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Computations in Large N Matrix Mechanics

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Abstract

The algebraic formulation of Large N matrix mechanics recently developed by Halpern and Schwartz leads to a practical method of numerical computation for both action and Hamiltonian problems. The new technique posits a boundary condition on the planar connected parts X_w, namely that they should decrease rapidly with increasing order. This leads to algebraic/variational schemes of computation which show remarkably rapid convergence in numerical tests on some many-matrix models. The method allows the calculation of all moments of the ground state, in a sequence of approximations, and excited states can be determined as well. There are two unexpected findings: a large d expansion and a new selection rule for certain types of interaction.

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1 Introduction

Large N matrix mechanics [1] differs from ordinary quantum mechanics in that the canonical commutator

$$i[p,q] = I, \tag{1.1}$$

in the one-matrix case, is replaced by the relation

$$i[\pi,\phi] = \mid 0\rangle\langle 0 \mid \tag{1.2}$$

where $\mid 0\rangle$ is the ground state in the reduced Hilbert space. The original matrix-valued coordinates $\phi_{rs}, \quad r,s = 1 \ldots N$ are represented by the single operator ϕ in this reduced Hilbert space. [2]

The solution of the one-matrix Large N Hamiltonian problem with an arbitrary potential $V(\phi)$ was given some years ago [3]; and only a couple of two-matrix problems in the action formalism have previously been solved. [4, 5]

The many-matrix problem involves several *noncommuting* operators ϕ_m and their conjugate momenta. Following Halpern and Schwartz [6], this system is described at equal times by a symmetric free algebra which involves a pair (tilde and untilde) for each hermitian operator

$$[\tilde{\phi}_m, \phi_n] = [\tilde{\pi}_m, \pi_n] = 0, \qquad m,n = 1 \ldots d \tag{1.3a}$$

$$i[\tilde{\pi}_m, \phi_n] = i[\pi_m, \tilde{\phi}_n] = \delta_{mn} \mid 0\rangle\langle 0 \mid \tag{1.3b}$$

$$\tilde{\phi}_m \mid 0\rangle = \phi_m \mid 0\rangle, \qquad \tilde{\pi}_m \mid 0\rangle = \pi_m \mid 0\rangle \tag{1.3c}$$

and the ground state energy is given by

$$E_0 = N^2\langle 0 \mid \frac{1}{2}\sum_{m=1}^{d} \pi_m\pi_m + V(\phi) \mid 0\rangle \tag{1.4}$$

where (ϕ) refers to the set of operators $\{\phi_m\}$. We shall use the summation convention in what follows.

In ordinary quantum mechanics systems of several interacting bodies are most commonly attacked from the Schrodinger equation in coordinate space, using the direct product basis $\mid q_1, q_2, \ldots q_d\rangle$. That approach is not available

in the Large N reduced Hilbert space because of the noncommutativity of the operators ϕ_m. A basis of states in this reduced space may be written as

$$\mid w\rangle \equiv \phi^w \mid 0\rangle \tag{1.5}$$

where we use the "word" notation for ordered products of operators

$$\phi^w = \phi_{m_1}\phi_{m_2}\ldots\phi_{m_n}, \qquad w = m_1 m_2 \ldots m_n, \qquad m_i = 1\ldots d \tag{1.6}$$

and we write $[w] = n$ for the length of the word w. See App. A for a collection of relevant definitions and formulas.

The new approximation technique presented in this paper lies close to the Heisenberg (matrix) formulation rather than the Schrodinger (wavefunction) formulation and makes use of the set of polynomials $T_w(\phi)$ introduced in Ref. [7].

$$(1 - \beta_m\phi_m + X(\beta))^{-1} \;=\; \sum_w \beta^w T_w(\phi) \tag{1.7a}$$

$$X(\beta) = \sum_w \beta^w X_w, \qquad X_0 \;=\; 0, \qquad \langle 0 \mid T_w(\phi) \mid 0\rangle = \delta_{w,0} \tag{1.7b}$$

where the β_m are a dummy set of (noncommuting) parameters and the numbers X_w were identified as the planar connected parts defined in earlier diagrammatic studies. [8] Various properties of these X_w are given in App. A, including their relation to the ordinary moments $Z_w \equiv \langle 0 \mid \phi^w \mid 0\rangle$ of the ground state.

The core idea of the present work is to truncate the set of these X's

$$set \quad X_w = 0 \quad for \;\; all \;\; [w] > n \tag{1.8}$$

and solve the (now finite) set of algebraic equations, calling this the "n-th order approximation". Then increase n, step by step, and see whether the numerical results appear to converge. This is an intuitive/experimental approach for now, since we have no mathematical proof that this method should work.

With even a small number of the X's determined, one can approximate all the moments of the ground state and the accuracy of these results increases systematically as one proceeds to higher orders of approximation. The excited states of a Hamiltonian system are also amenable to this method.

The recent algebraic developments by Halpern and Schwartz [6, 7] provide a wealth of formal definitions and relations for many-matrix problems, unifying the study of both action and Hamiltonian systems. These start with the definitions of generalized creation and annihilation operators in the reduced Hilbert space,

$$\pi_m \mid 0\rangle = iF_m(\phi) \mid 0\rangle, \qquad \langle 0 \mid \pi_m = -i\langle 0 \mid F_m(\phi) \tag{1.9a}$$

$$B_m = F_m(\phi) + i\pi_m, \qquad B_m \mid 0\rangle = \langle 0 \mid B_m^\dagger = 0 \tag{1.9b}$$

$$B_m B_n^\dagger = E_{mn}(\phi) \tag{1.9c}$$

$$E_{mn}(\phi) \mid 0\rangle = 2i[\tilde{\pi}_n, F_m(\phi)] \mid 0\rangle \tag{1.9d}$$

which is the Interacting Cuntz Algebra. (In the case of non-interacting harmonic oscillators, we have $E_{mn} \propto \delta_{mn}$ and Eqs. (1.9b, 1.9c) reduce to the original Cuntz algebra.)

In the practical work of this paper there is a basic distinction between the two types of problems. For action problems we start out knowing the functions $F_m(\phi)$ explicitly and this lets us work directly with the algebraic equations for the connected parts X_w derived in Ref. [7] (see Sec. 5). For Hamiltonian problems we do not know $F_m(\phi)$ beforehand and so part of the method presented here involves a constructive representation of these operators, for which task we use the polynomials $T_w(\phi)$ (see Sec. 6).

