Title

Deterministic and stochastic fluid-structure interaction

Author
Kuan, Jeffrey

Publication Date
2023

Peer reviewed|Thesis/dissertation

Deterministic and stochastic fluid-structure interaction

by

Jeffrey Kuan

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Sunčica Čanić, Chair
Professor Fraydoun Rezakhanlou
Professor Sung-Jin Oh

Spring 2023

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Abstract

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This thesis will study fluid-structure interaction (FSI), which describes the coupled multi-physical dynamical interaction between fluids and deformable structures. From modeling the flow of blood in compliant elastic arteries to modeling biomedical prostheses and large-scale structures such as wings, bridges, and dams, FSI is prevalent in science, making the rigorous analysis of such coupled fluid-structure systems important for continued technological development and progress in engineering. While prototypical models of FSI involving incompressible, viscous, Newtonian fluids interacting with elastic structures have been well-studied in the literature, the types of FSI models found in present-day real-life applications have unique and interesting features that require new mathematical methods for their analysis. The goal of this thesis will be to develop new tools for studying new complex FSI models of practical importance that extend past work on prototypical models of FSI. Motivated by real-life applications, we will study stochastic FSI systems involving coupled FSI dynamics under the additional influence of random noise in time, and fluid-poroelastic structure interaction (FPSI) which describes FSI systems in which the structure is poroelastic and hence admits fluid flow through its pores. In the study of stochastic FSI, we establish well-posedness for two models: (1) a reduced model where the full stochastic fluid-structure dynamics can be reduced to a single stochastic equation known as the stochastic viscous wave equation and (2) a fully coupled stochastic FSI system involving linear coupling between a Stokes flow through a channel and the stochastically forced elastic walls of the channel, where the full system is described by a stochastic system of PDEs. Next, we study deterministic nonlinearly coupled FPSI and consider a model in which a multilayered poroelastic structure consisting of a thin plate and a thick poroelastic medium, modeled by the Biot equations, interacts with an incompressible fluid modeled by the Navier-Stokes equation. We study well-posedness and consistency of this nonlinearly coupled FPSI model, which is especially challenging since the fluid and poroelastic structure domains are time-dependent and a priori unknown. To the best of our knowledge, the results in this thesis represent the first well-posedness results for stochastic fluid-structure systems and nonlinearly coupled FPSI with moving domains.

To the friends I made along the way

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Acknowledgments

As I write this dissertation and approach the end of my Ph.D., I have had a lot of time to reflect on my time at UC Berkeley, and how I got to where I am today.

I think about the people who have been there to help me and support me at every step of my Ph.D. and more generally at every step of my life. I think about the people who have been there through it all, and who have given me their unconditional support through all of the difficult times. I think about the people who have challenged me to grow in ways that I could have never expected when I first moved out to Berkeley on this journey. And all I can say is that I am eternally grateful: this dissertation is dedicated to all of you. Thank you all so much. No words can express how much I appreciate you all.

I would like to start by thanking my absolutely amazing advisor Professor Sunčica Čanić for her constant and unwavering support during my Ph.D. She has helped me grow as a mathematician and a person in so many ways during my time here at UC Berkeley, and I would definitely not be where I am today without her excellent mentorship and guidance. Professor Čanić never fails to amaze me with her mathematical prowess, her compassion for others, and her dedication to her students. From her mentorship, I have become a more confident, well-rounded mathematician and individual, and I am so grateful for her presence in my life. Thank you Professor Čanič, you are an inspiration!

I would also like to thank my amazing family, my mom, my dad, and my brother Johnathan, for always being there for me. Through all of the challenges, your unconditional support has gotten me through it all. No matter what challenges life may throw at me, I always know that I will have my family, and I know that they will always be there for me. They are amazing, and they are truly my biggest supporters.

I would like to also thank our amazing research group here at UC Berkeley, led by Professor Čanić. Thank you for creating a welcoming, productive, and fun environment for learning. You all are part of some of my best memories here at UC Berkeley. Thank you Professor Franziska Weber for being a friendly, welcoming, and inspiring presence in our research group. I look forward to continuing to learn from you! Thank you Krutika Tawri, for your keen mathematical insights. You are an inspiration to me: I have enjoyed our many discussions and I have learned so much from you this year! Thank you Amy Dai - I have enjoyed talking with you about math and thank you so much for giving me the opportunity to learn new and exciting mathematics alongside you! Finally, thank you Andrew Scharf for your constant support. Through both the ups and the downs, both the times of joy and the times of uncertainty, you have always been there for me, and I am so grateful for your friendship.

I would also like to thank the many professors who have supported me throughout my academic journey. Thank you Professor Boris Muha for all of your help throughout the years. I have learned so much from you and I am constantly inspired by you and your work. Thank you Professor Justin Webster for your kindness, and for always being a warm and welcoming presence throughout my academic journey. I am so grateful for all of your mathematical insights and for all of the advice that you have given me along the way. Thank you Professor Konstantina Trivisa for your help and support throughout the year. It has meant so much to me, and I am looking forward to continuing to work with you! Finally, thank you Professor Sung-Jin Oh and Professor Fraydoun Rezakhanlou for your insights along the way as I have progressed in my research and for serving on my dissertation committee.

Next, I would like to thank the Newman family: Heather, Kim, Jenna, and Loretta. What started off as a friendship from college has developed into a strong unbreakable bond that I know will last for years to come. Thank you for welcoming me into your family - you are all like my second family to me, and I know that I can go to you all with anything. I have shared so many amazing moments with you all and your presence in my life is truly irreplaceable.

Thank you also to the Peachin family: Antonia, Nora, Beate, and Mike. Throughout my years in college and graduate school, your hospitality and kindness has helped me through so much, and I always look forward to seeing you all. I always have such a great time whenever we are all together, and I am so thankful!

I would like to thank my friends who have been at my side throughout my life. In particular, thank you Surin Ahn, Nikhil Ayyala, Kathleen Bryant, Jack Burdick, Richard Chu, Izzy Farrell, Kevin Gu, Allison Halter, Alex Handin, Jeffmin Lin, Amy Mei, Isabel Rose, Aji Sjamsu, Ming-Ming Tran, Jeffrey Wang, and Justin Wong. These are people who have consistently been there for me, and I would not be where I am today without their friendship and support. I have made so many great memories over the years and I have learned so much from you all. I truly value each and every one of you, and your friendship means so much to me. I appreciate you all so much!

I would also like to thank my amazing Math 1A, Math 1B, Math N54, Math 54, Math 10A, and Math 10B students here at UC Berkeley. Teaching at UC Berkeley was an enriching experience, and I am so grateful to have had such amazing, hard-working, and kind students. These students have been so inspiring, and I am so thankful for all of the great times I have had as a graduate student instructor at UC Berkeley.

Finally, I would like to acknowledge the UC Berkeley Math Department faculty and staff for creating an amazing environment for my Ph.D. and I would like to acknowledge the Mathematical Sciences Research Institute (MSRI), for their incredible programs, which have been very informative and interesting. I would also like to acknowledge the National Science Foundation (NSF), which has supported this work through the grants DMS-1613757, DMS-1853340, DMS-2011319, and through funding via grant DMS-1928930 for an MSRI program "Mathematical Problems in Fluid Dynamics" that I participated in during Spring 2021. I would also like to acknowledge UC Berkeley, which supported parts of my Ph.D. work through the UC Berkeley start-up funds.

Chapter 1

Introduction and background

1.1 Fluid-structure interaction and its applications

Fluid-structure interaction (FSI) describes multiphysical systems consisting of fluids and deformable solids interacting dynamically with each other. These coupled fluid-structure systems are *multiphysical systems* in the sense that models of FSI involve partial differential equations (PDEs) describing fluid dynamics which are coupled to partial differential equations describing the elastodynamics of a elastic structure, where the coupling is a *two-way coupling* in which the fluid and structure dynamics mutually influence each other. Because of the two-way coupling between the fluid and structure, rigorously analyzing FSI systems is mathematically challenging, since FSI problems are described by systems of coupled PDEs, which are often of mixed hyperbolic-parabolic type. Furthermore, since the structure in such problems is often deformable, FSI problems can be moving boundary problems with additional geometric nonlinearities arising from the time-dependent fluid domain, which depends on the structure displacement and is hence not known beforehand.

Despite all of these mathematical difficulties associated with the study of FSI, understanding FSI dynamics is essential for advancing science and technology. In particular, FSI is widespread in applications to engineering since the interaction between fluids and structures is a commonplace phenomena in the natural world. Advancements in the mathematical theory and the development of stable and efficient numerical methods for FSI have contributed to significant accomplishments in a variety of applications to engineering. This includes advancements in the accurate simulation of real-life FSI systems in civil and mechanical engineering, and the development of new biomedical technologies that significantly improve standards of patient care. Hence, in addition to being of inherent mathematical interest, FSI is fundamentally relevant to science and engineering, which makes the development of new mathematical methods for analyzing FSI especially crucial for continued progress in science and engineering.

To emphasize the fundamental importance of FSI and the prevalence of coupled fluid-structure systems in science and engineering, we highlight some of the numerous applications of FSI to a broad variety of scientific disciplines below:

Biomedical engineering. Many physiological processes in the human body can be modeled using coupled fluid-structure systems. Fluids, such as blood and air, interact with a variety of tissues which can be modeled as elastic shells or solids. Numerical simulations of FSI have been used to investigate the effects of medical pathologies, such as aneurysms [17, 130] and plaque in arteries [106], on the human body via computational simulations. In recent years, there has also been substantial progress in engineering new biomedical technologies that substantially improve quality of life for patients. The development of robust numerical methods for FSI has played a crucial role in these technologies, as engineers can use simulations to study the effects of biomedical interventions. This allows for the computational study of prostheses in the cardiovascular system, such as prosthetic heart valves [16, 92] and vascular stents [33, 28], and new technologies such as a bioartificial pancreas [26], which is a revolutionary biomedical technology that would eliminate the need for long-term immunosuppressant therapy after organ transplantation.

Mechanical engineering and aeroelasticity. The operation of many mechanical systems involve the interaction between fluids and structures, and hence fluid-structure interaction is important in many mechanical engineering applications. The study of the interaction between solids and fluids has been applied to the simulation of various components of rockets, such as rocket engine nozzles [77], rocket engines [110], and rocket motors [179]. Another major application of fluid-structure interaction is the study of aeroelasticity, which studies the deformations of elastic solid structures in surrounding air flows. Numerical studies of aeroelasticity have included studies of wings, such as the AGARD 445.6 [104] and the ARW-2 wing [71]. Computational studies of insect flight [145] are also another application of aeroelasticity, and more generally, studies of flight involving both rigid and flexible wings have contributed to the study of micro air vehicles [47]. Another significant application of aeroelasticity found in the literature is the study of parachutes, which form a fluid-structure interaction system in which the shape of the parachute derforms under the influence of the surrounding air [76, 127, 168, 169, 176].

Civil engineering. Fluid-structure interaction is also particularly important in civil engineering, where it is important for engineering large-scale structures, such as bridges and dams, which are able to withstand external loading due to environmental factors, such as wind and water, which can be modeled using the equations of fluid dynamics. In these simulations, it is important to take into account the elastic and vibrational properties of these large-scale structures, in order to accurately assess their durability in response to external environmental loading. The study of fluid-structure interaction in the context of bridges interacting with surrounding airflow such as wind has been carried out mathematically in small-scale experiments such as wind tunnel experiments [78] and also numerically in large-scale simulations of bridges [133, 170]. Simulations of the robustness of bridges in response to external loading due to water waves, as a result of natural disasters such as tsunamis [96] and hurricanes [5], have also been conducted in the civil engineering literature. In addition, there has been a lot of work on the interaction between dams and water, for example in surrounding reservoirs, which has been carried out in works such as [2, 4, 46, 59, 122, 147, 178].

Because of the widespread application of FSI to science, developing a robust mathematical theory for quantitatively analyzing fluid-structure interaction systems is of fundamental importance for supporting continued progress in engineering. The mathematical study of FSI is multifaceted, integrating a broad set of mathematical approaches. This includes the **mathematical modeling** of FSI systems, which involves deriving equations describing the dynamics of fluid-structure systems from physical principles and asymptotic analysis. Upon obtaining such models, one can then study the rigorous mathematical **analysis** of the systems of fully coupled PDEs that describe these physical systems, by examining mathematical issues such as well-posedness (existence and uniqueness of weak or strong solutions) and long-time behavior of these systems. Upon obtaining models of fully coupled fluid-structure systems that are well-posed, one can then work on developing robust, stable, and accurate **numerical schemes** for numerically solving for the solutions to FSI dynamics. These numerical schemes are particularly valuable for practical applications, as they can be used to simulate real-life technologies involving FSI. In this thesis, the emphasis will be on the mathematical analysis of FSI systems, in terms of showing well-posedness for complex FSI systems that are inspired by real-life applications to engineering.

Before discussing the types of more complex FSI models that we will consider in this thesis, we will start by describing a *prototypical model of FSI*, consisting of an incompressible, viscous fluid interacting with an elastic plate or membrane. While this model has been extended to more complex settings, this prototypical model is a particularly useful benchmark problem for FSI and this prototypical model was where much of the work on the mathematical analysis of FSI began. In this prototypical model of FSI, the coupling between the fluid and structure can be of two types. First, the fluid and structure can be **linearly coupled**, which is a linearization of the full moving-boundary FSI problem with a time-dependent fluid domain around the state of zero structure displacement, where the fluid equations are evaluated on a fixed, reference fluid domain and the coupling between the fluid and structure occurs on the fixed, rather than time-dependent, fluid-structure interface. Second, the fluid and structure can be **nonlinearly coupled**, which gives rise to moving-boundary problems involving time-dependent fluid domains that are not known beforehand and must be solved for as part of the FSI problem. In particular, in the real-life dynamics of such a prototypical FSI system, the structure displacement at any given time determines the time-dependent fluid domain, and since the structure displacement is a priori unknown, the moving (time-dependent) fluid domain is not known a priori either. Hence, the equations for the dynamics of the fluid in nonlinearly coupled models of FSI are posed on a moving time-dependent fluid domain that is also an unknown of the problem, and the coupling conditions between the fluid and structure are evaluated on the time-dependent interface between the fluid and the structure. This gives rise to additional nonlinearities originating from the changing (time-dependent) geometry, which makes these nonlinearly coupled moving boundary problems particularly challenging.

While there have been many techniques developed for solving FSI problems both in the linearly coupled and nonlinearly coupled cases, one particular method for solving these fully coupled problems that has been particularly robust in handling both the prototypical model and its numerous extensions is the idea of *Lie operator splitting*. This involves semi-discretizing the problem in time, and splitting the fully coupled problem into a structure and a fluid subproblem on each time step, in order to construct approximate solutions. However, not every such splitting into a structure and fluid subproblem would necessarily work, and one must be careful to precisely split the problem in an appropriate way so as to respect the energy of the problem, in order to produce a stable scheme. In this sense, developing a proper splitting scheme is often the main challenge in this approach. However, this splitting scheme approach remains particularly useful because it explicitly constructs weak solutions to the FSI problem. *This thesis will discuss extensions of this splitting scheme approach to complex FSI problems involving stochasticity and more complex structures, such as poroelastic structures.* Hence, as a starting point, in the remainder of this section, we will first describe the prototypical benchmark FSI model in both the linearly coupled and nonlinearly coupled case, discuss the literature that has been done for these models, and finally outline the method of constructive existence for these models via Lie operator splitting.

1.2 A linearly coupled prototypical FSI model

As a preliminary step in the analysis of fluid-structure interaction, linearly coupled models of FSI were first analyzed, where the fluid domain is assumed to be fixed in time, so that the fluid equations are posed on a fixed reference fluid domain Ω_f, even though the structure is assumed to displace. Therefore, the fluid equations are formulated on a fixed domain that is not time-dependent, with the fluid-structure coupling occurring on a fixed fluid-structure interface, rather than a moving interface. Although these linearly coupled models are not moving-boundary problems, the analysis of these linearly coupled FSI models is still challenging due to the two-way coupling between the fluid and structure and the multiphysical nature of these problems. In this section, we begin by reviewing the important results in the study of linearly coupled models of FSI. We will then describe a prototypical model of linearly coupled FSI and describe a Lie operator splitting approach to analyzing this linearly coupled model. Though the Lie operator splitting approach was only later used in fluid-structure interaction in the context of moving boundary nonlinearly coupled models, see for example [30, 140], we summarize the application of the splitting scheme approach to the prototypical linearly coupled FSI model, since these splitting schemes will be important in analyzing linearly coupled models presented later in this thesis.

Literature review

The first mathematically rigorous works on linearly coupled fluid-structure interaction involve fluids and structures interacting with each other, where the fluids and structure are in domains of these same dimension. We will summarize the fundamental developments in linearly coupled models of FSI of this type, but note that the prototypical model that we will consider in the remainder of this section is of a slightly different type, where the structure is a lower-dimensional membrane that is on the boundary of the fluid domain.

The study of linearly coupled fluid-structure interaction was initiated in the work [66], where the linear Stokes equations posed on a fluid domain interact with the equations of linear elasticity on a solid domain, where the fluid and solid domain are in contact with each other along a common interface, and are both of the same dimension, either 2D or 3D. At the interface, there is a kinematic coupling condition prescribing matching of the fluid and structure velocity, and there is a dynamic coupling condition that prescribes continuity of normal stresses. This work proves well-posedness from the perspective of weak solutions, by using a Galerkin method and passing to the limit, and it shows that given initial structure displacement, structure velocity, and fluid velocity in $H^1(\Omega_s)$, $H^1(\Omega_s)$, and $H^1(\Omega_f)$ respectively, there exists a solution where the structure displacement, structure velocity, and fluid velocity are bounded in $H^1(\Omega_s)$, $L^2(\Omega_s)$, and $L^2(\Omega_f)$ for all time, while the fluid velocity is also in $L^2(0, T; H^1(\Omega_f))$. It is shown that with more regular initial data, one can obtain improved regularity for the solution, in addition to being able to recover the fluid pressure, which vanishes in the weak formulation when using divergence-free test functions.

After this initial work by [66], there was a growth in interest in linearly coupled models, where the solid and fluid have the same spatial dimension, specifically models in which an elastic solid is immersed in a fluid. For concreteness, in these models, we will refer to the structure displacement by $\boldsymbol{\eta}$ on the reference structure domain Ω_s, we will refer to the fluid velocity by $\boldsymbol{u}$ on the reference fluid domain Ω_f, and we will denote the reference configuration of the fluid-structure interface by Γ. In the work [10], a solid modeled by the linear damped wave equation $\boldsymbol{\eta}_{tt} - \Delta\boldsymbol{\eta} + \boldsymbol{\eta} = 0$ on Ω_s is immersed in a fluid modeled by the linear Stokes equations, where the solid and fluid are both 2D or are both 3D. The kinematic coupling condition is a no-slip condition

$$\boldsymbol{u} = \boldsymbol{\eta}_t, \qquad \text{on } \Gamma,$$

and the dynamic coupling condition is

$$\frac{\partial \boldsymbol{u}}{\partial \nu} - \frac{\partial \boldsymbol{\eta}}{\partial \nu} = p\nu, \quad \text{on } \Gamma, \tag{1.1}$$

where ν is the normal vector along Γ. As described in [10], this work improves upon the past results in [66], where an initial structure displacement, structure velocity, and fluid velocity in $H^1(\Omega_s)$, $H^1(\Omega_s)$, and $L^2(\Omega_f)$ give rise to a solution that is bounded in time in the weaker spaces $H^1(\Omega_s)$, $L^2(\Omega_s)$, and $L^2(\Omega_f)$. In contrast, this work [10] does not have this decrease in regularity from the initial data to the solutions. The main result of this work is that the linearly coupled problem has an associated evolution that is given by a strongly continuous C_0 semigroup, so that initial data in a finite energy space $\mathcal{H}$ give rise to solutions that are continuous in $\mathcal{H}$. In addition, the semigroup is strongly stable on $\mathcal{H}$ quotiented out by a one-dimensional space of stationary solutions, meaning that on this factor space, the operator norm of the semigroup converges to zero as $t \to \infty$, which implies some long-time decay of solutions. These results were obtained by reformulating the initial fluid-structure interaction problem as an abstract evolution equation, where the fluid pressure is eliminated from the equations by rewriting it using operators applied to the fluid velocity and structure displacement. The study of well-posedness of this system was continued in a later work [7], where it is shown that the solution possesses additional regularity if one assumes that the initial data is in the domain of the generator $\mathcal{A}$ of the associated semigroup and the initial structure displacement is in $H^2(\Omega_s)$, so that the initial data is more regular.

These asymptotic stability results about immersed structures in fluids were later improved in [11], where the same model as in [10] is considered, but instead of a no-slip condition $\boldsymbol{u} = \boldsymbol{\eta}_t$ on Γ, there is a boundary dissipation condition,

$$\boldsymbol{u} = \boldsymbol{\eta}_t + \alpha \frac{\partial \boldsymbol{\eta}}{\partial \nu} \quad \text{on } \Gamma, \tag{1.2}$$

where $\alpha = 1$ for concreteness. For this problem with boundary dissipation, the previous work [10] shows that the associated semigroup is strongly stable, so that the operator norm of the semigroup maps goes to zero as t goes to infinity. The work in [11] improves this result by showing *uniform stability* of the semigroup, which means that the operator norm of the semigroup maps more specifically decays *exponentially* as t goes to infinity.

We note that these past results on elastic structures immersed in fluids [7, 10, 11] involve damped wave equations. These results were generalized to elastic structures modeled by elasticity equations, specifically the Lamé equations of linear elasticity, interacting with incompressible Stokes flows in later works [8, 9]. These models involve boundary dissipation in the kinematic coupling condition so that

$$\boldsymbol{\eta}_t - \varphi \boldsymbol{\sigma}(\boldsymbol{\eta}) \cdot \boldsymbol{\nu}|_\Gamma = \boldsymbol{u} \quad \text{on } \Gamma,$$

where $\varphi \geqslant 0$ is a sufficiently regular and nonnegative function, for example in $C^1(\Gamma)$, and $\boldsymbol{\sigma}(\boldsymbol{\eta})$ is an elasticity stress tensor for the structure. The dynamic coupling condition, instead of being the simpler condition (1.1), is now the full continuity of normal stress between the fluid and structure:

$$2\boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\nu} = \boldsymbol{\sigma}(\boldsymbol{\eta}) \cdot \boldsymbol{\nu} + p\boldsymbol{\nu}, \qquad \text{on } \Gamma,$$

where $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla(\boldsymbol{u}) + (\nabla \boldsymbol{u})^t)$ is the symmetrized gradient for the fluid velocity. We refer the reader to [9] for the full details of the Stokes-Lamé model. In [9], it is shown that this Stokes-Lamé system has an associated strongly continuous C_0 semigroup, which establishes well-posedness, and strong stability properties of the semigroup are established. As was done for the model involving just the damped wave equation in [11], these strong stability results for the Stokes-Lamé system in [9] were strengthened to results on uniform stability of the semigroup in the work [8], in the sense of exponential decay of the operator norms of the semigroup, when the boundary dissipation function φ is strictly positive. The boundary dissipation function $\varphi > 0$ in this case is the analogue of the constant $\alpha = 1$ in (1.2) for the related model involving a linear damped wave equation interacting with Stokes flow.

Nonlinear variants of these models were also considered, in which the Navier-Stokes equations with the nonlinear advection term are considered for the fluid interacting with the immersed solid. We emphasize that these models are still *linearly coupled* because the Navier-Stokes equations are posed on a fixed reference fluid domain and the fluid-structure coupling takes place at a fixed reference fluid-structure interface. This model was considered in [12], where the continuity of normal stresses is modified into a *transmission boundary condition* in order to obtain energy balance, due to the effect of the nonlinear advection term in the Navier-Stokes equations and the fact that the nonlinear Navier-Stokes equations interacting with a deformable structure are being posed on a fixed reference fluid domain. In this work, existence of weak solutions which are defined globally in time is established in finite energy spaces by using semigroup methods and microlocal analysis of hyperbolic equations. The proof considers an auxiliary related nonlinear problem where the nonlinearity is of a suitably tractable form, and this auxiliary problem is solved using semigroup methods. The initial problem can be solved by truncating the Navier-Stokes advection nonlinearity to reduce the initial problem to this auxiliary problem, and then passing to the limit in the truncation parameter. However, an additional difficulty here is that to show that the solutions which are solutions in the semigroup (mild) formulation are also weak solutions satisfying a variational formulation, we must be able to interpret appropriate terms in the weak formulation, such as the trace of the normal stress of the structure on the boundary. While the finite energy space does not have a structure displacement and velocity with sufficient regularity to classically define a trace of the normal stress of the structure along the fluid-structure interface Γ, the work in [12] shows that one can use microlocal analysis to establish a *"hidden regularity"* result where if we are given initial structure displacement in $H^1(\Omega_s)$, initial structure velocity in $L^2(\Omega_s)$, and forcing in $L^2(0, T; H^{1/2}(\Omega_s))$, then the corresponding solution to the linear elasticity equation has a normal stress with a well-defined trace on the boundary in $L^2(0, T; H^{-1/2}(\Gamma))$. We remark that this "hidden regularity" result has been fundamental to the development of methods for linearly coupled FSI, as it has resulted in well-posedness results that do not require the addition of (potentilly artificial) higher order terms in the structure equations that regularize the structure dynamics.

The consideration of this linearly coupled model consisting of the nonlinear Navier-Stokes model interacting with an elastic immersed structure is continued in [13], where the existence and uniqueness of strong solutions is considered in 2D and 3D. In terms of strong solutions, there are global unique solutions in 2D, and in 3D, given an initial fluid velocity, structure displacement, and structure velocity in $H^2(\Omega_f)$, $H^2(\Omega_s)$, and $H^1(\Omega_s)$ respectively, one can obtain local existence of strong solutions. It is important to note also that because these solutions are strong, an important component of the analysis in [13] involves obtaining improved regularity estimates that will allow for the recovery of the fluid pressure function. These results on local existence of strong solutions were later improved in subsequent works [120], [118], and [119]. In [120], a variant of the model in [13] is considered with "slab" domains containing flat boundaries, and in this context, the results from [13] are improved. In [120], the regularity of the initial fluid velocity is reduced from H^2 to H^1 and later, in [118], the regularity of the initial structure displacement and velocity is reduced to $H^{\frac{3}{2}+k}$ and $H^{\frac{1}{2}+k}$ for sufficiently small $k > 0$ by using "hidden regularity" and fractional estimates. Finally, in [119], these well-posedness results for strong solutions are extended beyond the model with "slab" domains with a flat fluid-structure interface that was previously found in [120, 118] to a more general geometry involving a potentially non-flat fluid-structure interface.

Problem description

We start by describing a linearly coupled prototypical model, involving the interaction between a two-dimensional incompressible Newtonian fluid and a one-dimensional elastic structure. Define the two-dimensional reference domain for the fluid by

$$\Omega_f = [0, L] \times [0, R],$$

which represents half of a two-dimensional channel with elastic walls which contains the flow of an incompressible fluid. We will represent points in the two-dimensional fluid domain Ω_f using the coordinates $(z, r) \in \Omega_f$. The elastic channel will be open at the ends, so that we allow for inlet and outlet flow through

$$\Gamma_{in} = \{0\} \times [0, R] \qquad \text{and} \qquad \Gamma_{out} = \{L\} \times [0, R]$$

respectively. The bottom boundary

$$\Gamma_b = [0, L] \times \{0\},$$

represents the central axis of the channel, which will be a line of even symmetry, so that the dynamics on Ω_f can be reflected across the line Γ_b to retrieve the full (symmetric) dynamics in the full elastic channel. We represent the elastic walls of the channel using the top boundary, defined by

$$\Gamma = [0, L] \times \{R\},$$

which will be the reference configuration of an elastic structure representing the deformable walls of the channel. Hence, Γ is the reference configuration of the fluid-structure interface, which represents the physical location of the fluid-structure interface when the displacement of the elastic structure from its reference configuration is zero. See Figure 1.1.

Description of the fluid and structure dynamics

We now describe the subproblems for the fluid and structure, by explicitly specifying the partial differential equations governing the dynamics of the fluid and structure separately.

Structure subproblem. We will describe the elastic walls of the channel, with reference configuration Γ, by specifying the displacement of the structure from its reference configuration Γ. While one can consider general vector-valued displacements from the reference

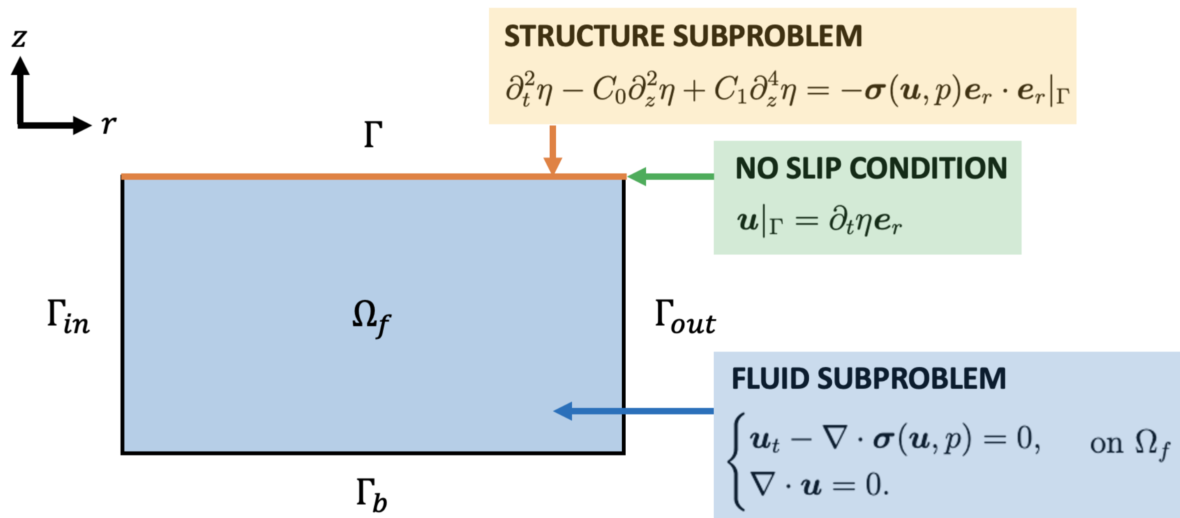

Figure 1.1: *The domain for the linearly coupled prototypical FSI problem in two dimensions, describing the interaction between a fluid modeled by the linear Stokes equation and an elastic structure. Though the structure displaces from its reference configuration, by the linear coupling, we evaluate the interface conditions on the reference fluid-structure interface* Γ *and pose the linear Stokes equations for the fluid on the fixed fluid reference domain* Ω_f.

configuration such as $\boldsymbol{\eta} : \Gamma \to \mathbb{R}^2$, we will consider a simplified case where the structure is assumed to displace only in the radial direction. In this case, we can keep track of just the *scalar* displacement of the structure in the radial direction from its reference configuration, which we will denote by the scalar function $\eta : \Gamma \to \mathbb{R}$. To describe the dynamics of the elastic structure, we will use a partial differential equation describing a Koiter shell to specify the evolution of the radial displacement of the elastic structure:

$$\partial_t^2\eta - C_0\partial_z^2\eta + C_1\partial_z^4\eta = F \qquad \text{on } \Gamma, \tag{1.3}$$

where C_0 and C_1 are positive (constant) elasticity coefficients and F is the source term describing the external load on the structure, to be specified later in the coupling conditions. We will assume that the ends of the elastic walls are clamped so that

$$\eta(0) = \eta(L) = \partial_z\eta(0) = \partial_z\eta(L),$$

where $z = 0$ and $z = L$ represent the left and right endpoints of Γ. We remark that because the structure equation is a fourth order equation in space, we need a total of four boundary conditions, which is why we must specify that both η and $\partial_z\eta$ are equal to zero at the left and right endpoints $z = 0$ and $z = L$ of Γ. For the structure subproblem, we are given initial data for the initial displacement $\eta_0 \in H_0^2(\Gamma)$ and the initial structure velocity $v_0 \in L^2(\Gamma)$.

Fluid subproblem. We model the fluid using the linear Stokes equations describing the dynamics of an incompressible, viscous, Newtonian fluid with constant density in terms of the fluid velocity $\boldsymbol{u} : \Omega_f \to \mathbb{R}^2$ and the fluid pressure $p : \Omega_f \to \mathbb{R}$:

$$\begin{cases} \boldsymbol{u}_t - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = 0, \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases}$$

The first balance of momentum equation describes the dynamics of the fluid via Newton's second law, with the Cauchy stress tensor being defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu_f \boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}, \qquad \text{where } \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}[(\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^t]$$

and $\mu_f > 0$ is the (constant) fluid viscosity. The second equation (known as the incompressibility condition) describing conservation of mass is an infinitesimal description of the fact that the flow (under the assumption of constant fluid density, as we assume here) preserves volume. We are given initial data $\boldsymbol{u}_0 \in L^2(\Omega_f)$ for the linear Stokes equations.

Next, we define the boundary conditions for the fluid subproblem. Because we are allowing for flow through the inlet and outlet of the channel, we will set

$$p|_{\Gamma_{in}} = P_{in}(t) \qquad \text{and} \qquad p|_{\Gamma_{out}} = P_{out}(t),$$

where $P_{in}(t)$ and $P_{out}(t)$ are time-dependent inlet and outlet pressure data, which drive the flow of the fluid through the channel. Furthermore, we will assume that the direction of the flow at the inlet and outlet is purely horizontal so that

$$u_r = 0, \qquad \text{on } \Gamma_{in} \cup \Gamma_{out}.$$

On the bottom boundary, because Γ_b is a line of even symmetry, we impose the following boundary conditions along Γ_b for the flow velocity $\boldsymbol{u}$, which essentially say that the flow velocity respects the even symmetry at the central axis Γ_b:

$$u_r = 0 \quad \text{and} \quad \partial_r u_z = 0, \qquad \text{on } \Gamma_b.$$

The coupling conditions. Now that we have separately described the fluid and structure dynamics, we need to couple them together to get a fully coupled FSI system. We emphasize that the coupling in this FSI system is a *two-way coupling*, in the sense that the motion of the structure and the dynamics of the fluid both mutually affect each other. There are two types of coupling conditions: one which specifies continuity of displacements (the kinematic coupling condition) and one which specifies the dynamic loading on the structure (the dynamic coupling condition).

We remark that in most of the FSI systems that we will consider, there will be two-way coupling described by a set of kinematic coupling conditions and dynamic coupling conditions, which need to be specified in such a way that allows for a proper energy estimate while respecting certain physically relevant considerations. In this prototypical model of linearly coupled FSI, we have the following coupling conditions, which are evaluated along the fixed reference fluid-structure interface Γ:

- **Kinematic coupling condition.** This describes continuity of the velocity at the interface, via the *no-slip condition*, which states that the particles of the fluid along the elastic walls have the same velocity as the moving structure. Since we assume that the structure displaces in only the radial direction, this kinematic coupling no-slip condition takes the form:
$$\boldsymbol{u}|_{\Gamma} = \partial_t \eta \boldsymbol{e}_r.$$

- **Dynamic coupling condition.** This condition specifies the fluid load on the structure. In this FSI system, the motion of the elastic structure is driven by the forcing exerted on the structure by the fluid. Mathematically, this amounts to specifying the source term on the elasticity equation describing the structure elastodynamics in (1.3). We specify the external fluid load on the structure via the Cauchy stress tensor of the fluid as
$$F = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{e}_r \cdot \boldsymbol{e}_r|_{\Gamma}.$$
Physically, this force F represents the radial component of the force that the fluid exerts on the elastic structure at the interface, where we are considering the fixed reference fluid-structure interface Γ due to the linear coupling. We take only the radial component of this force since we are assuming that the elastic walls of the channel are constrained to move in only the radial direction for simplicity. Therefore, the equation for the elastic structure reads
$$\partial_t^2 \eta - C_0 \partial_z^2 \eta + C_1 \partial_z^4 \eta = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{e}_r \cdot \boldsymbol{e}_r|_{\Gamma}.$$

A priori energy estimate and weak formulation

Our next step will be to formulate the definition of a weak solution to the linearly coupled prototypical FSI problem. This is done in a classical way, by testing against sufficiently regular test functions and integrating by parts, in order to move derivatives from the solution itself to the sufficiently regular test function. However, before deriving the *weak formulation* that a weak solution must satisfy, it is useful to first do an **a priori energy estimate**, which uses a formal calculation to show that we should expect the total energy of the system to be controlled by the initial and boundary data. Obtaining an energy estimate validates, at least on a preliminary level, that the mathematical model that we are considering is physically reasonable. In addition, obtaining an energy estimate gives valuable information about what *finite energy function spaces* we expect our weak solutions to belong to.

For the linearly coupled prototypical FSI model, we do not provide the details of the computation of the energy estimate, as a similar computation will be done in a later chapter, see Section 4.5 in Chapter 4. To get the energy estimate, we would test the structure equation by η_t, test the fluid equation by $\boldsymbol{u}$, and relate the two resulting sets of equations using the dynamic coupling condition. Define the norm

$$||\eta||_E^2 := \frac{1}{2} C_0 ||\partial_z \eta||^2_{L^2(\Gamma)} + \frac{1}{2} C_1 ||\partial_z^2 \eta||^2_{L^2(\Gamma)}. \tag{1.4}$$

Then, we obtain the following energy estimate for all $t \in [0, T]$, up to a fixed but arbitrary final time T:

$$\frac{1}{2}||\partial_t \eta(t)||^2_{L^2(\Gamma)} + ||\eta(t)||^2_E + \frac{1}{2}||\boldsymbol{u}(t)||^2_{L^2(\Omega_f)} + \mu_f \int_0^t ||\boldsymbol{D}(\boldsymbol{u})(s)||^2_{L^2(\Omega_f)} ds$$
$$\leqslant \frac{1}{2}||v_0||^2_{L^2(\Gamma)} + ||\eta_0||^2_E + \frac{1}{2}||\boldsymbol{u}_0||^2_{L^2(\Omega_f)} + C\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}||^2_{L^2(0,T)}\right),$$

where the constant C in the estimate above is independent of the final time T and depends only on the pre-specified geometry of the problem.

From this energy estimate, we can define the finite energy spaces for our weak solution, by noting which norms must be bounded as a result of energy considerations. For example, from the estimate above, we see that formally, we would expect the structure displacement η to have the norms $||\partial_t \eta(t)||^2_{L^2(\Gamma)}$ and $||\eta(t)||^2_E$ be bounded for (almost every) $t \in [0, T]$. Hence, we would define the finite energy solution space for the structure displacement η to be

$$V_s = W^{1,\infty}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; \mathcal{V}_s),$$

where V_s is the following fixed-time solution space which incorporates the clamped boundary conditions for the structure:

$$\mathcal{V}_s = H^2_0(\Gamma). \tag{1.5}$$

Similarly, we define the solution space for the fluid. For each fixed time, we expect the solution to take values in the following space of divergence-free H^1 functions respecting the boundary conditions of the fluid subproblem:

$$\mathcal{V}_f = \{\boldsymbol{u} \in H^1(\Omega_f) : \nabla \cdot \boldsymbol{u} = 0, u_r = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_b, u_z = 0 \text{ on } \Gamma\}. \tag{1.6}$$

Then, from the energy estimate, we have the following finite energy space for the fluid velocity $\boldsymbol{u}$:

$$V_f = L^\infty(0, T; L^2(\Omega_f)) \cap L^2(0, T; \mathcal{V}_f).$$

Now that we have derived an energy estimate and the finite energy solution spaces, we can define the weak formulation to the problem. We consider a test function $(\boldsymbol{q}, \psi)$ where $\boldsymbol{q}$ is the test function for the fluid velocity and ψ is the test function for the structure velocity, and where $\boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r$ (which is the kinematic coupling condition imposed on the test functions). We multiply the fluid equation by $\boldsymbol{q}$ and we multiply the structure equation by ψ, and then we integrate by parts. In fact, many of these calculations will be similar to the corresponding calculations used to obtain the a priori energy estimate. We then obtain that a weak solution to the linearly coupled prototypical FSI problem satisfies the following weak formulation.

Definition 1.2.1. A fluid velocity $\boldsymbol{u} \in V_f$ and a structure displacement $\eta \in V_s$ are a weak solution to the linearly coupled FSI problem if $\eta(0) = \eta_0$ and furthermore, for all test functions $(\boldsymbol{q}, \psi) \in C_c^\infty([0,T); \mathcal{V}_s \times \mathcal{V}_f)$ such that $\boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r$, the following weak formulation holds:

$$-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} + 2\mu_f \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) - \int_0^T \int_\Gamma \partial_t \eta \cdot \partial_t \psi + C_0 \int_\Gamma \nabla \eta \cdot \nabla \psi$$
$$+ C_1 \int_\Gamma \Delta \eta \cdot \Delta \psi = \int_0^T \int_{\Gamma_{in}} P_{in}(t) \cdot q_z - \int_0^T \int_{\Gamma_{out}} P_{out}(t) \cdot q_z + \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) + \int_\Gamma v_0 \cdot \psi(0).$$

If we substitute $(\boldsymbol{u}, \partial_t \eta)$ for the test function $(\boldsymbol{q}, \psi)$ in the weak formulation, note that we formally recover the a priori energy estimate above.

Construction of approximate solutions via Lie operator splitting

This linearly coupled FSI problem has a global weak solution, and we state this result in the following theorem.

Theorem 1.2.1. Suppose we have initial data $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $\eta_0 \in H_0^2(\Gamma)$, and $v_0 \in L^2(\Gamma)$. Then, there exists a unique weak solution $(\boldsymbol{u}, \eta)$ to the linearly coupled fluid-structure interaction problem above.

The method for showing uniqueness of weak solutions is handled using energy-type arguments, where we have to carefully consider the regularity of the weak solution and the test functions. We will not handle the proof of uniqueness now, as we will demonstrate the mathematical methodology for establishing uniqueness of weak solutions later in Chatper 4 (specifically, see Lemma 4.9.1), so we will instead focus on the existence result for now. In order to show this existence result, we use a splitting scheme that semidiscretizes in time and splits the structure and fluid apart into two separate subproblems on each time step, in such a way that the energy balance of the resulting scheme resembles the energy balance of the continuous problem, hence resulting in a stable scheme. We remark that this splitting must be done in a particular way in order to obtain a stable scheme, as one must carefully take into account the two-way coupling between the fluid and structure when splitting them apart from each other.

The splitting scheme that is used for this prototypical linearly coupled FSI model is as follows. We set a final target time T and for each positive integer N, we discretize the full time interval $[0,T]$ into N subintervals $[t_n, t_{n+1}]$ of length $\Delta t = T/N$, where the endpoints are defined by $t_n = n\Delta t$. On each time interval $[t_n, t_{n+1}]$, we run two subproblems, a structure subproblem and a fluid subproblem, and update an approximate vector

$$\begin{pmatrix} \eta_N^{n+\frac{i}{2}} \\ v_N^{n+\frac{i}{2}} \\ \boldsymbol{u}_N^{n+\frac{i}{2}}, \end{pmatrix}$$

consisting of the approximate state of the structure displacement, the structure velocity, and the fluid velocity at that current time step. Each of the subproblems, which we will now describe, will handle certain terms in the full weak formulation, so that when the weak formulations for each of the two subproblems are combined, we obtain a semidiscretized version of the full (continuous) weak formulation.

The structure subproblem. For the structure subproblem, we solve the weak formulation of the elasticity equation with zero external forcing. We keep the fluid velocity from the previous subproblem

$$\boldsymbol{u}_N^{n+\frac{1}{2}} = \boldsymbol{u}_N^n$$

and update the structure displacement $\eta_N^{n+\frac{1}{2}}$ and the structure velocity $v_N^{n+\frac{1}{2}}$. This entails finding $(\eta_N^{n+\frac{1}{2}}, v_N^{n+\frac{1}{2}}) \in H_0^2(\Gamma) \times H_0^2(\Gamma)$ such that the following system is solved in the weak formulation:

$$\begin{cases} \displaystyle\int_\Gamma \frac{\eta_N^{n+\frac{1}{2}} - \eta_N^n}{\Delta t}\varphi dz = \int_\Gamma v_N^{n+\frac{1}{2}}\varphi dz, \qquad \text{for all } \varphi \in L^2(\Gamma), \\ \displaystyle\int_\Gamma \frac{v_N^{n+\frac{1}{2}} - v_N^n}{\Delta t}\psi + C_0\int_\Gamma \partial_z\eta_N^{n+\frac{1}{2}}\partial_z\psi + C_1\int_\Gamma \partial_z^2\eta_N^{n+\frac{1}{2}}\partial_z^2\psi = 0, \qquad \text{for all } \psi \in H_0^2(\Gamma). \end{cases} \tag{1.7}$$

The fluid subproblem. For the fluid subproblem, we update both the structure velocity v_N^{n+1} and the fluid velocity $\boldsymbol{u}_N^{n+1}$, and we keep the structure displacement from the previous subproblem so that

$$\eta_N^{n+1} = \eta_N^{n+\frac{1}{2}}.$$

While the structure velocity is a quantity associated with the structure rather than the fluid, one must update both to obtain a stable scheme, since the structure velocity affects the fluid velocity through the kinematic coupling condition (since the trace of the fluid velocity along the fluid-structure interface Γ is the structure velocity). For the fluid subproblem, we want to find $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}_f \times L^2(\Gamma)$ satisfying $\boldsymbol{u}_N^{n+1}|_\Gamma = v_N^{n+1}$ such that

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{1}{2}}}{\Delta t}\cdot \boldsymbol{q} + 2\mu_f\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{1}{2}}}{\Delta t}\psi$$
$$= \int_{\Gamma_{in}} P_{N,in}^n q_z - \int_{\Gamma_{out}} P_{N,out}^n q_z,$$

where $P_{N,in}^n$ and $P_{N,out}^n$ are the numerical inlet and outlet pressures

$$P_{N,in/out}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P(t)dt.$$

Semidiscrete weak formulation and discrete energy estimates. On each time step, the approximate structure displacement, structure velocity, and fluid velocity satisfy the following semidiscrete weak formulation, obtained by adding the weak formulations for each of the structure and fluid subproblems:

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} + 2\mu_f \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t}\psi$$
$$+ C_0 \int_\Gamma \partial_z v_N^{n+\frac{1}{2}} \partial_z \psi + C_1 \int_\Gamma \partial_z^2 v_N^{n+\frac{1}{2}} \partial_z^2 \psi = \int_{\Gamma_{in}} P_{N,in}^n q_z - \int_{\Gamma_{out}} P_{N,out}^n q_z. \quad (1.8)$$

We establish discrete energy estimates for the approximate solutions on each time step, which can be derived from the weak formulations for the structure and fluid subproblems above. Define the discrete energy $E_N^{n+\frac{i}{2}}$ and the discrete dissipation D_N^n by

$$E_N^{n+\frac{i}{2}} = \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}_N^{n+\frac{i}{2}}|^2 + \frac{1}{2}\int_\Gamma |v_N^{n+\frac{i}{2}}|^2 + \frac{1}{2}C_0 \int_\Gamma |\partial_z \eta_N^{n+\frac{i}{2}}|^2 + \frac{1}{2}C_1 \int_\Gamma |\partial_z^2 \eta_N^{n+\frac{i}{2}}|^2.$$

$$D_N^n = \mu_f(\Delta t)\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^n)|^2.$$

We then obtain the following discrete energy estimates for the structure subproblem

$$E_N^{n+\frac{1}{2}} + \frac{1}{2}\int_\Gamma |v_N^{n+\frac{1}{2}} - v_N^n|^2 + \frac{1}{2}C_0 \int_\Gamma |\partial_z(\eta_N^{n+\frac{1}{2}} - \eta_N^n)|^2 + \frac{1}{2}C_1 \int_\Gamma |\partial_z^2(\eta_N^{n+\frac{1}{2}} - \eta_N^n)|^2 = E_N^n,$$

and the fluid subproblem

$$E_N^{n+1} + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{1}{2}}|^2 + \frac{1}{2}\int_\Gamma |v_N^{n+1} - v_N^{n+\frac{1}{2}}|^2$$
$$\leqslant E_N^{n+\frac{1}{2}} + C\left(||P_{in}(t)||^2_{L^2(n\Delta t,(n+1)\Delta t)} + ||P_{out}(t)||^2_{n\Delta t,(n+1)\Delta t)}\right),$$

where the additional terms that are not already included in $E_N^{n+\frac{i}{2}}$ and D_N^n are numerical dissipation terms, which arise as a result of the discretization in time.

Approximate solutions and uniform boundedness. The approximate quantities $\boldsymbol{u}_N^{n+\frac{i}{2}}$, $\eta_N^{n+\frac{i}{2}}$, and $v_N^{n+\frac{i}{2}}$ are defined on each time step, so in order to construct approximate solutions, we must glue together these approximate solutions on the individual time steps. We do this using piecewise constant functions and linear interpolations, as described below.

We define the following piecewise constant approximate solutions η_N, v_N, v_N^*, and $\boldsymbol{u}_N$, which are constant on each time step $[n(\Delta t), (n+1)\Delta t)$, with

$$\eta_N(t) = \eta_N^{n+1}, \quad v_N(t) = v_N^{n+1}, \quad v_N^* = v_N^{n+\frac{1}{2}}, \quad \boldsymbol{u}_N = \boldsymbol{u}_N^{n+1}, \quad \text{if } t \in [n(\Delta t), (n+1)\Delta t).$$

We also define the approximate solutions $\overline{\eta}_N$, $\overline{v}_N$, and $\overline{\boldsymbol{u}}_N$ by linearly interpolating values at the points $t_n = n\Delta t$, where the values of these linearly interpolated approximate solutions at these discrete time values t_n are

$$\overline{\eta}_N(n\Delta t) = \eta_N^n, \quad \overline{v}_N(n\Delta t) = v_N^n, \quad \overline{\boldsymbol{u}}_N(n\Delta t) = \boldsymbol{u}_N^n, \qquad \text{for } n = 0, 1, ..., N.$$

We remark that $\partial_t \overline{\eta}_N = v_N^*$. From the discrete energy estimates, we obtain the following result, which shows that the approximate solutions are uniformly bounded in the finite energy spaces.

Proposition 1.2.1. We have the following uniform boundedness results for the approximate solutions, uniformly in N.

1. **Structure displacement.** η_N is uniformly bounded in $L^\infty(0,T; H_0^2(\Gamma))$ and $\overline{\eta}_N$ is uniformly bounded in $W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H_0^2(\Gamma))$.
2. **Fluid velocity.** $\boldsymbol{u}_N$ is uniformly bounded in $L^\infty(0,T; L^2(\Omega_f)) \cap L^2(0,T; H^1(\Omega_f))$.
3. **Structure velocity.** v_N^* is uniformly bounded in $L^\infty(0,T; L^2(\Gamma))$. Using the continuous trace operator mapping from $H^1(\Omega_f) \to H^{1/2}(\Gamma)$, we have that $v_N = \boldsymbol{u}_N|_\Gamma \cdot \boldsymbol{e}_r$ is uniformly bounded in $L^\infty(0,T; L^2(\Gamma)) \cap L^2(0,T; H^{1/2}(\Gamma))$.

An important feature of linearly coupled FSI models is that having uniform boundedness of approximate solutions is sufficient for passing to the limit in approximate solutions in order to obtain a weak solution to the continuous problem. This is because uniform boundedness of approximate solutions in appropriate energy level function spaces allows us to obtain weak and weak star convergence of approximate solutions along appropriate subsequences. Because the weak formulation to linearly coupled problems involves linear terms, this weak and weak star convergence is sufficient for passing to the limit. From the uniform boundedness result above, we conclude that there is a limiting structure displacement η, fluid velocity $\boldsymbol{u}$, and structure velocity v, such that the following convergences hold:

$$\begin{aligned}
\eta_N &\rightharpoonup \eta \text{ weakly star in } L^\infty(0,T; H_0^2(\Gamma)), \\
\overline{\eta}_N &\rightharpoonup \eta \text{ weakly star in } W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H_0^2(\Gamma)), \\
v_N^* &\rightharpoonup v \text{ weakly star in } L^\infty(0,T; L^2(\Gamma)), \\
v_N &\rightharpoonup v \text{ weakly in } L^2(0,T; H^{1/2}(\Gamma)) \text{ and weakly star in } L^\infty(0,T; L^2(\Gamma)), \\
\boldsymbol{u}_N &\rightharpoonup \boldsymbol{u} \text{ weakly in } L^2(0,T; H^1(\Omega_f)) \text{ and weakly star in } L^\infty(0,T; L^2(\Omega_f)).
\end{aligned}$$

This is sufficient to pass to the limit in the semidiscrete formulation. We integrate the semidiscrete formulation in time from $t = 0$ to $t = T$ in order to obtain:

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} + 2\mu_f \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{q}) + \int_0^T \int_\Gamma \partial_t \overline{v}_N \cdot \psi + C_0 \int_0^T \int_\Gamma \partial_z v_N^* \cdot \partial_z \psi$$
$$+ C_1 \int_0^T \int_\Gamma \partial_z^2 v_N^* \cdot \partial_z^2 \psi = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Gamma_{in}} P_{N,in}^n \cdot q_z - \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Gamma_{out}} P_{N,out}^n \cdot q_z,$$

for all test functions $(\boldsymbol{q}, \psi) \in C_c^1([0,T); \mathcal{V}_f \times \mathcal{V}_s)$. From the weak convergence of the approximate structure velocities,

$$C_0 \int_0^T \int_\Gamma \partial_z v_N^* \cdot \partial_z \psi \to C_0 \int_0^T \int_\Gamma \partial_z v \cdot \partial_z \psi, \quad C_1 \int_0^T \int_\Gamma \partial_z^2 v_N^* \cdot \partial_z^2 \psi \to C_1 \int_0^T \int_\Gamma \partial_z^2 v \cdot \partial_z^2 \psi.$$

From the weak convergence of the approximate fluid velocities,

$$2\mu_f \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{q}) \to 2\mu_f \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}).$$

Finally, we use integration by parts in time, the weak convergence of the fluid velocities, and the fact that $\boldsymbol{q} \in C_c^1([0,T); \mathcal{V}_f)$ in order to deduce that

$$\begin{aligned}
\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} &= \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} \\
&= \sum_{n=0}^{N-1} \left(- \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_f} \overline{\boldsymbol{u}}_N \cdot \partial_t \boldsymbol{q} + \int_{\Omega_f} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{q}((n+1)\Delta t) - \int_{\Omega_f} \boldsymbol{u}_N^n \cdot \boldsymbol{q}(n\Delta t) \right) \\
&= - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) - \int_0^T \int_{\Omega_f} \overline{\boldsymbol{u}}_N \cdot \partial_t \boldsymbol{q} \to - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) - \int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q}.
\end{aligned}$$

An identical argument shows that

$$\int_0^T \int_\Gamma \partial_t \overline{v}_N \cdot \psi \to - \int_\Gamma v_0 \cdot \psi(0) - \int_0^T \int_\Gamma v \cdot \partial_t \psi.$$

Thus, the semidiscrete formulation converges to the weak formulation to the continuous problem as $N \to \infty$, which establishes the existence of a weak solution to the prototypical linearly coupled FSI problem, stated in Theorem 1.2.1.

1.3 A nonlinearly coupled prototypical FSI model

Next, we consider a prototypical *nonlinearly coupled* FSI problem, where the fluid domain is determined by the displacement of the elastic structure from its reference configuration, which gives rise to a moving boundary problem involving a coupled system of PDEs. While the Lie operator splitting approach also works well for this nonlinearly coupled prototypical FSI problem, there are many unique challenges that arise when considering the moving boundary, and we will emphasize the differences in the mathematical approach to the nonlinearly coupled problem in comparison to the linearly coupled problem that was just presented.

Literature review

Before we state the prototypical nonlinearly coupled model that we will consider in this introduction, we give a brief summary of the results for various types of prototypical nonlinearly coupled FSI models of types similar to the one that we will describe below.

The study of FSI involving time-dependent fluid domains is classical, and it arose first from the problem of considering the motion of a rigid body in a viscous incompressible Newtonian fluid, which is a problem that has been widely studied in the literature, see for example [49, 60, 61, 84, 87, 91, 146, 161, 171, 172, 173]. However, in many applications of FSI where the fluid occupies a time-dependent domain in space that evolves according to the movement of a structure, the structure is actually elastic or deformable, rather than rigid. This is the case for example in the study of blood flow in arteries where the arterial walls are modeled to be elastic structures, and models of FSI that take into account of the displacement of the arterial walls and other relevant biological properties of the blood flow system are commonplace in the literature, see for example [36, 73, 149, 151]. Hence, because of their relevance to applications, the development of rigorous mathematical techniques for analyzing the influence of the time-dependent nature of the fluid domain on nonlinearly coupled fluid-structure dynamics is essential. There was some work done in [62] on the interaction between an elastic structure and a fluid on a time-dependent domain modeled by the Navier-Stokes equations, where only finitely many eigenmodes in order to update the displacement of the structure. However, we want to more generally consider elastic structures modeled with full equations of elasticity, which determine the a priori unknown configuration of a time-dependent fluid domain in nonlinearly coupled models. In the literature, there are two main types of such prototypical nonlinearly coupled FSI models: (1) models where the structure is of lower dimension than the fluid, and is an elastic part of the boundary of the time-dependent fluid domain [18, 30, 42, 81, 82, 124, 125, 140] and (2) models where the structure has the same dimension as the fluid and hence can be thought of as an immersed elastic body within a surrounding fluid [44, 45, 51, 52, 93, 94, 117, 152].

The first results for nonlinearly coupled FSI were obtained for fluid-structure systems where the *structure is of lower dimension than the fluid domain.* One of the first well-posedness results for nonlinearly coupled fluid-structure interaction was obtained in [18], where existence of a local strong solution for sufficiently smooth and sufficiently small initial data is obtained for a nonlinearly coupled FSI model involving the interaction between a 1D viscoelastic structure described by the generalized string model interacting with the Navier-Stokes equations in a 2D domain. This local existence result was shown by using a fixed point argument to obtain a solution, after moving the problem on a moving domain to a fixed domain, and linearizing the resulting system appropriately. These local existence results for strong solutions were extended in [125], which considers a similar model involving a 1D structure modeled by the damped beam equation interacting with a fluid modeled by the Navier-Stokes equations. The smallness condition on the initial data from [18] is removed, so that in [125], local existence of strong solutions is established for general sufficiently regular initial data, under the additional assumption that the structure is a viscoelastic plate (so that it has fourth order spatial derivatives in the defining equations). Furthermore, if the initial data is sufficiently small, the result in [125] establishes global existence for small data, if the structure is a viscoelastic plate. These past results for strong solutions [18, 125] involve local existence of strong solutions and were recently improved in [82], where it is established that there is global existence of strong solutions to an FSI problem involving a viscoelastic plate interacting with a fluid modeled by the Navier-Stokes equations, assuming the initial data is sufficiently regular and does not contact the bottom boundary of the fluid domain at the initial time. In particular, this work [82] shows that if the structure is not touching the bottom of the fluid domain at the initial time, then the structure will never contact the bottom of the fluid domain, and hence the strong solution will exist globally in time.

These previously described results for nonlinearly coupled models involving elastic structures of lower dimension than the fluid are results involving strong solutions, but there has also been a lot of work from the perspective of weak solutions for this particular model from [18]. An existence result for weak solutions to such a model was first shown in [42], where it is established that a weak solution exists until the time of self-contact of the structure in a nonlinearly coupled FSI model involving a 3D fluid modeled by the Navier-Stokes equations interacting with a elastic plate with an additional regularization term. This result is extended in [81], where it is shown that a weak solution exists for this model when the coefficient in front of the regularizing term in the plate equation is zero. The study of weak solutions to nonlinearly coupled FSI was continued in [140], which considered a 2D fluid modeled by the Navier-Stokes equations interacting with a Koiter shell, where the flow is driven by inlet and outlet dynamic pressure data. This work [140] was important because it gave a new methodology for studying existence of weak solutions to nonlinearly coupled FSI problems, as it featured a new constructive proof based on using an operator splitting scheme to explicitly construct approximate solutions. These results on existence of weak solutions were extended to nonlinearly coupled FSI models with curved structures in [124], where a Koiter shell structure that is potentially curved is modeled by considering the contributions of the membrane energy and the bending energy to the dynamics of the elastic shell.

Another class of prototypical FSI models that has been considered are models where *the structure is of the same dimension as the fluid domain.* These are models in which the structure is an elastic solid that is modeled by the equations of linear elasticity, which is immersed in a surrounding incompressible viscous fluid modeled by the Navier-Stokes equations. The first well-posedness result for such a model was established in [51], where it is shown that for a nonlinearly coupled model involving a fluid modeled by the Navier-Stokes equations interacting with a linearly elastic solid, there is local existence of strong solutions given that the initial fluid velocity and structure velocity are in H^5 and H^2 respectively. This local existence result for strong solutions to such an FSI model was extended in [52] where an elastic solid modeled by quasilinear equations of elasticity is considered, [44] where the structure is a biofluid shell that is modeled by nonlinear equations, and [45] where the elastodynamics of the elastic shell is affected by inertial, membrane, and bending contributions. The result on local existence of strong solutions, which was established first in [51], was later improved in [117], where this result is shown for less regular initial data, where the initial data for the fluid and structure velocity are required only to be in H^3 and H^2. Further analysis of this model, in terms of a priori energy estimates that are necessary for studying well-posedness, was subsequently carried out in [94]. A variant of the initial model from [51], in which there is an additional transmission boundary condition that stabilizes the dynamics, is studied in [93], where it is shown that the damping and additional transmission boundary condition give rise to a global existence result for smooth solutions, under the assumption that the initial data is sufficiently small with respect to some higher regularity Sobolev norms. Finally, well-posedness of a model involving the Navier-Stokes equations interacting with a structure modeled by the Lamé equations of elasticity is considered in [152], where local existence of strong solutions is established with initial data requiring less regularity, compared to the corresponding local existence results for strong solutions in past works such as [51] and [117].

For further discussion of results for nonlinearly coupled FSI models, from the perspectives of both weak solutions and strong solutions, we refer the reader to the references [30, 83].

Description of the model

We now discuss the existence of weak solutions to a prototypical nonlinearly coupled FSI model, established using a splitting scheme that splits the fluid and structure dynamics to explicitly construct approximate solutions.

In the summary that we will present in the remainder of this section, we summarize the presentation of constructive existence for the prototypical nonlinearly coupled FSI model, in the reference [30]. We refer the reader to this reference [30] for full details, and we insert additional comments about the distinctions between the splitting scheme approach for the prototypical nonlinearly coupled case and the linearly coupled case discussed in the previous section.

First, we will define the prototypical nonlinearly coupled problem. As in the linearly coupled case, we will consider a two-dimensional fluid domain where the reference configuration for the fluid domain is $\Omega_f = [0, L] \times [0, R]$, and the variables describing this domain are $(z, r) \in \Omega_f$. As in the linearly coupled case, we have that the top boundary Γ of Ω_f is the reference configuration for an elastic structure, the left and right boundaries Γ_{in} and Γ_{out} of Ω_f are the inlet and outlet, and the bottom boundary Γ_b of Ω_f is a line of even symmetry. We denote the fluid velocity by $\boldsymbol{u}$ and we denote the structure displacement by η. As before, we assume that the structure displaces in only the radial direction, so that η is a scalar quantity that describes the radial displacement of the structure from its reference configuration Γ. Hence, the time-dependent configuration of the elastic structure is given by

$$\Gamma(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, r = R + \eta(t, z)\}.$$

Since we must consider the time-dependent configuration $\Gamma(t)$ of the elastic structure, we remark that assuming that the structure displaces in only the radial direction simplifies the analysis by preventing self-intersection of the structure with itself, so that domain degeneracy occurs only when $\eta(t, z) = -R$ at any point $z \in \Gamma$ for any time t. Since this is a nonlinearly coupled FSI model, we have to take into account the motion of the elastic structure when considering the fluid domain, which is now time-dependent. The time-dependent configuration of the structure, given by $\Gamma(t)$, will determine the top boundary of the time-dependent fluid domain $\Omega_f(t)$, so that the time-dependent fluid domain $\Omega_f(t)$ is defined by

$$\Omega_f(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant R, 0 \leqslant r \leqslant R + \eta(t, z)\}. \tag{1.9}$$

Note that the bottom boundary $\Omega_f(t)$ is fixed at Γ_b because Γ_b is a fixed line of even symmetry through the center of the full channel. See Figure 1.2.

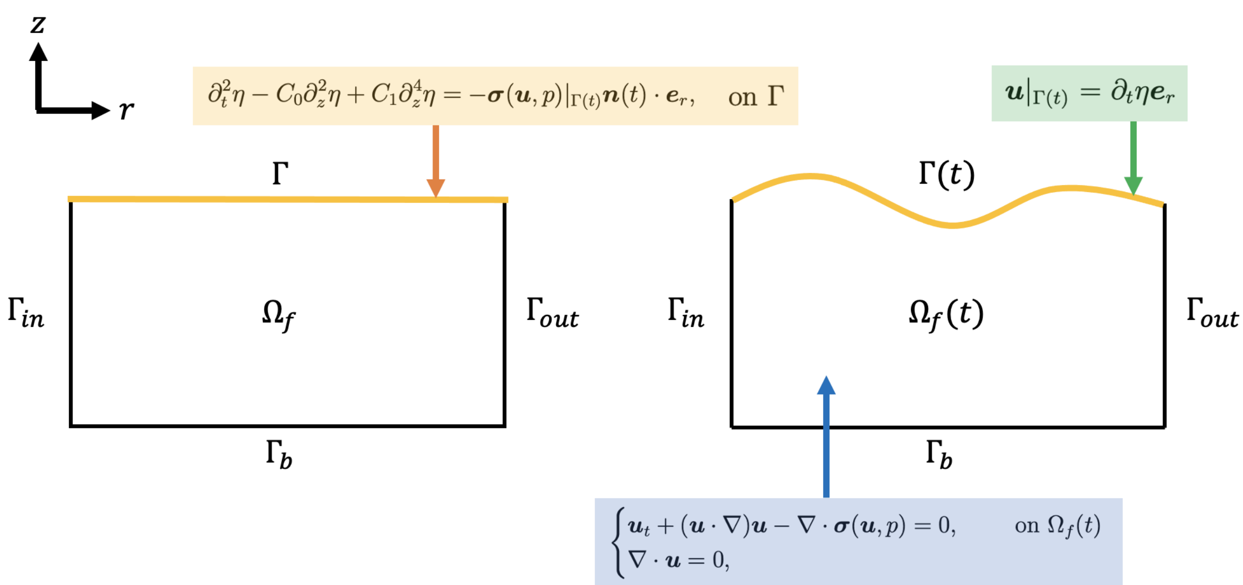

Figure 1.2: *Left: The reference configuration for the nonlinearly coupled prototypical FSI problem in two dimensions. Note that the structure equation is posed on the reference configuration Γ of the structure. Right: The moving fluid domain in physical space for the nonlinearly coupled prototypical FSI problem in two dimensions. Note that the Navier-Stokes equations are posed on the time-dependent a priori unknown fluid domain $\Omega_f(t)$, which is determined by the structure displacement.*

The fluid subproblem. In contrast to the linearly coupled prototypical FSI model, because of the nonlinear coupling present in this current model, we must pose the fluid equations on a *moving fluid domain*, whose time-dependent configuration $\Omega_f(t)$ depends on the structure displacement, as in (1.9). Because the fluid domain $\Omega_f(t)$ is now time-dependent, we must consider the full Navier-Stokes equations rather than just the linear Stokes equations, as the advection term is needed in order to obtain an energy estimate. Therefore, we model the fluid velocity $\boldsymbol{u} : \Omega_f(t) \to \mathbb{R}^2$ and the fluid pressure $p : \Omega_f(t) \to \mathbb{R}$ by the Navier-Stokes equations

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = 0, \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases} \qquad \text{on } \Omega_f(t), \tag{1.10}$$

where the Cauchy stress tensor is defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu_f \boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}, \qquad \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\left((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^t\right). \tag{1.11}$$

We have the following boundary conditions on the inlet and outlet:

$$u_r = 0, \qquad p + \frac{|\boldsymbol{u}|^2}{2} = P_{in}(t), \qquad \text{on } \Gamma_{in},$$

$$u_r = 0, \qquad p + \frac{|\boldsymbol{u}|^2}{2} = P_{out}(t), \qquad \text{on } \Gamma_{out}.$$

$$u_r = 0, \qquad \partial_r u_z = 0, \qquad \text{on } \Gamma_b.$$

We note that all of these boundary conditions are the same, except for the condition

$$p + \frac{|\boldsymbol{u}|^2}{2} = P_{in/out}(t), \qquad \text{on } \Gamma_{in/out},$$

which takes the form of $p = P_{in/out}$ on $\Gamma_{in/out}$ in the linearly coupled prototypical FSI model. Due to the advection term which is present in the Navier-Stokes equations but not the linear Stokes equations, we must include the additional $\frac{|\boldsymbol{u}|^2}{2}$ term in order to obtain an energy estimate, and we remark the term $p + \frac{|\boldsymbol{u}|^2}{2}$ is referred to as the dynamic pressure.

The structure subproblem. The PDE describing the structure subproblem is unchanged from the linearly coupled case:

$$\partial_t^2 \eta - C_0 \partial_z^2 \eta + C_1 \partial_z^4 \eta = F, \qquad \text{on } \Gamma. \tag{1.12}$$

The scalar quantity $\eta : \Gamma \to \mathbb{R}$ describes the radial displacement of the structure from its reference configuration Γ, and C_0 and C_1 are positive elasticity coefficients, as in the linearly coupled case. The external force F will be specified in the dynamic coupling condition later as the fluid load on the structure. Even though the structure has a time-dependent configuration described by

$$\Gamma(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, r = R + \eta(t, z)\},$$

the equations of elasticity for the structure are posed on the reference domain for the structure Γ, or in the *Eulerian formulation*, which is common in the mathematical study of elasticity.

The coupling conditions. Next, we will describe the two-way nonlinear coupling between the structure and the fluid via the kinematic coupling condition and the dynamic coupling condition. We emphasize that these coupling conditions are evaluated on the moving interface $\Gamma(t)$, in contrast to the linearly coupled case in which the coupling conditions are evaluated on the reference configuration Γ of the fluid-structure interface. The fact that the coupling conditions are evaluated on a moving interface introduces additional challenging nonlinearities arising from the time-dependent geometry of the FSI problem. We describe the two coupling conditions below:

- **The kinematic coupling condition.** The kinematic coupling condition, which is the no-slip condition on the fluid, now reads

$$\boldsymbol{u}(z, R + \eta(t, z)) = \partial_t \eta(t, z) \boldsymbol{e}_r.$$

  We will informally write this condition above as $\boldsymbol{u}|_{\Gamma(t)} = \partial_t \eta \boldsymbol{e}_r$. We remark that this informal way of writing this condition is not entirely precise, as $\partial_t \eta$ is defined on Γ rather than on $\Gamma(t)$, so we are implicitly composing with the map sending $(z, R) \in \Gamma$ to $(z, R + \eta(t, z)) \in \Gamma(t)$.

- **The dynamic coupling condition.** The dynamic coupling condition loads the elastodynamics of the structure via the external load of the fluid on the structure. Because we are evaluating the coupling conditions along the *moving interface* $\Gamma(t)$, this condition reads

$$F = -\mathcal{J}^\eta \boldsymbol{\sigma}(\boldsymbol{u}, p)|_{\Gamma(t)} \boldsymbol{n}(t) \cdot \boldsymbol{e}_r,$$

  where F is the external load on the structure on the right hand side of the structure equation (1.12), and $\boldsymbol{n}(t)$ is the outward unit normal vector to the time-dependent configuration $\Gamma(t)$ of the structure representing the elastic walls of the channel. The factor $\mathcal{J}^\eta = \sqrt{1 + (\partial_z \eta)^2}$ appears since the fluid-structure interface is given by $\Gamma(t)$ while the elastodynamics of the structure are expressed on the reference domain Γ, so $\mathcal{J}^\omega$ is the arc length measure for $\Gamma(t)$ to convert from lengths on the moving interface $\Gamma(t)$ to lengths on the reference configuration Γ.

A priori energy estimate and weak formulation

Next, we use an a priori energy estimate to show that this prototypical nonlinearly coupled mathematical FSI model is physically reasonable. Using the norm for the elastic energy defined in (1.4), we multiply the Navier-Stokes equation by $\boldsymbol{u}$ and we multiply the structure equation by η_t. After integrating by parts, we obtain the following formal energy estimate:

$$\frac{1}{2}||\partial_t \eta(t)||^2_{L^2(\Gamma)} + ||\eta(t)||^2_E + \frac{1}{2}||\boldsymbol{u}(t)||^2_{L^2(\Omega_f(t))} + \mu_f \int_0^t ||\boldsymbol{D}(\boldsymbol{u})(s)||^2_{L^2(\Omega_f(s))} ds$$
$$\leqslant \frac{1}{2}||v_0||^2_{L^2(\Gamma)} + ||\eta_0||^2_E + \frac{1}{2}||\boldsymbol{u}_0||^2_{L^2(\Omega_f(0))} + C\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right).$$

The only notable difference from the linearly coupled case is that the norms $||\boldsymbol{u}(t)||^2_{L^2(\Omega_f(t))}$ and $||\boldsymbol{D}(\boldsymbol{u})(s)||^2_{L^2(\Omega_f(t))}$ involve the moving time-dependent fluid domain, so that for example,

$$||\boldsymbol{u}(t)||^2_{L^2(\Omega_f(t))} = \int_{\Omega_f(t)} |\boldsymbol{u}(t)|^2.$$

This is in contrast to the linearly coupled case, where the integrals in space over the fluid domain were over the fixed reference fluid domain Ω_f.

This energy estimate allows us to define the finite energy spaces for the solutions. We define the solution space $\mathcal{V}_f(t)$ (for fixed but arbitrary time) for the fluid, which is time-dependent since the fluid domain is time dependent:

$$\mathcal{V}_f(t) = \{\boldsymbol{u} \in H^1(\Omega_f(t)) : \nabla \cdot \boldsymbol{u} = 0, u_r = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_b, u_z = 0 \text{ on } \Gamma\}. \quad (1.13)$$

Since the structure problem is posed on the fixed reference domain Γ, the finite energy space for the structure is the same as in the linearly coupled case

$$\mathcal{V}_s = H^2_0(\Gamma). \quad (1.14)$$

The solution spaces for the fluid velocity and the structure displacement in space and time are given by

$$V_f = L^\infty(0, T; L^2(\Omega_f(t))) \cap L^2(0, T; \mathcal{V}_f(t)), \quad (1.15)$$

$$V_s = W^{1,\infty}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; \mathcal{V}_s). \quad (1.16)$$

The full coupled solution space is

$$\mathcal{V} = \{(\boldsymbol{u}, \eta) \in V_f \times V_s : \boldsymbol{u}|_{\Gamma(t)} = \partial_t \eta \boldsymbol{e}_r\}, \quad (1.17)$$

and the test space is

$$\mathcal{Q} = \{(\boldsymbol{q}, \psi) \in C^\infty_c([0, T); \mathcal{V}_f(t) \times V_s) : \boldsymbol{q}|_{\Gamma(t)} = \psi \boldsymbol{e}_r\}. \quad (1.18)$$

We note that the test space depends on the *a priori unknown structure displacement* because the test space involves the function space $\mathcal{V}_f(t)$, which depends on $\Omega_f(t)$ and hence $\eta(t)$. The fact that the test functions cannot be defined a priori is one of the additional challenges that arises more generally in nonlinearly coupled FSI problems.

Now that we have defined the finite energy solution and test spaces, we can define the weak formulation. To derive the weak formulation, we consider a test function $(\boldsymbol{q}, \psi)$ satisfying $\boldsymbol{q}|_{\Gamma(t)} = \psi \boldsymbol{e}_r$ and use integration by parts to derive a weak formulation. In this case, the weak formulation has an additional symmetrized term arising from the advection term in the Navier-Stokes equation, and furthermore, the integrals over the fluid domain are over the time-dependent fluid domain $\Omega_f(t)$.

Definition 1.3.1. An ordered pair $(\boldsymbol{u}, \eta) \in \mathcal{V}$ is a weak solution to the prototypical nonlinearly coupled FSI problem if for all test functions $(\boldsymbol{q}, \psi) \in \mathcal{Q}$, the following weak formulation holds:

$$
\begin{aligned}
&-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} + \frac{1}{2}\int_0^T \int_{\Omega_f(t)} \Big([(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{q} - [(\boldsymbol{u}\cdot\nabla)\boldsymbol{q})\boldsymbol{u}]\Big) - \frac{1}{2}\int_0^T \int_\Gamma (\partial_t \eta)^2 \cdot \psi \\
&+ 2\mu_f \int_0^T \int_{\Omega_f(s)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) - \int_0^T \int_\Gamma \partial_t \eta \cdot \partial_t \psi + C_0 \int_0^T \int_\Gamma \partial_z \eta \cdot \partial_z \psi + C_1 \int_0^T \int_\Gamma \partial_z^2 \eta \cdot \partial_z^2 \psi \\
&\qquad = \int_0^T \int_{\Gamma_{in}} P_{in}(t) \cdot q_z - \int_0^T \int_{\Gamma_{out}} P_{out}(t)\cdot q_z + \int_{\Omega_f(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) + \int_\Gamma v_0 \cdot \psi(0).
\end{aligned}
$$

This is the weak formulation defined on the moving domain. However, when developing a splitting scheme to solve this problem, it is more convenient to solve the semidiscretized problem on a fixed reference domain Ω_f. Hnece, we will need to formulate an equivalent version of this weak formulation, defined on the fixed reference domain Ω_f for the fluid. To do this, we will need to discuss a mapping between the reference (Eulerian) fluid domain Ω_f and the time-dependent physical (Lagrangian) fluid domain $\Omega_f(t)$, known as the **Arbitrary Lagrangian-Eulerian (ALE) mapping**.

The ALE mapping and transformation of functions/derivatives. For the given two-dimensional geometry of this particular model, it is easy to define a bijective mapping between Ω_f and $\Omega_f(t)$, known as the Arbitrary Lagrangian-Eulerian (ALE) mapping, assuming sufficient regularity of η. We will denote this mapping by $\boldsymbol{\Phi}_f^\eta$, where the superscript of η emphasizes the dependency of the definition of this mapping on the structure displacement η. We define the ALE mapping $\boldsymbol{\Phi}_f^\eta : \Omega_f \to \Omega_f(t)$ by

$$
\boldsymbol{\Phi}_f^\eta(z, r) = \left(z, \left(1 + \frac{\eta}{R}\right) r\right), \qquad (z, r) \in \Omega_f,
$$

with an inverse $\left(\boldsymbol{\Phi}_f^\eta\right)^{-1} : \Omega_f(t) \to \Omega_f$ defined by

$$
\left(\boldsymbol{\Phi}_f^\eta\right)^{-1}(z, r) = \left(z, \frac{R}{R+\eta} r\right), \qquad (z, r) \in \Omega_f(t).
$$

We emphasize that as long as the structure displacement η is sufficiently regular (for example, if it is a continuous function on Γ) and $\eta > -R$ so that the structure does not come into contact with itself (since the bottom boundary Γ_b of the reference fluid domain Ω_f is a line of even symmetry), the ALE mapping is a bijective map with a Jacobian given by

$$
\mathcal{J}_f^\eta = 1 + \frac{\eta}{R},
$$

which is a positive quantity since we are assuming that the structure displacement η satisfies $\eta > -R$.

Next, we consider how functions and derivatives transform under this ALE mapping, so that we can translate between quantities defined on $\Omega_f(t)$ and Ω_f. First, for a function h defined on the physical domain $\Omega_f(t)$, we define the pullback of the function via composition with the ALE mapping as follows:

$$h^\eta(t,\cdot) = h(t, \boldsymbol{\Phi}_f^\eta(t,\cdot)).$$

Conversely, we also have that

$$h(t,\cdot) = h^\eta(t, (\boldsymbol{\Phi}_f^\eta)^{-1}(t,\cdot)).$$

Next, we want to consider how derivatives transform under the ALE mapping. To see why we need to do this, let us consider the fluid velocity $\boldsymbol{u}$, which is defined on $\Omega_f(t)$ and which is required to be divergence-free so that

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{on } \Omega_f(t).$$

If we consider the pullback via the ALE mapping, we have that the resulting fluid velocity $\boldsymbol{u}^\eta$ is defined on the fixed reference domain Ω_f. However, because the ALE mapping is a nonlinear map, it is no longer true that $\nabla \cdot \boldsymbol{u}^\eta = 0$ on Ω_f, where the divergence is taken with respect to the reference domain variables. Hence, we will need to find an appropriate differential operator that translates the divergence-free condition on the physical domain to an appropriate variant of the divergence-free condition on the fixed reference domain.

We use the chain rule to determine how derivatives will transform under the ALE mapping. For clarity, in the upcoming calculations, we will distinguish between the variables $(\tilde{z}, \tilde{r})$ on the reference domain Ω_f and the variables (z, r) on the physical domain $\Omega_f(t)$, though we will drop the tilde notation in the remainder of the section. Given a function h on the physical domain $\Omega_f(t)$, we have that

$$\begin{aligned}\frac{\partial}{\partial z} h(t,z,r) = \frac{\partial}{\partial z} h^\eta\left(t, z, \frac{R}{R+\eta(t,z)} r\right) &= \frac{\partial h^\eta}{\partial \tilde{z}} - \frac{R}{(R+\eta(t,\tilde{z}))^2}(r\partial_z \eta)\frac{\partial h^\eta}{\partial \tilde{r}} \\ &= \frac{\partial h^\eta}{\partial \tilde{z}} - \frac{\tilde{r}\partial_z\eta}{R+\eta(t,\tilde{z})}\frac{\partial h^\eta}{\partial \tilde{r}},\end{aligned}$$

where we used that $r = \left(1 + \dfrac{\eta}{R}\right)\tilde{r}$, and we also have that

$$\frac{\partial}{\partial r} h(t,z,r) = \frac{\partial}{\partial r} h^\eta\left(t, z, \frac{R}{R+\eta(t,z)} r\right) = \frac{R}{R+\eta(t,\tilde{z})} \cdot \frac{\partial h^\eta}{\partial \tilde{r}}.$$

So if we define the differential operator

$$\nabla^\eta = \left(\frac{\partial}{\partial \tilde{z}} - \frac{\tilde{r}\partial_z\eta}{R+\eta(t,\tilde{z})}, \frac{R}{R+\eta(t,\tilde{z})}\frac{\partial}{\partial \tilde{r}}\right),$$

we have that

$$\nabla^\eta h^\eta = (\nabla h) \circ \boldsymbol{\Phi}_f^\eta. \tag{1.19}$$

Similarly,

$$\nabla^\eta \cdot \boldsymbol{u}^\eta = (\nabla \cdot \boldsymbol{u}) \circ \boldsymbol{\Phi}_f^\eta,$$

and

$$\boldsymbol{D}^\eta(\boldsymbol{u}^\eta) = \boldsymbol{D}(\boldsymbol{u}) \circ \boldsymbol{\Phi}_f^\eta, \quad \text{where } \boldsymbol{D}^\eta(\boldsymbol{u}^\eta) = \frac{1}{2}\Big((\nabla^\eta \boldsymbol{u}^\eta) + (\nabla^\eta \boldsymbol{u}^\eta)^t\Big).$$

Next, we consider

$$\begin{aligned}\frac{\partial}{\partial t} h(t,z,r) = \frac{\partial}{\partial t} h^\eta\left(t, z\frac{R}{R+\eta(t,z)}r\right) &= \frac{\partial h^\eta}{\partial t} - \frac{R}{(R+\eta(t,\tilde{z}))^2} \cdot r\partial_t \eta \frac{\partial h^\eta}{\partial \tilde{r}} \\ &= \frac{\partial h^\eta}{\partial t} - \frac{\tilde{r}\partial_t \eta}{R+\eta(t,\tilde{z})}\frac{\partial h^\eta}{\partial \tilde{r}}.\end{aligned}$$

So if we define the vector

$$\boldsymbol{w}^\eta = \frac{\tilde{r}}{R}\partial_t \eta \boldsymbol{e}_r,$$

which represents the velocity of the ALE mapping so that $\boldsymbol{w}^\eta(\tilde{z},\tilde{r}) = \frac{\partial}{\partial t}\boldsymbol{\Phi}_f^\eta(\tilde{z},\tilde{r})$, we have that

$$\frac{\partial h}{\partial t} = \frac{\partial h^\eta}{\partial t} - (\boldsymbol{w} \cdot \nabla^\eta) h^\eta. \tag{1.20}$$

Using the transformation laws (1.19) and (1.20) for derivatives, we can rewrite the weak formulation for the prototypical nonlinearly coupled FSI problem on the fixed reference domain Ω_f. By transferring the weak formulation on the physical moving domain $\Omega_f(t)$ to the weak formulation involving integrals defined on the fixed reference domain Ω_f, we will introduce additional geometric nonlinearities in the problem, which arise from the Jacobian of the ALE mapping $\boldsymbol{\Phi}_f^\eta$ and from the transformation of derivatives via the ALE mapping. We will now state the weak formulation for the prototypical nonlinearly coupled FSI problem defined on the fixed reference domain for the fluid Ω_f.

We define the solution spaces for the fluid velocity and the structure displacement on the fixed reference domain Ω_f. Let us define the fluid velocity solution space on the fixed reference domain Ω_f by

$$V_f^\eta = L^\infty(0,T;L^2(\Omega_f)) \cap L^2(0,T;\mathcal{V}_f^\eta), \tag{1.21}$$

where

$$\mathcal{V}_f^\eta = \{\boldsymbol{u} \in H^1(\Omega_f) : \nabla^\eta \cdot \boldsymbol{u} = 0, u_r = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_b, u_z = 0 \text{ on } \Gamma\}.$$

For the structure displacement, we will use the same solution space V_s defined in (1.16) for the moving domain weak formulation. The full solution space is then

$$\mathcal{V}^\eta = \{(\boldsymbol{u},\eta) \in V_f^\eta \times V_s : \boldsymbol{u}|_\Gamma = \partial_t \eta \boldsymbol{e}_r.\}.$$

Similarly, the test space using the fixed reference fluid domain formulation is defined as

$$\mathcal{Q}^\eta = \{(\boldsymbol{q}, \psi) \in C_c^\infty([0,T); V_f^\eta \times V_s) : \boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r\}. \tag{1.22}$$

Then, we can define the weak formulation that a weak solution $(\boldsymbol{u}, \eta) \in \mathcal{V}^\eta$ defined on the fixed reference domain satisfies.

Definition 1.3.2. An ordered pair $(\boldsymbol{u}, \eta) \in \mathcal{V}^\eta$ with $\boldsymbol{u}$ defined on the fixed reference fluid domain Ω_f, is a weak solution in the fixed reference domain formulation of the nonlinearly coupled FSI problem if the following weak formulation is satisfied for all test functions $(\boldsymbol{q}, \psi) \in \mathcal{Q}^\eta$:

$$\begin{aligned}
-\int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \boldsymbol{u} \cdot \partial_t \boldsymbol{q} + \int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \Big([((\boldsymbol{u} - \boldsymbol{w}^\eta) \cdot \nabla^\eta)\boldsymbol{u}] \cdot \boldsymbol{q} - [((\boldsymbol{u} - \boldsymbol{w}^\eta) \cdot \nabla^\eta)\boldsymbol{q}] \cdot \boldsymbol{u}\Big) \\
- \int_0^T \int_{\Omega_f} \frac{1}{2R}(\partial_t \eta)\boldsymbol{u} \cdot \boldsymbol{q} + 2\mu_f \int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \boldsymbol{D}^\eta(\boldsymbol{u}) : \boldsymbol{D}^\eta(\boldsymbol{q}) - \int_0^T \int_\Gamma \partial_t \eta \cdot \partial_t \psi \\
+ C_0 \int_\Gamma \partial_z \eta \cdot \partial_z \psi + C_1 \int_\Gamma \partial_z^2 \eta \cdot \partial_z^2 \psi = \int_0^T \int_{\Gamma_{in}} P_{in}(t) \cdot q_z - \int_0^T \int_{\Gamma_{out}} P_{out}(t) \cdot q_z \\
+ \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) + \int_\Gamma v_0 \cdot \psi(0).
\end{aligned}$$

In the weak formulation above, there is an extra (yet important) term that arises due to the motion of the fluid domain. This term is the term $\displaystyle\int_0^T \int_{\Omega_f} \frac{1}{2R}(\partial_t \eta)\boldsymbol{u} \cdot \boldsymbol{q}$. This term arises as a result of the fact that the time derivative is applied to the fluid velocity defined on the fixed reference fluid domain Ω_f rather than the fluid velocity defined on the physical moving fluid domain $\Omega_f(t)$. More specifically, after formally integrating by parts in time, this term appears, since

$$-\int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \boldsymbol{u}^\eta \cdot \partial_t \boldsymbol{q} = \int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \partial_t \boldsymbol{u}^\eta \cdot \boldsymbol{q} + \int_0^T \int_{\Omega_f} \frac{\partial_t \eta}{R} \cdot \boldsymbol{u}^\eta \cdot \boldsymbol{q}.$$

For full details, we refer the reader to a similar computation in a later chapter, see specifically the computation in (5.47).

We emphasize that the fact that we must have a weak formulation defined for both the moving (physical) fluid domain and the reference fluid domain is an important distinction between the nonlinearly coupled and the linearly coupled models of FSI. The need to define a weak formulation where all integrals involving the fluid domain are defined on the fixed reference fluid domain Ω_f is due to the fact that the splitting scheme works better when using a fixed domain, as we will see in the next step of the constructive existence proof below.

Now that we have defined an appropriate weak formulation for the nonlinearly coupled prototypical FSI problem, we can state the main existence result for this model.

Theorem 1.3.1. Given an initial structure displacement $\eta_0 \in H_0^2(\Gamma)$, an initial structure velocity $v_0 \in L^2(\Gamma)$, and an initial fluid velocity $\boldsymbol{u}_0 \in L^2(\Omega_f(0))$, there exists a weak solution $(\boldsymbol{u}, \eta) \in \mathcal{V}$ that satisfies the weak formulation posed on the physical moving fluid domain given by Definition 1.3.1 on the time interval $[0, T]$. Furthermore, the solution can be maximally extended in time in the sense that T can be taken to be infinite, or else, $T = \sup\{t \geqslant 0 : R + \eta(t, z) > 0 \text{ for all } z \in [0, L]\}$ so that T is the first instance in time of domain degeneracy.

The splitting scheme for nonlinearly coupled FSI

We will define a Lie operator splitting using the weak formulation defined on the fixed reference domain. As in the linearly coupled case, we will develop a semidiscretized scheme, where we discretize in time by the time step $\Delta t = T/N$, where N is the number of discretized time subintervals. On each time subinterval $[t_n, t_{n+1}]$ for $t_n = n\Delta t$, we run two subproblems and update the following approximate solution vector at the nth time step for the time step $\Delta t = T/N$:

$$X_N^{n+\frac{i}{2}} = \begin{pmatrix} \boldsymbol{u}_N^{n+\frac{i}{2}} \\ \eta_N^{n+\frac{i}{2}} \\ v_N^{n+\frac{i}{2}} \end{pmatrix} \qquad \text{for } i = 1, 2 \text{ and } n = 0, 1, ..., N-1,$$

where $\boldsymbol{u}_N^{n+\frac{i}{2}}$ is the approximate fluid velocity defined on the *fixed reference domain* Ω_f, $\eta_N^{n+\frac{i}{2}}$ is the approximate structure displacement, and $v_N^{n+\frac{i}{2}}$ is the approximate structure velocity.

The structure subproblem. We keep the fluid velocity the same so that

$$\boldsymbol{u}_N^{n+\frac{1}{2}} = \boldsymbol{u}_N^n,$$

and we update the structure displacement and velocity by letting $\eta_N^{n+\frac{1}{2}}$ and $v_N^{n+\frac{1}{2}}$ be the unique solution that satisfies the following weak formulation. We remark that this weak formulation is identical to that in the structure subproblem for the prototypical linearly coupled FSI problem in (1.7), but we reproduce it here for the reader's convenience.

$$\begin{cases} \displaystyle\int_\Gamma \frac{\eta_N^{n+\frac{1}{2}} - \eta_N^n}{\Delta t} \varphi dz = \int_\Gamma v_N^{n+\frac{1}{2}} \varphi dz, \qquad \text{for all } \varphi \in L^2(\Gamma), \\ \displaystyle\int_\Gamma \frac{v_N^{n+\frac{1}{2}} - v_N^n}{\Delta t} \psi + C_0 \int_\Gamma \partial_z \eta_N^{n+\frac{1}{2}} \partial_z \psi + C_1 \int_\Gamma \partial_z^2 \eta_N^{n+\frac{1}{2}} \partial_z^2 \psi = 0, \qquad \text{for all } \psi \in H_0^2(\Gamma). \end{cases}$$

The fluid subproblem. For the prototypical nonlinearly coupled FSI problem, the fluid subproblem is more complex than the corresponding fluid subproblem for the prototypical linearly coupled FSI model, discussed in the previous section. This is because nonlinearly coupled FSI models have weak formulations which include additional terms that take into account the time-dependent geometry of the fluid domain, which must be handled properly in the fluid subproblem for the splitting scheme, so that the fluid subproblem indeed has a unique solution that exists, and so that the fluid subproblem has an appropriate semidiscrete energy estimate.

We will keep the structure displacement the same so that

$$\eta_N^{n+1} = \eta_N^{n+\frac{1}{2}},$$

and we will update the coupled fluid/structure velocities $\boldsymbol{u}_N^{n+1}$ and v_N^{n+1}, which will be in the following solution space:

$$\tilde{Q}_N^n = \{(\boldsymbol{u}, v) \in \mathcal{V}_f^{\eta_N^n} \times L^2(\Gamma) : \boldsymbol{u}|_\Gamma = v\boldsymbol{e}_r\}, \tag{1.23}$$

where we recall the definition of V_f^η for a structure displacement η from (1.21). We then let $\boldsymbol{u}_N^{n+1}$ and v_N^{n+1} be the unique solution in $\tilde{Q}_N^n$ to the following weak formulation, to be satisfied for all test functions $(\boldsymbol{q}, \psi) \in \tilde{Q}_N^n$:

$$\begin{aligned}
&\int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} \\
&+ \frac{1}{2}\int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \left(\left(\left(\boldsymbol{u}_N^n - v_N^{n+\frac{1}{2}} \frac{r}{R}\boldsymbol{e}_y \right) \cdot \nabla^{\eta_N^n} \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{q} - \left(\left(\boldsymbol{u}_N^n - v_N^{n+\frac{1}{2}} \frac{r}{R}\boldsymbol{e}_y \right) \cdot \nabla^{\eta_N^n} \boldsymbol{q} \right) \cdot \boldsymbol{u}_N^{n+1} \right) \\
&\quad + \int_{\Omega_f} \frac{1}{2R} v_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{q} + 2\mu_f \int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \boldsymbol{D}^{\eta_N^n}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}^{\eta_N^n}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{1}{2}}}{\Delta t}\psi \\
&\qquad = \int_{\Gamma_{in}} P_{N,in}^n(t) q_z - \int_{\Gamma_{out}} P_{N,out}^n(t) q_z,
\end{aligned}$$

where $P_{N,in/out}^n = \dfrac{1}{\Delta t}\displaystyle\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt$ is the discretized inlet/outlet pressure.

We have several remarks about this weak formulation, which we list below.

1. We note that in the original weak formulation for the problem defined on a fixed reference domain, we have the following terms:

$$-\int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \boldsymbol{u} \cdot \partial_t \boldsymbol{q} - \frac{1}{2R}\int_0^T \int_{\Omega_f} (\partial_t \eta)\boldsymbol{u} \cdot \boldsymbol{q}.$$

   While the signs in the above weak formulation of the fluid subproblem may seem incongruous with these terms at first, if we integrate by parts in time, we get that the above expression is equal to

$$\int_0^T \int_{\Omega_f} \left(1 + \frac{\eta}{R}\right) \partial_t \boldsymbol{u} \cdot \boldsymbol{q} + \frac{1}{2R}\int_0^T \int_{\Omega_f} (\partial_t \eta)\boldsymbol{u} \cdot \boldsymbol{q},$$

since the time derivative applies to both the term $\left(1+\frac{\eta}{R}\right)$ and $\boldsymbol{u}$.

2. Because η_N^n and $v_N^{n+\frac{1}{2}}$ have already been calculated previously in the splitting scheme, the weak formulation above is a linearized problem, which crucial for allowing us to find a unique solution (which can be done using a standard Lax-Milgram lemma argument). Because we cannot use more than one term involving an unknown quantity at the $n+1$ time step per integral, in order to have a linearized problem, we must lag the numerical fluid domains behind by one step when defining the weak formulations for the fluid subproblem. For example, for the first term can be rewritten as:

$$\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{q}=\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{q},$$

where $\Omega_{f,N}^n=\{(z,r)\in\mathbb{R}^2:0\leqslant z\leqslant L, 0\leqslant r\leqslant R+\eta_N^n(z)\}$. Note that this integral (when brought back to the physical domain) is an integral over $\Omega_{f,N}^n$ rather than $\Omega_{f,N}^{n+1}$ so that there is a lag in the fluid domain that is used in the weak formulation in the current step. This is because we cannot use the nonlinear term $\int_{\Omega_f}\left(1+\frac{\eta_N^{n+1}}{R}\right)\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{q}$ in the weak formulation.

3. Finally, we remark that the use of the term $v_N^{n+\frac{1}{2}}$ is due to energy considerations. If we substitute $\boldsymbol{q}=\boldsymbol{u}_N^{n+1}$,

$$\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{u}_N^{n+1}$$
$$=\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^{n+1}|^2-\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n|^2-\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^n|^2.$$

However, we want the term $\int_{\Omega_f}\left(1+\frac{\eta_N^{n+1}}{R}\right)|\boldsymbol{u}_N^{n+1}|^2$ instead of $\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^{n+1}|^2$, since $\left(1+\frac{\eta_N^{n+1}}{R}\right)$ is the Jacobian of the ALE mapping for the fluid domain associated with the *current* structure displacement η_N^{n+1}. It turns out that fortunately, the term $\frac{1}{2R}\int_{\Omega_f}v_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{q}$ compensates exactly for this, in the sense that upon substituting $\boldsymbol{q}=\boldsymbol{u}_N^{n+1}$, we have that

$$\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{u}_N^{n+1}+\frac{1}{2R}\int_{\Omega_f}v_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{q}$$
$$=\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^{n+1}}{R}\right)|\boldsymbol{u}_N^{n+1}|^2-\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n|^2-\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^n|^2,$$

since $v_N^{n+\frac{1}{2}} = \dfrac{\eta_N^{n+1} - \eta_N^n}{\Delta t}$ from the structure subproblem and the fact that $\eta_N^{n+1} = \eta_N^{n+\frac{1}{2}}$.

We note that the fluid subproblem has a unique solution $(\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ as long as $\eta_N^n > -R$. In particular, this is because the Jacobian of the ALE mapping, which is $\mathcal{J}_f^\eta = 1 + \dfrac{\eta}{R}$, would be less than or equal to zero when $\eta \leqslant -R$, which would prevent us from having the necessary coercivity in order to use the Lax-Milgram lemma. The fact that having $\eta_N^n \leqslant -R$ is problematic can also be seen by noting that the elastic structure comes into contact with itself when $\eta_N^n = -R$. Thus, the splitting scheme runs for each N as long as $\eta_N^n > -R$, and an important fact is that the splitting scheme indeed holds locally up to some time $T_0 > 0$ uniformly for all N (and hence Δt). The fact that the splitting scheme runs uniformly up to some common (local) time allows us to construct approximate solutions for each time step Δt that are all defined on a common time interval, and we will establish this essential fact by deriving uniform energy estimates for the approximate solution vectors that we have obtained from this splitting scheme.

Uniform energy estimates. By adding together the weak formulations for each of the subproblems, we obtain that the approximate solutions generated by the splitting scheme satisfy the following semidiscrete weak formulation for all test functions $(\boldsymbol{q}, \psi)$ in an appropriate test space Q_N^n:

$$\int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q}$$
$$+ \frac{1}{2} \int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \left(\left(\left(\boldsymbol{u}_N^n - \frac{r}{R} v_N^{n+\frac{1}{2}} \boldsymbol{e}_y \right) \cdot \nabla^{\eta_N^n} \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{q} - \left(\left(\boldsymbol{u}_N^n - \frac{r}{R} v_N^{n+\frac{1}{2}} \boldsymbol{e}_y \right) \cdot \nabla^{\eta_N^n} \boldsymbol{q} \right) \cdot \boldsymbol{u}_N^{n+1} \right)$$
$$+ \int_{\Omega_f} \frac{1}{2R} v_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{q} + 2\mu_f \int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) \boldsymbol{D}^{\eta_N^n}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}^{\eta_N^n}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi$$
$$+ C_0 \int_\Gamma \partial_z \eta_N^{n+\frac{1}{2}} \partial_z \psi + C_1 \int_\Gamma \partial_z^2 \eta_N^{n+\frac{1}{2}} \partial_z^2 \psi = \int_{\Gamma_{in}} P_{N,in}^n(t) q_z - \int_{\Gamma_{out}} P_{N,out}^n(t) q_z, \quad (1.24)$$

where the test space Q_N^n is

$$Q_N^n = \{(\boldsymbol{q}, \psi) \in \mathcal{V}_f^{\eta_N^n} \times H_0^2(\Gamma) : \boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r\}. \quad (1.25)$$

Next, we will derive uniform energy estimates for the approximate solutions, which will be used to show weak convergence of the approximate solutions along a subsequence. We define the total discrete energy

$$E_N^{n+\frac{i}{2}} = \frac{1}{2} \int_{\Omega_f} \left(1 + \frac{\eta_N^{n+\frac{i}{2}}}{R}\right) |\boldsymbol{u}_N^{n+\frac{i}{2}}|^2 + \frac{1}{2} \int_\Gamma |v_N^{n+\frac{i}{2}}|^2 + \frac{1}{2} C_0 \int_\Gamma |\partial_z \eta_N^{n+\frac{i}{2}}|^2 + \frac{1}{2} C_1 \int_\Gamma |\partial_z^2 \eta_N^{n+\frac{i}{2}}|^2,$$

and the total fluid dissipation

$$D_N^n = \mu_f \int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) |\boldsymbol{D}^{\eta_N^n}(\boldsymbol{u}_N^n)|^2.$$

We remark that these are the same expressions as for the linearly coupled case, except any integrals over the fluid domain are over $\Omega^n_{f,N}$ (when transferred to the physical domain) rather than Ω_f. We have the following discrete energy inequalities for the splitting scheme for the prototypical nonlinearly coupled FSI problem, which correspond to the structure and fluid subproblems respectively:

$$E_N^{n+\frac{1}{2}} = E_N^n + \frac{1}{2}\int_\Gamma |v_N^{n+\frac{1}{2}} - v_N^n|^2 + \frac{1}{2}C_0\int_\Gamma |\partial_z(\eta_N^{n+\frac{1}{2}} - \eta_N^n)|^2 + \frac{1}{2}C_1\int_\Gamma |\partial_z^2(\eta_N^{n+\frac{1}{2}} - \eta_N^n)|^2,$$

$$E_N^{n+1} \leqslant E_N^{n+\frac{1}{2}} + \frac{1}{2}\int_{\Omega_f}\left(1 + \frac{\eta_N^n}{R}\right)|\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{1}{2}}|^2 + C\left(||P_{in}(t)||^2_{L^2(n\Delta t,(n+1)\Delta t)} + ||P_{out}(t)||^2_{L^2(n\Delta t,(n+1)\Delta t)}\right).$$

Recall that the splitting scheme holds as long as $\eta_N^n > -R$, and hence these uniform energy estimates hold until this time step for which this condition no longer holds. From the uniform discrete energy estimates above, we note that $||\eta_N^n||_{H_0^2(\Gamma)}$ and $||v_N^{n+\frac{1}{2}}||_{L^2(\Gamma)}$ are both uniformly bounded in time by some time-dependent constant $C(t)$ depending on $||P_{in/out}||_{L^2(0,t)}$ that is independent of N, and the initial structure displacement η_0 is a continuous function satisfying $\eta_0 > -R$ by assumption. Therefore, because of these uniform estimates, we can deduce that there exists a sufficiently small $T_0 > 0$ for which $\eta_N^n > -R$ as long as $n\Delta t < T_0$. See Lemma 5.6.2 in Chapter 5 for an explicit demonstration of this argument. Without loss of generality, we will rename this time T_0 to just be the time T, for simplicity of notation.

Approximate solutions. As before in the prototypical linearly coupled FSI model, we define piecewise constant approximate solutions $\boldsymbol{u}_N$, η_N, v_N, and v_N^* such that

$$\eta_N(t) = \eta_N^{n+1}, \quad v_N(t) = v_N^{n+1}, \quad v_N^*(t) = v_N^{n+\frac{1}{2}}, \quad \boldsymbol{u}_N = \boldsymbol{u}_N^{n+1}, \text{ for } t \in (n\Delta t, (n+1)\Delta t].$$

We also define the linearly interpolations, which interpolate the values of these functions in the splitting scheme at the times $t_n = n\Delta t$. In particular, we will define the linear interpolation $\overline{\eta}_N$ of the structure displacements by linearly interpolating the following values

$$\overline{\eta}_N(n\Delta t) = \eta_N^n, \quad \text{for } n = 0, 1, ..., N,$$

and we note that $\partial_t\overline{\eta}_N = v_N^*$. We can similarly define the approximate solutions $\overline{v}_N$ and $\overline{\boldsymbol{u}}_N$ by linearly interpolating the values

$$v_N(n\Delta t) = v_N^n, \quad \boldsymbol{u}_N(n\Delta t) = \boldsymbol{u}_N^n, \quad \text{for } n = 0, 1, ..., N.$$

From the discrete uniform energy estimates, we have the following uniform boundedness result for the approximate solutions on the uniform time interval for which they are all defined. These follow from the fact that $||\eta_N||_{C(0,T;C(\Gamma))} \geqslant R_0 > -R$ for some R_0 that is independent of N and hence Δt. This allows us to uniformly bound the factor of $\left(1 + \frac{\eta_N^{n+1}}{R}\right)$ which appears in the discrete energy estimates. Note that the uniform boundedness results that follow hold with the approximate solutions for the fluid velocity $\boldsymbol{u}_N$ being defined on the fixed reference domain Ω_f (however, the divergence free condition on each time step will transform nonlinearly into the condition $\nabla^{\eta_N^n} \cdot \boldsymbol{u}_N^{n+1} = 0$ on Ω_f).

Lemma 1.3.1. The following uniform boundedness results hold for the approximate solutions constructed above:

- $\boldsymbol{u}_N$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega_f)) \cap L^2(0,T;H^1(\Omega_f))$.
- $\overline{\eta}_N$ is uniformly bounded in $W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^2(\Gamma))$.
- η_N is uniformly bounded in $L^\infty(0,T;H_0^2(\Gamma))$.
- v_N^* is uniformly bounded in $L^\infty(0,T;L^2(\Gamma))$.
- v_N is uniformly bounded in $L^\infty(0,T;L^2(\Gamma)) \cap L^2(0,T;H^{1/2}(\Gamma))$.

These uniform boundedness results give us the following weak convergences. There exist limiting solutions $\boldsymbol{u}$, η, and v such that

$$\begin{aligned}
&\boldsymbol{u}_N \rightharpoonup \boldsymbol{u} \text{ weakly-star in } L^\infty(0,T;L^2(\Omega_f)) \text{ and weakly in } L^2(0,T;H^1(\Omega_f)),\\
&\overline{\eta}_N \rightharpoonup \eta \text{ weakly-star in } W^{1,\infty}(0,T;L^2(\Gamma)) \text{ and weakly-star in } L^\infty(0,T;H_0^2(\Gamma)),\\
&\eta_N \rightharpoonup \eta \text{ weakly-star in } L^\infty(0,T;H_0^2(\Gamma)),\\
&v_N^* \rightharpoonup v \text{ weakly-star in } L^\infty(0,T;L^2(\Gamma)),\\
&v_N \rightharpoonup v \text{ weakly-star in } L^\infty(0,T;L^2(\Gamma)) \text{ and weakly in } L^2(0,T;H^{1/2}(\Gamma)).
\end{aligned}$$

Weak convergence is useful for passing to the limit in approximate solutions to linear problems, which is why at this point, we were able to pass to the limit in the semidiscrete formulation for the prototypical linearly coupled FSI problem to complete the proof of existence of a solution. However, with only weak convergences, we do not have strong enough convergence to pass to the limit in the semidiscrete formulation (1.24) for the prototypical nonlinearly coupled FSI problem. For example, if we consider the term

$$\int_{\Omega_f} \left(1 + \frac{\eta_N^n}{R}\right) [(\boldsymbol{u}_N^n \cdot \nabla^{\eta_N^n})\boldsymbol{u}_N^{n+1}] \cdot \boldsymbol{q}$$

for a fixed test function $\boldsymbol{q}$, we have a nonlinear term where we cannot just use weak convergence to pass to the limit, as we would need additional strong convergence results to take the limit in this nonlinear term. Thus, unlike in the linearly coupled case, we need to use compactness arguments to obtain stronger forms of convergence, in order to properly pass to the limit in the semidiscrete formulation.

A summary of compactness arguments. The general principle behind compactness arguments is that by obtaining uniform boundedness of spacetime functions in a particular function space and by obtaining additional uniform boundedness of the *time derivatives* of these functions in a weaker function space, we can obtain strong convergence of these functions to a limiting function. This is the fundamental principle behind the *Aubin-Lions compactness lemma*, which we will use to obtain strong convergence results for our approximate solutions. The Aubin-Lions compactness lemma relies on a chain of embeddings of function spaces in spatial variables:

$$Z \subset\subset Y \subset X,$$

where the embedding of Z into Y is a compact embedding. Then, for any $1 < p \leqslant \infty$, showing that the functions $\{f_n\}_{n=1}^{\infty}$ are uniformly bounded in $L^p(0,T;Z)$ and showing that their time derivatives are uniformly bounded in $L^p(0,T;X)$ implies that the functions f_n converge strongly along a subsequence in $L^p(0,T;Y)$ to a limiting function f. See [6, 129].

Note that the Aubin-Lions lemma immediately gives strong convergence of the structure displacements $\overline{\eta}_N$, via the chain of embeddings:

$$H_0^2(\Gamma) \subset\subset L^2(\Gamma) \subset L^2(\Gamma).$$

Since $\overline{\eta}_N$ are uniformly bounded in $L^\infty(0,T;H_0^2(\Gamma))$ and $\partial_t \overline{\eta}_N = v_N^*$ is uniformly bounded in $L^\infty(0,T;L^2(\Gamma))$, the Aubin-Lions compactness lemma implies that η_N converges along a subsequence strongly to a limiting function η in $L^\infty(0,T;L^2(\Gamma))$.

We will need a more involved argument to show the convergence of the fluid velocities and the structure velocities $(\boldsymbol{u}_N, v_N)$. In principle, we should be able to use an Aubin-Lions type argument to show strong convergence of $(\boldsymbol{u}_N, v_N)$ in an appropriate function space along a subsequence. To see this, let us return to the linearly coupled case for a brief moment, and consider what would happen in that case. In the linearly coupled case, the semidiscrete formulation reads:

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} + 2\mu_f \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t}\psi$$
$$+ C_0 \int_\Gamma \partial_z v_N^{n+\frac{1}{2}} \cdot \partial_z \psi + C_1 \int_\Gamma \partial_z^2 v^{n+\frac{1}{2}} \cdot \partial_z^2 \psi = \int_{\Gamma_{in}} P_{N,in}^n q_z - \int_{\Gamma_{out}} P_{N,out}^n q_z,$$

which we have reproduced from (1.8) for convenience, where $(\boldsymbol{q}, \psi)$ is a test function in a fixed test function space $\mathcal{Q} := \mathcal{V}_f \times \mathcal{V}_s$, where $\mathcal{V}_f$ and $\mathcal{V}_s$ are defined by (1.6) and (1.5) respectively. We can formally think of $\dfrac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t}$ and $\dfrac{v_N^{n+1} - v_N^n}{\Delta t}$ as a discretized form of the time derivative of $(\boldsymbol{u}_N, v_N)$, in which case we can use the weak formulation for fixed but arbitrary test functions $(\boldsymbol{q}, \psi)$ satisfying $||(\boldsymbol{q}, \psi)||_{\mathcal{Q}} \leqslant 1$ to show that the discretized time derivative of $(\boldsymbol{u}_N, v_N)$ is at least formally uniformly bounded in $L^2(0,T;\mathcal{Q}')$. In addition, $(\boldsymbol{u}_N, v_N)$ itself is uniformly bounded in $L^2(0,T;H^1(\Omega_f) \times H^{1/2}(\Gamma))$, and hence we conclude that the fluid velocities and structure velocities $(\boldsymbol{u}_N, v_N)$ are strongly precompact in $L^2(0,T;L^2(\Omega_f)\times L^2(\Gamma))$ via the chain of embeddings:

$$H^1(\Omega_f)\times H^{1/2}(\Gamma)\subset\subset L^2(\Omega_f)\times L^2(\Gamma)\subset \mathcal{Q}'.$$

Even though in principle, we should expect there to be a similar convergence of $(\boldsymbol{u}_N, v_N)$ in the nonlinearly coupled case, the same standard Aubin-Lions argument will not apply here. This is because the approximate fluid velocities $\boldsymbol{u}_N^n$ are defined on moving fluid domains $\Omega_{f,N}^n$, which are defined by the approximate structure displacements η_N^n via

$$\Omega_{f,N}^n = \{(z,r)\in\mathbb{R}^2 : 0\leqslant z\leqslant L, 0\leqslant r\leqslant R+\eta_N^n\}.$$

Therefore, all of the approximate fluid velocities $\boldsymbol{u}_N^n$ are defined on different fluid domains, and hence, we cannot create a uniform chain of embeddings needed for the standard Aubin-Lions compactness lemma.

We could try to pull back all of the approximate fluid velocities $\boldsymbol{u}_N^n$ to the fixed reference domain Ω_f so that all of the resulting approximate fluid velocities will be defined on the same domain. However, the difficulty here is that the divergence-free condition $\nabla\cdot\boldsymbol{u}_N^n = 0$ on the physical domain will transform nonlinearly as $\nabla^{\eta_N^n}\cdot(\boldsymbol{u}_N^{n+1}\circ\boldsymbol{\Phi}_f^{\eta_N^n}) = 0$, so that the divergence-free condition on the reference domain will change depending on n and N. This will make it hard to properly identify an appropriate test space for the semidiscrete formulations on the fixed reference domain, since the divergence-free condition on the fixed reference domain becomes a nonlinear condition.

Instead, we will take the approach of working on the physical fluid domains $\Omega_{f,N}^n$ so that the divergence-free condition remains the same for all of the approximate fluid velocities $\boldsymbol{u}_N^n$. However, the difficulty here is that the approximate fluid velocities $\boldsymbol{u}_N^n$ are still all defined on different domains $\Omega_{f,N}^n$. To address this difficulty, we will define the approximate fluid velocities $\boldsymbol{u}_N^n$ on a common maximal fluid domain Ω_f^M that contains all of the approximate fluid domains $\Omega_{f,N}^n$ by extending $\boldsymbol{u}_N^n$ by zero in $\Omega_f^M\backslash\Omega_{f,N}^n$. The existence of such a maximal fluid domain Ω_f^M can be shown by establishing the existence of a continuous function $M(z)$ satisfying $M(0) = M(L) = 0$ such that

$$\eta_N^n(z)\leqslant M(z),\qquad \text{for all } N \text{ and } n = 0,1,...,N,$$

which follows from the uniform boundedness properties of the structure displacements. Then, we can define the maximal fluid domain by

$$\Omega_f^M = \{(z,r)\in\mathbb{R}^2 : 0\leqslant z\leqslant L, 0\leqslant r\leqslant R+M(z)\},$$

and we can extend the fluid velocities $\boldsymbol{u}_N^n$ to Ω_f^M by keeping these fluid velocities the same on $\Omega_{f,N}^n\subset\Omega_f^M$ and then using an extension by zero outside of $\Omega_{f,N}^n$ so that

$$\boldsymbol{u}_N^n = 0,\qquad \text{on } \Omega_f^M\backslash\Omega_{f,N}^n.$$

This gives us approximate fluid velocities that still satisfy the divergence-free condition, but now all of the fluid velocities are defined on a common fluid domain Ω_f^M. While this extension by zero may introduce discontinuities into the functions $\boldsymbol{u}_N^n$, because of the fact that the structure displacements η_N^n are uniformly Lipschitz and the fact that the fluid velocities $\boldsymbol{u}_N$ are uniformly bounded in $L^2(0,T;H^1(\Omega_f(t)))$, we then have that the extended fluid velocities $\boldsymbol{u}_N$ are uniformly bounded in $L^2(0,T;H^s(\Omega_f^M))$ for $0 \leqslant s < 1/2$. This follows from results in [131] on extensions by zero of functions defined on Lipschitz domains, and we refer the reader to the arguments about extensions by zero to the maximal fluid domain in [30, 136] for more details.

Therefore, for the approximate fluid velocities $\boldsymbol{u}_N$ defined on the common maximal fluid domain Ω_f^M, we have that $\boldsymbol{u}_N$ are uniformly bounded in $L^2(0,T;H^s(\Omega_f^M))$ for $0 \leqslant s < 1/2$. However, for a Aubin-Lions type compactness argument to work, we need further bounds on the "time derivatives" of $\boldsymbol{u}_N$. We get estimates on the "time derivatives" of $\boldsymbol{u}_N$ by using the semidiscretized form of the weak formulation, but since we are working with fluid velocities that are defined on the physical domains $\Omega_{f,N}^n$, we must transfer the semidiscretized weak formulation (1.24) from the reference fluid domain Ω_f to the physical fluid domain $\Omega_{f,N}^n$, so that the fluid velocities satisfy

$$\begin{aligned}
&\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1} - \tilde{\boldsymbol{u}}_N^n}{\Delta t} \cdot \boldsymbol{q} \\
&+ \frac{1}{2}\int_{\Omega_{f,N}^n} \left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \frac{r}{R+\eta_N^n} v_N^{n+\frac{1}{2}} \boldsymbol{e}_r\right)\cdot \nabla \boldsymbol{u}_N^{n+1}\right)\cdot \boldsymbol{q}\right] - \left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \frac{r}{R+\eta_N^n} v_N^{n+\frac{1}{2}} \boldsymbol{e}_r\right)\cdot \nabla \boldsymbol{q}\right)\cdot \boldsymbol{u}_N^{n+1}\right] \\
&\quad + \int_{\Omega_{f,N}^n} \frac{1}{2(R+\eta_N^n)} v_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1}\cdot \boldsymbol{q} + 2\mu_f \int_{\Omega_{f,N}^n} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{1}{2}}}{\Delta t}\psi \\
&\qquad = \int_{\Gamma_{in}} P_{N,in}^n(t) q_z - \int_{\Gamma_{out}} P_{N,out}^n(t) q_z,
\end{aligned}$$

where $\tilde{\boldsymbol{u}}_N^n = \boldsymbol{u}_N^n \circ \boldsymbol{\Phi}_f^{\eta_N^n} \circ \left(\boldsymbol{\Phi}_f^{\eta_N^{n-1}}\right)^{-1}$ is defined on $\Omega_{f,N}^n$. Thus, at least on a formal level, we can estimate the discretized time derivative in a dual space, where the test functions are test functions defined on $\Omega_{f,N}^n$. However, because the solution space and test space for this semidiscretized weak formulation on the physical domains depend on n and N, we will have to use a *generalization of the Aubin-Lions compactness lemma* to handle the fact that the physical domains are changing, where we will have to verify additional conditions to show that there exist mappings between the different test and solution spaces that are uniformly bounded in an appropriate sense, which will allow us to be able to properly compare functions defined on different physical domains with each other in a uniform way. We refer the reader to [30, 136] for additional details. In the end, we will be able to obtain that along a subsequence, we get convergence of the fluid velocities

$$\boldsymbol{u}_N \to \boldsymbol{u}, \qquad \text{in } L^2(0,T;L^2(\Omega_f^M)),$$

as $N \to \infty$, as a result of Aubin-Lions compactness arguments, generalized to the context of moving domains.

Convergence of test functions. In contrast to the linearly coupled prototypical model, we have seen that for the nonlinearly coupled prototypical model, we had to show strong convergence of the approximate solutions, as weak and weak-star convergence is not sufficient to handle the nonlinear terms in the weak formulation. Even upon obtaining strong convergence of the approximate solutions, there are still additional difficulties in passing to the limit that arise from the time-dependent nature of the fluid domain. Recall the semidiscrete formulation for the approximate solutions, which was stated in (1.24). The approximate solutions constructed via the splitting scheme, for each N and corresponding time step Δt, satisfy the semidiscrete formulation in (1.24) for all appropriate test functions $(\boldsymbol{q}, \psi) \in Q_N^n$. Note that the test space Q_N^n depends on n and N, however, because the test function $\boldsymbol{q} \in H^1(\Omega_f)$ must satisfy $\nabla^{\eta_N^n} \cdot \boldsymbol{q} = 0$ on Ω_f. Therefore, even though the test functions in Q_N^n are all defined on the same domain $\Omega_f \times \Gamma$, the divergence free condition transforms differently back to the reference domain depending on η_N^n and hence, depending on n and N. The fact that the divergence-free condition for the test space transforms differently depending on n and N is a problem that directly arises from the fact that the moving domain $\Omega_f(t)$ depends on the displacement η of the structure.

Therefore, test functions that are admissible in the test space for a certain choice of n and N will not generally be admissible for all of the semidiscrete formulations for all n and N. This makes it difficult to compare the semidiscrete formulations for different values of n and N, and hence, we have to find a way to transfer test functions between the various semidiscrete formulations, in such a way that these test functions become close to each other in the limit as N goes to infinity. While it is nice to have test functions $(\boldsymbol{q}, \psi)$ that are defined on a common reference domain $\Omega_f \times \Gamma$, the divergence-free condition $\nabla^{\eta_N^n} \cdot \boldsymbol{q} = 0$ on Ω_f is a nonlinear condition that is highly nontrivial to work with. Hence, we will instead consider test functions on the physical domain, where the divergence-free condition always reads $\nabla \cdot \boldsymbol{q} = 0$ (albeit on a time-dependent domain), and then we will pull test functions back to the reference domain using the ALE mapping for the structure displacement η_N^n for a given value of n and N.

To do this, recall that we have defined a maximal domain by showing the existence of a function $M(z)$ for $z \in [0, L]$, satisfying $M(0) = M(L) = 0$ and

$$M(z) \geqslant \eta_N^n(z), \qquad \text{for all } z \in [0, L], \text{ positive integers } N, \text{ and } n = 0, 1, 2, ..., N.$$

We then defined the associated maximal fluid domain in the physical space, given by

$$\Omega_f^M = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, 0 \leqslant r \leqslant R + M(z)\}.$$

Note that for all n and N, $\Omega_{f,N}^n \subset \Omega_f^M$ and furthermore, $\Omega_f^\eta(t) \subset \Omega_f^M$ for all $t \in [0, T]$ by the convergence of the approximate structure displacements η_N to the limiting structure displacement η, where

$$\Omega_f^\eta(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, 0 \leqslant r \leqslant R + \eta(t, z)\}.$$

Recall that the test space for the problem on the fixed domain is given by $\mathcal{Q}^{\eta}$, defined in (1.22). Our approach to constructing suitable test functions will involve constructing an appropriate dense subset of test functions in $\mathcal{Q}^{\eta}$ defined on the fixed reference domain, for which we can find corresponding test functions for the semidiscrete weak formulations which converge appropriately to the limiting test function for the limiting weak formulation as N goes to infinity. We emphasize that we have already extracted the limiting structure displacement η as a limit of the approximate structure displacements η_N so that η is a known function. To construct appropriate test functions for the limiting weak formulation and for the semidiscrete weak formulations, consider the subset of spatially smooth functions $\hat{\mathcal{Q}}$ from $[0, T]$ to $\Omega_f^M \times \Gamma$ satisfying the following properties:

1. $\psi \in C_c^1([0, T); H_0^2(\Gamma))$
2. There exists a function $\eta_m(t, z) \in C_c^{\infty}([0, T); H_0^2(\Gamma))$ such that $\eta_m(t, 0) = \eta_m(t, L) = 0$, for all $t \in [0, T]$ and $-R < \eta_m(t, z) \leqslant \eta(t, z)$ for all $t \in [0, T]$ and $z \in [0, L]$, and furthermore,
$$\boldsymbol{q}(t) = \psi \boldsymbol{e}_r \qquad \text{in } \Omega_f^M \backslash \Omega_f^m(t),$$
where
$$\Omega_f^m(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, 0 \leqslant r \leqslant R + \eta_m(t, z)\}.$$
3. $\boldsymbol{q}(t, \cdot)$ is a divergence-free function in $H^1(\Omega_f^M)$ satisfying the boundary conditions:
$$q_r = 0, \qquad \text{on } \Gamma_{in/out},$$
$$q_r = 0, \qquad \text{on } \Gamma_b.$$
So $\boldsymbol{q}(t, \cdot)$ is a divergence-free extension of the constant vector field $\psi \boldsymbol{e}_r$ defined on $\Omega_f^M \backslash \Omega_f^m(t)$ to the entire maximal fluid domain Ω_f^M.

