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On Algebraic Methods in Quantum Theories

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by

Benjamin Henry Feintzeig

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2016

DEDICATION

To JAF and RAF

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ABSTRACT OF THE DISSERTATION

On Algebraic Methods in Quantum Theories

By

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Doctor of Philosophy in Philosophy

University of California, Irvine, 2016

Associate Professor James O. Weatherall, Chair

The received view in the philosophy of physics is that a quantum theory is given its mathematical formulation in a so-called Hilbert space, with the states of a quantum system given by the elements of this space and observables—measurable quantities of the system—given by so-called operators on this space. But the status of this fundamental postulate becomes threatened when one moves to complex systems, like the quantum fields underlying fundamental particle physics or statistical systems composed of infinite collections of quantum particles. In these more complex cases, one finds many inequivalent Hilbert space representations that appear to be competing quantum theories. In looking for a mathematical structure of the theory to interpret as representing the physical world, one is faced with a choice: either arbitrarily pick one Hilbert space representation (out of the many competing representations) as a *Hilbert Space Conservative*, or else look for the abstract algebraic structure that all of these competing representations have in common as an *Algebraic Imperialist*. This dissertation provides an extended argument for the second route.

Chapter 1

Introduction

Quantum field theory is our best theory of matter. It underlies all of modern particle physics and makes the predictions that we spend billions of dollars to test in collider experiments. But even in the most sophisticated works of theoretical and mathematical physicists, one finds disagreements, persisting for over 50 years, about what quantum field theory *is*.[1] Of course, working physicists have a collection of heuristic methods they use in their research, but the underlying mathematical foundation for these tools is still unclear. Finding a rigorous mathematical framework to underlie and support our best quantum field theories has been deemed so urgent that the Clay Institute for Mathematics lists it as one of their "Millennium Prize Problems" (Douglas, 2004). One reason a solution has proved so elusive is that there are many competing approaches to rigorizing quantum field theory, and the relations between these approaches as well as their advantages and disadvantages are not yet well understood. In particular, there are real interpretational ambiguities that arise when one attempts to understand the roles of physical states and physical quantities in competing frameworks for quantum theories (Arageorgis, 1995; Ruetsche, 2011). The goal of this dissertation is to gain a better understanding of how the mathematical tools in quantum theories are used to represent features of physical systems, and to argue for the appropriateness of a particular framework

[1] See, e.g., Haag and Kastler (1964); Haag (1992); Wald (1994); Hollands and Wald (2001); Giulini and Marolf (1999); Costello (2011); Brunetti and Fredenhagen (2009).

for constructing quantum theories that has come to be known as the algebraic approach.

Let's begin to look at the interpretational problems just alluded to. As we will see in the next chapter, quantum theories are ordinarily formulated in a mathematical setting known as Hilbert space. Physical states are represented by elements of the Hilbert space, and physical quantities are represented by objects known as operators on the Hilbert space. In the context of nonrelativistic quantum mechanics for finitely many particles, one can describe the procedure of constructing a quantum theory in a Hilbert space in perfect detail (Clifton and Halvorson, 2001; Petz, 1990; Ruetsche, 2011). But when one tries to generalize this procedure to apply it to quantum field theories, or even quantum statistical theories in the thermodynamic limit,[2] one finds an ambiguity in the Hilbert space one is using for the theory (Ruetsche, 2011; Segal, 1963; Emch, 1972). In fact, for these more complex systems, one finds many different Hilbert spaces as the result of this construction procedure, all of which appear to be competing versions of the quantum theory we are aiming at. Facing this situation, we have at least two options. The first is to find some way of choosing one from these many competing Hilbert space theories. But this may seem ad hoc, and as we will see in later chapters, it turns out there are real problems with representing enough physical states within the Hilbert space framework to reconstruct physically significant explanations. The second option is to find some other way of formulating a quantum theory that captures the structure these many different Hilbert spaces have in common. Indeed, there is an abstract algebraic structure that one can use for just this purpose (Haag and Kastler, 1964; Haag, 1992; Emch, 1972).

The first option of choosing a Hilbert space has come to be known as *Hilbert Space Conservatism* while the second option of using the abstract algebraic structure the Hilbert spaces have in common is known as *Algebraic Imperialism* (Arageorgis, 1995; Ruetsche, 2011). These two approaches lead to real physical and interpretational disagreements over, for example, the collection of physically possible states and the collection of physicaly significant observables. Whether one

[2]These theories have in common that they have infinitely many degrees of freedom.

goes down one route or the other depends at least in part on whether one sees these different Hilbert spaces as having genuinely distinct physical content (Baker, 2011; Earman, 2003; Ruetsche, 2011). As such, these issues in quantum theory are tied to deep questions concerning theoretical equivalence, the relationship of physical states with physical quantities, and the constraints we put on the construction of new physical theories. What follows in this dissertation is an extended argument for the use of algebraic methods by showing that algebraic methods provide the best route to answering these important foundational questions as well as the best route to understanding the explanations and descriptions that quantum theory provides.

1.1 Overview

Chapter 2 will provide the necessary background concerning abstract algebraic and Hilbert space tools as well as their application in the construction of quantum theories. This will set the stage for the distinction between the various interpretive stances one might take towards the mathematical tools used in quantum theory. It is the goal of the remainder of the dissertation to weigh the virtues and vices of these interpretive stances.

Chapter 3 argues against the interpretive position of Hilbert Space Conservatism. Ruetsche has previously argued that the Conservative cannot represent all of the physically possible states for paradigmatic examples of physical systems (Ruetsche, 2002, 2003, 2006). I show that this argument can be made even more forcefully by pulling it back to the simpler context of classical physics. I apply to classical physics the same mathematical tools of abstract algebras and Hilbert space representations that are used in quantum physics. I show that in this context Hilbert Space Conservatism reduces to absurdity because each Hilbert space representation can only describe a single state of the many states the theory deems physically possible. I argue that even though classical physics may deserve a different interpretation than quantum physics, the classical case drives home just how much physical content can be lost when one insists on the mathematical framework of a Hilbert space.

Chapter 4 focuses on the phenomenon of symmetry breaking, a supposedly novel feature of quantum physics in which inequivalent Hilbert space representations are claimed to play a fundamental role (Earman, 2003; Strocchi, 2008). While it has been claimed that the presence of inequivalent representations gives rise to a number of puzzles concerning equivalent and inequivalent physical possibilities (Baker, 2011; Baker and Halvorson, 2013), I argue that these puzzles dissolve when we compare quantum symmetry breaking with the same phenomenon in classical systems. In the process, I show that the Conservative's understanding of symmetry breaking is at best misleading.

Chapter 5 provides a solution to the primary outstanding problem with Algebraic Imperialism. Some claim that we need the mathematical tools of Hilbert spaces to construct approximations and idealizations that are essential for describing certain physical quantities, such as temperature and net magnetization—quantities that Ruetsche calls "parochial observables." Ruetsche (2003; 2011) claims that the Algebraic Imperialist does not have the resources to describe these physically significant quantities of quantum systems. However, I show to the contrary that the Algebraic Imperialist has access to analogous tools for constructing approximations and idealizations that allow her to describe these parochial observables without needing to make any choice of Hilbert space representation. Employing these tools clarifies how the notions of state and quantity are conceptually and mathematically intertwined in our best physical theories.

Chapter 6 aims to demonstrate virtues of algebraic methods in action in the context of the construction of even the simplest nonrelativistic quantum theories. While Halvorson (2004) has argued that one finds the same mathematical ambiguities that we found in quantum field theory and quantum statistical mechanics for even a single particle system, I show that detailed attention to the algebraic tools one uses to represent this system can help us choose the appropriate quantum theory. Specifically, I show that requiring a condition of continuity between the algebraic tools used to represent classical and quantum systems helps us to constrain and inform the quantum theories we end up with.

Chapter 7 provides some brief concluding discussion and points at open questions and future directions for this work.

Chapter 2

Background on Algebraic Quantum Theories

2.1 Algebraic Tools

A standard presentation of the nonrelativistic quantum theory of a single particle consists in a Hilbert space $\mathcal{H}$ containing operators P and Q, which represent the momentum and position of the particle, and which satisfy the canonical commutation relations.[1] The states of the theory are the vectors or density operators on $\mathcal{H}$ and the observables are self-adjoint operators on $\mathcal{H}$. A result known as the Stone-von Neumann theorem[2] (described below) tells us that this formulation is equivalent to a more abstract algebraic approach. The algebraic approach, which provides a useful way of generalizing to quantum field theory and quantum statistical mechanics, proceeds as follows. One begins with a C*-algebra[3] $\mathfrak{A}$ generated by canonical (anti-)commutation relations.

[1]See, e.g. Sakurai (1994), Mackey (1963), or Jordan (1969).

[2]For mathematical reviews of the Stone-von Neumann theorem, see Petz (1990) and Summers (1999). For philosophical discussions, see Ruetsche (2011, p.41) and Clifton and Halvorson (2001, p.427).

[3]For more on the theory of C*-algebras and their representations, see Kadison and Ringrose (1997); Bratteli and Robinson (1987); Emch (1972). For philosophical introductions, see Ruetsche (2011) and Halvorson (2006).

In a C*-algebra, we can add and multiply observables, and multiply observables by scalars. In addition, a C*-algebra has an operation of involution $^* : \mathfrak{A} \to \mathfrak{A}$ that is a generalization of complex conjugation. The observables of the theory are represented by the self-adjoint elements of $\mathfrak{A}$, i.e. elements $A \in \mathfrak{A}$ such that $A^* = A$. A C*-algebra $\mathfrak{A}$ comes equipped with a norm, which is required to satisfy the C*-identity:

$$\|A^*A\| = \|A\|^2$$

The norm defines a topology, called the *norm topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges to A in the norm topology[4] iff

$$\|A_i - A\| \to 0$$

where the convergence is now in the standard topology on $\mathbb{R}$. The C*-algebra $\mathfrak{A}$ is required to be complete with respect to this topology in the sense that for every Cauchy net $\{A_i\} \subseteq \mathfrak{A}$, i.e. for every net such that

$$\left\|A_i - A_j\right\| \to 0$$

as $i, j \to \infty$, there is an $A \in \mathfrak{A}$ such that $A_i \to A$ in the norm topology. Standard results in the theory of normed vector spaces tell us that every normed vector space has a unique completion.[5]

Since $\mathfrak{A}$ is a vector space, we can also consider the dual space $\mathfrak{A}^*$ of bounded (i.e. norm continuous) linear functionals $\rho : \mathfrak{A} \to \mathbb{C}$. States on a C*-algebra $\mathfrak{A}$ are particular elements of the dual space $\mathfrak{A}^*$—namely ones that are *positive* in the sense that $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and *normalized* in the sense that $\|\rho\| = 1$. A state is called *pure* if it cannot be written as a convex combination of distinct states. Otherwise, a state is called *mixed*. Pure states represent the possible states of an individual system while mixed states are typically taken to represent some sort of probabilistic combination (whether it be via an ensemble interpretation or mere epistemic uncertainty).

[4]One could restrict attention here to sequences because the norm topology is second countable, but for the weak topologies considered later, which are not second countable, one must work with arbitrary nets.

[5]See Reed and Simon (1980). A complete normed vector space is called a Banach space. A C*-algebra is thus a Banach algebra whose norm is, in a certain sense, compatible with multiplication and involution.

Importantly, this algebraic formalism can be translated back into the familiar Hilbert space theory through a representation. A *representation* of $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, where $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism into the bounded linear operators on some Hilbert space $\mathcal{H}$. A representation is called *irreducible* if the only subspaces of $\mathcal{H}$ invariant under $\pi(\mathfrak{A})$ are $\mathcal{H}$ and $\{0\}$. One may find representations of $\mathfrak{A}$ on different Hilbert spaces, and in this case one wants to know when these can be understood as "the same representation." This notion of "sameness" is given by the concept of unitary equivalence:[6] two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are *unitarily equivalent* if there is a unitary mapping $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the representations, i.e. for each $A \in \mathfrak{A}$,

$$U\pi_1(A) = \pi_2(A)U$$

The specified unitary mapping U sets up a way of translating between density operator states on $\mathcal{H}_1$ and density operator states on $\mathcal{H}_2$, and between observables in $\mathcal{B}(\mathcal{H}_1)$ and observables in $\mathcal{B}(\mathcal{H}_2)$. The translated states reproduce the same expectation values on the translated observables that the original states assigned to the original observables.

One of the most fundamental results in the theory of C*-algebras, known as the GNS theorem (Kadison and Ringrose, 1997, p. 278, Thm. 4.5.2), asserts that for each state ω on $\mathfrak{A}$, there exists a representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathfrak{A}$, known as the *GNS representation for* ω, and a (cyclic) vector $\Omega_\omega \in \mathcal{H}_\omega$ such that for all $A \in \mathfrak{A}$,

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$$

The GNS representation for a state ω is unique in the sense that any other representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ containing a cyclic vector corresponding to ω is unitarily equivalent to $(\pi_\omega, \mathcal{H}_\omega)$ (Kadison and Ringrose, 1997, p.279, Prop. 4.5.3).

[6]See Ruetsche (2011, Ch. 2.2), Ruetsche (2013), and Clifton and Halvorson (2001, Sec. 2.2-2.3) for more on unitary equivalence as a notion of "sameness of representations."

When one is working in the context of a particular Hilbert space $\mathcal{H}$, one can define the weak operator topology on $\mathcal{B}(\mathcal{H})$ by the following criterion for convergence (Reed and Simon, 1980, p. 183): a net $\{A_i\}$ converges in the weak operator topology to A just in case for all $\phi, \psi \in \mathcal{H}$,

$$\langle \phi, A_i \psi \rangle \to \langle \phi, A\psi \rangle$$

in $\mathbb{C}$. Recall that the GNS theorem allows us to take any C*-algebra of observables $\mathfrak{A}$ and, having chosen some state ω, represent it via the representation π_ω as a subalgebra $\pi_\omega(\mathfrak{A}) \subseteq \mathcal{B}(\mathcal{H}_\omega)$ for some Hilbert space $\mathcal{H}_\omega$. Using the weak operator topology on $\mathcal{B}(\mathcal{H}_\omega)$ as a physically relevant notion of approximation,[7] one can include in any algebra of observables the operators that are physically indistinguishable from or arbitrarily well approximated by the observables already picked out in our algebra. To do so, we take the weak operator closure (written $\overline{\pi_\omega(\mathfrak{A})}$) of our original algebra of observables $\pi_\omega(\mathfrak{A})$—this adds to our original algebra all limit points of weak operator converging nets. Any weak operator closed subalgebra of a Hilbert space is called a von Neumann algebra, so $\overline{\pi_\omega(\mathfrak{A})}$ will be called the von Neumann algebra affiliated with the representation π_ω. If the state ω that we used to take our GNS representation is pure, then, because the representation π_ω is irreducible (Kadison and Ringrose, 1997, p. 728, Thm. 10.2.3), it follows that $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$ (Sakai, 1971, p. 52, Prop. 1.21.9). So taking the GNS representation for a pure state and closing in the weak operator topology brings us back to the familiar situation where our observables are *all* of the bounded self-adjoint operators on a Hilbert space.

[7] The motivation for this standard practice is that in the weak operator topology, a net of observables well approximates (i.e. converges to) another observable just in case it approximates it with respect to all possible expectation values and transition probabilities, and hence with respect to the empirical predictions of the theory. We will discuss the significance of this in Ch. 5.

2.2 Quantum Observables

The Stone-von Neumann theorem tells us that for systems with finitely many degrees of freedom, the usual Schrödinger representation is the unique irreducible representation of the algebra of observables up to unitary equivalence.[8] This means that for ordinary, nonrelativistic quantum theories, it does not make any difference to physics whether we work in the abstract algebra or one of its representations because there is only one suitable representation available. It is worth taking a brief look at the details of the Stone-von Neumann theorem. To formulate it, we first need to know what algebra we are working with. The non-commutative quantum algebra of observables $\mathfrak{A}$ is supposed to be the result of implementing some form of the canonical commutation relations on our classical observables (see, e.g. Petz, 1990), thereby converting them into quantum observables.

Let's begin with the simplest case of a single particle moving in one dimension, with phase space $\mathcal{M} = \mathbb{R}^{2n}$. This phase space carries the structure of a Poisson manifold with Poisson bracket $\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ defined by (see, e.g. Landsman, 1998a):

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

for all smooth functions $f,g \in C^\infty(\mathcal{M})$, where (q,p) is a canonical coordinate system. The canonical commutation relations are given by:[9]

$$[Q(f), Q(g)] = iQ(\{f,g\})$$

for all $f,g \in C^\infty(\mathcal{M})$. We can see that the canonical commutation relations transform information encoded in the classical Poisson bracket into a constraint on the algebraic relations among the

[8] The Stone-von Neumann theorem carries other substantive assumptions as well. It assumes that the representation is continuous in an appropriate sense and that the configuration space of the classical theory is symplectic. See Ruetsche (2011, Ch. 2-3) and see Landsman (1998a) for a more general version of the theorem than presented here.

[9] Throughout, we work in units such that $\hbar = 1$.

quantum observables. [10]

We will put the canonical commutation relations in a slightly different form for the convenience of applying them to a specific collection of bounded observables[11] Recall that we want to find quantum observables Q and P to represent the quantities position and momentum; to do so we might start with the classical quantities q and p defined as unbounded functions on $\mathcal{M} = \mathbb{R}^{2n}$, and then set $Q = \mathcal{Q}(q)$ and $P = \mathcal{Q}(p)$. But we cannot do this directly with the tools that we've laid out in the previous section. After all, since both $\mathcal{Q}(p)$ and $\mathcal{Q}(q)$ are unbounded, neither can belong to a C*-algebra. As such, one typically considers (formally) the exponentiated observables $U_a = e^{ia\cdot\mathcal{Q}(q)}$ and $V_b = e^{ib\cdot\mathcal{Q}(p)}$, for $a, b \in \mathbb{R}^n$, which are bounded. The canonical commutation relations for q and p can be put in the (formally equivalent) Weyl form

$$U_a V_b = e^{-ia\cdot b} V_b U_a$$

with

$$U_a^* = U_{-a} \text{ and } V_b^* = V_{-b}$$

One then searches for a noncommutative C*-algebra $\mathfrak{A}$ to represent the quantum observables with the requirement that $\mathfrak{A}$ contain all the operators U_a, V_b satisfying the Weyl form of the canonical commutation relations. One choice, used often in the physics literature and which has now become standard in philosophy of physics, is the *Weyl algebra*, which is the smallest C*-algebra $\mathcal{W}$ containing all of the operators U_a, V_b.[12] To form the Weyl algebra, one first defines the Weyl operators

$$W_{a,b} = e^{ia\cdot b/2} V_b U_a$$

[10] This definition captures as a special case the usual commutation relations

$$[\mathcal{Q}(q), \mathcal{Q}(p)] = i$$

for the quantized position observable $\mathcal{Q}(q) = Q$ and the quantized momentum observable $\mathcal{Q}(p) = P$.

[11] See Petz (1990), Clifton and Halvorson (2001), Halvorson (2004), and Ruetsche (2011).

[12] The Weyl algebra can also be uniquely characterized through its Hilbert space representations (Clifton and Halvorson, 2001).

To simplify expressions, we note that the phase space $\mathcal{M} = \mathbb{R}^{2n}$ is a symplectic manifold, where the symplectic form $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is defined by

$$\sigma((a,b),(a',b')) = a' \cdot b - a \cdot b'$$

for all $a, b, a', b' \in \mathbb{R}^n$ generates the Poisson bracket on $\mathcal{M}$. Using the symplectic form, the Weyl form of the commutation relations is equivalent to

$$W_{a,b}W_{a',b'} = e^{-i\sigma((a,b),(a',b'))/2}W_{a+a',b+b'}$$

with

$$W^*_{a,b} = W_{-a,-b}$$

Now, the Weyl algebra $\mathcal{W}$ is generated from the operators $W_{a,b}$ as a basis (with multiplication and involution defined via the above formulas) by completing the resultant vector space in the norm topology.[13]

To state the Stone-von Neumann theorem, we need one further assumption to ensure that a representation of the Weyl algebra is appropriately continuous. We will say that a representation $(\pi, \mathcal{H})$ of $\mathcal{W}$ is *regular* if the one-parameter families $t \mapsto \pi(W_{ta,tb})$ are all weak operator continuous (Petz, 1990; Clifton and Halvorson, 2001). Whenever this is the case, Stone's theorem (Reed and Simon, 1980, p. 266) guarantees the existence of an unbounded self-adjoint operator $\Phi_{a,b}$ that generates the one-parameter family in the sense that $\pi(W_{ta,tb}) = e^{it\Phi_{a,b}}$, so that the operators of the form $\Phi_{a,b}$ correspond to generalized positions and momenta.

Now we can state the Stone-von Neumann theorem.[14] Let $(\pi_S, \mathcal{H}_S)$ be the usual Schrödinger

[13] The Weyl operators admit a unique maximal C*-norm (Manuceau et al., 1974).

[14] The version of the Stone-von Neumann theorem we present is adapted from Summers (1999), Petz (1990), and Ruetsche (2011). See Mackey (1949) and Rieffel (1972) for alternative versions of the uniqueness result.

representation of the Weyl algebra $\mathcal{W}$ over $\mathcal{M} = \mathbb{R}^{2n}$, i.e. let $\mathcal{H} = L^2(\mathbb{R}^n)$ and

$$\pi_S(W_{a,b})\psi(x) = e^{ia\cdot b/2}e^{ia\cdot x}\psi(x-b)$$

for all $\psi \in L^2(\mathbb{R}^n)$.

Theorem 2.2.1 (Stone-von Neumann)**.** *Let $(\pi, \mathcal{H})$ be any irreducible, regular representation of the Weyl algebra $\mathcal{W}$ over $\mathcal{M} = \mathbb{R}^{2n}$. Then $(\pi, \mathcal{H})$ is unitarily equivalent to the Schrödinger representation $(\pi_S, \mathcal{H}_S)$.*

Thus, by assuming our phase space is $\mathbb{R}^{2n}$ and that our representation is regular, we find a unique irreducible representation of the quantum algebra of observables in the Weyl algebra. All other (reducible) representations are then formed by direct sums of this unique irreducible representation.

But it is well known that the assumptions of the Stone-von Neumann theorem fail for the algebras used in quantum field theory and quantum statistical mechanics in the thermodynamic limit because these theories describe systems with infinitely many degrees of freedom. In this case, we generalize the construction of the Weyl algebra to an arbitrary classical phase space, which is now infinite dimensional and so does not take the form $\mathbb{R}^{2n}$. We still assume that the classical phase space carries the structure of a symplectic vector space: let $\mathcal{M}$ be a real symplectic vector space with symplectic form $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$. Such a vector space, which may be infinite-dimensional, can be used to represent the space of solutions to some (linear[15]) field equations. We define the Weyl algebra $\mathcal{W}$ over $\mathcal{M}$ again as the algebra generated by the Weyl operators $W_f, W_g, \ldots$ for $f, g \in \mathcal{M}$ as a basis, where these obey the commutation relations in the form

$$W_f W_g = e^{-i\sigma(f,g))/2} W_{f+g}$$

[15]For solutions to a nonlinear field equation, one would have to allow $\mathcal{M}$ to be a (possibly curved) symplectic manifold.

with

$$W_f^* = W_{-f}$$

Now when $\mathcal{M}$ is infinite dimensional, there exist many unitarily inequivalent irreducible (and even regular) representations of the Weyl algebra.[16] So what appears to be the very same construction procedure we used in the finite case does not yield a unique Hilbert space representation in the infinite case. These inequivalent representations appear to provide us with inequivalent formulations of the theory because the set of states that can be represented as density operators on a given irreducible Hilbert space representation is disjoint from the set of states that can be represented as density operators on a unitarily inequivalent irreducible Hilbert space representation (Kadison and Ringrose, 1997, p. 741). Different Hilbert spaces seem to specify completely different collections of ways the world might be.

Luckily, even for systems with infinitely many degrees of freedom, we always know that we can construct a Hilbert space theory if we desire through the GNS theorem. This means that one can always start with an algebraic theory and get back to an ordinary Hilbert space theory by using the GNS construction; for example, the Minkowski quanta Fock space representation of a quantum field system can be constructed as the GNS representation for the Minkowski vacuum state (Wald, 1994; Clifton and Halvorson, 2001). However, it is important to note that for systems with infinitely many degrees of freedom even the GNS representation does not give a unique way of generating a representation for the algebra $\mathfrak{A}$. Even though the GNS representation of $\mathfrak{A}$ for each state is unique, by cycling through the GNS representations of different states on the abstract algebra, one can generate unitarily inequivalent representations. For example, by taking the GNS representation for the Rindler vacuum state, one can construct a Fock space representation that is unitarily inequivalent to the Minkowski quanta Fock space representation.

How, then, are we supposed to interpret infinite quantum theories like quantum field theory and quantum statistical mechanics? According to Ruetsche, to interpret a physical theory is to spec-

[16]See Segal (1963), Emch (1972), Kay and Wald (1991), and Wald (1994).

ify the collection of worlds the theory deems possible (Ruetsche, 2011, p. 6). Although I do not endorse this view of interpretation, I will accept it provisionally to illustrate the different positions.[17] For our purposes then, an interpretation will consist in what Ruetsche calls a kinematic pair (Ruetsche, 2011, p. 35), specifying the physically measurable quantities and the physically possible states of a system. The presence of unitarily inequivalent representations in quantum theories with infinitely many degrees of freedom gives rise to the following three interpretations.[18,19]

2.3 Three Extremist Interpretations

2.3.1 Algebraic Imperialism

First, one can be an *Algebraic Imperialist* by asserting that a quantum theory is given in full by the abstract algebra of observables and the states on that algebra rather than its Hilbert space representations. The abstract algebra captures a structure that all Hilbert space representations have in common, so the Algebraic Imperialist chooses to focus only on this structure. To do so is to proclaim that all the work that has been done on interpreting the Hilbert space formalism for ordinary quantum mechanics with finitely many degrees of freedom cannot yield a complete and adequate interpretation for the case of infinitely many degrees of freedom. According to the Algebraic Imperialist, "the extra structure one obtains along with a concrete representation of [$\mathfrak{A}$] is extraneous." (Ruetsche, 2011, p. 132). All that matters is the abstract algebraic structure. The physically measurable quantities are given by the observables (self-adjoint elements) in $\mathfrak{A}$, and the physically possible states are given by the states on $\mathfrak{A}$.

[17] We'll revisit this notion of interpretation and consider alternatives in Ch. 7.

[18] For more on these positions and their advantages and disadvantages, see Arageorgis (1995), Ruetsche (2002; 2003; 2006; 2011). Of course, as Ruetsche describes, there are more subtle interpretive options, but we deal initially only with three of the simplest cases.

[19] Although I will not discuss it here, unitarily inequivalent representations are also related to particle interpretations and dynamics. For more, see Arageorgis et al. (2002); Earman and Fraser (2006); Fraser (2008).

2.3.2 Hilbert Space Conservatism

On the other hand, if one wants to be a *Hilbert Space Conservative* and maintain an interpretation via the Hilbert space formalism like those usually discussed for ordinary quantum mechanics, then one must pick a particular Hilbert space representation to interpret. To ensure that one gets an irreducible representation whose weak operator closure exhausts the operators on the Hilbert space chosen, the Hilbert Space Conservative may choose a pure state ω on $\mathfrak{A}$ and take its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$. For the Hilbert Space Conservative, one such particular irreducible Hilbert space representation of the abstract algebra specifies the physical possibilities. Following standard practice in ordinary quantum mechanics (for finitely many degrees of freedom), the Hilbert Space Conservative takes the physically measurable quantities to be the self-adjoint elements of the weak operator closure $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$ and the physically possible states to be density operators on $\mathcal{H}_\omega$.

Ruetsche argues that Hilbert Space Conservatism is inadequate because it does not give us access to enough states. In theories with infinitely many degrees of freedom, the existence of unitarily inequivalent representations entails that for any privileged irreducible representation $(\pi, \mathcal{H})$, there is some algebraic state that cannot be implemented as a density operator on $\mathcal{H}$. This would be fine if we only ever needed the density operator states on a single Hilbert space to accomplish the goals of physics, but there are instances in which we need to appeal to two states which cannot be represented as density operators on the same irreducible Hilbert space representation in the course of giving a single physically significant explanation. For example, states of different pure thermodynamic phases cannot be represented as density operators on the same irreducible Hilbert space representation. But certainly we need to be able to account for states of different pure thermodynamic phases as simultaneously physically possible in order to explain phase transitions in quantum statistical mechanics. So the Hilbert Space Conservative lacks the resources to recover such physically significant explanations.[20] We'll take a look at this argument in detail in Ch. 3, but for now note that this may push the Hilbert Space Conservative to try to find a Hilbert space

[20]This argument is elaborated in Ruetsche (2002; 2003; 2006; 2011).

on which *all* states can be represented as density operators. The *Universalist* seeks to do precisely this.

2.3.3 Universalism

The *Universalist* agrees with the Hilbert Space Conservative that we need a Hilbert space representation of the abstract algebra $\mathfrak{A}$ to interpret our quantum theory but disagrees that we need an irreducible representation. The Universalist holds that the *universal representation* is the privileged representation of the abstract algebra. Letting $\mathcal{S}_{\mathfrak{A}}$ denote the set of states on $\mathfrak{A}$ and letting $(\pi_\omega, \mathcal{H}_\omega)$ denote the GNS representation of any $\omega \in \mathcal{S}_{\mathfrak{A}}$, the universal representation is given by $(\pi_U, \mathcal{H}_U)$, where

$$\mathcal{H}_U = \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \mathcal{H}_\omega$$

is the universal Hilbert space and for each $A \in \mathfrak{A}$

$$\pi_U(A) = \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \pi_\omega(A)$$

The universal representation is universal in the sense that every other representation can be embedded within it (Kadison and Ringrose, 1997, p. 719, Thm. 10.1.12). The universal representation is guaranteed to be faithful (Kadison and Ringrose, 1997, p. 281, Remark 4.5.8) so each element $A \in \mathfrak{A}$ has a distinct counterpart in $\pi_U(\mathfrak{A})$. The Universalist, like the Hilbert Space Conservative, has access to more observables than the Algebraic Imperialist. The universal Hilbert space carries its own weak operator topology defined precisely as above, which allows us to take the weak operator closure $\overline{\pi_U(\mathfrak{A})}$ of our original algebra, thereby obtaining *the universal enveloping von Neumann algebra of* $\mathfrak{A}$. Since π_U is reducible, it follows that

$$\pi_U(\mathfrak{A}) \subseteq \overline{\pi_U(\mathfrak{A})} \subsetneq \mathcal{B}(\mathcal{H}_U)$$

For the Universalist, the physically measurable quantities are the self-adjoint elements of $\overline{\pi_U(\mathfrak{A})}$, and the physically possible states are density operators on $\mathcal{H}_U$.

Furthermore, every state on the abstract algebra $\mathfrak{A}$ can be represented by a density operator on the universal Hilbert space $\mathcal{H}_U$, so the Universalist has access to as many states as there are on the abstract algebra $\mathfrak{A}$ and avoids one of the pitfalls of the Hilbert Space Conservative.[21]

It is worth noting that although Universalism uses a Hilbert space representation to think about states and observables, it differs crucially from the ways we are accustomed to thinking about these mathematical tools in ordinary quantum mechanics, and thus it may not give the Hilbert Space Conservative everything she desired. First, in ordinary quantum mechanics we are accustomed to working in an irreducible representation of the algebra of observables. We have seen that the universal representation is manifestly reducible. Second, in ordinary quantum mechanics we are accustomed to thinking of vector states as pure states (because we work in an irreducible representation). Vector states on the universal representation, or universal enveloping von Neumann algebra, are not necessarily pure states. In fact, every mixed state on $\mathfrak{A}$ can be thought of as a vector state as well. Third, in ordinary quantum mechanics we are accustomed to working in a separable Hilbert space. Because the universal Hilbert space is the direct sum of an uncountable number of nontrivial Hilbert spaces, it follows that the universal Hilbert space is nonseparable. These are certainly important differences from the mathematical tools that we use in ordinary quantum mechanics, and I save further discussion of their significance for future work.