In Sec. 2 we test the idea on a simple example: a one-matrix action problem and in Sec. 3 we try to give some understanding of why this method apparently works well. Counting of the variables in many-matrix problems and making use of symmetry to keep things manageable is discussed in Sec 4, followed in Sec. 5 by some algebraic results for a model action problem with d interacting matrices. The plan of attack for many-matrix Hamiltonian problems is set out in Sec. 6 and numerical results for a set of model potentials are presented in Sec. 7. We note not only the extremely rapid convergence found in these examples but also an unexpected selection rule. Section 8 presents more details of this computational program; and a related method for calculating excited states is given in Sec. 9. Several appendices discuss further details and possible extensions of this work.

2 First Test: One-Matrix Action Problem

We start with a simple problem: a one-matrix action at Large N. As given in Ref. [7] for the quartic action ($F = \phi^3$), we have the following equation for the connected parts X_n:

$$X(X+1)^2 - \beta^2 X_2 - \frac{1}{2}\beta^4 = 0, \qquad X = \sum_{n>0} \beta^n X_n \tag{2.1}$$

which leads to the recursion formula

$$X_n = \frac{1}{2}\delta_{n,4} - \sum_{p=2}^{n-2} X_p \Big(\sum_{q=2}^{n-p-2} X_q X_{n-p-q} + 2X_{n-p} \Big), \qquad n = 4, 6, \ldots \ . \tag{2.2}$$

For one-matrix problems we replace the word label w by $n = [w]$. We can compare this with the Schwinger-Dyson equations for the ordinary moments $Z_n = \langle 0 \mid \phi^n \mid 0 \rangle$, which may be written as

$$2Z_{n+4} = \sum_{m=0}^{n} Z_m Z_{n-m}, \qquad Z_0 = 1 \tag{2.3}$$

and only even n enter because of the parity symmetry in this problem. If we have the value of $X_2 = Z_2$ (which we know from other analysis to be $(2/3)^{3/2}$), then we can compute all the higher ones from these recursion formulas. Table 1 shows some numerical results and we see that the ratio X_n/Z_n decreases fairly rapidly as n increases.

Table 1. X_n and Z_n for $F = \phi^3$ action problem

n	2	4	6	10	20
X_n	.544331	-.0925926	.0403208	.0143736	-.00311591
Z_n	.544331	.500000	.544331	.816497	3.95996
X_n/Z_n	1.00000	-.185185	.074074	.017604	.000787

Now we want to turn this process around and calculate the value of X_2 from the recursion formula (2.2) using the idea that X_n should decrease rapidly at large n – a sort of boundary condition. That is, we consider X_2 as an unknown parameter and then search for that value that will allow us to

truncate the equations (2.2) with $X_{n+2} = 0$; and then we step up the value of n and repeat the process. Table 2 contains the results of this computation and we see that the residual error at each level of approximation decreases quite rapidly as we increase n.

Table 2. Compute X_2 by truncation: $X_{n+2} = 0$

n+2	4	6	8	10	20
Approx. X_2	.500000	.534522	.541429	.543344	.544321
Error	-.044331	-.009809	-.002902	-.000987	-.000010

We view this as a sort of eigenvalue problem for the connected parts X_n and recognize a certain similarity here with the familiar procedure for numerical integration of the one-dimensional Schrodinger equation in some given potential. While that other problem involves a continuous variable $\psi(x)$ obeying a linear (differential) equation our current problem involves a discrete set X_n obeying a nonlinear (algebraic) equation.

3 Why Should this Method Work?

To understand what is going on here it may help to consider the ordinary moments

$$Z_n = \langle 0 \mid \phi^n \mid 0 \rangle = \int dq \; \rho(q) \; q^n \tag{3.1}$$

for a one-matrix problem. These Z_n, for a typical ground state density $\rho(q)$, are a rather monotonous sequence of numbers. The infinite set of coupled equations for these moments (Schwinger-Dyson equations in one language) contains all the information about the ground state; but one would not try to truncate this infinite system of equations by setting the Z_n equal to zero after some cutoff $n = n^*$.

(In earlier work [9] on moment equations for the one- and two-body Schrodinger equation, the asymptotic behavior of these moments as $n \to \infty$ was inferred from the differential equation for the wavefunction and this allowed a backward iteration procedure.)

Now, by contrast, observe the definition of the planar connected parts,

again for the one-matrix problem:

$$X_{n+1} = \langle 0 \mid \phi T_n(\phi) \mid 0 \rangle = \int dq\ \rho(q)\ q\ T_n(q) \tag{3.2}$$

where the polynomials T_n have the property

$$\langle 0 \mid T_n(\phi) \mid 0 \rangle = 0 \qquad n > 0. \tag{3.3}$$

Clearly, the X_n are just an algebraic combination of the Z_n. But Eq. (3.3) tells us that the polynomials T_n are oscillatory within the domain of integration; and this suggests that the X_n, given by (3.2), can be thought of as something like the Fourier coefficients of the density $\rho(q)$. Therefore, if the ground state is reasonably smooth and the polynomials T_n are reasonably "appropriate", then we would expect that the higher Fourier coefficients (the X_n) could decrease rapidly. This is the motivation to try a truncation scheme on the X's.

A further advantage of the X's is that they are directly sensitive to the interactions in many-matrix problems. In Ref. [7] it was shown that in many-matrix problems without interactions, the X_w vanish if there is any mixing of letters in the word w.

Once one has determined, approximately, even a small number of the X's, this allows one to give approximate values for *all* of the Z's in any one- or many-matrix problem by use of the general algebraic relation (A.2) between the generating functions for these two sets of parameters.

With these encouraging results, we go on to study the problems of many matrices in Large N action and Hamiltonian systems.

4 Many Matrices - Counting the Variables

With d matrices, the number of words of length n is d^n and this number grows very rapidly. If we have some symmetries in the action or the Hamiltonian, then we can reduce the number of independent variables X_w that we have to handle at each level of approximation. In this paper we consider model problems with the following invariance properties of the ground state $\mid 0\rangle$.

Parity Symmetry: Change the sign of ϕ_m (and π_m) for any m.

Permutation Symmetry: Make any permutation among the d labels $m, n, \ldots$.

In addition, there is the general invariance of the X_w (as of the Trace operation in the unreduced space) under a cyclic permutation of the letters in the word w.

With these conditions, the number of independent X_w's is greatly reduced - to what we shall call a set of "basic words" at each level n - as shown in Table 3.

Table 3. Count of $d^n \rightarrow$ basic words

n	d=2	d=3	d=5	d=9
2	4→1	9→1	25→1	81→1
4	16→3	81→3	625→3	6561→3
6	64→4	729→9	15625→9	531441→9
8	256→12	6561→41	390625→59	
10	1024→28	59049→257		
12	4096→94			

At each level of approximation (signified by the maximum word length n) we shall deal with a number of basic words (the dimension D of our parameter space). From Table 3 we read off these dimensions: for d=2, D=1,4,8,20,48,...; for d=3, D=1,4,13,54,...; for d=5, D=1,4,13,72,...; for d=9, D=1,4,13,.... The first task of the computer program is to make a table of all d^n words at each n, identify each word with an equivalence class according to the symmetries described above and assign one member of each class as a basic word w_i, $i = 1 \dots D$.

