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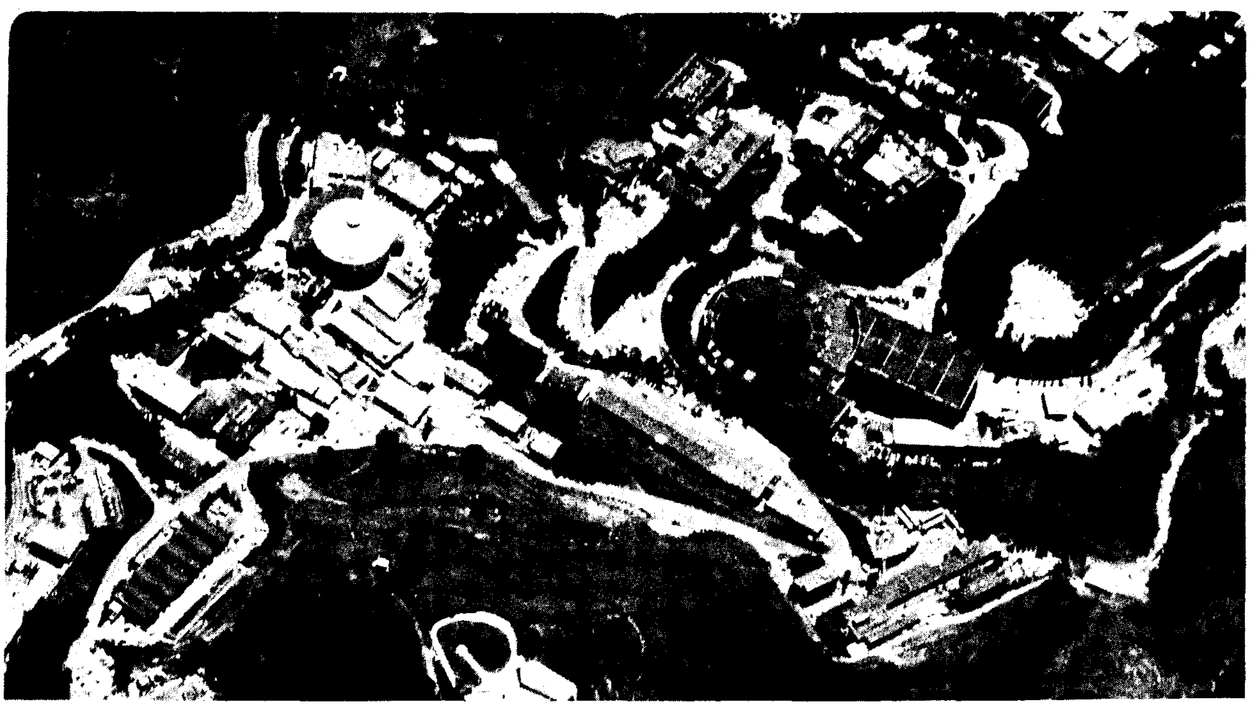

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NEW Δ BARYON PARAMETERS FROM A BARRELET ANALYSIS OF ELASTIC π^+p DATA BETWEEN 0.6 AND 2.3 GeV/c

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ABSTRACT

We have performed a Barrelet analysis of all available π^+p elastic scattering and polarization data. After selecting those with the smallest errors, we have determined the zeros of the amplitude closest to the physical region by applying principles of analyticity and causality; unitarity is used to determine the phase of the amplitude. The partial waves are obtained at $\simeq$ 50 independent energies by a projection on the orthonormal polynomial $R_{J\varepsilon}$, up to the order allowed by the statistical significance of the data. The mass, total and elastic widths are obtained by an energy dependent fit to Breit-Wigner formulae. Although most of the resonances we find have already been observed, we report striking differences in parameters, especially in our total widths which are systematically much narrower. We discuss possible reasons for the discrepancy with earlier analysis.

(*) On leave of absence from the University of Paris VI, Paris, France.

1. INTRODUCTION

Deduction of baryon parameters from experiment requires partial wave analysis. After obtaining the modulus of the amplitude from a fit to the experimental data, the phase has to be obtained from theoretical considerations. The quality of differential cross section data obviously limits both the accuracy of the determination of the amplitude modulus and the number of possible projections on orthonormal polynomials, allowing determination of only a limited number of partial waves at each energy. Another limitation comes from the need to resolve the discrete ambiguity, [1,2,3] which requires accurate determination of the critical points [1,2]. Extremely accurate polarization data is essential here(*). The lack of importance given in most partial wave analysis to resolution of the discrete ambiguity may explain the great spread in mass and width values found in the Particle Data tables.

We shall present here the results of a partial wave analysis based on principles of analyticity, unitarity and causality with the use of the Barrelet method of data analysis [1,2]. We have made amplitude determinations at about 50 independent laboratory momenta between 0.6 and 2.3 GeV/c, and thereby have achieved isospin 3/2 baryon resonance parameters with what we believe to be greater reliability than earlier partial-wave analysis.

2. THE PRINCIPLES OF OUR PARTIAL WAVE ANALYSIS

The major ingredients in our partial wave analysis have been described in ref. [3], although one of these ingredients was not entirely justified there by S - Matrix principles. We have added since ref. [3] a principle

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(*) Because the zero trajectories only cross the physical region when the polarization P is ± 1, and P = ± 1 occurs in πN interactions simultaneously with strong minima in the differential cross section [1,2].

of analyticity for modulus determination [2] and now understand how unitarity underlies our phase determination. We proceed to develop these two new points.

Because the amplitude is analytic in cosθ, so is its squared modulus. Therefore if we use the notations of ref. [1] and call F(w) the amplitude ($w = e^{i\theta}$), the "ideal" data obey the relation

$$\Sigma^{\pm}(w) = F(w)\,\overline{F(\bar{w}^{-1})} = \frac{d\sigma}{d\Omega}(1 \pm P) \tag{1}$$

with $0 \leqslant \theta \leqslant 2\pi$, the plus (minus) sign for $0 < \theta < \pi$ ($\pi < \theta < 2\pi$). The quantities $\Sigma^{\pm}(w)$, like the amplitude, may be approximated by a convergent series of polynomials; the zeros of the amplitude are also zeros of its squared modulus, the complete set of the latter's zeros being obtained by adding the complex conjugates of the amplitude zeros.

The method used to obtain the polynomial approximation to the analytic function constituted by the ideal data of eq. (1) is that of Barrelet moments [1]. The coefficients - all independent of each other and not depending on where the polynomial approximation is terminated(*) - are calculated as pseudopolynomial moments of the experimental distributions, namely for the ith order

$$a_i = \int \frac{d\sigma}{d\Omega} \cdot p_i(w) \tag{2}$$

$$b_i = \int P\,\frac{d\sigma}{d\Omega} \cdot q_i(w)$$

so that

$$\Sigma^{\pm} \simeq \sum_i (a_i\, p_i \pm b_i\, q_i). \tag{3}$$

Thus at each energy the data are represented in terms of "data zeros" by the polynomial form: $\text{constant} \times \prod_{i=1}^{N} (w - w_i)\overline{(\bar{w}^{-1} - \bar{w}_i^{-1})}$, N even. (4)

(*) Which is never the case, of course, for any minimum Chi-squared method.

