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Optimal Execution Strategy: Price Impact and Transaction Cost

by

Mauricio José Junca Peláez

A dissertation submitted in partial satisfaction of the
requirements for the degree of
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of the

University of California, Berkeley

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Professor Xin Guo, Chair
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Abstract

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We study a single risky financial asset model subject to price impact and transaction cost over infinite and finite horizon. An investor needs to execute a long position in the asset affecting the price of the asset and possibly incurring in fixed transaction cost. The objective is to maximize the discounted revenue obtained by this transaction. This problem is formulated first as an impulse control problem and we characterize the value function using the viscosity solutions framework. We establish an associated optimal stopping problem that provides bounds and in some cases the solution of the value function. We also analyze the case where there is no transaction cost and how this formulation relates with a singular control problem. A viscosity solution characterization is provided in this case as well. An explicit solution of the value function is calculated in a particular case. Numerical examples with different types of price impact conclude the discussion.

To my father.

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Chapter 1

Introduction

An important problem for stock traders is to unwind large block orders of shares. According to [He and Mamaysky, 2005] the market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying securities, either for informational or liquidity reasons. Several papers addressed this issue and formulated a hedging and arbitrage pricing theory for large investors under competitive markets. For example, in [Cvitanić and Ma, 1996] a forward-backward SDE is defined, with the price process being the forward component and the wealth process of the investor's portfolio being the backward component. In both cases, the drift and volatility coefficients depend upon the price of the stocks, the wealth of the portfolio and the portfolio itself. [Frey, 1998] describes the discounted stock price using a reaction function that depends on the position of the large trader. In [Bank and Baum, 2004, Çetin et al., 2004] the authors, independently, described the price impact by assuming a given family of continuous semi-martingales indexed by the number of shares held ([Bank and Baum, 2004]) and by the number of shares traded ([Çetin et al., 2004]).

The optimal execution problem has been studied in [Bertsimas and Lo, 1998, Almgren and Chriss, 2000] in a discrete-time framework and without any transaction cost. In both cases the dynamics of the price processes are arithmetic random walks affected by the trading strategy. In [Bertsimas and Lo, 1998], the impact is proportional to the amount of shares traded. In [Almgren and Chriss, 2000], the change in the price is twofold, a temporary impact caused by temporary imbalances in supply/demand dynamics and a permanent impact in the equilibrium or unperturbed price process due to the trading itself. Also, this work takes into account the variance of the strategy with a mean-variance optimization procedure. Later on, nonlinear price impact functions were introduced in [Almgren, 2003]. These ideas were adopted by more recent works under a continuous time framework. [Schied et al., 2010] proposes the problem within a regular control setting. The authors consider expected-utility maximization for CARA utility functions, that is, for exponential utility functions. The dynamics of the price and the market impact function are fairly general, and there is no transaction cost. [Schied and Schöneborn, 2009] is the only reference that considers an

infinite horizon model based on the original model in [Almgren and Chriss, 2000].

On the other hand, it is also well established that transaction costs in asset markets are an important factor in determining the trading behavior of market participants. Typically, two types of transaction costs are considered in the context of optimal consumption and portfolio optimization: proportional transaction costs [Davis and Norman, 1990, Øksendal and Sulem, 2002] using singular type controls and fixed transaction costs [Korn, 1998, Øksendal and Sulem, 2002] using impulse type controls. The market impact effect can be significantly reduced by splitting the order into smaller orders but this will increase the transaction cost effect. Thus, the question is to find optimal times and allocations for each individual placement such that the expected revenue after trading is maximized. The papers [He and Mamaysky, 2005, Ly Vath et al., 2007] include both permanent market price impact and transaction cost and assume that the unperturbed price process is a geometric Brownian motion process. The first one ([He and Mamaysky, 2005]) allows continuous and discrete trading (singular control setting) and assumes enough regularity in the value function to characterize it as the solution of a second order nonlinear partial differential equation.

Finally, [Subramanian and Jarrow, 2001] proposes a slightly different model which does not include any transaction cost but includes an execution lag associated with size of the discrete trades. It also considers the geometric Brownian motion case and does not discuss any viscosity solutions.

From the references above, there are two which ideas contributed to develop this dissertation both in the modeling part and the mathematical analysis part. In order to make these contributions clear we discuss those papers in more detail.

Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets [Schied and Schöneborn, 2009].

Let X_t be the strategy described by the number of shares held at time t, with $X_0 = x$ being the amount of shares to sell, and assume X_t be absolutely continuous. In the model the incremental order $\dot{X}_t$ induces a permanent price impact $\gamma \dot{X}_t dt$ which accumulates over time, and a temporary price impact $\lambda \dot{X}_t$ which vanishes instantaneously. When the investor is not active, the price process follows a Bachelier model with volatility σ (assume to have no drift). The resulting price dynamics are:

$$P_t = P_0 + \sigma B_t + \gamma(X_t - X_0) + \lambda \dot{X}_t.$$

The strategies are parametrized as $X_t = x - \int_0^t \xi_s ds$ with progressively measurable (or adapted) process ξ such that $\int_0^T \xi_s^2 ds < \infty$, where x is initial amount of shares. It is also assumed that the strategies are admissible in the sense that $X_t(\omega)$ is bounded uniformly in t and ω. Denote by $\mathcal{X}$ the class of admissible strategies. The investor is assumed to be an investor with a utility function u. Given the dynamics above, the revenue obtained when

the investor applies the strategy ξ is given by

$$\mathcal{R}_T(\xi) = r + \int_0^T P_t \xi_t dt$$

$$= r + P_0(X_0 - X_T) - \frac{\gamma}{2}(X_T - X_0)^2 + \sigma \int_0^T X_t dB_t - \sigma X_T B_T - \lambda \int_0^T \xi_t^2 dt.$$

