell)\mathbb{Z}^{n+1})$. Based on the choice of vertices on $\delta_\ell$, Equation (3.4) and (3.5), one can get $d_\ell = \mathrm{lcm}\{b_{i_l} | 1 \le l \le k\}$. The proposition then follows. □

Now we are ready to consider the Newton polygon of ordinary Laurent polynomials where $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$. The essential part of this work is to compute the Hodge numbers.

**Theorem 3.2.** *Let* $b_{ij} \in \mathbb{Z}_{>0}$ *and* $p \nmid b_{ij}$ $(1 \le i \le r,\ 1 \le j \le n_i)$. *The Hodge number* $H_\Delta(k)$ *is the coefficient of* $x^k$ *in* $G(x)$ *where*

$$G(x) = \left(1 - x^D\right)^{n-r} \prod_{i=1}^{r} \frac{1 - x^{\left(1+\sum_{j=1}^{n_i} \frac{1}{b_{ij}}\right) D}}{\prod_{j=1}^{n_i} \left(1 - x^{D/b_{ij}}\right)}. \tag{3.7}$$

Note that $b_{ij} = b_{N_{i-1}+j}$ where $N_i = \sum_{l=1}^{i} n_l$ for $1 \le i \le r$. The positive integral constant $b_{ij}$ $(1 \le i \le r, 1 \le j \le n_i)$ can be relabeled as $b_k$ $(1 \le k \le n)$.

*Proof.* By Definition 2.4 and 2.3, to obtain the Hodge number $H_\Delta(k)$, it suffices to consider

$$W_\Delta(k) = \#\left\{ u \in \mathbb{Z}^{n+1} | w(u) = \frac{k}{D} \right\} \tag{3.8}$$

where the weight function

$$w(u) = \inf_{\vec{c}} \left\{ \sum_{l=1}^{n+r+1} c_l \;\middle|\; \sum_{l=1}^{n+r+1} c_l V_l = u, c_l \in \mathbb{R}_{\ge 0} \right\}. \tag{3.9}$$

Let $u \in C(\Delta) \cap \mathbb{Z}^{n+1}$, expressed as a linear combination of the $n + r + 1$ vertices of $\Delta$

$$u = \sum_{l=1}^{n+r+1} c_l V_l, \tag{3.10}$$

in which $c_l \in \mathbb{R}_{\ge 0}$. By Formula (3.9), we need to discuss when $\sum_{l=1}^{n+r+1} c_l$ reaches its minimum. Following the partition of $n = \sum_{i=1}^{r} n_i$, we divide the first $n$ coordinates of $u$ into $r$ disjoint parts as follows,

$$u = (\underbrace{u_1, \cdots, u_{n_1}}_{n_1}, \cdots, \underbrace{u_{N_{i-1}+1}, \cdots, u_{N_i}}_{n_i}, \cdots, \underbrace{u_{N_{r-1}+1}, \cdots, u_n}_{n_r}, u_{n+1})$$

where the $i$-th part has $n_i$ elements $(1 \leq i \leq r)$. Let $M_i = \{V_{N_{i-1}+1}, \cdots, V_{N_i}\} \cup \{V_{n+1+i}\}$ and $S_i = \{c_{N_{i-1}+1}, \cdots, c_{N_i}\} \cup \{c_{n+1+i}\}$. By Formula (3.10), we know elements in $S_i$ are precisely the coefficients of vertices in $M_i$. Note that $|S_i| = n_i + 1$ and $S_{i_1} \cap S_{i_2} = \emptyset$ for $1 \leq i_1 \neq i_2 \leq r$. Combining Formula (3.10) with the coordinates of vertices, one can easily get the fact that $\{u_{N_{i-1}+1}, \cdots, u_{N_i}\}$, the $i$-th part of coordinates of $u$, are uniquely determined by the vertices in $M_i$ with the coefficients in $S_i$.

We claim that $\sum_{l=1}^{n+r+1} c_l$ reaches its minimum if and only if at least one of elements in $S_i$ is 0 for all $1 \leq i \leq r$. First we prove the necessary condition. If we focus on $n_i$ coordinates of $u$, we consider the corresponding $n_i$ coordinates of each vertex in $M_i$. For a vertex $V = (v_1, v_2, \cdots, v_{n+1})$, let

$$V^{(n_i)} = (\underbrace{v_{N_{i-1}+1}, \cdots, v_{N_i}}_{n_i})$$

denote the $i$-th part of coordinates of $V$. Let

$$M_i^{(n_i)} = \{V_{N_{i-1}+1}^{(n_i)}, \cdots, V_{N_i}^{(n_i)}\} \cup \{V_{n+1+i}^{(n_i)}\}.$$

If we take $V^{(n_i)}$ as a vector in $\mathbb{Z}^{n_i}$, $M_i^{(n_i)}$ is linearly dependent, while any proper subset of $M_i^{(n_i)}$is linearly independent. Therefore, if at least one of elements in $S_i$ vanishes, $\sum_{c_l \in S_i} c_l$ is unique and thus minimum. By the assumption that $c_l \in \mathbb{R}_{\geq 0}$, the sum $\sum_{c_l \in S_i} c_l$ reaches minimum for all $1 \leq i \leq r$ implies $\sum_{l=1}^{n+r+1} c_l$ reaches minimum. The sufficient condition holds as well and can be proved by contradiction. Suppose none of the elements in $S_i$ equal 0 for some $1 \leq i \leq r$. Let $c = \sum_{l=1}^{n+r+1} c_l$. By the selection of vertices in Equation (3.4), $u$ doesn't lie on any co-dimension 1 face $\delta$ of $c\Delta$, which implies that $\sum_{l=1}^{n+r+1} c_l$ can be reduced.

Then by Formula (3.8) and (3.9), $W_\Delta(k)$ is the number of non-negative solutions to

$$\sum_{l=1}^{n+r+1} c_l = k/D,$$

satisfying that $\sum_{l=1}^{n+r+1} c_l V_l \in \mathbb{Z}^{n+1}$ and $\prod_{c_l \in S_i} c_l = 0$ for $1 \le i \le r$.

By the coordinates of all vertices, $\sum_{l=1}^{n+r+1} c_l V_l \in \mathbb{Z}^{n+1}$ is equivalent to the condition that $c_l \in \frac{1}{b_l}\mathbb{Z}_{\ge 0}$ for $1 \le l \le n$ and $c_l \in \mathbb{Z}_{\ge 0}$ for $n+1 \le l \le n+r$. Consequently, $W_\Delta(k)$ equals the number of non-negative integer solutions to

$$\frac{D}{b_1}x_1 + \cdots + \frac{D}{b_n}x_n + Dx_{n+1} + \cdots + Dx_{n+r+1} = k, \tag{3.11}$$

satisfying that $x_{N_{i-1}+1} \cdots x_{N_i} x_{n+1+i} = 0$ for $1 \le i \le r$.

Now we determine the generating function for $W_\Delta(k)$ based on Formula (3.11).

$$\begin{aligned}
g(x) =& \frac{1}{\prod_{k=1}^{n}\left(1 - x^{D/b_k}\right)\left(1 - x^D\right)^{r+1}} - \sum_{i=1}^{r} \frac{\prod_{j=N_{i-1}+1}^{N_i} x^{D/b_j} x^D}{\prod_{k=1}^{n}\left(1 - x^{D/b_k}\right)\left(1 - x^D\right)^{r+1}} \\
&+ \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{r} \frac{\prod_{j=N_{i_1-1}+1}^{N_{i_1}} x^{D/b_j} x^D \prod_{j=N_{i_2-1}+1}^{N_{i_2}} x^{D/b_j} x^D}{\prod_{k=1}^{n}\left(1 - x^{D/b_k}\right)\left(1 - x^D\right)^{r+1}} - \cdots \\
&+ (-1)^r \frac{\prod_{k=1}^{n} x^{D/b_k} x^{rD}}{\prod_{k=1}^{n}\left(1 - x^{D/b_k}\right)\left(1 - x^D\right)^{r+1}} \\
=& \frac{\prod_{i=1}^{r}\left(1 - x^{\left(1 + \sum_{j=N_{i-1}+1}^{N_i} \frac{1}{b_j}\right)D}\right)}{\prod_{k=1}^{n}\left(1 - x^{D/b_k}\right)\left(1 - x^D\right)^{r+1}}.
\end{aligned}$$

For simplicity, we replace $b_k$ with $b_{ij}$ and obtain

$$g(x) = \left(1 - x^D\right)^{-r-1} \prod_{i=1}^{r} \frac{1 - x^{\left(1 + \sum_{j=1}^{n_i} \frac{1}{b_{ij}}\right)D}}{\prod_{j=1}^{n_i}\left(1 - x^{D/b_{ij}}\right)}. \tag{3.12}$$

Note that $W_\Delta(k)$ is the coefficient of $x^k$ in $g(x)$. By Equation (2.4), the generating function of $H_{\Delta}(k)$ is given as follows,

$$\begin{aligned} G(x) &= \left(1-x^{D}\right)^{n+1} g(x) \\ &= \left(1-x^{D}\right)^{n-r} \prod_{i=1}^{r} \frac{1-x^{\left(1+\sum_{j=1}^{n_i} \frac{1}{b_{ij}}\right)D}}{\prod_{j=1}^{n_i}\left(1-x^{D/b_{ij}}\right)}, \end{aligned}$$

where the $k$-th Hodge number $H_{\Delta}(k)$ equals the coefficients of $x^k$ in $G(x)$. □

As a corollary, we obtain a much simpler formula of $H_{\Delta}(k)$ when $b_{ij}=1$.