The pseudo-polynomials p_i and q_i (polynomials with both positive and negative powers of w so as to cover the complete w-plane in which the physical region is the unit circle) are chosen to be orthonomal on the measured cosθ interval, this last condition being necessary to insure convergence of the series (3). For such a convergent series the coefficients then possess, for ideal data, the property of decreasing exponentially with increasing order [1,2] for sufficiently high order. For data with statistical errors but of high density in cosθ and free from systematic error, the higher coefficients will, within statistics, be compatible with such an exponential decrease. Roughly speaking the series should be terminated at the first coefficient which becomes compatible with zero within its statistical error; the exponential decrease is only meaningful up to this point.

Now, with "real" data, one observes [2] that the coefficients do not always behave in this way: for some scattering experiments of very high statistics, one observes the exponential decrease to a plateau compatible with zero for the order between say six to ten, and then a rise of coefficients of higher order before returning to a plateau compatible with zero. When such "non-analytic" behaviour is observed, we assume systematic error of some data points to be responsible. By studying, for each individual point of the set of measurements, the chi-squared measure of deviation between the data and the curve whose order corresponds to the beginning of the first zero plateau, one is able sometimes to pin point which are the point(s) with a systematic error larger than statistical. By this criteria, we could also replace a collection of data points in a certain cosθ interval from one set of data by other data (for an example, see ref. [4] with respect to the data of refs 5(b) and 6). We judge an isolated point as responsible for "non-anlytic" behaviour if its removal restores "analytic" moment behaviour and its individual chi 2 is larger than 9.0 - the probability of it being on the "analytical" curve being therefore smaller than 0.3%. This method of identification of systematic errors larger than statistical [2] has allowed a severe selection of scattering data.

The surviving data used in this analysis are part of ref. [5] (scattering differential cross sections) and all of ref. [7] (polarization). These latter data having no competitors for accuracy and density in $\sqrt{s}$, but being of low density in $\cos\theta$ ($\lesssim$ 40 data points per energy except for a few sets), could not be tested for systematic error. Not all of the polarization measurements could be used in the determination of the zero trajectories (as we also required that the scattering data p_{lab} momentum not be more than 20 MeV/c away from the P_{lab} momentum of the polarization). However, the remaining polarization data sets proved to be very useful in the determination of possible critical points with one star [2,3].

We explain in the following section how we determine "data zeros" of the amplitude and their trajectories $w_i(s)$ in the w plane as a function of energy. One needs a knowledge of the Barrelet moments at each energy as well as a knowledge of the critical points where the polarization is ± 1. Barrelet [1] showed that up to a factor of the form $1/w^p$, where p is an integer, the w dependence of the amplitude is consistently approximated in terms of its zeros by the product

$$\prod_{i=1}^{N} (w - w_i) \, , \qquad (5)$$

where N is the order of the highest statistically significant moment. Subsequently Barrelet and Urban proved [10] on the basis of positivity for partial-wave imaginary parts, that the integer p is either N/2 or (N + 2)/2 if N is even(*) . We showed through an empirical study [3] that the choice must be N/2(**) .

(*) N must be even for πN interactions because - as observed in refs [1,2, 3,8,9] - one minimum in the differential cross section is always observed to be associated with at least two (sometimes 4) extrema in the polarization; giving rise to 2 (or 4) zeros. The total number of amplitude zeros can be 2 (4, 6 etc.) associated with one (2, 3 etc.) minima minimum (a) in the differential cross section.

(**) For example when N = 6, the choice p = 4 makes the F37 wave equal to zero, while the choice p = 3 makes the G37 wave zero. The ambiguity is thus related to that of Minami.

There remains the overall normalization of formula (4). For this we take the forward amplitude (modulus and phase) as determined from total cross sections and dispersion relations [11]. Our full expression for the amplitude at a given energy $\sqrt{s}$ is then

$$F(w) \simeq F(w_o)\left(\frac{w}{w_o}\right)^{-N/2} \prod_{i=1}^{N} \left(\frac{w - w_i}{w_o - w_i}\right) \qquad (6)$$

where w_o corresponds to some fixed angle ($\theta=o \rightarrow w_o=1$).

This expression(*) has been criticized as not explicitly recognizing constraints from analyticity in energy at non forward angles, (the forward amplitude being determined from dispersion relations is, of course, not controversial). Formule (6) does not however, ignore analyticity in energy. The notion that the zero trajectories $w_i(s)$ are analytic functions of s is heavily used in resolution of the discrete ambiguity. We do not, nevertheless, risk prejudicing the outcome by imposing an a priori form of s-dependence on our trajectories-- which would amount to a guess about the location and density of amplitude s-poles (resonances). We prefer to let the data give the answer.

A second observation is that the choice of the phase factor $w^{-N/2}$ in Formula (6) ensures compatibility with a partial wave expansion where the maximum angular momentum is $\frac{N+1}{2}$. If Formula (6) is projected onto the pseudopolynomials $R_{J\varepsilon}(w)$ to give partial waves $T_{J\varepsilon}$ (see next paragraph), then a partial wave expansion terminating at $J_{max} = \frac{N+1}{2}$ is precisely equivalent to Formula (6). We shall have more to say later about the phase of Formula (6) at non forward directions (when we try to extract the "continuous phase" from more theoretically oriented partial wave analysis ; see the last section).

(*) Pennington has proposed a corresponding prescription for analysis of reactions where the $\pi\pi$ Channel is involved. See for example Ann.Phys. 114 (1978),1.

3. "DATA ZEROS" OF THE AMPLITUDE AND THEIR TRAJECTORIES

Let us first discuss the results obtained for the zeros of the squared amplitude. From the analysis described above, applied to the selected data, we determine that the amplitude can be approximated by 6 zeros for $0.6 \leq P_{lab} \leq 1.2$ GeV/c and by 10 zeros in the range 1.2 - 2.3 GeV/c. These numbers are higher than in ref. [3] because of the higher precision of the data used here. As stated earlier, adding higher-order coefficients is meaningless when these coefficients are compatible with zero within their statistical errors. They correspond to "unstable zeros" [1] i.e. zeros whose positions cannot be followed from one energy to the next but rather seem random . When looking at data [5,6,7] one observes smooth variation in the shape of both $d\sigma/d\Omega$ and P. According to eq. (1) a polarization minimum (P near -1) is associated with a zero of Σ^+ and a polarization maximum (P near +1) with a zero of Σ^-. We designate the latter zeros, belonging to the amplitude angular interval $\pi < \theta < 2\pi$, by the letters (A), (C), (E), (G) and (I) while the former zeros, belonging to the interval $0 < \theta < \pi$, are labelled by the letters (B), (D), (F), (H) and (J). Because of the continuity in energy of the polarization maxima and minima, it is possible to follow each of these zeros from one energy to the next, defining a "trajectory".