5 Many-Matrix Action Problems

5.1 General algebraic machinery

For action problems, we have the dual basis system of equations derived by Halpern and Schwartz [7]:

$$B^\dagger_m = G_m(\phi) - E_{mn}(\phi)\bar{B}_n, \qquad \phi_m = \bar{B}_m(1 + \bar{X}(B^\dagger)) \tag{5.1a}$$

$$\bar{X}(B^\dagger) = \sum_w X_{\bar{w}} B^{\dagger\, w} = \sum_w X_w B^{\dagger\, \bar{w}}. \tag{5.1b}$$

Here, the operators $\bar{B}_m,\ B_m^\dagger$ obey the simple Cuntz algebra

$$\bar{B}_m B_n^\dagger = \delta_{mn} \tag{5.2}$$

and the role of these operators is to generate an infinite set of coupled algebraic equations for the connected parts X_w, as will be shown by example below. The functions $G_m = 2F_m$ and E_{mn}, defined earlier in (1.9a, 1.9d), are immediately known once we specify the action S. Then we shall proceed with the sequence of truncation approximations, generalizing the one-matrix example of Sec. 2.

5.2 A model problem

We take for our model problem here the d-matrix action

$$S = -\frac{1}{4N} \sum_{m,n=1}^{d} Tr([\phi_m, \phi_n])^2 \tag{5.3}$$

in the unreduced Hilbert space. This gives us the reduced operators,

$$G_m(\phi) = \sum_{n\neq m} (\phi_m\phi_n\phi_n + \phi_n\phi_n\phi_m - 2\phi_n\phi_m\phi_n) \tag{5.4a}$$

$$E_{mm}(\phi) = \sum_{n\neq m} (\phi_n\phi_n + X_{nn}) \tag{5.4b}$$

$$E_{m\neq n}(\phi) = \phi_m\phi_n - 2\phi_n\phi_m \tag{5.4c}$$

where we note that this S has the symmetries mentioned in the previous section and this leads to the simplifications $X_m = 0,\ \ X_{mn} = \delta_{mn}X_{11}$.

The equations (5.1a) now look like this

$$\begin{aligned}
B_m^\dagger = \sum_{n\neq m=1}^{d} \{ & \\
&(\bar{B}_n\bar{B}_n\bar{B}_m\bar{X} + \bar{B}_m(\bar{B}_n\bar{B}_n\bar{X} - X_{11}) - 2\bar{B}_n\bar{B}_m\bar{B}_n\bar{X}) + \\
&(\bar{B}_n\bar{X}\bar{B}_n\bar{B}_m\bar{X} + \bar{B}_m\bar{X}\bar{B}_n\bar{B}_n\bar{X} - 2\bar{B}_n\bar{X}\bar{B}_m\bar{B}_n\bar{X}) + \\
&(\bar{B}_n\bar{B}_n\bar{X}\bar{B}_m\bar{X} + \bar{B}_m\bar{B}_n\bar{X}\bar{B}_n\bar{X} - 2\bar{B}_n\bar{B}_m\bar{X}\bar{B}_n\bar{X}) + \\
&(\bar{B}_n\bar{X}\bar{B}_n\bar{X}\bar{B}_m\bar{X} + \bar{B}_m\bar{X}\bar{B}_n\bar{X}\bar{B}_n\bar{X} - 2\bar{B}_n\bar{X}\bar{B}_m\bar{X}\bar{B}_n\bar{X})\}.
\end{aligned} \tag{5.5}$$

This system of equations is equivalent to the Schwinger-Dyson set of equations but it is packaged to emphasize the role of the X's and it leads directly to our sequence of approximations. The first line of terms in (5.5) has only one $\bar{X}$ and its first few terms are

$$(X_{mnnp}+X_{nnmp}-2X_{nmnp})B_p^\dagger+(X_{mnnpqr}+X_{nnmpqr}-2X_{nmnpqr})B_r^\dagger B_q^\dagger B_p^\dagger \quad (5.6)$$

where the usual constraint on the sum ($n \neq m$) is understood. The second and third lines have two $\bar{X}$'s and their first few terms are

$$\begin{aligned} &2X_{11}X_{mp}B_p^\dagger + (2X_{11}X_{mpqr} + X_{np}X_{mnqr} + X_{mp}X_{nnqr} - \\ &2X_{np}X_{nmqr} + X_{nnpq}X_{mr} + X_{nmpq}X_{nr} - 2X_{mnpq}X_{nr})B_r^\dagger B_q^\dagger B_p^\dagger \end{aligned} \quad (5.7)$$

and the fourth line, with three $\bar{X}$'s, starts off as

$$(X_{np}X_{nq}X_{mr} + X_{mp}X_{nq}X_{nr} - 2X_{np}X_{mq}X_{nr})B_r^\dagger B_q^\dagger B_p^\dagger. \quad (5.8)$$

Collecting the linear terms in $B^\dagger$ gives us the equation

$$1 = 2(d-1)(X_{1122} - X_{1212} + X_{11}^2) \quad (5.9)$$

where we have used the symmetry properties to list the basic words: (11) at $n = 2$ and $(1111), (1122), (1212)$ at $n = 4$. This equation is exact and leads to our lowest (second) order approximation: we set all X's with word length greater than 2 equal to zero and we get

$$X_{11} \simeq 1/\sqrt{2(d-1)}. \quad (5.10)$$

Next, we collect the cubic terms in $B^\dagger$. For our fourth order approximation we drop all X_w's with $[w] > 4$:

$$\begin{aligned} 0 \;=\; & X_{11}\{(2d-2+\epsilon_{mp}+\epsilon_{mr})X_{mpqr} - 2\epsilon_{mp}X_{mqrp} - 2\epsilon_{mr}X_{mrpq} + \\ & (\delta_{mp}\delta_{qr} + \delta_{mr}\delta_{pq})[(d-1)X_{1122} + X_{1111} - X_{mmqq}]\} + \\ & X_{11}^3[\delta_{pq}\delta_{mr}\epsilon_{mp} + \delta_{qr}\delta_{mp}\epsilon_{mq} - 2\delta_{pr}\delta_{mq}\epsilon_{mp}] \end{aligned} \quad (5.11)$$

where $\epsilon_{pq} = 1 - \delta_{pq}$. These equations are now evaluated for varying choices of the labels m, p, q, r, which must be paired. We find

$$X_{1111} + X_{1122} = 0 \quad for\ m = p = q = r \quad (5.12a)$$

$$X_{1111} + 3(d-1)X_{1122} - 2X_{1212} + X_{11}^2 = 0 \quad for\ m = p \neq q = r \quad (5.12b)$$

$$2X_{1122} - dX_{1212} + X_{11}^2 = 0 \quad for\ m = q \neq p = r. \quad (5.12c)$$

The solution of this set of equations (for $d \neq 2$) is

$$X_{1111} = -X_{1122} = \frac{1}{3}X_{1212} = \frac{1}{3d+2}X_{11}^2 \tag{5.13}$$

and, putting these results back into (5.9), we find the fourth order approximation for X_{11}

$$X_{11} \simeq [2(d-1)(1 - \frac{4}{3d+2})]^{-\frac{1}{2}}. \tag{5.14}$$

For $d = 2$ the equations (5.12) are indeterminate; but for this case a scaling argument leads to the conclusion that the system is not bounded.