When then have

$$R_\infty^\xi := \lim_{T \to \infty} \mathcal{R}_T(\xi) = r + P_0 x - \frac{\gamma}{2}x^2 + \sigma \int_0^\infty X_t dB_t - \lambda \int_0^\infty \xi_t^2 dt$$

$$:= R + \sigma \int_0^\infty X_t dB_t - \lambda \int_0^\infty \xi_t^2 dt.$$

The objective of the investor is to maximize the expected utility of her revenue:

$$v(x, R) = \sup_{\xi \in \mathcal{X}} \mathbb{E}[u(R_\infty^\xi)].$$

The main result in this work is the following:

Theorem 1. *The value function v is a classical solution of the HJB equation*

$$\inf_c \left[-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial R^2} + \lambda \frac{\partial v}{\partial R}c^2 + \frac{\partial v}{\partial X}c \right] = 0$$

with boundary condition

$$v(0, R) = u(R) \; \forall R \in \mathbb{R}.$$

This characterization of the value function is not always possible as we will see later when we discuss continuous strategies.

A model of optimal portfolio selection under liquidity risk and price impact [Ly Vath et al., 2007].

Consider a continuous time price process of a risky asset (stock) P_t. Let R_t be the amount of money (cash) and X_t the number of shares in the stock held by an investor at time t. For a given process Y_t, we denote by Y_{t-} the left limit at time t. Let $T > 0$ the liquidation date. Only discrete trading on $[0, T)$ is accepted, this is modeled as an impulse control $(\tau_n, \zeta_n)_{1 \leq n \leq M}$, where $M \leq \infty$ is the number of trades, $\tau_1 \leq \cdots \leq \tau_n \leq \cdots < T$ are stopping times with respect to the filtration $(\mathcal{F}_t)$ that represent the times of the investor's trades, and ζ_n are real-valued $\mathcal{F}_{\tau_n}$-measurable random variables for all n representing the number shares

purchased (if negative) or sold (if positive) at the intervention times. The dynamics of X are given by

$$X_s = X_{\tau_n}, \text{ for } \tau_n \leq s < \tau_{n+1},$$
$$X_{\tau_{n+1}} = X_{\tau_n} - \zeta_{n+1}.$$

Additionally, in this work the authors allow for short selling, that is, the process X_t can be negative. The investor affects the price of the stock in the following way: The price goes up when the investor buys shares and go down when the investor sales shares. They consider the impact of the form:

$$\alpha(\zeta, p) = pe^{-\lambda\zeta}$$

where $\lambda > 0$. The price dynamics are given by:

$$dP_s = P_s(bds + \sigma dB_s), \text{ for } \tau_n \leq s < \tau_{n+1},$$
$$P_{\tau_n} = P_{\tau_{n-}}e^{-\lambda\zeta_n}.$$

Every time the investor trades ζ shares of the stock when the pre-trade price is p, the investor receives $\zeta pe^{-\lambda\zeta}$. In the absence of transactions the process R grows at a rate ρ. Also, the investor has to pay a fixed fee per trade $k > 0$. Therefore the cash holdings dynamics are:

$$dR_s = \rho R_s ds, \text{ for } \tau_n \leq s < \tau_{n+1},$$
$$R_{\tau_n} = R_{\tau_{n-}} + \zeta_n P_{\tau_{n-}}e^{-\lambda\zeta_n} - k.$$

In general, the impulse control formulation allows multiple actions at the same moment. Hence, in this case multiple trading is allowed and it could be optimal as well. The presence of transaction cost will off course forbid infinite trades but do not forbid that for some n we could have optimally $\tau_n = \tau_{n+1}$ and $\zeta_n, \zeta_{n+1} \neq 0$. The authors claim that multiple trading is not optimal: "The assumption that any trading incurs a fixed cost of money to be paid will rule out continuous trading, i.e., optimally, the sequence (τ_n, ζ_n) is not degenerate in the sense that for all n, $\tau_n < \tau_{n+1}$ and $\zeta_n \neq 0$ a.s." This claim will have an important impact in the terminal condition stated in the main result of the paper (see below). The proof of this condition (Proposition 4.16 in the paper) could be modified in order to avoid this assumption as we show later in this dissertation.

Suppose now, the investor has r units of cash and x number of shares of the stock at a price p. For such state value $z = (r, x, p) \in \mathbb{R}^2 \times (0, \infty)$ her liquidation value is given by

$$L(z) = \max\{L_0(z), r\}1_{x \geq 0} + L_0(z)1_{x < 0},$$

where

$$L_0(z) = r + xpe^{-\lambda x} - k.$$

The interpretation is that if the investor has a long position in stock, she can also choose to do a bin trade (that is, not to sell the shares).

Remark 2. *It should be also possible to not execute the whole position if is not profitable. That is, the function L_0 could be defined as*

$$L_0(z) = r + \max_{0 \leq \zeta \leq x} \zeta p e^{-\lambda \zeta} - k.$$

The solvency region is defined as

$$\mathcal{S} = \{z \in \mathbb{R}^2 \times (0, \infty) : L(z) > 0\}.$$

Also,

$$\partial \mathcal{S} = \{z \in \mathbb{R}^2 \times (0, \infty) : L(z) = 0\},$$
$$\overline{\mathcal{S}} = \mathcal{S} \cup \partial \mathcal{S}.$$