Some zeros are easier to follow than others. The easiest are those that come close to the physical region where P = ± 1. Those near the forward (G and H) or near the backward (I and J) directions are difficult to follow because here the polarization is necessarily small.

Zeros in $\Sigma(w)$ come in pairs, at w_i and $\overline{w_i^{-1}}$, each pair corresponding to one zero of the amplitude (see eq. 4). The problem of the "discrete ambiguity" is to determine at each energy which of the two paired zeros in $\Sigma(w)$ belongs to the amplitude $F(w)$. To resolve the discrete ambiguity we first determine all the possible "critical" points in s and θ where zeros trajectories cross the physical region. The polarization at critical points is exactly ± 1, but zero trajectories tend to move rapidly in w near critical points so it is necessary to have precise data, dense both in s and θ in order to be sure that the physical region has been crossed.

Because this step of the analysis is extremely important, being a major source of differences from earlier analysis, we have spent a long time checking the polarization information given by the polynomial representation of the data and the data itself. Critical points are labelled 4^* to 1^* in reliability according to the criteria described in ref. [2]. Our list of critical points is given in the first column of table 1.

One needs next to know for at least one energy for each trajectory whether the zero is located inside or outside the physical unit circle in the w plane. Here we partially rely on the causality principle that constrains certain trajectories near an isolated resonance. This principle ties down six trajectories (A, B, C, D, E and F) in the Δ(1900 region as described in ref. [2]. The Δ(1238) region provides a similar anchor point for two trajectories (B and C). A further helpful principle is that new zeros with increasing energy arise in pairs, one member of a pair outside the physical unit circle and the other inside(*). Combining these principles with our table of critical points allows us unambiguously to determine the first six trajectories.

What about the remaining four, G to J? We unfortunately could not find accurate scattering and polarization data above 2.3 GeV/c so we cannot use causality to anchor trajectories in the Δ(2400) region. However, the principle regulating the origin of new trajectories leaves only four ambiguities: (G) in, (H) out and (I) in/out, (J) out/in, or (G) out, H (in) and (I) in/out, (J) out/in. With the first six trajectories secure, we tried the four different hypotheses and observed what happened to the $J^P = 7/2^+$ resonance in the 1900 MeV mass region. Out of the four possibilities the first one gave clockwise phase behaviour for the $T_{7/2^+}$ partial wave, in disagreement with causality. Two of the three remaining possibilities for G, H, I, J: (out, in, out, in) and (in, out, in, out) show no sign of any Δ resonance. Only the solution (out, in, in, out) produced a resonating Δ(1900) as observed with six zeros only and lower statistical data [3], and so we kept it for the final solution. Table 1 lists the complete identity of the trajectories as we have found them together with their critical points: there remains lots of points to be checked with higher statistical experiments at greater density in s.

(*) A pseudo-polynomial, when its order is increased by one unit, adds two terms: w^r and w^{-r} respectively.

We would like at this point to make a remark about the location (in the w plane) of the "data zeros" according to (6) versus what can be called the "true" amplitude zeros.

Because the partial-wave expansion of the full amplitude converges only within a singularity-free ring surrounding the unit circle in the w plane, data zeros can approximate "true" amplitude zeros only within this ring, whose width shrinks with increasing momentum. The outer radius(*) of the singularity-free ring is roughly

$$1 + \frac{1}{\ell_{max}} \quad , \qquad \ell_{max} > 1$$

where $\ell_{max} \propto p_{cm}$ is the largest "important" orbital angular momentum. For data of normal accuracy, $\ell_{max} \simeq J_{max} - \frac{1}{2} = N/2$, because partial waves decrease exponentially with increasing ℓ for $\ell > \ell_{max}$. The usual result of Barrelet zero analysis is thus a set of N data zeros distributed throughout a ring of outer radius $\sim 1 + 2/N$. Those zeros well inside the ring ("nearby" zeros) approximate true amplitude zeros but a few are typically outside or on the fringe of the singularity free ring. These fringe zeros correspond to fringe angular momenta whose influence on the data is marginal. That is, if such fringe zeros were omitted in fitting the data (thereby decreasing N and eliminating fringe partial waves), there would be only a modest shift in the positions of the remaining zeros and a relatively small change in the surviving partial waves. Conversely, inclusion of fringe zeros (in order to fit all features of the available data), although these may not correspond to true amplitude zeros, does not undermine the reliability of the determination of "nearby" zeros and, correspondingly, of the large partial waves. The resonance characteristics (position and width) of such large waves is determined primarily by the trajectories (as a function of energy) of nearby zeros.

(*) The inner radius of the ring is the reciprocal of the outer radius.

4. RESULTS

For each energy where there exists (after our screening process) accurate scattering data and also polarization measurements (p_{lab} values for scattering and polarization measurements must be within 20 MeV of each other), we have built the amplitude F(s, w) by eq. (6). Its projection onto the orthonomal functions $R_{J\varepsilon}$ [1,3] then gives the partial waves

$$T_{J\varepsilon}(s) = \frac{\lambda^{-1}}{2(2J + 1)} \int_{(c)} F(s, w)\, \overline{R_{J\varepsilon}}(w)\, dw. \qquad (7)$$

As in ref. [3] statistical data errors [12] are translated linearly into errors of the Barrelet moments and thence into errors of the partial waves [13] . Our partial waves are shown in figs 1 to 4 for energies between 1500 and 2300 MeV. Visual exmaination of the correlation between Re $T_{J\varepsilon}$ and Im $T_{J\varepsilon}$ shows resonance behaviour in all partial waves with $\ell < 3$. For the four partial waves with $\ell = 4, 5$, only G_{37} gives a hint of a possible resonance. The clearly visible resonances are for $\ell = 2, 3$ (in decreasing order) being F_{37}, ($\sim$ 1900 MeV), F_{35} ($\sim$ 1800), D_{35} ($\sim$ 1900) and D_{33}(1600-1700 MeV). The P_{33} spectrum suggests a resonance in the $\simeq$ 2000 MeV region.

In the P_{31} and S_{31} spectra, real-part oscillations occur near the $\ell = 2, 3$ resonances ($\sim$ 1600-1700, 1800, 1900 MeV), corresponding to the fact that the zero trajectories are behaving smoothly, (near an isolated J = 1/2 resonance all zeros must recede either to infinity or to the origin of the w plane because $R_{\frac{1}{2}\varepsilon}(w)$ **has no zeros). Smoothness of zero** trajectories is well known to imply degeneracy between resonances in different partial waves [14].