2.4 Algebraic Quantum Field Theory

To demonstrate how one might use the above algebraic tools independently of any Hilbert space representation, we here briefly outline the starting point of algebraic quantum field theory, which

[21] In fact, every state can be represented in the universal representation as a vector and hence a *finite rank density operator*. See Kadison and Ringrose (1997, p. 281, Remark 4.5.8).

adds physical significance to the algebra of quantum observables described above by understanding the observables as "living in" a spacetime.[22] One begins with a net of algebras, which is an association of a unital C*-algebra $\mathfrak{A}(O)$, called a *local algebra of observables*, to each suitable[23] open bounded region O of some spacetime $\mathcal{M}$

$$O \mapsto \mathfrak{A}(O)$$

This net of C*-algebras bears the following initial interpretation: each self-adjoint element of $\mathfrak{A}(O)$ corresponds to a possible local observable, a quantity measurable within O. This net of algebras is required to satisfy a number of general axioms, including

Isotony: If $O_1 \subseteq O_2$, then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$.

Isotony guarantees that the net $\mathfrak{A}(O)$ has an inductive limit $\mathfrak{A}$, called the *quasilocal algebra*:

$$\mathfrak{A} = \overline{\bigcup_{O \subseteq \mathcal{M}} \mathfrak{A}(O)}$$

Global states of the system are understood as states on this quasilocal algebra, assigning expectation values to the observables in any spacetime region.

Now we restrict attention to a spacetime with an affine structure. For our purposes, this could be either relativistic Minkowski spacetime or classical Newtonian spacetime. All that matters is that we have an associated vector space G, which can be thought of as the translation group of that spacetime. To make the algebraic structure of the theory compatible with this affine structure of the spacetime, one employs a further axiom:[24]

[22]For more, see Haag and Kastler (1964), Summers and Werner (1987), Haag (1992), Roberts (1990), Baez et al. (1992), Wald (1994), and Halvorson (2006).

[23]For example, one might restrict attention to only open double cones (see, e.g. Halvorson, 2006, p. 742)

[24]For more on Translation Covariance and translation-invariant states, see (Ruetsche, 2011, p. 105-106) and (Halvorson, 2006, Sec. 2.2).

Translation Covariance: There is a representation $x \mapsto \alpha_x$ of the translation group G into the automorphism group of $\mathfrak{A}$ such that for all $O \subseteq \mathcal{M}$

$$\alpha_x(\mathfrak{A}(O)) = \mathfrak{A}(O + x)$$

Translation Covariance tells us that translating our algebra via the automorphism α_x will get us the same result as translating our region of spacetime and looking at the algebra associated with the new region.[25] A *vacuum state* is a state ω_0 on the quasilocal algebra $\mathfrak{A}$ that is invariant under the translation group, i.e. for each $x \in G$, $\omega_0 \circ \alpha_x = \omega_0$.[26] One often works in a *vacuum representation* $(\pi_{\omega_0}, \mathcal{H}_{\omega_0})$, the GNS representation of $\mathfrak{A}$ in a chosen vacuum state ω_0.

Although there are many further axioms concerning, for example, locality and gauge structure, that one might employ in algebraic quantum field theory, we will stop here, having provided only the most basic outline of the kinds of constructions one can use within the algebraic framework. This background will suffice for what I have to say in later chapters, and hopefully also demonstrate that one has the resources within the algebraic framework to say quite a bit about the physical content of an algebraic quantum theory within a spacetime setting.

[25] One can generalize Translation Covariance to curved space times as well (Brunetti and Fredenhagen, 2009).

[26] Note that translation-invariance is typically taken as only a necessary condition for being a vacuum state. What I say in later chapters about translation invariant states will therefore apply to any state that is even a candidate for being a vacuum state.

Chapter 3

Unitary Inequivalence in Classical Systems

3.1 Introduction

Ruetsche (2011) argues that a *problem of unitarily inequivalent representations* arises in quantum theories with infinitely many degrees of freedom. When one attempts to "quantize" a classical theory, i.e. formulate a quantum theory for the same system the classical theory was meant to describe, one is not guaranteed that the resulting quantum theory is unique.[1] For a classical theory with infinitely many degrees of freedom, e.g. a field theory or a statistical theory in the thermodynamic limit, there are many inequivalent theories that compete to be called its "quantization".

Work on the problem of unitarily inequivalent representations is done in the algebraic framework for quantum theories. One begins by representing physical observables as elements of an abstract C*-algebra, which captures the structure that all of the representations of the canonical commutation or anti-commutation relations have in common. One then looks for concrete representations of that algebra in the bounded operators on some Hilbert space. The problem of unitarily inequivalent

[1] This assumes that a quantum theory is a concrete Hilbert space representation of canonical commutation or anti-commutation relations. Responses to the problem of unitarily inequivalent representations may lead one to challenge this assumption and take on one of the alternative interpretive positions laid out in Ch. 2.

representations forces us to make a choice in interpreting the algebraic formalism. As we saw in Ch. 2, one can take the algebraic formulation as basic, and becomes an *Algebraic Imperialist*, or else one privileges a particular Hilbert space representation as a *Hilbert Space Conservative*.

Ruetsche (2002, 2003, 2006, 2011) provides arguments against both Algebraic Imperialism and Hilbert Space Conservatism, and this leads her to consider alternate kinds of interpretation altogether. Rejecting both of the most prominent interpretations of quantum field theory might seem radical, and indeed the philosophical community has not been quick to accept Ruetsche's conclusions. Even though Ruetsche's arguments are well known, some (Baker, 2011; Baker and Halvorson, 2013) continue to discuss Hilbert Space Conservatism as if it were a viable option. It is the purpose of this chapter to bolster Ruetsche's argument against Conservatism,[2] which asserts that the Hilbert Space Conservative cannot represent all of the physically significant states of a quantum theory with infinitely many degrees of freedom.

I will demonstrate the power of Ruetsche's argument by transplanting it from the already highly contentious arena of quantum theory to the simpler context of classical physics. As is already known,[3] one can use the very same algebraic formalism previously mentioned to describe classical theories as well as quantum ones. Since the interpretation of classical physics is at the very least better understood and better agreed upon than that of quantum mechanics, this suggests that we can use classical theories to probe our understanding of the abstract algebraic formalism. This paper uses an algebraic reformulation of classical field theory as a concrete tool in order to investigate the status of Ruetsche's argument.

I will show that unitarily inequivalent representations arise in the algebraic formulation of classical field theories. More specifically, I will show that in classical theories, the GNS representations for any two distinct pure states are unitarily inequivalent. This trivial mathematical result is well

[2]In this chapter, I will only briefly mention Ruetsche's argument against Algebraic Imperialism (amongst others) in section 3.4, and I will not discuss it in the quantum case. See Ch. 5 and Feintzeig (2016a) for a critical examination of Ruetsche's argument against Imperialism.

[3]See, e.g., Summers and Werner (1987), Landsman (1998a), Brunetti and Fredenhagen (2009), and Brunetti et al. (2012).

known among specialists, but I believe it is worth bringing to the fore to better understand unitarily inequivalent representations as they appear in quantum theories.

Naively, one might think that the presence of unitarily inequivalent representations leads to a problem in the classical case just as in the quantum case. But I will show that no such problem arises because Ruetsche's argument against Hilbert Space Conservatism translates immediately to the classical case: it is obvious that the Hilbert Space Conservative about classical theories does not have enough resources to represent all of the physically significant states of the theory.[4] One is not even tempted in the classical case to be a Hilbert Space Conservative, and so one does not have reason to think of unitarily inequivalent representations as competing theories. Hilbert Space Conservatism might have seemed attractive for quantum theories because Hilbert space methods have been enormously fruitful in nonrelativistic quantum mechanics, and extra complications of the quantum case might make one hesitant to move away from the (rightfully called) Conservative position. But the simplicity of the classical case demonstrates just how powerful and decisive Ruetsche's argument against Hilbert Space Conservatism is. I hope to show that the algebraic formulation of quantum theory is sufficiently like the algebraic formulation of classical field theory (where it is obvious that we should reject Hilbert Space Conservatism) that we are justified in following Ruetsche in her rejection of Hilbert Space Conservatism about quantum theories.

3.2 Ruetsche on Hilbert Space Conservatism

Quantizing a classical theory involves two steps: first, one isolates from the observables of the classical theory certain algebraic relations (usually commutation or anti-commutation relations)

[4] While this chapter provides an argument against Hilbert Space Conservatism about classical field theories, I do not claim to provide a comprehensive argument for Algebraic Imperialism about classical field theories. I will lay out some possible objections to Algebraic Imperialism about classical field theories and some possible responses in section 3.4. But whatever the outcome of these debates about Algebraic Imperialism, my central claim in this chapter is only that Hilbert Space Conservatism about classical field theories is untenable, and this teaches us about Hilbert Space Conservatism even in the quantum case.

and second, one finds a representation of the resulting algebra in the bounded operators on some Hilbert space. As we saw in Ch. 2, in the case where the original classical theory has finitely many degrees of freedom, the Stone-von Neumann theorem (Summers, 1999; Clifton and Halvorson, 2001; Ruetsche, 2011, p. 41) shows that all of the Hilbert space representations we end up with are unitarily equivalent and so the resulting quantum theory is unique. However, it is well known that theories with infinitely many degrees of freedom, including field theories and statistical theories in the thermodynamic limit, violate the assumptions of the Stone-von Neumann theorem. The algebra obtained in the first step of the quantization procedure will admit many unitarily inequivalent representations when we attempt to perform the second step (Ruetsche, 2011, Ch. 3.3). Because of this, it appears that many theories of interest for physics do not have a unique quantization.

Anyone hoping to understand quantum theories of infinite systems, like quantum field theory, must decide which quantum theory to consider. The positions of Algebraic Imperialism and Hilbert Space Conservatism laid out in Ch. 2.3 provide illustrative, albeit extreme, interpretive stances from which one can try to make sense of the formal apparatus used to construct quantum theories with infinitely many degrees of freedom.

Ruetsche argues at length against Hilbert Space Conservatism (2002, p. 364; 2003, p. 1338; 2006, p. 480) on the basis that the Hilbert Space Conservative cannot represent all of the physically significant states of the quantum theories at issue. Because the GNS representations of (for example) states corresponding to two different pure thermodynamic phases in quantum statistical mechanics turn out to be unitarily inequivalent, neither of these irreducible Hilbert space representations can represent the other state as a density operator, and so neither Hilbert space can represent the other state as a physical possibility. A Hilbert Space Conservative, having chosen one particular pure state through which to consider the GNS representation, must deny that states that cannot be represented on the Hilbert space of that representation are physically possible. Since the Hilbert Space Conservative cannot represent states of different thermodynamic phases in a single Hilbert space, she can never represent distinct thermodynamic phases as coexisting and hence, she can never give certain desirable explanations—e.g. of phase transitions (Ruetsche, 2002, 2003). The Hilbert Space Conservative cannot recover these physically significant explanations precisely because she cannot countenance all of the physically significant states.

Of course this argument is very abstract, and because the interpretation of quantum theory is already so controversial, one might try to find reason to either ignore or reject Ruetsche's conclusions. In what follows, I hope to show that Ruetsche's argument is indeed powerful and correct by bringing it to the context of classical field theory, where it becomes absolutely transparent.

3.3 Algebraic Classical Field Theory

We now apply the concepts and constructions of the algebraic formalism to classical field theories and show that the resulting structures are analogous to those of quantum field theory. This attempt to put classical field theory into the algebraic framework falls into a tradition in philosophy of physics of translating our previous theories into the language of our current theories in order to compare them. For example, many use Newton-Cartan theory (geometrized Newtonian gravitation) as a way of translating classical Newtonian gravitation into a framework in which one can compare it with general relativity (see, e.g. Weatherall, 2011). One can understand the algebraic formulation of classical field theory in much the same way as Newton-Cartan theory—as a tool for learning about the distinctive features of quantum field theory.

I do *not* claim that the algebraic formalism provides the best or most useful way of understanding classical field theory for all purposes, or even that it captures the full, rich structure of the classical theory. For example, we will see that the algebraic formalism I consider does not contain any information about dynamics or the classical Poisson bracket. As such, one might worry that the algebraic setting is artificial in the classical case, and that we might want to interpret the formalism differently here than in quantum theories. I will have more to say about the differences between the classical and quantum case below, but for now I note that all I require for my arguments is that quantum and classical theories share a background of very weak interpretive assumptions—namely, that observables correspond to quantities that we can measure of our system and states assign expectation values to those observables. We can set aside many details, including, e.g., dynamics, because the only point at issue is whether irreducible Hilbert space representations contain enough physically significant states.

This common interpretive background has made the practice of translating classical theories into the algebraic language common and fruitful among researchers investigating the boundary between the classical and the quantum. For example, Summers and Werner (1987) prove algebraic results about the status of the Bell inequalities when classical and quantum systems are coupled, and Landsman (1998a, 2006) reviews techniques for understanding algebraic classical theories as the appropriate limits of quantum theories, e.g. using deformation quantization. In order for this practice to make sense, we need to be able to understand classical and quantum theories in a unified framework. Putting these theories in a unified framework will give us the tools we need here to compare classical field theory with quantum field theory and in particular to analyze the significance of unitarily inequivalent representations.

We restrict ourselves for simplicity and concreteness to the algebraic formulation of the classical theory of a real scalar field[5] $\varphi : \mathcal{M} \to \mathbb{R}$. Each such smooth field $\varphi \in C^\infty(\mathcal{M})$ represents a possible configuration of our system. Thus, we take an appropriate subcollection[6] $\mathcal{U} \subseteq C^\infty(\mathcal{M})$ (say, of solutions to some partial differential equation) to be our configuration space. Observables in this theory will be functions $f : \mathcal{U} \to \mathbb{C}$ (Landsman, 1998a; Brunetti et al., 2012). Each such observable f corresponds to a possible measurable quantity of the system, and $f(\varphi)$ is the value that we would measure if the actual field configuration were φ. The bounded observables form a C*-algebra with

[5] One could generalize by repeating these constructions for smooth sections of an arbitrary vector bundle over $\mathcal{M}$.

[6] Some technical caveats about "appropriate" configuration spaces: first, we require that $\mathcal{U}$ be closed under translations (see footnote 8). Second, we also would like to put a topology on $\mathcal{U}$ so that we have the option of restricting attention to observables that are continuous functions on $\mathcal{U}$.

operations defined pointwise by:

$$f^*(\varphi) := \overline{f(\varphi)}$$
$$(fg)(\varphi) := f(\varphi) \cdot g(\varphi)$$
$$(f+g)(\varphi) := f(\varphi) + g(\varphi)$$
$$\|f\| := \sup_{\varphi \in \mathcal{U}} |f(\varphi)|$$

for all $f, g : \mathcal{U} \to \mathbb{C}$. One must restrict attention to bounded functions so that the norm is well-defined everywhere on the algebra. Furthermore, if one wishes, one may restrict attention to the continuous functions with respect to some appropriate topology. For example, one may use the compact-open topology or the Whitney topology on $\mathcal{U}$ (Brunetti et al., 2012). In this paper, I will remain noncommittal as to whether the algebra considered is the algebra of bounded continuous functions in some topology or merely the bounded (not necessarily continuous) functions. One reason for this indifference is as follows. Even if one wants to restrict attention to the C*-algebra of continuous bounded functions, it is standard practice in the algebraic framework to complete this algebra in the weak topology to form a W*-algebra. Completing in the weak topology in the classical case adds functions that are pointwise limits of our original observables, which means that even discontinuous functions will be included in this completion.[7] We can mostly set aside these technical complications because the distinction between these two algebras (the bounded, continuous functions and the merely bounded, not necessarily continuous functions) will not make a difference for my arguments except at only one point—in dealing with an objection in section 3.4—where I will discuss each of these two options for the abstract algebra separately.

The only property of this algebra that will matter in the next section is that it is abelian, i.e. commutative. It is worth noting that this is a property that all classical algebras of observables share in common, including the ones for field theories and for systems with finitely many degrees of freedom. In this section, we have added extra technical complications by dealing with field theories,

[7] See Chs. 5 and 6 and Feintzeig (2016a).

where the configuration space is not locally compact, whereas if we had worked in a theory with finitely many degrees of freedom, the configuration space would be locally compact. I will discuss some of the extra complications of field theory in more detail in the following sections. For now I note that the only reason I choose to work with a classical field theory here is that it allows us to construct explicit analogies with quantum field theory by associating observables with spacetime regions, as we do presently.

One may pick out the local observables to a given region $O \subseteq \mathcal{M}$ (which form a C*-algebra with the operations defined above suitably restricted), and thereby define the net of observables, according to the following rule:

Definition 3.3.1. *(Classical Net) An observable $f : \mathcal{U} \to \mathbb{C}$ is in $\mathfrak{A}(O)$, and said to be* local *to O, iff for all $\varphi, \varphi' \in \mathcal{U}$ such that $\varphi_{|\overline{O}} = \varphi'_{|\overline{O}}$, we have $f(\varphi) = f(\varphi')$.*

This definition fits with our intuitive understanding of what it means for an observable to be local—an observable f is local to O just in case the value of f on any field configuration φ depends only on the values φ takes within O (or on the boundary of O). The following proposition shows that one can take the inductive limit of the classical net to obtain the quasilocal algebra $\mathfrak{A}$.

Proposition 3.3.2. *The classical net satisfies Isotony, i.e. if $O_1 \subseteq O_2$, then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$.*

Proof. Suppose $O_1 \subseteq O_2$ and choose any $f \in \mathfrak{A}(O_1)$. For any $\varphi, \varphi' \in \mathcal{U}$, if $\varphi_{|\overline{O_2}} = \varphi'_{|\overline{O_2}}$, then $\varphi_{|\overline{O_1}} = \varphi'_{|\overline{O_1}}$, and therefore $f(\varphi) = f(\varphi')$, which shows that $f \in \mathfrak{A}(O_2)$. □

One can also define a representation of the translation group α_x on $\mathfrak{A}$ (via an intermediary group β_x acting on $\mathcal{U}$) as follows. For each $x \in G$, define $\beta_x : \mathcal{U} \to \mathcal{U}$ by

$$\beta_x(\varphi)(p) := \varphi(p + x)$$

for all $\varphi \in \mathcal{U}$ and $p \in \mathcal{M}$.[8] Then for each $x \in G$, define $\alpha_x : \mathfrak{A} \to \mathfrak{A}$ by

$$\alpha_x(f)(\varphi) := f(\beta_x(\varphi))$$

for all $f \in \mathfrak{A}$ and $\varphi \in \mathcal{U}$.

Proposition 3.3.3. *The classical net satisfies Translation Covariance, i.e.* $f \in \mathfrak{A}(O)$ *iff* $\alpha_x(f) \in \mathfrak{A}(O+x)$.

Proof. Suppose $f \in \mathfrak{A}(O)$. Then given any two $\varphi, \varphi' \in \mathcal{U}$, if $\varphi_{|\overline{O+x}} = \varphi'_{|\overline{O+x}}$, then for any $p \in O$,

$$\beta_x(\varphi)(p) = \varphi(p+x) = \varphi'(p+x) = \beta_x(\varphi')(p)$$

and so $\beta_x(\varphi)_{|\overline{O}} = \beta_x(\varphi')_{|\overline{O}}$. It follows that

$$\alpha_x(f)(\varphi) = f(\beta_x(\varphi)) = f(\beta_x(\varphi')) = \alpha_x(f)(\varphi')$$

which shows that $\alpha_x(f) \in \mathfrak{A}(O+x)$. The other direction follows similarly. $\square$

Given this representation of the translation group, we can identify translation invariant states. Each determinate field configuration $\varphi \in \mathcal{U}$ defines a state ω_φ on $\mathfrak{A}$ by

$$\omega_\varphi(f) := f(\varphi)$$

for all $f \in \mathfrak{A}$. If φ is a *constant* determinate field configuration, then it is invariant under the translation group β_x, because for all $p \in \mathcal{M}$, $\beta_x(\varphi)(p) = \varphi(p+x) = \varphi(p)$. It follows that ω_φ is invariant under the translation group α_x, because for all $f \in \mathfrak{A}$,

$$\omega_\varphi(\alpha_x(f)) = \alpha_x(f)(\varphi) = f(\beta_x(\varphi)) = f(\varphi) = \omega_\varphi(f)$$

[8] For this definition to make sense, we must require that $\mathcal{U}$ is closed under translations, i.e. if $\varphi \in \mathcal{U}$, then $\beta_x(\varphi) \in \mathcal{U}$.

Thus any constant determinate field configuration defines a translation invariant state on $\mathfrak{A}$, and is thus a candidate for being a vacuum state. One intuitive choice of vacuum state corresponds to the determinate constant field configuration $\varphi_0 = 0$.

We have seen that classical field theories can be fit into the algebraic framework and that the classical case is analogous enough to the quantum to allow us to perform the same mathematical operations. Just as in algebraic quantum field theory, once one has chosen a vacuum state on the classical algebra $\mathfrak{A}$ by choosing a constant field configuration $\varphi \in \mathcal{U}$, one can work in its corresponding vacuum representation. Now we can start to ask about the significance of such Hilbert space representations in the classical theory.[9]

3.4 Unitary Inequivalence

I will show that unitarily inequivalent representations are endemic to classical field theories (considered prior to any quantization procedure). Unitarily inequivalent representations might at first appear more problematic for classical field theories than quantum theories because, while in the quantum case at least the GNS representations for *some* pure states are unitarily equivalent, we will see that in the classical case the GNS representations of any two distinct pure states are *always* unitarily *in*equivalent. But unitarily inequivalent representations only provide an *apparent* problem for classical field theories. A closer examination shows that Ruetsche's argument immediately and decisively rules out Hilbert Space Conservatism for the classical case. Since Hilbert Space Conservatism provided the only viewpoint from which unitarily inequivalent representations seemed problematic (because they forced us to choose arbitrarily between competing theories), we are left with no problem in the classical case.

[9]There is another notion of representation one can use for abelian algebras (see Landsman, 1998a, p. 76) according to which observables are represented as functions on a manifold in much the same way we started this section. One *can* prove some uniqueness results about such representations (Landsman, 1998a, p. 80). While that notion of representation is fruitful for many purposes, I will not use it in this paper. The purpose of this paper is specifically to investigate the significance of *Hilbert space* representations of an algebra of observables.

Proposition 3.4.1. *Let $\varphi \in \mathcal{U}$ be a field configuration. Then the corresponding state ω_φ (defined in section 3.3) is pure.*

Proof. The state ω_φ is multiplicative: if $f, g \in \mathfrak{A}$, then

$$\omega_\varphi(fg) = (fg)(\varphi) = f(\varphi) \cdot g(\varphi) = \omega_\varphi(f) \cdot \omega_\varphi(g)$$

It follows immediately that ω_φ is a pure state (Kadison and Ringrose, 1997, p. 269, Thm. 4.4.1). □

In particular, any vacuum state corresponding to a constant determinate field configuration is pure. The rest of the argument follows merely from the fact that the quasilocal algebra $\mathfrak{A}$ is abelian.

Proposition 3.4.2. *Let $\mathfrak{A}$ be an abelian C*-algebra. Let ω be a pure state on $\mathfrak{A}$ and let $(\pi_\omega, \mathcal{H}_\omega)$ be the GNS representation of $\mathfrak{A}$ for ω. Then $\mathcal{H}_\omega$ is one dimensional.*

Proof. Because ω is pure, the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ for the state ω is irreducible (Kadison and Ringrose, 1997, p. 728, Thm. 10.2.3). Since $\mathfrak{A}$ is abelian, any irreducible representation of $\mathfrak{A}$ is on a one-dimensional Hilbert space (Kadison and Ringrose, 1997, p. 744). □

This means that the GNS representation for any pure state, and thus for the state corresponding to any determinate field configuration, is so weak that it has the power to represent *only a single state* as a density operator.

Proposition 3.4.3. *Let ω_1 and ω_2 be distinct pure states on an abelian C*-algebra $\mathfrak{A}$. Let $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ be the GNS representations of $\mathfrak{A}$ for the states ω_1 and ω_2 with corresponding cyclic vectors Ω_{ω_1} and Ω_{ω_2}, respectively. Then $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ are unitarily inequivalent.*

Proof. (See Kadison and Ringrose, 1997, p. 744) Suppose there is a unitary transformation U :

$\mathcal{H}_1 \to \mathcal{H}_2$ such that for all $A \in \mathfrak{A}$,

$$U\pi_{\omega_1}(A) = \pi_{\omega_2}(A)U$$

Then since the Hilbert spaces are one-dimensional, it follows that $U\Omega_{\omega_1} = e^{i\theta}\Omega_{\omega_2}$ for some $\theta \in \mathbb{R}$. So for all $A \in \mathfrak{A}$

$$\begin{aligned}
\omega_1(A) = \langle \Omega_{\omega_1}, \pi_{\omega_1}(A)\Omega_{\omega_1}\rangle &= \langle \Omega_{\omega_1}, U^{-1}\pi_{\omega_2}(A)U\Omega_{\omega_1}\rangle \\
&= \langle U\Omega_{\omega_1}, \pi_{\omega_2}(A)e^{i\theta}\Omega_{\omega_2}\rangle \\
&= \langle e^{i\theta}\Omega_{\omega_2}, e^{i\theta}\pi_{\omega_2}(A)\Omega_{\omega_2}\rangle \\
&= e^{-i\theta}e^{i\theta}\langle \Omega_{\omega_2}, \pi_{\omega_2}(A)\Omega_{\omega_2}\rangle = \langle \Omega_{\omega_2}, \pi_{\omega_2}(A)\Omega_{\omega_2}\rangle = \omega_2(A)
\end{aligned}$$

Therefore, $\omega_1 = \omega_2$. □

Thus, if determinate field configurations $\varphi, \varphi' \in \mathcal{U}$ determine distinct states ω_φ and $\omega_{\varphi'}$, then, since the quasilocal algebra $\mathfrak{A}$ of any classical field theory is abelian, the corresponding GNS representations for ω_φ and $\omega_{\varphi'}$ will be unitarily inequivalent. In particular, the vacuum representation mentioned above for the vacuum state determined by the constant zero field configuration is unitarily inequivalent to the GNS representation for any distinct state. This is, in a certain sense, worse than the quantum case because while in quantum field theory a representation may include many states as density operators, in the classical case, each representation includes only a single state.

It is worth looking in more detail to see why these unitarily inequivalent representations arise in the classical case. The quasilocal algebra $\mathfrak{A}$, since it is abelian, is *-isomorphic to $C(\mathcal{P}(\mathfrak{A}))$, the continuous functions on the compact Hausdorff space $\mathcal{P}(\mathfrak{A})$ of pure states of $\mathfrak{A}$ with the weak* topology (Kadison and Ringrose, 1997, p. 270, Thm. 4.4.3).[10] As such, each observable $f \in \mathfrak{A}$ corresponds to a function $\hat{f} \in C(\mathcal{P}(\mathfrak{A}))$ defined by

$$\hat{f}(\omega) = \omega(f)$$

[10]Recall that each determinate field configuration in $\mathcal{U}$ defines one of these pure states, so that $\mathcal{P}(\mathfrak{A})$ (with the weak* topology) may be understood as a kind of compactification of $\mathcal{U}$.

for each pure state $\omega \in \mathcal{P}(\mathfrak{A})$. Taking the GNS representation for a pure state ω amounts to choosing a measure on the space $\mathcal{P}(\mathfrak{A})$ (See Kadison and Ringrose 1997, p. 744; Landsman 1998a, p. 55), which defines an (L^2) inner product, hence constructing a Hilbert space as follows. By the Riesz-Markov theorem (Reed and Simon, 1980, p. 107, Thm. IV.14), each pure state ω on $\mathfrak{A}$ corresponds to a unique regular Borel measure μ_ω on $\mathcal{P}(\mathfrak{A})$ such that for all $f \in \mathfrak{A}$

$$\omega(f) = \int_{\mathcal{P}(\mathfrak{A})} \hat{f} d\mu_\omega$$

The GNS representation of $\mathfrak{A}$ for the pure state ω is unitarily equivalent to the representation[11] $(\pi_\omega, \mathcal{H}_\omega)$ on the Hilbert space $\mathcal{H}_\omega = L^2(\mathcal{P}(\mathfrak{A}), d\mu_\omega)$, with π_ω defined by

$$\pi_\omega : f \mapsto M_{\hat{f}}$$

where the operator $M_{\hat{f}}$ is defined as multiplication by the function $\hat{f}$, i.e. for any $\psi \in \mathcal{H}_\omega$,

$$M_{\hat{f}}\psi = \hat{f} \cdot \psi$$

Furthermore, when ω is pure,

$$\omega(f) = \hat{f}(\omega) = \int_{\mathcal{P}(\mathfrak{A})} \hat{f}\delta(\omega)$$

where $\delta(\omega)$ is the point mass or delta function centered on ω. It follows by the uniqueness clause of the Riesz-Markov theorem that $d\mu_\omega = \delta(\omega)$. Now every vector $\psi \in \mathcal{H}_\omega$ is defined by a single complex number—the value of ψ on $\omega \in \mathcal{P}(\mathfrak{A})$—which shows that $\mathcal{H}_\omega$ is one-dimensional (Prop. 3.4.2). Each observable $f \in \mathfrak{A}$ is represented on this Hilbert space via π_ω as the value that $\hat{f}$ takes at ω, i.e. the complex number $\hat{f}(\omega) = \omega(f)$. Notice that the representation π_ω will in general not be faithful, i.e. not one-to-one, because multiple observables may be assigned the same expectation value by the state ω. Choosing a distinct pure state with which to take the GNS

[11] Here, the relevant cyclic vector Ω_ω is the constant unit function.

representation amounts to choosing a distinct measure on $\mathcal{P}(\mathfrak{A})$, which means that some observable will be represented as a different complex number, and hence the representations will be unitarily inequivalent (Prop. 3.4.3).

Now that we have seen how unitarily inequivalent representations arise, we can translate Ruetsche's argument that Hilbert Space Conservatism cannot represent enough states into the classical context. In the classical case, the argument begins from a single, extremely weak premise: every classical field theory admits multiple distinct solutions to its governing equations of motion, yielding multiple possible determinate field configurations and hence, multiple distinct states. Each GNS representation for a pure state on $\mathfrak{A}$, because it is one-dimensional, has only the resources to represent a single state as a density operator; any other state can only be represented as a density operator on the Hilbert space of a unitarily *in*equivalent representation of the algebra. By focusing only on one particular representation, the Hilbert Space Conservative would have to deny the existence of multiple distinct states, which is absurd.

We see that the problem for the Hilbert Space Conservative about classical theories is just an exaggerated form of the problem for the Hilbert Space Conservative about quantum theories. Neither Conservative has access to resources powerful enough to represent all of the states that are physically possible or physically significant. The classical case illustrates this point clearly and emphatically. The obvious solution to the problem of unitarily inequivalent representations for the classical case is to not limit oneself to a particular irreducible representation, because no representation will suffice for representing more than a single state.