Given a function $(\boldsymbol{q}, \psi) \in \hat{\mathcal{Q}}$, we define test functions $(\tilde{\boldsymbol{q}}_N, \psi)$ for the semidiscrete weak formulations and the test function $(\tilde{\boldsymbol{q}}, \psi)$ for the limiting weak formulation by pulling the divergence-free fluid functions on the maximal fluid domain back to the reference fluid domain using the appropriate ALE mapping:

$$\tilde{\boldsymbol{q}}_N(t, \cdot) = \boldsymbol{q}|_{\Omega_N^{\eta_N}(t)} \circ \boldsymbol{\Phi}_f^{\eta_N}(t, \cdot), \quad \tilde{\boldsymbol{q}}(t, \cdot) = \boldsymbol{q}|_{\Omega_f^{\eta}(t)} \circ \boldsymbol{\Phi}_f^{\eta}(t, \cdot).$$

Note that $\nabla^{\eta_N} \cdot \tilde{\boldsymbol{q}}_N = 0$ and $\nabla^{\eta} \cdot \tilde{\boldsymbol{q}} = 0$ on Ω_f and in addition, by the way the functions $\boldsymbol{q} \in \hat{\mathcal{Q}}$ are defined, we also have that $\tilde{\boldsymbol{q}}_N|_{\Gamma} = \psi \boldsymbol{e}_r$ and $\tilde{\boldsymbol{q}}|_{\Gamma} = \psi \boldsymbol{e}_r$ as desired for all sufficiently large N, by the strong convergence of η_N to η in $C(0, T; C(\Gamma))$. We also remark that the test function $\tilde{\boldsymbol{q}}_N$ is actually discontinuous because of the fact that η_N itself is piecewise constant on the subintervals $[n\Delta t, (n+1)\Delta t)$ and hence, (potentially) discontinuous at the endpoints of each time step.

We then use the test functions $(\tilde{\boldsymbol{q}}_N, \psi_N)$ restricted to the subintervals $[n\Delta t, (n+1)\Delta t)$ as the test functions in the semidiscrete weak formulations and we use $(\tilde{\boldsymbol{q}}, \psi)$ as the test functions for the limiting weak formulation, whenever $(\boldsymbol{q}, \psi) \in \hat{\mathcal{Q}}$. The test functions $(\tilde{\boldsymbol{q}}, \psi)$ on $\Omega_f^\eta(t)$ that can be generated in this way are dense in the test space $\mathcal{Q}^\eta$. In general, the test functions $\tilde{\boldsymbol{q}}_N$ and $\tilde{\boldsymbol{q}}$ are different. However, we can show that they are close to each other for sufficiently large N, as

$$\tilde{\boldsymbol{q}}_N \to \tilde{\boldsymbol{q}} \text{ and } \nabla\tilde{\boldsymbol{q}}_N \to \nabla\tilde{\boldsymbol{q}} \quad \text{pointwise uniformly in } [0, T] \times \overline{\Omega_f}.$$

This convergence result for the test functions combined with the strong convergence results for the approximate solutions allows us to pass to the limit in the semidiscrete weak formulations to show that the limiting fluid velocity and structure displacement $(\boldsymbol{u}, \eta)$ satisfy the limiting weak formulation. We remark that to show that this weak solution can be defined on a maximal time interval until the time of domain degeneracy, one can use a standard argument introduced in pg. 397-398 of [42], which is also used for the constructive existence proof for the current nonlinearly coupled prototypical FSI model in the proof of Theorem 7.1 in [140]. This concludes the final step of the proof of the existence of a weak solution to the nonlinearly coupled prototypical FSI model.

1.4 Summary of the prototypical FSI model

In the previous two sections, we outlined the main steps of constructive existence for two prototypical models of FSI, consisting of a two-dimensional incompressible viscous fluid in a channel interacting with the elastic walls of the channel, with the coupled fluid-structure dynamics driven by inlet and outlet pressure data. The proof in both cases is a constructive existence proof, which establishes existence of a weak solution to the prototypical FSI model via a splitting scheme, where we split the structure and fluid subproblems and solve them each on discretized time steps, giving rise to a semidiscrete weak formulation satisfied by appropriate approximate solutions which are obtained from the constructive splitting scheme. The splitting scheme is constructed by assigning certain terms in the weak formulation to the structure and fluid subproblems, in a precise way that preserves the energy estimates in the discretized setting.

From here, in the linearly coupled model involving the linear Stokes equations for the fluid, uniform semidiscrete energy estimates obtained from the splitting scheme show that we have uniform boundedness of the approximate solutions in the finite energy spaces, which allows us to find weakly and weakly-star convergent subsequences of approximate solutions. In the linearly coupled case, since the weak formulation involves linear terms, we can then directly pass to the limit in the semidiscrete weak formulation to obtain existence of a weak solution that satisfies the continuous-in-time weak formulation of the full problem. While we also obtain uniform boundedness of approximate solutions in the nonlinearly coupled case, the analysis is more complex in this case because of the moving boundary, which introduces geometric nonlinearities into the weak formulation, and because of the nonlinear advection term in the full Navier-Stokes equations.

In contrast to the linearly coupled case, having weak and weak-star convergent subsequences of approximate solutions is not sufficient to conclude the proof for the nonlinearly coupled prototypical model. The additional nonlinearities which appear in the weak formulation require strong convergence of approximate solutions in order to pass to the limit, and hence, one must obtain additional estimates on the time derivatives or on the discretized time derivatives of the approximate solutions. By using the semidiscrete formulation, we can estimate the discretized time derivatives of the approximate solutions in appropriate dual spaces and obtain strong precompactness in appropriate function spaces by using Aubin-Lions type compactness arguments. One important novelty in the analysis of the prototypical model of nonlinearly coupled FSI is a new Aubin-Lions compactness argument for functions defined on moving domains, which is needed to handle the fact that the test functions and solution spaces for the semidiscretized problems depend on the structure displacement and the moving physical fluid domains, so this generalization of the Aubin-Lions compactness lemma to moving domains gives a way of uniformly comparing functions defined on different physical domains with each other. These Aubin-Lions type compactness arguments allow us to obtain the strong convergence results that will ultimately help us pass to the limit in the semidiscrete formulations as $N \to \infty$. The final ingredient needed in the constructive existence proof for the nonlinearly coupled prototypical FSI model is a careful analysis of the test functions in the semidiscrete formulations, as the test functions themselves depend on the structure displacement, through the transformation of the divergence-free condition on the physical fluid domain to a nonlinear divergence-free type condition on the reference fluid domain, due to the transformation of spatial derivatives under the ALE mapping. However, by considering an appropriate subset of test functions and pulling them back to the reference domain via the ALE mappings defined by the approximate structure displacements η_N, we can define a test function $(\tilde{\boldsymbol{q}}_N, \psi)$ for the semidiscrete weak formulation with parameter N that will converge in the limit to the limiting test function $(\tilde{\boldsymbol{q}}, \psi)$ with $\nabla^\eta \cdot \tilde{\boldsymbol{q}} = 0$ where η is the limiting structure displacement. This will allow us to complete the proof of constructive existence of a weak solution for the nonlinearly coupled prototypical FSI model.

1.5 Extensions of the splitting scheme for FSI

The splitting scheme approach to fluid-structure interaction that we have demonstrated in the context of the 2D prototypical model of FSI is robust, as it has been generalized to a variety of different physically relevant contexts. The mathematical framework for showing constructive existence of weak solutions provided by the splitting scheme approach has been useful in analyzing complex fluid-structure interaction models, and has been used to analyze coupled fluid-structure systems of real-life significance in applications. We outline the more complex models of FSI that have been analyzed using extensions of this Lie operator splitting approach to fluid-structure interaction, in order to demonstrate the robustness of this method, which will lay the foundation for the work outlined in this thesis.

3D-2D fluid-structure systems

The prototypical model of FSI described in both the linearly coupled and nonlinearly coupled case involves a 2D incompressible viscous fluid interacting with a 1D elastic structure. However, it is of practical interest to extend constructive existence results for weak solutions to coupled fluid-structure systems to the physically relevant dimensions. In particular, we would like to obtain corresponding results for incompressible viscous fluids in three dimensions interacting with two-dimensional elastic structures, such as membranes or shells.

We first note that in the linearly coupled prototypical model of FSI, the analysis would be unchanged for the most part if we considered a three-dimensional Stokes flow interacting with a two-dimensional elastic structure. This is because of the fact that in the linearly coupled case, the geometric configuration of the fluid domain in time is not taken into account, so the fluid equations, even in three dimensions, are posed on a fixed reference fluid domain Ω_f, which has a regular and well-behaved fixed geometry, even though it is a three-dimensional spatial domain.

However, there are additional challenges that appear when considering three-dimensional incompressible Navier-Stokes equations coupled to two-dimensional elastic structures, in the nonlinearly coupled prototypical model of FSI. The main difficulty lies in the fact that when considering a plate, we have that the structure displacement η from the reference configuration is in the finite energy space $W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^2(\Gamma))$. Because the problem is nonlinearly coupled, the regularity of the structure displacement η is especially important, as the structure displacement determines the time-dependent location of the structure $\Gamma(t)$, which in turn determines the three-dimensional time-dependent fluid domain $\Omega_f(t)$, which is a domain that is bounded by the inlet and outlet Γ_{in} and Γ_{out}, the bottom boundary Γ_b, and the time-dependent moving structure $\Gamma(t)$.

Because we are considering three spatial dimensions for the fluid and two spatial dimensions for the elastic structure, we have different Sobolev embeddings. For a one-dimensional structure as in the 2D-1D prototypical fluid-structure model described before, we have that for a one-dimensional structure, a structure displacement $\eta \in W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^2(\Gamma))$ defines a function that is uniformly Lipschitz continuous for all $t \in [0,T]$. However, for a two-dimensional structure, a function $\eta \in H_0^2(\Gamma)$ is not a Lipschitz continuous function, so the domain $\Omega_f(t)$ is not a Lipschitz domain. This has several important ramifications that complicate the analysis of three-dimensional nonlinearly coupled FSI, which was carried out in [137] and was later extended to three-dimensional FSI involving nonlinear elasticity in [141].

First, the standard *trace theorem* from the classical theory of Sobolev spaces states that functions in $H^1(\Omega)$ have well-defined traces in $L^2(\partial\Omega)$ and even $H^{1/2}(\partial\Omega)$ under the condition that Ω is a Lipschitz domain. However, because $\Omega_f(t)$ in the three-dimensional case of nonlinearly coupled FSI is no longer necessarily a Lipschitz domain due to the limited regularity of the structure displacement η, we must use an extension of the trace theory for Sobolev spaces. This extension of the Sobolev trace theory was developed in a note [135], which states an extension of the Sobolev trace theorem to functions defined on domains which are locally subgraphs of Hölder continuous functions. This is exactly what is required in the current scenario of three-dimensional nonlinearly coupled FSI, since structure displacements η which are in $H_0^2(\Gamma)$ are not Lipschitz continuous, but are α-Hölder continuous for $\alpha \in (0, 1)$. The result from [135] states that the trace of an $H^1(\Omega)$ function on a domain Ω that is locally a subgraph of an α-Hölder continuous function belongs to $H^s(\partial\Omega)$ for any $s \in (0, \alpha/2)$.

This result on the extension of the Sobolev trace theory, described in [135], is exactly what is needed to properly define the trace of the fluid velocity $\boldsymbol{u} \in L^2(0, T; H^1(\Omega_f(t)))$. This allows us to impose the boundary conditions on Γ_{in}, Γ_{out}, and Γ_b on $\boldsymbol{u}$, as the trace of $\boldsymbol{u}$ along the inlet, outlet, and bottom boundary are well-defined. The fact that we can define the trace of $\boldsymbol{u}$ in a well-defined manner allows us to also precisely make sense of the kinematic coupling condition $\boldsymbol{u}|_{\Gamma(t)} = \partial_t \eta \boldsymbol{e}_r$, which involves the trace of $\boldsymbol{u}$ along the time-dependent moving interface $\Gamma(t)$.

Nonlinear structures

The study of nonlinearly elastic structures, for example St. Venant-Kirchhoff elastic structures, nonlinearly elastic Koiter shells, and nonlinear biomembranes, has also been considered in the literature on FSI [141, 143, 44, 45, 52, 153, 74, 80]. The splitting scheme methodology of establishing existence of weak solutions to fluid-structure interaction has been successfully extended from the prototypical model of FSI to fluid-structure systems with nonlinear elastic structures in [141, 143]. These elastic structures with nonlinear behavior appear in many applications of real-life significance. For example, a computational study of fluid-structure interaction in a curved coronary artery [28] involves considering nonlinear equations of elasticity for the displacement of the curved cylindrical arterial walls from its reference configuration, where a cubic-type nonlinearity appears in the elasticity equations as a result of the curved cylindrical geometry of the coronary artery. The extension of the splitting scheme to FSI systems involving nonlinear structures was done first in [141], and splitting scheme methods have been used in FSI systems of practical significance involving nonlinearly elastic structures, see for example [143, 28].

We briefly summarize some of the new approaches needed in the splitting scheme to handle the unique challenges posed by nonlinear elastic structures. We describe these mathematical developments in the context of the study of FSI involving a nonlinear Koiter shell, in [141]. Before discussing the nonlinear Koiter shell theory, we briefly look back at the prototypical model of FSI, where in both the linearly coupled and the nonlinearly coupled case, the transverse (scalar) displacement η of the elastic structure is described by a linear Koiter shell equation

$$\partial_t^2 \eta - C_0 \partial_z^2 \eta + C_1 \partial_z^4 \eta = F, \qquad \text{on } \Gamma,$$

where in the prototypical model that we discussed, Γ was the one-dimensional reference configuration of a one-dimensional elastic structure. The linear Koiter shell equation is a fourth-order linear PDE that has the following associated elastic energy $||\eta||_E^2$, which arises from testing the equation with $\partial_t \eta$ in a formal energy estimate and integrating by parts:

$$||\eta||_E^2 = \frac{1}{2}\left(C_0||\partial_z \eta||^2_{L^2(\Gamma)} + C_1||\partial_z^2 \eta||^2_{L^2(\Gamma)}\right).$$

As seen above, the linear Koiter shell equation can be written as a PDE, which then gives rise to a corresponding associated elastic energy. Meanwhile, for the nonlinear Koiter shell equation, it is most natural to define the nonlinear elastic energy first, and then from there, one can derive an expression for the PDE governing the nonlinear Koiter shell dynamics. The nonlinear elastic energy of the nonlinear Koiter shell is given by

$$||\eta||_E^2 = \int_\Gamma \mathcal{A}\boldsymbol{G}(\eta) : \boldsymbol{G}(\eta),$$

where $\mathcal{A}$ describes the elasticity tensor, and $\boldsymbol{G}(\eta)$ describes the change in the metric due to the resulting curvature of the elastic shell arising from the displacement η of the shell. While this elastic energy is inherently associated with a nonlinear operator, it can be shown to be equivalent in norm to a standard Sobolev norm, which will be useful for the analysis of the problem. We refer the reader to [141] for full details.

We mention that the main difference in the splitting scheme is the structure subproblem. The nature of the splitting itself is similar in principle to the splitting scheme for the prototypical FSI problem with the linear Koiter shell, in the sense that the structure is split apart from the fluid, and the structure velocity is updated in both steps in order to account for the influence of both the structure and fluid dynamics on the resulting structure velocity. However, the main difference is the nature of the structure subproblem itself. The prototypical model of FSI in both the linearly and nonlinearly coupled case involves solving a time-discretized form of the linear Koiter shell with zero forcing, where the time derivative is discretized by using a backward Euler time discretization. This gives rise to a linear elliptic problem for the new structure displacement $\eta_N^{n+\frac{1}{2}}$ written in weak formulation, see (1.7), where the existence and uniqueness of the new updated structure displacement $\eta_N^{n+\frac{1}{2}}$ can be obtained using standard tools from the theory of linear elliptic PDEs, namely the Lax-Milgram theorem. However, in the case of a nonlinearly elastic Koiter shell, there are notable differences. First, one cannot use just a backwards Euler time discretization to discretize the structure subproblem, as a more involved time discretization is required in order to obtain a semidiscrete energy estimate which mirrors the corresponding continuous energy estimates for the nonlinear Koiter shell PDE. In addition, because the resulting problem that arises after time discretization is no longer a linear problem, the Lax-Milgram theorem is not sufficient for obtaining existence and uniqueness of a solution to the (nonlinear) weak formulation of the structure subproblem, and instead, one must invoke a fixed point argument using the Schaefer Fixed Point Theorem in order to obtain existence of an updated structure displacement $\eta_N^{n+\frac{1}{2}}$ from the structure subproblem. However, if one is able to obtain existence of a solution to the structure subproblem via a fixed point argument and if one is able to discretize time derivatives in an appropriate way that preserves the balance of energy of the fully coupled FSI dynamics, then the splitting scheme remains an effective method for studying well-posedness of FSI systems containing structures with nonlinear effects.

The consideration of splitting scheme methods for studying FSI with nonlinearly elastic structures was extended in recent work [143], in which a more general nonlinear Koiter shell considering contributions both membrane and bending energy is considered, whereas the past work in [141] does not include the fully nonlinear terms arising from the bending energy. The existence of a weak solution is established for this problem in [143] by using a splitting scheme approach to show existence of a weak solution to a form of the weak formulation containing a linear sixth order regularization of the structure dynamics, and then obtaining uniform estimates that are independent of the regularization parameter. In addition to showing well-posedness for this nonlinearly coupled FSI problem with a nonlinear Koiter shell, this work also establishes regularity results for the structure displacement η and the structure velocity $\partial_t \eta$.

Fluid-structure interaction with the Navier slip condition

In the prototypical model of fluid-structure interaction, we use the no-slip condition as a kinematic coupling condition to couple the fluid and structure dynamics together, and we hence impose continuity of velocities at the fluid-structure interface. In particular, we impose that the fluid velocity at the fluid-structure interface must be equal to the structure velocity, so that in the linearly coupled prototypical FSI model, we have that

$$\boldsymbol{u}|_{\Gamma} = (\partial_t \eta)\boldsymbol{e}_r,$$

and in the nonlinearly coupled prototypical FSI model, we have that

$$\boldsymbol{u}|_{\Gamma(t)} = (\partial_t \eta)\boldsymbol{e}_r.$$

In both the linearly coupled and nonlinearly coupled models, the no-slip condition implies that the fluid velocity along the fluid-structure interface is purely transversal (radial), since one of the assumptions in the prototypical model is that the structure displaces in only the transverse (radial) direction.

The no-slip condition is a useful condition to start with for mathematical analysis, but in many real-life systems, the no-slip condition is not sufficient to describe the coupled fluid-structure dynamics. In particular, in many fluid-structure systems, there is often tangential slip of the fluid along the fluid-structure interface, and one way of modeling this tangential slip is by imposing the **Navier slip condition**, which allows for a mismatch between the tangential components of the structure and fluid velocities. The extension of the Lie operator splitting method beyond the prototypical FSI model with the no-slip kinematic coupling condition to more complex FSI models involving the Navier slip condition was achieved in [139] and compactness arguments for this Navier slip nonlinearly coupled FSI model were carried out in [136] using the generalized Aubin-Lions compactness lemma for functions on moving domains.

The Navier slip condition will appear in an FSI model that we study later in this thesis (see the nonlinearly coupled model described in Chapter 5), so we briefly summarize the nonlinearly coupled FSI model involving the Navier slip condition from [139] and we discuss the ways in which the splitting scheme approach is adapted to handle the new features of this model. The fluid velocity and pressure are described as before by the nonlinear Navier-Stokes equations describing an incompressible, viscous Newtonian fluid, stated previously in (1.10) and (1.11), where the Navier-Stokes equations are posed on a time-dependent and a priori unknown fluid domain $\Omega_f(t)$, which will be two-dimensional. However, for the elastic structure, we will allow for *arbitrary vector-valued displacements* rather than considering only transverse (radial) structure displacements, as was done in the prototypical model of nonlinearly coupled FSI. Therefore, we will describe the structure displacement $\boldsymbol{\eta} : \Gamma \to \mathbb{R}^2$ from the reference configuration Γ by an elastodynamics equation governed by an elasticity operator $\mathcal{L}_e$:

$$\partial_{tt}\boldsymbol{\eta} + \mathcal{L}_e\boldsymbol{\eta} = F, \qquad \text{on } \Gamma,$$

where the elasticity operator $\mathcal{L}_e$ has the property that it will give rise to an elastic energy defined by the norm $||\boldsymbol{\eta}||_E^2 = (\mathcal{L}_e\boldsymbol{\eta}, \boldsymbol{\eta})_{L^2(\Gamma)}$, and this elastodynamics equation will be given appropriate clamped boundary conditions. The time-dependent location of the structure $\Gamma(t)$ will be described by

$$\Gamma(t) = \{(z, R) + \boldsymbol{\eta}(t, z) : z \in [0, L]\}$$

and since $\boldsymbol{\eta}(0) = \boldsymbol{\eta}(L)$, we can then define the time-dependent fluid domain $\Omega_f(t)$ to be the two-dimensional region that is bounded by Γ_{in}, Γ_{out}, Γ_b, and $\Gamma(t)$.

We will couple the fluid and structure dynamics via the kinematic coupling condition and the dynamic coupling condition, but these two coupling conditions will be different from the corresponding conditions in the prototypical model of nonlinearly coupled FSI. We describe these below:

- **Kinematic coupling condition.** Rather than imposing a single no-slip condition as in the prototypical model of FSI described previously, we will instead impose two kinematic coupling conditions: a condition for the normal component of the fluid velocity and another for the tangential component of the fluid velocity, where we consider the fluid velocity along the moving fluid-structure interface. For the normal component of the fluid velocity, we prescribe a **no penetration** condition, which states that there is no net leakage of fluid out of the domain in the sense that the normal components of the fluid velocity and the structure velocity along the interface $\Gamma(t)$ agree:

$$\boldsymbol{u}|_{\Gamma(t)} \cdot \boldsymbol{n}(t) = \partial_t\boldsymbol{\eta} \cdot \boldsymbol{n}(t),$$

  where $\boldsymbol{n}(t)$ is the outward pointing unit normal vector to the moving fluid-structure interface $\Gamma(t)$. For the tangential component of the fluid velocity, we prescribe the **Navier slip condition**, which states that the difference in the tangential components of the structure velocity and the fluid velocity along the interface is proportional to the tangential component of the normal stress of the fluid on the structure:

$$\beta(\partial_t \boldsymbol{\eta} - \boldsymbol{u}|_{\Gamma(t)}) \cdot \boldsymbol{\tau}(t) = \boldsymbol{\sigma}(\boldsymbol{u}, \pi)\boldsymbol{n}(t) \cdot \boldsymbol{\tau}(t)|_{\Gamma(t)},$$

where $\boldsymbol{\tau}(t)$ is the unit rightward pointing tangent vector to $\Gamma(t)$ and $\beta \geqslant 0$ is a non-negative Navier slip coefficient.

- **Dynamic coupling condition.** In principle, the dynamic coupling condition here is the same in the sense that it states that the structure subproblem is loaded by the fluid load onto the structure. The only change in the dynamic coupling condition from the prototypical model arises from the fact that we are considering general structure displacements, so instead of loading the structure with $-\mathcal{J}^\eta \boldsymbol{\sigma}(\boldsymbol{u}, \pi)\boldsymbol{n}(t) \cdot \boldsymbol{e}_r$ which would be the fluid load only in the transverse (radial) direction, the forcing on the structure equation here is given by the full fluid load on the structure:

$$\partial_{tt}\boldsymbol{\eta} + \mathcal{L}_e \boldsymbol{\eta} = -\mathcal{J}^\eta \boldsymbol{\sigma}(\boldsymbol{u}, \pi)\boldsymbol{n}(t)|_{\Gamma(t)}.$$

Now that we have finished the description of the nonlinearly coupled FSI problem, we now point out some of the key differences in the splitting scheme proof, and refer the reader to [136, 139] for full details. One of the first key differences is in the energy estimate, where we can see the impact of the Navier slip condition in an additional Navier slip dissipation term. In particular, the energy estimate for the nonlinearly coupled Navier slip FSI model is given by:

$$\begin{aligned}
&\frac{1}{2}||\partial_t \boldsymbol{\eta}(t)||^2_{L^2(\Gamma)} + ||\boldsymbol{\eta}(t)||^2_E + \frac{1}{2}||\boldsymbol{u}(t)||^2_{L^2(\Omega_f(t))} \\
&\qquad + \mu_f \int_0^t ||\boldsymbol{D}(\boldsymbol{u})(s)||^2_{L^2(\Omega_f(s))} ds + \beta \int_0^t ||(\partial_t \boldsymbol{\eta} - \boldsymbol{u})|_{\Gamma(t)} \cdot \boldsymbol{\tau}(s)||_{L^2(\Gamma(s))} ds \\
&\qquad \leqslant \frac{1}{2}||v_0||^2_{L^2(\Gamma)} + ||\boldsymbol{\eta}_0||^2_E + \frac{1}{2}||\boldsymbol{u}_0||^2_{L^2(\Omega_f(0))} + C\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right),
\end{aligned}$$

where the term involving β represents the frictional dissipation due to the tangential slip of the fluid along the structure interface.

When defining the weak formulation, there are also several important modifications. In the prototypical nonlinearly coupled no-slip model, the solution space incorporates a no-slip condition $\boldsymbol{u}|_{\Gamma(t)} = (\partial_t \eta)\boldsymbol{e}_r$ and the test space for the test functions $(\boldsymbol{q}, \psi)$ incorporates this condition also as $\boldsymbol{q}|_{\Gamma(t)} = \psi \boldsymbol{e}_r$. However, this changes slightly in the Navier slip case. For the model with the Navier slip condition, we note that the Navier slip condition will be incorporated into the weak formulation via a term with β, so the only condition that needs to be explicitly prescribed in the solution space and the test space is the no penetration condition. Therefore, we instead define the coupled solution space to be

$$\mathcal{V} = \{(\boldsymbol{u}, \boldsymbol{\eta}) \in V_f \times V_s : \boldsymbol{u} \cdot \boldsymbol{n}(t)|_{\Gamma(t)} = \partial_t \boldsymbol{\eta} \cdot \boldsymbol{n}(t)\}$$

for appropriate finite energy spaces V_f and V_s, and we define the test space to be

$$\mathcal{Q} = \{(\boldsymbol{q}, \boldsymbol{\psi}) \in C_c^\infty([0,T); \mathcal{V}_f(t) \times \mathcal{V}_s) : \boldsymbol{q} \cdot \boldsymbol{n}(t)|_{\Gamma(t)} = \boldsymbol{\psi} \cdot \boldsymbol{n}(t)\},$$

where $\mathcal{V}_f$ and $\mathcal{V}_s$ are defined similarly to (1.15) and (1.16) as in the prototypical nonlinearly coupled FSI model, but in the current context, the trace of the fluid velocity along $\Gamma(t)$ is no longer restricted to be purely radial and the structure displacement can more generally be vector-valued.

In this case, the weak formulation involves finding a solution $(\boldsymbol{u}, \boldsymbol{\eta}) \in \mathcal{V}$ such that for all test functions $(\boldsymbol{q}, \boldsymbol{\psi}) \in \mathcal{Q}$,

$$\begin{aligned}
-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} + \frac{1}{2} \int_0^T \int_{\Omega_f(t)} \Big([(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \boldsymbol{q} - [(\boldsymbol{u} \cdot \nabla)\boldsymbol{q})\boldsymbol{u}]\Big) - \int_0^T \int_{\Gamma(t)} (\partial_t \boldsymbol{\eta} \cdot \boldsymbol{n}(t))(\boldsymbol{u} \cdot \boldsymbol{q}) \\
+ \frac{1}{2} \int_0^T \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n}(t))(\boldsymbol{u} \cdot \boldsymbol{q}) - \beta \int_{\Gamma(t)} [(\partial_t \boldsymbol{\psi} - \boldsymbol{q}) \cdot \tau(t)] \cdot [(\partial_t \boldsymbol{\eta} - \boldsymbol{u}) \cdot \tau(t)] \\
+ 2\mu_f \int_0^T \int_{\Omega_f(s)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) - \int_0^T \int_\Gamma \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\psi} + \int_0^T (\boldsymbol{\eta}, \boldsymbol{\psi})_E \\
= \int_0^T \int_{\Gamma_{in}} P_{in}(t) \cdot q_z - \int_0^T \int_{\Gamma_{out}} P_{out}(t) \cdot q_z + \int_{\Omega_f(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) + \int_\Gamma v_0 \cdot \psi(0).
\end{aligned}$$

We conclude by making a few observations about this weak formulation. Rather than having a single term

$$-\int_0^T \int_\Gamma (\partial_t \eta)^2 \cdot \psi$$

in the weak formulation as in the no-slip prototypical model, the weak formulation instead has the terms

$$-\int_0^T \int_{\Gamma(t)} (\partial_t \boldsymbol{\eta} \cdot \boldsymbol{n}(t))(\boldsymbol{u} \cdot \boldsymbol{q}) + \frac{1}{2} \int_0^T \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n}(t))(\boldsymbol{u} \cdot \boldsymbol{q}),$$

which can also be expressed as the single term $-\frac{1}{2} \int_0^T \int_{\Gamma(t)} (\partial_t \boldsymbol{\eta} \cdot \boldsymbol{n}(t))(\boldsymbol{u} \cdot \boldsymbol{q})$ by the no penetration condition. Compared to the weak formulation for the prototypical model, there is also an additional dissipation term in the weak formulation, which is the term involving the Navier slip coefficient β. We note that although the Navier slip FSI model now allows for tangential fluid velocities, the problem is still well-defined in the sense that the tangential fluid velocities are taken into account in the weak formulation (so that the Navier slip condition on the tangential fluid and structure velocities is weakly imposed) and then the remaining normal component of the fluid velocity is handled explicitly in the definition of the solution and test spaces, where the matching of normal velocities of the fluid and structure at the interface is strongly imposed. A similar fluid-structure splitting scheme can be developed for this Navier slip model, and we refer the reader to [139] for a full exposition.

Multilayered structures in FSI

Another important extension of the prototypical FSI model that will play an important role in an FSI model considered later in Chapter 5 of this thesis is the extension to coupled fluid-structure systems involving *multilayered composite structures*, which are made up of different layers with varying properties. This is an important extension that is particularly relevant in many real-life applications of FSI, as many FSI systems, especially in biomedical applications, have structures that are made up of multiple layers with different physical properties. This includes for example, arterial walls, which are made up of several different layers of arterial tissue with different thicknesses and elastic properties, and bioartificial organs, which have different layers in the structural walls that have different functionalities [138]. The extension of the splitting scheme approach for FSI to the interaction between a fluid and a multilayered structure was first carried out in [138].

FSI with multilayered structures will play a role in a nonlinearly coupled FSI model that is discussed later in this thesis in Chapter 5, so we will describe the multilayered structure FSI model that is considered in [138]. In this model, the fluid is an incompressible, viscous Newtonian fluid which is flowing in an elastic channel where the walls of the channel are elastic and are made up of multiple layers. Thus, the structure is now a multilayered composite structure that consists of an external thick layer and an inner thin layer that is in contact with the fluid. The problem is posed on the following geometry: the fluid domain has a reference configuration of $\Omega_f = [0, L] \times [0, R]$, the thick structure has a reference configuration of $\Omega_s = [0, L] \times [R, R + h]$ where h is the thickness of the structure, and the thin structure is a one-dimensional elastic structure with a reference configuration of $\Gamma = [0, L] \times \{R\}$.

Let us define $\boldsymbol{d}$ to be the displacement of the two-dimensional thick structure from its reference configuration Ω_s, and let η be the displacement of the thin structure from its one-dimensional reference configuration Γ. The incompressible viscous Newtonian fluid is described by the Navier-Stokes equations (1.10) posed on a moving time-dependent fluid domain, where the moving fluid domain $\Omega_f(t)$ is determined by the position $\Gamma(t)$ of the inner thin structure layer in direct contact with the fluid so that

$$\Omega_f(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, 0 \leqslant r \leqslant R + \eta(t, z)\},$$

where we assume that the thin layer displaces in only the transverse (radial) direction so that

$$\Gamma(t) = \{(z, r) \in \mathbb{R}^2 : 0 \leqslant z \leqslant L, r = R + \eta(t, z)\}.$$

The elastodynamics of the **thin structure** is described by the wave equation, where the scalar transverse displacement η of the one-dimensional thin structure from its reference configuration Γ is given by

$$\partial_{tt}\eta - \Delta\eta = F, \qquad \text{on } \Gamma,$$

where the external load F on the thin structure will be described later in the dynamic coupling condition. The elastodynamics of the thick structure are described by the equations of elasticity for the (two-dimensional) vector-valued displacement $\boldsymbol{d}$ of the thick structure from its reference configuration Ω_s so that

$$\partial_{tt}\boldsymbol{d} - \nabla \cdot \boldsymbol{S}(\boldsymbol{d}) = 0, \qquad \text{on } \Omega_s.$$

The elasticity stress tensor $\boldsymbol{S}$ for the thick structure is given by the Piola-Kirchhoff stress tensor

$$\boldsymbol{S}(d) = 2\mu_e \boldsymbol{D}(\boldsymbol{d}) + \lambda(\nabla \cdot \boldsymbol{d}),$$

where $\boldsymbol{D}(\boldsymbol{d})$ is the symmetrized gradient.

The kinematic and dynamic coupling conditions must now account for the fact that there are three subproblems that must be coupled together appropriately.

- The kinematic coupling condition is given by a no-slip condition between the fluid and thin structure
$$\boldsymbol{u}|_{\Gamma(t)} = (\partial_t \eta)\boldsymbol{e}_r,$$
and a continuity of displacements condition which ensures that the thin structure and thick structure share the same displacement along their common interface so that the structure remains intact:
$$\boldsymbol{d}|_{\Gamma} = \eta \boldsymbol{e}_r, \quad \text{on } \Gamma.$$

- The dynamic coupling condition specifies the fluid load on the thin structure, which is in contact from one side with the fluid and which is in contact on the other side with the thick structure. Therefore, there are two contributions to the traction along the thin structure: the fluid load from the inside and the elastic loading from the outside thick structure. Therefore, the elastodynamics equation for the thin structure reads:
$$\partial_{tt}\eta - \Delta\eta = -\mathcal{J}^\eta \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{e}_r|_{\Gamma(t)} + \boldsymbol{S}(\boldsymbol{\eta})\boldsymbol{e}_r \cdot \boldsymbol{e}_r|_{\Gamma},$$
where $\mathcal{J}^\eta = \sqrt{1 + (\partial_z \eta)^2}$.

For the multilayered structure FSI problem, we get contributions to the energy estimate for the fully coupled problem that involve energies and dissipation for the fluid, and elastic energies for both the thin structure and the thick structure. We obtain an energy estimate of the following form for the nonlinearly coupled multilayered structure FSI problem described above:

$$\begin{aligned}
&\frac{1}{2}||\partial_t \eta(t)||^2_{L^2(\Gamma)} + ||\eta(t)||^2_{H^1(\Gamma)} + \frac{1}{2}||\boldsymbol{u}(t)||^2_{L^2(\Omega_f(t))} + \mu_f \int_0^t ||\boldsymbol{D}(\boldsymbol{u})(s)||^2_{L^2(\Omega_f(s))} ds \\
&\quad + \frac{1}{2}||\partial_t \boldsymbol{d}(t)||^2_{L^2(\Omega_s)} + \mu_e ||\boldsymbol{D}(\boldsymbol{d})(t)||^2_{L^2(\Omega_s)} + \frac{\lambda}{2}||\nabla \cdot \boldsymbol{d}(t)||^2_{L^2(\Omega_s)} \\
&\leqslant \frac{1}{2}||v_0||^2_{L^2(\Gamma)} + ||\eta_0||^2_{H^1(\Gamma)} + \frac{1}{2}||\boldsymbol{u}_0||^2_{L^2(\Omega_f(0))} + \frac{1}{2}||\partial_t \boldsymbol{d}(0)||^2_{L^2(\Omega_s)} \\
&\quad + \mu_e ||\boldsymbol{D}(\boldsymbol{d}_0)||^2_{L^2(\Omega_s)} + \frac{\lambda}{2}||\nabla \cdot \boldsymbol{d}_0||^2_{L^2(\Omega_s)} + C\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right).
\end{aligned}$$

As emphasized in the original manuscript [138], this energy estimate reveals one of the primary reasons why considering multilayered structures in FSI is mathematically advantageous, especially for problems involving thick structures that are of the same dimension as the fluid domain. Having a multilayered structure where there is a thin layer that has a lower dimension than the full composite structure/fluid regularizes the geometry of the problem in the following way. For a thick elastic structure, the structure displacement $\boldsymbol{d}$ belongs to the finite energy space $L^\infty(0,T;H^1(\Omega_s))$ where Ω_s is two-dimensional. If the FSI model just involved a two-dimensional fluid in direct contact with a two-dimensional thick structure, then the time-dependent location of the interface between the fluid and the thick structure would be determined by the trace of $\boldsymbol{d}$ along the bottom boundary of Ω_s, which would only be in $L^\infty(0,T;H^{1/2}(\Gamma))$ in the finite energy space. Even with the interface Γ between the thick structure and the fluid being one-dimensional, $H^{1/2}(\Gamma)$ is not enough Sobolev regularity to ensure that we even have a continuous moving interface, and this lack of boundary regularity of the structure displacement for thick structures is one of the predominant mathematical obstacles in FSI models between fluids and thick structures of the same dimension.

If we instead consider a composite multilayered structure interacting with a fluid as in the model described above, we see that the elastodynamics of the thin membrane separating the thick structure and the fluid regularize the time-dependent fluid-structure interface. In particular, even though $\boldsymbol{d}$ is still in $L^\infty(0,T;H^1(\Omega_s))$, we have by the kinematic coupling condition that the trace of $\boldsymbol{d}$ along Γ is equal to $\eta\boldsymbol{e}_r$, where by the energy estimates, $\eta \in L^\infty(0,T;H^1_0(\Gamma))$. Thus, the elastodynamics of η place the thin structure displacement in a higher regularity space $H^1(\Gamma)$ in contrast to the space $H^{1/2}(\Gamma)$ that would arise from the Sobolev trace theorem applied to $\boldsymbol{d}$. Since we have that $\boldsymbol{d}|_\Gamma = \eta\boldsymbol{e}_r$ by the kinematic coupling condition between the thick and thin structures, we see that the higher $H^1(\Gamma)$ regularity of the interface displacement (which is now the thin structure displacement) transfers to the trace of the thick structure displacement along the bottom boundary Γ of Ω_s. This allows us to define the fluid-structure interface $\Gamma(t)$ to be a continuous curve, which is advantageous for the mathematical analysis of the problem.

One can then extend the splitting scheme for the nonlinearly coupled prototypical model of FSI to this multilayered model by developing a similar fluid-structure splitting, where the structure subproblem involves updating both the thick layer and the thin layer elastodynamics, and the thin layer structure velocity is updated in both the fluid and structure subproblem, in order to reflect the fact that the thin layer structure velocity must match the fluid velocity along the moving fluid-structure interface $\Gamma(t)$. For full details on the extension of the Lie operator splitting scheme to the case of multilayered structures interacting with incompressible fluids, we refer the reader to the exposition in the manuscript [138].

FSI with mesh-supported elastic structures

We conclude this literature review by discussing one more extension of the prototypical FSI model that is motivated by recent progress in biomedical engineering. Atherosclerosis is a cardiovascular condition caused by the buildup of excess plaque on arterial vessel walls, which limits the resulting flow of blood through arteries [106, 151]. Recently, advances in biomedical technology have improved the quality of patient health in patients who are affected by atherosclerosis. One of the leading successful treatments for this condition is the implantation of prostheses, called stents, which are mesh-like structures which are inserted in arteries with plaque buildup to restore healthy circulation. The specific geometry of how the metallic rods in a stent are connected to each other to form the stent is essential to the performance of the stent. Constructing stents to test in real-life experiments can be costly and time-inefficient, which makes developing numerical methods for simulating stents computationally essential for assessing stent design.

In past works, stents have been modeled as three-dimensional rod-like structures, but because stents are comprised of slender metallic rods, finite element discretizations in three dimensions of stents are costly in time, and also can suffer from inaccuracies, see the discussions for example in [31, 32]. Instead, a new approach, introduced in [174], models these stents using a reduced-dimensional model, so that the stents are modeled as one-dimensional structures described by a given mesh or graph topology. The stent is thus modeled as a network of one-dimensional hyperbolic equations which are coupled to each other at junction points. The reduced equations modeling the stent are one-dimensional, and this dimensional reduction is accurate since the metallic rods of the stent are thin and slender. This gives rise to a set of equations that can be solved efficiently, and which accurately model stent dynamics. These governing equations are derived from a curved rod model, where the equations are a coupled hyperbolic system of PDEs describing the evolution of the displacement and rotation of the stent, in addition to the contact force and contact moment. It is shown that this new reduced one-dimensional model of stents shows good agreement in computational simulations with three-dimensional simulations of stent dynamics, see [32]. In addition, asymptotic studies that relate three-dimensional dynamics of linear elasticity to one-dimensional equations for stents in limiting regimes (for example, the radius of the cross section decreasing to zero) have mathematically justified the use of these one-dimensional equations, see [85, 100, 101, 102, 157].

Due to interest in simulating the behavior of stents on arterial walls, there was work on simulating Naghdi type shells to the one-dimensional equations for a stent in [34], where existence of a unique solution is established using the Lax-Milgram theorem. However, the work [34] does not consider the blood flow through the arteries through an additional coupling to the Navier-Stokes equations. Alternatively, there was also work on modeling stents in [27], where a splitting scheme was used to show existence of weak solutions to a nonlinearly coupled FSI problem involving an elastic shell and the Navier-Stokes equations, where the elasticity parameters and thickness of the elastic shell are allowed to have jump discontinuities and are hence only prescribed to be in L^{∞}, to model the influence of the stent on the elastic properties of the structure. However, this work does not couple the resulting fluid-structure system explicitly to the equations for stent dynamics.

A full FSI model with three subproblems considering a stent coupled to an elastic shell, which is further coupled to fluid flow, was considered in [31, 35]. The splitting scheme methodology has been used to analyze a linearly coupled version [35] and a nonlinearly coupled version [31] of an FSI involving mesh-supported elastic structures interacting with the flow of a viscous incompressible Newtonian fluid. A similar fluid-structure splitting scheme has been successfully employed in both the linearly coupled and nonlinearly coupled versions of this FSI model involving structures supported by stents in order to show constructive existence of weak solutions. This model is particularly interesting because it is multiphysical, involving the interaction between three different mediums across three different dimensions: a three-dimensional fluid described by parabolic-type fluid equations for an incompressible viscous Newtonian fluid, a two-dimensional elastic shell structure described by hyperbolic equations of elasticity, and a one-dimensional mesh-like structure described by a hyperbolic system of PDEs on a graph.

Since these first existence results via Lie operator splitting for this mesh-supported structure FSI model, there has been additional analysis of such models. Analysis of the regularity of weak solutions to the linearly coupled model was carried out in recent work in [75]. In addition, the mathematical analysis of mesh-supported structures in FSI has set the foundation for the development of accurate numerical schemes which can be used to model real-life stent dynamics and assess the performance of various stent designs through computational simulations. This is done in [28] for example, where four different stent geometries are assessed through computations that are performed using a numerical method, where the full dynamics of blood flow interacting with arterial walls supported by different stent geometries are computationally analyzed. We remark that the study of stents is now an active area of research with many important practical implications for biomedicine. As a further example, we refer the reader to work in [33], where a numerical solver is used to model drug-eluting stents, which are stents which release anti-inflammatory agents into surrounding tissue in order to reduce the risk of arterial reclosure after stent implantation. In addition, studies of optimal stent design have been carried out in order to inform decisions about the design of stents in bioengineering, using mathematical techniques for constrained optimization problems, see [37].

1.6 Outline of the thesis

In this chapter, we have discussed the major developments in the analysis of fluid-structure interaction, focusing in particular on a splitting scheme approach to FSI which uses a Lie operator splitting to separate the fluid and structure subproblems in a precise way to allow for constructive existence of weak solutions to FSI. We summarized the splitting scheme methodology in the context of a linearly coupled prototypical model and a nonlinearly coupled prototypical model of FSI describing the flow of a two-dimensional incompressible viscous Newtonian fluid in an elastic channel with one-dimensional elastic walls. In the linearly coupled prototypical model, we considered a problem where the coupling conditions are evaluated along a fixed reference fluid-structure interface and where the linear Stokes equations modeling the fluid are posed on a fixed reference fluid domain. The nonlinearly coupled prototypical model involved a moving boundary problem where the full Navier-Stokes equations are solved on a moving a priori unknown fluid domain and the coupling conditions are evaluated along the moving interface between the fluid and the structure.

This splitting scheme methodology for studying weak solutions to coupled fluid-structure systems has proven to be extremely robust, and the splitting scheme used for the prototypical model of FSI has been extended to a wide variety of contexts of practical significance in real-life applications, including higher-dimensional models involving three-dimensional fluid flow through two-dimensional elastic shells, FSI models which allow tangential slip of the fluid along the fluid-structure interface, FSI models involving multilayered structures, and FSI models involving the flow of fluids in mesh-supported structures. This thesis will be focused on *new applications of the splitting scheme methodology*, which involve new FSI models with unique mathematical challenges that are inspired by real-life applications to engineering. This thesis will initiate the study of FSI in brand new contexts, by generalizing a splitting scheme approach in order to mathematically analyze new FSI models. The splitting scheme approach to FSI will be generalized to two new contexts: FSI models involving stochasticity (random effects in time) and models of nonlinearly coupled fluid-poroelastic structure interaction. By demonstrating these new applications of the splitting scheme approach, this work will affirm the robustness and wide applicability of splitting schemes in the analysis of complex fluid-structure systems that arise in applications.

In Chapters 2-4, we focus on the study of fluid-structure interaction with stochastic effects, which involves fluid-structure systems where there are random effects which can affect the system in time. All of the past analysis of fluid-structure systems has involved *deterministic systems* and even though the study of stochastic PDEs is a rich field and an active area of research, stochastic fluid-structure systems have not been considered previously. In Chapter 2, we provide a brief summary of probabilistic preliminaries that will be useful for the analysis of stochastic fluid-structure systems, with a particular emphasis on tools for analyzing random variables which take values in Banach spaces. In Chapter 3, we describe the first progress in stochastic fluid-structure interaction, involving the analysis of a linearly coupled reduced model of stochastic FSI, where the model is reduced in the sense that the full fluid-structure dynamics can be fully described by a single self-contained stochastic PDE for the structure dynamics known as the *stochastic viscous wave equation*. Although the analysis of this stochastic viscous wave equation does not need the use of the splitting scheme and relies more on standard tools from stochastic PDEs, we include a description of the analysis of this equation in Chapter 3, since this equation was the first equation studied in stochastic FSI. In Chapter 4, we use a splitting scheme approach to analyze a prototypical linearly coupled model of *stochastic FSI*, which describes the two-dimensional flow of an incompressible fluid modeled by the linear Stokes equations through an elastic channel with stochastically perturbed walls. This is the first such well-posedness result for a stochastic fluid-structure system that is *fully coupled*, and the splitting scheme approach is shown to be robust for such stochastic FSI problems involving stochastic multiphysical systems of PDEs.

Next, we turn our attention back to the deterministic theory of FSI in Chapter 5 and discuss recent developments in the analysis of fluid-structure interaction involving poroelastic structures, known as fluid-poroelastic structure interaction (FPSI). Poroelastic structures are materials that are porous and have the flow of fluid through their pores coupled to their elastic properties, and they can be modeled by a set of parabolic-hyperbolic or elliptic-hyperbolic type PDEs known as the Biot equations. Many structures in applications which are deformable or elastic are not completely solid or impermeable, and this includes soils, tissues in the human body, bones, rocks, sponges, and media used in bioartificial prostheses such as bioartificial organs. Due to their practical significance in technological advancements that have positive and direct societal impacts, poroelastic materials are important to consider, and creating a mathematical methodology for studying coupled FPSI systems is essential. In Chapter 5, we describe a nonlinearly coupled model consisting of an incompressible fluid interacting with a multilayered structure consisting of a thick poroelastic medium and a thin plate. We use a spatial regularization to study a regularized nonlinearly coupled FPSI problem via a splitting scheme approach, and the work done in this chapter represents the first well-posedness result for a nonlinearly coupled moving boundary model of FPSI. We continue the analysis of this nonlinearly coupled FPSI problem by showing that the weak solutions that we have constructed to the regularized FPSI problem are consistent in the sense that they converge as the regularization parameter tends to zero to classical solutions of the original non-regularized FPSI problem when such classical solutions exist. This will show that the weak solutions to the regularized FPSI problem that we have considered are physically relevant to real-life FPSI dynamics. In Chapter 6, we give some concluding remarks to summarize the work done in this thesis.

Chapter 2

Probabilistic preliminaries

In Chapters 2-4, we will discuss *stochastic fluid-structure interaction*, which describes a new and emerging class of coupled systems where a fluid and an elastic structure interact dynamically under the additional influence of stochasticity, or randomness in time. The study of stochastic FSI is motivated by the fact that randomness is inherent in many real-life systems, and this randomness can take the form of random forcing on the elastic structure in an FSI system, random deviations in the fluid flow, or randomness in the inlet and outlet pressure that drives the flow of a fluid through an elastic channel. Because randomness is commonplace in real-life dynamics and because stochasticity can have strong effects on the resulting observed dynamics of a system, the study of stochastic PDEs has been an active area of research for many decades. Despite the significant progress made in the analysis of stochastic PDEs, the analysis of stochastic fluid-structure systems has not been considered until very recently. The goal of Chapters 3 and 4 is to discuss the recent progress in stochastic FSI, and these chapters will discuss the models of stochastic FSI which have been developed to initiate the study of stochastic fluid-structure systems. In order to prepare for the presentation of the results in these chapters on stochastic FSI, we will use this current chapter, Chapter 2, in order to summarize the necessary probabilistic background and results from stochastic analysis that will be useful for studying stochastic fluid-structure systems.

In this chapter, we will focus on probabilistic background, with an emphasis on considering random variables which take values in Banach spaces. This will be important since stochastic fluid-structure systems involve random quantities, such as the fluid/structure velocity and the structure displacement, which will now be random functions due to the stochasticity in the system. We will summarize results about random variables taking values in Banach spaces and discuss the three modes of probabilistic convergence of such random variables which will play a key role in our analysis of stochastic FSI: convergence in probability, weak convergence (or convergence in law or distribution), and convergence almost surely. We will discuss the relationships between these modes of convergence and review classical theorems that will be important in our analysis later. Finally, in order to develop a vocabulary for discussing stochasticity in random systems, we will review the concept of one-dimensional white noise and spacetime white noise, which are prototypical examples of random noise that arise in stochastic PDEs. We will conclude by discussing the notion of stochastic integration against these types of random noise, which will be important for quantifying the effects of randomness on PDE dynamics and for rigorously defining the notion of a solution to a stochastic PDE or stochastic system of PDEs.

Though we will provide a brief review of many of the foundational results from probability theory that we will need later in this thesis, we assume that the reader is acquainted with abstract probability spaces and is comfortable with real-valued random variables. Though we will review the basics of probability theory more generally for random variables taking values in Banach spaces, we note that there are many parallels with the case of real-valued random variables. We refer the reader to texts such as [67, 103] for a detailed account of measure theoretic probability theory, and we assume basic familiarity with the general theory of abstract measure theoretic probability theory and real-valued random variables in the remainder of this chapter. For more information about Brownian motion and stochastic integration, we refer the reader to [58, 103, 150, 155] and for more information about stochastic PDEs, we refer the reader to the introductory book [58], the following classic book on stochastic analysis in infinite dimensional spaces [150], and the references [90, 177]. For more details about random variables taking values in general infinite dimensional Banach spaces, we refer the reader to the reference [150].

2.1 Probability spaces and random variables

Recall that a probability space is an ordered triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a set of outcomes Ω, a sigma algebra of measurable sets $\mathcal{F}$ (which can be thought of as the collection of events), and a probability measure $\mathbb{P}$ on the measurable sets (events) in $\mathcal{F}$. Given a probability space, we can talk about the probability of an event $A \in \mathcal{F}$ happening, and we denote this probability by $\mathbb{P}(A)$.

Let $\mathcal{B}$ be a Banach space with a norm denoted by $||\cdot||_{\mathcal{B}}$. A **$\mathcal{B}$-valued random variable** $X(\omega)$ is a measurable map $X(\omega) : (\Omega, \mathcal{F}) \to (\mathcal{B}, \text{Borel}(\mathcal{B}))$, where $\text{Borel}(\mathcal{B})$ is the sigma algebra of Borel measurable subsets of the Banach space $\mathcal{B}$. For simplicity of notation, we usually omit the dependence of a random variable on the outcome $\omega \in \Omega$, and hence, we will denote a random variable by X instead of $X(\omega)$, whenever it is clear from context that X is random. In the case where $\mathcal{B} = \mathbb{R}$, this coincides with the usual definition of a real-valued random variable. Given a random variable $X(\omega) : \Omega \to \mathcal{B}$, we define the **law of** X to be the probability measure μ_X on $(\mathcal{B}, \text{Borel}(\mathcal{B}))$ defined by

$$\mu_X(A) = \mathbb{P}(X(\omega) \in A), \quad \text{for } A \in \text{Borel}(\mathcal{B}).$$

It will be important for us later to quantify the boundedness properties of random variables taking values in Banach spaces. A $\mathcal{B}$-valued random variable $X(\omega) : \Omega \to \mathcal{B}$ belongs to the space $L^p(\Omega; \mathcal{B})$ if

$$\mathbb{E}\Big(||X(\omega)||_{\mathcal{B}}^p\Big) < \infty, \text{ if } 1 \leqslant p < \infty,$$

$$||X(\omega)||_{\mathcal{B}} < C \text{ almost surely for some constant } C \geqslant 0, \text{ if } p = \infty.$$

Knowing that a random variable $X(\omega)$ is in $L^p(\Omega; \mathcal{B})$ gives important information about how likely it is for X to attain large values. This is quantified through **Chebychev's inequality**, which states that if $X \in L^p(\Omega; \mathcal{B})$ for $1 \leqslant p < \infty$, then

$$\mathbb{P}(||X||_{\mathcal{B}} \geqslant \lambda) \leqslant \frac{\mathbb{E}(||X||_{\mathcal{B}}^p)}{\lambda^p}, \qquad \text{for all } \lambda > 0.$$

See also Theorem 1.6.4 in [67] for a more general form of Chebychev's inequality.

2.2 Probabilistic convergence

Next, we discuss convergence of random variables, and review standard results about probabilistic convergence in the context of random variables taking values in Banach spaces. We discuss convergence in probability, weak convergence, and convergence almost surely, and state several fundamental results which relate these different types of convergence with each other and which will be useful in the future chapters about stochastic FSI.

Convergence almost surely

Convergence almost surely is the probabilistic analogy of pointwise convergence. Suppose that $\{X_n(\omega)\}_{n=1}^{\infty}$ and $X(\omega)$ are $\mathcal{B}$-valued random variables that are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $X_n(\omega)$ **converges to** $X(\omega)$ **almost surely** if

$$\mathbb{P}\left(\lim_{n\to\infty} ||X_n(\omega) - X(\omega)||_{\mathcal{B}} = 0\right) = 1,$$

or equivalently if $X_n(\omega) \to X(\omega)$ in $\mathcal{B}$ for all outcomes ω in some measurable set $\Omega_0 \in \mathcal{F}$ with $\Omega_0 \subset \Omega$, where $\mathbb{P}(\Omega_0) = 1$.

Convergence in probability

Next, we describe convergence in probability of $\mathcal{B}$-valued random variables, which states informally that random variables are arbitrarily close in value to a limiting random variable with arbitrarily high probabilities as $n \to \infty$. Given a sequence $\{X_n(\omega)\}_{n=1}^{\infty}$ of $\mathcal{B}$-valued random variables and a $\mathcal{B}$-valued random variable $X(\omega)$, all defined on the same probability space, we say that X_n **converges in probability** to X if

$$\lim_{n\to\infty} \mathbb{P}(||X_n - X||_{\mathcal{B}} \geqslant \epsilon) = 0 \qquad \text{for all } \epsilon > 0.$$

There is the following well-known connection between convergence almost surely and convergence in probability, see Theorem 2.3.2 in [67]. We note that the proof uses a well-known measure theoretic result known as the Borel-Cantelli lemma, see Theorem 2.3.1 in [67].

Proposition 2.2.1. Let $\{X_n(\omega)\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-valued random variables such that $X_n(\omega)$ converges in probability to a limiting $\mathcal{B}$-valued random variable $X(\omega)$. Then, there exists a subsequence such that

$$X_{n_k}(\omega) \to X(\omega) \quad \text{in } \mathcal{B} \text{ almost surely as } k \to \infty.$$

Weak convergence

Next, we will discuss weak convergence of random variables, which describes convergence of the expected values of observable quantities. For further discussion of weak convergence, we refer the reader to Section 3.2 in [67] for example.

A sequence of random variables $\{X_n(\omega)\}_{n=1}^{\infty}$ **converges weakly (or equivalently, converges in law or distribution)** to a limiting random variable $X(\omega)$ if for all continuous bounded functions $f : \mathcal{B} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)].$$

Here, we can intuitively think of the bounded continuous function $f : \mathcal{B} \to \mathbb{R}$ as some observable which depends on the random value of X_n or X.

We can also define the notion of weak convergence for probability measures too, and we will state this specifically for probability measures on Banach spaces. Suppose that $\{\mu_n\}_{n=1}^{\infty}$ and μ are probability measures on a common Banach space $(\mathcal{B}, \text{Borel}(\mathcal{B}))$. Then, the probability measures μ_n **converge weakly** to μ if for all continuous bounded functions $f : \mathcal{B} \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathcal{B}} f(x) d\mu_n(x) = \int_{\mathcal{B}} f(x) d\mu(x).$$

Note that if $\mathcal{B}$-valued random variables $\{X_n\}_{n=1}^{\infty}$ converge weakly to X, then their laws $\{\mu_{X_n}\}_{n=1}^{\infty}$ also converge weakly (as probability measures on $\mathcal{B}$) to μ_X.

Weak convergence is a particularly nice form of probabilistic convergence because there is a well-known criterion, known as **tightness**, which will allow us to extract a weakly convergent subsequence from a tight sequence of random variables, see Theorem 3.2.13 in [67] and Proposition 6.1 in [128]. A sequence of $\mathcal{B}$-valued random variables $\{X_n(\omega)\}_{n=1}^{\infty}$ is **tight** if for all $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \mathcal{B}$ depending on ϵ such that

$$\mathbb{P}(X_n(\omega) \in K_\epsilon) > 1 - \epsilon, \quad \text{for all } n.$$

Essentially, tightness is a condition that keeps the probabilistic mass of random variables from escaping out to infinity, by requiring the probabilistic mass of a collection of probability measures to be uniformly contained in compact sets. If we specialize to the case of *separable Banach spaces*, then tightness allows us to extract weakly convergent subsequences from tight sequences of random variables. This is the content of the following classical probability result, which is part of a more general result known as Prokhorov's theorem, stated for example in Proposition 6.1 in [128].

Theorem 2.2.1. If $\{X_n(\omega)\}_{n=1}^{\infty}$ is a tight sequence of random variables taking values in a separable Banach space $\mathcal{B}$, then there exists a subsequence and a $\mathcal{B}$-valued random variable $X(\omega)$ such that $X_{n_k}(\omega)$ converges weakly to $X(\omega)$ as $k \to \infty$.

We make some additional remarks about tightness, in terms of the connection between uniform boundedness of random variables and tightness, which will be essential in the study of stochastic PDEs. We first establish the following corollary to the previous theorem, which states that uniform boundedness of *real-valued random variables* is sufficient to establish tightness.

Corollary 2.2.1. Suppose that $\{X_n(\omega)\}_{n=1}^{\infty}$ is a sequence of real-valued random variables such that

$$\mathbb{E}(|X_n|^p) < C, \qquad \text{for all } n, \text{ for some } 1 \leqslant p < \infty.$$

Then, the sequence $\{X_n(\omega)\}_{n=1}^{\infty}$ is tight and hence, there exists a real-valued random variable $X(\omega)$ such that $X_{n_k}(\omega)$ converges weakly to $X(\omega)$ as $k \to \infty$, along some subsequence $\{n_k\}_{k=1}^{\infty}$.

Proof. Choose $\epsilon > 0$. Choose λ sufficiently large so that $\frac{C}{\lambda^p} < \epsilon$. Then, by Chebychev's inequality, if we set $K_\epsilon = [-\lambda, \lambda]$, we have that

$$\mathbb{P}(X_n(\omega) \in K_\epsilon) > 1 - \epsilon, \quad \text{for all } n.$$

Since K_ϵ is a closed and bounded subset of $\mathbb{R}$, it is a compact set in $\mathbb{R}$, which establishes that $\{X_n(\omega)\}_{n=1}^{\infty}$ is a tight sequence of real-valued random variables, which establishes the result. □

However, we note that a similar argument does not work for random variables X that take values in general (infinite-dimensional) Banach spaces. In particular, if $\{X_n(\omega)\}_{n=1}^{\infty}$ is a sequence of $\mathcal{B}$-valued random variables where $\mathcal{B}$ is a separable Banach space, then having a uniform bound of the form

$$\mathbb{E}(||X_n||_{\mathcal{B}}^p) < C, \quad \text{for all } n, \tag{2.1}$$

does *not* imply that the random variables $\{X_n(\omega)\}_{n=1}^{\infty}$ are tight. The reason the argument above involving Chebychev's inequality works for the case of real-valued random variables is because closed and bounded subsets of $\mathbb{R}$ are compact in $\mathbb{R}$. However, because general Banach spaces can be infinite dimensional, closed and bounded subsets of Banach spaces are no longer necessarily compact subsets of the Banach space. In particular, given the uniform bound (2.1) above, we can find a closed ball $\overline{B(R_\epsilon)}$ of radius R_ϵ in $\mathcal{B}$ for which

$$\mathbb{P}(X_n \in \overline{B(R_\epsilon)}) > 1 - \epsilon, \quad \text{for all } n.$$

However, this closed ball $\overline{B(R_\epsilon)}$ is not necessarily compact in $\mathcal{B}$, which prevents us from obtaining that the random variables $\{X_n(\omega)\}_{n=1}^{\infty}$ are tight. Hence, in order to deduce tightness from the uniform bound (2.1), one would need to embed the Banach space $\mathcal{B}$ compactly into another Banach space $\mathcal{B}_0$. Then, the closure in $\mathcal{B}_0$ of the image of $\overline{B(R_\epsilon)}$ under this compact embedding from $\mathcal{B}$ to $\mathcal{B}_0$ would be a compact subset of $\mathcal{B}_0$, in which case we can show that the random variables $\{X_n(\omega)\}$ considered instead as $\mathcal{B}_0$-valued random variables are tight.

Additional classical results about convergence

We will finish this discussion of probabilistic convergence by discussing two classical results that will be needed in later chapters on stochastic FSI. We will first discuss the Skorokhod representation theorem, which relates weak convergence and convergence almost surely. We will then discuss the Gyöngy-Krylov lemma, which relates weak convergence of appropriate joint laws of random variables to convergence in probability, and hence to almost sure convergence along a subsequence.

First, we discuss the Skorokhod representation theorem, which will give a way of "upgrading" weak convergence of random variables to almost sure convergence of random variables, at the expense of moving to a potentially different probability space. However, moving to a different probability space can be done while retaining the laws of the random variables, so that even on the new probability space, we can make statements about the distributions of the random variables on the initial probability space. The statement of the Skorokhod representation theorem, see Proposition 6.2 in [128] for example, is as follows.

Theorem 2.2.2. Suppose that $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of probability measures on a separable Banach space $\mathcal{B}$ that converges weakly to a limiting probability measure μ on $\mathcal{B}$. Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{B}$-valued random variables $\{\tilde{X}_n\}_{n=1}^{\infty}$ and $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{X}_n \to \tilde{X}, \qquad \tilde{\mathbb{P}}\text{-almost surely,}$$

and the laws of $\tilde{X}_n$ and $\tilde{X}$ for all positive integers n are μ_n and μ respectively.

Remark 2.2.1. Though we have stated the result above for a weakly convergent sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$, the Skorokhod representation theorem is often applied to weakly convergent sequences of random variables $\{X_n\}_{n=1}^{\infty}$ taking values in a separable Banach space $\mathcal{B}$, all defined on a common initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given such a sequence $\{X_n\}_{n=1}^{\infty}$ that converges weakly to a $\mathcal{B}$-valued random variable X also defined on $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and new $\mathcal{B}$-valued random variables $\{\tilde{X}_n\}_{n=1}^{\infty}$ and $\tilde{X}$ on this new probability space such that $\tilde{X}_n \to \tilde{X}$ $\tilde{\mathbb{P}}$-almost surely, and $\tilde{X}_n$ and $\tilde{X}$ have the same laws as X_n and X. This follows from the previous statement of the Skorokhod representation theorem by letting μ_n be the law of X_n and by letting μ be the law of X.

Example 2.2.1. Since the Skorokhod representation theorem may seem abstract initially, we give a concrete example to demonstrate how this theorem works. Consider the probability space $\Omega = [0,1]$, where $\mathcal{F}$ is the set of Borel measurable sets and $\mathbb{P}$ is the usual uniform measure on $[0,1]$ (Lebesgue measure dx). For each k and n, define the random variables

$$X_{n,k}(\omega) = 1_{[(k-1)/n,k/n]}(\omega), \qquad \text{for } \omega \in [0,1],$$

where $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Then, consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$, where if N is the largest integer such that $1+2+3+...+N < n$ and if $k = n-(1+2+3+...+N)$, then

$$X_n = X_{N+1,k}.$$

Explicitly, the sequence we are considering is

$$X_{1,1}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{3,3}, X_{4,1}, X_{4,2}, X_{4,3}, X_{4,4}, ...$$

One can verify that the sequence $\{X_n\}_{n=1}^{\infty}$ constructed above converges weakly to the zero random variable X which is zero for all $\omega \in [0,1]$, but X_n does not converge almost surely.

However, by the Skorokhod representation theorem, we can essentially rearrange how outcomes are mapped to realizations without changing laws to recover almost sure convergence. In the notation of the Skorokhod representation theorem, we will let the tilde probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be the same as the initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (However, we remark that in general, the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ can be different from $(\Omega, \mathcal{F}, \mathbb{P})$, but we do not need to do this for this particular example.) Then, we will define $\tilde{X}_n$ by letting N as before be the largest positive integer such that $1+2+...+N < n$, and then we will define

$$\tilde{X}_n = X_{N+1,1}.$$

Thus, the sequence of random variables $\{\tilde{X}_n\}_{n=1}^{\infty}$ is given by

$$X_{1,1}, X_{2,1}, X_{2,1}, X_{3,1}, X_{3,1}, X_{3,1}, X_{4,1}, X_{4,1}, X_{4,1}, X_{4,1}, ...$$

Note that $\tilde{X}_n$ has the same law of X_n for all n, but now we have that $\tilde{X}_n$ converges almost surely to the zero random variable $\tilde{X}$, whereas the original sequence $\{X_n\}_{n=1}^{\infty}$ converges weakly to the zero random variable X, but does not converge almost surely.

Next, we will discuss a result which uses weak convergence in order to show convergence in probability of random variables. This result is known as the Gyöngy-Krylov lemma, and it will be closely connected to the Skorokhod representation theorem. In stochastic PDE problems, what often happens is that one uses the Skorokhod representation theorem to upgrade weak convergence to almost sure convergence, but the challenge is that the random functions are now defined on an alternate tilde probability space. One often wants to see if the random solution that is constructed to a stochastic problem can be shown to exist on the *original probability space*, and to do this, one usually invokes a standard Gyöngy-Krylov argument which uses a uniqueness result for the stochastic problem to transfer the random functions on the tilde probability space back to the initial probability space. In Chapter 4, we will discuss the full so-called Gyöngy-Krylov *diagonal argument* in the context of fully coupled stochastic FSI problems, but here, we state the relevant lemma for future reference. This lemma can be thought of as a criterion for showing that random variables converge in probability, which relies on the weak convergence of the joint laws of ordered pairs of these random variables. For further discussion about the Gyöngy-Krylov lemma and its applications, see Lemma 1.1 in [88] and Proposition 6.3 in [128].

Theorem 2.2.3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables taking values in a separable Banach space $\mathcal{B}$. Denote the joint law of (X_m, X_n) by $\nu_{m,n}$, which is a probability measure on $\mathcal{B} \times \mathcal{B}$ defined by

$$\nu_{m,n}(A \times B) = \mathbb{P}(X_m \in A, X_n \in B), \quad \text{for } A, B \in \text{Borel}(\mathcal{B}),$$

and more generally,

$$\mu_{m,n}(A) = \mathbb{P}((X_m, X_n) \in A), \quad \text{for } A \in \text{Borel}(\mathcal{B} \times \mathcal{B}).$$

Consider the collection of joint laws $\{\nu_{m,n}\}_{m,n=1}^{\infty}$. Suppose that for any subsequence m_k, n_k where $\{m_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ are monotically increasing to infinity, we have that the sequence of probability measures $\{\nu_{m_k,n_k}\}_{k=1}^{\infty}$ converges weakly to a probability measure ν on $\mathcal{B} \times \mathcal{B}$ such that

$$\nu(\{(x, y) \in \mathcal{B} \times \mathcal{B} : x = y\}) = 1.$$

This condition is known as the *diagonal condition*, and if the diagonal condition is satisfied, then the sequence of $\mathcal{B}$-valued random variables $\{X_n\}_{n=1}^{\infty}$ converges in probability to a limiting $\mathcal{B}$-valued random variable X.

2.3 Stochastic integration

Next, we will discuss the types of random noise that will be relevant to our analysis of stochastic FSI later. We will begin with one-dimensional white noise, which is a type of random noise in time whose intensity at every point in time is independent from its intensity at every other point in time. We will formally denote the intensity of a one-dimensional white noise at a given time t by $\dot{W}(t)$, so that we can express this independence property of white noise formally as

$$\mathbb{E}[\dot{W}(s)\dot{W}(t)] = \delta_0(t - s), \qquad s, t \geqslant 0,$$

where δ_0 is the Dirac delta function.

As the notation suggests, we can think of white noise as the formal time derivative of a stochastic process $\{W(t)\}_{t \geqslant 0}$, known as a one-dimensional Brownian motion. In this section, we will review basic properties of stochastic processes, and review the definition and properties of a one-dimensional Brownian motion. We will then use these ideas to review the construction of the stochastic integral against one-dimensional white noise, which is the Itô integral.

Stochastic processes

While we described random variables in the previous section, because PDE dynamics are often evolving in time, we will have to consider more generally random quantities in time. Thus, we will introduce the notion of a *stochastic process*, or a random process in time. Stochastic processes can be indexed by discrete time steps (such as a random walk), or can be indexed by continuous time (such as a Brownian motion). We will focus on the case of stochastic processes indexed by continuous time, since this will be the case that is most relevant to our analysis later. In this case, a **stochastic process** on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables $\{X_t\}_{t\geqslant 0}$ indexed by time $t \geqslant 0$, where each random variable X_t is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. These random variables all take values in the same space, which can be assumed to be a Banach space (which is often just the real numbers $\mathbb{R}$, but can be more general Banach spaces too).

When we index the random variables $\{X_t\}_{t\geqslant 0}$ by time $t \geqslant 0$, it will be convenient to think of X_t as the random information that we observe at a time $t \geqslant 0$. Intuitively, when we have such a stochastic process, we can then think of observing X_t for increasing values of t as observing more information in time about a random system. We will mathematically encode this increase in observed information about a random system in time using the mathematical concept of a filtration.

Definition 2.3.1. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A **filtration** $\{\mathcal{F}_t\}_{t\geqslant 0}$ is an increasing family of sigma algebras, in the sense that $\mathcal{F}_t \subset \mathcal{F}$ for all $t \geqslant 0$, each $\mathcal{F}_t$ is a sigma algebra, and $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leqslant t$. This filtration is a **complete filtration** if every measurable set $A \subset \mathcal{F}$ with $\mathbb{P}(A) = 0$ is included in $\mathcal{F}_t$ for all $t \geqslant 0$. When we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a particular filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ on this probability space, we can refer to the probability space with filtration as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

Given a filtration, we can then discuss whether a given stochastic process is compatible with the information that is observable at each time $t \geqslant 0$, specified by the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ on the underlying probability space.

Definition 2.3.2. A stochastic process $\{X_t\}_{t\geqslant 0}$ defined on a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is **adapted** to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geqslant 0$.

By viewing a stochastic process $\{X_t\}_{t\geqslant 0}$ as a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed in time, we are viewing the stochastic process as a collection of random variables $X_t : \Omega \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space. However, instead of viewing time as just an index set, we can also view the entire stochastic process, which we will refer to as X, as a random function $X : [0, \infty) \times \Omega \to \mathcal{B}$, where $X(t, \omega)$ is a function of both time $t \in [0, \infty)$ and the outcome $\omega \in \Omega$. We can then define measurability properties of stochastic processes that refer to how the measurability properties of the stochastic process evolve in time and interact with the time parameter. We will consider two measurability properties in time that will be important in our future analysis: joint measurability and predictability.

Definition 2.3.3. Given a stochastic process $\{X_t\}_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Banach space $\mathcal{B}$, the stochastic process is **jointly measurable** if it is measurable as a function $X : [0,\infty) \times \Omega \to \mathcal{B}$, where $[0,\infty) \times \Omega$ is considered with the product sigma algebra consisting of the product of the Borel measurable subsets of $[0,\infty)$ and $\mathcal{F}$, and where $\mathcal{B}$ is considered with the sigma algebra of Borel measurable subsets of $\mathcal{B}$.

Definition 2.3.4. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, the **predictable sigma algebra** is the sigma algebra on $[0,\infty) \times \Omega$ generated by subsets of the form

$$\{[a,b) \times A : 0 \leqslant a < b \text{ and } A \in \mathcal{F}_a\}.$$

A stochastic process $\{X_t\}_{t\geqslant 0}$ is **predictable** if it is measurable as a function from $[0,\infty) \times \Omega$ to $\mathcal{B}$, where $[0,\infty) \times \Omega$ is considered with the predictable sigma algebra and $\mathcal{B}$ is considered with the sigma algebra of Borel measurable subsets of $\mathcal{B}$.

We can use the following criterion to classify stochastic processes as predictable, see Proposition 5.1 in Chapter IV, Section 5 of [155].

Proposition 2.3.1. Suppose that $\{X_t\}_{t\geqslant 0}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Suppose that $\{X_t\}_{t\geqslant 0}$ also has left continuous paths almost surely so that with probability one, $X(t,\omega) : [0,\infty) \to \mathcal{B}$ is left continuous. Then, $\{X_t\}_{t\geqslant 0}$ is predictable as a stochastic process taking values in $\mathcal{B}$.

We refer the reader to Chapter IV, Section 5 of [155] for example, for more details about predictable processes and the predictable sigma algebra.

One-dimensional Brownian motion

We will next discuss an important stochastic process, known as (one-dimensional) Brownian motion. Brownian motion is a nice prototypical stochastic process to consider in many applications since it has many desirable properties, such as having continuous paths almost surely and having stationary increments $W(t) - W(s)$, whose distribution depends only on the time difference $t - s$. We recall the following definition of a one-dimensional Brownian motion.

Definition 2.3.5. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process $\{W_t\}_{t\geqslant 0}$ taking values in $\mathbb{R}$ is a **one-dimensional Brownian motion** if the following properties are satisfied:

- The path $W(t,\omega) : [0,\infty) \to \mathbb{R}$ is a continuous path with $W(0) = 0$ almost surely (for all ω in a measurable set having probability one).
- The increments $W(t) - W(s)$ are distributed as $N(0, t-s)$, a normal distribution with mean zero and variance $t - s$.

- The independent increments property is satisfied: if $s_1 < t_1 \leqslant s_2 < t_2$, then $W(t_2) - W(s_2)$ and $W(t_1) - W(s_1)$ are independent random variables.

The independent increments property has the following direct corollary: if we define the natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ associated to the Brownian motion $\{W_t\}_{t\geqslant 0}$ by

$$\mathcal{F}_t = \sigma(\{W_s : 0 \leqslant s \leqslant t\}),$$

where $\sigma(\{W_s : 0 \leqslant s \leqslant t\}) \subset \mathcal{F}$ denotes the sigma algebra generated by the random variables W_s for $0 \leqslant s \leqslant t$, then the increment $W(t) - W(s)$ for $0 \leqslant s < t$ is independent of any random variable that is $\mathcal{F}_s$-measurable.

We can extend this independence criterion to general filtrations. Suppose that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t\}_{t\geqslant 0}$ on this probability space. Then, we say that a one-dimensional Brownian motion is more specifically a **Brownian motion with respect to the filtration** $\{\mathcal{G}_t\}_{t\geqslant 0}$ if

- W_t is adapted to $\mathcal{G}_t$ for all $t \geqslant 0$.
- $W(t) - W(s)$ is independent of all $\mathcal{G}_s$-measurable random variables whenever $0 \leqslant s < t$.

In particular, note that any one-dimensional Brownian motion $\{W_t\}_{t\geqslant 0}$ is a Brownian motion with respect to its natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ defined above.

Brownian motion has many interesting properties, and we refer the reader to the exposition in references, such as [155], Chapter 7 of [67], and Chapter 14 of [103], for more detailed information about Brownian motion and its relevant probabilistic properties. We will conclude our brief summary of Brownian motion by describing some of its fundamental path properties, in particular its properties when considered as a random function of time $t \in [0, \infty)$, that will play a role in describing some of the subtleties underlying the theory of stochastic integration.

We recall the following definitions to discuss the path properties of one-dimensional Brownian motion. Recall that a real-valued function $f : [0, \infty) \to \mathbb{R}$ is **locally α-Hölder continuous** for $\alpha \in (0, 1)$ if for all $T > 0$,

$$\sup_{s,t\in[0,T], s\neq t} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty.$$

Recall that a real-valued function $f : [a, b] \to \mathbb{R}$ is **of finite variation** if

$$\sup_{\mathcal{P}} \sum_{i=1}^{|\mathcal{P}|} |f(t_i) - f(t_{i-1})| < \infty,$$

where the supremum is over all finite partitions $\mathcal{P}$ of the interval $[a, b]$ consisting of points:

$$a = t_0 < t_1 < ... < t_{N-1} < t_N = b,$$

where $N = |\mathcal{P}|$ is the number of subintervals in the partition. We then have the following results about the path properties of one-dimensional Brownian motion. For more information about these properties, see Theorem 14.5 and Proposition 14.10 in [103] and Chapter I, Section 2 of [155].

Proposition 2.3.2. Let $\{W_t\}_{t \geqslant 0}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- With probability 1, the paths of Brownian motion are not differentiable for all $t \geqslant 0$.
- More generally, with probability 1, the paths of Brownian motion are not α-Hölder continuous on any closed interval for $\alpha \in [1/2, 1)$.
- The paths of Brownian motion are not of finite variation on any closed interval with probability 1.
- The paths of Brownian motion with probability 1 are locally α-Hölder continuous for $\alpha \in (0, 1/2)$.

Stochastic integration and the Itô integral

Next, we discuss the construction of the Itô integral, which is a way of integrating predictable processes against one-dimensional white noise $dW(t)$. For the construction of the Itô integral, we follow the presentation in Section 2.2 in [111], co-authored with Sunčica Čanić, which discusses stochastic integration more generally against spacetime white noise, and we specialize the discussion in [111] to the more specific case of stochastic integration against one-dimensional white noise in the current subsection. We refer the interested reader to Chapter IV of [155] for a more involved discussion of stochastic integration.

As we discussed in the previous subsection, Brownian motion is almost surely not differentaible at any time, so the formal expression $dW(t)$ for white noise does not make sense in a classical pathwise sense, meaning that with probability one, the path $W(t, \omega) : [0, \infty) \to \mathbb{R}$ is not differentiable at any time $t \geqslant 0$. This complicates the construction of the stochastic integral for the following reason. If we wanted to define the stochastic integral

$$\int_0^T X(t, \omega) dW(t),$$

where $X(t, \omega)$ is a predictable stochastic process, we might first try to follow the procedure in the case of Riemann integration and consider for each partition $\mathcal{P}$ consisting of points

$$0 = t_0 < t_1 < ... < t_{N-1} < t_N = T,$$

the following discrete sums:

$$\sum_{i=1}^{N} X(t_{i-1}, \omega) \cdot [W(t_i) - W(t_{i-1})],$$

which is a real-valued random quantity depending on ω, and then we might hope to pass to the limit almost surely in these random sums, so that we would pass to the limit pathwise for each ω in a probability one measurable subset of Ω. However, these random sums do not necessarily converge pathwise as $|\mathcal{P}| \to 0$, and this has to do with the fact that the paths of Brownian motion are not differentiable for any time $t \geqslant 0$, and are not even of finite variation on any finite closed interval, with probability one. In fact, the convergence of these random sums must be taken in probability, rather than almost surely, in order to obtain a reasonable limit, which reflects the fact that Brownian motion behaves well in a probabilitic sense but not path by path, see Proposition 2.13 in Chapter IV, Section 2 of [155].

We will thus use a different approach to define the stochastic integral, where we will define stochastic integration of elementary integrands and then use a density argument to extend the stochastic integral to a more general class of integrands. Consider a Brownian motion $\{W_t\}_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the natural filtration associated with the Brownian motion. Then, we define the collection $\mathcal{S}$ of **elementary predictable integrands**, which are finite sums of random functions in time of the following form:

$$f(t,\omega) = X(\omega)1_{(a,b]}(t),$$

where $X(\omega)$ is a real-valued $\mathcal{F}_a$-measurable random variable and $0 \leqslant a < b$. We define the stochastic integral of $f(t,\omega) = X(\omega)1_{(a,b]}(t)$ to be

$$\int_0^\infty f(t,\omega)dW(t) = X(\omega)\cdot[W(b) - W(a)],$$

and by linearity, we have more generally that for $f \in \mathcal{S}$ of the form

$$f(t,\omega) = \sum_{i=1}^{N} X_i(\omega)1_{(a_i,b_i]}(\omega), \tag{2.2}$$

where X_i is a real-valued $\mathcal{F}_{a_i}$-measurable random variable and $0 \leqslant a_1 < b_1 < a_2 < b_2 < ... < a_N < b_N$,

$$\int_0^\infty f(t,\omega)dW(t) = \sum_{i=1}^{N} X_i(\omega)\cdot[W(b_i) - W(a_i)].$$

We note that every elementary predictable integrand $f \in \mathcal{S}$ can be expressed uniquely in the form (2.2) since for $0 \leqslant s < t$, we have that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies $\mathcal{F}_s \subset \mathcal{F}_t$. For more on elementary integrands and elementary processes, see Chapter IV, Section 2 of [155].

Next, we will extend the definition of the stochastic Itô integral from elementary predictable integrands to more general integrands by establishing the following identity, which is known as the **Itô isometry**.

Proposition 2.3.3. For elementary predictable integrands $f \in \mathcal{S}$,

$$\mathbb{E}\left[\left(\int_0^\infty f(t,\omega)dW(t)\right)^2\right] = \mathbb{E}\left(\int_0^\infty |f(t,\omega)|^2 dt\right).$$

Proof. Consider $f \in \mathcal{S}$, which thus has the form

$$f(t,\omega) = \sum_{i=1}^{N} X_i(\omega) 1_{[a_i,b_i)}(t),$$

where X_i is $\mathcal{F}_{a_i}$-measurable and $a_1 < b_1 < a_2 < b_2 < \ldots < a_N < b_N$.

We first compute the left hand side and we use the definition of the stochastic integral to calculate

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty f(t,\omega)dW(t)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{N} X_i(\omega)\cdot[W(b_i)-W(a_i)]\right)^2\right] \\ &= \mathbb{E}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} X_i(\omega)\cdot X_j(\omega)\cdot[W(b_i)-W(a_i)]\cdot[W(b_j)-W(a_j)]\right).\end{aligned}$$

We will show that the cross terms $i \neq j$ will vanish. Suppose without loss of generality that $i < j$ so that we have $a_i < b_i < a_j < b_j$. In this case, $X_i(\omega)$, $X_j(\omega)$, and $W(b_i) - W(a_i)$ are all $\mathcal{F}_{a_j}$ measurable (by properties of filtrations), so they are all independent of the increment $W(b_j) - W(a_j)$. Hence, we have that for $i < j$,

$$\begin{aligned}&\mathbb{E}\Big(X_i(\omega)\cdot X_j(\omega)\cdot[W(b_i)-W(a_i)]\cdot[W(b_j)-W(a_j)]\Big) \\ &\qquad = \mathbb{E}\Big(X_i(\omega)\cdot X_j(\omega)\cdot[W(b_i)-W(a_i)]\Big)\cdot\mathbb{E}\Big(W(b_j)-W(a_j)\Big) = 0,\end{aligned}$$

since the increment $W(b_j) - W(a_j)$ has mean zero. Therefore, only the terms where $i = j$ remain and thus, using the fact that $|X_i(\omega)|^2$ is $\mathcal{F}_{a_i}$-measurable and is hence independent of $|W(b_i) - W(a_i)|^2$, we obtain that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty f(t,\omega)dW(t)\right)^2\right] &= \mathbb{E}\left(\sum_{i=1}^{N} |X_i(\omega)|^2\cdot|W(b_i)-W(a_i)|^2\right) \\ &= \sum_{i=1}^{N}\mathbb{E}(|X_i(\omega)|^2)\cdot\mathbb{E}(|W(b_i)-W(a_i)|)^2 = \sum_{i=1}^{N}(b_i-a_i)\cdot\mathbb{E}(|X_i(\omega)|^2).\end{aligned}$$

For the right hand side, we compute that

$$\mathbb{E}\left(\int_0^\infty |f(t,\omega)|^2 dt\right) = \mathbb{E}\left(\int_0^\infty \sum_{i=1}^{N} |X_i(\omega)|^2 1_{[a_i,b_i)}(t)\right) = \sum_{i=1}^{N}(b_i-a_i)\cdot\mathbb{E}(|X_i(\omega)|^2),$$

since the sets $(a_i, b_i]$ are all disjoint from each other. This establishes the desired identity. □

Using this isometry, we can extend the definition of the Itô integral to a broader class of integrands $\mathcal{P}$, which is the collection of jointly measurable integrands $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$ that can be obtained as the closure of $\mathcal{S}$ in the set of jointly measurable stochastic processes, under the norm $|| \cdot ||_{\mathcal{P}}$ defined by:

$$||f||_{\mathcal{P}}^2 = \mathbb{E}\left(\int_0^\infty |f(t, \omega)|^2 dt\right).$$

Since we have defined the Itô integral for elementary predictable integrands $f \in \mathcal{S}$, we will use the Itô isometry to extend the definition of the Itô integral to the broader class of integrands $f \in \mathcal{P}$ by using a density argument. Given an integrand $f \in \mathcal{P}$, we consider an approximating sequence $\{f_n\}_{n=1}^\infty$ of elementary predictable integrands $f_n \in \mathcal{S}$, where f_n converges to f in the norm of $\mathcal{P}$. Given such an approximating sequence, we define the Itô integral $\int_0^\infty f(t, \omega) dW(t)$ to be the real-valued random variable that is the limit in $L^2(\Omega)$ of the random variables $\int_0^\infty f_n(t, \omega) dW(t)$ for $f_n \in \mathcal{S}$, where the existence of this limit follows from the Itô isometry.

White noise and stochastic integration

We conclude this chapter by extending our past disucssion of one-dimensional white noise and stochastic integration to the context of spacetime white noise, which is random noise with an intensity that is formally independent at every point in space and time. The material from this section is adapted from Section 2.2 of the manuscript [111] written with Sunčica Čanić.

In this section we review the concept of spacetime white noise on $\mathbb{R}^+ \times \mathbb{R}^n$ and stochastic integration against white noise. This will be used throughout the rest of the manuscript. Note that we will use $\mathbb{R}^+$ to denote $[0, \infty)$, which represents the time variable. While we will be primarily concerned with dimensions $n = 1, 2$, we will define white noise in full generality, as the extension to higher dimensions is no more difficult.

We follow the exposition that can be found in [107] about martingale measures and refer the reader to the original reference by Walsh [177] for more details. We note that while the forthcoming analysis can be carried out more generally for martingale measures, we will restrict to the case of white noise for simplicity. The full martingale measure theory can be found in [177] and [107].

Recall that a *Gaussian process* is a process $\{G_i\}_{i \in I}$, such that the finite dimensional random vectors

$$(G_{i_1}, G_{i_2}, ..., G_{i_k}), \ i_1, i_2, ..., i_k \in I$$

have distributions that are multivariable Gaussian, for any finite collection of $i_1, i_2, ..., i_k \in I$.

We will define the *covariance function* to be the symmetric function $C : I \times I \to \mathbb{R}$ that gives the covariance of any two Gaussians G_{i_1} and G_{i_2},

$$C(i_1, i_2) = \mathbb{E}[(G_{i_1} - \mathbb{E}(G_{i_1}))(G_{i_2} - \mathbb{E}(G_{i_2}))].$$

For a mean zero Gaussian process, which is a Gaussian process $\{G_i\}_{i\in I}$ such that $\mathbb{E}[G_i] = 0$ for all $i \in I$, this reduces to the simpler formula

$$C(i_1, i_2) = \mathbb{E}[G_{i_1} G_{i_2}].$$

We will now define white noise as a Gaussian process, taking for granted the existence of such a process.

Definition 2.3.6 (White noise on $\mathbb{R}^+ \times \mathbb{R}^n$)**.** Let $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)$ denote the collection of all Borel subsets of $\mathbb{R}^+ \times \mathbb{R}^n$. White noise on $\mathbb{R}^+ \times \mathbb{R}^n$ is a *mean zero* Gaussian process $\{\dot{W}(A)\}_{A\in\mathcal{B}(\mathbb{R}^+\times\mathbb{R}^n)}$ indexed by the Borel subsets of $\mathbb{R}^+ \times \mathbb{R}^n$, with the covariance function

$$C(A, B) := \mathbb{E}[\dot{W}(A)\dot{W}(B)] = \lambda(A \cap B), \qquad \text{for } A, B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n), \tag{2.3}$$

where λ is Lebesgue measure in $\mathbb{R}^+ \times \mathbb{R}^n$.

Some basic facts about white noise that will be useful later are summarized in the following proposition.

Proposition 2.3.4. Let $\{\dot{W}(A)\}_{A\in\mathcal{B}(\mathbb{R}^+\times\mathbb{R}^n)}$ denote white noise. Then, the following holds true:

- For each bounded set $A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)$, $\dot{W}(A)$ is normally distributed with mean 0 and variance $\lambda(A)$, namely $\dot{W}(A) \sim N(0, \lambda(A))$. So $\dot{W}(A) \in L^2(\Omega)$, where Ω is the probability space.
- If $A \cap B = \varnothing$, then $\dot{W}(A)$ and $\dot{W}(B)$ are independent.
- Given $A, B \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)$, $\dot{W}(A \cup B) = \dot{W}(A) + \dot{W}(B) - \dot{W}(A \cap B)$, almost surely (a.s.), as random variables.
- White noise is a signed measure taking values in $L^2(\Omega)$, namely $\dot{W} : \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n) \to L^2(\Omega)$. Furthermore, white noise considered as a measure is σ-finite.

Proof. The first point follows from the fact that white noise is a mean zero Gaussian process, and $\mathbb{E}[(\dot{W}(A))^2] = \lambda(A \cap A) = \lambda(A)$ by (2.3). The second and third points are from Exercise 3.15 in [107]. The second point follows from the fact that $\dot{W}(A)$ and $\dot{W}(B)$ are mean zero Gaussians with zero covariance, by applying (2.3). The third fact follows from the computation of the expectation $\mathbb{E}[(\dot{W}(A \cup B) - \dot{W}(A) - \dot{W}(B) + \dot{W}(A \cap B))^2] = 0$. One can verify this by expanding the square and applying (2.3) repeatedly. Note that the third property gives the finite additivity properties of a measure. For the final property, one must check that white noise has the remaining properties of a measure, and we refer the reader to the proof of Proposition 5.1 in [107]. □

Remark 2.3.1. Heuristically, one thinks of white noise as random noise that is "independent" at every point in time and space. One can then interpret $\dot{W}(A)$ heuristically as being the net contribution of the noise in A. With this heuristic interpretation, it is at least intuitively reasonable that white noise has the properties of a measure. The fact that the noise is independent at every point in time and space is in accordance with the second property in Proposition 2.3.4.

Stochastic integration against white noise. We will first define integration of simple functions against white noise, and then proceed to the most general case by an approximation argument. For this purpose, we introduce the following nomenclature (see Sec. 5 of [107]):

- For any $A \in \mathcal{B}(\mathbb{R}^n)$ and $t \in \mathbb{R}^+$, we use $W_t(A)$ to denote $W_t(A) = \dot{W}([0,t] \times A)$, for $A \in \mathcal{B}(\mathbb{R}^n)$, so that $[0,t] \times A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^n)$.
- For $t > 0$, we consider the *filtration* $\mathcal{F}_t$ associated to white noise to be the σ-algebra generated by the collection of random variables $\{W_s(A) : s \in [0,t], A \in \mathcal{B}(\mathbb{R}^n)\}$.
- We use $\mathcal{S}$ to denote the space of simple functions, which are functions of the form

$$f(t,x,\omega) = \sum_{i=1}^{n} X_i(\omega) 1_{(a_i,b_i]}(t) 1_{A_i}(x), \tag{2.4}$$

  where X_i is a bounded, $\mathcal{F}_{a_i}$-measurable random variable with $0 \leqslant a_i < b_i$, and $A_i \in \mathcal{B}(\mathbb{R}^n)$ is bounded.

Definition 2.3.7. Let $f \in \mathcal{S}$ be a simple function. We define

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} f(s,x,\omega) W(dx,ds) &= \int_0^t \int_{\mathbb{R}^n} \sum_{i=1}^{n} X_i(\omega) 1_{(a_i,b_i]}(s) 1_{A_i}(x) W(dx,ds) \\ &:= \sum_{i=1}^{n} X_i(\omega)[W_{t\wedge b_i}(A_i) - W_{t\wedge a_i}(A_i)], \end{aligned} \tag{2.5}$$

where the "wedge" notation corresponds to $\alpha \wedge \beta = \min\{\alpha, \beta\}$.

It is easy to check that the definition of the integral in (2.5) is independent of the representation of the simple function as (2.4).

We have the following crucial isometry property for the stochastic integral of simple functions against spacetime white noise. This is an extension of the Itô isometry to the stochastic integral against spacetime white noise.

Proposition 2.3.5. For $f \in \mathcal{S}$,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega) W(dx,dt)\right)^2\right] = \mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} |f(t,x,\omega)|^2 dx dt\right). \tag{2.6}$$

Proof. In the case where $f(t,x,\omega)$ is a simple function of the form

$$f(t,x,\omega) = X(\omega)1_{(a,b]}(t)1_A(x),$$

one easily checks that

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega)W(dx,dt)\right)^2\right] = \mathbb{E}\left(X^2(\omega)\left[W_b(A) - W_a(A)\right]^2\right)$$
$$= \mathbb{E}\left(\mathbb{E}\left[X^2(\omega)\left(W_b(A) - W_a(A)\right)^2 |\mathcal{F}_a\right]\right) = \mathbb{E}\left(X^2(\omega)\mathbb{E}\left[\left(W_b(A) - W_a(A)\right)^2 |\mathcal{F}_a\right]\right),$$

where we used the fact that $X \in \mathcal{F}_a$ to take it out of the conditional expectation. Using the third property in Proposition 2.3.4,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega)W(dx,dt)\right)^2\right] = \mathbb{E}\left(X^2(\omega)\mathbb{E}\left[\dot{W}^2((a,b] \times A)|\mathcal{F}_a\right]\right).$$

Using the second property in Proposition 2.3.4 we deduce that this is equal to

$$= \mathbb{E}\left[X^2(\omega)\right]\mathbb{E}\left[\dot{W}^2((a,b] \times A))\right] = \lambda(A)(b-a)\mathbb{E}\left[X^2(\omega)\right] = \mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} |f(t,x,\omega)|^2 dxdt\right).$$

We note that (2.6) holds for general $f \in \mathcal{S}$, by choosing a representation (2.4) of an arbitrary simple function where the sets $(a_i \times b_i] \times A_i$ are disjoint, and then using the independence property in the second property listed in Proposition 2.3.4. □

Next, we want to extend the definition of the stochastic integral to more general integrands. For this purpose we recall the following definitions.

Definition 2.3.8. Let $f(t,x,\omega)$ be a real valued function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$.

1. We say that $f(t,x,\omega)$ is *adapted to the filtration* $\{\mathcal{F}_t\}_{t \geqslant 0}$ if the map $\omega \to f(t,x,\omega)$ is $\mathcal{F}_t$ measurable for each $x \in \mathbb{R}^n$ and $t \geqslant 0$.

2. We say that $f(t,x,\omega)$ is *jointly measurable* if it is measurable as a function in time, space, and the probability space, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$.

To define the stochastic integral, we must identify the class of admissible integrands, which will be called predictable processes [56]. To do that, we denote by $\mathcal{H}$ the set of all jointly measurable $f(t,x,\omega)$ such that

$$\mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega)^2 dxdt\right) < \infty.$$

Note that $\mathcal{S} \subset \mathcal{H}$.

Definition 2.3.9. Define $\mathcal{P}_W$ to be the the closure of $\mathcal{S} \subset \mathcal{H}$ under the norm

$$||f||^2_{\mathcal{P}_W} := \mathbb{E}\left(\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega)^2 dx dt\right) < \infty. \tag{2.7}$$

The elements of $\mathcal{P}_W$ are called *predictable processes*.

Finally, we define the stochastic integral for predicable processes, namely

$$\int_0^\infty \int_{\mathbb{R}^n} f(t,x,\omega) W(dx,dt), \qquad \text{for } f \in \mathcal{P}_W, \tag{2.8}$$

by utilizing a density argument that uses the Itô isometry. In particular, we use the fact that functions $\mathcal{S}$ are dense in $\mathcal{P}_W$ (see Proposition 2.3 in [177]). Hence, given $f \in \mathcal{P}_W$, there is a sequence $f_k \in \mathcal{S}$ such that $f_k \to f$ in $\mathcal{P}_W$, as $k \to \infty$. Using the isometry relation in Proposition 2.3.5, one can show that the sequence

$$\left\{\int_0^\infty \int_{\mathbb{R}^n} f_k(t,x,\omega) W(dx,dt)\right\}_{k=1}^\infty \tag{2.9}$$

is a Cauchy sequence in $L^2(\Omega)$.

Definition 2.3.10. The random variable obtained in the limit of integrals (2.9) is the *stochastic integral* (2.8).

We can also define the integral on bounded time intervals, by noting that

$$\int_0^T \int_{\mathbb{R}^n} f(t,x,\omega) W(dx,dt) = \int_0^\infty \int_{\mathbb{R}^n} 1_{(0,T]}(t) f(t,x,\omega) W(dx,dt).$$

Since the definition of the admissible integrands $\mathcal{P}_W$ is abstract, we list a set of criteria that will help us determine whether a given integrand is in $\mathcal{P}_W$ or not. Hence, we use the following proposition, which follows directly from Proposition 2 in [56].

Proposition 2.3.6. Let $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that the following conditions hold.

1. Joint measurability: $(t,x,\omega) \to u(t,x,\omega)$ is $\mathcal{B}([0,T]\times\mathbb{R}^n)\times\mathcal{F}_T$ measurable.
2. Finite second moments: $\mathbb{E}\left(|u(t,x)|^2\right) < \infty$ for all $t \in [0,T]$, $x \in \mathbb{R}^n$.
3. Continuity in $L^2(\Omega)$: The process u considered as a map $(t,x) \in [0,T]\times\mathbb{R}^n \to L^2(\Omega)$ is continuous in $L^2(\Omega)$.
4. Square integrability: $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^n} |u(s,y)|^2 dy ds\right) < \infty$.

Then, the stochastic integral

$$\int_0^t \int_{\mathbb{R}^n} u(s,y)W(ds,dy)$$

is defined for all $t \in [0,T]$.

Proof. This proposition follows from Proposition 2 of [56], and is Proposition 2 of [56] adapted to the current context. Though Proposition 2 of [56] is stated for the more general case of spatially homogeneous Gaussian noise, the statement of Proposition 2 of [56] specialized to the case of white noise reads as follows:

Let $\{u(t,x)\}_{t\in\mathbb{R}^+, x\in\mathbb{R}^n}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and define $\mathcal{F} = \bigcup_{t\geqslant 0} \mathcal{F}_t$. Suppose the following conditions hold:

1. Joint measurability: $(t,x,\omega) \to u(t,x,\omega)$ is $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n) \times \mathcal{F}$ measurable.
2. Finite second moments: $\mathbb{E}\left(|u(t,x)|^2\right) < \infty$ for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$.
3. Continuity in $L^2(\Omega)$: The process u considered as a map $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \to L^2(\Omega)$ is continuous in $L^2(\Omega)$.
4. Square integrability on a compact set and finite time: There exists a compact set $K \subset \mathbb{R}^n$ and $t_0 > 0$ such that
$$\mathbb{E}\left(\int_0^{t_0} \int_K |u(s,y)|^2 dyds\right) < \infty.$$

Then, $1_{[0,t_0]\times K}(t,x)u(t,x) \in \mathcal{P}_W$.

While the result in Proposition 2 of [56] is stated specifically for spatial dimension two, one can verify that it holds for arbitrary dimension.

To see that the statement of Proposition 2 of [56] implies the result in Proposition 2.3.6, let K_i be a sequence of compact sets that increase to $\mathbb{R}^n$, and consider $\{u(t,x)\}_{t\in[0,T], x\in\mathbb{R}^n}$ satisfying the four conditions in Proposition 2.3.6. We extend $\{u(x,t)\}_{t\in[0,T], x\in\mathbb{R}^n}$ to be defined on all of time $t \geqslant 0$ by defining

$$\tilde{u}(x,t) = u(x,t) \quad \text{if } t \in [0,T], \qquad \tilde{u}(x,t) = u(x,T) \quad \text{if } t \geqslant T.$$

Then, $\{\tilde{u}(t,x)\}_{t\in\mathbb{R}^+, x\in\mathbb{R}^n}$ along with $t_0 = T$ and each K_i satisfies the conditions in Proposition 2 in [56]. Therefore, $1_{[0,T]\times K_i}(t,x)u(t,x) = 1_{[0,T]\times K_i}(t,x)\tilde{u}(t,x) \in \mathcal{P}_W$.

Since the fourth condition of Proposition 2.3.6 states that

$$\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^n} |u(s,y)|^2 dyds\right) < \infty,$$

we have that $1_{[0,T]\times K_i}(t,x)u(t,x) \to 1_{[0,T]\times\mathbb{R}^n}(t,x)u(t,x)$ in the norm of $\mathcal{P}_W$, since K_i is a sequence of compact sets in $\mathbb{R}^n$ increasing to all of $\mathbb{R}^n$. Hence, $1_{[0,T]\times\mathbb{R}^n}(t,x)u(t,x) \in \mathcal{P}_W$ since $\mathcal{P}_W$ is complete with respect to its norm. □

A couple of remarks are in order. The first one uses the concept of *modification*, which we now recall.

Definition 2.3.11. Let $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ be a stochastic process. Then $\{\tilde{u}(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ is a *modification* of $\{u(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ if

$$\mathbb{P}(u(t,x)=\tilde{u}(t,x))=1, \qquad \text{for all } t\in[0,T], x\in\mathbb{R}^n.$$

Remark 2.3.2. The third condition in Proposition 2.3.6 implies that there is a jointly measurable modification (see the discussion on pg. 201 of [56], and the proof of Theorem 13 in [54]). Thus, in practice, one does not need to check the first condition, as by taking a modification, the third condition implies the first.

Finally, we recall the following useful inequality, which is a direct consequence of a classical result known as the BDG (Burkholder-Davis-Gundy) inequality, which will be used frequently, see [107] and Chapter IV, Section 4 of [155].

Theorem 2.3.1. For each $p \geqslant 2$, there exists a positive constant c_p depending only on p (and not on T) such that

$$\mathbb{E}\left(\left|\int_0^T\int_{\mathbb{R}^n} f(t,x,\omega)W(dx,dt)\right|^p\right) \leqslant c_p\mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}^n}|f(t,x,\omega)|^2dxdt\right)^{p/2}\right),$$

for all $f\in\mathcal{P}_W$.

Chapter 3

A reduced model of stochastic FSI

In this chapter, we begin the study of stochastic fluid-structure interaction by considering a *reduced* model of stochastic FSI, given by a stochastic viscous wave equation. The model under consideration is a fully coupled fluid-structure interaction model involving a viscous incompressible Newtonian fluid modeled by the stationary Stokes equations in the lower half space $(z < 0) \subset \mathbb{R}^3$ interacting dynamically with an elastic structure which displaces transversally from its reference configuration $(z = 0)$. There is two-way coupling between the fluid and structure, and the model is linearly coupled so that the stationary Stokes equations for the fluid are posed on the fixed reference domain for the fluid $\Omega_f = (z < 0)$. In order to consider the effects of randomness on the coupled fluid-structure dynamics, we consider spacetime white noise forcing, scaled by a nonlinear Lipschitz function of the structure displacement, acting on the elastic membrane, where the stochastic forcing on the membrane perturbs the coupled dynamics of the fluid and structure, as a result of the two-way coupling between the fluid and structure.

This model is a *reduced model* because even though the original model is described in terms of a fluid subproblem and a structure subproblem that are appropriately coupled through a kinematic and dynamic coupling condition, one can show that the full dynamics of the structure and fluid can be described by a single self-contained equation for the structure displacement, known as the stochastic viscous wave equation. This is a wave equation perturbed by stochastic spacetime white noise forcing, with an additional term $\sqrt{-\Delta}\eta_t$ arising from the Dirichlet-to-Neumann operator for the lower half space, which mathematically reflects the regularizing effects of the fluid viscosity on the structure dynamics. The original fully coupled FSI model, involving a coupled system of PDEs, can be reduced to a single self-contained stochastic equation for the structure displacement, since the fluid normal stress on the structure, which is the forcing on the elastodynamics equation for the structure via the dynamic coupling condition, can be expressed purely in terms of the structure displacement. This allows us to consider a single stochastic equation for the whole stochastic FSI model, and the specific geometry for this model allows us to consider a stochastic equation on $\mathbb{R}^2$ for the physical model, called the **stochastic viscous wave equation**. We can more generally consider this stochastic viscous wave equation for general spatial dimensions n, where we emphasize that $n = 2$ is the dimension of the physical FSI model from which we derived this equation.

Having a single self-contained stochastic equation on $\mathbb{R}^n$ allows us to analyze this fluid-structure interaction model by using classical methods from stochastic PDEs, that have been used to study stochastic heat and wave equations for example. This involves considering *mild solutions*, which are solutions that are defined via convolution of the fundamental solution with the spacetime white noise via an appropriately defined stochastic integral. In this chapter, we will derive the stochastic viscous wave equation, and analyze its well-posedness in various dimensions. The important result here will be well-posedness in terms of mild solutions in spatial dimensions $n = 1$ and $n = 2$, which improves upon classically known results for the stochastic heat and wave equations. We will also show that the stochastic viscous wave equation has better Hölder regularity of mild solutions in both spatial dimensions $n = 1$ and $n = 2$ than the stochastic heat and wave equations. The improvements in terms of well-posedness and regularity of mild solutions to the stochastic viscous wave equation when compared to the classical stochastic heat and wave equations with spacetime white noise arise from a combination of a favorable spacetime scaling in the equation combined with the regularizing effects of fluid viscosity on the structure. While the methods we use will be based on fundamental solutions, Fourier analysis, and stochastic integration rather than the splitting scheme approaches found in Chapter 1, we discuss this reduced model since the first well-posedness results for stochastic FSI were developed in the context of this reduced model, and led the way for further analysis of such stochastic fluid-structure systems. In addition, the methods used in the analysis of this reduced model and the stochastic viscous wave equation provide a nice application of the theory of spacetime white noise and stochastic integration.

We emphasize that the content in this chapter is adapted from the previously published paper [111], co-authored with Sunčica Čanić.

3.1 Introduction

We propose a stochastic model for fluid-structure interaction given by a stochastic wave equation augmented by dissipation associated with the effects of an incompressible, viscous fluid:

$$\eta_{tt} + 2\mu\sqrt{-\Delta}\eta_t - \Delta\eta = f(\eta)W(dt, dx), \qquad \text{in } \mathbb{R}^n. \tag{3.1}$$

The wave operator models the elastodynamics of a linearly elastic membrane, where η denotes membrane displacement, while the dissipative part, which is in the form of the Dirichlet-to-Neumann operator applied to the time derivative of displacement, accounts for dissipation due to fluid viscosity, where μ denotes the fluid viscosity coefficient. The equation is forced by spacetime white noise $W(dt, dx)$, which accounts for stochastic effects in real-life problems. The spacetime white noise is scaled by a nonlinear, Lipschitz function $f(\eta)$. We show below how this equation is derived from a coupled fluid-structure interaction problem involving the Stokes equations describing the flow of an incompressible, viscous fluid, and the wave equation modeling the elastodynamics of a (stretched) linearly elastic membrane. We consider equation (3.1) in $\mathbb{R}^n$ with $n = 1$ and $n = 2$, focusing primarily on $n = 2$, which is the physical dimension.

We prove the existence of a function-valued *mild solution* to a Cauchy problem for equation (3.1), which holds both in dimensions 1 and 2. Here, by "mild solution" we refer to a stochastic mild solution defined via stochastic integration involving the Green's function, specified below in Definition 3.3.1. This is interesting because our result contrasts the results that hold for the stochastic heat and wave equations: the stochastic heat and the stochastic wave equations do not have function-valued mild solutions in spatial dimension 2 or higher. Additionally, we prove that sample paths of the stochastic mild solution for the stochastic viscous wave equation are Hölder continuous with Hölder exponents $\alpha \in [0, 1)$ for $n = 1$, and $\alpha \in [0, 1/2)$ for $n = 2$.

Our results show that the viscous fluid dissipation in fluid-structure interaction is sufficient to smooth out the rough stochastic nature of the real-life data in the problem modeled by the spacetime white noise. In particular, the Dirichlet-to-Neumann operator controls the high frequencies in the structure (membrane) displacement that are driven by the spacetime white noise. To the best of our knowledge, this is the first result on stochastic fluid-structure interaction.

We begin by describing the fluid-structure interaction model from which the equation (3.1) arises. Consider a prestressed infinite elastic membrane surface, which is modeled by the linear wave equation

$$\eta_{tt} - \Delta\eta = F_s, \quad \text{on } \Gamma := \{(x_1, x_2, 0) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{R}^2\}, \tag{3.2}$$

where $\eta(x_1, x_2)$ denotes the transverse displacement (in the x_3 direction) of the elastic surface from its reference configuration Γ and the external load F_s will be specified later in the dynamic coupling condition. See Figure 3.1.

Beneath this elastic drum surface, we consider a viscous, incompressible fluid, which resides in the lower half-space in $\mathbb{R}^3$,

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}, \tag{3.3}$$

modeled by the stationary Stokes equations for an incompressible, viscous fluid:

$$\left.\begin{array}{rcl} \nabla \cdot \boldsymbol{\sigma}(\pi, \boldsymbol{u}) & = & 0, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}, \tag{3.4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and the unknown quantities are the fluid pressure $\pi : \Omega \to \mathbb{R}$ and the fluid velocity $\boldsymbol{v} : \Omega \to \mathbb{R}^3$. We will be assuming that the fluid is Newtonian, so that

$$\boldsymbol{\sigma} = -\pi \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}),$$

where μ denotes the fluid viscosity coefficient, $\boldsymbol{I}$ is the three by three identity matrix, and $\boldsymbol{D}(\boldsymbol{u})$ is the symmetrized gradient of fluid velocity $\boldsymbol{D}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$. Therefore, the

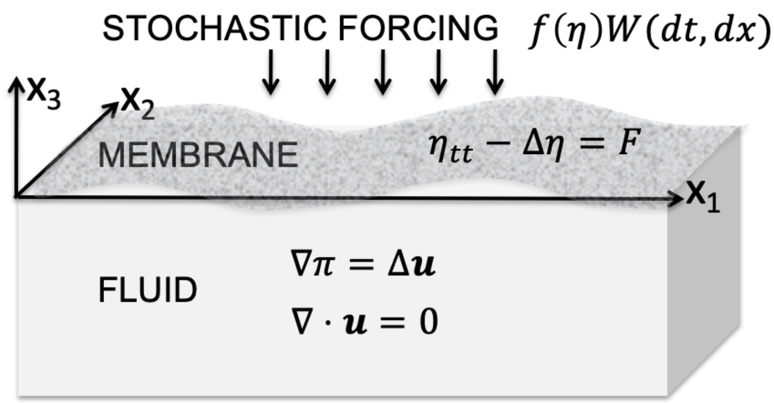

Figure 3.1: *A sketch of the fluid and structure domains.*

Stokes equations now read

$$\left.\begin{array}{rcl}\nabla\pi & = & \mu\triangle\boldsymbol{u},\\ \nabla\cdot\boldsymbol{u} & = & 0,\end{array}\right\} \quad \text{in } \Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : z < 0\}, \tag{3.5}$$

where we require that the fluid velocity is bounded in the lower half space, and the pressure $\pi \to 0$ as $|x| \to \infty$.

We consider the problem in which the elastic surface is displaced from its reference configuration Γ with some given initial displacement and velocity, allowing only vertical displacement, where the elastodynamics of the elastic surface is driven by the total force exerted onto the membrane, which comes from the fluid on one side, and an external stochastic forcing on the other. See Figure 4.1.

Inifinite domains are considered to simplify the analysis, since the main purpose of this work is to understand the interplay between the dispersion effects in the 2D wave equation, dissipation due to fluid viscosity, and stochasticity imposed by the external forcing, which can be related to the stochasticity of not only the external forcing, but also to the stochasticity of data (e.g., inlet/outlet data) in real-life applications, such as blood flow through arteries.

The fluid and the structure are coupled via two coupling conditions, the kinematic and dynamic coupling conditions, giving rise to the so-called two-way coupled fluid-structure interaction problem. The coupling conditions in the present study are evaluated at a *fixed (linearized) fluid-structure interface* corresponding to the structure's reference configuration Γ. This is known as *linear coupling*. The two conditions read:

- ***Kinematic coupling condition.*** The kinematic coupling condition describes the coupling between the kinematic quantities such as velocity. We will be assuming the no-slip condition, meaning that the fluid and structure velocities are continuous at the interface (there is no slip between the two):

$$\eta_t = \boldsymbol{u}|_\Gamma, \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2, t \geqslant 0. \tag{3.6}$$

- ***Dynamic coupling condition.*** The dynamic coupling condition describes the balance of forces at the fluid-structure interface Γ, namely, it states that the elastodynamics of the membrane is driven by the force corresponding to the jump in traction (normal stress) across the membrane. This specifies the external load F_s on the structure elastodynamics in the structure equation (3.2). On the fluid side, the traction (normal stress) at the interface is given by $-\boldsymbol{\sigma}\boldsymbol{e_{x_3}}$, where $\boldsymbol{e_{x_3}}$ is the normal vector to Γ, while on the other side, we are assuming a given loading $F_{ext}(\eta)$ to be a stochastic process $f(\eta)W(dt, dx)$ in the $\boldsymbol{e_{x_3}}$ direction. Examples of such a loading can be found in cardiovascular applications, see e.g., [132]. Since we assume that the structure only has transversal displacement, the dynamic coupling condition reads:

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_{x_3}} \cdot \boldsymbol{e_{x_3}} + f(\eta(t,x))W(dt, dx) \quad \text{where } x = (x_1, x_2) \in \mathbb{R}^2, t \geqslant 0. \tag{3.7}$$

In fluid-structure interaction problems and physical problems in general, physical phenomena are subject to small random deviations that cause deviations from deterministic behavior. The consideration of such stochastic effects in partial differential equations can give rise to new phenomena, and is an area of active research. Furthermore, in real-life data, one observes such stochastic noise both in terms of the force exerted onto the structure, as well as in the data that drives the problem. For example, the measured inlet/outlet pressure data in a fluid-structure interaction problem describing arterial blood flow, has similar stochastic noise deviations to $F_{ext}(\eta) = f(\eta(t,x))W(dt, dx)$. Here $W(dt, dx)$ is spacetime white noise in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$, whose properties we will recall in Sec. 3.2. We will assume that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. In particular, the case $f \not\equiv 1$ allows dependence of the magnitude of the stochastic noise at each point on the structure displacement $\eta(t, x)$ itself.

To derive equation (3.1) as a model which describes the fluid-structure interaction problem (3.2)-(3.7), we focus on the dynamic coupling condition (3.7). The goal is to try to express the effects of fluid normal stress via the Dirichlet-to-Neumann operator defined entirely in terms of η and/or its derivatives. In this derivation we also use the kinematic coupling condition (3.6) as explained below.

First notice that the right hand-side of (3.7) is given by

$$-\boldsymbol{\sigma}\boldsymbol{e_{x_3}} \cdot \boldsymbol{e_{x_3}} = \pi - 2\mu\frac{\partial u_{x_3}}{\partial x_3} \text{ on } \Gamma.$$

Since the tangential displacements are assumed to be zero, the kinematic coupling condition (3.6) implies that the x_1 and x_2 components of the fluid velocity $\boldsymbol{u}$ are zero on Γ, namely

$u_{x_1} = u_{x_2} = 0$ on Γ. By the divergence free condition, one immediately gets that $\partial u_{x_3}/\partial x_3 = 0$ on Γ. Therefore,

$$-\boldsymbol{\sigma} \boldsymbol{e_{x_3}} \cdot \boldsymbol{e_{x_3}} = \pi \quad \text{on } \Gamma, \tag{3.8}$$

where π is the fluid pressure given as a solution to the Stokes equations (3.5). So it remains to find an appropriate expression for π on Γ in terms of the structure displacement η. In fact, we will show that the following formula holds

$$\pi = -2\mu\sqrt{-\Delta}\eta_t \quad \text{on } \Gamma, \tag{3.9}$$

under the assumption that η and η_t, along with their spatial derivatives, are smooth functions that are rapidly decreasing at infinity. We will also impose the boundary conditions on (3.5), stating that the fluid velocity is bounded on the lower half space, and the pressure π has a limit equal to zero as $|x| \to \infty$ in the lower half space. Details of this calculation are presented in [112]. Here we present the main steps.

To derive the formula (3.9), we note that by taking the inner product of the first equation in (3.5) with $\boldsymbol{e_{x_3}}$, we obtain

$$\frac{\partial \pi}{\partial x_3} = \mu \Delta_{x_1,x_2} u_{x_3} + \mu \frac{\partial^2 u_{x_3}}{\partial x_3^2} = \mu \Delta u_{x_3}, \tag{3.10}$$

where $\Delta_{x_1,x_2} := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Furthermore, by taking the divergence of the first equation in (3.5), and by using the divergence-free condition, we get that the pressure π is harmonic. Thus, if we can compute the right hand side of (3.10) on Γ, we can recover π as the solution to a Neumann boundary value problem for Laplace's equation in the lower half space, with the boundary condition requiring that π goes to zero at infinity.

To compute the right hand side of (3.10), we need to compute v_{x_3}. Taking the Laplacian on both sides of (3.10), and using the fact that π is harmonic, we obtain

$$\Delta^2 u_{x_3} = 0, \qquad \text{on } \Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}. \tag{3.11}$$

Thus, u_{x_3} satisfies the biharmonic equation with the following two boundary conditions: from the kinematic coupling condition, we get

$$u_{x_3}(x_1, x_2, 0) = \eta_t(x_1, x_2, 0), \qquad \text{on } \Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \tag{3.12}$$

and from the fact that $u_{x_1} = u_{x_2} = 0$ on Γ, by the kinematic coupling condition and the fact that $\boldsymbol{u}$ is divergence free, we get

$$\frac{\partial u_{x_3}}{\partial x_3}(x_1, x_2, 0) = 0, \qquad \text{on } \Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}. \tag{3.13}$$

We solve (3.11) with boundary conditions (3.12) and (3.13) by taking a Fourier transform in the variables x_1 and x_2, but not in x_3. We will denote the Fourier variables associated with

x_1 and x_2 by ξ_1 and ξ_2, and we will denote $\xi = (\xi_1, \xi_2)$, $|\xi|^2 = \xi_1^2 + \xi_2^2$. The Fourier transform equation then reads:

$$|\xi|^4 \widehat{u_{x_3}}(\xi, x_3) - 2|\xi|^2 \frac{\partial^2}{\partial x_3^2} \widehat{u_{x_3}}(\xi, x_3) + \frac{\partial^4}{\partial x_3^4} \widehat{u_{x_3}}(\xi, x_3) = 0. \tag{3.14}$$

The solution is given by

$$\widehat{u_{x_3}}(\xi, x_3) = \widehat{\eta}_t(\xi) e^{|\xi| x_3} - |\xi| \widehat{\eta}_t(\xi) x_3 e^{|\xi| x_3}. \tag{3.15}$$

We can now compute the right hand side of (3.10). Taking the Fourier transform of (3.10) in the x_1 and x_2 variables, and evaluating the equation on Γ by using the kinematic coupling condition (3.6), we get

$$\frac{\partial \widehat{\pi}}{\partial x_3}(\xi, 0) = -\mu |\xi|^2 \widehat{\eta}_t(\xi) + \mu \frac{\partial^2 \widehat{u_{x_3}}}{\partial x_3^2}(\xi, 0) = -2\mu |\xi|^2 \widehat{\eta}_t(\xi), \tag{3.16}$$

where the last equality follows by using the explicit formula for $\widehat{u_{x_3}}(\xi, z)$ in (3.15).

We now know that the pressure π is a harmonic function in the lower half space, satisfying the Neumann boundary condition (3.16) given in Fourier space. To recover formula (3.9) we want π on Γ. This can be obtained via the Neumann to Dirichlet operator. It is well known that the *Dirichlet to Neumann* operator for Laplace's equation in the lower half space is given by $\sqrt{-\Delta}$, thereby having a Fourier multiplier $|\xi|$, see e.g., [29]. Therefore, the Neumann to Dirichlet operator for Laplace's equation in the lower half space (with the solution to Laplace's equation having a limit of zero at infinity) is a Fourier multiplier of the form $\frac{1}{|\xi|}$. Thus, the Neumann to Dirichlet operator applied to the Neumann data (3.16) gives the pressure as Dirichlet data:

$$\widehat{\pi}(\xi) = -2\mu |\xi| \widehat{\eta}_t(\xi) \qquad \text{on } \Gamma,$$

which establishes the desired formula (3.9).

The dynamic coupling condition (3.7), together with (3.9) gives the stochastic model

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = f(\eta) W(dt, dx) \quad \text{on } \mathbb{R}^2,$$

where we have set the fluid viscosity $\mu = 1/2$. We will refer to this model as the *stochastic viscous wave equation*, as it is a stochastic wave equation augmented by the viscous effects of the fluid, which are captured by the Dirichlet to Neumann operator acting on the structure velocity u_t.

Fluid-structure interaction systems have been considered extensively in the past mathematical literature, and we refer the reader to the references described for linearly coupled and nonlinearly coupled models of FSI in Chapter 1 of this thesis for an overview of what is known in the deterministic context. However, we emphasize that to the best of our knowledge, prior work in FSI has been exclusively for deterministic models of fluid-structure interaction.

On the other hand, the study of stochasticity in PDEs has been of recent interest. Most physical phenomena occurring in real-life applications feature the presence of some sort of random noise that perturbs the system from what may be deterministically expected. The types of noise added to the equation can vary, from the simplest spacetime white noise, which intuitively is noise that is "independent at each time and space", to forms of noise with smoother spatial correlation so that the noise is still independent at separate times, but the noise in space allows for correlation between points and is hence "smoother" than white noise.

Many classical partial differential equations, such as the heat and wave equations, have been studied with the addition of stochastic random forcing. There are many approaches to the study of such stochastic PDEs. One approach uses Walsh's theory of martingale measures, of which white noise is an example. The theory of integration against such martingale measures can be found in Walsh's work [177]. Upon defining such an appropriate theory of stochastic integration, one can define what is called a *mild solution* to a given stochastic partial differential equation by means of the Green's function.