**Corollary 3.3.** *If $b_{ij}=1$ for $1 \le i \le r$ and $1 \le j \le n_i$, $H_{\Delta}(k)$ is the coefficient of $x^k$ in $G_1(x)$ where*

$$G_1(x) = \prod_{i=1}^{r} \left(1+x+\cdots+x^{n_i}\right). \tag{3.13}$$

*In this case, we have $H_{\Delta}(0)=1$, $H_{\Delta}(1)=r$, $H_{\Delta}(n+1)=0$ and $H_{\Delta}(k)=H_{\Delta}(n-k)$ for $0 \le k \le n$.*

As a consequence of Theorem 3.2, we have determined the Hodge polygon HP($\Delta$), which is exactly the $q$-adic Newton polygon of $L(f,T)^{(-1)^n}$ if $p \equiv 1(\mathrm{mod} D)$. Furthermore, we have the following important theorem on the slopes of $L(\vec{a},T)^{(-1)^n}$.

**Theorem 3.3.** *Notations as above.*

1. *If $p \equiv 1(\mathrm{mod} D)$, for each $m \in \{0,1,\ldots,nD\}$, the number of reciprocal zeros of $L(\vec{a},T)^{(-1)^n}$ with $q$-adic slope $m/D$ is the coefficient of $x^{m+D}$ in the following generating function*

$$G(x) = \left(1-x^{D}\right)^{n-r} \prod_{i=1}^{r} \frac{1-x^{(1+\sum_{j=1}^{n_i} \frac{1}{b_{ij}})D}}{\prod_{j=1}^{n_i}\left(1-x^{D/b_{ij}}\right)},$$

*and for any rational number* $m \notin \{0, 1, \ldots, nD\}$*, there is no reciprocal zero of* $L(\vec{a}, T)^{(-1)^n}$ *with q-adic slope* $m/D$*.*

2. *In particular, assume* $b_{ij} = 1$ $(1 \leq i \leq r,\ 1 \leq j \leq n_i)$*. Then for any prime* $p$ *and for each* $m \in \{0, 1, 2, \ldots, n-1\}$*, the number of reciprocal zeros of* $L(\vec{a}, T)^{(-1)^n}$ *with q-adic slope* $m$ *is the coefficient of* $x^{m+1}$ *in the following generating function*

$$G_1(x) = \prod_{i=1}^{r} \left(1 + x + \cdots + x^{n_i}\right),$$

*and for any rational number* $m \notin \{0, 1, \ldots, n-1\}$*, there is no reciprocal zero of* $L(\vec{a}, T)^{(-1)^n}$ *with q-adic slope* $m$*.*

*Proof.* The first part follows from the relationship that

$$L(\vec{a}, T)^{(-1)^n} = \frac{1}{1 - T/q} L^*\left(f, T/q\right)^{(-1)^n}.$$

For the second part, we assume $b_{ij} = 1$ $(1 \leq i \leq r,\ 1 \leq j \leq n_i)$. In this case, $D = 1$ and $f$ is ordinary for any prime $p$. □

### 3.2.3 Trivial factors of $L(\vec{a}, T)^{(-1)^n}$

Based on Dwork trace formula and Wan's boundary decomposition theorem, we can get some trivial factors of the L-functions.

**Theorem 3.4.** *Notations as above. We have*

$$L^*(f, T)^{(-1)^n} = (1 - T)(1 - qT)^{r-1} \prod_{i=1}^{d-r} (1 - \beta_i T),$$

$$L(\vec{a}, T)^{(-1)^n} = (1 - T)^{r-1} \prod_{i=1}^{d-r} (1 - \alpha_i T),$$

*where* $d = \prod_{i=1}^{r}\left(1+\sum_{j=1}^{n_i} b_{ij}\right)$ *and* $\alpha_i = \beta_i/q$.

*Proof.* By Proposition 3.2 and Theorem 2.3, the associated L-function $L^*(f,T)^{(-1)^n}$ is a polynomial satisfying

$$L^*(f,T)^{(-1)^n} = \prod_{i=0}^{n+1} \det(I - Tq^i A_a(f))^{(-1)^i\binom{n+1}{i}}, \tag{3.14}$$

and

$$\det(I - TA_a(f)) = \prod_{i=0}^{\infty}\left(L^*(f, q^iT)^{(-1)^n}\right)^{\binom{n+i}{i}}. \tag{3.15}$$

For simplicity, let $D(T) = \det(I - TA_a(f))$. Recall that in Definition 2.13 (Wan's boundary decomposition),

$$B(\Delta) = \{\text{ the interior of a closed face in } C(\Delta) \text{ containing the origin}\}.$$

Depending on Theorem 2.6 (Wan's boundary decomposition theorem),

$$\det(I - TA_a(f)) = \prod_{\Sigma\in B(\Delta)} \det\left(I - TA_a(\Sigma, f^{\overline{\Sigma}})\right), \tag{3.16}$$

we can get some trivial factors of $D(T)$ and thus $L^*(f,T)^{(-1)^n}$.

Let $\Sigma_i \in B(\Delta)$ be the interior of an $i$-dimensional closed face and $N(i) = \#\{\Sigma_i\}$ be the number of such open cones. For Newton polyhedron $\Delta = \Delta(f)$, we have $N(0) = 1$ and $N(1) = n + r$. Let

$$D_i(T) = \det\left(I - TA_a(\Sigma_i, f^{\overline{\Sigma}_i})\right) \quad \text{and} \quad D_i'(T) = \det\left(I - TA_a(\overline{\Sigma}_i, f^{\overline{\Sigma}_i})\right).$$

where $\overline{\Sigma}_i$ is the closure of $\Sigma_i$. It is obvious that the unique 0-dimensional cone $\overline{\Sigma}_0$ is the origin and $D_0(T) = D_0'(T) = 1 - T$. When $i = 1$, each $f^{\overline{\Sigma}_1}$ can be normalized to $x$ through variable substitutions and it's easy to check that

$$L^*(f^{\overline{\Sigma}_1}, T) = L^*(x, T) = \exp\left(\sum_{k=1}^{\infty} -\frac{T^k}{k}\right) = 1 - T.$$

Based on Equation (3.15), we have

$$D_1'(T) = \prod_{i=0}^{\infty} \left(L^*\left(f^{\overline{\Sigma}_1}, q^i T\right)\right)^{\binom{i}{i}} = (1-T)(1-qT)\left(1-q^2T\right)\cdots .$$

Applying Equation (3.16) to the 1-dimensional case, we obtain the following equation

$$D_1(T) = \frac{D_1'(T)}{D_0(T)} = (1-qT)\left(1-q^2T\right)\prod_{i=3}^{\infty}\left(1-q^iT\right), \tag{3.17}$$

where $B(\overline{\Sigma}_1) = \{\Sigma_0, \Sigma_1\}$. Recall that $N(0) = 1$ and $N(1) = n + r$. Using Equation (3.16) again, we have

$$\begin{aligned} D(T) &= \prod_{i=1}^{n+1}\prod_{j=1}^{N(i)} D_i(T) = D_0(T)\left(D_1(T)^{n+r}\right)\prod_{i=2}^{n+1}\prod_{j=1}^{N(i)} D_i(T) \\ &= (1-T)(1-qT)^{n+r}\cdots . \end{aligned}$$

Note that 1 is the unique unit root of $D(T)$ by Proposition 2.3 and Theorem 3.2.