Since L is upper semi-continuous, $\overline{\mathcal{S}}$ is closed in $\mathbb{R}^2 \times (0, \infty)$, but $\mathcal{S}$ is not open. Given $t \in [0, T]$, $z \in \overline{\mathcal{S}}$ and the initial condition $Z_{t-} = z$, consider impulse controls (with the convention $\tau_0 = t$) such that $Z_s = (R_s, X_s, P_s) \in \overline{\mathcal{S}}$ for all $s \in [t, T]$. Denote by $\mathcal{A}(t, z)$ the set of all such controls. The authors also consider a smooth utility function U strictly increasing and concave and w.l.o.g $U(0) = 0$. Also, assume that there exists $K_1 \geq 0$ and $\gamma \in [0, 1)$ such that for all $w \geq 0$

$$U(w) \leq K_1 w^\gamma.$$

Then the value function defined for $(t, z) \in [0, T] \times \overline{\mathcal{S}}$ is

$$v(t, z) = \sup_{\mathcal{A}(t,z)} \mathbb{E}[U(L(Z_T))].$$

The impulse transaction function is defined by

$$\Gamma(z, \zeta) = (r + \zeta p e^{\lambda \zeta} - k, \ x - \zeta, \ p e^{\lambda \zeta})$$

for all $z = (r, x, p) \in \overline{\mathcal{S}}$ and $\zeta \in \mathbb{R}$. This corresponds to an immediate trading at time t of ζ shares, so that the state process jumps from $Z_{t-} = z$ to $Z_t = \Gamma(z, \zeta)$. Consider the set of admissible transactions

$$\mathcal{C}(z) = \{\zeta \in \mathbb{R} : L(\Gamma(z, \zeta)) \geq 0\}.$$

Then, define the intervention operator by

$$\mathcal{M}\varphi(z, t) = \sup_{\zeta \in \mathcal{C}(z)} \varphi(t, \Gamma(z, \zeta)),$$

for any measurable function φ. The infinitesimal generator associated with state variables when no trading is done is

$$\mathcal{L}\varphi = \rho r \frac{\partial \varphi}{\partial r} + bp \frac{\partial \varphi}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2},$$

for any C^2 function. The HJB equation that follows from the dynamic programming principle is then

$$\min\left\{-\frac{\partial v}{\partial t} - \mathcal{L}v, v - \mathcal{M}v\right\} = 0 \text{ in } [0, T) \times \mathcal{S}. \tag{1.1}$$

The main result stated in this paper is the following:

Theorem 3. *The value function v is continuous in $[0, T) \times \mathcal{S}$ and is the unique constrained viscosity solution to* (1.1) *satisfying*

$$\lim_{\substack{(t',z')\to(t,z) \\ z'\in\mathcal{S}}} v(t', z') = 0, \ \forall(t, z) \in [0, T) \times \{0\} \times \{0\} \times (0, \infty), \tag{1.2}$$

$$\lim_{\substack{(t,z')\to(T,z) \\ t<T, z'\in\mathcal{S}}} v(t, z') = \max\{U(L(z)), \mathcal{M}U(L(z))\}, \ \forall z \in \overline{\mathcal{S}}, \tag{1.3}$$

and the growth condition

$$|v(t, z)| \leq K\left(1 + \left(r + \frac{p}{\lambda}\right)^{\gamma}\right), \ \forall(t, z) \in [0, T) \times \mathcal{S}, \tag{1.4}$$

for some $K > 0$.

As mentioned before, the terminal condition is not the correct one. Also, there is a typo in the proof of Theorem 5.6 in the paper that could lead to not have the continuity of the value function: Consider the case where

$$z_0 \in \partial\mathcal{S} \setminus D_0 \cap \mathcal{K}.$$

After writing the inequalities derived from the viscosity subsolution property, consider the case where

$$-q_0 - \rho\hat{r}_iq_1 - b\hat{p}_iq_3 - \frac{1}{2}\sigma^2\hat{p}_i^2M_{33} \leq 0$$

and notice that the correct explicit form of q' is

$$q' = \frac{(\hat{z}_i - \hat{z}_i')}{\varepsilon_i} - \frac{4Dd(\hat{z}_i')}{d(z_i)}\left(\frac{d(\hat{z}_i')}{d(z_i)} - 1\right)^3.$$

It is not clear now how the second term above vanishes as i goes to ∞ since $d(z_i)$ goes to 0 as i goes to ∞. Similarly, in the explicit form of Q_i, factors of the form $\frac{1}{d(z_i)^2}$ and $\frac{1}{d(z_i)}$ are missing and again it is not clear how this term would vanish.

Intuitively the possibility of a bin trade when the number of shares x is nonnegative can create a discontinuity of the value function along the plane $\{x = 0\}$. Consider the point $z_0 = (r, 0, p)$ for r slightly bigger than k and given p. Then $L(z_0) = r > 0$ and for any t $v(t, z_0) \geq U(r)$. For ϵ small enough $z_\epsilon = (r, -\epsilon, p) \in \mathcal{S}$, but since the investor has to trade, almost all cash holdings will go to pay the transaction cost and the value function will be slightly bigger than zero by condition (1.2) for t close to T.

As seen later, the dynamics given in this paper are the starting point of this dissertation, but we consider a different objective function in order the avoid the problems mentioned above.

Main Contribution In this dissertation we study both infinite and finite horizon price impact models that include transaction cost under the setting of impulse control. Taking ideas of the [Ly Vath et al., 2007], we describe a general underlying price process and a general market impact that allows for either temporary or permanent impact but not both. With help of some classic results for optimal stopping problems and the discontinuous viscosity solutions theory for nonlinear partial differential equations, developed in references such as [Crandall et al., 1992, Ishii and Lions, 1990, Ishii, 1993, Fleming and Soner, 2006], we obtain a full characterization of the value function when the transaction cost is strictly positive and the price process satisfies some technical condition. These conditions are related with growth of an associated optimal stopping control problem and most of the price processes considered in the mathematical finance literature satisfy these conditions. Additionally, we are able to calculate the value function, in the finite horizon situation, as the expected value of certain measurable function of the price process at the expiration date.