The choice of the stochastic noise used in the PDE being studied is of utmost importance. White noise, which is noise that is intuitively "independent at all spaces and times", is a starting point for many studies. In formal mathematical notation, the time and space independence property of white noise is expressed via expectation as

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)\delta_0(x-y),$$

where δ_0 is the Dirac delta function, and $\dot{W}(t,x)$, or $W(dt,dx)$, denotes spacetime white noise. In the rest of this manuscript, we will be using $W(dt,dx)$ to denote spacetime white noise and stochastic integration against white noise.

The white noise perturbed heat and wave equations have interesting properties. When perturbed by white noise, the two equations:

$$\eta_t - \Delta\eta = W(dt,dx) \quad \text{and} \quad \eta_{tt} - \Delta\eta = W(dt,dx) \qquad \text{in } \mathbb{R}^n,$$

do not allow function-valued mild solutions in spatial dimensions two and higher, while they do in spatial dimension one. See, for example, [55] and [107], where questions of existence and uniqueness of mild solutions are addressed. This interesting property related to spatial dimensions two and higher is due to the lack of square integrability of the Green's function in time and space, as we discuss later. Because of this property, "smoother" types of stochastic noise, such as spatially homogeneous Gaussian noise, are used to perturb the stochastic heat and wave equations in higher dimensions in order to yield function-valued mild solutions. In formal mathematical notation, such spatially homogeneous Gaussian noise $\dot{F}(t,x)$ has a covariance structure

$$\mathbb{E}(\dot{F}(t,x)\dot{F}(s,y)) = \delta_0(t-s)k(x-y), \tag{3.17}$$

where $k : \mathbb{R}^n \to \mathbb{R}$. Note that the formal case of setting k to be the Dirac delta "function" recovers the previous white noise case, though choosing smoother functions k allows us to formulate "smoother" types of noise. See Sec. 3 of [55] and the work in [56] for more information about spatially homogeneous Gaussian noise. One of the key questions in studying stochastically perturbed PDEs is what conditions on k need to be imposed so that the resulting equation with the spatially homogeneous Gaussian noise with covariance structure (3.17) has function-valued mild solutions? See, for example [56], and for more general contexts [54] and [105].

In the current manuscript, we do not need the properties of general spatially homogeneous Gaussian noise. This is because, as we shall see below, the viscous wave equation

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = F, \qquad \text{on } \mathbb{R}^n, \tag{3.18}$$

considered in [112, 116] as a model for fluid-structure interaction, combines the following two desirable properties: the "right" spacetime scaling (c.f. wave equation), and adequate dissipative effects. The resulting behavior is "in between" the wave and heat equations. The viscous wave equation (3.18) turns out to have just the right scaling and dissipation to allow function-valued mild solutions even in spatial dimension two for the white noise perturbed equation

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = W(dt, dx) \qquad \text{in } \mathbb{R}^n. \tag{3.19}$$

This is of great interest, since equations (3.1) and (3.19) in two spatial dimensions correspond exactly to the physical fluid-structure interaction model we are considering, and hence have direct physical significance.

The main results of this work are: (1) the existence of a function-valued mild solution for the white noise perturbed viscous wave equation (3.1) (and (3.19)) for dimensions $n = 1$ and $n = 2$, and (2) Hölder continuity $C^{0,\alpha}$ with $\alpha \in [0, 1)$ for $n = 1$, and $\alpha \in [0, 1/2)$ for $n = 2$, of "every" realization of the displacement u, obtained as a mild solution to the randomly perturbed viscous wave equation.

In particular, in terms of Hölder continuity, our results imply that the stochastic mild solution to equation (3.1) with zero initial data has a continuous *modification* that is α-Hölder continuous in time and space, with $\alpha \in [0, 1)$ for $n = 1$, and $\alpha \in [0, 1/2)$ for $n = 2$. Here, a *modification* of a stochastic process $\{X_i\}_{i\in I}$ is defined to be a stochastic process $\{\tilde{X}_i\}_{i\in I}$ such that the probability $\mathbb{P}(\tilde{X}_i = X_i) = 1$, for all $i \in I$. Thus, we show that the stochastic function-valued mild solution has a modification that is a Hölder continuous function for every realization u of the displacement, obtained as a mild solution to the randomly perturbed viscous wave equation (3.1).

Even in dimension $n = 1$, this contrasts the results for the stochastically perturbed heat and wave equations:

$$\eta_t - \Delta\eta = f(\eta)W(dt, dx) \quad \text{and} \quad \eta_{tt} - \Delta\eta = f(\eta)W(dt, dx) \qquad \text{on } \mathbb{R}. \tag{3.20}$$

Namely, in $n = 1$, the function-valued mild solutions (up to modification) for zero initial data are α-Hölder continuous in time and β-Hölder continuous in space, where $\alpha \in [0, 1/4)$, $\beta \in [0, 1/2)$ for the stochastic heat equation, and $\alpha, \beta \in [0, 1/2)$ for the stochastic wave equation,

Stochastic equation with $W(dt, dx)$	Spatial dimension n	Regularity of mild solution
Stochastic heat equation	$n = 1$	$\alpha \in [0, 1/4)$, $\beta \in [0, 1/2)$
Stochastic wave equation	$n = 1$	$\alpha \in [0, 1/2)$, $\beta \in [0, 1/2)$
Stochastic viscous wave equation	$n = 1$	$\alpha \in [0, 1)$, $\beta \in [0, 1)$
Stochastic heat equation	$n = 2$	Mild solution does not exist
Stochastic wave equation	$n = 2$	Mild solution does not exist
Stochastic viscous wave equation	$n = 2$	$\alpha \in [0, 1/2)$, $\beta \in [0, 1/2)$

Table 3.1: This table shows the α-Hölder regularity in time and the β-Hölder regularity in space for the mild solutions of the various stochastic equations with spacetime white noise in spatial dimensions $n = 1$ and $n = 2$. Note that the regularity of the mild solutions is improved for the stochastic viscous wave equation over the classical stochastic heat and wave equations, where these equations are all considered with spacetime white noise forcing.

see [107], [55], and [90]. The difference in space and time Hölder regularity between the heat and wave equations is due to the scaling of space and time, where for the heat equation, one time derivative "corresponds" to two spatial derivatives, while for the wave equation, one time derivative "corresponds" to one spatial derivative. In the stochastic viscous wave equation, the additional regularizing effect of the fluid viscosity implies improved Hölder regularity. In spatial dimension one, the solution is Hölder continuous of order $\alpha \in [0, 1)$ in space and time, which is an improvement over the results for both the stochastic heat and wave equations. In spatial dimension two, the solution is Hölder continuous of order $\alpha \in [0, 1/2)$ in space and time, whereas the stochastic heat and wave equations do not have function-valued mild solutions in spatial dimension two. We summarize these Hölder continuity results for mild solutions with appropriate initial data in Table 3.1 below, for the three stochastic equations with spacetime white noise stochasticity that we have previously mentioned.

The literature on the Hölder continuity properties of the solutions to the heat equation and the wave equation with random noise in spatial dimensions two and higher, is an area of extensive study. However, we emphasize again that for these equations, the stochastic noise is not white noise, but something smoother, such as, e.g., spatially homogeneous Gaussian noise, as a function-valued mild solution does not exist with spacetime white noise in spatial dimensions two and higher for these equations. We refer the reader to [50], [57], [158] for more details.

This chapter is organized as follows. In Sec. 3.2, we recall the properties of white noise, stochastic integration, and the deterministic forms of the heat, wave, and viscous wave equation solutions that will be necessary to show the main result. In Sec. 3.3, we show the existence and uniqueness of a stochastic function-valued mild solution for equation (3.1), in dimensions $n = 1, 2$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function, and in Sec. 3.4, we study the Hölder continuity of sample paths of solutions to (3.1).

3.2 Preliminaries

In this section, we recall some basic facts about the deterministic linear heat, wave, and viscous wave equations that will be needed later in the analysis of the stochastic viscous wave equation.

The viscous wave equation

We begin by considering the (linear) viscous wave equation

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = 0, \tag{3.21}$$

with initial data

$$\eta(0,x) = g(x), \qquad \partial_t\eta(0,x) = h(x).$$

We recall some basic properties related to the analysis of this equation here, and recommend [112] for more information. Assuming that the initial data g, h are regular enough, for example $g, h \in \mathcal{S}(\mathbb{R}^n)$, we can explicitly solve this equation using the Fourier transform to obtain

$$\widehat{\eta}(t,\xi) = \widehat{g}(\xi)e^{-\frac{|\xi|}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right) + \widehat{h}(\xi)e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}. \tag{3.22}$$

From the Fourier representation one can see that this equation has both parabolic and wave-like properties. The wavelike behavior is represented by the presence of cosine and sine, and the strong parabolic dissipation is given by the exponential factor $e^{-\frac{|\xi|}{2}t}$, which causes damping of frequencies over time.

Of particular interest to this work is the solution to the general inhomogeneous problem

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = F, \tag{3.23}$$

with initial data $u(0,x) = g(x)$, $\partial_t u(0,x) = h(x)$, which can be obtained using Duhamel's principle:

$$\begin{aligned}\widehat{\eta}(t,\xi) &= \widehat{g}(\xi)e^{-\frac{|\xi|}{2}t}\left(\cos\left(\tfrac{\sqrt{3}}{2}|\xi|t\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}|\xi|t\right)\right) + \widehat{h}(\xi)e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} \\ &\quad + \int_0^t \widehat{F}(\tau,\xi)e^{-\frac{|\xi|}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-\tau)\right)}{\frac{\sqrt{3}}{2}|\xi|}d\tau. \end{aligned} \tag{3.24}$$

The inverse Fourier transform gives the solution u in physical space:

$$\begin{aligned}\eta(t,\cdot) &= e^{-\frac{\sqrt{-\Delta}}{2}t}\left(\cos\left(\tfrac{\sqrt{3}}{2}\sqrt{-\Delta}t\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)\right)g + e^{-\frac{\sqrt{-\Delta}}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}t\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}h \\ &\quad + \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-\tau)}\frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-\tau)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}}F(\tau,\cdot)d\tau. \end{aligned} \tag{3.25}$$

Of considerable importance in future sections will be the effect of an inhomogeneous source term on the linear operator. In particular, the solution to (3.23) with zero initial data is given by the formula:

$$\eta(t,\cdot) = \int_0^t e^{-\frac{\sqrt{-\Delta}}{2}(t-s)} \frac{\sin\left(\frac{\sqrt{3}}{2}\sqrt{-\Delta}(t-s)\right)}{\frac{\sqrt{3}}{2}\sqrt{-\Delta}} F(s,\cdot)d\tau. \tag{3.26}$$

By recalling that the Fourier transform interchanges multiplication of functions and convolution, we can rewrite the formula (3.26) in a more explicit manner. Let us define the kernel $K_t(x)$ by the inverse Fourier transform,

$$K_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi. \tag{3.27}$$

To take advantage of the scaling of this PDE, we introduce the unit scale kernel $K(x)$, defined by

$$K(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi, \tag{3.28}$$

which is just the kernel $K_t(x)$ at unit time $t = 1$. A simple change of variables shows the following crucial scaling relation:

$$K_t(x) = t^{1-n} K\left(\frac{x}{t}\right). \tag{3.29}$$

Equipped with the notation above, we can rewrite (3.26) in physical spatial variables as

$$\eta(t,\cdot) = \int_0^t K_{t-s}(\cdot) * F(s,\cdot)ds = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)F(s,y)dyds, \tag{3.30}$$

where the convolution operator $*$ denotes a convolution only in the spatial variables.

The importance of using the kernel $K_t(x)$ to express the solution to the viscous wave equation explicitly as (3.30) lies in the fact that we have the following strong estimate for the unit-scale kernel $K(x)$, which carries over to the general kernel $K_t(x)$ by the scaling relation (3.29). The following lemma reflects the strong dissipative effects of the fluid viscosity, represented by the presence of the Dirichlet to Neumann operator.

Lemma 3.2.1. For all dimensions n, the kernel $K(x)$ is in $L^q(\mathbb{R}^n)$ for all $1 \leqslant q \leqslant \infty$.

Proof. The proof of this lemma is by estimates using a repeated integration by parts. We refer the reader to the proof of Lemma 3.3 in [112]. □

This representation of solution will be important later in Section 3.3 when we discuss well-posedness of the *stochastic* viscous wave equation. To compare the stochastic viscous wave equation with the stochastic heat and the stochastic wave equations as will be done in Section 3.3, we now give the analogue of the above analysis using a convolution kernel for the heat and wave equations, focusing on the inhomogeneous forms of these equations with zero initial data.

First, we consider the inhomogeneous heat equation. Define the heat equation kernel and the corresponding unit scale kernel:

$$K_t^H(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-|\xi|^2 t}d\xi = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{x^2}{4t}}, \quad K^H(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-|\xi|^2}d\xi = \frac{1}{(4\pi)^{n/2}}e^{-\frac{x^2}{4}}. \tag{3.31}$$

A simple change of variables shows the following scaling relation for the heat equation kernel:

$$K_t^H(x) = t^{-\frac{n}{2}}K^H\left(\frac{x}{t^{1/2}}\right). \tag{3.32}$$

In terms of the heat equation kernel, the solution to the inhomogeneous heat equation

$$\eta_t - \Delta\eta = F$$

with zero initial data is given by the formula:

$$\eta(t,x) = \int_0^t e^{-(t-s)\Delta}F(s,\cdot)ds = \int_0^t K_{t-s}^H(\cdot) * F(s,\cdot)ds = \int_0^t\int_{\mathbb{R}^n} K_{t-s}^H(x-y)F(s,y)dyds. \tag{3.33}$$

Note that in all dimensions n, and for all times $t > 0$, the kernel $K_t^H(x)$ defined in (3.31), is function-valued and is, in fact, a Schwartz function.

Next, we carry out the same analysis for the inhomogeneous wave equation. The wave equation kernel and the corresponding unit scale kernel can be defined similarly as

$$K_t^W(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin(|\xi|t)}{|\xi|}d\xi, \ K^W(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin(|\xi|)}{|\xi|}d\xi. \tag{3.34}$$

The corresponding scaling relation is

$$K_t^W(x) = t^{1-n}K^W\left(\frac{x}{t}\right), \tag{3.35}$$

which is the same as the scaling (3.29) of the kernel for the viscous wave equation. The solution to the inhomogeneous wave equation

$$\eta_{tt} - \Delta\eta = F$$

with zero initial data is then given by the formula:

$$\eta(t,x) = \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s,\cdot)ds = \int_0^t K_{t-s}^W(\cdot) * F(s,\cdot)ds = \int_0^t\int_{\mathbb{R}^n} K_{t-s}^W(x-y)F(s,y)dyds. \tag{3.36}$$

It is important to note that unlike the viscous wave and heat equation, the kernel $K_t(x)$ is no longer necessarily function-valued. In fact, we have, for example, the following well-known formulas for the kernel $K_t(x)$, giving the fundamental solution for the wave equation:

$$K_t^W(x) = \begin{cases} \frac{1}{2} 1_{|x|<t} & \text{for } n = 1, \\ \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{|x|<t} & \text{for } n = 2, \\ \frac{1}{4\pi} \sigma_t(dx) & \text{for } n = 3, \end{cases} \tag{3.37}$$

where $\sigma_t(dx)$ in the last expression denotes the surface measure on the sphere of radius t centered at the origin. There are more complicated formulas for higher dimensions also, but $K_t^W(x)$ is function-valued only in dimensions one and two. In fact, $K_t^W(x)$ becomes increasingly singular as the dimension increases.

It is interesting to note that the kernel for the wave equation K_t^W can be tied to the kernel for the viscous wave equation K_t by the following result.

Proposition 3.2.1. The kernel $K_t(x)$ for the viscous wave equation in dimension n, defined by (3.27), is given by the convolution

$$K_t(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(t^2 + 4|x-y|^2)^{\frac{n+1}{2}}} \, K^W_{\frac{\sqrt{3}}{2}t}(y)dy,$$

where c_n is a constant depending only on the dimension n.

Proof. We use formula (3.27) and recall that the Fourier transform interchanges multiplication and convolutions. The inverse Fourier transform of $e^{-\frac{|\xi|}{2}t}$ is

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t} d\xi = \tilde{c}_n \frac{t}{(t^2 + 4|x|^2)^{\frac{n+1}{2}}},$$

where $\tilde{c}_n$ depends only on n. From the definition of K_t^W, we get

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|} d\xi = \frac{2}{\sqrt{3}} K^W_{\frac{\sqrt{3}}{2}t}(x). \tag{3.38}$$

The result then follows by using the fact that the Fourier transform interchanges multiplication and convolution, where we replaced the constant $\frac{2}{\sqrt{3}}\tilde{c}_n$ by c_n. □

We have so far considered the inhomogeneous viscous wave equation with zero initial data. However, we will consider eventually the stochastic form of this equation with continuous bounded initial data, and will hence have to consider the full form of the solution given in (3.25), which takes into account the possibility of nonzero initial displacement and velocity. Note that the convolution kernel $K(x)$ defined in (3.27) can be used to describe the effect of an inhomogeneous source term and an initial velocity, as seen in (3.25). However, the kernel $K(x)$ does not describe the effect of an initial displacement g on the solution. For this reason, we introduce the corresponding convolution kernel $J_t(x)$ and the respective unit scale kernel $J(x)$ associated to the propagation of $g(x)$ in (3.25):

$$J_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}t} \left(\cos\left(\frac{\sqrt{3}}{2}|\xi|t\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)\right) d\xi, \tag{3.39}$$

$$J(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}|\xi|\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}|\xi|\right)\right) d\xi. \tag{3.40}$$

A change of variables shows that

$$J_t(x) = t^{-n} J\left(\frac{x}{t}\right). \tag{3.41}$$

We can then write the representation formula (3.25) for the solution of the general viscous wave equation with nonzero initial data $\eta(0,x) = g(x)$ and $\partial_t\eta(0,x) = h(x)$ and inhomogeneous source term F, as

$$\eta(t,x) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)F(s,y)dsdy. \tag{3.42}$$

In analogy to Lemma 3.2.1, one can show the following lemma, which shows that the unit scale kernel $J(x)$ has strong integrability properties.

Lemma 3.2.2. For all dimensions n, the kernel $J(x)$ is in $L^q(\mathbb{R}^n)$ for all $1 \leqslant q \leqslant \infty$. Furthermore, we have the estimate,

$$|J(x)| \leqslant C_N |x|^{-1-n\left(\frac{N-1}{N}\right)},$$

for any $N \geqslant n+1$, where C_N is a constant depending on N.

Proof. We refer the reader to the proof of Lemma 3.3 in [112]. While the proof there is for the slightly different unit kernel $K(x)$, the corresponding proof for $J(x)$ is just a slight modification of the proof given there. □

Finally, we establish a final lemma in this section, which shows the effect of the viscous wave operator on initial data $\eta(0,x) = g(x)$ and $\partial_t\eta(0,x) = h(x)$, where g and h are the continuous versions of functions in $H^2(\mathbb{R}^n)$. This will be useful when showing existence and uniqueness of a mild solution to the stochastic viscous wave equation with initial data g and h, which are the continuous versions of functions in $H^2(\mathbb{R}^n)$, see Section 3.3.

Lemma 3.2.3. Let $n = 1$ or $n = 2$, and let g and h be the continuous versions of functions in $H^2(\mathbb{R}^n)$. Then, for any positive time $T > 0$, the solution to

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = 0,$$

with initial data

$$\eta(0,x) = g(x), \qquad \partial_t \eta(0,x) = h(x),$$

is a bounded, continuous function on $[0,T] \times \mathbb{R}^n$. Furthermore, the solution $\eta(t,x)$ has the following Hölder continuity properties depending on the dimension n:

- If $n = 1$, then for every $\rho \in [0,1]$, there exists a constant $C_{\rho,T}$ depending only on ρ and T such that for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}$,

$$|\eta(t,x) - \eta(t,x')| \leqslant C_{\rho,T}|x - x'|^{\rho}, \qquad |\eta(t,x) - \eta(t',x)| \leqslant C_{\rho,T}|t - t'|^{\rho}.$$

- If $n = 2$, then for every $\rho \in [0,1)$, there exists a constant $C_{\rho,T}$ depending only on ρ and T such that for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^2$,

$$|\eta(t,x) - \eta(t,x')| \leqslant C_{\rho,T}|x - x'|^{\rho}, \qquad |\eta(t,x) - \eta(t',x)| \leqslant C_{\rho,T}|t - t'|^{\rho/2}.$$

Remark 3.2.1. By Sobolev embedding, every function in $H^2(\mathbb{R}^n)$ for $n = 1, 2$ is a continuous function, up to a set of measure zero. Because we need the initial data g, h to be continuous for our arguments, we state that g, h are the *continuous versions* of functions in $H^2(\mathbb{R}^n)$, in order to indicate that $g, h \in H^2(\mathbb{R}^n)$ are potentially redefined on a set of measure zero so that they are also continuous functions. We will use this terminology for the remainder of the chapter.

Proof. First we show that $\eta(t,x)$ is bounded. By Lemma 3.2.1 and Lemma 3.2.2, $J, K \in L^1(\mathbb{R}^n)$. Therefore, by using the scaling relations (3.41) and (3.29), we have that for $t \in [0,T]$,

$$\int_{\mathbb{R}^n} |J_t(x)|dx = t^{-n} \int_{\mathbb{R}^n} \left|J\left(\frac{x}{t}\right)\right| dx = ||J||_{L^1} < \infty, \tag{3.43}$$

$$\int_{\mathbb{R}^n} |K_t(x)|dx = t^{1-n} \int_{\mathbb{R}^n} \left|K\left(\frac{x}{t}\right)\right| dx = t||K||_{L^1} \leqslant T||K||_{L^1} < \infty. \tag{3.44}$$

Using these facts along with the fact that g, h are bounded (by Sobolev embedding since they are in $H^2(\mathbb{R}^n)$), the explicit formula

$$\eta(t,x) = \int_{\mathbb{R}^n} J_t(y)g(x-y)dy + \int_{\mathbb{R}^n} K_t(y)h(x-y)dy \tag{3.45}$$

implies that $\eta(t,x)$ is bounded on $[0,T] \times \mathbb{R}^n$.

Next, we establish continuity. First, we consider spatial increments. Since $g, h \in H^2(\mathbb{R}^n)$ are continuous, they are ρ-Hölder continuous for $\rho \in [0,1)$ by Sobolev embedding, and in fact also Lipschitz continuous in dimension one. Then, for $x, x' \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned} |\eta(t,x) - \eta(t,x')| &\leqslant \int_{\mathbb{R}^n} |J_t(y)| \cdot |g(x-y) - g(x'-y)|dy + \int_{\mathbb{R}^n} |K_t(y)| \cdot |h(x-y) - h(x'-y)|dy \\ &\leqslant C_\rho |x - x'|^{\rho} \left(\int_{\mathbb{R}^n} |J_t(y)|dy + \int_{\mathbb{R}^n} |K_t(y)|dy \right) \leqslant C_{\rho,T}|x - x'|^{\rho}, \end{aligned} \tag{3.46}$$

where $\rho = [0, 1]$ if $n = 1$, and $\rho \in [0, 1)$ if $n = 2$, with C_ρ depending on ρ. In particular, the Lipschitz or Hölder continuity of the initial data is propagated in time on a finite time interval. We have also shown that $u(t, x)$ at each fixed time is a continuous function.

Next, we consider time increments. Consider $0 \leqslant t' < t \leqslant T$. We want to estimate the quantity $|\eta(t, x) - \eta(t', x)|$ for arbitrary $x \in \mathbb{R}^n$. We consider the two cases of $n = 1$ and $n = 2$ separately.

Case 1: If $n = 1$, then we have that $(\eta, \eta_t) \in C([0, T]; H^2(\mathbb{R})) \times C([0, T]; H^1(\mathbb{R}))$. Since $H^1(\mathbb{R})$ embeds continuously into the bounded continuous functions on $\mathbb{R}$, η_t is bounded and continuous on $[0, T] \times \mathbb{R}$. Hence, $|\eta_t(t, x)| \leqslant C_T$, $\forall x \in \mathbb{R}$ and $t \in [0, T]$. By the fundamental theorem of calculus,

$$|\eta(t, x) - \eta(t', x)| \leqslant \int_{t'}^{t} |\eta_t(s, x)| ds \leqslant C_T |t - t'|. \tag{3.47}$$

Case 2: If $n = 2$, by uniqueness of the solution in $C([0, T]; H^2(\mathbb{R}^2)) \times C([0, T]; H^1(\mathbb{R}^2))$, we can consider $\eta(t', x)$ and $\partial_t \eta(t', x)$ as initial data at time t', to get

$$|\eta(t, x) - \eta(t', x)| = \left| \int_{\mathbb{R}^n} J_{t-t'}(y) \eta(t', x - y) dy - \eta(t', x) + \int_{\mathbb{R}^n} K_{t-t'}(y) \partial_t \eta(t', x - y) dy \right|.$$

Since $\int_{\mathbb{R}^n} J_t(y) dy = \widehat{J}_t(\xi = 0) = 1$, the following estimate holds:

$$\begin{aligned} |\eta(t, x) - \eta(t', x)| &\leqslant \int_{\mathbb{R}^n} |J_{t-t'}(y)| \cdot |\eta(t', x - y) - \eta(t', x)| dy + \int_{\mathbb{R}^n} |K_{t-t'}(y)| \cdot |\partial_t \eta(t', x - y)| dy \\ &:= I_1 + I_2. \end{aligned} \tag{3.48}$$

To complete the estimate, we first consider integral I_1. We break up the integral I_1 into two parts,

$$\begin{aligned} I_1 &= \int_{|y| \leqslant |t-t'|^{1/2}} |J_{t-t'}(y)| \cdot |\eta(t', x - y) - \eta(t', x)| dy + \int_{|y| > |t-t'|^{1/2}} |J_{t-t'}(y)| \cdot |\eta(t', x - y) - \eta(t', x)| dy \\ &= I_{1,1} + I_{1,2}. \end{aligned} \tag{3.49}$$

Using the Hölder continuity in space from (3.46), and using the estimate (3.43), we get for $\rho \in [0, 1)$,

$$I_{1,1} \leqslant C_{\rho,T} \int_{|y| \leqslant |t-t'|^{1/2}} |J_{t-t'}(y)| \cdot |y|^\rho dy \leqslant C_{\rho,T} |t - t'|^{\rho/2} \int_{|y| \leqslant |t-t'|^{1/2}} |J_{t-t'}(y)| dy \leqslant C_{\rho,T} |t - t'|^{\rho/2}. \tag{3.50}$$

To estimate $I_{1,2}$, we recall that we already showed that $\eta(t, x)$ is bounded on $[0, T] \times \mathbb{R}^n$ by some constant M_T. Therefore, by the scaling relation (3.41), and by using a change of variables, we get

$$\begin{aligned} I_{1,2} \leqslant 2M_T \int_{|y| > |t-t'|^{1/2}} |J_{t-t'}(y)| dy &= 2M_T (t - t')^{-n} \int_{|y| > |t-t'|^{1/2}} \left| J\left(\frac{y}{t - t'}\right) \right| dy \\ &= 2M_T \int_{|z| > |t-t'|^{-1/2}} |J(z)| dz. \end{aligned} \tag{3.51}$$

To estimate the last integral, we recall the estimate stated in Lemma 3.2.2 and choose an N in that estimate (which depends on ρ) $N_\rho \geqslant n+1$, sufficiently large so that

$$1 + n\left(\frac{N_\rho - 1}{N_\rho}\right) \geqslant n + \rho, \quad \text{or equivalently} \quad \rho \leqslant 1 - \frac{n}{N_\rho}, \tag{3.52}$$

for arbitrary $\rho \in [0,1)$. Then, continuing from (3.51) and switching to polar coordinates, the estimate from Lemma 3.2.2, together with the inequality (3.52), imply

$$\begin{aligned} I_{1,2} &\leqslant 2M_T C_{N_\rho} \int_{|z|>|t-t'|^{-1/2}} \frac{1}{|z|^{1+n\left(\frac{N_\rho-1}{N_\rho}\right)}} dz = 2M_T C_{N_\rho} |\mathbb{S}^{n-1}| \int_{|t-t'|^{-1/2}}^{\infty} r^{-1-n\left(\frac{N_\rho-1}{N_\rho}\right)} r^{n-1} dr \\ &= 2M_T C_{N_\rho} |\mathbb{S}^{n-1}| \int_{|t-t'|^{-1/2}}^{\infty} r^{-2+\frac{n}{N_\rho}} dr = C_{\rho,T} |t-t'|^{\frac{1}{2}\left(1-\frac{n}{N_\rho}\right)} \leqslant C_{\rho,T} |t-t'|^{\rho/2}, \end{aligned} \tag{3.53}$$

for $\rho \in [0,1)$, where C_{N_ρ} denotes the constant for $N = N_\rho$ in the inequality in Lemma 3.2.2. In the last inequality, we used the fact that t', t belong to a bounded interval $[0,T]$, and in the last step, with a slight abuse of notation, we used the same notation for the constant $C_{T,\rho}$.

Finally, we estimate

$$I_2 := \int_{\mathbb{R}^n} |K_{t-t'}(y)| \cdot |\partial_t \eta(t', x-y)| dy.$$

Since $(\eta, \partial_t \eta) \in C([0,T]; H^2(\mathbb{R}^n)) \times C([0,T]; H^1(\mathbb{R}^n))$, we have that $\partial_t \eta(t,\cdot)$ is uniformly bounded in $H^1(\mathbb{R}^n)$ for $t \in [0,T]$. We note that for $n = 2$, $H^1(\mathbb{R}^n)$ embeds into $L^q(\mathbb{R}^n)$ for all $2 \leqslant q < \infty$. This is because for general dimension n, if a function $f \in H^{n/2}(\mathbb{R}^n)$, we can show that for all $1 < p \leqslant 2$, $\widehat{f} \in L^p(\mathbb{R}^n)$, which implies the result by the Hausdorff-Young inequality. Using Hölder's inequality with the conjugate exponents $2/p$ and $2/(2-p)$, one can compute:

$$\begin{aligned} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^p d\xi &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{-np/4} (1+|\xi|^2)^{np/4} |\widehat{f}(\xi)|^p d\xi \\ &\leqslant \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{np}{2(2-p)}} d\xi\right)^{\frac{2-p}{2}} \cdot ||f||^p_{H^{n/2}(\mathbb{R}^n)} = C_p ||f||^p_{H^{n/2}(\mathbb{R}^n)}, \end{aligned}$$

since $\frac{p}{2-p} > 1$ for $1 < p \leqslant 2$. Hence, for q such that $2 \leqslant q < \infty$, by the Hausdorff-Young inequality, we have that for conjugate exponents p and q,

$$||f||_{L^q(\mathbb{R}^n)} \leqslant ||\widehat{f}||_{L^p(\mathbb{R}^n)} \leqslant C_p ||f||_{H^{n/2}(\mathbb{R}^n)},$$

so that $H^1(\mathbb{R}^2)$ embeds continuously into $L^q(\mathbb{R}^2)$ for $2 \leqslant q < \infty$.

Hence, since $\partial_t \eta(t, \cdot)$ is uniformly bounded in $H^1(\mathbb{R}^n)$ on $t \in [0, T]$, we have that

$$\sup_{0 \leqslant t \leqslant T} ||\partial_t \eta(t, \cdot)||_{L^q(\mathbb{R}^2)} \leqslant C_{T,q}, \qquad \text{for } 2 \leqslant q < \infty. \tag{3.54}$$

In addition, we use the scaling property for the kernel $K_t(x)$ given by (3.29) to deduce that

$$\begin{aligned} ||K_t(x)||_{L^p(\mathbb{R}^n)} &= t^{1-n} \left(\int_{\mathbb{R}^n} \left| K\left(\frac{x}{t}\right) \right|^p dx \right)^{1/p} = t^{1-n+\frac{n}{p}} \left(\int_{\mathbb{R}^n} |K(y)|^p dy \right)^{1/p} \\ &= t^{1-\frac{n}{q}} ||K||_{L^p(\mathbb{R}^n)} = C_p t^{1-\frac{n}{q}}, \end{aligned} \tag{3.55}$$

where p and q are conjugate exponents. Therefore, by (3.54) and (3.55),

$$I_2 \leqslant \tilde{C}_{T,q} |t - t'|^{1-\frac{n}{q}}, \tag{3.56}$$

for $2 \leqslant q < \infty$, where $n = 2$. By choosing $2 \leqslant q < \infty$ appropriately, this implies the desired estimate

$$I_2 \leqslant C_{T,\rho} |t - t'|^{\rho/2}$$

for $\rho \in [0, 1)$.

By combining (3.48), (3.49), (3.50), (3.53), and (3.56), we get the desired result:

$$|\eta(t, x) - \eta(t', x)| \leqslant C_{\rho,T} |t - t'|^{\rho/2}, \tag{3.57}$$

for $\rho \in [0, 1)$ in the case $n = 2$.

The spatial estimate (3.46), and the time estimates (3.47) for $n = 1$ and (3.57) for $n = 2$, which are *uniform on* $[0, T] \times \mathbb{R}^n$, establish the continuity of $\eta(t, x)$ on $[0, T] \times \mathbb{R}^n$. □

3.3 The stochastic viscous wave equation in dimensions $n = 1, 2$

We are now in the position to study the stochastic viscous wave equation:

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = f(\eta) W(dt, dx), \qquad \text{in } \mathbb{R}^n, \tag{3.58}$$

with initial data:

$$\eta(0, x) = g(x), \qquad \partial_t \eta(0, x) = h(x). \tag{3.59}$$

For simplicity, we assume that g and h are the continuous versions of functions in $H^2(\mathbb{R}^n)$, $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, and $W(dt, dx)$ is spacetime white noise. In particular, since f is Lipschitz continuous, there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leqslant L|x - y|, \qquad \text{for all } x, y \in \mathbb{R},$$

$$|f(x)| \leqslant L(1+|x|), \qquad \text{for all } x \in \mathbb{R}. \tag{3.60}$$

We will show that the Cauchy problem (3.58), (3.59) has a *mild solution* in the sense of a stochastic process satisfying a stochastic integral equation (see Definition 3.3.1 below), which is function-valued in dimensions $n = 1, 2$. This is in contrast to the corresponding stochastic heat and wave equations,

$$\eta_t - \Delta\eta = f(\eta)W(dt, dx), \qquad \text{in } \mathbb{R}^n, \tag{3.61}$$

$$\eta_{tt} - \Delta\eta = f(\eta)W(dt, dx), \qquad \text{in } \mathbb{R}^n, \tag{3.62}$$

which have function-valued mild solutions only in dimension $n = 1$.

In the next section, we review the concept of mild solution for the stochastic heat and wave equations, and demonstrate the well-known fact that there are function-valued mild solutions only in dimension one. We then consider the concept of mild solutions for the stochastic viscous wave equation, showing heuristically why we will be able to consider such solutions in dimension two. In Section 3.3 we rigorously prove existence and uniqueness of a mild solution to (3.58), (3.59) using a Picard iteration argument to deal with the nonlinearity $f(\eta)$.

The concept of mild solution

To define the concept of mild solution for the *stochastic* viscous wave equation (3.58), we first recall the solution for the *deterministic* inhomogeneous problem (3.23). Namely, as shown earlier, the solution to the *deterministic* inhomogeneous problem (3.23) with initial data $\eta(0, x) = g(x)$ and $\partial_t\eta(0, x) = h(x)$, is given by the formula

$$\eta(t,x) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)F(s,y)dyds, \tag{3.63}$$

where $J_t(x)$ is defined by (3.39), and $K_t(x)$ by (3.27). For the general *stochastic* case (3.58) with initial data (3.59), we can formally regard the stochastic forcing $f(\eta)W(dt, dx)$ as the forcing term F in the deterministic equation, and formally require that the solution u to the stochastic viscous wave equation satisfy the *stochastic integral equation* alla (3.63):

$$\eta(t,x,\omega) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)f(\eta(s,y,\omega))W(ds,dy).$$

The result of this formal argument gives rise to the concept of a mild solution.

Definition 3.3.1. A stochastic process $\eta(t, x)$ is a *mild solution* to the stochastic viscous wave equation (3.58) with initial data (3.59) if $\eta(t, x)$ is jointly measurable and adapted to the filtration $\mathcal{F}_t$ with

$$\eta(t,x,\omega) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)f(\eta(s,y,\omega))W(ds,dy), \tag{3.64}$$

and the stochastic integral on the right hand side of (3.64) is defined.

Remark 3.3.1 (Probabilistic notation)**.** In the remainder of this manuscript, we will generally follow the probabilistic convention of not writing the explicit ω dependence of random variables and stochastic processes. In particular, while we wrote out the explicit ω dependence in the stochastic process $u(t, x, \omega)$ in (3.64), we will henceforth omit the explicit ω dependence when it is clear from context that the mathematical quantity involved is a random variable. For example, we would write

$$\eta(t,x) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)f(\eta(s,y))W(dy,ds),$$

for the full expression in (3.64).

We can define the concept of a mild solution to the stochastic heat and the stochastic wave equations (3.61) and (3.62) in the same way using the deterministic heat and wave equation representation formulas for the solutions of the corresponding inhomogeneous equation, given in (3.33) and (3.36).

As mentioned earlier, the existence of a function-valued mild solution to the stochastic heat and wave equations (3.61) and (3.62), defined this way, can be obtained only in dimension $n = 1$, as was discussed in [55] and [107]. However, we will be able to prove the existence of a function-valued mild solution to the stochastic viscous wave equation (3.58) in both dimensions $n = 1, 2$. To give an idea of why we might expect this to be true, we present the following heuristic argument.

A heuristic argument for the existence of a mild solution to (3.58)**,** (3.59) **in** $n = 2$**.** For simplicity, let $f : \mathbb{R} \to \mathbb{R}$ on the right hand-side in (3.58), (3.61), and (3.62) be identically equal to 1, so that we can just consider the case of additive noise. Therefore, we consider the equations

$$\eta_t - \Delta\eta = W(dt, dx), \qquad \text{on } \mathbb{R}^n, \tag{3.65}$$

$$\eta_{tt} - \Delta\eta = W(dt, dx), \qquad \text{on } \mathbb{R}^n, \tag{3.66}$$

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = W(dt, dx), \qquad \text{on } \mathbb{R}^n. \tag{3.67}$$

Furthermore, for simplicity, we consider *zero initial data* for the purposes of this heuristic argument.

For the stochastic heat equation with additive noise (3.65), we have an explicit formula for the solution as a stochastic integral,

$$\eta(t,x) = \int_0^t \int_{\mathbb{R}^n} K^H_{t-s}(x-y)W(ds,dy),$$

where the kernel K^H_t is defined by (3.31). For this stochastic integral to make sense, we must have

$$\int_0^t \int_{\mathbb{R}^n} |K^H_{t-s}(x-y)|^2 dyds < \infty. \tag{3.68}$$

Using the scaling relation (3.32), we can rewrite condition (3.68) in terms of the unit kernel as

$$\int_0^t \int_{\mathbb{R}^n} |K^H_{t-s}(x-y)|^2 dyds = \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n} \left| K^H \left(\frac{x-y}{(t-s)^{1/2}} \right) \right|^2 dyds$$
$$= \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n/2} |K^H(y)|^2 dyds = \left(\int_0^t (t-s)^{-n/2} ds \right) ||K^H||^2_{L^2(\mathbb{R}^n)} < \infty. \quad (3.69)$$

Because K^H is a Gaussian in all dimensions n, we have that $||K^H||^2_{L^2(\mathbb{R}^n)} < \infty$ for all n. However, the time integral $\int_0^t (t-s)^{-n/2} ds$ only converges in dimension $n = 1$. This is the reason why a function-valued mild solution to the stochastic heat equation with additive noise (3.65) exists only in dimension 1.

Let us carry out a similar analysis for the stochastic wave equation with additive noise (3.66). A mild solution, if it exists, must be given by the stochastic integral

$$\eta(t,x) = \int_0^t \int_{\mathbb{R}^n} K^W_{t-s}(x-y) W(ds,dy),$$

where K^W_t is defined by (3.34). This stochastic integral exists only if the following integrability condition is satisfied:

$$\int_0^t \int_{\mathbb{R}^n} |K^W_{t-s}(x-y)|^2 dyds < \infty. \quad (3.70)$$

By using the scaling relation (3.35), this condition can be rewritten as

$$\int_0^t \int_{\mathbb{R}^n} |K^W_{t-s}(x-y)|^2 dyds = \int_0^t \int_{\mathbb{R}^n} (t-s)^{2-2n} \left| K^W \left(\frac{x-y}{t-s} \right) \right|^2 dyds$$
$$= \int_0^t \int_{\mathbb{R}^n} (t-s)^{2-n} |K^W(y)|^2 dyds = \left(\int_0^t (t-s)^{2-n} ds \right) ||K^W||^2_{L^2(\mathbb{R}^n)} < \infty. \quad (3.71)$$

Note here that the time integral $\displaystyle\int_0^t (t-s)^{2-n}$ converges for $n = 1, 2$. However, it is easy to check from the explicit form of the fundamental solution in (3.37) that K^W is in $L^2(\mathbb{R}^n)$ only for dimension $n = 1$. This is the reason why a function-valued mild solution to the stochastic wave equation with additive noise (3.66) exists only in dimension one.

The stochastic viscous wave equation with additive noise (3.67) is exactly in between these two cases, with both factors of the unit kernel integrable in both $n = 1$ and $n = 2$. Namely, a function-valued mild solution to (3.67) would be defined by

$$\eta(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) W(ds,dy),$$

where $K_t(x)$ is the kernel given by (3.27). Using the scaling relation (3.29) involving the unit kernel (3.28), we compute, similarly as in the previous examples, that this integral exists only if the following integrability condition is satisfied:

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds = \left(\int_0^t (t-s)^{2-n} ds\right) ||K||^2_{L^2(\mathbb{R}^n)} < \infty. \tag{3.72}$$

However, by Lemma 3.2.1, $K \in L^2(\mathbb{R}^n)$ for all n. Therefore, this integrability condition is satisfied in both dimensions one and two. Thus, the stochastic viscous wave equation with additive noise (3.67) has a function-valued mild solution *both in dimensions one and two.*

Finally, we note that the nature of the parabolic damping is essential for the stochastic viscous wave equation to have a function-valued mild solution in dimension two also. In particular, the stochastic damped wave equation

$$\eta_{tt} + c\eta_t - \Delta\eta = W(dx, dt), \qquad \text{on } \mathbb{R}^n \tag{3.73}$$

with $c > 0$ and zero initial data, has a function-valued mild solution only in dimension one. To see this, we use the explicit formula for the fundamental solution from [54] in frequency space,

$$\widehat{K}_t^{DW}(\xi) = (c^2 - |\xi|^2)^{-1/2} e^{-ct} \sinh\left(t\sqrt{c^2 - |\xi|^2}\right), \tag{3.74}$$

so that the mild solution if it exists for (3.73) must be given by

$$\eta(t, x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}^{DW}(x-y) W(ds, dy).$$

Here, the superscript DW indicates that we are considering the fundamental solution for the damped wave equation. Thus, the integrability condition for the mild solution to exist is that

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^{DW}(x-y)|^2 dyds < \infty.$$

This is equivalent, by Plancherel's theorem, to

$$\int_0^t \int_{\mathbb{R}^n} |K_{t-s}^{DW}(x-y)|^2 dyds = \int_0^t \int_{\mathbb{R}^n} |K_{t-s}^{DW}(y)|^2 dyds = \int_0^t \int_{\mathbb{R}^n} |\widehat{K}_{t-s}^{DW}(\xi)|^2 d\xi ds < \infty. \tag{3.75}$$

However, the condition (3.75) holds only in dimension $n = 1$. This is because $\widehat{K}_t(\xi) \notin L^2(\mathbb{R}^n)$ for $n \geqslant 2$ for all $t > 0$. To see this, note that for $|\xi| > c$, the explicit formula (3.74) gives that

$$\widehat{K}_t^{DW}(\xi) = (|\xi|^2 - c^2)^{-1/2} e^{-ct} \sin\left(t\sqrt{|\xi|^2 - c^2}\right), \qquad \text{for } |\xi| > c.$$

We then compute that

$$||\widehat{K}_t^{DW}||^2_{L^2(\mathbb{R}^n)} \geqslant e^{-2ct} \int_{|\xi|>c} (|\xi|^2 - c^2)^{-1} \sin^2\left(t\sqrt{|\xi|^2 - c^2}\right) d\xi$$
$$= \alpha_{n-1} e^{-2ct} \int_c^\infty \frac{r^{n-1}}{r^2 - c^2} \sin^2\left(t\sqrt{r^2 - c^2}\right) dr,$$

where α_{n-1} denotes the surface area of the sphere $S^{n-1} \subset \mathbb{R}^n$. We use a change of variables,

$$\rho = \sqrt{r^2 - c^2} \iff r = \sqrt{\rho^2 + c^2} \qquad d\rho = \frac{r}{\sqrt{r^2 - c^2}} dr.$$

Then,

$$||\widehat{K}_t^{DW}||^2_{L^2(\mathbb{R}^n)} \geqslant \alpha_{n-1} e^{-2ct} \int_0^\infty \frac{\sin^2(t\rho)}{\rho} (\rho^2 + c^2)^{\frac{n}{2}-1} d\rho.$$

So for $n \geqslant 2$, we have that $\frac{n}{2} - 1 \geqslant 0$ and hence for any $t > 0$,

$$||\widehat{K}_t^{DW}||^2_{L^2(\mathbb{R}^n)} \geqslant \alpha_{n-1} e^{-2ct} c^{n-2} \int_0^\infty \frac{\sin^2(t\rho)}{\rho} d\rho = \infty \qquad \text{for } n \geqslant 2,$$

since this integral diverges. Thus, the integrability condition (3.75) does not hold in dimensions two and higher and holds only in dimension one. So the stochastic damped wave equation has a function-valued mild solution only in dimension one.

Existence and uniqueness for the stochastic viscous wave equation

While the heuristic argument above was done for a simpler case of additive white noise when $f(\eta) = 1$, we can get an existence and uniqueness result for the more general equation (3.58) with a general, Lipschitz $f(\eta)$ in dimensions one and two, by a standard Picard iteration procedure, and by estimates of the kernel. We then obtain estimates on the higher moments of the solution for later use in Section 3.4. Such a Picard iteration and higher moment bound procedure are standard in the stochastic PDE literature [55, 107, 177, 54]. More precisely, we have the following main result.

Theorem 3.3.1 (**Existence and uniqueness**)**.** Let $n = 1$ or $n = 2$, and let g and h be the continuous versions of functions in $H^2(\mathbb{R}^n)$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Then, there exists a function-valued mild solution to the equation

$$\eta_{tt} + \sqrt{-\Delta}\eta_t - \Delta\eta = f(\eta)W(dt, dx) \qquad \text{on } \mathbb{R}^n \tag{3.76}$$

with initial data $\eta(0, x) = g(x)$, $\partial_t \eta(0, x) = h(x)$, which is unique up to stochastic modification.

Proof. To establish existence, we use Picard iterations. We begin by setting the first iterate η_0 to be the deterministic function

$$\eta_0(t,x) = \int_{\mathbb{R}^n} J_t(x-y)g(y)dy + \int_{\mathbb{R}^n} K_t(x-y)h(y)dy, \tag{3.77}$$

which is the solution to the deterministic linear homogeneous viscous wave equation with initial data given by g and h. By Lemma 3.2.3, $\eta_0(t,x)$ is a bounded, continuous function on $[0,T]\times\mathbb{R}^n$.

Then, define the Picard iterates η_k for $k \geqslant 1$ inductively by

$$\eta_k(t,x) = \eta_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta_{k-1}(s,y)) W(ds,dy), \tag{3.78}$$

where η_0 captures the deterministic evolution of the initial data g and h. However, we must check that the stochastic integral on the right hand side makes sense. This is the content of the following lemma.

Lemma 3.3.1. The Picard iteration procedure (3.78) is well-defined at each step. Furthermore,

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^n} \mathbb{E}\left(|\eta_k(t,x)|^2\right) < \infty, \qquad \text{for all } k \geqslant 0, T \geqslant 0.$$

The proof of Lemma 3.3.1 is given in the Appendix.

Remark 3.3.2. Note that because random variables are only defined up to a measure zero set, the kth Picard iterate $\{\eta_k(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ is defined only up to stochastic modification. However, as we show in the proof of Lemma 3.3.1, there exists a modification of $\{\eta_k(t,x)\}_{t\in[0,T],x\in\mathbb{R}^n}$ for which the stochastic integral

$$\int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta_k(s,y)) W(dy,ds)$$

is defined, where this stochastic integral is needed to obtain the next iterate η_{k+1}. Hence, when defining η_k at each point $(t,x) \in [0,T]\times\mathbb{R}^n$ in (3.78), we choose the modification that allows the stochastic integral needed for the next step of the Picard iteration to be defined. Note that all of the arguments that follow are well suited to the fact that the Picard iterates η_k are defined only up to stochastic modification. For example, we will later consider the quantity for each k and $t \in [0,T]$,

$$\sup_{0\leqslant s\leqslant t, x\in\mathbb{R}^n} \mathbb{E}[(\eta_k - \eta_{k-1})^2(s,x)],$$

when studying convergence of the iterates, and this quantity is unchanged by stochastic modification of any of the individual iterates.

The next step is to show that $\eta_k(t,x)$ converge in an appropriate sense as $k \to \infty$, and that the limit is a unique mild solution to the stochastic viscous wave equation (3.76).

Convergence. We start by considering the difference between consecutive iterates:

$$\eta_k(t,x) - \eta_{k-1}(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)[f(\eta_{k-1}(s,y)) - f(\eta_{k-2}(s,y))]W(ds,dy), \ k \geqslant 2. \tag{3.79}$$

Using the Itö isometry (2.6) and the fact that f is Lipschitz continuous with a global Lipschitz constant L, we obtain

$$\begin{aligned}\mathbb{E}[(\eta_k - \eta_{k-1})^2(t,x)] &\leqslant L^2 \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y)|\eta_{k-1}(s,y) - \eta_{k-2}(s,y)|^2 dyds\right) \\ &= L^2 \int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y)\mathbb{E}[(\eta_{k-1} - \eta_{k-2})^2(s,y)]dyds,\end{aligned}$$

where we used Fubini's theorem in the last step.

Let

$$H_k^2(t) := \sup_{0 \leqslant s \leqslant t, x \in \mathbb{R}^n} \mathbb{E}[(\eta_k - \eta_{k-1})^2(s,x)]. \tag{3.80}$$

We want to show that for every $t \geqslant 0$, $\sum_{k=1}^{\infty} H_k(t) < \infty$, as this would imply that $\{\eta_k(t,x)\}_{k=0}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$ for each $t \geqslant 0, x \in \mathbb{R}^n$. Indeed, first notice that the following inequality holds:

$$H_k^2(t) \leqslant L^2 \int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y) H_{k-1}^2(s) dyds. \tag{3.81}$$

To further estimate the right hand side, we estimate the kernel using a calculation as in (3.72) for dimensions $n = 1$ and 2 to obtain

$$\int_{\mathbb{R}^n} K_{t-s}^2(x-y)dy = (t-s)^{2-n}||K||^2_{L^2(\mathbb{R}^n)} = c_n(t-s)^{2-n} \leqslant c_n t^{2-n}, \tag{3.82}$$

for some constant c_n depending only on n, for $s \in [0,t]$. Combining this estimate with (3.81), one obtains

$$H_k^2(t) \leqslant c_{n,t} \int_0^t H_{k-1}^2(s)ds, \ k = 2, 3, \ldots \tag{3.83}$$

for a finite constant $c_{n,t}$ that depends only on t and the dimension $n = 1, 2$. We will use this inequality inductively, for $k = 2, 3, \ldots$ to obtain the desired result. For this purpose, we must first show that $H_1(t)$ is finite. In particular, recalling (3.80) and using the result in Lemma 3.3.1, we have

$$H_1^2(t) \leqslant 2\left(\sup_{0 \leqslant s \leqslant t, x \in \mathbb{R}^n} \mathbb{E}\left(|\eta_1(s,x)|^2\right) + \sup_{0 \leqslant s \leqslant t, x \in \mathbb{R}^n} \mathbb{E}\left(|\eta_0(s,x)|^2\right)\right) = A_t < \infty,$$

where A_t is a constant depending only on t. Hence, by inductively using (3.83), we have that

$$H_k^2(t) \leqslant \frac{A_t \cdot (c_{n,t})^k t^k}{k!}. \tag{3.84}$$

Thus,

$$\sum_{k=1}^{\infty} H_k(t) \leqslant A_t^{1/2} \sum_{k=0}^{\infty} \frac{(c_{n,t})^{k/2} t^{k/2}}{(k!)^{1/2}} < \infty, \tag{3.85}$$

as this series converges. Recalling the definition of $H_k^2(t)$ in (3.80), we conclude that $\{\eta_k(t,x)\}_{k=0}^{\infty}$ for each $t \geqslant 0, x \in \mathbb{R}^n$ is a Cauchy sequence in $L^2(\Omega)$. Hence, $\eta_k(t,x)$ converges in $L^2(\Omega)$ to some $\eta(t,x)$ for each $t \geqslant 0, x \in \mathbb{R}^n$.

Existence of a mild solution. We now show that the limit $\eta(t,x)$ is a mild solution to (3.76). Indeed, after passing to the limit on both sides of (3.78) we immediately see that the left hand side of (3.78) converges to $\eta(t,x)$ in $L^2(\Omega)$. To deal with the limit on the right hand side of (3.78), we first calculate the following estimate: by the Lipschitz property of f and the Itô isometry (2.6) we have

$$\begin{aligned} &\left\| \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)[f(\eta_{k-1}(s,y)) - f(\eta(s,y))] W(ds,dy) \right\|_{L^2(\Omega)} \\ &\qquad \leqslant L^2 \int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y) \mathbb{E}\left(|\eta_{k-1}(y,s) - \eta(y,s)|^2 \right) dyds. \end{aligned} \tag{3.86}$$

To further estimate the right hand side, we recall the convergence of the series (3.85) and the definition (3.80) of $H_k^2(t)$, to conclude:

$$\mathbb{E}(|\eta_{k-1}(s,y) - \eta(s,y)|^2) \to 0 \qquad \text{as } k \to \infty, \qquad \text{uniformly for } 0 \leqslant s \leqslant t \text{ and } y \in \mathbb{R}^n. \tag{3.87}$$

Additionally, by recalling (3.82), we get:

$$\int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y) dyds = \frac{c_n}{3-n} t^{3-n}. \tag{3.88}$$

Therefore, combining this equality with (3.87), and using it in the right hand side of (3.86), we obtain that for every fixed $t \geqslant 0$, the following convergence result holds:

$$\int_0^t \int_{\mathbb{R}^n} K_{t-s}(x,y) f(\eta_{k-1}(s,y)) W(ds,dy) \to \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x,y) f(\eta(s,y)) W(ds,dy) \qquad \text{in } L^2(\Omega).$$

This shows that η satisfies (3.64).

To complete the proof that η is a mild solution, we must show according to Definition 3.3.1 that $\{\eta(t,x)\}_{t \in \mathbb{R}^+ \times \mathbb{R}^n}$ is jointly measurable and adapted to $\mathcal{F}_t$. Since each η_k is adapted to $\mathcal{F}_t$, so is the limit u. In addition, by the uniform convergence (3.87), η is continuous in $L^2(\Omega)$

on $\mathbb{R}^+ \times \mathbb{R}^n$ since each η_k has this property, by the proof of Lemma 3.3.1 in the Appendix. Hence, by Remark 2.3.2, η has a stochastic modification that is jointly measurable. This completes the proof that u is a mild solution.

Uniqueness. Uniqueness follows from Gronwall's inequality. More precisely, suppose that η and ϕ are both mild solutions with the same initial data (3.59). Then, their difference $\psi := \eta - \phi$ satisfies the following stochastic integral equation:

$$\psi(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)[f(\eta(s,y)) - f(\phi(s,y))]W(dy,ds).$$

Taking the $L^2(\Omega)$ norm of both sides, we get that

$$\mathbb{E}[\psi^2(t,x)] \leqslant L^2 \int_0^t \int_{\mathbb{R}^n} K_{t-s}^2(x-y)\mathbb{E}\left(|\eta(s,y) - \phi(s,y)|^2\right) dyds.$$

So defining

$$H(t) := \sup_{0 \leqslant s \leqslant t, x \in \mathbb{R}^n} \mathbb{E}[\psi^2(s,x)],$$

we get after using (3.82), the following inequality:

$$H(t) \leqslant L^2 \int_0^t H(s) \left(\int_{\mathbb{R}^n} K_{t-s}^2(x-y)dy \right) ds = c_{n,t} L^2 \int_0^t H(s)ds.$$

Since $H(0) = 0$, using Gronwall's inequality then implies that $H(t)$ is identically zero for all t. In particular, η is unique up to stochastic modification since the expectation $\mathbb{E}[\psi^2(t,x)] = 0$ for all $t \geqslant 0$ and $x \in \mathbb{R}^n$. This completes the uniqueness proof, and the proof of Theorem 3.3.1. □

Now that we have shown an appropriate notion of existence and uniqueness of a mild solution for (3.58) in dimensions one and two, we would like to understand more details of the solution behavior. In particular, we would like to study the *Hölder continuity of the sample paths*, defined below in Section 3.4. In order to do that, it is useful to obtain uniform boundedness of L^p moments of the unique mild solution, for $p \geqslant 2$, uniformly in space and time on a bounded time interval. The proof of this result will rely on the BDG inequality, stated in Theorem 2.3.1.

Theorem 3.3.2. Let $n = 1$ or 2, and let g and h be the continuous versions of functions in $H^2(\mathbb{R}^n)$. Let $\eta(t,x)$ be the unique function-valued mild solution to (3.58) with initial data (3.59). Then, for each $T > 0$ and $p \geqslant 2$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^n} \mathbb{E}(|\eta(t,x)|^p) < \infty. \tag{3.89}$$

Proof. Note that we have already established this result for $p = 2$ by using Lemma 3.3.1 and the uniform convergence in $L^2(\Omega)$ given by (3.87). To prove the higher moment bound (3.89), we reexamine our Picard iterates (3.78):

$$\eta_k(t,x) - \eta_{k-1}(t,x) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)[f(\eta_{k-1}(s,y)) - f(\eta_{k-2}(s,y))]W(ds,dy).$$

Using the BDG inequality stated in Theorem 2.3.1, for $k \geqslant 2$ we get

$$\mathbb{E}(|\eta_k(t,x) - \eta_{k-1}(t,x)|^p) \leqslant c_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)|f(\eta_{k-1}(s,y)) - f(\eta_{k-2}(s,y))|^2 dyds\right)^{p/2}\right].$$

Since f is Lipschitz, we can further estimate the right hand side to obtain:

$$\mathbb{E}(|\eta_k(t,x) - \eta_{k-1}(t,x)|^p) \leqslant c_p L^p \cdot \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)|\eta_{k-1}(s,y) - \eta_{k-2}(s,y)|^2 dyds\right)^{p/2}\right]. \tag{3.90}$$

We would like to move the expectation inside the integral sign on the right hand side, but we cannot do this yet because of the exponent of $p/2$. To handle this, we will separate $K^2_{t-s}(x-y)$ into

$$K^2_{t-s}(x-y) = |K_{t-s}(x-y)|^{\frac{2p-4}{p}} \cdot |K_{t-s}(x-y)|^{\frac{4}{p}}. \tag{3.91}$$

We then apply Hölder's inequality with the conjugate exponents $p/2$ and $p/(p-2)$ in (3.90) to obtain

$$\begin{aligned} &\mathbb{E}(|\eta_k(t,x) - \eta_{k-1}(t,x)|^p) \\ &\leqslant c_p L^p \left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)dyds\right)^{\frac{p}{2}-1} \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)|\eta_{k-1}(s,y) - \eta_{k-2}(s,y)|^p dyds\right) \\ &= c_p L^p \left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)dyds\right)^{\frac{p}{2}-1} \left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)\mathbb{E}\left(|\eta_{k-1}(s,y) - \eta_{k-2}(s,y)|^p\right) dyds\right). \end{aligned} \tag{3.92}$$

Therefore, defining

$$J^p_k(t) := \sup_{0 \leqslant s \leqslant t, x \in \mathbb{R}^n} \mathbb{E}(|\eta_k(s,x) - \eta_{k-1}(s,x)|^p),$$

we get that

$$J^p_k(t) \leqslant c_p L^p \left(\int_0^t \int_{\mathbb{R}^n} K^2_{t-s}(x-y)dyds\right)^{\frac{p}{2}-1} \int_0^t J^p_{k-1}(s) \left(\int_{\mathbb{R}^n} K^2_{t-s}(x-y)dy\right) ds. \tag{3.93}$$

Using (3.82) and (3.88), we obtain the following recursive inequality:

$$J^p_k(t) \leqslant c_{p,n} t^{(3-n)\left(\frac{p}{2}-1\right)} \int_0^t J^p_{k-1}(s)(t-s)^{2-n} ds \leqslant c_{p,n} t^{(3-n)\frac{p}{2}-1} \int_0^t J^p_{k-1}(s) ds. \tag{3.94}$$

Note that $J_1^p(t)$ is finite. Namely, by using the BDG inequality from Theorem 2.3.1 one obtains

$$\begin{aligned}\mathbb{E}(|\eta_1(t,x)-\eta_0(t,x)|^p) &= \mathbb{E}\left(\left|\int_0^t\int_{\mathbb{R}^n} K_{t-s}(x-y)f(\eta_0(s,y))W(ds,dy)\right|^p\right)\\ &\leqslant c_p\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^n} K_{t-s}^2(x-y)|f(\eta_0(s,y))|^2dyds\right)^{p/2}\right]\\ &\leqslant c_pL^p\cdot\left[\left(\int_0^t\int_{\mathbb{R}^n} K_{t-s}^2(x-y)(1+|\eta_0(s,y)|)^2dyds\right)^{p/2}\right],\end{aligned}$$

where we eliminated the expectation because $\eta_0(t,x)$ is deterministic. We then use the splitting from (3.91) above, and the same Hölder inequality argument as before, to obtain

$$\mathbb{E}(|\eta_1(t,x)-\eta_0(t,x)|^p) \leqslant c_pL^p\left(\int_0^t\int_{\mathbb{R}^n} K_{t-s}^2(x-y)dyds\right)^{\frac{p}{2}-1}\left(\int_0^t\int_{\mathbb{R}^n} K_{t-s}^2(x-y)(1+|\eta_0(s,y)|)^pdyds\right).$$

The right hand-side is uniformly bounded for $t\in[0,T]$ and $x\in\mathbb{R}^n$ by Lemma 3.2.3 and (3.72). So $J_1^p(t)$ is finite for each $t\geqslant 0$.

The recursive inequality (3.94) implies that for any fixed $T>0$, we have that for all $0\leqslant t\leqslant T$ and $k\geqslant 2$,

$$J_k^p(t)\leqslant C_{T,p,n}\int_0^t J_{k-1}^p(s)ds. \tag{3.95}$$

Since $J_1^p(t)$ is finite and bounded by a constant A_T for all $0\leqslant t\leqslant T$, we then apply (3.95) inductively to conclude that

$$\sum_{k=1}^{\infty} J_k(t)\leqslant A_T^{1/p}\sum_{k=0}^{\infty}\frac{(C_{T,p,n})^{k/p}t^{k/p}}{(k!)^{1/p}}<\infty, \qquad \text{for all } t\in[0,T]. \tag{3.96}$$

Since η_0 is bounded on $[0,T]\times\mathbb{R}^n$ for all $T>0$, deterministic, and continuous by Lemma 3.2.3, we have that $\eta_0\in C([0,T]\times\mathbb{R}^n;L^p(\Omega))$. Since $\sum_{k=1}^{\infty}J_k(T)<\infty$, we have that the sequence $\eta_k, k=1,2,\ldots$ is a Cauchy sequence in the complete space of bounded functions on $[0,T]\times\mathbb{R}^n$, taking values in $L^p(\Omega)$, equipped with the appropriate supremum norm:

$$||f|| = \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^n}\left(\mathbb{E}(|f(t,x)|^p)\right)^{1/p}.$$

Hence, the sequence η_k converges in this space as $k\to\infty$, and the limit must be u. Thus, $||\eta|| = \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^n}\left(\mathbb{E}(|\eta(t,x)|^p)\right)^{1/p}$ is bounded. □

3.4 Hölder continuity of sample paths

In this section we investigate additional properties of our unique mild solution by focusing on what the sample paths of the solution look like. In particular, we study Hölder continuity of sample paths. Because we are working with a stochastic process, we have to precisely define what we mean by Hölder continuity of the sample paths.

For this purpose, we recall the notion of a *modification*. If $\{X_i\}_{i\in I}$ where I is an index set is a stochastic process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then $\{\tilde{X}_i\}$ is a modification if $\mathbb{P}(\tilde{X}_i = X_i) = 1, \forall i \in I$.

We also recall that, given a stochastic process $\{X_i\}_{i\in I}$, the *finite dimensional distributions* are the distributions of the random vectors $(X_{i_1}, X_{i_2}, ..., X_{i_k})$ for all finite collections $(i_1, i_2, ..., i_k)$ of indices in I.

Note that $\{X_i\}_{i\in I}$ and a modification have the same finite dimensional distributions. Because the uniqueness result for the equation (3.58) is up to modification, we will show that (3.58) has a suitable modification such that the sample paths are Hölder continuous with a certain degree of Hölder regularity.

Theorem 3.4.1 (Hölder continuity of sample paths). Let g, h be the continuous versions of functions in $H^2(\mathbb{R}^n)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. For each $\alpha \in [0, 1)$ in the case of $n = 1$ and for each $\alpha \in [0, 1/2)$ in the case of $n = 2$, the mild solution to (3.58) has a modification that is (locally) α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ in space and time.

Remark 3.4.1. There are analogous results for the stochastic heat and wave equations (3.61) and (3.62), but only in one dimension, since existence and uniqueness hold only in one dimension. For the stochastic *heat* equation in $n = 1$ (3.61), there is a modification that is α-Hölder continuous in time and β-Hölder continuous in space for each $\alpha \in [0, 1/4)$ and each $\beta \in [0, 1/2)$. For the stochastic *wave* equation in $n = 1$ (3.62), there is a modification that is α-Hölder continuous in time and space for $\alpha \in [0, 1/2)$. The difference in the degree of Hölder regularity is due to the differences in spacetime scaling. We emphasize that our result for the stochastic viscous wave equation (3.58) considers both $n = 1$ and $n = 2$.

The proof of Theorem 3.4.1 follows from a version of the Kolmogorov continuity criterion (see e.g. Theorem 2.1 in Chapter I, Section 2 of Revuz and Yor [155]).

Theorem 3.4.2 (Kolmogorov continuity criterion). Let $\{X_i\}_{i\in[0,1]^N}$ be a real-valued stochastic process. If there exist two positive constants γ and ϵ such that

$$\mathbb{E}(|X_{i_1} - X_{i_2}|^\gamma) \leqslant C|i_1 - i_2|^{N+\epsilon},$$

then for each α such that $0 \leqslant \alpha < \dfrac{\epsilon}{\gamma}$, the stochastic process $\{X_i\}_{i\in[0,1]^N}$ has a modification that is α-Hölder continuous.

We can extend this to stochastic processes on unbounded Euclidean domains. In particular, we will reframe the Kolmogorov continuity criterion for our current case of a stochastic process indexed by $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$. This is similar to Theorem 2.5.1 in [108].

Corollary 3.4.1. Let $\{X(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ be a real-valued stochastic process. If there exist two positive constants γ and ϵ such that for each compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^n$,

$$\mathbb{E}(|X(t,x) - X(s,y)|^\gamma) \leqslant C_K |(t,x) - (s,y)|^{n+1+\epsilon} \qquad \text{for all } (t,x), (s,y) \in K,$$

where C_K can depend on K, then for each α such that $0 \leqslant \alpha < \dfrac{\epsilon}{\gamma}$, the stochastic process $\{X(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ has a modification $\{\tilde{X}(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ that is locally α-Hölder continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.

Proof. Since the Kolmogorov continuity theorem appears more often in the form listed in Theorem 3.4.2, we provide an explicit proof of Corollary 3.4.1, using an idea called a patching argument, as described on pg. 160 of [108]. The corollary follows from the Kolmogorov continuity criterion in Theorem 3.4.2 by considering compact cubes, for example

$$A_k := [0,k] \times [-k/2, k/2]^n \subset \mathbb{R}^+ \times \mathbb{R}^n,$$

that increase to all of $\mathbb{R}^+ \times \mathbb{R}^n$. We will construct the desired modification $\{\tilde{X}(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ as the limit $k \to \infty$ of α-Hölder continuous modifications $\{X_k(t,x)\}$ defined for $(t,x) \in A_k$, constructed as follows.

Fix α such that $0 \leqslant \alpha < \frac{\epsilon}{\gamma}$. By the usual Kolmogorov continuity criterion given in Theorem 3.4.2, we can construct a modification $\{X_k(t,x)\}_{(t,x)\in A_k}$ of $\{X(t,x)\}_{(t,x)\in A_k}$ that is α-Hölder continuous on A_k. The modifications $\{X_k(t,x)\}_{(t,x)\in A_k}$ in particular are continuous.

We claim that any two of these modifications $X_k(t,x)$ and $X_l(t,x)$ must agree with probability one on their overlap because they are continuous modifications. Otherwise there exists a ball with rational radius and center with rational coordinates on which the two modifications have disjoint range with positive probability. More precisely, consider $k \leqslant l$ so that $A_k \subset A_l$ is the overlap. We claim that

$$\mathbb{P}(X_k(t,x) = X_l(t,x) \text{ for all } (t,x) \in A_k) = 1.$$

We argue by contradiction. Suppose that $\mathbb{P}(X_k(t_0,x_0) \neq X_l(t_0,x_0) \text{ for some } (t_0,x_0) \in A_k) > 0$. Since $X_k(t,x)$ and $X_l(t,x)$ are continuous on A_k, for every outcome $\omega \in \Omega$ for which $X_k(t_0,x_0) \neq X_l(t_0,x_0)$ for some $(t_0,x_0) \in A_k$, we can find an open ball $B_r(q)$ with rational radius r centered at a point $q = (t,x) \in \mathbb{Q}^+ \times \mathbb{Q}^n \cap A_k \subset \mathbb{R}^+ \times \mathbb{R}^n$, such that X_k and X_l have *"disjoint" range* on the ball $B_r(q) \cap A_k$ in the sense that there exist two closed intervals $K_1 \subset \mathbb{R}$ and $K_2 \subset \mathbb{R}$ such that

$$X_k(B_r(q) \cap A_k) \subset K_1, \qquad X_l(B_r(q) \cap A_k) \subset K_2, \qquad K_1 \cap K_2 = \varnothing.$$

Hence,

$$\mathbb{P}\Big(\bigcup_{r\in\mathbb{Q}, q\in\mathbb{Q}^+\times\mathbb{Q}^n\cap A_k} \{X_k \text{ and } X_l \text{ have "disjoint" range on } B_r(q) \cap A_k\}\Big)$$
$$\geqslant \mathbb{P}(X_k(t_0,x_0) \neq X_l(t_0,x_0) \text{ for some } (t_0,x_0) \in A_k) > 0.$$

Therefore, by the countability of the index set, there exist $r_0 \in \mathbb{Q}^+$ and $q_0 \in \mathbb{Q}^+ \times \mathbb{Q}^n \cap A_k$ such that

$$\mathbb{P}(X_k \text{ and } X_l \text{ have “disjoint” range on } B_{r_0}(q_0) \cap A_k) > 0.$$

But this implies that $\mathbb{P}(X_k(q_0) \neq X_l(q_0)) > 0$, which contradicts that $\{X_k(t,x)\}_{(t,x)\in A_k}$ and $\{X_l(t,x)\}_{(t,x)\in A_l}$ are modifications of the same stochastic process on A_k and A_l respectively, since $q_0 \in A_k \subset A_l$. Therefore, $X_k = X_l$ on A_k almost surely.

This implies that, with probability one (up to a null set), any two modifications from this collection of modifications $\{X_k(t,x)\}_{(t,x)\in A_k}$ on increasing cubes must agree.

To define the desired modification $\tilde{X}(t,x)$ we now focus on the null sets $E_{k,l}$ for $k < l$ on which the modifications $\{X_k(t,x)\}_{(t,x)\in A_k}$ and $\{X_l(t,x)\}_{(t,x)\in A_l}$ do not agree on A_k. Define

$$E = \bigcup_{k<l \text{ and } k,l\in\mathbb{Z}^+} E_{k,l}$$

and note that $\mathbb{P}(E) = 0$. Then, the desired modification $\{\tilde{X}(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ is

$$\tilde{X}(t,x,\omega) = \lim_{k\to\infty} X_k(t,x,\omega) \qquad \text{for } \omega \in E^c,$$
$$\tilde{X}(t,x,\omega) = 0, \qquad \text{for } \omega \in E.$$

This limit exists since the sequence $X_k(t,x,\omega)$ for $\omega \in E^c$ is eventually constant, because the modifications $\{X_k(t,x)\}_{(t,x)\in A_k}$ all agree pairwise on their common domains for $\omega \in E^c$. It is easy to check that $\{\tilde{X}(t,x)\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}$ is a modification that is α-Hölder continuous, by using the properties that each of the $\{X_k(t,x)\}_{(t,x)\in A_k}$ are modification on A_k that are α-Hölder continuous. This completes the proof of the corollary. □

Proof of Theorem 3.4.1

We will prove the theorem for $n = 1$ and $n = 2$. Though the specific estimates will be slightly different for each dimension, the general computations are the same for both and hence we prove the results for $n = 1$ and $n = 2$ simultaneously.

By Corollary 3.4.1, it follows that to prove Theorem 3.4.1, it suffices to show that for all $T > 0$, for all $\delta \in (0,1)$, and for all $p \geqslant 2$, there exists a constant $C_{T,p,\delta}$ depending on T, p, and δ such that:

1. The following two estimates hold if $n = 1$:

$$\mathbb{E}(|\eta(t,x) - \eta(t',x)|^p) \leqslant C_{T,p,\delta}|t-t'|^{\frac{(1+\delta)p}{2}}, \text{ for all } t,t' \in [0,T], \text{ and } x \in \mathbb{R}, \tag{3.97}$$

$$\mathbb{E}(|\eta(t,x) - \eta(t,x')|^p) \leqslant C_{T,p,\delta}|x-x'|^{\frac{(1+\delta)p}{2}}, \text{ for all } t \in [0,T], \text{ and } x,x' \in \mathbb{R}. \tag{3.98}$$

2. The following two estimates hold if $n = 2$:

$$\mathbb{E}(|\eta(t,x) - \eta(t',x)|^p) \leqslant C_{T,p,\delta}|t-t'|^{\frac{\delta p}{2}}, \text{ for all } t,t' \in [0,T], \text{ and } x \in \mathbb{R}^2, \tag{3.99}$$

$$\mathbb{E}(|\eta(t,x) - \eta(t,x')|^p) \leqslant C_{T,p,\delta}|x-x'|^{\frac{\delta p}{2}}, \text{ for all } t \in [0,T], \text{ and } x,x' \in \mathbb{R}^2. \tag{3.100}$$

Estimate for the time increments. To prove estimates (3.97) and (3.99), we consider for $p \geqslant 2$,

$$\mathbb{E}(|\eta(t,x) - \eta(t',x)|^p), \qquad \text{for } t, t' \in [0,T],$$

for $n = 1, 2$, where $T > 0$ is fixed but arbitrary. Recall from the definition of a mild solution (3.64) that

$$\eta(t,x) = \eta_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta(s,y)) W(ds,dy),$$

where $\eta_0(t,x)$ is the deterministic function solving the homogeneous deterministic viscous wave equation with initial data g, h. We assume that $0 \leqslant t' < t \leqslant T$ and express the time increment as

$$\eta(t,x) - \eta(t',x) = \eta_0(t,x) - \eta_0(t',x)$$
$$+ \int_0^{t'} \int_{\mathbb{R}^n} [K_{t-s}(x-y) - K_{t'-s}(x-y)] f(\eta(s,y)) W(ds,dy) + \int_{t'}^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta(s,y)) W(ds,dy).$$

Next, we use the BDG inequality from Theorem 2.3.1, along with $(a+b+c)^p \leqslant c_p(|a|^p + |b|^p + |c|^p)$ for $a, b, c \geqslant 0$, to obtain

$$\begin{aligned} &\mathbb{E}\left(|\eta(t,x) - \eta(t',x)|^p\right) \\ &\leqslant c_p \Bigg[|\eta_0(t,x) - \eta_0(t',x)|^p + \mathbb{E}\left(\int_0^{t'} \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t'-s}(x-y)|^2 |f(\eta(s,y))|^2 dyds \right)^{p/2} \\ &+ \mathbb{E}\left(\int_{t'}^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 |f(\eta(s,y))|^2 dyds \right)^{p/2} \Bigg] := c_p(I_1 + I_2 + I_3). \end{aligned} \tag{3.101}$$

By Lemma 3.2.3, there exists C_T such that

$$I_1 = |\eta_0(t,x) - \eta_0(t',x)|^p \leqslant C_T |t-t'|^p, \qquad \text{in the case of } n = 1, \tag{3.102}$$

and for every $\delta \in (0,1)$, there exists a constant $C_{T,\delta}$ depending only on $T > 0$ and δ such that

$$I_1 = |\eta_0(t,x) - \eta_0(t',x)|^p \leqslant C_{T,\delta} |t-t'|^{\frac{\delta p}{2}}, \qquad \text{in the case of } n = 2. \tag{3.103}$$

For the integral in I_2 defined in (3.101), we use the same idea as in (3.91) and separate the term involving the kernel into two factors by using Hölder's inequality with $p/2$ and $p/(p-2)$ to obtain

$$\begin{aligned} I_2 \leqslant &\left(\int_0^{t'} \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t'-s}(x-y)|^2 dyds \right)^{\frac{p}{2}-1} \\ &\cdot \mathbb{E}\left(\int_0^{t'} \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t'-s}(x-y)|^2 |f(\eta(s,y))|^p dyds \right). \end{aligned}$$

In the second factor, we can move the expectation into the integrand, and use the Lipschitz property of f to obtain that for all $s \in [0, t'], y \in \mathbb{R}^n$, the following estimate holds:

$$\mathbb{E}\left(|f(\eta(s, y))|^p\right) \leqslant 2^p L^p \mathbb{E}(1 + |\eta(s, y)|^p) = C_{T,p} < \infty, \tag{3.104}$$

where the last inequality follows from the boundedness of pth moments in (3.89). Therefore,

$$I_2 \leqslant C_{T,p} \left(\int_0^{t'} \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t'-s}(x-y)|^2 dy ds \right)^{\frac{p}{2}}.$$

Using Plancherel's theorem and absorbing constants into $C_{T,p}$,

$$I_2 \leqslant C_{T,p} \left(\int_0^{t'} \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|(t-s)}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right)}{|\xi|} - e^{-\frac{|\xi|(t'-s)}{2}} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t'-s)\right)}{|\xi|} \right|^2 d\xi ds \right)^{\frac{p}{2}}. \tag{3.105}$$

Continuing to absorb constants into $C_{T,p}$ as necessary, we separate this into

$$I_2 \leqslant C_{T,p}(J_1 + J_2)^{\frac{p}{2}}, \tag{3.106}$$

where

$$J_1 = \int_0^{t'} \int_{\mathbb{R}^n} \frac{\sin^2\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right)}{|\xi|^2} \left| e^{-\frac{|\xi|(t-s)}{2}} - e^{-\frac{|\xi|(t'-s)}{2}} \right|^2 d\xi ds, \tag{3.107}$$

$$J_2 = \int_0^{t'} \int_{\mathbb{R}^n} e^{-|\xi|(t'-s)} \left(\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right)}{|\xi|} - \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t'-s)\right)}{|\xi|} \right)^2 d\xi ds. \tag{3.108}$$

To estimate J_1, we first simplify J_1 to get

$$J_1 = \int_0^{t'} \int_{\mathbb{R}^n} e^{-|\xi|(t'-s)} \frac{\sin^2\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right)}{|\xi|^2} \left(1 - e^{-\frac{|\xi|(t-t')}{2}}\right)^2 d\xi ds.$$

Next, we use the fact that there exists a uniform constant C such that

$$0 \leqslant 1 - e^{-r} \leqslant \min(1, r), \qquad \text{for all } r \geqslant 0. \tag{3.109}$$

In addition, there exists a uniform constant depending only on $\delta \in (0, 1)$ such that

$$r^\delta e^{-r} \leqslant C_\delta, \qquad \text{for all } r > 0, \delta \in (0, 1). \tag{3.110}$$

Thus, for each $\delta \in (0,1)$ we have:

$$\begin{aligned}
J_1 &\leqslant C_\delta \int_0^{t'} \frac{1}{(t'-s)^\delta} \int_{\mathbb{R}^n} \frac{1}{|\xi|^\delta} \cdot \frac{1}{|\xi|^2} \min\left(1, \frac{1}{4}|\xi|^2(t-t')^2\right) d\xi ds \\
&\leqslant C_\delta \int_0^{t'} \frac{1}{(t'-s)^\delta} \int_{\mathbb{R}^n} \frac{1}{|\xi|^{2+\delta}} \min(1, |\xi|^2(t-t')^2) d\xi ds \\
&= C_\delta \left(\int_0^{t'} \frac{1}{(t'-s)^\delta} \left(\int_{|\xi|\leqslant(t-t')^{-1}} \frac{1}{|\xi|^\delta}(t-t')^2 d\xi + \int_{|\xi|\geqslant(t-t')^{-1}} \frac{1}{|\xi|^{2+\delta}} d\xi \right) ds \right), \qquad (3.111)
\end{aligned}$$

where $C_{T,\delta}$ is a constant depending on $\delta \in (0,1)$, and on the fixed but arbitrary $T > 0$. Note that we have restricted δ to the range of $\delta \in (0,1)$ so that the appropriate integrals converge in both spatial dimension $n = 1$ and $n = 2$. Computing the integrals in (3.111) gives

$$J_1 \leqslant C_\delta \left(\int_0^{t'} \frac{1}{(t'-s)^\delta}(t-t')^{1+\delta} ds \right) = C_{T,\delta}|t-t'|^{1+\delta}, \qquad \text{for } n = 1, \qquad (3.112)$$

$$J_1 \leqslant C_\delta \left(\int_0^{t'} \frac{1}{(t'-s)^\delta}(t-t')^{\delta} ds \right) = C_{T,\delta}|t-t'|^{\delta}, \qquad \text{for } n = 2. \qquad (3.113)$$

We now consider J_2 as defined in (3.108). By the mean value theorem,

$$\left| \sin\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right) - \sin\left(\frac{\sqrt{3}}{2}|\xi|(t'-s)\right) \right| \leqslant \min\left(2, \frac{\sqrt{3}}{2}|\xi|(t-t')\right) \leqslant \min(2, |\xi|(t-t')). \qquad (3.114)$$

Combining the estimates (3.110) and (3.114) gives for arbitrary $\delta \in (0,1)$,

$$J_2 \leqslant C_\delta \int_0^{t'} \frac{1}{(t'-s)^\delta} \int_{\mathbb{R}^n} \frac{1}{|\xi|^\delta} \cdot \frac{1}{|\xi|^2} \min\left(1, |\xi|^2(t-t')^2\right) d\xi ds.$$

The rest proceeds exactly as for the computation for J_1, see (3.111), and thus we obtain

$$J_2 \leqslant C_{T,\delta}|t-t'|^{1+\delta} \qquad \text{for } n = 1, \qquad J_2 \leqslant C_{T,\delta}|t-t'|^{\delta} \qquad \text{for } n = 2,$$

for $C_{\delta,T}$ depending only on $\delta \in (0,1)$ and T. Substituting into (3.106), we have for arbitrary $\delta \in (0,1)$,

$$I_2 \leqslant C_{T,p,\delta}|t-t'|^{\frac{(1+\delta)p}{2}} \qquad \text{for } n = 1, \qquad I_2 \leqslant C_{T,p,\delta}|t-t'|^{\frac{\delta p}{2}} \qquad \text{for } n = 2. \quad (3.115)$$

For I_3 as defined in (3.101), we use the idea from (3.91), combined with the Lipschitz property of f, the boundedness of pth moments of $u(t,x)$ on finite time intervals, and a calculation similar to (3.72) to obtain for $n = 1, 2$,

$$\begin{aligned}
I_3 &\leqslant c_p \left(\int_{t'}^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds \right)^{\frac{p}{2}-1} \left(\int_{t'}^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 \mathbb{E}(|f(\eta(s,y))|^p) dyds \right) \\
&\leqslant c_{T,p} \left(\int_{t'}^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 dyds \right)^{\frac{p}{2}} = c_{T,p} \left(\int_{t'}^t (t-s)^{2-n} ds \right)^{p/2} ||K||^p_{L^2(\mathbb{R}^n)} = c_{T,p}|t-t'|^{\frac{p}{2}(3-n)}. \qquad (3.116)
\end{aligned}$$

The estimates (3.102), (3.103), (3.115), (3.116) for I_1, I_2, and I_3 and (3.101) establish the desired time increment estimates (3.97) for $n = 1$ and (3.99) for $n = 2$.

Estimate for the spatial increments. We examine the spatial regularity of the stochastic solution $u(t, x)$ by establishing (3.98) and (3.100). For $0 \leqslant t \leqslant T$ and $x, x' \in \mathbb{R}^n$, we have that

$$\eta(t,x) - \eta(t,x') = (\eta_0(t,x) - \eta_0(t,x')) + \int_0^t \int_{\mathbb{R}^n} (K_{t-s}(x-y) - K_{t-s}(x'-y)) f(\eta(s,y)) W(ds,dy),$$

and hence, for $p \geqslant 2$,

$$\begin{aligned} &\mathbb{E}(|\eta(t,x) - \eta(t,x')|^p) \\ &= c_p \left(|\eta_0(t,x) - \eta_0(t,x')|^p + \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}^n} (K_{t-s}(x-y) - K_{t-s}(x'-y)) f(\eta(s,y)) W(ds,dy) \right|^p \right) \right) \\ &:= c_p(I_4 + I_5). \end{aligned} \tag{3.117}$$

We bound I_4 by using Lemma 3.2.3 to obtain

$$I_4 = |\eta_0(t,x) - \eta_0(t,x')|^p \leqslant C_T |x - x'|^p \qquad \text{for } n = 1, \tag{3.118}$$

and for arbitrary $\delta \in (0,1)$,

$$I_4 = |\eta_0(t,x) - \eta_0(t,x')|^p \leqslant C_{T,\delta} |x - x'|^{\frac{\delta p}{2}}, \qquad \text{for } n = 2. \tag{3.119}$$

To estimate I_5, we use the BDG inequality stated in Theorem 2.3.1 to obtain

$$I_5 \leqslant \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t-s}(x'-y)|^2 |f(\eta(s,y))|^2 dyds \right)^{\frac{p}{2}}.$$

By using the same computation as in (3.91),

$$\begin{aligned} I_5 \leqslant &\left(\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t-s}(x'-y)|^2 dyds \right)^{\frac{p}{2}-1} \\ &\cdot \left(\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t-s}(x'-y)|^2 \mathbb{E}\left(|f(\eta(s,y))|^p\right) dyds \right). \end{aligned}$$

Using the higher moment bound on u as in (3.104), we have

$$\begin{aligned} I_5 &\leqslant C_{T,p} \left(\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y) - K_{t-s}(x'-y)|^2 dyds \right)^{\frac{p}{2}} \\ &= C_{T,p} \left(\int_0^t \int_{\mathbb{R}^n} |K_{t-s}(y) - K_{t-s}(y + x' - x)|^2 dyds \right)^{\frac{p}{2}}. \end{aligned}$$

Absorbing constants into $C_{T,p}$ as necessary and using Plancherel's formula gives that

$$I_5 \leqslant C_{T,p} \left(\int_0^t \int_{\mathbb{R}^n} \left| e^{-\frac{|\xi|}{2}(t-s)} \frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|(t-s)\right)}{|\xi|} (1 - e^{i\xi\cdot(x'-x)}) \right|^2 d\xi ds \right)^{\frac{p}{2}}$$
$$= C_{T,p} \left(\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|(t-s)} \frac{1}{|\xi|^2} \left[1 - \cos(\xi \cdot (x'-x))\right] d\xi ds \right)^{\frac{p}{2}}.$$

We use the inequality $1 - \cos(\xi \cdot (x'-x)) \leqslant \min(2, |\xi|^2|x'-x|^2)$ and (3.110) to obtain for $\delta \in (0,1)$,

$$I_5 \leqslant C_{T,p} \left(\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|(t-s)} \frac{1}{|\xi|^2} \left[1 - \cos(\xi \cdot (x'-x))\right] d\xi ds \right)^{\frac{p}{2}}$$
$$\leqslant C_{T,p,\delta} \left(\int_0^t \frac{1}{(t-s)^\delta} \int_{\mathbb{R}^n} \frac{1}{|\xi|^{2+\delta}} \min(2, |\xi|^2|x'-x|^2) d\xi ds \right)^{\frac{p}{2}}$$
$$\leqslant C_{T,p,\delta} \left(\int_0^t \frac{1}{(t-s)^\delta} \left(\int_{|\xi| \leqslant |x-x'|^{-1}} \frac{1}{|\xi|^\delta} |x'-x|^2 d\xi + \int_{|\xi| \geqslant |x-x'|^{-1}} \frac{2}{|\xi|^{2+\delta}} d\xi \right) ds \right)^{\frac{p}{2}}.$$

These integrals converge for $n = 1, 2$ since $\delta \in (0,1)$. We can compute these integrals to obtain

$$I_5 \leqslant C_{T,p,\delta} \left(\int_0^t \frac{1}{(t-s)^\delta} |x-x'|^{1+\delta} ds \right)^{\frac{p}{2}} = C_{T,p,\delta} |x-x'|^{\frac{(1+\delta)p}{2}}, \qquad \text{for } n = 1, \tag{3.120}$$

$$I_5 \leqslant C_{T,p,\delta} \left(\int_0^t \frac{1}{(t-s)^\delta} |x-x'|^{\delta} ds \right)^{\frac{p}{2}} = C_{T,p,\delta} |x-x'|^{\frac{\delta p}{2}}, \qquad \text{for } n = 2. \tag{3.121}$$

The estimates (3.117), (3.118), (3.119), (3.120), and (3.121) establish the required spatial increment estimates (3.98) for $n = 1$ and (3.100) for $n = 2$. This completes the proof of Theorem 3.4.1.

3.5 Conclusion

We have shown that a Cauchy problem for the stochastically perturbed viscous wave equation (3.1) has a unique (up to a modification) mild solution in both $n = 1$ and $n = 2$, and that the stochastic mild solution has a modification which is α-Hölder continuous, where α-Hölder continuity is up to $\alpha = 1$ in $n = 1$, and up to $\alpha = 1/2$ in $n = 2$. This result is significant, especially for $n = 2$, since it indicates that stochastically perturbed fluid-structure interaction problems involving viscous, incompressible fluids at low-to-medium Reynolds numbers, will have Hölder continuous solutions for almost all realizations of sample paths, even in the case when the stochasticity in the forcing (or data) is represented by the very rough spacetime white noise. We remark that this would not be the case if the structure itself, modeled by the stochastically perturbed wave equation in $n = 2$, were considered without the fluid, as it is well known that for the spacetime white noise perturbed wave and heat equations, stochastic mild solutions do not exist in dimensions $n = 2$ and higher. It is the coupled problem that provides the right scaling and sufficient dissipation that damps high-order frequencies exponentially fast in time, thereby allowing a unique stochastic, Hölder continuous mild solution to exist for almost all realizations.

3.6 Appendix

Proof of Lemma 3.1. To prove this lemma we use induction, presented in several steps below.

Step 1. For the inductive step, suppose that the following properties of u_{k-1} are satisfied:

1. η_{k-1} is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$,
2. η_{k-1} is jointly measurable,
3. η_{k-1} satisfies for every $T > 0$,

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^n}\mathbb{E}\left(|\eta_{k-1}(s,y))|^2\right) := C_{k-1,T} < \infty. \tag{3.122}$$

4. η_{k-1} is continuous as a map from $(t, x) \in [0, T] \times \mathbb{R}^n$ to $L^2(\Omega)$, for arbitrary $T > 0$.

Certainly, the base case holds. This is because η_0 is deterministic, hence it immediately satisfies the adaptedness and joint measurability conditions. For (3.122), we can get rid of the expectation, since η_0 is deterministic. Then, (3.122) follows from the fact that η_0 is bounded by Lemma 3.2.3. The $L^2(\Omega)$ continuity, since η_0 is deterministic, follows from the continuity statement in Lemma 3.2.3.

Step 2. We want to show that with this inductive assumption, the stochastic integral in (3.78) is well-defined. So given arbitrary $t > 0$, we must show that the integrand $K_{t-s}(x - y)f(\eta_{k-1}(y, s))$ for $s \in [0, t), y \in \mathbb{R}^n$, satisfies the conditions in Proposition 2.3.6. Recall from (3.78) that (t, x) is a fixed but arbitrary point in $\mathbb{R}^+ \times \mathbb{R}^n$ and (s, y) here indicates the variables that are integrated against the spacetime white noise. Since the kernel $K_{t-s}(x-y)$ is singular at $s = t$ and $x = y$, we first show that the conditions in Proposition 2.3.6 hold for $s \in [0, t)$ and then also in the limit $s \to t$. We start by showing that the conditions in Proposition 2.3.6 hold for $s \in [0, t)$:

- Since $\eta_{k-1}(s, y)$ for $s \in [0, t), y \in \mathbb{R}^n$ is adapted, so is $K_{t-s}(x, y)f(\eta_{k-1}(s, y))$ since f is continuous.

- For each $s \in [0, t)$, $y \in \mathbb{R}^n$, we have

$$\mathbb{E}\left(|K_{t-s}(x-y)f(u_{k-1}(s,y))|^2\right) \leqslant 2L^2C_{t-s}\left(1+\mathbb{E}\left(|\eta_{k-1}(s,y)|^2\right)\right) < \infty,$$

by the inductive assumption (3.122). Here, L is the Lipschitz constant for f, and we used the fact that $K_{t-s}(\cdot)$ is bounded by a finite constant C_{t-s} depending on the parameter $t-s$.

- To show that $K_{t-s}(x-y)f(\eta_{k-1}(y,s))$ is $L^2(\Omega)$-continuous for $s \in [0,t)$ and $y \in \mathbb{R}^n$, we fix $s_0 \in [0,t)$ and $y_0 \in \mathbb{R}^n$ and compute

$$\begin{aligned}&\mathbb{E}(|K_{t-s_1}(x-y_1)f(\eta_{k-1}(s_1,y_1)) - K_{t-s_0}(x-y_0)f(\eta_{k-1}(s_0,y_0))|^2)\\ &\quad\leqslant 2\mathbb{E}\left(|K_{t-s_1}(x-y_1)f(\eta_{k-1}(s_1,y_1)) - K_{t-s_1}(x-y_1)f(\eta_{k-1}(s_0,y_0))|^2\right)\\ &\quad+ 2\mathbb{E}\left(|K_{t-s_1}(x-y_1)f(\eta_{k-1}(s_0,y_0)) - K_{t-s_0}(x-y_0)f(\eta_{k-1}(s_0,y_0))|^2\right).\end{aligned}$$

Using the Lipschitz condition for f in the first term on the right hand side, and the fact that f is linearly bounded by $|f(x)| \leqslant L(1+|x|)$ in the second term on the right hand side,

$$\begin{aligned}&\mathbb{E}(|K_{t-s_1}(x-y_1)f(\eta_{k-1}(s_1,y_1)) - K_{t-s_0}(x-y_0)f(\eta_{k-1}(s_0,y_0))|^2)\\ &\quad\leqslant 2L^2|K_{t-s_1}(x-y_1)|^2\mathbb{E}\left(|u_{k-1}(s_1,y_1) - u_{k-1}(s_0,y_0)|^2\right)\\ &\quad+ 4|K_{t-s_1}(x-y_1) - K_{t-s_0}(x-y_0)|^2\mathbb{E}\left(L^2(1+|u_{k-1}(s_0,y_0)|^2)\right).\end{aligned}$$

By using the inductive assumption (3.122), in the second term above we can bound the expectation of $|\eta_{k-1}(s_0,y_0)|^2$ to obtain the following estimate:

$$\begin{aligned}&\mathbb{E}(|K_{t-s_1}(x-y_1)f(\eta_{k-1}(y_1,s_1)) - K_{t-s_0}(x-y_0)f(\eta_{k-1}(s_0,y_0))|^2)\\ &\quad\leqslant 2L^2|K_{t-s_1}(x-y_1)|^2\mathbb{E}\left(|\eta_{k-1}(s_1,y_1) - \eta_{k-1}(s_0,y_0)|^2\right)\\ &\quad+ \tilde{C}_{k-1,t}|K_{t-s_1}(x-y_1) - K_{t-s_0}(x-y_0)|^2,\end{aligned} \tag{3.123}$$

for some constant $\tilde{C}_{k-1,t}$ depending only on $k-1$ and t. To show continuity, we want to make the right hand-side of (3.123) arbitrarily small whenever $|(s_1,y_1)-(s_0,y_0)|$ is small. Indeed, in the first term on the right hand-side, $K_{t-s}(x-y)$ is locally bounded for $s \in [0,t)$ and $y \in \mathbb{R}^n$, and u_{k-1} is $L^2(\Omega)$ continuous by the inductive assumption, so the first term on the right hand-side of (3.123) can be made arbitrarily small for $|(s_1,y_1)-(s_0,y_0)| < \delta$, for δ sufficiently small. This is also true for the second term on the right hand side because $K_{t-s}(x-y)$ is continuous for $s \in [0,t)$ and $y \in \mathbb{R}^n$. This establishes the claim.

- To check the square integrability condition, we compute

$$\begin{aligned}&\mathbb{E}\int_0^t\int_{\mathbb{R}^n}|K_{t-s}(x-y)|^2|f(\eta_{k-1}(s,y))|^2dyds\\ &\quad\leqslant 2L^2\int_0^t\int_{\mathbb{R}^n}|K_{t-s}(x-y)|^2\left(1+\mathbb{E}\left(|\eta_{k-1}(s,y)|^2\right)\right)dyds\\ &\quad= 2L^2\left(1+\sup_{s'\in[0,t],y'\in\mathbb{R}^n}\mathbb{E}\left(|\eta_{k-1}(s',y')|\right)^2\right)\left(\int_0^t(t-s)^{2-n}ds\right)||K||^2_{L^2(\mathbb{R}^n)} < \infty,\end{aligned} \tag{3.124}$$

where we used the identity in (3.72), the fact that $n = 1$ or 2, Lemma 3.2.1, and the inductive assumption (3.122).

To show that the stochastic integral in (3.78) is still well-defined for $s \in [0, t]$, we claim that this stochastic integral can be defined as the $L^2(\Omega)$ limit of stochastic integrals whose integrands are explicitly in the admissible class $\mathcal{P}_W$ of integrands. To see this, choose an increasing sequence $t_i, i = 1, 2, \ldots$ of positive real numbers such that $t_i \to t$ as $i \to \infty$. Note that

$$\int_0^{t_i} \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta_{k-1}(s,y)) W(dy, ds) \tag{3.125}$$

is a well-defined stochastic integral by the properties verified above, by Proposition 2.3.6. By (3.124),

$$\mathbb{E} \int_{t_i}^{t} \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 |f(\eta_{k-1}(s,y))|^2 dyds \to 0, \text{ as } t_i \to t.$$

Hence, since $\mathcal{P}_W$ is a closed Banach space, the integrand in (3.78) is in $\mathcal{P}_W$, and can be defined rigorously as the limit of the Cauchy sequence (3.125) in $L^2(\Omega)$, by the Itô isometry and the finiteness of the quantity in (3.124).

Step 3. It remains to show that $u_k(t,x)$ satisfies the conditions in the inductive assumption in Step 1. Indeed, $\eta_k(t,x)$ is adapted by the construction of the stochastic integral. Joint measurability (up to modification) will follow from the later verification of continuity in $L^2(\Omega)$, as noted in Remark 2.3.2. Thus, properties 1 and 2 in Step 1 are verified.

To verify property 3 in Step 1, we check that for each $T > 0$,

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^n} \mathbb{E}\left(|\eta_k(t,x)|^2\right) = C_{k,T} < \infty. \tag{3.126}$$

This follows by direct calculation. Fix arbitrary $T > 0$ and consider $t \in [0, T]$, $x \in \mathbb{R}^n$. By (2.6) and (3.78), we get

$$\begin{aligned} \mathbb{E}\left(|\eta_k(t,x)|^2\right) &= 2\mathbb{E}(|\eta_0(t,x)|^2) + 2\mathbb{E} \int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 |f(\eta_{k-1}(s,y))|^2 dyds \\ &= 2|\eta_0(t,x)|^2 + 2\mathbb{E} \int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x-y)|^2 |f(\eta_{k-1}(s,y))|^2 dyds. \end{aligned} \tag{3.127}$$

Note that by Lemma 3.2.3, $\eta_0(t,x)$ is bounded on $t \in [0,T]$, $x \in \mathbb{R}^n$. So we consider the remaining term. Using the calculation in (3.124) and the bound $|f(x)| \leqslant L(1 + |x|)$ for some L by the Lipschitz condition,

$$\begin{aligned} &\mathbb{E} \int_0^t \int_{\mathbb{R}^n} |K_{t-s}(x,y)|^2 |f(\eta_{k-1}(s,y))|^2 dyds \\ &\qquad \leqslant 2L^2 \left(1 + \sup_{s' \in [0,t], y' \in \mathbb{R}^n} \mathbb{E}\left(|\eta_{k-1}(s',y')|\right)^2\right) \left(\int_0^t (t-s)^{2-n} ds\right) ||K||^2_{L^2(\mathbb{R}^n)} \leqslant \tilde{C}_{k,T}, \end{aligned}$$

where $\tilde{C}_{k,T}$ is the finite constant, independent of $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$\tilde{C}_{k,T} := 2L^2 \left(1 + \sup_{s' \in [0,T], y' \in \mathbb{R}^n} \mathbb{E}\left(|\eta_{k-1}(s', y')|\right)^2\right) \left(\int_0^T (T-s)^{2-n} ds\right) ||K||^2_{L^2(\mathbb{R}^n)},$$

which is finite by the inductive assumption (3.122), Lemma 3.2.1, and the fact that $n = 1$ or 2. This verifies (3.126).

Finally, we show that property 4 in Step 1 holds, namely that the mapping $(t,x) \mapsto \eta_k(t,x)$ taking values in $L^2(\Omega)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. We decompose $u_k(t,x)$ in (3.78) as

$$\eta_k(t,x) = \eta_0(t,x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(\eta_{k-1}(s,y)) W(ds,dy) := \eta_0(t,x) + \eta_k^{stoch}(t,x).$$

Because $\eta_0(t,x)$ is deterministic and continuous by Lemma 3.2.3, it suffices to show that $\eta_k^{stoch}(t,x)$ is continuous in $L^2(\Omega)$. Consider $t_0 > 0$ and $x_0 \in \mathbb{R}^n$. (The argument for $t_0 = 0$ is similar.) Let

$$S_\delta = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : |t - t_0| < \delta, |x - x_0| < \delta\}. \tag{3.128}$$

Continuity would follow if we can show that given arbitrary $\epsilon > 0$, there exists $\delta > 0$ sufficiently small such that

$$\mathbb{E}\left(|\eta_k^{stoch}(t,x_0) - \eta_k^{stoch}(t_0,x_0)|^2\right) < \epsilon, \qquad \text{for } |t - t_0| < \delta, \tag{3.129}$$

$$\mathbb{E}\left(|\eta_k^{stoch}(t,x_1) - \eta_k^{stoch}(t,x_0)|^2\right) < \epsilon, \qquad \text{for all } (t,x_1), (t,x_0) \in S_\delta. \tag{3.130}$$

Denote

$$T^* = t_0 + 1. \tag{3.131}$$

Let us show the first part of the continuity estimate (3.129). For every $\epsilon > 0$, we need to find a $\delta > 0$ such that (3.129) holds. We begin by first assuming that $\delta > 0$ is such that

$$\delta < \min\{1, \frac{t_0}{2}\} = \frac{t_0}{2} \wedge 1 \tag{3.132}$$

(the reason for this choice will be clear later), and we denote

$$\tau_m = t \wedge t_0 > 0, \qquad \tau_M = t \vee t_0 > 0, \tag{3.133}$$

where $t \vee t_0 := \max\{t, t_0\}$. By using a change of variables,

$$\begin{aligned}
&|\eta_k^{stoch}(t,x_0) - \eta_k^{stoch}(t_0,x_0)| \\
&= \left|\int_0^{\tau_m} \int_{\mathbb{R}^n} K_{\tau_m - s}(x-y)[f(\eta_{k-1}(s + \tau_M - \tau_m, x_0)) - f(\eta_{k-1}(s,x_0)) W(ds,dy)\right| \\
&+ \left|\int_0^{\tau_M - \tau_m} \int_{\mathbb{R}^n} K_{\tau_M - s}(x-y) f(\eta_{k-1}(s,x_0)) W(ds,dy)\right|.
\end{aligned}$$

Using the Lipschitz condition and the growth condition (3.60) on f, together with the Itô isometry (2.6), we can bound the expectation $\mathbb{E}(|\eta_k^{stoch}(t,x_0)-\eta_k^{stoch}(t_0,x_0)|^2)$ by two integrals, J_1 and J_2, one integrated from 0 to τ_m and the other from τ_m to τ_M:

$$\begin{aligned}
&\mathbb{E}(|\eta_k^{stoch}(t,x_0)-\eta_k^{stoch}(t_0,x_0)|^2)\\
&\quad\leqslant 2L^2\int_0^{\tau_m}\int_{\mathbb{R}^n}|K_{\tau_m-s}(x-y)|^2\mathbb{E}\left(|\eta_{k-1}(s+\tau_M-\tau_m,x_0)-\eta_{k-1}(s,x_0)|^2\right)dyds\\
&\quad+4L^2\int_0^{\tau_M-\tau_m}\int_{\mathbb{R}^n}|K_{\tau_M-s}(x-y)|^2\left(1+\mathbb{E}\left(|\eta_{k-1}(s,x_0)|^2\right)\right)dyds := 2L^2(J_1+2J_2).
\end{aligned} \tag{3.134}$$

To handle J_1, as long as the condition (3.132) on δ is satisfied, we have $\tau_m \leqslant T^*$, where T^* is defined in (3.131). Hence, by (3.72),

$$\int_0^{\tau_m}\int_{\mathbb{R}^n}|K_{\tau_m-s}(x-y)|^2dyds \leqslant \left(\int_0^{T^*}(T^*-s)^{2-n}ds\right)\cdot||K||^2_{L^2(\mathbb{R}^n)} := C_1.$$

Since continuous functions are uniformly continuous on compact sets, by using the fact that $u_{k-1}(t,x)$ is $L^2(\Omega)$ continuous, along with $0<\tau_m<\tau_M\leqslant T^*$ and $|\tau_M-\tau_m|<\delta$, we can make

$$J_1 < \frac{\epsilon}{4L^2}, \tag{3.135}$$

by choosing $\delta < \frac{t_0}{2}\wedge 1$ sufficiently small so that

$$\mathbb{E}\left(|\eta_{k-1}(t_1,x_0)-\eta_{k-1}(t_2,x_0)|^2\right) < C_1^{-1}\frac{\epsilon}{4L^2},\ \text{whenever } |t_1-t_2|<\delta \text{ and } t_1,t_2\in[0,T^*].$$

To handle J_2, we note that by (3.126) and a calculation similar to (3.72),

$$\begin{aligned}
J_2 \leqslant (1+C_{k-1,T^*})\int_0^{\tau_M-\tau_m}\int_{\mathbb{R}^n}|K_{\tau_M-s}(x-y)|^2dyds &= \tilde{C}_{k-1,T^*}\int_0^{\tau_M-\tau_m}(\tau_M-s)^{2-n}ds\\
&= \frac{1}{3-n}\tilde{C}_{k-1,T^*}(\tau_M^{3-n}-\tau_m^{3-n}).
\end{aligned}$$

Therefore, because $|\tau_M-\tau_m|<\delta$ and $0\leqslant\tau_m\leqslant\tau_M<T^*$ by (3.131) and (3.132), we can choose δ satisfying condition (3.132) sufficiently small such that

$$J_2 < \frac{\epsilon}{8L^2}. \tag{3.136}$$

So by (3.134), (3.135), (3.136), we can choose δ sufficiently small so that (3.129) holds.

Next, we verify (3.130). By the Itô isometry (2.6) and the bound in Lemma 3.3.1,

$$\begin{aligned}
\mathbb{E}\left(|\eta_k^{stoch}(t,x_1)-\eta_k^{stoch}(t,x_0)|^2\right) &= \int_0^t\int_{\mathbb{R}^n}|K_{t-s}(x_1-y)-K_{t-s}(x_0-y)|^2\mathbb{E}\left(|\eta_{k-1}(s,y)|^2\right)dyds\\
&\leqslant C_{k-1,T^*}\int_0^t\int_{\mathbb{R}^n}|K_{t-s}(x_1-y)-K_{t-s}(x_0-y)|^2dyds\\
&= C_{k-1,T^*}\int_0^t\int_{\mathbb{R}^n}|K_s(y)-K_s(y+x_0-x_1)|^2dyds.
\end{aligned}$$

Recall that the Fourier transform of $K_t(x)$ is $e^{-\frac{|\xi|}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|t\right)}{\frac{\sqrt{3}}{2}|\xi|}$. Therefore, by Plancherel's formula,

$$\begin{aligned}
\mathbb{E}(|\eta_k^{stoch}(t,x_1)-\eta_k^{stoch}(t,x_0)|^2) &\leqslant C_{k-1,T^*}\int_0^t\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2|1-e^{i(x_0-x_1)\cdot\xi}|^2 d\xi ds\\
&= 2C_{k-1,T^*}\int_0^t\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2[1-\cos((x_0-x_1)\cdot\xi)]d\xi ds\\
&\leqslant 2C_{k-1,T^*}\int_0^{T^*}\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2[1-\cos((x_0-x_1)\cdot\xi)]d\xi ds\\
&\leqslant 4C_{k-1,T^*}\int_0^{\tau}\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2 d\xi ds\\
&+ 2C_{k-1,T^*}\int_{\tau}^{T^*}\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2[1-\cos((x_0-x_1)\cdot\xi)]d\xi ds\\
&:= 4C_{k-1,T^*}J_3+2C_{k-1,T^*}J_4, \qquad (3.137)
\end{aligned}$$

where $\tau > 0$ will be chosen later. We have repeatedly used the fact that as long as δ is chosen so that it is also less than one (see (3.132)), then $t \in [0, T^*]$ for $t \in S_\delta$. Note that

$$\int_0^{T^*}\int_{\mathbb{R}^n}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2 d\xi ds = \int_0^{T^*}\int_{\mathbb{R}^n}|K_s(y)|^2 dy ds < \infty,$$

by a calculation similar to (3.72). Therefore, by choosing $\tau \in (0, T^*)$ sufficiently small, we can make

$$4C_{k-1,T^*}J_3 < \frac{\epsilon}{2}. \qquad (3.138)$$

Now that we have fixed a choice of τ, we consider J_4. We split it into two integrals, one over the frequencies ξ such that $|\xi| > Ms^{-1}$, and the other over $|\xi| \leqslant Ms^{-1}$, where $M > 0$ we will be chosen later:

$$J_4 \leqslant \int_{\tau}^{T^*}\int_{|\xi|>Ms^{-1}}\cdot+\int_{\tau}^{T^*}\int_{|\xi|\leqslant Ms^{-1}}\left|e^{-\frac{|\xi|}{2}s}\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|}\right|^2[1-\cos((x_0-x_1)\cdot\xi)]d\xi ds.$$

By noting that $\frac{\sin\left(\frac{\sqrt{3}}{2}|\xi|s\right)}{\frac{\sqrt{3}}{2}|\xi|} \leqslant s \leqslant T^*$ and $0 \leqslant 1 - \cos((x_0 - x_1) \cdot \xi) \leqslant 2$ in the first integral, and $0 \leqslant 1 - \cos(\theta) \leqslant \frac{1}{2}\theta^2$ in the second integral, we get:

$$\begin{aligned} J_4 &\leqslant 2(T^*)^2 \int_\tau^{T^*} \int_{|\xi|>Ms^{-1}} e^{-|\xi|s} d\xi ds + \int_\tau^{T^*} \int_{|\xi|\leqslant Ms^{-1}} e^{-|\xi|s} |x_0 - x_1|^2 d\xi ds \\ &= 2(T^*)^2 \int_\tau^{T^*} s^{-n} \int_{|\alpha|>M} e^{-|\alpha|} d\alpha ds + |x_0 - x_1|^2 \int_\tau^{T^*} s^{-n} \int_{|\alpha|\leqslant M} e^{-|\alpha|} d\alpha ds \\ &\leqslant 2T^*\tau^{-n} \left((T^*)^2 \int_{|\alpha|>M} e^{-|\alpha|} d\alpha + |x_0 - x_1|^2 \int_{\mathbb{R}^n} e^{-|\alpha|} d\alpha \right). \end{aligned}$$

By taking M sufficiently large such that

$$\int_{|\alpha|>M} e^{-|\alpha|} d\alpha < \frac{1}{2C_{k-1,T^*}} \frac{1}{2(T^*)^3\tau^{-n}} \frac{\epsilon}{4},$$

and then taking $\delta > 0$ sufficiently small satisfying the condition (3.132), such that

$$2T^*\tau^{-n}\delta^2 \int_{\mathbb{R}^n} e^{-|\eta|} d\eta < \frac{1}{2C_{k-1,T^*}} \frac{\epsilon}{4},$$

we have that $2C_{k-1,T^*} J_4 < \frac{\epsilon}{2}$ whenever $(t, x_0), (t, x_1) \in S_\delta$ with $|x_0 - x_1| < \delta$. Using this fact along with (3.138) in (3.137) establishes the desired result (3.130). □

Acknowledgements

This work was partially supported by the National Science Foundation under grants DMS-1613757, DMS-1853340, and DMS-2011319. This material is based upon work supported by the National Science Foundation under grant DMS-1928930 while the authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

Chapter 4

A fully coupled model of stochastic FSI

In this chapter, we continue the study of stochastic fluid-structure interaction by considering a more general approach to stochastic FSI. In the past chapter, we showed well-posedness of a reduced model of stochastic FSI represented by a stochastic viscous wave equation, which is a single self-contained equation that describes the full dynamics of a linearly coupled stochastic fluid-structure system in an unbounded domain. The special geometry of the stochastic reduced model found in the previous chapter, which consists of a fluid residing in the lower half space $(z < 0) \subset \mathbb{R}^3$ interacting with an elastic membrane with reference configuration $(z = 0)$, was essential for reducing the equations down to a single self-contained equation. In addition, the fact that we have a final equation, the stochastic viscous wave equation, posed on $\mathbb{R}^2$ allowed us to appeal to Fourier analysis and fundamental solutions in order to define the solution via a mild formulation, where the solution is defined by convolving the fundamental solution with the stochastic spacetime white noise. In fact, the use of Fourier analysis on the space $\mathbb{R}^2$ allowed for a very specific characterization of the fundamental solution for the viscous wave operator, which was essential for our analysis of the reduced stochastic model. However, the unbounded geometry of the model is not feasible for use in real-life applications, which often involve problems posed on finite bounded domains, and attempting to do the same model reduction in the case of even a linearly coupled fluid-structure model posed on a finite domain would be challenging.

Thus, our goal in the current chapter is to develop a more general framework for analyzing stochastic FSI models posed on more general domains. In this case, we will use the operator splitting method in order to construct random approximate solutions to our linearly coupled fluid-structure problem, where we will more generally consider a 2D linear Stokes flow describing an incompressible viscous Newtonian fluid (rather than a stationary Stokes flow as in the previous chapter) interacting with a stochastically forced 1D elastic membrane whose transverse displacement is described by a wave equation. The notion of solution that we will develop here is the notion of a weak solution analytically, where the solution will satisfy an appropriate weak formulation almost surely for every admissible deterministic test function, and we will constructively generate this analytically weak random solution through an operator splitting method where we split the problem into three subproblems: a structure, stochastic, and fluid subproblem, which each will handle different parts of the full weak formulation. We will then pass to the limit in these random approximate solutions using probabilistic methods, which differ from the methods used for the deterministic problem. In particular, because the uniform boundedness of approximate solutions in finite energy spaces is only in expectation rather than pathwise, we will use compactness arguments to show that the laws of the approximate solutions converge weakly to a limiting law. These compactness arguments are used to establish weak convergence of the laws of the approximate solutions since one must verify a condition called tightness to obtain a limiting law as a weak limit along a subsequence of laws of the approximate solutions. Since weak convergence of laws is not sufficient to conclude almost sure convergence of the approximate solutions, which is what is needed to pass to the limit in the semidiscrete weak formulations for the approximate solutions, we will need to use probabilistic techniques such as the Skorokhod representation theorem and the Gyöngy-Krylov lemma (see Chapter 2), to strengthen the convergence of our approximate solutions to almost sure convergence along a subsequence. This chapter discusses a new general framework for analyzing general stochastic fluid-structure systems based on an operator splitting approach, and represents the first application of this approach to a stochastic FSI problem, which highlights the versatility of the operator splitting method.

We emphasize that the material discussed in this chapter is adapted from the manuscript [113], co-authored with Sunčica Čanić.

4.1 Introduction

In this chapter, we introduce a constructive approach to study solutions of stochastic fluid-structure interaction (SFSI) with stochastic noise. This chapter is written as an introduction to the use of stochastic techniques to study SFSI, and is aimed at audiences that have experience with deterministic FSI, but may be new to stochastic analysis. We focus on a benchmark problem in which a stochastically forced linearly elastic membrane interacts with the flow of a viscous incompressible Newtonian fluid in two spatial dimensions. The membrane is modeled by the linear wave equation, while the fluid is modeled by the 2D time-dependent Stokes equations. The problem is forced by a "rough" stochastic forcing given by a time-dependent white noise $\dot{W}(t)$, where W is a given one-dimensional Brownian motion with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with complete filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. The fluid and the membrane are coupled via a two-way coupling describing continuity of fluid and structure velocities at the fluid-structure interface, and continuity of contact forces at the interface. The coupling is calculated at the linearized, fixed interface, rendering this problem a linear stochastic fluid-structure interaction problem. The goal is to show that despite the rough white noise, the resulting problem is well-posed, showing that the underlying deterministic fluid-structure interaction problem is robust to noise. Indeed, we prove the existence of a unique weak solution in the probabilistically strong sense (see Definition 4.4.2

in Section 4.4) to this stochastic fluid-structure interaction problem. This means that there exist unique random variables (stochastic processes), describing the fluid velocity $\boldsymbol{u}$, the structure velocity v, and the structure displacement η, such that those stochastic processes are adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, i.e., they only depend on the past history of the processes up to time t and not on the future, which satisfy the weak formulation of the original problem almost surely. This is the main result of this manuscript.

To prove the existence of a unique weak solution in the probabilistically strong sense, we design a constructive existence proof. The constructive existence proof is based on semi-discretizing the problem in time by dividing the time interval $(0, T)$ into N subintervals of width $\Delta t = T/N$, and combining a fluid-structure splitting introduced in [140] with a stochastic splitting up method introduced in [20], to construct approximate solutions. The goal is to show that the approximate solutions converge almost surely with respect to a certain topology, to the unique weak solution as Δt goes to zero. In contrast to the deterministic case, see the works of Muha and Čanić in [140, 138, 137], where a time-discretization via operator splitting approach was used to study existence of weak solutions, we propose an alternative splitting scheme in Section 4.6, where the stochastic part is considered separately from the deterministic part, and the fluid and structure problems are split and solved in a particular order so that the resulting stochastic integrals involving the stochastic noise increments can be evaluated and estimated to prove stability. See Remark 4.6.1 in Section 4.6. More precisely, along each time sub-interval $(t_N^n, t_N^{n+1}), n = 0, \dots, N-1$, the following three sub-problems are solved to obtain approximate solutions consisting of the fluid and structure velocities, and the structure displacement, $(\boldsymbol{u}, v, \eta)$. First, in **Step 1**, the structure displacement and structure velocity are updated using only the structure displacement and structure velocity from the previous time step. The resulting random variables are measurable with respect to the sigma algebra $\mathcal{F}_{t_N^n}$. Then, in **Step 2**, which is the stochastic step, the structure velocity is updated by adding to the structure velocity calculated in Step 1 the stochastic noise increment from time step t_N^n to time step t_N^{n+1}. Since the structure velocity obtained in Step 1 is a random variable that is measurable with respect to the sigma algebra $\mathcal{F}_{t_N^n}$, and the stochastic increment from t_N^n to t_N^{n+1} is independent of it, we will be able to obtain boundedness of the stochastic integral involving these two quantities by using their independence. This will lead to stability. The resulting updated structure velocity is a random variable that is measurable with respect to the sigma algebra $\mathcal{F}_{t_N^{n+1}}$. Finally, in **Step 3**, the fluid and structure velocities are updated by using the information from the just calculated structure velocity in Step 2. This gives rise to random variables that are measurable with respect to the sigma algebra $\mathcal{F}_{t_N^{n+1}}$. We would like to show that the sequence or a subsequence of random variables constructed this way converges in a certain topology to a weak solution in the probabilistically strong sense of the coupled SFSI problem.

Based on this splitting scheme, uniform energy estimates in terms of expectation can be derived. In addition to estimating the expectation of the kinetic and elastic energy of the problem, it is important to get a uniform bound on the expectation of the numerical dissipation, to show that the numerical dissipation is bounded and that it in fact, approaches zero as the time step Δt goes to zero, which is crucial in the convergence proof. This is provided in Proposition 4.6.7. Furthermore, another interesting observation is that the energy estimates will have an extra term on the right-hand side which accounts for the energy pumped into the problem by the stochastic noise. This is in addition to the energy/work contributions by the initial and boundary data. These energy estimates define an energy function space for the unknown functions $(\boldsymbol{u}, v, \eta)$. A separable Banach space containing the energy space, specified in (4.34) in Section 4.8 is called a *phase space*, and is denoted by $\mathcal{X}$, and will be the space in which we will consider our approximate solutions.

For this linearly coupled prototypical model of stochastic FSI, we want to pass to the limit in our random approximate solutions. Our main goal in developing a way of passing to the limit is to develop a robust methodology that generalizes well to a wide class of stochastic FSI problems. In particular, we want to establish a mathematical framework that generalizes both to the case where the stochastic noise $dW(t)$ is scaled by a nonlinear function f that can depend on the solution itself, as in the model discussed in Chapter 3, and the case where the stochastic FSI system is nonlinearly coupled so that we are considering the Navier-Stokes equations posed on an a priori random time-dependent domain. Thus, we will consider probabilistic compactness arguments for passing to the limit, by invoking probabilistic tools which show that the laws of the approximate solutions on the phase space $\mathcal{X}$ converge as the time discretization parameter goes to 0, along a subsequence. We anticipate that these probabilistic arguments which we apply to this prototypical linearly coupled stochastic FSI system will generalize well to the aforementioned models of stochastic FSI containing nonlinearities, both in the intensity of the noise and in the fluid-structure coupling.

We remark that while it may be possible to use a generalization of the so-called Skorokhod representation theorem to Jakubowski spaces [99] to pass to the limit, instead of probabilistic compactness arguments, this methodology would not generalize well to other more complex cases of interest. This methodology however would work well to handle convergence in Banach spaces equipped with the topology of only weak or weak star convergence rather than strong convergence, and we refer the reader to the discussions in Appendix A in both papers [166, 167] for further discussion of more general Jakubowski spaces, which generalize Banach spaces. This will however not be the approach that we will take here, and we will use a more standard framework of well-known probabilistic compactness arguments that generalizes well to more complex stochastic FSI models and gives a broad overview of some key techniques from stochastic analysis that are commonly used in the study of stochastic PDEs.

Hence, will develop a robust mathematical method of establishing probabilistic convergence of our (random) approximate solutions, and the first step in doing this is to show that the laws of the approximate solutions converge weakly. To establish weak convergence of probability measures, one must show that the probability measures are **tight**. More precisely, one must show that for each $\epsilon > 0$, there exists a **compact set** in the phase space $\mathcal{X}$ of displacements and fluid and structure velocities, such that the probability that our approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ live in that compact set is greater than $1 - \epsilon$. See Definition 4.8.1 for tightness of measures. The proof of tightness of the sequence of probability measures μ_N corresponding to the laws of the approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ will follow from a *deterministic* compactness argument alla Aubin-Lions. The compactness argument will establish the existence of a compact subset of the phase space $\mathcal{X}$ that contains the approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ with probability greater than $1 - \epsilon$, thus verifying the tightness property.