The following formula then follows from Equation (3.14),

$$L^*(f, T)^{(-1)^n} = \frac{D(T)D(q^2T)^{\binom{n+1}{2}}\cdots}{D(qT)^{n+1}D(q^3T)^{\binom{n+1}{3}}\cdots} = (1-T)(1-qT)^{r-1}\prod_{i=1}^{d-r}(1-\beta_iT),$$

where $d = \prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} b_{ij}\right)$. Together with Formula (3.3), we get the factorization for $L(\vec{a}, T)^{(-1)^n}$. $\square$

**Corollary 3.4.** *Assume $b_{ij} = 1$ and $n_i$ is even $(1 \le i \le r, 1 \le j \le n_i)$. Let $d = \prod_{i=1}^{r}(1 + n_i)$*

*and* $m = d - \frac{r^2+3r+2}{2}$*. Then the polynomial* $L^*(f,T)^{(-1)^n}$ *of degree* $d$ *can be expressed in the following form,*

$$L^*(f,T)^{(-1)^n} = (1-T)(1-qT)^{r-1}(1-\gamma_0'T)(1-q^2T)^{\frac{r^2-r}{2}}\prod_{i=1}^{r}(1-\gamma_i'T)\prod_{j=1}^{m}(1-\beta_jT),$$

*where* $|\gamma_i'| = |\beta_j| = q^{\frac{n+1}{2}}$ $(0 \leq i \leq r, 1 \leq j \leq r+1)$ *and* $|\beta_j| \leq q^{\frac{n+1}{2}}$ $(r+2 \leq j \leq m)$*. For the slopes of the reciprocal roots,* $\operatorname{ord}_q \gamma_0' = 1$*,* $\operatorname{ord}_q \gamma_i' = 2$ $(1 \leq i \leq r)$*,* $\operatorname{ord}_q \beta_j = n-1$ $(1 \leq j \leq r)$*,* $\operatorname{ord}_q \beta_{r+1} = n$ *and* $2 < \operatorname{ord}_q \beta_j < n-1$ $(r+2 \leq j \leq m)$*.*

*Consequently, we have a similar formula for* $L(\vec{a},T)^{(-1)^n}$*,*

$$L(\vec{a},T)^{(-1)^n} = (1-T)^{r-1}(1-\gamma_0T)(1-qT)^{\frac{r^2-r}{2}}\prod_{i=1}^{r}(1-\gamma_iT)\prod_{j=1}^{m}(1-\alpha_jT),$$

*where* $|\gamma_i| = |\alpha_j| = q^{\frac{n-1}{2}}$ $(0 \leq i \leq r, 1 \leq j \leq r+1)$ *and* $|\alpha_j| \leq q^{\frac{n-1}{2}}$ $(r+2 \leq j \leq m)$*. For the slopes of the reciprocal roots,* $\operatorname{ord}_q \gamma_0 = 0$*,* $\operatorname{ord}_q \gamma_i = 1$ $(1 \leq i \leq r)$*,* $\operatorname{ord}_q \alpha_j = n-2$ $(1 \leq j \leq r)$*,* $\operatorname{ord}_q \alpha_{r+1} = n-1$ *and* $1 < \operatorname{ord}_q \alpha_j < n-2$ $(r+2 \leq j \leq m)$*.*

*Proof.* With the assumption $b_{ij} = 1$, $f$ is ordinary and the Newton polygon is determined by Corollary 3.3. Proof of this corollary follows from the proof of theorem 3.4. Similarly, when $i = 2$, we have $L^*(f^{\overline{\Sigma}_2},T) = (1-T)^{-1}$ which implies

$$\begin{aligned} D_2'(T) &= \prod_{i=0}^{\infty}\left(L^*\left(f^{\overline{\Sigma}_2}, q^iT\right)\right)^{-\binom{1+i}{i}} \\ &= (1-T)\,(1-qT)^2\left(1-q^2T\right)^3\left(1-q^3T\right)^4\cdots. \end{aligned}$$

Since the boundary decomposition of $\overline{\Sigma}_2$ consists of $\Sigma_0$ and two distinct $\Sigma_1$s, we obtain

$$D_2(T) = \frac{D_2'(T)}{D_0(T)D_1^2(T)} = \left(1-q^2T\right)\prod_{i=3}^{\infty}\left(1-q^iT\right)^{i-1}. \tag{3.18}$$

By Theorems 2.4 and 3.3, we have

$$D(T) = \prod_{i=0}^{\infty} (1 - \lambda_i T), \tag{3.19}$$

where $\#\{\lambda_i \mid \mathrm{ord}_q \lambda_i = k\} = W_\Delta(k)$ for $k \in \mathbb{Z}_{\geq 0}$. Note that $W_\Delta(1) = n + r + 1$ by Equation (3.12) and $N(2) = \binom{n+r}{2}$. Based on Equations (3.16), (3.14), (3.17) and (3.18), we have

$$\begin{aligned} D(T) &= \prod_{i=1}^{n+1} \left(D_i(T)^{N(i)}\right) = D_0(T) \left(D_1(T)^{n+r}\right) \left(D_2(T)^{\binom{n+r}{2}}\right) \prod_{i=3}^{n+1} \left(D_i(T)^{N(i)}\right) \\ &= (1-T)(1-qT)^{n+r}(1-\gamma_0' T)(1-q^2T)^{\binom{n+r}{2}+n+r} h_1(T), \\ L^*(f,T)^{(-1)^n} &= (1-T)(1-qT)^{r-1}(1-q^2T)^{\frac{r^2-r}{2}} \frac{1-\gamma_0' T}{(1-\gamma_0' qT)^{n+1}} h_2(T), \end{aligned}$$

where $|\gamma_0'|_p = q^{-1}$ and the slopes of reciprocal roots of $h_2(T)$ are greater than 1.

Now we claim that $L^*(f,T)^{(-1)^n}$ has at least $2r+2$ non-real reciprocal roots of weight $n+1$ including $\gamma_0'$. Let $\beta_i$ be a reciprocal root of $L^*(f,T)^{(-1)^n}$. Under the assumption that $b_{ij} = 1$ and $n_i$ is even for $1 \leq i \leq r,\ 1 \leq j \leq n_i$, $f$ is an odd function, i.e., $f(-x) = -f(x)$ which implies

$$L^*(f,T) = \overline{L^*(f,T)}.$$

In that case, the conjugate $\overline{\beta_i}$ is also a reciprocal root of $L^*(f,T)^{(-1)^n}$. As stated in Theorem 2.1, $|\beta_i| \leq q^{(n+1)/2}$. So if $\mathrm{ord}_q\, \beta_i > (n+1)/2$, then $\beta_i$ should be non-real satisfying

$$\mathrm{ord}_q\, \beta_i + \mathrm{ord}_q\, \overline{\beta_i} = \omega_i \in \mathbb{Z} \cap [0, n+1]. \tag{3.20}$$

By Corollary 3.3, the Hodge number $H_\Delta(0) = H_\Delta(n) = 1$, $H_\Delta(1) = H_\Delta(n-1) = r$ and $H_\Delta(2) = r + \frac{r^2-r}{2}$. Restricted by formula (3.20), $\gamma_0'$ is non-real and its conjugate $\overline{\gamma_0'}$ is the unique reciprocal root of $L^*(f,T)^{(-1)^n}$ with slope $n$. Furthermore, $L^*(f,T)^{(-1)^n}$ has exactly $r$ non-real slope-2 reciprocal roots $\gamma_i'$ $(1 \le i \le r)$ whose conjugates $\overline{\gamma_i'}$ are reciprocal roots of slope $n-1$. These $2r+2$ non-real reciprocal roots are purely of weight $n+1$. $\square$

Then it would be interesting to consider how to get an optimal ordinary condition in terms of congruence equations of $p$ and the parameter $b_{ij}$s. Note that it is generally hard to obtain a clean simple result if we don't restrict the parameter $b_{ij}$s. In addition, through adding some more suitable conditions on the family or applying some other methods concerning $\ell$-adic cohomology and local monodromy theory, one may be able to completely determine the weights of all reciprocal roots. We leave these to interested readers.

### 3.2.4 Special case when $n = 4$, $n_i = 2$ and $b_{ij} = 1$

When $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \le i \le 2, 1 \le j \le 2)$, the Laurent polynomial

$$f(x_1, x_2, \dots, x_5) = x_1 + x_2 + x_3 + x_4 + x_5\left(\frac{a_1}{x_1x_2} + \frac{a_2}{x_3x_4} - 1\right)$$

where $a_1, a_2 \in \mathbb{F}_q^*$. In this case, the Newton polyhedron $\Delta$ is generated by 8 vertices, $V_0 = (0,0,0,0,0)$, $V_1 = (1,0,0,0,0)$, $V_2 = (0,1,0,0,0)$, $V_3 = (0,0,1,0,0)$, $V_4 = (0,0,0,1,0)$, $V_5 = (0,0,0,0,1)$, $V_6 = (-1,-1,0,0,1)$ and $V_7 = (0,0,-1,-1,1)$.