As mentioned, the previous characterization is not complete when there is no transaction cost. By analyzing the Hamilton-Jacobi-Bellman (HJB) equation obtained before, we formulate a singular control model to include this case. For this new formulation we are able to completely characterize the value function. Even though any impulse control is a singular control, in general the expected revenue obtained when applying the same impulse control in both formulation is different. However, the value function may be the same. We are able to show that this is the case for a special case and provide the explicit solution. We also consider a regular control formulation in the spirit of [Schied and Schöneborn, 2009] and show that is not appropriate for our model of the optimal execution problem.

The structure of the dissertation is as follows: Chapter 2 include some technical background needed to develop the mathematical analysis of this dissertation. The first part reviews the theory of stochastic optimal control and the second part the theory of viscosity solutions. Chapter 3 describes the impulse control model both in the infinite horizon and finite horizon settings. In both cases we characterize the value function of the problem as a viscosity solution of the HJB equation and show uniqueness when the fixed transaction cost is strictly positive and the price process satisfies certain conditions. An important special case is considered in finite horizon situation. Chapter 4 considers the case where there is no transaction cost in the infinite horizon framework and shows the connection with a singular control model. For this model we also characterize the value function. We also describe a regular control formulation and show the problems associated with this model. Finally, we present numerical results of the value function and the optimal strategy for different underlying stochastic processes that allow to model permanent or temporary price impacts. Chapter 5 presents some conclusions and future work.

Chapter 2

Technical Review

In this chapter we include some technical background needed to develop the mathematical analysis of this dissertation. The first section reviews some aspects of the stochastic optimal control theory and the second section reviews the theory of viscosity solutions of fully nonlinear partial differential equations.

2.1 Stochastic Optimal Control

This section will include some mathematical background and literature review about stochastic control theory. Stochastic controls literature include the optimal stopping problem and problems involving different types of controls such as regular, singular, impulse and switching. Thorough this section we assume a probability space $(\Omega, \mathcal{F}, P)$ and $(B_t)_{t \geq 0}$ a m-dimensional Brownian motion in that space. Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by B and assume that satisfies the usual hypothesis.

2.1.1 Stochastic Differential Equations

Stochastic differential equations are the tool that allows to describe the dynamics of the stochastic process. We start with the following important theorem:

Theorem 4 (Existence and Uniqueness for Stochastic Differential Equations [Øksendal, 1998]). *Let $T > 0$ and $b : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$ be measurable function such that*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \ x \in \mathbb{R}^n, t \in [0, T]$$

for some constant C, and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \ x \in \mathbb{R}^n, t \in [0, T]$$

for some constant D. Let Z be a r.v. which is independent of the σ-algebra generated by the Brownian motion B and such that $\mathbb{E}[|Z|^2] < \infty$. Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \ 0 \le t \le T, X_0 = Z \tag{2.1}$$

has a unique continuous solution X such that is adapted to the filtration generated by B and Z, which satisfies the stochastic integral equation

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

for $t \in [0, T]$, and

$$\mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

The solution is unique up to indistinguishability, that is, if $\hat{X}$ and X are both solutions of (2.1), then $P(X_t = \hat{X}_t, \forall t \in [0, T]) = 1$.

The above solution is called a *strong* solution because the Brownian motion is given. There are also weak solutions to (2.1) but we are not going to consider them. Stochastic process that are strong solutions of stochastic differential equations that satisfy the assumptions of the theorem above are called *Itô diffusion*. Let's give now some important properties about diffusions. Let $\mathbb{E}^x$ be the expected value with respect to the probability law $Q^x = P(\cdot | X_0 = x)$, for $x \in \mathbb{R}^n$.

Theorem 5 (Itô's Formula [Øksendal, 1998])**.** *Let X be a solution of (2.1) and $f : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be of class $C^{1,2}$. Then for all $t \in [0, T]$,*

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^n b_i(t, X_t)\frac{\partial f}{\partial x_i}(t, X_t)dt$$

$$+ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) \sum_{j=1}^m \sigma_{ij}(t, X_t)dB_t^j$$

$$+ \frac{1}{2}\sum_{i,k}^n \frac{\partial^2 f}{\partial x_i \partial x_k}(t, X_t)(\sigma\sigma^T)_{ij}(t, X_t)dt.$$

Let's consider the case where the coefficients b and σ in (2.1) do not depend on t.

Definition 6. *Let X_t be a Itô diffusion in $\mathbb{R}^n$. The (infinitesimal) generator A of X_t is defined by*

$$Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}$$

for each $x \in \mathbb{R}^n$. The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$, while $\mathcal{D}_A$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

Theorem 7 ([Øksendal, 1998])**.** *Let X_t be the Itô diffusion*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

If $f \in C_0^2(\mathbb{R}^n)$, then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Using Itô's formula we have the following important identity which provides an important way of linking PDE and SDE:

Theorem 8 (Dynkin's formula [Øksendal, 1998])**.** *Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a $\mathcal{F}_t^{(m)}$- stopping time, $\mathbb{E}^x[\tau] < \infty$. Then*

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right].$$

2.1.2 Dynamic Programming

A stochastic optimal control problem is composed of a diffusion system, described by Itô SDE; alternative decisions of certain type than can affect the dynamics of the system; constraints on the decisions and/or the state of the system; and a criterion that measures the performance of the system under the decisions. We will focus on the most studied type of control, called regular control, to describe some of the methods to solve optimal control problems. Other types of formulations will be discussed later.