Once we have established the existence of a subsequence of probability measures μ_N that converges weakly to some probability measure μ as $N \to \infty$, or equivalently, as $\Delta t \to 0$, we would like to show that on a further subsequence, the *random variables* $(\boldsymbol{u}, v, \eta)_N$ will converge almost surely to a random variable with the law μ, with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Showing this almost sure convergence with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, however, has to be done in two parts. In the first part, we get a hold of a subsequence of approximate solutions that converge almost surely **but** on another probability space, and then use this information in the second part to construct a convergent subsequence of approximate solutions that converge on the *original* probability space. The following is a more detailed albeit succinct description of the two parts.

Part 1. We use the Skorokhod representation theorem to deduce that there exists a sequence of random variables $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_N$, defined on **a** probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, which is *not necessarily the same as* the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the laws of $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_N$ are μ_N, and $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_N$ converge almost surely to a random variable $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ with the law μ, on the "tilde" probability space. On this "tilde" probability space we also show that the almost sure limit $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ satisfies the weak formulation of the original problem almost surely, **but** with respect to the "tilde" probability space. This means that this limit is a **weak solution to the original problem in the probabilistically *weak* sense**, see Definition 4.4.1. This result will be useful in showing the existence of a unique weak solution in the probabilistically *strong* sense on the *original* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, discussed in the second part.

Part 2. We would like to be able to prove that our sequence of approximate solutions $(\boldsymbol{u}, v, \eta)_N$, obtained using our time-discretization via operator splitting approach described above, converges almost surely to a random variable $(\boldsymbol{u}, v, \eta)$ on the original probability space, and satisfies the weak formulation almost surely on the original probability space. Namely, we would like to prove that the limit is a **weak solution to the original problem in the probabilistically *strong* sense**. If we could obtain that the sequence $(\boldsymbol{u}, v, \eta)_N$ converges **in probability** to a random variable on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, namely $(\boldsymbol{u}, v, \eta)_N \xrightarrow{p} (\boldsymbol{u}, v, \eta)$, then the almost sure convergence along a subsequence will follow immediately. To obtain convergence *in probability* of $(\boldsymbol{u}, v, \eta)_N$, we will invoke a standard Gyöngy-Krylov argument [88].

More precisely, to prove that $X_N = (\boldsymbol{u}, v, \eta)_N$ converge *in probability* to some random variable $X^* = (\boldsymbol{u}, v, \eta)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, $X_N \xrightarrow{p} X^*$, based on the Gyöngy-Krylov lemma [88], we need to show that for every two subsequences X_l and X_m, there exists a subsequence $x_k = (X_{l_k}, X_{m_k})$ such that the following two properties hold:

1. The joint laws $\nu_{X_{l_k}, X_{m_k}}$ of the subsequence x_k converge to some probability measure ν

as $k \to \infty$;

2. The limiting law is supported on the diagonal: $\nu(\{(X,Y) : X = Y\}) = 1$.

The first property will follow from the tightness of measures μ_l and μ_m, which are the laws associated with the random variables $X_l = (\boldsymbol{u}, v, \eta)_l$ and $X_m = (\boldsymbol{u}, v, \eta)_m$. The tightness of the measures μ_l and μ_m implies tightness of the joint measures ν_{X_l,X_m} as well. To show that the second property holds, we will use the result of Part 1 above, combined with a *deterministic uniqueness* argument. Namely, Part 1 gives us the existence of the almost surely convergent subsequences $\tilde{X}_l = (\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_l$ and $\tilde{X}_m = (\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_m$ on the "tilde" probability space that have the same laws μ_l and μ_m as $X_l = (\boldsymbol{u}, v, \eta)_l$ and $X_m = (\boldsymbol{u}, v, \eta)_m$. Those two "tilde" subsequences of random variables converge to the limits $\tilde{X}^1$ and $\tilde{X}^2$, respectively, each of which has the law μ, and a joint law of $(\tilde{X}^1, \tilde{X}^2)$ equal to ν from Property 1 above. Recall, from Step 1, that both $\tilde{X}^1$ and $\tilde{X}^2$ are weak solutions in the probabilistically weak sense. To show that this joint law ν is supported on the diagonal, namely, to show Property 2 above, it is sufficient to show that $\tilde{X}^1$ is equal to $\tilde{X}^2$ almost surely, namely it will be sufficient to show that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$. Indeed, proving the diagonal condition from the Gyöngy-Krylov lemma is associated with proving pathwise uniqueness of weak solutions, which we present in Section 4.9.

Once the properties from the Gyöngy-Krylov lemma have been verified, we can conclude that there exists a subsequence of $(\boldsymbol{u}, v, \eta)_N$, which we continue to denote by N, such that $(\boldsymbol{u}, v, \eta)_N \xrightarrow{p} (\boldsymbol{u}, v, \eta)$, which implies almost sure convergence along a subsequence on the original probability space. This is presented in Section 4.9.

Finally, the proof that the limiting function $(\boldsymbol{u}, v, \eta)$ recovered above is a weak solution in the probabilistically strong sense is presented in Section 4.9.

To the best of our knowledge, this is the first well-posedness result in the context of stochastic fluid-structure interaction. The result shows that our deterministic benchmark FSI model is robust to stochastic noise, even in the presence of rough white noise in time. This proof combines stochastic PDE analysis tools with deterministic FSI approaches. Additionally, the constructive proof lays out a framework for the development of a numerical scheme for this class of SFSI problems.

In the next section, we provide a brief review of the related literature.

4.2 Literature review

The mathematical analysis of deterministic fluid-structure interaction began around twenty years ago by focusing on rigorous well-posedness for linearly coupled fluid-structure interaction models. Linearly coupled FSI models are models where the fluid and structure coupling conditions are evaluated along a fixed fluid-structure interface, and the fluid equations are posed on a fixed fluid domain, even though the structure is assumed to be elastic and displaces from its reference configuration. The results concerning these linearly coupled models typically deal with establishing existence/uniqueness of weak or strong solutions. The existence and uniqueness of a weak solution to a linearly coupled model involving an interaction between the linear Stokes equations and the equations of linear elasticity was established in [66] using a Galerkin method. The Navier-Stokes equations for an incompressible, viscous fluid linearly coupled to immersed elastic solids were considered in [12, 13, 118]. In particular, the work in [12] deals with showing the existence of energy-level weak solutions, by a careful examination of the trace regularity of the hyperbolic structure dynamics in terms of the normal stress at the fluid-structure interface. The results in [13, 118] deal with establishing sufficient regularity of initial data that provides existence of strong solutions of the corresponding linearly coupled systems.

The well-posedness analysis of deterministic FSI models was extended later to nonlinearly coupled models, where the fluid domain changes in time according to the structure displacement, and hence the problem is a moving boundary problem where the fluid domain is not known a priori. There is by now an extensive mathematical literature dealing with the well-posedness of such models, see the discussion in Chapter 1. Of these references, we note that the approach outlined in [86, 137, 138, 139, 140, 141] is closely related to the approach used in the current manuscript, in that we use a splitting scheme in order to obtain a constructive existence proof.

In particular, the approach is based on using a splitting scheme, known as the Lie operator splitting scheme, that discretizes the nonlinearly coupled problem in time by a time step Δt, and separates the coupled problem into fluid and structure subproblems. Then, compactness arguments of Aubin-Lions type (see [6, 129, 136]) are used to pass to the limit as $\Delta t \to 0$ in the approximate weak formulations satisfied by the approximate solutions, in order to obtain a *constructive existence proof* for weak solutions to nonlinearly coupled fluid-structure interaction problems. This approach proved to be quite robust for *deterministic* fluid-structure interaction problems, since it provided existence of weak solutions for several different scenarios involving thin, thick, and multi-layered structures coupled to the flow of an incompressible, viscous fluid via the no-slip or Navier slip boundary conditions, see [137, 138, 139, 140, 141].

In the present work, a version of this approach is extended to deal with *stochastic* fluid-structure interaction problems, by combining stochastic calculus with stochastic operator splitting approaches introduced in [20] and analyzed in [89]. More precisely, we design a time-discretized, operator splitting method in just the right way so that all the stochastic integrals are well-defined, and the resulting time-discretized scheme is stable, allowing us to show, using stochastic calculus, an almost sure convergence of approximate solutions to a weak solution in the probabilistically strong sense of the coupled fluid-structure interaction problem. To the best of our knowledge, this is the first well-posedness result on fully coupled *stochastic fluid-structure interaction.* Our result builds on recent developments in the area of *stochastic partial differential equations* (SPDEs).

Stochastic partial differential equations are PDEs that feature some sort of random noise forcing, such as white noise forcing in either time, or both time and space, or spatially homogeneous Gaussian noise that is independent at every time but potentially correlated in space. They are motivated by the fact that many real-life systems modeled by PDEs exhibit some type of random noise, which can significantly impact the resulting dynamics of the system. The current manuscript considers a stochastic linearly coupled fluid-structure interaction model involving the interaction between a fluid modeled by the linear Stokes equations and an elastic membrane modeled by the wave equation. Although the coupled stochastic FSI model has not been previously considered in the stochastic PDE literature, there are many works that study either stochastic fluid dynamics or stochastic wave equations separately, as we summarize below.

In terms of stochastic fluid equations, the consideration of stochastic Navier-Stokes equations is an active area of research, see e.g., [21, 38, 72, 121]. The study of stochastic Navier-Stokes equations was initiated in the work of [21], which considered an abstract stochastic equation of Navier-Stokes type, with an additive random noise forcing in time, and a random initial condition. It was shown that there exists a solution that satisfies the problem almost surely in a distributional sense. In the works of [38, 72], this abstract equation of Navier-Stokes type is extended to more general settings where there is nonlinear dependence of the intensity of the random noise forcing on the actual solution itself. These two works consider different abstract conditions on this nonlinear dependence and prove existence of martingale, or probabilistically weak, solutions to the resulting stochastic equations. Both of these works use a Galerkin scheme to construct solutions and obtain existence by establishing uniform bounds on the sequence of random functions satisfying the finite-dimensional Galerkin problems. We note that passing to the limit in the Galerkin solutions in [38, 72] was done by using standard probabilistic methods, such as establishing tightness of laws, showing weak convergence in law, and invoking the Skorokhod representation theorem, which are standard techniques that we will employ for our current problem as well. While there are many works on stochastic fluid dynamics, we mention in particular a recent work [128], which establishes the existence of local martingale solutions, which are martingale solutions up to some stopping time, for a system of one layer shallow water equations for fluid velocity and water depth in two spatial dimensions, driven by random noise forcing described by cylindrical Wiener processes. We remark that [128] employs similar probabilistic methods in passing to the limit in a sequence of random approximate solutions (obtained by a Galerkin method) that motivated many of the probabilistic arguments in this manuscript, though the methods used in this current manuscript for constructing approximate solutions are different, as they are based on time discretization using an operator splitting approach, and not spatial discretization using a Galerkin method. One reason for the use of time-discretization via operator splitting, versus a Galerkin approach, is a possible extension to the moving boundary case. In the Galerkin case, the basis functions for the moving boundary case will depend on the *random* solution itself, which is difficult to deal with.

In terms of stochastic wave equations, there is extensive work on well-posedness and properties of solutions. It is classically well-known that the stochastic wave equation with spacetime white noise has a mild solution only in dimension one, but not in dimensions two and higher (see for example [55]). This is due to the fact that the fundamental solution of the linear wave equation in dimension two is not square integrable in spacetime, and in higher dimensions, it is not even function-valued. Hence, work on the stochastic wave equation in dimensions two and higher, focuses on considering stochastic wave equations with a more general type of noise, such as spatially homogeneous Gaussian noise (see for example [150]) which is independent in time but correlated in space. In particular, the authors of [54, 56, 105] consider conditions for this spatially homogeneous Gaussian noise, such that the resulting stochastic wave equation has a solution that is function-valued (rather than just a distribution) in dimensions two and higher. Existence results for such stochastic wave equations in higher dimensions are also considered in [50], and the Hölder continuity and regularity properties of stochastic wave equations in higher dimensions are considered in [50, 57].

As discussed in Chapter 3, some past progress in stochastic FSI [111] involves a *stochastic viscous wave equation*, which was derived as a reduced model for a stochastic linearly coupled fluid-structure interaction problem, where the entire fluid-structure system can be modeled by a single stochastic viscous wave equation, describing the random displacement of the structure from its reference configuration. The work in [111] considers well-posedness for the stochastic viscous wave equation and establishes existence and uniqueness of a mild solution in spatial dimensions one and two, in addition to improved Hölder regularity properties. While the results in [111] provide an insight into the behavior of solutions to stochastic FSI, they are restricted by the fact that the stochastic viscous wave equation is not a fully coupled model, it is defined in a special geometry on the entire $\mathbb{R}^2$, and it does not include the fluid inertia effects. This allowed the use of mathematical techniques that are not available in the fully coupled case of stochastic FSI. The goal of the current manuscript is to develop techniques for studying *fully coupled* stochastic fluid-structure interaction systems, defined on physically relevant geometries, including fluid inertia effects described by the time-dependent Stokes equations.

4.3 Description of the model

The model problem considered here is defined on a fixed fluid domain, which is a rectangle $\Omega_f = [0, L] \times [0, R]$. The boundary $\partial\Omega_f$ of the fluid domain consists of four parts: the moving boundary part denoted by Γ (it is the reference configuration of the moving boundary), the bottom of the "channel" denoted by Γ_b, and the inlet and outlet parts of the boundary Γ_{in} and Γ_{out} where the pressure data is prescribed. The flow in the fluid domain Ω_f is driven by the inlet and outlet pressure data, and by the motion of the moving boundary. See Fig. 4.1. We will use $\boldsymbol{x} = (z, r)$ to denote the coordinates of points in the fluid domain.

The **fluid flow** in Ω_f will be modeled by the time-dependent Stokes equations for an incompressible, viscous fluid:

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} & = & \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f, \tag{4.1}$$

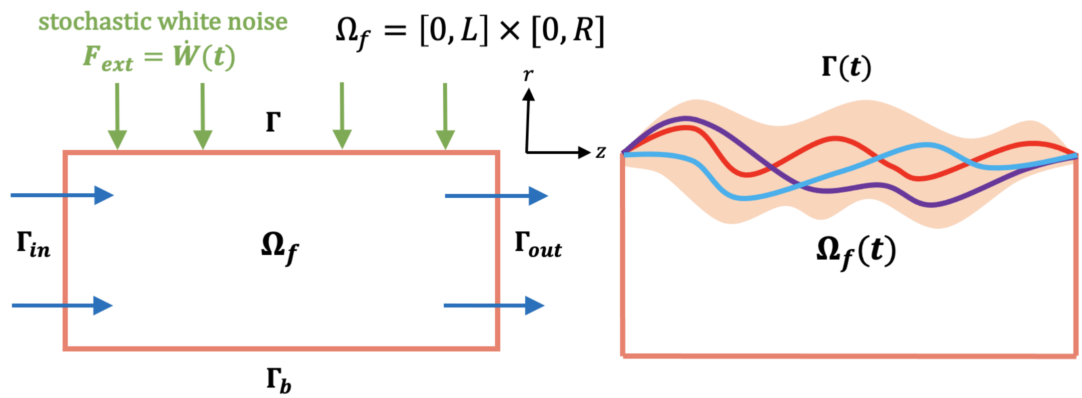

Figure 4.1: *Left: A sketch of the linearly coupled stochastic FSI problem, with Ω_f denoting the reference fluid domain, Γ denoting the reference configuration of the structure, and $\dot{W}(t)$ denoting stochastic white noise forcing on the structure. Right: The different colors represent different possible outcomes for the random configuration $\Gamma(t)$ of the structure at some time t. The lightly shaded region represents a confidence interval of where the structure is likely to be.*

where $\boldsymbol{u}(t, \boldsymbol{x}) = (u_z(t, \boldsymbol{x}), u_r(t, \boldsymbol{x}))$ is the fluid velocity, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u})$ is the Cauchy stress tensor describing a Newtonian fluid, and p is the fluid pressure. This gives rise to the following system:

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p &=& 0, \\ \nabla \cdot \boldsymbol{u} &=& 0, \end{array}\right\} \quad \text{in } \Omega_f. \tag{4.2}$$

At the top boundary Γ of the fluid domain, an elastic membrane interacts with the fluid flow. We assume that this elastic structure experiences displacement only in the vertical direction from its reference configuration Γ, and we denote the magnitude of this displacement by $\eta(t, z)$. The **elastodynamics of the structure** will be modeled by the wave equation:

$$\eta_{tt} - \Delta\eta = f, \qquad \text{on } \Gamma, \tag{4.3}$$

where f is an external forcing term.

The fluid and structure are coupled via two sets of coupling conditions, the kinematic and dynamic coupling conditions, which are evaluated along the *fixed interface*. This is known as **linear coupling**. The **kinematic coupling condition** considered in this work describes the continuity of velocities at the fluid-structure interface

$$\boldsymbol{u} = \eta_t \boldsymbol{e}_r, \qquad \text{on } \Gamma, \tag{4.4}$$

also known as the *no-slip* condition. The **dynamic coupling condition** describes balance of forces at the interface. Namely, it states that the elastodynamics of the thin elastic structure is driven by the jump in the force acting on the structure, coming from the normal component of the normal fluid stress $\boldsymbol{\sigma e_r} \cdot \boldsymbol{e_r}$ on one side, and the external forcing F_{ext} on the other:

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma e_r} \cdot \boldsymbol{e_r} + F_{ext}, \qquad \text{on } \Gamma,$$

where $\boldsymbol{e_r}$ is the unit outer normal to the fixed fluid-structure interface Γ.

In this manuscript, we consider the external force F_{ext} to be a stochastic force. In particular, as a start, we consider

$$F_{ext} = \dot{W}(t),$$

where W is a one-dimensional Brownian motion in time. Note that the stochastic force is constant on the whole structure at each time. As a result, the stochastic noise is rough temporally but is constant spatially. We remark that although this is a simplified model, we use it to demonstrate the difficulties present in the stochastic case in the simplest possible setting.

More precisely, we let W denote a one-dimensional Brownian motion with respect to an underlying probability space with filtration, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, in which case dW is formally the derivative of this Brownian motion. This is a purely formal notation that we will give precise meaning to later, as Brownian motion is almost surely nowhere differentiable.

In addition, we will assume that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a **complete filtration**, which means that $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$ for every $t \geqslant 0$, where a **null set** is defined to be any measurable set in $\mathcal{F}$ that has probability zero. This technical assumption will be useful to pass to the limit in our analysis of the stochastic problem above, as it allows us to bypass technicalities regarding null sets when considering almost sure limits of stochastic processes. *In particular, the almost sure limit of $\mathcal{F}_t$ measurable random variables for any arbitrary $t \geqslant 0$ is still $\mathcal{F}_t$ measurable under the assumption of a complete filtration.* This is not a restrictive assumption, as one can complete a filtration by simply adding all null sets to $\mathcal{F}_t$ for all $t \geqslant 0$, and W will still be a Brownian motion with respect to the completed filtration. See Section 1.4 in Revuz and Yor [155] for more information about complete filtrations.

In summary, the coupled stochastic fluid-structure interaction problem studied in this manuscript, supplemented with initial and boundary data, is given by the following: *Find* $(\boldsymbol{u}, \eta)$ *such that*

$$\left.\begin{array}{rcl} \eta_{tt} - \Delta\eta &=& -\boldsymbol{\sigma e_r} \cdot \boldsymbol{e_r} + dW(t), \\ \boldsymbol{u} &=& \eta_t \boldsymbol{e_r}, \end{array}\right\} \quad \text{on } \Gamma, \qquad \left.\begin{array}{rcl} \partial_t \boldsymbol{u} &=& \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \boldsymbol{u} &=& 0, \end{array}\right\} \quad \text{in } \Omega_f, \tag{4.5}$$

with boundary data:

$$\left.\begin{array}{rcl} u_r &=& 0, \\ p &=& P_{in/out}(t), \end{array}\right\} \quad \text{on } \Gamma_{in/out}, \qquad u_r = \partial_r u_z = 0, \qquad \text{on } \Gamma_b, \tag{4.6}$$

and the following deterministic initial data:

$$\boldsymbol{u}(0, z, r) = \boldsymbol{u}_0(z, r), \qquad \eta(0, z, R) = \eta_0(z), \qquad \partial_t \eta(0, z, R) = v_0(z), \tag{4.7}$$

where $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $\eta_0 \in H_0^1(\Gamma)$, *and* $v_0 \in L^2(\Gamma)$, *and* W *is a given one-dimensional Brownian motion with respect to the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with complete filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Thus, the problem is driven by deterministic inlet and outlet pressure data $P_{in/out}(t)$ prescribed on $\Gamma_{in/out}$, with the flow symmetry condition imposed at the bottom boundary Γ_b. Notice that throughout this manuscript, we will be using Ω to denote the underlying probability space, while Ω_f denotes the fluid domain.

4.4 Definition of a weak solution and main result

To define the space of weak solutions to the above problem, we first introduce the function space for the fluid velocity:

$$\mathcal{V}_F = \{\boldsymbol{u} = (u_z, u_r) \in H^1(\Omega_f)^2 : \nabla \cdot \boldsymbol{u} = 0,\ u_z = 0 \text{ on } \Gamma,\ u_r = 0 \text{ on } \partial\Omega_f \backslash \Gamma\}. \tag{4.8}$$

Since the structure subproblem is given by the wave equation with clamped ends, the natural space of functions for the structure is

$$\mathcal{V}_S = H_0^1(\Gamma). \tag{4.9}$$

Motivated by the energy inequality presented in Sec. 4.5, we introduce the following solution spaces in time for the fluid and structure subproblems:

$$\mathcal{W}_F(0,T) = L^2(\Omega; L^\infty(0,T; L^2(\Omega_f))) \cap L^2(\Omega; L^2(0,T;\mathcal{V}_F)). \tag{4.10}$$

$$\mathcal{W}_S(0,T) = L^2(\Omega; W^{1,\infty}(0,T; L^2(\Gamma))) \cap L^2(\Omega; L^\infty(0,T;\mathcal{V}_S)). \tag{4.11}$$

We emphasize that $\boldsymbol{u}$ and η are random variables, and that the $L^2(\Omega)$ part of the solution spaces reflects the fact that the energy estimate will hold in expectation.

Finally, we introduce the solution space for the *stochastic coupled FSI problem*:

$$\mathcal{W}(0,T) = \{(\boldsymbol{u},\eta) \in \mathcal{W}_F(0,T) \times \mathcal{W}_S(0,T) : \boldsymbol{u}|_\Gamma = \eta_t \boldsymbol{e_r} \text{ for almost every } t \in [0,T], \text{ a.s.}\}. \tag{4.12}$$

Notice that in this solution space, the kinematic coupling condition is enforced strongly.

As in the deterministic case, we define weak solutions by integrating in space and time against an appropriate space of test functions, which we define to be:

$$\mathcal{Q}(0,T) = \{(\boldsymbol{q},\psi) \in C_c^1([0,T); \mathcal{V}_F \times \mathcal{V}_S) : \boldsymbol{q}(t,z,R) = \psi(t,z)\boldsymbol{e_r}\}. \tag{4.13}$$

These test functions are deterministic functions. Because the fluid domain does not change in time with the assumption of linear coupling, we can define

$$\mathcal{Q} = \{(\boldsymbol{q},\psi) \in \mathcal{V}_F \times \mathcal{V}_S : \boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r\}, \tag{4.14}$$

and hence view the test functions as differentiable, compactly supported functions on $[0, T)$ that take values in the fixed function space $\mathcal{Q}$.

To motivate the definition of a weak solution, we will proceed as in [140]. For the purposes of the derivation of the weak solution, we consider, for the moment, the case of a general deterministic external force $F_{ext}(t)$ in place of $\dot{W}(t)$, so that the first equation for the structure becomes

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r} + F_{ext}(t).$$

We will derive the standard deterministic partial differential equation definition of a weak solution, assuming that $F_{ext}(t)$ is a purely deterministic function in time, and then generalize this to the stochastic case.

We start by taking a test function $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0, T)$, and multiplying the linear Stokes equation by $\boldsymbol{q}$ and integrating in space and time. We obtain

$$\int_0^T \int_{\Omega_f} \partial_t \boldsymbol{u} \cdot \boldsymbol{q} d\boldsymbol{x} dt = \int_0^T \int_{\Omega_f} (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{q} d\boldsymbol{x} dt.$$

By integrating the first term by parts in time, we obtain:

$$\begin{aligned}\int_0^T \int_{\Omega_f} \partial_t \boldsymbol{u} \cdot \boldsymbol{q} d\boldsymbol{x} dt &= \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{q} d\boldsymbol{x} \Big|_{t=0}^{t=T} - \int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt \\ &= -\int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} - \int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt.\end{aligned}$$

By integrating the second term by parts in space and using the divergence free condition on $\boldsymbol{q}$, we obtain:

$$\int_{\Omega_f} (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{q} d\boldsymbol{x} = \int_{\partial\Omega_f} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{q} dS - 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x},$$

where $\boldsymbol{D}(\boldsymbol{u})$ and $\boldsymbol{D}(\boldsymbol{q})$ represent the symmetrized gradient. Using the definition of the Cauchy stress tensor, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u})$, and integrating in time, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega_f} (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{q} d\boldsymbol{x} dt &= \int_0^T \int_{\Gamma_{in}} p q_z dr dt - \int_0^T \int_{\Gamma_{out}} p q_z dr dt - 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt \\ - \int_0^T \int_{\Gamma} \nabla\eta \cdot \nabla\psi dz dt &+ \int_0^T \int_{\Gamma} \partial_t \eta \partial_t \psi dz dt + \int_{\Gamma} v_0 \psi(0) dz + \int_0^T \left(\int_{\Gamma} \psi dz \right) F_{ext}(t) dt.\end{aligned}$$

Putting this all together, we get that

$$\begin{aligned}-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_{\Gamma} \partial_t \eta \partial_t \psi dz dt \\ + \int_0^T \int_{\Gamma} \nabla\eta \cdot \nabla\psi dz dt = \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt \\ + \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_{\Gamma} v_0 \psi(0) dz + \int_0^T \left(\int_{\Gamma} \psi dz \right) F_{ext}(t) dt,\end{aligned}$$

where we used the fact that $P_{in/out}(t) = p$ on $\Gamma_{in/out}$.

Now, we formally substitute $F_{ext}(t) = \dot{W}(t)$, into the definition of the deterministic weak solution, to get that the term containing $F_{ext}(t)$ can be interpreted in the stochastic case as:

$$\int_0^T \left(\int_\Gamma \psi dz \right) dW(t).$$

Since W is a one dimensional Brownian motion and since $\int_\Gamma \psi dz$ is a deterministic function in time, we can interpret this term as a stochastic integral.

Before we give the definition of a weak solution to the stochastic FSI problem above, we recall the definition of a **stochastic basis**. A **stochastic basis** $\mathcal{S}$ is an ordered quintuple (see [128] for the notation)

$$\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P}, W),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a complete filtration with respect to this probability space, and W is a one-dimensional Brownian motion on the probability space with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, meaning that: (1) W has continuous paths, almost surely, (2) W is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, and (3) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t \geqslant s$ and $W(t) - W(s) \sim N(0, t-s)$ for all $0 \leqslant s \leqslant t$, where N denotes the normal distribution.

We will define two notions of solution: (1) a weak solution in a probabilistically weak sense, and (2) a weak solution in a probabilistically strong sense. The second one is stronger than the first, but we will need the first to be able to prove the existence of a weak solution in a probabilistically strong sense.

Definition 4.4.1. An ordered triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{u}}, \tilde{\eta})$ is a *weak solution in a probabilistically* ***weak*** *sense* if there exists a stochastic basis

$$\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geqslant 0}, \tilde{\mathbb{P}}, \tilde{W})$$

and $(\tilde{\boldsymbol{u}}, \tilde{\eta}) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$, which satisfies:

- $(\tilde{\boldsymbol{u}}, \tilde{\eta})$ is adapted to the filtration $\{\tilde{\mathcal{F}}_t\}_{t\geqslant 0}$,
- $\tilde{\eta}(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \tilde{\eta} \partial_t \psi dz dt$$
$$+ \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt = \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt$$
$$+ \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W},$$

almost surely.

Definition 4.4.2. An ordered pair $(\boldsymbol{u}, \eta)$ is a *weak solution in a probabilistically* ***strong*** *sense* if $(\boldsymbol{u}, \eta) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$, satisfies:

- $(\boldsymbol{u}, \eta)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$
- $\eta(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$
\begin{aligned}
-\int_0^T\int_{\Omega_f} \boldsymbol{u}\cdot\partial_t\boldsymbol{q}\,d\boldsymbol{x}dt + 2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt - \int_0^T\int_\Gamma \partial_t\eta\partial_t\psi dzdt \\
+\int_0^T\int_\Gamma \nabla\eta\cdot\nabla\psi dzdt = \int_0^T P_{in}(t)\left(\int_{\Gamma_{in}} q_z dr\right)dt - \int_0^T P_{out}(t)\left(\int_{\Gamma_{out}} q_z dr\right)dt \\
+\int_{\Omega_f}\boldsymbol{u_0}\cdot\boldsymbol{q}(0)d\boldsymbol{x} + \int_\Gamma v_0\psi(0)dz + \int_0^T\left(\int_\Gamma \psi dz\right)dW.
\end{aligned}
$$

almost surely.

In a probabilistically strong solution as in the second definition above, we have a random solution satisfying the initial conditions on the originally given (arbitrary) probability space with a one dimensional Brownian motion with respect to a complete filtration. In a probabilistically weak solution, we have a weaker requirement that the random solution exists on a *particular (not arbitrary) probability space*, where the initial conditions are satisfied "in law". We will show the existence of a weak solution in the probabilistically strong sense. However, to get to that solution, we will first show existence of a convergent subsequence of probability measures corresponding to the laws of the approximate solutions, then construct a weak solution in the probabilistically weak sense using the Skorokhod representation theorem, and then use the Gyöngy-Krylov argument [88] to get to a weak solution in the probabilistically strong sense.

The main result of this work is stated in the following theorem.

Theorem 4.4.1 (Main Result). Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H_0^1(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0,\infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a given complete filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense to the given stochastic fluid-structure interaction problem (4.5)–(4.7).

4.5 A priori energy estimate

We derive a formal energy estimate by assuming that the solution is pathwise regular enough to justify the integration by parts. We use $\|\cdot\|_{L^2(\Gamma)}$ and $(\cdot,\cdot)$ to denote the norm and inner product on $L^2(\Gamma)$, and $\|\cdot\|_{L^2(\Omega_f)}$ and $\langle\cdot,\cdot\rangle$ to denote the norm and inner product on $L^2(\Omega_f)$.

We define the total energy at time T by

$$E(T) := \frac{1}{2}\int_\Gamma |\nabla\eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x} = \frac{1}{2}\left(\|\nabla\eta\|^2_{L^2(\Gamma)} + \|v\|^2_{L^2(\Gamma)} + \|\boldsymbol{u}\|^2_{L^2(\Omega_f)}\right),$$

and the total dissipation by time T by

$$D(T) = \int_0^T \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}.$$

To estimate the total energy and dissipation for the stochastic processes $\boldsymbol{u}$, v and η, we rewrite the stochastic fluid-structure interaction problem in the following stochastic differential formulation:

$$\begin{aligned} d\eta &= v dt, \\ dv &= (\Delta\eta - \boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r})dt + dW, \\ d\boldsymbol{u} &= (\nabla\cdot\boldsymbol{\sigma})dt. \end{aligned}$$

Notice that the first equation implies $d(\nabla\eta) = (\nabla v)dt$. To obtain an energy estimate, we first apply Itö's formula to express the differentials of the L^2-norms of the stochastic processes that define the total energy of the problem:

$$\begin{aligned} d(\|\nabla\eta\|^2_{L^2(\Gamma)}) &= 2(\nabla\eta, \nabla v)dt, \\ d(\|v\|^2_{L^2(\Gamma)}) &= [2(\Delta\eta, v) - 2(\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r}, v) + L]dt + 2(1, v)dW, \\ d(\|\boldsymbol{u}\|^2_{L^2(\Omega_f)}) &= 2\langle\nabla\cdot\boldsymbol{\sigma}, \boldsymbol{u}\rangle dt. \end{aligned}$$

By adding these equations together, we obtain that the differential of the total energy satisfies:

$$d(\|\nabla\eta\|^2_{L^2(\Gamma)} + \|v\|^2_{L^2(\Gamma)} + \|\boldsymbol{u}\|^2_{L^2(\Omega_f)}) = [2\langle\nabla\cdot\boldsymbol{\sigma}, \boldsymbol{u}\rangle - 2(\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r}, v) + L]dt + 2(1, v)dW,$$

where we have used that $(\Delta\eta, v) = -(\nabla\eta, \nabla v)$ under the assumption that η and v are smooth and vanish at the endpoints of Γ. Recalling the kinematic coupling condition $\boldsymbol{u}|_\Gamma = v$, we obtain that

$$\int_{\Omega_f} (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{u} d\boldsymbol{x} = \int_{\Gamma_{in}} p u_z dr - \int_{\Gamma_{out}} p u_z dr + \int_\Gamma (\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r}) v dz - 2\mu\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x},$$

which implies

$$\begin{aligned} &d\left(\frac{1}{2}\int_\Gamma |\nabla\eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}\right) \\ &\qquad = \left(\frac{L}{2} - 2\mu\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_{\Gamma_{in}} p u_z dr - \int_{\Gamma_{out}} p u_z dr\right) dt + \left(\int_\Gamma v dz\right) dW. \end{aligned}$$

Therefore, after integration, for all $T \geqslant 0$, we have

$$E(T) + 2\mu \int_0^T \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} dt$$
$$= E_0 + \frac{LT}{2} + \int_0^T \int_{\Gamma_{in}} P_{in}(t) u_z dr dt - \int_0^T \int_{\Gamma_{out}} P_{out}(t) u_z dr dt + \int_0^T \left(\int_\Gamma v dz \right) dW. \quad (4.15)$$

We estimate the terms on the right hand side of (4.15) as follows. For the pressure term we use Hölder's inequality, the trace inequality, Poincaré's inequality, and Korn's inequality [109] to get

$$\left| \int_0^T \left(\int_{\Gamma_{in}} u_z dr \right) P_{in}(t) dt \right| \leqslant C \left| \int_0^T \left(\int_{\Gamma_{in}} |u_z|^2 dr \right)^{1/2} P_{in}(t) dt \right| \leqslant C \left| \int_0^T ||\nabla \boldsymbol{u}||_{L^2(\Omega_f)} P_{in}(t) dt \right|$$
$$\leqslant C \left| \int_0^T ||\boldsymbol{D}(\boldsymbol{u})||_{L^2(\Omega_f)} P_{in}(t) dt \right| \leqslant C(D(T))^{1/2} ||P_{in}(t)||_{L^2(0,T)} \leqslant \epsilon D(T) + C(\epsilon) ||P_{in}(t)||^2_{L^2(0,T)}. \quad (4.16)$$

We note that the constant $C(\epsilon)$ depends only on ϵ and the parameters of the problem. The same computation holds for the outlet pressure.

For the stochastic integral, we bound the expectation $\mathbb{E}\left(\max_{0 \leqslant \tau \leqslant T} \left| \int_0^\tau \left(\int_\Gamma \partial_t \eta dz \right) dW \right|\right)$ since the final energy estimate will be given in terms of expectation of the total energy and dissipation at time T. To bound this quantity, we use the Burkholder-Davis-Gundy (BDG) inequality (see Theorem 2.3.1) under the assumption that the process $\partial_t \eta$ is a predictable stochastic process with respect to the given filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$:

$$\mathbb{E}\left(\max_{0 \leqslant s \leqslant T} \left| \int_0^s \left(\int_\Gamma \partial_t \eta dz \right) dW \right| \right) \leqslant \mathbb{E}\left(\left| \int_0^T \left(\int_\Gamma \partial_t \eta dz \right)^2 dt \right|^{1/2} \right) \leqslant C\mathbb{E}\left(\left| \int_0^T ||\partial_t \eta||^2_{L^2(\Gamma)} dt \right|^{1/2} \right)$$
$$\leqslant C \left(\mathbb{E} \left| \int_0^T ||\partial_t \eta||^2_{L^2(\Gamma)} dt \right| \right)^{1/2} \leqslant CT^{1/2} \cdot \left[\mathbb{E}\left(\max_{0 \leqslant t \leqslant T} ||\partial_t \eta(t, \cdot)||^2_{L^2(\Gamma)} \right) \right]^{1/2}$$
$$\leqslant C(\epsilon) T + \epsilon \mathbb{E}\left(\max_{0 \leqslant t \leqslant T} ||\partial_t \eta(t, \cdot)||^2_{L^2(\Gamma)} \right) \leqslant C(\epsilon) T + \epsilon \mathbb{E}\left(\max_{0 \leqslant t \leqslant T} E(t) \right). \quad (4.17)$$

Now, we first use (4.16) in (4.15) to obtain

$$E(T) + 2\mu D(T) \leqslant E(0) + \frac{LT}{2} + 2\epsilon D(T) + C(\epsilon) \left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)} \right) + \int_0^T \left(\int_\Gamma v dz \right) dW,$$

and then choose $\epsilon < \frac{\mu}{2}$ and $\epsilon < \frac{1}{2}$ to get

$$E(T) + \mu D(T) \leqslant E(0) + \frac{LT}{2} + C(\epsilon) \left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)} \right) + \int_0^T \left(\int_\Gamma v dz \right) dW.$$

Taking a maximum over times $t \in [0, T]$, taking an expectation, and then using the estimate in (4.17), we obtain the following *a priori* energy estimate for the coupled problem

(4.5)–(4.7):

$$\mathbb{E}\left(\max_{0\leqslant t\leqslant T} E(t) + \mu\int_0^t\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}ds\right) \leqslant C\left(T + E(0) + ||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right),$$

where C is independent of T, depending only on the parameters of the problem.

Remark 4.5.1. The right hand side of the energy estimate shows the four sources of energy input into the system: $E(0)$ represents the initial kinetic and potential energy, the two final terms represent the energy input from the inlet and outlet pressure, and CT represents the energy input from the stochastic forcing on the structure.

4.6 The splitting scheme

To prove the existence of a weak solution to the given stochastic FSI problem we adapt a Lie operator splitting scheme that was first designed in the context of nonlinear fluid-structure interaction by Muha and Čanić in [140]. See also [86]. To modularize the problem which now involves a fluid, structure, and stochastic effects, we use a stochastic splitting introduced in [20], which has been used in stochastic differential equations to split stochastic effects from all other deterministic effects, and combine it with classical fluid-structure splitting such as the one introduced in [140]. We design a three part splitting scheme that involves a structure subproblem, a stochastic subproblem, and a fluid subproblem, **which gives rise to a stable and convergent scheme**, as we show below.

Given a fixed time $T > 0$, for each positive integer N, let $\Delta t = \frac{T}{N}$ denote the associated time step, and let $t_N^n = n\Delta t$ denote the discrete times for $n = 0, 1, ..., N-1, N$. At each time step, we update the following vector using a three step method described below:

$$\boldsymbol{X}_N^{n+\frac{i}{3}} = \left(\boldsymbol{u}_N^{n+\frac{i}{3}}, v_N^{n+\frac{i}{3}}, \eta_N^{n+\frac{i}{3}}\right)^T, \quad n = 0, 1, ..., N-1, \quad i = 1, 2, 3,$$

where $i = 1$ corresponds to the result after updating the structure subproblem, $i = 2$ corresponds to the stochastic subproblem, and $i = 3$ corresponds to the fluid subproblem, with the initial data $\boldsymbol{X}_N^0 = (\boldsymbol{u}_0, v_0, \eta_0)^T$ for each N.

The structure subproblem

In this subproblem, we keep the fluid velocity fixed, so that

$$\boldsymbol{u}_N^{n+\frac{1}{3}} = \boldsymbol{u}_N^n,$$

and update the structure displacement and the structure velocity by having $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ satisfy the following first order system in weak variational form:

$$\int_\Gamma \frac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t}\phi dz = \int_\Gamma v_N^{n+\frac{1}{3}}\phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

$$\int_\Gamma \frac{v_N^{n+\frac{1}{3}} - v_N^n}{\Delta t}\psi dz + \int_\Gamma \nabla \eta_N^{n+\frac{1}{3}} \cdot \nabla \psi dz = 0, \qquad \text{for all } \psi \in H_0^1(\Gamma), \tag{4.18}$$

where this system is solved pathwise for each $\omega \in \Omega$ separately. We note that $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a random variable taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$. To verify this, we must check that it is a measurable function of the probability space.

Proposition 4.6.1. Suppose that η_N^n and v_N^n are $\mathcal{F}_{t_N^n}$ measurable random variables taking values in $H_0^1(\Gamma)$ and $L^2(\Gamma)$ respectively. Then, the structure problem (4.18) has a unique solution $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$, which is a random variable taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$ that is measurable with respect to $\mathcal{F}_{t_N^n}$.

Proof. Let $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ be the deterministic linear map that sends deterministic data $(\eta_N^n, v_N^n) \in H_0^1(\Gamma) \times L^2(\Gamma)$ to the unique solution $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}}) \in H_0^1(\Gamma) \times H_0^1(\Gamma)$ satisfying the weak formulation (4.18) as a deterministic problem. We must show that this deterministic linear map $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a continuous (or equivalently, bounded) linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$. To do this, we must show that for given deterministic functions η_N^n and v_N^n in $H_0^1(\Gamma)$ and $L^2(\Gamma)$, there is a unique solution $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ to the above problem in $H_0^1(\Gamma) \times H_0^1(\Gamma)$, and that the solution map is a bounded linear map.

The existence of a unique weak solution follows from the Lax-Milgram lemma. Namely, by plugging the first equation in (4.18) into the second equation, we see that $\eta_N^{n+\frac{1}{3}}$ must satisfy the following weak formulation:

$$\int_\Gamma \eta_N^{n+\frac{1}{3}} \psi dz + (\Delta t)^2 \int_\Gamma \nabla \eta_N^{n+\frac{1}{3}} \cdot \nabla \psi dz = (\Delta t) \int_\Gamma v_N^n \psi dz + \int_\Gamma \eta_N^n \psi dz, \qquad \text{for all } \psi \in H_0^1(\Gamma). \tag{4.19}$$

The bilinear form $B : H_0^1(\Gamma) \times H_0^1(\Gamma) \to \mathbb{R}$ (defined by the left-hand side)

$$B(\eta, \psi) := \int_\Gamma \eta \psi dz + (\Delta t)^2 \int_\Gamma \nabla \eta \cdot \nabla \psi dz,$$

is clearly coercive and continuous, and furthermore, for any fixed but arbitrary $\eta_N^n \in H_0^1(\Gamma)$ and $v_N^n \in L^2(\Gamma)$, the map

$$\psi \to (\Delta t) \int_\Gamma v_N^n \psi dz + \int_\Gamma \eta_N^n \psi dz$$

is a linear functional on $H_0^1(\Gamma)$. So the existence of a unique $\eta_N^{n+\frac{1}{3}} \in H_0^1(\Gamma)$ satisfying the weak formulation above in (4.19) is given by the Lax-Milgram lemma applied to $H_0^1(\Gamma)$. One then recovers

$$v_N^{n+\frac{1}{3}} = \frac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t} \in H_0^1(\Gamma).$$

To show that the linear map $F^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a bounded linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$, we note that by substituting $\psi = \eta_N^{n+\frac{1}{3}}$ in (4.19), we obtain

$$||\eta_N^{n+\frac{1}{3}}||^2_{H_0^1(\Gamma)} \leqslant C_N \left((\Delta t) \int_\Gamma v_N^n \cdot \eta_N^{n+\frac{1}{3}} dz + \int_\Gamma \eta_N^n \cdot \eta_N^{n+\frac{1}{3}} dz \right)$$
$$\leqslant C_N \left(||\eta_N^n||_{H_0^1(\Gamma)} + ||v_N^n||_{L^2(\Gamma)} \right) ||\eta_N^{n+\frac{1}{3}}||_{H_0^1(\Gamma)}.$$

Hence, $||\eta_N^{n+\frac{1}{3}}||_{H_0^1(\Gamma)} \leqslant C_N \left(||\eta_N^n||_{H_0^1(\Gamma)} + ||v_N^n||_{L^2(\Gamma)} \right)$ for a constant C_N depending only on N.

Then, by the fact that $v_N^{n+\frac{1}{3}} = \dfrac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t}$, we also have

$$||v_N^{n+\frac{1}{3}}||_{H_0^1(\Gamma)} \leqslant C_N \left(||\eta_N^n||_{H_0^1(\Gamma)} + ||v_N^n||_{L^2(\Gamma)} \right).$$

Thus, $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a bounded linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$, and so the result of the structure subproblem, which consists of the random functions $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}}) = F_N^n \circ (\eta_N^n, v_N^n)$, is a pair of $\mathcal{F}_{t_N^n}$ measurable random variables, taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$. □

To show that the approximate solutions defined by the subproblems converge to the weak solution of the continuous problem as $\Delta t \to 0$, we will need uniform bounds on the approximating sequences, which will follow from the uniform bounds on the discrete energy of the problem. For this purpose, we define the discrete energy at time t_n by

$$E_N^{n+\frac{i}{3}} = \frac{1}{2} \left(\int_{\Omega_f} |\boldsymbol{u}_N^{n+\frac{i}{3}}|^2 d\boldsymbol{x} + ||v_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} + ||\nabla \eta_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} \right), \tag{4.20}$$

and we define the fluid dissipation at time t_n by

$$D_N^n = (\Delta t) \mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^n)|^2 d\boldsymbol{x}. \tag{4.21}$$

We emphasize that these are random variables.

Proposition 4.6.2. The following discrete energy equality is satisfied pathwise:

$$E_N^{n+\frac{1}{3}} + \frac{1}{2} \left(||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} \right) + \frac{1}{2} \left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)} \right) = E_N^n.$$

Proof. Because $v_N^{n+\frac{1}{3}} \in H_0^1(\Gamma)$, we can substitute $\psi = v_N^{n+\frac{1}{3}}$ in the weak formulation to obtain that pathwise,

$$\int_\Gamma (v_N^{n+\frac{1}{3}} - v_N^n) \cdot v_N^{n+\frac{1}{3}} dz + (\Delta t) \int_\Gamma \nabla \eta_N^{n+\frac{1}{3}} \cdot \nabla v_N^{n+\frac{1}{3}} dz = 0.$$

By using the identity $(a-b)\cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$, along with the fact that $v_N^{n+\frac{1}{3}} = \frac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t}$, we obtain that the following identity holds pathwise:

$$\frac{1}{2}||v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla\eta_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{2}||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla\eta_N^{n+\frac{1}{3}} - \nabla\eta_N^n||^2_{L^2(\Gamma)}$$
$$= \frac{1}{2}||v_N^n||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla\eta_N^n||^2_{L^2(\Gamma)}.$$

The result follows once we note that $\boldsymbol{u}_N^{n+\frac{1}{3}} = \boldsymbol{u}_N^n$. □

The stochastic subproblem

In this subproblem, we incorporate only the effects of the stochastic forcing, which appears in only the structure equation. In this step, we keep the structure displacement and fluid velocity fixed

$$\eta_N^{n+\frac{2}{3}} = \eta_N^{n+\frac{1}{3}}, \qquad \boldsymbol{u}_N^{n+\frac{2}{3}} = \boldsymbol{u}_N^{n+\frac{1}{3}},$$

and only update the structure velocity as

$$v_N^{n+\frac{2}{3}} = v_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]. \tag{4.22}$$

In particular, we are splitting the stochastic part of the structure problem from the deterministic part. This is necessary to obtain a stable scheme. We state the following simple proposition.

Proposition 4.6.3. Suppose that $v_N^{n+\frac{1}{3}}$ is an $\mathcal{F}_{t_N^n}$ measurable random variable taking values in $H_0^1(\Gamma)$. Then, $v_N^{n+\frac{2}{3}}$ is an $\mathcal{F}_{t_N^{n+1}}$ measurable random variable taking values in $H^1(\Gamma)$.

Notice that the solution $v_N^{n+\frac{2}{3}}$ to the stochastic subproblem taking values in $H^1(\Gamma)$, satisfies pathwise the following integral equality, which will be useful later:

$$\int_\Gamma \frac{v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}}{\Delta t}\psi dz = \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t}\psi dz, \qquad \text{for all } \psi \in H_0^1(\Gamma). \tag{4.23}$$

Proposition 4.6.4. The following discrete energy identity holds pathwise:

$$E_N^{n+\frac{2}{3}} = E_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]\int_\Gamma v_N^{n+\frac{1}{3}} dz + \frac{L}{2}[W((n+1)\Delta t) - W(n\Delta t)]^2,$$

Proof. From $v_N^{n+\frac{2}{3}} = v_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]$, we get that

$$\frac{1}{2}|v_N^{n+\frac{2}{3}}|^2 = \frac{1}{2}|v_N^{n+\frac{1}{3}}|^2 + v_N^{n+\frac{1}{3}}\cdot[W((n+1)\Delta t) - W(n\Delta t)] + \frac{1}{2}[W((n+1)\Delta t) - W(n\Delta t)]^2.$$

Therefore, after integrating over Γ, one gets the desired energy equality, after recalling that η and $\boldsymbol{u}$ do not change in this subproblem. □

The fluid subproblem

In this subproblem, we keep the structure displacement fixed

$$\eta_N^{n+1} = \eta_N^{n+\frac{2}{3}},$$

and update the fluid and structure velocities. To define the problem satisfied by the fluid and structure velocities, we introduce the following notation for the corresponding fixed time function spaces:

$$\mathcal{V} = \{(\boldsymbol{u}, v) \in \mathcal{V}_F \times L^2(\Gamma) : \boldsymbol{u}|_\Gamma = v\boldsymbol{e_r}\}, \quad \mathcal{Q} = \{(\boldsymbol{q}, \psi) \in \mathcal{V}_F \times H_0^1(\Gamma) : \boldsymbol{q}|_\Gamma = \psi\boldsymbol{e}_r\},$$

where $\mathcal{V}_F$ is defined by (4.8). Note that the definition of $\mathcal{V}$ and $\mathcal{Q}$ does not depend on N or n.

Then, the fluid subproblem is to find $(\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ taking values in $\mathcal{V}$ pathwise, such that

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{2}{3}}}{\Delta t} \psi dz$$
$$= P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \quad \forall (\boldsymbol{q}, \psi) \in \mathcal{Q}, \quad (4.24)$$

pathwise for each outcome $\omega \in \Omega$, where $P_{N,in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt$.

Proposition 4.6.5. Suppose that $\boldsymbol{u}_N^{n+\frac{2}{3}}$ and $v_N^{n+\frac{2}{3}}$ are $\mathcal{F}_{t_N^{n+1}}$ measurable random variables taking values in $\mathcal{V}_F$ and $H^1(\Gamma)$ respectively. Then, the fluid subproblem (4.24) has a unique solution $(\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ that is an $\mathcal{F}_{t_N^{n+1}}$ measurable random variable taking values in $\mathcal{V}$.

Proof. We establish this result using the Lax-Milgram lemma. Let $T_N^n : \mathcal{V}_F \times H^1(\Gamma) \times \mathbb{R} \times \mathbb{R} \to \mathcal{V}$ be the deterministic map that sends deterministic data $(\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}}, P_{N,in}^n, P_{N,out}^n) \in \mathcal{V}_F \times H^1(\Gamma) \times \mathbb{R} \times \mathbb{R}$ to the unique solution $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$ satisfying the deterministic form of the weak formulation (4.24). We want to show that the deterministic linear map $T_N^n : (\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}}, P_{N,in}^n, P_{N,out}^n) \to (\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ is a continuous map. We start by showing that the bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ given by

$$B((\boldsymbol{u}, v), (\boldsymbol{q}, \psi)) = \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu(\Delta t) \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma v\psi dz,$$

is coercive and continuous. Coercivity follows from the Korn equality (see for example, Lemma 6 on pg. 377 in [42]), applied to

$$B((\boldsymbol{u}, v), (\boldsymbol{u}, v)) = \int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x} + 2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_\Gamma v^2 dz,$$

to obtain $||\nabla \boldsymbol{u}||^2_{L^2(\Omega_f)} = 2||\boldsymbol{D}(\boldsymbol{u})||^2_{L^2(\Omega_f)}$. Continuity of the bilinear form B follows from an application of the Cauchy-Schwarz inequality.

Next, one can verify that the map sending

$$(\boldsymbol{q}, \psi) \to \int_{\Omega_f} \boldsymbol{u}_N^{n+\frac{2}{3}} \cdot \boldsymbol{q} d\boldsymbol{x} + \int_{\Gamma} v_N^{n+\frac{2}{3}} \psi dz + (\Delta t) \left(P^n_{N,in} \int_0^R (q_z)|_{z=0} dr - P^n_{N,out} \int_0^R (q_z)|_{z=L} dr \right),$$

is a continuous linear functional on $\mathcal{V}$. Thus, the existence of a unique $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$ satisfying (4.24) with the larger space of test functions $(\boldsymbol{q}, \psi) \in \mathcal{V}$ is guaranteed by the Lax-Milgram lemma. Note that $\mathcal{V}$ is a larger space than the space $\mathcal{Q}$ required for the test functions in the fluid subproblem (4.24). However, we still have the desired uniqueness of the solution in $\mathcal{V}$ if we restrict the test functions to $\mathcal{Q}$ as in (4.24) because $\mathcal{Q}$ is dense in $\mathcal{V}$.

Then, using coercivity, the trace inequality for $\boldsymbol{u} \in H^1(\Omega_f)$, and the fact that

$$\begin{aligned} &B((\boldsymbol{u}_N^{n+1}, v_N^{n+1}), (\boldsymbol{u}_N^{n+1}, v_N^{n+1})) \\ &= \int_{\Omega_f} \boldsymbol{u}_N^{n+\frac{2}{3}} \cdot \boldsymbol{u}_N^{n+1} d\boldsymbol{x} + \int_{\Gamma} v_N^{n+\frac{2}{3}} \cdot v_N^{n+1} dz + (\Delta t) \left(P^n_{N,in} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P^n_{N,out} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr \right), \end{aligned}$$

we obtain the continuity of the map T^n.

Thus, since $\boldsymbol{u}_N^{n+\frac{2}{3}}$ and $v_N^{n+\frac{2}{3}}$ are $\mathcal{F}_{t_N^{n+1}}$ measurable by assumption, the random functions $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) = T_N^n \circ (\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}})$, which solve the fluid subproblem, are $\mathcal{F}_{t_N^{n+1}}$ measurable random variables also. □

Proposition 4.6.6. The following discrete energy identity holds pathwise:

$$\begin{aligned} E_N^{n+1} + 2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x} + \frac{1}{2} \left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \right) + \frac{1}{2} \left(||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) \\ = E_N^{n+\frac{2}{3}} + (\Delta t) \left(P^n_{N,in} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P^n_{N,out} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr \right). \end{aligned}$$

Proof. We can substitute $(\boldsymbol{q}, \psi) = (\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ into the weak formulation of the fluid subproblem since we showed in Proposition 4.6.5 that (4.24) holds more generally for test functions in $\mathcal{V}$. We obtain

$$\begin{aligned} \int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}}{\Delta t} \cdot \boldsymbol{u}_N^{n+1} d\boldsymbol{x} + 2\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x} + \int_{\Gamma} \frac{v_N^{n+1} - v_N^{n+\frac{2}{3}}}{\Delta t} \cdot v_N^{n+1} dz \\ = P^n_{N,in} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P^n_{N,out} \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr. \end{aligned}$$

The desired equality follows after multiplication by Δt, and by using the identity $(a-b)\cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$. □

The full, coupled semidiscrete problem

By adding the weak formulations of the stochastic and fluid subproblems (4.23) and (4.24), and the second equation in the structure subproblem (4.18), we have that the solution to the full semidiscrete problem is $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$, and $(v_N^{n+\frac{1}{3}}, \eta_N^{n+\frac{1}{3}}) \in H_0^1(\Gamma) \times H_0^1(\Gamma)$, satisfying the following equality pathwise:

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \psi dz$$
$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \ \forall (\boldsymbol{q}, \psi) \in \mathcal{Q},$$

$$\int_\Gamma \frac{\eta_N^{n+1} - \eta_N^n}{\Delta t} \phi dz = \int_\Gamma v_N^{n+\frac{1}{3}} \phi dz, \quad \forall \phi \in L^2(\Gamma), \tag{4.25}$$

where $P_{N,in/out}^n = \dfrac{1}{\Delta t} \displaystyle\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt$. Note that $\eta_N^{n+1} = \eta_N^{n+\frac{1}{3}}$ by the way we constructed the splitting scheme.

The following proposition provides uniform estimates on the expectation of the kinetic and elastic energy for the full, semidiscrete coupled problem (uniform in the number of time steps N, or equivalently, uniform in Δt), as well as uniform estimates on the expectation of the numerical dissipation.

Proposition 4.6.7. Let $N > 0$ and let $\Delta t = \frac{T}{N}$. There exists a constant C independent of N and depending only on the initial data, the parameters of the problem, and $||P_{in/out}||_{L^2(0,T)}^2$, such that the following uniform energy estimates hold:

1. ***Uniform semidiscrete kinetic energy and elastic energy estimates:***

$$\mathbb{E}\left(\max_{n=0,1,\ldots,N-1} E_N^{n+\frac{1}{3}}\right) \leqslant C, \quad \mathbb{E}\left(\max_{n=0,1,\ldots,N-1} E_N^{n+\frac{2}{3}}\right) \leqslant C, \quad \mathbb{E}\left(\max_{n=0,1,\ldots,N-1} E_N^{n+1}\right) \leqslant C.$$

2. ***Uniform semidiscrete viscous fluid dissipation estimate:***

$$\sum_{j=1}^N \mathbb{E}(D_N^j) \leqslant C.$$

3. ***Uniform numerical dissipation estimates:***

$$\sum_{n=0}^{N-1} \left(\mathbb{E}\left(||v_N^{n+\frac{1}{3}} - v_N^n||_{L^2(\Gamma)}^2\right) + \mathbb{E}\left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||_{L^2(\Gamma)}^2\right)\right) \leqslant C.$$

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||_{L^2(\Gamma)}^2\right) \leqslant C.$$

$$\sum_{n=0}^{N-1}\left(\mathbb{E}\left(||\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right)+\mathbb{E}\left(||v_N^{n+1}-v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right)\leqslant C.$$

Proof. First, recall the definitions of the discrete energy $E_N^{n+\frac{i}{3}}$ and the discrete fluid dissipation D_N^n from (4.20) and (4.21). We start with the second uniform numerical dissipation estimate. This estimate follows directly from the stochastic subproblem (4.22) after integration

$$\int_\Gamma |v_N^{n+\frac{2}{3}}-v_N^{n+\frac{1}{3}}|^2dz = L\cdot[W((n+1)\Delta t)-W(n\Delta t)]^2,$$

and summation of the expectations of both sides:

$$\sum_{n=0}^{N-1}\mathbb{E}\left(||v_N^{n+\frac{2}{3}}-v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)}\right)=\sum_{n=0}^{N-1}\mathbb{E}\left(L\cdot[W((n+1)\Delta t)-W(n\Delta t)]^2\right)=LT.$$

We now verify the remaining uniform energy estimates. By summing the structure, stochastic, and fluid discrete energy identities, we obtain

$$\begin{aligned}
E_N^{n+1}&+\sum_{k=0}^{n}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2d\boldsymbol{x}+\frac{1}{2}\left(||\boldsymbol{u}_N^{k+1}-\boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right)+\frac{1}{2}\left(||v_N^{k+1}-v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right)\\
&\qquad+\sum_{k=0}^{n}\left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}}-v_N^k||^2_{L^2(\Gamma)}\right)+\frac{1}{2}\left(||\nabla\eta_N^{k+\frac{1}{3}}-\nabla\eta_N^k||^2_{L^2(\Gamma)}\right)\right)\\
&=E_0+(\Delta t)\sum_{k=0}^{n}\left(P_{N,in}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr-P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right)\\
&\quad+\sum_{k=0}^{n}\left([W((k+1)\Delta t)-W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz+\frac{L}{2}[W((k+1)\Delta t)-W(k\Delta t)]^2\right),
\end{aligned}\tag{4.26}$$

$$\begin{aligned}
E_N^{n+\frac{2}{3}}&+\sum_{k=0}^{n-1}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2d\boldsymbol{x}+\frac{1}{2}\left(||\boldsymbol{u}_N^{k+1}-\boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right)+\frac{1}{2}\left(||v_N^{k+1}-v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right)\\
&\qquad+\sum_{k=0}^{n}\left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}}-v_N^k||^2_{L^2(\Gamma)}\right)+\frac{1}{2}\left(||\nabla\eta_N^{k+\frac{1}{3}}-\nabla\eta_N^k||^2_{L^2(\Gamma)}\right)\right)\\
&=E_0+(\Delta t)\sum_{k=0}^{n-1}\left(P_{N,in}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr-P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right)\\
&\qquad+\sum_{k=0}^{n}\left([W((k+1)\Delta t)-W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz+\frac{L}{2}[W((k+1)\Delta t)-W(k\Delta t)]^2\right),
\end{aligned}$$

$$
\begin{aligned}
E_N^{n+\frac{1}{3}} + \sum_{k=0}^{n-1} \left(2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x} + \frac{1}{2}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)} \right) + \frac{1}{2}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) \right) \\
+ \sum_{k=0}^{n} \left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}} - v_N^k||^2_{L^2(\Gamma)} \right) + \frac{1}{2}\left(||\nabla \eta_N^{k+\frac{1}{3}} - \nabla \eta_N^k||^2_{L^2(\Gamma)} \right) \right) \\
= E_0 + (\Delta t) \sum_{k=0}^{n-1} \left(P_{N,in}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0} dr - P_{N,out}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=L} dr \right) \\
+ \sum_{k=0}^{n-1} \left([W((k+1)\Delta t) - W(k\Delta t)] \int_\Gamma v_N^{k+\frac{1}{3}} dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2 \right),
\end{aligned}
$$

for $n = 0, 1, ..., N-1$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left(\max_{i=1,2,3} \left[\max_{n=0,1,...,N-1} E_N^{n+\frac{i}{3}} \right] \right) + \sum_{k=0}^{N-1} \left[\mathbb{E}\left(2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x} \right) + \frac{1}{2}\mathbb{E}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)} \right) \right. \\
\left. + \frac{1}{2}\mathbb{E}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) + \left(\frac{1}{2}\mathbb{E}\left(||v_N^{k+\frac{1}{3}} - v_N^k||^2_{L^2(\Gamma)} \right) + \frac{1}{2}\mathbb{E}\left(||\nabla \eta_N^{k+\frac{1}{3}} - \nabla \eta_N^k||^2_{L^2(\Gamma)} \right) \right) \right] \\
\leqslant 2E_0 + 2\mathbb{E}\left[\max_{n=0,1,...,N-1} (\Delta t) \sum_{k=0}^{n} \left(P_{N,in}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0} dr - P_{N,out}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=L} dr \right) \right] \\
+ 2\mathbb{E}\left[\max_{n=0,1,...,N-1} \sum_{k=0}^{n} \left([W((k+1)\Delta t) - W(k\Delta t)] \int_\Gamma v_N^{k+\frac{1}{3}} dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2 dr \right) \right].
\end{aligned}
$$

What is left is to bound the quantities

$$
I_1 := \mathbb{E}\left[\max_{n=0,1,...,N-1} (\Delta t) \sum_{k=0}^{n} \left(P_{N,in}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0} dr - P_{N,out}^k \int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=L} dr \right) \right],
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\max_{n=0,1,...,N-1} \sum_{k=0}^{n} \left([W((k+1)\Delta t) - W(k\Delta t)] \int_\Gamma v_N^{k+\frac{1}{3}} dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2 dr \right) \right] \\
\leqslant \mathbb{E}\left[\max_{n=0,1,...,N-1} \sum_{k=0}^{n} \left([W((k+1)\Delta t) - W(k\Delta t)] \int_\Gamma v_N^{k+\frac{1}{3}} dz \right) \right] + \frac{L}{2}\mathbb{E}\left(\sum_{k=0}^{N-1} [W((k+1)\Delta t) - W(k\Delta t)]^2 \right) \\
:= I_2 + I_3.
\end{aligned}
$$

Bound for I_1***:*** The same argument will work for $P_{N,in}^k$ and $P_{N,out}^k$ so without loss of generality, we perform the bounds below for $P_{N,in}^k$. We recall that $P_{N,in}^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} P_{in}(t)dt$, where $P_{N,in}^k$ is deterministic. Therefore, we have the following bound, for the term in I_1 that involves $P^k_{N,in}$:

$$\mathbb{E}\left[\max_{n=0,1,\dots,N-1}(\Delta t)\sum_{k=0}^{n}\left(P^k_{N,in}\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right)\right] \leqslant \mathbb{E}\left(\sum_{k=0}^{N-1}(\Delta t)|P^k_{N,in}|\left|\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right|\right)$$
$$\leqslant \sum_{k=0}^{N-1}\mathbb{E}\left[(\Delta t)\frac{1}{4\epsilon}|P^k_{N,in}|^2+\epsilon(\Delta t)\left(\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right)^2\right]$$
$$\leqslant \sum_{k=0}^{N-1}\mathbb{E}\left[\frac{1}{4\epsilon}\cdot\frac{1}{\Delta t}\left(\int_{k\Delta t}^{(k+1)\Delta t}P_{in}(t)dt\right)^2+C\epsilon(\Delta t)\int_0^R(\boldsymbol{u}_N^{k+1})_z^2|_{z=0}dr\right]$$
$$\leqslant \sum_{k=0}^{N-1}\mathbb{E}\left[\frac{1}{4\epsilon}||P_{in}||^2_{L^2(k\Delta t,(k+1)\Delta t)}+C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2d\boldsymbol{x}\right]$$
$$= \frac{1}{4\epsilon}||P_{in}||^2_{L^2(0,T)}+\sum_{k=0}^{N-1}\mathbb{E}\left(C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2d\boldsymbol{x}\right),$$

where we used Korn's inequality in the last line. Therefore,

$$I_1 \leqslant \frac{1}{4\epsilon}||P_{in}||^2_{L^2(0,T)}+\frac{1}{4\epsilon}||P_{out}||^2_{L^2(0,T)}+\sum_{k=0}^{N-1}\mathbb{E}\left(2C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2d\boldsymbol{x}\right).$$

Note that the constant C is independent of Δt and N. It is the geometric constant arising from the application of the Poincaré inequality on the fluid domain Ω_f.

Bound for I_2: Next, we examine I_2 and start with an estimate involving the absolute values:

$$I_2 \leqslant \mathbb{E}\left(\max_{n=0,1,\dots,N-1}\left|\sum_{k=0}^{n}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot[W((k+1)\Delta t)-W(k\Delta t)]\right|\right).$$

Next, we consider the expression under the absolute value sign, and consider it as the following *stochastic integral*:

$$\sum_{k=0}^{n}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot[W((k+1)\Delta t)-W(k\Delta t)] = \int_0^{(n+1)\Delta t}f(t)dW(t),$$

where $f(t)$ is the random function on $[0,T]$ defined by:

$$f(t)=\sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot 1_{(k\Delta t,(k+1)\Delta t]}(t). \tag{4.27}$$

Because $v_N^{k+\frac{1}{3}}$ is $\mathcal{F}_{t^k_N}$ measurable, this integrand is predictable. *This is a direct consequence of how we split the stochastic part of the problem from the structure subproblem.* Without such a splitting, we would not be able to make the same conclusion. Hence, since the stochastic integral is a continuous process in time, we have

$$I_2 \leqslant \mathbb{E}\left(\max_{0\leqslant s\leqslant T}\left|\int_0^s f(t)dW\right|\right).$$

Using the BDG inequality, we obtain that

$$\begin{aligned}
I_2 &\leqslant \mathbb{E}\left[\left(\int_0^T |f(t)|^2 dt\right)^{1/2}\right] = \mathbb{E}\left[(\Delta t)^{1/2}\left(\sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}} dz\right)^2\right)^{1/2}\right] \\
&\leqslant \epsilon(\Delta t)\mathbb{E}\sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}} dz\right)^2 + \frac{1}{4\epsilon} \leqslant \epsilon L(\Delta t)\mathbb{E}\sum_{k=0}^{N-1}||v_N^{k+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{4\epsilon} \\
&\leqslant \epsilon LN(\Delta t)\mathbb{E}\left(\max_{k=0,1,\dots,N-1}||v_N^{k+\frac{1}{3}}||^2_{L^2(\Gamma)}\right) + \frac{1}{4\epsilon} \leqslant 2\epsilon LN(\Delta t)\mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{1}{3}}\right) + \frac{1}{4\epsilon}.
\end{aligned}$$

Bound for I_3***:*** Finally, by using the properties of Brownian motion, we immediately deduce that

$$I_3 := \frac{L}{2}\mathbb{E}\left(\sum_{k=0}^{N-1}[W((k+1)\Delta t) - W(k\Delta t)]^2\right) = \frac{LT}{2}.$$

Conclusion: From the above estimates, we conclude that

$$\begin{aligned}
&\mathbb{E}\left(\max_{i=1,2,3}\left[\max_{n=0,1,\dots,N-1} E_N^{n+\frac{i}{3}}\right]\right) + \sum_{k=0}^{N-1}\Bigg[\mathbb{E}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right) + \frac{1}{2}\mathbb{E}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) \\
&\quad + \frac{1}{2}\mathbb{E}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right) + \left(\frac{1}{2}\mathbb{E}\left(||v_N^{k+\frac{1}{3}} - v_N^k||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\mathbb{E}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^k||^2_{L^2(\Gamma)}\right)\right)\Bigg] \\
&\leqslant 2E_0 + \frac{1}{2\epsilon}||P_{in}||^2_{L^2(0,T)} + \frac{1}{2\epsilon}||P_{out}||^2_{L^2(0,T)} + \sum_{k=0}^{N-1}\mathbb{E}\left(4C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right) \\
&\qquad + 4\epsilon LT\cdot\mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{1}{3}}\right) + \frac{1}{2\epsilon} + LT.
\end{aligned}$$

We note that the constant C depends only on the fluid domain Ω_f and not on Δt or N. The result follows once we fix $\epsilon > 0$, independent of Δt, such that $4C\epsilon < \mu$ and $4\epsilon LT < \frac{1}{2}$, and move the associated terms from the right hand side to the left hand side. We emphasize that this gives a uniform energy estimate because the choice of ϵ is independent of Δt and hence N.

□

Remark 4.6.1. We remark that it is in these energy estimates that one can see the importance of using our particular splitting strategy to obtain a stable scheme. Namely, this splitting strategy enabled us to estimate the terms involving the white noise as stochastic integrals, such as the second to last term in estimate (4.26). Because $v_N^{k+\frac{1}{3}}$ is $\mathcal{F}_{t_N^k}$ measurable, the stochastic increment $[W((k+1)\Delta t) - W(k\Delta t)]$ is independent of the integral of $v_N^{k+\frac{1}{3}}$, and hence, we were able to rewrite this term as a stochastic integral, see (4.27).

4.7 Approximate solutions

We use the solutions at fixed times of our semidiscrete scheme, $\boldsymbol{u}_N^{n+\frac{i}{3}}$, $\eta_N^{n+\frac{i}{3}}$, and $v_N^{n+\frac{i}{3}}$ for $i = 1, 2, 3$, to create approximate solutions for the given stochastic FSI problem in time on the time interval $[0, T]$, for each N, which we will need to pass to the limit as $\Delta t \to 0$. The approximate solutions will be defined as piecewise functions in time. However, we must be careful in this construction of approximate solutions to make sure that they are adapted to the given filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ associated to the given Brownian motion.

Definition of approximate solutions

We start with the fluid. We define the approximate random function $\boldsymbol{u}_N$ on $[0, T] \times \Omega_f$ to be the piecewise constant function

$$\boldsymbol{u}_N(t, \cdot) = \boldsymbol{u}_N^{n-1}, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t].$$

Note that because $\boldsymbol{u}_N^n$ is $\mathcal{F}_{t_N^n}$ measurable, the choice of $\boldsymbol{u}_N^{n-1}$ instead of $\boldsymbol{u}_N^n$ above is used so that the resulting process $\boldsymbol{u}_N$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$.

Next, we consider the functions associated with the structure. Note that η_N^n, $\eta_N^{n+\frac{1}{3}}$, and $v_N^{n+\frac{1}{3}}$ are $\mathcal{F}_{t_N^n}$ measurable while $v_N^{n+\frac{2}{3}}$ is $\mathcal{F}_{t_N^{n+1}}$ measurable. It turns out that we will not need to keep track of $v_N^{n+\frac{2}{3}}$ when passing to the limit, since it does not appear in (4.25). So it suffices to define

$$\eta_N(t, \cdot) = \eta_N^{n-1}, \qquad v_N(t, \cdot) = v_N^{n-1}, \qquad v_N^*(t, \cdot) = v_N^{n-\frac{2}{3}}, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t],$$

and these are all adapted to the given filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. Note that v_N defined on $[0, T] \times \Gamma$ is pathwise the trace of the fluid velocity $\boldsymbol{u}_N$ defined on $[0, T] \times \Omega_f$ for all $t \in [0, T]$, but this is not true for v_N^*, since v_N^* is the structure velocity obtained after the structure subproblem in the semidiscrete scheme, which does not update the fluid velocity directly.

We also introduce a piecewise linear interpolation $\overline{\eta}_N$ of η_N to add additional regularity to the structure displacement, since we will want the structure displacement to be in $W^{1,\infty}(0, T; L^2(\Gamma))$ almost surely in the limit as $\Delta t \to 0$. Thus, $\overline{\eta}_N$ is piecewise linear such that

$$\overline{\eta}_N(n\Delta t) = \eta_N^n, \qquad \text{for } n = 0, 1, ..., N. \tag{4.28}$$

Note that $\overline{\eta}_N$ has Lipschitz continuous paths in time, and furthermore,

$$\partial_t \overline{\eta}_N = v_N^*. \tag{4.29}$$

Because both η_N^n and η_N^{n+1} are $\mathcal{F}_{t_N^n}$ adapted, $\overline{\eta}_N$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. We will also introduce a piecewise constant function $\eta_N^{\Delta t}$ for the structure displacement, given by

$$\eta_N^{\Delta t}(t, \cdot) = \eta_N^n, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t]. \tag{4.30}$$

Note that $\eta_N^{\Delta t}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and is a time-shifted version of η_N, which is emphasized in the notation by the superscript of Δt. This time-shifted structure displacement will be useful for passing to the limit in Section 4.8.

We will also consider the corresponding piecewise linear interpolations for the fluid velocity and structure velocity, which satisfy

$$\overline{\boldsymbol{u}}_N(n\Delta t) = \boldsymbol{u}_N^n, \qquad \overline{v}_N(n\Delta t) = v_N^n, \qquad \text{for } n = 0, 1, ..., N. \tag{4.31}$$

We will need to consider $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$ because we will express the discrete time derivatives $\frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t}$ and $\frac{v_N^{n+1} - v_N^n}{\Delta t}$ in the semidiscrete formulation (4.25) in terms of the time derivatives of $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$. We will also need to consider piecewise constant time-shifted functions $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$ for the fluid velocity and the structure velocity, defined by

$$\boldsymbol{u}_N^{\Delta t}(t, \cdot) = \boldsymbol{u}_N^n, \qquad v_N^{\Delta t}(t, \cdot) = v_N^n, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t]. \tag{4.32}$$

We note that $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$ are time-shifted versions of $\boldsymbol{u}_N$ and v_N. We will need these time-shifted functions because the fluid dissipation estimate in Proposition 4.6.7 implies that $\boldsymbol{u}_N^{\Delta t}$, rather than $\boldsymbol{u}_N$, is uniformly bounded in $L^2(\Omega; L^2(0, T; H^1(\Omega_f)))$. See Proposition 4.7.2.

We make the following important observation. Unlike $\overline{\eta}_N$, we note that $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$ are *not* necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, even though they can still be considered as random variables taking values in their appropriate path spaces. Similarly, $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$, unlike $\boldsymbol{u}_N$ and v_N, are *not* necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. However, this will not be an issue, because we will see later in Lemma 4.8.3 that $\overline{\boldsymbol{u}}_N$, $\boldsymbol{u}_N^{\Delta t}$, $\overline{v}_N$, and $v_N^{\Delta t}$ are almost surely "close to" the random processes $\boldsymbol{u}_N$ and $\boldsymbol{v}_N$, which are adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, as $N \to \infty$ along a subsequence.

We summarize some of the previously discussed measure theoretic properties of the stochastic approximate solutions in the following proposition, for future reference.

Proposition 4.7.1. Recall that W is a one dimensional Brownian motion with respect to the probability space with complete filtration, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. For all $N \in \mathbb{N}$, $\boldsymbol{u}_N$, v_N, v_N^*, η_N, and $\overline{\eta}_N$ are adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ with left continuous paths, with $\overline{\eta}_N$ having continuous paths. In addition, for some fixed $t > 0$ and for each N, define $n_0 = \lfloor \frac{t}{\Delta t} \rfloor + 1$. Then, $W_\tau - W_t$ is independent of each of the random variables in the following collection of random variables for each N and for each $\tau > t$:

$$\{\boldsymbol{u}_N^{n-1}, v_N^{n-1}, v_N^{n-\frac{2}{3}} : 1 \leqslant n \leqslant n_0\}, \{\eta_N^n : 0 \leqslant n \leqslant n_0\}, \{\overline{\eta}_N(s) : s \in [0, n_0\Delta t]\}.$$

Uniform boundedness of approximate solutions

Using the previous discrete energy estimates, we establish uniform boundedness of the approximate solutions in the following proposition. We note that in contrast to the case of deterministic FSI, the uniform boundedness of these (random) approximate solutions is only in *expectation*.

Proposition 4.7.2. The following uniform boundedness results hold:

- $(\eta_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; H_0^1(\Gamma)))$.
- $(v_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$.
- $(v_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0,T; H^{1/2}(\Gamma)))$.
- $(v_N^*)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$.
- $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; L^2(\Omega_f)))$.
- $(\boldsymbol{u}_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0,T; H^1(\Omega_f)))$.

Proof. The only part of this result that does not follow directly from Proposition 4.6.7 is to show that $(\boldsymbol{u}_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0,T; H^1(\Omega_f)))$. We compute

$$||\boldsymbol{u}_N^{\Delta t}||^2_{L^2(\Omega;L^2(0,T;H^1(\Omega_f)))} = \mathbb{E}\left(\int_0^T ||\boldsymbol{u}_N^{\Delta t}||^2_{H^1(\Omega_f)} dt\right) = (\Delta t)\mathbb{E}\left(\sum_{k=1}^N ||\boldsymbol{u}_N^k||^2_{H^1(\Omega_f)}\right)$$
$$\leqslant C(\Delta t)\mathbb{E}\left(\sum_{k=1}^N ||\boldsymbol{D}(\boldsymbol{u}_N^k)||^2_{L^2(\Omega_f)}\right).$$

The result follows from the uniform boundedness of the sum of the dissipation terms (recall that the (Δt) term is included in the definition of the energy dissipation (4.21)). By taking the trace of the r component of the fluid velocity $\boldsymbol{u}_N^n$, which is in $H^{1/2}(\Gamma)$, we get the corresponding boundedness of $(v_N^{\Delta t})_{N\in\mathbb{N}}$ in $L^2(\Omega; L^2(0,T; H^{1/2}(\Gamma)))$. □

We also state the corresponding uniform boundedness property for the linear interpolations $(\overline{\eta}_N)_{N\in\mathbb{N}}$. Note that in terms of distributional derivatives, $\partial_t\overline{\eta}_N = v_N^*$ holds pathwise for $\omega \in \Omega$. Therefore, we have:

Proposition 4.7.3. The sequence of linear interpolations of the structure displacements $(\overline{\eta}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0,T; H_0^1(\Gamma))) \cap L^2(\Omega; W^{1,\infty}(0,T; L^2(\Gamma)))$.

Remark 4.7.1. To be very precise, one must check that the stochastic approximate solutions are measurable, as random variables taking values in a given path space. The measurability of these stochastic processes is easy to see by using the measurability properties of the functions $\boldsymbol{u}_N^n$, $v_N^{n+\frac{i}{3}}$, and η_N^n. For example, η_N is measurable as a map from the probability space Ω to

$L^\infty(0,T;H_0^1(\Gamma))$ because η_N can be considered as the composition of a measurable map F_1 with a continuous map F_2. F_1 is the map from $\omega \in \Omega$ to the space of bounded sequences of length N with values in $H_0^1(\Gamma)$, given by $F_1 : \omega \to (\eta_N^0, \eta_N^1, ..., \eta_N^{N-1})$, which is measurable by the measurability properties of each η_N^n. F_2 is the map from the space of bounded sequences of length N with values in $H_0^1(\Gamma)$ to $L^\infty(0,T;H_0^1(\Gamma))$, given by $F_2 : (\eta_N^0, \eta_N^1, ..., \eta_N^{N-1}) \to \sum_{k=0}^{N-1} \eta_N^k \cdot 1_{(k\Delta t,(k+1)\Delta t]}(t)$, which is continuous.

4.8 Passage to the limit

We would like to show that our approximate solution sequences converge in a certain sense, to a weak solution of the original problem. While one could use a generalized Skorokhod-type argument for general Jakubowski spaces (see [99, 166, 167]), we employ a standard probabilistic compactness argument which has the advantage of generalizing to a broad class of more complex nonlinear stochastic FSI systems of interest. In this probabilistic compactness methodology, we will show the existence of a convergent subsequence of the *probability measures* which describe the *laws* or equivalently, the *distributions* of the approximate solutions. From here, we will eventually be able to get to *almost sure* convergence of the stochastic approximate solutions themselves.

We start by designing compactness arguments that will provide weak convergence of the probability measures describing the laws of our random approximate solutions.

Weak convergence of measures

We first show that along subsequences, the probability measures, or the laws describing the distributions of our stochastic approximate solutions constructed earlier, converge to a probability measure, as the time step $\Delta t \to 0$, or $N \to \infty$. For this purpose, we recall that we are given a probability space with complete filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, with a one dimensional Brownian motion W with respect to the given filtration. For each N, we define the probability measure (or the law) μ_N:

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\eta_N^{\Delta t}} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}} \times \mu_W, \quad (4.33)$$

defined on the phase space $\mathcal{X}$:

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^3 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^4 \times C(0,T;\mathbb{R}). \quad (4.34)$$

Here, μ_{η_N} denotes the law of η_N on $L^2(0,T;L^2(\Gamma))$, $\mu_{\boldsymbol{u}_N}$ denotes the law of $\boldsymbol{u}_N$ on $L^2(0,T;L^2(\Omega_f))$, μ_W denotes the law of W on $C(0,T;\mathbb{R})$, and so on. Thus, μ_N is the joint law of the random variables η_N, $\overline{\eta}_N$, $\eta_N^{\Delta t}$, $\boldsymbol{u}_N$, v_N, $\boldsymbol{u}_N$,v_N^*, $\overline{\boldsymbol{u}}_N$, $\overline{v}_N$, $\boldsymbol{u}_N^{\Delta t}$, $v_N^{\Delta t}$, and W. As we shall see below, it is easier to work with the fluid velocity and the structure velocity in pairs, which is the reason why in (4.33) above, we consider $(\mu_{\boldsymbol{u}_N}, \mu_{v_N})$, $(\mu_{\boldsymbol{u}_N}, \mu_{v_N^*})$, $(\mu_{\overline{\boldsymbol{u}}_N}, \mu_{\overline{v}_N})$, and $(\mu_{\boldsymbol{u}_N^{\Delta t}}, \mu_{v_N^{\Delta t}})$. The main result of this subsection is the following.

Theorem 4.8.1. Along a subsequence (which we will continue to denote by N), μ_N converges weakly as probability measures to a probability measure μ on $\mathcal{X}$.

To show weak convergence of these probability measures along a subsequence, stated in Theorem 4.8.1, we must show that the probability measures are **tight**.

Definition 4.8.1. The probability measures μ_N are **tight** if for every $\epsilon > 0$, there exists a compact set A_ϵ, compact in $\mathcal{X}$, such that

$$\mu_N(A_\epsilon) > 1 - \epsilon, \qquad \text{for all } N.$$

To get a hold of the compact subset A_ϵ, we will need the following two deterministic compactness results for the structure displacements $\{\eta_N(\omega)\}$ and for the fluid and structure velocities $\{\boldsymbol{u}_N(\omega)\}$ and $\{v_N(\omega)\}$. The two results are obtained in the following two lemmas.

The first lemma, which will be applied to the structure displacements $\{\eta_N(\omega)\}$, is a direct consequence of the classical Aubin-Lions compactness lemma [6, 129]:

Lemma 4.8.1. The following holds:

$$[W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma))] \subset\subset L^\infty(0,T;L^2(\Gamma)).$$

The Aubin-Lions compactness lemma actually gives a stronger compact embedding of $W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma))$ into $C(0,T;L^2(\Gamma))$, but since we want η_N and $\overline{\eta}_N$ to take values in the same path space, we use $L^\infty(0,T;L^2(\Gamma))$ since η_N is not continuous.

To handle the compactness argument for the structure and fluid velocities, we consider the subsets $\mathcal{K}$ and $\mathcal{K}_R$ in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$, defined as follows.

Definition 4.8.2 (Definition of $\mathcal{K}$ and $\mathcal{K}_R$). The sets $\mathcal{K}$ and $\mathcal{K}_R$ of paths (or realizations) are defined as follows. For the pathwise left continuous approximate functions $\boldsymbol{u}_N(\omega), v_N(\omega)$ on $[0,T]$, we define:

$$\mathcal{K} = \{(\boldsymbol{u}, v) \in L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) : \boldsymbol{u} = \boldsymbol{u}_N(\omega) \text{ and } v = v_N(\omega) \text{ for some } \omega \in \Omega \text{ and } N \in \mathbb{N}\}.$$

For any arbitrary positive constant R, define $\mathcal{K}_R$ to be the subset of paths $(\boldsymbol{u}_N(\omega), v_N(\omega)) \in \mathcal{K}$ where ω and N satisfy the following properties.

1. *Uniform boundedness:*

$$||(\boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t})||_{L^2(0,T;H^1(\Omega_f))\times L^2(0,T;H^{1/2}(\Gamma))} \leqslant R, \quad ||\boldsymbol{u}_N||_{L^\infty(0,T;L^2(\Omega_f))} \leqslant R,$$
$$||v_N||_{L^\infty(0,T;L^2(\Gamma))} \leqslant R, \quad ||\eta_N||_{L^\infty(0,T;H_0^1(\Gamma))} \leqslant R.$$

2. *Boundedness of numerical dissipation:*

$$\sum_{n=0}^{N-1} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \leqslant R, \quad \sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} \leqslant R,$$
$$\sum_{n=0}^{N-1} ||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} \leqslant R, \quad \sum_{n=0}^{N-1} ||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \leqslant R.$$

3. *Boundedness of fluid dissipation:*

$$(\Delta t)\sum_{n=1}^{N}\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^n)|^2 d\boldsymbol{x} \leqslant R.$$

4. *Boundedness of* $1/4$*-Hölder exponent of Brownian motion:*

$$\sup_{s,t\in[0,T],s\neq t}\frac{|W(t)-W(s)|}{|t-s|^{1/4}} \leqslant R.$$

Remark 4.8.1. In the fourth condition above, any positive Hölder exponent that is strictly less than $1/2$ would suffice, since Brownian motion is "almost" $1/2$-Hölder continuous, but we have fixed $1/4$ for concreteness.

The following lemma provides the desired compactness result for $(\boldsymbol{u}_N, v_N)$.

Lemma 4.8.2. The set $\mathcal{K}_R$ is precompact in $L^2(0,T;L^2(\Omega_f))\times L^2(0,T;L^2(\Gamma))$ for any $R>0$.

Proof. We use the Simon's compactness theorem [165, 136]. According to Simon's theorem, it suffices to check two conditions.

First condition: We show that for any $0<t_1<t_2<T$, the collection $\left\{\int_{t_1}^{t_2} f(t)dt : f\in\mathcal{K}_R\right\}$ is relatively compact in $L^2(\Omega_f)\times L^2(\Gamma)$. Consider a sequence $\{f_m(t,\cdot)\}_{m=1}^{\infty}$ in $\mathcal{K}_R$, where $f_m(t,\cdot)=(\boldsymbol{u}_m(t,\cdot), v_m(t,\cdot))$. We want to show that there is a subsequence $\left\{\int_{t_1}^{t_2} f_{m_k}(t)dt\right\}_{k=1}^{\infty}$ that converges in $L^2(\Omega_f)\times L^2(\Gamma)$.

For each m, there exists some N_m and $\omega_m\in\Omega$ (both depending on m) such that

$$\boldsymbol{u}_m(t)=\boldsymbol{u}_0\cdot 1_{t\in[0,(\Delta t)_m]}+\sum_{n=1}^{N_m-1}\boldsymbol{u}_{N_m}^n(\omega_m)\cdot 1_{t\in(n(\Delta t)_m,(n+1)(\Delta t)_m]},$$
$$v_m(t)=v_0\cdot 1_{t\in[0,(\Delta t)_m]}+\sum_{n=1}^{N_m-1}v_{N_m}^n(\omega_m)\cdot 1_{t\in(n(\Delta t)_m,(n+1)(\Delta t)_m]},$$

where $(\Delta t)_m = T/N_m$. Therefore, we have that

$$\int_{t_1}^{t_2}\boldsymbol{u}_m(t)dt=a_m\boldsymbol{u}_0+\int_{\max(t_1,(\Delta t)_m)}^{\max(t_2,(\Delta t)_m)}\boldsymbol{u}_m(t)dt,\quad \int_{t_1}^{t_2}v_m(t)dt=a_m v_0+\int_{\max(t_1,(\Delta t)_m)}^{\max(t_2,(\Delta t)_m)}v_m(t)dt,$$

where $a_m=\max(0,(\Delta t)_m-t_1)$. Because $a_m\in[0,T]$, we can find a subsequence $\{m_k\}_{k=1}^{\infty}$ such that $a_{m_k}\to a$ as $k\to\infty$, for some $a\in[0,T]$. Because $\boldsymbol{u}_0$ and v_0 are the fixed initial data for the fluid velocity and the structure velocity, $a_{m_k}\boldsymbol{u}_0$ and $a_{m_k}v_0$ converge along this subsequence in $L^2(\Omega_f)$ and $L^2(\Gamma)$.

It remains to show that the sequences in k given by

$$\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} \boldsymbol{u}_{m_k}(t)dt \qquad \text{and} \qquad \int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} v_{m_k}(t)dt \tag{4.35}$$

converge in $L^2(\Omega_f)$ and $L^2(\Gamma)$ respectively along a further subsequence. Because of the compact embedding $H^1(\Omega_f) \times H^{1/2}(\Gamma) \subset\subset L^2(\Omega_f) \times L^2(\Gamma)$, it suffices to show that the two sequences in k given in (4.35) are uniformly bounded in $H^1(\Omega_f)$ and $H^{1/2}(\Gamma)$. This can be easily verified by using the uniform boundedness property of functions in $\mathcal{K}_R$ in Definition 4.8.2:

$$\begin{aligned}
&\left|\left|\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} \boldsymbol{u}_{m_k}(t)dt\right|\right|_{H^1(\Omega_f)} + \left|\left|\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} v_{m_k}(t)dt\right|\right|_{H^{1/2}(\Gamma)} \\
&\leqslant \int_{(\Delta t)_{m_k}}^T ||\boldsymbol{u}_{m_k}(t)||_{H^1(\Omega_f)}dt + \int_{(\Delta t)_{m_k}}^T ||v_{m_k}(t)||_{H^{1/2}(\Gamma)}dt \\
&\leqslant T^{1/2}\left(\int_{(\Delta t)_{m_k}}^T ||\boldsymbol{u}_{m_k}(t)||^2_{H^1(\Omega_f)}dt\right)^{1/2} + T^{1/2}\left(\int_{(\Delta t)_{m_k}}^T ||v_{m_k}(t)||^2_{H^{1/2}(\Gamma)}dt\right)^{1/2} \leqslant 2T^{1/2}R.
\end{aligned}$$

So we can further refine the subsequence $\{m_k\}_{k=1}^\infty$ to obtain that $\left\{\int_{t_1}^{t_2} (\boldsymbol{u}_{m_k}(t), v_{m_k}(t))\, dt\right\}_{k=1}^\infty$ converges in $L^2(\Omega_f)\times L^2(\Gamma)$, where we continue to denote the refined subsequence by $\{m_k\}_{k=1}^\infty$.

Second condition: We must show that $||\tau_h f - f||_{L^2(h,T;L^2(\Omega_f)\times L^2(\Gamma))} \to 0$ uniformly for all $f = (\boldsymbol{u}, v) \in \mathcal{K}_R$, as $h \to 0$. Here τ_h for $h > 0$ denotes the time shift map $(\tau_h f)(t,\cdot) = f(t-h,\cdot)$. Consider an arbitrary $\epsilon > 0$. We want to find $h > 0$ sufficiently small such that

$$||\tau_h \boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} < \epsilon \quad \text{and} \quad ||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} < \epsilon \quad \forall (\boldsymbol{u}, v) \in \mathcal{K}_R.$$

To verify this, we can write $h = l(\Delta t) + s$, for each $\Delta t = \frac{T}{N}$, where $0 \leqslant s < \Delta t$, so that

$$\begin{aligned}
||\tau_h \boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} &\leqslant ||\tau_s\tau_{l\Delta t}\boldsymbol{u} - \tau_{l\Delta t}\boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))}, \\
||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} &\leqslant ||\tau_s\tau_{l\Delta t}v - \tau_{l\Delta t}v||_{L^2(h,T;L^2(\Gamma))} + ||\tau_{l\Delta t}v - v||_{L^2(h,T;L^2(\Gamma))}.
\end{aligned}$$

The above estimates require one estimate for the small s time shift, and one for the larger $l\Delta t$ time shift. We will handle the first time shift estimate using the numerical dissipation estimate holding for $\mathcal{K}_R$, specified in Definition 4.8.2, and we will handle the second time shift estimate using an Ehrling property.

Estimate for time shift by s: Consider arbitrary $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$. Recalling that $0 \leqslant s < \Delta t$, we compute

$$\begin{aligned}
||\tau_s\tau_{l\Delta t}\boldsymbol{u}_N - \tau_{l\Delta t}\boldsymbol{u}_N||^2_{L^2(h,T;L^2(\Omega_f))} &= s\sum_{n=0}^{N-l-2} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)} \\
&\leqslant s\sum_{n=0}^{N-1} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)} \leqslant sR,
\end{aligned}$$

where we used that $\boldsymbol{u}_N^n = \boldsymbol{u}_N^{n+\frac{2}{3}}$ and the numerical dissipation estimate in the last inequality. Similarly,

$$||\tau_s\tau_{l\Delta t}v_N - \tau_{l\Delta t}v_N||^2_{L^2(h,T;L^2(\Gamma))} = s\sum_{n=0}^{N-l-2}||v_N^{n+1}-v_N^n||^2_{L^2(\Gamma)} \leqslant s\sum_{n=0}^{N-1}||v_N^{n+1}-v_N^n||^2_{L^2(\Gamma)}$$
$$\leqslant 3s\left(\sum_{n=0}^{N-1}||v_N^{n+\frac{1}{3}}-v_N^n||^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1}||v_N^{n+\frac{2}{3}}-v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1}||v_N^{n+1}-v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)}\right) \leqslant 9sR.$$

Recalling that $h = s + l\Delta t$ so that $0 < s \leqslant h$, we can make these quantities arbitrarily small by taking h sufficiently small, since R is a fixed arbitrary positive constant.

Estimate for time shift by $l\Delta t$: Consider arbitrary $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$. We want to estimate

$$||\tau_{l\Delta t}\boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}v_N - v_N||_{L^2(h,T;L^2(\Gamma))}.$$

This is identically zero if $h < \Delta t$, so we assume for the following estimate that $h \geqslant \Delta t$. We use the chain of embeddings $H^1(\Omega_f)\times H^{1/2}(\Gamma) \subset\subset L^2(\Omega_f)\times L^2(\Gamma) \subset \mathcal{Q}'$, where $\mathcal{Q}$ is the test space defined in (4.14). Applying the uniform Ehrling property, see e.g., [154, 136], we obtain

$$||\tau_{l\Delta t}\boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}v_N - v_N||_{L^2(h,T;L^2(\Gamma))}$$
$$\leqslant 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;L^2(\Omega_f)\times L^2(\Gamma))}$$
$$\leqslant 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + \delta||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))}$$
$$+ C(\delta)||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;\mathcal{Q}')} := I_1 + I_2 + I_3.$$

To estimate I_1, we use the triangle inequality, the assumption that $h \geqslant \Delta t$, and the uniform boundedness property of $\mathcal{K}_R$ in Definition 4.8.2:

$$I_1 \leqslant 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + 2||(\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} \leqslant 8(\Delta t)^{1/2}R \leqslant 8h^{1/2}R.$$

To estimate I_2, we use the triangle inequality and the uniform boundedness property of $\mathcal{K}_R$ in Definition 4.8.2:

$$I_2 \leqslant \delta\left(||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} + ||(\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))}\right)$$
$$\leqslant 2\delta||(\boldsymbol{u}_N, v_N)||_{L^2(\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} \leqslant 2\delta R.$$

To estimate I_3, we multiply the first equation in the weak formulation (4.25) by Δt to obtain:

$$\int_{\Omega_f}(\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n)\cdot\boldsymbol{q}d\boldsymbol{x} + \int_\Gamma(v_N^{n+l} - v_N^n)\psi dz = \int_\Gamma[W((n+l)\Delta t) - W(n\Delta t)]\psi dz$$
$$+ (\Delta t)\sum_{k=1}^{l}\left(P_{N,in}^{n+k-1}\int_0^R(q_z)|_{z=0}dr - P_{N,out}^{n+k-1}\int_0^R(q_z)|_{z=L}dr\right.$$
$$\left. - 2\mu\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}_N^{n+k}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x} - \int_\Gamma\nabla\eta_N^{n+k}\cdot\nabla\psi dz\right), \qquad \forall(\boldsymbol{q},\psi)\in\mathcal{Q}.$$

We estimate the terms on the right hand side as follows. For $(\boldsymbol{q},\psi)\in\mathcal{Q}$, where $\mathcal{Q}$ is defined in (4.14), with $||(\boldsymbol{q},\psi)||_{\mathcal{Q}} \leqslant 1$, we have the following estimates.

- Using Cauchy-Schwarz and the boundedness of the 1/4-Holder exponent of Brownian motion in the definition of $\mathcal{K}_R$, see Definition 4.8.2, we obtain

$$\left|\int_\Gamma [W((n+l)\Delta t) - W(n\Delta t)]\psi dz\right| \leqslant \left(\int_\Gamma |W((n+l)\Delta t) - W(n\Delta t)|^2 dz\right)^{1/2}$$
$$\leqslant \left(\int_\Gamma \left|R(l\Delta t)^{1/4}\right|^2 dz\right)^{1/2} \leqslant C(l\Delta t)^{1/4}.$$

- We recall the definition of the discretized pressure $P^n_{N,in/out} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt$, and use the trace inequality on the integral involving q_z to obtain

$$(\Delta t)\left|\sum_{k=1}^{l} P^{n+k-1}_{N,in}\int_0^R (q_z)|_{z=0}dr\right| = (\Delta t)\left|\sum_{k=1}^{l} P^{n+k-1}_{N,in}\right| \cdot \left|\int_0^R (q_z)|_{z=0}dr\right|$$
$$\leqslant C(\Delta t)\left|\sum_{k=1}^{l} P^{n+k-1}_{N,in}\right| = C\left|\int_{n\Delta t}^{(n+l)\Delta t} P_{in}(t)dt\right|$$
$$\leqslant C(l\Delta t)^{1/2}||P_{in}||_{L^2(n\Delta t,(n+l)\Delta t)} \leqslant C(l\Delta t)^{1/2}||P_{in}||_{L^2(0,T)} = C(l\Delta t)^{1/2}.$$

The same estimate holds for the outlet pressure term.

- Using Cauchy-Schwarz and the uniform fluid dissipation estimate in Definition 4.8.2 of $\mathcal{K}_R$, we get

$$(\Delta t)\left|\sum_{k=1}^{l} 2\mu\int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}^{n+k}_N) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}\right| \leqslant C(\Delta t)\sum_{k=1}^{l}\left(\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}^{n+k}_N)|^2 d\boldsymbol{x}\right)^{1/2}$$
$$\leqslant Cl^{1/2}(\Delta t)\left(\sum_{k=1}^{l}\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}^{n+k}_N)|^2 d\boldsymbol{x}\right)^{1/2}$$
$$\leqslant Cl^{1/2}(\Delta t)\left(\sum_{k=1}^{N}\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}^{k}_N)|^2 d\boldsymbol{x}\right)^{1/2} \leqslant C(l\Delta t)^{1/2}.$$

- Using Cauchy-Schwarz and the uniform boundedness of η_N in Definition 4.8.2 of $\mathcal{K}_R$, we get:

$$(\Delta t)\left|\sum_{k=1}^{l}\int_\Gamma \nabla\eta^{n+k}_N \cdot \nabla\psi dz\right| \leqslant (\Delta t)\sum_{k=1}^{l}\int_\Gamma |\nabla\eta^{n+k}_N \cdot \nabla\psi| dz \leqslant C(\Delta t)\sum_{k=1}^{l} ||\eta^{n+k}_N||_{H^1_0(\Gamma)} \leqslant Cl(\Delta t).$$

Here, all constants C are independent of n, l, and Δt and hence N, but can depend on the fixed, arbitrary constant R, and on the given parameters of the problem. Combining all of these estimates together, we obtain that

$$||(\boldsymbol{u}^{n+l}_N, v^{n+l}_N) - (\boldsymbol{u}^n_N, v^n_N)||_{\mathcal{Q}'} \leqslant C(l\Delta t)^{1/4}, \tag{4.36}$$

where we use the estimate $0 \leqslant l(\Delta t) \leqslant T$ to reduce all exponents on $(l\Delta t)$ to the smallest one, which is 1/4. Hence,

$$\begin{aligned}
&||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||^2_{L^2((l+1)\Delta t, T; \mathcal{Q}')} \\
&= (\Delta t) \sum_{n=1}^{N-1-l} ||(\boldsymbol{u}_N^{n+l}, v_N^{n+l}) - (u_N^n, v_N^n)||^2_{\mathcal{Q}'} \leqslant C(\Delta t) \sum_{n=0}^{N-1} (l\Delta t)^{1/2} \leqslant C(l\Delta t)^{1/2}.
\end{aligned}$$

and so $I_3 := C(\delta)||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t, T; \mathcal{Q}')} \leqslant C(\delta)(l\Delta t)^{1/4}$.

Combining the estimates for I_1, I_2, and I_3, we obtain

$$||\tau_{l\Delta t}\boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}v_N - v_N||_{L^2(h,T;L^2(\Gamma))} \leqslant 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4}.$$

We can now conclude the verification of the second condition of Simon's compactness result. Namely, we have shown that

$$\begin{aligned}
||\tau_h \boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} &\leqslant (sR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4}, \\
||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} &\leqslant 3(sR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4}.
\end{aligned}$$

Now, since $h = s + l\Delta t$ and $s, l\Delta t \in [0, h]$, we get

$$\begin{aligned}
||\tau_h \boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} &\leqslant (hR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)h^{1/4}, \\
||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} &\leqslant 3(hR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)h^{1/4}.
\end{aligned}$$

Therefore, given $\epsilon > 0$, we can first choose $\delta > 0$ so that $2\delta R < \frac{\epsilon}{2}$, which fixes a value for $C(\delta)$. Then, we can choose $h > 0$ sufficiently small so that

$$3(hR)^{1/2} + 8h^{1/2}R + C(\delta)h^{1/4} < \frac{\epsilon}{2}.$$

This establishes the desired equicontinuity estimate, and hence Lemma 4.8.2 follows from Simon's compactness theorem. □

Finally, we note that we have obtained compactness results only for the velocity approximate function v_N and not v_N^*. In addition, when passing to the limit, we will consider the linear interpolations and time-shifted versions of the fluid velocity and of the structure displacement and velocity. We recall that the linear interpolations are piecewise linear functions defined by (4.28), (4.31), and the time-shifted functions are piecewise constant functions defined by (4.30), (4.32). Hence, we will need the following result.

Lemma 4.8.3. For an appropriate subsequence, which we continue to denote by N,

$$
\begin{aligned}
||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||v_N - \overline{v}_N||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||v_N - v_N^{\Delta t}||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\boldsymbol{u}_N - \overline{\boldsymbol{u}}_N||_{L^2(0,T;L^2(\Omega_f))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\boldsymbol{u}_N - \boldsymbol{u}_N^{\Delta t}||_{L^2(0,T;L^2(\Omega_f))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\eta_N - \overline{\eta}_N||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\eta_N - \eta_N^{\Delta t}||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely.}
\end{aligned}
$$

Proof. We start by showing the first convergence result. To do that, we introduce the events

$$E_{j,N} = \left\{ ||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \leqslant \frac{1}{j} \right\}, \quad j \geqslant 1,$$

and show that the probability that the complements of $E_{j,N}$ occur for infinitely many j, is zero. Indeed, by multiplying by Δt the uniform numerical dissipation estimate from Proposition 4.6.7 and keeping only the first term on the left hand side, we obtain

$$\mathbb{E}\left(\Delta t \sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)}\right) = \mathbb{E}\left(||v_N - v_N^*||^2_{L^2(0,T;L^2(\Gamma))}\right) \leqslant C(\Delta t). \tag{4.37}$$

By Chebychev's inequality, we get $\mathbb{P}(E^c_{j,N}) \leqslant C(\Delta t) j^2 = CTN^{-1}j^2$. Thus, for the events $E_{j,N=j^4}$, we have $\sum_{j=1}^{\infty} \mathbb{P}(E^c_{j,N=j^4}) \leqslant CT \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$. Therefore, by the Borel-Cantelli lemma,

$$\mathbb{P}\left(E^c_{j,N=j^4} \text{ occurs for infinitely many } j\right) = 0.$$

This implies that for almost every $\omega \in \Omega$, there exists $j_0(\omega)$ such that $||v_{N_j} - v^*_{N_j}||_{L^2(0,T;L^2(\Gamma))} \leqslant \frac{1}{j}$ for all $j \geqslant j_0(\omega)$, where $N_j := j^4$, which implies the desired result, where our subsequence N_j will continue to be denoted by N for simplicity of notation.

To show the the remaining convergence results, we use Proposition 4.6.7 to conclude that there exists a uniform constant C independent of N such that

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \leqslant C, \quad \sum_{n=0}^{N-1} \mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leqslant C, \quad \sum_{n=0}^{N-1} \mathbb{E}\left(||\nabla\eta_N^{n+1} - \nabla\eta_N^n||^2_{L^2(\Gamma)}\right) \leqslant C,$$

where we recall that $\boldsymbol{u}_N^{n+\frac{2}{3}} = \boldsymbol{u}_N^n$ and $\eta_N^{n+\frac{1}{3}} = \eta_N^{n+1}$, and where we used the triangle inequality to obtain the first estimate. Then, the same argument as above gives the desired result, once we note that

$$\mathbb{E}\left(||\overline{v}_N - v_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leqslant (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.38}$$

$$\mathbb{E}\left(||v_N^{\Delta t} - v_N||^2_{L^2(0,T;L^2(\Gamma))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.39}$$

$$\mathbb{E}\left(||\overline{\boldsymbol{u}}_N - \boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leqslant (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.40}$$

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t} - \boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.41}$$

$$\mathbb{E}\left(||\overline{\eta}_N - \eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leqslant (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\eta_N^{n+1} - \eta_N^n||^2_{L^2(\Gamma)}\right) \leqslant C'(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.42}$$

$$\mathbb{E}\left(||\eta_N^{\Delta t} - \eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\eta_N^{n+1} - \eta_N^n||^2_{L^2(\Gamma)}\right) \leqslant C'(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{4.43}$$

where we used Poincaré's inequality to deduce (4.42) and (4.43). □

Notice that this result follows from the numerical dissipation estimates in Proposition 4.6.7, which imply convergence to zero in expectation, of the numerical dissipation terms, shown in (4.38), (4.39), (4.40), (4.41), (4.42), and (4.43), from which we were able to deduce the almost sure convergence.

Proof of Theorem 4.8.1. To show weak convergence of probability measures along a subsequence, we must show that the probability measures μ_N are tight, see Definition 4.8.1. Here, we note that for reasons that will be clear later (see Step 2 below), we will take N to be the subsequence provided by Lemma 4.8.3 and begin with this indexing convention of N.

Step 1: Weak convergence of $\mu_{\overline{\eta}_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ along a subsequence. We show this by showing that $\mu_{\overline{\eta}_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ are tight. To show the tightness of $\mu_{\overline{\eta}_N}$, we define the set

$$A_R = \{\overline{\eta} \in W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma)) : ||\overline{\eta}||_{W^{1,\infty}(0,T;L^2(\Gamma))} \leqslant R, ||\overline{\eta}||_{L^\infty(0,T;H_0^1(\Gamma))} \leqslant R\}.$$

By Lemma 4.8.1, $\overline{A_R}$ is a compact set in $L^2(0,T;L^2(\Gamma))$ since $L^\infty(0,T;L^2(\Gamma))$ embeds continuously into $L^2(0,T;L^2(\Gamma))$, where the closure is taken in the topology of $L^2(0,T;L^2(\Gamma))$.

So by Chebychev's inequality and the previous uniform boundedness results, we have that for an arbitrary $\epsilon > 0$,

$$\mu_{\overline{\eta}_N}(\overline{A_R}) > 1 - \epsilon,$$

if R is chosen sufficiently large. So there exists a subsequence, which we continue to denote by N, for which $\mu_{\overline{\eta}_N}$ converges weakly to some probability measure μ_η on $L^2(0,T;L^2(\Gamma))$.

To show the tightness of $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$, recall the definition of the set $\mathcal{K}_R$, and note that by Lemma 4.8.2, $\overline{\mathcal{K}_R}$ is a compact set in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Furthermore, using the uniform boundedness estimates from Proposition 4.7.2 combined with Chebychev's inequality, we have that for any $\epsilon > 0$, we can find R sufficiently large such that

$$(\mu_{\boldsymbol{u}_N} \times \mu_{v_N})(\overline{\mathcal{K}_R}) > 1 - \epsilon.$$

Hence, there exists a subsequence, which we continue to denote by N, for which the measures $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ converge weakly to some limiting probability measure on $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$, which we denote by $\mu_{\boldsymbol{u}} \times \mu_v$.

Step 2: Weak convergence of $\mu_{\boldsymbol{u}_N} \times \mu_{v_N^*}$, $\mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N}$, $\mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}}$, μ_{η_N}, and $\mu_{\eta_N^{\Delta t}}$ along the subsequence obtained from Step 1. Since $\mu_{\boldsymbol{u}_N} \times \mu_{v_N} \Longrightarrow \mu_{\boldsymbol{u}} \times \mu_v$, by the definition of weak convergence, we have

$$\mathbb{E}[f(\boldsymbol{u}_N, v_N)] \to \int_{L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))} f d(\mu_{\boldsymbol{u}} \times \mu_v),$$

for all bounded, Lipschitz continuous functions $f : L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) \to \mathbb{R}$. However, because $||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \to 0$ a.s. due to Lemma 4.8.3, we have that by the Lipschitz continuity of f,

$$|f(\boldsymbol{u}_N, v_N) - f(\boldsymbol{u}_N, v_N^*)| \leqslant \text{Lip}(f)||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \to 0, \quad \text{a.s. as } N \to \infty.$$

Hence, by the bounded convergence theorem, $\mathbb{E}[f(\boldsymbol{u}_N, v_N)] - \mathbb{E}[f(\boldsymbol{u}_N, v_N^*)] \to 0$, as $N \to \infty$, and hence,

$$\mathbb{E}[f(\boldsymbol{u}_N, v_N^*)] \to \int_{L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))} f d(\mu_{\boldsymbol{u}} \times \mu_v),$$

for all bounded, Lipschitz continuous functions $f : L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) \to \mathbb{R}$. Thus, along the subsequence generated from Step 1, we have that both $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N^*}$ converge weakly to the same limiting probability measure $\mu_{\boldsymbol{u}} \times \mu_v$ on $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$.

The same argument can be used to show that $\mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N}$ and $\mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}}$ also converge weakly to $\mu_{\boldsymbol{u}} \times \mu_v$. This follows from the result on a.s. convergence of $||\overline{\boldsymbol{u}}_N - \boldsymbol{u}_N||_{L^2(0,T;L^2(\Omega_f))}$, $||\overline{v}_N - v_N||_{L^2(0,T;L^2(\Gamma))}$, $||\boldsymbol{u}_N^{\Delta t} - \boldsymbol{u}_N||_{L^2(0,T;L^2(\Omega_f))}$, and $||v_N^{\Delta t} - v_N||_{L^2(0,T;L^2(\Gamma))}$ in Lemma 4.8.3.

Finally, we have from Step 1 that $\mu_{\overline{\eta}_N}$ converges weakly to some probability measure μ_η, as probability measures on $L^2(0,T;L^2(\Gamma))$. Then, the weak convergence of μ_{η_N} and $\mu_{\eta_N^{\Delta t}}$ to this same weak limit μ_η follows from the same argument as above, and the result from

Lemma 4.8.3 that $||\overline{\eta}_N - \eta_N||_{L^2(0,T;L^2(\Gamma))} \to 0$ and $||\eta_N^{\Delta t} - \eta_N||_{L^2(0,T;L^2(\Gamma))} \to 0,$ as $N \to \infty$, a.s.

Step 3: Tightness of full measures μ_N along the subsequence obtained from Step 1. We now consider the full probability measures μ_N specified in (4.33) on the phase space $\mathcal{X}$ specified in (4.34). We want to show that these probability measures μ_N are tight along the subsequence N constructed as a result of Step 1.

Consider $\epsilon > 0$. We want to construct a compact set in the phase space $\mathcal{X}$ for which the probability measure μ_N has probability greater than $1-\epsilon$ on this compact set, for all N. We will construct this compact set component-wise, using $\pi_1, \ldots, \pi_{12}$ to denote the projections onto the components 1 through 12 of μ_N.

By the weak convergence of the measures $\mu_{\overline{\eta}_N}$, μ_{η_N}, and $\mu_{\eta_N^{\Delta t}}$, by Prohorov's theorem (see for example Proposition 6.1 in [128]), there exist compact sets B_1, B_2, and B_3 in $L^2(0,T;L^2(\Gamma))$ such that

$$\pi_1(\mu_N)(B_1) > 1 - \frac{\epsilon}{8}, \qquad \pi_2(\mu_N)(B_2) > 1 - \frac{\epsilon}{8}, \qquad \pi_3(\mu_N)(B_3) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N.$$

Similarly, because $(\boldsymbol{u}_N, v_N)$, $(\boldsymbol{u}_N, v_N^*)$, $(\overline{\boldsymbol{u}}_N, \overline{v}_N)$, and $(\boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t})$ converge weakly along this subsequence N by Step 1 and Step 2, there exist compact sets $B_{4,5}$, $B_{6,7}$, $B_{8,9}$, and $B_{10,11}$ in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$ such that

$$\begin{aligned} &\pi_{4,5}(\mu_N)(B_{4,5}) > 1 - \frac{\epsilon}{8}, \qquad \pi_{6,7}(\mu_N)(B_{6,7}) > 1 - \frac{\epsilon}{8}, \\ &\pi_{8,9}(\mu_N)(B_{8,9}) > 1 - \frac{\epsilon}{8}, \qquad \pi_{10,11}(\mu_N)(B_{10,11}) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N. \end{aligned}$$

Finally, the last component of μ_N, which is μ_W, is constant in N. Hence, the probability measures $\pi_{12}(\mu_N)$ defined on $C(0,T;\mathbb{R})$ are trivially, weakly compact. Therefore, the collection $\pi_{12}(\mu_N)$ for all N is tight, and hence, there exists a compact set $B_{12} \subset C(0,T;\mathbb{R})$ such that

$$\pi_{12}(\mu_N)(B_{12}) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N.$$

Based on this construction, we have the set $M_\epsilon := B_1 \times B_2 \times B_3 \times B_{4,5} \times B_{6,7} \times B_{8,9} \times B_{10,11} \times B_{12}$, which is a compact subset of the phase space $\mathcal{X}$, satisfying $\mu_N(M_\epsilon) > 1 - \epsilon$, for all N. This establishes the desired tightness of the probability measures, and completes the proof of Proposition 4.8.1. □

Continuity properties of the weak limit

To be able to prove appropriate measure theoretic properties of the limiting solutions, we need to establish continuity properties of the limiting solution. This is because many measure theoretic properties are simpler for stochastic processes with continuous paths in time. This is simple to do for the structure displacements, since the approximate structure displacements $\overline{\eta}_N$ all have Lipschitz continuous paths. However, because the approximate fluid and structure velocities $\boldsymbol{u}_N$ and v_N have paths that are not continuous, we want to establish that the limiting solutions for the fluid and structure velocities have continuous paths in time, with an appropriate notion of continuity.

First, we introduce the following definition, which will be used throughout the remainder of the manuscript.

Definition 4.8.3. Let B be an arbitrary Banach space and let $f, g : [0, T] \to B$. The function $g : [0, T] \to B$ is a **version** of f if $f = g$ a.e. on $[0, T]$.

The goal is to show that the limit function $(\boldsymbol{u}, v)$ is in $C(0, T; \mathcal{Q}')$ almost surely, or more precisely, that the limiting measure μ is supported on a measurable subset of phase space $\mathcal{X}$, with the projections onto the functions involving the fluid and structure velocities being in $C(0, T; \mathcal{Q}')$ almost surely. This continuity property will allow us to conclude later that the resulting limit process is well-behaved in a stochastic measure theoretic sense.

To do this, we will use the idea of p-variation for functions in time taking values in $\mathcal{Q}'$. The notion of considering total variations of functions is a classical idea [144], [181]. We remark however that our definition below differs slightly from classical definitions of total p-variation.

Definition 4.8.4. For any real number $p \geqslant 1$ and any $\delta > 0$, we define the *p-variation of length scale* δ of a given function $(\boldsymbol{u}, v) : [0, T] \to \mathcal{Q}'$ by

$$V_p^{\delta}(\boldsymbol{u}, v) = \sup_{|P| \leqslant \delta} \sum_{i=1}^{M} ||(\boldsymbol{u}(t_i), v(t_i)) - (\boldsymbol{u}(t_{i-1}), v(t_{i-1}))||_{\mathcal{Q}'}^{p},$$

where P denotes a partition $0 \leqslant t_0 < t_1 < ... < t_M \leqslant T$ for some positive integer M, and the condition $|P| \leqslant \delta$ means that $|t_i - t_{i-1}| \leqslant \delta$ for all $i = 1, 2, ..., M$.

We introduce this definition of the p-variation of length scale δ because we will invoke estimates on the time shifts, as in (4.36), in order to deduce continuity in $\mathcal{Q}'$. The strategy will be to show that almost surely, the limiting fluid velocity and structure velocity, denoted by the pair $(\boldsymbol{u}, v)$, has a variation that goes to zero as the length scale δ goes to zero, which would imply that the pair $(\boldsymbol{u}, v)$ cannot have any discontinuities and is hence continuous in $\mathcal{Q}'$. We hence want to define and examine the subset of functions whose p-variation of length scale δ is bounded above by a certain parameter ϵ. We do this in the following lemma.

Lemma 4.8.4. Let $A_{p,\delta,\epsilon}$ be the set of functions $(\boldsymbol{u}, v) : [0, T] \to \mathcal{Q}'$ in

$$X = L^2(0, T; L^2(\Omega_f)) \times L^2(0, T; L^2(\Gamma))$$

such that the following properties hold:

1. $(\boldsymbol{u}, v)$ has a version that is left continuous on $[0, T]$ as a function of time, taking values in $\mathcal{Q}'$.

2. This version of $(\boldsymbol{u}, v)$ is also right continuous at $t = 0$ as a function taking values in $\mathcal{Q}'$.

3. For this (necessarily unique) left continuous version, $V_p^\delta(\boldsymbol{u}, v) \leqslant \epsilon$.

Then, for any $p \geqslant 1$, $\delta > 0$, and $\epsilon > 0$, $A_{p,\delta,\epsilon}$ is a closed set in X.

Proof. To show that $A_{p,\delta,\epsilon}$ is a closed set in X, we consider a sequence $\{(\boldsymbol{u}_n, v_n)\}_{n=1}^\infty$ in $A_{p,\delta,\epsilon}$ that converges to some element $(\boldsymbol{u}, v) \in X$ in the norm of X. We claim that $(\boldsymbol{u}, v) \in A_{p,\delta,\epsilon}$.

We start by showing Property 3 above, namely $V_p^\delta(\boldsymbol{u}, v) \leqslant \epsilon$. Because $(\boldsymbol{u}_n, v_n) \to (\boldsymbol{u}, v)$ in $L^2(0, T; L^2(\Omega_f) \times L^2(\Gamma))$, we have that along a subsequence, which we will continue to denote by the same index, we have $(\boldsymbol{u}_n(t), v_n(t)) \to (\boldsymbol{u}(t), v(t))$ in $L^2(\Omega_f) \times L^2(\Gamma)$, for a.e. $t \in [0, T]$. Since $L^2(\Omega_f) \times L^2(\Gamma)$ embeds continuously into $\mathcal{Q}'$,

$$(\boldsymbol{u}_n(t), v_n(t)) \to (\boldsymbol{u}(t), v(t)), \text{ in } \mathcal{Q}', \ \forall t \in S, \tag{4.44}$$

where S is some measurable subset of $[0, T]$, which consists of the points $t \in [0, T]$ for which the convergence above holds. Note that S has almost every $t \in [0, T]$.

Consider any partition P with $|P| \leqslant \delta$, *consisting only of points in* S. Now, from the fact that $V_p^\delta(\boldsymbol{u}_n, v_n) \leqslant \epsilon$ for all n, we have that $\sum_{i=1}^M ||(\boldsymbol{u}_n(t_i), v_n(t_i)) - (\boldsymbol{u}_n(t_{i-1}), v_n(t_{i-1}))||_{\mathcal{Q}'}^p \leqslant \epsilon$. Because the partition is finite and because the partition consists of points in S for which the convergence (4.44) holds, in the limit as $n \to \infty$:

$$\sum_{i=1}^M ||(\boldsymbol{u}(t_i), v(t_i)) - (\boldsymbol{u}(t_{i-1}), v(t_{i-1}))||_{\mathcal{Q}'}^p \leqslant \epsilon. \tag{4.45}$$

This verifies Property 3 for partitions $|P| \leqslant \delta$ consisting of points in S. To show Properties 1 and 2, we use the above inequality (4.45) and construct a version of $(\boldsymbol{u}, v)$ which will satisfy all the properties of the set $A_{p,\delta,\epsilon}$. Then, we will conclude the proof by verifying Property 3 for this new version and extending the verification of Property 3 to general partitions $|P| \leqslant \delta$ consisting of any points in $[0, T]$. We start with Property 1 above, namely that $(\boldsymbol{u}, v)$ must have a version that is left continuous. We do this in the following steps.

Step 1: First, we show that at each point $t \in [0, T]$, the left and right limits of $(\boldsymbol{u}, v)$ along points in S must exist. This will be useful, as S is dense in $[0, T]$. In addition, the density of S in $[0, T]$ means that for all $t \in [0, T]$, the notion of a left and right limit along points in S makes sense.

To show this, consider any point $t_0 \in [0, T]$. We emphasize that t_0 is not necessarily in the set S. We claim that the left and right limit at t_0 along points in S must exist. In particular, for any sequence $\{t_n\}_{n=1}^\infty$ with $t_n \in S$ that increases to t_0 or decreases to t_0, we claim that $\lim_{n\to\infty}(\boldsymbol{u}(t_n), v(t_n))$ exists in $\mathcal{Q}'$.