**Theorem 3.5.** *Notations as above. If $n = 4$, $n_i = 2$ and $b_{ij} = 1$ $(1 \le i \le r, 1 \le j \le n_i)$, then*

$$L^*(f,T) = (1-T)(1-qT)(1-q^2T)\prod_{i=1}^{6}(1-\beta_iT),$$

*where $|\beta_i| = q^{5/2}$ for $1 \le i \le 6$. The weights and slopes of $\beta_i$ are given by Table 3.1, where the reciprocal roots $\beta_i$ are enumerated with respect to their $q$-adic slopes.*

| $\beta_i$ | $\{\beta_1, \beta_6\}$ | $\{\beta_2, \beta_4\}$ | $\{\beta_3, \beta_5\}$ |
|---|---|---|---|
| Slopes | $\{1, 4\}$ | $\{2, 3\}$ | $\{2, 3\}$ |
| Weights | 5 | 5 | 5 |
| $\lvert\beta_i\rvert_p$ | $\{q^{-1}, q^{-4}\}$ | $\{q^{-2}, q^{-3}\}$ | $\{q^{-2}, q^{-3}\}$ |
| $\lvert\beta_i\rvert$ | $\{q^{\frac{5}{2}}, q^{\frac{5}{2}}\}$ | $\{q^{\frac{5}{2}}, q^{\frac{5}{2}}\}$ | $\{q^{\frac{5}{2}}, q^{\frac{3}{2}}\}$ |

Table 3.1: Slopes and Weights of $\beta_i$

*Proof.* If $n = 4$, $n_i = 2$ and $b_{ij} = 1$, the generating function in Corollary 3.3 has a much simpler form,

$$G_1(x) = \prod_{i=1}^{2} \left(1 + x + x^2\right) = 1 + 2x + 3x^2 + 2x^3 + x^4,$$

where the Hodge number $H_\Delta(k)$ is the coefficient of $x^k$ in $G_1(x)$. Here $H_\Delta(k)$ represents the number of $\beta_i$ whose slope is $k$. The weight computation follows directly from Corollary 3.4 when $r = 2$ and $m = 3$. $\square$

Adolphson and Sperber [3] propose a simple conjecture about the number of weight $k$ reciprocal roots of an $n$-dimensional polytope ($k \leq n$) that is disproved by Denef and Loeser [12]. In the disproof, Denef and Loeser show the existence of counterexample for $n = 5$, but they are not able to provide an explicit one. Our example in Theorem 3.5 is the first explicit counterexample for Adolphson-Sperber's conjecture.

**Conjecture 3.1** (Adolphson and Sperber, [3]). *Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be non-degenerate and suppose $\dim \Delta(f) = n$. Let $w_k$ be the number of roots of $L^*(f, T)$ with weight $k$, where $0 \leq k \leq n$. Then*

$$w_k = (-1)^k \sum_{\substack{0 \in \sigma \text{ face of } \Delta(f) \\ \dim \sigma \leq k}} (-1)^{\dim \sigma} \left( F_\sigma(k) + F_\sigma(k-1) - \binom{n - \dim \sigma}{n - k + 1} \right) (\dim \sigma)! V(\sigma),$$

*where $V(\sigma)$ is the volume of $\sigma$ normalized by the assumption that a fundamental domain of the lattice $\mathbb{Z}^n \cap$ (affine space of $\sigma$) has unit volume and $F_\sigma(k)$ is the number of $k$-dimensional* *faces of* $\Delta(f)$ *that contain* $\sigma$*.*

Let 0 be the origin. Then $F_0(k)$ is the number of $k$-dimensional faces of $\Delta(f)$ that contain the origin. The detailed information of the Newton polyhedron $\Delta = \Delta(f)$ in Theorem 3.5 is given in Table 3.2.

We compute $w_5$ for our example using Adolphson-Sperber's Conjecture,

$$\begin{aligned} w_5 =&(-1)^5 \sum_{\substack{0\in\sigma \text{ face of } \Delta \\ \dim\sigma\le 5}} (-1)^{\dim\sigma}\left(F_\sigma(5)+F_\sigma(4)-\binom{5-\dim\sigma}{1}\right)(\dim\sigma)!V(\sigma) \\ =&9F_0(5)-F_0(4)(1+1-1)+F_0(3)(1+2-2)-(6(1+3-3)+9(1+4-3)) \\ &+F_0(1)(1+6-4)-F_0(0)(1+9-5) \\ =&7\neq 6. \end{aligned}$$

| $k$ | Vertices of $k$-dimensional faces of $\Delta$ that contain 0 | $F_0(k)$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | $\{0,V_1\},\{0,V_2\},\{0,V_3\},\{0,V_4\},\{0,V_6\},\{0,V_7\}$ | 6 |
| 2 | $\{0,V_1,V_2\},\{0,V_1,V_3\},\{0,V_1,V_4\},\{0,V_1,V_6\},\{0,V_1,V_7\},$ $\{0,V_2,V_3\},\{0,V_2,V_4\},\{0,V_2,V_6\},\{0,V_2,V_7\},\{0,V_3,V_4\},$ $\{0,V_3,V_6\},\{0,V_3,V_7\},\{0,V_4,V_6\},\{0,V_4,V_7\},\{0,V_6,V_7\}$ | 15 |
| 3 | $\{0,V_1,V_2,V_3\},\{0,V_1,V_2,V_4\},\{0,V_1,V_2,V_7\},\{0,V_1,V_3,V_4\},$ $\{0,V_1,V_3,V_6\},\{0,V_1,V_3,V_7\},\{0,V_1,V_4,V_6\},\{0,V_1,V_4,V_7\},$ $\{0,V_1,V_6,V_7\},\{0,V_2,V_3,V_4\},\{0,V_2,V_3,V_6\},\{0,V_2,V_3,V_7\},$ $\{0,V_2,V_4,V_6\},\{0,V_2,V_4,V_7\},\{0,V_2,V_6,V_7\},\{0,V_3,V_4,V_6\},$ $\{0,V_3,V_6,V_7\},\{0,V_4,V_6,V_7\}$ | 18 |
| 4 | $\{0,V_1,V_2,V_3,V_4\},\{0,V_1,V_2,V_3,V_7\},\{0,V_1,V_2,V_4,V_7\},$ $\{0,V_1,V_3,V_4,V_6\},\{0,V_1,V_3,V_6,V_7\},\{0,V_1,V_4,V_6,V_7\},$ $\{0,V_2,V_3,V_4,V_6\},\{0,V_2,V_3,V_6,V_7\},\{0,V_2,V_4,V_6,V_7\}$ | 9 |

Table 3.2: Vertices of $k$-dimensional faces of $\Delta$ that contain 0

This result contradicts with the one we obtained in Theorem 3.5 which implies Conjecture 3.1 fails in our example.

# Chapter 4

# Type II exponential sums

In this chapter, we study the $p$-adic estimate of the exponential sums associated to a certain family of Laurent polynomials over $\mathbb{F}_q$, denoted by $\mathcal{F}$:

$$f(x_1, x_2 \ldots, x_{n+1}) = \sum_{i=1}^{n} a_i x_{n+1} \left( x_i + \frac{1}{x_i} \right) + a_{n+1} x_{n+1} + \frac{1}{x_{n+1}}$$

where $a_i \in \mathbb{F}_q^*$, $i = 1, 2, \ldots, n+1$. The Newton polyhedron $\Delta(f)$ is a $(n+1)$-dimensional polytope in $\mathbb{R}^{n+1}$. All the results of type II exponential sums comes from my own published paper [8]. In this chapter, let $\Delta_n = \Delta(f)$.

The estimate of the exponential sums associated to $f \in \mathcal{F}$ applies to many critical problems in analytic number theory. For instance, in Iwaniec's work [33] on small eigenvalues of Laplacian for $\Gamma_0(p)$, he improves the lower bound of eigenvalues conjectured by Selberg via the estimate for the specific exponential sum $S_1^*(f)$ where $f \in \mathcal{F}$ is in 3 variables.

For non-degenerate Laurent polynomial $f \in \mathcal{F}$ in $n+1$ variables, the associated L-function $L^*(f, T)^{(-1)^n}$ is a polynomial of degree $2^{n+1}$ by Adolphson-Sperber's theorem [2]. Let $\vec{a} = (a_1, \ldots, a_{n+1})$ denote the coefficients of a non-degenerate Laurent polynomial $f \in \mathcal{F}$. Based

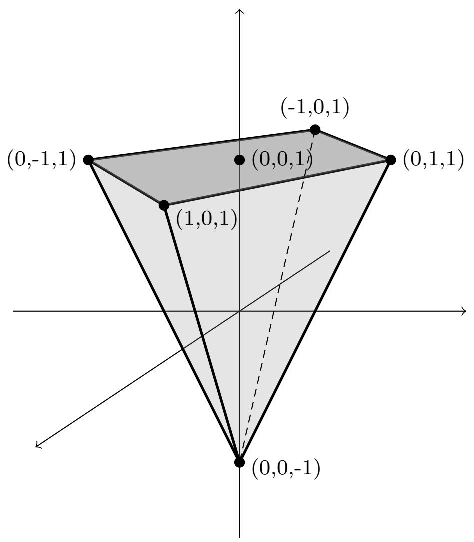

Figure 4.1: Newton Polyhedron $\Delta_2$

on Wan's decomposition theorems [44], Zhang and Feng [47] compute the Hodge polygon $\mathrm{HP}(\Delta_n)$ and prove that this family $\mathcal{F}$ is generically ordinary for any prime $p$, i.e., there exists a non-zero polynomial $h_p(\Delta_n)(\vec{a}) \in \mathbb{F}_p[a_1, \ldots, a_{n+1}]$ satisfying: if $h_p(\Delta_n)(\vec{a}) \neq 0$, the Newton polygon of $L^*(f,T)^{(-1)^n}$ coincides with its lower bound $\mathrm{HP}(\Delta_n)$. By Definition 2.8, $h_p(\Delta_n)(\vec{a})$ is called a *Hasse polynomial* which is the main object of study in this chapter.