Under the same probability space as before, we consider the following stochastic controlled system for $t \in [0, T]$:

$$dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \tag{2.2}$$

with $X_0 = x$, where $X_t \in \mathbb{R}^n$, $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$. $u_t \in U \subset \mathbb{R}^k$ Borel set, is the control of the process. u_t is assumed to be $\mathcal{F}_t$-adapted. Assume that the process X_t satisfying (2.2) exists. Define the performance function

$$J(u) = \mathbb{E}\left[\int_0^T f(t, X_t, u_t)dt + g(X_T)\right],$$

which is assumed to be integrable, where $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given continuous functions. The optimal control problem is then to minimize the performance function over all admissible strategies u and find the optimal control $\bar{u}$ that minimize the functional, if it exists.

One approach to solve this problem is called Stochastic Maximum Principle (analogous to Pontryagin Maximum Principle for deterministic systems). This method introduces the

so-called adjoint variables that satisfy a system of backward (in time) SDE defined in terms of the Hamiltonian

$$H(t, x, u, p, q) = p^T b(t, x, u) + \text{tr}(q^T \sigma(t, x, u)) - f(t, x, u),$$

with $(t, x, u, p, q) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times m}$, and the terminal condition defined in terms of the function h. Since this is not the approach taken in this work we refer the reader to [Yong and Zhou, 1999] for more information about this method.

The other common approach to solve stochastic control problems is Dynamic Programming Principle (DPP). This is the approach taken in this work. The idea of this method is to solve the problem simultaneously for different initial times and states, establishing a relationship among these problems trough a nonlinear partial differential equation called Hamilton-Jacobi-Bellman (HJB) equation. Consider the same SDE (2.2) but only for $t \in [s, T]$, $0 \le s \le T$, with $X_s = x$. We redefine the performance functional

$$J(s, x; u) = \mathbb{E}\left[\int_s^T f(t, X_t, u_t)dt + g(X_T)\right].$$

Now, we define the value function for all $(s, x) \in [0, T) \times \mathbb{R}^n$

$$V(s, x) = \inf_u J(s, x; u)$$

with the terminal condition

$$V(T, x) = g(x),$$

for $x \in \mathbb{R}^n$. Under certain regularity conditions on the functions involved in the problem, we have Bellman's principle of optimality or DPP:

Theorem 9 ([Yong and Zhou, 1999]). *For any $(s, x) \in [0, T) \times \mathbb{R}^n$*

$$V(s, x) = \inf_u \mathbb{E}\left[\int_s^{s'} f(t, X_t, u_t)dt + V(s', X_{s'})\right],$$

for all $s \le s' \le T$.

Different forms of DPP and the corresponding principle for other type of stochastic optimal control problems are available in the literature. We will state later the appropriate form important to us. Recent work [Bouchard and Touzi, 2009] has proved a weak version of DPP for some cases which avoids the difficulties related to measurability arguments. This weak form is enough to be able to derive the HJB equation in the sense of viscosity solutions, which is the ultimate use of the optimality principle. For the problem in hand, the DPP allows to derive, at least formally, the following PDE:

$$\begin{cases} -v_t + H(t, x, Dv, D^2 v) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n \\ v(T, x) = g(x), & x \in \mathbb{R}^n, \end{cases}$$

where

$$H(t, x, p, P) = \sup_{u \in U} -\frac{1}{2}\text{tr}(P\sigma(t, x, u)\sigma(t, x, u)^T) - p^T b(t, x, u) - f(t, x, u),$$

for $(t, x, p, P) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$. When the value function V is smooth enough, then [Yong and Zhou, 1999] shows that V solves the above HJB. Unfortunately, this is not always the case and the value function could not be smooth enough, or the HJB may not have a classical solution. Here is where the theory of viscosity solutions helps to overcome this difficulty. The second part of this review will be dedicated to some features of this theory.

On the other hand, when the Hamiltonian do not satisfy some regularity conditions, for example when it is not locally bounded, we can have a situation where the value function does not satisfy the HJB equation. In section 4.4.1 we have an example where this occurs.

2.1.3 Two Formulations

In this work we concentrate in two types of stochastic control: Impulse and singular control. We now review some history and results of these control problems.

Impulse control

Impulse control is perhaps the less studied type of stochastic control, especially in terms of the analytic properties of the value function and the optimal policy. One of the first references is [Bensoussan and Lions, 1982] which gives a connection between weak solutions of quasivariational inequalities and stochastic impulse control. It considers the following infinite horizon problem: In the absence of control, X_t is governed by the Itô's stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, X_0 = x.$$

An impulse control for the system is the sequence $v = (\tau_n, \zeta_n)_{0 \leq n \leq M}$ for $M \leq \infty$, where $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ are stopping times and $\zeta_1, \zeta_2, \ldots$ are the interventions at these times, which are $\mathcal{F}_{\tau_n}$-measurable for all n. If a control $v = (\tau_n, \zeta_n)$ is adopted, then X_t evolves as

$$dX_t^v = b(X_t^v)dt + \sigma(X_t^v)dB_t + \sum_i \delta(t - \tau_i)\zeta_i,$$

where δ is the Dirac delta function. The performance value is define as

$$J^v(x) := \mathbb{E}^x \left[\int_0^\tau e^{-rt}f(X_t)dt + \sum_{\tau_i \leq \tau} e^{-r\tau_i}(k + B(\zeta_i)) \right].$$

Here, f is the running cost, $k > 0$, B is the transaction cost function, $r > 0$ is the discount factor and τ is the first exit time of X from a given open bounded set $\mathcal{O}$. [Bensoussan and