We show this by contradiction. Suppose there exists a strictly increasing sequence $\{t_n\}_{n=1}^\infty$ with $t_n \in S$ and $t_n \nearrow t_0$, such that $\lim_{n\to\infty}(\boldsymbol{u}(t_n), v(t_n))$ does not exist in $\mathcal{Q}'$. The same argument will hold in the case of a decreasing sequence. This implies that $\{(\boldsymbol{u}(t_n), v(t_n))\}_{n=1}^\infty$

does not converge in $\mathcal{Q}'$, and hence is not a Cauchy sequence. Thus, there exists $\epsilon_0 > 0$, such that given any N, there exists $n_1, n_2 \geqslant N$ such that

$$||(\boldsymbol{u}(t_{n_1}), v(t_{n_1})) - (\boldsymbol{u}(t_{n_2}), v(t_{n_2}))||_{\mathcal{Q}'} \geqslant \epsilon_0.$$

Note that we have called this constant ϵ_0 to distinguish it from the ϵ in the definition of $A_{p,\delta,\epsilon}$. Now, choose M sufficiently large such that

$$M(\epsilon_0)^p > \epsilon.$$

Choose a partition P consisting of points $s_0, ..., s_{2M-1}$ in S with the following properties.

1. For each $i = 0, 1, ..., 2M-1$, we have that $t_0 - \delta < s_i < t_0$.
2. The sequence $s_0, ..., s_{2M-2}, s_{2M-1}$ is strictly increasing.
3. For even $i = 0, 2, ..., 2M-2$, $||(\boldsymbol{u}(s_i), v(s_i)) - (\boldsymbol{u}(s_{i+1}), v(s_{i+1}))||_{\mathcal{Q}'} \geqslant \epsilon_0$. This can be accomplished by using the non-convergent sequence $\{(\boldsymbol{u}(t_n), v(t_n))\}_{n=1}^{\infty}$ and the definition of ϵ_0 to choose the s_i from the sequence $\{t_n\}_{n=1}^{\infty}$, as $t_n \in S$ for all n.

Since M was chosen so that $M(\epsilon_0)^p > \epsilon$, for this partition P, consisting of points in S with $|P| \leqslant \delta$, we have

$$\sum_{i=1}^{2M-1} ||(\boldsymbol{u}(s_i), v(s_i)) - (\boldsymbol{u}(s_{i-1}), v(s_{i-1}))||_{\mathcal{Q}'}^p > \epsilon$$

which is a contradiction.

Furthermore, the left and right limits along points in S are well-defined. Suppose for contradiction that

$$\lim_{n\to\infty} (\boldsymbol{u}(s_n), v(s_n)) = L_1 \neq L_2 = \lim_{n\to\infty} (\boldsymbol{u}(t_n), v(t_n)),$$

for two increasing sequences $s_n \nearrow t_0$ and $t_n \nearrow t_0$, consisting of points in S. Then, we can construct a new sequence $\{r_k\}_{k=1}^{\infty}$, where we set $r_0 = s_0$. Then, we set r_1 to be any t_n for which $t_n \in (s_0, t_0)$. We continue, creating an interlaced sequence where all odd indices of r_k come from the sequence of s_n points, and all even indices of r_k come from the sequence of t_n points, where along the odd sequence, the indices of the corresponding s_n points is strictly increasing, and similarly for the even sequence of the t_n points. We can also perform this construction so that the points in r_k are strictly increasing to t_0. However, one can see that $\lim_{n\to\infty}(\boldsymbol{u}(r_n), v(r_n))$ does not exist, which contradicts our earlier result. So the left and right limits along points in S are well-defined.

Step 2: In Step 1, we have shown that $\lim_{t\to t_0^-, t\in S}(\boldsymbol{u}(t), v(t))$ and $\lim_{t\to t_0^+, t\in S}(\boldsymbol{u}(t), v(t))$ both exist for all $t \in [0, T]$, where these are limits in $\mathcal{Q}'$. We show that there can only be countably many points $t_0 \in (0, T)$ for which these limits, which take values in $\mathcal{Q}'$, do not agree.

To do this, we argue by contradiction. Suppose that there are uncountably many points in $(0,T)$ for which these limits do not agree. Then, there exists $\rho > 0$ sufficiently small such that there are infinitely many points $t_0 \in (0,T)$ for which

$$||\lim_{t\to t_0^-, t\in S}(\boldsymbol{u}(t), v(t)) - \lim_{t\to t_0^+, t\in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} \geqslant \rho.$$

Let M be sufficiently large such that $M\left(\frac{\rho}{2}\right)^p > \epsilon$ and select $t_1, ..., t_M$ points of discontinuity in $(0,T)$ with

$$||\lim_{t\to t_n^-, t\in S}(\boldsymbol{u}(t), v(t)) - \lim_{t\to t_n^+, t\in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} \geqslant \rho, \qquad \text{for } n = 1, 2, ..., M.$$

We can order these points as $t_1 < t_2 < ... < t_M$, and select $2M$ points $\{s_{n,i}\}_{1\leqslant n\leqslant M, i=1,2}$ in S, such that

1. $s_{1,1} < s_{1,2} < s_{2,1} < s_{2,2} < ... < s_{M,1} < s_{M,2}$.
2. For each $n = 1, 2, ..., M$, $t_n - \frac{\delta}{2} < s_{n,1} < t_n < s_{n,2} < t_n + \frac{\delta}{2}$.
3. For each $n = 1, 2, ..., M$, $||(\boldsymbol{u}(s_{n,1}), v(s_{n,1})) - \lim_{t\to t_n^-, t\in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} < \frac{\rho}{4}$, and $||(\boldsymbol{u}(s_{n,2}), v(s_{n,2})) - \lim_{t\to t_n^+, t\in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} < \frac{\rho}{4}$.

Then, we can form a partition of points in S that interlaces the sequence $s_{1,1} < s_{1,2} < s_{2,1} < s_{2,2} < ... < s_{M,1} < s_{M,2}$ with additional points so that the resulting partition P has $|P| < \delta$, since S is dense in $[0,T]$. We can do this in a way that keeps the points $s_{n,i}$ for $i = 1,2$ consecutive in the partition for each $n = 1, 2, ..., M$. Since $M\left(\frac{\rho}{2}\right)^p > \epsilon$, we have that the variation for this resulting partition is greater than ϵ, which is a contradiction.

The same argument as above implies that *there are only countably many points $t_0 \in S$ for which*

$$\lim_{t\to t_0^-, t\in S}(\boldsymbol{u}(t), v(t)) \neq (\boldsymbol{u}(t_0), v(t_0)).$$

Thus, we define S^* to be the set of points $t_0 \in S$ for which

$$\lim_{t\to t_0^-, t\in S}(\boldsymbol{u}(t), v(t)) = (\boldsymbol{u}(t_0), v(t_0)).$$

Since countable sets have measure zero, S^* still has the property that $[0,T]-S^*$ is of measure zero. So in particular, S^* is still dense in $[0,T]$. *We emphasize that now, $(\boldsymbol{u}(t), v(t))$ has the useful property that it is left continuous on S^*.*

Step 3: Because $S^* \subset S$ and is still a dense set in $[0,T]$, the result from Step 1 implies that:

$$\lim_{t\to t_0^-, t\in S^*}(\boldsymbol{u}(t), v(t)) \text{ and } \lim_{t\to t_0^+, t\in S^*}(\boldsymbol{u}(t), v(t)) \text{ exist for all } t_0 \in [0,T].$$

However, these limits are only along points in S^*. By the density of S^* in $[0, T]$ and the fact that $[0, T] - S^*$ has measure zero, we can redefine $(\boldsymbol{u}, v)$ up to a version, so that

$$(\boldsymbol{u}(t_0), v(t_0)) \text{ is unchanged if } t_0 \in S^*, \text{ and } (\boldsymbol{u}(t_0), v(t_0)) = \lim_{t \to t_0^-, t \in S^*} (\boldsymbol{u}(t), v(t)) \text{ if } t_0 \in [0, T] - S^*. \tag{4.46}$$

For the remainder of this proof, $(\boldsymbol{u}, v)$ will denote this newly defined version in (4.46). We then claim that for this version,

$$\lim_{t \to t_0^-, t \in S^*} (\boldsymbol{u}(t), v(t)) = \lim_{t \to t_0^-} (\boldsymbol{u}(t), v(t)) \qquad \text{and} \qquad \lim_{t \to t_0^+, t \in S^*} (\boldsymbol{u}(t), v(t)) = \lim_{t \to t_0^+} (\boldsymbol{u}(t), v(t)), \tag{4.47}$$

for all $t \in [0, T]$. We will just prove the first statement, for the limit from the left, as the statement for the limit from the right is proved analogously. To see this, note that by the definition of the version and by the definition of S^* in Step 2,

$$(\boldsymbol{u}(t_0), v(t_0)) = \lim_{t \to t_0^-, t \in S^*} (\boldsymbol{u}(t), v(t)), \qquad \text{for all } t_0 \in [0, T]. \tag{4.48}$$

So given any strictly increasing sequence $t_n \nearrow t_0$ where t_n is not necessarily in S^*, we want to show that

$$\lim_{n \to \infty} (\boldsymbol{u}(t_n), v(t_n)) = \lim_{t \to t_0^-, t \in S^*} (\boldsymbol{u}(t), v(t)).$$

To do this, we use the density of S^* in $[0, T]$ along with (4.48) to construct a sequence s_n such that

1. $s_0 \leqslant t_0$ and $t_{n-1} < s_n \leqslant t_n$ for all $n \geqslant 1$.

2. $|s_n - t_n| < 2^{-n}$ for all n.

3. $||(\boldsymbol{u}(s_n), v(s_n)) - (\boldsymbol{u}(t_n), v(t_n))||_{\mathcal{Q}'} < 2^{-n}$ for all n.

4. $s_n \in S^*$ for all n.

This is possible because S^* is dense in $[0, T]$, and shows the desired result, as s_n is a strictly increasing sequence converging to t_0 by Property 1 and 2, and by Property 3 and 4 we have

$$\lim_{n \to \infty} (\boldsymbol{u}(t_n), v(t_n)) = \lim_{n \to \infty} (\boldsymbol{u}(s_n), v(s_n)) = \lim_{t \to t_0^-, t \in S^*} (\boldsymbol{u}(t), v(t)).$$

Note that this version of $(\boldsymbol{u}(t), v(t))$ on $[0, T]$ is left continuous by (4.47) and (4.48), with only countably many points of discontinuity by Step 2.

Conclusion: We have constructed a left continuous version of $(\boldsymbol{u}(t), v(t))$ on $[0, T]$ taking values in $\mathcal{Q}'$ in Step 3. At the left boundary, $t = 0$, we can set the version of $(\boldsymbol{u}, v)$ so that $(\boldsymbol{u}(0), v(0)) = \lim_{t \to 0^+} (\boldsymbol{u}(t), v(t))$, so that we have right continuity at $t = 0$. This is possible since this limit exists by Step 1 and (4.47). For the newly defined version of $(\boldsymbol{u}(t), v(t))$, we have that

$$\sum_{i=1}^{N} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^{p} \leqslant \epsilon,$$

for all partitions P consisting of points in S^* with $|P| \leqslant \delta$, since we did not change the original $(\boldsymbol{u}(t), v(t))$ on points of S^*, which is a subset of S. We can now show that this p-variation inequality holds more generally for all partitions P with points in $[0, T]$ with $|P| \leqslant \delta$. To do this, we note that since S^* is dense in $[0, T]$, we can approximate any partition P of arbitrary points in $[0, T]$ with $|P| \leqslant \delta$ by a sequence of partitions $\{P_k\}_{k \geqslant 1}$ of points in S^* with $|P_k| \leqslant \delta$ containing the same number of points as P. We can do this by approaching any partition points of P in $(0, T]$ from the left by points in S^*, and approaching $t = 0$ from the right by points in S^* if $t = 0$ is a partition point in P. We then obtain the desired result by taking the limit in k as the partitions P_k approach P. This process of taking the limit uses the fact that the version of $(\boldsymbol{u}, v)$ as defined in Step 3 is left continuous on $[0, T]$ and right continuous at $t = 0$. Therefore, we conclude that $V_p^{\delta}(\boldsymbol{u}, v) \leqslant \epsilon$. □

The next lemma, along with the weak convergence of the laws μ_N, will allow us to use the result above to prove almost sure continuity in $\mathcal{Q}'$ of the limiting fluid and structure velocity. In particular, this next lemma will show that if the length scale δ is chosen appropriately, then eventually, for large enough N (or equivalently small enough Δt), the approximate solutions will have p-variation (for $p > 4$) with length scale δ bounded above uniformly with high probability. This is to be expected, due to the time shift estimate (4.36), which is *independent of* N.

For the following results, we recall the definition of μ_N on the phase space $\mathcal{X}$ from (4.33) and (4.34), and we denote by $\pi_{4,5}\mu_N$ the projection onto the fourth and fifth components of $\mathcal{X}$, which gives the law of $(\boldsymbol{u}_N, v_N)$ on $L^2(0, T; L^2(\Omega_f)) \times L^2(0, T; L^2(\Gamma))$.

Lemma 4.8.5. For any $p > 4$ and any $\epsilon > 0$, there exists $\delta_0 > 0$ sufficiently small and N_0 sufficiently large such that for all $0 < \delta \leqslant \delta_0$,

$$\pi_{4,5}\mu_N(A_{p,\delta,\epsilon}) > 1 - \epsilon, \qquad \text{for all } N \geqslant N_0.$$

Proof. We start by first introducing a set $\mathcal{K}_{R,N}$. Let $\mathcal{K}_{R,N}$ be the collection of paths in $\mathcal{K}_R$ (introduced in Definition 4.8.2), corresponding to path realizations of the random variables $(\boldsymbol{u}_N, v_N)$ for fixed N, satisfying the properties in the definition of $\mathcal{K}_R$. In particular, $\mathcal{K}_R = \bigcup_{N=1}^{\infty} \mathcal{K}_{R,N}$. Notice that we can choose R large enough so that

$$\pi_{4,5}\mu_N(\overline{\mathcal{K}_{R,N}}) > 1 - \epsilon, \qquad \text{for all } N,$$

where the closure is taken in $L^2(0, T; L^2(\Omega_f)) \times L^2(0, T; L^2(\Gamma))$. Recall from (4.36) that

$$||(\boldsymbol{u}_N^{n+l}, v_N^{n+l}) - (\boldsymbol{u}_N^{n}, v_N^{n})||_{\mathcal{Q}'} \leqslant C_R(l\Delta t)^{1/4},$$

where C_R is a constant depending only on R for all $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$. In particular, C_R is independent of n, l, and N and hence Δt. We will use this estimate on the increments in $\mathcal{Q}'$ to choose $\delta_0 > 0$ and N_0 such that $\mathcal{K}_{R,N} \subset A_{p,\delta,\epsilon}$, for all $N \geqslant N_0$ and $0 < \delta \leqslant \delta_0$, and ultimately, $\overline{\mathcal{K}_{R,N}} \subset A_{p,\delta,\epsilon}$, for all $N \geqslant N_0$ and $0 < \delta \leqslant \delta_0$, from which the result $\pi_{4,5}\mu_N(A_{p,\delta,\epsilon}) > 1 - \epsilon$, for all $N \geqslant N_0$ and $0 < \delta \leqslant \delta_0$ will follow. Indeed, for any given partition P with $|P| \leqslant \delta$, the following estimate holds:

$$\sum_{i=1}^{M} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^p \leqslant C_R \sum_{k=1}^{l} n_k (k\Delta t)^{p/4},$$

for any $(\boldsymbol{u}, v) \in \mathcal{K}_R$, where n_k is the number of increments that have indices k apart and l is the maximum integer for which $l\Delta t < \delta + \Delta t$. This is true by the fact that the paths $(\boldsymbol{u}, v)$ in $\mathcal{K}_R$ are defined as piecewise constant functions taking values $(\boldsymbol{u}_N^n, v_N^n)$, and by inequality (4.36). Because the partition P has $|P| \leqslant \delta$, we have that l must satisfy

$$l\Delta t < \delta + \Delta t = \delta + N^{-1}T \quad \text{and} \quad \sum_{k=1}^{l} n_k(k\Delta t) \leqslant (N-1)\Delta t = T - \Delta t. \tag{4.49}$$

Therefore, since $p > 4$, we have that for any partition P with $|P| \leqslant \delta$ and for any $(\boldsymbol{u}, v) \in \mathcal{K}_R$,

$$\begin{aligned}\sum_{i=1}^{M} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^p &\leqslant C_R \sum_{k=1}^{l} n_k (k\Delta t)^{p/4} \leqslant C_R \left(\sum_{k=1}^{l} n_k (k\Delta t)\right) (l\Delta t)^{\frac{p}{4}-1} \\ &\leqslant C_R T (l\Delta t)^{\frac{p}{4}-1} \leqslant C_R T (\delta + N^{-1}T)^{\frac{p}{4}-1},\end{aligned}$$

where we used (4.49). The proof is complete once we choose $\delta_0 > 0$ sufficiently small and N_0 sufficiently large such that

$$C_R T (\delta_0 + N_0^{-1} T)^{\frac{p}{4}-1} < \epsilon.$$

Therefore, for $(\boldsymbol{u}_N, v_N)$ in $\mathcal{K}_R$ for any $N \geqslant N_0$ and $0 < \delta \leqslant \delta_0$, we have $V_p^\delta(\boldsymbol{u}_N, v_N) \leqslant \epsilon$. Thus,

$$\mathcal{K}_{R,N} \subset A_{p,\delta,\epsilon}, \qquad \text{for all } N \geqslant N_0 \text{ and } 0 < \delta \leqslant \delta_0.$$

Since $A_{p,\delta,\epsilon}$ is closed in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$ by Lemma 4.8.4, we conclude that

$$\overline{\mathcal{K}_{R,N}} \subset A_{p,\delta,\epsilon}, \qquad \text{for all } N \geqslant N_0 \text{ and } 0 < \delta \leqslant \delta_0,$$

where the closure is taken with respect to the norm of $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Since $\pi_{4,5}\mu_N(\overline{\mathcal{K}_{R,N}}) > 1 - \epsilon$ for all positive integers N by the initial choice of R, this implies the result. □

Lemma 4.8.6. For the weak limit μ,

$$\pi_{4,5}\mu(X \cap C(0,T;\mathcal{Q}')) = 1,$$

where $X := L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Furthermore, $\pi_{4,5}\mu$ is supported on a Borel measurable subset of X such that every function has a version in $C(0,T;\mathcal{Q}')$ that is equal to $(\boldsymbol{u}_0, v_0)$ at $t = 0$.

Remark 4.8.2. We remark that $X \cap C(0,T;\mathcal{Q}')$ is a Borel measurable subset of X, and hence the statement above makes sense. To see this, note that the inclusion map

$$\iota : X \to L^2(0,T;\mathcal{Q}')$$

is continuous since $L^2(\Omega_f) \times L^2(\Gamma)$ embeds continuously into $\mathcal{Q}'$. It suffices to show that $C(0,T;\mathcal{Q}')$ is a Borel measurable subset of $L^2(0,T;\mathcal{Q}')$. Then, $X \cap C(0,T;\mathcal{Q}')$ is the preimage of $C(0,T;\mathcal{Q}')$ under ι, and is hence measurable in X, which is the desired result.

To show that $C(0,T;\mathcal{Q}')$ is a Borel measurable subset of $L^2(0,T;\mathcal{Q}')$, we note that a closed ball of arbitrary radius R in $L^\infty(0,T;\mathcal{Q}')$ is Borel measurable in $L^2(0,T;\mathcal{Q}')$ since it is a closed set in $L^2(0,T;\mathcal{Q}')$. Since one can express an open ball as a countable union of closed balls, an open ball of arbitrary radius R in $L^\infty(0,T;\mathcal{Q}')$ is also measurable in $L^2(0,T;\mathcal{Q}')$. Since $C(0,T;\mathcal{Q}')$ is closed in $L^\infty(0,T;\mathcal{Q}')$ in the topology of $L^\infty(0,T;\mathcal{Q}')$, and since closed and open balls of $L^\infty(0,T;\mathcal{Q}')$ are Borel measurable in $L^2(0,T;\mathcal{Q}')$, this implies that $C(0,T;\mathcal{Q}')$ is Borel measurable in $L^2(0,T;\mathcal{Q}')$.

Proof. Fix $p > 4$ and set $\epsilon_k = 2^{-k}$. Then, by Lemma 4.8.5, there exists a decreasing sequence of positive real numbers $\{\delta_k\}_{k=1}^\infty$ and an increasing sequence of positive integers $\{N_k\}_{k=1}^\infty$, such that

$$\pi_{4,5}\mu_N(A_{p,\delta_k,\epsilon_k}) > 1 - \epsilon_k, \qquad \text{for all } N \geqslant N_k, \text{ and } k \in \mathbb{Z}^+.$$

Note that since μ_N converges weakly to μ, we have that $\pi_{4,5}\mu_N$ converges weakly to $\pi_{4,5}\mu$. For each fixed positive integer k, since $A_{p,\delta_k,\epsilon_k}$ is a closed set in X, we have by Portmanteau's theorem for weak convergence of probability measures that

$$\pi_{4,5}\mu(A_{p,\delta_k,\epsilon_k}) \geqslant \limsup_{N\to\infty} \pi_{4,5}\mu_N(A_{p,\delta_k,\epsilon_k}) \geqslant 1 - \epsilon_k.$$

By the Borel Cantelli lemma and by the choice of $\epsilon_k = \frac{1}{2^k}$ so that $\sum_{k=1}^\infty \frac{1}{2^k} < \infty$,

$$\pi_{4,5}\mu(A^c_{p,\delta_k,\epsilon_k} \text{occurs infinitely often in } k \in \mathbb{Z}^+) = 0.$$

So almost surely, $\pi_{4,5}\mu$ takes values in the set $\{A_{p,\delta_k,\epsilon_k} \text{occurs for infinitely many } k\}$. However, one can show that

$$\{A_{p,\delta_k,\epsilon_k} \text{ occurs for infinitely many } k\} \subset X \cap C(0,T;\mathcal{Q}'), \tag{4.50}$$

which then implies the result. To see why this is true, suppose that

$$(\boldsymbol{u}, v) \in \{A_{p,\delta_k,\epsilon_k} \text{ occurs for infinitely many } k\}.$$

By the fact that $V_p^{\delta_k}(\boldsymbol{u}, v) \leqslant \epsilon_k$, we must have that for every $t_0 \in [0,T]$ (modified appropriately for the endpoint cases $t_0 = 0$ and $t_0 = T$),

$$||(\boldsymbol{u}(t), v(t)) - (\boldsymbol{u}(t_0), v(t_0))||_{\mathcal{Q}'} \leqslant \epsilon_k^{1/p}, \qquad \text{for all } t \in (t_0 - \delta_k, t_0 + \delta_k) \cap [0,T],$$

for any k such that $(\boldsymbol{u}, v) \in A_{p,\delta_k,\epsilon_k}$. But since $(\boldsymbol{u}, v) \in A_{p,\delta_k,\epsilon_k}$ for infinitely many k and since $\epsilon_k = 2^{-k} \to 0$ as $k \to \infty$, this implies that

$$\lim_{t \to t_0} (\boldsymbol{u}(t), v(t)) \text{ exists and equals } (\boldsymbol{u}(t_0), v(t_0)).$$

This shows the desired result in (4.50). Therefore, we have shown the first part of the lemma, that $\pi_{4,5}\mu(X \cap C(0,T;\mathcal{Q}')) = 1$.

It remains to show that $\pi_{4,5}\mu$ is supported more specifically on a Borel measurable subset of X that consists entirely of functions that have a version that is in $C(0,T;\mathcal{Q}')$ with value $(\boldsymbol{u}_0, v_0)$ at time $t = 0$. Define the set B_R to be the set of functions $(\boldsymbol{u}, v) \in L^2(0,T;L^2(\Omega_f) \times L^2(\Gamma))$ such that

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \leqslant C_R h^{3/4}, \qquad \text{for all } 0 < h \leqslant T, \tag{4.51}$$

where C_R is the constant from the estimate (4.36).

One can check that for every $R > 0$, every element of $\mathcal{K}_R$ satisfies (4.51) and hence is in B_R. This is because by using (4.36), which states that

$$||(\boldsymbol{u}_N^{n+l}, v_N^{n+l}) - (\boldsymbol{u}_N^n, v_N^n)||_{\mathcal{Q}'} \leqslant C_R(l\Delta t)^{1/4}, \qquad \text{for all } (\boldsymbol{u}_N, v_N) \in \mathcal{K}_R,$$

we have that for all $0 < h \leqslant T$ and for all $(\boldsymbol{u}, v) \in \mathcal{K}_R$,

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||^2_{L^2(0,h;\mathcal{Q}')} \leqslant \int_0^h ||(\boldsymbol{u}(s), v(s)) - (\boldsymbol{u}_0, v_0)||^2_{\mathcal{Q}'} ds \leqslant C_R^2 h \cdot \left(h^{1/4}\right)^2 = C_R^2 h^{3/2}.$$

Furthermore, one checks easily that B_R is closed in $L^2(0,T;L^2(\Omega_f) \times L^2(\Gamma))$ since a sequence that converges in $L^2(0,T;L^2(\Omega_f) \times L^2(\Gamma))$ also converges in $L^2(0,T;\mathcal{Q}')$, in which case one can take the limit in (4.51) to get the corresponding property for the limit function. Since $\mathcal{K}_R \subset B_R$ and B_R is closed in X, we obtain that

$$\overline{\mathcal{K}_R} \subset B_R \subset X, \qquad \text{for all } R > 0.$$

Consider any $\epsilon > 0$. Choose R sufficiently large so that

$$\pi_{4,5}\mu_N(\overline{K_R}) > 1 - \epsilon, \qquad \text{for all } N.$$

Then, by Portmanteau's theorem,

$$\pi_{4,5}\mu(B_R) \geqslant \limsup_{N \to \infty} \pi_{4,5}\mu_N(B_R) \geqslant \limsup_{n \to \infty} \pi_{4,5}\mu_N(\overline{K_R}) \geqslant 1 - \epsilon.$$

So there exists an increasing sequence $\{R_k\}_{k=1}^{\infty}$ such that $\pi_{4,5}\mu\left(\bigcup_{k=1}^{\infty} B_{R_k}\right) = 1$. Thus,

$$\pi_{4,5}\mu\left[\left(\bigcup_{k=1}^{\infty} B_{R_k}\right) \cap C(0,T;\mathcal{Q}')\right] = 1, \text{ where } \left(\bigcup_{k=1}^{\infty} B_{R_k}\right) \cap C(0,T;\mathcal{Q}') = \left(\bigcup_{k=1}^{\infty} B_{R_k}\right) \cap X \cap C(0,T;\mathcal{Q}')$$

is a Borel measurable subset of X. However, we note that any function in $\left(\bigcup_{k=1}^{\infty} B_{R_k}\right) \cap C(0,T;\mathcal{Q}')$ must have the property that its (unique) continuous version taking values in $\mathcal{Q}'$ must be equal to $(\boldsymbol{u}_0, v_0)$ at $t = 0$. To see this, if instead, $(\boldsymbol{u}(0), v(0)) \neq (\boldsymbol{u}_0, v_0)$, let

$$d = ||(\boldsymbol{u}(0), v(0)) - (\boldsymbol{u}_0, v_0)||_{\mathcal{Q}'} > 0.$$

Then, one can show that there exists h_0 such that for all $0 < h \leqslant h_0$,

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \geqslant \frac{d}{2} h^{1/2}.$$

Therefore, this function cannot satisfy an estimate of the type

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \leqslant Ch^{3/4}, \qquad \text{for all } 0 < h \leqslant h_0,$$

for any C, and so this function cannot be in any B_R. This completes the proof. □

Skorokhod representation theorem

We now use the classical Skorokhod representation theorem posed for random variables taking values in *separable Banach spaces* to translate weak convergence of probability measures to almost sure convergence of random variables, which will allow us to pass to the limit in the semidiscrete weak formulation. However, this will be at the expense of working on a different probability space. Namely, the Skorokhod representation theorem provides the *existence* of *a probability space*, on which we will have almost sure convergence of new random variables with the same laws as the original approximate solutions, to a weak solution with the law μ from Theorem 4.8.1. This probability space is not necessarily the same as the original probability space on which our problem is posed. Nevertheless, we can get back to the original probability space by using another result, known as the Gyöngy-Krylov lemma, see Section 4.9, to show that along a subsequence, the original approximate solutions on the original probability space converge almost surely to a limit with the same law μ from Theorem 4.8.1.

More precisely, showing convergence of our approximate solutions almost surely to a weak solution on the original probability space, consists of two steps. First, we use the Skorokhod representation theorem to show that *there exists* a probability space, which we denote by "tilde", on which a sequence of random variables that are equal to our approximate solutions *in law* converges almost surely in $\mathcal{X}$ as $N \to \infty$, to a weak solution on the "tilde" probability space, where the law of this weak solution is equal to μ, obtained in Theorem 4.8.1. Thus, in this step, we prove the existence of a weak solution in a probabilistically *weak* sense, see Definition 4.4.1. Then, in step two, we show using the Gyöngy-Krylov lemma, that we can bring that weak solution back to the original probability space, implying that we will have constructed a weak solution in a probabilistically strong sense, see Definition 4.4.2, of the original continuous problem. This will complete the existence proof, which is the main result of this manuscript.

To achieve these goals, we first obtain almost sure convergence along a subsequence of approximate solutions on a "tilde" probability space using Skorokhod's theorem. A statement of the Skorokhod representation theorem, which holds for probability measures on complete separable metric spaces, can be found in Theorem 2.2.2.

Before we state the result, we introduce the notation "$=_d$" to denote random variables that are "equal in distribution" i.e., the random variables have the same laws as random variables taking values on the same given phase space $\mathcal{X}$. Namely, we will say that a random variable X is equal in distribution (or equal in law) to the random variable $\tilde{X}$, and denote

$$X =_d \tilde{X} \quad \text{if} \quad \mu_X = \mu_{\tilde{X}},$$

where μ_X for example is the probability measure on $\mathcal{X}$ describing the law of the random variable X on $\mathcal{X}$.

Recall again the definition of the laws corresponding to the approximate solutions (4.33), and the definition of the corresponding phase space (4.34).

Lemma 4.8.7. Let μ denote the probability measure obtained as a weak limit of the measures μ_N from Theorem 4.8.1. Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{X}$-valued random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$:

$$(\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W}), \text{ and } (\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N),$$

for each N, such that

$$(\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N) =_d (\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W),$$

for all N, and

$$(\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N) \to (\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W}), \tag{4.52}$$

a.s. in $\mathcal{X}$, as $N \to \infty$, where the law of $(\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W})$ is μ.

Furthermore, the following properties hold:

1. $\tilde{\boldsymbol{u}}_N = \tilde{\boldsymbol{u}}_N^*$, $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}^* = \tilde{\overline{\boldsymbol{u}}} = \tilde{\boldsymbol{u}}^{\Delta t}$ almost surely, $\tilde{v} = \tilde{v}^* = \tilde{\overline{v}} = \tilde{v}^{\Delta t}$ almost surely, and $\tilde{\eta} = \tilde{\overline{\eta}} = \tilde{\eta}^{\Delta t}$ almost surely.

2. $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^2(0,T; H^1(\Omega_f)) \cap L^\infty(0,T; L^2(\Omega_f)))$, $\tilde{v} \in L^2(\tilde{\Omega}; L^\infty(0,T; L^2(\Gamma)))$, and $\tilde{\eta} \in L^2(\tilde{\Omega}; W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H_0^1(\Gamma)))$.

3. $\tilde{\eta}(0) = \eta_0$ almost surely.

4. $\partial_t \tilde{\eta} = \tilde{v}$ almost surely.

5. $(\tilde{\boldsymbol{u}}, \tilde{v}) \in C(0,T; \mathcal{Q}')$ and $(\tilde{\boldsymbol{u}}, \tilde{\eta}) \in \mathcal{W}(0,T)$ almost surely.

6. Define the filtration
$$\tilde{\mathcal{F}}_t = \sigma(\tilde{\eta}(s), \tilde{\boldsymbol{u}}(s), \tilde{v}(s) : 0 \leqslant s \leqslant t). \tag{4.53}$$
Then $\tilde{W}$ is a Brownian motion with respect to $\tilde{\mathcal{F}}_t$.

7. $(\tilde{\boldsymbol{u}}, \tilde{\eta}, \tilde{v})$ is a predictable process with respect to the filtration $\{\tilde{\mathcal{F}}_t\}_{0 \leqslant t \leqslant T}$.

Proof. The existence of the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the given random variables follows from the previous result on weak convergence in Theorem 4.8.1 and the Skorokhod representation theorem. So it suffices to prove the given properties.

Property 1: Because $(\tilde{\boldsymbol{u}}_N, \tilde{\boldsymbol{u}}_N^*) =_d (\boldsymbol{u}_N, \boldsymbol{u}_N)$, we have that $\tilde{\boldsymbol{u}}_N - \tilde{\boldsymbol{u}}_N^* =_d 0$ as random variables taking values in $L^2(0,T;L^2(\Omega_f))$, so $\tilde{\boldsymbol{u}}_N = \tilde{\boldsymbol{u}}_N^*$ a.s. for all N. Hence, by taking the limit as $N \to \infty$, we obtain $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}^*$ a.s., since $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}_N^* \to \tilde{\boldsymbol{u}}^*$ in $L^2(0,T;L^2(\Omega_f))$ a.s.

Because $\boldsymbol{u}_N$ and $\overline{\boldsymbol{u}}_N$ actually have different laws from each other, we must use a different argument to conclude that $\tilde{\boldsymbol{u}} = \tilde{\overline{\boldsymbol{u}}}$ a.s. However, we recall the following fact (4.40) from the proof of Lemma 4.8.3,

$$\mathbb{E}\left(||\boldsymbol{u}_N - \overline{\boldsymbol{u}}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \to 0, \qquad \text{as } N \to \infty.$$

Hence, by the equivalence of laws,

$$\tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N - \tilde{\overline{\boldsymbol{u}}}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \to 0, \qquad \text{as } N \to \infty.$$

Therefore, along a further subsequence, $||\tilde{\boldsymbol{u}}_N - \tilde{\overline{\boldsymbol{u}}}_N||^2_{L^2(0,T;L^2(\Omega_f))} \to 0$ almost surely, by a standard Borel Cantelli lemma argument. Since $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ and $\tilde{\overline{\boldsymbol{u}}}_N \to \tilde{\overline{\boldsymbol{u}}}$ in $L^2(0,T;L^2(\Omega_f))$, we conclude that $\tilde{\boldsymbol{u}} = \tilde{\overline{\boldsymbol{u}}}$ a.s.

The remaining statements follow from the same argument as above. In particular, by using the estimates (4.37)–(4.43) from the proof of Lemma 4.8.3, the equivalence of laws, and the almost sure convergence of the "tilde" random variables in (4.52), we obtain the desired result.

Property 2: These properties will all be handled similarly. By the uniform energy estimates in Lemma 4.7.2 and Lemma 4.7.3, we have that

$$\mathbb{E}\left(||\overline{\eta}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \leqslant C,\ \mathbb{E}\left(||\overline{\eta}_N||^2_{L^\infty(0,T;H_0^1(\Gamma))}\right) \leqslant C,$$

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t}||^2_{L^2(0,T;H^1(\Omega_f))}\right) \leqslant C,\ \mathbb{E}\left(||\boldsymbol{u}_N||^2_{L^\infty(0,T;L^2(\Omega_f))}\right) \leqslant C,\ \mathbb{E}\left(||v_N||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \leqslant C,$$

for a constant C that is independent of N. Therefore, by the equivalence of laws, we have that these uniform estimates hold for the random variables on the new probability space, so that

$$\tilde{\mathbb{E}}\left(||\tilde{\overline{\eta}}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \leqslant C,\ \tilde{\mathbb{E}}\left(||\tilde{\overline{\eta}}_N||^2_{L^\infty(0,T;H_0^1(\Gamma))}\right) \leqslant C,$$

$$\tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N^{\Delta t}||^2_{L^2(0,T;H^1(\Omega_f))}\right) \leqslant C,\ \tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N||^2_{L^\infty(0,T;L^2(\Omega_f))}\right) \leqslant C,\ \tilde{\mathbb{E}}\left(||\tilde{v}_N||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \leqslant C,$$

for a constant C that is independent of N. Therefore, by this uniform boundedness, we conclude for example that $\tilde{\overline{\eta}}_N$ converges weakly star in $L^2(\tilde{\Omega}; W^{1,\infty}(0,T;L^2(\Gamma)))$ and weakly star in $L^2(\tilde{\Omega}; L^\infty(0,T; H_0^1(\Gamma)))$. Since we already have that $\tilde{\overline{\eta}}_N$ converges to $\tilde{\overline{\eta}}$ almost surely in $L^2(0,T;L^2(\Gamma))$ and $\tilde{\overline{\eta}} = \tilde{\eta}$ almost surely by Property 1, by the uniqueness of this limit, we conclude that $\tilde{\overline{\eta}}_N \rightharpoonup \tilde{\eta}$, weakly star in $L^2(\tilde{\Omega}; W^{1,\infty}(0,T;L^2(\Gamma)))$ and $L^2(\tilde{\Omega}; L^\infty(0,T;H_0^1(\Gamma)))$.

Similarly, $\tilde{\boldsymbol{u}}_N^{\Delta t} \rightharpoonup \tilde{\boldsymbol{u}}^{\Delta t}$, weakly $L^2(\tilde{\Omega}; L^2(0,T;H^1(\Omega_f)))$, $\tilde{\boldsymbol{u}}_N \rightharpoonup \tilde{\boldsymbol{u}}$ weakly star in the space $L^2(\tilde{\Omega}; L^\infty(0,T;L^2(\Omega_f)))$, and $\tilde{v}_N \rightharpoonup \tilde{v}$ weakly star in $L^2(\tilde{\Omega}; L^\infty(0,T;L^2(\Gamma)))$. This establishes Property 2.

Property 3: Since $\tilde{\eta} = \tilde{\overline{\eta}}$ almost surely, it suffices to show that $\tilde{\overline{\eta}}(0) = \eta_0$ almost surely. To do this, we use a method similar to the method in the proof of Lemma 4.8.6. We define

$$D_M = \{\eta \in L^2(0,T;L^2(\Gamma)) : ||\eta(\cdot) - \eta_0||_{L^2(0,h,L^2(\Gamma))} \leqslant Mh^{3/2}, \text{ for all } 0 < h \leqslant T\}. \quad (4.54)$$

Because of the uniform bound $\mathbb{E}\left(||\overline{\eta}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \leqslant C$ for all N, from Lemma 4.7.3, we have that

$$\mathbb{P}(\overline{\eta}_N \in D_M) \geqslant 1 - \frac{C}{M^2} \qquad \text{for all } M \text{ and } N,$$

by using Chebychev's inequality. This is because if $||\overline{\eta}_N||_{W^{1,\infty}(0,T;L^2(\Gamma))} \leqslant M$, then from the fact that $\overline{\eta}_N(0) = \eta_0$ for all $\omega \in \Omega$ and N, we have that

$$||\overline{\eta}(\cdot) - \eta_0||_{L^2(0,h,L^2(\Gamma))} = \left(\int_0^h ||\overline{\eta}(s) - \eta_0||^2_{L^2(\Gamma)} ds\right)^{1/2} \leqslant \left(\int_0^h (Ms)^2 ds\right)^{1/2} \leqslant Mh^{3/2}.$$

Then, by equivalence of laws,

$$\tilde{\mathbb{P}}(\tilde{\overline{\eta}}_N \in D_M) \geqslant 1 - \frac{C}{M^2} \qquad \text{for all } M \text{ and } N.$$

Because D_M is a closed set in $L^2(0,T;L^2(\Gamma))$ and $\tilde{\overline{\eta}}_N \to \tilde{\overline{\eta}}$ in $L^2(0,T;L^2(\Gamma))$ a.s., we conclude that

$$\tilde{\mathbb{P}}(\tilde{\overline{\eta}} \in D_M) \geqslant \limsup_{N\to\infty} \tilde{\mathbb{P}}(\tilde{\overline{\eta}}_N \in D_M) \geqslant 1 - \frac{C}{M^2} \text{ for all } M, \text{ which implies } \tilde{\mathbb{P}}\left(\tilde{\overline{\eta}} \in \bigcup_{M=1}^\infty D_M\right) = 1.$$

Because $\tilde{\overline{\eta}}$ is almost surely continuous on $[0,T]$ taking values in $L^2(\Gamma)$ by Property 2, we obtain $\tilde{\overline{\eta}}(0) = \eta_0$ almost surely. This is because if a continuous function η on $[0,T]$ taking values in $L^2(\Gamma)$ has $\eta(0) \neq \eta_0$, then

$$||\eta(\cdot) - \eta_0||_{L^2(0,h;L^2(\Gamma))} \geqslant \frac{d}{2} h^{1/2},$$

for all h sufficiently small where $d = ||\eta(0) - \eta_0||_{L^2(\Gamma)}$, and hence η cannot belong to $\bigcup_{M=1}^\infty D_M$.

Property 4: To prove this property, we recall from the second equation in the semidiscrete formulation (4.25) that

$$\int_\Gamma \frac{\eta_N^{n+1} - \eta_N^n}{\Delta t} \phi dz = \int_\Gamma v_N^{n+\frac{1}{3}} \phi dz,$$

almost surely for all $\phi \in L^2(\Gamma)$. Integrating in time, we obtain for all N that

$$\int_0^T \int_\Gamma \partial_t \overline{\eta}_N \phi dz dt = \int_0^T \int_\Gamma v_N^* \phi dz dt, \qquad \text{for all } \phi \in C^1([0,T); L^2(\Gamma)),$$

almost surely. Because each $\overline{\eta}_N$ is almost surely a piecewise linear continuous function satisfying $\overline{\eta}(0) = \eta_0$, we obtain by integration by parts that almost surely, for all $\phi \in C^1([0,T); L^2(\Gamma))$,

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \overline{\eta}_N \partial_t \phi dz dt = \int_0^T \int_\Gamma v_N^* \phi dz dt,$$

and hence, by equivalence of laws,

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \tilde{\overline{\eta}}_N \partial_t \phi dz dt = \int_0^T \int_\Gamma \tilde{v}_N^* \phi dz dt.$$

Passing to the limit, we obtain

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \tilde{\eta} \partial_t \phi dz dt = \int_0^T \int_\Gamma \tilde{v} \phi dz dt,$$

for all $\phi \in C^1([0,T); L^2(\Gamma))$, almost surely. This implies that $\partial_t \tilde{\eta} = \tilde{v}$ holds almost surely for the limiting solution, since we showed in Property 3 that $\tilde{\eta}(0) = \eta_0$ almost surely.

Property 5: The fact that $(\tilde{\boldsymbol{u}}, \tilde{v}) \in C(0,T; \mathcal{Q}')$ almost surely follows from Lemma 4.8.6, since the limiting random variables with the tildes have their law given by the probability measure μ. So it remains to show that $(\tilde{\boldsymbol{u}}, \tilde{v}) \in \mathcal{W}(0,T)$, where $\mathcal{W}(0,T)$ is defined in (4.12).

To establish this result, first notice that we already know from Property 2 that $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^\infty(0,T; L^2(\Omega_f)))$ and $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^2(0,T; H^1(\Omega_f)))$, and Property 2 already gives the desired result for the structure. Thus, it remains to show that $\tilde{\boldsymbol{u}} \in L^2(0,T; \mathcal{V}_F)$ almost surely, where $\mathcal{V}_F$ is defined in (4.8), and that the kinematic coupling condition holds. By Property 4, we must show in particular that $\tilde{\boldsymbol{u}} = \tilde{v}\boldsymbol{e}_r$ a.s. on Γ.

To do this, define the deterministic function space

$$\mathcal{H} = \{(\boldsymbol{u}, v) \in L^2(0,T; \mathcal{V}_F) \times L^2(0,T; L^2(\Gamma)) : \boldsymbol{u} = v\boldsymbol{e_r} \text{ for almost every } t \in [0,T]\}.$$

One can check that the linear subspace $\mathcal{H} \subset L^2(0,T; H^1(\Omega_f)) \times L^2(0,T; L^2(\Gamma))$ is closed in the Hilbert space $L^2(0,T; H^1(\Omega_f)) \times L^2(0,T; L^2(\Gamma))$, and hence $\mathcal{H}$ is a Hilbert space with the inner product of $L^2(0,T; H^1(\Omega_f)) \times L^2(0,T; L^2(\Gamma))$. By equivalence of laws and the uniform boundedness in Lemma 4.7.2, $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N)$ is uniformly bounded in $L^2(\tilde{\Omega}; \mathcal{H})$, and hence converges weakly to $(\tilde{\boldsymbol{u}}, \tilde{v}) \in L^2(\tilde{\Omega}; \mathcal{H})$ by uniqueness of the limit, since we already have that $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N)$ converges almost surely to $(\tilde{\boldsymbol{u}}, \tilde{v})$ in $L^2(0,T; L^2(\Omega_f)) \times L^2(0,T; L^2(\Gamma))$. This gives the desired result.

Property 6: First, we sketch the idea. By construction, we have that on the original probability space, $W(t) - W(s)$ is independent of $\sigma(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau), \text{ for } 0 \leqslant \tau \leqslant s)$, where we recall that these processes $(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau))$ are piecewise constant on intervals of length $\Delta t = T/N$. This is because for a given time $\tau \in [0, T]$, $(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau))$ depends only on the values of the Brownian motion at time $\lfloor \frac{\tau}{\Delta t} \rfloor \Delta t$ or earlier, from which the claim follows by the independent increments property of Brownian motion. The idea will be to transfer this independence property over to the new random variables $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\eta}_N)$ on the new probability space $(\tilde{\Omega}, \tilde{F}, \tilde{\mathbb{P}})$ and then take a limit as $N \to \infty$ to get the desired independence in the limit.

Note that the definition of $\tilde{\mathcal{F}}_t$ as

$$\tilde{\mathcal{F}}_t = \sigma(\tilde{\boldsymbol{u}}(s), \tilde{v}(s), \tilde{\eta}(s), \text{ for } 0 \leqslant s \leqslant t)$$

makes sense, since by the above properties, $\tilde{\eta}$ and $(\tilde{\boldsymbol{u}}, \tilde{v})$ are continuous on $[0, T]$ in time, taking values in $L^2(\Gamma)$ and $\mathcal{Q}'$ respectively. So it makes sense to refer to values pointwise at specific times, for example as in $\tilde{\boldsymbol{u}}(\tau)$ for a given $\tau \in [0, T]$. However, it is not clear yet, for example, what $\tilde{\boldsymbol{u}}_N(\tau)$ would be, since a priori, we only know that $\tilde{\boldsymbol{u}}_N \in L^2(0, T; L^2(\Omega_f))$, and hence, each path of $\tilde{\boldsymbol{u}}_N$ is only defined up to a version for $t \in [0, T]$.

To handle this, define the set K_N of all functions in $L^2(0, T; L^2(\Omega_f))$ that have a version that is piecewise constant on the intervals of the form $[0, \Delta t]$ and $(n\Delta t, (n+1)\Delta t]$ for $1 \leqslant n \leqslant N-1$, where $\Delta t = T/N$. Note that K_N is a closed subset of $L^2(0, T; L^2(\Omega_f))$, so by equivalence of laws,

$$\tilde{\mathbb{P}}(\tilde{\boldsymbol{u}}_N \in K_N) = \mathbb{P}(\boldsymbol{u}_N \in K_N) = 1.$$

Therefore, $\tilde{\boldsymbol{u}}_N$ is almost surely piecewise constant on $[0, \Delta t]$ and $(n\Delta t, (n+1)\Delta t]$ for $1 \leqslant n \leqslant N-1$. The same argument shows that $\tilde{v}_N$ and $\tilde{\eta}_N$ also almost surely have versions that are piecewise constant on these same intervals, since v_N and η_N on the original probability space almost surely have this property too.

Therefore, for each N, up to taking a version of $\tilde{\boldsymbol{u}}_N$, $\tilde{v}_N$, and $\tilde{\eta}_N$, we can define random variables $\tilde{\boldsymbol{u}}_N^n$, $\tilde{v}_N^n$, and $\tilde{\eta}_N^n$ for $0 \leqslant n \leqslant N-1$, satisfying

$$\begin{aligned}
\tilde{\boldsymbol{u}}_N(t, \omega) &= \tilde{\boldsymbol{u}}_N^0(\omega), \quad \text{if } 0 \leqslant t \leqslant \Delta t \text{ and } \tilde{\boldsymbol{u}}_N(t, \omega) = \tilde{\boldsymbol{u}}_N^n(\omega), \quad \text{if } n\Delta t < t \leqslant (n+1)\Delta t, \\
\tilde{v}_N(t, \omega) &= \tilde{v}_N^0(\omega), \quad \text{if } 0 \leqslant t \leqslant \Delta t \text{ and } \tilde{v}_N(t, \omega) = \tilde{v}_N^n(\omega), \quad \text{if } n\Delta t < t \leqslant (n+1)\Delta t, \\
\tilde{\eta}_N(t, \omega) &= \tilde{\eta}_N^0(\omega), \quad \text{if } 0 \leqslant t \leqslant \Delta t \text{ and } \tilde{\eta}_N(t, \omega) = \tilde{\eta}_N^n(\omega), \quad \text{if } n\Delta t < t \leqslant (n+1)\Delta t.
\end{aligned}$$

Furthermore, by the equivalence of laws, the joint distribution of $\tilde{\boldsymbol{u}}_N^n, \tilde{v}_N^n, \tilde{\eta}_N^n$ for $0 \leqslant n \leqslant N-1$ is the same as that of $\boldsymbol{u}_N^n, v_N^n, \eta_N^n$ for $0 \leqslant n \leqslant N-1$. Therefore, we can now make sense of $\tilde{\boldsymbol{u}}_N(\tau)$ for example for any $\tau \in [0, T]$, by considering the piecewise constant versions of these stochastic processes as given above. When we refer to $\tilde{\boldsymbol{u}}_N$, $\tilde{v}_N$, and $\tilde{\eta}_N$, we will refer to the piecewise constant versions defined above.

We now show the desired independence. We consider $\tau_0 \in [0, s]$ and $0 \leqslant s \leqslant t$, and show that $\tilde{\boldsymbol{u}}(\tau_0)$ and $\tilde{W}(t) - \tilde{W}(s)$ are independent. The same argument will work for $v(\tau_0)$ and $\eta(\tau_0)$, so it suffices to show the independence of $\tilde{W}(t) - \tilde{W}(s)$ and $\tilde{\boldsymbol{u}}(\tau_0)$ for arbitrary $\tau_0 \in [0, s]$ and $0 \leqslant s \leqslant t$.

Recall that $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ almost surely in $L^2(0,T;L^2(\Omega_f))$. Define the set

$$E_{N,n} = \{(t,\omega) \in [0,T] \times \tilde{\Omega} : ||\tilde{\boldsymbol{u}}(t,\omega,\cdot) - \tilde{\boldsymbol{u}}_N(t,\omega,\cdot)||_{L^2(\Omega_f)} \geqslant 2^{-n}\}.$$

For each positive integer n, we can choose $N := N(n)$ sufficiently large such that $N(n) > N(n-1)$ for $n \geqslant 2$, and

$$(dt \times \tilde{\mathbb{P}})(E_{N(n),n}) \leqslant 2^{-n}. \tag{4.55}$$

To see this, one selects $N(n)$ sufficiently large so that

$$\tilde{\mathbb{P}}\left(||\tilde{u} - \tilde{u}_{N(n)}||_{L^2(0,T;L^2(\Omega_f))} \leqslant 2^{-2n}\right) \geqslant 1 - 2^{-2n},$$

and then apply Chebychev's inequality in time. Then, by applying the Borel Cantelli lemma to (4.55), we obtain that

$$\tilde{\boldsymbol{u}}_N(t,\omega,\cdot) \to \tilde{\boldsymbol{u}}(t,\omega,\cdot) \qquad \text{in } L^2(\Omega_f), \tag{4.56}$$

for all $(t,\omega) \in S \subset [0,T] \times \tilde{\Omega}$ for a set S satisfying $(dt \times \tilde{\mathbb{P}})(S) = T$, where we continue to denote the new subsequence $N(n)$ by N. Thus, $([0,T] \times \tilde{\Omega}) - S$ has measure zero with respect to the measure $(dt \times \tilde{\mathbb{P}})$.

Let $S_0 \subset [0,T]$ be the set of all $t \in [0,T]$ such that $\tilde{\mathbb{P}}((t,\omega) \in S) = 1$. By Fubini's theorem, S_0 is a measurable subset of $[0,T]$ for which $[0,T] - S_0$ has measure zero. Note that for each $t \in S_0$, $\tilde{\boldsymbol{u}}_N(t,\cdot) \to \tilde{\boldsymbol{u}}(t,\cdot)$ almost surely as random variables taking values in $L^2(\Omega_f)$.

So if $\tau_0 \in S_0$, we deduce the independence of $\tilde{\boldsymbol{u}}(\tau_0)$ and $\tilde{W}(t) - \tilde{W}(s)$ as follows. By the fact that $\boldsymbol{u}_N(\tau_0)$ and $W(t) - W(s)$ are independent, we have by equivalence of laws that

$$\tilde{\boldsymbol{u}}_N(\tau_0) \text{ and } \tilde{W}_N(t) - \tilde{W}_N(s) \text{ are independent.}$$

Here, N denotes the subsequence $N(n)$ we used to define S and S_0. However, since $\tau_0 \in S_0$, we have that $\tilde{\boldsymbol{u}}(\tau_0)$ is the almost sure limit of $\tilde{\boldsymbol{u}}_N(\tau_0)$, and furthermore, $\tilde{W}(t) - \tilde{W}(s)$ is the almost sure limit of $\tilde{W}_N(t) - \tilde{W}_N(s)$. So since the almost sure limits of independent random variables are independent, this gives the desired result.

If $\tau_0 \notin S_0$, since $[0,T] - S_0$ has measure zero in $[0,T]$, there exists a sequence $\tau_i \in S_0$ that converges to τ_0 as $i \to \infty$, where $\tau_i \in [0,s]$. Then, since $\tilde{\boldsymbol{u}}(\tau_i)$ and $\tilde{W}(t) - \tilde{W}(s)$ are independent for each i and since $\tilde{\boldsymbol{u}}(\tau_i) \to \tilde{\boldsymbol{u}}(\tau_0)$ almost surely by continuity, the result follows. (For the case of $\tau_0 = 0$, we recall from Lemma 4.8.6, that $(\tilde{\boldsymbol{u}}(0), \tilde{v}(0)) = (\boldsymbol{u}_0, v_0)$ almost surely.)

We use the equivalence of laws to verify the remaining properties of Brownian motion. In particular, we just need to show that $\tilde{W}(t) - \tilde{W}(s)$ is distributed as $N(0, t-s)$. By the equivalence of laws and the fact that W is originally a Brownian motion, $\tilde{W}_N(t) - \tilde{W}_N(s) =_d W(t) - W(s)$, so that $\tilde{W}_N(t) - \tilde{W}_N(s)$ is distributed as $N(0, t-s)$. Since $\tilde{W}_N \to \tilde{W}$ a.s. in $C(0,T;\mathbb{R})$, we obtain that $\tilde{W}_N(t) - \tilde{W}_N(s) \to \tilde{W}(t) - \tilde{W}(s)$ almost surely, so that $\tilde{W}(t) - \tilde{W}(s)$ is the almost sure limit of random variables distributed as $N(0, t-s)$. Thus, we conclude that $\tilde{W}(t) - \tilde{W}(s)$ must also be distributed as $N(0, t-s)$, which concludes the proof of Property 6.

Property 7: By the definition of $\tilde{\mathcal{F}}_t$, the process $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ is adapted to $\tilde{\mathcal{F}}_t$. By Property 2, $\tilde{\eta}$ almost surely has continuous paths on $[0, T]$, taking values in $L^2(\Omega_f)$. By Property 5, $(\tilde{\boldsymbol{u}}, \tilde{v})$ almost surely has continuous paths on $[0, T]$, taking values in $\mathcal{Q}'$. Since a continuous adapted process is predictable (see Proposition 5.1 in Chapter IV of Revuz and Yor [155]), this establishes the desired property.

This completes the proof of Lemma 4.8.7. □

Passing to the limit

We now consider the approximate solutions defined as random variables on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, discussed in Lemma 4.8.7, and show that the almost sure limit obtained in Lemma 4.8.7, satisfies the weak formulation stated in Definition 4.4.1, almost surely on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For this purpose, we recall the semidiscrete formulation of the problem from (4.25), given by

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \psi dz$$
$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \ \forall (\boldsymbol{q}, \psi) \in \mathcal{Q},$$

$$\int_\Gamma \frac{\eta_N^{n+1} - \eta_N^n}{\Delta t} \phi dz = \int_\Gamma v_N^{n+\frac{1}{3}} \phi dz, \qquad \forall \phi \in L^2(\Gamma),$$

where $P_{N,in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt$. Notice that as stated, this semidiscrete formulation refers to the original variables, defined on the original probability space. Given a general $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0, T)$, we use the semidiscrete formulation at each fixed time and integrate in time from 0 to T to obtain for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0, T)$,

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt + \int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dz dt$$
$$+ \int_0^T \int_\Gamma \nabla \eta_N^{\Delta t} \cdot \nabla \psi dz dt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz dt$$
$$+ \sum_{n=0}^{N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n \int_0^R (q_z)|_{z=0} dr dt - \int_{n\Delta t}^{(n+1)\Delta t} P_{N,out}^n \int_0^R (q_z)|_{z=L} dr dt \right),$$

$$\int_0^T \int_\Gamma \partial_t \overline{\eta}_N \phi dz dt = \int_0^T \int_\Gamma v_N^* \phi dz dt, \quad \forall \phi \in C^1(0, T; L^2(\Gamma)),$$

where $\overline{\boldsymbol{u}}_N, \overline{v}_N$ and $\overline{\eta}_N$ are the piecewise linear approximations, given by (4.28) and (4.31), and $\boldsymbol{u}_N^{\Delta t}$ and $\eta_N^{\Delta t}$ are the piecewise constant time shifted functions, given by (4.30) and

(4.32). Now, we convert to the new probability space by noticing that the same identities hold for the new random variables defined on the "tilde" probability space since the two sets of random variables have the same law on $\mathcal{X}$. So for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$, on the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the filtration $\{\tilde{\mathcal{F}}_t\}_{t\geqslant 0}$ defined in (4.53), we obtain

$$\int_0^T \int_{\Omega_f} \partial_t \tilde{\overline{\boldsymbol{u}}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt + \int_0^T \int_\Gamma \partial_t \tilde{\overline{v}}_N \psi dz dt$$
$$+ \int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t} \psi dz dt$$
$$+ \sum_{n=0}^{N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - \int_{n\Delta t}^{(n+1)\Delta t} P_{N,out}^n \int_0^R (q_z)|_{z=L} dr dt \right),$$

$$\int_0^T \int_\Gamma \partial_t \tilde{\overline{\eta}}_N \phi dz dt = \int_0^T \int_\Gamma \tilde{v}_N^* \phi dz dt \quad \forall \phi \in C^1(0,T; L^2(\Gamma)).$$

We can now pass to the limit in all of the integrals, and use the almost sure convergence of the "tilde" random variables as follows.

First term: For the functions on the original probability space, note that because $\boldsymbol{q}(T) = 0$, we can integrate by parts to obtain

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x} dt = -\int_0^T \int_{\Omega_f} \overline{\boldsymbol{u}}_N \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) d\boldsymbol{x}.$$

By equivalence of laws, this identity also holds with $\tilde{\overline{\boldsymbol{u}}}_N$ in place of $\overline{\boldsymbol{u}}_N$. Then, because $\tilde{\overline{\boldsymbol{u}}}_N \to \tilde{\boldsymbol{u}}$ almost surely in $L^2(0,T; L^2(\Omega_f))$, we can pass to the limit to obtain the desired almost sure convergence,

$$\int_0^T \int_{\Omega_f} \partial_t \tilde{\overline{\boldsymbol{u}}}_N \cdot \boldsymbol{q} d\boldsymbol{x} \to -\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) d\boldsymbol{x}.$$

Third term: For the third term, we use an argument similar to that for the first term. Since $\psi(T) = 0$, we can integrate by parts,

$$\int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dz dt = -\int_0^T \int_\Gamma \overline{v}_N \partial_t \psi dz dt - \int_\Gamma v_0 \psi(0) dz.$$

This holds with $\tilde{\overline{v}}_N$ in place of $\overline{v}_N$ too by equivalence of laws. Since $\tilde{\overline{v}}_N \to \tilde{v}$ in $L^2(0,T; L^2(\Gamma))$ almost surely, we have the desired almost sure convergence:

$$\int_0^T \int_\Gamma \partial_t \tilde{\overline{v}}_N \psi dz dt = -\int_0^T \int_\Gamma \tilde{\overline{v}}_N \partial_t \psi dz dt - \int_\Gamma v_0 \psi(0) dz \to -\int_0^T \int_\Gamma \tilde{v} \partial_t \psi dz dt - \int_\Gamma v_0 \psi(0) dz.$$

Second and fourth term with smooth test function: For the second and fourth term, we have to use an approximation argument, since we only have estimates of convergence of $\tilde{\boldsymbol{u}}_N$ and $\tilde{\boldsymbol{u}}_N^{\Delta t}$ in $L^2(0,T;L^2(\Omega_f))$ and $\tilde{v}_N$ in $L^2(0,T;L^2(\Gamma))$.

We will first show the desired convergence under the assumption that $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$ is spatially smooth at each time in $[0,T]$. Then, on the original probability space,

$$2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}_N^{\Delta t}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt = \mu\int_0^T\int_{\Omega_f}\nabla\boldsymbol{u}_N^{\Delta t}:\nabla\boldsymbol{q}d\boldsymbol{x}dt = -\mu\int_0^T\int_{\Omega_f}\boldsymbol{u}_N^{\Delta t}\cdot\Delta\boldsymbol{q}d\boldsymbol{x}dt,$$

where the last integration by parts has no boundary terms due to the properties of the solution space and test space for the fluid. Then, by the uniform dissipation estimate in Proposition 4.6.7, $\sum_{n=0}^{N-1}\mathbb{E}\left(||\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leqslant C$, we have that

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t}-\boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N\to\infty.$$

By equivalence of laws, the above identities and estimates hold for $\tilde{\boldsymbol{u}}_N^{\Delta t}$ in place of $\boldsymbol{u}_N^{\Delta t}$. By the Borel-Cantelli lemma, we have that

$$||\tilde{\boldsymbol{u}}_N^{\Delta t}-\tilde{\boldsymbol{u}}_N||_{L^2(0,T;L^2(\Omega_f))} \to 0 \text{ almost surely as } N\to\infty,$$

taking a subsequence if needed. Because $\tilde{\boldsymbol{u}}_N$ converges to $\tilde{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega_f))$ as $N\to\infty$, we also have that

$$||\tilde{\boldsymbol{u}}_N^{\Delta t}-\tilde{\boldsymbol{u}}||_{L^2(0,T;L^2(\Omega_f))} \to 0 \text{ almost surely as } N\to\infty$$

along this subsequence, which allows us to pass to the limit to obtain

$$\begin{aligned}2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt &= -\mu\int_0^T\int_{\Omega_f}\tilde{\boldsymbol{u}}_N^{\Delta t}\cdot\Delta\boldsymbol{q}d\boldsymbol{x}dt \\ \to -\mu\int_0^T\int_{\Omega_f}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{q}d\boldsymbol{x}dt &= 2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}(\tilde{\boldsymbol{u}}):\boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt.\end{aligned} \tag{4.57}$$

For the fourth term, one can use a similar argument under the assumption that the test function $(\boldsymbol{q},\psi)$ is spatially smooth. On the original probability space,

$$\int_0^T\int_\Gamma\nabla\eta_N^{\Delta t}\cdot\nabla\psi dzdt = -\int_0^T\int_\Gamma\eta_N^{\Delta t}\cdot\Delta\psi dzdt.$$

By the numerical dissipation estimate from Lemma 4.6.7, $\sum_{n=0}^{N-1}\mathbb{E}\left(||\nabla\eta_N^{n+\frac{1}{3}}-\nabla\eta_N^n||^2_{L^2(\Gamma)}\right) \leqslant C$, so we obtain, by Poincaré's inequality, that

$$\mathbb{E}\left(||\eta_N^{\Delta t}-\eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leqslant C(\Delta t) \to 0, \qquad \text{as } N\to\infty.$$

These estimates hold on the new probability space with $\tilde{\eta}_N$ in place of η_N. By the Borel-Cantelli lemma and the convergence of $\tilde{\eta}_N$ to $\tilde{\eta}$ in $L^2(0,T;L^2(\Gamma))$,

$$||\tilde{\eta}_N^{\Delta t} - \tilde{\eta}||_{L^2(0,T;L^2(\Gamma))} \to 0, \qquad \text{almost surely as } N \to \infty,$$

taking a subsequence. This allows us to pass to the limit to obtain the almost sure convergence,

$$\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt = -\int_0^T \int_\Gamma \tilde{\eta}_N^{\Delta t} \cdot \Delta \psi dz dt \tag{4.58}$$
$$\to -\int_0^T \int_\Gamma \tilde{\eta} \cdot \Delta \psi dz dt = \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt, \qquad \text{as } N \to \infty.$$

Second and fourth term with general test function: To show the almost sure convergence in the previous step, we assumed that $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$ was spatially smooth. To get the general convergence, we use an approximation argument. Suppose that $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$ is not smooth spatially. It suffices to show that $\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt \to \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt$ in probability, and $\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt \to \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt$ in probability (see below for the precise definition), as we would get the desired result from the fact that we then have almost sure convergence along a subsequence. So given any $\epsilon > 0$ and $\delta > 0$, we must show that there exists N_0 such that for all $N \geqslant N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt\right| > \delta\right) \leqslant \epsilon, \tag{4.59}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt - \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt\right| > \delta\right) \leqslant \epsilon. \tag{4.60}$$

To show this, observe that by the uniform dissipation estimate in Proposition 4.6.7, we have that

$$\mathbb{E}\left(\sum_{n=0}^{N-1} (\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x}\right) = \mathbb{E}\left(||\boldsymbol{D}(\boldsymbol{u}_N^{\Delta t})||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leqslant C.$$

and hence by equivalence of laws,

$$\tilde{\mathbb{E}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t})||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leqslant C,$$

for a uniform constant C. Since $\tilde{\boldsymbol{u}} \in L^2(\Omega; L^2(0,T;H^1(\Omega_f)))$ by Property 2 of Lemma 4.8.7, we conclude that there exists a sufficiently large positive constant M such that for all N,

$$\tilde{\mathbb{P}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t})||_{L^2(0,T;L^2(\Omega_f))} \geqslant M\right) \leqslant \frac{\epsilon}{3}, \qquad \tilde{\mathbb{P}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}})||_{L^2(0,T;L^2(\Omega_f))} \geqslant M\right) \leqslant \frac{\epsilon}{3}. \tag{4.61}$$

For the fourth term involving structure displacements, recall from Lemma 4.6.7 that

$$\mathbb{E}\left(||\nabla \eta_N^{\Delta t}||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \leqslant C,$$

and by Property 2 in Lemma 4.8.7, $\tilde{\eta} \in L^2(\tilde{\Omega}; L^\infty(0,T; H_0^1(\Gamma)))$. So using equivalence of laws, M can also be chosen sufficiently large so that for all N,

$$\tilde{\mathbb{P}}\left(||\nabla \tilde{\eta}_N^{\Delta t}||_{L^\infty(0,T;L^2(\Gamma))} \geqslant M\right) \leqslant \frac{\epsilon}{3}, \qquad \tilde{\mathbb{P}}\left(||\nabla \tilde{\eta}||_{L^\infty(0,T;L^2(\Gamma))} \geqslant M\right) \leqslant \frac{\epsilon}{3}. \tag{4.62}$$

Then, choose $(\widehat{\boldsymbol{q}}, \widehat{\psi}) \in \mathcal{Q}(0,T)$ that are smooth spatially at all times in $[0,T]$, such that

$$||\boldsymbol{D}(\boldsymbol{q}) - \boldsymbol{D}(\widehat{\boldsymbol{q}})||_{L^2(0,T;L^2(\Omega_f))} \leqslant \frac{\delta}{3M}, \qquad ||\nabla\psi - \nabla\widehat{\psi}||_{L^1(0,T;L^2(\Gamma))} \leqslant \frac{\delta}{3M}. \tag{4.63}$$

Then, the almost sure convergences (4.57) and (4.58), which hold for this smoother $(\widehat{\boldsymbol{q}}, \widehat{\psi})$, allow us to choose N_0 sufficiently large such that for all $N \geqslant N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x}dt - \int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x}dt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}, \tag{4.64}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_\Gamma \nabla\tilde{\eta}_N^{\Delta t} \cdot \nabla\widehat{\psi} dzdt - \int_0^T\int_\Gamma \nabla\tilde{\eta} \cdot \nabla\widehat{\psi} dzdt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}. \tag{4.65}$$

Furthermore, the choice of $(\widehat{\boldsymbol{q}}, \widehat{\psi})$ in (4.63) and the choice of M in (4.61) and (4.62) give that for all N,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x}dt - \int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x}dt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}, \tag{4.66}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_\Gamma \nabla\tilde{\eta}_N^{\Delta t} \cdot \nabla\psi dzdt - \int_0^T\int_\Gamma \nabla\tilde{\eta}_N^{\Delta t} \cdot \nabla\widehat{\psi} dzdt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}, \tag{4.67}$$

and

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x}dt - \int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x}dt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}, \tag{4.68}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_\Gamma \nabla\tilde{\eta} \cdot \nabla\psi dzdt - \int_0^T\int_\Gamma \nabla\tilde{\eta} \cdot \nabla\widehat{\psi} dzdt\right| > \frac{\delta}{3}\right) \leqslant \frac{\epsilon}{3}. \tag{4.69}$$

Combining the estimates (4.64), (4.65), (4.66), (4.67), (4.68), and (4.69) establishes the desired estimates (4.59) and (4.60), and hence proves the desired convergence in probability.

Passing to the limit in the stochastic integral. We want to pass to the limit in the stochastic integral and show that for arbitrary ψ such that $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$\sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t} \psi dz dt \to \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W}, \quad \text{a.s. as } N \to \infty.$$

Note that because ψ is deterministic, we can express the right hand side as a stochastic integral,

$$\sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t} \psi dz dt$$
$$= \int_0^T \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) d\tilde{W}_N(t).$$

Because convergence in probability implies convergence almost surely along a subsequence, it thus suffices to prove that

$$\int_0^T \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) d\tilde{W}_N \to \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W},$$

as $N \to \infty$ in probability. So we must show that given any $\delta > 0$ and any $\epsilon > 0$, there exists N_0 sufficiently large such that for all $N \geqslant N_0$

$$\tilde{\mathbb{P}} \left(\left| \int_0^T \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) d\tilde{W}_N - \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W} \right| > \delta \right) < \epsilon.$$

We accomplish this through two estimates. We claim that we can choose N_0 sufficiently large such that

$$\tilde{\mathbb{P}} \left(\left| \int_0^T \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) d\tilde{W}_N - \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W}_N \right| > \frac{\delta}{2} \right) < \frac{\epsilon}{2}, \tag{4.70}$$

and

$$\tilde{\mathbb{P}} \left(\left| \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W}_N - \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W} \right| > \frac{\delta}{2} \right) < \frac{\epsilon}{2}, \tag{4.71}$$

for all $N \geqslant N_0$.

For the first estimate (4.70), it suffices to use the Itô isometry along with the fact that

$$\left\| \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) - \int_\Gamma \psi dz \right\|_{L^2(0,T)} \to 0, \qquad \text{as } N \to \infty,$$

to conclude that

$$\tilde{\mathbb{E}} \left(\left| \int_0^T \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)(\Delta t)} \int_\Gamma \psi(s,z) dz ds \right) 1_{t \in (n\Delta t, (n+1)\Delta t]}(t) d\tilde{W}_N - \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W}_N \right|^2 \right) \to 0,$$

as $N \to \infty$. The first estimate (4.70) thus follows from taking N_0 sufficiently large to make this expectation sufficiently small, and then using Chebychev's inequality.

For the second estimate, note that we can approximate $\int_\Gamma \psi(t,z)dz := g(t)$ by a *deterministic* step function

$$g_m(t) = g\left(\frac{kT}{m}\right) \qquad \text{if } \frac{kT}{m} < t \leqslant \frac{(k+1)T}{m}.$$

By the continuity of $g(t)$, we can select m sufficiently large such that

$$||g(t) - g_m(t)||^2_{L^2(0,T)} < \frac{\epsilon}{6} \cdot \left(\frac{\delta}{6}\right)^2.$$

Then, by the Itô isometry and Chebychev's inequality,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \left(\int_\Gamma \psi dz\right) d\tilde{W}_N - \int_0^T g_m(t) d\tilde{W}_N\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}, \tag{4.72}$$

for all N, and

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \left(\int_\Gamma \psi dz\right) d\tilde{W} - \int_0^T g_m(t) d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}. \tag{4.73}$$

So it remains to choose N_0 sufficiently large such that for all $N \geqslant N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T g_m(t) d\tilde{W}_N - \int_0^T g_m(t) d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}. \tag{4.74}$$

We note that $|g_m(t)| \leqslant K$ for some constant K that is deterministic, as $g_m(t)$ is a deterministic function of time. Also, note that

$$\int_0^T g_m(t) d\tilde{W}_N = \sum_{k=0}^{m-1} g\left(\frac{kT}{m}\right) \cdot \left(\tilde{W}_N\left(\frac{(k+1)T}{m}\right) - \tilde{W}_N\left(\frac{kT}{m}\right)\right),$$

with an analogous formula for the integration against $\tilde{W}$. Hence,

$$\left|\int_0^T g_m(t) d\tilde{W}_N - \int_0^T g_m(t) d\tilde{W}\right|$$
$$\leqslant \sum_{k=0}^{m-1} \left|g\left(\frac{kT}{m}\right) \cdot \left[\left(\tilde{W}_N\left(\frac{(k+1)T}{m}\right) - \tilde{W}_N\left(\frac{kT}{m}\right)\right) - \left(\tilde{W}\left(\frac{(k+1)T}{m}\right) - \tilde{W}\left(\frac{kT}{m}\right)\right)\right]\right|$$
$$\leqslant \sum_{k=0}^{m-1} 2K||\tilde{W} - \tilde{W}_N||_{C(0,T;\mathbb{R})} \leqslant 2Km \cdot ||\tilde{W} - \tilde{W}_N||_{C(0,T;\mathbb{R})}.$$

Because $\tilde{W}_N \to \tilde{W}$ in $C(0,T;\mathbb{R})$ almost surely, there exists N_0 sufficiently large such that

$$\tilde{\mathbb{P}}\left(||\tilde{W} - \tilde{W}_N||_{C(0,T;\mathbb{R})} > \frac{\delta}{12Km}\right) < \frac{\epsilon}{6}, \qquad \text{for all } N \geqslant N_0.$$

Therefore,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T g_m(t)d\tilde{W}_N - \int_0^T g_m(t)d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}, \qquad \text{for all } N \geqslant N_0.$$

The estimates (4.72), (4.73), and (4.74) thus imply the desired estimate in (4.71).

Convergence of the pressure term. Finally, we show that

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P^n_{N,in}\left(\int_0^R (q_z)|_{z=0}dr\right)dt \to \int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt, \qquad \text{as } N\to\infty. \tag{4.75}$$

The same argument will work for the outlet pressure term.

Define the following piecewise approximation of the test function $\boldsymbol{q}$,

$$\boldsymbol{q}^m(t,\cdot) = \boldsymbol{q}\left(\frac{kT}{m},\cdot\right), \qquad \text{if } \frac{kT}{m} < t \leqslant \frac{(k+1)T}{m}.$$

For any positive integer N,

$$\begin{aligned}\int_0^T P_{in}(t)\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt - \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P^n_{N,in}\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt\\ = \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}(P_{in}(t)-P^n_{N,in})\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt\\ = \sum_{n=0}^{N-1}\left(\int_0^R (q^N_z)|_{z=0}dr\right)\int_{n\Delta t}^{(n+1)\Delta t}(P_{in}(t)-P^n_{N,in})dt = 0.\end{aligned}$$

To establish (4.75), it suffices to show that

$$\int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \int_0^T P_{in}(t)\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt \to 0, \qquad \text{as } N\to\infty, \tag{4.76}$$

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P^n_{N,in}\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P^n_{N,in}\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt \to 0, \quad \text{as } N\to\infty. \tag{4.77}$$

For (4.76), we compute

$$\begin{aligned}&\left|\int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \int_0^T P_{in}(t)\left(\int_0^R (q^N_z)|_{z=0}dr\right)dt\right|\\ &= \left|\int_0^T P_{in}(t)\left(\int_0^R (q_z - q^N_z)|_{z=0}dr\right)dt\right| \leqslant ||P_{in}||_{L^2(0,T)}\left(\int_0^T\left(\int_0^R (q_z-q^N_z)|_{z=0}dr\right)^2 dt\right)^{1/2}\\ &\leqslant C||P_{in}||_{L^2(0,T)}\left(\int_0^T ||\boldsymbol{q}-\boldsymbol{q}^N||^2_{H^1(\Omega_f)}dt\right)^{1/2}.\end{aligned} \tag{4.78}$$

Because $\boldsymbol{q}$ is continuous taking values in $\mathcal{V}_F$ equipped with the norm of $H^1(\Omega_f)$, we have that $||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)} \to 0$ uniformly on $[0,T]$ as $N \to \infty$, which establishes the desired limit. Similary, to estabish (4.77) we calculate

$$\begin{aligned}
&\left|\sum_{n=0}^{N-1} P_{N,in}^n \int_{n\Delta t}^{(n+1)\Delta t} \left(\int_0^R (q_z - q_z^N)|_{z=0} dr\right) dt\right| \\
&\quad \leqslant \left|\sum_{n=0}^{N-1} (\Delta t)^{1/2} P_{N,in}^n \left(\int_{n\Delta t}^{(n+1)\Delta t} \left(\int_0^R (q_z - q_z^N)|_{z=0} dr\right)^2 dt\right)^{1/2}\right| \\
&\quad \leqslant C\left|\sum_{n=0}^{N-1} (\Delta t)^{1/2} P_{N,in}^n \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)}^2 dt\right)^{1/2}\right| \\
&\quad \leqslant C\left(\sum_{n=0}^{N-1} (\Delta t)^{1/2} |P_{N,in}^n|\right) \cdot \max_{0\leqslant n\leqslant N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)}^2 dt\right)^{1/2} \\
&\quad \leqslant C\left(\sum_{n=0}^{N-1} \frac{1}{(\Delta t)^{1/2}} \int_{n\Delta t}^{(n+1)\Delta t} |P_{in}(t)| dt\right) \cdot \max_{0\leqslant n\leqslant N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)}^2 dt\right)^{1/2} \\
&\quad \leqslant C||P_{in}||_{L^2(0,T)} \cdot \max_{0\leqslant n\leqslant N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)}^2 dt\right)^{1/2}.
\end{aligned}$$

Again, because $\boldsymbol{q}$ is continuous taking values in $\mathcal{V}_F$ equipped with the norm of $H^1(\Omega_f)$, we have that $||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)} \to 0$ uniformly on $[0,T]$ as $N \to \infty$, which establishes the desired limit.

We have, therefore, established the existence of a weak solution to the stochastic fluid-structure interaction problem in a probabilistically weak sense, as in Definition 4.4.1.

4.9 Return to the original probability space

We have thus constructed a stochastic process $(\tilde{\boldsymbol{u}}, \tilde{\eta})$, which satisfies the weak formulation of the continuous problem almost surely on the "tilde" probability space determined by the Skorokhod representation theorem. However, we want to bring the solution back to the original probability space. In particular, we must get convergence of the original approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the original given complete filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and the original Brownian motion $W(t)$.

Recall from Theorem 2.2.3 that to show that a sequence of random variables X_n converges in probability, we must verify that for every two subsequences X_{l_k} and X_{m_k}, there exists a further subsequence such that the joint probability measures associated with $x_k = (X_{l_k}, X_{m_k})$ on $B \times B$, defined by

$$\nu_{x_k} = \nu_{X_{l_k}, X_{m_k}}(A_1 \times A_2) = \mathbb{P}(X_{l_k} \in A_1, X_{m_k} \in A_2), \quad A_1, A_2 \in \mathcal{B}(B),$$

where $\mathcal{B}(B)$ is the Borel sigma algebra on B, converge weakly along this *further subsequence* to some probability measure ν, where ν is such that

$$\nu(\{(x,y) \in B \times B : x = y\}) = 1. \tag{4.79}$$

Thus, the limits of any two convergent subsequences have to be "the same" with probability 1.

Once we show convergence in probability of our original sequence using the Gyöngy-Krylov lemma, we will have almost sure convergence along a subsequence of our approximate solutions on the *original probability space*. Then, using the fact that our approximate solutions converge almost surely along a subsequence on the original probability space, we can adapt the arguments in Section 4.8 in order to show that the limiting weak solution on the original probability space satisfies the weak form of the continuous problem almost surely, so that the limiting solution is a weak solution in a probabilistically strong sense.

Thus, what remains to be shown is that the diagonal condition in the Gyöngy-Krylov lemma holds. Since our problem is linear and the stochastic noise is additive, using the Skorokhod representation theorem, one can show that the diagonal condition is equivalent to showing *deterministic uniqueness* holding pathwise. To demonstrate this, we first prove deterministic uniqueness, and then use it to show how this implies the diagonal condition.

Uniqueness of the deterministic linear problem

Lemma 4.9.1 (Uniqueness for the deterministic problem)**.** Suppose we have $(\boldsymbol{u}, v, \eta)$ with $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega_f)) \cap L^2(0,T;\mathcal{V}_F)$, $\eta \in W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;\mathcal{V}_S)$, and $\boldsymbol{u}|_\Gamma = \partial_t \eta \boldsymbol{e}_r$. Suppose also that $(\boldsymbol{u}, \partial_t\eta) \in C(0,T;\mathcal{Q}')$, with $\eta(0) = 0$. If for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \eta \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \eta \cdot \nabla \psi dz dt = 0,$$

then $(\boldsymbol{u}, \eta) = 0$.

Proof. Observe first that to get the usual energy equality, we would want to formally substitute in $(\boldsymbol{u}, \partial_t \eta)$ for $(\boldsymbol{q}, \psi)$. However, since $(\boldsymbol{q}, \psi)$ must have $\psi(t) \in H_0^1(\Gamma)$ by the definition of the test space $\mathcal{Q}(0,T)$, we do not have enough regularity to do this. Therefore, we use a different approach of taking an antiderivative, which is an approach used for example in establishing uniqueness of weak solutions for general hyperbolic equations (see Section 7.2 in [70]).

Consider an arbitrary s such that $0 \leqslant s \leqslant T$. We use the following test function,

$$(\boldsymbol{q}_0(t), \psi_0(t)) = \begin{cases} \left(\int_t^s \left(\int_0^\tau \boldsymbol{u}(\sigma) d\sigma\right) d\tau, \int_t^s \eta(\tau) d\tau\right) & \text{if } 0 \leqslant t \leqslant s, \\ (0,0) & \text{if } s \leqslant t \leqslant T. \end{cases}$$

Recall that $\eta(0) = 0$ by assumption. Note that since

$$\int_0^\tau \boldsymbol{u}(\sigma) d\sigma \Big|_\Gamma = \int_0^\tau \partial_t \eta(\sigma) d\sigma = \eta(\tau)$$

for all $\tau \in [0,T]$, the function $(\boldsymbol{q}_0, \psi_0)$ satisfies the necessary kinematic coupling condition for $\mathcal{Q}(0,T)$. While this test function is only piecewise differentiable, it is easy to show by an approximation argument that the weak formulation should still hold with this test function by approximating it with differentiable functions. For notational simplicity, we define

$$\boldsymbol{U}(t) = \int_0^t \boldsymbol{u}(\sigma) d\sigma.$$

Substituting the test function into the weak formulation, we obtain for all $s \in [0,T]$,

$$\int_0^s \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{U} d\boldsymbol{x} dt + 2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt + \int_0^s \int_\Gamma \partial_t \eta \cdot \eta dz dt + \int_0^s \int_\Gamma \nabla \eta \cdot \nabla \psi_0 dz dt = 0,$$

where we note that $\partial_t \boldsymbol{q}_0(t) = -U(t)$ and $\partial_t \psi_0(t) = -\eta(t)$, for $t \in [0,s)$. We handle the four terms on the left hand side as follows.

- **First term:** We note that $\boldsymbol{u} = \partial_t \boldsymbol{U}$. Hence, using the fact that $\boldsymbol{U}(0) = 0$, we get

$$\int_0^s \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{U} d\boldsymbol{x} dt = \int_0^s \frac{d}{dt} \left(\frac{1}{2} ||\boldsymbol{U}||^2_{L^2(\Omega_f)} \right) dt = \frac{1}{2} ||\boldsymbol{U}(s)||^2_{L^2(\Omega_f)}.$$

- **Second term:** For the second term, we again use that $\boldsymbol{u} = \partial_t \boldsymbol{U}$. Therefore,

$$2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt = 2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\partial_t \boldsymbol{U}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt.$$

  We integrate by parts in time. Note that $\boldsymbol{U}(0) = 0$ and $\boldsymbol{q}_0(s) = 0$, so there are no boundary terms from the integration by parts. Hence, using the fact that $\partial_t \boldsymbol{q}_0 = -\boldsymbol{U}$, we obtain

$$2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt = -2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{U}) : \boldsymbol{D}(\partial_t \boldsymbol{q}_0) d\boldsymbol{x} dt = 2\mu \int_0^s \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{U})|^2 d\boldsymbol{x} dt.$$

- **Third term:** We immediately have that

$$\int_0^s \int_\Gamma \partial_t \eta \cdot \eta dz dt = \frac{1}{2} ||\eta(s)||^2_{L^2(\Gamma)} - \frac{1}{2} ||\eta(0)||^2_{L^2(\Gamma)} = \frac{1}{2} ||\eta(s)||^2_{L^2(\Gamma)}.$$

- **Fourth term:** Since $\eta = -\partial_t \psi_0$, we have that

$$\int_\Gamma \nabla \eta \cdot \nabla \psi_0 dz = -\frac{1}{2} \frac{d}{dt} \left(||\nabla \psi_0||^2_{L^2(\Gamma)} \right),$$

  and hence, using the fact that $\psi_0(s) = 0$, we get that

$$\int_0^s \int_\Gamma \nabla \eta \cdot \nabla \psi_0 dz dt = \frac{1}{2} ||\nabla \psi_0(0)||^2_{L^2(\Gamma)}.$$

Therefore, for all $0 \leqslant s \leqslant T$, the entire expression (energy) can now be written as

$$\frac{1}{2}||\boldsymbol{U}(s)||^2_{L^2(\Omega_f)} + 2\mu \int_0^s \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{U})|^2 d\boldsymbol{x}dt + \frac{1}{2}||\eta(s)||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla\psi_0(0)||^2_{L^2(\Gamma)} = 0.$$

Thus, we conclude that $\boldsymbol{U}(s) = 0$ and $\eta(s) = 0$ for all $s \in [0, T]$. From the definition of $\boldsymbol{U}$, we conclude that $\boldsymbol{u}(t) = \partial_t \boldsymbol{U}(t) = 0$ for all $t \in [0, T]$ also, which completes the proof. □

Verifying the diagonal condition of the Gyöngy-Krylov lemma

Now that we have established a uniqueness result, we can construct a solution on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by invoking a standard argument involving the Gyöngy-Krylov argument (Theorem 2.2.3), to show that the random variables $(\eta_N, \boldsymbol{u}_N, v_N)$ defined on the original probability space converge in probability, and hence converge almost surely along a subsequence *in the original topology.*

Because we have already shown deterministic uniqueness in Sec. 4.9, it only remains to demonstrate how the Skorokhod representation theorem can be used to show that the diagonal condition (4.79) from the Gyöngy-Krylov lemma is equivalent to showing deterministic uniqueness.

For this purpose, denote by $\{X^1_{M_k}\}_{k=1}^\infty$ and $\{X^2_{N_k}\}_{k=1}^\infty$ any two subsequences of our random variables (approximate solutions) defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{aligned} X^1_{M_k} &= (\eta_{M_k}, \overline{\eta}_{M_k}, \eta^{\Delta t}_{M_k}, \boldsymbol{u}_{M_k}, v_{M_k}, \boldsymbol{u}_{M_k}, v^*_{M_k}, \overline{\boldsymbol{u}}_{M_k}, \overline{v}_{M_k}, \boldsymbol{u}^{\Delta t}_{M_k}, v^{\Delta t}_{M_k}, W), \\ X^2_{N_k} &= (\eta_{N_k}, \overline{\eta}_{N_k}, \eta^{\Delta t}_{N_k}, \boldsymbol{u}_{N_k}, v_{N_k}, \boldsymbol{u}_{N_k}, v^*_{N_k}, \overline{\boldsymbol{u}}_{N_k}, \overline{v}_{N_k}, \boldsymbol{u}^{\Delta t}_{N_k}, v^{\Delta t}_{N_k}, W). \end{aligned}$$

Recall that the laws corresponding to each of these these two sequences of random variables *individually* converge to the law μ. However, to verify the diagonal condition in the Gyöngy-Krylov lemma, we must examine the *joint laws* of these random variables $(X^1_{M_k}, X^2_{N_k})$.

Hence, we consider the joint probability measures (or joint laws) $\{\nu_{X^1_{M_k}, X^2_{N_k}}\}_{k=1}^\infty$ on $\mathcal{X} \times \mathcal{X}$, associated with the subsequence $(X^1_{M_k}, X^2_{N_k})$. By the tightness of the original probability measures μ_N, established in the proof of Theorem 4.8.1, we have that the collection of joint laws $\{\nu_{X^1_{M_k}, X^2_{N_k}}\}_{k=1}^\infty$ is also tight, and hence converges weakly to a probability measure ν on $\mathcal{X} \times \mathcal{X}$ along a further subsequence, which we will continue to denote by the same indexing for notational simplicity. Then, by the Skorokhod representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables

$$\begin{aligned} \tilde{X}^1_{M_k} &= (\tilde{\eta}^1_{M_k}, \tilde{\overline{\eta}}^1_{M_k}, \tilde{\eta}^{\Delta t,1}_{M_k}, \tilde{\boldsymbol{u}}^1_{M_k}, \tilde{v}^1_{M_k}, \tilde{\boldsymbol{u}}^{*,1}_{M_k}, \tilde{v}^{*,1}_{M_k}, \tilde{\overline{\boldsymbol{u}}}^1_{M_k}, \tilde{\overline{v}}^1_{M_k}, \tilde{\boldsymbol{u}}^{\Delta t,1}_{M_k}, \tilde{v}^{\Delta t,1}_{M_k}, \tilde{W}^1_{M_k}), \\ \tilde{X}^2_{N_k} &= (\tilde{\eta}^2_{N_k}, \tilde{\overline{\eta}}^2_{N_k}, \tilde{\eta}^{\Delta t,2}_{N_k}, \tilde{\boldsymbol{u}}^2_{N_k}, v^2_{N_k}, \tilde{\boldsymbol{u}}^{*,2}_{N_k}, \tilde{v}^{*,2}_{N_k}, \tilde{\overline{\boldsymbol{u}}}^2_{N_k}, \tilde{\overline{v}}^2_{N_k}, \tilde{\boldsymbol{u}}^{\Delta t,2}_{N_k}, \tilde{v}^{\Delta t,2}_{N_k}, \tilde{W}^2_{N_k}), \end{aligned}$$

such that

$$(\tilde{X}^1_{M_k}, \tilde{X}^2_{N_k}) =_d (X^1_{M_k}, X^2_{N_k}), \tag{4.80}$$

and $(\tilde{X}^1_{M_k}, \tilde{X}^2_{N_k}) \to (\tilde{X}^1, \tilde{X}^2)$ in $\mathcal{X} \times \mathcal{X}$ almost surely as $k \to \infty$, where

$$\tilde{X}^1 = (\tilde{\eta}^1, \tilde{\bar{\eta}}^1, \tilde{\eta}^{\Delta t,1}, \tilde{\boldsymbol{u}}^1, \tilde{v}^1, \tilde{\boldsymbol{u}}^{*,1}, \tilde{v}^{*,1}, \tilde{\bar{\boldsymbol{u}}}^1, \tilde{\bar{v}}^1, \tilde{\boldsymbol{u}}^{\Delta t,1}, \tilde{v}^{\Delta t,1}, \tilde{W}^1),$$
$$\tilde{X}^2 = (\tilde{\eta}^2, \tilde{\bar{\eta}}^2, \tilde{\eta}^{\Delta t,2}, \tilde{\boldsymbol{u}}^2, \tilde{v}^2, \tilde{\boldsymbol{u}}^{*,2}, \tilde{v}^{*,2}, \tilde{\bar{\boldsymbol{u}}}^2, \tilde{\bar{v}}^2, \tilde{\boldsymbol{u}}^{\Delta t,2}, \tilde{v}^{\Delta t,2}, \tilde{W}^2),$$

are random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and ν is the law of $(\tilde{X}_1, \tilde{X}_2)$.

We want to show that ν is supported on the diagonal. It suffices to show that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$. We do this in three steps.

Step 1. First we notice that $\tilde{X}^1$ *is a weak solution in a probabilistically weak sense with respect to the stochastic basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^1_t\}_{t\geqslant 0}, \tilde{\mathbb{P}}, \tilde{W}_1)$ in the sense of Definition 4.4.1. This follows from the results of Lemma 4.8.7. Namely, the results of Lemma 4.8.7 imply that $\tilde{\eta}^1 = \tilde{\bar{\eta}}^1 = \tilde{\eta}^{\Delta t,1}$, $\tilde{\boldsymbol{u}}^1 = \tilde{\boldsymbol{u}}^{*,1} = \tilde{\bar{\boldsymbol{u}}}^1 = \tilde{\boldsymbol{u}}^{\Delta t,1}$, $\tilde{v}^1 = \tilde{v}^{*,1} = \tilde{\bar{v}}^1 = \tilde{v}^{\Delta t,1}$, and $\partial_t \tilde{\eta}^1 = \tilde{v}^1$ almost surely. Furthermore, $(\tilde{\boldsymbol{u}}^1, \tilde{\eta}^1) \in \mathcal{W}(0,T)$ and $(\tilde{\boldsymbol{u}}^1, \tilde{v}^1) \in C(0,T;\mathcal{Q}')$, satisfying the initial condition $\tilde{\eta}^1(0) = \eta_0$ almost surely. Furthermore, $\tilde{X}^1$ is a weak solution in a probabilistically weak sense with respect to the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^1_t\}_{t\geqslant 0}, \tilde{\mathbb{P}}, \tilde{W}_1)$ in the sense of Definition 4.4.1. The same is true for the components of $\tilde{X}^2$, with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^2_t\}_{t\geqslant 0}, \tilde{\mathbb{P}}, \tilde{W}_2)$. Here, the filtrations $\{\tilde{\mathcal{F}}^1_t\}_{t\geqslant 0}$ and $\{\tilde{\mathcal{F}}^2_t\}_{t\geqslant 0}$ are defined by (4.53) with the appropriate limiting random variables with superscripts "1" and "2" respectively.

Step 2. Here we notice that the limiting white noise satisfies $\tilde{W}_1 = \tilde{W}_2$. This follows directly from (4.80), which implies $\tilde{W}^1_{M_k} = \tilde{W}^2_{N_k}$ almost surely, since the law of $(\tilde{W}^1_{M_k}, \tilde{W}^2_{N_k})$ is the same as that of (W, W). Thus, by the convergence of $\tilde{W}^1_{M_k}$ and $\tilde{W}^2_{N_k}$ in $C(0,T;\mathbb{R})$ almost surely to $\tilde{W}^1$ and $\tilde{W}^2$, we have that $\tilde{W}^1 = \tilde{W}^2$ almost surely in $C(0,T;\mathbb{R})$. This will allow us to make sense of the difference of the stochastic integrals with respect to $\tilde{W}_1$ and $\tilde{W}_2$ in the weak formulations on the "tilde" probability space.

Step 3. Finally, we use deterministic uniqueness to obtain the diagonal condition. We consider the difference $(\tilde{\eta}^1 - \tilde{\eta}^2, \tilde{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^2)$. By subtracting the weak formulations defining $(\tilde{\boldsymbol{u}}^1, \tilde{\eta}^1)$ and $(\tilde{\boldsymbol{u}}^2, \tilde{\eta}^2)$ as probabilistically weak solutions, given in Definition 4.4.1, and by using the result of Step 2 above, we obtain that $(\tilde{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^2, \tilde{\eta}^1 - \tilde{\eta}^2)$ almost surely satisfies for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_1 - \boldsymbol{u}_2) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt$$
$$-\int_0^T \int_\Gamma \partial_t(\eta_1 - \eta_2)\partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla(\eta_1 - \eta_2) \cdot \nabla \psi dz dt = 0,$$

with $\tilde{\eta}^1 - \tilde{\eta}^2 = 0$ almost surely. Therefore, by using the uniqueness result in Lemma 4.9.1, we conclude that $\tilde{\eta}^1 = \tilde{\eta}^2$ and $\tilde{\boldsymbol{u}}^1 = \tilde{\boldsymbol{u}}^2$ almost surely. Since $\tilde{v}^1 = \partial_t \tilde{\eta}^1$ and $\tilde{v}^2 = \partial_t \tilde{\eta}^2$, we also obtain that $\tilde{v}^1 = \tilde{v}^2$ almost surely. This allows us to conclude that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$, which implies that the limiting joint probability measure (or law) ν is supported on the diagonal.

This completes the verification of the diagonal condition of the Gyöngy-Krylov lemma.

Existence of a weak solution in a probabilistically strong sense

The existence of a weak solution in a probabilistically strong sense, given by Definition 4.4.2, now follows from the Gyöngy-Krylov lemma in Lemma 2.2.3. More precisely, by the Gyöngy-Krylov lemma, the original sequence $(\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W)$ converges in probability to some random variable $(\eta, \overline{\eta}, \eta^{\Delta t}, \boldsymbol{u}, v, \boldsymbol{u}^*, v^*, \overline{\boldsymbol{u}}, \overline{v}, \boldsymbol{u}^{\Delta t}, v^{\Delta t}, W)$, where the last component must be W up to a null set, since the limit in probability of any constant sequence is almost surely exactly that constant.

Since convergence in probability implies almost sure convergence along a subsequence, we conclude that along a subsequence which we continue to denote by N, we have that

$$(\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W) \to (\eta, \overline{\eta}, \eta^{\Delta t}, \boldsymbol{u}, v, \boldsymbol{u}^*, v^*, \overline{\boldsymbol{u}}, \overline{v}, \boldsymbol{u}^{\Delta t}, v^{\Delta t}, W), \tag{4.81}$$

almost surely in $\mathcal{X}$. To show that this limit is a weak solution in the sense of Definition 4.4.2, we use the same arguments as in Lemma 4.8.7. All of the properties from Definition 4.4.2 follow from Lemma 4.8.7, except for uniqueness and showing that $(\boldsymbol{u}, v, \eta)$ is $\mathcal{F}_t$-adapted.

Uniqueness follows from the deterministic uniqueness result of Lemma 4.9.1.

$\mathcal{F}_t$-adaptedness of $(\boldsymbol{u}, v, \eta)$: Note that this is not provided by Lemma 4.8.7, as we want to show that this solution is adapted to the *original* filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, while the filtration defined in (4.53) is not necessarily the same filtration.

To verify this, we note that by construction, $(\boldsymbol{u}_N, v_N, \eta_N)$ is adapted to the given complete filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. We want to pass to the limit as $N \to \infty$. By the convergence in (4.81),

$$\begin{aligned} \boldsymbol{u}_N &\to \boldsymbol{u}, && \text{almost surely in } L^2(0,T;L^2(\Omega_f)), \\ v_N &\to v, && \text{almost surely in } L^2(0,T;L^2(\Gamma)), \\ \eta_N &\to \eta, && \text{almost surely in } L^2(0,T;L^2(\Gamma)). \end{aligned}$$

By the same argument used to establish (4.56) for example, we obtain that for a measurable set $S \subset [0,T] \times \Omega$ with $(dt \times \mathbb{P})(S) = T$,

$$\boldsymbol{u}_{N_k}(t,\omega,\cdot) \to \boldsymbol{u}(t,\omega,\cdot) \text{ in } L^2(\Omega_f), \quad v_{N_k}(t,\omega,\cdot) \to v(t,\omega,\cdot), \ \eta_{N_k}(t,\omega,\cdot) \to \eta(t,\omega,\cdot) \text{ in } L^2(\Gamma) \tag{4.82}$$

along a common subsequence N_k. In particular, $([0,T] \times \Omega) - S$ has measure zero with respect to the product measure $dt \times \mathbb{P}$.

Define $S_0 \subset [0,T]$ to be all times $t \in [0,T]$ for which $\mathbb{P}((t,\omega) \in S) = 1$, so that the time slice at time t has full measure in probability. S_0 is measurable in $[0,T]$ and contains almost every time in $[0,T]$ by Fubini's theorem. So for all $t \in S_0$, the convergences (4.82) are almost sure convergences.

Because $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a complete filtration by assumption, the almost sure limit of $\mathcal{F}_t$-measurable random variables must also be $\mathcal{F}_t$-measurable, since $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$. So for all $t \in S_0$, $\boldsymbol{u}(t)$, $v(t)$, and $\eta(t)$ are $\mathcal{F}_t$-measurable since $\boldsymbol{u}_{N_k}(t)$, $v_{N_k}(t)$, and $\eta_{N_k}(t)$ are $\mathcal{F}_t$-measurable by construction.

To show $\boldsymbol{u}(t)$, $v(t)$, and $\eta(t)$ are $\mathcal{F}_t$-measurable for $t \notin S_0$, we use the fact that S_0 has full measure in $[0, T]$ and is hence dense. We can assume $t \neq 0$, since at $t = 0$, $(\boldsymbol{u}(0), v(0), \eta(0)) = (\boldsymbol{u}_0, v_0, \eta_0)$ almost surely so the result holds. So for $t \notin S_0$ and $t \neq 0$, we can construct $t_n \in S_0$ such that $t_n \nearrow t$. By the fact that $(\boldsymbol{u}, v) \in C(0, T; \mathcal{Q}')$ and η is Lipschitz continuous almost surely, we have that $(\boldsymbol{u}(t), v(t), \eta(t))$ is the almost sure limit of $(\boldsymbol{u}(t_n), v(t_n), \eta(t_n))$, which are $\mathcal{F}_t$-measurable since $\mathcal{F}_{t_n} \subset \mathcal{F}_t$, as $t_n \leqslant t$. This establishes the adaptedness of $(\boldsymbol{u}, v, \eta)$ to the given complete filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$.

In conclusion, we have now shown that $(\boldsymbol{u}, v, \eta)$ has all of the required properties needed to be a weak solution in a probabilistically strong sense to the given fluid-structure interaction problem with respect to the Brownian motion W with complete filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$, as in Definition 4.4.2. This completes the proof of the main result, stated in Theorem 4.4.1, and restated here:

Theorem 4.9.1 (**Main Result**). Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H_0^1(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0, \infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a given complete filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense to the given stochastic fluid-structure interaction problem (4.5)–(4.7).

4.10 Conclusions

In this manuscript, we presented a constructive proof of the existence of a weak solution in a probabilistically strong sense, to a benchmark stochastic fluid-structure interaction (SFSI) problem (4.5)–(4.7). Our well-posedness result indicates that stochastic FSI models are robust in the sense that a unique weak solution in the sense of Definition 4.4.2 will exist even when the problem is stochastically forced by a rough time-dependent white noise, as considered in this work.

In addition to the importance of this work in terms of modeling real-life fluid structure interaction phenomena with stochastic noise, to the best of our knowledge the results of this work present a first constructive existence proof of a unique weak solution in a probabilistically strong sense to a stochastically forced and fully coupled FSI problem, as defined in Definition 4.4.2.

In contrast to the deterministic case, the proof based on the operator splitting strategy presented in this work has several new interesting components, which we summarize below.

1. The energy estimates are given in expectation, and do not necessarily hold pathwise. Furthermore, the energy estimate has an extra term that accounts for the energy pumped into the problem by the stochastic forcing in expectation.

2. We can modularize the fully coupled problem into three separate subproblems via an operator splitting scheme, where the three separate subproblems must be solved in the "correct" order to obtain a *stable scheme*. In particular, the order is: (1) the structure

subproblem, (2) the stochastic subproblem, and (3) the fluid subproblem. With this order, we can properly interpret the terms involving time increments of the stochastic forcing as a stochastic integral, due to the measurability properties of the approximate solutions, which allows us to show stability.

3. To establish weak convergence of probability measures, one can show that the probability measures are **tight**, which requires the use of a *compactness result* alla Aubin-Lions. This is a robust approach to showing convergence of the (random) approximate solutions that generalizes well to problems with nonlinear scaling in the intensity of the stochastic noise and to stochastic FSI problems with nonlinear coupling, where the fluid domain is determined by the random structure displacement and hence the fluid equations are posed on time-dependent random (and a priori unknown) domains. The consideration of such complex nonlinearly coupled FSI models is work in progress [175].

4. Once weak convergence of the probability measures (laws) associated with the approximate solutions is established, probabilistic techniques based on the Skorokhod representation theorem and the Gyöngy-Krylov lemma can be employed to obtain almost sure convergence along a subsequence to a weak solution.

Acknowledgements

This work was partially supported by the National Science Foundation under grants DMS-1853340 and DMS-2011319.

Chapter 5

Fluid-poroelastic structure interaction

This chapter will be devoted to the study of nonlinearly coupled fluid-poroelastic structure interaction (FPSI), which describes the coupled dynamical interaction between poroelastic structures and fluids. Poroelastic materials are porous materials with elastic properties that are coupled to the flow of fluid through their pores. These materials are described by a set of equations known as the **Biot equations**, which are a pair of coupled equations for the displacement of the poroelastic material from its reference configuration and the pore pressure. In this chapter, we will consider a nonlinearly coupled FPSI system consisting of an incompressible fluid modeled by the Navier-Stokes equations interacting with a multilayered poroelastic structure, consisting of a thin reticular plate and a thick Biot poroelastic material modeled by the Biot equations. This FPSI system will be nonlinearly coupled, so that both the fluid domain and the Biot domain are moving domains, where the time-dependent configuration of the fluid domain is determined by the thin plate displacement and the time-dependent configuration of the Biot domain is determined by the displacement of the Biot material from its reference configuration. The nonlinearly coupled nature of this problem and the additional geometric nonlinearities arising from considering the Biot equations and the Navier-Stokes equations on moving domains make the analysis of this problem complicated. In this chapter, our goal is to develop a well-posedness theory for this problem, involving weak solutions to a regularized FPSI problem. We will then show that our well-posedness theory is compatible with real-life dynamics, by establishing a weak-classical consistency result that states that weak solutions to the regularized FPSI problem coincide with smooth solutions to the original problem when smooth solutions exist, as the regularization parameter goes to zero. We begin by describing the Biot equations on a fixed domain in the introduction to this chapter, and we give a brief literature review of the Biot equations and FPSI. We then precisely state the specific nonlinearly coupled FPSI model that we will consider. We then carry out a priori estimates and define the concept of a weak solution to a regularized form of the problem. We use a splitting scheme to construct approximate solutions and then use compactness arguments for functions defined on moving domains, where there are additional subtleties that make this constructive existence proof distinct from the corresponding existence proof for the prototypical model of nonlinearly coupled FSI. We end this chapter by stating and establishing the weak-classical consistency result for the regularized FPSI problem.

5.1 Introduction

The Biot equations on a fixed domain. In this introduction, we first define the Biot equations for poroelasticity. These equations were first formulated in the context of geoscience by Maurice A. Biot in the seminal works [22] and [23], where these equations for poroelasticity arose from Biot's interest in modeling soil consolidation. Our nonlinearly coupled model of FPSI will involve the Biot equations posed on a *moving (time-dependent) domain*. However, it will be useful to first discuss the Biot equations defined on a fixed domain Ω_b, where Ω_b is a bounded domain in $\mathbb{R}^n$. The Biot equations are a set of two coupled PDEs for the displacement $\boldsymbol{\eta} : \Omega_b \to \mathbb{R}^n$ of a poroelastic material from its reference configuration Ω_b and its pore pressure $p : \Omega_b \to \mathbb{R}$, given by

$$\rho_b \partial_{tt} \boldsymbol{\eta} - \nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}, p) = 0, \quad \text{in } \Omega_b, \tag{5.1}$$

$$c_0 \partial_t p + \alpha (\nabla \cdot \boldsymbol{\eta})_t - \nabla \cdot (\kappa \nabla p) = 0, \quad \text{in } \Omega_b, \tag{5.2}$$

where the Cauchy stress tensor is given by

$$\boldsymbol{\sigma}(\nabla \boldsymbol{\eta}, p) = \boldsymbol{\sigma}_E(\nabla \boldsymbol{\eta}, p) - \alpha p \boldsymbol{I}.$$

Here, $\boldsymbol{\sigma}_E(\nabla \boldsymbol{\eta}, p)$ is the elastic part of the full stress tensor, which can be given for example by the usual Piola-Kirchhoff stress tensor in linear elasticity.

The first equation (5.1) describes the elastodynamics of the poroelastic material, arising from balance of momentum, where the influence of the pore pressure on the elastodynamics appears in the term $\alpha p \boldsymbol{I}$ in the stress tensor $\boldsymbol{\sigma}(\nabla \boldsymbol{\eta}, p)$ for the poroelastic material. The second equation (5.2) describes conservation of mass, where the quantity $\kappa \nabla p$ is related to the filtration velocity $\boldsymbol{q}$, or the velocity of fluid flow through the pores of the poroelastic material, via **Darcy's law**:

$$\boldsymbol{q} = -\kappa \nabla p, \tag{5.3}$$

where $\kappa > 0$ is a positive constant. If we use Darcy's law to rewrite the term $-\nabla \cdot (\kappa \nabla p)$ appearing in the second equation (5.2) as $\nabla \cdot \boldsymbol{q}$, we see that the second equation (5.2) relates the local change in the volume occupied by the fluid in the pores of the material to the local expansion/contraction of the poroelastic material and the change in the pore pressure.

We make several remarks about the values of the physical constants which appear in the Biot equations (5.1) and (5.2). The constant ρ_b is nonnegative, and if $\rho_b = 0$, then the resulting equations are known as the **quasistatic Biot equations**, which describe the situation when the inertial effects of the structure are negligible. Generally, one can take $\alpha > 0$, $\kappa > 0$, and $c_0 \geqslant 0$, though in many works, it is assumed that c_0 is a strictly positive constant.

Literature on the Biot equations. Now that we have stated the Biot equations on a fixed domain, we describe some of the literature about these equations. One of the first well-posedness analyses of the Biot equations was carried out in [183], where the quasi-static Biot equations with $\rho_b = 0$ in addition to $c_0 = 0$ are analyzed from the perspective of solutions satisfying a variational formulation. The existence of weak (variational) solutions that are in addition unique is established in this work by discretizing the problem in time using a backwards Euler scheme and by discretizing the problem in space using finite elements, and then passing to the limit in the time and space approximation parameters. The result in [183] is later extended in [148], where the same quasistatic Biot system with $\rho_b = c_0 = 0$ is considered, but more regularity is established for the resulting weak solutions. The Biot displacement and pore pressure are shown to be continuous as functions taking values in $H^1(\Omega_b)$, with the Biot displacement additionally having a time derivative in $L^2(0, T; H^1(\Omega_b))$ in [148], via a Galerkin method combined with energy estimates.

Variational solutions to the Biot equations are also considered in the later work [15], where the authors consider the Biot equations with so-called *secondary consolidation*. In this case, the second equation (5.2) remains the same and the first equation has a secondary consolidation term involving a parameter $\lambda^* \geqslant 0$ that reads

$$\rho_b \boldsymbol{\eta}_{tt} - \nabla(\lambda^*(\nabla \cdot \boldsymbol{\eta})_t) - \nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}, p) = F \quad \text{ on } \Omega_b,$$

where F is an external forcing term. This work [15] considers several regimes, and shows the existence of a weak solution via a Galerkin approach if $\rho > 0$ and $\lambda^* > 0$. By taking the limit as $\lambda^* = 0$, this work also obtains results for the so-called *thermoelastic case* where $\rho > 0$ and $\lambda^* = 0$, which has also been considered in the different context of linear thermoelasticity in [53] (where in thermoelasticity, the elastodynamics equation takes a similar form to the first equation in the Biot equations, and the temperature equation takes a similar form to the pressure equations in the Biot equations). Finally, the work in [15] also considers the limiting case where $\rho = 0$ and $\lambda^* > 0$ by taking the limit as ρ goes to zero to obtain results for the quasi-static Biot equations with secondary consolidation. Work on variational solutions to Biot equations was extended to nonlinear Biot equations on fixed domains in [14], where a semilinear Biot equation in 1D is considered. This equation involves the same second equation (5.2), but adds an additional semilinear nonlinearity in addition to secondary consolidation to the first equation (5.1), which now reads

$$\rho_b \partial_{tt}\eta - \lambda^* \partial_t \partial_x^2 \eta - \partial_x \sigma(\nabla \eta, p) - \mu^* \partial_x(|\partial_x \eta|^{q-2} \partial_x u) = f \quad \text{ on } \Omega_b,$$

for some $q > 1$.

Another approach to studying the Biot equations is to reframe the Biot equations as an abstract evolution equation, for which the tools of implicit/degenerate evolution equation theory can be applied in order to obtain well-posedness results. This is the approach taken in [162], where a quasistatic Biot model with $c_0 \geqslant 0$, and potentially with secondary consolidation, is considered. This framework of considering abstract evolution equations is extended in the work [164] to a quasistatic nonlinear Biot model on a fixed domain, where there are nonlinear relationships between the pore pressure and physical quantities in the problem. The work in [164] studies a problem, which has the same relation for the first equation (5.1) with $\rho_b = 0$, but the second equation is now replaced by the nonlinear equation

$$(b(p) + \alpha\nabla \cdot \boldsymbol{\eta})_t - \nabla \cdot (\kappa\nabla p + \boldsymbol{g}(p)) = F \quad \text{on } \Omega_b,$$

where b and $\boldsymbol{g}$ are nonlinear functions describing the relationship between pore pressure, and density and gravitational effects respectively.

Finally, there has been recent work on challenging Biot models involving nonlinear permeability in Darcy's law. This work was initiated in [25], where quasistatic Biot models with $c_0 = 0$ are considered with poroelasticity and poroviscoelasticity. The model here was

$$\nabla \cdot \boldsymbol{\sigma}(\nabla\boldsymbol{\eta}, p) = -F,$$

$$\alpha(\nabla \cdot \boldsymbol{\eta})_t - \nabla \cdot (\kappa(\nabla \cdot \boldsymbol{\eta})\nabla p) = S,$$

on a fixed domain Ω_b, where the permeability κ depends on $\nabla \cdot \boldsymbol{\eta}$. The stress tensor is given by

$$\boldsymbol{\sigma}(\boldsymbol{\eta}, p) = 2\mu_e \boldsymbol{D}(\boldsymbol{\eta}) + \lambda_e(\nabla \cdot \boldsymbol{\eta})\boldsymbol{I} + 2\mu_v \boldsymbol{D}(\boldsymbol{\eta}_t) + \lambda_v(\nabla \cdot \boldsymbol{\eta}_t) - \alpha p\boldsymbol{I},$$

and the problem is purely poroelastic when $\mu_v = 0, \lambda_v = 0$ and poroviscoelastic when μ_v and λ_v are positive. This work [25] discretizes in space and time, as in [183], and passes to the limit in the discretization parameters, in order to establish existence of a solution under appropriate assumptions on the nonlinear permeability. This work was extended in [24], where a more general quasistatic case where $c_0 \geqslant 0$ is considered, rather than the case of $c_0 = 0$ in [25]. In this case, the permeability κ is not just a function of $\nabla \cdot \boldsymbol{\eta}$, but it is more generally a function of $c_0 p + \alpha\nabla \cdot \boldsymbol{\eta}$. The well-posedness analysis in this work [24] is based on only spatial discretization rather than discretization in both time and space, and it relies on analyzing a linear problem and using a fixed point argument for multi-valued maps, since it is not known if weak solutions are unique for the specific linear system that is needed for this fixed point argument.

Literature on FPSI. Now that we have discussed the literature for the Biot equations, we discuss the literature for fluid-poroelastic structure interaction (FPSI), which couples the Biot equations to the equations of fluid flow in order to describe a fluid interacting with a poroelastic structure. Poroelastic structures are commonplace in applications, for example in applications to geoscience [79, 126] and in applications to modeling biological tissues in the human bodies [40, 180]. We remark that despite the importance of FPSI in engineering applications, there are few works analyzing the well-posedness of fluids interacting with poroelastic structures, and the past work in FPSI has only considered linearly coupled FPSI models, leaving the consideration of nonlinearly coupled FPSI models with moving fluid and Biot domains open.

The abstract evolution equation approach from the Biot equations on a fixed domain [162, 164] has been considered for linearly coupled FPSI in [163], where an FPSI system involving the Biot equations interacting with slightly compressible (rather than incompressible) Stokes equations is considered from an abstract evolution equation framework. Next, a linearly coupled FPSI system involving a poroelastic material modeled by the Biot equations interacting with an incompressible fluid modeled by the Navier-Stokes equations driven by inlet flow is considered in [41], and it is shown that there exists a unique weak solution to this problem given sufficiently small external forcing and inlet pressure. To show this result, a Galerkin method is used to obtain existence of solutions to a weak formulation, where the test functions for the fluid velocity are divergence-free test functions in $H^1(\Omega_f)$. The fluid pressure is recovered using an inf-sup argument, where the fluid pressure along with the other quantities satisfy a weak formulation where the fluid velocity test functions are not necessarily divergence-free functions in $H^1(\Omega_f)$ and the divergence-free condition on the fluid velocity is enforced by testing this divergence-free condition with test functions in $L^2(\Omega_f)$. The analysis of linearly coupled FPSI has been extended to the interaction between Biot poroelastic materials and non-Newtonian fluids in [3], where the generalized Stokes equations model the fluid dynamics. In this case, the fluid viscosity is a nonlinear function of the strain $\boldsymbol{D}(\boldsymbol{u})$, which is a behavior that can be observed in non-Newtonian fluids such as shear-thinning fluids. In addition, the effective viscosity in the Biot poroelastic material is also a nonlinear function of the filtration velocity $\boldsymbol{q}$, so that Darcy's law (5.3) now reads

$$\nu_{eff}(\boldsymbol{q})\kappa^{-1}\boldsymbol{q} + \nabla p = 0.$$

Finally, we mention the recent work in [26], which considers a novel FPSI model involving the coupled interaction between an incompressible fluid modeled by the Stokes equations and a *multilayered poroelastic structure* consisting of a thin poroelastic plate and a thick Biot poroelastic material. This model was motivated by applications to biomedical engineering in the context of developing bioartificial organs. A well-posedness result in the context of weak solutions is obtained for this problem by discretizing the problem in time in order to construct approximate solutions and then passing to the limit in the time discretization.

While the study of fluids interacting with poroelastic materials is an emerging, yet important, field of mathematical research, we note that the study of FPSI has arisen naturally from past studies of incompressible Newtonian fluids interacting with porous materials [63, 123, 156] and more generally non-Newtonian fluids interacting with porous materials [39, 64, 68, 69], where the porous material dynamics is modeled using the Darcy equation for porous media flow.

Next, we discuss the nonlinearly coupled FPSI problem that we will consider. We emphasize that this chapter is adapted from a forthcoming manuscript co-authored with Sunčica Čanić and Boris Muha [115], and features work from the paper [114], co-authored with Sunčica Čanić and Boris Muha also.

5.2 Statement of the problem and motivation

Description of the model

In this chapter, we study the nonlinearly coupled evolution of a Biot poroviscoelastic material with an incompressible fluid modeled by the Navier-Stokes equations, separated by a plate interface. Such problems are of interest in biomedical applications, in which biofluids interact with solid media. While past studies of fluid-structure interaction problems arising in biological applications model such solid structures using equations of elasticity (such as the equations for a Koiter shell), many biological tissues and substances are actually porous in nature, admitting fluid flow through their pores. This makes such moving boundary fluid-poroelastic structure interaction (FPSI) problems of interest in modeling real-life phenomena.

We begin by describing the model. We will consider a two-dimensional model for this problem. Although eventually, it will be of interest to extend such a model to three spatial dimensions, there are already significant mathematical difficulties arising from the moving domains that occur in even the two-dimensional case. We thus begin with the simplest possible geometric configuration of two-dimensional rectangles as the reference configuration for the whole problem, where the entire two-dimensional reference domain $\hat{\Omega}$ can be decomposed into a reference domain for the fluid $\hat{\Omega}_f$, a reference domain for the Biot poroviscoelastic material $\hat{\Omega}_b$, and the interface $\hat{\Gamma}$ separating these two media, which will be the reference configuration of the elastic plate that is between the fluid and the poroviscoelastic structure:

$$\hat{\Omega} = \hat{\Omega}_b \cup \hat{\Omega}_f \cup \hat{\Gamma}.$$

We will consider a simple geometric configuration where

$$\hat{\Omega}_b = (0, L) \times (0, R), \qquad \hat{\Gamma} = (0, L) \times \{0\}, \qquad \hat{\Omega}_f = (0, L) \times (-R, 0). \tag{5.4}$$

We use the variables x and y as coordinates, where $x \in [0, L]$, $y \in [-R, R]$. Because we are considering a problem with nonlinear coupling, all of these regions on the physical time-dependent domain will evolve in time, giving rise to time-dependent $\Omega(t) = \Omega_b(t) \cup \Omega_f(t) \cup \Gamma(t)$. See Figure 5.1. On each of these regions, we will pose a different subproblem, and we will have three subproblems, one for the fluid, the Biot media, and the plate separately. These will then be coupled using appropriate coupling conditions, to be described later. We emphasize that we will be using the notational convention, where we will denote objects associated with the reference domain with a hat, and we will denote objects associated with the physical domain with no hat. We begin by first describing each of the subproblems separately.