As introduced in Section 2.4, there exists a general method to directly calculate the Hasse polynomials but the method becomes insufficient for higher dimensional Newton polyhedrons. Based on this method, Zhang and Feng [47] obtain an explicit formula of Hasse polynomials in low dimensions, i.e., $n \leq 3$. Recall that $h_p(\Delta_n, \leq 1)$ is a factor of $h_p(\Delta_n)$ satisfying: if $h_p(\Delta_n, \leq 1) \neq 0$, the Newton polygon coincides with $\mathrm{HP}(\Delta_n)$ for all sides of slope $\leq 1$. In this chapter, we will give an explicit formula of the partial Hasse polynomial $h_p(\Delta_n, \leq 1)$ that holds for arbitrary natural number $n$.

In addition, Zhang-Feng's formula for the full Hasse polynomial $h_p(\Delta_3)$ when $n = 3$ is very complicated involving the determinant of a $4 \times 4$ matrix whose entries are all polynomials. In this chapter, we provide a much simpler formula for the $n = 3$ case based on Denef-Loeser's theorem [12] and the symmetric property of $\mathrm{NP}(f)$, which answers an open question of Zhang and Feng [47].

## 4.1 Previous results

Based on Wan's decomposition theorems [41], Zhang and Feng [47] prove the following results.

**Proposition 4.1** (Zhang and Feng, [47]). *Let $f \in \mathcal{F}$ with $\Delta_n = \Delta(f)$.*

1. *Then $f$ is non-degenerate if and only if $\pm 2a_1 \pm 2a_2 \ldots \pm 2a_n + a_{n+1} \neq 0$, i.e., $\mathcal{M}_p(\Delta_n) = \{\vec{a} = (a_1, \ldots, a_{n+1}) \in \bar{\mathbb{F}}_p^{n+1} \mid a_1 \cdots a_{n+1} \prod(\pm 2a_1 \pm 2a_2 \ldots \pm 2a_n + a_{n+1}) \neq 0\}$.*
2. *If $f \in \mathcal{M}_p(\Delta_n)$, the associated L-function $L^*(f,T)^{(-1)^n}$ is a polynomial of degree $2^{n+1}$, i.e., $(n+1)!\mathrm{Vol}(\Delta_n) = 2^{n+1}$.*
3. *The Hodge polygon $HP(\Delta_n)$ is a lower convex polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and*
$$\left( \sum_{m=0}^{k} \binom{n+1}{m}, \sum_{m=0}^{k} m \binom{n+1}{m} \right), \ k = 0, 1, \ldots, n+1,$$
*i.e., $D = 1$ and $H_{\Delta_n}(m) = \binom{n+1}{m}$ for $m = 0, 1, \ldots, n+1$.*

   *That is, the Hodge polygon $HP(\Delta_n) \in \mathbb{R}^2$ consists of $n+2$ line segments starting from initial point $(0,0)$ and has sides of increasing slopes, i.e., the $j$-th segment has slope $j-1$ with horizontal length $\binom{n+1}{j-1}$, $j = 1, \ldots, n+2$.*
4. *This family is generically ordinary for any prime $p$, i.e., there exists a non-zero polynomial $h_p(\Delta_n)(\vec{a}) \in \mathbb{F}_p[a_1, \ldots, a_{n+1}]$ satisfying: if $h_p(\Delta_n)(\vec{a}) \neq 0$, the Newton polygon of $L^*(f,T)^{(-1)^n}$ coincides with its lower bound $HP(\Delta_n)$.*

Note that when $n = 1$, the non-degenerate Laurent polynomial is always ordinary which means the Hasse polynomial is a nonzero constant. So it suffices to compute the Hasse polynomial for $n \geq 2$.

**Proposition 4.2** (Zhang and Feng, [47])**.** *Notations as above.*

1. *When $n = 2$, a Hasse polynomial can be taken to be*

$$h_p(\Delta_2)(a_1, a_2, a_3) = \sum_{\substack{0 \leq v_1 + v_2 \leq \frac{p-1}{2} \\ v_1, v_2 \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} a_3^{p-1-2(v_1+v_2)}}{(v_1! v_2!)^2 (p - 1 - 2(v_1 + v_2))!}.$$

*where $\vec{a} = (a_1, a_2, a_3) \in \mathcal{M}_p(\Delta_2)$.*

2. *When $n = 3$, a Hasse polynomial can be taken to be*

$$h_p(\Delta_3)(\vec{a}) = T(\vec{a}) \sum_{\substack{0 \leq v_1 + v_2 + v_3 \leq \frac{p-1}{2} \\ v_1, v_2, v_3 \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} a_3^{2v_3} a_4^{p-1-2(v_1+v_2+v_3)}}{(v_1! v_2! v_3!)^2 (p - 1 - 2(v_1 + v_2 + v_3))!},$$

*where $\vec{a} = (a_1, a_2, a_3, a_4) \in \mathcal{M}_p(\Delta_3)$ and $T(\vec{a})$ is the determinant of a $4 \times 4$ matrix whose entries are all polynomials.*

## 4.2 Main results

**Theorem 4.1.** *Let $f(x_1, \ldots, x_{n+1}) \in \mathcal{F}$ be a non-degenerate Laurent polynomial with $\Delta_n = \Delta(f)$. When $n \geq 2$, a Hasse polynomial of slope at most one side can be taken to be,*

$$h_p(\Delta_n, \leq 1)(\vec{a}) = \sum_{\substack{0 \leq v_1 + \ldots + v_n \leq \frac{p-1}{2} \\ v_1, \ldots, v_n \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)}}{(v_1! v_2! \ldots v_n!)^2 (p - 1 - 2(\sum_{i=1}^n v_i))!}.$$

**Theorem 4.2.** *For $n = 3$, let $f(x_1, \ldots, x_4) \in \mathcal{F}$ be a non-degenerate Laurent polynomial* *with* $\Delta(f)=\Delta_3$*. A Hasse polynomial of* $\Delta_3$ *can be written as,*

$$h_p(\Delta_3)(\vec{a}) = h_p(\Delta_3, \leq 1)(\vec{a}) = \sum_{\substack{0\leq v_1+v_2+v_3\leq \frac{p-1}{2} \\ v_1,v_2,v_3\in\mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1}a_2^{2v_2}a_3^{2v_3}a_4^{p-1-2(\sum_{i=1}^{3} v_i)}}{(v_1!v_2!v_3!)^2(p-1-2(\sum_{i=1}^{3} v_i))!}.$$

*In particular, NP*$(f)$*=HP*$(\Delta_3)$ *if and only if* $h_p(\Delta_3)(\vec{a}) \not\equiv 0 (\mathrm{mod}\ p)$ *where* $\vec{a}$ *is the vector of coefficients of* $f$*.*

Furthermore, we are interested in the irreducibility of the Hasse polynomials. For $n\geq 3$ and $p\leq 7$, we prove that the Hasse polynomial $h_p(\Delta_n,\leq 1)(\vec{a})$ is irreducible over $\overline{\mathbb{F}}_p$. Based on this fact, we have the following hypothesis of the irreducibility.

**Conjecture 4.1.** *Let* $p$ *be an odd prime and* $n\geq 3$*. Then the Hasse polynomial* $h_p(\Delta_n,\leq 1)(\vec{a})$ *is irreducible over* $\overline{\mathbb{F}}_p$*.*

## 4.3 Proof of the main theorems

In this section, we give the proof of our main theorems.

**Theorem 4.3.** *For* $n\geq 2$*. Let* $f$ *be a non-degenerate Laurent polynomial defined as*

$$f(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n} a_i x_{n+1}\left(x_i+\frac{1}{x_i}\right) + a_{n+1}x_{n+1} + \frac{1}{x_{n+1}}$$

*where* $a_i\in\mathbb{F}_q^*$*,* $i=1,2,\ldots,n+1$*. For* $\Delta_n=\Delta(f)$*, we have*

$$h_p(\Delta_n,\leq 1)(\vec{a}) = \sum_{\substack{0\leq v_1+\ldots+v_n\leq \frac{p-1}{2} \\ v_1,\ldots,v_n\in\mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1}a_2^{2v_2}\ldots a_n^{2v_n}a_{n+1}^{p-1-2(\sum_{i=1}^{n} v_i)}}{(v_1!v_2!\ldots v_n!)^2(p-1-2(v_1+v_2+\ldots+v_n))!} \tag{4.1}$$