We have made minimum chi-squared fits [15] to our partial-wave amplitudes using for each apparent resonance a Breit-Wigner formula plus linear background, together with a threshold factor (*). The resulting resonance parameters are shown in table 2. A variety of differences are found from the results of analyses which amalgate data from different energies in the process of determining amplitudes and which consequently make a different resolution of the discrete ambiguity. We now review the novel aspects of our results.

With respect to the well-established resonances in F_{37}, F_{35}, D_{35} and D_{33}, we find moderate mass shifts (between 50 and 100 MeV) and more importantly, widths narrower by a factor 2-3 than given by previous analysis [16] . Regarding the low-ℓ partial waves S_{31} and P_{31} we have explored an arbitrary 3-resonance fit over the 1600-2000 MeV interval, but the sparcity of data makes it impossible to determine so many parameters.

We do not confirm the existence of the P_{33} ($\sim$ 1700) as necessary for an adequate fit in the 1500-1800 MeV region as we observe first the indication of a P_{33} resonance at $\sim$ 2000 MeV (Fig 2b). The fit with two resonances gives very little improvenent in the 1500-1800 MeV mass region over the fit with one resonance only. This latter on the other hand leads to a very narrow resonance which may be due to a wrong parametrisation of the background after the Δ(1232) resonance.

In summary we observe the already well established ℓ = 2, 3 Δ resonances but generally with lower masses and much narrower widths than previously reported, the widths in fact being of the same order of magnitude as that of the Δ(1232). For the ℓ = 0, 1, J = 1/2 partial waves we find evidence for systematic resonance degeneracy with higher ℓ values. We observe a P_{33} resonance near 2000 MeV. FInally we confirm the already-reported indications of resonances in the 2100-2200 MeV mass region. We believe the H311 partial wave to be larger that it should be because of the impossibility (from a statistical point of view) of determining higher partial waves from the data.

(*) We have used the form $(1 + 1/(qr)^2)^{-\ell-\frac{1}{2}}$, where r is a "radius" parameter determined by the best fit and q is the C.M. momentum.

5. COMPARISON OF FORMULA (6) WITH PREVIOUS PARTIAL WAVE ANALYSIS

In this section, we present reasons for believing that formula (6) used in conjunction with careful data selection and analysis may be more reliable than amplitude construction procedures which use explicit energy dispersions relations as a supplementary condition ([17]).

As pointed out above, besides the possible systematic degeneracy of S31 and P31 resonances with resonances having $\ell \neq o$, -- which we cannot prove completly due to the insufficient quality of the data -- our main new result is the narrow width of the F37, F33 and D35 resonances. We believe the selection of data made possible by Barrelet analysis, as well as the resolution of the discrete ambiguity, to be responsible.

However, it has been suggested ([18]) that our disagreement with other analysis is because we omitted in Formula (6) a phase factor that varies with w. This is the celebrated "continuum ambiguity"--- the possibility of adding to Formula (2) a phase factor depending exponentially on w and thus without zeros(*). Such a factor has no effect on the data but changes the amplitude and thus, in principle, affects the determination of resonance parameters. It has been suggested ([18]) that if the continuum ambiguity is ignored, large errors may results in resonance width determinations. Energy dispersion relations have been proposed as a necessary device to determine the unknown phase.

(*) The possibility of such a phase, $\phi = \phi_o(1 + A\cos\theta + B\sin\theta)$-- with A and B real (to satisfy (2)) and allowed to vary in order to study the uncertainties due to the continuous ambiguity, has been empirically tested near the $\Delta(1232)$. Formula (6) without an extra phase factor satisfies (elastic) unitarity in the region of the $\Delta(1232)$; only for A (or B) $<$ 1% can unitarity continue to be satisfied within the statistical errors.

The continuum ambiguity, however, can only arise from the contribution of partial waves with $J > \frac{N+1}{2}$. Is it possible for such waves to contribute a large change of phase without a large change in the N lower waves (and a corresponding large change in the modulus of the amplitude)? Given that the individual moduli of such partial waves must decrease exponentially with increasing J, we find such a circumstance difficult to imagine.

As a test of such general considerations we have studied the amplitude that was determined in Ref. (17) through the aid of energy dispersion relations, an amplitude

$$F(w) = \sum_{J\varepsilon} T_{J\varepsilon} \quad R_{J\varepsilon}(w)$$

with many more partial waves than ours and whose resonance widths are much broader. Do the extra partial waves correspond to the phase factor which may be missing from Formula (6) ?

We have been able to establish in the neighborhood of 1900 MeV c.m. energy, where the sequence of large partial waves terminates at JP = 7/2 +, that indeed the amplitude of Ref. (17) has six stable(*) nearby zeros, confirming the expected connection between N and J_{max}. For certain of these zeros the discrete ambiguity has been resolved oppositely to our resolution, a point to which we return below, but so far as representation of the data is concerned, it appears that there is approximate correspondence between the six dominant zeros of Ref. (17) and our six leading zeros. Now we find that, to achieve an adequate fit to all the angular structure observed in the data at these energies, four additional zeros are needed. At certain energies it is possible to identify four additional zeros from Ref. (17) with our four. We then asked the question: Do the remaining zeros of Ref. (17), which represent their extra partial waves, correspond at least approximately to a phase factor?

(*) The positions of these zeros remain almost unchanged when the partial-wave expansion is truncated at a total of 7, 11, 15, 19 partial waves.

The answer is negative. When we divide the Ref.(17) amplitude by the product of their first ten zeros, the modulus of the quotient has a large dependence on angle. (Fig. (4) shows 2 examples around $E_{cm} \simeq 1920$ MeV). The extra partial waves are not merely contributing a phase factor; they are apparently compensating differences in location between our last four zeros and the corresponding four zeros of Ref. (17).

Our position is that no more than ten zeros are needed to fit the data. If extra zeros could be added so as effectively to contribute no more than a phase factor, we should find it difficult to argue that our phase is correct. However, the amplitude of Ref. (17) fails to provide an example of a phase that does not come from "data zeros" -- zeros which generate visible angular structure.

Can the use of energy dispersion relations make it difficult to find narrow resonances ? In Ref. (17) the energy dependence of the amplitude was represented by a polynomial of fixed order. Such a representation obviously cannot accommodate zero -width resonances and will have difficulty with extremely narrow resonances. If at higher energy, the resonances remain sharp (widths $\lesssim$ 100 MeV) and at the same time increase in density, the representation of Ref. (17) is bound to encounter difficulty.

The use of energy dispersion relations is appropriate for deducing average amplitude characteristics such as Regge trajectories, but is unsuited to finding individual nearby poles on the unphysical sheet. Here the phase variation of the amplitude in the immediate neighborhood of the pole is essential. In Formula (6) this variation arises from the zero trajectories as a function of energy. We independently determine the positions of the zeros at each energy where data is available, so the energy dependence of our phase is unprejudiced by parameterization. We suspect that the parameterization of Ref. (17) may also have led for certain zeros to an incorrect resolution of the discrete ambiguity.