The Biot poroviscoelastic structure subproblem

We consider a 2D Biot poroviscoelastic material with reference configuration $\hat{\Omega}_b$. We will denote by $\hat{\boldsymbol{\eta}} : [0, T] \times \hat{\Omega}_b \to \mathbb{R}^2$ the displacement of the material from the reference config-

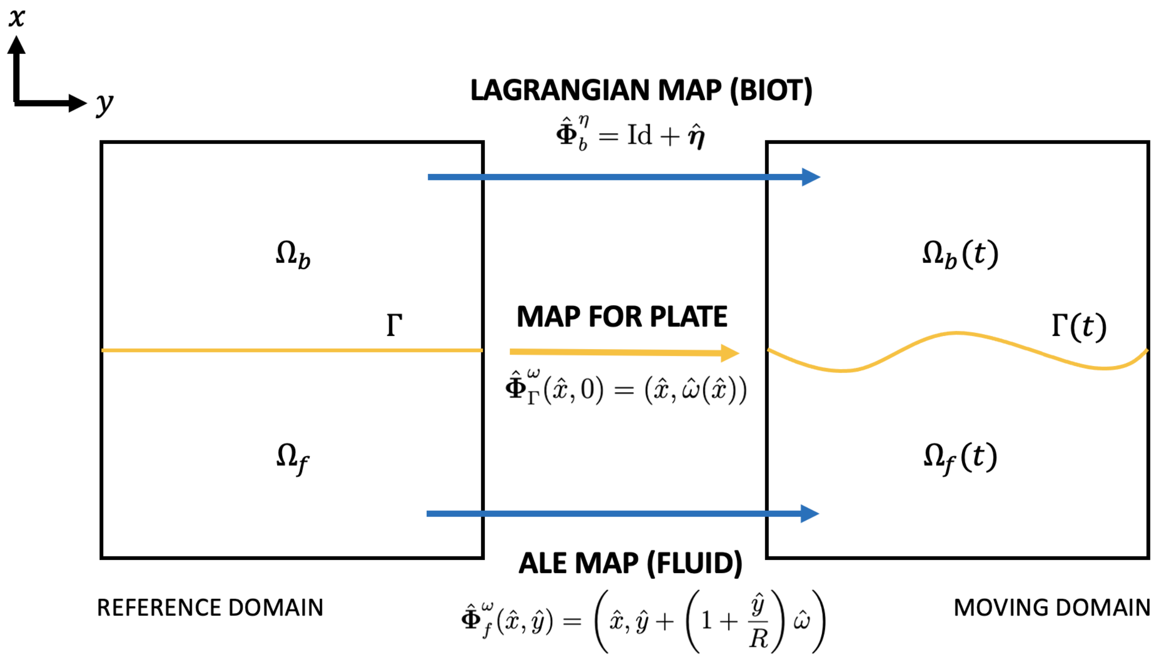

Figure 5.1: *A sketch of the nonlinearly coupled FPSI problem domains, with the reference domain on the left and the moving time-dependent fluid/Biot domains $\Omega_f(t)$ and $\Omega_b(t)$, and time-dependent interface $\Gamma(t)$ on the right. We also illustrate the various maps between the reference domains and the moving domains, including the Lagrangian map for the Biot medium and the Arbitrary Lagrangian-Eulerian (ALE) map for the fluid.*

uration $\hat{\Omega}_b$ and we will denote the pore pressure by $\hat{p} : \hat{\Omega}_b \to \mathbb{R}$. We denote the Lagrangian map by

$$\hat{\mathbf{\Phi}}_b^\eta(t, \cdot) = \text{Id} + \hat{\boldsymbol{\eta}}(t, \cdot) : \hat{\Omega}_b \to \Omega_b(t), \tag{5.5}$$

and we denote its inverse by $(\mathbf{\Phi}_b^\eta)^{-1}(t, \cdot) : \Omega_b(t) \to \hat{\Omega}_b$. We model the poroviscoelastic structure by the nonlinear moving-domain Biot equations, given by

$$\rho_b \partial_{tt} \hat{\boldsymbol{\eta}} = \hat{\nabla} \cdot \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}), \qquad \text{in } \hat{\Omega}_b, \tag{5.6}$$

$$\frac{c_0}{[\det(\hat{\nabla}\hat{\mathbf{\Phi}}_b^\eta)] \circ (\mathbf{\Phi}_b^\eta)^{-1}} \frac{D}{Dt} p + \alpha \nabla \cdot \frac{D}{Dt}\boldsymbol{\eta} - \nabla \cdot (\kappa \nabla p) = 0, \qquad \text{in } \Omega_b(t), \tag{5.7}$$

where the Piola-Kirchhoff stress tensor is given by

$$\hat{S}_b(\nabla\hat{\boldsymbol{\eta}}, \hat{p}) = 2\mu_e \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) + \lambda_e(\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})\boldsymbol{I} + 2\mu_v \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}_t) + \lambda_v(\hat{\nabla} \cdot \hat{\boldsymbol{\eta}}_t)\boldsymbol{I} - \alpha \det(\hat{\nabla}\hat{\mathbf{\Phi}}_b^\eta)p(\hat{\nabla}\hat{\mathbf{\Phi}}_b^\eta)^{-t}. \tag{5.8}$$

In the Piola-Kirchhoff stress tensor, $\boldsymbol{D}$ denotes the symmetrized gradient, μ_e and λ_e are parameters related to the elastic stress, μ_v and λ_v are parameters related to the viscoelastic stress, and $\hat{\mathbf{\Phi}}_b^\eta$ is the Lagrangian map defined above. Recall the convention that variables with hats are defined on the reference domain $\hat{\Omega}_b$, and variables without hats are defined on the physical domain $\Omega_b(t)$. Thus, for example, in the second equation (5.7), the Biot material displacement $\boldsymbol{\eta}$ and the pore pressure p on the physical domain $\Omega_b(t)$ are defined as

$$\boldsymbol{\eta}(t,\cdot) = \hat{\boldsymbol{\eta}}(t, (\mathbf{\Phi}_b^\eta)^{-1}(t,\cdot)), \qquad p(t,\cdot) = \hat{p}(t, (\mathbf{\Phi}_b^\eta)^{-1}(t,\cdot)).$$

We remark that in the definition of the Piola-Kirchhoff stress tensor (5.8), we have used the Piola transform. The Piola transform is a transformation that maps tensors in Lagrangian coordinates to corresponding tensors in Eulerian coordinates, in such a way that divergence-free tensors in Lagrangian coordinates are still divergence free when transferred to Eulerian coordinates. In particular, given a tensor $\boldsymbol{T}(x)$ on the moving domain $\Omega_b(t)$ and the map $\hat{\mathbf{\Phi}}_b^\eta : \hat{\Omega}_b \to \Omega_b(t)$, the Piola transform is defined by

$$\boldsymbol{T}(x) \to \hat{\boldsymbol{T}}(\hat{x}) = \det(\hat{\nabla}\hat{\mathbf{\Phi}}_b^\eta)\boldsymbol{T}(\hat{\mathbf{\Phi}}_b^\eta(\hat{x}))(\hat{\nabla}\hat{\mathbf{\Phi}}_b^\eta)^{-t}.$$

See Section 1.7 of [48] for more details about the Piola transform.

The first equation (5.6) describes the elastodynamics of the structure, while the second equation (5.7) describes the change in the fluid content of the pores. We emphasize that while the first equation is defined on the fixed domain $\hat{\Omega}_b$, the second equation is defined on the moving domain, where

$$\Omega_b(t) = \hat{\mathbf{\Phi}}_b^\eta(t, \hat{\Omega}_b). \tag{5.9}$$

These nonlinear Biot equations for a moving poroelastic structure have been introduced in [160, 182].

A priori, we note that the notion of $\Omega_b(t)$ is not entirely clear, unless $\hat{\boldsymbol{\eta}}$ is sufficiently regular, and furthermore, the formulation of this problem makes sense only if the map $\hat{\mathbf{\Phi}}_b^\eta = \mathrm{Id} + \hat{\boldsymbol{\eta}}$ is an injective map from $\hat{\Omega}_b$ to $\Omega_b(t)$. For the purposes of defining the problem, we do not consider these mathematical difficulties yet, and remark that we will handle these important issues later.

The plate subproblem

We give the equations describing the plate separating the fluid and the Biot poroviscoelastic medium. We assume for simplicity that the plate experiences displacement only in the transverse y direction from the reference configuration $\hat{\Gamma}$, and we denote this displacement by $\hat{\boldsymbol{\omega}} = \hat{\omega}\boldsymbol{e}_y$. Making this assumption avoids technical issues related to potential self-intersection of the plate that would arise if we consider general vector displacements $\hat{\boldsymbol{\omega}}$. The equation for the plate displacement $\hat{\omega}$ in the radial direction is given by

$$\hat{\rho}_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2\hat{\omega} = \hat{F}_p, \qquad \text{on } \hat{\Gamma}, \tag{5.10}$$

where $\hat{\rho}_p$ is the plate density coefficient and $\hat{F}_p$ is the external forcing on the plate in the y direction, to be specified later as the difference in normal stress on both sides from the fluid and the Biot medium. We emphasize that the plate equation is posed on the reference configuration $\hat{\Gamma}$ of the plate.

The time-dependent configuration of the plate

$$\Gamma(t) = \{(x, y) : 0 < x < L,\ y = \hat{\omega}(t, x)\},$$

forms the bottom boundary of the moving domain $\Omega_b(t)$, and the remaining left, top, and right boundaries of $\Omega_b(t)$ are fixed in time. This reflects the fact that the plate displaces in only the transverse y direction. Hence, if the Biot structure displacement $\boldsymbol{\eta}$ is sufficiently regular, then we can describe the moving domain $\Omega_b(t)$, defined in (5.9), equivalently as

$$\Omega_b(t) = \{(x, y) : 0 < x < L, \hat{\omega}(t, x) < y < R\},$$

since on the left, top, and right boundary of $\Omega_b(t)$, we assume that $\boldsymbol{\eta} = 0$ as a boundary condition, see Sec. 5.2.

The fluid subproblem

We model the incompressible fluid by the Navier-Stokes equations, given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \cdot \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi), \qquad \text{in } \Omega_f(t), \tag{5.11}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in } \Omega_f(t), \tag{5.12}$$

where $\boldsymbol{u}$ is the fluid velocity and π is the fluid pressure. The Cauchy stress tensor is given by

$$\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) = 2\nu \boldsymbol{D}(\boldsymbol{u}) - \pi \boldsymbol{I},$$

where π is the fluid pressure and ν is the kinematic viscosity coefficient. The moving fluid domain $\Omega_f(t)$ is determined by the plate displacement $\hat{\omega}$, as follows:

$$\Omega_f(t) = \{(x, y) : 0 < x < L, -R < y < \hat{\omega}(t, x)\}.$$

The first equation describes the balance of forces in the fluid, whereas the second equation is the incompressibility condition for the fluid.

The coupling conditions

Next, we describe the coupling conditions that will couple these three subproblems together. To state the coupling conditions, we introduce the following notation. The flow of fluid through the poroviscoelastic medium is given by the Darcy velocity $\boldsymbol{q}$, defined by

$$\boldsymbol{q} = -\kappa \nabla p \qquad \text{on } \Omega_b(t), \tag{5.13}$$

where κ is a positive permeability constant.

We also define the Biot Cauchy stress tensor on the physical domain, by applying the Piola transform to the Biot Cauchy stress tensor on the reference domain $\hat{S}_b(\nabla\boldsymbol{\eta}, p)$. In particular,

$$S_b(\nabla\boldsymbol{\eta}, p) = [\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^{-1}\hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^t] \circ (\boldsymbol{\Phi}_b^{\eta})^{-1}$$
$$= \left(\frac{1}{\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})}\left[2\mu_e\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) + \lambda_e(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}) + 2\mu_v\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}_t) + \lambda_v(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}_t)\right](\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^t\right) \circ (\boldsymbol{\Phi}_b^{\eta})^{-1} - \alpha\hat{p}\boldsymbol{I}. \tag{5.14}$$

We define the Eulerian structure velocity of the Biot poroviscoelastic material, which gives the structure velocity at each point of the physical domain $\Omega_b(t)$, by

$$\boldsymbol{\xi}(t,\cdot) = \partial_t\hat{\boldsymbol{\eta}}\left(t, (\boldsymbol{\Phi}_b^{\eta})^{-1}(t,\cdot)\right). \tag{5.15}$$

Finally, we define $\boldsymbol{n}(t)$ to be the normal unit vector to the moving interface $\Gamma(t)$ and we define $\hat{\boldsymbol{n}}$ to be the normal unit vector to the reference configuration of the interface Γ. Note that $\hat{\boldsymbol{n}} = \boldsymbol{e}_y$. We will follow the convention that $\boldsymbol{n}(t)$ and $\hat{\boldsymbol{n}}$ will point away from $\Omega_f(t)$ and Ω_f, and inward towards $\Omega_b(t)$ and Ω_b.

With all of this notation, we can now describe the coupling conditions:

- Conservation of mass of the fluid,

$$\boldsymbol{u}\cdot\boldsymbol{n}(t) = (\boldsymbol{q} + \boldsymbol{\xi})\cdot\boldsymbol{n}(t), \qquad \text{on } (0,T)\times\Gamma(t).$$

- Continuity of the displacement (kinematic coupling condition),

$$\hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y, \qquad \text{on } (0,T)\times\hat{\Gamma}.$$

- Beavers-Joseph-Saffman condition describing tangential fluid slip on the interface,

$$\beta(\boldsymbol{\xi} - \boldsymbol{u})\cdot\boldsymbol{\tau}(t) = \boldsymbol{\sigma}_f\boldsymbol{n}(t)\cdot\boldsymbol{\tau}(t), \qquad \text{on } (0,T)\times\Gamma(t), \tag{5.16}$$

  where $\beta \geqslant 0$ is a constant and $\boldsymbol{\tau}(t)$ is the rightward pointing unit tangent vector to $\Gamma(t)$. The Beavers-Joseph-Saffman coupling condition was rigorously justified using homogenization theory in the context of Stokes-Darcy coupling in seminal works of Jäger and Mikelić [97, 98].

- Equality of forces (dynamic coupling condition) describing the external forcing on the plate as the difference between the external and internal load,

$$\hat{F}_p = -\det(\nabla\hat{\boldsymbol{\Phi}}_f^{\omega})[\boldsymbol{\sigma}_f(\nabla\boldsymbol{u}, \pi)\circ\hat{\boldsymbol{\Phi}}_f^{\omega}](\nabla\hat{\boldsymbol{\Phi}}_f^{\omega})^{-t}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}} + \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}|_{\hat{\Gamma}}, \qquad \text{on } \hat{\Gamma}, \tag{5.17}$$

  where $\hat{\boldsymbol{\Phi}}_f^{\omega} : \hat{\Omega}_f \to \Omega_f(t)$ is the Arbitrary Lagrangian-Eulerian (ALE) map for the fluid

$$\hat{\boldsymbol{\Phi}}_f^{\omega}(\hat{x}, \hat{y}) = \left(\hat{x}, \hat{y} + \left(1 + \frac{\hat{y}}{R}\right)\hat{\omega}\right),$$

which maps the reference fluid domain to the moving fluid domain, $\hat{\mathcal{J}}_\Gamma^\omega$ is the arc length measure

$$\hat{\mathcal{J}}_\Gamma^\omega = \sqrt{1 + |\partial_z \hat{\omega}|^2} \tag{5.18}$$

and $(\boldsymbol{\Phi}_\Gamma^\omega)^{-1} : \Gamma(t) \to \hat{\Gamma}$ is the inverse of the Lagrangian map

$$\hat{\boldsymbol{\Phi}}_\Gamma^\omega(\hat{x}, 0) = (\hat{x}, \hat{\omega}(\hat{x})), \qquad \text{on } \hat{\Gamma}. \tag{5.19}$$

- Balance of pressure at the interface,

$$-\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n}(t) \cdot \boldsymbol{n}(t) + \frac{1}{2}|\boldsymbol{u}|^2 = p, \qquad \text{on } (0, T) \times \Gamma(t). \tag{5.20}$$

The boundary conditions

Furthermore, we will impose the following boundary conditions. We recall that the dynamics are occurring in a domain with solid walls, given by $[0, L] \times [-R, R]$. The physical fluid domain is given by

$$\Omega_f(t) = \{(x, y) : 0 < x < L, -R < y < \hat{\omega}(t, x)\},$$

so that the portion of the boundary given by $\partial\Omega_f(t) \backslash \Gamma(t)$ consists of rigid walls. Thus, for the fluid, we will impose a no-slip condition that

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega_f(t) \backslash \Gamma(t).$$

Similarly, we will assume that the boundaries of the Biot poroviscoelastic medium, excluding the interface $\Gamma(t)$, are rigid walls. Thus, we will also impose a Dirichlet condition on the left, top, and right boundaries of the Biot poroviscoelastic domain $\hat{\Omega}_b$, so that

$$\hat{\boldsymbol{\eta}} = 0 \quad \text{and} \quad \hat{p} = 0, \qquad \text{on } \partial\hat{\Omega}_b \backslash \hat{\Gamma}.$$

Statement of the main results and overview of the proof strategy

The goal of this chapter is to study weak solutions to the presented FPSI problem. We will work with class of finite energy weak solutions analogous to the Leray-Hopf class for the Navier-Stokes equations. Such solutions have been widely studied in the context of FSI when the structure is lower dimensional, i.e. described by plate/shell type of the model, see for example the references for the prototypical model of FSI described in Chapter 1. However, in the case of FSI problems with bulk elasticity (i.e. when the dimension of fluid domain is the same as the dimension of the structure domain) the existence of a weak solution is still open. The issue is that the energy inequality does not provide enough regularity of the interface to define the moving domain, the corresponding traces of functions along the interface, and integrals over the interface in the weak formulation. To the best of our knowledge, in the current literature, there are currently only two approaches to circumvent this issue. The first one is to consider the elastic interface with mass [138]. The elastic interface with mass regularizes the problem [134], which makes it possible to define and construct a weak solution. The second one is to consider a nonlinear second order viscoelastic model with energy that is coercive in $W^{1,p}$, $p > 3$, see [19]. Our approach is closer to the former. In our FPSI problem, the issue is even more difficult because equation (5.7) is given in Eulerian coordinates and therefore, the weak formulation has terms involving integrals over $\Omega_b(t)$ which are not well defined (see Section 5.3 for more a precise discussion and a formal derivation of the weak formulation). Consequently, in the analysis of moving boundary FPSI problems it is not enough to regularize the interface, since one needs more regularity of the displacement in the whole domain. Our approach is to define weak solutions to a regularized form of the problem by using a suitably constructed convolution in the space variables, see Section 5.4 for more details. We regularize only the "problematic" terms, i.e. the terms that are not well defined in the finite energy regularity class. We emphasize that the regularized problem is still a nonlinear moving boundary problem of FPSI type and is thus very challenging. Our first main result is the existence of a weak solution to the regularized FPSI problem. It holds for both elastic and visocelastic case for the Biot material. Here we state the theorem informally and refer reader to Theorem 5.5.1 for the precise statement.

Theorem 5.2.1 (Existence of a weak solution)**.** Let $\varrho_b, \mu_c, \lambda_e, \alpha, \hat{\rho}_b, \nu > 0$ and $\mu_v, \lambda_b \geqslant 0$. Moreover, assume that initial data are in finite energy class and that initially, the interface does not touch the outer boundary and certain compatibility condition are satisfied. Then for every regularization parameter $\delta > 0$, there exists $T > 0$ (potentially depending on $\delta > 0$) such that there is a weak solution to the regularized problem with regularization parameter δ on $[0, T]$.

The proof of this theorem is carried out in Sections 5.5, 5.6, 5.7, and 5.8. In Section 5.5, we define a splitting scheme that will be used in the construction of approximate solutions. The scheme is combination of a semi-discretization in the time variable and a Lie operator splitting. Even though this scheme has already been successfully used in analysis and numerics for FSI problems, we needed to adapt it to the setting of FPSI problems. The main benefit of this approach is that it enables us to decouple the interface dynamics from the Biot-Navier-Stokes coupling so that we can treat both subproblems separately. The approximate solutions are constructed in Section 5.6, where it is also proved that the approximate solutions satisfy energy estimates which are uniform in the discretization parameter. In Section 5.7, we study compactness properties of the sequences of approximate solutions, i.e. we show that the approximate solutions converge strongly in a suitable topology. This is the most delicate part of the existence proof. Here, we use two abstract compactness results which are generalizations of the Aubin-Lions theorem for piecewise constant functions [65] and moving domains [136]. Here, we note that the compactness proof differs from the standard FSI cases due to the fact that in the test space, the fluid and the structure test functions are not coupled. Therefore, we can divide the compactness proof into several distinct parts. Finally, in Section 5.8, we pass to the limit in the approximate solutions to show that the limit of the approximate solutions is indeed a weak solution.

Our second main result is a weak-classical consistency result. Namely, in order to justify our regularization procedure and the corresponding definition of weak solutions to the regularized problem, we prove that weak solutions to the regularized problem indeed converge to the solution to the original FPSI problem. More precisely, we prove the following result (for the full statement, see Theorem 5.9.1)

Theorem 5.2.2 (Weak-classical consistency)**.** Assume that a classical (smooth) solution to the FPSI with a Biot poroviscoelastic medium exists (so that $\mu_v, \lambda_v > 0$). Then, there exists $T > 0$ such that every sequence of weak solutions to the regularized problem with regularization parameter $\delta > 0$ converge to the classical solution on $[0, T]$ as the regularization parameter δ converges to 0. In particular, there exists $\delta_0 > 0$ such that regularized solutions exist on $[0, T]$ for $\delta < \delta_0$, i.e. provided that a classical solution exists, the time interval of existence for the weak solutions to the regularized problem is uniform in regularization parameter.

The heart of the proof of this theorem is a bootstrap argument presented in Section 5.9. Namely, the main issue is that geometric quantities, such as the determinant of the displacement, cannot be estimated by the energy and thus are not uniformly bounded in the regularization parameter δ. We derive appropriate bounds by using a bootstrap argument in combination with optimal convergence rate estimates for the convolution regularization. The main technical issue in comparing the classical solution with weak solutions to the regularized problem is the fact that they are defined on different domains. Therefore, we use a change of variables that transfers fluid velocities as vector fields and preserves the divergence-free condition. This transformation was introduced by [95] and was used in proving weak-strong type of results in the context of FSI in [43, 142, 159]. The corresponding estimates are carried out in Section 5.10.

5.3 Definition of a weak solution

Maps between reference and physical domains

Because this problem is nonlinearly coupled, the fluid domain $\Omega_f(t)$ and the Biot poroviscoelastic domain $\Omega_b(t)$ in physical space are time-dependent and hence are not known apriori. This is a mathematical difficulty that is seen even in prototypical fluid-structure interaction models.

To handle the moving domains, it will be useful to work on the fixed reference domains $\hat{\Omega}_b$, $\hat{\Gamma}$, and $\hat{\Omega}_f$, which are defined by (5.4). Hence, we need a map that maps the fixed reference domains onto the appropriate time-dependent domains. We will denote these maps for the Biot medium, plate, and fluid by

$$\hat{\mathbf{\Phi}}_b^{\eta}(t,\cdot) : \hat{\Omega}_b \to \Omega_b(t), \qquad \hat{\mathbf{\Phi}}_\Gamma^{\omega}(t,\cdot) : \hat{\Gamma} \to \Gamma(t), \qquad \hat{\mathbf{\Phi}}_f^{\omega}(t,\cdot) : \hat{\Omega}_f \to \Omega_f(t).$$

See Figure 5.1. We emphasize that these maps are *time-dependent*, even though in the rest of this manuscript, we will not explicitly notate this time dependence for ease of notation.

On the reference domain for the Biot poroviscoelastic material $\hat{\Omega}_b = (0, L) \times (0, R)$, we define $\hat{\boldsymbol{\Phi}}_b$ on $\hat{\Omega}_b^\eta$, as before in (5.5), by

$$\hat{\boldsymbol{\Phi}}_b^\eta = \text{Id} + \hat{\boldsymbol{\eta}}.$$

On the reference domain $\hat{\Gamma}$ for the plate, we define $\hat{\boldsymbol{\Phi}}_\Gamma^\omega$ on $\hat{\Gamma}$ as in (5.19) by

$$\hat{\boldsymbol{\Phi}}_\Gamma^\omega(\hat{x}, 0) = (\hat{x}, \hat{\omega}(\hat{x})).$$

On the reference domain for the fluid $\hat{\Omega}_f = (0, L) \times (-R, 0)$, we define $\hat{\boldsymbol{\Phi}}_f^\omega$ on $\hat{\Omega}_f$ by using the standard Arbitrary Lagrangian-Eulerian (ALE) mapping

$$\hat{\boldsymbol{\Phi}}_f^\omega(\hat{x}, \hat{y}) = \left(\hat{x}, \hat{y} + \left(1 + \frac{\hat{y}}{R}\right)\hat{\omega}\right), \qquad (\hat{x}, \hat{y}) \in \hat{\Omega}_f, \tag{5.21}$$

where we are using the variables $(\hat{x}, \hat{y})$ to denote the coordinates on the reference domain and we are using the variables (x, y) to denote the coordinates on the physical domain. We can express the inverse of this map as

$$(\boldsymbol{\Phi}_f^\omega)^{-1}(x, y) = \left(x, -R + \frac{R}{R + \hat{\omega}}(R + y)\right), \qquad (x, y) \in \Omega_f(t).$$

In the analysis of the full FPSI problem, it is necessary to consider functions on both the reference and the physical domains, and hence, we must examine how functions and derivatives transform under $\hat{\boldsymbol{\Phi}}_f^\omega$ and $\hat{\boldsymbol{\Phi}}_b^\eta$. We will first focus on the behavior of functions under the transformation $\hat{\boldsymbol{\Phi}}_f^\omega$ on $\hat{\Omega}_f$, considering in particular the fluid velocity $\boldsymbol{u}$.

First, we note that on $\hat{\Omega}_f$, the Jacobian of $\hat{\boldsymbol{\Phi}}_f^\omega$ is given by $\hat{\mathcal{J}}_f^\omega = \left|1 + \frac{\hat{\omega}}{R}\right|$. Because our results will hold up until the time of domain degeneracy when $|\hat{\omega}| \geqslant R$, we can get rid of the absolute values and just write

$$\hat{\mathcal{J}}_f^\omega = 1 + \frac{\hat{\omega}}{R}. \tag{5.22}$$

We will furthermore denote the fluid velocity $\boldsymbol{u}$ defined on $\Omega_f(t)$, transferred to the fixed reference domain $\hat{\Omega}_f$, by

$$\hat{\boldsymbol{u}}(t, \hat{x}, \hat{y}) = \boldsymbol{u} \circ \hat{\boldsymbol{\Phi}}_f, \qquad \text{for } (\hat{x}, \hat{y}) \in \hat{\Omega}_f.$$

Recall that on the moving domain $\Omega_f(t)$, the fluid velocity $\boldsymbol{u}$ is divergence free, so that $\nabla \cdot \boldsymbol{u} = 0$. However, when we pull the fluid velocity back to the reference domain, $\hat{\boldsymbol{u}}$ is not necessarily divergence free on $\hat{\Omega}_f$. Hence, we want to reformulate the divergence free condition on the fixed reference domain. To do this, we look at how derivatives transform under the map $\hat{\boldsymbol{\Phi}}_f^\omega$.

In particular, note that for any function g defined on $\Omega_f(t)$,

$$\nabla g = \nabla\left(\hat{g} \circ (\boldsymbol{\Phi}_f^{\omega})^{-1}\right) = (\hat{\nabla}_f^{\omega}\hat{g}) \circ (\boldsymbol{\Phi}_f^{\omega})^{-1}$$

for the differential operator

$$\hat{\nabla}_f^{\omega} = \begin{pmatrix} \partial_{\hat{x}} - (R+y)\partial_{\hat{x}}\hat{\omega}\frac{R}{(R+\hat{\omega})^2}\partial_{\hat{y}} \\ \frac{R}{R+\hat{\omega}}\partial_{\hat{y}} \end{pmatrix} = \begin{pmatrix} \partial_{\hat{x}} - \frac{(R+\hat{y})\partial_{\hat{x}}\hat{\omega}}{R+\hat{\omega}}\partial_{\hat{y}} \\ \frac{R}{R+\hat{\omega}}\partial_{\hat{y}} \end{pmatrix}, \tag{5.23}$$

where we used $y = \hat{y} + \left(1 + \frac{\hat{y}}{R}\right)\hat{\omega}$. Therefore, the divergence free condition on the fixed reference domain $\hat{\Omega}_f$ is $\hat{\nabla}_f^{\omega} \cdot \hat{\boldsymbol{u}} = 0$ and the symmetrized gradient transforms as $\hat{\boldsymbol{D}}_f^{\omega}(\hat{\boldsymbol{u}}) = \frac{1}{2}\left(\hat{\nabla}_f^{\omega}\hat{\boldsymbol{u}} + (\hat{\nabla}_f^{\omega}\hat{\boldsymbol{u}})^t\right)$.

Next, we consider how time derivatives are transformed under the map $\hat{\boldsymbol{\Phi}}_f^{\omega}$. We have that

$$\partial_t \boldsymbol{u}(t,x,y) = \partial_t(\hat{\boldsymbol{u}}(t,(\boldsymbol{\Phi}_f^{\omega})^{-1}(\hat{x},\hat{y})) = \partial_t\hat{\boldsymbol{u}} - \partial_{\hat{y}}\hat{\boldsymbol{u}}\cdot(R+y)\cdot\frac{R}{(R+\hat{\omega})^2}\partial_t\hat{\omega} = \partial_t\hat{\boldsymbol{u}} - \partial_{\hat{y}}\hat{\boldsymbol{u}}\frac{(R+\hat{y})\partial_t\hat{\omega}}{R+\hat{\omega}}.$$

So defining

$$\hat{\boldsymbol{w}} = \frac{R+\hat{y}}{R}\partial_t\hat{\omega}\boldsymbol{e}_y, \tag{5.24}$$

we have that

$$\partial_t\boldsymbol{u} = \partial_t\hat{\boldsymbol{u}} - (\hat{\boldsymbol{w}}\cdot\hat{\nabla}_f^{\omega})\hat{\boldsymbol{u}}. \tag{5.25}$$

The weak formulation will also involve integrals with a time-dependent domain for the Biot equations, $\Omega_b(t)$, so we will examine the transformation of spatial derivatives via the map $\hat{\boldsymbol{\Phi}}_b^{\eta}$ on $\hat{\Omega}_b$. Recall that

$$\hat{\boldsymbol{\Phi}}_b^{\eta}(\hat{x},\hat{y}) = (\hat{x},\hat{y}) + \hat{\boldsymbol{\eta}}(\hat{x},\hat{y}), \qquad \text{for } (\hat{x},\hat{y}) \in \hat{\Omega}_b,$$

where we recall that the displacement $\hat{\boldsymbol{\eta}}$ is defined on the fixed reference domain $\hat{\Omega}_b$. So given a scalar function g defined on $\Omega_b(t)$, we want to see how ∇g transforms when pulled back to the reference domain. We define the pull back of g to the reference domain $\hat{\Omega}_b$ by

$$\hat{g} = g \circ \hat{\boldsymbol{\Phi}}_b^{\eta}.$$

We claim that for some differential operator $\hat{\nabla}_b^{\eta}$,

$$\nabla g = \nabla\left(\hat{g} \circ (\boldsymbol{\Phi}_b^{\eta})^{-1}\right) = (\hat{\nabla}_b^{\eta}\hat{g}) \circ (\boldsymbol{\Phi}_b^{\eta})^{-1}.$$

We emphasize that ∇ is a gradient on the physical domain, $\hat{\nabla}$ is a gradient on the reference domain, and $\hat{\nabla}_b^{\eta}$ is a differential operator (different from $\hat{\nabla}$) on the reference domain. For any function g defined on the physical domain, we have that

$$\hat{\nabla}\left(g \circ \hat{\boldsymbol{\Phi}}_b^{\eta}\right) = [(\nabla g) \circ \hat{\boldsymbol{\Phi}}_b^{\eta}] \cdot (\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}).$$

Hence, for

$$\hat{\nabla}_b^\eta \hat{g} = (\nabla g) \circ \hat{\boldsymbol{\Phi}}_b^\eta,$$

we have the following explicit formula for the differential operator $\hat{\nabla}_b^\eta$ on the reference domain, given by

$$\hat{\nabla}_b^\eta \hat{g} = \left(\frac{\partial \hat{g}}{\partial \hat{x}}, \frac{\partial \hat{g}}{\partial \hat{y}}\right) \cdot (\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}})^{-1}. \tag{5.26}$$

We finally note that the Jacobian of the map $\hat{\boldsymbol{\Phi}}_b^\eta$ on $\hat{\Omega}_b$ is

$$\hat{\mathcal{J}}_b^\eta = \det(\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}). \tag{5.27}$$

We remark that the invertibility of the matrix $\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}$ will be related to whether the map $(\hat{x}, \hat{y}) \to (\hat{x}, \hat{y}) + \hat{\boldsymbol{\eta}}(\hat{x}, \hat{y})$ is a bijection between $\hat{\Omega}_b$ and $\Omega_b(t)$.

We now derive the definition of a weak solution to the given FPSI problem, by means of the following formal calculation. We start with the fluid equations and multiply by a test function $\boldsymbol{v}$. Recall the definition of the Eulerian structure velocity $\boldsymbol{\xi}$ from (5.15). For the acceleration term of the Navier-Stokes equations, we obtain by the Reynold's transport theorem and integration by parts,

$$\begin{aligned}\int_{\Omega_f(t)} (\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u})) \cdot \boldsymbol{v} &= \frac{d}{dt}\int_{\Omega_f(t)} \boldsymbol{u} \cdot \boldsymbol{v} - \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} - \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\ &\quad + \frac{1}{2}\int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\ = \frac{d}{dt}\int_{\Omega_f(t)} \boldsymbol{u} \cdot \boldsymbol{v} - \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} &+ \frac{1}{2}\int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v}.\end{aligned}$$

For the diffusive term of the Navier Stokes equations, we integrate by parts to obtain

$$-\int_{\Omega_f(t)} (\nabla \cdot \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)) \cdot \boldsymbol{v} = 2\nu \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{v},$$

where we used the fact that the test function $\boldsymbol{v}$ is divergence free to eliminate the pressure from the integral over $\Omega_f(t)$, and we use that the test function satisfies $\boldsymbol{v} = 0$ on $\partial\Omega_f(t)\backslash\Gamma(t)$ due to the boundary conditions for the fluid velocity $\boldsymbol{u}$.

Next, we multiply the structure equation by a test function $\hat{\boldsymbol{\psi}}$ to obtain

$$\begin{aligned}\int_{\hat{\Omega}_b} (\rho_b \partial_{tt}\hat{\boldsymbol{\eta}} - \hat{\nabla} \cdot \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})) \cdot \hat{\boldsymbol{\psi}} &= \rho_b \left(\frac{d}{dt}\int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{\psi}} - \int_{\Omega_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}}\right) \\ + \int_{\hat{\Omega}_b} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}) : \hat{\nabla}\hat{\boldsymbol{\psi}} + \int_{\hat{\Gamma}} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})\boldsymbol{e}_y \cdot \hat{\boldsymbol{\psi}} &= \rho_b \left(\frac{d}{dt}\int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{\psi}} - \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}}\right) \\ + \int_{\hat{\Omega}_b} (2\mu_e \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_e(\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) &+ 2\mu_v \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v(\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}})) \\ &- \alpha \int_{\Omega_b(t)} p(\nabla \cdot \boldsymbol{\psi}) + \int_{\hat{\Gamma}} \hat{S}_b(\nabla \hat{\boldsymbol{\eta}}, \hat{p})\boldsymbol{e}_r \cdot \hat{\boldsymbol{\psi}}.\end{aligned}$$

Except on $\hat{\Gamma}$, there are no boundary terms, because $\hat{\boldsymbol{\eta}} = 0$ on the left, top, and right boundaries of $\hat{\Omega}_b$, and hence the same condition holds for the corresponding test function $\hat{\boldsymbol{\psi}}$. Note that in the integral over $\Omega_b(t)$, $\boldsymbol{\psi} := \hat{\boldsymbol{\psi}} \circ (\boldsymbol{\Phi}_b^\eta)^{-1}$.

Finally, we test the second equation corresponding to the evolution of the pore pressure for the Biot poroviscoelastic medium with a test function r, where we recall the definition of the Darcy velocity $\boldsymbol{q}$ from (5.13) and we emphasize that $\boldsymbol{n}$ is the *inward* normal vector to $\Omega_b(t)$.

$$\int_{\Omega_b(t)} \left(\frac{c_0}{[\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^\eta)] \circ (\boldsymbol{\Phi}_b^\eta)^{-1}} \frac{D}{Dt} p + \alpha \nabla \cdot \frac{D}{Dt}\boldsymbol{\eta} - \nabla \cdot (\kappa \nabla p) \right) r$$

$$= \int_{\hat{\Omega}_b} c_0 \partial_t \hat{p} \cdot \hat{r} + \int_{\Omega_b(t)} \alpha \left(\nabla \cdot \frac{D}{Dt}\boldsymbol{\eta} \right) r + \int_{\Omega_b(t)} \kappa \nabla p \cdot \nabla r - \int_{\Gamma(t)} (\boldsymbol{q} \cdot \boldsymbol{n}) r$$

$$= \frac{d}{dt} \int_{\hat{\Omega}_b} c_0 \hat{p} \cdot \hat{r} - \int_{\hat{\Omega}_b} c_0 \hat{p} \cdot \partial_t \hat{r} - \int_{\Omega_b(t)} \alpha \frac{D}{Dt}\boldsymbol{\eta} \cdot \nabla r - \alpha \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n}) r + \int_{\Omega_b(t)} \kappa \nabla p \cdot \nabla r - \int_{\Gamma(t)} (\boldsymbol{q} \cdot \boldsymbol{n}) r.$$

There are no boundary terms except on $\Gamma(t)$ from the integration by parts in the integral involving α and in the integral involving κ because we have from the Dirichlet boundary condition that the test function satisfies $r = 0$ (since $p = 0$) on the left, top, and right boundaries of $\hat{\Omega}_b$.

We recall the definition of $\hat{\boldsymbol{\Phi}}_\Gamma^\omega$ in (5.19) and the arc length measure $\hat{\mathcal{J}}_\Gamma^\omega$ (5.18) and we sum the two terms

$$- \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{v} + \int_{\hat{\Gamma}} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})\boldsymbol{e}_y \cdot \hat{\boldsymbol{\psi}}$$

$$= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} (\hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})\boldsymbol{e}_y - \hat{\mathcal{J}}_\Gamma^\omega \cdot (\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n}|_{\Gamma(t)} \circ \boldsymbol{\Phi}_\Gamma^\omega) \cdot \hat{\boldsymbol{\psi}}.$$

Since the displacement of the plate is only in the y direction so that $\hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y$ on $\hat{\Gamma}$, the test function $\hat{\boldsymbol{\psi}}$ points in the y direction on $\hat{\Gamma}$ as well. We will denote by $\hat{\varphi}$ the magnitude of $\hat{\boldsymbol{\psi}}|_{\hat{\Gamma}}$ so that $\hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y$ on $\hat{\Gamma}$. By the dynamic coupling condition (5.17), we have that the previous expression is equal to

$$
\begin{aligned}
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} \hat{F}_p \cdot \hat{\varphi} = \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{n}(\psi_n - v_n) + \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{n}(\psi_n - v_n) + \int_{\Gamma(t)} \beta(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \int_{\Gamma(t)} \beta(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) \\
&\quad + \frac{d}{dt}\left(\int_{\hat{\Gamma}} \rho_p \partial_t \hat{\omega} \cdot \hat{\varphi}\right) - \int_{\hat{\Gamma}} \rho_p \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega} \cdot \hat{\Delta}\hat{\varphi},
\end{aligned}
$$

where we used the coupling conditions (5.16) and (5.20) in the last step.

We sum everything together to get the following definition of a weak solution. Define the transverse velocity of the plate by the variable $\hat{\zeta}$, so that

$$\partial_t \hat{\omega} = \hat{\zeta}, \tag{5.28}$$

and let $\zeta = \hat{\zeta} \circ (\boldsymbol{\Phi}_\Gamma^\omega)^{-1}$. The ordered four-tuple $(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, p)$ is a weak solution if for every test function $(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, r)$ that is C_c^1 in time on $[0, T]$ taking values in the test space, satisfying $\hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y$ on $\hat{\Gamma}$, we have that

$$
\begin{aligned}
&-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^T \int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_0^T \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\zeta \boldsymbol{e}_y \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\
&+ 2\nu \int_0^T \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^T \int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \beta \int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{\tau} \\
&- \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega} \cdot \hat{\Delta}\hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&+ \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) \\
&- \alpha \int_0^T \int_{\Omega_b(t)} p \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p} \cdot \partial_t \hat{r} - \alpha \int_0^T \int_{\Omega_b(t)} \frac{D}{Dt}\boldsymbol{\eta} \cdot \nabla r - \alpha \int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}) r \\
&+ \kappa \int_0^T \int_{\Omega_b(t)} \nabla p \cdot \nabla r - \int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta \boldsymbol{e}_y) \cdot \boldsymbol{n}) r \\
&= \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot \boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t \hat{\omega}(0) \cdot \hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}}(0) \cdot \hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0) \cdot \hat{r}(0).
\end{aligned} \tag{5.29}
$$

We remark however that the above weak formulation *is inadequate for the regularity of finite-energy solutions* for the following reason. By the energy estimates (see Section 5.4), the regularity of the structure displacement $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$ in the finite energy space is $L^\infty(0, T, H^1(\hat{\Omega}_b))$, which is not enough regularity to interpret the term

$$\alpha \int_{\Omega_b(t)} p \nabla \cdot \boldsymbol{\psi},$$

since the test function has regularity $\hat{\boldsymbol{\psi}} \in H^1(\hat{\Omega}_b)$ on the *fixed reference domain*, due to the corresponding finite energy regularity of $\hat{\boldsymbol{\eta}}$. Hence, after changing variables, which adds an extra factor of $\det(\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}})$ arising from the Jacobian (which is only in $L^\infty(0, T; L^1(\hat{\Omega}_b))$ in two dimensions), there is not enough regularity to guarantee that this integral is finite.

Therefore, we cannot interpret the above notion of weak solution properly in the space of finite energy solutions, as the finite energy space does not have enough regularity to make sense of certain integrals in the weak formulation, involving the deformed domain $\Omega_b(t)$.

5.4 Regularized weak solution

We note that all of these mathematical challenges, which are related to the inability to properly interpret all of the terms in the weak solution, arise fundamentally from the *lack of regularity* of $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$. Therefore, we modify our weak formulation appropriately to give $\hat{\boldsymbol{\eta}}$ more regularity, by convolving with a smooth compactly supported function with support on the order of δ. This allows us to develop an appropriate *regularized weak formulation* of the original FPSI problem. To show that the weak solutions we construct to the regularized problem are physically relevant, we show that as $\delta \to 0$, given a sufficiently smooth classical solution to the original FPSI problem, the weak solutions to the regularized problem will converge to this classical solution as $\delta \to 0$. See Section 5.9.

Therefore, we will define a regularized version of the structure displacement, $\hat{\boldsymbol{\eta}}^\delta$, which is spatially smooth. We do this by convolution with a smooth compactly supported function. However, because we are working on a bounded domain $\hat{\Omega}_b$, we must be careful to do this convolution in a way that preserves the Dirichlet condition on the left, top, and right boundaries of $\hat{\Omega}_b$.

To do this, we define an extended domain $\tilde{\Omega}_b$ as follows. Recalling that the reference configuration of the Biot domain is given by $\hat{\Omega}_b = (0, L) \times (0, R)$, we define

$$\tilde{\Omega}_b = [-L, 2L] \times [-R, 2R].$$

Assuming that $\delta < \min(L, R)$, then the convolution of a function on $\tilde{\Omega}_b$ with a smooth function of compact support in the closed ball of radius δ gives a function defined on $\hat{\Omega}_b$.

We must extend the function $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$ to the larger domain $\tilde{\Omega}_b$ in such a way so that the resulting spatial convolution has the desired properties. To do this, we will use an odd extension along the lines $\hat{x} = 0$, $\hat{x} = L$, $\hat{y} = 0$ and $\hat{y} = R$. Thus, we will introduce the following definition.

Definition 5.4.1. Given $\hat{\boldsymbol{\eta}}$ defined on $\hat{\Omega}_b$ satisfying $\hat{\boldsymbol{\eta}} = 0$ on $\hat{x} = 0$, $\hat{x} = L$, and $\hat{y} = R$ and $\hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y$ on $\hat{y} = 0$, we define the **odd extension of $\hat{\boldsymbol{\eta}}$ to $\tilde{\Omega}_b$** by keeping $\hat{\boldsymbol{\eta}}$ the same on $[0, L] \times [0, R]$ and defining $\hat{\boldsymbol{\eta}}$ outside of the closure of $\hat{\Omega}_b$ as follows:

1. On $[0, L] \times [-R, 0]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = \hat{\omega}(\hat{x})\boldsymbol{e}_y + (\hat{\omega}(\hat{x})\boldsymbol{e}_y - \hat{\boldsymbol{\eta}}(\hat{x}, -\hat{y}))$.
2. On $[0, L] \times [R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(\hat{x}, 2R - \hat{y})$.
3. On $[-L, 0] \times [-R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(-\hat{x}, \hat{y})$.
4. On $[L, 2L] \times [-R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(2L - \hat{x}, \hat{y})$.

Let σ be a radially symmetric function on $\mathbb{R}^2$ with compact support in the closed ball of radius one, with the property that $\displaystyle\int_{\mathbb{R}^2} \sigma = 1$. Let

$$\sigma_\delta = \delta^{-2}\sigma(\delta^{-1}\boldsymbol{x}), \qquad \text{on } \mathbb{R}^2.$$

We regularize in space, and define

$$\hat{\boldsymbol{\eta}}^\delta = \hat{\boldsymbol{\eta}} * \sigma_\delta, \qquad \text{on } \hat{\Omega}_b, \tag{5.30}$$

and we note that these regularized functions are spatially smooth on the closure of $\hat{\Omega}_b$. We define

$$\hat{\boldsymbol{\Phi}}_b^{\eta^\delta} = \text{Id} + \hat{\boldsymbol{\eta}}^\delta, \tag{5.31}$$

and we define the regularized moving Biot domain by

$$\Omega_b^\delta(t) = \hat{\boldsymbol{\Phi}}_b^{\eta^\delta}(\hat{\Omega}_b). \tag{5.32}$$

Note that even though the kinematic coupling condition holds for $\hat{\boldsymbol{\eta}}$ in the sense that $\hat{\boldsymbol{\eta}}|_{\hat{\Gamma}} = \hat{\omega}\boldsymbol{e}_y$, it is *not necessarily true* that $\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}} = \hat{\omega}\boldsymbol{e}_y$. Therefore, we will also define the moving interface by

$$\Gamma^\delta(t) = \hat{\boldsymbol{\Phi}}_b^{\eta^\delta}(\hat{\Gamma}).$$

Alternatively, $\hat{\Gamma}^\delta$ is the plate interface if it were displaced from the reference configuration $\hat{\Gamma}$ in the direction $\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}}$, which is a purely transverse y displacement, as one can verify.

Note that by the way we extended $\hat{\boldsymbol{\eta}}$ to the larger domain $\tilde{\Omega}_b$ before doing the spatial convolution, we have that

$$\hat{\boldsymbol{\eta}}^\delta = 0,$$

on $\partial\hat{\Omega}_b \backslash \hat{\Gamma}$.

With these regularized versions of the Biot structure displacement and velocity, we now define the notion of a *regularized weak solution with regularization parameter* δ to the nonlinearly coupled FPSI problem. We start by defining the solution and test space, which are motivated by the energy estimates in Section 5.4, and then we state the regularized weak formulation. We will formulate the solution space, test space, and regularized weak formulation for both the moving fluid domain and the fixed reference fluid domain.

Definition 5.4.2. (Solution and test space)

- *Fluid function space (moving domain).*

$$V_f(t) = \{\boldsymbol{u} = (u_x, u_y) \in H^1(\Omega_f(t)) : \nabla \cdot \boldsymbol{u} = 0, \text{ and } \boldsymbol{u} = 0 \text{ when } x = 0, x = L, y = -R\}, \tag{5.33}$$

$$\mathcal{V}_f = L^\infty(0,T; L^2(\Omega_f(t))) \cap L^2(0,T; V_f(t)). \tag{5.34}$$

- *Fluid function space (fixed domain).*

$$V_f^\omega = \{\hat{\boldsymbol{u}} = (\hat{u}_x, \hat{u}_y) \in H^1(\hat{\Omega}_f) : \hat{\nabla}_f^\omega \cdot \hat{\boldsymbol{u}} = 0, \text{ and } \hat{\boldsymbol{u}} = 0 \text{ when } \hat{x} = 0, \hat{x} = L, \hat{y} = -R\}, \tag{5.35}$$

$$\mathcal{V}_f^\omega = L^\infty(0,T; L^2(\hat{\Omega}_f)) \cap L^2(0,T; V_f^\omega). \tag{5.36}$$

- *Plate function space.*

$$\mathcal{V}_\omega = W^{1,\infty}(0,T; L^2(\hat{\Gamma})) \cap L^\infty(0,T; H_0^2(\hat{\Gamma})). \tag{5.37}$$

- *Biot displacement function space.*

$$V_d = \{\hat{\boldsymbol{\eta}} = (\hat{\eta}_x, \hat{\eta}_y) \in H^1(\hat{\Omega}_b) : \hat{\boldsymbol{\eta}} = 0 \text{ for } \hat{x} = 0, \hat{x} = L, \hat{y} = R, \text{ and } \hat{\eta}_x = 0 \text{ on } \hat{\Gamma}\}, \tag{5.38}$$

$$\mathcal{V}_b = W^{1,\infty}(0,T; L^2(\hat{\Omega}_b)) \cap L^\infty(0,T; V_d) \cap H^1(0,T; V_d). \tag{5.39}$$

- *Biot pore pressure function space.*

$$V_p = \{\hat{p} \in H^1(\hat{\Omega}_b) : \hat{p} = 0 \text{ for } \hat{x} = 0, \hat{x} = L, \hat{y} = R\}, \tag{5.40}$$

$$\mathcal{Q}_b = L^\infty(0,T; L^2(\hat{\Omega}_b)) \cap L^2(0,T; V_p). \tag{5.41}$$

- *Weak solution space (moving domain).*

$$\mathcal{V}_{\text{sol}} = \{(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}_f \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{5.42}$$

- *Weak solution space (fixed domain).*

$$\mathcal{V}_{\text{sol}}^\omega = \{(\hat{\boldsymbol{u}}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}_f^\omega \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{5.43}$$

- *Test space (moving domain).*

$$\mathcal{V}_{\text{test}} = \{(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in C_c^1([0,T); V_f(t) \times H_0^2(\hat{\Gamma}) \times V_d \times V_p) : \hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{5.44}$$

- *Test space (fixed domain).*

$$\mathcal{V}_{\text{test}}^\omega = \{(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in C_c^1([0,T); V_f^\omega \times H_0^2(\hat{\Gamma}) \times V_d \times V_p) : \hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{5.45}$$

Remark 5.4.1. Because $\hat{\Gamma}$ is one dimensional, for plate displacements $\hat{\omega} \in \mathcal{V}_\omega$, we have that $\hat{\omega} \in C(0, T; C^1(\hat{\Gamma}))$ and hence, there is a one-to-one correspondence between functions in $\mathcal{V}_{\text{sol}}$ and $\mathcal{V}_{\text{sol}}^\omega$ and functions in $\mathcal{V}_{\text{test}}$ and $\mathcal{V}_{\text{test}}^\omega$, given by composition with the ALE mapping (5.21) for the fluid domain, $\hat{\boldsymbol{\Phi}}_f^\omega : \hat{\Omega}_f \to \Omega_f(t)$, for the component involving fluid velocities.

We now state the regularized weak formulation in the moving domain formulation. We note that the regularization that we have used in regularizing the Biot domain is minimal, in the sense that it only affects four terms in the weak formulation, and we use the regularization in only the terms in which it is strictly necessary. In the following statement of the regularized weak formulation, we recall the definition (5.28) of the plate velocity $\hat{\zeta}$.

Definition 5.4.3. (Weak solution to the regularized problem, moving fluid domain formulation) An ordered four-tuple $(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, p) \in \mathcal{V}_{\text{sol}}$ is a *weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter* δ if for every test function $(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in \mathcal{V}_{\text{test}}$,

$$\begin{aligned}
&-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} + \frac{1}{2} \int_0^T \int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2} \int_0^T \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{v} \\
&+ 2\nu \int_0^T \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^T \int_{\Gamma(t)} \left(\frac{1}{2} |\boldsymbol{u}|^2 - p \right) (\psi_n - v_n) + \beta \int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y - \boldsymbol{u}) \cdot \boldsymbol{\tau} (\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{\tau} \\
&- \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta} \hat{\omega} \cdot \hat{\Delta} \hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&+ \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) \\
&- \alpha \int_0^T \int_{\Omega_b^\delta(t)} p \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p} \cdot \partial_t \hat{r} - \alpha \int_0^T \int_{\Omega_b^\delta(t)} \frac{D^\delta}{Dt} \boldsymbol{\eta} \cdot \nabla r - \alpha \int_0^T \int_{\Gamma^\delta(t)} (\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}^\delta) r \\
&+ \kappa \int_0^T \int_{\Omega_b^\delta(t)} \nabla p \cdot \nabla r - \int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta \boldsymbol{e}_y) \cdot \boldsymbol{n}) r \\
&= \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot \boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t \hat{\omega}(0) \cdot \hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}}(0) \cdot \hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0) \cdot \hat{r}(0), \quad (5.46)
\end{aligned}$$

where $\frac{D^\delta}{Dt} = \frac{d}{dt} + (\boldsymbol{\xi}^\delta \cdot \nabla)$ with $\boldsymbol{\xi}^\delta = \partial_t \boldsymbol{\eta}^\delta$ is the material derivative with respect to the regularized displacement, $\boldsymbol{n}$ denotes the upward pointing normal vector to $\Gamma(t)$, and $\boldsymbol{n}^\delta$ denotes the upward pointing normal vector to $\Gamma^\delta(t)$.

Remark 5.4.2 (Remark on notation in the weak formulation)**.** We will omit explicit mention of function compositions with the mappings $\boldsymbol{\Phi}_f$ and $\boldsymbol{\Phi}_b^\delta$ (defined in (5.21) and (5.31)), and their inverses throughout the chapter, as the function compositions needed will be clear from the context. In particular, we follow this convention in the weak formulation above. For example, since the pore pressure p and the test function $\boldsymbol{\psi}$ are defined on the reference domain $\hat{\Omega}_b$, the integral

$$-\alpha \int_0^T \int_{\Omega_b^\delta(t)} p \nabla \cdot \boldsymbol{\psi},$$

means

$$-\alpha \int_0^T \int_{\Omega_b^\delta(t)} \left(\hat{p} \circ (\boldsymbol{\Phi}_b^{\eta^\delta})^{-1}\right) \nabla \cdot \left(\hat{\boldsymbol{\psi}} \circ (\boldsymbol{\Phi}_b^{\eta^\delta})^{-1}\right).$$

As another example, the integral

$$-\int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta \boldsymbol{e}_y) \cdot \boldsymbol{n}) r$$

means

$$-\int_0^T \int_{\Gamma(t)} \left(\left(\boldsymbol{u} - \left(\hat{\zeta} \circ (\boldsymbol{\Phi}_\Gamma^\omega)^{-1}\right) \boldsymbol{e}_y\right) \cdot \boldsymbol{n}\right) \left(\hat{r} \circ (\boldsymbol{\Phi}_b^\eta)^{-1}\right).$$

Our goal will be to reformulate the definition of a regularized weak solution on the fixed reference domain. In particular, we will need to handle any integrals dealing with time-dependent domains by using a change of variables. We recall the factors that appear upon using a change of variables to the reference domain for the fluid, the Biot medium, and the moving interface, which are given by $\hat{\mathcal{J}}_f^\omega$, $\hat{\mathcal{J}}_b^\eta$, and $\hat{\mathcal{J}}_\Gamma^\omega$ respectively in (5.22), (5.27), and (5.18).

We handle the terms in (5.46) as follows. For the first term, we use the formula for the transformation of time derivatives, given by (5.24) and (5.25). Furthermore, we assume that $|\hat{\omega}| < R$ so that there is no domain degeneracy. Using (5.24) and (5.23), we then have that

$$\begin{aligned}
\int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} &= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot [(\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega) \hat{\boldsymbol{v}}] \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{R} \int_{\hat{\Omega}_f} \hat{\boldsymbol{u}} \cdot [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{v}}] \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{2R} \int_{\hat{\Omega}_f} \hat{\boldsymbol{u}} \cdot [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{v}}] + \frac{1}{2R} \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} \\
&\quad + \frac{1}{2R} \int_{\hat{\Omega}_f} [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{u}}] \cdot \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega} \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) [((\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega) \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{u}} - ((\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega) \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{v}}] \\
&\quad + \frac{1}{2R} \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega}, \qquad (5.47)
\end{aligned}$$

where we integrated by parts in the $\hat{y}$ direction. Recall that $\hat{\boldsymbol{w}}$ is defined by (5.24). We also note that the final term in (5.47) will combine with the following term in (5.46):

$$\int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{v} = \int_0^T \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega}. \tag{5.48}$$

This is clear, once we observe that the normal vector to the interface is $\boldsymbol{n} = \frac{1}{\hat{\mathcal{J}}_\Gamma^\omega}(-\partial_{\hat{x}}\hat{\omega}, 1)$, and $\zeta \boldsymbol{e}_y|_{\Gamma(t)} = \partial_t \hat{\omega} \boldsymbol{e}_y$, which establishes the equality (5.48). Because the transformation from $\Gamma(t)$ to $\hat{\Gamma}$ cancels out the factor of $\hat{\mathcal{J}}_\Gamma^\omega$ in the unit normal vector, it will be useful to define the following renormalized normal and tangent vectors:

$$\hat{\boldsymbol{n}}^\omega = (-\partial_{\hat{x}}\hat{\omega}, 1), \qquad \hat{\boldsymbol{\tau}}^\omega = (1, \partial_{\hat{x}}\hat{\omega}). \tag{5.49}$$

We will similarly define

$$\hat{\boldsymbol{n}}^{\omega^\delta} = (-\partial_{\hat{x}}(\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}}), 1). \tag{5.50}$$

We can now define a weak solution to the regularized problem on the fixed reference domain as follows, where we recall that we use the variable $\hat{\zeta}$ to denote the plate velocity on Γ, see (5.28).

Definition 5.4.4. (Weak solution to the regularized problem, fixed fluid domain formulation) An ordered four-tuple $(\hat{\boldsymbol{u}}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}_{\text{sol}}^\omega$ is a *weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter* δ if for all test functions $(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in \mathcal{V}_{\text{test}}^\omega$, the following equality holds:

$$\begin{aligned}
&-\int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} + \frac{1}{2}\int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) [((\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) \cdot \hat{\nabla}_f^\omega \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{v}} - ((\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) \cdot \hat{\nabla}_f^\omega \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{u}}] \\
&- \frac{1}{2R}\int_0^T \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} + \frac{1}{2}\int_0^T \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}^\omega - \hat{\zeta} \boldsymbol{e}_y \cdot \hat{\boldsymbol{n}}^\omega) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} + 2\nu \int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{D}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{v}}) \\
&\quad + \int_0^T \int_{\hat{\Gamma}} \left(\frac{1}{2}|\hat{\boldsymbol{u}}|^2 - \hat{p}\right)(\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}^\omega + \frac{\beta}{\hat{\mathcal{J}}_\Gamma^\omega}\int_0^T \int_{\hat{\Gamma}} (\hat{\zeta} \boldsymbol{e}_y - \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{\tau}}^\omega (\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{\tau}}^\omega \\
&- \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega} \cdot \hat{\Delta}\hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&+ \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) \\
&\quad - \alpha \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^\delta} \hat{p} \hat{\nabla}_b^{\eta^\delta} \cdot \hat{\boldsymbol{\psi}} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p} \cdot \partial_t \hat{r} - \alpha \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^\delta} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\nabla}_b^{\eta^\delta} \hat{r} \\
&\quad - \alpha \int_0^T \int_{\hat{\Gamma}} (\hat{\zeta} \boldsymbol{e}_y \cdot \hat{\boldsymbol{n}}^{\omega^\delta}) \hat{r} + \kappa \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^\delta} \hat{\nabla}_b^{\eta^\delta} \hat{p} \cdot \hat{\nabla}_b^{\eta^\delta} \hat{r} - \int_0^T \int_{\hat{\Gamma}} ((\hat{\boldsymbol{u}} - \hat{\zeta} \boldsymbol{e}_y) \cdot \hat{\boldsymbol{n}}^\omega) \hat{r} \\
&= \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot \boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t \hat{\omega}(0) \cdot \hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}}(0) \cdot \hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0) \cdot \hat{r}(0).
\end{aligned} \tag{5.51}$$

We list the definitions of all of the relevant expressions below.

$$\hat{\boldsymbol{w}} = \frac{R+\hat{y}}{R}\partial_t\hat{\omega}\boldsymbol{e}_y, \qquad \hat{\nabla}_f^{\omega} = \left(\partial_{\hat{x}} - \frac{(R+\hat{y})\partial_{\hat{x}}\hat{\omega}}{R+\hat{\omega}}\partial_{\hat{y}}, \frac{R}{R+\hat{\omega}}\partial_{\hat{y}}\right), \qquad \hat{\nabla}_b^{\eta^\delta}\hat{g} = \left(\frac{\partial \hat{g}}{\partial \hat{x}}, \frac{\partial \hat{g}}{\partial \hat{y}}\right)\cdot(\boldsymbol{I}+\hat{\nabla}\hat{\boldsymbol{\eta}}^\delta)^{-1}$$

$$\hat{\boldsymbol{n}}^{\omega} = (-\partial_{\hat{x}}\hat{\omega}, 1), \qquad \hat{\boldsymbol{\tau}}^{\omega} = (1, \partial_{\hat{x}}\hat{\omega}), \qquad \hat{\boldsymbol{n}}^{\omega^\delta} = (-\partial_{\hat{x}}(\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}}), 1),$$
$$\hat{\mathcal{J}}_b^{\eta^\delta} = \det(\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}^\delta), \qquad \hat{\mathcal{J}}_\Gamma^{\omega} = \sqrt{1+|\partial_{\hat{x}}\hat{\omega}|^2}.$$

Formal energy inequality

In this subsection, we show that our regularization is defined in a way that preserves the variational structure of the problem. More precisely, we formally prove that a weak solution to the regularized problem satisfies an energy inequality. To do this, we recall the regularized weak formulation (5.51) defined on the fixed reference domain and we formally substitute

$$(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) = (\hat{\boldsymbol{u}}, \hat{\zeta}, \partial_t\hat{\boldsymbol{\eta}}, \hat{p})$$

in for the test function. We verify that

$$\frac{1}{2}\int_{\hat{\Gamma}}(\hat{\boldsymbol{u}} - \hat{\zeta}\boldsymbol{e}_y)\cdot\hat{\boldsymbol{n}}^{\omega}|\hat{\boldsymbol{u}}|^2 + \int_{\hat{\Gamma}}\left(\frac{1}{2}|\hat{\boldsymbol{u}}|^2 - \hat{p}\right)(\hat{\zeta}\boldsymbol{e}_y - \hat{\boldsymbol{u}})\cdot\hat{\boldsymbol{n}}^{\omega} - \int_{\hat{\Gamma}}((\hat{\boldsymbol{u}} - \hat{\zeta}\boldsymbol{e}_y)\cdot\hat{\boldsymbol{n}}^{\omega})\hat{p} = 0.$$

Furthermore, by integration by parts, we have that

$$\alpha\left(\int_{\hat{\Omega}_b}\hat{\mathcal{J}}_b^{\eta^\delta}\hat{p}\hat{\nabla}_b^{\eta^\delta}\cdot\partial_t\hat{\boldsymbol{\eta}} + \int_{\hat{\Omega}_b}\hat{\mathcal{J}}_b^{\eta^\delta}\partial_t\hat{\boldsymbol{\eta}}\cdot\hat{\nabla}_b^{\eta^\delta}\hat{p} + \int_{\hat{\Gamma}}(\hat{\zeta}\boldsymbol{e}_y\cdot\hat{\boldsymbol{n}}^{\omega^\delta})\hat{p}\right)$$
$$= \alpha\left(\int_{\Omega_b^\delta(t)} p\nabla\cdot\boldsymbol{\xi} + \int_{\Omega_b^\delta(t)}\boldsymbol{\xi}\cdot\nabla p + \int_{\Gamma^\delta(t)}(\zeta\boldsymbol{e}_y\cdot\boldsymbol{n}^\delta)p\right) = 0,$$

where we recall that $\boldsymbol{n}^\delta$ is the upward pointing unit normal vector to $\Gamma^\delta(t)$. Finally, by the Reynold's transport theorem, we have that

$$\int_0^T\int_{\Omega_f(t)}\boldsymbol{u}\cdot\partial_t\boldsymbol{u} + \frac{1}{2}\int_0^T\int_{\Gamma(t)}(\zeta\boldsymbol{e}_y\cdot\boldsymbol{n})|\boldsymbol{u}|^2 = \frac{1}{2}\int_{\Omega_f(T)}|\boldsymbol{u}|^2 - \frac{1}{2}\int_{\Omega_f(0)}|\boldsymbol{u}|^2.$$

We then obtain the final energy estimate:

$$\frac{1}{2}\int_{\Omega_f(T)}|\boldsymbol{u}(T)|^2 + 2\nu\int_0^T\int_{\Omega_f(t)}|\boldsymbol{D}(\boldsymbol{u})|^2 + \beta\int_0^T\int_{\Gamma(t)}|(\boldsymbol{\xi}-\boldsymbol{u})\cdot\boldsymbol{\tau}|^2 + \frac{1}{2}\rho_p\int_{\hat{\Gamma}}|\partial_t\hat{\omega}(T)|^2 + \int_{\hat{\Gamma}}|\hat{\Delta}\hat{\omega}(T)|^2$$
$$+ \frac{1}{2}\rho_b\int_{\hat{\Omega}_b}|\partial_t\hat{\boldsymbol{\eta}}(T)|^2 + 2\mu_e\int_{\hat{\Omega}_b}|\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}})(T)|^2 + 2\lambda_e\int_{\hat{\Omega}_b}|\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}(T)|^2 + 2\mu_v\int_0^T\int_{\hat{\Omega}_b}|\hat{\boldsymbol{D}}(\partial_t\hat{\boldsymbol{\eta}})|^2$$
$$+ \lambda_v\int_0^T\int_{\hat{\Omega}_b}|\hat{\nabla}\cdot\partial_t\hat{\boldsymbol{\eta}}|^2 + \frac{1}{2}c_0\int_{\hat{\Omega}_b}|\hat{p}(T)|^2 + \kappa\int_0^T\int_{\Omega_b^\delta(t)}|\nabla p|^2 = \frac{1}{2}\int_{\Omega_f(0)}|\boldsymbol{u}(0)|^2 + \frac{1}{2}\rho_p\int_{\hat{\Gamma}}|\partial_t\hat{\omega}(0)|^2$$
$$+ \int_{\hat{\Gamma}}|\hat{\Delta}\hat{\omega}(0)|^2 + \frac{1}{2}\rho_b\int_{\hat{\Omega}_b}|\partial_t\hat{\boldsymbol{\eta}}(0)|^2 + 2\mu_e\int_{\hat{\Omega}_b}|\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}})(0)|^2 + 2\lambda_e\int_{\hat{\Omega}_b}|\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}(0)|^2 + \frac{1}{2}c_0\int_{\hat{\Omega}_b}|\hat{p}(0)|^2.$$

5.5 The existence result and splitting scheme

To show the existence of a weak solution to the regularized problem, we use a constructive existence proof, which involves a splitting scheme. This is an approach that has been used for constructive existence of weak solutions for a large variety of FSI problems, see for example [140]. The goal of this section is to define the splitting scheme, which consists of two subproblems: one for the fluid/Biot medium, and one for the plate.

An important notational convention. For notational simplicity, we will no longer distinguish between functions on the reference domain and functions on the physical domain, as we have in previous sections. Specifically, we will no longer use the "hat" notation to distinguish between functions and domains in the physical or reference configuration: for example, we will denote both the pore pressure p on $\Omega_b(t)$ and $\hat{p}$ on $\hat{\Omega}_b$ by p, as the distinction between these two will be clear from context. In addition, we will remove the "hat" convention from the reference domains, and for example, we will denote the reference domain $\hat{\Omega}_b$ for the Biot medium by Ω_b. We will follow this notational convention for the rest of the manuscript.

We now state the main result.

Theorem 5.5.1. Let $\mu_e, \lambda_e > 0$, and let μ_v, λ_v both be positive or both be zero. Consider initial data for the plate displacement $\omega_0 \in H_0^2(\Gamma)$, plate velocity $\zeta_0 \in L^2(\Gamma)$, Biot displacement $\boldsymbol{\eta}_0 \in H^1(\Omega_b)$, Biot velocity $\boldsymbol{\xi}_0 \in L^2(\Omega_b)$, Biot pore pressure $p_0 \in L^2(\Omega_b)$, and fluid velocity $\boldsymbol{u}_0 \in H^1(\Omega_f(0))$, where $\boldsymbol{u}_0$ is divergence-free. Suppose further that $|\omega_0| \leqslant R_0 < R$ for some R_0 and $\boldsymbol{\eta}_0|_\Gamma = \omega_0 \boldsymbol{e}_y$, and for some arbitrary but fixed regularization parameter $\delta > 0$, suppose that $\text{Id} + (\boldsymbol{\eta}_0)^\delta$ is an invertible map with $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) > 0$. Then, there exists a weak solution $(\boldsymbol{u}, \omega, \boldsymbol{\eta}, p)$ to the regularized FPSI problem with regularization parameter δ on some time interval $[0, T]$, for some $T > 0$.

Remark 5.5.1. The result above is a local result, since it holds up to some time $T > 0$, which will need to be sufficiently small, as seen in the proof. However, it is easy to show that this $T > 0$ can be made to be maximal, in the sense that it is either infinite or it is the finite time at which $\text{Id} + (\boldsymbol{\eta}_0)^\delta$ fails to be invertible, or $|\hat{\omega}(T, \hat{x})| = R$ for some $\hat{x} \in [0, L]$ so that the plate collides with either the top or bottom boundary of the domain. This will not be the focus of the current work, as this is done by a standard method. See for example pg. 397-398 of [42] or the proof of Theorem 7.1 in [140].

We now define the splitting scheme. We will semi-discretize the problem in time with time step $\Delta t = T/N$, and we will keep track of the fluid velocity, plate displacement and velocity, and Biot poroviscoelastic material displacement and pressure. We will denote these by

$$(\boldsymbol{u}_N^{n+\frac{i}{2}}, \omega_N^{n+\frac{i}{2}}, \zeta_N^{n+\frac{i}{2}}, \boldsymbol{\eta}_N^{n+\frac{i}{2}}, p_N^{n+\frac{i}{2}}), \qquad \text{for } n = 0, 1, ..., N \text{ and } i = 0, 1.$$

Note that the plate displacement $\omega_N^{n+\frac{i}{2}}$ and the plate velocity $\zeta_N^{n+\frac{i}{2}}$ are scalars, because we are assuming that the plate displaces in only the transverse y direction. For the splitting scheme, we will work on the fixed reference domain and hence, we will semi-discretize the regularized weak formulation (5.51) on the fixed reference domain. We will use the backwards Euler discretization to approximate time derivatives, and we will use the following shorthand notation:

$$\dot{f}_N^{n+\frac{i}{2}} = \frac{f_N^{n+\frac{i}{2}} - f_N^{n+\frac{i}{2}-1}}{\Delta t}.$$

Before stating the splitting scheme, we remark that the existence result above holds for both the purely poroelastic case and the poroviscoelastic case. In the poroviscoelastic case where μ_v and λ_v are both strictly positive, we remark that $\boldsymbol{\xi} \in L^2(0,T;H^1(\Omega_b))$ and hence we can interpret the trace of $\boldsymbol{\xi}$ along Γ as the plate velocity ζ. While this is no longer true in the purely poroelastic case $\mu_v = \lambda_v = 0$, where $\boldsymbol{\xi}$ is only in $L^\infty(0,T;L^2(\Omega_b))$, the constructive existence proof via splitting scheme still works since the time discretized Biot velocity $\frac{\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n}{\Delta t}$ still retains $H^1(\Omega_b)$ regularity due to the time discretization, and hence it can still be used as a test function in the weak formulation for the Biot subproblem. Thus, we can get discrete energy estimates as usual and the rest of the argument works. We remark that we will carry out the constructive existence proof in the specific case of a Biot poroviscoelastic medium but we emphasize that the techniques that follow will generalize to the purely poroelastic case as well.

The plate subproblem. For the plate subproblem, we will update only the plate displacement and velocity, to obtain the updated time-dependent domains to be used for the fluid/Biot subproblem in the next time step. Therefore,

$$\boldsymbol{u}_N^{n+\frac{1}{2}} = \boldsymbol{u}_N^n, \qquad \boldsymbol{\eta}_N^{n+\frac{1}{2}} = \boldsymbol{\eta}_N^n, \qquad p_N^{n+\frac{1}{2}} = p_N^n,$$

and we will update $\omega_N^{n+\frac{1}{2}}$ and $\zeta_N^{n+\frac{1}{2}}$. We write the structure subproblem in the following weak formulation: find $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ and $\zeta_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$, such that

$$\int_\Gamma \left(\frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t}\right) \cdot \phi = \int_\Gamma \zeta_N^{n+\frac{1}{2}} \cdot \phi, \qquad \text{for all } \phi \in L^2(\Gamma), \tag{5.52}$$

$$\rho_p \int_\Gamma \left(\frac{\zeta_N^{n+\frac{1}{2}} - \zeta_N^n}{\Delta t}\right) \cdot \varphi + \int_\Gamma \Delta\omega_N^{n+\frac{1}{2}} \cdot \Delta\varphi = 0, \qquad \text{for all } \varphi \in H_0^2(\Gamma). \tag{5.53}$$

When $n = 0$, we set $\omega_N^{-\frac{1}{2}} = \omega(0)$ and $\zeta_N^0 = \zeta(0)$. In particular, $\omega(0)\boldsymbol{e}_y = \boldsymbol{\eta}(0)|_\Gamma$ and $\zeta(0)\boldsymbol{e}_y = \boldsymbol{\xi}(0)$.

We must show that this subproblem written in weak formulation has a unique solution. To see this, we use the fact that

$$\zeta_N^{n+\frac{1}{2}} = \frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t}$$

so that we must find $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ such that the following is satisfied in weak formulation:

$$\rho_p \int_\Gamma \omega_N^{n+\frac{1}{2}} \cdot \varphi + (\Delta t)^2 \int_\Gamma \Delta \omega_N^{n+\frac{1}{2}} \cdot \Delta\varphi = \rho_p \int_\Gamma (\omega_N^{n-\frac{1}{2}} + (\Delta t)\zeta_N^n) \cdot \varphi, \qquad \text{for all } \varphi \in H_0^2(\Gamma).$$

We consider the bilinear form

$$B[\omega, \varphi] = \rho_p \int_\Gamma \omega \cdot \varphi + (\Delta t)^2 \int_\Gamma \Delta\omega \cdot \Delta\varphi.$$

It is clear that this bilinear form is coercive on $H_0^2(\Gamma)$. In addition,

$$\varphi \to \rho_p \int_\Gamma \left(\omega_N^{n-\frac{1}{2}} + (\Delta t)\zeta_N^n\right) \cdot \varphi$$

is a continuous linear functional on $H_0^2(\Gamma)$, since we will have $\omega_N^{n-\frac{1}{2}} \in H_0^2(\Gamma)$ and $\zeta_N^n \in L^2(\Gamma)$ by the way our splitting scheme is defined. So this guarantees the existence of a unique solution $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ by the Lax-Milgram theorem. We then recover

$$\zeta_N^{n+\frac{1}{2}} = \frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t} \in H_0^2(\Gamma).$$

To obtain an energy inequality, we then substitute $\varphi = \zeta_N^{n+\frac{1}{2}} = \frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t} \in H_0^2(\Gamma)$ and use the identity

$$(a-b) \cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2).$$

We obtain the following energy equality:

$$\frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+\frac{1}{2}}|^2 + \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+\frac{1}{2}} - \zeta_N^n|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_N^{n+\frac{1}{2}}|^2 + \frac{1}{2}\int_\Gamma |\Delta(\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}})|^2$$
$$= \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n-\frac{1}{2}}|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_N^{n-\frac{1}{2}}|^2. \quad (5.54)$$

The fluid and Biot subproblem. For the fluid and Biot subproblem, we update the quantities related to the fluid and the Biot medium. Due to the kinematic coupling between the Biot medium displacement and the plate displacement, we must also update the plate velocity, as the dynamics of the Biot medium hence affect the kinematics of the plate via the continuity of displacements. In this step, we will not update the plate displacement, so that

$$\omega_N^{n+1} = \omega_N^{n+\frac{1}{2}}.$$

We will use ω_N^n calculated from the previous step for the ALE mapping for the fluid.

The weak formulation of the fluid and Biot subproblem reads as follows. First, we define the solution space as

$$\mathcal{V}_N^{n+1} = \{(\boldsymbol{u}, \beta, \boldsymbol{\eta}, p) \in \mathcal{V}_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p\}, \tag{5.55}$$

and the test space as

$$\mathcal{Q}_N^{n+1} = \{(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in V_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma\}, \tag{5.56}$$

where V_f^{ω}, V_d, and V_p are defined as in (5.35), (5.38), and (5.40).

The weak formulation of the fluid/Biot subproblem is: find $(\boldsymbol{u}_N^{n+1}, \zeta_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_N^{n+1}$ defined on the reference domain, such that for all test functions $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{Q}_N^{n+1}$ defined on the reference domain, the following holds:

$$\begin{aligned}
&\int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \dot{\boldsymbol{u}}_N^{n+1} \cdot \boldsymbol{v} \\
&+ \frac{1}{2} \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left[\left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{v} \right) \cdot \boldsymbol{u}_N^{n+1} \right] \\
&\quad + \frac{1}{2R} \int_{\Omega_f} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{v} + \frac{1}{2} \int_\Gamma (\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n}^{\omega_N^n} (\boldsymbol{u}_N^n \cdot \boldsymbol{v}) \\
&+ 2\nu \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + \int_\Gamma \left(\frac{1}{2} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{u}_N^n - p_N^{n+1} \right) (\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{n}^{\omega_N^n} \\
&+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} \int_\Gamma (\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1}) \cdot \boldsymbol{\tau}^{\omega_N^n} (\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} \left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n}{\Delta t} \right) \cdot \boldsymbol{\psi} \\
&+ \rho_p \int_\Gamma \left(\frac{\zeta_N^{n+1} - \zeta_N^{n+\frac{1}{2}}}{\Delta t} \right) \varphi + 2\mu_e \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_e \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) \\
&+ 2\mu_v \int_{\Omega_b} \boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_{\Omega_b} (\nabla \cdot \dot{\boldsymbol{\eta}}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} \\
&+ c_0 \int_{\Omega_b} \frac{p_N^{n+1} - p_N^n}{\Delta t} r - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \dot{\boldsymbol{\eta}}_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha \int_\Gamma (\dot{\boldsymbol{\eta}}_N^{n+1} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r \\
&\quad + \kappa \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \int_\Gamma [(\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n}^{\omega_N^n}] r = 0,
\end{aligned} \tag{5.57}$$

and

$$\int_\Gamma \left(\frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t} \right) \cdot \boldsymbol{\phi} = \int_\Gamma \zeta_N^{n+1} \boldsymbol{e}_y \cdot \boldsymbol{\phi}, \qquad \text{for all } \boldsymbol{\phi} \in L^2(\Gamma). \tag{5.58}$$

Here, $\boldsymbol{n}^{\omega_N^n}$ and $\boldsymbol{n}^{(\omega_N^n)^\delta}$ are the renormalized normal vectors $(-\partial_x \omega_N^n, 1)$ and $\left(-\partial_x \left((\boldsymbol{\eta}_N^n)^\delta|_\Gamma \right), 1\right)$ and $\boldsymbol{\tau}^{\omega_N^n}$ is the renormalized tangent vector $(1, \partial_x \omega_N^n)$.

We consider the scheme under the following **two essential assumptions**, which are *necessary for the scheme to successfully produce a numerical solution*:

1. **Assumption 1A**: *Boundedness of the plate displacement away from R.* There exists a positive constant R_{max} such that

$$|\omega_N^{k+\frac{i}{2}}| \leqslant R_{max} < R, \qquad \text{for all } k = 0, 1, 2, ..., n-1 \text{ and } i = 0, 1. \tag{5.59}$$

2. **Assumption 2A:** *Invertibility of the map from fixed to moving Biot domain.* The map
$$\text{Id} + (\boldsymbol{\eta}_N^n)^\delta : \Omega_b \to (\Omega_b)_N^{n,\delta} \qquad \text{is invertible,} \tag{5.60}$$
where $(\Omega_b)_N^{n,\delta}$ is defined as the image of the reference Biot domain Ω_b under the map $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta$.

We will show that the above subproblem (5.57) and (5.58) in weak formulation has a unique solution, under these two assumptions 1A and 2A.

To do this, we first rewrite the weak formulation so that all of the functions on the $n+1$ time step are on the left hand side while all other quantities are on the right hand side. In addition, we can rewrite the variable ζ_N^{n+1} in terms of $\boldsymbol{\eta}_N^n$ and $\boldsymbol{\eta}_N^{n+1}$ by using (5.58):

$$\zeta_N^{n+1}\boldsymbol{e}_y = \frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\Big|_\Gamma.$$

The resulting weak formulation does not involve a coercive bilinear form because of a mismatch in scaling. However, we can recover a coercive structure when we transform the weak formulation using the following rescaling of test functions:

$$\boldsymbol{v} \to (\Delta t)\boldsymbol{v}, \qquad r \to (\Delta t)r.$$

This scaling of the test functions is valid because if $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, v) \in \mathcal{Q}_N^{n+1}$, then the rescaled test function satisfies $((\Delta t)^{-1}\boldsymbol{v}, \varphi, \boldsymbol{\psi}, (\Delta t)^{-1}r) \in \mathcal{Q}_N^{n+1}$ also. After performing this rescaling, the weak formulation involves the following coercive and continuous bilinear form:

$$\begin{aligned}
B[\boldsymbol{u},\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\psi},p,r] &:= (\Delta t)^2\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\boldsymbol{u}\cdot\boldsymbol{v} \\
&+ \frac{1}{2}(\Delta t)^3\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left[\left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla^{\omega_N^n}\boldsymbol{u}\right)\cdot\boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla^{\omega_N^n}\boldsymbol{v}\right)\cdot\boldsymbol{u}\right] \\
&+ (\Delta t)^3\cdot\frac{1}{2R}\int_{\Omega_f}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{v} + \frac{1}{2}(\Delta t)^3\int_\Gamma(\boldsymbol{u}-(\Delta t)^{-1}\boldsymbol{\eta})\cdot\boldsymbol{n}^{\omega_N^n}(\boldsymbol{u}_N^n\cdot\boldsymbol{v}) \\
&+ 2\nu(\Delta t)^3\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}):\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + (\Delta t)^2\int_\Gamma\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}_N^n - p\right)(\boldsymbol{\psi}-(\Delta t)\boldsymbol{v})\cdot\boldsymbol{n}^{\omega_N^n} \\
&+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}(\Delta t)^2\int_\Gamma[(\Delta t)^{-1}\boldsymbol{\eta}-\boldsymbol{u}]\cdot\boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi}-(\Delta t)\boldsymbol{v})\cdot\boldsymbol{\tau}^{\omega_N^n} + \rho_b\int_{\Omega_b}\boldsymbol{\eta}\cdot\boldsymbol{\psi} + \rho_p\int_\Gamma\boldsymbol{\eta}\cdot\boldsymbol{\psi} \\
&+ (2\mu_e(\Delta t)^2 + 2\mu_v(\Delta t))\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}):\boldsymbol{D}(\boldsymbol{\psi}) + (\lambda_e(\Delta t)^2+\lambda_v(\Delta t))\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\psi}) \\
&- \alpha(\Delta t)^2\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi} + c_0(\Delta t)^2\int_{\Omega_b}pr - \alpha(\Delta t)^2\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\boldsymbol{\eta}\cdot\nabla_b^{(\eta_N^n)^\delta}r \\
&- \alpha(\Delta t)^2\int_\Gamma(\boldsymbol{\eta}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta})r + \kappa(\Delta t)^3\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p\cdot\nabla_b^{(\eta_N^n)^\delta}r \\
&- (\Delta t)^3\int_\Gamma[(\boldsymbol{u}-(\Delta t)^{-1}\boldsymbol{\eta})\cdot\boldsymbol{n}^{\omega_N^n}]r.
\end{aligned}$$

The weak formulation is: find functions $(\boldsymbol{u}_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$ such that for all test functions $(\boldsymbol{v}, \boldsymbol{\psi}, r) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$,

$$B[\boldsymbol{u}_N^{n+1}, \boldsymbol{v}, \boldsymbol{\eta}_N^{n+1}, \boldsymbol{\psi}, p_N^{n+1}, r] = (\Delta t)^2 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{u}_N^n \cdot \boldsymbol{v} - \frac{1}{2}(\Delta t)^2 \int_\Gamma \boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{\omega_N^n} (\boldsymbol{u}_N^n \cdot \boldsymbol{v})$$
$$+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} (\Delta t) \int_\Gamma \boldsymbol{\eta}_N^n \cdot \boldsymbol{\tau}^{\omega_N^n} (\boldsymbol{\psi} - (\Delta t)\boldsymbol{v}) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} (2\boldsymbol{\eta}_N^n - \boldsymbol{\eta}_N^{n-1}) \cdot \boldsymbol{\psi} + \rho_p \int_\Gamma (\boldsymbol{\eta}_N^n + (\Delta t)\zeta_N^{n+\frac{1}{2}} \boldsymbol{e}_y) \cdot \boldsymbol{\psi}$$
$$+ 2\mu_v(\Delta t) \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N^n) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v(\Delta t) \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N^n)(\nabla \cdot \boldsymbol{\psi})$$
$$+ c_0(\Delta t)^2 \int_{\Omega_b} p_N^n r - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta}_N^n \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha(\Delta t)^2 \int_\Gamma (\boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r + (\Delta t)^2 \int_\Gamma (\boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{\omega_N^n}) r. \tag{5.61}$$

We verify that the bilinear form $B[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\psi}, p, r]$ is coercive and continuous as a bilinear form on the Hilbert space

$$\mathcal{V}_f^{\omega_N^n} \times V_d \times V_p,$$

with the inner product given by

$$\langle (\boldsymbol{u}, \boldsymbol{\eta}, p), (\boldsymbol{v}, \boldsymbol{\psi}, r) \rangle = \int_{\Omega_f} (\boldsymbol{u} \cdot \boldsymbol{v} + \nabla \boldsymbol{u} : \nabla \boldsymbol{v}) + \int_{\Omega_b} (\boldsymbol{\eta} \cdot \boldsymbol{\psi} + \nabla \boldsymbol{\eta} : \nabla \boldsymbol{\psi}) + \int_{\Omega_b} (p \cdot r + \nabla p \cdot \nabla r).$$

Proving that this bilinear form is continuous is standard, so we focus on establishing coercivity. To compute $B[\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\eta}, p, p]$, we note that by integration by parts,

$$-\alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\eta} - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta} \cdot \nabla_b^{(\eta_N^n)^\delta} p - \alpha(\Delta t)^2 \int_\Gamma \left(\boldsymbol{\eta} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}\right) p = 0.$$

To do this, we bring the integrals back to the time-dependent physical domain, **which we can do as long as $(\boldsymbol{\eta}_N^n)^\delta$ is a bijection from Ω_b to $(\Omega_b)_N^{n,\delta}$**, which we are assuming in Assumption 2A (5.60). We compute

$$- \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\eta} - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta} \cdot \nabla_b^{(\eta_N^n)^\delta} p - \alpha(\Delta t)^2 \int_\Gamma \left(\boldsymbol{\eta} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}\right) p$$
$$= -\alpha(\Delta t)^2 \left(\int_{(\Omega_b)_N^{n,\delta}} p \nabla \cdot \boldsymbol{\eta} + \int_{(\Omega_b)_N^{n,\delta}} \boldsymbol{\eta} \cdot \nabla p + \int_{\Gamma_N^{n,\delta}} (\boldsymbol{\eta} \cdot \boldsymbol{n}) p \right) = 0,$$

by integration by parts, once we recall that $\boldsymbol{n}$ points outwards from Ω_f and hence inwards towards Ω_b, and also once we recall that $\boldsymbol{\eta} = 0$ on the left, right, and top boundaries of Ω_b by the Dirichlet boundary condition. Combining this with the fact that $(\Delta t)\zeta_N^{n+\frac{1}{2}} =$

$\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}} = \omega_N^{n+1} - \omega_N^n$, we compute that

$$B[\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\eta}, p, p] := (\Delta t)^2 \int_{\Omega_f} \left(1 + \frac{\omega_N^n + \omega_N^{n+1}}{2R}\right) |\boldsymbol{u}|^2 + 2\nu(\Delta t)^3 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left|\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u})\right|^2$$
$$+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}(\Delta t) \int_\Gamma \left|(\boldsymbol{\eta} - (\Delta t)\boldsymbol{u}) \cdot \boldsymbol{\tau}^{\omega_N^n}\right|^2 + \rho_b \int_{\Omega_b} |\boldsymbol{\eta}|^2 + \rho_p \int_\Gamma |\boldsymbol{\eta}|^2 + (2\mu_e(\Delta t)^2 + 2\mu_v(\Delta t)) \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2$$
$$+ (\lambda_e(\Delta t)^2 + \lambda_v(\Delta t)) \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}|^2 + c_0(\Delta t)^2 \int_{\Omega_b} |p|^2 + \kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2.$$

We must show that the symmetrized gradient can be expressed in terms of the usual gradient. It is well-known that one can do this for the fluid, and one even has a Korn *equality* in this case. However, this is not immediate for the Biot material. We thus prove the following explicit Korn inequality for the Biot domain.

Proposition 5.5.1 (Korn inequality for the Biot poroviscoelastic domain)**.** For all $\boldsymbol{\eta} \in V_d$,

$$\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 \geqslant \frac{1}{2} \int_{\Omega_b} |\nabla \boldsymbol{\eta}|^2.$$

Proof. By a standard approximation argument, it suffices to assume that $\boldsymbol{\eta}$ is smooth. Because $\eta_x = 0$ on Γ and because $\boldsymbol{\eta} = 0$ on the left, top, and right boundaries of Ω_b, we have from integration by parts, that

$$\int_{\Omega_b} \frac{\partial \eta_x}{\partial y} \frac{\partial \eta_y}{\partial x} = -\int_{\Omega_b} \eta_x \frac{\partial^2 \eta_y}{\partial x \partial y} = \int_{\Omega_b} \frac{\partial \eta_x}{\partial x} \frac{\partial \eta_y}{\partial y}.$$

Therefore,

$$\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 = \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial \eta_x}{\partial y} + \frac{\partial \eta_y}{\partial x}\right)^2$$
$$= \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial \eta_x}{\partial y}\right)^2 + \left(\frac{\partial \eta_y}{\partial x}\right)^2\right] + \frac{\partial \eta_x}{\partial y}\frac{\partial \eta_y}{\partial x}$$
$$= \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \frac{\partial \eta_x}{\partial x}\frac{\partial \eta_y}{\partial y} + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial \eta_x}{\partial y}\right)^2 + \left(\frac{\partial \eta_y}{\partial x}\right)^2\right] \geqslant \frac{1}{2}\int_{\Omega_b} |\nabla \boldsymbol{\eta}|^2,$$

by using the inequality $a^2 + 2ab + b^2 \geqslant 0$. □

We then deduce coercivity from the fact that $|\omega_N^{k+\frac{i}{2}}| < R$ (see Assumption 1A in (5.59)) and the preceding Korn inequality, once we handle the last term and show that

$$\kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2 \geqslant c \int_{\Omega_b} |\nabla p|^2,$$

for some positive constant $c > 0$. Recall from (5.27) and (5.26) that

$$\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta), \qquad \nabla_b^{(\eta_N^n)^\delta} p = \nabla p \cdot (\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}.$$

Therefore, letting $|\cdot|$ denote the matrix norm, we have that

$$\kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2 \geqslant \kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^{-2} |\nabla p|^2. \tag{5.62}$$

We recall from Assumption 2A (5.60) that $\boldsymbol{I} + (\boldsymbol{\eta}_N^n)^\delta$ is an invertible map from Ω_b to $(\Omega_b)_N^{n,\delta}$, and note that $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|$ is continuous on $\overline{\Omega_b}$ and hence is bounded from above. Thus, $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^{-2} \geqslant c_0 > 0$ for some positive constant c_0. The assumption that $\boldsymbol{I} + (\boldsymbol{\eta}_N^n)^\delta$ is invertible implies that $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) > 0$. However, since this determinant is a continuous function on the compact set $\overline{\Omega_b}$, we conclude that there exists a positive constant $c_1 > 0$ such that $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) \geqslant c_1 > 0$. This establishes coercivity.

We obtain the existence of a unique solution $(\boldsymbol{u}_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$ to the weak formulation (5.61) from the Lax-Milgram lemma. We recover ζ_N^{n+1}, by using $\zeta_N^{n+1}\boldsymbol{e}_y = \dfrac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\Big|_\Gamma$. Note that we have that $\dfrac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\Big|_\Gamma$ points in the y direction, because the trace of any function $\boldsymbol{\eta} \in V_d$ on Γ points in the y direction by definition (see (5.38)).

Energy equality: For the fluid and Biot subproblem, we substitute $\boldsymbol{v} = \boldsymbol{u}_N^{n+1}$, $\varphi = \zeta_N^{n+1}$, $\boldsymbol{\psi} = \dot{\boldsymbol{\eta}}_N^{n+1}$, and $r = p_N^{n+1}$, and we use the identity

$$(a - b) \cdot a = \frac{1}{2}(|a|^2 + |a - b|^2 - |b|^2).$$

We will substitute this test function into the original form of the weak formulation (5.57), without the time scaling. Since $\omega_N^{n+1} = \omega_N^{n+\frac{1}{2}}$ and $(\Delta t)\zeta_N^{n+\frac{1}{2}} = \omega_N^{n+\frac{1}{2}} - \omega_N^n$, we obtain the following energy equality for the fluid subproblem:

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_f} \left(1 + \frac{\omega_N^{n+1}}{R}\right) |\boldsymbol{u}_N^{n+1}|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N^{n+1}|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1})|^2 + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_N^{n+1}|^2 \\ &+ \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+1}|^2 + 2\mu_v(\Delta t) \int_{\Omega_b} |\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1})|^2 + \lambda_v(\Delta t) \int_{\Omega_b} |\nabla \cdot \dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \kappa(\Delta t) \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \\ &+ \frac{\beta(\Delta t)}{\mathcal{J}_\Gamma^{\omega_N^n}} \int_\Gamma |(\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1}) \cdot \boldsymbol{\tau}^{\omega_N^n}|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N^{n+1} - p_N^n|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2 \\ &+ \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot (\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2 = \frac{1}{2}\int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) |\boldsymbol{u}_N^n|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N^n|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_N^n)|^2 \\ &\qquad + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_N^n|^2 + \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+\frac{1}{2}}|^2 \end{aligned}$$

Note that the following terms cancel out, by bringing the integrals back to the time-dependent domain and integrating by parts, recalling that the normal vector points inward towards the Biot domain.

$$-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\dot{\boldsymbol{\eta}}_N^{n+1}-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}-\alpha\int_\Gamma\left(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta}\right)p_N^{n+1}$$
$$=-\alpha\int_{(\Omega_b)_N^{n,\delta}}p_N^{n+1}(\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1})-\alpha\int_{(\Omega_b)_N^{n,\delta}}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla p_N^{n+1}-\alpha\int_{\Gamma_N^{n,\delta}}(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n})p_N^{n+1}=0.$$

The semidiscrete problem. We express the entire scheme in the following weak semidiscrete formulation. The following equality holds for all test functions $(\boldsymbol{v},\varphi,\boldsymbol{\psi},r)\in\mathcal{Q}_N^{n+1}$, where $\mathcal{Q}_N^{n+1}$ is defined by (5.56).

$$\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\dot{\boldsymbol{u}}_N^{n+1}\cdot\boldsymbol{v}$$
$$+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left[\left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac12}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{v}-\left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac12}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{v}\right)\cdot\boldsymbol{u}_N^{n+1}\right]$$
$$+\frac{1}{2R}\int_{\Omega_f}\zeta_N^{n+\frac12}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{v}+\frac12\int_\Gamma(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}^{\omega_N^n}(\boldsymbol{u}_N^n\cdot\boldsymbol{v})$$
$$+2\nu\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1}):\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v})+\int_\Gamma\left(\frac12\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{u}_N^n-p_N^{n+1}\right)(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{n}^{\omega_N^n}$$
$$+\frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{\tau}^{\omega_N^n}+\rho_b\int_{\Omega_b}\left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n}{\Delta t}\right)\cdot\boldsymbol{\psi}$$
$$+\rho_p\int_\Gamma\left(\frac{\zeta_N^{n+1}-\zeta_N^n}{\Delta t}\right)\varphi+2\mu_e\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi})+\lambda_e\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta}_N^{n+1})(\nabla\cdot\boldsymbol{\psi})$$
$$+2\mu_v\int_{\Omega_b}\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi})+\lambda_v\int_{\Omega_b}(\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1})(\nabla\cdot\boldsymbol{\psi})-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}$$
$$+c_0\int_{\Omega_b}\frac{p_N^{n+1}-p_N^n}{\Delta t}r-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r-\alpha\int_\Gamma(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta})r$$
$$+\kappa\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r-\int_\Gamma[(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}^{\omega_N^n}]r+\int_\Gamma\Delta\omega_N^{n+\frac12}\cdot\Delta\varphi=0,\quad(5.63)$$

$$\int_\Gamma\left(\frac{\omega_N^{n+\frac12}-\omega_N^{n-\frac12}}{\Delta t}\right)\phi=\int_\Gamma\zeta_N^{n+\frac12}\phi,\quad\int_\Gamma\left(\frac{\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n}{\Delta t}\right)\cdot\boldsymbol{\phi}=\int_\Gamma\zeta_N^{n+1}\boldsymbol{e}_y\cdot\boldsymbol{\phi},\quad\text{for all }\phi,\boldsymbol{\phi}\in L^2(\Gamma).\tag{5.64}$$

Uniform energy estimates. Define the following discrete energy and discrete dissipation:

$$E_N^{n+\frac{i}{2}}=\frac12\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)|\boldsymbol{u}_N^{n+\frac{i}{2}}|^2+\frac12\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+\frac{i}{2}}|^2+\frac12c_0\int_{\Omega_b}|p_N^{n+\frac{i}{2}}|^2+\mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+\frac{i}{2}})|^2$$
$$+\frac12\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^{n+\frac{i}{2}}|^2+\frac12\rho_p\int_\Gamma|\zeta_N^{n+\frac{i}{2}}|^2+\frac12\int_\Gamma|\Delta\omega_N^{n+\frac{i}{2}}|^2.\quad(5.65)$$

$$D_N^{n+1} = 2\nu(\Delta t)\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1})\right|^2 + 2\mu_v(\Delta t)\int_{\Omega_b}|\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1})|^2 + \lambda_v(\Delta t)\int_{\Omega_b}|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}|^2$$
$$+ \kappa(\Delta t)\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}|\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}|^2 + \frac{\beta(\Delta t)}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma\left|(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}\right|^2. \quad (5.66)$$

We then have the following energy equality.

$$E_N^{n+\frac{1}{2}} + \frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+\frac{1}{2}}-\zeta_N^n\right|^2 + \frac{1}{2}\int_\Gamma\left|\Delta(\omega_N^{n+\frac{1}{2}}-\omega_N^{n-\frac{1}{2}})\right|^2 = E_N^n \quad (5.67)$$

$$E_N^{n+1}+D_N^{n+1}+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n\right|^2+\frac{1}{2}\rho_b\int_{\Omega_b}\left|\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n\right|^2+\frac{1}{2}c_0\int_{\Omega_b}\left|p_N^{n+1}-p_N^n\right|^2$$
$$+\mu_e\int_{\Omega_b}\left|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n)\right|^2+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n)\right|^2+\frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+1}-\zeta_N^{n+\frac{1}{2}}|^2 = E_N^{n+\frac{1}{2}}. \quad (5.68)$$

The remaining terms not included in the definition of the discrete energy and discrete dissipation are numerical dissipation terms.

These discrete energy estimates immediately imply that $E_N^{n+\frac{i}{2}}$ for $n+\frac{i}{2}=0,\frac{1}{2},1,\frac{3}{2},...,N-\frac{1}{2},N$ and $\sum_{n=1}^N D_N^n$ are uniformly bounded by a constant C independent of n and N.

5.6 Approximate solutions

Now that we have defined the numerical solutions at each time step, we collect the solutions into approximate solutions defined on the whole time interval $[0,T]$, for which we will obtain uniform estimates from our previous energy estimates.

We define the following piecewise constant approximate solutions

$$\boldsymbol{u}_N(t)=\boldsymbol{u}_N^n,\quad \boldsymbol{\eta}_N(t)=\boldsymbol{\eta}_N^n,\quad p_N(t)=p_N^n,\quad \omega_N(t)=\omega_N^{n-\frac{1}{2}},\quad \zeta_N(t)=\zeta_N^{n-\frac{1}{2}},\quad \zeta_N^*(t)=\zeta_N^n,$$
$$\text{for } (n-1)\Delta t < t \leqslant n\Delta t.$$

To obtain an estimate on time derivatives for the compactness arguments, we also define the following approximate solutions, which are linear interpolations of the following points:

$$\overline{\boldsymbol{\eta}}_N(n\Delta t)=\boldsymbol{\eta}_N^n,\qquad \overline{p}_N(n\Delta t)=p_N^n,\qquad \overline{\omega}_N(n\Delta t)=\omega_N^{n-\frac{1}{2}},\qquad \text{for } n=0,1,...,N,$$

where we formally set $\omega_N^{-\frac{1}{2}}=\omega_0$. Note that by construction, we have that

$$\partial_t\overline{\omega}_N=\zeta_N,\qquad \partial_t\overline{\boldsymbol{\eta}}_N|_\Gamma=\zeta_N^*\boldsymbol{e}_y.$$

From the preceding energy estimates, we have the following lemma on uniform boundedness.

Lemma 5.6.1 (Uniform boundedness of approximate solutions)**.** Consider the following three assumptions:

1. **Assumption 1B**: *Uniform boundedness of plate displacements.* There exists a positive constant R_{max} such that for all N,

$$|\omega_N^{n-\frac{1}{2}}| \leqslant R_{max} < R, \qquad \text{for all } n = 0, 1, ..., N, \tag{5.69}$$

$$\left|(\boldsymbol{\eta}_N^n)^\delta|_\Gamma\right| \leqslant R_{max} < R, \qquad \text{for all } n = 0, 1, ..., N. \tag{5.70}$$

2. **Assumption 2B**: *Uniform invertibility of the ALE mapping (Jacobian).* There exists a positive constant c_0 such that for all N,

$$\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) \geqslant c_0 > 0, \qquad \text{on } \overline{\Omega}_b \text{ for all } n = 0, 1, ..., N. \tag{5.71}$$

3. **Assumption 2C**: *Uniform boundedness of the ALE mapping (matrix norm).* There exists positive constants c_1 and c_2 such that for all N,

$$|(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}| \leqslant c_1, \qquad |\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta| \leqslant c_2, \qquad \text{for all } n = 0, 1, ..., N. \tag{5.72}$$

If these three assumptions hold, then we have the following uniform boundedness results for all N:

- $\boldsymbol{u}_N$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega_f))$ and $L^2(0, T; H^1(\Omega_f))$.
- $\boldsymbol{\eta}_N$ is uniformly bounded in $L^\infty(0, T; H^1(\Omega_b))$.
- p_N is uniformly bounded in $L^\infty(0, T; L^2(\Omega_b))$ and $L^2(0, T; H^1(\Omega_b))$.
- ω_N is uniformly bounded in $L^\infty(0, T; H_0^2(\Gamma))$.

In addition, we have the following estimates on the linear interpolations.

- $\overline{\boldsymbol{\eta}}_N$ is uniformly bounded in $W^{1,\infty}(0, T; L^2(\Omega_b))$.
- $\overline{\boldsymbol{\omega}}_N$ is uniformly bounded in $W^{1,\infty}(0, T; L^2(\Gamma))$.

Remark 5.6.1 (A crucial remark about invertibility)**.** At first, it would appear that to show the uniform boundedness results above, we also need to have a fourth assumption, which is Assumption 2A (5.60) from before: that the map $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta : \Omega_b \to \mathbb{R}^2$ is injective (and is hence a bijection onto its image), for each $n = 0, 1, ..., N$ and for all N. However, this is implied by an injectivity theorem (see Ciarlet [48] Theorem 5-5-2). Note also that Assumption 1A (5.59) from before is automatically satisfied once we verify Assumption 1B (5.69).

In particular, this injectivity theorem is as follows. Since $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) > 0$ by Assumption 2B (5.71), it suffices to show that $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta = \boldsymbol{\varphi}_0$ on $\partial\Omega_b$, for some injective mapping $\boldsymbol{\varphi}_0$ from $\overline{\Omega_b} \to \mathbb{R}^2$. Furthermore, this would imply the very useful fact that $(\text{Id} + (\boldsymbol{\eta}_N^n)^\delta)(\overline{\Omega_b}) = \boldsymbol{\varphi}_0(\overline{\Omega_b})$, so that the *deformed configuration is fully determined by the behavior on the boundary.*

To construct the mapping $\boldsymbol{\varphi}_0$, we use a standard ALE mapping. Because $(\boldsymbol{\eta}_N^n)^\delta = \omega \boldsymbol{e}_y$ on Γ for some function ω with $|\omega| \leqslant R_{max} < R$ (by Assumption 1B (5.69)) and $\omega = 0$ on $\partial\Gamma$, with $(\boldsymbol{\eta}_N^n)^\delta$ satisfying Dirichlet boundary conditions on all other parts of the boundary $\partial\Omega_b$, we can define the following injective mapping $\boldsymbol{\varphi}_0$ satisfying the necessary conditions on $\Omega_b = (0, L) \times (0, R)$:

$$\boldsymbol{\varphi}_0(z, r) = \left(x, y + \left(1 - \frac{y}{R}\right)\omega\right).$$

Note also that ω, which is the trace of $(\boldsymbol{\eta}_N^n)^\delta$ on Γ, is a continuous function. We observe that this map $\boldsymbol{\varphi}_0$ is an injective map. Thus, we conclude that $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta$ is injective for all $n = 0, 1, ..., N$ and for all N, if Assumptions 1B and 2B, given by (5.69), (5.70), (5.71), hold.

Proof. This result follows from the uniform energy estimates. To establish the uniform boundedness of $\boldsymbol{u}_N$ in $L^\infty(0, T; L^2(\Omega_f))$, we use Assumption 1B (5.69). To establish the uniform boundedness of $\boldsymbol{u}_N$ in $L^2(0, T; H^1(\Omega_f))$, we use Korn's inequality on the fluid domain. To establish the uniform boundedness of $\boldsymbol{\eta}_N$ in $L^\infty(0, T; H^1(\Omega_b))$, we combine the uniform energy estimates with Korn's inequality, stated in Proposition 5.5.1. To establish the uniform boundedness of p_N in $L^2(0, T; H^1(\Omega_b))$, we recall that by the uniform dissipation estimate,

$$\sum_{n=1}^{N} \kappa(\Delta t) \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C,$$

for some constant C uniform in N, where

$$\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta),$$

and

$$\nabla_b^{(\eta_N^n)^\delta} r = \nabla r \cdot (\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1} \quad \text{on } \Omega_b.$$

By Assumption 2B (5.71), we conclude that

$$(\Delta t) \sum_{n=1}^{N} \int_{\Omega_b} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C.$$

Since on Ω_b, we have that $\nabla p_N^{n+1} = \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot (\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)$, we use the fact from Assumption 2C (5.72) that $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta| \leqslant c_2$ to obtain the estimate

$$(\Delta t) \sum_{n=1}^{N} \int_{\Omega_b} |\nabla p_N^{n+1}|^2 \leqslant |\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^2 \cdot (\Delta t) \sum_{n=1}^{N} \int_{\Omega_b} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C,$$

for a constant C independent of N, which gives the final estimate that p_N is uniformly bounded in $L^2(0, T; H^1(\Omega_b))$.

□

We emphasize that later, we will have to establish that the three assumptions listed above are in fact true, in order to use this uniform boundedness result. Using the above uniform boundedness result, we can obtain the following weak convergences.

Proposition 5.6.1. Assuming that the three assumptions listed in Lemma 5.6.1 hold, we conclude that along an appropriate subsequence, we have the following weak convergences to limiting functions $\boldsymbol{u}$, p, $\boldsymbol{\eta}$, and ω:

- $\boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ weakly* in $L^\infty(0, T; L^2(\Omega_f))$.
- $\boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ weakly in $L^2(0, T; H^1(\Omega_f))$.
- $\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta}$ weakly* in $L^\infty(0, T; H^1(\Omega_b))$.
- $\overline{\boldsymbol{\eta}}_N \rightharpoonup \overline{\boldsymbol{\eta}}$ weakly* in $W^{1,\infty}(0, T; L^2(\Omega_b))$.
- $p_N \rightharpoonup p$ weakly* in $L^\infty(0, T; L^2(\Omega_b))$.
- $p_N \rightharpoonup p$ weakly in $L^2(0, T; H^1(\Omega_b))$.
- $\omega_N \rightharpoonup \omega$ weakly* in $L^\infty(0, T; H_0^2(\Gamma))$.
- $\overline{\omega}_N \rightharpoonup \overline{\omega}$ weakly* in $W^{1,\infty}(0, T; L^2(\Gamma))$.

Furthermore, $\boldsymbol{\eta} = \overline{\boldsymbol{\eta}}$ and $\omega = \overline{\omega}$.

Because the uniform boundedness result requires the three assumptions listed in Lemma 5.6.1, we must verify that these assumptions hold. We note that these assumptions in Lemma 5.6.1 imply the assumptions needed for the splitting scheme to successfully go through, in Assumptions 1A (5.59) and 2A (5.60). This is because Assumptions 1B and 2B in Lemma 5.6.1 imply Assumption 2A (5.60) that $\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta$ is invertible, by the previous discussion in Remark 5.6.1.

So in particular, verifying these assumptions is essential for showing that our approximate solutions can actually be constructed. We verify the assumptions in Lemma 5.6.1 in the lemma below.

Lemma 5.6.2. Suppose that the initial data satisfies $|\omega_0| \leqslant R_0 < R$ for some R_0, and $\boldsymbol{\eta}_0$ has the property that $\text{Id} + (\boldsymbol{\eta}_0)^\delta$ is invertible with $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) > 0$ on $\overline{\Omega_b}$ for some positive constant c_0. Then, there exists a sufficiently small time $T > 0$ such that for all N, the splitting scheme successfully runs until time T and all three assumptions (Assumptions 1B, 2B, and 2C) in Lemma 5.6.1 hold.

Proof. Consider $T > 0$ sufficiently small that we will choose later in the proof. It suffices to show that the three assumptions in Lemma 5.6.1 hold, as this will ensure that the splitting scheme will successfully go through.

First, we note that the three assumptions clearly hold for $n = 0$, because $\omega_N^0 = \omega_0$ and $\boldsymbol{\eta}_N^0 = \boldsymbol{\eta}_0$ for all N, and because of the assumptions on the initial data. In particular, there exist constants α_0, α_1, and α_2 such that

$$\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) \geqslant \alpha_0 > 0, \tag{5.73}$$

$$|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta| \geqslant \alpha_1 > 0, \qquad |(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta)^{-1}| \geqslant \alpha_2 > 0. \tag{5.74}$$

This is because $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta)$, $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta|$, and $|(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta)^{-1}|$ are positive continuous functions on the compact set $\overline{\Omega_b}$.

Next, we want to choose an appropriate time T such that the three assumptions hold uniformly for all N and $n\Delta t$ up to time T. To do this, we use the energy estimates. Define the initial energy determined by the initial data by E_0. Recall the definition of the discretized energy in (5.65). Then, by the uniform energy estimates, we have that

$$E_N^{k+\frac{1}{2}} \leqslant E_0, \qquad E_N^{k+1} \leqslant E_0.$$

Therefore, after completing both subproblems of the scheme on the time step $[k\Delta t, (k+1)\Delta t]$, we obtain that

$$||\dot{\boldsymbol{\eta}}_N^n||_{L^2(\Omega_b)} \leqslant C, \qquad \text{for } n = 0, 1, ..., k+1, \tag{5.75}$$

$$||\omega_N^{n+\frac{1}{2}}||_{H_0^2(\Gamma)} \leqslant C, \qquad \text{for } n = 0, 1, ..., k, \tag{5.76}$$

$$||\zeta_N^{n+\frac{i}{2}}||_{L^2(\Gamma)} \leqslant C, \qquad \text{for } 0 \leqslant n + \frac{i}{2} \leqslant k+1 \qquad \text{and } i = 0, 1, \tag{5.77}$$

for a constant C depending only on the initial energy E_0.

Step 1. Let us first find a condition on T that will imply that Assumption 1B (5.69) and (5.70) will hold up to time T, if the splitting scheme runs until time T. Suppose that the linear interpolation $\overline{\omega}_N$ is defined up to time $(k+1)\Delta t$. Then, by (5.76) and (5.77), it satisfies

$$||\overline{\omega}_N||_{W^{1,\infty}(0,(k+1)\Delta t; L^2(\Gamma))} \leqslant C, \qquad ||\overline{\omega}_N||_{L^\infty(0,(k+1)\Delta t; H_0^2(\Gamma))} \leqslant C, \tag{5.78}$$

where C depends only on E_0 and is independent of N. Thus, following the method in [140], we obtain by an interpolation inequality that for all $t, t+\tau \in [0, (k+1)\Delta t]$ with $\tau > 0$,

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^1(\Gamma)} \leqslant C||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{L^2(\Gamma)}^{1/2}||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^2(\Gamma)}^{1/2}. \tag{5.79}$$

Here, we used a Sobolev interpolation inequality, see for example Theorem 4.17 (pg. 79) of [1]. By the Lipschitz continuity of $\overline{\omega}_N$ taking values in $L^2(\Gamma)$ and the boundedness of $\overline{\omega}_N$ in $H_0^2(\Gamma)$,

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^1(\Gamma)} \leqslant C \cdot \tau^{1/2} \tag{5.80}$$

for a constant C depending only on E_0 (and in particular, not depending on k or N). Therefore, setting $t = 0$ and $\tau = (k+1)\Delta t$ and using the continuous embedding of $H^1(\Gamma)$ into $C(\Gamma)$,

$$||\omega_N^{k+1} - \omega_0||_{C(\Gamma)} \leqslant C \cdot [(k+1)t]^{1/2} \leqslant C \cdot T^{1/2}, \tag{5.81}$$

where C depends only E_0. Because $|\omega_0| < R$, we can choose $T > 0$ sufficiently small so that

$$C \cdot T^{1/2} < R - ||\omega_0||_{C(\Gamma)}. \tag{5.82}$$

This will give the first part of Assumption 1B, which is (5.69).

Step 2. Next, we show the remaining assumptions by controlling the behavior of the structure displacement $\boldsymbol{\eta}$. To do this, note that

$$||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\Omega_b)} \leqslant (\Delta t) \sum_{n=1}^{k+1} ||\dot{\boldsymbol{\eta}}_N^n||_{L^2(\Omega_b)} \leqslant C(k+1)(\Delta t) \leqslant CT,$$

for C depending only on E_0. So by the odd extension defined in Definition 5.4.1,

$$||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\tilde{\Omega}_b)} \leqslant C\left(||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\Omega_b)} + ||\omega_N^{k+1} - \omega_0||_{L^2(\Gamma)}\right) \leqslant CT,$$

for a constant C depending only on E_0, where the estimate $||\omega_N^{k+1} - \omega_0||_{L^2(\Gamma)} \leqslant CT$ follows from the bound (5.78). By regularization, we then have that for a constant depending only on δ and E_0,

$$||(\boldsymbol{\eta}_N^{k+1})^\delta - (\boldsymbol{\eta}_0)^\delta||_{H^3(\Omega_b)} \leqslant C(\delta, E_0) \cdot T.$$

By using the trace theorem and the continuous embedding of $H^2(\Gamma)$ into $C(\Gamma)$, we thus conclude that

$$||(\boldsymbol{\eta}_N^{k+1})^\delta|_\Gamma - (\boldsymbol{\eta}_0)^\delta|_\Gamma||_{C(\Gamma)} \leqslant C(\delta, E_0) \cdot T. \tag{5.83}$$

Since $H^2(\Omega_b)$ embeds continuously into $C(\Omega_b)$, we also have that

$$||\nabla(\boldsymbol{\eta}_N^{k+1})^\delta - \nabla(\boldsymbol{\eta}_0)^\delta||_{C(\Omega_b)} \leqslant C(\delta, E_0) \cdot T. \tag{5.84}$$

Note that $\det(\boldsymbol{I} + \boldsymbol{A})$ is a continuous function of the entries of $\boldsymbol{A}$. Also note that the matrix norms $|\boldsymbol{I}+\boldsymbol{A}|$ and $|(\boldsymbol{I}+\boldsymbol{A})^{-1}|$ are continuous functions of the matrix $\boldsymbol{A}$. Furthermore, we emphasize that the constant $C(\delta, E_0)$ depends only on δ and E_0 and hence is independent of k and N. This dependence on δ is allowable, since for this existence proof, δ is an arbitrary but fixed regularization parameter.

Thus, there exists T sufficiently small so that by (5.83) and (5.84), the remaining assumptions (5.70), (5.71), and (5.72) are satisfied, since these assumptions are all satisfied for the initial displacement $\boldsymbol{\eta}_0$. Furthermore, we can choose the constants c_0, c_1, c_2, and R_{max} (defined in the statement of those assumptions) independently of N and $n = 0, 1, ..., N$, because of the fact that the constant $C(\delta, E_0)$ in our estimates does not depend on k (satisfying $(k+1)\Delta t \leqslant T$) or N. □

5.7 Compactness arguments

We next want to pass to the limit in the semidiscrete formulation for the approximate solutions, stated in (5.63) and (5.64). Because this is a nonlinear problem with geometric nonlinearities, we must obtain stronger convergence than just weak and weak* convergence in Proposition 5.6.1, in order to pass to the limit. To do this, we will use compactness arguments of two types: the classical Aubin-Lions compactness theorem for functions defined on fixed domains, and generalized Aubin-Lions compactness arguments for functions defined on moving domains (using methods from [140], [136]). We will first deal with compactness arguments for the plate displacement and the Biot domain displacement. Then, we will deal with compactness arguments for the fluid velocity defined on moving domains.

Compactness for Biot poroelastic medium displacement

We begin with compactness arguments for the quantities associated with the Biot medium. In particular, we will show strong convergence of the Biot structure displacements $\overline{\boldsymbol{\eta}}_N$. Since the Biot poroviscoelastic structure displacement is defined on the fixed domain Ω_b, this will be achieved by using a standard Aubin-Lions compactness argument. In particular, we have the following strong convergence result for the Biot medium displacement:

Lemma 5.7.1. We have the following compact embedding:

$$W^{1,\infty}(0,T;L^2(\Omega_b)) \cap L^\infty(0,T;H^1(\Omega_b)) \subset\subset C(0,T;L^2(\Omega_b)).$$

Hence, there exists a subsequence such that $\overline{\boldsymbol{\eta}}_N \to \boldsymbol{\eta}$ strongly in $C(0,T;L^2(\Omega_b))$.

Proof. The compact embedding above is a direct consequence of the standard Aubin-Lions compactness lemma [6, 129] in the case of $p = \infty$, which gives a stronger compact embedding into $C(0,T;L^2(\Omega_b))$ rather than just $L^\infty(0,T;L^2(\Omega_b))$. The fact that we can find a strongly convergent subsequence follows from this compact embedding, once we recall that $\{\overline{\boldsymbol{\eta}}_N\}_{N=1}^\infty$ are uniformly bounded in the Banach space $W^{1,\infty}(0,T;L^2(\Omega_b)) \cap L^\infty(0,T;H^1(\Omega_b))$ by the uniform energy estimates. □

Compactness for the plate displacement

Next, we show strong convergence of the approximate plate displacements ω_N. We have that the linear interpolation $\overline{\eta}_N$ of the plate displacement is bounded in $W^{1,\infty}(0,T;L^2(\Gamma))$ and $L^\infty(0,T;H_0^2(\Gamma))$. We use this to establish the following convergence result.

Proposition 5.7.1. Given arbitrary $0 < s < 2$, there exists a subsequence such that the following strong convergences hold:

$$\overline{\omega}_N \to \omega, \qquad \text{in } C(0,T;H^s(\Gamma)),$$

$$\omega_N \to \omega, \qquad \text{in } L^\infty(0,T;H^s(\Gamma)).$$

Proof. For the linear interpolations $\overline{\omega}_N$, we have the uniform estimate for $\tau > 0$, $t, t+\tau \in [0,T]$, that

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^{2\alpha}(\Gamma)} \leqslant C\tau^{1-\alpha}, \qquad \text{for } 0 < \alpha < 1,$$

where the constant C is independent of N (but can depend on the choice of α). This follows from the same argument used in Step 1 of the proof of Lemma 5.6.2. Because the constant C in the estimate above is independent of N, the estimate implies that for a given arbitrary $\alpha \in (0, 1)$, the functions $\overline{\omega}_N$ are uniformly bounded as functions in $C^{0,1-\alpha}(0, T; H^{2\alpha}(\Gamma))$. Hence, the strong convergence of $\overline{\omega}_N$ follows directly from the Arzela-Ascoli theorem and the fact that $H^{2\alpha}$ embeds compactly into any $H^{2\alpha-\epsilon}$ for $\epsilon > 0$, once we choose $\alpha \in (0, 1)$ and $\epsilon > 0$ appropriately so that $2\alpha - \epsilon = s$ for a given arbitrary $0 < s < 2$. Hence, we obtain the desired strong convergence, as the equicontinuity condition for the Arzela-Ascoli theorem follows from the above estimate.

To show a similar strong convergence result for ω_N, we must show that

$$||\omega_N(t) - \overline{\omega}_N(t)||_{L^\infty(0,T;H^s(\Gamma))} \to 0,$$

for arbitrary $0 < s < 2$. Once we observe that $\overline{\omega}_N(n\Delta t) = \omega_N(t)$ for $n\Delta t \leqslant t < (n+1)\Delta t$, this follows immediately from the above Hölder continuity estimate, as

$$||\omega_N(t) - \overline{\omega}_N(t)||_{L^\infty(0,T;H^s(\Gamma))} \leqslant C(\Delta t)^{1-\frac{s}{2}} \to 0, \qquad \text{as } N \to \infty.$$

Thus, ω_N and $\overline{\omega}_N$ have the same limit in $L^\infty(0, T; H^s(\Gamma))$ for $0 < s < 2$.

□

Compactness arguments for the Biot velocity and plate velocity

Next, we will obtain compactness for the Biot velocity, plate velocity, pore pressure, and fluid velocity. Because the test space (5.56) has the pore pressure and fluid velocity decoupled from the Biot/plate velocity, we can handle the compactness argument for each of these quantities separately. In particular, we recall the definition of the discrete test space from (5.56):

$$\mathcal{Q}_N^{n+1} = \{(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in V_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma\}.$$

We note that we can decouple this test space into three smaller test spaces, one for the Biot/plate displacement/velocity, one for the pore pressure, and one for the fluid velocity. In this section, we will start by obtaining compactness results for the Biot/plate velocity, which must be treated together since they are coupled by a kinematic coupling condition at the plate interface Γ. We will show the following result:

Theorem 5.7.1. For $-1/2 < s < 0$, there exists a subsequence such that

$$(\boldsymbol{\xi}_N, \zeta_N) \to (\boldsymbol{\xi}, \zeta) \qquad \text{strongly in } L^2(0, T; H^{-s}(\Omega_b) \times H^{-s}(\Gamma)).$$

Proof. We will establish this result by using a compactness criterion for piecewise constant functions due to Dreher and Jüngel [65]. To simplify arguments, we define a slightly more regular Biot/plate velocity test space:

$$\mathcal{Q}_v = \{(\boldsymbol{\psi}, \varphi) \in (V_d \cap H^2(\Omega_b)) \times H_0^2(\Gamma) : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma\}. \tag{5.85}$$

We will use the following chain of embeddings

$$L^2(\Omega_b) \times L^2(\Gamma) \subset\subset H^{-s}(\Omega_b) \times H^{-s}(\Gamma) \subset \mathcal{Q}_v',$$

where the first embedding is compact, in the Dreher-Jüngel compactness criterion [65].

Let $\tau_{\Delta t}$ denote the time shift $\tau_{\Delta t} f(t, \cdot) = f(t - \Delta t, \cdot)$ for a function f defined on $[0, T]$. We must verify that there exists a uniform constant C such that for all $\Delta t = T/N$,

$$\left|\left|\frac{\tau_{\Delta t}(\boldsymbol{\xi}_N, \zeta_N) - (\boldsymbol{\xi}_N, \zeta_N)}{\Delta t}\right|\right|_{L^1(\tau, T; \mathcal{Q}_v')} + ||(\boldsymbol{\xi}_N, \zeta_N)||_{L^\infty(0,T; L^2(\Omega_b) \times L^2(\Gamma))} \leqslant C. \tag{5.86}$$

We have that $(\boldsymbol{\xi}_N, \zeta_N)$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega_b) \times L^2(\Gamma))$ by Lemma 5.6.1.