It was very pleasing to find, in the fourth order calculation above, that the number of independent equations was just equal to the number of unknowns and we found a unique solution. Will this circumstance continue at higher orders of approximation? I have no general answer.

One should program a computer to carry the above sequence of approximations to higher order; only algebraic work is required at each step. I have not done this yet, giving priority to the more difficult Hamiltonian problems, reported in Sec. 6.

5.3 A large d expansion

From the result above one is led to speculate that this truncation sequence of approximations may be related to a "large d" expansion. The algebraic calculations described above have been carried out to the sixth order, with 9 equations in 9 unknowns, and solved in the approximation that $d >> 1$. This leads to the following result:

$$(X_{11})^{-2} = 2(d-1)[1 - \frac{4}{3d+2} - \frac{185}{81d^2} + O(d^{-3})]. \tag{5.15}$$

We do not have a systematic theory of such a large d approximation but the following crude attempt may be instructive. Look back at the formula for G_m, Eq. (5.4a), and replace the operator pair $\phi_n\phi_n$ by its ground state average, which is X_{11}. This butchered G_m is then

$$G_m \sim 2\omega\phi_m, \qquad \omega = (d-1)X_{11} \tag{5.16}$$

which is the formula for a system of noninteracting harmonic oscillators. The oscillator result $X_{mn} = \delta_{mn}/(2\omega)$ then gives immediately the leading term

in (5.15). The higher order terms in $1/d$ are then expected to come from a perturbation theory expansion about this oscillator approximation. Also, if one looks at the computer results for the Hamiltonian problems (Sec. 7) one may discern a suggestion of more rapid convergence for larger values of d.

6 Many-Matrix Hamiltonian Problems

6.1 Choosing the model problems

We shall study the Hamiltonians for d bosonic matrices, given in the unreduced Hilbert space as

$$H = \frac{1}{2}\sum_{m=1}^{d} Tr(\pi_m \pi_m) + NTr(V(\frac{\phi}{\sqrt{N}})) \tag{6.1}$$

with the following choices of the potential:

$$V_1(\phi) = \frac{1}{4}\sum_{m=1}^{d} \phi_m^4 \tag{6.2a}$$

$$V_2(\phi) = \frac{1}{4}(\sum_{m=1}^{d} \phi_m^2)^2 \tag{6.2b}$$

$$V_3(\phi) = \frac{1}{4}\sum_{m<n=1}^{d} \phi_m^2 \phi_n^2 \tag{6.2c}$$

$$V_4(\phi) = -\frac{1}{8}\sum_{m<n=1}^{d} [\phi_m, \phi_n]^2 \tag{6.2d}$$

or, if desired, any linear combination of them. The first potential, which is just the non-interacting case, is used for verification of the computational procedure. The third and fourth potentials have "flat directions", which make them particularly interesting. (Will the calculations converge nicely, indicating a bound state, or will they not?) All four potentials have the symmetries (parity and permutation) described in Sec. 4. The additional SO(d) symmetry of V_2 and V_4 is not used at the outset but will be noted in the results.

The following subsections outline the method and further details are given in Sec. 8 and in Apps. A and B.

6.2 Construction of $F_m(\phi)$

A central construct of our previous work [6, 7] is the reduced operator $F_m(\phi)$, defined in (1.9a). We will represent this quantity by a finite linear expansion in the polynomials $T_w(\phi)$

$$F_m(\phi) = \sum_w R_w^{(m)} T_w(\phi) \tag{6.3}$$

at each level of approximation and then see how to determine the coefficients R. (See Subsec 8.1 for more details.).

For any reduced operator A which depends on the ϕ's one has the identity

$$2\langle 0 \mid A(\phi) F_m(\phi) \mid 0\rangle = \langle 0 \mid i[\tilde{\pi}_m, A(\phi)] \mid 0\rangle \tag{6.4}$$

which is proved using the definition (1.9a) and (1.3c). Choosing $A = T_{w'}$ and using the formulas (A.7) and (1.7b) this gives

$$\langle 0 \mid T_{w'}(\phi) F_m(\phi) \mid 0\rangle = \frac{1}{2}\delta_{w',m} \tag{6.5}$$

for any word w'. We impose these relations on the approximate expansion (6.3) and obtain

$$\sum_w K_{w',w} R_w^{(m)} = \frac{1}{2}\delta_{w',m} \tag{6.6}$$

where

$$K_{w',w} \equiv \langle 0 \mid T_{w'}(\phi) T_w(\phi) \mid 0\rangle. \tag{6.7}$$

This matrix K is numerically evaluated in terms of the X's, as detailed in App. B, and then we determine the expansion coefficients R from a straightforward matrix inversion calculation. Of course, we make this a square (and positive) matrix, as detailed in Eqs. (8.5, 8.6). This completes the first part of the fitting problem, which we would term the kinematic part since it assures that we are doing our best, at any given level of approximation, to represent the basic commutator algebra (1.3b).

Now we turn to the second part, which involves the dynamics of any particular Hamiltonian.

6.3 Minimizing the Energy

The kinetic energy of the ground state can be expressed as

$$\begin{aligned} K.E./N^2 &= \frac{1}{2}\langle 0 \mid \pi_m \pi_m \mid 0\rangle = \frac{1}{2}\langle 0 \mid F_m F_m \mid 0\rangle \\ &= \frac{1}{4}\langle 0 \mid i[\tilde{\pi}_m, F_m] \mid 0\rangle = \frac{1}{4}R_m^{(m)} = \frac{d}{4}R_1^{(1)} \end{aligned} \tag{6.8}$$

using the methods and results of the previous subsection.