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REFERENCES

[1] E. Barrelet, Thesis (unpublished), Paris (1970), Nuovo Cimento 8a. 331 (1972).

[2] D.M. Chew, Phys. Rev. D18, 2368 (1978).

[3] D.M. Chew, M. Urban, LBL-4656 (June 1976), Proceedings of the Topical Conference on Baryon Resonances (Oxford, 5-6 July 1976), edited by R.T. Ross and D.H. Saxon, p.156-174.

[4] D.M. Chew, LBL-6102 (March 1977).

[5] Elastic π+p scattering data
(a) C. Hugues, Thesis (1972).
(b) K. Abe et al., Phys. Rev. D10, 3556 (1974).
(c) D.J. Bardsley et al., in Proceedings of the Topical Conference on Baryon Resonances (op. cited) P66.

[6] Elastic π+p scattering data in the backward direction.
(a) A.S. Carroll et al., Phys. Rev. 177, 2047 (1968).
(b) R.J. Ott et al., Phys. Letters 42B, 133 (1972).

[7] Elastic π+p polarization data
J. F. Martin et al., Nucl. Phys. B89, 253 (1975)

[8] E. Barrelet Proceedings of the Topical Conference on Baryon Resonances (op. cit.) p 175.

[9] (a) M. Urban, Thesis of 3rd cycle, Paris (1973).
(b) M. Urban, Thesis (unpublished), Paris (Nov. 1977).

[10] E. Barrelet et M. Urban, see Ref 9b, p 115.

[11] G. Hohler, M.P. Jacob and F. Kaiser, tables of pion-nucleon forward amplitudes, TPK 7616, University of Karlsruhe (May 1976).

[12] E. Barrelet and M.G. Fouque, Program "Moulin", Ecole Polytechnique Paris (1969), completed for the errors by the information of ref. 13 .

[13] M. Urban, LBL-4809 (Feb. 1976); journal de Physique 38 629 (1977).

[14] R. Odorico, Lett. Nuovo Cimento, 2, 655 (1969)

[15] We have used the Program Minuit, version of 8/75 (F. James et al.,) CERN.

[16] Particle Date Group Tables - Phys. Let. Vol 75B (June 78).

[17] The Karlsruhe - Helsinki 1978 partial wave Analysis.
see G. Hohler, F. Kaiser, R. Koch and E. Pietarinen, JSSN 0344-8401 (1978).

[18] G. Hohler and J. Sabba-Stefanesco (private communication).

TABLE CAPTIONS

Table I : list of the critical points and our resolution of the discrete ambiguity. The conventions for the attribution of 1* up to 4* to the critical points are identical to those of Refs 2 and 3.

Table II : parameters of the Δ resonances as determined in this analysis.

FIGURE CAPTIONS

1. The curves represent the energy-dependent fits to a Breit-Wigne Formula with a threshold-dependent factor, the results of which are in Table I, for the waves:

   (a) F37

   (b) F35

   (c) D35

   (d) D33

   (e) P33

2. Partial waves (a) S31, (b) P31

3. Partial waves (a) G37. (b) G39, (c) H39 and (d) H311.

4. Examples of contribution to the modulus of the amplitude from extra zeros determined from fixed-t dispersion relations [17]

   (a) at P_{lab} = 1.473 GeV/c ($\sqrt{s}$ = 1917.4 MeV)

   (b) at P_{lab} = 1.505 GeV/c ($\sqrt{s}$ = 1932.9 MeV)

TABLE I

Critical points	P_{lab} (GeV/c) $\sqrt{s}$ (MeV)	A	B	C	D	E	F	G	H	I	J (†)
		0	0	I	I	I	0	0	I	I	0
$C^{(****)}$, $F^{(**)}$	(.753)1529 ± 10										
		0	0	0	I	I	I	0	I	I	0
$E^{(**)}$	(.776)1545 ±										
		0	0	0	I	0	I	0	I	I	0
$C^{(*)}$	(.820)1520 ±										
		0	0	I	I	0	I	0	I	I	0
$E^{(**)}$	(.849)1586										
		0	0	I	I	I	I	0	I	I	0
$E^{(*)}$	(1.097)1720										
		0	0	I	I	0	I	0	I	I	0
$E^{(*)}$	(1.200)1780										
		0	0	I	I	I	I	0	I	I	0
$B^{(****)}$	(1.310)1820 ±										
		0	I	I	I	I	I	0	I	I	0
$A^{(****)}$	(1.332)1848 ±										
		I	I	I	I	I	I	0	I	I	0
$A^{(****)}$	(1.475)1918 ±										
		0	I	I	I	I	I	0	I	I	0
$E^{(*)}$	(1.480)										
		0	I	I	I	0	I	0	I	I	0
$B^{(****)}$	(1.491)1926 ±										
		0	0	I	I	0	I	0	I	I	0
$E^{(*)}$	(1.705)2025 ±										
		0	0	I	I	I	I	0	I	I	0
$A^{(*)}$	(1.729)2038 ±										
		I	0	I	I	I	I	0	I	I	0
$A^{(*)}$	(1.793)2068 ±										
		0	0	I	I	I	I	0	I	I	0
$E^{(*)}$	(1.800)2071 ±										
		0	0	I	I	0	I	0	I	I	0
$F^{(****)}$	(1.890)2110 ±										
		0	0	I	I	0	0	0	I	I	0
$E^{(****)}$	(1.926)2127 ±										
		0	0	I	I	I	0	0	I	I	0
$J^{(**)}$	(2.091)2198 ±										
		0	0	I	I	I	0	0	I	I	I
$E^{(*)}$	(2.100)										
		0	0	I	I	0	0	0	I	I	I
$F^{(****)}$	(2.110)2206 ±										
		0	0	I	I	0	I	0	I	I	I
$C^{(*)}$	(2.150)2220 ±										
		0	0	0	I	0	I	0	I	I	I

(†) Where 0(I) stands for Outside (Inside) the unit circle in the w-plane according to the side of the zero trajectory in between 2 critical points.

TABLE II : Δ RESONANCE PARAMETERS

RESONANCE	CENTER OF MASS ENERGY INTERVAL (MeV)	NUMBER OF DATA POINTS USED IN THE FIT	$M \pm \delta M^*$ (MeV)	Γ_{TOTAL}^* (MeV)	$\Gamma_{ELASTIC}^*$ (MeV)	$x = \frac{\Gamma_e}{\Gamma_t}$
F37	1600 - 2100	39	1876.8 ± 0.2	113.3 ± 0.8	62.7 ± 0.2	.55
F35	1640 - 2290	49	1802.6 ± 0.9	101.6 ± 1.2	11.4 ± 0.2	.11
D35	1500 - 2290	58	1901.2 ± 5.2	100.2 ± 16.0	18.2 ± 2.3	.18
D33	1500 - 2290	58	1722.6 ± 2.4	137.4 ± 11.0	14.9 ± 1.5	.11
P33	1525 - 2200	53	1955.0 ± 1.4	55.0 ± 2.6	14.0 ± 0.7	.25

* errors correspond to a change of one unit in the function of the chi2 to be minimized.