For the other term, we use the semidiscrete formulation. Because we are considering only the Biot and plate velocities, we can set the test functions $\boldsymbol{v}$ and r for the fluid velocity and Biot pore pressure to be zero. We obtain that for all test functions $(\boldsymbol{\psi}, \varphi) \in \mathcal{Q}_v$, where $\mathcal{Q}_v$ is defined in (5.85),

$$\begin{aligned}
&\rho_b \int_{\Omega_b} \left(\frac{\boldsymbol{\xi}_N^{n+1} - \boldsymbol{\xi}_N^n}{\Delta t}\right) \cdot \boldsymbol{\psi} + \rho_p \int_\Gamma \left(\frac{\zeta_N^{n+1} - \zeta_N^n}{\Delta t}\right) \cdot \varphi \\
&\quad = -\int_\Gamma \left(\frac{1}{2} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{u}_N^n - p_N^{n+1}\right)(\boldsymbol{\psi} \cdot \boldsymbol{n}^{\omega_N^n}) - \int_\Gamma \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} (\zeta_N^{n+1} \boldsymbol{e}_y - \boldsymbol{u}_N^{n+1}) \cdot \boldsymbol{\tau}^{\omega_N^n} (\boldsymbol{\psi} \cdot \boldsymbol{\tau}^{\omega_N^n}) \\
&\quad - 2\mu_e \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) - \lambda_e \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) - 2\mu_v \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) \\
&\qquad - \lambda_v \int_{\Omega_b} (\nabla \cdot \boldsymbol{\xi}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) + \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} - \int_\Gamma \Delta \omega_N^{n+\frac{1}{2}} \cdot \Delta \varphi.
\end{aligned}$$

Consider an arbitrary $||(\boldsymbol{\psi}, \varphi)||_{\mathcal{Q}_v} \leqslant 1$, so that $||\boldsymbol{\psi}||_{H^2(\Omega_b)} \leqslant 1$ and $||\varphi||_{H_0^2(\Gamma)} \leqslant 1$. By the uniform estimates in Lemma 5.6.1 and the regularity of the test functions in (5.85), it is clear that the terms on the right hand side are all uniformly bounded by a constant C, independent of $||(\boldsymbol{\psi}, \varphi)||_{\mathcal{Q}_v} \leqslant 1$, so that

$$\left|\left|\frac{(\boldsymbol{\xi}_N^{n+1}, \zeta_N^{n+1}) - (\boldsymbol{\xi}_N^n, \zeta_N^n)}{\Delta t}\right|\right|_{\mathcal{Q}_v'} \leqslant C, \qquad \text{for a constant } C \text{ that is independent of } n \text{ and } N.$$

The only term that requires some care is the term

$$\alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi}.$$

For this term, we recall that by definition,

$$\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta), \qquad \nabla^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} = \text{tr}\left[\nabla \boldsymbol{\psi} \cdot (I + \nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}\right].$$

By assumption 2C (5.72) and the fact that $||\boldsymbol{\psi}||_{H^1(\Omega_b)} \leqslant 1$, we have that $||\nabla^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi}||_{L^2(\Omega_b)}$ is uniformly bounded, while by the boundedness of $\boldsymbol{\eta}_N^n$ in $H^1(\Omega_b)$, we have that $|\mathcal{J}_b^{(\eta_N^n)^\delta}| \leqslant C$. Therefore, using the fact that p_N is uniformly bounded in $L^\infty(0,T;L^2(\Omega_b))$, we can estimate

$$\left|\alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi}\right| \leqslant C.$$

Since

$$\sum_{n=1}^{N-1} (\Delta t) \left|\left| \frac{(\boldsymbol{\xi}_N^{n+1}, \zeta_N^{n+1}) - (\boldsymbol{\xi}_N^n, \zeta_N^n)}{\Delta t} \right|\right|_{\mathcal{Q}_v'} \leqslant (\Delta t) \sum_{n=1}^{N-1} C \leqslant CT,$$

we conclude that (5.86) holds for a uniform constant C. This establishes the desired result. □

Compactness arguments for the pore pressure

We show a similar compactness result for the Biot pore pressure in this section.

Theorem 5.7.2. There exists a subsequence such that

$$p_N \to p, \qquad \text{strongly in } L^2(0,T;L^2(\Omega_b)).$$

We proceed using similar arguments based on the Dreher-Jüngel compactness criterion for piecewise constant functions [65]. We observe that the approximate solutions for the pore pressure satisfy the following weak formulation for all test functions $r \in V_p$, where V_p is defined by (5.40):

$$\begin{aligned} c_0 \int_{\Omega_b} \left(\frac{p_N^{n+1} - p_N^n}{\Delta t}\right) \cdot r - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \dot{\boldsymbol{\eta}}_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha \int_\Gamma (\dot{\boldsymbol{\eta}}_N^{n+1} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r \\ + \kappa \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \int_\Gamma [(\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n}^{\omega_N^n}] r = 0. \end{aligned} \tag{5.87}$$

Proof. By the Dreher-Jüngel compactness criterion [65], it suffices to show that for a constant C independent of N,

$$\left|\left| \frac{\tau_{\Delta t} p_N - p_N}{\Delta t} \right|\right|_{L^1(\Delta t, T; (V_p \cap H^2(\Omega_b))')} + ||p_N||_{L^2(0,T;H^1(\Omega_b))} \leqslant C, \tag{5.88}$$

since we have the chain of embeddings $(V_p \cap H^2(\Omega_b))' \subset\subset L^2(\Omega_b) \subset H^1(\Omega_b)$. We use more regularity for the test space $V_p \cap H^2(\Omega_b)$ to make the following estimates simpler.

We compute that for any $r \in V_p \cap H^2(\Omega_b)$,

$$\begin{aligned} c_0 \int_{\Omega_b} \left(\frac{p_N^{n+1} - p_N^n}{\Delta t}\right) \cdot r = \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\xi}_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r + \alpha \int_\Gamma (\zeta_N^{n+1} \boldsymbol{e}_y \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r \\ - \kappa \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r + \int_\Gamma [(\boldsymbol{u}_N^{n+1} - \zeta_N^{n+1} \boldsymbol{e}_y) \cdot \boldsymbol{n}^{\omega_N^n}] r. \end{aligned}$$

We estimate the right hand side for $||r||_{V_p\cap H^2(\Omega_b)} \leqslant 1$. Recall that $\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)$,

$$\nabla_b^{(\eta_N^n)^\delta} r = \left(\frac{\partial r}{\partial \tilde{x}}, \frac{\partial r}{\partial \tilde{y}}\right)\cdot(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}, \quad \text{and} \quad \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} = \left(\frac{\partial p_N^{n+1}}{\partial \tilde{x}}, \frac{\partial p_N^{n+1}}{\partial \tilde{y}}\right)\cdot(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}.$$

We have by Assumption 2C (5.72) that $|(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}|$ is uniformly bounded, and furthermore, $\mathcal{J}_b^{(\eta_N^n)^\delta}$ is positive and bounded above. Therefore, combining these facts with standard estimates, we conclude that

$$\left|\left|\frac{p_N^{n+1}-p_N^n}{\Delta t}\right|\right|_{(V_p\cap H^2(\Omega_b))'} \leqslant C, \qquad \text{for a constant } C \text{ that is independent of } n \text{ and } N.$$

Combining this with the fact that p_N is uniformly bounded in $L^2(0,T;H^1(\Omega_b))$, this gives the desired estimate in (5.88). □

Compactness arguments for the fluid velocity

We will obtain convergence of the fluid velocity along a subsequence, by using a generalized Aubin-Lions compactness theorem for functions defined on moving domains. The reason we must use a *generalized Aubin-Lions compactness theorem* is that the approximate fluid velocities are defined on different time-dependent fluid domains. Thus, we need to introduce a maximal domain that contains all of the possible moving fluid domains, extend the fluid velocities to be defined on this maximal domain, and then apply a generalized form of the Aubin-Lions compactness theorem for problems on moving domains. For these compactness arguments, we can modify the argument found in [136] since we are still considering a moving fluid domain with a boundary determined by the time-dependent configuration of an elastic plate, with additional arguments which are needed to handle the unique form of the weak formulation corresponding to this specific FPSI model.

We first recall the semidiscrete formulation in (5.63) and rewrite it so that we can obtain an equation for the fluid velocities defined on the physical domain, rather than the reference domain. Since the fluid velocities are decoupled from the remaining physical quantities, we can simplify the semidiscrete weak formulation (5.63) by taking the fluid velocity test function to be the only nonzero test function. Our arguments will require us to consider the physical fluid domain $\Omega_{f,N}^n$ rather than the fixed reference fluid domain Ω_f, where we define

$$\Omega_{f,N}^n = \{(x,y)\in\mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega_N^n(x)\}.$$

We redefine the fluid velocity solution space and test space as follows:

$$V_N^{n+1} = \{\boldsymbol{u}\in H^1(\Omega_{f,N}^n) : \nabla\cdot\boldsymbol{u} = 0 \text{ on } \Omega_{f,N}^n, \boldsymbol{u} = 0 \text{ on } \partial\Omega_{f,N}^n\backslash\Gamma_N^n\}. \tag{5.89}$$

$$Q_N^n = V_N^{n+1} \cap H^3(\Omega_{f,N}^n). \tag{5.90}$$

We then obtain the following semidiscrete formulation on the physical fluid domain: the approximate fluid velocities $\boldsymbol{u}_N^{n+1} \in V_N^{n+1}$ satisfy the following equality for all test functions $\boldsymbol{v} \in Q_N^n$:

$$\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1} - \tilde{\boldsymbol{u}}_N^n}{\Delta t} \cdot \boldsymbol{v} + 2\nu \int_{\Omega_{f,N}^n} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{v})$$
$$+ \frac{1}{2} \int_{\Omega_{f,N}^n} \left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R+\omega_N^n} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R+\omega_N^n} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{v} \right) \cdot \boldsymbol{u}_N^{n+1} \right]$$
$$+ \frac{1}{2R} \int_{\Omega_{f,N}^n} \frac{R}{R+\omega_N^n} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{v} + \frac{1}{2} \int_{\Gamma_N^n} (\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n} (\tilde{\boldsymbol{u}}_N^n \cdot \boldsymbol{v})$$
$$- \int_{\Gamma_N^n} \left(\frac{1}{2} \boldsymbol{u}_N^{n+1} \cdot \tilde{\boldsymbol{u}}_N^n - p_N^{n+1} \right) (\boldsymbol{v} \cdot \boldsymbol{n}) - \beta \int_{\Gamma_N^n} (\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1}) \cdot \boldsymbol{\tau} (\boldsymbol{v} \cdot \boldsymbol{\tau}) = 0, \quad (5.91)$$

where we recall that $\boldsymbol{u}_N^n$ is originally defined on the domain $\Omega_{f,N}^{n-1}$, and hence we define

$$\tilde{\boldsymbol{u}}_N^n = \boldsymbol{u}_N^n \circ \boldsymbol{\Phi}_f^{\omega_N^{n-1}} \circ (\boldsymbol{\Phi}_f^{\omega_N^n})^{-1},$$

where the ALE map $\boldsymbol{\Phi}_f^{\omega_N^n} : \Omega_f \to \Omega_{f,N}^n$ is defined by (5.21).

Extension to maximal domain

We first uniformly bound the physical fluid domains $\Omega_{f,N}^n$, as this will allow us to extend the approximate fluid velocity functions to a common maximal domain. To do this, we will use the following proposition, which is Lemma 2.5 in [30] and Lemma 4.5 in [136], which was established in the context of nonlinearly coupled FSI between an incompressible viscous Newtonian fluid and an elastic Koiter shell.

Proposition 5.7.2. There exists smooth functions $m(x)$ and $M(x)$ defined on $\Gamma = [0, L]$, such that

$$m(x) \leqslant \omega_N^n(x) \leqslant M(x), \qquad \text{for all } x \in [0, L], N, \text{ and } n = 0, 1, ..., N.$$

Furthermore, there exist smooth functions $m_N^{n,l}(x)$ and $M_N^{n,l}(x)$ defined for positive integers N, $n = 0, 1, ..., N-1$ and $l = 0, 1, ..., N-n$, such that

1. $m_N^{n,l}(x) \leqslant \omega_N^{n+i}(x) \leqslant M_N^{n,l}(x), \qquad$ for all $x \in [0, L]$ and $i = 0, 1, ..., l$.
2. $M_N^{n,l}(x) - m_N^{n,l}(x) \leqslant C\sqrt{l\Delta t}, \qquad$ for all $x \in [0, L]$.
3. $||M_N^{n,l}(x) - m_N^{n,l}(x)||_{L^2(\Gamma)} \leqslant C(l\Delta t)$,

where C is independent of n, l, and N. Finally, the functions $M_N^{n,l}(x)$ and $m_N^{n,l}(x)$ for all n, l, and N, are Lipschitz continuous with a Lipschitz constant that is uniformly bounded above by some constant $L > 0$ independent of n, l, and N.

The result in Proposition 5.7.2 allows us to define a maximal domain Ω_f^M defined by the function $M(x)$, containing all of the physical approximate fluid domains $\Omega_{f,N}^n$:

$$\Omega_f^M = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant M(x)\}.$$

We can extend the fluid velocities $\boldsymbol{u}_N^n$ on $\Omega_{f,N}^n$ to this common maximal domain Ω_f^M, by extending by zero in $\Omega_f^M \cap (\Omega_{f,N}^n)^c$. We have the following result, which follows from the fact that the functions $\eta_N^n(x)$ are all uniformly Lipschitz and a result on extensions by zero of H^1 functions defined on Lipschitz domains, see [131] and the corresponding discussions in [30, 136].

Lemma 5.7.2. The approximate fluid velocities $\{\boldsymbol{u}_N\}_{N=1}^{\infty}$ defined on the maximal fluid domain Ω_f^M by extension by zero are uniformly bounded in $L^2(0,T;H^s(\Omega_f^M))$ for $s \in (0,1/2)$.

The generalized Aubin-Lions compactness lemma

We now pass to the limit along a subsequence in the approximate fluid velocities $\boldsymbol{u}_N$, which are now functions in time defined on the fixed maximal domain Ω_f^M. We have the following convergence result.

Proposition 5.7.3. The sequence $\boldsymbol{u}_N$ is relatively compact in $L^2(0,T;L^2(\Omega_f^M))$.

The rest of this section will be devoted to using a generalized Aubin-Lions compactness theorem for problems on moving domains, see [136, 30], to verify this convergence result. We first define the relevant function spaces

$$H = L^2(\Omega_f^M), \qquad V = H^s(\Omega_f^M), \qquad \text{for } 0 < s < 1/2,$$

where we note that $V \subset\subset H$. Note that $V_N^n \times Q_N^n$ defined by (5.89) and (5.90) embeds continuously into $V \times V$ by the extension by zero operator to the maximal domain Ω_f^M, uniformly in n and N.

We want to verify the conditions of the generalized Aubin-Lions compactness theorem [136] (Properties A, B, C1, C2, and C3), for the approximate solutions $\boldsymbol{u}_N$, which we recall are defined by

$$\boldsymbol{u}_N = \boldsymbol{u}_N^n, \qquad \text{on } ((n-1)\Delta t, n\Delta t], \text{ for } n = 1,2,...,N.$$

The proofs of Properties A, C1, C2, and C3 are analogous to the corresponding proofs in [136] (Section 4.2). So it suffices to verify Property B.

Property B. There exists a constant C independent of n and N, a constant $1 \leqslant p < 2$, and for each N, a sequence of nonnegative $\{a_N^n\}_{n=0}^{N-1}$ satisfying $(\Delta t)\sum_{n=0}^{N-1}|a_N^n|^2 \leqslant C$ uniformly in N, such that

$$\left\|P_N^n \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t}\right\|_{(Q_N^n)'} \leqslant C\left(a_N^n + ||\boldsymbol{u}_N^n||_{V_N^n} + ||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right)^p, \qquad \text{for all } n = 0,1,...,N-1, \tag{5.92}$$

where P_N^n denotes the orthogonal projection onto the closed subspace $\overline{Q_N^n}^H$ of the Hilbert space H.

Remark 5.7.1. In the original reference [136], there is a different statement of Property B, which is that there exists a constant $C > 0$ independent of N such that

$$\left|\left|P_N^n \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t}\right|\right|_{(Q_N^n)'} \leqslant C\left(1 + ||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right), \qquad \text{for all } n = 0, 1, ..., N-1. \tag{5.93}$$

The above version of Property B can be seen as a generalization of this, and this generalized version of Property B is needed, due to the appearance of terms in the weak formulation that do not allow us to show that (5.93) holds. With the generalized form of Property B as above in (5.92), the generalized Aubin-Lions compactness theorem stated in [136] for moving domains still holds, as we still have the essential equicontinuity estimate needed in the proof. In particular, for the original form of Property B in (5.93), one has from Lemma 3.1 in [136] the following equicontinuity estimate for a constant $C > 0$ that is independent of N:

$$||P_{\Delta t}^{n,l}(\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n)||_{(Q_N^{n,l})'} \leqslant C\sqrt{l\Delta t}.$$

With the generalized form of Property B that we use above in (5.92), the same arguments as in the proof of Lemma 3.1 in [136] will still give rise to the following equicontinuity estimate for a constant $C > 0$ that is independent of N:

$$||P_{\Delta t}^{n,l}(\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n)||_{(Q_N^{n,l})'} \leqslant C(l\Delta t)^{1-\frac{p}{2}},$$

where the generalized Aubin-Lions compactness theorem on moving domains still holds with this new equicontinuity estimate, since $1 \leqslant p < 2$ and hence, $C(l\Delta t)^{1-\frac{p}{2}}$ still converges to zero as $\Delta t \to 0$.

Proof of Property B. We consider

$$\tilde{\boldsymbol{u}}_N^n = \boldsymbol{u}_N^n \circ \boldsymbol{\Phi}_f^{\omega_N^{n-1}} \circ (\boldsymbol{\Phi}_f^{\omega_N^n})^{-1},$$

and use the semidiscrete formulation for the fluid velocity on the physical domain. By definition,

$$\left|\left|P_N^n \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t}\right|\right|_{(Q_N^n)'} = \max_{||\boldsymbol{v}||_{Q_N^n} \leqslant 1} \left|\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right|. \tag{5.94}$$

We then estimate

$$\left|\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}^n}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right| \leqslant \left|\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1} - \tilde{\boldsymbol{u}}_N^n}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right| + \left|\int_{\Omega_{f,N}^n} \frac{\tilde{\boldsymbol{u}}_N^n - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right|. \tag{5.95}$$

By the semidiscrete formulation (5.91), we have that

$$\left|\int_{\Omega_{f,N}^n} \frac{\boldsymbol{u}_N^{n+1}-\tilde{\boldsymbol{u}}_N^n}{\Delta t}\cdot \boldsymbol{v} d\boldsymbol{x}\right| \leqslant 2\nu \left|\int_{\Omega_{f,N}^n} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{v})\right|$$
$$+\frac{1}{2}\left|\int_{\Omega_{f,N}^n}\left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla \boldsymbol{u}_N^{n+1}\right)\cdot \boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla \boldsymbol{v}\right)\cdot \boldsymbol{u}_N^{n+1}\right]\right|$$
$$+\frac{1}{2R}\left|\int_{\Omega_{f,N}^n}\frac{R}{R+\omega_N^n}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot \boldsymbol{v}\right| + \frac{1}{2}\left|\int_{\Gamma_N^n}(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot \boldsymbol{n}(\tilde{\boldsymbol{u}}_N^n\cdot \boldsymbol{v})\right|$$
$$+\left|\int_{\Gamma_N^n}\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\tilde{\boldsymbol{u}}_N^n - p_N^{n+1}\right)(\boldsymbol{v}\cdot\boldsymbol{n})\right| + \beta\left|\int_{\Gamma_N^n}(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}(\boldsymbol{v}\cdot\boldsymbol{\tau})\right|.$$

We can bound the terms uniformly in n, N, and $||\boldsymbol{v}||_{Q_N^n} \leqslant 1$ as follows. By the boundedness of $\boldsymbol{u}_N^{n+1}$ in the uniform energy estimates, we have

$$2\nu\left|\int_{\Omega_{f,N}^n}\boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{v})\right| \leqslant C||\boldsymbol{u}_N^{n+1}||_{H^1(\Omega_{f,N}^n)}.$$

Because $||\boldsymbol{v}||_{Q_N^n} \leqslant 1$, by the definition of Q_N^n in (5.90), we have that $\boldsymbol{v}$ is bounded in $H^3(\Omega_{f,N}^n)$, and hence, $\boldsymbol{v}$ and $\nabla \boldsymbol{v}$ are bounded pointwise. Furthermore, by the boundedness of the fluid velocity $\boldsymbol{u}_N^n$ on the reference domain by the uniform energy estimates, and the uniform boundedness of the Jacobian of the ALE map $\Phi_f^{\omega_N^n}$, we obtain the following bound:

$$\frac{1}{2}\left|\int_{\Omega_{f,N}^n}\left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla \boldsymbol{u}_N^{n+1}\right)\cdot \boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla \boldsymbol{v}\right)\cdot \boldsymbol{u}_N^{n+1}\right]\right|$$
$$\leqslant C\left(||\tilde{\boldsymbol{u}}_N^n||_{L^2(\Omega_{f,N}^n)} + ||\zeta^{n+\frac{1}{2}}||_{L^2(\Gamma)}\right)||\boldsymbol{u}_N^{n+1}||_{H^1(\Omega_{f,N}^n)}\cdot||\boldsymbol{v}||_{H^3(\Omega_{f,N}^n)} \leqslant C||\boldsymbol{u}_N^{n+1}||_{H^1(\Omega_{f,N}^n)}.$$

The next term is bounded similarly by

$$\frac{1}{2R}\left|\int_{\Omega_{f,N}^n}\frac{R}{R+\omega_N^n}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{v}\right| \leqslant C||\zeta_N^{n+\frac{1}{2}}||_{L^2(\Gamma)}||\boldsymbol{u}_N^{n+1}||_{L^2(\Omega_{f,N}^n)}\cdot||\boldsymbol{v}||_{H^3(\Omega_{f,N}^n)} \leqslant C||\boldsymbol{u}_N^{n+1}||_{L^2(\Omega_{f,N}^n)}.$$

We observe that $||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}$ is bounded uniformly and furthermore, the arc length element on Γ_N^n is uniformly bounded pointwise since η_N^n is uniformly bounded in $H_0^2(\Gamma)$.

Therefore, by using the trace inequality on Ω_f, we have that

$$\begin{aligned}
&\frac{1}{2}\left|\int_{\Gamma_N^n}(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}(\tilde{\boldsymbol{u}}_N^n\cdot\boldsymbol{v})\right| \\
&\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{L^4(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{L^4(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}+||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{L^4(\Gamma)}\cdot||\boldsymbol{v}||_{L^4(\Gamma)}\right) \\
&\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{H^{1/4}(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{H^{1/4}(\Gamma)}\cdot||\boldsymbol{v}||_{H^1(\Omega_f)}+||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{H^{1/4}(\Gamma)}\cdot||\boldsymbol{v}||_{H^{1/4}(\Gamma)}\right) \\
&\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{H^{3/4}(\Omega_f)}\cdot||\boldsymbol{u}_N^n||_{H^{3/4}(\Omega_f)}+||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{H^{3/4}(\Omega_f)}\right)\cdot||\boldsymbol{v}||_{H^3(\Omega_f)} \\
&\leqslant C\left(||\boldsymbol{u}_N^{n+1}||^{1/4}_{L^2(\Omega_f)}||\boldsymbol{u}_N^{n+1}||^{3/4}_{H^1(\Omega_f)}||\boldsymbol{u}_N^n||^{1/4}_{L^2(\Omega_f)}||\boldsymbol{u}_N^n||^{3/4}_{H^1(\Omega_f)}+||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}||\boldsymbol{u}_N^n||^{1/4}_{L^2(\Omega_f)}||\boldsymbol{u}_N^n||^{3/4}_{H^1(\Omega_f)}\right) \\
&\leqslant C\left(||\boldsymbol{u}_N^{n+1}||^{3/4}_{H^1(\Omega_f)}\cdot||\boldsymbol{u}_N^n||^{3/4}_{H^1(\Omega_f)}+||\boldsymbol{u}_N^n||^{3/4}_{H^1(\Omega_f)}\right)\leqslant C\left[1+\left(||\boldsymbol{u}_N^n||_{V_N^n}+||\boldsymbol{u}_N^{n+1}||_{V_N^n}\right)^{3/2}\right].
\end{aligned}$$

We also estimate

$$\begin{aligned}
&\Big|\int_{\Gamma_N^n}\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\tilde{\boldsymbol{u}}_N^n-p_N^{n+1}\right)(\boldsymbol{v}\cdot\boldsymbol{n})\Big| \\
&\qquad\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{L^4(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{L^4(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}+||p_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}\right) \\
&\qquad\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{H^{1/4}(\Gamma)}\cdot||\boldsymbol{u}_N^n||_{H^{1/4}(\Gamma)}\cdot||\boldsymbol{v}||_{H^1(\Omega_f)}+||p_N^{n+1}||_{H^1(\Omega_b)}\cdot||\boldsymbol{v}||_{H^1(\Omega_f)}\right) \\
&\qquad\leqslant C\left(||\boldsymbol{u}_N^{n+1}||_{H^{3/4}(\Omega_f)}\cdot||\boldsymbol{u}_N^n||_{H^{3/4}(\Omega_f)}+||p_N^{n+1}||_{H^1(\Omega_b)}\right) \\
&\qquad\leqslant C\left(||\boldsymbol{u}_N^{n+1}||^{1/4}_{L^2(\Omega_f)}||\boldsymbol{u}_N^{n+1}||^{3/4}_{H^1(\Omega_f)}\cdot||\boldsymbol{u}_N^n||^{1/4}_{L^2(\Omega_f)}||\boldsymbol{u}_N^n||^{3/4}_{H^1(\Omega_f)}+||p_N^{n+1}||_{H^1(\Omega_b)}\right) \\
&\qquad\leqslant C\left[1+\left(||p_N^{n+1}||_{H^1(\Omega_b)}+||\boldsymbol{u}_N^n||_{V_N^n}+||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right)^{3/2}\right].
\end{aligned}$$

Finally, we estimate

$$\begin{aligned}
\beta\left|\int_{\Gamma_N^n}(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}(\boldsymbol{v}\cdot\boldsymbol{\tau})\right| &\leqslant C\left(||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}+||\boldsymbol{u}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}\right) \\
&\leqslant C\left(1+||\boldsymbol{u}_N^{n+1}||_{H^1(\Omega_f)}\right).
\end{aligned}$$

Therefore, we obtain the final estimate that for a constant C independent of n and N,

$$\max_{||\boldsymbol{v}||_{Q_N^n}\leqslant 1}\left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\tilde{\boldsymbol{u}}_N^n}{\Delta t}\cdot\boldsymbol{v}d\boldsymbol{x}\right|\leqslant C\left(a_N^n+||\boldsymbol{u}_N^n||_{V_N^n}+||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right)^{3/2},$$

$$\text{for } a_N^n:=1+||p_N^{n+1}||_{H^1(\Omega_b)},\ \text{where } (\Delta t)\sum_{n=0}^{N-1}|a_N^n|^2\leqslant 2\left[(\Delta t)N+||p_N||^2_{L^2(0,T;H^1(\Omega_b))}\right]\leqslant C. \tag{5.96}$$

From the inequality (5.95), it remains to estimate $\left|\int_{\Omega^n_{f,N}} \frac{\tilde{\boldsymbol{u}}^n_N - \boldsymbol{u}^n_N}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right|$. The same estimates as in [136] show that there exists a constant C independent of n and N, such that

$$\max_{||\boldsymbol{v}||_{\mathcal{Q}^n_N} \leqslant 1} \left|\int_{\Omega^n_{f,N}} \frac{\tilde{\boldsymbol{u}}^n_N - \boldsymbol{u}^n_N}{\Delta t} \cdot \boldsymbol{v} d\boldsymbol{x}\right| \leqslant C. \tag{5.97}$$

Combining (5.96) and (5.97) with (5.94) and (5.95) establishes Property B. □

This completes the proof of Proposition 5.7.3.

5.8 Passing to the limit

With the convergences that we have established, we now pass to the limit in the semidiscrete formulation. The main difficulty in passing to the limit will be the test functions for the fluid velocity. In particular, on the fixed reference domain Ω_f for the fluid, we note that the test functions for the fluid velocity in (5.45) satisfy $\nabla^\omega_f \cdot \boldsymbol{v} = 0$ on Ω_f, where ω is the solution for the plate displacement. However, the test functions for the fluid velocity in the semidiscrete formulation in the semidiscrete test space $\mathcal{Q}^{n+1}_N$, defined by (5.56), satisfy $\nabla^{\omega^n_N} \cdot \boldsymbol{v} = 0$ on Ω_f. Hence, we need a way of comparing test functions in $\mathcal{Q}^{n+1}_N$ to test functions in the actual test space $\mathcal{V}^\omega_{\text{test}}$.

To do this, recall that we have defined the maximal domain Ω^M_f that contains all of the numerical fluid domains $\Omega^n_{f,N}$. Note that the maximal domain Ω^M_f is fixed in time. Hence, we can consider the following test space $\mathcal{X}$, which consists of functions $\boldsymbol{v}$ defined on $C^1_c([0,T); H^1(\Omega^M_f))$, satisfying the following properties for each $t \in [0,T)$:

1. For each $t \in [0,T)$, $\boldsymbol{v}(t)$ is a smooth vector-valued function on Ω^M_f.
2. $\nabla \cdot \boldsymbol{v}(t) = 0$ on Ω^M_f for all $t \in [0,T)$.
3. $\boldsymbol{v}(t) = 0$ on $\partial\Omega^M_f \backslash \Gamma_M$ for all $t \in [0,T)$, where $\Gamma_M = \{(x, M(x)) : 0 \leqslant x \leqslant L\}$ is the top boundary of the maximal fluid domain Ω^M_f.

We note that restricting functions in $\mathcal{X}$ to the physical domain defined by the plate displacement ω and composing with the ALE mapping Φ^ω_f defined in (5.21) gives a space of test functions $\mathcal{X}^\omega_f$ that is dense in $\mathcal{V}^\omega_f$, which is the fluid velocity component of the full test space (5.45). We emphasize that in the definition of the full test space $\mathcal{V}^\omega_{\text{test}}$ in (5.45), the only component of the test space whose definition depends on the plate displacement is the fluid velocity, and fortunately, this fluid velocity component of the test space is *decoupled from the other components of the test space.*

Then, given a function $\boldsymbol{v} \in \mathcal{X}$, we can construct a fluid velocity that is in the limiting test space $\mathcal{V}^\omega_f$ and the semidiscrete test space $\mathcal{Q}^n_N$. We transfer the function by the ALE mapping, and define

$$\tilde{\boldsymbol{v}}(t,\cdot) = \boldsymbol{v}(t,\cdot)|_{\Omega^\omega_f(t)} \circ \Phi^\omega_f(t,\cdot),$$

where $\Omega_f^\omega(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega(t,x)\}$. To construct approximate test functions that are admissible in the semidiscrete formulation, we define

$$\tilde{\boldsymbol{v}}_N(t,\cdot) = \boldsymbol{v}(t,\cdot)|_{\Omega_f^{\omega_N}(t)} \circ \Phi_f^{\omega_N}(t,\cdot),$$

where we analogously define $\Omega_f^{\omega_N}(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega_N^n(x)\}$. We note that $\tilde{\boldsymbol{v}}_N \in Q_N^n$ on $[n\Delta t, (n+1)\Delta t)$. Therefore, we can use the test function $\tilde{\boldsymbol{v}}_N$ in the semidiscrete formulation on the time interval $[n\Delta t, (n+1)\Delta t)$. We emphasize that $\tilde{\boldsymbol{v}}_N$ is discontinuous at time, due to the jumps in ω_N at each $n\Delta t$. We have the following convergence result, see Lemma 7.1 in [140] and Lemma 2.8 in [30], which shows that the test functions for the semidiscretized problem $\tilde{v}_N$ given $\boldsymbol{v} \in \mathcal{X}$ converge to the test function $\tilde{\boldsymbol{v}} \in \mathcal{X}_f^\omega$ for the limiting weak formulation.

Proposition 5.8.1. For $\boldsymbol{v} \in \mathcal{X}$, $\tilde{\boldsymbol{v}}_N \to \tilde{\boldsymbol{v}}$, and $\nabla\tilde{\boldsymbol{v}}_N \to \nabla\tilde{\boldsymbol{v}}$ pointwise uniformly on $[0,T]\times\overline{\Omega_f}$, as $N \to \infty$.

We use this convergence result to pass to the limit in the semidiscrete formulation. We have from (5.63) that for all $(\tilde{\boldsymbol{v}}_N, \varphi, \boldsymbol{\psi}, r)$ in the test space with $\boldsymbol{v} \in \mathcal{X}$,

$$\begin{aligned}
&\int_0^T\int_{\Omega_f}\left(1+\frac{\tau_{\Delta t}\omega_N}{R}\right)\partial_t\overline{\boldsymbol{u}}_N\cdot\tilde{\boldsymbol{v}}_N + \frac{1}{2}\int_0^T\int_{\Omega_f}\left(1+\frac{\tau_{\Delta t}\omega_N}{R}\right)\Bigg[\left(\left(\tau_{\Delta t}\boldsymbol{u}_N - \zeta_N\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\tau_{\Delta t}\omega_N}\boldsymbol{u}_N\right)\cdot\tilde{\boldsymbol{v}}_N \\
&\quad - \left(\left(\tau_{\Delta t}\boldsymbol{u}_N - \zeta_N\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\tau_{\Delta t}\omega_N}\tilde{\boldsymbol{v}}_N\right)\cdot\boldsymbol{u}_N\Bigg] + \frac{1}{2R}\int_0^T\int_{\Omega_f}\zeta_N\boldsymbol{u}_N\cdot\tilde{\boldsymbol{v}}_N \\
&+ \frac{1}{2}\int_0^T\int_\Gamma(\boldsymbol{u}_N - \zeta_N^*\boldsymbol{e}_y)\cdot\boldsymbol{n}^{\tau_{\Delta t}\omega_N}(\tau_{\Delta t}\boldsymbol{u}_N\cdot\tilde{\boldsymbol{v}}_N) + 2\nu\int_0^T\int_{\Omega_f}\left(1+\frac{\tau_{\Delta t}\omega_N}{R}\right)\boldsymbol{D}_f^{\tau_{\Delta t}\omega_N}(\boldsymbol{u}_N):\boldsymbol{D}_f^{\tau_{\Delta t}\omega_N}(\tilde{\boldsymbol{v}}_N) \\
&+ \int_0^T\int_\Gamma\left(\frac{1}{2}\boldsymbol{u}_N\cdot\tau_{\Delta t}\boldsymbol{u}_N - p_N\right)(\boldsymbol{\psi}-\tilde{\boldsymbol{v}}_N)\cdot\boldsymbol{n}^{\tau_{\Delta t}\omega_N} + \frac{\beta}{\mathcal{J}_\Gamma^{\tau_{\Delta t}\omega_N}}\int_0^T\int_\Gamma(\zeta_N^*\boldsymbol{e}_y - \boldsymbol{u}_N)\cdot\boldsymbol{\tau}^{\tau_{\Delta t}\omega_N}(\boldsymbol{\psi}-\tilde{\boldsymbol{v}}_N)\cdot\boldsymbol{\tau}^{\tau_{\Delta t}\omega_N} \\
&\quad + \rho_b\int_0^T\int_{\Omega_b}\left(\frac{\boldsymbol{\xi}_N - \tau_{\Delta t}\boldsymbol{\xi}_N}{\Delta t}\right)\cdot\boldsymbol{\psi} + \rho_p\int_0^T\int_\Gamma\partial_t\overline{\zeta}_N\cdot\varphi + 2\mu_e\int_0^T\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}_N):\boldsymbol{D}(\boldsymbol{\psi}) \\
&\quad + \lambda_e\int_0^T\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta}_N)(\nabla\cdot\boldsymbol{\psi}) + 2\mu_v\int_0^T\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\xi}_N):\boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v\int_0^T\int_{\Omega_b}(\nabla\cdot\boldsymbol{\xi}_N)(\nabla\cdot\boldsymbol{\psi}) \\
&\quad - \alpha\int_0^T\int_{\Omega_b}\mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta}p_N\nabla_b^{(\tau_{\Delta t}\eta_N)^\delta}\cdot\boldsymbol{\psi} + c_0\int_0^T\int_{\Omega_b}\partial_t\overline{p}_N\cdot r - \alpha\int_0^T\int_{\Omega_b}\mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta}\boldsymbol{\xi}_N\cdot\nabla_b^{(\tau_{\Delta t}\eta_N)^\delta}r \\
&\quad - \alpha\int_0^T\int_\Gamma(\zeta_N^*\boldsymbol{e}_y\cdot\boldsymbol{n}^{(\tau_{\Delta t}\omega_N)^\delta})r + \kappa\int_0^T\int_{\Omega_b}\mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta}\nabla_b^{(\tau_{\Delta t}\eta_N)^\delta}p_N\cdot\nabla_b^{(\tau_{\Delta t}\eta_N)^\delta}r \\
&\quad - \int_0^T\int_\Gamma[(\boldsymbol{u}_N - \zeta_N^*\boldsymbol{e}_y)\cdot\boldsymbol{n}^{\tau_{\Delta t}\omega_N}]r + \int_0^T\int_\Gamma\Delta\omega_N\cdot\Delta\varphi = 0.
\end{aligned}$$

We summarize the main strong convergences that we have obtained.

$$\boldsymbol{\eta}_N \to \boldsymbol{\eta}, \quad \text{in } C(0,T;L^2(\Omega_b)),$$

$$\omega_N \to \omega, \quad \text{in } L^\infty(0,T;H^s(\Gamma)) \text{ for } 0 < s < 2,$$

$$\zeta_N^* \to \zeta, \quad \text{in } L^2(0,T;H^{-s}(\Gamma)), \text{ for } -1/2 < s < 0,$$
$$\zeta_N \to \zeta, \quad \text{in } L^2(0,T;H^{-s}(\Gamma)), \text{ for } -1/2 < s < 0,$$
$$\boldsymbol{\xi}_N \to \boldsymbol{\xi}, \quad \text{in } L^2(0,T;H^{-s}(\Omega_b)), \text{ for } -1/2 < s < 0,$$
$$\boldsymbol{u}_N \to \boldsymbol{u}, \quad \text{in } L^2(0,T;L^2(\Omega_f^M)), \qquad p_N \to p, \quad \text{in } L^2(0,T;L^2(\Omega_b)),$$

where ζ_N^* and ζ_N converge to the same limit in $L^2(0,T;H^{-s}(\Gamma))$ for $-1/2 < s < 0$ due to the numerical dissipation estimates $\sum_{n=1}^N ||\zeta_N^n - \zeta_N^{n-\frac{1}{2}}||^2_{L^2(\Gamma)} \leqslant C$, which imply that $||\zeta_N - \zeta_N^*||_{L^2(0,T;L^2(\Gamma))} \to 0$.

Due to the presence of terms in the weak formulation involving the trace of the fluid velocity along Γ, we will also need stronger convergence results for the trace of the fluid velocities along Γ. In particular, we have the following convergence result.

Proposition 5.8.2. We have that

$$\hat{\boldsymbol{u}}_N|_\Gamma \to \hat{\boldsymbol{u}}|_\Gamma, \qquad \text{in } L^2(0,T;H^{s-\frac{1}{2}}(\Gamma)), \qquad \text{for } s \in (0,1),$$

where $\hat{\boldsymbol{u}}_N = \boldsymbol{u}_N \circ \Phi_f^{\tau_{\Delta t}\omega_N}$ and $\hat{\boldsymbol{u}} = \boldsymbol{u} \circ \Phi_f^\omega$.

To prove Proposition 5.8.2, we will use the following elementary lemma.

Lemma 5.8.1. Suppose that the functions $\{f_n\}_{n=1}^\infty$ and f are all uniformly bounded in $L^2(0,T;H^1(\Omega_f))$ and $f_n \to f$ in $L^2(0,T;L^2(\Omega_f))$. Then, $f_n \to f$ in $L^2(0,T;H^s(\Omega_f))$ and hence $f_n|_\Gamma \to f|_\Gamma$ in $L^2(0,T;H^{s-\frac{1}{2}}(\Gamma))$ for $s \in (0,1)$.

Proof of Lemma 5.8.1. For $s \in (0,1)$, we compute using the trace lemma and Sobolev interpolation that

$$||f_n|_\Gamma - f|_\Gamma||^2_{L^2(0,T;H^{s-\frac{1}{2}}(\Gamma))} \leqslant ||f_n - f||^2_{L^2(0,T;H^s(\Omega_f))} = \int_0^T ||(f_n - f)(t)||^2_{H^s(\Omega_f)} dt$$
$$\leqslant \int_0^T ||(f_n - f)(t)||^{2(1-s)}_{L^2(\Omega_f)} \cdot ||(f_n - f)(t)||^{2s}_{H^1(\Omega_f)} dt \leqslant ||f_n - f||^{2(1-s)}_{L^2(0,T;L^2(\Omega_f))} \cdot ||f_n - f||^{2s}_{L^2(0,T;H^1(\Omega_f))}.$$

The result then follows from the fact that $||f_n - f||_{L^2(0,T;H^1(\Omega_f))} \leqslant C$ for a constant C that does not depend on N and the assumption that $||f_n - f||_{L^2(0,T;L^2(\Omega_f))} \to 0$ as $N \to \infty$. □

We can use the elementary lemma above to show the desired strong convergence of the fluid velocity traces.

Proof of Proposition 5.8.2. We have that $\boldsymbol{u}_N \to \boldsymbol{u}$ in $L^2(0,T;L^2(\Omega_f^M))$ on the physical maximal fluid domain and we want to combine this with the fact that $\boldsymbol{u}_N$ for all N and $\boldsymbol{u}$ are all uniformly bounded in $L^2(0,T;H^1(\Omega_f(t)))$ in order to deduce strong convergence of the trace of the fluid velocities using the previous elementary lemma. We do this in the following steps.

Step 1. Consider the fluid velocities $\hat{\boldsymbol{u}}_N$ and $\hat{\boldsymbol{u}}$ defined on the reference fluid domain. We claim that $\hat{\boldsymbol{u}}_N \to \hat{\boldsymbol{u}}$ on $L^2(0,T;L^2(\Omega_f))$. To show this, we recall that the original functions $\boldsymbol{u}_N$ and $\boldsymbol{u}$ are defined on the maximal domain Ω_f^M and we compute that

$$||\hat{\boldsymbol{u}}_N - \hat{\boldsymbol{u}}||^2_{L^2(0,T;L^2(\Omega_f))} = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}_N\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\omega\right)\right|^2 \leqslant 2(I_1+I_2),$$

where

$$I_1 = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}_N\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right)\right|^2,$$

$$I_2 = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\omega\right)\right|^2.$$

Since $1+\frac{\omega_N^n}{R}$ is uniformly bounded from above by a positive constant, we have that

$$I_1 = \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\left(1+\frac{\omega_N^n}{R}\right)\int_{\Omega_{f,N}^n}|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}|^2 \leqslant C\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Omega_{f,N}^n}|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}|^2 \leqslant C||\boldsymbol{u}_N-\boldsymbol{u}||^2_{L^2(0,T;L^2(\Omega_f^M))} \to 0,$$

as $N\to\infty$, since Ω_f^M contains all of the domains $\Omega_{f,N}^n$. For I_2, we break up the integral into two parts:

$$I_2 = I_{2,1} + I_{2,2},$$

where

$$I_{2,1} = \int_0^T\int_0^L\int_{-R}^{\min(0,y^*(t,x))}\left|\boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\omega\right)\right|^2,$$

$$I_{2,2} = \int_0^T\int_0^L\int_{\min(0,y^*(t,x))}^0\left|\boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right)\right|^2,$$

for $y^*(t,x) = \frac{\omega-\tau_{\Delta t}\omega_N}{R+\tau_{\Delta t}\omega_N}$. We can interpret $y^*(t,x)$ as the y value for which $y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N = \omega$. Note that

$$I_{2,1} \leqslant \int_0^T\int_0^L\int_{-R}^{\min(0,y^*(t,x))}\left(\int_{y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N}^{y+\left(1+\frac{y}{R}\right)\omega}|\partial_y\boldsymbol{u}(t,x,y')|dy'\right)^2$$
$$\leqslant \int_0^T\int_0^L\int_{-R}^{\min(0,y^*(t,x))}\left(\int_{y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N}^{y+\left(1+\frac{y}{R}\right)\omega}|\partial_r\boldsymbol{u}(t,x,y')|^2dy'\right)\cdot\left(1+\frac{y}{R}\right)\cdot|\omega-\tau_{\Delta t}\omega_N|.$$

By Proposition 5.7.1, $\overline{\omega}_N \to \omega$ in $C(0,T;H^s(\Gamma))$ for $0<s<2$, and combining this with the estimate (5.80), we obtain that $\tau_{\Delta t}\omega_N \to \omega$ pointwise uniformly on $[0,T]\times\Gamma$ as $N\to\infty$. Combining this with the fact that $||\nabla \boldsymbol{u}||_{L^2(0,T;L^2(\Omega_f^\omega(t)))}$ is bounded, we have that $I_{2,1}\to 0$ as $N\to\infty$.

Next, by Poincare's inequality,

$$I_{2,2} \leqslant \int_0^T \int_0^L |\min(0,y^*(t,x))| \cdot \max_{w\in[-R,\omega(t,x)]} |\boldsymbol{u}(t,x,w)|^2 \leqslant \int_0^T \int_0^L |\min(0,y^*(t,x))| \cdot \int_{-R}^{\omega(t,x)} |\partial_r \boldsymbol{u}(t,x,y')|^2 dy',$$

so we conclude that $I_{2,2}\to 0$ as $N\to\infty$ by the fact that $|\min(0,y^*(t,x))|\to 0$ uniformly on $[0,T]\times\Gamma$, and by the boundedness of $||\nabla\boldsymbol{u}||_{L^2(0,T;L^2(\Omega_f^\omega(t))}$. Thus, we have that $||\hat{\boldsymbol{u}}_N - \hat{\boldsymbol{u}}||_{L^2(0,T;L^2(\Omega_f)))} \to 0$.

Step 2. We claim that the functions $\hat{\boldsymbol{u}}_N$ for positive integers N and $\hat{\boldsymbol{u}}$ are all uniformly bounded in $L^2(0,T;H^1(\Omega_f))$. Recall from Lemma 5.6.1 that the approximate solutions $\hat{\boldsymbol{u}}_N$ are uniformly bounded in $L^2(0,T;H^1(\Omega_f))$. Since $\hat{\boldsymbol{u}}$ is the strong limit of $\hat{\boldsymbol{u}}_N$ in $L^2(0,T;L^2(\Omega_f))$ and $\hat{\boldsymbol{u}}_N$ converge weakly in $L^2(0,T;H^1(\Omega_f))$ along a subsequence to a weak limit which hence must also be $\hat{\boldsymbol{u}}$, we conclude that $\hat{\boldsymbol{u}}$ is also in $L^2(0,T;H^1(\Omega_f))$, which establishes the desired result of this step.

Step 3. From Step 1, we have that $\hat{\boldsymbol{u}}_N \to \hat{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega_f))$ and from Step 2, the functions $\hat{\boldsymbol{u}}_N$ and $\hat{\boldsymbol{u}}$ are bounded in $L^2(0,T;H^1(\Omega_f))$ independently of N, so we can conclude the proof of Proposition 5.8.2 by using Lemma 5.8.1. □

Using these strong convergences, in addition to the previously established weak convergences in Proposition 5.6.1, we can pass to the limit in all of the terms in the semidiscrete weak formulation except those involving time derivatives. However, we can handle these by a discrete integration by parts. For example, for the first integral, we can use a discrete integration by parts to obtain that

$$\int_0^T \int_{\Omega_f} \left(1+\frac{\tau_{\Delta t}\omega_N}{R}\right)\partial_t \overline{\boldsymbol{u}}_N \cdot \tilde{\boldsymbol{v}}_N \to -\int_0^T\int_{\Omega_f}\left(1+\frac{\omega}{R}\right)\boldsymbol{u}\cdot\partial_t\tilde{\boldsymbol{v}} - \frac{1}{R}\int_0^T\int_{\Omega_f}(\partial_t\omega)\boldsymbol{u}\cdot\tilde{\boldsymbol{v}} - \int_{\Omega_f}\left(1+\frac{\omega_0}{R}\right)\boldsymbol{u}(0)\cdot\tilde{\boldsymbol{v}}(0),$$

where $\tilde{\boldsymbol{v}}_N = \boldsymbol{v}\circ\Phi_f^{\tau_{\Delta t}\omega_N}$ and $\tilde{\boldsymbol{v}} = \boldsymbol{v}\circ\Phi_f^\omega$ for $\boldsymbol{v}\in\mathcal{X}$. See for example pg. 79-81 in [30].

Thus, we conclude that the weak formulation holds for all test functions in the smooth test space $\mathcal{X}_f^\omega$ for the fluid velocities, which consists of all test functions of the form $\tilde{\boldsymbol{v}} = \boldsymbol{v}\circ\Phi_f^\omega$ for $\boldsymbol{v}\in\mathcal{X}$. We can extend to the more general test space $\mathcal{V}_{\text{test}}^\omega$ defined in (5.45) by using a density argument. This completes the proof of the existence of a weak solution to the regularized nonlinearly coupled FPSI problem.

We conclude this section by making the important observation that the weak solution that we have constructed to the regularized FPSI problem satisfies the desired energy estimate. This will be important for showing weak-classical consistency in the next section, and can be shown easily by using the discrete energy estimate for the approximate solutions.

Proposition 5.8.3. The weak solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ constructed from the splitting scheme as the limit of approximate solutions satisfies the following energy estimate

$$\frac{1}{2}\int_{\Omega_f(t)} |\boldsymbol{u}|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 \\ + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}|^2 + \frac{1}{2}\rho_p \int_{\Gamma} |\zeta|^2 + \frac{1}{2}\int_{\Gamma} |\Delta \omega|^2 + 2\nu \int_0^t \int_{\Omega_f(s)} |\boldsymbol{D}(\boldsymbol{u})|^2 \\ + 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi})|^2 + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}|^2 + \kappa \int_0^t \int_{\Omega_b^\delta(s)} |\nabla p|^2 + \beta \int_0^t \int_{\Gamma(s)} |(\zeta \boldsymbol{e}_y - \boldsymbol{u}) \cdot \boldsymbol{\tau})|^2 \leqslant E_0, \tag{5.98}$$

for almost every $t \in [0, T]$.

Proof. The approximate solutions $(\boldsymbol{u}_N, \boldsymbol{\eta}_N, p_N, \omega_N)$ satisfy the following energy inequality:

$$\frac{1}{2}\int_{\Omega_{f,N}(t)} |\boldsymbol{u}_N|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}_N|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_N)|^2 + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_N|^2 \\ + \frac{1}{2}\rho_p \int_{\Gamma} |\zeta_N|^2 + \frac{1}{2}\int_{\Gamma} |\Delta \omega_N|^2 + 2\nu \int_0^t \int_{\Omega_{f,N}(s)} |\boldsymbol{D}(\boldsymbol{u}_N)|^2 + 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_N)|^2 \\ + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}_N|^2 + \kappa \int_0^t \int_{\Omega_{b,N}^\delta(s)} |\nabla p_N|^2 + \beta \int_0^t \int_{\Gamma(s)} |(\zeta_N^* \boldsymbol{e}_y - \boldsymbol{u}_N) \cdot \boldsymbol{\tau}|^2 \leqslant E_0,$$

where E_0 is the initial energy of the problem. We can then use the weak and weak-star convergences of the approximate solutions, stated in Proposition 5.6.1, and lower semicontinuity in order to pass to the limit in the energy inequality. □

5.9 Weak-classical consistency

Statement of the result and notation

We have now shown the existence of weak solutions to the regularized FPSI problem (5.46). However, it is not clear that the solutions to this regularized problem are physically relevant, since the regularized weak formulation is not equivalent to the original weak formulation without the regularization. However, we will demonstrate a weak-classical consistency result in this section: given a spatially and temporally smooth solution $(\boldsymbol{u}_1, \boldsymbol{\eta}_1, p_1, \omega_1)$ to the FPSI problem, then the weak solutions to the regularized problem with regularization parameter δ, which we will denote by $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$, converge to the smooth solution as $\delta \to 0$.

Recall that the regularized weak formulation uses a spatial convolution with respect to a smooth compactly supported function to regularize the Biot displacement $\boldsymbol{\eta}$. For ease of notation, for the regularized solutions, we will use a more compact notation for the spatial convolution, defined by (5.30). In particular, we will use the new notation in this section, that

$$\tilde{\boldsymbol{\eta}}_{2,\delta} = (\boldsymbol{\eta}_{2,\delta})^{\delta} := \delta^{-2}\boldsymbol{\eta}_{2,\delta} * \sigma(x/\delta).$$

Because we will later have to use the spatial convolution of the strong solution $\boldsymbol{\eta}_1$ with the convolution kernel, we will adopt a similar notation:

$$\tilde{\boldsymbol{\eta}}_1 = (\boldsymbol{\eta}_1)^{\delta} = \delta^{-2}\boldsymbol{\eta}_1 * \sigma(x/\delta).$$

We also use this notation for the domain $\tilde{\Omega}_{b,2,\delta}(t)$, which denotes the physical Biot domain under the regularized displacement. In particular,

$$\tilde{\Omega}_{b,2,\delta}(t) = (\boldsymbol{I} + \tilde{\boldsymbol{\eta}}_{2,\delta}(t))(\Omega_b). \tag{5.99}$$

For convenience, we reproduce the weak formulation and the regularized weak formulations below. Furthermore, we note that even though the weak formulation (5.29) and the regularized weak formulation (5.46) are stated up until a fixed final time T, we can reformulate the weak formulation for almost every time $\tau \in [0, T]$ by using a cutoff function (see for example the proof of Lemma 5.12.2 in the appendix where this is done explicitly). Thus, we have that the classical solution $(\boldsymbol{u}_1, \boldsymbol{\eta}_1, p_1, \omega_1)$ satisfies the following (non-regularized) weak formulation for almost all $\tau \in [0, T]$, for all test functions $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{V}_{\text{test}}$ with the (moving domain) test space $\mathcal{V}_{\text{test}}$ defined in (5.44):

$$\begin{aligned}
&-\int_0^\tau\int_{\Omega_{f,1}(t)} \boldsymbol{u}_1\cdot\partial_t\boldsymbol{v} + \frac{1}{2}\int_0^\tau\int_{\Omega_{f,1}(t)}[((\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1)\cdot\boldsymbol{v} - ((\boldsymbol{u}_1\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{u}_1] + \frac{1}{2}\int_0^\tau\int_{\Gamma_1(t)}(\boldsymbol{u}_1\cdot\boldsymbol{n} - 2\boldsymbol{\xi}_1\cdot\boldsymbol{n})\boldsymbol{u}_1\cdot\boldsymbol{v}\\
&+ 2\nu\int_0^\tau\int_{\Omega_{f,1}(t)}\boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{v}) + \int_0^\tau\int_{\Gamma_1(t)}\left(\frac{1}{2}|\boldsymbol{u}_1|^2 - p_1\right)(\psi_n - v_n) + \beta\int_0^\tau\int_{\Gamma_1(t)}(\boldsymbol{\xi}_1 - \boldsymbol{u}_1)\cdot\boldsymbol{\tau}(\psi_\tau - v_\tau)\\
&\quad - \rho_p\int_0^\tau\int_\Gamma\partial_t\omega_1\cdot\partial_t\varphi + \int_0^\tau\int_\Gamma\Delta\omega_1\cdot\Delta\varphi - \rho_b\int_0^\tau\int_{\Omega_b}\partial_t\boldsymbol{\eta}_1\cdot\partial_t\boldsymbol{\psi} + 2\mu_e\int_0^\tau\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}_1):\boldsymbol{D}(\boldsymbol{\psi})\\
&\quad + \lambda_e\int_0^\tau\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta}_1)(\nabla\cdot\boldsymbol{\psi}) + 2\mu_v\int_0^\tau\int_{\Omega_b}\boldsymbol{D}(\partial_t\boldsymbol{\eta}_1):\boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v\int_0^\tau\int_{\Omega_b}(\nabla\cdot\partial_t\boldsymbol{\eta}_1)(\nabla\cdot\boldsymbol{\psi})\\
&\quad - \alpha\int_0^\tau\int_{\Omega_b(t)}p_1\nabla\cdot\boldsymbol{\psi} - c_0\int_0^\tau\int_{\Omega_b}p_1\partial_t r - \alpha\int_0^\tau\int_{\Omega_b(t)}\frac{D}{Dt}\boldsymbol{\eta}_1\cdot\nabla r - \alpha\int_0^\tau\int_{\Gamma(t)}(\boldsymbol{\xi}_1\cdot\boldsymbol{n})r\\
&\qquad + \kappa\int_0^\tau\int_{\Omega_b(\tau)}\nabla p_1\cdot\nabla r - \int_0^\tau\int_{\Gamma_1(t)}((\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot\boldsymbol{n})r\\
&\quad = -\int_{\Omega_{f,1}(t)}\boldsymbol{u}_1(\tau)\cdot\boldsymbol{v}(\tau) - \rho_p\int_\Gamma\beta_1(\tau)\cdot\boldsymbol{\psi}(\tau) - \rho_b\int_{\Omega_b}\boldsymbol{\xi}_1(\tau)\cdot\boldsymbol{\psi}(\tau) - c_0\int_{\Omega_b}p_1(\tau)\cdot r(\tau)\\
&\qquad + \int_{\Omega_f(0)}\boldsymbol{u}_0\cdot\boldsymbol{v}(0) + \rho_p\int_\Gamma\beta_0\cdot\boldsymbol{\psi}(0) + \rho_b\int_{\Omega_b}\boldsymbol{\xi}_0\cdot\boldsymbol{\psi}(0) + c_0\int_{\Omega_b}p_0\cdot r(0).
\end{aligned}\tag{5.100}$$

We also have that for each $\delta > 0$ and for almost every $\tau \in [0, T_\delta]$ where the final time T_δ potentially depends on δ, the corresponding solution $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\psi}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$ to the regularized FPSI problem with regularization parameter δ satisfies the following regularized weak formulation for every test function $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{V}_{\text{test}}$:

$$
\begin{aligned}
&-\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [((\boldsymbol{u}_{2,\delta} \cdot \nabla)\boldsymbol{u}_{2,\delta}) \cdot \boldsymbol{v} - ((\boldsymbol{u}_{2,\delta} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}_{2,\delta}] + \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta} \cdot \boldsymbol{n} - 2\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n})\boldsymbol{u}_{2,\delta} \cdot \boldsymbol{v} \\
&+ 2\nu \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} \left(\frac{1}{2}|\boldsymbol{u}_{2,\delta}|^2 - p_{2,\delta}\right)(\psi_n - v_n) + \beta \int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} - \boldsymbol{u}_{2,\delta}) \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) \\
&- \rho_p \int_0^\tau \int_\Gamma \partial_t \omega_{2,\delta} \cdot \partial_t \varphi + \int_0^\tau \int_\Gamma \Delta\omega_{2,\delta} \cdot \Delta\varphi - \rho_b \int_0^\tau \int_{\Omega_b} \partial_t \boldsymbol{\eta}_{2,\delta} \cdot \partial_t \boldsymbol{\psi} + 2\mu_e \int_0^\tau \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{\psi}) \\
&+ \lambda_e \int_0^\tau \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_{2,\delta})(\nabla \cdot \boldsymbol{\psi}) + 2\mu_v \int_0^\tau \int_{\Omega_b} \boldsymbol{D}(\partial_t \boldsymbol{\eta}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_0^\tau \int_{\Omega_b} (\nabla \cdot \partial_t \boldsymbol{\eta}_{2,\delta})(\nabla \cdot \boldsymbol{\psi}) \\
&- \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} p_{2,\delta} \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^\tau \int_{\Omega_b} p_{2,\delta} \partial_t r - \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \frac{D^\delta}{Dt} \boldsymbol{\eta}_{2,\delta} \cdot \nabla r - \alpha \int_0^\tau \int_{\tilde{\Gamma}_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}) r \\
&+ \kappa \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \nabla p_{2,\delta} \cdot \nabla r - \int_0^\tau \int_{\Gamma_{2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta}) \cdot \boldsymbol{n}) r \\
&= -\int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau) \cdot \boldsymbol{v}(\tau) - \rho_p \int_\Gamma \beta_{2,\delta}(\tau) \cdot \varphi(\tau) - \rho_b \int_{\Omega_b} \boldsymbol{\xi}_{2,\delta}(\tau) \cdot \boldsymbol{\psi}(\tau) - c_0 \int_{\Omega_b} p_{2,\delta}(\tau) \cdot r(\tau) \\
&+ \int_{\Omega_f(0)} \boldsymbol{u}_0 \cdot \boldsymbol{v}(0) + \rho_p \int_\Gamma \beta_0 \cdot \varphi(0) + \rho_b \int_{\Omega_b} \boldsymbol{\xi}_0 \cdot \boldsymbol{\psi}(0) + c_0 \int_{\Omega_b} p_0 \cdot r(0), \qquad (5.101)
\end{aligned}
$$

where $\frac{D^\delta}{Dt}$ is the material derivative with respect to the regularized Biot displacement. We remark that while our existence proof in the previous sections holds for both a purely elastic and viscoelastic Biot medium, our weak-classical consistency result will hold in the specific case of a Biot *poroviscoelastic* medium so that the viscoelasticity parameters μ_v and λ_v are strictly positive, and hence, the plate velocity $\zeta_{2,\delta}\boldsymbol{e}_y$ in the weak formulation is equivalently the trace of the Biot medium velocity $\boldsymbol{\xi}_{2,\delta} \in L^2(0,T; H^1(\Omega_b))$ along Γ. We need viscoelasticity in the Biot medium because we will need to estimate certain terms involving the trace of the Biot velocity along the moving interface, which will require having extra spatial regularity on the Biot velocity defined on Ω_b.

In the remainder of the manuscript, we will prove the weak-classical consistency result, which we will state at the end of this subsection. However, before stating the result, we need to introduce some additional notation. To motivate why we need this notation, we note that to prove this weak-classical consistency, we will subtract the weak formulations for the two solutions and test formally with the difference of the two solutions. Hence, for the fluid part of the weak formulation, we formally want to test with a fluid test function $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}$. However, the functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_{2,\delta}$ are defined on different domains, and hence, the difference $\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}$ is not well-defined. Therefore, we will have to use a transformation to bring a divergence-free function defined on one fluid domain to a divergence-free function on another fluid domain.

Consider the two fluid domains

$$\Omega_{f,1}(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega_1(t,x)\},$$

$$\Omega_{f,2,\delta}(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega_{2,\delta}(t,x)\},$$

that are associated to the plate displacements ω_1 and $\omega_{2,\delta}$. We define a map between $\Omega_{f,1}(t)$ and $\Omega_{f,2,\delta}(t)$, and a transformation that sends functions on one domain to functions on the other domain as follows. We denote

$$\gamma_\delta(t,x) = \frac{R+\omega_1(t,x)}{R+\omega_{2,\delta}(t,x)}, \tag{5.102}$$

and we define the map $\psi_\delta(t) : \Omega_{f,2,\delta}(t) \to \Omega_{f,1}(t)$ by

$$\psi_\delta(t,x,y) = (t, x, \gamma_\delta(t,x)(R+y) - R). \tag{5.103}$$

However, we cannot use ψ_δ to move fluid velocity functions from one fluid domain to the other because of the fact that we want the fluid velocity to remain divergence free under such a transformation. Therefore, we define the matrices

$$J_\delta(t,x,y) = \begin{pmatrix} 1 & 0 \\ (R+y)\partial_x\gamma_\delta(t,x) & \gamma_\delta(t,x) \end{pmatrix}, \tag{5.104}$$

$$\tilde{J}_\delta = J_\delta \circ \psi_\delta^{-1} = \begin{pmatrix} 1 & 0 \\ (R+y)\gamma_\delta^{-1}\partial_x\gamma_\delta(t,x) & \gamma_\delta(t,x) \end{pmatrix}. \tag{5.105}$$

Note that the Jacobian matrix J_δ defined in (5.104) is associated with a change of variables under the map ψ_δ of the gradient of functions, since for a function $\boldsymbol{u}$ defined on $\Omega_{f,1}$,

$$\nabla(\boldsymbol{u} \circ \psi_\delta) = [(\nabla \boldsymbol{u}) \circ \psi_\delta] J_\delta. \tag{5.106}$$

Given a divergence-free function $\boldsymbol{u}_1$ on $\Omega_{f,1}(t)$, we can define a function $\widehat{\boldsymbol{u}}_1$ on $\Omega_{f,2,\delta}(t)$ by

$$\widehat{\boldsymbol{u}}_1 = \gamma_\delta J_\delta^{-1} \cdot (\boldsymbol{u}_1 \circ \psi_\delta), \tag{5.107}$$

and given a divergence-free function $\boldsymbol{u}_{2,\delta}$ defined on $\Omega_{f,2,\delta}(t)$, we can define a function $\check{\boldsymbol{u}}_{2,\delta}$ on $\Omega_{f,1}(t)$ by

$$\check{\boldsymbol{u}}_{2,\delta} = \gamma_\delta^{-1} \tilde{J}_\delta \cdot (\boldsymbol{u}_{2,\delta} \circ \psi_\delta^{-1}). \tag{5.108}$$

We remark that even though the definition of $\widehat{\boldsymbol{u}}_1$ depends on δ, we will not explicitly notate this dependence, as δ will be clear from the context, since we will be considering $\boldsymbol{u}_1$ and $\boldsymbol{u}_{2,\delta}$ together for a specific but arbitrary choice of δ whenever this notation appears. The resulting functions are both also divergence free on their respective fluid domains. Note further that both of these transformations preserve the trace of the function along Γ also.

We now state the weak-classical consistency result.

Theorem 5.9.1. Consider smooth initial data $(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0, p_0, \omega_0, \beta_0, \boldsymbol{u}_0)$ to the given nonlinearly coupled FPSI problem. Suppose $(\boldsymbol{\eta}_1, \omega_1, p_1, \boldsymbol{u}_1)$ is a classical solution to the given FPSI problem on the time interval $[0,T]$ with this initial data, that is smooth in space and time. Then, if $(\boldsymbol{\eta}_{2,\delta}, \omega_{2,\delta}, p_{2,\delta}, \boldsymbol{u}_{2,\delta})$ denotes the weak solution to the regularized FPSI problem with regularity parameter δ, then $(\boldsymbol{\eta}_{2,\delta}, \omega_{2,\delta}, p_{2,\delta}, \boldsymbol{u}_{2,\delta})$ can be uniformly defined on the time interval $[0,T]$ for all $\delta > 0$, and furthermore,

$$E_\delta(t) \to 0, \qquad \text{for all } t \in [0,T], \qquad \text{as } \delta \to 0,$$

where

$$\begin{aligned} E_\delta(t) := &||(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})(t)||^2_{L^2(\Omega_{f,2,\delta}(t))} + \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})(s)||^2_{L^2(\Omega_{f,2,\delta}(s))} ds \\ &+ ||(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})(t)||^2_{L^2(\Gamma)} + ||(\omega_1 - \omega_{2,\delta})(t)||^2_{H^2(\Gamma)} + ||(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})(t)||^2_{L^2(\Omega_b)} \\ &+ ||\boldsymbol{D}(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta})(t)||^2_{L^2(\Omega_b)} + ||(\nabla \cdot (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta}))(t)||_{L^2(\Omega_b)} + \int_0^t ||\boldsymbol{D}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})(s)||^2_{L^2(\Omega_b)} ds \\ &+ \int_0^t ||\nabla \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})(s)||_{L^2(\Omega_b)} + ||(p_1 - p_{2,\delta})(t)||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla (p_1 - p_{2,\delta})(s)||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(s))}. \end{aligned} \tag{5.109}$$

The general strategy

We want to estimate the energy difference between $(\boldsymbol{u}_1, \boldsymbol{\eta}_1, p_1, \omega_1)$ and $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$, defined in (5.109). We want to obtain an estimate for $E_\delta(t)$ in terms of $E_\delta(0)$, the integral of $E_\delta(s)$ for times $s \in [0,t]$, and other terms that have sufficiently strong convergence in δ as $\delta \to 0$ (see Lemma 5.9.1 in the next subsection). To do this, we will test the weak formulations for $\boldsymbol{u}_1$ and $\boldsymbol{u}_{2,\delta}$ with appropriate test functions and use the energy inequality, as described formally in the following steps:

1. Since $(\boldsymbol{u}_1, \boldsymbol{\eta}_1, p_1, \omega_1)$ is a classical solution to the non-regularized FPSI problem, we test the non-regularized weak formulation with the "difference" of $(\boldsymbol{u}_1, \partial_t \boldsymbol{\eta}_1, p_1, \partial_t \omega_1)$ and $(\boldsymbol{u}_{2,\delta}, \partial_t \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \partial_t \omega_{2,\delta})$, where this notion of the difference between these two solutions will be made precise in Section 5.9.

2. We test the weak formulation for $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$ with $(\boldsymbol{u}_1, \partial_t \boldsymbol{\eta}_1, p_1, \partial_t \omega_1)$.

3. We rewrite the energy inequality for $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$ so that it parallels the terms in the weak formulation.

4. We combine the equations from Step 1, Step 2, and Step 3. This will give us an expression that we can analyze term by term in order to obtain an estimate for the energy difference $E_\delta(t)$ by Gronwall's inequality.

Most of the proof, carried out in Section 5.10, will involve estimating, term by term, the various quantities that arise from combining the weak formulations in Steps 1 and 2, and the energy estimate in Step 3. As noted in Step 4 above, this will allow us to obtain an inequality for the energy $E_\delta(t)$ that can be used to conclude the proof of Theorem 5.9.1 by an application of Gronwall's inequality. We emphasize that the outline of the proof given above is formal, and needs to be rigorously justified due to the following two mathematical difficulties:

1. The regularized weak formulation involves integrals on the physical time-dependent Biot domain $\tilde{\Omega}_{b,2,\delta}(t)$, which give an extra factor of $\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})$ in the integrand from the Jacobian, when the integrals are transferred to the fixed reference Biot domain Ω_b. This factor cannot be estimated in the finite energy space, where $\boldsymbol{\eta}_{2,\delta}$ is only bounded uniformly in δ in the function space $L^\infty(0, T; H^1(\Omega_b))$. Thus, we need to use a *bootstrap argument* to estimate this determinant, as discussed in Section 5.9.

2. We want to test with the "difference" of $(\boldsymbol{u}_1, \partial_t\boldsymbol{\eta}_1, p_1, \partial_t\omega_1)$ and $(\boldsymbol{u}_{2,\delta}, \partial_t\boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \partial_t\omega_{2,\delta})$. However, this is formal because the test functions in $\mathcal{V}_{\text{test}}$, defined in (5.44), must be continuously differentiable in time, and furthermore, for the fluid velocities, the difference between $\boldsymbol{u}_1$ and $\boldsymbol{u}_{2,\delta}$ does not make sense, since these functions are defined on different fluid domains. Thus, we must carefully define which test functions we will use, see Section 5.9.

Setting up the bootstrap argument

We note that one recurring challenge in showing weak-classical consistency involves the terms in the regularized weak formulation (5.46), which are integrals over $\tilde{\Omega}_{b,2,\delta}(t)$. This is because if we use a change of variables to rewrite these as integrals on the fixed reference domain Ω_b, we obtain an additional factor of $\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})$ in the integrand from the Jacobian.

This factor of $\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})$ is problematic, because although this factor is a smooth function due to the spatial convolution, it is not uniformly bounded in the appropriate function space. In particular, we only have that $\boldsymbol{\eta}_{2,\delta}$ is uniformly bounded in $L^\infty(0, T; H^1(\Omega_b))$ in δ. Therefore, $\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})$ is only uniformly bounded in $L^\infty(0, T; L^1(\Omega_b))$, which is insufficient for estimating any integrands with this factor, since any explicit estimate on this Jacobian incurs a substantial loss in spatial integrability.

Hence, we will use an alternative strategy. In particular, we recall that by the way we constructed the weak solution to the regularized problem by the splitting scheme, we have that there exists a sufficiently small constant c (uniform in δ) such that

$$\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta}) \geqslant c > 0, \qquad \text{for all } t \in [0, T_\delta], \tag{5.110}$$

for $T_\delta > 0$ potentially depending on δ. This estimate holds at least locally (though not locally uniformly) for each $\delta > 0$. However, we would ideally want to show a *stronger result*, which states that

$$\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta}) \geqslant c > 0, \qquad \text{for all } t \in [0, T], \tag{5.111}$$

for some time $T > 0$ that is *independent of* δ.

To do this, the strategy will be to use a bootstrap argument. We note that like the estimate in (5.110), the following three estimates hold locally (though not locally uniformly in δ), with positive constants c and C that are independent of δ:

$$\det(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta}) \geqslant c, \tag{5.112}$$

$$0 < c \leqslant |\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta}| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}, \tag{5.113}$$

$$|\nabla\tilde{\boldsymbol{\eta}}_{2,\delta}| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}. \tag{5.114}$$

We emphasize that the time interval for which these estimates hold may depend on δ. Note that the previous three estimates imply (after potentially making $c > 0$ smaller, if necessary) that

$$0 < C^{-1} \leqslant |(\boldsymbol{I} + \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^{-1}| \leqslant c^{-1}.$$

We can choose c and C so that these estimates also hold for the classical solution $\boldsymbol{\eta}$ up to the time T, where the classical solution exists on the time interval $[0, T]$.

We will use Gronwall's inequality to get an estimate that will hold as long as the assumptions (5.112), (5.113), and (5.114) are valid. The resulting estimate we obtain will formally be of the form

$$E_\delta(t) \leqslant C_1 \int_0^t ||(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(s)||^2_{H^1(\Omega_b)} ds + C_2 \int_0^t E_\delta(s) ds,$$

where the constants C_1 and C_2 are independent of δ and $E_\delta(t)$ is the energy difference between the classical solution and the weak solution to the regularized problem with regularity parameter δ, defined by (5.109).

As we will prove in the upcoming lemma,

$$||\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}, \qquad \text{for all } t \in [0, T],$$

since the classical solution $\boldsymbol{\eta}_1$ is spatially smooth. This is an essential observation, as the Gronwall estimate we obtain would give that

$$E_\delta(t) \leqslant C_1 \left(\int_0^t ||(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(s)||^2_{H^1(\Omega_b)} ds \right) e^{C_2 t} \leqslant C_1 \left(\int_0^T ||(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(s)||^2_{H^1(\Omega_b)} ds \right) e^{C_2 t} \sim C\delta^3 e^{C_2 t}.$$

By the definition of $E_\delta(t)$ and an application of Poincare's and Korn's inequality on Ω_b (see Proposition 5.5.1), this implies that $||(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta})(t)||_{H^1(\Omega_b)}$ and $||(\omega_1 - \omega_{2,\delta})(t)||_{H^2(\Gamma)}$, which are terms in $E_\delta(t)$, converges to zero as $\delta \to 0$ at a rate of $\delta^{3/2}$, as long as the assumptions (5.112), (5.113), and (5.114) hold. Therefore, by Holder's inequality, this gives the estimate for sufficiently small δ that

$$|(\nabla\tilde{\boldsymbol{\eta}}_1 - \nabla\tilde{\boldsymbol{\eta}}_{2,\delta})(t, x)| = \left| \int_{\tilde{\Omega}_b} (\nabla\boldsymbol{\eta}_1 - \nabla\boldsymbol{\eta}_{2,\delta})(t, y)\sigma_\delta(x - y)dy \right| \sim \delta^{3/2} \cdot \delta^{-1} \to 0,$$
$$\text{pointwise uniformly in } [0, T] \times \Omega_b \text{ as } \delta \to 0. \tag{5.115}$$

Here, we recall that the convolution is defined using an odd extension as in Definition 5.4.1, and by the definition of the odd extensions of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_{2,\delta}$ to the larger domain $\tilde{\Omega}_b$,

$$||\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta}||_{H^1(\tilde{\Omega}_b)} \leqslant C \left(||\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta}||_{H^1(\Omega_b)} + ||\omega_1 - \omega_{2,\delta}||_{H^1(\Gamma)} \right).$$

In addition, since we have extended the functions $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_{2,\delta}$ to the larger domain $\tilde{\Omega}_b$, the estimate (5.115) holds for all δ such that $\{(x,y) \in \mathbb{R}^2 : \text{dist}((x,y), \Omega_b) \leqslant \delta\} \subset \tilde{\Omega}_b$. In addition, from the upcoming lemma,

$$|(\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_1)(t,x)| \leqslant C\delta \to 0, \qquad \text{pointwise uniformly in } [0,T] \times \Omega_b \text{ as } \delta \to 0.$$

So combining this with (5.115), we have that $|(\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})(t,x)| \to 0$ pointwise uniformly in $[0,T] \times \Omega_b$ as $\delta \to 0$. So we can use a bootstrap argument on $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}_{2,\delta})$ by continuity, since we have that $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}_1) \geqslant c > 0$ up to a final time $T > 0$ (which depends only on the classical solution and has no dependence on δ). Similarly, for all sufficiently small δ, the assumptions (5.113) and (5.114) will also hold up to the final time $T > 0$, as we can also bootstrap these two conditions (5.113) and (5.114) similarly. This closes the bootstrap and so we obtain that the estimate (5.112), and similarly the estimates (5.113) and (5.114), hold uniformly up to the final time $T > 0$ uniformly in δ.

We end this section by proving the following lemma, which establishes convergence of the spatial convolution of the classical solution $\boldsymbol{\eta}_1$ in $H^1(\Omega_b)$, which is needed for the bootstrap argument described above.

Lemma 5.9.1. Let $\boldsymbol{\eta}_1 \in L^\infty(0,T;V_d)$ be an arbitrary but fixed smooth function in time and space on $[0,T] \times \overline{\Omega_b}$, where V_d is defined in (5.38). Then, there exists a constant C independent of $\delta > 0$, depending only on $\boldsymbol{\eta}_1$, such that

$$\max_{t \in [0,T]} ||\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1||_{H^1(\Omega_b)} \leqslant C\delta^{3/2},$$

and $|\nabla \tilde{\boldsymbol{\eta}}_1 - \nabla \boldsymbol{\eta}_1| \leqslant C\delta$ for all $x \in \overline{\Omega_b}$ and for all $t \in [0,T]$.

Remark 5.9.1. More generally, if f is a smooth function on $\mathbb{R}^2$ with sufficient decay at infinity, such as a Schwartz function, then the argument shown below shows that the function $\tilde{f}$ defined by

$$\tilde{f} = f * \sigma_\delta \qquad \text{on } \mathbb{R}^2$$

would satisfy $||\tilde{f} - f||_{H^1(\Omega_b)} \leqslant C\delta^2$ for a constant C. However, because we are working on a bounded domain Ω_b, we must use an odd extension to define the spatial convolution of $\boldsymbol{\eta}_1$. Since the odd extension of $\boldsymbol{\eta}_1$ to the larger domain $\tilde{\Omega}_b$ is not necessarily smooth on $\tilde{\Omega}_b$ even if $\boldsymbol{\eta}_1$ is a smooth function on $\overline{\Omega_b}$, we incur a loss in our estimate due to potentially irregularities of the odd extension due to the behavior of the initial function $\boldsymbol{\eta}_1$ near the boundary $\partial\Omega_b$, which gives rise to the convergence rate $\delta^{3/2}$ instead of the optimal rate of convergence δ^2.

Proof. We separate the domain $\Omega_b = (0,L) \times (0,R)$ into two parts:

$$\Omega_{b,1} = (\delta, L-\delta) \times (\delta, R-\delta), \qquad \Omega_{b,2} = \Omega_b \backslash \Omega_{b,1}.$$

For $\boldsymbol{x} \in \Omega_{b,1}$, we note that because the convolution kernel σ_δ is radially symmetric,

$$(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(\boldsymbol{x}) = \int_{\Omega_b} \left(\frac{1}{2}\boldsymbol{\eta}_1(\boldsymbol{x} + \boldsymbol{x}') - \boldsymbol{\eta}_1(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\eta}_1(\boldsymbol{x} - \boldsymbol{x}') \right) \sigma_\delta(\boldsymbol{x}') d\boldsymbol{x}',$$

$$(\nabla\tilde{\boldsymbol{\eta}}_1 - \nabla\boldsymbol{\eta}_1)(\boldsymbol{x}) = \int_{\Omega_b} \left(\frac{1}{2}\nabla\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \nabla\boldsymbol{\eta}_1(\boldsymbol{x}) + \frac{1}{2}\nabla\boldsymbol{\eta}_1(\boldsymbol{x}-\boldsymbol{x}')\right)\sigma_\delta(\boldsymbol{x}')d\boldsymbol{x}'.$$

For $\boldsymbol{x} \in \Omega_{b,1}$, we have that these points are at least δ away from the boundary, so that we have the following estimate for the discretized second derivative:

$$\left|\frac{1}{2}\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \boldsymbol{\eta}_1(x) + \frac{1}{2}\boldsymbol{\eta}_1(\boldsymbol{x}-\boldsymbol{x}')\right| \leqslant C\delta^2 \quad \text{for } |\boldsymbol{x}'| \leqslant \delta,$$

and similarly for $\nabla\boldsymbol{\eta}_1$, by using the fact that $\boldsymbol{\eta}_1$ is spatially smooth in $\overline{\Omega_b}$. So we conclude that

$$|(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(\boldsymbol{x})| \leqslant C\delta^2, \quad |(\nabla\tilde{\boldsymbol{\eta}}_1 - \nabla\boldsymbol{\eta}_1)(\boldsymbol{x})| \leqslant C\delta^2, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,1}, \tag{5.116}$$

for a constant C depending only on $\boldsymbol{\eta}_1$.

For $\boldsymbol{x} \in \Omega_{b,2}$, we cannot use the same estimate, since after extending $\boldsymbol{\eta}_1$ to the larger domain $\tilde{\Omega}_b$, the extended function on $\tilde{\Omega}_b$ does not necessarily have a continuous second derivative, as a result of the properties of odd extension, and in fact, there may be discontinuities of the second derivative along the boundary $\partial\Omega_b$. However, $\nabla\boldsymbol{\eta}_1$ on the larger domain $\tilde{\Omega}_b$ is still *Lipschitz continuous* so we instead use the equations:

$$(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(\boldsymbol{x}) = \int_{\Omega_b} (\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \boldsymbol{\eta}_1(\boldsymbol{x}))\sigma_\delta(\boldsymbol{x}')d\boldsymbol{x}',$$

$$(\nabla\tilde{\boldsymbol{\eta}}_1 - \nabla\boldsymbol{\eta}_1)(\boldsymbol{x}) = \int_{\Omega_b} (\nabla\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \nabla\boldsymbol{\eta}_1(\boldsymbol{x}))\sigma_\delta(\boldsymbol{x}')d\boldsymbol{x}'.$$

Since $\boldsymbol{x} \in \Omega_{b,2}$, even if $|\boldsymbol{x}'| \leqslant \delta$, we may have that $\boldsymbol{x}+\boldsymbol{x}'$ is outside of Ω_b. However, due to the Lipschitz continuity of $\nabla\boldsymbol{\eta}_1$ on the larger domain $\tilde{\Omega}_b$, we still have the estimates

$$|\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \boldsymbol{\eta}_1(\boldsymbol{x})| \leqslant C\delta, \quad |\nabla\boldsymbol{\eta}_1(\boldsymbol{x}+\boldsymbol{x}') - \nabla\boldsymbol{\eta}_1(\boldsymbol{x})| \leqslant C\delta, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,2}, |\boldsymbol{x}'| \leqslant \delta,$$

which gives

$$|(\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)(\boldsymbol{x})| \leqslant C\delta, \quad |(\nabla\tilde{\boldsymbol{\eta}}_1 - \nabla\boldsymbol{\eta}_1)(\boldsymbol{x})| \leqslant C\delta, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,2}. \tag{5.117}$$

The area of $\Omega_{b,2}$ is $\leqslant (2R+2L)\delta$, so by (5.116) and (5.117), we have $||\tilde{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}$ for a spatially smooth function $\boldsymbol{\eta}_1$ on $\overline{\Omega_b}$, where C depends only on the norms of up to the second spatial derivative of $\boldsymbol{\eta}_1$ on $\overline{\Omega_b}$. The generalization of this result to a function $\boldsymbol{\eta}_1$ that also depends on time and is spatially smooth in both space and time follows analogously.

□

The test functions

As described in Section 5.9, we want to test the non-regularized weak formulation formally with the difference between $(\boldsymbol{u}_1, \partial_t \boldsymbol{\eta}_1, p_1, \partial_t \omega_1)$ and $(\boldsymbol{u}_{2,\delta}, \partial_t \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \partial_t \omega_{2,\delta})$. However, there are two reasons why this is not rigorously justified. First, $\partial_t \boldsymbol{\eta}_1 - \partial_t \boldsymbol{\eta}_{2,\delta}$ is not a continuously differentiable function in time as is required for the test functions, and hence, we must first use a *convolution in time* and pass to the limit as the convolution parameter goes to zero. Second, the fluid velocities give an additional difficulty, as the fluid velocities are defined on *time-dependent moving domains*. Thus, we must transfer the fluid velocities between different time-dependent domains in order to make sense of the "difference" between $\boldsymbol{u}_1$ and $\boldsymbol{u}_{2,\delta}$ as a test function, and the way in which we do this transformation and the way in which we perform the convolution in time must both respect the divergence-free nature of the fluid velocity on the time-dependent domain. We will address both of these difficulties in this section, using a transformation that preserves the divergence-free condition from [95] and additional mathematical techniques for regularizing the test functions found in [159].

We address the first difficulty by defining a convolution in time. This will allow us to regularize $\partial_t(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta}) = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}$, $p_1 - p_{2,\delta}$, and $\partial_t(\omega_1 - \omega_{2,\delta}) = \beta_1 - \beta_{2,\delta}$ so that these functions are continuously differentiable in time. Since the classical solution is already continuously differentiable in time, we only need to regularize the weak solutions to the regularized problem. Because these differences are all defined on fixed domains, we can use a standard convolution in time. We let $j(\cdot) : \mathbb{R} \to \mathbb{R}$ be a compactly supported even function with $\text{supp}(j) \subset [-1, 1]$ and $\displaystyle\int_{\mathbb{R}} j = 1$, and we define $j_\alpha(t) = \alpha^{-1} j(\alpha^{-1} t)$, where $\alpha > 0$ is the convolution parameter in time.

Consider $\alpha > 0$. We extend $\boldsymbol{\xi}_{2,\delta}$, $p_{2,\delta}$, and $\zeta_{2,\delta}$ to the larger interval $[-\alpha, T + \alpha]$ by reflecting across $t = 0$ and $t = T$. For example, we define

$$\boldsymbol{\xi}_{2,\delta}(t) = \boldsymbol{\xi}_{2,\delta}(-t), \qquad \text{for } t \in [-\alpha, 0],$$
$$\boldsymbol{\xi}_{2,\delta}(t) = \boldsymbol{\xi}_{2,\delta}(2T - t), \qquad \text{for } t \in [T, T + \alpha].$$

We then convolve in time and define for $t \in [0, T]$,

$$(\boldsymbol{\xi}_{2,\delta})_\alpha(t) = \boldsymbol{\xi}_{2,\delta}(t, \cdot) * j_\alpha = \int_{\mathbb{R}} \boldsymbol{\xi}_{2,\delta}(s) j_\alpha(t - s) ds.$$

We define $(p_{2,\delta})_\alpha$ and $(\zeta_{2,\delta})_\alpha$ similarly. We will hence test with $\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha$, $p_1 - (p_{2,\delta})_\alpha$, and $\zeta_1 - (\zeta_{2,\delta})_\alpha$.

Because the fluid velocities are defined on moving time-dependent domains, we cannot directly apply a convolution in time. Before we can convolve in time, we must be able to transform fluid velocities from one domain to another, while preserving the divergence-free condition. We do this by using the following matrix:

$$K(s, t, x, y) = \begin{pmatrix} \frac{R+\omega(s,x)}{R+\omega(t,x)} & 0 \\ -(R + y)\partial_x \left(\frac{R+\omega(s,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix}. \tag{5.118}$$

This matrix has the following essential property. If $\boldsymbol{u}(x,y)$ is a divergence-free function defined on the domain $\Omega_f(s)$ defined by the structure displacement $\omega(s,x)$, then the function

$$K(s,t,x,y)\boldsymbol{u}\left(x, \frac{R+\omega(s,x)}{R+\omega(t,x)}(R+y)-R\right)$$

is a divergence-free vector field defined on $\Omega_f(t)$ defined by the structure displacement $\omega(t,x)$. We can therefore use this transformation to convolve in time, as follows. We extend $\boldsymbol{u}_{2,\delta}$ to $[-\alpha, T+\alpha]$ by reflection, as above, and define, for $t \in [0,T]$,

$$(\boldsymbol{u}_{2,\delta})_\alpha(t) = \int_{\mathbb{R}} K_{2,\delta}(s,t,x,y)\boldsymbol{u}_{2,\delta}\left(s, x, \frac{R+\omega_{2,\delta}(s,x)}{R+\omega_{2,\delta}(t,x)}(R+y)-R\right) j_\alpha(t-s)ds. \tag{5.119}$$

For a divergence-free function $\boldsymbol{v}$, extended as above in time to $[-\alpha, T+\alpha]$, we can define $\boldsymbol{v}_\alpha$ on $\Omega_{f,1}(t)$ analogously by

$$\boldsymbol{v}_\alpha(t) = \int_{\mathbb{R}} K_1(s,t,x,y)\boldsymbol{v}\left(s, x, \frac{R+\omega_1(s,x)}{R+\omega_1(t,x)}(R+y)-R\right) j_\alpha(t-s)ds.$$

Here, $K_1(s,t,x,y)$ and $K_{2,\delta}(s,t,x,y)$ are defined as $K(s,t,x,y)$ with the choices of $\omega = \omega_1$ and $\omega = \omega_{2,\delta}$ respectively. One such function $\boldsymbol{v}$ which will be convenient to consider on $\Omega_{f,1}(t)$ is the function $\breve{\boldsymbol{u}}_{2,\delta}$ defined on $\Omega_{f,1}(t)$, which is the function $\boldsymbol{u}_{2,\delta}$ defined on $\Omega_{f,2,\delta}(t)$ transferred in a divergence-free manner, as described above, onto the domain $\Omega_{f,1}(t)$. Specifically,

$$\breve{\boldsymbol{u}}_{2,\delta}(t,x,y) = \begin{pmatrix} \frac{R+\omega_{2,\delta}(t,x)}{R+\omega_1(t,x)} & 0 \\ -(R+y)\partial_x\left(\frac{R+\omega_{2,\delta}(t,x)}{R+\omega_1(t,x)}\right) & 1 \end{pmatrix} \cdot \boldsymbol{u}\left(x, \frac{R+\omega_{2,\delta}(t,x)}{R+\omega_1(t,x)}(R+y)-R\right).$$

We collect some properties of $(\boldsymbol{u}_{2,\delta})_\alpha$ in the proposition below, which are from the reference [159], and which are a specific case of Lemma 2.6 in [159].

Proposition 5.9.1. Fix an arbitrary $\delta > 0$. Given $\boldsymbol{u}_{2,\delta} \in L^2(0,T;H^1(\Omega_{f,2,\delta}(t))$ and $\omega_1, \omega_{2,\delta} \in H_0^2(\Gamma)$ with $|\omega_1| \leqslant R_0 < R$ and $|\omega_{2,\delta}| \leqslant R_0 < R$, $\text{div}\,[(\boldsymbol{u}_{2,\delta})_\alpha] = 0$ and $\text{div}[(\breve{\boldsymbol{u}}_{2,\delta})_\alpha] = 0$ for all $t \in [0,T]$ and for all $\alpha > 0$. In addition,

$$(\boldsymbol{u}_{2,\delta})_\alpha \to \boldsymbol{u}_{2,\delta} \quad \text{strongly in } L^p(0,T;L^q(\Omega_{f,2,\delta})), \quad \text{for all } p \in [1,\infty), q \in [1,2),$$

$$(\breve{\boldsymbol{u}}_{2,\delta})_\alpha \to \breve{\boldsymbol{u}}_{2,\delta} \quad \text{strongly in } L^p(0,T;L^q(\Omega_{f,1})), \quad \text{for all } p \in [1,\infty), q \in [1,2),$$

and

$$(\boldsymbol{u}_{2,\delta})_\alpha \rightharpoonup \boldsymbol{u}_{2,\delta} \quad \text{weakly in } L^2(0,T;W^{1,p}(\Omega_{f,2,\delta})), \quad \text{for all } p \in [1,2),$$
$$(\breve{\boldsymbol{u}}_{2,\delta})_\alpha \rightharpoonup \breve{\boldsymbol{u}}_{2,\delta} \quad \text{weakly in } L^2(0,T;W^{1,p}(\Omega_{f,1})), \quad \text{for all } p \in [1,2).$$

5.10 Proof of weak-classical consistency: obtaining a Gronwall estimate

In the next subsections, we will carry out the strategy outlined in the previous section by obtaining an appropriate Gronwall estimate involving the energy difference $E_\delta(t)$ defined in (5.109) between the classical solution $\boldsymbol{u}_1$ and the weak solution $\boldsymbol{u}_{2,\delta}$ to the regularized problem with regularization parameter δ, under the assumption that the three conditions (5.112), (5.113), and (5.114) hold. We will test the weak formulation (5.100) for the classical solution $(\boldsymbol{u}_1, \boldsymbol{\eta}_1, p_1, \omega_1)$ to the original non-regularized problem with

$$\boldsymbol{v} = \boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha, \quad \varphi = \zeta_1 - (\zeta_{2,\delta})_\alpha, \quad \boldsymbol{\psi} = \boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha, \quad r = p_1 - (p_{2,\delta})_\alpha. \tag{5.120}$$

We test the regularized weak formulations (5.101) for the weak solutions $(\boldsymbol{u}_{2,\delta}, \boldsymbol{\eta}_{2,\delta}, p_{2,\delta}, \omega_{2,\delta})$ with

$$\boldsymbol{v} = \widehat{\boldsymbol{u}}_1, \quad \varphi = \zeta_1, \quad \boldsymbol{\psi} = \boldsymbol{\xi}_1, \quad r = p_1. \tag{5.121}$$

Finally, we note that the energy estimate in Proposition 5.8.3 holds for the function $\boldsymbol{u}_{2,\delta}$. We will rewrite the energy inequality for $\boldsymbol{u}_{2,\delta}$ in a more convenient form by adding extra terms that will cancel out, in order to have the energy inequality parallel the weak formulation term by term. In particular, we have that for almost every $\tau \in [0, T_\delta]$,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |\boldsymbol{u}_{2,\delta}|^2 + \frac{1}{2}\int_0^\tau \int_{\Omega_f(t)} [((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{u}_{2,\delta})\cdot \boldsymbol{u}_{2,\delta} - ((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{u}_{2,\delta}] + \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n} - 2\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{u}_{2,\delta} \\
&+ 2\nu\int_0^\tau\int_{\Omega_{f,2,\delta}(t)} |\boldsymbol{D}(\boldsymbol{u}_{2,\delta})|^2 + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} \left(\frac{1}{2}|\boldsymbol{u}_{2,\delta}|^2 - p_{2,\delta}\right)(\boldsymbol{\xi}_{2,\delta} - \boldsymbol{u}_{2,\delta})\cdot\boldsymbol{n} + \beta\int_0^T\int_{\Gamma_{2,\delta}(t)} |(\boldsymbol{\xi}_{2,\delta} - \boldsymbol{u}_{2,\delta})\cdot\boldsymbol{\tau})|^2 \\
&\qquad + \frac{1}{2}\rho_p\int_\Gamma |\boldsymbol{\xi}_{2,\delta}|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_{2,\delta}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b} |\boldsymbol{\xi}_{2,\delta}|^2 + \mu_e\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_{2,\delta})(\tau)|^2 \\
&\qquad + \frac{1}{2}\lambda_e\int_{\Omega_b} |\nabla\cdot\boldsymbol{\eta}_{2,\delta}(\tau)|^2 + 2\mu_v\int_0^\tau\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_{2,\delta})|^2 + \lambda_v\int_0^\tau\int_{\Omega_b} |\nabla\cdot\boldsymbol{\xi}_{2,\delta}|^2 \\
&- \alpha\int_0^\tau\int_{\tilde{\Omega}_{b,2,\delta}(t)} p_{2,\delta}\nabla\cdot\boldsymbol{\xi}_{2,\delta} + \frac{1}{2}c_0\int_{\Omega_b} |p_{2,\delta}(\tau)|^2 - \alpha\int_0^\tau\int_{\tilde{\Omega}_{b,2,\delta}(t)} \frac{\tilde{D}}{Dt}\boldsymbol{\eta}_{2,\delta}\cdot\nabla p_{2,\delta} - \alpha\int_0^\tau\int_{\tilde{\Gamma}_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})p_{2,\delta} \\
&\quad + \kappa\int_0^T\int_{\tilde{\Omega}_{b,2,\delta}(t)} |\nabla p_{2,\delta}|^2 - \int_0^\tau\int_{\Gamma_{2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot\boldsymbol{n})p_{2,\delta} \leqslant \frac{1}{2}\int_{\Omega_f(0)} |\boldsymbol{u}_0|^2 + \frac{1}{2}\rho_p\int_\Gamma |\boldsymbol{\xi}_0|^2 \\
&\qquad + \frac{1}{2}\int_\Gamma |\Delta\omega_0|^2 + \frac{1}{2}\rho_b\int_{\Omega_b} |\boldsymbol{\xi}_0|^2 + \mu_e\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_0)|^2 + \frac{1}{2}\lambda_e\int_{\Omega_b} |\nabla\cdot\boldsymbol{\eta}_0|^2 + \frac{1}{2}c_0\int_{\Omega_b} |p_0|^2.
\end{aligned} \tag{5.122}$$

We will then combine these estimates together by taking the weak formulation for $\boldsymbol{u}_1$ tested with (5.120), subtracting the regularized weak formulation for $\boldsymbol{u}_{2,\delta}$ tested with (5.121), and adding the energy estimate for $\boldsymbol{u}_{2,\delta}$ in (5.122). After doing this, we will obtain an expression of the form

$$\sum T_i \leqslant 0, \tag{5.123}$$

where each T_i contains the terms relating to a particular term in the weak formulation. We will then take the limit in the resulting expression as $\alpha \to 0$ in order to obtain a Gronwall estimate for the energy difference $E_{2,\delta}$ defined in (5.109). The terms T_i in the expression (5.123) are as follows.

$$
\begin{aligned}
T_1 = &-\int_0^\tau \int_{\Omega_{f,1}(t)} \boldsymbol{u}_1 \cdot \partial_t \left[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\right] - \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot [\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha] + \int_{\Omega_{f,1}(\tau)} \boldsymbol{u}_1(\tau)\cdot[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha](\tau) \\
&- \int_{\Omega_f(0)} \boldsymbol{u}_1(0)\cdot[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha](0) - \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}\cdot \partial_t \widehat{\boldsymbol{u}}_1 - \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta}\cdot \boldsymbol{n}_{2,\delta})\boldsymbol{u}_{2,\delta}\cdot \widehat{\boldsymbol{u}}_1 \\
&+ \int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau)\cdot \widehat{\boldsymbol{u}}_1(\tau) - \int_{\Omega_f(0)} \boldsymbol{u}_{2,\delta}(0)\cdot\widehat{\boldsymbol{u}}_1(0) + \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |\boldsymbol{u}_{2,\delta}(\tau)|^2 - \frac{1}{2}\int_{\Omega_{f,2,\delta}(0)} |\boldsymbol{u}_0|^2
\end{aligned} \tag{5.124}
$$

$$
\begin{aligned}
T_2 = &\frac{1}{2}\int_0^\tau\int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1)\cdot[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha] - \frac{1}{2}\int_0^\tau\int_{\Omega_{f,1}(t)} (\boldsymbol{u}_1\cdot\nabla)[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha]\cdot\boldsymbol{u}_1 \\
&- \frac{1}{2}\int_0^\tau\int_{\Omega_{f,2,\delta}(t)} ((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{u}_{2,\delta})\cdot(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + \frac{1}{2}\int_0^\tau\int_{\Omega_{f,2,\delta}(t)} ((\boldsymbol{u}_{2,\delta}\cdot\nabla)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}))\cdot\boldsymbol{u}_{2,\delta}
\end{aligned} \tag{5.125}
$$

$$
\begin{aligned}
T_3 = &\frac{1}{2}\int_0^\tau\int_{\Gamma_1(t)} (\boldsymbol{u}_1\cdot\boldsymbol{n}_1 - \boldsymbol{\xi}_1\cdot\boldsymbol{n}_1)\boldsymbol{u}_1\cdot[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha] - \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n}_{2,\delta} - \boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n}_{2,\delta})\boldsymbol{u}_{2,\delta}\cdot\widehat{\boldsymbol{u}}_1 \\
&+\frac{1}{2}\int_0^\tau\int_{\Gamma_1(t)} |\boldsymbol{u}_1|^2(\boldsymbol{\xi}_1\cdot\boldsymbol{n}_1 - \boldsymbol{u}_1\cdot\boldsymbol{n}_1) - \frac{1}{2}\int_0^\tau\int_{\Gamma_1(t)} |\boldsymbol{u}_1|^2[(\boldsymbol{\xi}_{2,\delta})_\alpha\cdot\boldsymbol{n}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\cdot\boldsymbol{n}_1] - \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} |\boldsymbol{u}_{2,\delta}|^2(\boldsymbol{\xi}_1\cdot\boldsymbol{n}_{2,\delta} - \widehat{\boldsymbol{u}}_1\cdot\boldsymbol{n}_{2,\delta}) \\
&+ \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n} - \boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})|\boldsymbol{u}_{2,\delta}|^2 + \frac{1}{2}\int_0^\tau\int_{\Gamma_{2,\delta}(t)} |\boldsymbol{u}_{2,\delta}|^2(\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n} - \boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n})
\end{aligned} \tag{5.126}
$$

$$
T_4 = 2\nu\int_0^\tau\int_{\Omega_{f,1}(t)} \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha) - 2\nu\int_0^\tau\int_{\Omega_{f,2,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_{2,\delta}) : \boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) \tag{5.127}
$$

$$
T_5 = \beta\int_0^\tau\int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 - \boldsymbol{u}_1)_\tau[(\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha)_\tau - (\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha)_\tau] - \beta\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} - \boldsymbol{u}_{2,\delta})_\tau[(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})_\tau - (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})_\tau] \tag{5.128}
$$

$$
\begin{aligned}
T_6 = &-\rho_p\int_0^\tau\int_\Gamma \zeta_1\cdot\partial_t\left[\zeta_1 - (\zeta_{2,\delta})_\alpha\right] + \rho_p\int_\Gamma \zeta_1(\tau)\cdot[\zeta_1(\tau) - (\zeta_{2,\delta})_\alpha(\tau)] - \rho_p\int_\Gamma \zeta_1(0)\cdot[\zeta_1(0) - (\zeta_{2,\delta})_\alpha(0)] \\
&+ \rho_p\int_0^\tau\int_\Gamma \zeta_{2,\delta}\cdot\partial_t\zeta_1 - \rho_p\int_\Gamma \zeta_{2,\delta}(\tau)\cdot\zeta_1(\tau) + \rho_p\int_\Gamma \zeta_{2,\delta}(0)\cdot\zeta_1(0) + \frac{1}{2}\rho_p\int_\Gamma |\zeta_{2,\delta}(\tau)|^2 - \frac{1}{2}\rho_p\int_\Gamma |\zeta_0|^2
\end{aligned} \tag{5.129}
$$

$$
T_7 = \int_0^\tau\int_\Gamma \Delta\omega_1\cdot\Delta\left[\zeta_1 - (\zeta_{2,\delta})_\alpha\right] - \int_0^\tau\int_\Gamma \Delta\omega_{2,\delta}\cdot\Delta\zeta_1 + \frac{1}{2}\int_\Gamma |\Delta\omega_{2,\delta}(\tau)|^2 - \frac{1}{2}\int_\Gamma |\Delta\omega_0|^2 \tag{5.130}
$$

$$
\begin{aligned}
T_8 = &-\rho_b\int_0^\tau\int_{\Omega_b} \partial_t\boldsymbol{\eta}_1\cdot\partial_t\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right] + \rho_b\int_{\Omega_b} \boldsymbol{\xi}_1(\tau)\cdot\left[\boldsymbol{\xi}_1(\tau) - (\boldsymbol{\xi}_{2,\delta})_\alpha(\tau)\right] - \rho_b\int_{\Omega_b} \boldsymbol{\xi}_1(0)\cdot\left[\boldsymbol{\xi}_1(0) - (\boldsymbol{\xi}_{2,\delta})_\alpha(0)\right] \\
&+ \rho_b\int_0^\tau\int_{\Omega_b} \partial_t\boldsymbol{\eta}_{2,\delta}\cdot\partial_t\boldsymbol{\xi}_1 - \rho_b\int_{\Omega_b} \boldsymbol{\xi}_{2,\delta}(\tau)\cdot\boldsymbol{\xi}_1(\tau) + \rho_b\int_{\Omega_b} \boldsymbol{\xi}_{2,\delta}(0)\cdot\boldsymbol{\xi}_1(0) + \frac{1}{2}\rho_b\int_{\Omega_b} |\boldsymbol{\xi}_{2,\delta}(\tau)|^2 - \frac{1}{2}\rho_b\int_{\Omega_b} |\boldsymbol{\xi}_0|^2
\end{aligned} \tag{5.131}
$$

$$
T_9 = 2\mu_e\int_0^\tau\int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_1) : \boldsymbol{D}\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right] - 2\mu_e\int_0^\tau\int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{\xi}_1) + \mu_e\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_{2,\delta})(\tau)|^2 - \mu_e\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_0)|^2 \tag{5.132}
$$

$$T_{10} = \lambda_e \int_0^\tau \int_{\Omega_b} (\nabla\cdot\boldsymbol{\eta}_1)\left(\nabla\cdot\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right]\right) - \lambda_e \int_0^\tau \int_{\Omega_b} (\nabla\cdot\boldsymbol{\eta}_{2,\delta})(\nabla\cdot\boldsymbol{\xi}_1) + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla\cdot\boldsymbol{\eta}_{2,\delta}(\tau)|^2 - \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla\cdot\boldsymbol{\eta}_0|^2 \tag{5.133}$$

$$T_{11} = 2\mu_v \int_0^\tau \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}_1) : \boldsymbol{D}\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right] - 2\mu_v \int_0^\tau \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{\xi}_1) + 2\mu_v \int_0^\tau \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_{2,\delta})|^2. \tag{5.134}$$

$$T_{12} = \lambda_v \int_0^\tau \int_{\Omega_b} (\nabla\cdot\boldsymbol{\xi}_1)\left(\nabla\cdot\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right]\right) - \lambda_v \int_0^\tau \int_{\Omega_b} (\nabla\cdot\boldsymbol{\xi}_{2,\delta})(\nabla\cdot\boldsymbol{\xi}_1) + \lambda_v \int_0^\tau \int_{\Omega_b} |\nabla\cdot\boldsymbol{\xi}_{2,\delta}|^2. \tag{5.135}$$

$$T_{13} = -\alpha \int_0^\tau \int_{\Omega_{b,1}(t)} p_1\left(\nabla\cdot\left[\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha\right]\right) + \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} p_{2,\delta}\left(\nabla\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})\right) \tag{5.136}$$

$$\begin{aligned} T_{14} = -c_0 \int_0^\tau \int_{\Omega_b} p_1 \cdot \partial_t\left[p_1 - (p_{2,\delta})_\alpha\right] + c_0 \int_{\Omega_b} p_1(\tau)\cdot[p_1(\tau) - (p_{2,\delta})_\alpha(\tau)] - c_0 \int_{\Omega_b} p_0 \cdot [p_1(0) - (p_{2,\delta})_\alpha(0)] \\ + c_0 \int_0^\tau \int_{\Omega_b} p_{2,\delta}\cdot\partial_t p_1 - c_0 \int_{\Omega_b} p_{2,\delta}(\tau)\cdot p_1(\tau) + c_0 \int_{\Omega_b} |p_0|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_{2,\delta}(\tau)|^2 - \frac{1}{2}c_0 \int_{\Omega_b} |p_0|^2. \end{aligned} \tag{5.137}$$

$$T_{15} = -\alpha \int_0^\tau \int_{\Omega_{b,1}(t)} \boldsymbol{\xi}_1 \cdot \nabla\left[p_1 - (p_{2,\delta})_\alpha\right] + \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \boldsymbol{\xi}_{2,\delta}\cdot\nabla(p_1 - p_{2,\delta}) \tag{5.138}$$

$$T_{16} = -\alpha \int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1\cdot\boldsymbol{n})\left[p_1 - (p_{2,\delta})_\alpha\right] + \alpha \int_0^\tau \int_{\tilde{\Gamma}_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})(p_1 - p_{2,\delta}) \tag{5.139}$$

$$T_{17} = \kappa \int_0^\tau \int_{\Omega_{b,1}(t)} \nabla p_1 \cdot \nabla\left[p_1 - (p_{2,\delta})_\alpha\right] - \kappa \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \nabla p_{2,\delta}\cdot\nabla(p_1 - p_{2,\delta}) \tag{5.140}$$

$$\begin{aligned} T_{18} = \int_0^\tau \int_{\Gamma_1(t)} p_1(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot\boldsymbol{n} - \int_0^\tau \int_{\Gamma_1(t)} p_1[(\boldsymbol{u}_{2,\delta})_\alpha - (\boldsymbol{\xi}_{2,\delta})_\alpha]\cdot\boldsymbol{n} - \int_0^\tau \int_{\Gamma_{2,\delta}(t)} p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot\boldsymbol{n} \\ + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} p_{2,\delta}(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot\boldsymbol{n} - \int_0^\tau \int_{\Gamma_1(t)} ((\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot\boldsymbol{n})[p_1 - (p_{2,\delta})_\alpha] + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot\boldsymbol{n})(p_1 - p_{2,\delta}) \end{aligned} \tag{5.141}$$

We will estimate each of the terms in this list in the subsequent subsections.

Term 1

In the weak formulation (5.100) for $\boldsymbol{u}_1$, we test with $\boldsymbol{v} = \boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha$ and obtain the terms:

$$\begin{aligned} T_{1,1} = -\int_0^\tau \int_{\Omega_{f,1}(t)} \boldsymbol{u}_1 \cdot \partial_t\left[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\right] - \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1\cdot\boldsymbol{n}_1)\boldsymbol{u}_1 \cdot \left[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\right] \\ + \int_{\Omega_{f,1}(\tau)} \boldsymbol{u}_1(\tau)\cdot\left[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\right](\tau) - \int_{\Omega_f(0)} \boldsymbol{u}_1(0)\cdot\left[\boldsymbol{u}_1 - (\breve{\boldsymbol{u}}_{2,\delta})_\alpha\right](0), \end{aligned}$$

where $\Omega_f(0)$ is the fluid domain corresponding to the initial structure displacement ω_0. We note that $\boldsymbol{u}_1$ is smooth in time and $(\check{\boldsymbol{u}}_{2,\delta})_\alpha$ as a result of the time convolution is differentiable in time. Thus, by the Reynold's transport theorem,

$$T_{1,1} = \int_0^\tau \int_{\Omega_{f,1}(t)} \partial_t \boldsymbol{u}_1 \cdot [\boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha] + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot [\boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha].$$

Because $\boldsymbol{u}_1$ is smooth and by the weak convergence properties of $(\check{\boldsymbol{u}}_{2,\delta})_\alpha$ in Proposition 5.9.1,

$$T_{1,1} = \int_0^\tau \int_{\Omega_{f,1}(t)} \partial_t \boldsymbol{u}_1 \cdot [\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}] + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot [\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}] + K_{1,1,\alpha},$$

where $K_{1,1,\alpha} \to 0$ as $\alpha \to 0$. Using estimates as found in [159], we can transfer the first integral from $\Omega_{f,1}(t)$ to $\Omega_{f,2,\delta}(t)$ at the cost of an additional term, so that

$$T_{1,1} = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \partial_t \widehat{\boldsymbol{u}}_1 \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}) + R_{1,1,\delta} + K_{1,1,\alpha},$$

where

$$\begin{aligned}|R_{1,1,\delta}| \leqslant \epsilon \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2.\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))} \\ + C(\epsilon)\left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\partial_t\omega_1 - \partial_t\omega_{2,\delta}||^2_{L^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).\end{aligned}$$

Thus, by using Proposition 5.9.1 again,

$$T_{1,1} = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \partial_t \widehat{\boldsymbol{u}}_1 \cdot (\widehat{\boldsymbol{u}}_1 - (\boldsymbol{u}_{2,\delta})_\alpha) + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha) + R_{1,1,\delta} + \tilde{K}_{1,1,\alpha}, \tag{5.142}$$

where $\tilde{K}_{1,1,\alpha} \to 0$ as $\alpha \to 0$.

Next, we test the regularized weak formulation for $\boldsymbol{u}_{2,\delta}$ with $\widehat{\boldsymbol{u}}_1$ and obtain the following terms:

$$\begin{aligned}T_{1,2} = -\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot \partial_t \widehat{\boldsymbol{u}}_1 - \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})\boldsymbol{u}_{2,\delta} \cdot \widehat{\boldsymbol{u}}_1 \\ + \int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau) \cdot \widehat{\boldsymbol{u}}_1(\tau) - \int_{\Omega_f(0)} \boldsymbol{u}_{2,\delta}(0) \cdot \widehat{\boldsymbol{u}}_1(0).\end{aligned}$$

We want to integrate by parts in time, but $\boldsymbol{u}_{2,\delta}$ is not necessarily smooth in time. Thus, we replace $\boldsymbol{u}_{2,\delta}$ by its time regularization $(\boldsymbol{u}_{2,\delta})_\alpha$ at the cost of a term $K_{1,2,\alpha}$ which goes to zero as $\alpha \to 0$ by Proposition 5.9.1. Combining this with the Reynold's transport theorem,

$$T_{1,2} = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \partial_t \left[(\boldsymbol{u}_{2,\delta})_\alpha\right] \cdot \widehat{\boldsymbol{u}}_1 + \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})(\boldsymbol{u}_{2,\delta})_\alpha \cdot \widehat{\boldsymbol{u}}_1 + K_{1,2,\alpha}, \tag{5.143}$$

where $K_{1,2,\alpha} \to 0$ as $\alpha \to 0$.

From the energy inequality, we obtain the terms

$$T_{1,3} = \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |\boldsymbol{u}_{2,\delta}(\tau)|^2 - \frac{1}{2}\int_{\Omega_{f,2,\delta}(0)} |\boldsymbol{u}_{2,\delta}(0)|^2. \tag{5.144}$$

Using the Reynold's transport theorem, the total contribution $T_1 = T_{1,1} - T_{1,2} + T_{1,3}$ is

$$\begin{aligned} T_1 = \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |\widehat{\boldsymbol{u}}_1(\tau)|^2 - \frac{1}{2}\int_{\Omega_{f,2,\delta}(0)} |\widehat{\boldsymbol{u}}_1(0)|^2 - \int_{\Omega_{f,2,\delta}(\tau)} (\widehat{\boldsymbol{u}}_1 \cdot (\boldsymbol{u}_{2,\delta})_\alpha)(\tau) + \int_{\Omega_{f,2,\delta}(0)} (\widehat{\boldsymbol{u}}_1 \cdot (\boldsymbol{u}_{2,\delta})_\alpha)(0) \\ + \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |\boldsymbol{u}_{2,\delta}(\tau)|^2 - \frac{1}{2}\int_{\Omega_{f,2,\delta}(0)} |\boldsymbol{u}_{2,\delta}(0)|^2 - \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})\widehat{\boldsymbol{u}}_1 \cdot (\widehat{\boldsymbol{u}}_1 - (\boldsymbol{u}_{2,\delta})_\alpha) \\ + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha) + R_{1,1,\delta} + \tilde{K}_{1,1,\alpha} + K_{1,2,\alpha}. \end{aligned}$$

By Proposition 5.9.1, $(\boldsymbol{u}_{2,\delta})_\alpha$ and $(\check{\boldsymbol{u}}_{2,\delta})_\alpha$ converge weakly to $\boldsymbol{u}_{2,\delta}$ and $\check{\boldsymbol{u}}_{2,\delta}$ respectively, weakly in $L^2(0, T, W^{1,p}(\Omega_{f,2,\delta}))$ and $L^2(0, T, W^{1,p}(\Omega_{f,1}))$ for all $p \in [1, 2)$. Furthermore, we have that

$$\int_{\Omega_{f,2,\delta}(0)} (\widehat{\boldsymbol{u}}_1 \cdot (\boldsymbol{u}_{2,\delta})_\alpha)(0) \to \int_{\Omega_{f,2,\delta}(0)} (\widehat{\boldsymbol{u}}_1 \cdot \boldsymbol{u}_{2,\delta})(0), \quad \int_{\Omega_{f,2,\delta}(\tau)} (\widehat{\boldsymbol{u}}_1 \cdot (\boldsymbol{u}_{2,\delta})_\alpha)(\tau) \to \int_{\Omega_{f,2,\delta}(\tau)} (\widehat{\boldsymbol{u}}_1 \cdot \boldsymbol{u}_{2,\delta})(\tau). \tag{5.145}$$

We defer the technical proof of this statement to the appendix, see Lemma 5.12.5. Thus, taking the limit as $\alpha \to 0$, the contribution of this term is now

$$\begin{aligned} T_1 = \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})(\tau)|^2 - \frac{1}{2}\int_{\Omega_{f,2,\delta}(0)} |(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})(0)|^2 \\ - \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})\widehat{\boldsymbol{u}}_1 \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}) + R_{1,1,\delta}. \end{aligned}$$

Since $\widehat{\boldsymbol{u}}_1(0) = \boldsymbol{u}_{2,\delta}(0) = \boldsymbol{u}_0$, we obtain after some standard estimates that

$$T_1 = \frac{1}{2}\int_{\Omega_{f,2,\delta}(\tau)} |(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})(\tau)|^2 + R_{1,\delta},$$

where

$$\begin{aligned} |R_{1,\delta}| \leqslant \epsilon \int_0^T ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2.\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))} \\ + C(\epsilon)\left(\int_0^T ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^T ||\partial_t\omega_1 - \partial_t\omega_{2,\delta}||^2_{L^2(\Gamma)} + \int_0^T ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right). \end{aligned}$$

Term 2

Because $(\breve{\boldsymbol{u}}_{2,\delta})_\alpha$ converges weakly to $\breve{\boldsymbol{u}}_{2,\delta}$ in $L^2(0,T;W^{1,p}(\Omega_{f,1}))$ for $p \in [1,2)$ by Proposition 5.9.1, and because $\boldsymbol{u}_1$ is smooth, as $\alpha \to 0$, we have that T_2 converges to

$$T_2 := \frac{1}{2}\int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1) \cdot (\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}) - \frac{1}{2}\int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1 \cdot \nabla)(\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta})) \cdot \boldsymbol{u}_1$$
$$- \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} \cdot \nabla)\boldsymbol{u}_{2,\delta}) \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} \cdot \nabla)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})) \cdot \boldsymbol{u}_{2,\delta}.$$

We note that the quantity

$$\frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} \cdot \nabla)\boldsymbol{u}_{2,\delta}) \cdot \boldsymbol{u}_{2,\delta},$$

is well-defined because $\boldsymbol{u}_{2,\delta} \in L^\infty(0,T;L^2(\Omega_{f,2,\delta})) \cap L^2(0,T;H^1(\Omega_{f,2,\delta}))$, which by interpolation is in $L^4(0,T;H^{1/2}(\Omega_{f,2,\delta}))$, which embeds into $L^4(0,T;L^4(\Omega_{f,2,\delta}))$.

We want to transfer the integrals

$$\int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1) \cdot (\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}), \qquad \int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1 \cdot \nabla)(\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta})) \cdot \boldsymbol{u}_1, \tag{5.146}$$

to integrals on $\Omega_{f,2,\delta}(t)$ by using the map $\psi_\delta : \Omega_{f,2,\delta}(t) \to \Omega_{f,1}(t)$ defined by (5.103). We use

$$\widehat{\boldsymbol{u}}_1 = \gamma_\delta J_\delta^{-1} \cdot (\boldsymbol{u}_1 \circ \psi_\delta), \qquad \widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta} = \gamma_\delta J_\delta^{-1} \cdot ((\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}) \circ \psi_\delta),$$

where we recall the definitions of the appropriate terms from (5.102), (5.103), (5.104), (5.107), and (5.108).

Following arguments found in [159], we obtain the following estimates. We have, using (5.106), that

$$\int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1) \cdot (\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}) = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \gamma_\delta [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta)) J_\delta^{-1}(\boldsymbol{u}_1 \circ \psi_\delta)] \cdot (\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}) \circ \psi_\delta$$
$$= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot [\gamma_\delta^{-1} J_\delta (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})]$$
$$= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) - \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot [(I - \gamma_\delta^{-1} J_\delta)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})]$$
$$= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ((\nabla \widehat{\boldsymbol{u}}_1)\widehat{\boldsymbol{u}}_1) \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla((I - \gamma_\delta J_\delta^{-1})(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1) \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})$$
$$- \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot [(I - \gamma_\delta^{-1} J_\delta)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})]$$
$$= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\widehat{\boldsymbol{u}}_1 \cdot \nabla)\widehat{\boldsymbol{u}}_1] \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) + R_1, \tag{5.147}$$

where

$$R_1 = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla((I-\gamma_\delta J_\delta^{-1})(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1) \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) \\ - \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot [(I-\gamma_\delta^{-1}J_\delta)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})].$$

In the following estimates, we will repeatedly use the following inequalities, which hold for a constant C that is independent of δ:

$$|\gamma_\delta^{-1}J_\delta - I| \leqslant C(|\gamma_\delta^{-1} - 1| + |\nabla\gamma_\delta|) \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)},$$
$$|\gamma_\delta J_\delta^{-1} - I| \leqslant C(|\gamma_\delta - 1| + |\nabla\gamma_\delta|) \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)},$$
$$|\nabla(\gamma_\delta J_\delta^{-1})| \leqslant C(|\partial_x\gamma_\delta| + |\partial_{xx}\gamma_\delta|) \leqslant C(||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} + |\partial_{xx}(\omega_1 - \omega_{2,\delta})|), \quad (5.148)$$

so that

$$||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}. \quad (5.149)$$

To obtain (5.148), we estimate $|\partial_{xx}\gamma_\delta|$ by using the fact that ω_1 is smooth so that $|\partial_{xx}\omega_1| \leqslant C$ and a direct computation of $\partial_{xx}\gamma_\delta$.