*where* $\vec{a}=(a_1,\ldots,a_{n+1})\in\mathcal{M}_p(\Delta_n)$*.*

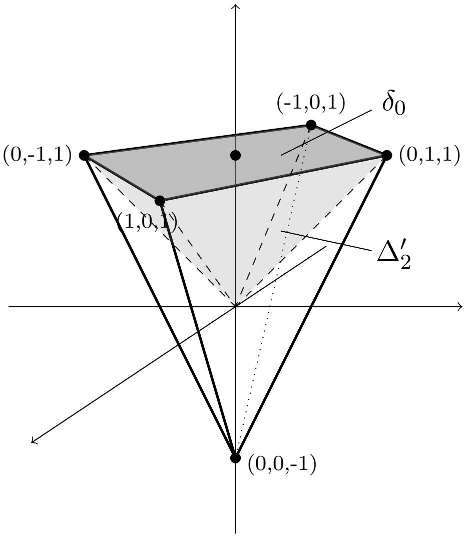

Figure 4.2: Newton Polyhedron $\Delta_2'$

*Proof.* The Newton polyhedron of $f$, $\Delta_n = \Delta(f)$, has $2^n + 1$ co-dimension 1 faces denoted by $\delta_i$ ($0 \leq i \leq 2^n$). Let $\delta_0 : x_{n+1} = 1$ be the top face. Since the origin is an interior point of the polytope, these co-dimension 1 faces don't contain the origin. By Theorem 2.5 (Wan's facial decomposition theorem), $f$ is ordinary if and only if $f^{\delta_i}$ is ordinary for $0 \leq i \leq 2^n$. For $1 \leq i \leq 2^n$, $f^{\delta_i}$ is diagonal, i.e., the polytope generated by $\delta_i$ and the origin is a simplex. More specifically, the vertex matrix of $f^{\delta_i}$ ($1 \leq i \leq 2^n$) belongs to the special linear group $SL(n+1, \mathbb{Z})$. From Proposition 2.5 (local diagonal theory), $f^{\delta_i}$ is ordinary for $1 \leq i \leq 2^n$. Thus $f$ is ordinary if and only if $f^{\delta_0}$ is ordinary, i.e., NP($f$) coincides with its lower bound at the $k$-th vertex if and only if NP($f^{\delta_0}$) coincides with its lower bound at the $k$-th vertex.

If we restrict $f$ to the co-dimension 1 face $\delta_0 : x_{n+1} = 1$, we will get the following new Laurent polynomial

$$g = f^{\delta_0} = \sum_{i=1}^{n} a_i x_{n+1} \left( x_i + \frac{1}{x_i} \right) + a_{n+1} x_{n+1}.$$

Let $\Delta_n = \Delta(f)$ and $\Delta_n' = \Delta(g)$. To obtain $h_p(\Delta_n, \leq 1)$, it suffices to compute $h_p(\Delta_n', \leq 1)$.

From Formulas (2.5), (2.6) and (2.9),

$$h_p(\Delta_n', \leq 1) = h_p(\Delta_n', 1) = \det \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \bmod p.$$

where $A_{00} = 1$ and $A_{10} = (0, 0, \ldots, 0)^T$. So $h_p(\Delta_n', \leq 1) = \det A_{11} \bmod p$. Based on Zhang and Feng's calculation[47], $W_{\Delta_n'}(1) = 2n+1$. So $A_{11}$ is a matrix of size $(2n+1)\times(2n+1)$. For simplicity, we enumerate the vertices of $\Delta_n'$ in the following way. Let $V_1 = (1, 0, 0, \ldots, 0, 1)$, $V_2 = (-1, 0, 0, \ldots, 0, 1)$, $V_3 = (0, 1, 0, \ldots, 0, 1)$, $\ldots$, $V_{2n} = (0, 0, \ldots, 0, -1, 1)$, $V_{2n+1} = (0, 0, \ldots, 0, 1)$. If the vertices are suitably arranged, $A_{11}$ is a lower triangular matrix of the following form,

$$A_{11} = \begin{array}{c} \\ V_{2n+1} \\ V_1 \\ V_2 \\ \vdots \\ V_{2n} \end{array} \begin{array}{c} \begin{array}{ccccc} V_{2n+1} & V_1 & V_2 & \cdots & V_{2n} \end{array} \\ \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix} \end{array} = (b_{ij})_{1\leq i,j\leq 2n+1}.$$

So $h_p(\Delta_n', \leq 1) = \det A_{11} \bmod p = \prod_{k=1}^{2n+1} b_{kk} \bmod p$. By Formulas (2.5) and (2.6),

$$\begin{aligned} b_{11} &= \frac{1}{p} F_{(0,\ldots,0,p-1)}(g) \\ &= \sum_{\substack{0 \leq v_1 + \ldots + v_n \leq \frac{p-1}{2} \\ v_1, v_2, \ldots, v_n \in \mathbb{Z}_{\geq 0}}} \frac{1}{(v_1!)^2} \cdots \frac{1}{(v_n!)^2} \frac{1}{(p-1-2(\sum_{i=1}^n v_i))!} a_1^{2v_1} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)} \end{aligned}$$

and

$$b_{ii} = \begin{cases} \frac{1}{(p-1)!} a_k^{p-1} & \text{if } i = 2k, \\ \frac{1}{(p-1)!} a_k^{p-1} & \text{if } i = 2k+1, \end{cases} \tag{4.2}$$

where $1 \leq k \leq n$. Since $a_k \in \mathbb{F}_q^*$ and $\mathrm{ord}_p(\frac{1}{(p-1)!}) = 0$, we can conclude $b_{ii} (i \geq 2)$ are trivial factors for the Hasse polynomial and get the following equation,

$$\begin{aligned} h_p(\Delta_n, \leq 1) &= h_p(\Delta'_n, \leq 1) \\ &= \sum_{\substack{0 \leq \sum_{i=1}^n v_i \leq \frac{p-1}{2} \\ v_1, v_2, \ldots, v_n \in \mathbb{Z}_{\geq 0}}} \lambda_{v_1}^2 \lambda_{v_2}^2 \ldots \lambda_{v_n}^2 \lambda_{p-1-2(\sum_{i=1}^n v_i)} a_1^{2v_1} a_2^{2v_2} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)} \\ &= \sum_{\substack{0 \leq \sum_{i=1}^n v_i \leq \frac{p-1}{2} \\ v_1, v_2, \ldots, v_n \in \mathbb{Z}_{\geq 0}}} \frac{1}{(v_1!)^2} \cdots \frac{1}{(v_n!)^2} \frac{a_1^{2v_1} a_2^{2v_2} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)}}{(p-1-2(v_1 + \ldots + v_n))!} \end{aligned}$$

□

In the second part of this section, we will give a formula of the full Hasse polynomial $h_p(\Delta_3)$ which is much simpler compared to Zhang and Feng's result [47], stated in Theorem 4.2. Before proving the formula, we introduce a lemma which will be used in the proof. This lemma follows from Denef-Loeser's weight formula [12].

**Lemma 4.1.** *Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ be a non-degenerate Laurent polynomial. Assume that the Newton polyhedron of $f$ is an $(n+1)$-dimensional polytope in $\mathbb{R}^{n+1}$, denoted by $\Delta$. If the origin is an interior point of $\Delta$, then the associated L-function is purely of weight $n+1$, i.e.,*

$$L^*(f, T)^{(-1)^n} = \prod_{i=1}^{(n+1)!\mathrm{Vol}(\Delta)} (1 - \alpha_i T)$$

*where $|\alpha_i| = q^{(n+1)/2}$, $i = 0, 1, \ldots, (n+1)!\mathrm{Vol}(\Delta)$.*

Then we have the following formula based on the symmetric property of the Newton polygons of L-functions derived from Lemma 4.1.

**Theorem 4.4.** *For $n=3$, let $f(x_1,\ldots,x_4)\in\mathcal{F}$ be a non-degenerate Laurent polynomial with $\Delta_3=\Delta(f)$. A Hasse polynomial of $\Delta_3$ can be written as,*

$$h_p(\Delta_3)(\vec{a}) = h_p(\Delta_3,\leq 1)(\vec{a}) = \sum_{\substack{0\leq v_1+v_2+v_3\leq\frac{p-1}{2}\\ v_1,v_2,v_3\in\mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1}a_2^{2v_2}a_3^{2v_3}a_4^{p-1-2(\sum_{i=1}^{3}v_i)}}{(v_1!v_2!v_3!)^2(p-1-2(\sum_{i=1}^{3}v_i))!}$$

*where $\vec{a}=(a_1,a_2,a_3,a_4)\in\mathcal{M}_p(\Delta_3)$.*

*In particular, NP($f$)=HP($\Delta_3$) if and only if $h_p(\Delta_3,\leq 1)(\vec{a})\not\equiv 0 (\mathrm{mod}\ p)$ where $\vec{a}$ is the vector of coefficients of $f$.*

*Proof.* When $n=3$, the non-degenerate Laurent polynomial is given by

$$f(x_1,x_2,x_3,x_4) = \sum_{i=1}^{3} a_i x_4\left(x_i+\frac{1}{x_i}\right) + a_4x_4 + \frac{1}{x_4},$$

where $a_i\in\mathbb{F}_q^*, i=1,2,3,4$. From Proposition 4.1, HP($\Delta_3$) has 6 vertices (0,0), (1,0), (5,4), (11,16), (15, 28) and (16, 32).