One might at this point retreat from the original Conservative position, which required irreducible representations, and instead allow her representations to be reducible so that they might be made faithful. I'll consider three possible alternative representations that are both faithful and reducible. First, one might use the Koopman-von Neumann formalism (Koopman, 1931), where we take some predefined, perhaps canonical, measure μ_K (e.g. Lebesgue measure) on our original configuration space $\mathcal{U}$ and consider the representation $(\pi_K, \mathcal{H}_K)$ on the Hilbert space $\mathcal{H}_K = L^2(\mathcal{U}, d\mu_K)$ with π_K

defined by

$$\pi_K : f \to M_f$$

where the operator M_f is defined, as above, as multiplication by the function f. This representation is obviously reducible because for any subset $X \subseteq \mathcal{U}$ (not necessarily a singleton), $L^2(X, d\mu_K)$ is a subspace of $\mathcal{H}_K$ left invariant under the action of $\pi_K(\mathfrak{A})$. Notice, however, that while this formalism is easily understandable for systems with finitely many degrees of freedom, where the configuration space $\mathcal{U}$ is locally compact and the Lebesgue measure is readily available, this procedure does not easily generalize to the field theoretic case where $\mathcal{U}$ is not locally compact and there may be no canonical measure on $\mathcal{U}$. But we can set this issue aside and focus on systems with finitely many degrees of freedom where $\mathcal{U}$ is locally compact and μ_K is just the Lebesgue measure. In this case, the Hilbert Space Conservative who takes the density operators on $\mathcal{H}_K$ to represent the physically possible states still fails to represent all of the physical possibilities. Even though this Hilbert Space Conservative does better by gaining more than one state, she fails to represent, for example, any state ω_φ corresponding to a determinate configuration φ as a density operator because of the familiar fact that $\mu_K(\{\varphi\}) = 0$. Thus, this position also reduces to absurdity because every classical theory obviously allows determinate configurations to be physically possible states.

A second possible route is to use the reduced atomic representation, which is defined as follows (Kadison and Ringrose, 1997, p. 741). Define an equivalence relation $\sim$ on the pure state space $\mathcal{P}(\mathfrak{A})$ by $\omega \sim \rho$ iff the GNS representations $(\pi_\omega, \mathcal{H}_\omega)$ and $(\pi_\rho, \mathcal{H}_\rho)$ are unitarily equivalent. Let $[\omega] \in \mathcal{P}(\mathfrak{A})/\sim$ denote the equivalence class containing ω under this equivalence relation. Then the reduced atomic representation is given by $(\pi_R, \mathcal{H}_R)$, where

$$\mathcal{H}_R = \bigoplus_{[\omega]\in\mathcal{P}(\mathfrak{A})/\sim} \mathcal{H}_\omega$$

and

$$\pi_R(A) = \bigoplus_{[\omega]\in\mathcal{P}(\mathfrak{A})/\sim} \pi_\omega(A)$$

for all $A \in \mathfrak{A}$, where again $(\pi_\omega, \mathcal{H}_\omega)$ is the GNS representation for ω. The reduced atomic representation is the direct sum of a maximal family of pairwise inequivalent irreducible representations of $\mathfrak{A}$. Suppose a Hilbert Space Conservative takes the density operators on $\mathcal{H}_R$ to represent the physically possible states of our classical system. Just as above, even though this Hilbert Space Conservative can represent more than one state, she cannot represent all physically significant states. In particular, she cannot represent the intrinsically mixed states (Ruetsche, 2004), i.e. the states given by suitably "continuous" probability distributions over pure states. In classical systems with finitely many degrees of freedom, these are just the probability measures on configuration space $\mathcal{U}$ that are absolutely continuous with respect to the Lebesgue measure. These states are obviously physically significant as they are used often in classical statistical mechanics. Thus, even this weakened Hilbert Space Conservative fails to represent all physically significant states.

Finally, one might use the *universal representation*, leading to the position of *Universalism* from Ch. 2.3. If one retreats from the original Conservative position this far, then one runs the risk of giving up the game. I'll argue in Ch. 5 (see also Feintzeig, 2016a) that the Universalist's position is actually equivalent to Algebraic Imperialism, which means that the Universalist has retreated so far that the Hilbert space is not essential to her interpretation.

Hence, I claim this argument rules out Hilbert Space Conservatism about classical theories, whether we maintain the requirement that representations be irreducible or weaken it to allow for faithful representations. But even if I am correct, this does not immediately lead one to Algebraic Imperialism. I will now deal with two possible objections to Algebraic Imperialism.

First, Ruetsche provides an important argument against Algebraic Imperialism in the quantum case, which leads her to an alternative "adulterated interpretation" of quantum theories (Ruetsche, 2011).[12] One might wonder whether Ruetsche's argument against Algebraic Imperialism can be translated to provide an obstacle for Imperialism in the classical case as well. I will show that Ruetsche's argument against Imperialism does not apply to classical theories.

[12] See Ch. 5 and Feintzeig (2016a) for more on this argument against Algebraic Imperialism.

Ruetsche asserts that the Algebraic Imperialist (in the quantum case) does not have the resources to represent all of the physically possible observables. The Hilbert Space Conservative, having privileged some pure state ω and its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$, acquires all of the observables (which Ruetsche calls *parochial observables*) in $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$. Many of the observables on the boundary of $\overline{\pi_\omega(\mathfrak{A})}$ have real physical import (e.g., the temperature observable) but have no analogue in the abstract algebra because they do not reside in $\pi_\omega(\mathfrak{A})$. So the Hilbert Space Conservative gains access to more observables than the Algebraic Imperialist, and these observables are physically significant, e.g. for giving explanations of thermodynamic phase transitions. According to Ruetsche, the Algebraic Imperialist runs into a problem because she cannot recognize these operators as physically possible observables, and so cannot vindicate such explanations.

However, this problem of parochial observables does not apply in the classical case. In the classical case, no such parochial observables appear in the GNS representation of any pure state. Since the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ for any pure state ω on the classical algebra $\mathfrak{A}$ is one-dimensional, and since $\pi_\omega(\mathfrak{A})$ contains the identity operator and is closed under scalar multiplication by complex numbers, it follows that $\pi_\omega(\mathfrak{A}) = \mathbb{C}I = \mathcal{B}(\mathcal{H}_\omega)$, and so the representation exhausts the possible observables on $\mathcal{H}_\omega$. In this case, $\overline{\pi_\omega(\mathfrak{A})} = \pi_\omega(\mathfrak{A})$, which means that there are no parochial observables. In other words every observable the Hilbert Space Conservative recognizes has an analogue in the abstract algebra, so the Algebraic Imperialist recognizes that observable too.[13]

A further objection to Algebraic Imperialism comes from comparing the algebraic formulation of our classical field theory with its standard formulation.[14] One might object that our algebraic formulation cannot capture the entire classical field theory because the state space of the algebra is too small; while the pure state space $\mathcal{P}(\mathfrak{A})$ for our algebraic theory is compact in the weak* topology, the configuration space $\mathcal{U}$ in the standard formulation will fail to be even locally compact in any vector space topology. So it looks like the configuration space of the standard formulation

[13] Admittedly, this argument works *only* for irreducible representations in the classical case. See Feintzeig (2016a) and Ch. 5, Appendix B for a general solution.

[14] Thanks to an anonymous referee for bringing this objection to my attention.

is much larger than the pure state space of our algebraic theory, and thus that the algebraic theory is missing much of the information about physically significant states.

However, this gloss is misleading because the topologies used on $\mathcal{P}(\mathfrak{A})$ and $\mathcal{U}$—according to which one might judge $\mathcal{P}(\mathfrak{A})$ to be smaller than $\mathcal{U}$—do not provide the only relevant information about relative size. There is another natural sense in which $\mathcal{P}(\mathfrak{A})$ is itself larger than $\mathcal{U}$: namely, there is a natural mapping of $\mathcal{U}$ in $\mathcal{P}(\mathfrak{A})$ given by

$$\varphi \mapsto \omega_\varphi$$

where ω_φ is, as above, the pure state on $\mathfrak{A}$ determined by φ. When we compare the standard and algebraic formulations with respect to this mapping we find that, regardless of what the topological properties tell us, the pure state space $\mathcal{P}(\mathfrak{A})$ contains every state that corresponds to a configuration in $\mathcal{U}$.

But there are some technical complications. Whether this mapping is injective depends on the algebra we chose to use in section 3.3. Recall that the two options I have considered are the bounded continuous functions and the bounded (not necessarily continuous) functions on $\mathcal{U}$. If we choose the algebra of bounded continuous functions (in some vector space topology), then the map $\varphi \mapsto \omega_\varphi$ fails to be injective. So if one insisted on using the algebra of bounded continuous functions, then in order to make the case that the state space of the abstract algebra captures all of the physical states of the theory one would have to argue that any configurations that get mapped to the same state on the abstract algebra are really physically indistinguishable (perhaps because these states are similar to each other to arbitrarily high accuracy, according to the topology). But if one instead used the other option, the algebra of bounded (not necessarily continuous) functions, then the map $\varphi \mapsto \omega_\varphi$ *is* injective because, for example, characteristic functions of singleton sets can be used to distinguish any two states we like. This amounts to what is referred to as the Stone-Čech compactification (although that term is sometimes reserved for completely regular spaces). In this case, the algebraic formulation straightforwardly contains all of the states of the original theory.

I think this shows that there are real choices to be made concerning the algebraic structure of classical field theory that deserve further investigation (see also footnote 18). Which states we take to be physically possible and physically distinct will guide our choice of what algebra to use in the construction of the theory. I do not wish to take a stance concerning which algebra (and consequently, which state space) an Algebraic Imperialist ought to use; I only wish to lay out some possible options. There may be major drawbacks to either or both of the options I have discussed here. For example, one might worry that the Stone-Čech compactification has undesirable properties that make it unsuitable for describing the state space of a physical system. I leave further investigation of these options for future work. What matters most for the purposes of this paper is that *it is possible* for at least one abstract algebraic formulation of classical field theory (using the bounded, not necessarily continuous, functions) to capture all of the states of the classical theory.

Hence, even while I do not claim that the algebraic formulation captures all of the information about the classical theory (for example, not the dynamical information—see section 3.5), I do believe the algebraic formulation can be used to capture at least as many states as the standard formulation of a classical theory. This is all that matters for our purposes because the only thing at issue in this paper is whether the Algebraic Imperialist and Hilbert Space Conservative interpretations can represent the physically significant states and observables.

The road is now paved to be an Algebraic Imperialist about classical theories. By this, I do not mean that I have provided a comprehensive argument in favor of Algebraic Imperialism. I have first and foremost only provided an argument against its main competitor, Hilbert Space Conservatism. Additionally, I have shown both that the most prominent objection to Algebraic Imperialism that arises in the quantum case does not apply to the classical case and that new objections that one might pose to Algebraic Imperialism in the classical case can be surmounted. I think this makes Algebraic Imperialism about classical field theories a viable option worthy of further investigation.

Of course, the stance of Algebraic Imperialism may be much easier to take in the classical case because we have no prior reason to regard unitarily inequivalent representations as competing theories in the classical context. One might say it was never a good idea to look for Hilbert space representations of classical theories in the first place, so the problem should never even have arisen. On the other hand, one might think we have good reason to look for Hilbert space representations for quantum theories because quantum theories are different. In the next section we will examine differences between classical and quantum theories and argue that they are similar enough for my purposes. But first, I want to look closer to help us understand the role that representations play in the theory.

In the classical case, the presence of unitarily inequivalent irreducible representations signifies the presence of a *superselection rule*. In general, a superselection rule obtains between two states just in case there is a nontrivial element of the center of $\mathfrak{A}$, i.e. an observable commuting with all other observables, that receives different expectation values in those two states (see Earman, 2008). One way of explaining the physical significance of this statement is that when there exists a superselection rule between two states, they cannot be coherently formed into a superposition. For example, there is a superselection rule in quantum mechanics between states with integer and half-integer angular momentum. In the classical case, since $\mathfrak{A}$ is abelian, it is its own center, and it follows that there is a superselection rule between *any* distinct pure states, i.e. between any pure states whose GNS representations are unitarily inequivalent. This corresponds to the fact that classical physics does not allow for coherent superpositions, or in other words that superposition is a strictly quantum phenomenon (Landsman, 1991, p. 5354).[15] The above propositions show a sense in which one can *derive* this fact—that classical and quantum physics differ in allowing coherent superpositions—from the basic structure of the relevant algebras of observables.

Thus, the classical case helps us to see that a representation is a tool that focuses in on some particular states while ignoring others. Namely, a representation focuses in on a set of states that can

[15] Superselection sectors are also sometimes thought to have some extra dynamical significance. In the classical case, unitarily inequivalent representations are only significant for the notion of superposition and not for dynamics.

be formed into coherent superpositions. In the quantum case, because there are nontrivial superpositions, we end up with representations on a nontrivial Hilbert space; but in the classical case, because there are no superpositions, taking a representation focuses us on only a single state. Since in both cases representations are giving us physical information about when superpositions can be formed, this gives us reason to believe that representations have a similar physical significance in the classical and the quantum case—it just so happens that classical states behave differently from quantum ones in that they cannot be formed into superpositions. The above propositions then show that we should not think of a representation as a full theory as the Hilbert Space Conservative would have us do. Rather, we should think of a representation as some part of the theory containing only a subcollection of all of the states the theory deems possible—namely, a subcollection of states that can be coherently superposed with each other.

3.5 Are quantum theories different?

Even if Hilbert Space Conservatism is untenable in the classical case, one might still contend that the quantum case is sufficiently different that Conservatism is tenable there. For example, one might point to the fact that the algebras of observables used in quantum theories (namely, the Weyl algebra) are typically *simple*, i.e. they have no non-trivial two-sided ideals.[16] This implies that every irreducible representation of the algebra is faithful,[17] which is in stark contrast to the classical case in which (as we saw above) every irreducible representation fails to be faithful because the Hilbert space is one-dimensional. Hence, one might contend that even if my argument is sound in the classical case, the quantum case is different enough that the argument will not apply.

The quantum case is certainly different in many respects, but (as mentioned in 3.3) we have reason to want to understand classical and quantum theories as being part of a unified framework, as

[16] A *two-sided ideal* is a subalgebra $\mathfrak{I} \subseteq \mathfrak{A}$ such that for all $A \in \mathfrak{I}$ and $B \in \mathfrak{A}$, we have $AB \in \mathfrak{I}$ and $BA \in \mathfrak{I}$.

[17] Thanks to an anonymous referee for bringing this to my attention.

many of the practitioners of the theory do (Summers and Werner, 1987; Landsman, 2006). Insofar as we understand algebraic classical and quantum theories to fit into this unified framework with at least some basic shared interpretive assumptions, we can ask about the significance of certain mathematical operations that we can perform on both theories.

However, one might worry that the algebraic framework is not unified *enough* to allow for this kind of analysis. The pointwise multiplication operation of the classical algebra of observables leaves out much of the information encoded in the classical theory—specifically the classical structure of a Poisson bracket on the relevant phase space and the dynamical information, which may be encoded (for example) in a Hamiltonian. On the other hand, the quantum algebra contains at least some of this information because the noncommutative multiplication operation is defined by the canonical commutation relations, which in turn derive from the classical Poisson bracket. I think this is a real difference between the physical significance of the classical and quantum algebras, but it does not change the status of the arguments given in this paper. I have not claimed that the abelian algebra $\mathfrak{A}$ captures the entire classical theory. Here I have considered the unified algebraic framework only as a tool for representing the possible states and observables of the theory, without any consideration of the Poisson bracket or the dynamics. *This is all the information we need* to answer basic questions about the status of unitarily inequivalent representations (as in, e.g. Clifton and Halvorson, 2001; Baker, 2011), and in particular the question of whether the Hilbert Space Conservative can represent all physically significant states, which is the central question of this paper. One can fully specify the states and observables of a classical theory with the algebraic structures considered here, even though this will not include a specification of the Poisson bracket or the dynamics. This objection—that the classical algebra does not contain all of the information of the classical theory—is irrelevant because the arguments here only depend on representing the classical states and observables, for which the abstract algebra suffices.[18]

[18] As noted in section 3.4, there may be real issues involved in choosing the abstract algebra to use in constructing the theory. For simpler classical systems with finitely many degrees of freedom (i.e., for which the configuration space $\mathcal{U}$ is locally compact), the choice of algebra is more obvious. One standardly uses $C_0(\mathcal{U})$ (Landsman, 1998a), the continuous functions on $\mathcal{U}$ that vanish at infinity, because the pure state space of this algebra corresponds exactly to the collection of field configurations $\mathcal{U}$ (see Ch. 6). However, for the case we have considered of classical field theory

So given that quantum and classical theories do fit into a unified framework, and given that this framework suffices for many scientific purposes, the analogy is strong enough that we can use algebraic methods in classical theories to shed light on the quantum case. We have seen that Hilbert Space Conservatism is untenable in the classical case for reasons that are absolutely transparent. An irreducible representation focuses in on only a single classical state, and a single classical state is not the entire theory. Precisely because the situation is clear in classical physics and controversial in quantum physics, we ought to use the classical case to guide our understanding of the formalism.

But I do not claim that the argument presented here in the classical context is what rules out Conservatism about quantum theories. I only claim that the classical case helps us to understand an argument that Ruetsche *has already given*. Ruetsche has already shown that the Hilbert Space Conservative about quantum theories cannot represent enough states because the Conservative cannot (for example) simultaneously represent states of different thermodynamic phases (Ruetsche, 2002, 2003). In the case of quantum statistical mechanics, an irreducible representation focuses in on only the states of a single thermodynamic phase, and the states of this single thermodynamic phase do not exhaust all the states of the theory. The role of the classical case is to help us grasp how the Hilbert Space Conservative could possibly fail to represent enough states. Indeed, the classical case helps us see that Ruetsche's reasons for rejecting Conservatism are very good ones.

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with infinitely many degrees of freedom, things are not so simple because $\mathcal{U}$ is not locally compact. I have restricted attention to two choices: the algebra of bounded continuous functions (in some appropriate topology) or the algebra of bounded (not necessarily continuous functions) on $\mathcal{U}$. I do not claim that these are the only choices one could make. For now, I only claim that using one or the other of the algebras listed above allows us to represent *at least as many states as we would like*. This suffices for the conclusions I wish to draw here.

Moreover, I think it is important to investigate the possible algebraic options further for a complete understanding of the relationship between classical field theory and quantum field theory. For example, one might be able to apply the methods of deformation quantization to an algebraic formulation of the field theoretic case to illuminate conceptual issues in both classical and quantum field theory.

3.6 Conclusion

We have seen that we can formulate classical theories in the algebraic framework, and when we do so we find that the GNS representations of any two distinct pure states are unitarily inequivalent. The Hilbert Space Conservative uses only one of these GNS representations so she can only represent a single state as physically possible. Since all classical theories contain more than one state, this implies that the Hilbert Space Conservative cannot represent all possible states of the theory. This argument is analogous to ones that Ruetsche gives in the quantum case (Ruetsche, 2002, 2003, 2006, 2011), and I believe my arguments only bolster her conclusion. This is important because even though the arguments in the quantum case are well known, some (Baker, 2011; Baker and Halvorson, 2013) continue to discuss Conservatism as if it were a viable option. Here, I hope to have shown, by consideration of the classical case, that Ruetsche's arguments against Hilbert Space Conservatism are already strong enough to defeat that position. Her arguments really do show, as is evident from the analogous argument in the classical case, that the Hilbert Space Conservative fails to represent all physically significant states, and it follows that Conservatism is untenable.

Chapter 4

On Broken Symmetries and Classical Systems

4.1 Introduction

Baker (2011) argues that broken symmetries play a crucial role in guiding our interpretation of quantum field theory. Baker and others (Baker and Halvorson, 2013; Earman, 2003; Ruetsche, 2011) approach the topic of symmetry breaking through the algebraic formalism, in which one begins by representing physical quantities as elements of an abstract C*-algebra, and then one looks for concrete representations of that algebra in the bounded operators on some Hilbert space. Whenever a symmetry is broken, there are multiple *unitarily inequivalent representations* of the abstract algebra. According to Baker, the appearance of unitarily inequivalent representations in quantum cases of symmetry breaking gives rise to a number of puzzles—puzzles that don't appear in the classical case. He believes we must solve these puzzles to arrive at an adequate interpretation of quantum field theory.[1]

[1] As we will see, Baker ultimately believes these puzzles can be solved, so in that regard my conclusions may not differ substantially from where Baker ends up. Still, we take very different routes to that position, and I think there is

This chapter, like Ch. 3, proposes to pull the discussion back from the context of quantum physics—the interpretation of which is extremely controversial—to that of classical physics—which is at the very least better understood. I will analyze classical cases of symmetry breaking in order to compare their features with the quantum cases. The basic strategy of this chapter is to compare the mathematical features of symmetry breaking in quantum and classical theories by putting both kinds of theories on common mathematical ground, using the algebraic formulation of classical physics from Ch. 3. Given this algebraic reformulation of classical physics one can compare in detail classical cases of symmetry breaking with quantum cases.

In this chapter, I will show that classical cases of symmetry breaking—when translated into the algebraic formalism—give rise to unitarily inequivalent representations. I will illustrate this with two simple explicit examples of classical symmetry breaking: the classical real scalar field and the classical spin chain. All parties agree that the classical cases of symmetry breaking pose no interpretive puzzles.[2] To the extent that this is correct, it means that the presence of unitarily inequivalent representations is *not in itself* puzzling. This is not to say that there is nothing interesting about broken symmetries. Rather, my claim is that if there is something philosophically or physically interesting to be learned from broken symmetries, it must involve features beyond the mere presence of unitarily inequivalent representations.

insight to be gained from studying the examples here.

[2]For the main argument of this chapter, it does not matter what reasons—mathematical, metaphysical, or otherwise—one has for believing that classical cases of symmetry breaking pose no interpretive puzzles. (Baker, 2011, p. 132) provides arguments that the classical case is well understood, and I will provide some remarks from a somewhat different perspective (section 4.3.2) to suggest this is correct. If one rejects Baker's reasons or my own, then one can substitute whatever reasons he or she prefers and my argument remains intact. My central claim here is only that the very same feature of unitarily inequivalent representations that Baker finds puzzling in quantum theories appears again in my classical examples.

4.2 Baker's Puzzles

Informally, a symmetry is broken when the system has multiple possible ground states that are related by a symmetry.[3] To understand this definition and Baker's puzzles, we will need some brief background. First, a *ground state* is a state of lowest energy. While there are many ways to determine the ground state of a quantum system in the algebraic framework (see Bratteli and Robinson, 1996, pp. 97-98), these all rely on the fact that the Hamiltonian of the system is the generator *in an algebraic sense* of the dynamics. While the Hamiltonian of a classical system can also be understood to generate the dynamical evolution in a geometrical sense, it is important to note that the relation of the Hamiltonian to the dynamics is different in classical theories and quantum theories. As such, the definition of ground state employed in quantum theory will not apply to our discussion of classical physics later on. Instead, we will take a ground state in a classical theory to be (following standard practice in classical physics) a minimum of the Hamiltonian, understood as a scalar function on phase space.

A general *symmetry* is represented in the algebraic framework by an automorphism α of the algebra of observables $\mathfrak{A}$.[4] A symmetry acts on states by the transformation $\omega \mapsto \omega \circ \alpha^{-1}$. A symmetry α is *broken* just in case there is some ground state ω which is not invariant under α, i.e. $\omega \neq \omega \circ \alpha^{-1}$. When a symmetry α is broken for a ground state ω in a model of quantum field theory or quantum statistical mechanics, the GNS representations for ω and $\omega \circ \alpha^{-1}$ are unitarily inequivalent (Earman, 2003; Baker and Halvorson, 2013). To see why, simply notice that each Hilbert space representation can have at most one ground state as a vector state (Halvorson, 2006, Sec. 2.2). So if $\omega \circ \alpha^{-1}$ is a distinct ground state from ω, then $\omega \circ \alpha^{-1}$ can only be a vector state—as it must be in its own GNS representation—in a distinct (i.e. unitarily inequivalent) Hilbert space representation

[3] For more on the notion of symmetry breaking and the role of this phenomenon in quantum statistical mechanics and quantum field theory, see Fraser (2012) and Strocchi (2008).

[4] One might want to put further restrictions on which automorphisms count as symmetries of the theory, perhaps by looking only at dynamical symmetries, i.e. ones that commute with the dynamics (Baker, 2011, footnote 1). The symmetries considered in section 4.3 are all dynamical symmetries by virtue of being induced by symmetries of a Lagrangian or Hamiltonian.

from the GNS representation of ω. Baker (2011) argues that the presence of unitarily inequivalent representations when a symmetry is broken leads to a number of puzzles, which we turn to now.

4.2.1 Wigner Unitary

Baker and Halvorson (2013) argue that there is a prima facie puzzle to understanding how the GNS representations of two symmetry related states can be unitarily inequivalent. Let $(\pi_\omega, \mathcal{H}_\omega)$ be the GNS representation of a C* algebra $\mathfrak{A}$ for a state ω and let $(\pi_{\omega'}, \mathcal{H}_{\omega'})$ be the GNS representation of $\mathfrak{A}$ for the symmetry transformed state $\omega' = \omega \circ \alpha^{-1}$, where α is a symmetry of $\mathfrak{A}$. Let us suppose that the symmetry α is broken by ω so that $\omega \neq \omega'$. The puzzle arises, they claim, because the symmetry α gives rise to a transformation from vectors in $\mathcal{H}_\omega$ to vectors in $\mathcal{H}_{\omega'}$ which preserves all inner products. It follows from a result known as Wigner's theorem, that this transformation is given by a unitary operator $W : \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ between the two Hilbert spaces, and hence the symmetry is, in a certain sense, implemented by a unitary operator. Given the guaranteed existence of this unitary operator, how could it possibly be that the GNS representations of $\mathfrak{A}$ for ω and ω' are unitarily inequivalent?

To see why Baker and Halvorson find this puzzling, one simply has to note that the "Wigner unitary" W has many nice properties that make it act like a symmetry on the algebra of observables $\mathfrak{A}$. Namely, Baker and Halvorson show in their "Wigner representation theorem" that W satisfies

$$W\pi_\omega(\mathfrak{A}) = \pi_{\omega'}(\mathfrak{A})W \tag{4.1}$$

which means that W can be thought of as mapping the observables in one representation onto the observables in the other, on the whole. And furthermore, for every $A \in \mathfrak{A}$,

$$W\pi_\omega(\alpha^{-1}(A)) = \pi_{\omega'}(A)W \tag{4.2}$$

which means that W maps symmetry related observables in the different representations to each other. More specifically, W maps the representation of $\alpha^{-1}(A)$ in the the GNS representation of ω to the representation of A in the GNS representation of $\omega' = \omega \circ \alpha^{-1}$. How can W, which seems to implement the symmetry as a unitary operator, fail to be a unitary equivalence?

Baker and Halvorson resolve this puzzle by showing that this "Wigner unitary" W is not a unitary equivalence when $\omega \neq \omega'$. Even though W has the nice properties above allowing us to say that it implements the symmetry α, it is nevertheless *not* the case that W intertwines π_ω and $\pi_{\omega'}$. In other words, there is some $A \in \mathfrak{A}$ such that

$$W\pi_\omega(A) \neq \pi_{\omega'}(A)W \tag{4.3}$$

The puzzle then dissolves, but Baker and Halvorson assert that their results have foundational significance by teaching us about the notion of *physical equivalence* in algebraic quantum theories.

4.2.2 Physical Equivalence

According to Baker (2011), it is a characteristic feature of symmetry transformations that symmetry-related models of a theory are physically equivalent. On his view, symmetry transformations are just like changes of coordinates in that they do not change any of the physically significant information about a system. Baker asserts that this follows from each of the two most commonly held views about ontology and symmetries for theories of spacetime: Relationism and Sophisticated Substantivalism (see Baker, 2011, p.132). Both views imply that even when a symmetry α is broken for a ground state ω, the states ω and $\omega \circ \alpha^{-1}$ are physically equivalent (and, according to Baker, their GNS representations are physically equivalent as well).

Baker goes on to say that a necessary condition for ω and $\omega \circ \alpha^{-1}$ to be physically equivalent is that we can come up with a translation scheme between their respective GNS representations.

Drawing on results from Clifton and Halvorson (2001), Baker repeats a commonly held view that two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of an algebra $\mathfrak{A}$ are intertranslatable only if they are unitarily equivalent. In this case, one can use the unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ implementing the unitary equivalence to translate observables $\pi_1(A)$ in $\mathcal{B}(\mathcal{H}_1)$ to observables $\pi_2(A) = U\pi_1(A)U^{-1}$ in $\mathcal{B}(\mathcal{H}_2)$ and to translate vector states ψ_1 in $\mathcal{H}_1$ to vector states $\psi_2 = U\psi_1$ in $\mathcal{H}_2$ (and by extension, density operator states).

When the symmetry α is broken for a ground state ω, Baker claims there is an apparent dilemma: the GNS representations for ω and $\omega \circ \alpha^{-1}$ should be physically equivalent by our best understanding of ontology and symmetries, but we have no translation scheme between them because they are unitarily *in*equivalent. So it appears they cannot be physically equivalent after all.

To dissolve this puzzle, Baker (2011) argues that unitary equivalence is not necessary for a translation scheme and hence not necessary for physical equivalence. He claims that the arguments that led philosophers to believe that unitary equivalence was necessary for physical equivalence were founded on false premises. But without unitary equivalence as our standard of physical equivalence, Baker believes we are "left adrift" (Baker, 2011, p. 146) in our interpretation of quantum theories.

In what follows, I will not be concerned with answering Baker's plea for a notion of physical equivalence. Rather, I will argue to the contrary that the presence of unitarily inequivalent representations in cases of symmetry breaking does not "leave us adrift" at all. I take it to be uncontentious (as even Baker agrees (2011, p. 132)) that symmetry breaking in the classical case does not give rise to puzzles, whatever your view on physical equivalence, symmetries, and ontology may be.[5] Yet I will show that analogous technical results arise in classical theories as in quantum theories—namely, in classical theories, symmetry-related ground states have unitarily inequivalent

[5]Baker claims that one avoids the puzzle of physical equivalence in the classical case by taking on either of the metaphysical positions just mentioned: Relationism or Sophisticated Substantivalism. For the purposes of this chapter, I will not put the discussion in those terms (specifically, in section 4.3.2). If one prefers to understand classical symmetry breaking according to those views, it will make no difference to my argument here (see footnote 14).

GNS representations. To the extent that this does not puzzle us in the classical case, it should not puzzle us in the quantum case either.

4.3 Symmetry Breaking in Classical Systems

As outlined in Ch. 3, to describe a classical system in algebraic terms, one first specifies a configuration space $\mathcal{U}$ of the classical system. A classical observable is then a function $f : \mathcal{U} \to \mathbb{C}$ (Landsman, 1998a; Brunetti et al., 2012). The value of a classical observable f on a configuration $\varphi \in \mathcal{U}$ represents the determinate value one would get as the outcome to an experiment one could perform for measuring f. The bounded observables form an abelian C*-algebra $\mathfrak{A}$ with the usual operations of pointwise addition and multiplication of functions (see Ch. 3 and Feintzeig (2016b)).[6] We can easily recognize states of $\mathfrak{A}$ that are familiar from classical physics. Corresponding to any determinate configurations $\varphi \in \mathcal{U}$ of the classical system, we can define a state

$$\omega_\varphi(f) := f(\varphi)$$

I do *not* claim that this algebraic formulation of classical theories provides the best or most useful understanding of classical physics for all purposes, or even that it captures the full, rich structure of classical theories. For example, this algebraic formulation does not contain any information about the Poisson bracket associated with the classical phase space. But all I require is that quantum and classical theories share a background of very weak interpretive assumptions, namely, that observables correspond to quantities that we can measure of our system and states assign expectation values to those observables.[7] This is enough to allow us to take Hilbert space representations and

[6] One can impose a topological or differentiable structure on $\mathcal{U}$, which can be used to limit oneself to only smooth or continuous functions. For example, one may use the compact-open topology to pick out continuous functions or one may use the Whitney topology to induce a manifold structure (Brunetti et al., 2012). One can also restrict attention to either functions vanishing at infinity or functions with compact support, if the configuration space allows.

[7] For more on the analogies and disanalogies between classical and quantum systems in the algebraic framework, see section 3.5.

analyze their physical significance, which is all that I propose to do in this chapter.

Recall that Baker's puzzles about broken symmetries in quantum theories all revolved around the presence of unitarily inequivalent representations. Recall the following three propositions, presented in Ch. 3 and restated here,[8] which show that unitarily inequivalent representations abound in classical theories too. These trivial results are well known by specialists. And it is precisely because the mathematical features that Baker points to are so obviously trivial in classical systems that they give rise to no puzzles, as I will argue. After presenting the results very generally, I'll show how they apply to cases of broken symmetry.