Using these estimates, the Leibniz rule, and the smoothness of $\boldsymbol{u}_1$,

$$\left|\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla((I-\gamma_\delta J_\delta^{-1})(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1) \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})\right| \\ \leqslant C\int_0^\tau ||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C\left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

By using (5.106), and using the fact that $|J_\delta| \leqslant C$ is uniformly bounded, due to the fact that $|J_\delta| \leqslant C(1 + ||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}) \leqslant C$ is uniformly bounded, we obtain a similar estimate that

$$\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\boldsymbol{u}_1 \circ \psi_\delta))\widehat{\boldsymbol{u}}_1] \cdot [(I-\gamma_\delta^{-1}J_\delta)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})] \\ \leqslant C\left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

Thus, we obtain

$$|R_1| \leqslant C\left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right). \quad (5.150)$$

Handling the second integral in (5.146), we have by using (5.106) that

$$\begin{aligned}
\int_0^\tau \int_{\Omega_{f,1}(t)} ((\boldsymbol{u}_1\cdot\nabla)(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta}))\cdot\boldsymbol{u}_1 &= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \gamma_\delta[(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))J_\delta^{-1}(\boldsymbol{u}_1\circ\psi_\delta)]\cdot(\boldsymbol{u}_1\circ\psi_\delta) \\
&= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))\widehat{\boldsymbol{u}}_1]\cdot(\gamma_\delta^{-1}J_\delta\widehat{\boldsymbol{u}}_1) \\
= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))\widehat{\boldsymbol{u}}_1]\cdot\widehat{\boldsymbol{u}}_1 &- \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))\widehat{\boldsymbol{u}}_1]\cdot[(I-\gamma_\delta^{-1}J_\delta)\widehat{\boldsymbol{u}}_1] \\
= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})\widehat{\boldsymbol{u}}_1)\cdot\widehat{\boldsymbol{u}}_1 &+ \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla[(I-\gamma_\delta J_\delta^{-1})((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)]\widehat{\boldsymbol{u}}_1)\cdot\widehat{\boldsymbol{u}}_1 \\
&- \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))\widehat{\boldsymbol{u}}_1]\cdot[(I-\gamma_\delta^{-1}J_\delta)\widehat{\boldsymbol{u}}_1] \\
&= \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ((\widehat{\boldsymbol{u}}_1\cdot\nabla)(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}))\cdot\widehat{\boldsymbol{u}}_1 + R_2, \quad (5.151)
\end{aligned}$$

where

$$\begin{aligned}
R_2 := \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} &(\nabla[(I-\gamma_\delta J_\delta^{-1})((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)]\widehat{\boldsymbol{u}}_1)\cdot\widehat{\boldsymbol{u}}_1 \\
&- \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta))\widehat{\boldsymbol{u}}_1]\cdot[(I-\gamma_\delta^{-1}J_\delta)\widehat{\boldsymbol{u}}_1].
\end{aligned}$$

To estimate R_2, we will use the following inequalities:

$$|(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi| = |\gamma_\delta^{-1}J_\delta\cdot(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})| \leqslant C|\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}|,$$

$$\begin{aligned}
|\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)| = |\nabla(\gamma_\delta^{-1}J_\delta\cdot(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}))| &\leqslant |\nabla(\gamma_\delta^{-1}J_\delta)|\cdot|\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}|+|\gamma_\delta^{-1}J_\delta|\cdot|\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})| \\
&\leqslant C(|\nabla(\gamma_\delta^{-1}J_\delta)|\cdot|\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}|+|\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})|).
\end{aligned}$$

Therefore, we estimate, using $\max\left(|I-\gamma_\delta^{-1}J_\delta|, |I-\gamma_\delta J_\delta^{-1}|\right) \leqslant C\min\left(1, ||\omega_1-\omega_{2,\delta}||_{H^2(\Gamma)}\right)$,

$$\begin{aligned}
|R_2| \leqslant C\Bigg(\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} |\nabla(\gamma_\delta J_\delta^{-1})|\cdot|(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta| &+ \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} |I-\gamma_\delta J_\delta^{-1}|\cdot|\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)| \\
&+ \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} |I-\gamma_\delta^{-1}J_\delta|\cdot|\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)|\Bigg) \\
\leqslant C\Bigg(\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \left(|\nabla(\gamma_\delta J_\delta^{-1})|+|\nabla(\gamma_\delta^{-1}J_\delta)|\right)\cdot|\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}| &+ \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} ||\omega_1-\omega_{2,\delta}||_{H^2(\Gamma)}\cdot|\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})|\Bigg) \\
\leqslant \epsilon\int_0^\tau ||\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon)&\left(\int_0^\tau ||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right). \quad (5.152)
\end{aligned}$$

In the last line, we use the following estimates, derived similarly as for (5.149),

$$|\nabla(\gamma_\delta^{-1}J_\delta)| \leqslant C(|\partial_x(\gamma_\delta^{-1})| + |\partial_x\gamma_\delta| + |\partial_{xx}\gamma_\delta|) \leqslant C(||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} + |\partial_{xx}(\omega_1 - \omega_{2,\delta})|),$$

$$||\nabla(\gamma_\delta^{-1}J_\delta)||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}.$$

Therefore, in the expression in (5.125), after transferring the integrals (5.147) and (5.151) and estimating R_1 (5.150) and R_2 (5.152), the remaining terms we have to handle are as follows:

$$\frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\widehat{\boldsymbol{u}}_1 \cdot \nabla)\widehat{\boldsymbol{u}}_1] \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) - [(\widehat{\boldsymbol{u}}_1 \cdot \nabla)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})] \cdot \widehat{\boldsymbol{u}}_1$$
$$- \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t))} [(\boldsymbol{u}_{2,\delta} \cdot \nabla)\boldsymbol{u}_{2,\delta}] \cdot (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) - [(\boldsymbol{u}_{2,\delta} \cdot \nabla)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})] \cdot \boldsymbol{u}_{2,\delta}$$
$$= \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [((\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) \cdot \nabla)\boldsymbol{u}_{2,\delta}] \cdot \widehat{\boldsymbol{u}}_1 - \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [((\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) \cdot \nabla)\widehat{\boldsymbol{u}}_1] \cdot \boldsymbol{u}_{2,\delta}.$$

In absolute values, we can bound this by

$$\leqslant \epsilon \int_0^\tau ||\nabla(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}.$$

Thus, combining this with (5.150) and (5.152), we obtain

$$|T_2| \leqslant \epsilon \int_0^\tau ||\nabla(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon)\left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

Term 3

Because $\boldsymbol{u}_1$ and $\boldsymbol{\xi}_1$ are smooth, we can pass to the limit as $\alpha \to 0$ using Proposition 5.9.1 and the fact that $(\boldsymbol{\xi}_{2,\delta})_\alpha \to \boldsymbol{\xi}_{2,\delta}$ strongly in $L^2(0, T; H^1(\Omega_b))$, so that we can ultimately just test with $\boldsymbol{v} = \boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}$ and $\boldsymbol{\psi} = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}$. In the regularized weak formulation for $\boldsymbol{u}_{2,\delta}$, we test with $\boldsymbol{u}_1$ and $\boldsymbol{\xi}_1$. Note that both test functions $\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}$ and $\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}$ have the same trace along $\Gamma_1(t)$ and $\Gamma_{2,\delta}(t)$ respectively, which we will formally denote by $\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}$ along the reference configuration of the interface Γ. Combining the resulting expressions, we have the following contribution of T_3 in the limit as $\alpha \to 0$:

$$T_3 = \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{u}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}) - \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta} - \boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})\boldsymbol{u}_{2,\delta} \cdot \widehat{\boldsymbol{u}}_1$$
$$+ \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} |\boldsymbol{u}_1|^2(\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_1 \cdot \boldsymbol{n}_1) - \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} |\boldsymbol{u}_1|^2(\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_1 - \check{\boldsymbol{u}}_{2,\delta} \cdot \boldsymbol{n}_1)$$
$$- \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} |\boldsymbol{u}_{2,\delta}|^2(\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_{2,\delta} - \widehat{\boldsymbol{u}}_1 \cdot \boldsymbol{n}_{2,\delta}) = \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_1 \cdot \boldsymbol{n}_1)\boldsymbol{u}_1 \cdot \check{\boldsymbol{u}}_{2,\delta}$$
$$- \frac{1}{2}\int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_1 - \check{\boldsymbol{u}}_{2,\delta} \cdot \boldsymbol{n}_1)|\boldsymbol{u}_1|^2 - \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_{2,\delta} - \widehat{\boldsymbol{u}}_1 \cdot \boldsymbol{n}_{2,\delta})|\boldsymbol{u}_{2,\delta}|^2$$
$$+ \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta} - \boldsymbol{u}_{2,\delta} \cdot \boldsymbol{n}_{2,\delta})\boldsymbol{u}_{2,\delta} \cdot \widehat{\boldsymbol{u}}_1 = R_1 + R_2,$$

where

$$R_1 = \frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{\xi}_1 - \boldsymbol{u}_1)_y \boldsymbol{u}_{2,\delta} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}) - \frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{\xi}_{2,\delta} - \boldsymbol{u}_{2,\delta})_y \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}),$$

$$R_2 = \frac{1}{2}\int_0^\tau \int_\Gamma \partial_x \omega_1 (\boldsymbol{u}_1)_x \boldsymbol{u}_1 \cdot \boldsymbol{u}_{2,\delta} - \frac{1}{2}\int_0^\tau \int_\Gamma \partial_x \omega_1 (\boldsymbol{u}_{2,\delta})_x |\boldsymbol{u}_1|^2$$
$$- \frac{1}{2}\int_0^\tau \int_\Gamma \partial_x \omega_{2,\delta} (\boldsymbol{u}_1)_x |\boldsymbol{u}_{2,\delta}|^2 + \frac{1}{2}\int_0^\tau \int_\Gamma \partial_x \omega_{2,\delta} (\boldsymbol{u}_{2,\delta})_x \boldsymbol{u}_1 \cdot \boldsymbol{u}_{2,\delta}.$$

We estimate R_1 as follows. We decompose R_1 as $R_1 = R_{11} + R_{12}$, where

$$R_{11} = -\frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{\xi}_1)_y (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}) + \frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})_y \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}),$$

$$R_{12} = \frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{u}_1)_y (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}) - \frac{1}{2}\int_0^\tau \int_\Gamma (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta})_y \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}).$$

By interpolation,

$$|R_{11}| \leqslant C\left(\int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^{1/2}_{L^2(\Omega_{f,2,\delta}(t))} ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^{3/2}_{H^1(\Omega_{f,2,\delta}(t))} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||_{L^2(\Gamma)} ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||_{H^1(\Omega_{f,2,\delta}(t))}\right)$$
$$\leqslant \epsilon \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))} + C(\epsilon)\left(\int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Gamma)}\right).$$

By using the same interpolation inequality, we obtain

$$|R_{12}| \leqslant \epsilon \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}.$$

For R_2, we rewrite R_2 as

$$R_2 = -\frac{1}{2}\int_0^\tau\int_\Gamma(\partial_x\omega_1-\partial_x\omega_{2,\delta})(\boldsymbol{u}_1)_x\boldsymbol{u}_1\cdot(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})-\frac{1}{2}\int_0^\tau\int_\Gamma\partial_x\omega_{2,\delta}(\boldsymbol{u}_1)_x(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})\cdot(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})$$
$$+\frac{1}{2}\int_0^\tau\int_\Gamma(\partial_x\omega_1-\partial_x\omega_{2,\delta})(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})_x|\boldsymbol{u}_1|^2+\frac{1}{2}\int_0^\tau\int_\Gamma\partial_x\omega_{2,\delta}(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})_x\boldsymbol{u}_1\cdot(\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta}).$$

By interpolation, the boundedness of $|\partial_x\omega_1|$ and $|\partial_x\omega_{2,\delta}|$, and the smoothness of $\boldsymbol{u}_1$,

$$|R_2| \leqslant \epsilon\int_0^\tau||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))}+C(\epsilon)\left(\int_0^\tau||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)}+\int_0^\tau||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

Hence, we obtain the final result that

$$|T_3| \leqslant \epsilon\int_0^\tau||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{H^1(\Omega_{f,2,\delta}(t))}$$
$$+C(\epsilon)\left(\int_0^\tau||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)}+\int_0^\tau||\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Gamma)}+\int_0^\tau||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

Term 4

As usual, we use Proposition 5.9.1 to pass to the limit as $\alpha\to 0$ so that the contribution from T_4 is

$$T_4 := 2\nu\int_0^\tau\int_{\Omega_{f,1}(t)}\boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})-2\nu\int_0^\tau\int_{\Omega_{f,2,\delta}(t)}\boldsymbol{D}(\boldsymbol{u}_{2,\delta}):\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}). \quad (5.153)$$

We want to transfer the integral on $\Omega_{f,1}(t)$ to $\Omega_{f,2,\delta}(t)$. Recalling (5.106), we have that

$$\int_0^\tau\int_{\Omega_{f,1}(t)}\boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})=\int_0^\tau\int_{\Omega_{f,2,\delta}(t)}\gamma_\delta[\nabla(\boldsymbol{u}_1\circ\psi_\delta)J_\delta^{-1}]^s:[\nabla((\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})\circ\psi_\delta)J_\delta^{-1}]^s,$$

where the superscript ‘s’ notation denotes a symmetrization. Following estimates in [159], we break up the integral as

$$\int_0^\tau\int_{\Omega_{f,1}(t)}\boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{u}_1-\breve{\boldsymbol{u}}_{2,\delta})=\int_0^\tau\int_{\Omega_{f,2,\delta}(t)}\boldsymbol{D}(\widehat{\boldsymbol{u}}_1):\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})+R_1+R_2+R_3+R_4, \quad (5.154)$$

where

$$R_1=\int_0^\tau\int_{\Omega_{f,2,\delta}(t)}(\nabla(\boldsymbol{u}_1\circ\psi_\delta)J_\delta^{-1})^s:[\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})(J_\delta^{-1}-I)+(J_\delta-I)\nabla(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})J_\delta^{-1}]^s,$$

$$R_2=\int_0^\tau\int_{\Omega_{f,2,\delta}(t)}[(I-\gamma_\delta J_\delta^{-1})\nabla(\boldsymbol{u}_1\circ\psi_\delta)+\nabla(\boldsymbol{u}_1\circ\psi_\delta)(J_\delta^{-1}-I)]^s:\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}),$$

$$R_3 = \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} (\nabla(\boldsymbol{u}_1 \circ \psi_\delta) J_\delta^{-1})^s : (\gamma_\delta \nabla(\gamma_\delta^{-1} J_\delta)(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}) J_\delta^{-1})^s,$$

$$R_4 = -\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [(\nabla(\gamma_\delta J_\delta^{-1}))\boldsymbol{u}_1 \circ \psi_\delta]^s : \boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}).$$

To verify this equality, one can use the Leibniz rule, the definition $\widehat{\boldsymbol{u}}_1 = \gamma_\delta J_\delta^{-1} \cdot (\boldsymbol{u}_1 \circ \psi_\delta)$, and the identity $\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta} = \gamma_\delta J_\delta^{-1} \cdot ((\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}) \circ \psi_\delta)$.

Recalling the definition of J_δ from (5.104), we have the following inequalities

$$|J_\delta^{-1}| \leqslant C(1 + |\partial_x \gamma_\delta|), \qquad |J_\delta^{-1} - I| \leqslant C(|\gamma_\delta^{-1} - 1| + |\partial_x \gamma_\delta|),$$

$$|J_\delta - I| \leqslant C(|\gamma_\delta - 1| + |\partial_x \gamma_\delta|), \qquad |\gamma_\delta J_\delta^{-1} - I| \leqslant C(|\gamma_\delta - 1| + |\partial_x \gamma_\delta|).$$

Recalling the definition (5.102),

$$|\gamma_\delta - 1| \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}, \qquad |\gamma_\delta^{-1} - 1| \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)},$$

$$|\partial_x \gamma_\delta| \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}, \qquad |\partial_x(\gamma_\delta^{-1})| \leqslant C||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}.$$

Because $|J_\delta^{-1}| \leqslant C(1 + |\partial_x \gamma_\delta|) \leqslant C$ since $|\partial_x \gamma_\delta|$ is bounded, and because $\boldsymbol{u}_1$ is smooth,

$$|R_1| \leqslant C \int_0^\tau ||\nabla(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||_{L^2(\Omega_{f,2,\delta}(t))} (||\gamma_\delta^{-1} - 1||_{L^2(\Omega_{f,2,\delta}(t))} + ||\gamma_\delta - 1||_{L^2(\Omega_{f,2,\delta}(t))} + ||\partial_x \gamma_\delta||_{L^2(\Omega_{f,2,\delta}(t))})$$
$$\leqslant \epsilon \int_0^\tau ||\nabla(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon) \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)}.$$

We also have that

$$|R_2| \leqslant \epsilon \int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon) \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)}.$$

For R_3 and R_4, we compute that

$$\nabla(\gamma_\delta^{-1} J_\delta) = \nabla \begin{pmatrix} \gamma_\delta^{-1} & 0 \\ (R+y)\gamma_\delta^{-1} \partial_x \gamma_\delta & 1 \end{pmatrix}, \qquad \nabla(\gamma_\delta J_\delta^{-1}) = \nabla \begin{pmatrix} \gamma_\delta & 0 \\ -(R+y)\partial_x \gamma_\delta & 1 \end{pmatrix}.$$

Therefore,

$$|\nabla(\gamma_\delta^{-1} J_\delta)| \leqslant C(|\partial_x(\gamma_\delta^{-1})| + |\partial_x \gamma_\delta| + |\partial_{xx} \gamma_\delta|), \qquad |\nabla(\gamma_\delta J_\delta^{-1})| \leqslant C(|\partial_x \gamma_\delta| + |\partial_{xx} \gamma_\delta|),$$

where we can estimate

$$|\partial_{xx} \gamma_\delta| \leqslant C(||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} |\partial_{xx} \omega_1| + |\partial_{xx}(\omega_1 - \omega_{2,\delta})| + ||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)}).$$

So since $||\partial_{xx} \omega_1||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C$ since ω_1 is uniformly bounded in $H^2(\Gamma)$, we have that

$$|R_3| \leqslant C \int_0^\tau ||\nabla(\gamma_\delta^{-1} J_\delta)||_{L^2(\Omega_{f,2,\delta}(t))} ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||_{L^2(\Omega_{f,2,\delta}(t))}$$
$$\leqslant C \int_0^\tau ||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||_{L^2(\Omega_{f,2,\delta}(t))}$$
$$\leqslant C \left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))} \right).$$

Similarly, using $||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C||\omega_1-\omega_{2,\delta}||_{H^2(\Gamma)}$, we have the following estimate for R_4:

$$|R_4| \leqslant C\int_0^\tau ||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,2,\delta}(t))}||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})||_{L^2(\Omega_{f,2,\delta}(t))}$$
$$\leqslant \epsilon\int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)}.$$

Therefore, using (5.153) and (5.154), we have that the total contribution of this term is

$$T_4 = 2\nu\int_0^\tau\int_{\Omega_{f,2,\delta}(t)} |\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})|^2 + R,$$

where

$$|R| \leqslant \epsilon\int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon)\left(\int_0^\tau ||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta}||^2_{L^2(\Omega_{f,2,\delta}(t))}\right).$$

Term 5

After passing to the limit as $\alpha \to 0$, the contribution of this term is

$$T_5 = \beta\int_0^\tau\int_{\Gamma_1(t)} (\boldsymbol{\xi}_1-\boldsymbol{u}_1)_\tau[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta})_\tau - (\boldsymbol{u}_1-\check{\boldsymbol{u}}_{2,\delta})_\tau] - \beta\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta}-\boldsymbol{u}_{2,\delta})_\tau[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta})_\tau - (\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})_\tau].$$

We note that when we test the weak formulation for $\boldsymbol{u}_1$ with $\boldsymbol{v} = \boldsymbol{u}_1 - (\check{\boldsymbol{u}}_{2,\delta})_\alpha$ and $\boldsymbol{\psi} = \boldsymbol{\xi}_1 - (\boldsymbol{\xi}_{2,\delta})_\alpha$, we can pass to the limit as $\alpha \to 0$ to obtain the first term in T_5 above, by using similar arguments involving Proposition 5.9.1, as for the previously considered terms.

We can rewrite this term as

$$T_5 = \beta\int_0^\tau\int_{\Gamma_{2,\delta}(t)} |(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta})_\tau - (\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})_\tau|^2 + R_5,$$

where

$$R_5 = \beta\int_0^\tau\int_{\Gamma_1(t)} (\boldsymbol{\xi}_1-\boldsymbol{u}_1)_\tau[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta})_\tau - (\boldsymbol{u}_1-\check{\boldsymbol{u}}_{2,\delta})_\tau] - \beta\int_0^\tau\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{\xi}_1-\widehat{\boldsymbol{u}}_1)_\tau[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta})_\tau - (\widehat{\boldsymbol{u}}_1-\boldsymbol{u}_{2,\delta})_\tau].$$

We denote the arc length elements of $\Gamma_t(t)$ and $\Gamma_{2,\delta}(t)$ respectively by $\mathcal{J}_\Gamma^{\omega_1} = \sqrt{1+|\partial_x\omega_1|^2}$ and $\mathcal{J}_\Gamma^{\omega_{2,\delta}} = \sqrt{1+|\partial_x\omega_{2,\delta}|^2}$, and we denote the tangent vectors to $\Gamma_1(t)$ and $\Gamma_{2,\delta}(t)$ respectively by $\boldsymbol{\tau}_1 = \frac{1}{\mathcal{J}_\Gamma^{\omega_1}}(1,\partial_x\omega_1)$ and $\boldsymbol{\tau}_{2,\delta} = \frac{1}{\mathcal{J}_\Gamma^{\omega_{2,\delta}}}(1,\partial_x\omega_{2,\delta})$.

We can rewrite R_5 by writing everything in terms of the x and y components, where we recall that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_{2,\delta}$ along the interface displace in only the y direction. We formally express the common trace of $\boldsymbol{u}_1 - \check{\boldsymbol{u}}_{2,\delta}$ and $\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}$ along the reference configuration of the interface Γ by $\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta}$. Thus,

$$R_5 = \beta\int_0^\tau\int_\Gamma (\boldsymbol{\xi}_1-\boldsymbol{u}_1)\cdot(1,\partial_x\omega_1)[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta}) - (\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})]\cdot\boldsymbol{\tau}_1$$
$$-\beta\int_0^\tau\int_\Gamma (\boldsymbol{\xi}_1-\boldsymbol{u}_1)\cdot(1,\partial_x\omega_{2,\delta})[(\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,\delta}) - (\boldsymbol{u}_1-\boldsymbol{u}_{2,\delta})]\cdot\boldsymbol{\tau}_{2,\delta}.$$

In the previous step, we use the fact that when transferred back to the reference configuration Ω_f, $\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta}$ and $\boldsymbol{u}_1 - \breve{\boldsymbol{u}}_{2,\delta}$ have the same trace along Γ. Thus, $R_5 = R_{5,1} + R_{5,2}$, where

$$R_{5,1} = \beta \int_0^\tau \int_\Gamma (\boldsymbol{\xi}_1 - \boldsymbol{u}_1)_y (\partial_x \omega_1 - \partial_x \omega_{2,\delta})[(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) - (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta})] \cdot \boldsymbol{\tau}_1,$$

$$R_{5,2} = \beta \int_0^\tau \int_\Gamma (\boldsymbol{\xi}_1 - \boldsymbol{u}_1) \cdot (1, \partial_x \omega_{2,\delta})[(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) - (\boldsymbol{u}_1 - \boldsymbol{u}_{2,\delta})] \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_{2,\delta}).$$

We can use the fact that $|\partial_x \omega_1|$ and $|\partial_x \omega_{2,\delta}|$ are uniformly bounded to obtain the following estimates:

$$|R_{5,1}| \leqslant \epsilon \int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon) \left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Gamma)} \right),$$

where we used the trace inequality, Poincare's inequality, and Korn's inequality for the fluid. For the second term $R_{6,2}$, we use the estimate $|\boldsymbol{\tau}^{\omega_1} - \boldsymbol{\tau}^{\omega_{2,\delta}}| \leqslant C|\partial_x \omega_1 - \partial_x \omega_{2,\delta}|$ to obtain

$$|R_{5,2}| \leqslant \epsilon \int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon) \left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Gamma)} \right).$$

Hence,

$$T_5 = \beta \int_0^\tau \int_{\Gamma_{2,\delta}(t)} |(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})_\tau - (\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})_\tau|^2 + R_6,$$

where

$$|R_5| \leqslant \epsilon \int_0^\tau ||\boldsymbol{D}(\widehat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||_{L^2(\Omega_{f,2,\delta}(t))} + C(\epsilon) \left(\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Gamma)} \right).$$

Terms 6-8

We will consider Term 6, as the same procedure will hold for Terms 7 and 8. For Term 6, we note that ζ_1 and $\zeta_{2,\delta}$ are weakly continuous in $L^2(\Gamma)$, by the weak formulation. Hence,

$$\begin{aligned} &\int_0^\tau \int_\Gamma \zeta_1 \cdot \partial_t \left[(\zeta_{2,\delta})_\alpha\right] + \int_0^\tau \int_\Gamma \zeta_{2,\delta} \cdot \partial_t \zeta_1 \\ &\quad = \int_0^\tau \int_\Gamma \zeta_1 \cdot \partial_t \left[(\zeta_{2,\delta})_\alpha\right] + \int_0^\tau \int_\Gamma \zeta_{2,\delta} \cdot \partial_t \left[(\zeta_1)_\alpha\right] - \int_0^\tau \int_\Gamma \zeta_{2,\delta} \cdot \partial_t \left[(\zeta_1)_\alpha - \zeta_1\right] \\ &\quad\quad \to \int_\Gamma \zeta_1(\tau) \cdot \zeta_{2,\delta}(\tau) - \int_\Gamma |\zeta_0|^2. \end{aligned}$$

This is because by Lemma 2.5 in [159],

$$\int_0^\tau \int_\Gamma \zeta_1 \cdot \partial_t \left[(\zeta_{2,\delta})_\alpha\right] + \int_0^\tau \int_\Gamma \zeta_{2,\delta} \cdot \partial_t \left[(\zeta_1)_\alpha\right] \to \int_\Gamma \zeta_1(\tau) \cdot \zeta_{2,\delta}(\tau) - \int_\Gamma |\zeta_0|^2, \qquad \text{as } \alpha \to 0,$$

and since ζ_1 is smooth in space and time, we have that

$$\int_0^\tau \int_\Gamma \zeta_{2,\delta} \cdot \partial_t \left[(\zeta_1)_\alpha - \zeta_1\right] \to 0, \qquad \text{as } \alpha \to 0.$$

Furthermore, because $\zeta_1(0) = \zeta_{2,\delta}(0) = \zeta_0$, we have that $\int_\Gamma \zeta_1(0) \cdot [\zeta_1(0) - (\zeta_{2,\delta})_\alpha(0)] \to 0$ as $\alpha \to 0$ by the weak continuity of $\zeta_{2,\delta}$ at $t = 0$ and similarly, $\int_\Gamma \zeta_1(\tau) \cdot [\zeta_1(\tau) - (\zeta_{2,\delta})_\alpha(\tau)] \to 0$ as $\alpha \to 0$ for almost every $\tau \in [0, T]$. Hence, as $\alpha \to 0$, the contribution from T_6 is

$$T_6 = \frac{1}{2}\rho_p \int_\Gamma |(\zeta_1 - \zeta_2)(\tau)|^2.$$

Similarly, the contributions from Terms 7 and 8 as $\alpha \to 0$ are

$$T_7 = \frac{1}{2}\int_\Gamma |\Delta(\omega_1 - \omega_2)(\tau)|^2, \qquad T_8 = \frac{1}{2}\rho_b \int_{\Omega_b} |(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)(\tau)|^2.$$

Terms 9-12

Because $\boldsymbol{\xi}_{2,\delta} \in L^2(0,T;H^1(\Omega_b))$ where Ω_b is a fixed domain, we have that $(\boldsymbol{\xi}_{2,\delta})_\alpha \to \boldsymbol{\xi}_{2,\alpha}$ strongly in $L^2(0,T;H^1(\Omega_b))$. Hence, we have that Terms 9-12 converge to the following as $\alpha \to 0$.

$$T_9 = \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_{2,\delta})(\tau)|^2, \qquad T_{10} = \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)(\tau)|^2,$$

$$T_{11} = 2\mu_v \int_0^\tau \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})|^2, \qquad T_{12} = \lambda_v \int_0^\tau \int_{\Omega_b} |\nabla \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})|^2.$$

Term 13

By taking the limit as $\alpha \to 0$, we have that

$$T_{13} = -\alpha \int_0^\tau \int_{\Omega_{b,1}(t)} p_1 \nabla \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) + \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} p_{2,\delta} \nabla \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}).$$

Then, we estimate using (5.26) and the matrix identity $\boldsymbol{B}^{-1} = \frac{1}{\det(\boldsymbol{B})}\boldsymbol{B}^C$,

$$|T_{13}| = \alpha \left| \int_0^\tau \int_{\Omega_b} \mathcal{J}_b^{\eta_1} p_1 \nabla_b^{\eta_1} \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) - \int_0^\tau \int_{\Omega_b} \mathcal{J}_b^{\tilde{\eta}_{2,\delta}} p_{2,\delta} \nabla_b^{\tilde{\eta}_{2,\delta}} \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \right|$$
$$= \alpha \left| \int_0^\tau \int_{\Omega_b} p_1 \cdot \text{tr}\left(\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^C\right) - \int_0^\tau \int_{\Omega_b} p_{2,\delta} \cdot \text{tr}\left(\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C\right) \right| \leqslant I_1 + I_2,$$

where the superscript "C" denotes the cofactor matrix and where

$$I_1 = \alpha \left| \int_0^\tau \int_{\Omega_b} p_1 \cdot \text{tr}\left(\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot (\nabla(\boldsymbol{\eta}_1 - \tilde{\boldsymbol{\eta}}_{2,\delta}))^C\right) \right|,$$

$$I_2 = \alpha \left| \int_0^\tau \int_{\Omega_b} (p_1 - p_{2,\delta}) \cdot \text{tr} \left(\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C \right) \right|.$$

In the previous calculations, we observe that the cofactor matrix operation is linear when the matrices are two by two. Using the fact that p_1 is smooth, the assumption (5.114), and the fact that

$$||\nabla \tilde{\boldsymbol{\eta}}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}||_{L^2(\Omega_b)} \leqslant C||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_2||_{L^2(\tilde{\Omega}_b)} \leqslant C \left(||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_{2,\delta}||_{L^2(\Omega_b)} + ||\omega_1 - \omega_{2,\delta}||_{H^2(\Gamma)} \right) \tag{5.155}$$

for a constant C independent of δ, by Young's convolution inequality and the definition of odd extension to the larger domain $\tilde{\Omega}_b$ in Definition 5.7.2, we obtain

$$\begin{aligned} I_1 &\leqslant \epsilon \int_0^\tau ||\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})||^2_{L^2(\Omega_b)} + C(\epsilon) \left(\int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}||^2_{L^2(\Omega_b)} \right) \\ &\leqslant C(\epsilon) \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \epsilon \int_0^\tau ||\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})||^2_{L^2(\Omega_b)} \\ &\quad + C(\epsilon) \left(\int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} \right), \end{aligned}$$

and

$$I_2 \leqslant \epsilon \int_0^\tau ||\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})||^2_{L^2(\Omega_b)} + C(\epsilon) \left(\int_0^\tau ||p_1 - p_{2,\delta}||^2_{L^2(\Omega_b)} \right).$$

Therefore, we conclude that

$$\begin{aligned} |T_{13}| &\leqslant C(\epsilon) \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \epsilon \int_0^\tau ||\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})||^2_{L^2(\Omega_b)} \\ &\quad + C(\epsilon) \left(\int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||p_1 - p_{2,\delta}||^2_{L^2(\Omega_b)} \right). \end{aligned}$$

Term 14

This term can be handled in the same way as Terms 6-8. We obtain that in the limit as $\alpha \to 0$, the contribution from this term is

$$T_{14} = \frac{1}{2} c_0 \int_{\Omega_b} |(p_1 - p_{2,\delta})(\tau)|^2.$$

Term 15

Upon passing to the limit as $\alpha \to 0$,

$$T_{15} = -\alpha \int_0^\tau \int_{\Omega_{b,1}(t)} \frac{D}{Dt} \boldsymbol{\eta}_1 \cdot \nabla(p_1 - p_{2,\delta}) + \alpha \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \frac{\tilde{D}}{Dt} \boldsymbol{\eta}_{2,\delta} \cdot \nabla(p_1 - p_{2,\delta}).$$

By pulling back to the reference domain, we can estimate using (5.26) and the cofactor formula for the matrix inverse:

$$|T_{15}| = \alpha\left|\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\eta_1}\partial_t\boldsymbol{\eta}_1\cdot\nabla_b^{\eta_1}(p_1-p_{2,\delta}) - \int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}\partial_t\boldsymbol{\eta}_{2,\delta}\cdot\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})\right|$$
$$= \alpha\left|\int_0^\tau\int_{\Omega_b}\partial_t\boldsymbol{\eta}_1\cdot\left[\nabla(p_1-p_{2,\delta})\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_1)^C\right] - \int_0^\tau\int_{\Omega_b}\partial_t\boldsymbol{\eta}_{2,\delta}\cdot\left[\nabla(p_1-p_{2,\delta})\cdot(\boldsymbol{I}+\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C\right]\right| \leqslant I_1+I_2,$$

where

$$I_1 = \alpha\left|\int_0^\tau\int_{\Omega_b}\partial_t\boldsymbol{\eta}_1\cdot\left[\nabla(p_1-p_{2,\delta})\cdot(\nabla\boldsymbol{\eta}_1-\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C\right]\right|,$$
$$I_2 = \alpha\left|\int_0^\tau\int_{\Omega_b}(\partial_t\boldsymbol{\eta}_1-\partial_t\boldsymbol{\eta}_{2,\delta})\cdot\left[\nabla(p_1-p_{2,\delta})\cdot(\boldsymbol{I}+\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C\right]\right|.$$

For I_1, by using (5.112), (5.113), and the convolution inequality (5.155), we have

$$I_1 \leqslant \epsilon\int_0^\tau ||\nabla(p_1-p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\nabla\boldsymbol{\eta}_1-\nabla\tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)}$$
$$+ C(\epsilon)\left(\int_0^\tau ||\nabla\boldsymbol{\eta}_1-\nabla\boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)}\right).$$

Here, we used the following estimate on the norm of the gradient of the pressure on the reference domain and on the moving domain. We observe, using (5.112), (5.113), and (5.26), that

$$||\nabla(p_1-p_{2,\delta})(t)||^2_{L^2(\Omega_b)} = \int_{\Omega_b}|\nabla(p_1-p_{2,\delta})|^2 = \int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}|\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})\cdot(\boldsymbol{I}+\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})|^2\cdot(\mathcal{J}_b^{\tilde{\eta}_{2,\delta}})^{-1}$$
$$\leqslant C\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}|\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})|^2 = C||\nabla(p_1-p_{2,\delta})(t)||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))}, \quad (5.156)$$

for a constant C independent of δ and the time $t \in [0, T_\delta]$.

For I_2, we have

$$I_2 \leqslant \epsilon\int_0^\tau ||\nabla(p_1-p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\partial_t\boldsymbol{\eta}_1-\partial_t\boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)}.$$

Thus, we have that

$$|T_{15}| \leqslant \epsilon\int_0^\tau ||\nabla(p_1-p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} + C(\epsilon)\int_0^\tau ||\nabla\boldsymbol{\eta}_1-\nabla\tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)}$$
$$+ C(\epsilon)\left(\int_0^\tau ||\nabla\boldsymbol{\eta}_1-\nabla\boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1-\omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\partial_t\boldsymbol{\eta}_1-\partial_t\boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)}\right).$$

Term 16

After passing to the limit as $\alpha \to 0$, we want to estimate the quantity

$$T_{16} = -\alpha \int_0^\tau \int_{\Gamma_1(t)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}_1)(p_1 - p_{2,\delta}) + \alpha \int_0^\tau \int_{\tilde{\Gamma}_{2,\delta}(t)} (\boldsymbol{\xi}_{2,\delta} \cdot \tilde{\boldsymbol{n}}_{2,\delta})(p_1 - p_{2,\delta}),$$

where $\tilde{\boldsymbol{n}}_{2,\delta}$ is the upward pointing normal vector to $\tilde{\Gamma}_{2,\delta}(t)$. We integrate by parts to obtain that $|T_{16}| \leqslant I_1 + I_2$, where

$$I_1 := \alpha \left| \int_0^\tau \int_{\Omega_{b,1}(t)} (\nabla \cdot \boldsymbol{\xi}_1)(p_1 - p_{2,\delta}) - \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} (\nabla \cdot \boldsymbol{\xi}_{2,\delta})(p_1 - p_{2,\delta}) \right|,$$

$$I_2 := \alpha \left| \int_0^\tau \int_{\Omega_{b,1}(t)} \boldsymbol{\xi}_1 \cdot \nabla(p_1 - p_{2,\delta}) - \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} \boldsymbol{\xi}_{2,\delta} \cdot \nabla(p_1 - p_{2,\delta}) \right|.$$

By using (5.26) and the bootstrap assumption (5.114), we have that

$$\begin{aligned}
I_1 &= \alpha \left| \int_0^\tau \int_{\Omega_b} \mathcal{J}_b^{\eta_1} (\text{tr}(\nabla_b^{\eta_1} \boldsymbol{\xi}_1))(p_1 - p_{2,\delta}) - \int_0^\tau \int_{\Omega_b} \mathcal{J}_b^{\tilde{\eta}_{2,\delta}} (\text{tr}(\nabla_b^{\tilde{\eta}_{2,\delta}} \boldsymbol{\xi}_{2,\delta}))(p_1 - p_{2,\delta}) \right| \\
&= \alpha \left| \int_0^\tau \int_{\Omega_b} \text{tr}(\nabla \boldsymbol{\xi}_1 \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^C)(p_1 - p_{2,\delta}) - \int_0^\tau \int_{\Omega_b} \text{tr}(\nabla \boldsymbol{\xi}_{2,\delta} \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C)(p_1 - p_{2,\delta}) \right| \\
&\leqslant \alpha \left| \int_0^\tau \int_{\Omega_b} \text{tr}(\nabla \boldsymbol{\xi}_1 \cdot (\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C)(p_1 - p_{2,\delta}) \right| + \alpha \left| \int_0^\tau \int_{\Omega_b} \text{tr}(\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C)(p_1 - p_{2,\delta}) \right| \\
&\leqslant C \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}||_{L^2(\Omega_b)} \cdot ||p_1 - p_{2,\delta}||_{L^2(\Omega_b)} + C \int_0^\tau ||\nabla \boldsymbol{\xi}_1 - \nabla \boldsymbol{\xi}_{2,\delta}||_{L^2(\Omega_b)} \cdot ||p_1 - p_{2,\delta}||_{L^2(\Omega_b)}.
\end{aligned}$$

For the second term, we compute that

$$\begin{aligned}
I_2 &= \alpha \left| \int_0^\tau \int_{\Omega_b} \boldsymbol{\xi}_1 \cdot \left[\nabla(p_1 - p_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^C\right] - \int_0^\tau \int_{\Omega_b} \boldsymbol{\xi}_{2,\delta} \cdot \left[\nabla(p_1 - p_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C\right] \right| \\
&\leqslant \alpha \left| \int_0^\tau \int_{\Omega_b} \boldsymbol{\xi}_1 \cdot \left[\nabla(p_1 - p_{2,\delta}) \cdot (\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C\right] \right| + \alpha \left| \int_0^\tau \int_{\Omega_b} (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}) \cdot \left[\nabla(p_1 - p_{2,\delta}) \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^C\right] \right| \\
&\leqslant C \int_0^\tau ||\nabla p_1 - \nabla p_{2,\delta}||_{L^2(\Omega_b)} \cdot ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}||_{L^2(\Omega_b)} + C \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||_{L^2(\Omega_b)} \cdot ||\nabla p_1 - \nabla p_{2,\delta}||_{L^2(\Omega_b)}.
\end{aligned}$$

By the convolution inequality (5.155) and the previous estimate on the gradient of the pressure (5.156), we conclude that

$$\begin{aligned}
|T_{16}| &\leqslant \epsilon \left(\int_0^\tau ||\nabla \boldsymbol{\xi}_1 - \nabla \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\nabla p_1 - \nabla p_{2,\delta}||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} \right) + C(\epsilon) \left(\int_0^\tau ||p_1 - p_{2,\delta}||^2_{L^2(\Omega_b)} \right. \\
&\left. + \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} + \int_0^\tau ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta}||^2_{L^2(\Omega_b)} \right).
\end{aligned}$$

Term 17

We want to estimate

$$T_{17}=\kappa\int_0^\tau\int_{\Omega_{b,1}(t)}\nabla p_1\cdot\nabla(p_1-p_{2,\delta})-\kappa\int_0^\tau\int_{\tilde{\Omega}_{b,2,\delta}(t)}\nabla p_{2,\delta}\cdot\nabla(p_1-p_{2,\delta}).$$

Using (5.26), we compute that

$$T_{17}=\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\eta_1}\nabla_b^{\eta_1}p_1\cdot\nabla_b^{\eta_1}(p_1-p_{2,\delta})-\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}\nabla_b^{\tilde{\eta}_{2,\delta}}p_{2,\delta}\cdot\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})$$

$$=\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})\cdot\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta})+I_1+I_2=\kappa\int_0^\tau\int_{\tilde{\Omega}_{b,2,\delta}(t)}|\nabla(p_1-p_{2,\delta})|^2+I_1+I_2,$$

where

$$I_1=\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\eta_1}\nabla_b^{\eta_1}p_1\cdot\nabla_b^{\eta_1}(p_1-p_{2,\delta})-\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}\nabla_b^{\eta_1}p_1\cdot\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta}),$$

$$I_2=\kappa\int_0^\tau\int_{\Omega_b}\mathcal{J}_b^{\tilde{\eta}_{2,\delta}}(\nabla_b^{\eta_1}p_1-\nabla_b^{\tilde{\eta}_{2,\delta}}p_1)\cdot\nabla_b^{\tilde{\eta}_{2,\delta}}(p_1-p_{2,\delta}).$$

We estimate I_1 as follows. Using (5.26), we compute

$$I_1=\kappa\int_0^\tau\int_{\Omega_b}\nabla_b^{\eta_1}p_1\cdot\Big(\nabla(p_1-p_{2,\delta})\cdot\big[(\boldsymbol{I}+\nabla\boldsymbol{\eta}_1)^C-(\boldsymbol{I}+\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C\big]\Big).$$

Because $\boldsymbol{\eta}_1$ is smooth, $|\nabla_b^{\eta_1}p_1|\leqslant C$ uniformly in space and time. Therefore,

$$|I_1|\leqslant C\int_0^\tau||\nabla(p_1-p_{2,\delta})||_{L^2(\Omega_b)}\cdot||(\nabla\boldsymbol{\eta}_1-\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C||_{L^2(\Omega_b)}.$$

Using the estimate in (5.156), we obtain the desired estimate that

$$|I_1|\leqslant\epsilon\int_0^\tau||\nabla(p_1-p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))}+C(\epsilon)\int_0^\tau||\nabla\boldsymbol{\eta}_1-\nabla\tilde{\boldsymbol{\eta}}_{2,\delta}||^2_{L^2(\Omega_b)}.$$

For I_2, we use the bootstrap assumption (5.114) that there exists a constant C (independent of δ) such that $|\nabla\tilde{\boldsymbol{\eta}}_{2,\delta}|\leqslant C$ pointwise for $t\in[0,T_\delta]$. Therefore, $|(\boldsymbol{I}+\nabla\boldsymbol{\eta}_{2,\delta})^C|$ is is pointwise uniformly bounded in space and time on the time interval $[0,T_\delta]$. Thus, by (5.26),

$$I_2=\kappa\int_0^\tau\int_{\Omega_b}(\nabla_b^{\eta_1}p_1-\nabla_b^{\tilde{\eta}_{2,\delta}}p_1)\cdot\big[\nabla(p_1-p_{2,\delta})\cdot(\boldsymbol{I}+\nabla\tilde{\boldsymbol{\eta}}_{2,\delta})^C\big]$$

and hence,

$$|I_2|\leqslant C\int_0^\tau||\nabla_b^{\eta_1}p_1-\nabla_b^{\tilde{\eta}_{2,\delta}}p_1||_{L^2(\Omega_b)}\cdot||\nabla(p_1-p_{2,\delta})||_{L^2(\Omega_b)}.$$

Using (5.26), we compute that

$$||\nabla_b^{\eta_1} p_1 - \nabla_b^{\tilde{\eta}_{2,\delta}} p_1||^2_{L^2(\Omega_b)} = \int_{\Omega_b} \left|\nabla p_1 \cdot \left[(\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^{-1} - (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^{-1}\right]\right|^2$$
$$= \int_{\Omega_b} \left|\nabla p_1 \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^{-1} [(\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})(\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^{-1} - \boldsymbol{I}]\right|^2$$
$$= \int_{\Omega_b} \left|\nabla p_1 \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^{-1} [(\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}) - (\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)](\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^{-1}\right|^2$$
$$= \int_{\Omega_b} \left|\nabla p_1 \cdot (\boldsymbol{I} + \nabla \tilde{\boldsymbol{\eta}}_{2,\delta})^{-1} (\nabla \tilde{\boldsymbol{\eta}}_{2,\delta} - \nabla \boldsymbol{\eta}_1)(\boldsymbol{I} + \nabla \boldsymbol{\eta}_1)^{-1}\right|^2.$$

Using the fact that p_1 is smooth and the bootstrap assumption (5.113), we have that

$$||\nabla_b^{\eta_1} p_1 - \nabla_b^{\tilde{\eta}_{2,\delta}} p_1||^2_{L^2(\Omega_b)} \leqslant C||\nabla \tilde{\boldsymbol{\eta}}_{2,\delta} - \nabla \boldsymbol{\eta}_1||^2_{L^2(\Omega_b)}.$$

Therefore, combining this with the previous estimate (5.156), we obtain

$$I_2 \leqslant \epsilon \int_0^\tau ||\nabla(p_1 - p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} + C(\epsilon) \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_{2,\delta}||^2_{L^2(\Omega_b)}.$$

By applying the previous convolution inequality (5.155), we thus obtain the final estimate that

$$T_{17} \leqslant \kappa \int_0^\tau \int_{\tilde{\Omega}_{b,2,\delta}(t)} |\nabla(p_1 - p_{2,\delta})|^2 + R,$$

where the remainder is bounded by

$$|R| \leqslant \epsilon \int_0^\tau ||\nabla(p_1 - p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))}$$
$$+ C(\epsilon) \left(\int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \int_0^\tau ||\nabla \boldsymbol{\eta}_1 - \nabla \boldsymbol{\eta}_{2,\delta}||^2_{L^2(\Omega_b)} + \int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)} \right).$$

Term 18

We want to estimate

$$\begin{aligned}
T_{18} = &\int_0^\tau \int_{\Gamma_1(t)} p_1(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot \boldsymbol{n}_1 - \int_0^\tau \int_{\Gamma_1(t)} p_1(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot \boldsymbol{n}_1 \\
&- \int_0^\tau \int_{\Gamma_{2,\delta}(t)} p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot \boldsymbol{n}_{2,\delta} + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} p_{2,\delta}(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot \boldsymbol{n}_{2,\delta} \\
&- \int_0^\tau \int_{\Gamma_1(t)} ((\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot \boldsymbol{n}_1)(p_1 - p_{2,\delta}) + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot \boldsymbol{n}_{2,\delta})(p_1 - p_{2,\delta}) \\
&= -\int_0^\tau \int_{\Gamma_1(t)} p_1(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot \boldsymbol{n}_1 - \int_0^\tau \int_{\Gamma_{2,\delta}(t)} p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot \boldsymbol{n}_{2,\delta} \\
&\quad + \int_0^\tau \int_{\Gamma_1(t)} ((\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot \boldsymbol{n}_1)p_{2,\delta} + \int_0^\tau \int_{\Gamma_{2,\delta}(t)} ((\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot \boldsymbol{n}_{2,\delta})p_1.
\end{aligned}$$

By bringing all of the integrals back to the reference domain Γ,

$$\begin{aligned}
T_{18} = &-\int_0^\tau \int_\Gamma p_1(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot(-\partial_x\omega_1, 1) - \int_0^\tau \int_\Gamma p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot(-\partial_x\omega_{2,\delta}, 1) \\
&+ \int_0^\tau \int_\Gamma p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)\cdot(-\partial_x\omega_1, 1) + \int_0^\tau \int_\Gamma p_1(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})\cdot(-\partial_x\omega_{2,\delta}, 1) \\
&= \int_0^\tau \int_\Gamma p_1(\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})_x\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta}) - \int_0^\tau \int_\Gamma p_{2,\delta}(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta}) \\
&= -\int_0^\tau \int_\Gamma p_1[(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x - (\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})_x]\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta}) + \int_0^\tau \int_\Gamma (p_1 - p_{2,\delta})(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta}).
\end{aligned}$$

We obtain that in absolute values, this quantity is bounded as follows.

$$\begin{aligned}
&\left|\int_0^\tau \int_\Gamma p_1[(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x - (\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})_x]\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta})\right| + \left|\int_0^\tau \int_\Gamma (p_1 - p_{2,\delta})(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x\cdot(\partial_x\omega_1 - \partial_x\omega_{2,\delta})\right| \\
&\leqslant C\left(\int_0^\tau ||(\boldsymbol{u}_1 - \boldsymbol{\xi}_1)_x - (\boldsymbol{u}_{2,\delta} - \boldsymbol{\xi}_{2,\delta})_x||_{L^2(\Gamma)}||\partial_x\omega_1 - \partial_x\omega_{2,\delta}||_{L^2(\Gamma)} + \int_0^\tau ||p_1 - p_{2,\delta}||_{L^2(\Gamma)}||\partial_x\omega_1 - \partial_x\omega_{2,\delta}||_{L^2(\Gamma)}\right).
\end{aligned}$$

We thus use the trace theorem, Poincare's inequality, and Korn's inequality to conclude that

$$\begin{aligned}
|T_{18}| \leqslant \epsilon\left(\int_0^\tau ||\boldsymbol{D}(\hat{\boldsymbol{u}}_1 - \boldsymbol{u}_{2,\delta})||^2_{L^2(\Omega_{f,2,\delta}(t))} + \int_0^\tau ||\nabla(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,\delta})||_{L^2(\Omega_b)} + \int_0^\tau ||\nabla(p_1 - p_{2,\delta})||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))}\right) \\
+ C(\epsilon)\int_0^\tau ||\omega_1 - \omega_{2,\delta}||^2_{H^2(\Gamma)}.
\end{aligned}$$

5.11 The Gronwall estimate

Combining all of the previous estimates, we have that the following integral inequality holds for almost all $\tau \in [0, T_\delta]$, as long as the three assumptions (5.112), (5.113), and (5.114) hold:

$$E_\delta(\tau) \leqslant C\left(\int_0^\tau ||\nabla\boldsymbol{\eta}_1 - \nabla\tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \int_0^\tau E_\delta(t)dt\right), \tag{5.157}$$

where $E_\delta(t)$ is defined by (5.109). We have by Lemma 5.9.1 that

$$\int_0^t ||\nabla\boldsymbol{\eta}_1 - \nabla\tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} \leqslant C\delta^3.$$

So we rewrite (5.157) as

$$E_\delta(t) \leqslant C\delta^3 + C\int_0^t E_\delta(s)ds.$$

So by Gronwall's inequality, for all $t \in [0, T_\delta]$,

$$E_\delta(t) \leqslant C\delta^3 e^{Ct},$$

as long as the three conditions (5.112), (5.113), and (5.114) hold, where the constant C is independent of δ. By using the bootstrap argument described in Section 5.9, we complete the proof of the weak-classical consistency result described in the statement of Theorem 5.9.1.

5.12 Appendix

In this appendix, we show a result related to weak continuity of solutions to the regularized FPSI problem. This result will allow us to verify the claims from before that as $\alpha \to 0$,

$$\int_{\Omega_{f,2}(0)} \widehat{\boldsymbol{u}}_1(0)\cdot(\boldsymbol{u}_{2,\delta})_\alpha(0) \to \int_{\Omega_{f,2}(0)} |\boldsymbol{u}_0|^2, \quad \text{and} \quad \int_{\Omega_{f,2}(t)} \widehat{\boldsymbol{u}}_1(t)\cdot(\boldsymbol{u}_{2,\delta})_\alpha(t) \to \int_{\Omega_{f,2}(t)} \widehat{\boldsymbol{u}}_1(t)\cdot\boldsymbol{u}_{2,\delta}(t),$$

for almost all points $0 < t \leqslant T$. This result was used in (5.145), for estimating the first term T_1 given by (5.124) in order to show the weak-classical consistency result. We will accomplish this through the following series of lemmas.

Lemma 5.12.1. Let $\omega \in L^\infty(0,T;H^2_0(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))$ with

$$\min_{t\in[0,T], x\in[0,L]} R + \omega(t,x) > 0,$$

define the moving fluid domain $\Omega_f^\omega(t)$. Given $\boldsymbol{u} \in L^2(0,T;H^1(\Omega_f^\omega(t))) \cap L^\infty(0,T;L^2(\Omega_f^\omega(t)))$ where $\Omega_f^\omega(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega(t,x)\}$, we have that

$$||\boldsymbol{u}_\alpha(t,x,y) - \boldsymbol{u}(t,x,y)||_{L^2(\Omega_f^\omega(t))} \to 0 \qquad \text{as } \alpha \to 0,$$

for almost all $t \in [0,T]$.

Proof. Recall that in the case of real-valued functions, one shows convergence of the convolution to the function itself almost everywhere by using the Lebesgue differentiation theorem, see Section C.5 in [70]. To apply the theorem in this context, we need to apply it to a function taking values in a *fixed* Banach space rather than a time-dependent Banach space.

As a result, we consider the following function,

$$\boldsymbol{v}(t,x,y) = K(t,0,x,y)\boldsymbol{u}\left(t,x,\frac{R+\omega(t,x)}{R+\omega(0,x)}(R+y)-R\right),$$

where we have pulled the fluid velocity back to the fixed initial domain $\Omega_f^\omega(0)$. We recall the definition of $K(s,t,z,r)$ from (5.118) and its inverse:

$$K(s,t,x,y) = \begin{pmatrix} \frac{R+\omega(s,x)}{R+\omega(t,x)} & 0 \\ -(R+y)\partial_x\left(\frac{R+\omega(s,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix},$$

$$K^{-1}(s,t,x,y) = \begin{pmatrix} \frac{R+\omega(t,x)}{R+\omega(s,x)} & 0 \\ (R+y)\frac{R+\omega(t,x)}{R+\omega(s,x)}\partial_x\left(\frac{R+\omega(s,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix}.$$

By the uniform boundedness of $R+\omega(t,x)$ and $|\partial_x\omega(t,x)|$, and $\min_{t\in[0,T],x\in[0,L]} R+\omega(t,x) > 0$, it is immediate to see that $\boldsymbol{v}(t,z,r)$ is in $L^\infty(0,T;L^2(\Omega_f^\omega(0)))$, where we emphasize that $L^2(\Omega_f^\omega(0))$ is a fixed function space that no longer depends on time.

By Lebesgue's differentiation theorem, almost every $t \in [0,T]$ is a Lebesgue point satisfying

$$\lim_{\alpha\to 0}\frac{1}{2\alpha}\int_{t-\alpha}^{t+\alpha} ||\boldsymbol{v}(t,\cdot)-\boldsymbol{v}(s,\cdot)||_{L^2(\Omega_f^\omega(0))}ds \to 0. \tag{5.158}$$

Recall that by definition (5.119),

$$\boldsymbol{u}_\alpha(t,x,y) = \int_{\mathbb{R}} K(s,t,x,y)\boldsymbol{u}\left(s,x,\frac{R+\omega(s,x)}{R+\omega(t,x)}(R+y)-R\right)j_\alpha(t-s)ds.$$

Thus, we compute that

$$\begin{aligned}\boldsymbol{u}_\alpha(t,x,y)-\boldsymbol{u}(t,x,y) = \int_{\mathbb{R}}\left(K(s,t,x,y)\boldsymbol{u}\left(s,x,\frac{R+\omega(s,x)}{R+\omega(t,x)}(R+y)-R\right)-\boldsymbol{u}(t,x,y)\right)\cdot j_\alpha(t-s)ds \\ := I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 = \int_{\mathbb{R}} K^{-1}\left(t,0,x,\frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y)-R\right)\cdot \\ \left(\boldsymbol{v}\left(s,x,\frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y)-R\right)-\boldsymbol{v}\left(t,x,\frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y)-R\right)\right)j_\alpha(t-s)ds,\end{aligned}$$

$$I_2 = \int_{\mathbb{R}} \left(K(s,t,x,y) K^{-1} \left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right) \cdot \boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) j_\alpha(t-s).$$

We estimate each of these terms as follows. For I_1, we compute that

$$K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) = \begin{pmatrix} \frac{R+\omega(0,x)}{R+\omega(t,x)} & 0 \\ (R+y)\left(\frac{R+\omega(0,x)}{R+\omega(t,x)}\right)^2 \partial_x \left(\frac{R+\omega(t,x)}{R+\omega(0,x)}\right) & 1 \end{pmatrix},$$

which we note is uniformly bounded on $[0,T]$. Hence, using the fact that $|j_\alpha(t-s)| \leqslant \frac{1}{\alpha}$,

$$\begin{aligned} &||I_1||_{L^2(\Omega_f^\omega(t))} \\ &\leqslant C \cdot \frac{1}{\alpha} \int_{t-\alpha}^{t+\alpha} \left\| \boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - \boldsymbol{v}\left(t, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right\|_{L^2(\Omega_f^\omega(t))} ds \\ &\leqslant C \cdot \frac{1}{\alpha} \int_{t-\alpha}^{t+\alpha} \left(\frac{R+\omega(t,x)}{R+\omega(0,x)} \right)^{1/2} ||\boldsymbol{v}(s,x,y) - \boldsymbol{v}(t,x,y)||_{L^2(\Omega_f^\omega(0))} ds \to 0, \end{aligned}$$

as $\alpha \to 0$ if t is a Lebesgue point, by (5.158) and the uniform boundedness of $\frac{R+\omega(t,x)}{R+\omega(0,x)}$ on $[0,T]$.

To estimate I_2, we can use the continuity in time of ω and $\partial_x \omega$ to calculate that

$$\left| K(s,t,x,y) K^{-1} \left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right| \to 0,$$

uniformly in (x,y) as $s \to t$. We estimate

$$\begin{aligned} &||I_2||_{L^2(\Omega_f^\omega(t))} \\ &\leqslant \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1} \left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right| \\ &\qquad \cdot \left\| \boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right\|_{L^2(\Omega_f^\omega(t))} \cdot j_\alpha(t-s) ds \\ &\leqslant \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1} \left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right| \\ &\qquad \cdot \left(\frac{R+\omega(t,x)}{R+\omega(0,x)} \right)^{1/2} \cdot ||\boldsymbol{v}(s,x,y)||_{L^2(\Omega_f^\omega(0))} \cdot j_\alpha(t-s) ds \\ &\leqslant C \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1} \left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R \right) \right| \\ &\qquad \cdot j_\alpha(t-s) ds, \end{aligned}$$

where we used the fact that $\boldsymbol{v} \in L^\infty(0,T; L^2(\Omega_f^\omega(0)))$. Thus, we conclude that $||I_2||_{L^2(\Omega_f^\omega(t))} \to 0$ as $\alpha \to 0$. This completes the proof. □

We also have a weak continuity lemma, which states that the value of $\boldsymbol{u}_{2,\delta}$ tested against any function in the fluid function space has a continuity property as $t \to 0$.

Lemma 5.12.2. Consider an arbitrary $\boldsymbol{q} \in C^1(0,T;V_{f,2,\delta}(t))$ and the weak solution $\boldsymbol{u}_{2,\delta}$ to the regularized problem for arbitrary δ, where $V_{f,2,\delta}(t)$ is defined by the displacement $\omega_{2,\delta}$ and (5.33). There exists a measure zero subset S of $[0,T]$ (depending on δ) such that

$$\lim_{t\to 0, t\in[0,T]\cap S^c} \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t)\cdot\boldsymbol{q}(t) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0\cdot\boldsymbol{q}(0).$$

Proof. Consider the following function for each $\tau \in [0,T]$ and $\alpha > 0$, given by

$$J_{\tau,\alpha}(t) = 1 - \int_0^t j_\alpha(s-\tau)ds, \tag{5.159}$$

and note that $J'_{\tau,\alpha}(t) = -j_\alpha(t-\tau)$. We want to test the regularized weak formulation for $\boldsymbol{u}_{2,\delta}$ with the test function $J_{\tau,\alpha}(t)\boldsymbol{q}$ for certain admissible choices of τ. To see which τ we want to choose, we define the function

$$\boldsymbol{w}(t,x,y) = \frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(0)}\cdot\boldsymbol{u}_{2,\delta}\left(t,x,\frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(0)}(R+y)-R\right)\cdot\boldsymbol{q}\left(t,x,\frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(0)}(R+y)-R\right).$$

We claim that $\boldsymbol{w} \in L^\infty(0,T;L^1(\Omega_{f,2,\delta}(0)))$. To see this, we compute by a change of variables that

$$||\boldsymbol{w}(t,x,y)||_{L^1(\Omega_{f,2,\delta}(0))} = \int_{\Omega_{f,2,\delta}(t)} |\boldsymbol{u}_{2,\delta}(t,x,y)\cdot\boldsymbol{q}(t,x,y)|,$$

and we then use the fact that $\boldsymbol{u}_{2,\delta}, \boldsymbol{q} \in L^\infty(0,T;L^2(\Omega_{f,2,\delta}(t)))$.

Hence, by the Lebesgue differentiation theorem, there exists a measurable subset $S \subset [0,T]$ of measure zero such that every point in $[0,T]\cap S^c$ is a Lebesgue point of $\boldsymbol{w}$, in the sense that

$$\lim_{\alpha\to 0}\frac{1}{2\alpha}\int_{\tau-\alpha}^{\tau+\alpha} ||\boldsymbol{w}(\tau,\cdot)-\boldsymbol{w}(s,\cdot)||_{L^1(\Omega_{f,2,\delta}(0))}ds \to 0. \tag{5.160}$$

for every $\tau \in [0,T]\cap S^c$. These are the τ for which we will consider the test function $J_{\tau,\alpha}(t)\boldsymbol{q}$. For the test functions for the Biot medium and the plate, we will take these test functions to be zero. Hence, in the regularized weak formulation (5.101), we will test with $(\boldsymbol{v},\varphi,\boldsymbol{\psi},r) = (J_{\tau,\alpha}(t)\boldsymbol{q},0,0,0)$.

Hence, we obtain the following equality:

$$\begin{aligned} -\int_0^T\int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}\cdot\partial_t(J_{\tau,\alpha}(t)\boldsymbol{q}) + \frac{1}{2}\int_0^T\int_{\Omega_{f,2,\delta}(t)} [((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{u}_{2,\delta})\cdot(J_{\tau,\alpha}(t)\boldsymbol{q}) - ((\boldsymbol{u}_{2,\delta}\cdot\nabla)(J_{\tau,\alpha}(t)\boldsymbol{q}))\cdot\boldsymbol{u}_{2,\delta}] \\ + \frac{1}{2}\int_0^T\int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n} - 2\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})\boldsymbol{u}_{2,\delta}\cdot(J_{\tau,\alpha}(t)\boldsymbol{q}) \\ + 2\nu\int_0^T\int_{\Omega_{f,2,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_{2,\delta}):\boldsymbol{D}(J_{\tau,\alpha}(t)\boldsymbol{q}) - \int_0^T\int_{\Gamma_{2,\delta}(t)}\left(\frac{1}{2}|\boldsymbol{u}_{2,\delta}|^2 - p_{2,\delta}\right)J_{\tau,\alpha}(t)q_n \\ - \beta\int_0^T\int_{\Gamma_{2,\delta}(t)} [(\xi_{2,\delta})_\tau - (u_{2,\delta})_\tau]\cdot J_{\tau,\alpha}(t)q_\tau = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0\cdot J_{\tau,\alpha}(0)\boldsymbol{q}(0). \end{aligned}$$

Consider $\tau \in (0,T) \cap S^c$. We want to pass to the limit as $\alpha \to 0$, and then pass to the limit as $\tau \to 0$, in order to obtain the desired result.

First, we pass to the limit as $\alpha \to 0$. We handle the convergences as follows.

First term: We will show that because τ is a Lebesgue point of $\boldsymbol{w}$,

$$-\int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot \partial_t (J_{\tau,\alpha}(t)\boldsymbol{q}) \to \int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau)\boldsymbol{q}(\tau) - \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot \partial_t \boldsymbol{q}, \qquad \text{as } \alpha \to 0.$$

We compute that

$$-\int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot \partial_t (J_{\tau,\alpha}(t)\boldsymbol{q}) = \int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot j_\alpha(t-\tau)\boldsymbol{q} - \int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot J_{\tau,\alpha}(t)\partial_t \boldsymbol{q}.$$

It is easy to see that

$$\int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} J_{\tau,\alpha}(t)\partial_t \boldsymbol{q} \to \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}\partial_t \boldsymbol{q}.$$

So it remains to show that

$$\int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot j_\alpha(t-\tau)\boldsymbol{q} \to \int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau)\boldsymbol{q}(\tau), \qquad \text{as } \alpha \to 0.$$

By a change of variables, we compute that

$$\begin{aligned}
&\int_0^T \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta} \cdot j_\alpha(t-\tau)\boldsymbol{q} \\
&= \int_0^T \int_{\Omega_{f,2,\delta}(\tau)} \frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(\tau)} \cdot \boldsymbol{u}_{2,\delta}\left(t, x, \frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(\tau)}(R+y) - R\right) \cdot j_\alpha(t-\tau)\boldsymbol{q}\left(t, x, \frac{R+\omega_{2,\delta}(t)}{R+\omega_{2,\delta}(\tau)}(R+y) - R\right) \\
&= \int_0^T \int_{\Omega_{f,2,\delta}(\tau)} \frac{R+\omega_{2,\delta}(0)}{R+\omega_{2,\delta}(\tau)} \boldsymbol{w}\left(t, x, \frac{R+\omega_{2,\delta}(0)}{R+\omega_{2,\delta}(\tau)}(R+y) - R\right) \cdot j_\alpha(t-\tau) = \int_0^T \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{w}(t,x,y) \cdot j_\alpha(t-\tau).
\end{aligned}$$

By (5.160), we have that

$$\int_0^T \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{w}(t,x,y) \cdot j_\alpha(t-\tau) \to \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{w}(\tau,x,y) = \int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau) \cdot \boldsymbol{q}(\tau),$$

which establishes the desired convergence.

Final term: It is immediate to see that for all sufficiently small $\alpha > 0$,

$$\int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot J_{\tau,\alpha}(0)\boldsymbol{q}(0) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0).$$

Passing to the limit in the remaining terms as $\alpha \to 0$, we obtain that for any $\tau \in (0,T) \cap S^c$,

$$\int_{\Omega_{f,2,\delta}(\tau)} \boldsymbol{u}_{2,\delta}(\tau)\cdot\boldsymbol{q}(\tau) - \int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}\cdot\partial_t\boldsymbol{q} + \frac{1}{2}\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} [((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{u}_{2,\delta})\cdot\boldsymbol{q} - ((\boldsymbol{u}_{2,\delta}\cdot\nabla)\boldsymbol{q})\cdot\boldsymbol{u}_{2,\delta}]$$
$$+ \frac{1}{2}\int_0^\tau \int_{\Gamma_{2,\delta}(t)} (\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{n} - 2\boldsymbol{\xi}_{2,\delta}\cdot\boldsymbol{n})\boldsymbol{u}_{2,\delta}\cdot\boldsymbol{q} + 2\nu\int_0^\tau \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_{2,\delta}) : \boldsymbol{D}(\boldsymbol{q})$$
$$- \int_0^\tau \int_{\Gamma_{2,\delta}(t)} \left(\frac{1}{2}|\boldsymbol{u}_{2,\delta}|^2 - p_{2,\delta}\right) q_n - \beta\int_0^\tau \int_{\Gamma_{2,\delta}(t)} [(\xi_{2,\delta})_\tau - (u_{2,\delta})_\tau]\cdot q_\tau = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0\cdot\boldsymbol{q}(0).$$

Passing to the limit as $\tau \to 0$ with $\tau \in (0,T) \cap S^c$ gives the desired result.

□

Lemma 5.12.3. Recall the definition (5.118)

$$K_\delta(s,t,x,y) = \begin{pmatrix} \frac{R+\omega_{2,\delta}(s,x)}{R+\omega_{2,\delta}(t,x)} & 0 \\ -(R+y)\nabla\left(\frac{R+\omega_{2,\delta}(s,x)}{R+\omega_{2,\delta}(t,x)}\right) & 1 \end{pmatrix},$$

and consider the function

$$\tilde{\boldsymbol{q}}(t,x,y) = K_\delta(0,t,x,y)\boldsymbol{u}_0\left(x, \frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(t,x)}(R+y) - R\right). \tag{5.161}$$

for $\boldsymbol{u}_0$ which is divergence free and smooth on $\overline{\Omega_f(0)}$. There exists a sequence of functions $\tilde{\boldsymbol{q}}_m \in C_c^1(0,T;V_{f,2,\delta}(t))$, with $V_{f,2,\delta}(t)$ determined by the plate displacement $\omega_{2,\delta}$ via the definition (5.33), such that

$$\max_{0\leqslant t\leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,2,\delta}(t))} \to 0, \qquad \text{as } m \to \infty.$$

Proof. There exists a rectangular two-dimensional maximal domain Ω_M of the form $[0,L] \times [-R, R_{max}]$ for some positive constant R_{max} that contains all of the domains $\Omega_{f,2,\delta}(t)$ for $t \in [0,T]$. We will extend $\tilde{\boldsymbol{q}}$ to the maximal spacetime domain $[0,T] \times \Omega_M$ by extending vertically in the radial direction by the trace of $\tilde{\boldsymbol{q}}$ along $\Gamma_{2,\delta}(t)$. In particular, we define

$$\tilde{\boldsymbol{q}}(t,x,y) = K(0,t,x,\omega_{2,\delta}(t,x))\boldsymbol{u}_0\left(x,\omega_{2,\delta}(0,x)\right),$$
$$\text{for } (t,x,y) \in ([0,T]\times\Omega_M) - ([0,T]\times\Omega_{f,2,\delta}(t)). \tag{5.162}$$

Note that this extension preserves the *divergence free* property.

We have the following two claims about the extended function, considered as a function on the *fixed* maximal domain Ω_M. First, we claim that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$. Second, we claim that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$. To see that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$, we note that $\omega_{2,\delta}$ and $\partial_x\omega_{2,\delta}$ are bounded uniformly pointwise, and furthermore $\boldsymbol{u}_0$ and its first spatial derivatives are bounded by assumption. In addition, $\partial_x^2 \omega_{2,\delta} \in L^\infty(0,T;L^2(\Gamma))$, which allows us to conclude that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$.

Next, we want to verify that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$. Consider any $t \in [0,T]$ and consider any $s \in [0,T]$ with $s \neq t$. We define the following regions:

$$A(s,t) = \Omega_f^M \cap (\Omega_{f,2,\delta}(s) \cup \Omega_{f,2,\delta}(t))^c,$$

$$B(s,t) = [\Omega_{f,2,\delta}(s) \cap (\Omega_{f,2,\delta}(t))^c] \cup [(\Omega_{f,2,\delta}(s))^c \cap \Omega_{f,2,\delta}(t)],$$

$$C(s,t) = \Omega_{f,2,\delta}(s) \cap \Omega_{f,2,\delta}(t).$$

Consider $\epsilon > 0$. We want to find $h > 0$ such that

$$||\tilde{\boldsymbol{q}}(t,\cdot) - \tilde{\boldsymbol{q}}(s,\cdot)||^2_{L^2(\Omega_M)} \leqslant \epsilon, \qquad \text{for all } s \in (t-h,t+h) \cap [0,T]. \tag{5.163}$$

We compute that

$$\begin{aligned} &||\tilde{\boldsymbol{q}}(t,\cdot) - \tilde{\boldsymbol{q}}(s,\cdot)||^2_{L^2(\Omega_M)} \\ &= \int_{A(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 + \int_{B(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 + \int_{C(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 \\ &= I_A + I_B + I_C. \end{aligned} \tag{5.164}$$

We estimate each of the terms I_A, I_B, and I_C separately.

For I_A, we recall that we are extending by the trace as in (5.162) on $A(s,t)$, so we have that

$$I_A = \int_{A(s,t)} |K_\delta(0,t,x,\omega_{2,\delta}(t,x)) - K_\delta(0,s,x,\omega_{2,\delta}(s,x))|^2 \cdot |\boldsymbol{u}_0(x,\omega_{2,\delta}(0,x))|^2.$$

We have that $|\boldsymbol{u}_0(x,\omega_{2,\delta}(0,x))| \leqslant M_1$ for some constant M_1 by the fact that $\boldsymbol{u}_0$ is continuous on $\overline{\Omega_f(0)}$. By continuity, we can choose $h > 0$ sufficiently small so that

$$|K_\delta(0,t,x,\omega_{2,\delta}(t,x)) - K_\delta(0,s,x,\omega_{2,\delta}(s,x))|^2 < \frac{\epsilon}{3M_1^2(R+R_{max})L}, \quad \text{for all } s \in (t-h,t+h) \cap [0,T].$$

Thus, for all $s \in (t-h,t+h) \cap [0,T]$,

$$I_A \leqslant |A(s,t)| \cdot \frac{\epsilon}{3(R+R_{max})L} \leqslant \frac{\epsilon}{3}.$$

For I_B, we will use the fact that $\omega_{2,\delta}$ does not change much in time over small time intervals, by continuity. We note that there exists a uniform constant M_2 such that $|\tilde{\boldsymbol{q}}| \leqslant M_2$ on $[0,T] \times \Omega_M$. Hence,

$$I_B = \int_{B(s,t)} |\tilde{\boldsymbol{q}}(t,z,r) - \tilde{\boldsymbol{q}}(s,z,r)|^2 \leqslant |B(s,t)| \cdot 4M_2^2 = 4M_2^2 \int_0^L |\omega_{2,\delta}(t,x) - \omega_{2,\delta}(s,x)| dx.$$

Because $\omega_{2,\delta} \in L^\infty(0,T;H_0^2(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))$, there exists $h > 0$ sufficiently small such that

$$|\omega_{2,\delta}(t,x) - \omega_{2,\delta}(s,x)| \leqslant \frac{\epsilon}{12M_2^2 L}, \qquad \text{for all } x \in [0,L] \text{ and } s \in (t-h,t+h) \cap [0,T].$$

This allows us to conclude that $I_B \leqslant \frac{\epsilon}{3}$, for all $s \in (t-h,t+h) \cap [0,T]$.

For I_C, we refer to the definition of $\tilde{\boldsymbol{q}}$ in (5.161) and note that $K_\delta(0,t,x,y)$ is continuous in time uniformly in $(x,y) \in [0,L] \times [-R, R_{max}]$, $\boldsymbol{u}_0$ is uniformly continuous as a function on $\overline{\Omega_f(0)}$, and $\omega_{2,\delta}(t,x)$ is continuous in time uniformly in $x \in [0,L]$. Hence, there exists $h > 0$ sufficiently small such that

$$|\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 \leqslant \frac{\epsilon}{3(R+R_{max})L}, \quad \text{for all } (x,y) \in C(s,t) \text{ and } s \in (t-h,t+h) \cap [0,T],$$

which gives the desired result that $I_C \leqslant \frac{\epsilon}{3}$ for all $s \in (t-h,t+h) \cap [0,T]$. Thus, by using (5.164), we have established (5.163).

Since $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M)) \cap C(0,T;L^2(\Omega_M))$, we can extend $\tilde{\boldsymbol{q}}$ to a continuous function on all of $\mathbb{R}$ as follows. We can find an increasing sequence T_m with $T_m \to T$ as $m \to \infty$, such that $\tilde{\boldsymbol{q}}(T_m) \in H^1(\Omega_M)$ for all m. Define an extension $\hat{\boldsymbol{q}}_m$ for each m to all of $\mathbb{R}$ by $\hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}$ if $t \in [0,T_m]$,

$$\hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}(0), \qquad \text{if } t < 0, \qquad \hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}(T_m), \qquad \text{if } t > T_m.$$

Define

$$\tilde{\boldsymbol{q}}_m = \hat{\boldsymbol{q}}_m * j_{1/m},$$

where the convolution is a convolution in time with j_α for $\alpha = 1/m$. Because $\hat{\boldsymbol{q}}_m \in L^\infty(0,T;H^1(\Omega_M)) \cap C(0,T;L^2(\Omega_M))$ with $\hat{\boldsymbol{q}}_m$ being divergence free for every $t \in [0,T]$, we have that $\tilde{\boldsymbol{q}}_m$ restricted to $\bigcup_{t\in[0,T]}\{t\} \times \Omega_{f,2,\delta}(t)$ gives a function in $C^1([0,T);V_{f,2,\delta}(t))$, where $V_{f,2,\delta}(t)$ is the space defined in (5.33) with the plate displacement $\omega_{2,\delta}$. The fact that

$$\max_{0\leqslant t\leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,2,\delta}(t))} \to 0, \qquad \text{as } m \to \infty,$$

follows from the uniform continuity of $\tilde{\boldsymbol{q}}$ on $[0,T]$ as a function taking values in $L^2(\Omega_M)$, convergence properties of convolutions, and the fact that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$ which gives the convergence

$$\max_{t\in[T_m,T]} ||\tilde{\boldsymbol{q}}(T) - \tilde{\boldsymbol{q}}(t)||_{L^2(\Omega_M)} \to 0, \qquad \text{as } m \to \infty.$$

□

Lemma 5.12.4. For the function $\tilde{\boldsymbol{q}}$ defined in (5.161), there exists a measure zero subset S of $[0,T]$ such that

$$\lim_{t\to 0, t\in[0,T]\cap S^c} \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}(t) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0).$$

Proof. Note that because $\partial_t \tilde{\boldsymbol{q}}$ is not necessarily in $H^1(\Omega_{f,2,\delta}(t))$, $\tilde{\boldsymbol{q}}$ is not a valid test function. Thus, we use the sequence $\tilde{\boldsymbol{q}}_m \in C^1(0, T; \mathcal{V}_{f,2,\delta}(t))$ from Lemma 5.12.3, which satisfies

$$\max_{0 \leqslant t \leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,2,\delta}(t))} \to 0, \qquad \text{as } m \to \infty.$$

We can then apply Lemma 5.12.2 to each of the test functions $\tilde{\boldsymbol{q}}_m$, to deduce that there exists a measure zero subset S_m of $[0, T]$ such that

$$\lim_{t \to 0, t \in [0,T] \cap S_m^c} \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}_m(t) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_m(0).$$

In addition, by uniform boundedness, $\boldsymbol{u}_{2,\delta} \in L^\infty(0, T; L^2(\Omega_{f,2,\delta}(t)))$, and hence, there exists a measure zero subset S_0 of $[0, T]$, and a positive constant C such that $||\boldsymbol{u}_0||_{L^2(\Omega_{f,2,\delta}(0))} \leqslant C$, and

$$||\boldsymbol{u}_{2,\delta}(t)||_{L^2(\Omega_{f,2,\delta}(t))} \leqslant C, \qquad \text{for all } t \in S_0^c. \tag{5.165}$$

Define $S = S_0 \cup \bigcup_{m \geqslant 1} S_m$, which is also a measure zero subset of $[0, T]$. Then, for each m,

$$\lim_{t \to 0, t \in [0,T] \cap S^c} \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}_m(t) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_m(0). \tag{5.166}$$

By passing to the limit in m, we claim that in addition,

$$\lim_{t \to 0, t \in [0,T] \cap S^c} \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}(t) = \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0).$$

To see this, consider $\epsilon > 0$. We claim that there exists $h > 0$ sufficiently small such that for all $t \in (0, h) \cap S^c$,

$$\left| \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) \right| < \epsilon.$$

We can choose M sufficiently large such that $\max_{0 \leqslant t \leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_M||_{L^2(\Omega_{f,2,\delta}(t))} < \frac{\epsilon}{3C}$, where C is defined by (5.165). Therefore, for all $t \in [0, T] \cap S^c$,

$$\left| \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}_M(t) \right| < \frac{\epsilon}{3}.$$

In addition,

$$\left| \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) - \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_M(0) \right| < \frac{\epsilon}{3}.$$

By applying (5.166) with $m = M$, we can choose $h > 0$ sufficiently small such that for all $t \in (0, h) \cap S^c$,

$$\left| \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}_M(t) - \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_M(0) \right| < \frac{\epsilon}{3}.$$

Thus, by applying the triangle inequality, we have that for all $t \in (0, h) \cap S^c$,

$$\left| \int_{\Omega_{f,2,\delta}(t)} \boldsymbol{u}_{2,\delta}(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,2,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) \right| < \epsilon,$$

which establishes the desired result.

□

We can now prove the final result of this appendix. We recall the definition of $\widehat{\boldsymbol{u}}_1$ from (5.107).

Lemma 5.12.5. As $\alpha \to 0$,

$$\int_{\Omega_{f,2,\delta}(0)} \widehat{\boldsymbol{u}}_1(0) \cdot (\boldsymbol{u}_{2,\delta})_\alpha(0) \to \int_{\Omega_{f,2,\delta}(0)} |\boldsymbol{u}_0|^2, \text{ and } \int_{\Omega_{f,2,\delta}(t)} \widehat{\boldsymbol{u}}_1(t) \cdot (\boldsymbol{u}_{2,\delta})_\alpha(t) \to \int_{\Omega_{f,2,\delta}(t)} \widehat{\boldsymbol{u}}_1(t) \cdot \boldsymbol{u}_{2,\delta}(t),$$

for almost all points $t \in (0, T]$.

Proof. The second convergence for almost all points $t \in (0, T]$ follows directly from Lemma 5.12.1 and the fact that $\widehat{\boldsymbol{u}}_1 \in L^\infty(0, T; L^2(\Omega_{f,2,\delta}(t)))$.

So we just need to verify the convergence at $t = 0$. To do this, we note that $\widehat{\boldsymbol{u}}_1(0) = \boldsymbol{u}_0$. Hence,

$$\begin{aligned}
&\int_{\Omega_{f,2,\delta}(0)} \widehat{\boldsymbol{u}}_1(0) \cdot (\boldsymbol{u}_{2,\delta})_\alpha(0) \\
&= \int_{\Omega^{\omega_0}} \left(\int_{\mathbb{R}} K_\delta(s, 0, x, y) \boldsymbol{u}_{2,\delta} \left(s, x, \frac{R + \omega_{2,\delta}(s, x)}{R + \omega_{2,\delta}(0, x)}(R + y) - R \right) j_\delta(t - s) ds \right) \boldsymbol{u}_0(x, y) dx dy \\
&= \int_{\mathbb{R}} \left(\int_{\Omega^{\omega_0}} K_\delta(s, 0, x, y) \boldsymbol{u}_{2,\delta} \left(s, x, \frac{R + \omega_{2,\delta}(s, x)}{R + \omega_{2,\delta}(0, x)}(R + y) - R \right) \cdot \boldsymbol{u}_0(x, y) dx dy \right) j_\alpha(t - s) ds \\
&= \int_{\mathbb{R}} \left(\int_{\Omega_{f,2,\delta}(s)} \boldsymbol{u}_{2,\delta}(s, x, y) \cdot \frac{R + \omega_{2,\delta}(0, x)}{R + \omega_{2,\delta}(s, x)} K^t_\delta \left(s, 0, x, \frac{R + \omega_{2,\delta}(0, x)}{R + \omega_{2,\delta}(s, x)}(R + y) - R \right) \right. \\
&\qquad \left. \cdot \boldsymbol{u}_0 \left(x, \frac{R + \omega_{2,\delta}(0, x)}{R + \omega_{2,\delta}(s, x)}(R + y) - R \right) dx dy \right) j_\alpha(t - s) ds.
\end{aligned}$$

We compute that

$$\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\cdot K^t_\delta\left(s,0,x,\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}(R+y)-R\right)=\begin{pmatrix}1 & (R+y)\nabla\left(\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\right)\\ 0 & \frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)} & 0\\ -(R+y)\nabla\left(\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\right) & 1\end{pmatrix}+\begin{pmatrix}1-\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)} & (R+y)\nabla\left(\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\right)\\ (R+y)\nabla\left(\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}\right) & \frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}-1\end{pmatrix}$$

$$:=K_\delta(0,s,x,y)+R_\delta(0,s,x,y).$$

Hence,

$$\int_{\Omega_{f,2,\delta}(0)}\hat{\boldsymbol{u}}_1(0)\cdot(\boldsymbol{u}_{2,\delta})_\alpha(0)$$

$$=\int_{\mathbb{R}}\left(\int_{\Omega_{f,2,\delta}(s)}\boldsymbol{u}_{2,\delta}(s,x,y)\cdot K_\delta(0,s,x,y)\boldsymbol{u}_0\left(x,\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}(R+y)-R\right)dxdy\right)j_\alpha(t-s)ds$$

$$+\int_{\mathbb{R}}\left(\int_{\Omega_{f,2,\delta}(s)}\boldsymbol{u}_{2,\delta}(s,x,y)\cdot R_\delta(0,s,x,y)\boldsymbol{u}_0\left(x,\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}(R+y)-R\right)dxdy\right)j_\alpha(t-s)ds=I_{K,\delta}+I_{R,\delta}.$$

Note that

$$I_{K,\delta}=\int_{\mathbb{R}}\left(\int_{\Omega_{f,2,\delta}(s)}\boldsymbol{u}_{2,\delta}(s,x,y)\cdot\tilde{\boldsymbol{q}}(s,x,y)dxdy\right)j_\alpha(t-s)ds$$

where $\tilde{\boldsymbol{q}}$ is defined by (5.161). Since $\boldsymbol{u}_{2,\delta}(s)=\boldsymbol{u}_{2,\delta}(-s)$ so that $\omega_{2,\delta}(s)=\omega_{2,\delta}(-s)$ for $s\leqslant 0$ (see the extension procedure in Section 5.9), we conclude by Lemma 5.12.4 that

$$I_{K,\delta}\to\int_{\Omega_{f,2,\delta}(0)}\boldsymbol{u}_0\cdot\tilde{\boldsymbol{q}}(0)=\int_{\Omega_{f,2,\delta}(0)}|\boldsymbol{u}_0|^2,\qquad\text{as }\alpha\to 0.$$

So it suffices to show that $I_{R,\delta}\to 0$ as $\alpha\to 0$. This follows from the fact that $|R_\delta|\to 0$ uniformly as $s\to 0$. In particular,

$$\int_{\Omega_{f,2,\delta}(s)}\left|\boldsymbol{u}_{2,\delta}(s,x,y)\cdot\boldsymbol{u}_0\left(x,\frac{R+\omega_{2,\delta}(0,x)}{R+\omega_{2,\delta}(s,x)}(R+y)-R\right)\right|dxdy\leqslant C,\quad\text{for almost all }s\in[0,T],$$

by the boundedness of $\boldsymbol{u}_{2,\delta}\in L^\infty(0,T;L^2(\Omega_{f,2,\delta}(t))$ and the fact that $\boldsymbol{u}_0$ is uniformly bounded. In addition, by the continuity properties of $\omega_{2,\delta}$ in time, we have that

$$\max_{(x,y)\in\Omega_{f,2,\delta}(s)}|R_\delta(0,s,x,y)|\to 0,\qquad\text{as }s\to 0,$$

which implies that $I_{R,\delta}\to 0$ as $\alpha\to 0$. This completes the proof.

□

Chapter 6

Concluding remarks

In this thesis, we have studied several important extensions of FSI models that are motivated by real-life applications in engineering. These include two main types of models: (1) stochastic FSI models that take into account the influence of stochasticity and randomness on the fully coupled fluid-structure dynamics, and (2) fluid-poroelastic structure interaction systems (FPSI), specifically FPSI in the context of nonlinearly coupled models in which the fluid and Biot domains are time-dependent and a priori unknown. While there has been significant progress in prototypical models of fluid-structure interaction and several extensions of this model, as discussed in Chapter 1, the models considered in this thesis are novel problems that require the development of new mathematical techniques for their analysis.

The work in this thesis has initiated the study of stochastic fluid-structure interaction by considering two models of stochastic FSI. The first model that was considered in the field of stochastic FSI was the model described in Chapter 3, which is a reduced model whose dynamics are described by a stochastic viscous wave equation. This stochastic viscous wave equation is a single self-contained stochastic equation for the structure displacement in a linearly coupled FSI model where a fluid modeled by the stationary Stokes equations interacts with an elastic membrane modeled by the wave equation, under the additional influence of stochastic forcing in space and time acting on the elastic membrane. Due to the special geometry of this model, the full fluid-structure dynamics can be captured in just a single stochastic equation rather than a stochastic system of equations, which allows us to invoke classical techniques from the theory of stochastic analysis to establish the existence and uniqueness of a mild solution. Furthermore, we also established regularity of sample paths of the mild solution and showed that when compared to classical stochastic heat and wave equations with the same type of random noise, the stochastic viscous wave equation arising in stochastic FSI has improved existence and sample path properties.

We then extended the study of stochastic FSI to a stochastic fully coupled model involving a linear Stokes flow through a channel with elastic and stochastically forced walls. The goal of this work, described in Chapter 4, was to develop a new methodology for studying stochastic systems of PDEs, as this problem cannot be reduced to a single equation, and hence must be considered as a coupled stochastic system of fluid and structure equations.

Here, we used an operator splitting scheme for a constructive existence proof, which explicitly constructs random approximate solutions to the problem, and we introduce a methodology for passing to the limit in the random approximate solutions that combines compactness arguments with tools from stochastic analysis and stochastic PDEs. This involves showing weak convergence of the approximate solutions, using the Skorokhod representation theorem to upgrade this weak convergence to almost sure convergence to obtain probabilistically weak solutions, and then using the Gyöngy-Krylov lemma to show that we can obtain probabilistically strong solutions on the initially given probability space. We anticipate that this constructive existence scheme that we have developed for fully coupled stochastic FSI will apply more generally to more challenging problems of interest, including the study of stochastic nonlinearly coupled FSI, where the fluid domain is not only time-dependent and a priori unknown, but also random. Work on the well-posedness of these stochastic nonlinearly coupled FSI models is ongoing [175].

In this thesis, we have also studied deterministic FPSI, with particular attention given to nonlinearly coupled FPSI. As discussed in the introduction to Chapter 5, there have been a few works that have studied the well-posedness of linearly coupled FPSI systems, where the fluid and Biot domains are fixed in time for the purposes of defining the problem. The goal of the work in this thesis was to extend well-posedness results to the context of FPSI problems where we take into account the time-dependent nature of the Biot and fluid domains. The particular challenge here is that the Biot displacement determines the moving Biot domain in time via the Lagrangian map, but the structure displacement $\boldsymbol{\eta}$ of the Biot material does not possess enough spatial regularity to make proper sense of the moving Biot domain, or to guarantee that the integrals over the moving Biot domain appearing in the weak formulation are well-defined. Therefore, we have developed a new mathematical framework for analyzing these nonlinearly coupled FPSI problems which involves minimally regularizing the FPSI problem using a spatial regularization of the structure displacement $\boldsymbol{\eta}$ of the Biot medium. This regularization is minimal in the sense that we try to regularize as few terms as possible in the regularized weak formulation of the FPSI problem. We apply this regularization technique to a model of nonlinearly coupled FPSI involving a multilayered structure consisting of a thin plate and a thick Biot poroelastic medium interacting with a fluid described by the Navier-Stokes equations. In this case, the regularization must be performed in a specific way by using an odd extension to extend the structure displacement to a larger domain and then convolving spatially, since we are working on bounded domains. We use a splitting scheme to show constructive existence of weak solutions to the regularized FPSI problem and in order to show that these resulting solutions to the regularized problem are physically reasonable, we have also established a weak-classical consistency result in the case of a poroviscoelastic Biot medium. This result shows that given a classical (smooth) solution to the original FPSI problem without regularization, the weak solutions to the regularized FPSI problem will converge to the classical solution as the regularization parameter tends to zero. This method of using regularization to study FPSI problems on moving domains shows significant promise, and we hope to extend these methods in order to analyze and simulate more complex FPSI systems, in particular those that are directly relevant to real-life applications.

Bibliography

[1] R. A. Adams. *Sobolev spaces*. Vol. 65. Pure and Applied Mathematics. New York-London: Academic Press, 1975.

[2] M. Akköse et al. "Elasto-plastic earthquake response of arch dams including fluid-structure interaction by the Lagrangian approach". In: *Appl. Math. Model.* 32.11 (2008), pp. 2396–2412.

[3] I. Ambartsumyan et al. "A nonlinear Stokes-Biot model for the interaction of a non-Newtonian fluid with poroelastic media". In: *ESAIM Math. Model. Numer. Anal.* 53.6 (2019), pp. 1915–1955.

[4] T. B. Amina et al. "Fluid-structure interaction of Brezina arch dam: 3D modal analysis". In: *Eng. Struct.* 84 (2015), pp. 19–28.

[5] N. Ataei and J. E. Padgett. "Influential fluid-structure interaction modelling parameters on the response of bridges vulnerable to coastal storms". In: *Struct. Infrastruct. Eng.* 11.3 (2015), pp. 321–333.

[6] J.-P. Aubin. "Un théorème de compacité". In: *C. R. Acad. Sci. Paris* 256.24 (1963), pp. 5042–5044.

[7] G. Avalos, I. Lasiecka, and R. Triggiani. "Higher regularity of a coupled parabolic-hyperbolic fluid-structure interactive system". In: *Georgian Math. J.* 15.3 (2008), pp. 403–437.

[8] G. Avalos and R. Triggiani. "Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes-Lamé PDE system". In: *J. Evol. Equ.* 9.2 (2009), pp. 341–370.

[9] G. Avalos and R. Triggiani. "Semigroup well-posedness in the energy space of a parabolic-hyperbolic coupled Stokes-Lamé PDE system of fluid-structure interaction". In: *Discrete Contin. Dyn. Syst. Ser. S* 2.3 (2009), pp. 417–447.

[10] G. Avalos and R. Triggiani. "The coupled PDE system arising in fluid/structure interaction. Part I: Explicit semigroup generator and its spectral properties". In: *Fluids and waves*. Ed. by F. Botelho, T. Hagen, and J. Jamison. Vol. 440. Contemporary Mathematics. Providence, Rhode Island: American Mathematical Society, 2007, pp. 15–54.

[11] G. Avalos and R. Triggiani. "Uniform stabilization of a coupled PDE system arising in fluid-structure interaction with boundary dissipation at the interface". In: *Discrete Contin. Dyn. Syst.* 22.4 (2008), pp. 817–833.

[12] V. Barbu et al. "Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model". In: *Fluids and waves*. Ed. by F. Botelho, T. Hagen, and J. Jamison. Vol. 440. Contemporary Mathematics. Providence, Rhode Island: American Mathematical Society, 2007, pp. 55–82.

[13] V. Barbu et al. "Smoothness of weak solutions to a nonlinear fluid-structure interaction model". In: *Indiana Univ. Math. J.* 57.3 (2008), pp. 1173–1207.

[14] H. Barucq, M. Madaune-Tort, and P. Saint-Macary. "On nonlinear Biot's consolidation models". In: *Nonlinear Anal.* 63.5 (2005), e985–e995.

[15] H. Barucq, M. Madaune-Tort, and P. Saint-Macary. "Theoretical aspects of wave propagation for Biot's consolidation problem". In: *Monografías del Seminario Matemático García de Galdeano* 31 (2004), pp. 449–458.

[16] A. M. Bavo et al. "Fluid-structure interaction simulation of prosthetic aortic valves: comparison between immersed boundary and Arbitrary Lagrangian-Eulerian techniques for the mesh representation". In: *PLoS ONE* 11.4 (2016), e0154517.

[17] Y. Bazilevs et al. "A fully-coupled fluid-structure interaction simulation of cerebral aneurysms". In: *Comput. Mech.* 46.1 (2010), pp. 3–16.

[18] H. Beirão da Veiga. "On the existence of strong solutions to a coupled fluid-structure evolution problem". In: *J. Math. Fluid Mech.* 6.1 (2004), pp. 21–52.

[19] B. Benešová, M. Kampschulte, and S. Schwarzacher. *A variational approach to hyperbolic evolutions and fluid-structure interactions*. 2020. arXiv: `2008.04796 [math.AP]`.

[20] A. Bensoussan, R. Glowinski, and A. Răşcanu. "Approximation of some stochastic differential equations by the splitting up method". In: *Appl. Math. Optim.* 25.1 (1992), pp. 81–106.

[21] A. Bensoussan and R. Temam. "Equations stochastiques du type Navier-Stokes". In: *J. Funct. Anal.* 13.2 (1973), pp. 195–222.

[22] M. A. Biot. "General theory of three-dimensional consolidation". In: *J. Appl. Phys.* 12.2 (1941), pp. 155–164.

[23] M. A. Biot. "Theory of elasticity and consolidation for a porous anisotropic solid". In: *J. Appl. Phys.* 26.2 (1955), pp. 182–185.

[24] L. Bociu and J. T. Webster. "Nonlinear quasi-static poroelasticity". In: *J. Differential Equations* 296 (2021), pp. 242–278.

[25] L. Bociu et al. "Analysis of nonlinear poro-elastic and poro-visco-elastic models". In: *Arch. Ration. Mech. Anal.* 222.3 (2016), pp. 1445–1519.

[26] L. Bociu et al. “Multilayered poroelasticity interacting with Stokes flow”. In: *SIAM J. Math. Anal.* 53.6 (2021), pp. 6243–6279.

[27] M. Bukač, S. Čanić, and B. Muha. “A nonlinear fliud-structure interaction problem in compliant arteries treated with vascular stents”. In: *Appl. Math. Optim.* 73.3 (2016), pp. 433–473.

[28] M. Bukač et al. “Fluid-structure interaction between pulsatile blood flow and a curved stented coronary artery on a beating heart: a four stent computational study”. In: *Comput. Methods Appl. Mech. Engrg.* 350 (2019), pp. 679–700.

[29] L. Caffarelli and L. Silvestre. “An extension problem related to the fractional Laplacian”. In: *Comm. Partial Differential Equations* 32.8 (2007), pp. 1245–1260.

[30] S. Čanić. “Fluid-structure interaction with incompressible fluids”. In: *Progress in mathematical fluid dynamics*. Ed. by L. C. Berselli and M. Růžička. Vol. 2272. Lecture Notes in Mathematics. Springer, 2020, pp. 15–87.

[31] S. Čanić, M. Galić, and B. Muha. “Analysis of a 3D nonlinear, moving boundary problem describing fluid-mesh-shell interaction”. In: *Trans. Amer. Math. Soc.* 373.9 (2020), pp. 6621–6681.

[32] S. Čanić and J. Tambača. “Cardiovascular stents as PDE nets: 1D vs. 3D”. In: *IMA J. Appl. Math.* 77.6 (2012), pp. 748–770.

[33] S. Čanić, Y. Wang, and M. Bukač. “A next-generation mathematical model for drug-eluting stents”. In: *SIAM J. Appl. Math.* 81.4 (2021), pp. 1503–1529.

[34] S. Čanić et al. “A dimension-reduction based coupled model of mesh-reinforced shells”. In: *SIAM J. Appl. Math.* 77.2 (2017), pp. 744–769.

[35] S. Čanić et al. “Analysis of a linear 3D fluid-mesh-shell interaction problem”. In: *Z. Angew. Math. Phys.* 70.2 (2019), pp. 1–38.

[36] S. Čanić et al. “Modeling viscoelastic behavior of arterial walls and their interaction with pulsatile blood flow”. In: *SIAM J. Appl. Math.* 67.1 (2006), pp. 164–193.

[37] S. Čanić et al. “Optimal design of vascular stents using a network of 1D slender curved rods”. In: *Comput. Methods Appl. Mech. Engrg.* 394 (2022), p. 114853.

[38] M. Capiński and D. Gatarek. “Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension”. In: *J. Funct. Anal.* 126.1 (1994), pp. 26–35.

[39] S. Caucao et al. “A fully-mixed finite element method for the Navier-Stokes/Darcy coupled problem with nonlinear viscosity”. In: *J. Numer. Math.* 25.2 (2017), pp. 55–88.

[40] P. Causin et al. “A poroelastic model for the perfusion of the lamina cribrosa in the optic nerve head”. In: *Math. Biosci.* 257 (2014), pp. 33–41.

[41] A. Cesmelioglu. “Analysis of the coupled Navier-Stokes/Biot problem”. In: *J. Math. Anal. Appl.* 456.2 (2017), pp. 970–991.

[42] A. Chambolle et al. “Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate”. In: *J. Math. Fluid Mech.* 7.3 (2005), pp. 368–404.

[43] N. V. Chemetov, Š. Nečasová, and B. Muha. “Weak-strong uniqueness for fluid-rigid body interaction problem with slip boundary condition”. In: *J. Math. Phys.* 60.1 (2019), p. 011505.

[44] C. H. A. Cheng, D. Coutand, and S. Shkoller. “Navier-Stokes equations interacting with a nonlinear elastic biofluid shell”. In: *SIAM J. Math. Anal.* 39.3 (2007), pp. 742–800.

[45] C. H. A. Cheng and S. Shkoller. “The interaction of the 3D Navier-Stokes equations with a moving nonlinear Koiter elastic shell”. In: *SIAM J. Math. Anal.* 42.3 (2010), pp. 1094–1155.

[46] N. Cheraghi-Shirazi, A. R. Kabiri-Samani, and B. Boroomand. “Numerical analysis of rubber dams using fluid-structure interactions”. In: *Flow Meas. Instrum.* 40 (2014), pp. 91–98.

[47] S. K. Chimakurthi et al. “Computational aeroelasticity framework for analyzing flapping wing micro air vehicles”. In: *AIAA Journal* 47.8 (2009), pp. 1865–1878.

[48] P. G. Ciarlet. *Mathematical Elasticity. Volume I: Three-Dimensional Elasticity.* Vol. 20. Studies in Mathematics and Its Applications. Amsterdam: Elsevier Science Publishers B.V., 1988.

[49] C. Conca, J. San Martín H., and M. Tucsnak. “Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid”. In: *Comm. Partial Differential Equations* 25.5–6 (2000), pp. 1019–1042.

[50] D. Conus and R. C. Dalang. “The non-linear stochastic wave equation in high dimensions”. In: *Electron. J. Probab.* 13 (2008), pp. 629–670.

[51] D. Coutand and S. Shkoller. “Motion of an elastic solid inside an incompressible viscous fluid”. In: *Arch. Ration. Mech. Anal.* 176.1 (2005), pp. 25–102.

[52] D. Coutand and S. Shkoller. “The interaction between quasilinear elastodynamics and the Navier-Stokes equations”. In: *Arch. Ration. Mech. Anal.* 179.3 (2006), pp. 303–352.

[53] C. M. Dafermos. “On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity”. In: *Arch. Ration. Mech. Anal.* 29 (1968), pp. 241–271.

[54] R. C. Dalang. “Extending martingale measure stochastic integral with applications to spatially homogeneous S.P.D.E.’s”. In: *Electron. J. Probab.* 4 (1999), pp. 1–29.

[55] R. C. Dalang. "The stochastic wave equation". In: *A minicourse on stochastic partial differential equations*. Ed. by D. Khoshnevisan and F. Rassoul-Agha. Vol. 1962. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 2009.

[56] R. C. Dalang and N. E. Frangos. "The stochastic wave equation in two spatial dimensions". In: *Ann. Probab.* 26.1 (1998), pp. 187–212.

[57] R. C. Dalang and M. Sanz-Solé. "Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three". In: *Mem. Amer. Math. Soc.* 199.931 (2009), pp. vi–70.

[58] R. C. Dalang et al. *A minicourse on stochastic partial differential equations*. Ed. by D. Khoshnevisan and F. Rassoul-Agha. Vol. 1962. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 2009.

[59] A. Demir et al. "Numerical and experimental investigation of damping in a dam-break problem with fluid-structure interaction". In: *J. Zhejiang Univ.-Sci. A (Appl. Phys. & Eng.)* 20.4 (2019), pp. 258–271.

[60] B. Desjardins and M. J. Esteban. "Existence of weak solutions for the motion of rigid bodies in a viscous fluid". In: *Arch. Ration. Mech. Anal.* 146.1 (1999), pp. 59–71.

[61] B. Desjardins and M. J. Esteban. "On weak solutions for fluid-rigid structure interaction: compressible and incompressible models". In: *Comm. Partial Differential Equations* 25.7–8 (2000), pp. 263–285.

[62] B. Desjardins et al. "Weak solutions for a fluid-elastic structure interaction model". In: *Rev. Mat. Complut.* 14.2 (2001), pp. 523–538.

[63] M. Discacciati, E. Miglio, and A. Quarteroni. "Mathematical and numerical models for coupling surface and groundwater flows". In: *Appl. Numer. Math.* 43.1–2 (2002), pp. 57–74.

[64] S. Domínguez et al. "A primal-mixed formulation for the strong coupling of quasi-Newtonian fluids with porous media". In: *Adv. Comput. Math.* 42.3 (2016), pp. 675–720.

[65] M. Dreher and A. Jüngel. "Compact families of piecewise constant functions in $L^p(0,T;B)$". In: *Nonlinear Anal.* 75.6 (2012), pp. 3072–3077.

[66] Q. Du et al. "Analysis of a linear fluid-structure interaction problem". In: *Discrete Contin. Dyn. Syst.* 9.3 (2003), pp. 633–650.

[67] R. Durrett. *Probability: Theory and examples*. Fifth edition. Vol. 49. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press, 2019.

[68] V. J. Ervin, E. W. Jenkins, and S. Sun. "Coupled generalized nonlinear Stokes flow with flow through a porous medium". In: *SIAM J. Numer. Anal.* 47.2 (2009), pp. 929–952.

[69] V. J. Ervin, E. W. Jenkins, and S. Sun. “Coupling nonlinear Stokes and Darcy flow using mortar finite elements”. In: *Appl. Numer. Math.* 61.11 (2011), pp. 1198–1222.

[70] L. C. Evans. *Partial differential equations.* Second edition. Vol. 19. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2010.

[71] C. Farhat, M. Lesoinne, and P. Le Tallec. “Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: momentum and energy conservation, optimal discretization and application to aeroelasticity”. In: *Comput. Methods Appl. Mech. Engrg.* 157.1–2 (1998), pp. 95–114.

[72] F. Flandoli and D. Gatarek. “Martingale and stationary solutions for stochastic Navier-Stokes equations”. In: *Probab. Theory Related Fields* 102.3 (1995), pp. 367–391.

[73] L. Formaggia et al. “On the coupling of 3D and 1D Navier-Stokes equations for flow problems in compliant vessels”. In: *Comput. Methods Appl. Mech. Engrg.* 191.6 (2001), pp. 561–582.

[74] G. P. Galdi and M. Kyed. “Steady flow of a Navier-Stokes liquid past an elastic body”. In: *Arch. Ration. Mech. Anal.* 194.3 (2009), pp. 849–875.

[75] M. Galić. “Regularity of a weak solution to a linear fluid-composite structure interaction problem”. In: *Glas. Mat.* 56.2 (2021), pp. 407–440.

[76] X. Gao, Q. Zhang, and Q. Tang. “Fluid-structure interaction analysis of parachute finite mass inflation”. In: *Int. J. Aerosp. Eng.* 2016 (2016), pp. 1–8.

[77] L. Garelli, R. R. Paz, and M. A. Storti. “Fluid-structure interaction study of the start-up of a rocket engine nozzle”. In: *Comput. & Fluids* 39.7 (2010), pp. 1208–1218.

[78] F. Gazzola, V. Pata, and C. Patriarca. *Attractors for a fluid-structure interaction problem in a time-dependent phase space.* Preprint, 2022.

[79] V. Girault et al. “A lubrication fracture model in a poro-elastic medium”. In: *Math. Models Methods Appl. Sci.* 25.4 (2015), pp. 587–645.

[80] C. Grandmont. “Existence for a three-dimensional steady state fluid-structure interaction problem”. In: *J. Math. Fluid Mech.* 4.1 (2002), pp. 76–94.

[81] C. Grandmont. “Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate”. In: *SIAM J. Math. Anal.* 40.2 (2008), pp. 716–737.

[82] C. Grandmont and M. Hillairet. “Existence of global strong solutions to a beam-fluid interaction system”. In: *Arch. Ration. Mech. Anal.* 220.3 (2016), pp. 1283–1333.

[83] C. Grandmont, M. Lukáčová-Medvid’ová, and Š. Nečasová. “Mathematical and numerical analysis of some FSI problems”. In: *Fluid-structure interaction and biomedical applications.* Ed. by T. Bodnár, G. P. Galdi, and Š. Nečasová. Advances in Mathematical Fluid Mechanics. Switzerland: Springer Basel AG, 2014, pp. 1–77.

[84] C. Grandmont and Y. Maday. "Existence for an unsteady fluid-structure interaction problem". In: *ESAIM Math. Model. Numer. Anal.* 34.3 (2000), pp. 609–636.

[85] G. Griso. "Asymptotic behavior of structures made of curved rods". In: *Anal. Appl.* 6.1 (2008), pp. 11–22.

[86] G. Guidoboni et al. "Stable loosely-coupled-type algorithm for fluid-structure interaction in blood flow". In: *J. Comput. Phys.* 228.18 (2009), pp. 6916–6937.

[87] M. D. Gunzburger, H.-C. Lee, and G. A. Seregin. "Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions". In: *J. Math. Fluid Mech.* 2.3 (2000), pp. 219–266.

[88] I. Gyöngy and N. Krylov. "Existence of strong solutions for Itô's stochastic equations via approximations". In: *Probab. Theory Related Fields* 105.2 (1996), pp. 143–158.

[89] I. Gyöngy and N. Krylov. "On the splitting-up method and stochastic partial differential equations". In: *Ann. Probab.* 31.2 (2003), pp. 564–591.

[90] M. Hairer. *An introduction to stochastic PDEs.* 2009. arXiv: `0907.4178 [math.PR]`.

[91] K.-H. Hoffmann and V. N. Starovoitov. "On a motion of a solid body in a viscous fluid. Two-dimensional case". In: *Adv. Math. Sci. Appl.* 9 (1999), pp. 633–648.

[92] M.-C. Hsu et al. "Fluid-structure interaction analysis of bioprosthetic heart valves: significance of arterial wall deformation". In: *Comput. Mech.* 54.4 (2014), pp. 1055–1071.

[93] M. Ignatova et al. "On well-posedness and small data global existence for an interface damped free boundary fluid-structure model". In: *Nonlinearity* 27.3 (2014), pp. 467–499.

[94] M. Ignatova et al. "On well-posedness for a free boundary fluid-structure model". In: *J. Math. Phys.* 53.11 (2012), p. 115624.

[95] A. Inoue and M. Wakimoto. "On existence of solutions of the Navier-Stokes equation in a time dependent domain". In: *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 24.2 (1977), pp. 303–319.

[96] D. Istrati and I. G. Buckle. "Effect of fluid-structure interaction on connection forces in bridges due to tsunami loads". In: Proceedings of the 30th U.S.-Japan Bridge Engineering Workshop. UJNR Panel on Wind and Seismic Effects. Washington D.C., USA, 2014.

[97] W. Jäger and A. Mikelić. "On the boundary conditions at the contact interface between a porous medium and a free fluid". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 23.3 (1996), pp. 403–465.

[98] W. Jäger and A. Mikelić. "On the interface boundary condition of Beavers, Joseph, and Saffman". In: *SIAM J. Appl. Math.* 60.4 (2000), pp. 1111–1127.

[99] A. Jakubowski. “The almost sure Skorokhod representation for subsequences in nonmetric spaces”. In: *Teor. Veroyatnost. i Primenen.* 42.1 (1997), pp. 209–216.

[100] R. Jamal and É. Sanchez-Palencia. “Théorie asymptotique des tiges courbes anisotropes”. In: *C. R. Acad. Sci. Paris, Série I* 322.11 (1996), pp. 1099–1106.

[101] M. Jurak and J. Tambača. “Derivation and justification of a curved rod model”. In: *Math. Models Methods Appl. Sci.* 9.7 (1999), pp. 991–1014.

[102] M. Jurak and J. Tambača. “Linear curved rod model: General curve”. In: *Math. Models Methods Appl. Sci.* 11.7 (2001), pp. 1237–1252.

[103] O. Kallenberg. *Foundations of modern probability.* Third edition. Vol. 99. Probability theory and stochastic modelling. Switzerland: Springer, 2021.

[104] R. Kamakoti and W. Shyy. “Fluid-structure interaction for aeroelastic applications”. In: *Prog. Aerosp. Sci.* 40.8 (2004), pp. 535–558.

[105] A. Karczewska and J. Zabczyk. “Stochastic PDEs with function-valued solutions”. In: *Infinite dimensional stochastic analysis.* Ed. by Ph. Clément et al. Proceedings of the Colloquium of the Royal Netherlands Academy of Arts and Sciences. Amsterdam, 1999, pp. 197–216.

[106] A. Karimi et al. “A computational fluid-structure interaction model for plaque vulnerability assessment in atherosclerotic human coronary arteries”. In: *J. Appl. Phys.* 115.14 (2014), p. 144702.

[107] D. Khoshnevisan. “A primer on stochastic partial differential equations”. In: *A mini-course on stochastic partial differential equations.* Ed. by D. Khoshnevisan and F. Rassoul-Agha. Vol. 1962. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 2009.

[108] D. Khoshnevisan. *Multiparameter processes: an introduction to random fields.* New York: Springer, 2002.

[109] A. Korn. “Solution générale du problème d’équilibre dans la théorie de l’élasticité, dans le cas òu les effets sont donnés à la surface”. In: *Ann. Fac. Sci. Toulouse Math.* 10 (1908), pp. 165–269.

[110] D. Kowollik et al. “3D fluid-structure interaction analysis of a typical liquid rocket engine cycle based on a novel viscoplastic damage model”. In: *Int. J. Numer. Methods Eng.* 94.13 (2013), pp. 1165–1190.

[111] J. Kuan and S. Čanić. “A stochastically perturbed fluid-structure interaction problem modeled by a stochastic viscous wave equation”. In: *J. Differential Equations* 310 (2022), pp. 45–98.

[112] J. Kuan and S. Čanić. “Deterministic ill-posedness and probabilistic well-posedness of the viscous nonlinear wave equation describing fluid-structure interaction”. In: *Trans. Amer. Math. Soc.* 374.8 (2021), pp. 5925–5994.

[113] J. Kuan and S. Čanić. *Well-posedness of solutions to stochastic fluid-structure interaction.* Submitted 2022. arXiv: `2203.16109 [math.AP]`.

[114] J. Kuan, S. Čanić, and B. Muha. "Existence of a weak solution to a regularized moving boundary fluid-structure interaction problem with poroelastic media". In: *C. R. Mec.* (Accepted 2023).

[115] J. Kuan, S. Čanić, and B. Muha. *Well-posedness of a nonlinearly coupled fluid-poroviscoelastic structure interaction problem.* In preparation.

[116] J. Kuan, T. Oh, and S. Čanić. "Probabilistic global well-posedness for a viscous nonlinear wave equation modeling fluid-structure interaction". In: *Appl. Anal.* 101.12 (2022), pp. 4349–4373.

[117] I. Kukavica and A. Tuffaha. "Solutions to a fluid-structure interaction free boundary problem". In: *Discrete Contin. Dyn. Syst.* 32.4 (2012), pp. 1355–1389.

[118] I. Kukavica, A. Tuffaha, and M. Ziane. "Strong solutions for a fluid structure interaction system". In: *Adv. Differential Equations* 15.3-4 (2010), pp. 231–254.

[119] I. Kukavica, A. Tuffaha, and M. Ziane. "Strong solutions to a Navier-Stokes-Lamé system on a domain with a non-flat boundary". In: *Nonlinearity* 24.1 (2011), pp. 159–176.

[120] I. Kukavica, A. Tuffaha, and M. Ziane. "Strong solutions to a nonlinear fluid structure interaction system". In: *J. Differential Equations* 247.5 (2009), pp. 1452–1478.

[121] I. Kukavica, F. Xu, and M. Ziane. "Global existence for the stochastic Navier-Stokes equations with small L^p data". In: *Stochastics Partial Differ. Equations: Anal. Comput.* 10.1 (2022), pp. 160–189.

[122] J. S.-H. Kuo. *Fluid-structure interactions: added mass computations for incompressible fluid.* Tech. rep. UCB/EERC-82/09. Berkeley, California: University of California, Aug. 1982.

[123] W. J. Layton, F. Schieweck, and I. Yotov. "Coupling fluid flow with porous media flow". In: *SIAM J. Numer. Anal.* 40.6 (2003), pp. 2195–2218.

[124] D. Lengeler and M. Růžička. "Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell". In: *Arch. Ration. Mech. Anal.* 211.1 (2014), pp. 205–255.

[125] J. Lequeurre. "Existence of strong solutions to a fluid-structure system". In: *SIAM J. Math. Anal.* 43.1 (2011), pp. 389–410.

[126] M. Lesinigo, C. D'Angelo, and A. Quarteroni. "A multiscale Darcy-Brinkman model for fluid flow in fractured porous media". In: *Numer. Math.* 117.4 (2011), pp. 717–752.

[127] Y. Li et al. "Study of parachute inflation process using fluid-structure interaction method". In: *Chin. J. Aeronaut.* 27.2 (2014), pp. 272–279.

[128] J. Link, P. Nguyen, and R. Temam. “Local martingale solutions to the stochastic one layer shallow water equations”. In: *J. Math. Anal. Appl.* 448.1 (2017), pp. 93–139.

[129] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Vol. 31. Paris: Dunod, 1969.

[130] E. S. Di Martino et al. “Fluid-structure interaction within realistic three-dimensional models of the aneurysmatic aorta as a guidance to assess the risk of rupture of the aneurysm”. In: *Med. Eng. Phys.* 23.9 (2001), pp. 647–655.

[131] S. E. Mikhailov. “Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains”. In: *J. Math. Anal. Appl.* 378.1 (2011), pp. 324–342.

[132] P. Moireau et al. “External tissue support and fluid-structure simulation in blood flows”. In: *Biomech. Model. Mechanobiol.* 11.1-2 (2012), pp. 1–18.

[133] G. Morgenthal and A. McRobie. “A comparative study of numerical methods for fluid structure interaction analysis in long-span bridge design”. In: *Wind Struct.* 5.2–4 (2002), pp. 101–114.

[134] B. Muha. “A note on optimal regularity and regularizing effects of point mass coupling for a heat-wave system”. In: *J. Math. Anal. Appl.* 425.2 (2015), pp. 1134–1147.

[135] B. Muha. “A note on the Trace Theorem for domains which are locally subgraph of a Hölder continuous function”. In: *Netw. Heterog. Media* 9.1 (2014), pp. 191–196.

[136] B. Muha and S. Čanić. “A generalization of the Aubin-Lions-Simon compactness lemma for problems on moving domains”. In: *J. Differential Equations* 266.12 (2019), pp. 8370–8418.

[137] B. Muha and S. Čanić. “A nonlinear, 3D fluid-structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof”. In: *Commun. Inf. Syst.* 13.3 (2013), pp. 357–397.

[138] B. Muha and S. Čanić. “Existence of a solution to a fluid-multi-layered-structure interaction problem”. In: *J. Differential Equations* 256.2 (2014), pp. 658–706.

[139] B. Muha and S. Čanić. “Existence of a weak solution to a fluid-elastic structure interaction problem with the Navier slip boundary condition”. In: *J. Differential Equations* 260.12 (2016), pp. 8550–8589.

[140] B. Muha and S. Čanić. “Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls”. In: *Arch. Ration. Mech. Anal.* 207.3 (2013), pp. 919–968.

[141] B. Muha and S. Čanić. “Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy”. In: *Interfaces Free Bound.* 17.4 (2015), pp. 465–495.

[142] B. Muha, Š. Nečasová, and A. Radošević. “A uniqueness result for 3D incompressible fluid-rigid body interaction problem”. In: *J. Math. Fluid Mech.* 23.1 (2021), pp. 1–39.

[143] B. Muha and S. Schwarzacher. “Existence and regularity of weak solutions for a fluid interacting with a non-linear shell in three dimensions”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 39.6 (2022), pp. 1369–1412.

[144] J. Musielak and W. Orlicz. “On generalized variations (I)”. In: *Studia Math.* 18.1 (1959), pp. 11–41.

[145] T. Nakata and H. Liu. “A fluid-structure interaction model of insect flight with flexible wings”. In: *J. Comput. Phys.* 231.4 (2012), pp. 1822–1847.

[146] Š. Nečasová and J. Wolf. “On the existence of global strong solutions to the equations modeling a motion of a rigid body around a viscous fluid”. In: *Discrete Contin. Dyn. Syst.* 36.3 (2016), pp. 1539–1562.

[147] O. Omidi and V. Lotfi. “A symmetric implementation of pressure-based fluid-structure interaction for nonlinear dynamic analysis of arch dams”. In: *J. Fluids Struct.* 69 (2017), pp. 34–55.

[148] S. Owczarek. “A Galerkin method for Biot consolidation model”. In: *Math. Mech. Solids* 15.1 (2010), pp. 42–56.

[149] G. Pontrelli. “A mathematical model of flow in a liquid-filled viscoelastic tube”. In: *Med. Biol. Eng. Comput.* 40.5 (2002), pp. 550–556.

[150] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*. Second edition. Vol. 152. Encyclopedia of Mathematics and Its Applications. Cambridge: Cambridge University Press, 2014.

[151] A. Quarteroni, M. Tuveri, and A. Veneziani. “Computational vascular fluid dynamics: problems, models and methods”. In: *Comput. Vis. Sci.* 2.4 (2000), pp. 163–197.

[152] J.-P. Raymond and M. Vanninathan. “A fluid-structure model coupling the Navier-Stokes equations and the Lamé system”. In: *J. Math. Pures Appl.* 102.3 (2014), pp. 546–596.

[153] A. Remond-Tiedrez and I. Tice. “The viscous surface wave problem with generalized surface energies”. In: *SIAM J. Math. Anal.* 51.6 (2019), pp. 4894–4952.

[154] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*. Second edition. Vol. 13. Texts in Applied Mathematics. New York: Springer, 2004.

[155] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Third edition. Vol. 293. A Series of Comprehensive Studies in Mathematics. Berlin Heidelberg: Springer-Verlag, 1999.

[156] B. Rivière and I. Yotov. “Locally conservative coupling of Stokes and Darcy flows”. In: *SIAM J. Numer. Anal.* 42.5 (2005), pp. 1959–1977.

[157] J. Sanchez-Hubert and É. Sanchez-Palencia. “Statics of curved rods on account of torsion and flexion”. In: *Eur. J. Mech. A Solids* 18.3 (1999), pp. 365–390.

[158] M. Sanz-Solé and M. Sarrà. "Hölder continuity for the stochastic heat equation with spatially correlated noise". In: *Seminar on Stochastic Analysis, Random Fields and Applications III*. Ed. by R. C. Dalang, M. Dozzi, and F. Russo. Vol. 52. Progress in Probability. Basel: Birkhäuser, 2002, pp. 259–268.

[159] S. Schwarzacher and M. Sroczinski. "Weak-strong uniqueness for an elastic plate interacting with the Navier-Stokes equation". In: *SIAM J. Math. Anal.* 54.4 (2022), pp. 4104–4138.

[160] A. Seboldt et al. "Numerical modeling of the fluid-porohyperelastic structure interaction". In: *SIAM J. Sci. Comput.* 43.4 (2021), A2923–A2948.

[161] D. Serre. "Chute libre d'un solide dans un fluide visqueux incompressible. Existence." In: *Japan J. Appl. Math.* 4.1 (1987), pp. 99–110.

[162] R. E. Showalter. "Diffusion in poro-elastic media". In: *J. Math. Anal. Appl.* 251.1 (2000), pp. 310–340.

[163] R. E. Showalter. "Poroelastic filtration coupled to Stokes flow". In: *Control theory of partial differential equations*. Ed. by G. Leugering et al. First edition. Vol. 242. A Series of Lecture Notes in Pure and Applied Mathematics. New York: Chapman and Hall/CRC, 2005, pp. 229–242.

[164] R. E. Showalter and N. Su. "Partially saturated flow in a poroelastic medium". In: *Discrete Contin. Dyn. Syst. Ser. B* 1.4 (2001), pp. 403–420.

[165] J. Simon. "Compact sets in the space $L^p(0,T;B)$". In: *Ann. Mat. Pura Appl.* 146 (1986), pp. 65–96.

[166] S. A. Smith. "Random perturbations of viscous, compressible fluids: global existence of weak solutions". In: *SIAM J. Math. Anal.* 49.6 (2017), pp. 4521–4578.

[167] S. A. Smith and K. Trivisa. "The stochastic Navier-Stokes equations for heat-conducting, compressible fluids: global existence of weak solutions". In: *J. Evol. Equ.* 18.2 (2018), pp. 411–465.

[168] K. Stein et al. "Parachute fluid-structure interactions: 3-D computation". In: *Comput. Methods Appl. Mech. Engrg.* 190.3 (2000), pp. 373–386.

[169] K. R. Stein et al. "Fluid-structure interactions of a round parachute: modeling and simulation techniques". In: *J. Aircr.* 38.5 (2001), pp. 800–808.

[170] G. Szabó and J. Györgyi. "Three-dimensional fluid-structure interaction analysis for bridge aeroelasticity". In: vol. 2. Proceedings of the World Congress on Engineering and Computer Science. WCECS. San Francisco, USA, 2009.

[171] T. Takahashi. "Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain". In: *Adv. Diffential Equations* 8.12 (2003), pp. 1499–1532.

[172] T. Takahashi. "Existence of strong solutions for the problem of a rigid-fluid system". In: *C. R. Math.* 336.5 (2003), pp. 453–458.

[173] T. Takahashi and M. Tucsnak. "Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid". In: *J. Math. Fluid Mech.* 6.1 (2004), pp. 53–77.

[174] J. Tambača et al. "Mathematical modeling of vascular stents". In: *SIAM J. Appl. Math.* 70.6 (2010), pp. 1922–1952.

[175] K. Tawri and S. Čanić. *Existence of martingale solutions to a nonlinearly coupled stochastic fluid-structure interaction problem*. In preparation.

[176] T. Tezduyar and Y. Osawa. "Fluid-structure interactions of a parachute crossing the far wake of an aircraft". In: *Comput. Methods Appl. Mech. Engrg.* 191.6 (2001), pp. 717–726.

[177] J. B. Walsh. "An introduction to stochastic partial differential equations". In: *École d'Été de Probabilités de Saint Flour XIV - 1984*. Ed. by P. L. Hennequin. Vol. 1180. Lecture Notes in Mathematics. Berlin: Springer, 1986, pp. 265–439.

[178] K. Wang et al. "Analysis of hydraulic fracturing in concrete dam considering fluid-structure interaction using XFEM-FVM model". In: *Eng. Fail. Anal.* 57 (2015), pp. 399–412.

[179] H. Q. Yang, J. West, and R. E. Harris. "Coupled fluid-structure interaction analysis of solid rocket motor with flexible inhibitors". In: *J. Spacecr. Rockets* 55.2 (2018), pp. 303–314.

[180] J. Young et al. "A mathematical model of intestinal oedema formation". In: *Math. Med. Biol.* 31.1 (2014), pp. 1–15.

[181] L. C. Young. "An inequality of the Hölder type, connected with Stieltjes integration". In: *Acta Math.* 67 (1936), pp. 251–282.

[182] R. Zakerzadeh and P. Zunino. "A computational framework for fluid-porous structure interaction with large structural deformation". In: *Meccanica* 54.1–2 (2019), pp. 101–121.

[183] A. Ženíšek. "The existence and uniqueness theorem in Biot's consolidation theory". In: *Aplikace Matematiky* 29.3 (1984), pp. 194–211.