The potential energy of the ground state is expressed directly in terms of the X's using (A.3b):

$$\langle 0 \mid \phi_m^4 \mid 0\rangle = X_{1111} + 2X_{11}^2 \tag{6.9a}$$

$$\langle 0 \mid \phi_m^2 \phi_n^2 \mid 0\rangle = X_{1122} + X_{11}^2, \quad m \neq n \tag{6.9b}$$

$$\langle 0 \mid \phi_m \phi_n \phi_m \phi_n \mid 0\rangle = X_{1212}, \quad m \neq n \tag{6.9c}$$

where we have used the specified symmetries to write these formulas in terms of the four basic words at the second and fourth orders.

We program the computer to evaluate the ground state energy $E = E_0/N^2$ at the n-th order approximation with any assigned numerical values for the quantities X_w for $[w] \leq n$. The final step of this scheme is to vary this set of X's so as to minimize E. This procedure is without mathematical justification; it just seems like the natural thing to do.

What is more, this part of the method is far from straightforward as a computational task because the energy E is a very nonlinear function of the many variables X. In Subsec. 8.2 we describe the techniques used to search for this minimum. The numerical results are presented next.

7 Numerical Results

The Tables that follow give the outputs of the computations and are designed to show at a glance the convergence of the approximation scheme described above.

Table 4 shows the energy (E/d) calculated for the potential V_2, for several values of d and at several levels of approximation and Table 5 gives the corresponding values of $X_{11} = \langle 0 \mid \phi_1^2 \mid 0\rangle$.

Table 4. Calculated values of E/d for potential V_2

n	D	d=2	d=3	d=5	d=9
2	1	.429	.472	.5408	.6412
4	4	.42672	.47035	.53921	.64007
6	8,13	.426672	.4703152	.539189	.640058
8	20,54	.42667093	.47031461		
10	48	.426670885			

Table 5. Calculated values of X_{11} for potential V_2

n	D	d=2	d=3	d=5	d=9
2	1	.437	.397	.347	.292
4	4	.4428	.4010	.34912	.29365
6	8,13	.443007	.401106	.349171	.293667
8	20,54	.4430170	.4011103		
10	48	.44301744			

We note how rapidly these numbers converge as one goes down each column in the tables. For each step increasing the order of approximation, we see *one or two orders of magnitude* increase in accuracy, somewhat better for E than for X. Also, one sees in these tables that the first approximation (a 'back of the envelope' computation) is accurate to about one percent. Such is the power of the X. For comparison, Table 6 presents results for the one-matrix problem, d=1 and V_1, computed by the same program. We see that the results of the many-matrix computations (above) converge about as rapidly as the one-matrix results, although the amount of work required to obtain the former is much greater.

Table 6. Computed results for the one-matrix problem: V_1

n	D	E	X_{11}
2	1	.375	.50
4	2	.3717	.5100
6	3	.371638	.51057
8	4	.3716339	.510611
10	5	.37163373	.5106136

Table 7 gives the E/d results computed for the potential V_3 and one sees rapid convergence here as well.

Table 7. Calculated values of E/d for potential V_3

n	D	d=2	d=3	d=5	d=9
2	1	.236	.298	.375	.4725
4	4	.2312	.29470	.373207	.471358
6	8,13	.231036	.294625	.3731823	.47134965
8	20,54	.2310258	.29462242		
10	48	.23102504			

In Table 8 we see the results for the potential V_4, which has the greatest amount of "flat directions" among our models. Here the rate of convergence is noticeably slower than in the previous models, but still looks convincingly good.

Table 8. Calculated values of E/d for potential V_4

n	D	d=2	d=3	d=5	d=9
2	1	.24	.30	.38	.47
4	4	.224	.289	.370	.4690
6	8,13	.2232	.2890	.36944	.468940
8	20,54,72	.22299	.28895	.369431	
10	48	.222964			

Also, in the several tables above, one sees a suggestion of more rapid convergence for larger values of d; see the discussion of the large d expansion in Subsec. 5.3.

In another experiment, we studied the one-matrix problem with potential

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{g}{4}\phi^4 \tag{7.1}$$

as the parameter g approached the value $\sqrt{8}/3\pi$ where the bound state disappears. The numerical procedure searching to minimize the energy worked well until one approached very close to this critical value; then it failed dramatically.

Other X_w values are also produced in these computations, albeit with a somewhat lesser accuracy. Table 9 has some of these for the potential V_2.

Table 9. Computed values of some other X_w for V_2

X_w	d=2	d=3	d=5	d=9
X_{1111}	-.0132659	-.0082358	-.004201	-.001798
X_{1122}	-.0066329	-.0041179	-.002101	-.000899
X_{1212}	0.0	0.0	0.0	0.0

If there is rotational symmetry in the ground state, one can derive the following relation among the fourth order X's,

$$X_{1111} = 2X_{1122} + X_{1212} \tag{7.2}$$

and the data in Table 9 satisfy this relation, as does the corresponding data for the potential V_4, which is also rotationally invariant.

There is another, unexpected, phenomenon seen in the data of Table 9: namely that $X_{1212} = 0$. An increasing number of other X_w's also vanish when one looks at higher orders. This result also appears for the potential V_3, but not for V_4. When a particular X_w goes to zero, so does the corresponding coefficient R_w. The empirical rule is this: Write out the word w and remove any pair of matching adjacent letters; repeat this process; the X_w will vanish unless this process can reduce the original word to null. I do not have a full explanation for this newly discovered selection rule but it appears to be related to the fact that these potentials (see (6.2b) and (6.2c)) involve only pairs $(\phi_m\phi_m)$ of each operator. This new symmetry is particular to Large N matrix mechanics with its noncommuting coordinate operators; it would not arise in ordinary quantum mechanics.

From an experimental (numerical) perspective, but lacking any formal proof, it appears that these types of large N problems are now solvable. It will be important for others to repeat this work independently in order to verify these results.

8 Details of the Computational Program

8.1 The Full F_m

The expression (6.3) for $F_m(\phi)$ needs to be refined. The motivation for what follows comes from App. E in Ref. [6] where the ground state wavefunction (the action) is modeled and one sees the consequent structure of $F_m(\phi)$.

Corresponding to each basic word w_i we want to have a group of terms (in the $T_w(\phi)$) with a common coefficient $R_i^{(m)}$:

$$F_m(\phi) = \sum_{i=1}^{D} R_i^{(m)} F_{m,i}(\phi). \tag{8.1}$$

For the first stage in this construction we define

$$\partial_m T_w(\phi) \equiv \sum_{w \sim mw'} T_{w'}(\phi) \tag{8.2}$$

which, one can show, will guarantee that the flatness condition [6]

$$[\tilde{\pi}_m, F_n(\phi)] - [\pi_n, \widetilde{F_m(\phi)}] = 0 \tag{8.3}$$

is satisfied.