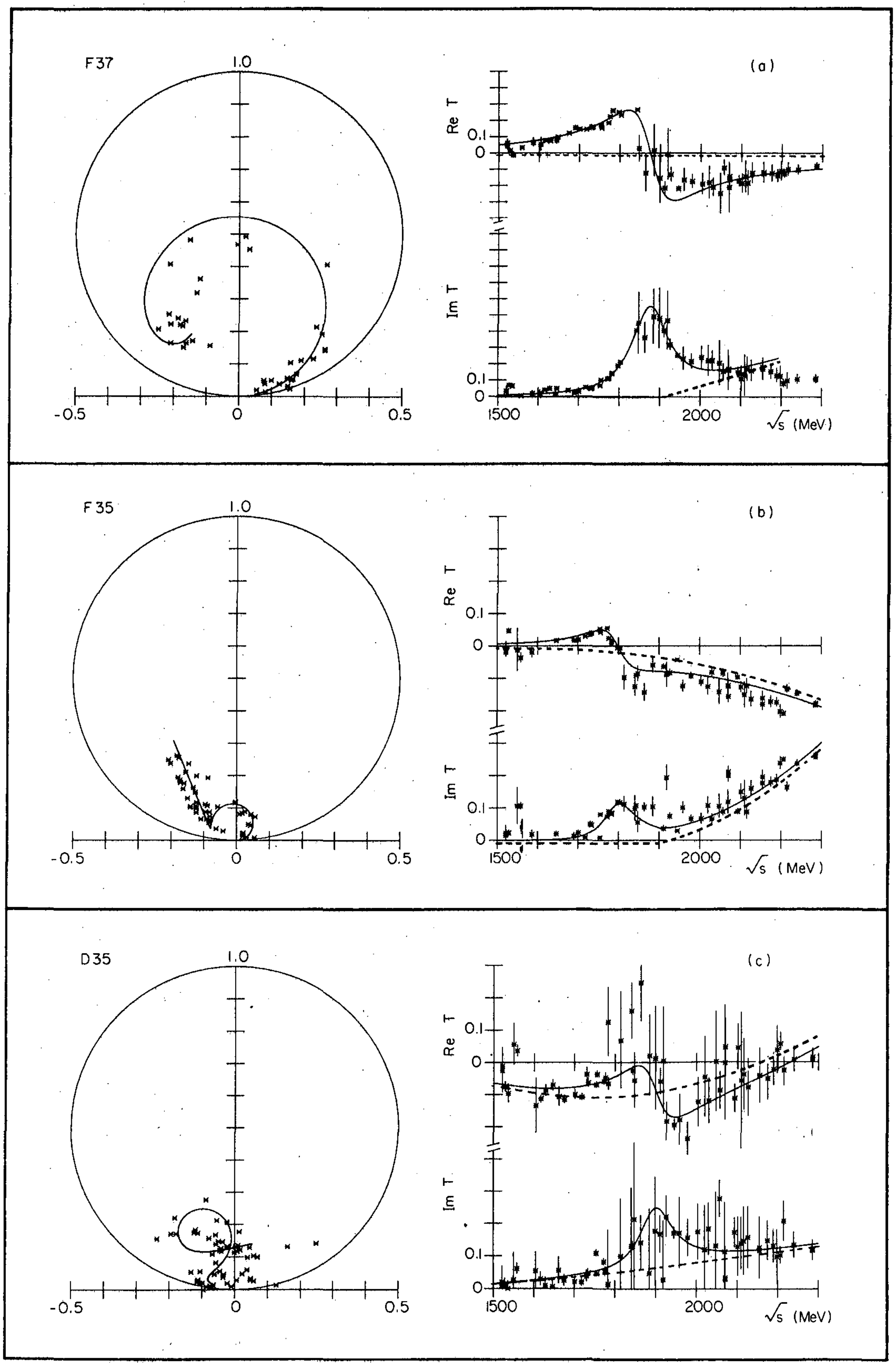
F37
1.0
(a)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)
F35
1.0
(b)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)
D35
1.0
(c)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)

Figure 1

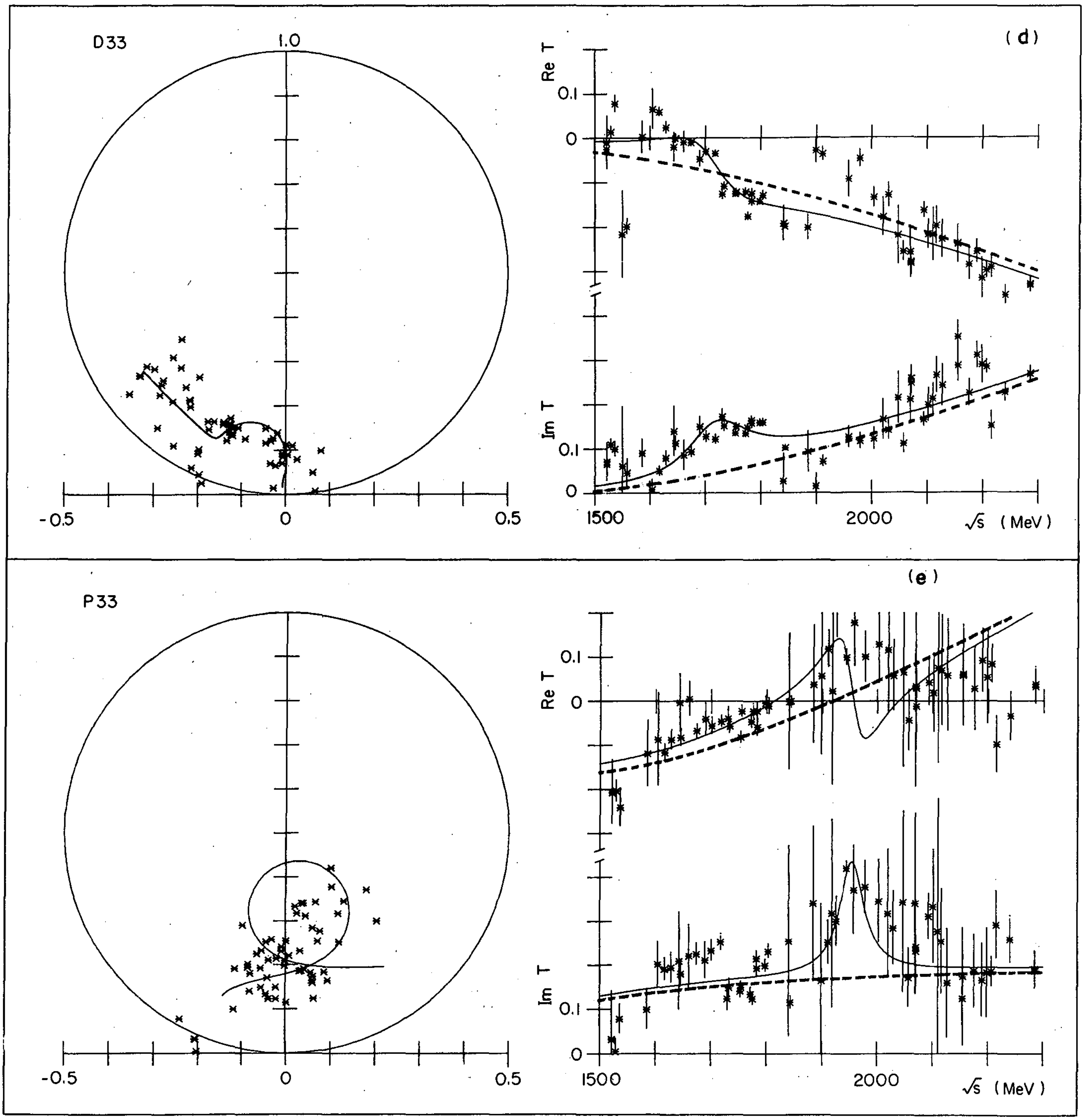
D33
1.0
(d)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)
(e)
P33
0.1
0
Re T
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)