Lions, 1982] shows that under some assumptions the value function

$$u(x) = \inf_v J^v(x),$$

is the unique solution in the Sobolev space $H_0^1(\mathcal{O})$ that satisfies the quasivariational inequality

$$\begin{cases} \langle Au, v - u \rangle \geq \langle f, v - u \rangle, & \forall v \in H_0^1(\mathcal{O}), v \leq \mathcal{M}v \\ 0 \leq u \leq \mathcal{M}u, \end{cases}$$

where the intervention operator is defined as

$$\mathcal{M}v(x) = k + \inf_{\xi \geq 0, x + \xi \in \bar{\mathcal{O}}} v(x + \xi) + B(\xi).$$

The recent paper [Guo and Wu, 2009] studies a similar problem, with some differences in the transaction cost function. Using the theory of viscosity solutions, the authors show that u is the unique viscosity solution in the set of uniformly continuous functions bounded by below with domain $\mathbb{R}^n$, $UC_{bb}(\mathbb{R}^n)$, (note that the domain is an unbounded set) of the HJB equation

$$\max\{Av(x) - f(x), v(x) - \mathcal{M}v(x)\} = 0, \ \forall x \in \mathbb{R}^n.$$

In fact, the paper goes beyond and shows that $u \in C^1(\mathbb{R}^n)$ provided σ is differentiable and its derivatives are Lipschitz.

The general formulation of the stochastic impulse control problem is the following (see [Øksendal and Sulem, 2005]): Suppose the state of the system Y is described, when no action is taken, by the Itô diffusion process

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \ Y_{0-} = y \in \mathbb{R}^k.$$

Suppose that at any time t and any state y we can intervene the system and give an impulse $\zeta \in \mathcal{Z} \subset \mathbb{R}^p$, where $\mathcal{Z}$ is the set of admissible impulse values. When an impulse ζ is given, the state variable $y = Y_{t-}$ jumps to $Y_t = \Gamma(y, \zeta) \in \mathbb{R}^k$ where the function $\Gamma : \mathbb{R}^k \times \mathcal{Z} \to \mathbb{R}^k$ is given. Therefore, if the impulse control v is applied, the dynamics of Y^v are:

- $Y_{0-}^v = y$

- $Y_t^v = Y_t$ for $0 < t < \tau_1$

- $Y_{\tau_n}^v = \Gamma(Y_{\tau_n-}^v, \zeta_n)$ for $n = 1, 2, \ldots$

- $dY_t^v = b(Y_t^v)dt + \sigma(Y_t^v)dB_t$ for $\tau_n < t < \tau_{n+1}$.

Define the performance value as

$$J^v(y) = \mathbb{E}^y \left[\int_0^{\tau_G} f(Y_t^v)dt + g(Y_{\tau_G}^v)1_{\{\tau_G < \infty\}} + \sum_{\tau_n \leq \tau_G} K(Y_{\tau_n-}^v, \zeta_n) \right],$$

where τ_G is the first exit time from the open set G. Let Υ be the family of admissible impulse controls such that there is a unique solution Y^v, the performance function is integrable and, when $M = \infty$ a.s., $\lim_{n\to\infty} \tau_n = \tau_G$ a.s. The problem, as usual, is to find $\Phi(y)$ and if possible $v^* \in \Upsilon$ such that

$$\Phi(y) = \sup_{v\in\Upsilon} J^v(y) = J^{v^*}(y).$$

For $n = 1, 2, \ldots$ let Υ_n denote the set of all $v \in \Upsilon$ such that $v = (\tau_1, \ldots, \tau_n; \zeta_1, \ldots, \zeta_n)$, that is, v has at most n interventions. Then for all n

$$\Upsilon_n \subset \Upsilon_{n+1} \subset \Upsilon.$$

Define

$$\Phi_n(y) = \sup_{v\in\Upsilon_n} J^v(y).$$

Then $\Phi_n \leq \Phi_{n+1} \leq \Phi$. Moreover,

Lemma 10 ([Øksendal and Sulem, 2005]). *Suppose $g \geq 0$. Then for all $y \in G$*

$$\lim_{n\to\infty} \Phi_n(y) = \Phi(y).$$

We now define formally the intervention operator:

Definition 11. *Let $\mathcal{H}$ be the space of all measurable functions $h : G \to \mathbb{R}$. The* intervention operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ *is defined by*

$$\mathcal{M}h(y) = \sup_{\substack{\zeta\in\mathcal{Z} \\ \Gamma(y,\zeta)\in G}} \{h(\Gamma(y, \zeta)) + K(y, \zeta)\}.$$

We will be mostly interested in applying $\mathcal{M}$ to Φ. In this case, $\mathcal{M}\Phi(y)$ represents the value of the strategy that consists of taking the best immediate action in state y and behaving optimally afterwards.

Singular control

One of the first papers that solved a singular control problem by ad hoc methods is [Benes et al., 1980]. They consider the problem

$$V(y) = \inf_{\xi^+,\xi^-} \mathbb{E}\left[\int_0^\infty e^{-\rho t}(B_t + Y_t)dt\right],$$

where $\rho > 0$ and $Y_t = y + \xi_t^+ - \xi_t^-$. The control (ξ^+, ξ^-) is a pair of adapted, nondecreasing and càglàd processes starting at 0. Later, in [Karatzas and Shreve, 1984] the authors considered

a more general problem in a finite horizon setting and established a connection between the singular control problem

$$V(T, y) = \inf_{\xi = \xi^+ - \xi^-} \mathbb{E}\left[\int_0^T h(t, B_t + Y_t)dt + \int_{[0,T)} f(t)d\xi_t + g(B_T + Y_T)\right],$$

and the optimal stopping problem

$$u(T, y) = \inf_{0 \leq \tau \leq T} \mathbb{E}\left[\int_0^\tau \frac{\partial h}{\partial y}(t, y + B_t)dt + f(\tau)1_{\{\tau < T\}} + g'(y + B_T)\right].$$

The authors show that under some conditions, if an optimal control exists then $u = \dfrac{\partial V}{\partial x}$. More recent works [Guo and Tomecek, 2009, 2008] connect a similar problem, with an Itô diffusion instead of the Brownian motion, with a switching control problem. Using this connection the authors establish some regularity properties of the value function. Singular control problems have also played an important role in portfolio selection with consumption and proportional transaction costs. Some important works in this topic are [Davis and Norman, 1990, Shreve and Soner, 1994], where explicit solutions are found. It is worth mentioning that the second reference relies on the concept of viscosity solutions.