Proposition 4.3.1. *Let $\varphi \in \mathcal{U}$ be a configuration. Then the corresponding state ω_φ is pure.*

Proposition 4.3.2. *Let $\mathfrak{A}$ be an abelian C*-algebra. Let ω be a pure state on $\mathfrak{A}$ and let $(\pi_\omega, \mathcal{H}_\omega)$ be the GNS representation of $\mathfrak{A}$ for ω. Then $\mathcal{H}_\omega$ is one dimensional.*

Since the the GNS representation for any pure state—e.g. a state corresponding to a determinate configuration—is one dimensional, it only has the resources to represent a single state as a density operator. This implies that any other state must be represented as a density operator on a different, i.e. unitarily inequivalent, Hilbert space representation.

Proposition 4.3.3. *Let ω_1 and ω_2 be distinct pure states on an abelian C*-algebra $\mathfrak{A}$. Let $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ be the GNS representations of $\mathfrak{A}$ for the states ω_1 and ω_2. Then $(\pi_{\omega_1}, \mathcal{H}_{\omega_1})$ and $(\pi_{\omega_2}, \mathcal{H}_{\omega_2})$ are unitarily inequivalent.*

It follows that the GNS representations for any two distinct states corresponding to determinate configurations are unitarily inequivalent.

If a symmetry α is broken for a ground state ω on $\mathfrak{A}$, then $\omega \neq \omega \circ \alpha^{-1}$. Suppose (as will be the case in the examples below) that ω is a pure state, which entails that $\omega \circ \alpha^{-1}$ is pure as well. Then it follows immediately from the preceding proposition that the GNS representations for ω

[8]See (Kadison and Ringrose, 1997, p. 744), Feintzeig (2016b), or Ch. 3 for proofs.

and $\omega \circ \alpha^{-1}$ are unitarily inequivalent. Recall that Baker's puzzles in quantum cases of symmetry breaking are based solely on the presence of unitarily inequivalent representations. This means that the very same feature that appears at the heart of philosophical discussions of symmetry breaking in quantum systems rears its head in the classical case as well.

To be clear, I am *not* claiming that unitarily inequivalent representations have special physical significance in classical cases of symmetry breaking. We will see in what follows that the unitarily inequivalent representations that appear in classical theories are utterly trivial and not puzzling. Each representation, because it is one dimensional, has the role of encoding only one state and its expectation values, with no reference to the rest of the theory.

I will now present two examples of simple and mundane classical systems in which broken symmetries lead to unitarily inequivalent representations, but no puzzles arise. In the first, I will identify the relevant 'Wigner unitary' in order to show that its existence is not puzzling. In the second, I will briefly illustrate why we need not worry about Baker's puzzle of physical equivalence.

4.3.1 Scalar Field

Let $\mathcal{M}$ be Minkowski spacetime.[9] Consider the classical theory of a real smooth scalar field $\varphi : \mathcal{M} \to \mathbb{R}$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\nabla_a \varphi)(\nabla^a \varphi)$$

The corresponding Euler-Lagrange equation of motion governing the field is

$$0 = \nabla_a \nabla^a \varphi$$

Such a theory may represent, for example, a massless classical Klein-Gordon field. Our configuration space is the subset $\mathcal{U} \subseteq C^\infty(\mathcal{M})$ of smooth solutions to the field equation. The algebra of

[9] In this section we work with the metric signature (+,-,-,-).

classical observables $\mathfrak{A}$ is then given, as above, by some (perhaps appropriately restricted) collection of (bounded) functions $f : \mathcal{U} \to \mathbb{C}$.

To express the Hamiltonian, we choose a foliation into spacelike hypersurfaces relative to a choice of inertial scalar coordinate fields (t, x, y, z). The Hamiltonian is then given by

$$H = \frac{1}{2}\Big[\Big(\frac{\partial\varphi}{\partial t}\Big)^2 + \Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2 + \Big(\frac{\partial\varphi}{\partial z}\Big)^2\Big]$$

Since the Hamiltonian is nonnegative, and takes the value zero for any constant scalar field φ, it follows that the state corresponding to any constant scalar field is a state of lowest energy.

For any constant C, the transformation $\beta : \varphi \mapsto \varphi - C$ is a symmetry of the Lagrangian and the Hamiltonian, and thus of the equations of motion. Hence, the transformation $\alpha : f \mapsto f \circ \beta$ is a symmetry, i.e. an automorphism, of the classical algebra $\mathfrak{A}$. Let φ be a determinate field configuration. Then for all $f \in \mathfrak{A}$,

$$\omega_\varphi \circ \alpha^{-1}(f) = \alpha^{-1}(f)(\varphi) = f(\varphi + C) = \omega_{\varphi+C}(f)$$

which shows that the symmetry transforms the pure state ω_φ to $\omega_{\varphi+C}$.

Suppose we take the configuration $\varphi_0 = 0$. Applying the symmetry to the ground state ω_0 corresponding to φ_0 yields a new ground state $\omega_0 \circ \alpha^{-1} = \omega_C$ corresponding to the configuration $\varphi_C = C$. Let us suppose that, for some C, the symmetry is broken.[10] This means that $\omega_C \neq \omega_0$, or in other words there is an observable $f \in \mathfrak{A}$ that distinguishes φ_0 from φ_C. It follows immediately from Prop. 4.3.3 that the GNS representations of $\mathfrak{A}$ for ω_0 and $\omega_C = \omega_0 \circ \alpha^{-1}$ are unitarily inequivalent.

What does Baker and Halvorson's puzzle about the Wigner unitary look like in this context? There *is* a unitary transformation between the Hilbert spaces of the two relevant GNS representations.

[10]If $\mathfrak{A}$ consists of all (bounded) maps $f : \mathcal{U} \to \mathbb{C}$, then the supposition is automatically true. In general, however, it may not be true. This supposition amounts to the requirement that the algebra contains *enough* observables to distinguish the two ground states considered.

Let $(\pi_0, \mathcal{H}_0)$ be the GNS representation of $\mathfrak{A}$ for the state ω_0 with corresponding cyclic vector Ω_0. Consider the representation $(\pi_0 \circ \alpha^{-1}, \mathcal{H}_0)$ of $\mathfrak{A}$. Notice that for any $f \in \mathfrak{A}$

$$\langle \Omega_0, \pi_0 \circ \alpha^{-1}(f)\Omega_0 \rangle = \omega_0(\alpha^{-1}(f)) = \alpha^{-1}(f)(\varphi_0) = f(\varphi_C) = \omega_C(f)$$

Since Ω_0 is also cyclic for the representation $\pi_0 \circ \alpha^{-1}(\mathfrak{A})$, it follows from the uniqueness clause of the GNS theorem that the representation $(\pi_0 \circ \alpha^{-1}, \mathcal{H}_0)$ is unitarily equivalent to the GNS representation of $\mathfrak{A}$ for the state ω_C. Henceforth, we will rename the GNS representation of ω_C as $\pi_C := \pi_0 \circ \alpha^{-1}$. Notice that in this case the very same vector Ω_0 corresponds to two different states ω_0 and ω_C on the respective representations π_0 and π_C.

Of course the identity operator $I : \mathcal{H}_0 \to \mathcal{H}_0$ suffices as a unitary transformation between the Hilbert spaces of the representations, and indeed I is a "Wigner unitary" in the terminology of Baker and Halvorson, meaning it implements the symmetry α. We can see this because the represented algebras are identical on the whole: (compare with eq. (4.1))

$$I\pi_0(\mathfrak{A}) = \pi_0(\mathfrak{A}) = \pi_0 \circ \alpha^{-1}(\mathfrak{A}) = \pi_C(\mathfrak{A})I$$

Furthermore, symmetry related observables are equal in the two different representations: for all $f \in \mathfrak{A}$, (compare with eq. (4.2))

$$I\pi_0(\alpha^{-1}(f)) = \pi_0(\alpha^{-1}(f))I = \pi_C(f)I$$

But it nevertheless follows from Prop. 4.3.3 that the two representations $(\pi_0, \mathcal{H}_0)$ and $(\pi_0 \circ \alpha^{-1}, \mathcal{H}_0)$ are unitarily inequivalent. One might initially be puzzled at how this could be. How is it possible to have unitarily inequivalent representations that nevertheless are related by a unitary operator implementing a symmetry?

Since $\mathcal{H}_0$ is one-dimensional by Prop. 4.3.2, we can take elements of $\mathcal{H}_0$ to simply be complex numbers, i.e. $\mathcal{H}_0 \cong \mathbb{C}$. Then the bounded operators acting on $\mathcal{H}_0$ will also correspond to complex numbers acting by ordinary multiplication. Each representation is an assignment to each observable $f \in \mathfrak{A}$ of a complex number in $\mathcal{B}(\mathcal{H}_0)$. Different representations may assign different complex numbers to the same observable. For each observable $f \in \mathfrak{A}$, the representation π_0 assigns to f the complex number $\pi_0(f) = \omega_0(f)$, which is the expectation value of f in the state ω_0. Similarly, for each observable $f \in \mathfrak{A}$, the representation $\pi_C = \pi_0 \circ \alpha^{-1}$ assigns to f the complex number $\pi_C(f) = \pi_0 \circ \alpha^{-1}(f) = \omega_0 \circ \alpha^{-1}(f) = \omega_C(f)$, which is the expectation value of f in the state ω_C. Since ω_0 and ω_C are distinct states, some observable must be represented by different numbers in the different representations. Now we can see that even though the automorphism α is implemented by the unitary I, this unitary transformation does not intertwine the two representations because there is an $f \in \mathfrak{A}$ such that $\omega_0(f) \neq \omega_C(f)$ and hence (compare with eq. (4.3))

$$I\pi_0(f) = \pi_0(f) \neq \pi_C(f) = \pi_C(f)I$$

Although there is a "Wigner unitary" transformation between the Hilbert spaces of the two representations, it does not implement a unitary equivalence.

However, one should not be surprised or puzzled by this situation. Each representation focuses our gaze on only a single state—π_0 takes us to a Hilbert space that contains only ω_0 as a density operator state, and similarly for π_C with the state ω_C. If the symmetry α is broken, then these are distinct states, and it follows immediately that the representations must be unitarily inequivalent, even though they are clearly systematically related.

Moreover, in the finite dimensional case, every Hilbert space is related to every other of the same dimension by a unitary transformation. A symmetry will correspond to some such transformation. But to achieve a unitary equivalence, one must satisfy the further requirement that the symmetry transforms operators to ones that receive the same expectation value in the chosen state. We have no reason to expect this further requirement to be satisfied; in fact, the definition of a broken symmetry demands that it is not.

Finally, notice that we can redescribe the situation in classical terms in a straightforward way without the use of the algebraic framework. The fields φ_0 and φ_C are two different possible configurations of the system that happen to evolve in the same way under the dynamics given by the Lagrangian. This is all it means for the corresponding states ω_0 and ω_C to be related by a symmetry transformation. We have found an interesting feature of this classical system, but it poses no problems or puzzles.

4.3.2 Spin Chain

Consider a denumerable[11] chain of systems, each of which has two possible states: $+1$ and -1. I will refer to the -1 state as the "spin down" state and the $+1$ state as the "spin up" state, although these systems will differ from spin-1/2 quantum particles in that their spin will always take on a definite value.[12] The entire chain will be thought of as living on the lattice $\mathbb{Z}$, so that states of the composite system are given by functions $\varphi : \mathbb{Z} \to \{-1, 1\}$ assigning a definite spin value to each subsystem in the chain. The relevant configuration space $\mathcal{U}$ is the set (or some appropriately restricted subset) of all such states of the entire spin chain. The algebra of observables $\mathfrak{A}$ will then be, as before, some (perhaps appropriately restricted) collection of the functions $f : \mathcal{U} \to \mathbb{C}$. The Hamiltonian of the system is given by the following expression for nearest neighbor interactions in the absence of an external magnetic field (compare with (Ruetsche, 2006, p. 479)):

$$H = -\sum_k \varphi(k) \cdot \varphi(k+1)$$

By direct inspection of the Hamiltonian, we can immediately identify two pure ground states

[11]Everything I say in this section applies to finite classical spin chains as well.

[12]This example is meant to be a classical analogue to the quantum infinite spin chain described in Ruetsche (2006). It is sometimes known as the classical Ising model.

(which are analogous to the constant field configurations considered in the previous section). First, we have the state ω_+ corresponding to the determinate field configuration φ_+, which we define as

$$\varphi_+(z) := +1$$

for all $z \in \mathbb{Z}$. Second, we have the state ω_- corresponding to the determinate field configuration φ_-, which we define as

$$\varphi_-(z) := -1$$

for all $z \in \mathbb{Z}$. The state ω_+ represents the spins of all of the subsystems as aligned pointing up, while the state ω_- represents the spins of all of the subsystems as aligned pointing down.

The map $\beta : \mathcal{U} \to \mathcal{U}$ given by $\varphi \mapsto -\varphi$ is a symmetry of the Hamiltonian, which reflects that the choice of which state is called "up" $(+1)$ and which is called "down" (-1) does not make a difference to the dynamics of the system. One can then define a symmetry, i.e. an automorphism, of the quasilocal algebra $\alpha : \mathfrak{A} \to \mathfrak{A}$ by $f \mapsto f \circ \beta$. The symmetry β takes φ_+ to φ_- and hence for all $f \in \mathfrak{A}$,

$$\omega_+ \circ \alpha^{-1}(f) = \alpha^{-1}(f)(\varphi_+) = f(\beta(\varphi_+)) = f(\varphi_-) = \omega_-(f)$$

which shows that the symmetry transforms the pure state ω_+ to the pure state ω_-.

Let us again suppose that the symmetry is broken. This means that $\omega_+ \neq \omega_-$ or in other words that there is an observable $f \in \mathfrak{A}$ that distinguishes φ_+ from φ_-. In this case, we can give explicit examples of observables that are physically significant and distinguish φ_+ from φ_-. First, define the *central spin value observable* O by $O(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{U}$. The central spin value observable represents the direction of the spin of the subsystem at the center point of the lattice (i.e. $0 \in \mathbb{Z}$). It follows immediately from the definition that $O(\varphi_+) \neq O(\varphi_-)$. Second, define the nth *polarization* *observable* m_n by[13]

$$m_n = \frac{1}{2n+1} \sum_{i=-n}^{+n} \varphi(i)$$

for all $\varphi \in \mathcal{U}$. The nth polarization observable represents the average direction of spin of the $2n+1$ subsystems symmetrically arranged about the center point of the lattice (i.e. the subsystems $i = -n, ..., 0, ..., +n$). Again, it follows immediately from the definition that for each of these polarization observables, $m_n(\varphi_+) \neq m_n(\varphi_-)$ because $m_n(\varphi_+) = +1$ and $m_n(\varphi_-) = -1$ for all n. If the observables O and m_n are included in $\mathfrak{A}$ (and they ought to be because they are physically significant!), then because these observables distinguish φ_+ from φ_-, it follows that ω_+ and ω_- will be distinct states. And then it is an immediate consequence of Prop. 4.3.3 that the GNS representations of $\mathfrak{A}$ for ω_+ and ω_- will be unitarily inequivalent.

By now this should be unsurprising; the situation is exactly analogous to that of the real scalar field. Each of the aforementioned GNS representations will be one-dimensional and hence will only be able to represent a single state as a density operator. The GNS representation for ω_+ cannot represent the state ω_- as a density operator, and vice versa.

What happens to Baker's puzzle of physical equivalence in this context? In the example, we obviously have two distinct ground states[14] ω_+ and ω_-. Of course, one could have used the opposite labeling convention and called ω_+ the state ω_- instead, and vice versa. But, having fixed a convention, we can distinguish these two states (according to their expectation values on some observables) and identify when one or the other obtains.[15] This poses no problem because the value of an observable on a particular state is in general dependent on the convention we choose for which direction is up. For example, one explicitly defines the polarization observables by looking at the spin values on lattice sites, which of course would have been the opposite of what they actually are

[13] Compare with Ruetsche (2006, p. 477).

[14] By asserting this, I am disagreeing with Baker's claim that symmetry-related states are physically equivalent. I say this as one possible way of seeing that there is no puzzle in the classical case, but it is not necessary for the argument that follows. All I require is that, whatever your view on symmetries, the classical case is not puzzling. It makes no difference for the rest of the argument of this chapter whether one arrives at the conclusion that the classical case is not puzzling for the reasons I give in this paragraph or for the reasons given by Baker (2011, p. 132).

[15] For a related view, see Weatherall (2016).

had we labelled our directions differently. In this way, the values of the observables are akin to the values of observables considered *in a particular coordinate system*.

Having fixed a convention for which direction is up, we noticed that the GNS representation for ω_+ does not have the resources to represent the other state ω_-. On the other hand, had we chosen the opposite convention, the state that we currently call ω_- would have been called ω_+ and would have been represented in that very same GNS representation. With a fixed convention, the existence of two unitarily inequivalent GNS representations corresponds to the existence of two distinct states, each of which cannot be represented as a density operator on the other Hilbert space. But we could have used the very same GNS representation to represent the opposite state had our convention been different. The symmetry and "Wigner unitary" transformation reflect this fact by allowing us to translate observables from the GNS representation for ω_+ to the GNS representation of ω_- even when the two representations are unitarily inequivalent.

We can sum this up by saying that there is a sense in which the GNS representations for ω_+ and $\omega_-$are equivalent, and another sense in which they are inequivalent. The sense in which they are equivalent is that any system which can be modeled by the GNS representation for ω_+ could equally as well have been modeled using the GNS representation for ω_- by changing our conventions about which direction is up. The sense in which they are inequivalent is that, having fixed a convention for which direction is up, the states ω_+ and ω_- correspond to two distinct states, and because their GNS representations are one-dimensional, neither can be used to represent the other state. Once we recognize this, there is no puzzle about physical equivalence in the classical spin chain. The presence of unitarily inequivalent representations is not troubling in the classical case.

4.4 Conclusion

We have seen two examples of symmetry breaking in classical physics. First, the "shift" symmetry may be broken by a real scalar field on flat space-time. Second, the "flip" symmetry may be broken by a classical spin chain. We showed in these examples that when a symmetry α is broken for a ground state ω, the GNS representation of ω is unitarily inequivalent to the GNS representation of $\omega \circ \alpha^{-1}$. In other words, symmetry breaking gives rise to precisely the same mathematical feature in classical systems that Baker finds puzzling in quantum systems. We looked at Baker's puzzles concerning the "Wigner unitary" and physical equivalence—we saw that a "Wigner unitary" appears in the classical cases without giving rise to any puzzles, and we saw that one can understand distinct states related by a symmetry transformation without being puzzled about physical equivalence.

Again, even Baker agrees that the classical case is unproblematic (Baker, 2011, p. 132). Yet Baker sees a difference between the quantum and the classical case only because he has not considered classical theories in the algebraic framework. When we put the classical examples into the algebraic formalism, there are no relevant differences between the classical and quantum cases with respect to Baker's arguments. To the extent that the presence of unitarily inequivalent representations in the classical case does not puzzle us, it should not puzzle us in the quantum case either.

I do not claim that there is nothing interesting or new about symmetry breaking in quantum theories. There very well may be features of philosophical and physical significance in quantum systems with broken symmetries, perhaps related to the dynamical behavior of such systems (Strocchi, 2008) and the approximation of infinite systems by finite ones (Butterfield, 2011a,b; Landsman, 2013). My claim is only that no puzzles arise from the mere presence of unitarily inequivalent representations on their own. I hope that readers interested in systems with broken symmetries will look deeper into the physics of such systems than I have been able to here.

I would like to conclude by venturing a tentative diagnosis of what may be leading to such confusion about unitarily inequivalent representations in quantum theories. Baker's puzzle about physical equivalence stems from thinking about a quantum theory as given by an irreducible representation of the algebra of observables. After all, physically equivalent representations are supposed to be equivalent *as physical theories.* But should we really think of a quantum theory as just a single irreducible representation? In classical theories, we have seen that this interpretation is not available to us, for immediate and obvious reasons. Since each irreducible representation of the classical algebra of observables is one-dimensional, it only has the resources to represent one state. We are not even tempted to say that a representation is the whole theory—it is undeniably only a small part.[16] So, I suggest, we do not in general have good reason to view two unitarily inequivalent representations as competing theories. From this point of view, not only does Baker's puzzle of physical equivalence not arise, but we simply find ourselves with no idea what to make of Baker's claim that representations related by a symmetry, or a Wigner unitary, are physically equivalent. I hope that reflecting on the classical examples in this chapter clarifies the status and significance of unitarily inequivalent representations.

[16]See Ch. 3 and Feintzeig (2016b) for more on this view.

Chapter 5

Toward an Understanding of Parochial Observables

5.1 Introduction

Ruetsche (2011) argues that both the Algebraic Imperialist, who takes an abstract C*-algebra of observables to provide the physical content of a quantum theory, and the Universalist, who takes the universal representation of that algebra to do so (see Ch. 2), fail to provide adequate interpretations of quantum theories with infinitely many degrees of freedom. Ruetsche's argument is what I will call *the problem of parochial observables*—she claims to show that there are certain observables, such as particle number and net magnetization, that the Hilbert Space Conservative acquires within the Hilbert space setting and employs in physically significant explanations, but which the Imperialist and Universalist do not have access to. These are the ones she calls *parochial observables*. Since the Imperialist and Universalist do not have access to these observables, they cannot recover physically significant explanations of (for example) particle content, phase transitions, and symmetry breaking. Since an interpretation is adequate only insofar as it can recover physically significant explanations, Ruetsche argues that Imperialism and Universalism are unacceptable interpretations.

I will argue that the problem of parochial observables is only an apparent problem, which disappears once one recognizes that the Imperialist and Universalist have additional resources for representing parochial observables. Parochial observables arise for the Conservative as approximations to or idealizations from observables in a particular Hilbert space representation (once we have chosen a physically relevant notion of idealization). I will show that the Imperialist and the Universalist can use the same sorts of tools in their respective settings to acquire the parochial observables. Namely, the Imperialist can define a physically relevant notion of approximation or idealization on the abstract algebra and the Universalist can similarly define a physically relevant notion of approximation or idealization on the bounded operators on the universal Hilbert space. According to both of these notions, parochial observables arise as idealizations from the observables we began with in the abstract algebra or the universal representation, respectively. When the Imperialist and Universalist are allowed the same mathematical tools that the Hilbert Space Conservative uses for representing approximations and idealizations, the problem of parochial observables does not arise. The Imperialist and Universalist each have access to all of the observables they need.

Furthermore, when we allow the Imperialist and the Universalist access to these idealized observables, one can show a precise sense in which they are equivalent interpretations. There is a sense in which the Imperialist and the Universalist subscribe to the same physical possibilities; they simply use different mathematical tools to describe these possibilities. Thus, Imperialism and Universalism amount to the same interpretation, and it is one which avoids Ruetsche's problem of parochial observables.

5.2 The Problem of Parochial Observables

Ruetsche's *problem of parochial observables* begins as an argument against Algebraic Imperialism and at times is extended to an argument against Universalism. The basic claim of the argument is that the Algebraic Imperialist does not have the resources to represent all of the physically significant observables (Ruetsche 2002, p. 367; Ruetsche 2003, p. 1330). As always, we begin with some algebra of observables $\mathfrak{A}$ representing the physical quantities, or observables, of our system. We have seen (in Ch. 2) that the Hilbert Space Conservative, having privileged some pure state ω and its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$, acquires all of the observables in $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega)$. The new observables on the boundary $\overline{\pi_\omega(\mathfrak{A})} \setminus \pi_\omega(\mathfrak{A})$ are the ones that Ruetsche calls *parochial observables*; these are limit points in the weak operator topology of nets of observables from the original algebra and so they can be thought of as approximations to or idealizations from the observables we already recognized.[1] Many of these observables have real physical import but have no analogue in the abstract algebra. So the Hilbert Space Conservative gains access to more observables than the Algebraic Imperialist. These observables are physically significant for giving explanations of thermodynamic phase transitions, for just one example. So, according to Ruetsche, the Algebraic Imperialist runs into a problem because she cannot recognize these operators as physically possible observables, and so cannot vindicate such explanations.

It may be helpful to see the problem of parochial observables in some more detail in the context of an example: the infinite one-dimensional spin chain. To construct the spin chain, one assigns Pauli spin operators $\sigma_x^k, \sigma_y^k, \sigma_z^k$ to each point k in the lattice $\mathbb{Z}$ of the integers. These operators are required

[1]By saying this, I do not mean to take a stance on whether the limit points are bona fide observables or "merely" idealizations, and I also do not mean to take a stance on whether such idealizations here are indispensable (see, e.g., Callender, 2001; Batterman, 2005, 2009). I hope only to invoke the notion that a topology captures a notion of similarity or resemblance and that limit points can be thought of as relevantly resembling elements of the original algebra to arbitrarily high accuracy so that we are licensed to use them for some scientific purposes. For more on limiting relations capturing a notion of similarity, resemblance, and approximation, see Butterfield (2011a,b) and Fletcher (2016).

If one is bothered by my use of the terms idealization and approximation, which are highly contested in the philosophical literature, one can view this discussion as merely being about the mathematical tools for taking certain limits, which end up being physically significant.

to satisfy the usual canonical anti-commutation relations. The relevant algebra $\mathfrak{A}$ for the total system is the inductive limit (quasilocal algebra) of algebras of spin operators assigned to finite regions of the chain (Ruetsche, 2006, p. 477). In order to describe and explain the phenomenon of broken symmetry of the spin chain—the fact that nature appears to select a preferred direction in space despite the invariance of the laws governing the system under rotations—one appeals to the global magnetization observable (which specifies the aforementioned preferred direction in space). We can define the global magnetization in, say, the z-direction by starting with the average magnetization in the z-direction over finite regions

$$m_n = \frac{1}{2n+1} \sum_{k=-n}^{k=+n} \sigma_z^k$$

Taking the limit as n goes to infinity yields the desired global observable. The Algebraic Imperialist hits a roadblock here because this sequence does not converge in the norm topology on $\mathfrak{A}$. The reason is that there are some states on $\mathfrak{A}$ of wildly oscillating average magnetization as n increases (see Ruetsche, 2013, p. 499). So there is no global magnetization observable in the abstract algebra. On the other hand, the global magnetization does converge in the weak operator topology of some representations, because these representations focus us on a privileged, and oftentimes well-behaved, subcollection of states on the algebra. For example, consider the GNS representation of the "up" state representing determinate spins aligned in the positive z-direction; the global magnetization does converge in the weak operator topology of this representation because its representing Hilbert space $\mathcal{H}$ only contains vectors representing states which differ from the "up" state by a finite number of spin flips and superpositions thereof (Ruetsche, 2006, 2013). And so it follows that the global magnetization does reside in $\mathcal{B}(\mathcal{H})$ for this "up" representation. Since only the Hilbert Space Conservative has access to the weak operator topology on a representation, it appears that we need to use the resources of the Hilbert Space Conservative in order to understand the global magnetization as a limit of observables we already recognized.

I will argue, contra Ruetsche, that there is a way for both the Imperialist and the Universalist to account for parochial observables. The problem of parochial observables only appears because we have given the Conservative more tools than the Imperialist—specifically, tools for representing idealizations and approximations. Once we give the Imperialist the analogous tools on the abstract algebra, she has no trouble representing the parochial observables. We will see that when the Imperialist is allowed access to a topology that is analogous to the weak operator topology induced by a representation, she too gains access to all of the parochial observables by using the relevant limiting procedures.

5.3 Two Solutions

5.3.1 Parochial Observables for the Imperialist

When one thinks of an abstract C*-algebra, one usually thinks of it as coming equipped with the topology induced by its norm. This of course corresponds to the uniform topology of a concrete C*-algebra of operators acting on a Hilbert space. But just as one can consider alternative topologies, like the weak operator topology, on concrete algebras of Hilbert space operators, one can consider alternative topologies on the abstract algebra *prior to taking a representation*. One of the alternative topologies on the abstract algebra corresponds in a certain sense to the algebraic translation of the weak operator topology. To motivate this, we must first think about the significance of the weak operator topology. We saw in Ch. 2 that the weak operator topology gives us a criterion of convergence based on expectation values and transition probabilities, and so gives us a notion of approximation relevant to the empirical content of the theory. But this can be made more precise. The following proposition[2] shows that the weak operator topology of a representation is the topology for convergence of expectation values with respect to a privileged collection of states—namely, the finite rank density operators on that Hilbert space representation.

[2]See Appendix A for proofs of propositions 1, 2, 3, 4, and 6. The proofs of propositions 5 and 7 are included in the references and are not reproduced here.

Proposition 5.3.1. *Let $(\pi, \mathcal{H})$ be a representation of a C*-algebra $\mathfrak{A}$. Let $\{A_i\} \subseteq \pi(\mathfrak{A})$ be a net of operators. Then the following are equivalent:*

(1) The net $\{A_i\}$ converges to $A \in \mathcal{B}(\mathcal{H})$ in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

(2) For all states ϕ on $\mathfrak{A}$ implementable by a finite rank density operator Φ on $\mathcal{H}$, $\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$.

One remark before we proceed: in clause (2) the expectation values converge as complex numbers even though there is in general no element of the abstract algebra that the net of observables converges to. This is because the parochial observable (the limit point of the net) in general will not have an analog in the abstract algebra.

Prop. 5.3.1 shows that the weak operator topology on a representation of $\mathfrak{A}$ gives us a notion of approximation relevant to only those states that can be implemented as finite rank density operators in this representation. A Hilbert Space Conservative might have reason to restrict attention to these states, but an Algebraic Imperialist does not. The Algebraic Imperialist sees all states on the abstract algebra as on a par and believes that the empirical content of the theory comes from all expectation values.

However, the Algebraic Imperialist has access to a different topology that defines a notion of convergence using the expectation values of all of the states she deems physically possible. The *weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology*[3] *on an abstract C*-algebra* $\mathfrak{A}$ is defined as follows. Let $\mathfrak{A}^*$ denote the dual space of bounded linear functionals on the Banach space $\mathfrak{A}$. A net $\{A_i\} \in \mathfrak{A}$ converges in the weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology to $A \in \mathfrak{A}$ just in case for all $\phi \in \mathfrak{A}^*$

$$\phi(A_i) \to \phi(A)$$

[3] The nomenclature here is a bit unfortunate. The weak topology (sometimes called the weak Banach space topology) on an abstract C*-algebra is to be distinguished from the weak operator topology on its particular representations. We will see the difference in what follows.

in $\mathbb{C}$. The notation $\sigma(\mathfrak{A}, \mathfrak{A}^*)$ signifies that the weak topology is the weakest (i.e., coarsest) topology on $\mathfrak{A}$ that makes all of the bounded linear functionals in $\mathfrak{A}^*$ continuous. Now we see from Prop. 5.3.1 that the notion of convergence given by the weak operator topology on a representation is simply the notion of convergence that we get by using the weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology with attention restricted to a particular set of states—those that can be represented by finite rank density operators. Likewise, the weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology is the analogue of the weak operator topology for the abstract algebra in the sense that it is an appropriate generalization to a notion of convergence with respect to the expectation values of all states on the abstract algebra.

Insofar as the Hilbert Space Conservative is justified in using the weak operator topology as a physically relevant standard of approximation or idealization, the Algebraic Imperialist is justified in using the weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology as a physically relevant standard of approximation or idealization too. The Imperialist is concerned with approximation and idealization with respect to the expectation values of *all states on the abstract algebra*.[4]

There is a sense in which the abstract algebra $\mathfrak{A}$ and its representations are not complete with respect to the weak and weak operator topologies, respectively (even though they are both complete with respect to the norm topology).[5] There are nets of observables whose relevant expectation values converge in $\mathbb{C}$ but which have no limit point in the abstract algebra. To find those limit points, we must think of the algebra of observables as living in some kind of ambient space of observables; for the Hilbert Space Conservative, this is just the collection $\mathcal{B}(\mathcal{H})$ of operators on some Hilbert space that the Imperialist eschews. But the Imperialist also has access to an ambient space of observables of her own, and we will see that this ambient space allows us to find the limit points of nets of observables in the weak ($\sigma(\mathfrak{A}, \mathfrak{A}^*)$) topology just as $\mathcal{B}(\mathcal{H})$ allows the Hilbert Space Conservative to find the limit points of nets of observables in the weak operator topology.