For the second stage we take all permutations among the $m = 1 \ldots d$ letters that occur in the basic words w_i.

$$F_{m,i}(\phi) = \frac{1}{c(w_i)\ (d-1)!}\ \partial_m \sum_{perm's} Permute\ T_{w_i}(\phi) \tag{8.4}$$

where the constant $c(w)$, the number of subcycles in the word w, is defined in App. A. The normalization constants used above are convenient but not essential.

Now we construct the matrix elements

$$t_{i,j} = \langle 0 \mid F_{m,i}^{\dagger}(\phi) F_{m,j}(\phi) \mid 0 \rangle \quad (no\ sum) \tag{8.5}$$

where these are linear combinations of the $K_{w,w'}$ defined in (6.7) and Eq.(6.6) is replaced by

$$\sum_{j=1}^{D} t_{i,j}\ R_j^{(m)} = \frac{1}{2}\delta_{i,1}, \qquad i = 1 \ldots D\ . \tag{8.6}$$

In order to save computing time in evaluating each $t_{i,j}$ it is important to find and to count repeated evaluations of the same K elements. I am not sure that I have done this job completely in my program.

8.2 Searching

The hardest part of this program is searching for the minimum energy in the parameter space of the basic word connected parts: $X_{w_i} \equiv x_i,\ i = 1 \ldots D$. The first method used fits a quadratic function to E(x) evaluated at D(D+1)/2 nearby points and then finds the extremum:

$$b_i = E(x_i + \delta) - E(x), \qquad a_{i,j} = E(x_i + \delta, x_j + \delta) - E(x) - b_i - b_j \tag{8.7a}$$

$$\sum_{j=1}^{D}\ a_{i,j}\ v_j = b_i - \frac{1}{2}a_{i,i}, \qquad x_i' = x_i - \delta v_i. \tag{8.7b}$$

If one is close enough to the minimum, iterating this procedure should converge rapidly. For most of the data presented in Sec. 7 this method worked, although I am sure that more sophisticated techniques could have been more efficient. For the largest size computation carried out (d=5, n=8, D=72) the time for each evaluation of the energy was about one minute and each iteration of this search procedure took about 44 hours on a common desktop microcomputer.

Sometimes, however, this approach failed. For the potential V_4, beyond the sixth order calculation (for d=2 and d=3) this method diverged or led to impossible output (see the next subsection). What succeeded in those cases was a second method: start by solving the numerical problem for some other potential (like V_3 where the first search method worked well) and then gradually change a coupling constant g inserted into the potential and solve again, repeating in small steps until one arrives at the desired result. At each new step one can start efficiently with a sort of perturbation theory

$$\sum_{j=1}^{D} a_{i,j}\Delta_j = \delta^2\ \frac{\partial^2 E(x)}{\partial g\ \partial x_i}, \qquad x_i' = x_i - (\delta g)\Delta_i \tag{8.8}$$

which involves the matrix $a_{i,j}$ (8.7a) which one has already calculated at the previous step.

Just because the numerical search appears to converge is no proof that we have found the correct solution. In work on the potential V_4 for $d = 2$ we had some results at the sixth order (D=8) which first appeared well converged by the first searching method; but a later check on the rotational symmetry (7.2) showed that this was a false solution. Repeating this calculation using the second search method described above led to satisfactory results. The fact that the false energy value was off only in the fifth decimal place stands as a cautionary note on this new numerical technique.

Another numerical searching procedure is suggested by the algebraic work in Sec. 5. One could vary only the subset of X_w's with $[w] = n^*$, keeping all others fixed; then cycle through the choices of n*.

It should be repeated that this is all experimental work that is in need of sound mathematical justification and guidance. The multidimensional energy surface $E(x)$ is a very complicated nonlinear function of the parameters x. In fact, there are singularities which may lie not far away from the desired minimum. One can see the simplest example of this situation in the 2x2 matrix equation (8.6) for the d=1 case.

8.3 Constraints

The quantities X_w cannot be regarded as completely independent variables. For example, in the one-matrix case one has

$$\langle 0 \mid (\phi^2 - < \phi^2 >)^2 \mid 0 \rangle \geq 0 \tag{8.9}$$

which leads to the inequality $X_4 \geq -(X_2)^2$.

Using the general Schwarz inequality, we can write

$$|\langle 0 \mid T_w T_{w'} \mid 0 \rangle|^2 \leq \langle 0 \mid T_{\bar{w}} T_w \mid 0 \rangle \langle 0 \mid T_{\bar{w}'} T_{w'} \mid 0 \rangle \tag{8.10}$$

for all words w and w'. This implies many constraints upon the allowed values of the X parameters as we search to minimize the energy. It is unclear how best to implement these constraints; in the computations reported here I only checked that the matrix (8.5) satisfied

$$|t_{i,j}|^2 \leq t_{i,i}\, t_{j,j}, \qquad t_{i,i} > 0 \quad \forall i, j \tag{8.11}$$

at each evaluation. A failure of this test signals that the search has strayed into forbidden territory.

An entirely different sort of constraint comes from the use of a purely real (rather than complex) representation for the ϕ operators. This implies that we should have $X_w = X_w^* = X_{\bar{w}}$. With the extensive symmetry of the problems studied here many of these constraints are automatic; but at the 10th order for $d = 2$ and at the 8th order for $d > 2$, one finds some basic words that do not satisfy $\bar{w} \approx w$. Rather than imposing this constraint, we are satisfied to find that this equality comes out in the numerical results.

9 Excited States

After the ground state problem is solved, we consider excited (adjoint) states in the reduced Hilbert space:

$$H \mid 0\rangle = E_0 \mid 0\rangle, \qquad H \mid E\rangle = E \mid E\rangle \tag{9.1}$$

where it should be remembered that we do not know the form of the reduced Hamiltonian H [6] but only that it generates time translations. With the postulate

$$\mid E\rangle = U \mid 0\rangle \tag{9.2}$$

for some operator U we find the identity

$$(E - E_0)\langle 0 \mid U^\dagger U \mid 0\rangle = -i\langle 0 \mid U^\dagger \dot{U} \mid 0\rangle. \tag{9.3}$$

Now we make the construction, as with F before,

$$U = \sum_w r_w T_w(\phi), \qquad U^\dagger = \sum_w r_w^* T_{\bar{w}}(\phi) \tag{9.4}$$

and we have, using (A.9),

$$\dot{U} = \sum_w r_w \sum_{w=w_1 m w_2} T_{w_1} \pi_m T_{w_2} \tag{9.5}$$

where the r_w are as yet undetermined constants.