Figure 1

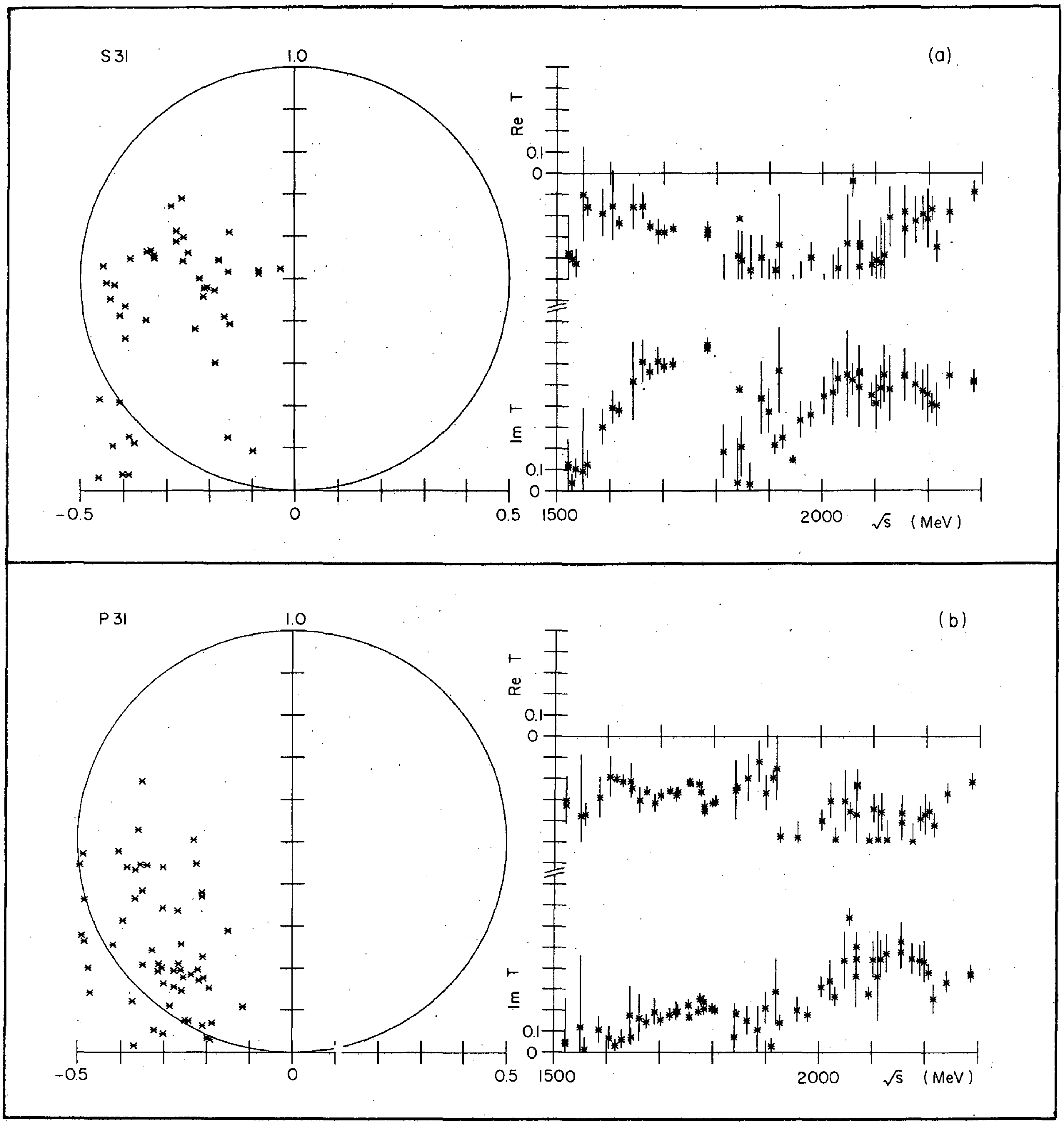
S 31
1.0
(a)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)
P 31
1.0
(b)
Re T
0.1
0
Im T
0.1
0
-0.5
0
0.5
1500
2000
√s (MeV)