[4] I do not claim that the weak topology is the "right" one for the Algebraic Imperialist or that there even is a "right" topology to use. Just as the Hilbert Space Conservative may have access to multiple topologies on $\mathcal{B}(\mathcal{H})$, the Algebraic Imperialist may have access to multiple topologies on $\mathfrak{A}$. My claim is simply that the weak topology on $\mathfrak{A}$ is analogous to the weak operator topology on $\mathcal{B}(\mathcal{H})$ in the sense that they derive from the same physical motivations for approximation and idealization, and they have analogous conditions for convergence.

[5] See Appendix C for more on the notion of completion.

Recall that the dual of any Banach space X, written X^*, is the set of continuous linear functionals on X (in the norm topology), and the *bidual* of X, written X^{**} is the set of continuous linear functionals on X^*. The original space X can be embedded in X^{**} via the canonical evaluation map $J : X \to X^{**}$ given by $x \in X \mapsto \hat{x} \in X^{**}$, where we define $\hat{x}$ by

$$\hat{x}(l) = l(x)$$

for all $l \in X^*$. When X is finite-dimensional, J always provides an isomorphism, but in general when X is infinite-dimensional J may not be an isomorphism because it may not be surjective.

The bidual $\mathfrak{A}^{**}$ of the abstract algebra $\mathfrak{A}$ provides the ambient space of observables in which to look for limit points or idealizations of observables from the original algebra. The elements of $\mathfrak{A}^{**}$ can be thought of as observables because for each state ω on $\mathfrak{A}$, we can think of the expectation value of any $A \in \mathfrak{A}^{**}$ as the value $A(\omega)$. Notice that this makes sense because for any element of our original algebra $A \in \mathfrak{A}$, letting $\hat{A} = J(A) \in \mathfrak{A}^{**}$, we see that $\hat{A}(\omega) = \omega(A)$ is the expectation value of A in the state ω. Thinking of observables and expectation values in the bidual like this suggests a way of extending the weak topology of $\mathfrak{A}$ to the bidual $\mathfrak{A}^{**}$: focus on convergence of expectation values with respect to the same collection of states now considered as states on an enlarged algebra of observables. This brings us to the *weak* ($\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$) topology on the bidual* $\mathfrak{A}^{**}$. A net $\{A_i\} \in \mathfrak{A}^{**}$ converges in the weak* ($\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$) topology to $A \in \mathfrak{A}^{**}$ just in case for all $\phi \in \mathfrak{A}^*$

$$A_i(\phi) \to A(\phi)$$

in $\mathbb{C}$. The notation $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$ signifies that the weak* topology on the bidual is the weakest (i.e., coarsest) topology that makes all of the bounded linear functionals in $\mathfrak{A}^*$ continuous when considered as linear functionals on $\mathfrak{A}^{**}$ (i.e. when the elements of $\mathfrak{A}^*$ are considered as elements of $\mathfrak{A}^{***}$ by the canonical evaluation map). The weak* topology on $\mathfrak{A}^{**}$ is the natural extension of the weak topology on $\mathfrak{A}$ because it makes precisely the same linear functionals, and more specifically states, continuous—namely, the linear functionals and states in $\mathfrak{A}^*$. Furthermore, each state on

$\mathfrak{A}$ has a unique continuous extension to a state on $\mathfrak{A}^{**}$ in the weak* topology. The following proposition shows that every element of the bidual $\mathfrak{A}^{**}$ can be understood as a limit point of a net of observables in the abstract algebra $\mathfrak{A}$ in the weak* ($\sigma(\mathfrak{A}^{**}, \mathfrak{A}^{*})$) topology on the bidual $\mathfrak{A}^{**}$. This shows that elements of the bidual can be thought of as approximations to or idealizations from our original observables in a topology that the Algebraic Imperialist deems physically significant.

Proposition 5.3.2. *Let $\mathfrak{A}$ be a C*-algebra, $\mathfrak{A}^{**}$ be its bidual, and $J : \mathfrak{A} \to \mathfrak{A}^{**}$ be the canonical evaluation map. Then $J(\mathfrak{A})$ is dense in $\mathfrak{A}^{**}$ in the weak* ($\sigma(\mathfrak{A}^{**}, \mathfrak{A}^{*})$) topology.*

Just as the Hilbert Space Conservative, upon adding limit points in the weak operator topology of some irreducible representation, arrives at the algebra of operators $\mathcal{B}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$, the Algebraic Imperialist, upon adding limit points in the weak (or really weak*) topology arrives at the collection of observables $\mathfrak{A}^{**}$. When allowed the same methods for constructing idealizations, the Algebraic Imperialist gains access to more observables than reside in the abstract algebra $\mathfrak{A}$.

Now that we have given the Algebraic Imperialist access to more observables than reside in the abstract algebra $\mathfrak{A}$, we must ask: does the Algebraic Imperialist gain access to the observables that Ruetsche argued are physically significant for giving explanations? Recall that the reason the problem of parochial observables was supposed to be a *problem* is that the Algebraic Imperialist appears to not allow us to reconstruct the physics of, say, phase transitions and symmetry breaking. The idealizations the Hilbert Space Conservative constructs are in a certain sense the "right" ones because they allow us to give these physically significant explanations. We need to check that the observables the Algebraic Imperialist constructs by moving to the bidual have enough structure to be able to recover those physically significant explanations, too. First, notice that although $\mathfrak{A}^{**}$ is initially only a Banach space, it can be made into a C*-algebra by defining multiplication and involution operations. We get these operations by Prop. 5.3.2 as the unique extensions of the multiplication and involution operations on the algebra $J(\mathfrak{A})$ (with algebraic structure inherited from $\mathfrak{A}$) such that multiplication is separately continuous in its individual arguments in the weak* topology[6] and involution is continuous in its only argument in the weak* topology. When we refer to C*-algebraic structure on $\mathfrak{A}^{**}$ in what follows, these are the operations we will have in mind. The following proposition shows that the Algebraic Imperialist, using this structure on the bidual, has access to every idealized parochial observable that every possible Hilbert Space Conservative (for each possible distinct privileged representation) has access to.

Proposition 5.3.3. *If π is a representation of a C*-algebra $\mathfrak{A}$, then there is a central*[7] *projection $P \in \mathfrak{A}^{**}$ and a *-isomorphism α from $\mathfrak{A}^{**}P$ onto $\overline{\pi(\mathfrak{A})}$ such that $\pi(A) = \alpha(J(A)P)$ for all $A \in \mathfrak{A}$.*

So the von Neumann algebra affiliated with any representation of $\mathfrak{A}$ is canonically isomorphic to a subalgebra of $\mathfrak{A}^{**}$. This shows that every parochial observable can be thought of as an element of the bidual $\mathfrak{A}^{**}$ and so can be thought of as an idealization from observables in the abstract algebra that we arrive at by a notion of approximation or idealization that is physically relevant by the Imperialist's lights. The Algebraic Imperialist, when allowed the same tools for constructing idealizations that we allowed the Hilbert Space Conservative, has access to all of the parochial observables, like net magnetization and particle number, that we need to give the kinds of physically significant explanations Ruetsche is worried about.[8,9]

Having presented the central technical results that provide the Algebraic Imperialist with the parochial observables, it's worth pausing for a moment to remark on the kind of Imperialism we've ended up with. One might want to distinguish between two different kinds of Algebraic Imperialists specifying different interpretations via different kinematic pairs. One Algebraic Imperialist believes the physically significant observables reside in $\mathfrak{A}$ and another who believes the physically significant observables reside in $\mathfrak{A}^{**}$. This paper provides an argument that the latter algebraic

[6] In general, multiplication is not jointly continuous in the weak* topology.

[7] The center of a C*-algebra $\mathfrak{A}$ consists in those elements $A \in \mathfrak{A}$ such that for all $B \in \mathfrak{A}$, $AB = BA$.

[8] See Kronz and Lupher (2005) and Lupher (2008), who also assert that the bidual contains all of the parochial observables. Here, I have added a precise characterization of how the parochial observables arise from the original algebra via limiting relations.

[9] One might worry that the observables the Algebraic Imperialist gets in Prop. 5.3.3 (or the analogous observables for the Universalist in the next section) are somehow not the *real* or *fundamental* net magnetization and particle number. I do not claim that the observables the Imperialist acquires are fundamental—all I claim is that these observables suffice to give the kinds of explanations that Ruetsche worries about.

Imperialist, who chooses $\mathfrak{A}^{**}$ as her algebra of observables,[10] overcomes the problem of parochial observables.

In this respect, my preferred interpretation differs from that of many of the original Algebraic Imperialists in the physics community who explicitly disavowed parochial observables (e.g, Haag and Kastler, 1964, p. 853). For example, rather than seeing all representations as specifying the same physical possibilities as did Haag and Kastler (1964, p. 851),[11] on my view a representation focuses our gaze on a particular collection of states and observables constituting only a small subset of the physically possible states and observables of the theory.[12] My interpretation also differs from that of Segal (1959, pp. 348-349), who presents a version of Algebraic Imperialism in which the norm topology on $\mathfrak{A}$ is centrally important because of its operational significance. I have shown that even if one takes the norm topology to be physically significant, one has reason to take the weak Banach space topology on $\mathfrak{A}$ as physically (or even operationally) significant as well. Because of this, the Algebraic Imperialist need not restrict attention to the algebra $\mathfrak{A}$, which is only complete in the norm topology but not the weak topology. I believe Prop. 5.3.2 justifies the Algebraic Imperialist in moving to the larger algebra $\mathfrak{A}^{**}$ for exactly the same reasons the Hilbert Space Conservative is justified in moving from $\pi(\mathfrak{A})$ to the larger algebra $\overline{\pi(\mathfrak{A})}$.[13] And we have seen that the Algebraic Imperialist acquires all parochial observables as elements of $\mathfrak{A}^{**}$.

5.3.2 Parochial Observables for the Universalist

When we think about Universalism, we do not need to go through the rigamarole of defining a new kind of topology as we did for Imperialism in the previous section. The universal representation already comes with a topology that is precisely analogous to the weak operator topology used by

[10]This is the same variant on Algebraic Imperialism that Lupher (2008) dubs *Bidualism*.

[11]See also Robinson (1966, pp. 487-489) and Ruetsche (2011, p. 135).

[12]For more on this interpretation of states for the Algebraic Imperialist, see Chs. 3 and 4 and Feintzeig (2015, 2016b).

[13]I also believe that one can make precise a sense in which this move from $\mathfrak{A}$ to $\mathfrak{A}^{**}$ does not add any information or content to the theory—I save this for future work (see Appendix C and Ch. 7).

the Hilbert Space Conservative: namely the weak operator topology on the collection $\mathcal{B}(\mathcal{H}_U)$ of bounded operators on the universal Hilbert space $\mathcal{H}_U$. But how does the weak operator topology of the Universalist compare to the weak operator topology of the Hilbert Space Conservative? While the Hilbert Space Conservative restricts attention to a privileged collection of states from the larger collection of states on the abstract algebra, the Universalist considers all states on the abstract algebra to be physically possible because they can all be implemented as vectors, and hence finite rank density operators, on the universal Hilbert space. In this sense, the Universalist is just like the Imperialist, so we expect their notions of approximation with respect to the empirical content of the theory to match up. As the following proposition shows, the notion of convergence of the weak operator topology on the universal representation corresponds exactly to the notion of convergence of the weak topology on the abstract algebra (or weak* topology on its bidual).

Proposition 5.3.4. *Let $(\pi_U, \mathcal{H}_U)$ be the universal representation of $\mathfrak{A}$. Let $\{A_i\} \subseteq \pi_U(\mathfrak{A})$ be a net of operators. Then the following are equivalent:*

(1) The net $\{A_i\}$ converges to $A \in \mathcal{B}(\mathcal{H}_U)$ in the weak operator topology on $\mathcal{B}(\mathcal{H}_U)$

(2) For all states ϕ on $\mathfrak{A}$, $\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$, where Φ is any finite rank density operator (and there is always at least one) implementing the state ϕ on $\pi_U(\mathfrak{A})$.

Just as for Prop. 5.3.1, in clause (2) the expectation values converge as complex numbers even though there is no element of the abstract algebra that the net is converging to. The collection of bounded operators on the universal Hilbert space provides the ambient space of observables in which to think about these limit points or idealizations. This allows us to construct the universal enveloping von Neumann algebra, which contains the idealizations of our original observables with respect to a notion of idealization that is physically relevant by the lights of the Universalist.

We also note that Prop. 5.3.4 shows a sense in which the universal representation is privileged if one wants to think algebraically—its weak operator topology reproduces the precise condition of convergence of the weak topology on the abstract algebra $\mathfrak{A}$. Restricting attention to finite rank density operator states on the universal representation amounts to no restriction at all because *every state is implementable as a finite rank density operator on the universal representation.*

We already knew that the Universalist could acquire more observables by using the universal enveloping von Neumann algebra, but as in the previous section, we still need to ask whether the Universalist acquires the "right" ones. Does the Universalist have access to parochial observables like net magnetization and particle number? The following proposition shows that the Universalist has access to every parochial observable that every possible (for each distinct privileged representation) Hilbert Space Conservative has access to (compare with Prop. 5.3.3).

Proposition 5.3.5. *If π is a representation of a C*-algebra $\mathfrak{A}$ and π_U is the universal representation of $\mathfrak{A}$, then there is a central projection P in $\overline{\pi_U(\mathfrak{A})}$ and a *-isomorphism α from the von Neumann algebra $\overline{\pi_U(\mathfrak{A})P}$ onto $\overline{\pi(\mathfrak{A})}$ such that $\pi(A) = \alpha(\pi_U(A)P)$ for all $A \in \mathfrak{A}$.*[14]

The von Neumann algebra affiliated with any representation is canonically isomorphic to a subalgebra of the universal enveloping von Neumann algebra. This shows that every parochial observable can be thought of as an element of the universal enveloping von Neumann algebra and so can be thought of as an idealization from observables in the universal representation with respect to a notion of idealization that is physically relevant by the lights of the Universalist. Just as in the previous section, the Universalist gains access to all of the observables—including net magnetization and particle number—that we need to recover the physically significant explanations that Ruetsche is worried about.

Ruetsche, however, presents a number of objections to the claim that the universal representation contains all parochial observables. I will consider what I take to be two of the most prominent objections here and argue that they fail. Seeing why they fail illustrates how the universal representation gives us access to all parochial observables.

First, Ruetsche asserts (2011, p. 145, fn. 10) that in the universal representation we would expect

[14] See Kadison and Ringrose (1997, p. 719, Thm. 10.1.12) for a proof.

an observable $\pi_\omega(A)$ from the GNS representation for ω to be implemented in the universal representation by a "portmanteau" operator of the form $\pi_\omega(A) \oplus_{\phi \neq \omega} 0$, and similarly for the parochial observables from the representation π_ω which will be weak operator limits of these "portmanteau" operators. But, Ruetsche correctly argues that these operators may not be contained in the universal enveloping von Neumann algebra because it follows from Thm. 10.3.5 of Kadison and Ringrose (1997, p. 738) that $\overline{\pi_U(\mathfrak{A})} \neq \bigoplus_{\omega \in \mathcal{S}_{\mathfrak{A}}} \overline{\pi_\omega(\mathfrak{A})}$.

This objection, however, fades in light of Prop. 5.3.5. The "portmanteau" operators are obviously intended to capture the observables from the GNS representation for ω in the universal representation. Prop. 5.3.5 explicitly asserts that we can think of any observable $\pi_\omega(A)$ in the GNS representation for ω as the observable $\pi_U(A)P$ in the universal representation (and similarly for parochial observables), where P is a central projection in the universal enveloping von Neumann algebra. Because of the presence of the projection P, such an observable acts nontrivially only on some subspace of the universal Hilbert space $\mathcal{H}_U$ and acts like the zero operator everywhere else. This means that the operator $\pi_U(A)P$ has some properties analogous to the operator $\pi_\omega(A) \oplus_{\phi \neq \omega} 0$. But since operators of the latter form are not required to belong to the universal enveloping von Neumann algebra, it follows that P may not be the projection onto the GNS representation for ω. I will show explicitly that P is not the projection onto the GNS representation for ω and that we should not be surprised by this fact.[15]

Intuitively, the observable $\pi_U(A)P$ ought to act like $\pi_\omega(A)$ on all of the vector states that $\pi_\omega(A)$ acts on. But in general there will be many vectors in the universal Hilbert space $\mathcal{H}_U$ that implement any vector state ψ corresponding to the vector $\Psi \in \mathcal{H}_\omega$, which is the sort of vector that $\pi_\omega(A)$ acts on. For example, the vector $\Psi \oplus_{\phi \neq \omega} 0$ will implement the state ψ in the universal representation. But so will the vector $\Omega_\psi \oplus_{\phi \neq \psi} 0$, where Ω_ψ is the cyclic vector implementing ψ in the GNS representation for ψ. In fact, there will be a vector implementing ψ in each direct summand

[15] See Thm. 6.8.8 and Thm. 10.1.12 of Kadison and Ringrose (1997, p. 443 and p. 719) for the general construction of P. It is easy to show that P is the projection whose range is the class of all subrepresentations of π_U quasiequivalent to π_ω, but this is beyond the scope of this paper.

of the universal representation that corresponds to a GNS representation unitarily equivalent to π_ω. And furthermore, if ϕ is any mixture having nonzero component on the state ω, then the GNS representation of ϕ will be a direct sum containing the GNS representation of ω as one of its summands. It follows by similar considerations that there will be some vector in the GNS representation of ϕ (and hence in another summand of the universal representation) that implements the state ψ. The following proposition shows that the range of the projection P must contain *all* of these vectors implementing the state ψ.

Proposition 5.3.6. *Let $(\pi, \mathcal{H})$ be a representation of a C*-algebra $\mathfrak{A}$, let $(\pi_U, \mathcal{H}_U)$ be the universal representation of $\mathfrak{A}$, and let P be the corresponding central projection in Prop. 5.3.5. Choose an arbitrary unit vector $\Psi \in \mathcal{H}$. Then for any vector $\Phi \in \mathcal{H}_U$ implementing the vector state Ψ in the sense that*

$$\langle \Phi, \pi_U(A)\Phi \rangle = \langle \Psi, \pi(A)\Psi \rangle$$

for all $A \in \mathfrak{A}$, it follows that $P\Phi = \Phi$.

This shows that the operators in the universal representation that correspond to operators from some particular GNS representation must take a more complicated form than the "portmanteau" operators Ruetsche suggests. In the universal representation, the correct operator must act on a whole host of vectors implementing the states from the GNS representation that we started with, and many of these vectors will lie elsewhere in the universal representation, outside of the GNS representation we began with. However, it is most important to recognize that even though the elements of the universal representation do not take the simple form we might have expected, Prop. 5.3.5 guarantees us that there is always some operator in the universal enveloping von Neumann algebra corresponding to the observable we are interested in.

Ruetsche's second objection (2011, p. 283) claims that parochial observables (in particular, phase observables like the net magnetization of a ferromagnet (Ruetsche, 2006, p. 478) may have a domain that is a proper subset of $\mathcal{S}_{\mathfrak{A}}$ and that no observable qua bounded operator on the entire universal Hilbert space can have this property. But we have already seen that parochial observables, by virtue of Prop. 5.3.5, can be thought of as acting on the subspace of $\mathcal{H}_U$ that is the range of the projection P given in that proposition. Parochial observables act like the zero operator everywhere else in $\mathcal{H}_U$ and so we can think of the orthogonal complement of the range of P as the subspace generated by the collection of states that are outside the domain of that parochial observable. Hence, we have a way of making sense in the universal representation of Ruetsche's claim that the domain of parochial observables may be smaller than $\mathcal{S}_{\mathfrak{A}}$.

It may be helpful to consider a particular example. (Ruetsche, 2011, p. 226) considers and rejects the universal representation as a way of accounting for inequivalent particle notions in quantum field theory.[16,17] The universal representation gives rise to a total number operator representing the number of particles of any variety as an observable in the universal enveloping von Neumann algebra.[18] Ruetsche argues that this total number operator does not suffice for doing physics because it identifies states with the same total number of quanta spread among the different varieties, and of course a state with n quanta of one variety is different from a state with n quanta of a different variety. For example, the total number operator cannot distinguish the state with one Minkowski quantum from the state with one Rindler quantum because it tells us that both states simply have one total quantum.[19] But Prop. 5.3.5 tells us that the universal representation will contain (in addition to the total number operator for all varieties) the Minkowski quanta total number operator and the Rindler quanta total number operator, which are parochial observables to particular GNS representations of the abstract algebra. The Minkowski quanta total number operator and the

[16] I am taking Ruetsche's claims out of context here. Really, she rejects the universalized particle notion as a "fundamental" particle notion (see Ruetsche, 2011, Ch. 9 for more detail.). However, considering her remarks as an objection to the view outlined here is illustrative.

[17] See also Clifton and Halvorson (2001) for more on inequivalent particle notions.

[18] Because the number operator is unbounded, it cannot strictly speaking belong to the universal enveloping von Neumann algebra. Instead, the number operator is affiliated with the algebra in the sense that any bounded function of it belongs to the algebra. We ignore this complication in what follows.

[19] Really, Ruetsche makes this claim in the context of the reduced atomic representation (see Kadison and Ringrose, 1997, pp. 740-741). But her objection may be carried over to the universal representation and my response may be carried back to the reduced atomic representation as well because Thm. 10.3.10 of Kadison and Ringrose (1997, p. 741) shows that the von Neumann algebra affiliated with the reduced atomic representation contains all of the parochial observables for all irreducible representations just as the universal enveloping von Neumann algebra contains the parochial observables associated with all (not just irreducible) representations.

Rindler quanta total number operator will have different expectation values in the state with one Minkwoski quantum and the state with one Rindler quantum. Hence, they will distinguish between these two distinct states. The universal representation, by virtue of containing all of the parochial observables, gives us the ability to make as many distinctions between states as we might like.

I hope that these remarks concerning Ruetsche's objections suffice to show that all parochial observables really are contained in the universal representation, or more specifically the universal enveloping von Neumann algebra. Now I want to remark upon the fact that the technical results I have presented for the Imperialist's and Universalist's solution to the problem of parochial observables appear so similar—this is no coincidence. We have already mentioned that the universal representation (or really universal enveloping von Neumann algebra) and the abstract algebra (or really its bidual) share the same topological structure in the sense that the notion of convergence provided by the weak topology on $\mathfrak{A}$ (or weak* topology on $\mathfrak{A}^{**}$) reproduces precisely the notion of convergence provided by the weak operator topology on $\mathcal{B}(\mathcal{H}_U)$. But these objects share much more structure than that. The following proposition shows that the bidual of a C*-algebra in a certain sense carries the same algebraic structure as the universal enveloping von Neumann algebra.

Proposition 5.3.7. *There is a *-isomorphism*[20] α *from the bidual* $\mathfrak{A}^{**}$ *of a C*-algebra* $\mathfrak{A}$ *to its universal enveloping von Neumann algebra* $\overline{\pi_U(\mathfrak{A})}$ *such that* $\pi_U(A) = \alpha(J(A))$ *for all* $A \in \mathfrak{A}$.[21]

Since the Imperialist and the Universalist invoke the same algebraic and topological structure to represent quantum systems, they end up believing in the same physically significant observables and the same physically possible states while using the same notion of approximation or idealization. This shows a sense in which Algebraic Imperialism and Universalism amount to the same position. Of course I do not claim that Imperialism and Universalism are equivalent with respect to

[20] Prop. 5.3.4 shows that this *-isomorphism is a W*-isomorphism in the sense of Sakai (1971, p. 40), i.e. it is also a homeomorphism in the weak* and weak operator topologies, respectively. This is a relevant notion of isomorphism because both $\mathfrak{A}^{**}$ and $\overline{\pi_U(\mathfrak{A})}$ are W*-algebras (the abstract version of von Neumann algebras, see Ch. 6).

[21] Adapted from Kadison and Ringrose (1997, p. 726, Prop. 10.1.21) and Emch (1972, pp. 121-2, Thm. 11). See those sources for a proof and see Lupher (2008, p. 95) for more discussion.

every purpose we might put quantum theory to, but at least they are equivalent with respect to the interpretive uses just outlined. One may have pragmatic reasons for choosing one or the other—for example, one might want to use the universal representation if one is familiar with interpreting Hilbert spaces from ordinary quantum mechanics. Nevertheless, whatever quantum states and observables the Imperialist can represent, the Universalist can represent too (and vice versa). So the Imperialist and the Universalist, when allowed the same tools as the Hilbert Space Conservative for representing idealizations, come up with the *same* solution to the problem of parochial observables.

5.4 Conclusion

I have argued that the Algebraic Imperialist and the Universalist both have solutions to Ruetsche's problem of parochial observables[22] and that these are in a certain sense equivalent interpretations of quantum theories. Prop. 5.3.3 shows that every parochial observable from every possible representation can be thought of as an element of the bidual $\mathfrak{A}^{**}$ and Prop. 5.3.2 shows that each of these elements of $\mathfrak{A}$ can be acquired as a limit (in the weak* topology) of some net of observables the Algebraic Imperialist already recognized. Similarly, Prop. 5.3.5 shows that the universal enveloping von Neumann algebra $\overline{\pi_U(\mathfrak{A})}$ contains every parochial observable from every possible representation, and since all elements of $\overline{\pi_U(\mathfrak{A})}$ can be constructed as the limit of some net of observables in $\pi_U(\mathfrak{A})$ in the weak operator topology of the universal representation, the Universalist also acquires all parochial observables. Furthermore, Prop. 5.3.7 shows that these extended spaces of observables—$\mathfrak{A}^{**}$ for the Imperialist and $\overline{\pi_U(\mathfrak{A})}$ for the Universalist—are isomorphic by the relevant notion of isomorphism, and so they specify the same physically measurable quantities for the system. This shows that Algebraic Imperialism and Universalism are equivalent interpretations,

[22]Note that although Algebraic Imperialism and Universalism are what Ruetsche (2011) calls pristine interpretations, the arguments of this paper do not vindicate the ideal of pristine interpretation, which is Ruetsche's main target. The reason is that none of my maneuvers for saving Imperialism and Universalism depend on that ideal. While my arguments here make Imperialism and Universalism viable interpretations, I believe the considerations of approximation and idealization in this paper show that thinking of these as pristine interpretations is misleading at best. I will provide some further remarks concerning this issue in section 7.1.

and both of them give us the tools to represent all of the parochial observables.

Since the problem of parochial observables is Ruetsche's main reason for rejecting Algebraic Imperialism and Universalism, and since we have seen that both of these positions can adequately solve this problem, this leaves the path open for Imperialism and Universalism as viable interpretations of quantum theories with infinitely many degrees of freedom.

Appendix A: Proofs of Propositions

This appendix contains the proofs of propositions from the body of the paper. Props. 5.3.1 and 5.3.4 follow immediately from the following lemma, which is a restatement of facts described in Reed and Simon (1980, p. 213):

Lemma 5.4.1. *Let $\mathcal{H}$ be a Hilbert space and let $\{A_i\} \subseteq \mathcal{B}(\mathcal{H})$ be a net of operators. Then the following are equivalent:*

(1) The net $\{A_i\}$ converges to $A \in \mathcal{B}(\mathcal{H})$ in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

(2) For all finite rank density operators ρ on $\mathcal{H}$, $Tr(\rho A_i)$ converges to $Tr(\rho A)$ in $\mathbb{C}$.

Proposition 5.3.1. *Let $(\pi, \mathcal{H})$ be a representation of a C*-algebra $\mathfrak{A}$. Let $\{A_i\} \subseteq \pi(\mathfrak{A})$ be a net of operators. Then the following are equivalent:*

(1) The net $\{A_i\}$ converges to $A \in \mathcal{B}(\mathcal{H})$ in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

(2) For all states ϕ on $\mathfrak{A}$ implementable by a finite rank density operator Φ on $\mathcal{H}$, $\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$.

Proof. $(1 \Rightarrow 2)$ This follows immediately by Lemma 5.4.1.

$(2 \Rightarrow 1)$ Suppose that for all states ϕ on $\mathfrak{A}$ implementable by a finite rank density operator Φ on $\mathcal{H}$,

$\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$. Every finite rank density operator ρ on $\mathcal{H}$ defines a state on $\mathfrak{A}$ implementable by a finite rank density operator. So it follows from the assumption that $Tr(\rho A_i)$ converges to $Tr(\rho A)$ for all finite rank density operators and hence A_i converges to A by Lemma 5.4.1. □

Proposition 5.3.4. *Let $(\pi_U, \mathcal{H}_U)$ be the universal representation of $\mathfrak{A}$. Let $\{A_i\} \subseteq \pi_U(\mathfrak{A})$ be a net of operators. Then the following are equivalent:*

(1) The net $\{A_i\}$ converges to $A \in \mathcal{B}(\mathcal{H}_U)$ in the weak operator topology on $\mathcal{B}(\mathcal{H}_U)$

(2) For all states ϕ on $\mathfrak{A}$, $\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$, where Φ is any finite rank density operator (and there is always at least one) implementing the state ϕ on $\pi_U(\mathfrak{A})$.

Proof. $(1 \Rightarrow 2)$ Suppose $A_i \to A$ in the weak operator topology on $\mathcal{B}(\mathcal{H}_U)$. Let Φ be any finite rank density operator (there is always at least one) implementing a state ϕ on $\pi_U(\mathfrak{A})$. Then by Prop. 5.3.1, $\phi(A_i) = Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$.

$(2 \Rightarrow 1)$ Suppose that for all states ϕ on $\mathfrak{A}$, $Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$, where Φ is a finite rank density operator implementing ϕ. Let Φ be any finite rank density operator on $\mathcal{H}$. Then $X \mapsto Tr(\Phi X)$ for all $X \in \pi_U(\mathfrak{A})$ defines a state on $\pi_U(\mathfrak{A})$ and hence $Tr(\Phi A_i)$ converges to $Tr(\Phi A)$ in $\mathbb{C}$, which shows by Prop. 5.3.1 that A_i converges in the weak operator topology to A. □

Prop. 5.3.2 is an immediate corollary of the following elementary lemma about Banach spaces.

Lemma 5.4.2. *Let X be a Banach space, X^{**} its bidual, and $J : X \to X^{**}$ the canonical evaluation map. Then $J(X)$ is dense in X^{**} in the weak* ($\sigma(X^{**}, X^*)$) topology.*

Proof. Let $\tilde{x} \in X^{**}$ and let U be an open neighborhood of $\tilde{x}$. We will show that U contains some element $y = J(x) \in J(X)$.

By the definition of the $\sigma(X^{**},X^*)$ topology, there are linear functionals $l_1,...,l_n \in X^*$ and $\varepsilon_1,...,\varepsilon_n > 0$ such that

$$\bigcap_{i=1}^{n}(\tilde{x}+N(l_i,\varepsilon_i)) \subseteq U$$

where $\tilde{x}+N(l_i,\varepsilon_i) = \{\tilde{y} \in X^{**} : |\tilde{y}(l_i) - \tilde{x}(l_i)| < \varepsilon_i\}$.

Now let $N'(l_i,\varepsilon_i) = \{x \in X : |l_i(x) - \tilde{x}(l_i)| < \varepsilon_i\}$. We will show that

$$\bigcap_{i=1}^{n} N'(l_i,\varepsilon_i) \subseteq \bigcap_{i=1}^{n}(\tilde{x}+N(l_i,\varepsilon_i)) \cap J(X)$$

is non-empty, or in other words we will show that there is a $y = J(x) \in J(X)$ such that for all $1 \leq i \leq n$,

$$|y(l_i) - \tilde{x}(l_i)| = |l_i(x) - \tilde{x}(l_i)| < \varepsilon_i \tag{5.1}$$

It suffices to consider only a linearly independent subset of the linear functionals $l_1,...,l_n \in X^*$ forming a basis for the subspace of X^* spanned by these functionals: if $x \in X$ satisfies the above inequality for this linearly independent subset of $l_1,...,l_n \in X^*$, then it must also satisfy the inequalities for the rest of the linear functionals, or else the inequalities would contradict each other and then $\bigcap_{i=1}^{n}(\tilde{x}+N(l_i,\varepsilon_i))$ would have to be empty, which it is not because it contains $\tilde{x}$.