We can now write (9.3) as

$$E - E_0 = (\sum_{w,w'} r_w^* L_{\bar{w},w'} r_{w'})/(\sum_{w,w'} r_w^* K_{\bar{w},w'} r_{w'}) \tag{9.6}$$

where the matrix $K_{w,w'}$ was defined earlier and from (A.10) we have

$$L_{w,w'} \equiv -i\langle 0 \mid T_w \dot{T}_{w'} \mid 0\rangle = \frac{1}{2} \sum_{w=umv} \sum_{w'=u'mv'} K_{u,v'} K_{u',v}. \tag{9.7}$$

Finally, vary the coefficients r to find stationary values of (9.6) and we get a traditional linear matrix problem, where $E - E_0$ is an eigenvalue of the matrix L with respect to the metric matrix K.

The evaluation of the matrix K and thus also of L is done entirely in terms of the X_w's, which were already solved with the ground state problem. Thus (although I have not done any explicit numerical calculations for excited states) the complete spectrum of H can be calculated. The lowest order approximation, $U = T_m(\phi)$, gives $E_m - E_0 = 1/(2X_{mm})$.

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Appendix A. Useful Formulas Old and New

Further conventions on the word notation:
$w = 0$ is the null word.
$w = m$ means that the word w consists of a single letter m.
$w \sim w'$ means that the two words differ by at most a cyclic permutation of their letters.
$w \approx w'$ means that the two words are equivalent under some larger symmetry.
$w_1 w_2 = w_3$ means that the second word is appended to the first word and the result is the third word.
$w = umv$ means that the word w is decomposed as indicated.
$\bar{w}$ is the word formed by reversing the sequence of letters in the word w.
$c(w)$, the number of subcycles in the word w, is defined as the largest integer k such that $w = u^k$ for any word u with $[u] > 0$.

Basic relations among $T(\phi)$ and X [7]:

$$T_{mw} = \phi_m T_w - \sum_{w=w_1 w_2} X_{m w_1} T_{w_2} \tag{A.1a}$$

$$T_{wm} = T_w\phi_m - \sum_{w=w_1 w_2} T_{w_1} X_{w_2 m} \tag{A.1b}$$

$$X_{mw} = \langle 0 \mid \phi_m T_w \mid 0\rangle = \langle 0 \mid T_w \phi_m \mid 0\rangle = X_{wm} \tag{A.1c}$$

$$T_w^\dagger = T_{\bar{w}}, \qquad X_w^* = X_{\bar{w}}. \tag{A.1d}$$

Relation between X and Z:

$$Z(j) = 1 + X(jZ(j)). \tag{A.2}$$

Examples (for the case of parity symmetry, which means that each letter must appear an even number of times or else the Z and X vanish):

$$Z_{mn} = \langle 0 \mid \phi_m \phi_n \mid 0\rangle = \delta_{mn} X_{mm} \tag{A.3a}$$

$$Z_{mnpq} = Z_{npqm} = \begin{cases} X_{mmmm} + 2X_{mm}^2 & if \;\; m = n = p = q \\ X_{mmpp} + X_{mm}X_{pp} & if \;\; m = n \neq p = q \\ X_{mnmn} & if \;\; p = m \neq n = q. \end{cases} \tag{A.3b}$$

For one-matrix problems the label w is replaced by $n = [w]$. For systems with parity selection rule:

$$T_0 = 1, \qquad T_1 = \phi, \qquad T_2 = \phi^2 - X_2, \qquad X_2 =< \phi^2 > \tag{A.4a}$$

$$T_3 = \phi^3 - 2\phi X_2, \qquad T_4 = \phi^4 - 3\phi^2 X_2 - X_4 + X_2^2 \tag{A.4b}$$

$$X_4 =< \phi^4 > -2X_2^2, \qquad X_6 =< \phi^6 > -6X_4X_2 - 5X_2^3. \tag{A.4c}$$

Below are some new relations involving $T(\phi)$ that are used in the present work. Start with the generating function

$$Y = 1/(1 - \beta_m\phi_m + X(\beta)) = \sum_w \beta^w T_w(\phi) \tag{A.5}$$

and calculate the commutator,

$$i[\tilde{\pi}_m, Y] = Y\beta_m \mid 0\rangle\langle 0 \mid Y. \tag{A.6}$$

Now expand in powers of β and match terms to find:

$$i[\tilde{\pi}_m, T_w(\phi)] = \sum_{w=w_1 m w_2} T_{w_1}(\phi) \mid 0\rangle\langle 0 \mid T_{w_2}(\phi). \tag{A.7}$$

The other version of this relation

$$i[\pi_m, \tilde{T}_w] = \sum_{w=w_1 m w_2} \tilde{T}_{w_2} \mid 0\rangle\langle 0 \mid \tilde{T}_{w_1} \tag{A.8}$$

comes from Eq.(D.11) in Ref. [7]. In a very similar way one gets the time derivative equation

$$\frac{d}{dt}T_w(\phi) = \sum_{w=w_1 m w_2} T_{w_1}\pi_m T_{w_2} \tag{A.9}$$

where we have used $\frac{d}{dt}\phi_m = \pi_m$. Combining the last two equations leads to

$$i\langle 0 \mid T_{w'}\frac{d}{dt}T_w \mid 0\rangle = -\frac{1}{2}\sum_{w=umv}\sum_{w'=u'mv'}\langle 0 \mid T_u T_{v'} \mid 0\rangle\langle 0 \mid T_{u'}T_v \mid 0\rangle \tag{A.10}$$

which is surprisingly simple.

Appendix B. Evaluating $< T_w T_{w'} >$

We seek some recursive procedure for evaluation of the matrix elements

$$K_{w,w'} = \langle 0 \mid T_w(\phi)T_{w'}(\phi) \mid 0\rangle = K_{w',w} \tag{B.1}$$

in terms of the connected parts X_w. Using equations (A.1a) and (A.1b) it is relatively easy to find the following relations

$$K_{wm,w'} = K_{w,mw'} + \sum_{w'=uv} X_{mu}K_{w,v} - \sum_{w=uv} X_{vm}K_{u,w'} \tag{B.2}$$

with the boundary counditions $K_{w,0} = K_{0,w} = \delta_{w,0}$. This looks very nice as a recursive computer program but it turns out to be expensive: the time required grows exponentially as one increases the size of the words. One could save time by building a table of all the K matrix elements one might need, but that requires enormous amounts of space.

An alternative method is given by the following formula

$$K_{w,w'} = \sum_{w=uv}\sum_{w'=u'v'} K_{u,v'}X_{vu'}, \quad [v] > 0, \quad [u'] > 0, \qquad K_{0,0} = 1 \tag{B.3}$$

which may be derived by combining equation (A.1a) with the expansion

$$\phi_m = \sum_w X_{mw}G_{\bar{w}}(\phi) \tag{B.4}$$

from Ref. [7] and also using the identity

$$\langle 0 \mid T_w T_{w'} G_{\bar{w}''} \mid 0 \rangle = \sum_{w''=uv} \langle 0 \mid T_{w_1} T_{w_2} \mid 0 \rangle \; \delta_{w,uw_1} \; \delta_{w',w_2 v} \tag{B.5}$$

which is similar to the Ward Identities derived in App. E of Ref. [7].