Figure 2

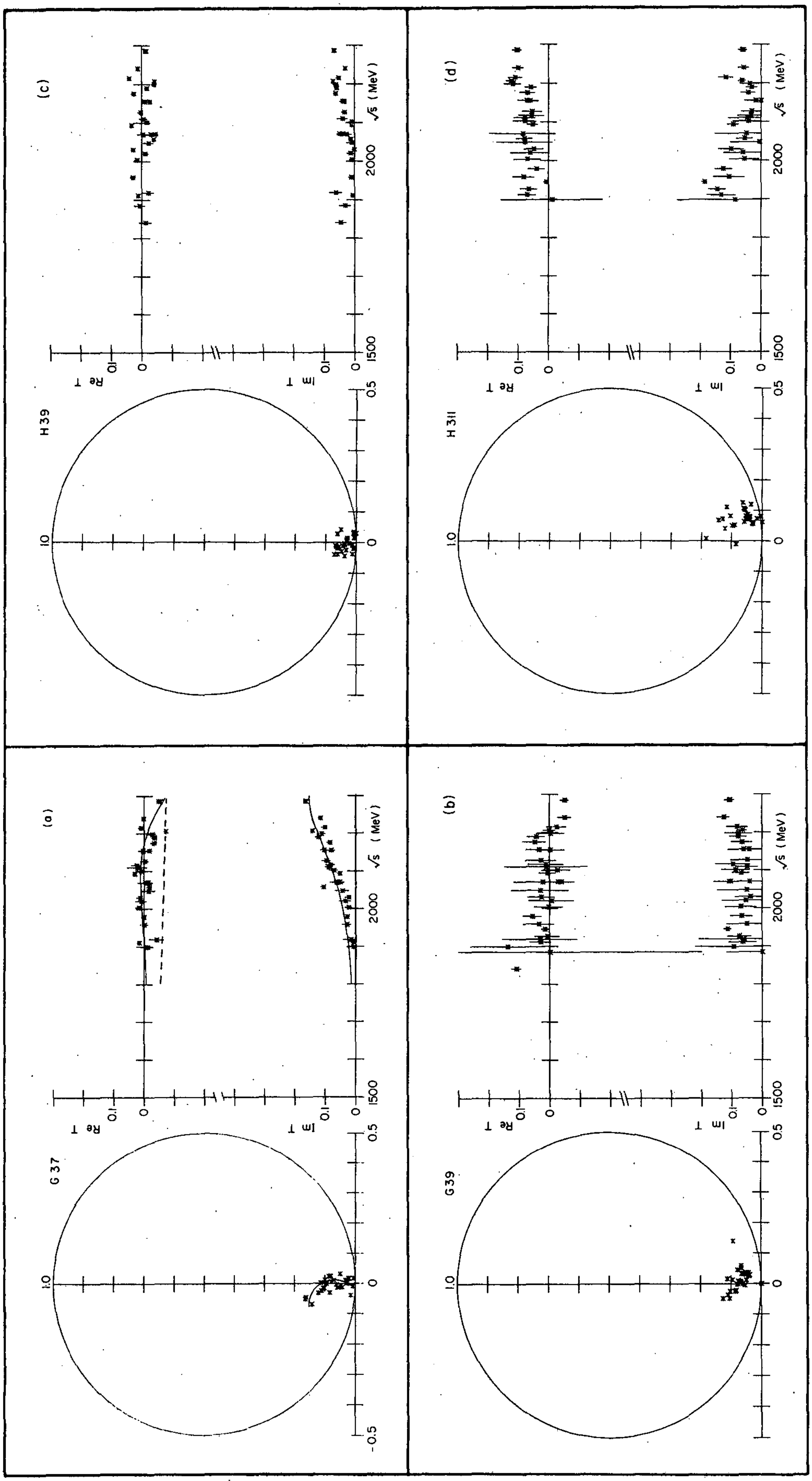

Figure 3

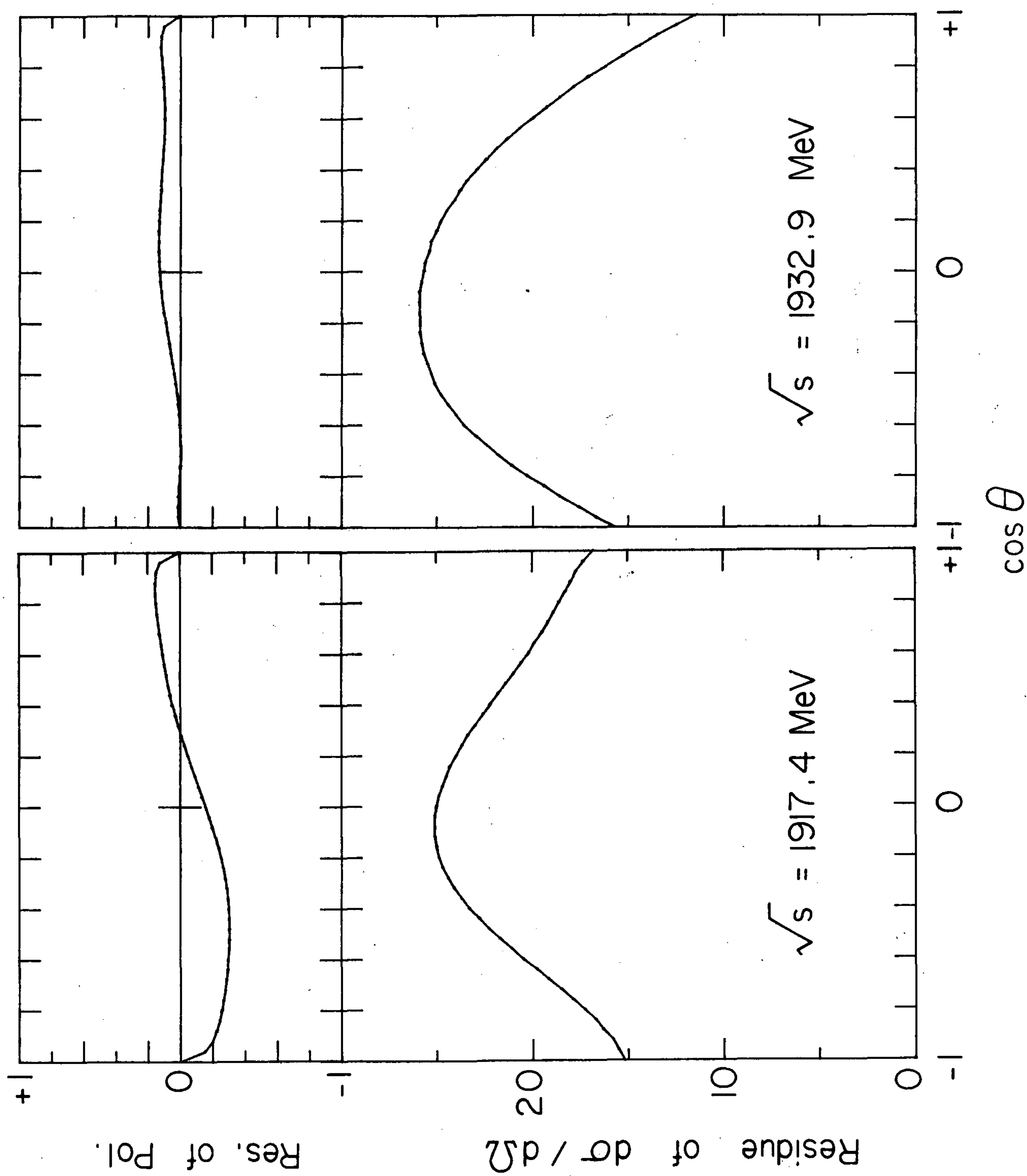
Res. of Pol.
Residue of dσ / dΩ
√s = 1917.4 MeV
√s = 1932.9 MeV
cos θ
+1
0
-1
20
10
0

Figure 4

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