Choose this linearly independent set of functionals $l_{k_1},...,l_{k_m}$. We know (Schechter, 2001, p. 93, Lemma 4.14) that there exists a dual basis $e_1,...,e_m \in X$ for a subspace of X such that $l_{k_i}(e_j) = \delta_{ij}$ for all $1 \leq i,j \leq m$. Consider the vector $x = \sum_{i=1}^{m} \tilde{x}(l_{k_i})e_i$. We have for all $1 \leq j \leq m$,

$$l_j(x) = \sum_{i=1}^{m} \tilde{x}(l_{k_i}) l_j(e_i) = \tilde{x}(l_{k_j})$$

Hence, x satisfies the above inequalities in eq. (5.1) and it follows that

$$y = J(x) \in \bigcap_{i=1}^{n} N'(l_i,\varepsilon_i) \subseteq \bigcap_{i=1}^{n}(\tilde{x}+N(l_i,\varepsilon_i)) \subseteq U$$

Therefore, since $U \cap J(X) \neq \varnothing$, $J(X)$ is dense in X^{**}. $\square$

Proposition 5.3.2. *Let $\mathfrak{A}$ be a C*-algebra, $\mathfrak{A}^{**}$ be its bidual, and $J : \mathfrak{A} \to \mathfrak{A}^{**}$ be the canonical evaluation map. Then $J(\mathfrak{A})$ is dense in $\mathfrak{A}^{**}$ in the weak* ($\sigma(\mathfrak{A}^{**}, \mathfrak{A}^{*})$) topology.*

Proof. Since every C*-algebra $\mathfrak{A}$ is a Banach space, this follows immediately from Lemma 5.4.2. $\square$

Prop. 5.3.3 is an immediate corollary of Prop. 5.3.5 and Prop. 5.3.7, whose proofs are contained in (Kadison and Ringrose, 1997, p. 719, Thm.10.1.12 and p. 726, Prop. 10.1.21, respectively).

Proposition 5.3.3. *If π is a representation of a C*-algebra $\mathfrak{A}$, then there is a central projection $P \in \mathfrak{A}^{**}$ and a *-isomorphism α from $\mathfrak{A}^{**}P$ onto $\overline{\pi(\mathfrak{A})}$ such that $\pi(A) = \alpha(J(A)P)$ for all $A \in \mathfrak{A}$.*

Proof. By Prop. 5.3.5, there is a projection $\tilde{P}$ in $\overline{\pi_U(\mathfrak{A})}$ in the center of $\overline{\pi_U(\mathfrak{A})}$ and a *-isomorphism α_1 from $\overline{\pi_U(\mathfrak{A})}\tilde{P}$ to $\pi(\mathfrak{A})$ such that $\pi(A) = \alpha_1(\pi_U(A)\tilde{P})$ for all $A \in \mathfrak{A}$. By Prop. 5.3.7, there is a *-isomorphism α_2 from the bidual $\mathfrak{A}^{**}$ to $\overline{\pi_U(\mathfrak{A})}$ such that $\pi_U(A) = \alpha_2(J(A))$ for all $A \in \mathfrak{A}$. The projection $P = \alpha_2^{-1}(\tilde{P})$ and the *-isomorphism $\alpha = \alpha_1 \circ (\alpha_2)_{|\mathfrak{A}^{**}P}$ serve as a witness to the current theorem, because for any $A \in \mathfrak{A}$,

$$\alpha(J(A)P) = \alpha_1 \circ \alpha_2(J(A)P) = \alpha_1[\alpha_2(J(A)) \cdot \alpha_2(P)] = \alpha_1(\pi_U(A)\tilde{P}) = \pi(A)$$

$\square$

Proposition 5.3.6. *Let $(\pi, \mathcal{H})$ be a representation of a C*-algebra $\mathfrak{A}$ and let $(\pi_U, \mathcal{H}_U)$ be the universal representation of $\mathfrak{A}$. Let P be the central projection in Prop. 5.3.5, and choose some unit vector $\Psi \in \mathcal{H}$. Then for any vector $\Phi \in \mathcal{H}_U$ implementing the vector state Ψ in the sense that*

$$\langle \Phi, \pi_U(A)\Phi \rangle = \langle \Psi, \pi(A)\Psi \rangle$$

for all $A \in \mathfrak{A}$, it follows that $P\Phi = \Phi$.

Proof. By construction (See Kadison and Ringrose, 1997, p. 443, Thm. 6.8.8 and p. 719, Thm. 10.1.12) , the projection P takes the form

$$P = I - E$$
$$E = \bigvee_{T \in Ker(\overline{\beta})} R(T^*)$$

where $\overline{\beta}$ is the ultraweakly continuous extension of the map $\beta = \pi \circ \pi_U^{-1}$ and $R(T^*)$ is the projection onto the range of T^*. With Φ and Ψ as above,

$$P\Phi = \Phi - E\Phi$$

It suffices to show that $\langle \Phi, E\Phi \rangle = 0$, which shows that the second term above is zero.

We know that $E = \bigvee_{T \in Ker(\overline{\beta})} R(T^*) \in Ker(\overline{\beta})$, so it follows that

$$\langle \Psi, \overline{\beta}(E)\Psi \rangle = 0$$

Choose a net $A_i \in \mathfrak{A}$ such that $\pi_U(A_i)$ converges in the weak operator topology on $\mathcal{B}(\mathcal{H}_U)$ to E. It follows immediately that $\pi(A_i) = \overline{\beta}(\pi_U(A_i))$ converges in the weak operator topology on $\mathcal{B}(\mathcal{H})$ to $\overline{\beta}(E)$, so

$$\begin{aligned}
\langle \Phi, E\Phi \rangle &= \langle \Phi, w\text{-}\lim(\pi_U(A_i))\Phi \rangle \\
&= \lim \langle \Phi, \pi_U(A_i)\Phi \rangle \\
&= \lim \langle \Psi, \pi(A_i)\Psi \rangle \\
&= \langle \Psi, w\text{-}\lim(\pi(A_i))\Psi \rangle \\
&= \langle \Psi, \overline{\beta}(E)\Psi \rangle = 0
\end{aligned}$$

where w-lim denotes the weak operator limit in the relevant Hilbert space. □

Appendix B: An Illustration in Classical Systems

Suppose that the system under consideration is classical so that $\mathfrak{A}$ is abelian. Examining the weak topology and weak operator topologies of representations of this algebra illustrates the concepts of section 5.3 in a somewhat more familiar and concrete setting (although admittedly not the most familiar or concrete!).[23] This section aims at an audience who is familiar with the different notions of convergence on spaces of functions; I show that the topologies on C*-algebras and their representations are simply extensions of these familiar notions of convergence to the noncommutative setting. Recall that when $\mathfrak{A}$ is abelian, it is *-isomorphic to $C(\mathcal{P}(\mathfrak{A}))$, the continuous functions on the compact Hausdorff space $\mathcal{P}(\mathfrak{A})$ of pure states of $\mathfrak{A}$ with the weak* ($\sigma(\mathcal{P}(\mathfrak{A}), \mathfrak{A})$) topology (Kadison and Ringrose, 1997, p. 270, Thm. 4.4.3). As such, each observable $A \in \mathfrak{A}$ corresponds to a function $\hat{A} \in C(\mathcal{P}(\mathfrak{A}))$ defined by

$$\hat{A}(\omega) = \omega(A)$$

for each pure state $\omega \in \mathcal{P}(\mathfrak{A})$.

Taking a representation of $\mathfrak{A}$ amounts to choosing a measure on the space $\mathcal{P}(\mathfrak{A})$ (See Kadison and Ringrose 1997, p. 744; Landsman 1998a, p. 55), which defines an (L^2) inner product, hence constructing a Hilbert space as follows. By the Riesz-Markov theorem (Reed and Simon, 1980, p. 107, Thm. IV.14), each state ω on $\mathfrak{A}$ corresponds to a unique regular Borel measure μ_ω on $\mathcal{P}(\mathfrak{A})$ such that for all $A \in \mathfrak{A}$

$$\omega(A) = \int_{\mathcal{P}(\mathfrak{A})} \hat{A}\, d\mu_\omega$$

The GNS representation of $\mathfrak{A}$ for the state ω is unitarily equivalent to the representation[24] $(\pi_\omega, \mathcal{H}_\omega)$

[23] Chs. 3 and 4 and Feintzeig (2015, 2016b) similarly use the classical case to gain insight about interpreting the algebraic formalism. This section can be understood as adding to that project.

[24] Here, the relevant cyclic vector Ω_ω is the constant unit function.

on the Hilbert space $\mathcal{H}_\omega = L^2(\mathcal{P}(\mathfrak{A}), d\mu_\omega)$, with π_ω defined by

$$\pi_\omega : A \mapsto M_{\hat{A}}$$

where the operator $M_{\hat{A}}$ is defined as multiplication by the function $\hat{A}$, i.e. for any $\psi \in \mathcal{H}_\omega$,

$$M_{\hat{A}}\psi = \hat{A} \cdot \psi$$

Now, we can pull the discussion of topologies on $\mathfrak{A}$ back to the more familiar topologies on ordinary functions on $\mathcal{P}(\mathfrak{A})$. We recall these topologies now before proceeding. Let f_n be a net of functions on $\mathcal{P}(\mathfrak{A})$. We say that f_n converges to the function f *uniformly* if $\sup_{\psi \in \mathcal{P}(\mathfrak{A})} |f_n(\psi) - f(\psi)|$ converges to zero in $\mathbb{C}$. We say that f_n converges to the function f *pointwise* if $f_n(\psi)$ converges to $f(\psi)$ in $\mathbb{C}$ for each $\psi \in \mathcal{P}(\mathfrak{A})$. And finally, we say that f_n converges to the function f *pointwise almost everywhere* with respect to a measure μ on $\mathcal{P}(\mathfrak{A})$ if $f_n(\psi)$ converges to $f(\psi)$ in $\mathbb{C}$ for all $\psi \in \mathcal{P}(\mathfrak{A})$, except possibly on a set of measure zero with respect to μ. Now it is easy to see that the norm topology on $\mathfrak{A}$ is the topology of uniform convergence, and it is similarly easy to show that for the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$, the weak operator topology on $\mathcal{B}(\mathcal{H}_\omega)$ is the topology of pointwise convergence almost everywhere with respect to the measure μ_ω. This implies that while $\pi_\omega(\mathfrak{A})$ is the collection of multiplication operators by *continuous functions* (which is uniformly closed), its weak operator closure $\pi_\omega(\mathfrak{A})$ will be the collection of multiplication operators by *essentially bounded measurable functions* with respect to the measure μ_ω (i.e., $L^\infty(\mathcal{P}(\mathfrak{A}), d\mu_\omega)$).

In some cases, taking the weak operator closure (and hence, moving to the essentially bounded measurable functions) does not give rise to any new parochial observables. When ω is pure,

$$\omega(A) = \hat{A}(\omega) = \int_{\mathcal{P}(\mathfrak{A})} \hat{A}\delta(\omega)$$

for all $A \in \mathfrak{A}$, where $\delta(\omega)$ is the point mass or delta function centered on ω. It follows by the uniqueness clause of the Riesz-Markov theorem that $d\mu_\omega = \delta(\omega)$. So every vector $\psi \in \mathcal{H}_\omega$ will be defined by a single complex number—the value of ψ on $\omega \in \mathcal{P}(\mathfrak{A})$ and $\mathcal{H}_\omega$ will be one-dimensional. Hence, $\mathcal{B}(\mathcal{H}_\omega)$ will be one-dimensional and since $\pi_\omega(\mathfrak{A})$ contains the identity and is closed under scalar multiplication, it follows that $\overline{\pi_\omega(\mathfrak{A})} = \mathcal{B}(\mathcal{H}_\omega) = \pi_\omega(\mathfrak{A})$. This means that there are no parochial observables in this representation. Furthermore, even the GNS representations for many mixed states do not give rise to new parochial observables. Consider an arbitrary state ω on $\mathfrak{A}$ such that μ_ω has "support" on only a countable subset of $\mathcal{P}(\mathfrak{A})$.[25] In such a special state, since the measure focuses our attention on only a countable subset of $\mathcal{P}(\mathfrak{A})$, the continuous functions coincide with the essentially bounded measurable functions—every discontinuous but essentially bounded measurable function is equivalent to a continuous function when we ignore differences on sets of measure zero. In other words, focusing only on a countable subset of $\mathcal{P}(\mathfrak{A})$ does not allow one to distinguish between continuous and merely bounded functions. So it similarly follows that $\overline{\pi_\omega(\mathfrak{A})} = \pi_\omega(\mathfrak{A})$ and there are no parochial observables in this representation.

However, we can also have an arbitrary mixed state ω on $\mathfrak{A}$ such that μ_ω has "support" on an uncountable subset of $\mathcal{P}(\mathfrak{A})$.[26] In this case, we may acquire new parochial observables. The weak operator closure will include even discontinuous functions like characteristic functions (projection operators), where the original algebra did not. Since each one of these essentially bounded measurable functions is the weak operator limit point of a collection of continuous functions, we can understand them as idealizations from or approximations to collections of our original observables. So the essentially bounded (but discontinuous) measurable functions with respect to some measure are the parochial observables for an algebra of continuous functions in the representation defined by that measure.

But, comes the obvious retort, in a similar sense *every bounded Borel measurable function* (without considering any particular measure) can be considered as an idealization from or approximation to

[25] Although the support of a measure is not defined in general, all I mean by the support of μ_ω being a countable subset is that there exists some countable subset of $\mathcal{P}(\mathfrak{A})$ to which μ_ω assigns measure 1.

[26] As in footnote 25, I take μ_ω having support on an uncountable subset just to mean that there is no countable subset of $\mathcal{P}(\mathfrak{A})$ to which μ_ω assigns measure 1.

a collection of our original observables without appeal to any measure, and hence without appeal to any representation. The sense in which this is true uses the weak topology on $\mathfrak{A}$. In particular, every bounded (Borel) function is the pointwise limit of a collection of continuous functions. The topology of pointwise convergence for functions is just the weak topology on $\mathfrak{A}$ (really, extended to the weak* topology on $\mathfrak{A}^{**}$), so $\mathfrak{A}^{**}$ is just the collection of bounded (Borel) functions on $\mathcal{P}(\mathfrak{A})$. Since every parochial observable qua essentially bounded function is equivalent to a bounded (Borel) function (ignoring differences on sets of measure zero), it follows that every parochial observable can be thought of as the weak (pointwise) limit of observables in the abstract algebra. This provides our algebraic route to all of the parochial observables at once, without reference to any representation.

On the other hand, we can gather all the parochial observables in a single representation as before by taking the universal representation (see Kadison and Ringrose, 1997, p. 746). The universal representation acts on the Hilbert space

$$\mathcal{H}_U = \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} L^2(\mathcal{P}(\mathfrak{A}), d\mu_\omega)$$

by the representation

$$\pi_U(A) = \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} M_{\hat{A}}$$

where $M_{\hat{A}}$ is the multiplication operator by $\hat{A}$. Notice that each summand only multiplies by $\hat{A}$ *almost everywhere with respect to some measure* μ_ω. But the full operator $\pi_U(A)$ in a certain sense amounts to the multiplication operator by $\hat{A}$ *everywhere* because we have taken the direct sum over spaces with all possible regular normalized Borel measures and so for each point $\omega \in \mathcal{P}(\mathfrak{A})$ there is some summand of π_U in which $\{\omega\}$ gets assigned nonzero measure. This is why π_U gives a faithful representation whereas the individual GNS representations need not be faithful (because functions that disagree only on a set of measure zero are mapped to the same multiplication operator by that GNS representation). Weak operator convergence in the universal representation is just pointwise convergence everywhere again because each point gets assigned nonzero measure by at least one of the regular normalized Borel measures defining the Hilbert space summands. Thus the weak operator closure in the universal representation gives us back all of the bounded (Borel) functions as direct sums of essentially bounded measurable functions considered over all possible regular normalized Borel measures.

The upshot is that the abstract algebra and its universal representation give us two routes to acquiring all of the parochial observables, which in this case are the bounded (Borel) functions. One can stay with the abstract algebra and use the weak topology, which defines the topology of pointwise convergence. Or one can use the universal representation and its weak operator topology, which similarly defines the topology of pointwise convergence. Either way, we start with the continuous functions and construct certain discontinuous idealizations from them. I hope that this special example may take some of the mystery out of the parochial observables and where they come from.

Appendix C: Completion and the Bidual

The aim of this appendix is to show that the Algebraic Imperialist's move from a C*-algebra $\mathfrak{A}$ with its weak topology to its bidual $\mathfrak{A}^{**}$ is *canonical* in a certain sense. More specifically, this move corresponds to taking the completion of $\mathfrak{A}$ as a locally convex vector space with the weak topology (see Reed and Simon, 1980, Ch. V, p. 125). Completion is a procedure that all functional analysts and interpreters of quantum mechanics are familiar with: we use it all the time when we move from a pre-Hilbert space to a Hilbert space. The only difference in the present case is that the topology at issue is not as nice as the usual norm topology on a Hilbert space—the weak topology on $\mathfrak{A}$ is not even metrizable. Nevertheless, we shall show that we can think of the bidual $\mathfrak{A}^{**}$ in precisely the same way we think of a Hilbert space as the completion of a pre-Hilbert space.

A locally convex vector space X with topology generated by a family of semi-norms L is said to be *complete* if every Cauchy net converges to an element of the space (Reed and Simon, 1980, p. 125). Here a net $\{y_\beta\} \subseteq X$ is *Cauchy* just in case for all $\varepsilon > 0$ and all $l \in L$, there is a β_0 such that $\left|l(y_\beta) - l(y_\gamma)\right| < \varepsilon$ for all $\beta, \gamma \succ \beta_0$. A C*-algebra $\mathfrak{A}$ is obviously a locally convex vector space with the weak topology, which is generated by the family of semi-norms $L = \mathfrak{A}^*$. Similarly, the bidual $\mathfrak{A}^{**}$ is a locally convex vector space with the weak* topology, which is generated by the same family of semi-norms $L = \mathfrak{A}^*$. The following proposition shows that the bidual is always complete in this sense in the weak* topology.

Proposition 5.4.3. *Given any Banach space X, its dual X^* is complete in the weak* topology as a locally convex vector space.*

Proof. Suppose $\{y_\beta\}$ is a Cauchy net in X^*. Then, Define $y : X \to \mathbb{C}$ by

$$y(x) = \lim y_\beta(x)$$

for all $x \in X$. We know that this limit exists because for any $x \in X$, $y_\beta(x)$ is a Cauchy net in $\mathbb{C}$, which means it must converge because $\mathbb{C}$ is complete.

Now we must show that $y \in X^*$, i.e. that y is linear and bounded. The functional y is linear because for any $x, x' \in X$ and $\alpha \in \mathbb{C}$,

$$y(x + \alpha x') = \lim y_\beta(x + \alpha x') = \lim y_\beta(x) + \alpha \lim y_\beta(x') = y(x) + \alpha y(x')$$

Notice that because $\left|y_\beta(x)\right|$ is bounded for each $x \in X$, it follows from the principle of uniform boundedness (Reed and Simon, 1980, p. 81, Thm. III.9) that $\left\|y_\beta\right\|$ is bounded. Hence, y is bounded with norm $\|y\| \leq \sup_\beta \left\|y_\beta\right\|$

Finally, we must show that y_β converges to y in the weak* topology on X^*. But this holds by construction because for any $x \in X$, $(y - y_\beta)(x)$ converges to zero in $\mathbb{C}$. $\square$

Since the bidual $\mathfrak{A}^{**}$ of a C*-algebra $\mathfrak{A}$ is the dual of a Banach space, we have the following immediate corollary.

Corollary 5.4.4. *The bidual $\mathfrak{A}^{**}$ of any C*-algebra is complete in the weak* topology.*

Now we know that if we are given a C*-algebra $\mathfrak{A}$, then there always exists a completion of $\mathfrak{A}$ in the weak topology, i.e. a C*-algebra that is complete in the weak* topology and in which $\mathfrak{A}$ can be densely embedded isomorphically and homeomorphically (via the canonical evaluation map $J : \mathfrak{A} \to \mathfrak{A}^{**}$). But could we have completed $\mathfrak{A}$ in the weak* topology in some other way? The following proposition shows that the answer is no; the bidual $\mathfrak{A}^{**}$ is the unique completion of $\mathfrak{A}$ in the weak* topology (see also Takesaki, 1979, p. 121-2).

Proposition 5.4.5. *Suppose X is a Banach space and $J : X \to X^{**}$ is the canonical evaluation embedding of X in its bidual. Suppose we are given another faithful linear embedding $K : X \to Y$ of X in a complete locally convex vector space Y such that $K(X)$ is dense in Y in the locally convex vector space topology on Y and K is a homeomorphism from X to $K(X)$ in the weak topology on X and the subspace topology on $K(X)$ generated by the locally convex vector space topology on Y.[27] Then there is a vector space isomorphism $\varphi : Y \to X^{**}$ that is a homeomorphism in the locally convex vector space topology on Y and the weak* topology on X^{**} and such that $J = \varphi \circ K$.*

Proof. Suppose $K : X \to Y$ is such an embedding. By Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15), the maps $\varphi_0 = J \circ K^{-1}$ and $\psi_0 = K \circ J^{-1}$ extend uniquely to continuous linear maps $\varphi : Y \to X^{**}$ and $\psi : X^{**} \to Y$ in the weak* topology on X^{**} and the locally convex vector space topology on Y. Since $\varphi_0 \circ \psi_0$ and $\psi_0 \circ \varphi_0$ are the identity operators on $J(X)$ and $K(X)$, respectively, it follows that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity operators on X^{**} and Y, respectively. Thus, φ is an isomorphism and a homeomorphism in the weak* topology on X^{**} and the locally convex vector space topology on Y. By construction, we have $\varphi \circ K = \varphi_0 \circ K = J \circ K^{-1} \circ K = J$. □

[27] These constraints on J are generalizations of the condition that the analogous map preserve norm in the case of the completion of a normed vector space into a Banach space.

Again, since a C*-algebra $\mathfrak{A}$ is a Banach space, we have the following corollary, showing that the bidual is the unique choice of completion of $\mathfrak{A}$ in the weak topology.

Corollary 5.4.6. *Let $K : \mathfrak{A} \to Y$ be any faithful *-homomorphism from a C*-algebra $\mathfrak{A}$ to a C*-algebra Y that is a complete locally convex vector space satisfying the conditions of Prop. 5.4.5. Then there is a *-isomorphism $\varphi : Y \to \mathfrak{A}^{**}$ that is a homeomorphism in the locally convex vector space topology on Y and the weak* topology on $\mathfrak{A}^{**}$ such that $J = \varphi \circ K$.*

Thus, if the Algebraic Imperialist is justified in using the weak (or weak*) topology as a physical standard of idealization or approximation (perhaps because of the analogies with the weak operator topology defined in a representation outlined above), then the Algebraic Imperialist is certainly justified in using the bidual $\mathfrak{A}^{**}$ as her algebra of observables for absolutely familiar reasons.

Chapter 6

Continuity from Classical to Quantum

6.1 Introduction

The process of *quantization*—constructing a quantum theory from a classical theory for some system—is full of technical and conceptual problems. As we have seen, philosophers of physics have recently created a stir because there appear to be many inequivalent quantization procedures for theories of great physical interest, including quantum field theory and quantum statistical mechanics. And if there are many inequivalent quantization procedures, how are we supposed to know which to use to construct our quantum theories? The purpose of this chapter is to argue that careful attention to the mathematical tools of classical physics can help us narrow the playing field and choose from among the different quantization methods.

Both classical and quantum theories come equipped with topologies on their classes of physical observables. These topologies are used, for example, to represent how an observable can be approximated in a certain limit, with different topologies corresponding to different notions of approximation. One can constrain the construction of quantum theories by requiring that the topology on classical observables be preserved in a quantization procedure. Inequivalent quantizations appear in standard approaches to even the quantization of nonrelativistic particle systems, which only require that a particular topology—called the *norm topology*—is preserved. But there are other topologies on the algebras of observables that we can use—in particular the *weak topologies* of Ch. 5. I will argue that requiring preservation of the weak topologies[1] rules out one source of inequivalent quantizations.[2]

As we are by now familiar with, quantizing a classical theory typically involves two steps: first, one constructs an abstract algebra of observables, and second, one represents those observables as operators on a Hilbert space. In Chs. 3, 4, and 5, we have been concerned with the ambiguity inherent in the second step—in quantum field theory and quantum statistical mechanics, there are many inequivalent Hilbert space representations of the abstract algebra of observables. But even the first step of this procedure involves some ambiguity (see Ashtekar and Isham, 1992; Emch, 1997)—there are in fact many different abstract algebras one might choose to represent the observables of the theory, although this has played almost no role in the philosophical discussions. This chapter proposes a change in perspective from the standard philosophical literature (in light of the arguments of the previous chapters) by focusing on the construction of the abstract algebra rather than inequivalent Hilbert space representations. I claim that this perspective lends some insight to the physical significance of states in quantum theories and their relation to classical physics. One can take this chapter to relate to the discussion of the previous chapters by demonstrating the utility of algebraic methods and attention to algebraic tools in the construction of quantum theories.

Philosophers of physics consistently use only one algebra of observables in the first step of quantization—the *Weyl algebra* defined in Ch. 2. In this chapter, I will show that the Weyl algebra has a topology that is, in a certain sense, incompatible with the classical theory it derives from. Specifically, I will show that the weak topology on the algebra of observables of a quantum theory can be understood to have physical significance through its origins in the manifestly physically significant notion of

[1]As we will see later, I specifically propose the requirement that the weak topology on a quantum algebra of observables preserves the information encoded in the classical topology of pointwise convergence, which can be understood itself as a weak topology.

[2]In particular, this rules out nonregular representations of the Weyl algebra.

pointwise approximation from the weak topology on the classical observables. I will argue that the weak topology on the Weyl algebra fails to carry important information about approximation by expectation values of states because it fails to preserve the notion of pointwise approximation from classical physics, and this is precisely the information that the weak topology is designed to encode. In other words, the weak topology on the Weyl algebra fails to capture the physically significant notion of approximation that philosophers of physics have assumed it represents. Thus, I will argue that topological considerations count against the Weyl algebra. This demonstrates how certain aspects of current theories provide methodological tools for the construction of new physical theories.

6.2 Preliminaries

6.2.1 Algebraic Tools

As before, we use a C*-algebra to represent the bounded observables of our quantum theory. We will briefly amalgamate the tools necessary to approach quantization from an algebraic perspective that have been developed in Ch. 5. Recall from Ch. 5 that the dual space $\mathfrak{A}^*$ to $\mathfrak{A}$ can be used to define an alternative to the norm topology on $\mathfrak{A}$, called the *weak topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges in the weak topology to $A \in \mathfrak{A}$ iff for every $\rho \in \mathfrak{A}^*$,

$$\rho(A_i) \to \rho(A)$$

where the convergence is now in the standard topology on $\mathbb{C}$. The weak topology is the coarsest topology on $\mathfrak{A}$ with respect to which all of the linear functionals in $\mathfrak{A}^*$ are continuous.

A C*-algebra $\mathfrak{A}$ need not be complete with respect to its weak topology; there may be nets $\{A_i\} \subseteq$

$\mathfrak{A}$ that are Cauchy in the sense that

$$\rho(A_i - A_j) \to 0$$

as $i, j \to \infty$ for every $\rho \in \mathfrak{A}^*$ without the net having a limit point $A \in \mathfrak{A}$ such that $A_i \to A$ in the weak topology. However, a C*-algebra can always be completed in its weak topology to form a *W*-algebra*, the abstract version of a von Neumann algebra.[3] A W*-algebra is a C*-algebra $\mathfrak{R}$ with a *predual* i.e. a vector space $\mathfrak{R}_*$ such that $(\mathfrak{R}_*)^* = \mathfrak{R}$. We can understand the elements of the predual as canonically embedded in the dual space by $\rho \in \mathfrak{R}_* \mapsto \hat{\rho} \in \mathfrak{R}^*$ with $\hat{\rho}$ defined by

$$\hat{\rho}(A) = A(\rho)$$

for all $A \in \mathfrak{R}$. The elements of the predual $\mathfrak{R}_*$ define a further topology on $\mathfrak{R}$, called the weak* topology, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{R}$ converges in the weak* topology to $A \in \mathfrak{R}$ iff for every $\rho \in \mathfrak{R}_*$,

$$A_i(\rho) \to A(\rho)$$

The weak* topology is the coarsest topology that makes every element of the predual continuous when considered as a linear functional on $\mathfrak{R}$. One can show that every W*-algebra is complete in its weak* topology (see Prop. 5.4.3 in Appendix C of Ch. 5).

Of course, more nets converge in the weak* topology on $\mathfrak{R}$ than in the weak topology because it only requires convergence of expectation values on a subspace of the dual space $\mathfrak{R}^*$. Nevertheless, the weak* topology is a natural generalization of the weak topology in the special case where the predual of $\mathfrak{R}$ is itself the dual space of a C*-algebra. In this case, one has a C*-algebra $\mathfrak{A}$, its dual space $\mathfrak{A}^*$, and a W*-algebra $\mathfrak{A}^{**}$ called the *bidual*. The original algebra $\mathfrak{A}$ is canonically embedded

[3] See Sakai (1971) for more on W*-algebras, abstractly. The results of Ch. 5 (specifically Prop. 5.3.2 and Appendix C) concern the completion of a C*-algebra into a W*-algebra, which Corollaries 5.4.4 and 5.4.6 show to be unique.

in its bidual as above by $A \in \mathfrak{A} \mapsto \hat{A} \in \mathfrak{A}^{**}$, with $\hat{A}$ defined by

$$\hat{A}(\rho) = \rho(A)$$

for all $\rho \in \mathfrak{A}^*$. With respect to this embedding, the W*-algebra $\mathfrak{A}^{**}$ is the completion of $\mathfrak{A}$ in its weak topology, which is the subspace topology of the weak* topology on $\mathfrak{A}^{**}$. The weak* topology on the bidual $\mathfrak{A}^{**}$ corresponds precisely to the extension of the condition of convergence for the weak topology on $\mathfrak{A}$ to the larger algebra $\mathfrak{A}^{**}$. In particular, the weak* topology on $\mathfrak{A}^{**}$ is the coarsest topology on $\mathfrak{A}^{**}$ that makes every linear functional in $\mathfrak{A}^*$ continuous (when considered as a linear functional on $\mathfrak{A}^{**}$ by the canonical embedding above).

6.2.2 Classical and Quantum Algebras

The state space (phase space) of a classical theory is represented by a manifold $\mathcal{M}$, points of which may represent, for example, the positions and momenta of particles.[4] The observables of a classical theory are given by functions $f : \mathcal{M} \to \mathbb{C}$.[5] Each observable represents a physical quantity, with the value $f(x)$ on any state $x \in \mathcal{M}$ representing the value of the quantity f in the determinate state x. For simplicity and concreteness, let us restrict attention to the case of a single particle moving in one-dimension, with phase space given by $\mathcal{M} = \mathbb{R}^2$ and canonical coordinates (q, p), representing the position q and momentum p of the particle. Everything that follows will also hold more generally for theories with finitely many degrees of freedom and no topological effects, i.e. for which the phase space $\mathcal{M}$ is locally compact and simply connected.[6]

For the observables of the classical theory, one can use the C*-algebra $C_0(\mathcal{M})$, the collection

[4] See Landsman (1998a, 2006) for a description of classical and quantum theories in the algebraic framework and a detailed investigation of quantization.