The program uses (B.3) to build a small table of K's each time one of them is called for and the time for this grows as n^4 rather than exponentially. Still, this is the main time consuming part of the computations.

Appendix C. Some Alternative Computational Schemes

One alternative scheme is to start out by fitting the quantity $E_{mn}(\phi)$ instead of $F_m(\phi)$,

$$E_{mn}(\phi) = \sum_w R_w^{(mn)} T_w(\phi). \tag{C.1}$$

The definition (1.9d) is

$$E_{mn}(\phi) \mid 0\rangle = 2i[\tilde{\pi}_n, F_m(\phi)] \mid 0\rangle \tag{C.2}$$

and using (A.7), we find

$$R_w^{(mn)} = 2R_{wn}^{(m)} \tag{C.3}$$

upon comparison with (6.3). Next, we use the formal expansion from Ref. [7]

$$(E^{-1})_{mn} = \sum_w X_{wmn} G_{\bar{w}}(\phi) \tag{C.4}$$

to write the system of conditions

$$\langle 0 \mid T_{w'}(E_{mn}(E^{-1})_{np} - \delta_{m,p}) \mid 0\rangle = 0 \tag{C.5a}$$

$$\sum_{w,w''} X_{npw''} \langle 0 \mid T_{w'} T_w G_{\bar{w}''} \mid 0\rangle R_{wn}^{(m)} = \frac{1}{2} \delta_{m,p} \delta_{w',0} \tag{C.5b}$$

and one can show, using (B.3), that this reduces to equations identical to (6.6). So this method is not alternative at all.

A second alternative scheme does away with minimizing the energy and works instead from the equations of motion:

$$\dot{\pi}_m \mid 0\rangle = i\dot{F}_m(\phi) \mid 0\rangle = -V'_m(\phi) \mid 0\rangle. \tag{C.6}$$

Using the representation (6.3) for F_m, this leads us to a new system of equations

$$-i\sum_w \langle 0 \mid T_{w'}\dot{T}_w \mid 0\rangle R_w^{(m)} = \langle 0 \mid T_{w'}V'_m \mid 0\rangle \tag{C.7}$$

where the matrix elements on the left side are the quantities $L_{w',w}$ defined in Sec. 9. One now has *two* sets of matrix equations - (6.6) and (C.7) - determining the same set of expansion coefficients: call the solutions R and R'. One would now seek a set of values for the parameters X_{w_i} that would make these two sets of solutions the same. Computationally, the way to do this would presumably be to minimize the error,

$$Error \ = \sum_i |R_i - R'_i|^2 \tag{C.8}$$

and this defines another nonlinear search procedure. But what weight function ought optimally to be put into this error calculation?

A third alternative is to use the monomials ϕ^w instead of the polynomials $T_w(\phi)$ as a basis for the fitting of the operators F_m or U. This leads to much simpler formulas for the matrix elements of K and L, expressed in terms of the moments $Z_w = \langle 0 \mid \phi^w \mid 0\rangle$. Then one would use the relation (A.2) to evaluate each Z_w in terms of the chosen set of parameters X_w. I believe that this approach has drawbacks in both speed and numerical accuracy; but it should be explored.

Appendix D. Is this Method Useful in Ordinary QM?

With the apparent success of this approximation method in Large N matrix mechanics, one goes back to ordinary quantum mechanics to see if we have a new useful calculational technique. The formalism developed in Ref. [7] is easily modified to fit the standard commutation relation

$$i[p_i, q_j] = \delta_{ij} I \tag{D.1}$$

with the following construction:

$$Y = e^{\beta_i q_i - X(\beta)} = \sum_\mu T_\mu(q) \tag{D.2a}$$

$$X(\beta) = \sum_\mu C_\mu \beta^\mu X_\mu \tag{D.2b}$$

$$Z(\beta) = \langle 0 \mid e^{\beta_i q_i} \mid 0 \rangle = \sum_\mu C_\mu \beta^\mu \langle 0 \mid q^\mu \mid 0 \rangle. \tag{D.2c}$$

Here μ represents the unordered set of occupation numbers $\{\mu_i\}$ (remember that the q_i's commute with one another now) and

$$C_\mu = 1/\prod_i (\mu_i)! \; . \tag{D.3}$$

In the simple one-matrix case we have

$$Z(\beta) = \sum_{n=0}^{\infty} \beta^n Z_n/n!, \qquad Z_n = \langle 0 \mid q^n \mid 0 \rangle = \int dq \; q^n \rho(q) \tag{D.4a}$$

$$X(\beta) = \sum_{n=1}^{\infty} \beta^n X_n/n! = ln(Z(\beta)) \tag{D.4b}$$

and we want to test whether the ratio X_n/Z_n decreases rapidly with n, as we saw for the Large N situation in Sec. 2. For the case of a harmonic oscillator, we have the same result in both theories, namely X_n vanishes for $n > 2$.

One simple (non-oscillator) model that allows analytic calculations is a constant density $\rho(q)$ over some finite range of q. Here we find that the ratio X_n/Z_n decays rapidly with n for the Large N situation but this ratio grows very rapidly for the ordinary quantum mechanics situation.

We have also applied the method of this paper to the quantum mechanical nonlinear oscillator,

$$H = \frac{1}{2}p^2 + \frac{1}{4}q^4. \tag{D.5}$$

Numerical results for the ground state are shown in Table 10. The convergence seen here is fairly good, although not as good as for the similar Large N problem, shown in Table 6. (The accuracy shown here is comparable to that obtained with conventional variational calculations of this Schrodinger equation, at the same levels of approximation.)

Table 10. Results for the Schrodinger equation (D.5)

n	D	E	X_{11}
2	1	.429	.437
4	2	.4217	.4525
6	3	.4210	.45512
8	4	.42086	.45571

It must be reported, however, that the results shown in Table 10 were not obtained easily. The problem of nearby singularities in the energy surface, mentioned in Subsec. 8.2, was more severe in this ordinary quantum mechanics problem than in the Large N problems. For the calculations through $D = 3$ I used the second searching method, starting from the harmonic oscillator and then moving gradually to the quartic potential in steps of size 1/8. For $D = 4$ I had to decrease the step size to 1/16; and for $D = 5$ I gave up after failing in the search procedure with step size 1/32.

In conclusion, I am still in doubt about the answer to the question posed in the heading of this appendix.

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