The general formulation of the problem is the following (see [Øksendal and Sulem, 2005]) : Let $\kappa \in \mathbb{R}^{k \times p}$ and $\theta \in \mathbb{R}^p$ be constants. Suppose the state of the system Y_t is described by

$$dY_t = b(Y_t, u_t)dt + \sigma(Y_t, u_t)dB_t + \kappa d\xi_t, \ Y_{0-} = y \in \mathbb{R}^k,$$

where $\xi_t \in \mathbb{R}^p$ is an adapted càdlàg finite variation process and $\xi_{0-} = 0$. Note that $d\xi_t$ may be singular with respect to Lebesgue measure dt (and hence the name). The process u is an adapted regular control with values in U as before. Define the performance functional as

$$J^{u,\xi}(y) = \mathbb{E}^y\left[\int_0^{\tau_G} f(Y_t, u_t)dt + g(Y_{\tau_G})1_{\{\tau_G < \infty\}} + \int_0^{\tau_G} \theta^T d\xi_t\right].$$

Let $\mathcal{A}$ be the family of admissible controls (u, ξ) such that a unique strong solution Y_t exists. The problem is to find the value function $\Phi(y)$ and an optimal control $(u^*, \xi^*) \in \mathcal{A}$ such that

$$\Phi(y) = \sup_{(u,\xi) \in \mathcal{A}} J^{u,\xi}(y) = J^{u^*,\xi^*}(y).$$

2.2 Viscosity Solutions

One of the most important references about viscosity solutions is [Crandall et al., 1992]. From there we quote the following paragraph that explains the importance of this theory: *The primary virtues of this theory are that it allows merely continuous functions to be solutions of fully nonlinear equations of second order, that it provides very general existence and*

uniqueness theorems and that it yields precise formulations of general boundary conditions. Although this reference considers only continuous viscosity solutions, actually, the theory extends to discontinuous viscosity solutions when there is no a priori knowledge of the solution of the equation (see [Fleming and Soner, 2006]). We will use this notion hereafter.

Definition 12. *Let W be an extended real-valued function on some open set $\mathcal{D} \subset \mathbb{R}^n$.*

(i) The upper semi-continuous envelope *of W is*

$$W^*(x) = \lim_{r \downarrow 0} \sup_{\substack{|x'-x| \leq r \\ x' \in \mathcal{D}}} W(x'), \ \forall x \in \mathcal{D}.$$

(ii) The lower semi-continuous envelope *of W is*

$$W_*(x) = \lim_{r \downarrow 0} \inf_{\substack{|x'-x| \leq r \\ x' \in \mathcal{D}}} W(x'), \ \forall x \in \mathcal{D}.$$

Note that W^* is the smallest upper semi-continuous function which is greater than or equal to W, and similarly for W_*. Now we define the notion of discontinuous viscosity solutions that apply to value functions of impulse control problems:

Definition 13. *Given an equation of the form*

$$\min \left\{ F(x, \varphi(x), D\varphi(x), D^2\varphi(x)), \varphi - \mathcal{M}\varphi \right\} = 0 \ \text{in } \mathcal{D}, \tag{2.3}$$

where $\mathcal{M}$ is defined in 11, a locally bounded function W on $\mathcal{D}$ is a:

(i) Viscosity subsolution of (2.3) in $\mathcal{D}$ if for each $\varphi \in C^2(\bar{\mathcal{D}})$,

$$\min \left\{ F(x_0, W(x_0), D\varphi(x_0), D^2\varphi(x_0)), W^*(x_0) - \mathcal{M}W^*(x_0) \right\} \leq 0$$

at every $x_0 \in \mathcal{D}$ which is a maximizer of $W^* - \varphi$ on $\bar{\mathcal{D}}$ with $W^*(x_0) = \varphi(x_0)$.

(ii) Viscosity supersolution of (2.3) in $\mathcal{D}$ if for each $\varphi \in C^2(\bar{\mathcal{D}})$,

$$\min \left\{ F(x_0, W(x_0), D\varphi(x_0), D^2\varphi(x_0)), W_*(x_0) - \mathcal{M}W_*(x_0) \right\} \geq 0$$

at every $x_0 \in \mathcal{D}$ which is a minimizer of $W_* - \varphi$ on $\bar{\mathcal{D}}$ with $W_*(x_0) = \varphi(x_0)$.

(iii) Viscosity solution of (2.3) in $\mathcal{D}$ if it is both a viscosity subsolution and a viscosity supersolution of (2.3) in $\mathcal{D}$.

2.2.1 Comparison principle

In order to prove uniqueness results for viscosity solutions a common technique, called comparison principle, is used. This principle is somehow an extension of the maximum principle for semi-continuous functions. Suppose the set $\mathcal{D}$ above is open and bounded. Let $u \in USC(\bar{\mathcal{D}})$ be a subsolution of (2.3) and $v \in LSC(\bar{\mathcal{D}})$ be a supersolution of (2.3) such that $u \leq v$ on $\partial\mathcal{D}$. We say that (2.3) satisfies the comparison principle if $u \leq v$ in $\bar{\mathcal{D}}$. Now suppose that W_1 and W_2 are viscosity solutions of (2.3) such that $W_{1*} = W_1^* = W_{2*} = W_2^*$ on $\partial\mathcal{D}$. Therefore

$$W_1^* \leq W_{2*} \leq W_2^* \leq W_{1*} \text{ in } \bar{\mathcal{D}},$$

which implies not only uniqueness of the solution but also continuity. When the open set $\mathcal{D}$ is unbounded we have to be more careful and assume some growth conditions in the sub and super solutions to prove the comparison principle. This is the case in this work.