[5] We include complex-valued functions for generality. One typically restricts to the self-adjoint functions, which are real-valued, for describing measurable quantities.

[6] For the case where $\mathcal{M}$ is not simply connected, as for a charged particle moving in an external Yang-Mills field, see Landsman (1990, 1993b,a, 1998b).

of continuous functions that vanish at infinity, equipped with algebraic operations of pointwise multiplication, addition, and complex conjugation.[7] The norm of the algebra is given by the usual supremum function norm:

$$\|f\| = \sup_{x \in \mathcal{M}} |f(x)|$$

Convergence in the norm topology corresponds precisely with the familiar notion of uniform function convergence (see Appendix B of Ch. 5).

One can provide at least two considerations in favor of using this algebra of observables. First, the pure state space of $C_0(\mathcal{M})$ coincides with the points of the manifold $\mathcal{M}$ (Fell and Doran, 1988), which are precisely what we would like to call the pure states of the physical theory. And, as expected, mixed states on $C_0(\mathcal{M})$ correspond to probability measures on $\mathcal{M}$, mapping each observable to the expectation value or average of the function integrated with respect to that measure. Second, $C_c^\infty(\mathcal{M})$, the collection of smooth functions of compact support, is norm dense in $C_0(\mathcal{M})$ (Landsman, 2006). This means that anyone willing to admit $C_c^\infty(\mathcal{M})$ as their observables will be able to approximate functions in $C_0(\mathcal{M})$ arbitrarily well in the norm topology. And it seems that even an operationalist who insisted that we only ever observe a finite (compact) region of phase space should admit the elements of $C_c^\infty(\mathcal{M})$ as observables.[8]

To quantize a classical theory, one looks for a norm continuous surjective linear *quantization map* $\mathcal{Q}$ from the observables of the classical theory into a (non-commutative) C*-algebra $\mathfrak{A}$. Norm continuity of $\mathcal{Q}$ ensures that the norm topology on the C*-algebra carries the same physical information about "global" approximation that the classical norm topology carries. Specifically, norm continuity of $\mathcal{Q}$ guarantees that whenever a net of classical observables $\{f_i\} \subseteq C_0(\mathcal{M})$ converges uniformly to $f \in C_0(\mathcal{M})$, it follows that the quantized observables $\{\mathcal{Q}(f_i)\} \subseteq \mathfrak{A}$ converge in norm

[7] Of course there are other possible choices for the algebra of classical observables that one may use if one wanted to admit unbounded observables. Here, I'll discuss the relative merits of only algebras of bounded observables.

[8] I do not mean to assert here that the operationalist will see the norm topology as the relevant standard of approximation; one might think, for example, that the weak topology is the relevant operational standard of approximation. However, my point is that because the norm topology is the finest topology of the ones typically considered, if one can approximate observables in norm, then the operationalist will be able to approximate those observables in whatever topology she deems appropriate.

to $Q(f) \in \mathfrak{A}$. Surjectivity of Q ensures that each quantum observable can be given physical significance by tracing its roots back to a classical quantity we already understand. One can always guarantee that Q is surjective by restricting attention to the subalgebra that is the range of Q, which constitutes precisely the collection of quantities whose origin traces back to classical physics.

Before we continue, I would like to make some brief remarks about how this chapter fits into the arguments of the previous chapters and the existing philosophical literature on algebraic methods in quantum theories. The results of this chapter can be seen as supporting the interpretive position of Algebraic Imperialism. I will demonstrate how a certain perspective on algebraic methods helps us to understand theory construction by helping us understand how the collection of states deemed physically possible by our theory is intimately related to what we deem to be our algebra of physically significant observables and the physically significant topologies on that algebra.[9] One might object that in precisely the same way that inequivalent Hilbert space representations pose a problem for the Hilbert Space Conservative, inequivalent observable algebras pose a problem for the Algebraic Imperialist because we have to choose just one of these algebras in an ad hoc fashion. However, I believe the freedom in choice of algebra is something the Imperialist should embrace. This chapter shows that while the question of which algebra to use is obviously an important question for the Algebraic Imperialist, she can indeed bring appropriate and non-ad hoc considerations to bear on this question. In particular, I will show that the Imperialist has tools from classical physics that she can use to constrain, inform, or perhaps even justify her choice of an algebra of observables.

[9] As I'll remark upon in Ch. 7 and as we saw in Ch. 5 (e.g., footnote 22), my understanding of the use of algebraic methods may differ in important ways from the interpretive position Ruetsche and Arageorgis define as Algebraic Imperialism.

6.3 Significance of the Weak Topologies

Just as the norm topology on the C*-algebra of classical observables $C_0(\mathcal{M})$ corresponds to a familiar notion of 'global' or uniform approximation, so too does the weak topology on $C_0(\mathcal{M})$ correspond to a familiar notion of approximation. The weak topology on $C_0(\mathcal{M})$ is equivalent to the topology of pointwise convergence of nets with bounded norm. More precisely, a net $\{f_i\} \subseteq C_0(\mathcal{M})$ converges to $f \in C_0(\mathcal{M})$ in the weak topology iff (a) for all $x \in \mathcal{M}$, $f_i(x) \to f(x)$ in $\mathbb{C}$ and (b) the net of real numbers $\{\|f_i\|\}$ is bounded (Reed and Simon, 1980, p. 1112). The weak topology provides a natural way of capturing this pointwise notion of approximation within the abstract algebraic framework.[10]

Completing our C*-algebra $C_0(\mathcal{M})$ in the weak topology allows us to include as observables of our theory functions that can be pointwise approximated by our original observables. The bidual of the classical algebra of observables is $C_0(\mathcal{M})^{**} = B(\mathcal{M})$, the collection of bounded Borel functions on $\mathcal{M}$. This is just a restatement of the familiar fact that while uniform limits of continuous functions must always be continuous, pointwise limits of continuous functions may be discontinuous, as long as they remain measurable. It immediately follows that the weak* topology on $B(\mathcal{M})$ is also precisely the topology of pointwise convergence of nets with bounded norm. This provides further reason for using the algebras $C_0(\mathcal{M})$ and $B(\mathcal{M})$—their naturally defined topologies correspond with familiar topologies on classes of functions that represent notions of approximation that are obviously physically significant.

One might object that not all measurable functions in $B(\mathcal{M})$ are physically significant in the clas-

[10]One gets this notion of pointwise convergence immediately for the abelian C*-algebra of all continuous functions on a compact Hausdorff space (see Kadison and Ringrose, 1997, p. 270). However, when we consider functions on a locally compact (but not compact simpliciter) manifold $\mathcal{M}$, the correspondence of weak convergence and pointwise convergence on $\mathcal{M}$ only holds for the algebra $C_0(\mathcal{M})$, rather than the algebra of all continuous functions, $C(\mathcal{M})$. The reason is that it is only for $C_0(\mathcal{M})$ that the pure states correspond to precisely the points in $\mathcal{M}$. Other algebras of bounded continuous functions allow for "states at infinity" (see Gamelin (1969, p. 16) and Halvorson (2004, p. 52))—pure states that cannot be represented as points of $\mathcal{M}$—which forces the weak topology to diverge from the topology of pointwise convergence on $\mathcal{M}$. Another way to state the main issue of this paper is as the question of whether these "states at infinity" are physically significant.

sical theory because this algebra includes many discontinuous functions. To alleviate this worry, notice that $B(\mathcal{M})$ is generated by its projections, the characteristic functions of measurable sets in $\mathcal{M}$. The usual interpretation of measurable sets is that they correspond to propositions, or "yes-no" questions, that have manifest physical or empirical significance (e.g. "do the position and momentum of the particle fall in this range of values?"). To the extent that this interpretation of measurable sets is appropriate, the characteristic functions of such sets have manifest physical significance as well for representing the very same propositions. The entire algebra $B(\mathcal{M})$ is then generated by composing the characteristic functions with the usual algebraic relations and taking pointwise limits, which gives an effective procedure for measuring any observable in $B(\mathcal{M})$: first measure the projection observables, then compose their values via the well-defined algebraic relations and pointwise limiting procedures. Hence, every observable in $B(\mathcal{M})$ carries at least a derived physical significance.[11]

One might think that the weak topology on a quantum algebra also carries physical significance by encoding a notion of approximation by expectation values. After all, the very definition of the weak topology encodes a notion of pointwise approximation of expectation values of states. But this will not do—given a non-commutative C*-algebra $\mathfrak{A}$ satisfying some form of the canonical commutation relations, we do not know *a priori* that all states on $\mathfrak{A}$, and hence all expectation values, are physically significant. There may be pathological states on $\mathfrak{A}$ that we want to rule out for the purposes of physical approximation.[12] If there are such unphysical states, then the weak topology provides a notion of approximation with respect to these unphysical states, and so does not carry the physical interpretation we have assumed. To be sure that one has an appropriate physically significant quantum state space, one can use a quantization map to connnect the quantum and classical algebras. The following proposition shows that a quantization map $\mathcal{Q}$ can be used to characterize the physical significance of the weak topology of a quantum algebra of observables.

[11]On the other hand, if we started with a C*-algebra other than $C_0(\mathcal{M})$, then completing in the weak topology would give rise to functions that cannot even be understood as measurable functions on $\mathcal{M}$ because of their values "at infinity" (cf. footnote 10).

[12]See Teller (1979), Halvorson (2001), and Halvorson (2004) for just such a dispute about when states are unphysical or pathological.

Proposition 6.3.1. *Suppose* $Q : C_0(\mathcal{M}) \to \mathfrak{A}$ *is a norm continuous surjective linear mapping onto a C*-algebra* $\mathfrak{A}$*. Then the weak topology on* $\mathfrak{A}$ *is the coarsest topology that makes continuous every linear functional* $\rho : \mathfrak{A} \to \mathbb{C}$ *whose composition with* Q*,* $\rho \circ Q : C_0(\mathcal{M}) \to \mathbb{C}$ *is continuous in the topology of pointwise convergence on* $C_0(\mathcal{M})$*.*

Proof. By the definition of the weak topology, it suffices to show that if a linear functional $\rho : \mathfrak{A} \to \mathbb{C}$ has a norm continuous composition with Q, $\rho \circ Q : C_0(\mathcal{M}) \to \mathbb{C}$, then ρ is norm continuous on $\mathfrak{A}$. (The converse is trivial.)

Let O be an open set in $\mathbb{C}$. Since $\rho \circ Q$ is continuous in the norm topology on $C_0(\mathcal{M})$, $(\rho \circ Q)^{-1}[O]$ is open in the norm topology on $C_0(\mathcal{M})$. Since Q is a norm continuous surjective mapping between Banach spaces, by the open mapping theorem (Reed and Simon, 1980, p. 82, Thm. III.10) we know

$$Q[(\rho \circ Q)^{-1}[O]] = Q[Q^{-1}[\rho^{-1}[O]]] = \rho^{-1}[O]$$

is open in the norm topology on $\mathfrak{A}$. Hence ρ is continuous in the norm topology on $\mathfrak{A}$. □

This shows that when a quantum algebra is constructed by a quantization map Q, then the weak topology on the quantum algebra captures a natural notion of approximation that respects the structure of the classical theory it derives from. Recall that the definition of the weak topology ensured that it was the coarsest topology that made certain maps continuous that ought to be continuous. The above proposition shows that the maps that are made continuous are precisely the ones that behave well with respect to the classical algebra and the quantization map.

Of course, the collection of functionals whose composition with Q is continuous turns out to be all of the norm continuous linear functionals on $\mathfrak{A}$, which is why we end up providing simply another characterization of the weak topology. But note that it is absolutely crucial that the quantum algebra be a quantization of $C_0(\mathcal{M})$ in order for its weak topology to gain physical significance from the notion of pointwise approximation of the classical theory. If we set things up differently, say by letting the classical algebra to be quantized (the domain of the quantization map Q) be some other algebra of functions besides $C_0(\mathcal{M})$, then we would allow for many states on $\mathfrak{A}$ whose composition with Q is not continuous in the topology of pointwise convergence (so-called "states at infinity"). In other words, as we will see in the next section, if one tries to quantize an algebra other than $C_0(\mathcal{M})$, one is liable to produce a theory with unphysical states, whose weak topology does not capture the proper notion of pointwise approximation (cf. footnotes 10 and 11).

It follows immediately from the above proposition that we can also characterize the physical significance of the weak* topology of the bidual of a quantum algebra of observables by reference to the classical theory and quantization map.

Corollary 6.3.2. *Suppose $Q : C_0(\mathcal{M}) \to \mathfrak{A}$ is a norm continuous surjective linear mapping onto a C*-algebra $\mathfrak{A}$. Then the weak* topology on $\mathfrak{A}^{**}$ is the coarsest topology τ such that for every linear functional $\rho : \mathfrak{A} \to \mathbb{C}$ whose composition with Q, $\rho \circ Q : C_0(\mathcal{M}) \to \mathbb{C}$ is continuous in the topology of pointwise convergence on $C_0(\mathcal{M})$, the map $\hat{\rho} : \mathfrak{A}^{**} \to \mathbb{C}$ defined by $\hat{\rho}(A) = A(\rho)$ for all $A \in \mathfrak{A}^{**}$ is continuous with respect to τ.*

This shows that the quantum algebras of observables $\mathfrak{A}$ and $\mathfrak{A}^{**}$ should be understood as analogous to $C_0(\mathcal{M})$ and $B(\mathcal{M})$. $\mathfrak{A}$ should be understood as a restricted subclass of bounded observables, and $\mathfrak{A}^{**}$ should be understood as a larger class of bounded observables that can be approximated by those in $\mathfrak{A}$ in a topology analogous to the topology of pointwise convergence from our classical theory. The problem that I point to in the next section is that one needs to be careful to ensure this correspondence holds—if one makes an unpropitious choice of the quantum algebra of observables, then one cannot understand the weak topology of that algebra as having physical significance, or equivalently, one cannot (at least in the absence of further argument) understand all states of the algebra as having physical significance.

Given this physical interpretation of the weak topologies, one can do real theoretical work by imposing continuity constraints on Q. Namely, one would like to be able to extend the quantization map $\mathcal{Q}$ from $C_0(\mathcal{M})$ to its weak completion $B(\mathcal{M})$.[13] It follows from Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15) that if $\mathcal{Q}$ is continuous in the weak topology on $C_0(\mathcal{M})$ and the weak topology on $\mathfrak{A}$, then $\mathcal{Q}$ has a unique extension $\tilde{\mathcal{Q}} : B(\mathcal{M}) \to \mathfrak{A}^{**}$ that is continuous in the weak* topology on $B(\mathcal{M})$ and the weak* topology on $\mathfrak{A}^{**}$.[14] Of course, this extension is unique *only if* one imposes the continuity condition; otherwise one could extend $\mathcal{Q}$ in any which way. As a precondition for imposing the continuity condition, we must already require that $\mathcal{Q}$ is continuous in the weak topologies on $C_0(\mathcal{M})$ and $\mathfrak{A}$. Thus, it is desirable to use a quantization map that is weakly continuous. The weak topologies have real import for physics by constraining which maps $\mathcal{Q}$ we use and how we extend them to larger algebras.

6.4 Against the Weyl Algebra

In this section, we show that if the Weyl algebra $\mathcal{W}$ is used as the quantum algebra of observables $\mathfrak{A}$, the quantization map $\mathcal{Q}$ fails to be continuous in the appropriate topologies. Thus, we conclude that the weak topology on $\mathcal{W}$ is, in a sense, incompatible with the classical theory $\mathcal{W}$ derives from. However, there are alternatives that fare better; we show that Berezin quantization, which uses the algebra of compact operators on a Hilbert space as its quantum algebra, succeeds in being weakly continuous.

First, notice that the generators of the Weyl algebra were defined (formally) as functions of position and momentum: $U_a = e^{ia\mathcal{Q}(q)} (:= W_{a,0})$ and $V_b = e^{ib\mathcal{Q}(p)} (:= W_{0,b})$. We can consider these operators as the quantization of functions $u_a, v_b : \mathcal{M} \to \mathbb{C}$ in the classical theory, defined analogously by $u_a(q,p) = e^{iaq}$ and $v_b(q,p) = e^{ibp}$.[15] We should already be wary because u_a and v_b do not vanish at

[13] This is necessary to ensure that the algebra of quantum observables contains projections associated with measurements of position and momentum. As Halvorson (2004) shows, this extension is not always possible if one does not work with an appropriate algebra.

[14] Another way to state Cor. 6.3.2 is that given this extended quantization map $\tilde{\mathcal{Q}} : B(\mathcal{M}) \to \mathfrak{A}^{**}$ to the W*-algebra $\mathfrak{A}^{**}$ of quantum observables, the weak* topology on $\mathfrak{A}^{**}$ is the coarsest topology that makes continuous every linear functional $\rho : \mathfrak{A}^{**} \to \mathbb{C}$ whose composition with $\tilde{\mathcal{Q}}$ is continuous in the topology of pointwise convergence.

[15] Even though $\mathcal{Q}$ is only required to be linear and not to be a homomorphism (i.e., it is not required to preserve

infinity, and thus these observables lie outside of $C_0(\mathcal{M})$. Instead we have $u_a, v_b \in B(\mathcal{M}) \setminus C_0(\mathcal{M})$. Let us form the classical analog $\mathcal{W}_C$ of the Weyl algebra, the smallest commutative C*-algebra generated by the functions u_a, v_b for all $a, b \in \mathbb{R}$.[16]

One especially important feature of the functions u_a, v_b is that they form pointwise continuous one-parameter unitary groups in the sense that $u_a, v_b \to 1$ in the topology of pointwise convergence as $a, b \to 0$, where 1 is the identity element of $B(\mathcal{M})$, i.e. the constant unit function. This fact allows us to reconstruct the observables q and p as the generators of the one-parameter families u_a, v_b. However, in the Weyl algebra, the one-parameter families U_a, V_b fail to be continuous in the weak topology. It is well known that there exists a state ω on $\mathcal{W}$ such that $\omega(U_a) \nrightarrow \omega(1)$ as $a \to 0$, and similarly there exists a state σ on $\mathcal{W}$ such that $\sigma(V_b) \nrightarrow \sigma(1)$ as $a \to 0$ (see Halvorson, 2004).[17] This immediately implies the following proposition.

Proposition 6.4.1. *Let $Q : \mathcal{W}_C \to \mathcal{W}$ be a mapping such that*

$$Q(u_a) = U_a \;\; \text{and} \;\; Q(v_b) = V_b$$

Then Q is not continuous in the topology of pointwise convergence on $\mathcal{W}_C$ and the weak topology on $\mathcal{W}$.

This shows that the weak topology on the Weyl algebra fails to capture the notion of pointwise approximation that was inherent in the weak topology on the classical algebra $C_0(\mathcal{M})$.

One might object that we should not be looking at the topology of pointwise convergence, but rather the weak topology on $\mathcal{W}_C$, which is distinct. But this would miss the point—$\mathcal{W}_C$ is unlike $C_0(\mathcal{M})$ in that it contains "states at infinity," which can be thought of as idealizations from the

multiplication and thus not required to preserve functional relations *in general*), it seems natural to expect Q to preserve functional relations on abelian subalgebras of its range. In other words, it seems natural to expect $Q(g(f)) = g(Q(f))$ for $f \in C_0(\mathcal{M})$ and any Borel function $g : \mathbb{R} \to \mathbb{R}$, as in the usual functional calculus.

[16] One can use the functions $w_{a,b}(q, p) = e^{i(a \cdot q + b \cdot p)}$ as a basis for generating the classical Weyl algebra $\mathcal{W}_C$ in analogy with the Weyl operators $W_{a,b}$ of Ch. 2. Then one has $u_a = w_{a,0}$ and $v_b = w_{0,b}$.

[17] These are known as *non-regular states* (Petz, 1990; Summers, 1999; Halvorson, 2004). See Ch. 2 and the Appendix to this chapter.

pure states in $\mathcal{M}$.[18] The weak topology of $\mathcal{W}_C$ captures a notion of approximation that includes approximation of values of these idealized states. Thus, it is not at all clear that the weak topology on $\mathcal{W}_C$ is physically significant in the classical case. So I suggest that we ought to at least aim for a quantum analog to the classical topology of pointwise convergence, which would capture a notion of approximation of the values of manifestly significant physical states.

Lack of continuity in the weak topologies poses real problems. When $\mathcal{Q}$ fails to be continuous in the appropriate topologies, we do not have a unique way of extending $\mathcal{Q}$ to the rest of the classical observables. This provides *prima facie* reason to look for alternatives to the Weyl algebra.

We do have other quantization procedures that stand in contrast by using a different quantum algebra for the range of $\mathcal{Q}$. In particular, Landsman (1998a, 2006) explicitly uses as the quantum algebra of observables the C*-algebra $\mathfrak{A} = K(\mathcal{H})$ of compact operators[19] on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e. the square integrable functions on $\mathbb{R}$.[20] Landsman (1998a, 2006) gives multiple examples of quantization maps $\mathcal{Q} : C_0(\mathcal{M}) \to K(\mathcal{H})$. Here we prove that one of those procedures, called *Berezin quantization*, results in a weakly continuous quantization map.

Following (Landsman, 2006, p. 460), Berezin quantization is given by the mapping $\mathcal{Q}^B : C_0(\mathcal{M}) \to K(\mathcal{H})$, which is defined as

$$\mathcal{Q}^B(f)\Phi(x) = \int_{\mathbb{R}^3} \frac{dp\, dq\, dy}{2\pi} f(q,p)\overline{\Psi_{(q,p)}(y)}\Phi(y)\Psi_{(q,p)}(x)$$

[18] In the Appendix of this chapter, I show a precise sense in which the non-normal states on a W*-algebra can be thought of as idealizations from the normal states by showing that they are limit points in a relevant topology.

[19] Importantly, one can also understand the algebra of compact operators to be defined abstractly as in Landsman (1990) and Fell and Doran (1988), without reference to a particular Hilbert space. This fact shows that the algebraic perspective can do real work for us without reference to Hilbert spaces, even though in this section I have used Hilbert spaces as a way of presenting the relevant algebras.

[20] This is meant only for the case of finitely many degrees of freedom on a simply connected phase space. It is well known that for systems with infinitely many degrees of freedom or with superselection rules, one requires a different algebra. But as Landsman (1990, 1993b,a, 1998b) shows, even in the case where the phase space of a system is not simply connected, one can use an algebra that may be understood as a generalization of the compact operators.

for all $f \in C_0(\mathcal{M})$ and $\Phi(x) \in \mathcal{H}$, where $\Psi_{(q,p)}(x) \in \mathcal{H}$ is a so-called "coherent state," given by

$$\Psi_{(q,p)}(x) = \pi^{-1/4} e^{-ipq/2} e^{ipx} e^{-(x-q)^2/2}$$

The weak topology on $K(\mathcal{H})$ is familiar—because the pure states on $K(\mathcal{H})$ are precisely the vector states in $\mathcal{H}$, the weak topology corresponds with the weak operator topology. In other words, a net $\{A_i\} \subseteq K(\mathcal{H})$ converges to $A \in K(\mathcal{H})$ in the weak topology iff $\langle \psi, A_i \varphi \rangle \to \langle \psi, A\varphi \rangle$ for all $\psi, \varphi \in \mathcal{H}$. We can use this fact to prove that Berezin quantization is weakly continuous.

Proposition 6.4.2. *The Berezin quantization map $Q^B : C_0(\mathcal{M}) \to K(\mathcal{H})$ is continuous in the topology of pointwise convergence on $C_0(\mathcal{M})$ and the weak topology on $K(\mathcal{H})$.*

Proof. It suffices to show that if $\{f_i\} \subseteq C_0(\mathcal{M})$ is a net that converges pointwise to $f \in C_0(\mathcal{M})$, then $Q^B(f_i)$ converges in the strong operator topology to $Q^B(f)$, which implies that it converges in the weak operator topology and hence in the weak topology on $K(\mathcal{H})$. Hence, we will show that for all $\Phi(x) \in \mathcal{H}$, $Q^B(f_i)\Phi(x) \to Q^B(f)\Phi(x)$ in the norm topology on $\mathcal{H}$. Straightforward calculation shows that

$$\begin{aligned} Q^B(f_i)\Phi(x) &= \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^3} dp\, dq\, dy\, f_i(q,p)\Phi(y) e^{ip(x-y)} e^{-(y-q)^2-(x-q)^2/2} \\ &= \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^3} dp\, dq\, dy\, f_i(q,p) g(x,y,q,p) \end{aligned}$$

where $g(x,y,q,p) = \Phi(y)e^{ip(x-y)}e^{-(y-q)^2-(x-q)^2/2}$ is in $L^2(\mathbb{R}^4)$, and similarly for $Q^B(f)\Phi(x)$. It follows that

$$\left\| Q^B(f_i)\Phi(x) - Q^B(f)\Phi(x) \right\|^2 = \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^4} dx\, dp\, dq\, dy\, \left| f_i(q,p) - f(q,p) \right|^2 \left| g(x,y,q,p) \right|^2$$

The dominated convergence theorem for L^p spaces (Simonnet, 1996, p. 100, Thm. 5.2.2) implies that the expression on the right hand side approaches zero as $i \to \infty$, which shows that $Q^B(f_i)$ converges in the strong operator topology, and hence in the weak topology, to $Q^B(f)$. □

This shows that the weak topology on $K(\mathcal{H})$ preserves the notion of approximation from the classical theory we began with. Or in other words, this shows that the states on $K(\mathcal{H})$ have physical significance because we can trace the topology they define back to the topology defined by the manifestly physically significant classical states in $\mathcal{M}$. Thus, the states on $K(\mathcal{H})$ have a much more secure status than states on $\mathcal{W}$.

Moreover, we can use the weak topology on $K(\mathcal{H})$ to do real work. The completion of $K(\mathcal{H})$ in the weak topology is $K(\mathcal{H})^{**} = \mathcal{B}(\mathcal{H})$, the collection of all bounded operators on $\mathcal{H}$. Thus, the quantization map Q^B extends uniquely to a map $\tilde{Q}^B : B(\mathcal{M}) \to \mathcal{B}(\mathcal{H})$ that is continuous in the topology of pointwise convergence on $B(\mathcal{M})$ and the weak operator topology (which is just the weak* topology) on $\mathcal{B}(\mathcal{H})$. This brings us back to the familiar situation for the quantum mechanics of a single particle, where our bounded observables correspond to all bounded operators on a Hilbert space. It is my contention that the weak continuity of the quantization map Q plays a fundamental role in this story. My suggestion is that even in more complex quantum theories, including ones with infinitely many degrees of freedom like field theories, we ought to prefer weakly continuous quantizations in the absence of arguments to the contrary.

6.5 Discussion

We have seen that the weak (respectively, weak*) topology on a C*-algebra (respectively, W*-algebra) of quantum observables can be given physical signifcance by connecting it to the manifestly physically significant weak topology on $C_0(\mathcal{M})$. That is, a quantization map Q allows us to capture the notion of pointwise approximation on classical states within the quantum algebra. However, the weak topology on the Weyl algebra fails to capture this notion of approximation, or in other words fails to be related to the classical theory by an appropriate quantization procedure. I believe this provides reason to use an alternative algebra for our quantization procedures, one in which the preservation of the notion of pointwise approximation from classical physics ensures that we focus on states that are physically significant. As we saw, we already possess such an algebra in the form of the compact operators, which succeeds in capturing the topological information at issue, and hence the physicality of certain states.

I admit, though, that none of the considerations in this chapter are decisive. In practice, when one uses the Weyl algebra, one restricts attention to Hilbert space representations satisfying the condition of *regularity* (see Ch. 2 or Petz (1990); Halvorson (2004)), which in turn guarantees that the weak operator topology on the Hilbert space has the appropriate relation to the weak topology of the classical theory. The problem is not that using the Weyl algebra constitutes some sort of error, but rather that using the Weyl algebra makes it easy to be misled. For example, one may be misled into thinking that the regularity condition is an unjustified assumption, as in Halvorson (2001, 2004). From the point of view of this paper, Halvorson's position gets things backwards. The regularity assumption is necessary to recover the topological information from the classical theory we began with. Thus, it would be a mistake to do away with the regularity assumption because one would lose important topological information, and thus important information about which states are physical and which are idealizations. Using the algebra of compact operators rather than the Weyl algebra is a way to safeguard against making this mistake (see the Appendix).

I believe that the discussion of this paper demonstrates how mathematical aspects of our current theories can be used to constrain future scientific theorizing. I have assumed that the goal of quantization is to alter a classical theory *as little as possible* in order to make it into a quantum theory. This means that one should preserve, e.g. the topological structure of the classical theory, while altering minimally the algebraic structure by imposing the canonical commutation relations. Of course, one is free to question this assumption; one might assert that theorists should be free to explore whatever avenues seem profitable for constructing new quantum theories. Some have in fact argued that discontinuous quantization procedures are fruitful in developing theories of quantum gravity (Ashtekar, 2009; Corichi et al., 2007). I do not wish to argue against such approaches; I only wish to point out that they are far more radical than they might at first seem. Rather than finding unexplored options already available in the standard approach to quantum mechanics (as these authors sometimes present their work), theorists who pursue discontinuous quantization procedures are proposing a major departure from the quantum theories we currently possess.

The fact that the structure of our current physical theories can be used as a methodological tool for constraining future theorizing, even in the absence of any new data, ought to have implications for general philosophy of science. But there are still many open questions that need to be discussed to understand these implications. Does the fact that a future theory is constructed in order to respect the structure of current physics entail that we ought to put more credence in that new theory? Or does the fact that our theorizing relies so heavily on current physics only show that we are more likely to leave important alternatives unconceived? And can new physical theories constitute Kuhnian paradigm shifts if they are constructed directly out of the materials of our current theories? I hope that philosophers of science continue to think about these important questions. I also hope that philosophers of physics engage with these issues in the context of quantization, where their conclusions may make a real difference for the future of physics.

Appendix: Regular and Normal States

The purpose of this appendix is to provide two additional results that may help to better understand the role of non-normal and non-regular states on the Weyl algebra, and their relation to the weak topologies discussed here. First, some definitions. We denote the states on a C*-algebra $\mathfrak{A}$ by $\mathcal{S}(\mathfrak{A})$. We call states falling in the predual $\mathfrak{R}_*$ of a W*-algebra $\mathfrak{R}$ *normal states*, and we denote these $\mathcal{S}_N(\mathfrak{R})$. Normal states, considered as elements of the dual space of $\mathfrak{R}$, are just the elements that are continuous in the weak* topology on $\mathfrak{R}$. If we start with a C*-algebra $\mathfrak{A}$ and take its bidual $\mathfrak{A}^{**}$, then the normal states on the bidual are just the original states on $\mathfrak{A}$, but of course there are many more non-normal states on $\mathfrak{A}^{**}$. If we take our specification of the original C*-algebra $\mathfrak{A}$ to specify our physically realizable states, or if we desire states that are continuous in the weak* topology, then we may have reason to restrict attention to normal states on $\mathfrak{A}^{**}$. A state ω on the Weyl algebra $\mathcal{W}$ is said to be *regular* if its GNS representation π_ω is regular in the sense of Ch. 2, i.e. if the one-parameter unitary families $\pi_\omega(U_a)$ and $\pi_\omega(V_b)$ are weak operator continuous, where, as before, $U_a = W_{a,0}$ and $V_b = W_{0,b}$.

We remarked in footnoes 10, 11, and 12 that one can understand the dispute over the Weyl algebra to concern whether all states on this algebra should be thought of as physically possible, or whether only some subset of states, like the regular states, ought to be thought of as physical. To say that only the regular states are physical is to say that the non-regular states are unphysical or pathological in some sense. To see why one might think the non-regular states are pathological, I first show that if one uses a weakly continuous quantization map, one that is continuous in the weak* topology and the topology of pointwise convergence, then every normal state on the quantum algebra is regular. Second, I will show that the non-regular (or non-normal) states can be thought of as, in a certain sense, idealizations from the regular (or normal) states.