Similar to the proof of Theorem 4.3, let $g=\sum_{i=1}^{3}a_ix_4(x_i+\frac{1}{x_i})+a_4x_4$ and $\Delta_3'=\Delta(g)$. We know that NP($f$) coincides with HP($\Delta_3$) at the $k$-th vertex if and only if NP($g$) coincides with HP($\Delta'$) at the $k$-th vertex. In addition, $h_p(\Delta_3',0)=\det A_{00}=1\not\equiv 0(\mathrm{mod}\ p)$ which shows that NP($g$) always pass through (1,0). So NP($f$) coincides with HP($\Delta_3$) at the first break point (1,0).

For $f\in\mathcal{F}$ in $n+1$ variables, we claim that NP($f$) is symmetric, i.e., if NP($f$) has a side of a slope $s$ with the horizontal length $l_s$, if and only if it has a side of slope $n+1-s$ with the

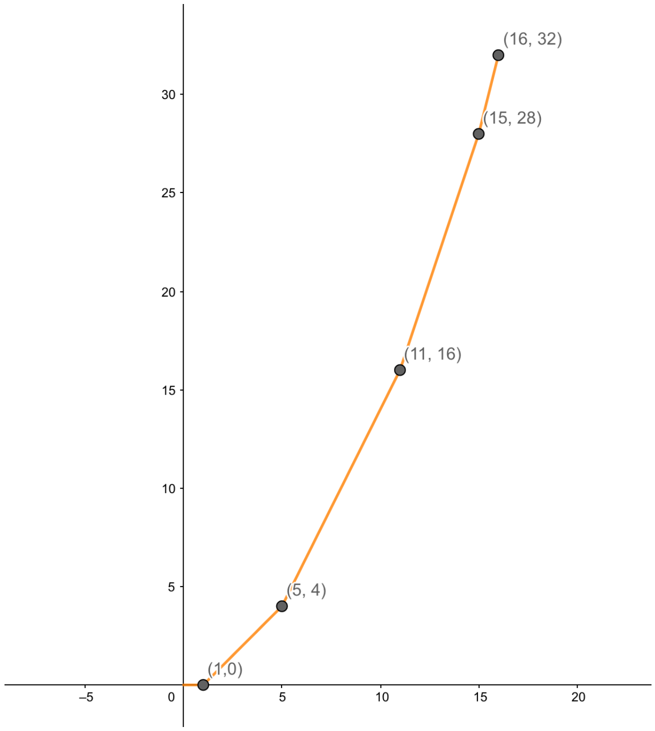

Figure 4.3: Hodge polygon $\mathrm{HP}(\Delta_3)$

same horizontal length $l_s$, $s = 0, 1, \ldots, n+1$. Recall that

$$L^*(f,T) = \exp(\sum_{k=1}^{\infty} S_k^*(f)\frac{T^k}{k})$$

where $S_k^*(f) = \sum_{x_i \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr_k} f(x_1,\ldots,x_{n+1})}$. Then

$$L^*(-f,T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(-f)\frac{T^k}{k}\right) = \overline{L^*(f,T)}.$$

Since the origin is an interior point of $\Delta(f)$, this family of L-functions is purely of weight $n+1$ by Lemma 4.1, i.e.,

$$L^*(f,T)^{(-1)^n} = \prod_{i=1}^{2^{n+1}}(1-\alpha_i T)$$

where $|\alpha_i| = q^{(n+1)/2}$. So we have

$$L^*(-f,T)^{(-1)^n} = \prod_{i=1}^{2^{n+1}} (1-\overline{\alpha_i}T) = \prod_{i=1}^{2^{n+1}} \left(1 - \frac{q^{n+1}}{\alpha_i}T\right).$$

From Lemma 2.1, we know that NP($f$) has a line segment of slope $s$ with horizontal length $l_s$ if and only if NP($-f$) has a line segment of slope $n+1-s$ with length $l_s$. In addition, it's easy to check that NP($f$)=NP($-f$). Consequently, we can get the symmetric property for the Newton polygon: NP($f$) has a side of slope $s$ with the horizontal length $l_s$ if and only if it also has a side of slope $n+1-s$ with the same horizontal length $l_s$.

Since NP($f$) is symmetric, NP($f$) coincides with HP($\Delta_n$) at the $k$-th vertex if and only if they coincide at the $(n+1-k)$-th vertex. For $n = 3$, NP($f$) and HP($\Delta_3$) share the same end points (0,0) and (16,32). As proved in Theorem 4.3, we know that NP($f$) matches HP($\Delta_3$) at point (1,0). By the symmetric property of NP($f$), (15,28) is a also break point on NP($f$). So NP($f$)=HP($\Delta_3$) if and only if NP($f$) passes through (5,4) if and only if $h_p(\Delta_3, \leq 1) \not\equiv 0 (\text{mod } p)$. □

## 4.4 Irreducibility of Hasse polynomial and open problems

In this section, we focus on the irreducibility of the partial Hasse polynomial $h_p(\Delta_n, \leq 1)$. Recall that a Laurent polynomial $f \in \mathcal{F}$ is defined as

$$f(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n} a_i x_{n+1}\left(x_i + \frac{1}{x_i}\right) + a_{n+1}x_{n+1} + \frac{1}{x_{n+1}}$$

where $a_i \in \mathbb{F}_q^*$, $i = 1, 2, \dots, n+1$. Let $\Delta_n = \Delta(f)$. From previous sections, we know its Hasse polynomial of slope at most one side $h_p(\Delta_n, \leq 1)$ for $n \geq 2$ is given by the following formula,

$$h_p(\Delta_n, \leq 1) = \sum_{\substack{0 \leq v_1 + \ldots + v_n \leq \frac{p-1}{2} \\ v_1, \ldots v_n \in \mathbb{Z}_{\geq 0}}} \frac{a_1^{2v_1} a_2^{2v_2} \ldots a_n^{2v_n} a_{n+1}^{p-1-2(\sum_{i=1}^n v_i)}}{(v_1! v_2! \ldots v_n!)^2 (p - 1 - 2(\sum_{i=1}^n v_i))!}$$

To study the irreducibility of a polynomial, we first consider the following lemma.

**Lemma 4.2.** *Assume $n \geq 2$. Let $\mathbb{P}^n$ denote the projective $n$-space over $\bar{\mathbb{F}}_p$ and $H$ be the zero locus of a homogeneous polynomial $h$ in $\mathbb{P}^n$, i.e., $H = Z(h)$. Let Sing$(H)$ denote the set of singular points of $H$. If* $\dim(Sing(H)) \leq n - 3$, *then $h$ is irreducible over $\bar{\mathbb{F}}_p$. Here* $\dim(Sing(H)) = -1$ *means* $Sing(H) = \varnothing$.

*Proof.* We give a simple proof by contradiction. Assume $h$ is reducible and $h = h_1 h_2$ where $h_1, h_2 \in \bar{\mathbb{F}}_p[x_1, \ldots, x_{n+1}]$. To get $\mathrm{Sing}(H)$, we compute

$$\frac{\partial h}{\partial x_i} = \frac{\partial h_1}{\partial x_i} h_2 + \frac{\partial h_2}{\partial x_i} h_1$$

where $i = 1, 2, \ldots, n+1$. So $Z(h_1, h_2) \subseteq \mathrm{Sing}(H)$. Combining $\dim(Z(h_1, h_2)) \geq n - 2$ and $\dim(\mathrm{Sing}(H)) \leq n - 3$, we have $n - 2 \leq \dim(Z(h_1, h_2)) \leq n - 3$ which leads to a contradiction. $\square$

**Lemma 4.3.** *Notations as above. Assume $m \geq 2$. If $h_p(\Delta_m, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_p$, then $h_p(\Delta_n, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_p$ for any positive integer $n > m$.*

*Proof.* It's easy to check that $h_p(\Delta_m, \leq 1) \equiv h_p(\Delta_n, \leq 1) \bmod (a_{m+1}, \ldots, a_n)$. If $h_p(\Delta_n, \leq 1) = g_n l_n$, then $h_p(\Delta_n, \leq 1) \equiv \bar{g}_n \bar{l}_n \bmod (a_{m+1}, \ldots, a_n)$. Since $h_p(\Delta_m, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_p$, we can conclude $\bar{g}_n$ or $\bar{l}_n$ is a constant. Based on the fact that factors of a homogenous polynomial are still homogenous, either $g_n$ or $l_n$ is a constant which implies $h_p(\Delta_n, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_p$. $\square$

Let's first compute three simple cases when $p = 3, 5$ and $7$.