As before, let $\mathcal{W}_C$ denote the classical Weyl algebra. Let $\mathcal{Q} : \mathcal{W}_C \to \mathfrak{R}$ be a map to a W*-algebra $\mathfrak{R}$.[21] One might consider the following constraints on $\mathcal{Q}$:

1. (*Unitarity*) $\mathcal{Q}(u_a)$ and $\mathcal{Q}(v_b)$ are one (real) parameter unitary groups.
2. (*Weak Continuity*) $\mathcal{Q}$ is continuous in the topology of pointwise convergence on $\mathcal{W}_C$ and the weak* topology on $\mathfrak{A}$.[22]

We will extend the definition of regularity and call a state ω on $\mathfrak{R}$ *regular* just in case in its GNS representation π_ω, the one parameter families $\pi_\omega(\mathcal{Q}(u_a))$ and $\pi_\omega(\mathcal{Q}(v_b))$ are weak operator continuous.

[21] We do not even need to assume now that $\mathcal{Q}$ satisfies the conditions of a quantization map, i.e. that it is linear or surjective.

[22] It is crucial here to notice that the topology of pointwise convergence on $\mathcal{W}_C$ is not the weak topology on $\mathcal{W}_C$ because the classical Weyl algebra allows states at infinity. Nevertheless, as in the body of the paper, I take the topology of pointwise convergence to have a much more secure physical significance than the weak topology on $\mathcal{W}_C$.

Proposition 6.5.1. *Let $Q : \mathcal{W}_C \to \mathfrak{R}$ be a map from the classical Weyl algebra $\mathcal{W}_C$ to a W*-algebra $\mathfrak{R}$ satisfying conditions 1 and 2. Then every normal state ω on $\mathfrak{R}$ is regular.*

Proof. The proof proceeds by straightforward, but tedious, $\varepsilon - \delta$ constructions. Let ω be a normal state on $\mathfrak{R}$. Let $\Omega_\omega \in \mathcal{H}_\omega$ be the cyclic vector representing ω in its GNS representation $(\pi_\omega, \mathcal{H}_\omega)$. Let $U_a = Q(u_a)$ and $V_b = Q(v_b)$.

Choose any $\psi, \phi \in \mathcal{H}_\omega$. We need to show that the map

$$a \mapsto \langle \psi, U_a \phi \rangle$$

is continuous. First notice that, because Ω_ω is cyclic for $\pi_\omega(\mathfrak{A})$, there are sequences $A_n, B_m \in \pi_\omega(\mathfrak{A})$ such that

$$\|A_n\Omega_\omega - \psi\| \to 0 \text{ and } \|B_m\Omega_\omega - \phi\| \to 0$$

Choose any $\varepsilon > 0$, and choose any $0 < \varepsilon_0 < \varepsilon$. We will show that there is a $\delta > 0$ such that $|\langle \psi, U_a\phi\rangle - \langle \psi, \phi\rangle| < \varepsilon$ whenever $|a| < \delta$.

Let $\varepsilon_1 = \frac{\varepsilon - \varepsilon_0}{4\|\phi\|}$. Then there is an N such that for all $n \geq N$, $\|A_n\Omega_\omega - \psi\| < \varepsilon_1$. Hence, for all $n \geq N$,

$$|\langle A_n\Omega_\omega, \phi\rangle - \langle \psi, \phi\rangle| = |\langle A_n\Omega_\omega - \psi, \phi\rangle| \leq \|A_n\Omega_\omega - \psi\|\,\|\phi\| < \varepsilon_1\,\|\phi\| = \frac{\varepsilon - \varepsilon_0}{4}$$

Similarly, we have that for all $n \geq N$

$$|\langle \psi, U_a\phi\rangle - \langle A_n\Omega_\omega, U_a\phi\rangle| = |\langle \psi - A_n\Omega_\omega, U_a\phi\rangle| \leq \|\psi - A_n\Omega_\omega\|\,\|U_a\|\,\|\phi\| < \varepsilon_1\,\|\phi\| = \frac{\varepsilon - \varepsilon_0}{4}$$

Now let $\varepsilon_2 = \frac{\varepsilon - \varepsilon_0}{4\|A_N\|}$. Then there is an M such that for all $m \geq M$, $\|B_m\Omega_\omega - \phi\| < \varepsilon_2$. Hence, for all

$m \geq M$,

$$|\langle A_N\Omega_\omega, U_a\phi\rangle - \langle A_N\Omega_\omega, U_aB_m\Omega_\omega\rangle| = |\langle A_N\Omega_\omega, U_a(\phi - B_m\Omega_\omega)\rangle|$$
$$\leq \|A_N\|\,\|\Omega_\omega\|\,\|U_a\|\,\|\phi - B_m\Omega_\omega\| < \|A_N\|\,\varepsilon_2 = \frac{\varepsilon - \varepsilon_0}{4}$$

Similarly, we have that for all $m \geq M$,

$$|\langle A_N\Omega_\omega, B_m\Omega_\omega\rangle - \langle A_N\Omega_\omega, \phi\rangle| = |\langle A_N\Omega_\omega, B_m\Omega_\omega - \phi\rangle|$$
$$\leq \|A_N\|\,\|\Omega_\omega\|\,\|B_m\Omega_\omega - \phi\| < \|A_N\|\,\varepsilon_2 = \frac{\varepsilon - \varepsilon_0}{4}$$

And finally, we know that the map

$$a \mapsto \langle A_N\Omega_\omega, U_aB_M\Omega_\omega\rangle = \omega(A_N^* Q(u_a)B_M)$$

is continuous because $Q \circ u$ is the composition of continuous maps in the relevant topologies, multiplication is separately continuous in the weak* topology, and ω is normal. Hence there is a $\delta > 0$ such that whenever $|a| < \delta$,

$$|\langle A_N\Omega_\omega, U_aB_M\Omega_\omega\rangle - \langle A_N\Omega_\omega, B_M\Omega_\omega\rangle| < \varepsilon_0$$

By the triangle inequality, we have that whenever $|a| < \delta$,

$$\begin{aligned}
|\langle \psi, U_a\phi\rangle - \langle \psi, \phi\rangle| &\leq |\langle \psi, U_a\phi\rangle - \langle A_N\Omega_\omega, U_a\phi\rangle| \\
&\quad + |\langle A_N\Omega_\omega, U_a\phi\rangle - \langle A_N\Omega_\omega, U_aB_M\Omega_\omega\rangle| \\
&\quad + |\langle A_N\Omega_\omega, U_aB_M\Omega_\omega\rangle - \langle A_N\Omega_\omega, B_M\Omega_\omega\rangle| \\
&\quad + |\langle A_N\Omega_\omega, B_M\Omega_\omega\rangle - \langle A_N\Omega_\omega, \phi\rangle| \\
&\quad + |\langle A_N\Omega_\omega, \phi\rangle - \langle \psi, \phi\rangle| \\
&< \frac{\varepsilon - \varepsilon_0}{4} + \frac{\varepsilon - \varepsilon_0}{4} + \varepsilon_0 + \frac{\varepsilon - \varepsilon_0}{4} + \frac{\varepsilon - \varepsilon_0}{4} = \varepsilon
\end{aligned}$$

An exactly analogous argument shows that $\pi_\omega(V_b)$ is weak operator continuous. □

This proposition shows that if we use a weakly continuous quantization map, then the normal states on the resulting algebra will automatically be regular. Notice that here we are not talking about normal states on the weak completion of the Weyl algebra because, as we saw in this chapter, no quantization map to the Weyl algebra can be weakly continuous. So such a weakly continuous quantization map will have to get us to a different quantum algebra. And if we think that the resulting algebra is one on which the physical states are the normal states, then the physical states will all be regular.

For example, if one uses the Berezin quantization map $Q_B : C_0(\mathcal{M}) \to K(\mathcal{H})$, and extends it continuously in the relevant weak topologies from $C_0(\mathcal{M})$ to all bounded Borel functions $B(\mathcal{M})$,[23] then one has a a quantization map whose domain includes the classical Weyl algebra and satisfies the weak continuity constraints discussed. This means that $\tilde{Q}^B_{|\mathcal{W}_C}$—i.e., the weakly continuous extension of the Berezin quantization map restricted to the classical Weyl algebra—is an example of a map Q satisfying the conditions of the preceding proposition. It follows that normal states on the quantum algebra, i.e. the codomain of this quantization map $\tilde{Q}^B$, are all regular. So if we think that the states on $K(\mathcal{H})$ we get from the Berezin quantization map, which are just the normal states on the weak completion $K(\mathcal{H})^{**}$, are the physical states, then the physical states are all regular.

Next we show by extending the approximation result of Prop. 5.3.2 that all states on the W*-algebra given by the bidual $\mathfrak{A}^{**}$ of a C*-algebra $\mathfrak{A}$ can be understood as idealizations or approximations from the states on $\mathfrak{A}$, which are normal states on $\mathfrak{A}^{**}$. We can think of this as showing how the non-normal or non-regular states might arise even if we consider only the normal or regular states to be physical. I take this to alleviate much of the tension in the debate between Teller (1979) and Halvorson (2001; 2004) concerning the existence of determinate position states in quantum mechanics—even though determinate position states are not regular states, we can still

[23]We must extend the Berezin quantization map Q^B in order to talk about regularity because the Weyl unitaries are not compact operators and so are not contained in the range of Q^B.

understand them as approximations or idealizations from the other states allowed by the theory. For what follows, we will consider $\mathcal{S}(\mathfrak{A}^{**})$ and $\mathcal{S}_N(\mathfrak{A}^{**})$ as subsets of $\mathfrak{A}^{***}$. Let $J : \mathfrak{A}^* \to \mathfrak{A}^{***}$ be the canonical embedding of $\mathfrak{A}^*$ in its bidual $\mathfrak{A}^{***}$. The following proposition shows that every element of $\mathcal{S}(\mathfrak{A}^{**})$ can be understood as a limit of normal states in the weak* topology.

Proposition 6.5.2. *$\mathcal{S}_N(\mathfrak{A}^{**})$ is dense in $\mathcal{S}(\mathfrak{A}^{**})$ in the weak* ($\sigma(\mathfrak{A}^{***},\mathfrak{A}^{**})$) topology on $\mathfrak{A}^{***}$.*

Proof. This follows as just a particular case of Fell's theorem (Fell, 1960; Clifton and Halvorson, 2001, p. 428) by using the faithful universal representation of $\mathfrak{A}$, whose universal enveloping von Neumann algebra is W*-isomorphic to $\mathfrak{A}^{**}$. We also provide an alternative, purely algebraic proof here, which is an adaptation of the proof of Prop. 5.3.2.

Let $\tilde{\omega} \in \mathcal{S}(\mathfrak{A}^{**})$ and let U be an open neighborhood of $\tilde{\omega}$. By the definition of the weak* topology, there are $\tilde{A}_1,...,\tilde{A}_n \in \mathfrak{A}^{**}$ and $\varepsilon_1,...,\varepsilon_n > 0$ such that

$$\bigcap_{i=1}^{n}(\tilde{\omega}+N(\tilde{A}_i,\varepsilon_i)) \subseteq U$$

where $\tilde{\omega}+N(\tilde{A}_i,\varepsilon_i) = \{\tilde{\varphi} \in \mathfrak{A}^{***} : \left|\tilde{\varphi}(\tilde{A}_i) - \tilde{\omega}(\tilde{A}_i) < \varepsilon\right|\}$.

Let $N'(\tilde{A}_i,\varepsilon) = \{\tilde{\varphi} \in \mathfrak{A}^{***} : \tilde{\varphi} = J(\varphi)$ for some $\varphi \in \mathfrak{A}^*$ and $\left|\tilde{A}_i(\varphi) - \tilde{\omega}(\tilde{A}_i)\right| < \varepsilon_i\}$. We will show that

$$\bigcap_{i=1}^{n} N'(\tilde{A}_i,\varepsilon_i) \subseteq \bigcap_{i=1}^{n}(\tilde{\omega}+N(\tilde{A}_i,\varepsilon_i)) \cap J(\mathfrak{A}^*)$$

is non-empty and contains a state in $J(\mathfrak{A}^*)$. In other words, we will show that there is a state $\Omega = J(\omega) \in J(\mathcal{S}(\mathfrak{A})) = \mathcal{S}_N(\mathfrak{A}^{**})$ such that for all $1 \leq i \leq n$

$$\left|\Omega(\tilde{A}_i) - \tilde{\omega}(\tilde{A}_i)\right| = \left|\tilde{A}_i(\omega) - \tilde{\omega}(\tilde{A}_i)\right| < \varepsilon_i \tag{6.1}$$

It suffices only to consider a linearly independent subset of $\tilde{A}_1,...,\tilde{A}_n \in \mathfrak{A}^{**}$ forming a basis for the subspace spanned by $\tilde{A}_1,...,\tilde{A}_n \in \mathfrak{A}^{**}$. If ω satisfies the above inequalities for this linearly independent subset of $\tilde{A}_1,\ldots,\tilde{A}_n \in \mathfrak{A}^{**}$, then it must also satisfy the above inequalities for the rest of the $\tilde{A}_i$'s, or else the inequalities would contradict each other and then $\bigcap_{i=1}^n(\tilde{\omega}+N(\tilde{A}_i,\varepsilon_i))$ would have to be empty which it is not because it contains $\tilde{\omega}$.

Choose a linearly independent set of functionals and adjoin to it the unit $\tilde{I} \in \mathfrak{A}^{**}$ so that we have the set consisting of $\tilde{A}_{k_1},\ldots,\tilde{A}_{k_m} \in \mathfrak{A}^{**}$, where $\tilde{A}_{k_1} = \tilde{I}$. We know (Schechter, 2001, p. 93, Lemma 4.14) that there is a dual basis $\varphi_1,\ldots,\varphi_m \in \mathfrak{A}^*$ for a subspace of $\mathfrak{A}^*$ such that $\tilde{A}_{k_i}(\varphi_j) = \delta_{ij}$ for all $1 \leq i,j \leq m$. Applying the same lemma again, we find a dual basis $A_1,\ldots,A_m$ for a subspace of $\mathfrak{A}$ such that $\varphi_i(A_j) = \delta_{ij}$ for all $1 \leq i,j \leq m$.

Let $\omega = \sum_{i=1}^m \tilde{\omega}(\tilde{A}_{k_i})\varphi_i$ be a linear functional on the subspace of $\mathfrak{A}$ spanned by $A_1,\ldots,A_m$. It follows that for all $1 \leq j \leq m$,

$$\tilde{A}_{k_j}(\omega) = \sum_{i=1}^m \tilde{\omega}(\tilde{A}_{k_i})\tilde{A}_{k_j}(\varphi_i) = \tilde{\omega}(\tilde{A}_{k_j})$$

This shows that ω satisfies the inequalities in eq. (6.1). Furthermore, letting $I \in \mathfrak{A}$ denote the unit of $\mathfrak{A}$, we have

$$\omega(I) = \tilde{I}(\omega) = \sum_{i=1}^m \tilde{\omega}(\tilde{A}_{k_i})\tilde{I}(\varphi_i) = \tilde{\omega}(\tilde{I}) = 1$$

We also know that for any A in the span of $A_1,\ldots,A_n \in \mathfrak{A}$ given by $A = \sum_{j=1}^m a_jA_j$, we have

$$\begin{aligned}
\omega(A) &= \sum_{i=1}^m \tilde{\omega}(\tilde{A}_{k_i})\varphi_i(\sum_{j=1}^m a_jA_j) \\
&= \sum_{i=1}^m \tilde{\omega}(\tilde{A}_{k_i})\sum_{j=1}^m a_j\varphi_i(A_j) \\
&= \sum_{i=1}^m a_i\tilde{\omega}(\tilde{A}_{k_i}) \\
&= \tilde{\omega}(\sum_{i=1}^m a_i\tilde{A}_{k_i})
\end{aligned}$$

It follows that $\|\omega\| \leq \|\tilde{\omega}\| = 1$. Now we have that $\|\omega\| = \omega(I) = 1$. By the Hahn-Banach theorem (Reed and Simon, 1980, p. 76, Thm. III.6), we can extend ω to all of $\mathfrak{A}$ without increasing its norm. Then it follows from Thm. 4.3.2 of Kadison and Ringrose (1997, p. 256) that ω is positive and hence is a state on $\mathfrak{A}$, which means that $\Omega = J(\omega)$ is a normal state on $\mathfrak{A}^{**}$. □

This shows that we can think of the regular states as the physical states, and still have access to the non-regular states as idealizations or approximations. However, one must be careful because if one chooses the wrong algebra of quantum observables, then one might be misled into thinking that these idealized states are physical. For example, if one uses the Weyl algebra, then one might think the states on the Weyl algebra, which include non-regular states, are physical. But if one instead uses a weakly continuous quantization map and uses the compact operators $K(\mathcal{H})$ as the algebra of quantum observables, then all states can be thought of as regular in the sense that they extend continuously in the weak* topology to normal states on $K(\mathcal{H})^{**}$, which are regular by Prop. 6.5.1. The above proposition shows that the non-regular states on the Weyl algebra, which can also be thought of as non-normal states[24] on $K(\mathcal{H})^{**}$, can all be thought of as idealizations or approximations from the normal, and hence regular states.

[24]Thinking of a non-regular states ρ on the Weyl algebra as non-normal states on $K(\mathcal{H})^{**}$ requires extending them to a different algebra. One can do this either by considering any Hahn-Banach extension of ρ (not necessarily continuous) from $\pi(\mathcal{W})$ to $\mathcal{B}(\mathcal{H})$, where $(\pi, \mathcal{H})$ is the usual Schrödinger representation. Or alternatively, one can extend ρ continuously in the weak topology on the Weyl algebra to a state $\hat{\rho}$ on the bidual $\mathcal{W}^{**}$ and then restrict back down to $K(\mathcal{H})^{**}$, understood in the natural way as a subalgebra of $\mathcal{W}^{**}$ (e.g., the closure of a subrepresentation of the universal representation of $\mathcal{W}$). Either way, the resulting states must turn out to be non-normal on $K(\mathcal{H})^{**}$.

Chapter 7

Conclusion

I have now presented the central arguments of the dissertation. In Ch. 2, I presented background on quantization in the algebraic approach to quantum theory. In Chs. 3 and 4, I argued that Hilbert Space Conservatism is inadequate as an interpretation of the formalism, and that it misleads us into thinking that inequivalent representations present a puzzle for understanding symmetry breaking. In Ch. 5, I provided a solution to the main extant problem for Algebraic Imperialism by showing how the Imperialist can account for parochial observables. And in Ch. 6, I demonstrated the usefulness of Algebraic Imperialism by showing how it gives rise to a new perspective on the process of quantization and focuses our attention on different questions (concerning the construction of algebras of quantum observables) that I take to be important both for understanding quantum field theory and for constructing future quantum theories. Now, I will remark on the interpretation of the algebraic formalism that we have ended up with in section 7.1, and then point to open questions that remain and future work to be done in section 7.2.

7.1 On Pristine and Adulterated Interpretations

As Ruetsche (2011) describes them, Algebraic Imperialism, Hilbert Space Conservatism, and Universalism are all instances of what she calls *pristine interpretation*. It is the ideal of pristine interpretation that is her main target in arguing against these interpretations of quantum theories with infinitely many degrees of freedom. Pristine interpretations are first of all characterized as ones that articulate the content of a theory by identifying the physically possible worlds according to the theory. And, in order to be pristine, such an interpretation must specify those physically possible worlds once and for all, using only general principles rather than contingent facts that may vary with the context of different specific applications. Ruetsche's work presents an extended argument against this ideal through the consideration of infinite quantum systems; Ruetsche argues that we should reject pristine interpretation because none of the pristine interpretations of infinite quantum theory are adequate on their own. Ruetsche's arguments concerning the unavailability of enough states in an irreducible Hilbert space representation (see Chs. 3 and 4) comes in service to this more general general project as a reason for rejecting pristine Hilbert Space Conservatism. And the problem of parochial observables (see Ch. 5) comes in service to this project as a reason for rejecting pristine Imperialism and Universalism. By dispatching with the main pristine options, Ruetsche aspires to reject pristine interpretation for these quantum theories full stop.

I have argued for a kind of Algebraic Imperialism (and by extension a kind of Universalism), but as the reader may have noticed from the discussion of the previous chapters, my adaptations of these interpretations differ quite a bit from Ruetsche's and Arageorgis' original descriptions of Imperialism in that I do not take my interpretation to be pristine. And furthermore, I do not take any of the arguments of this dissertation to constitute a defense of pristine interpretation. For example, in showing in Ch. 5 that one can use the abstract algebra or its universal representation to adequately recover the parochial observables I did not rely on the fact that Imperialism and Universalism are pristine interpretations at all. In particular, none of my arguments depended on thinking of theories as circumscribing a set of physically possible worlds by appealing to general principles. To see this, I will outline two alternative adulterated interpretations that the arguments given in the previous chapters support at least as well.

In fact, some (myself included) might think my arguments support the adulterated interpretations over the pristine ones in light of the comments of Chs. 5 and 6. Specifically, footnotes 1, 4, 9, and the discussion at the end of 5.3.1 of Ch. 5 make caveats to avoid questions about the "right" topology, the "fundamental" observables, and the "reality" of idealizations, which the pristine interpreter might demand an answer to. I am not optimistic that one could provide answers to satisfy her. This, however, does not undermine the central arguments of that chapter because an adulterated interpreter of the sort I am about to describe simply *does not need to answer those questions* for her purposes. The interpreter who favors algebraic methods for recovering parochial observables does not claim to have specified once and for all a set of physically possible worlds, but rather sees herself as providing a collection of tools sufficient for accomplishing certain purposes in scientific practice.

Similarly, one might think that the central plot line of Ch. 6—concerning the difficult choices one must make between different algebras of observables—undermines the Imperialist's ability to specify once and for all a set of possible worlds because which set of worlds we end up with would depend on which algebra we choose. But the point of that chapter was to show that algebraic tools can help us pose and attempt to answer deep and interesting questions concerning the nature of physical and unphysical states and their relationship to collections of observables and standards of approximation on those observables. It is this general perspective—that the algebraic tools are incredibly useful for posing and answering important questions—that my central arguments support. Even though one ought not to understand this perspective as a pristine interpretation, I have called it Algebraic Imperialism because of its affinity with the Imperialist's use of abstract algebras.

Let me briefly lay out what I take to be the core of the adulterated interpretations that I support. The idea lurking behind these interpretations is that it is misleading to think that our scientific theories aim at specifying a set of physically possible worlds in the first place. Our scientific theories aim at providing a formalism or collection of tools that we can use for a variety of different purposes: making predictions, constructing explanations, modeling particular systems, generating new theories, examining relationships with past theories, and more.[1] This is not some sort of simple operationalism or instrumentalism, but rather a way of saying that we should take the rich and variegated practices of science seriously when interpreting our theories.

Whereas the Algebraic Imperialist would force algebraic methods upon everyone as the correct ones to describe the physical possibilities, her adulterated counterpart—one might call her the *Algebraic Colonialist*—merely asserts that we can use the abstract algebra to accomplish all our scientific goals. We can, for example, think of any two states—even states describing different thermodynamic phases of a statistical system—as states on the abstract algebra. As Prop. 5.3.3 shows, we can also think of any observable—even a parochial observable like temperature or net magnetization—as a limit point of a net of observables in the abstract algebra with respect to the weak (or really weak*) topology. Hence, we can think of any observable as an idealization from a collection of observables in the abstract algebra. If we desired, we could also focus attention on a particular collection of states on the abstract algebra and use the weakest topology that makes those states continuous. That is essentially what we do when we take a Hilbert space representation and use the weak operator topology on that Hilbert space. For some scientific purposes, restricting our attention to certain states and changing our topology may turn out to be fruitful.[2] All the Algebraic Colonialist claims is that this practice can be understood algebraically and that we need not take representations at all to give the desired explanations, e.g. of phase transitions.

Similarly, whereas the Universalist claims that the universal representation once and for all specifies the physical possibilities, her adulterated counterpart—one might call her the *Unitarian Universalist*—is a bit more lax. The Unitarian Universalist simply claims that we can use the universal

[1] See Stein (1989) for a statement of a similar view and how it bears on the scientific realism debate. Also compare this view with the alternative adulterated interpretation of Ruetsche (2004, 2011).

[2] See Fletcher (2016) for an argument that we ought to allow ourselves access to different topologies for different scientific purposes, at least in the case of general relativity.

representation to accomplish all of our scientific goals. We can, for example, think of any two states as density operators (or even vectors!) on the Hilbert space $\mathcal{H}_U$ of the universal representation. And, as Prop. 5.3.5 shows, we can think of any observable—again, even a parochial observable—as being contained in the universal enveloping von Neumann algebra. Hence, we can think of any observable as well approximated by or physically indistinguishable from a collection of observables with respect to the weak operator topology of the universal representation. Similarly, if we desired, we could restrict attention to some subrepresentation of the universal representation π_U on a subspace of $\mathcal{H}_U$, thereby restricting our attention to a particular collection of density operator and vector states and defining a new topology by using the weak operator topology on this subspace. Again, this may be useful for many scientific purposes, but the universal representation gives us all of the resources we need to accomplish these tasks.

We have seen that Algebraic Imperialism and Universalism amount to the same position, and for precisely the same reasons Algebraic Colonialism and Unitarian Universalism are equivalent as well. Prop. 5.3.4 shows that the weak topology on the abstract algebra yields the same criterion for convergence as the weak operator topology on the universal representation. And Prop. 5.3.7 shows that the bidual of the abstract algebra, which is just the original algebra with the addition of its limit points in the weak topology, can be thought of as the same collection of observables as the universal enveloping von Neumann algebra, which of course is a faithful representation of the original algebra with the addition of its limit points in the weak operator topology of that representation. So the abstract algebra and the universal representation allow us to countenance the same physically significant states and observables and to use what amount to the same mathematical tools in our scientific investigations.

Even though I have targeted and attacked some crucial pieces of Ruetsche's argument against pristine interpretation, none of what I have said in this dissertation provides support for that ideal. All I have done is show that the use of algebraic methods and the universal representation are equivalent and adequate for interpreting infinite quantum theories. Both of these sets of tools—abstract algebras and universal representations—allow us to accomplish the same tasks, and in particular to begin making sense of scientific practice in quantum field theory and quantum statistical mechanics.

Although throughout this dissertation I have used the terminology of Algebraic Imperialism and Universalism, and I will continue to use that terminology in section 7.2, I really mean something more like the roughly sketched views of Colonialism and Unitarian Universalism just outlined. As should be clear from Ch. 6 and the projects I am about to outline, the central arguments of Chs. 3, 4, and 5 should be thought of as the prolegomenon to future work on quantum field theory. Now that we have established a collection of algebraic tools and gained some basic understanding of how to use them, we have a framework that allows us to pick out the important interpretive questions and then go on to apply the tools in our toolbox as we search for answers. None of this requires us to be pristine interpreters—in fact, I agree with Ruetsche that the adulterated interpretations are better suited for this task.

7.2 Future Directions

Having reflected on the kind of interpretation that we have arrived at through the arguments of the previous chapters, I would like now to briefly remark upon future work to be done to better understand the role of algebraic tools in our interpretation of quantum theory.

First, in Ch. 5 I argued that one is justified in moving from a C*-algebra $\mathfrak{A}$ representing the algebra of quantum observables associated with some system to its bidual $\mathfrak{A}^{**}$, which is a W*-algebra containing all of the observables that can be approximated in the weak topology by nets of observables in $\mathfrak{A}$. The question that immediately arises at this point is whether one should consider our use of the algebra $\mathfrak{A}^{**}$ to somehow constitute a new theory. For example, does $\mathfrak{A}^{**}$ contain more information or structure than $\mathfrak{A}$? One can show quite easily that $\mathfrak{A}^{**}$ is, in a certain sense, the unique completion of $\mathfrak{A}$ in the weak topology (see Appendix C of Ch. 5). But this does not yet fully answer the question concerning structure. In the language of category theory, the morphisms of our mathematical objects play a crucial role in determining the (relative) amount of structure they contain. So one still has an open question concerning whether the morphisms of C*-algebras can be appropriately mapped onto the morphisms of W*-algebras, which is equivalent to the problem of characterizing the functor from the category of C*-algebras to the category of W*-algebras that sends each C*-algebra to its unique completion. Answering this question would determine whether the theory that uses $\mathfrak{A}$ is, in a certain sense, equivalent or intertranslatable with the theory that uses $\mathfrak{A}^{**}$.

Another way to approach these questions concerning the viability of Imperialism is through the interpretation of Universalism, which we showed in Ch. 5 is, in a certain sense, equivalent. Recall that there are a number of properties of the universal representation of a C*-algebra that suggest its interpretation may differ from that of a Hilbert space representation for the Hilbert Space Conservative. Specifically, the universal representation is nonseparable and reducible. But it is an open question whether these properties ought to affect our interpretation of the universal representation. My inclination is that they should not—I cannot find convincing reason to restrict attention to separable Hilbert spaces because most of the technical results that undergird our interpretation of a Hilbert space (including the spectral theorem) still hold in some form or other in nonseparable Hilbert spaces, as one can see from the generality of the results concerning C*-algebras and W*-algebras abstractly—results which are independent of any representation, separable and nonseparable alike.

And furthermore, I believe the standard arguments for using irreducible representations—that reducible representations contain excess or redundant structure (Ruetsche, 2011)—simply fail because the "excess" structure of reducible representations is not necessarily redundant. I believe that the reason one is tempted to conclude that reducible representations contain redundancies comes only from looking at atypical examples of reducible direct sum representations of the form

$\pi(\mathfrak{A}) \oplus \pi(\mathfrak{A})$, which contains a redundancy only because one repeats the same representation π. If instead one considers unitarily inequivalent irreducible representations π_1 and π_2, then the reducible direct sum $\pi_1(\mathfrak{A}) \oplus \pi_2(\mathfrak{A})$ does not contain a similar redundancy; this direct sum contains much information beyond the irreducible representation π_1 or the irreducible representation π_2 alone, specifically in the form of more density operator states than either π_1 or π_2 individually allow. I believe that getting clear on the interpretation of the universal Hilbert space would help to better understand the relationship between Universalism, Imperialism, and Conservatism, as well as the viability of the first two listed positions.

On a similar note, there is a further objection against Algebraic Imperialism made by Arageorgis (1995) that, while the Hilbert Space Conservative does not allow for enough states, the Imperialist allows for *too many*. What precisely is meant by "too many" is not perfectly clear to me (especially if one takes on an adulterated interpretation of the form outlined in section 7.1), but I believe that this objection raises some interesting questions concerning how one constructs quantum algebras of observables. I am currently working to show that if one specifies (within some constraints) a privileged subset of the state space of our quantum algebra of observables, then one can systematically change the algebra to restrict attention to only this state space. This would show that, in a certain sense, specifying an algebra of observables amounts to just specifying a space of quantum states. And such a result would give one quite a bit of freedom in choosing and changing the algebra of observables (or alternatively the state space). I believe that accomplishing this with algebraic tools would provide a mark in favor of Imperialism, but this discussion also shows that the most interesting questions to be asking at this point concern which algebra we ought to use to represent the observables of quantum systems. As in Chapter 6, there will in general be many candidate algebras that we must choose between. But I believe that we can develop algebraic resources to help us decide on the appropriateness of different algebras for representing physical systems by paying special attention to how the quantum observables relate to a classical theory (e.g., by using a quantization map). This of course requires much further technical and conceptual work.

Finally, I would like to conclude by simply reemphasizing how important I take these topics to be for both our conceptual and philosophical understanding of quantum field theory and for the progress of physics. Although this is speculative, I believe that pursuing the algebraic approach to quantum field theory may provide a route to rigorizing non-perturbative quantum Yang-Mills theory. I also believe that all of this work may help provide a route toward making sense of theories of quantum gravity—by better understanding how quantization works in the cases where it is well understood, and in the field theories where it is not yet fully understood, I think we may find a more principled approach to constructing future theories. I believe that careful attention to interpretive issues concerning the role of physical observables and physical states is absolutely essential in all of this work, and especially for understanding the way in which our current best physical theories represent the world.

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