**Example 4.1.** *For $p = 3$, we have $h_3(\Delta_n, \leq 1) = a_1^2 + a_2^2 + \ldots + a_n^2 + \frac{1}{2}a_{n+1}^2$. Since Sing$(H) = \varnothing$ in $\mathbb{P}^n$, $h_3(\Delta_n, \leq 1) = a_1^2 + a_2^2 + \ldots + a_n^2 + \frac{1}{2}a_{n+1}^2$ is irreducible over $\overline{\mathbb{F}}_3$ for $n \geq 2$.*

**Example 4.2.** *Suppose $p = 5$ and $n \geq 2$. Then $h_5(\Delta_n, \leq 1)$ is irreducible over $\overline{\mathbb{F}}_5$.*

*Proof.* Combining Sing$(h_5(\Delta_2, \leq 1)) = \varnothing$ with Lemma 4.2 and Lemma 4.3, we get $h_5(\Delta_n, \leq 1)$ is irreducible $\overline{\mathbb{F}}_5$ for $n \geq 2$. □

**Example 4.3.** *For $p = 7$ and $n \geq 3$, $h_7(\Delta_n, \leq 1)$ is irreducible over $\overline{\mathbb{F}}_7$.*

*Proof.* It's easy to check that Sing$(h_7(\Delta_3, \leq 1)) = \{[1 : \pm 1 : \pm 1 : \pm 3], [1 : 0 : 0 : \pm 2], [0 : 1 : 0 : \pm 2], [0 : 0 : 1 : \pm 2]\}$. So dim(Sing$(H)$)$= 0 \leq n - 3$ for $n = 3$ which means $h_7(\Delta_3, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_7$. By Lemma 4.3, $h_7(\Delta_n, \leq 1)$ is irreducible over $\overline{\mathbb{F}}_7$ for $n \geq 3$. □

Based on these examples, we give our hypothesis.

**Conjecture 4.2.** *Assume $n \geq 3$. Let $p$ be an odd prime and $f$ be a Laurent polynomial in $\mathcal{F}$ with $\Delta_n = \Delta(f)$. The Hasse polynomial of the slope at most one side $h_p(\Delta_n, \leq 1)$ is irreducible over $\bar{\mathbb{F}}_p$.*

To prove the conjecture, it's sufficient to check if $h_p(\Delta_3, \leq 1)(\vec{a})$ is irreducible over $\bar{\mathbb{F}}_p$. We leave this to interested readers.

# Chapter 5

# Extension to unit root L-functions and Dwork's deformation theory

A classical L-function is generally not pure (the zeros and poles have distinct absolute values) and it can be decomposed into a product of pure pieces consisting of all the zeros and poles with the same absolute value. To further understand how each pure piece vary with respect to the parameters, one naturally considers the pure L-function constructed as an Euler product in terms of the reciprocal zeros and poles lying in the same piece. It would be interesting to consider the meromorphic continuation and the finer Riemann hypothesis of the pure L-functions. For complex or $\ell$-adic absolute value ($\ell \neq p$), the pure L-function from algebraic geometry, which can be identified as the L-function of a certain geometric constructible étale sheaf, is always rational and the situation is similar to the original L-function [29, 11].

The most complicated part is the $p$-adic decomposition. In that case, a pure L-function may not be rational any more [39]. Dwork [18] conjectures that a pure L-function from algebraic geometry is $p$-adic meromorphic. Wan [42, 43] proves this conjecture by relating the pure L-function to symmetric, tensor and exterior power L-function via Adams operations and then employing a suitable $p$-adic limiting argument which avoids the excellent lifting. Unfortunately, only a few examples of the pure L-function of slope zero piece have been studied so far. Higher slope portion is much more difficult. In this chapter, we focus on the unit root L-function, i.e., the pure L-function of the slope zero part. Let $\vec{a} = (a_1, \ldots, a_s)$ be the coefficients of a non-degenerate Laurent polynomial $f$ in $n+1$ variables. The corresponding L-function $L^*(f,T)^{(-1)^n}$ has a unique $p$-adic unit root [4], denoted by $\alpha_0(\vec{a})$, which is also a 1-unit. The unit root L-function of this family is defined as

$$L^*_{\text{unit}}(\kappa, T) = \prod_{\vec{a} \in |\mathbb{G}_m^s/\mathbb{F}_q|} \frac{1}{1 - \alpha_0(\vec{a})^\kappa T^{\deg(\vec{a})}} \tag{5.1}$$

where $\kappa \in \mathbb{Z}_p$, the coefficient vector $\vec{a}$ runs over the closed points of $\mathbb{G}_m^s/\mathbb{F}_q$ and $\deg(\vec{a}) = [\mathbb{F}_q(\vec{a}) : \mathbb{F}_q]$. Based on Wan's proof of Dwork's conjecture, this unit root L-function is $p$-adic meromorphic in $T$. The variation of the unit root L-function with respect to the parameter of the algebraic family is connected to the Gouvêa-Mazur conjecture on the dimension variation of classical and $p$-adic modular forms [10, 27]. The meromorphic continuation of the unit root L-functions also implies the existence of a weak $p$-adic equi-distribution theorem [37] for the $p$-adic angle of the zeros of the associated L-functions of certain exponential sums. However, little is known about the zeros and poles of the unit root L-functions. The $p$-adic Riemann hypothesis for such a pure L-function is extremely mysterious. Extending Wan's techniques, Sperber and Haessig [30] show the function $L^*_{\text{unit}}(\kappa, T)^{(-1)^{s+1}}$ is $p$-adic meromorphic with a unique $p$-adic unit root.

Back to the two types of exponential sums discussed in this dissertation, one may naturally consider their unit foot L-functions. For the type I family, the unique unit root of $L^*(f,T)^{(-1)^n}$ is 1. In that case, the unit root L-function is a classical zeta function which have been fully discussed. So we would focus on the type II family where the unique unit root is a non-real algebraic integer with weight $n+1$. Based on Sperber and Haessig's theorems [30], together with Adolphson and Sperber's formula [4] for the unique unit root of a classical toric L-function, we have the following result.

**Theorem 5.1.** *Assume $p$ is a prime and $q = p^d$ for some positive integer $d$. Let $f$ be a non-degenerate Laurent polynomial defined as*

$$f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} a_i x_{n+1} \left( x_i + \frac{1}{x_i} \right) + a_{n+1} x_{n+1} + \frac{1}{x_{n+1}}$$

*where $\vec{a} = (a_1, \ldots, a_{n+1}) \in (\mathbb{F}_q^*)^{n+1}$. Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_{n+1})$ denote the Teichmüller lifting of $\vec{a}$ in $\overline{\mathbb{Q}}_p^{n+1}$.*

1. *The toric L-function $L^*(f,T)^{(-1)^n}$ has a unique $p$-adic unit root given by*

$$\alpha_0(\hat{a}) = \prod_{i=0}^{d-1} \frac{F_0\left(\pi \hat{a}^{p^i}\right)}{F_0\left(\pi \hat{a}^{p^{i+1}}\right)} \tag{5.2}$$

*where $\pi^{p-1} = -p$ and*

$$F_0(\hat{a}) = \sum_{\substack{u_1, u_2, \ldots, u_{n+2} \in \mathbb{Z}_{\geq 0} \\ u_{n+2} = 2(\sum_{i=1}^{n} u_i) + u_{n+1}}} \frac{\hat{a}_1^{2u_1} \hat{a}_2^{2u_2} \cdots \hat{a}_n^{2u_n} \hat{a}_{n+1}^{u_{n+1}}}{(u_1! \cdots u_n!)^2 (u_{n+1}!)(u_{n+2}!)}. \tag{5.3}$$

*The formula of $\alpha_0(\hat{a})$ converges $p$-adically on the closed unit disc.*

2. *The unit root L-function $L^*_{\text{unit}}(\kappa, T)^{(-1)^n}$ of this family is $p$-adic meromorphic with a unique $p$-adic unit zero given by $\beta_0(\hat{a}) = (\alpha_0(\hat{a}))^\kappa$.*

Note that the function $F_0(\hat{a})$ is generally called hypergeometric function which is a solution to certain hypergeometric system of differential equations. So in fact, the unit zeros of the associated L-functions are closely related to $p$-adically bounded local solutions to certain differential equations. Dwork [16] is the first to obtain $p$-adic analytic formulas for eigenvalues of Frobenius. In particular, one of Dwork's fundamental discoveries [15, 16] is the existence of explicit $p$-adic limit formulas for certain unit roots of zeta functions of certain families of varieties over finite fields. He [16] identifies the reciprocal zeros of the zeta function for the family of K-3 surfaces explicitly by studying $p$-adic solutions of the Picard Fuchs equations. This analysis motivates Dwork's general study of $p$-adic periods and the Frobenius structures of Picard Fuchs equations, identified as Dwork's deformation equations [19].

In the cases arising via Dwork's cohomology theory, there's a semi-linear Frobenius action on the solution space of deformation equation which preserves $p$-adic growth conditions. Dwork [20] conjectures that such strong Frobenius structures of differential equations exist quite widely. This raises the general question of the extent of the domain of definition of a $p$-adic cohomology theory. Actually, Sperber's work [40], together with Wang and Yang's work [45], provide some evidence for Dwork's conjecture where they establish the strong structures of the deformation equations for some specific families of algebraic varieties.

Then a further interesting problem is to compute the Dwork's deformation equation for Type I and Type II families discussed in the previous two chapters. Using the theory of GKZ system [25, 22, 23, 26], one may obtain the specific forms of the deformation equations and check if they support Dwork's conjecture. This would also be helpful to get the explicit analytic formulas for other nontrivial reciprocal zeros of the corresponding L-functions.

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