The key result in order to prove a comparison principle is Ishii's Lemma. Before stating it we need a definition.

Definition 14. *Let $\mathcal{O} \subset \mathbb{R}^N$ arbitrary and $u : \mathcal{O} \mapsto \mathbb{R}$. Let $\hat{x} \in \mathcal{O}$ and $(p, X) \in \mathbb{R}^N \times \mathcal{S}(N)$. We say that $(p, X) \in \mathcal{J}_{\mathcal{O}}^{2,+} u(\hat{x})$ (the second order superjet of u at $\hat{x}$) if*

$$u(x) \leq u(\hat{x}) + \langle p, x \rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \text{ as } \mathcal{O} \ni x \to \hat{x}.$$

From the definition we have that $\mathcal{J}_{\mathcal{O}}^{2,+} u(\hat{x})$ is the same for all sets $\mathcal{O}$ where $\hat{x}$ is an interior point. This common value is noted as $\mathcal{J}^{2,+} u(\hat{x})$. We also have that $\mathcal{J}_{\mathcal{O}}^{2,-} u(\hat{x}) = -\mathcal{J}_{\mathcal{O}}^{2,+}(-u)(\hat{x})$. An equivalent definition of viscosity solutions can be stated in terms of superjets. We will use this definition later in uniqueness proofs (see [Crandall et al., 1992]). Now, we can state the key lemma.

Theorem 15 (Ishii's Lemma, Theorem 3.2 in [Crandall et al., 1992]). *Let $\mathcal{O}_i$ be a locally compact subset of $\mathbb{R}^{N_i}$ for $i = 1, \ldots, k$,*

$$\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_k,$$

$u_i \in USC(\mathcal{O}_i)$, and φ be twice continuously differentiable in a neighborhood of $\mathcal{O}$. Set

$$w(x) = u_1(x_1) + \cdots + u_k(x_k) \text{ for } x = (x_1, \ldots, x_k) \in \mathcal{O},$$

and suppose $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k) \in \mathcal{O}$ is a local maximum of $w - \varphi$ relative to $\mathcal{O}$. Then for each $\epsilon > 0$ there exists $X_i \in \mathcal{S}(N_i)$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \bar{\mathcal{J}}_{\mathcal{O}_i}^{2,+} u_i(\hat{x}_i) \text{ for } i = 1, \ldots, k,$$

and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\epsilon} + \|A\|\right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq A + \epsilon A^2$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{S}(N)$, $N = N_1 + \ldots + N_k$.

Chapter 3

Impulse Control Model

In this chapter we introduce a new model, inspired by the model described in [Ly Vath et al., 2007], for the Optimal Execution Problem. The main difference is that we do not allow short-selling and therefore we do not consider any liquidation value function. Our objective function is just the discounted revenue. In this way we overcome the problems found in [Ly Vath et al., 2007] mentioned in Chapter 1. The first section describes the model of price impact and transaction cost. For this model we consider infinite horizon and finite horizon in sections 2 and 3 respectively. In both cases we provide a characterization of the value function as viscosity solutions of fully nonlinear PDEs.

3.1   Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, \mathbb{P})$ be a probability space which satisfies the usual conditions and B_t be a one-dimensional Brownian motion adapted to the filtration. We consider a continuous time process adapted to the filtration denoting the price of a risky asset P_t. The unperturbed price dynamics are given by:

$$dP_s = \mu(P_s)ds + \sigma(P_s)dB_s, \tag{3.1}$$

where μ and σ satisfy regular conditions such that there is a unique strong solution of this SDE (i.e. Lipschitz continuity). We are mainly interested in dynamics such that the price process is always non-negative, thus we assume that P is absorbed as soon as it reaches 0. Also the initial price p is non-negative. We consider price impact functions such that the price goes up when the investor buys shares and goes down when the investor sells shares. Also, the greater the volume of the trade, the grater the impact in the price process. The number of shares in the asset held by the investor at time t is denoted by X_t and it is up to the investor to decide how to unwind the shares. Different models and formulations will define the admissible strategies for the investor. At the beginning the investor has $x \geq 0$ number of shares and we only allow strategies such that $X_t \geq 0$ for all $t \geq 0$. Since the

investor's interest is to execute the position, we don't allow to buy shares, that is X_t is a non-increasing process. Hence, we can see that $\mathbb{R}_+ \times \mathbb{R}_+ = \bar{\mathcal{O}}$ (with interior $\mathcal{O}$) is the state space of the problem. The goal of the investor is to maximize the expected discounted profit obtained by selling the shares before the deadline $T \geq 0$ (possibly infinity).

In this formulation we assume that the investor can only trade discretely over the time horizon. This is modeled with the impulse control $\nu = (\tau_n, \zeta_n)_{1 \leq n \leq M}$, where the random variable $M < \infty$ is the number of trades, (τ_n) are stopping times with respect to the filtration $(\mathcal{F}_t)$ such that $0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \cdots \leq \tau_M \leq T$ that represent the times of the investor's trades, and (ζ_n) are real-valued $\mathcal{F}_{\tau_n}$-measurable random variables that represent the number of shares sold at the intervention times. Note that any control policy ν fully determines M. We will consider both infinite horizon $(T = \infty)$ and finite horizon.

Given any strategy ν, the dynamics of X are given by

$$X_s = X_{\tau_n}, \text{ for } \tau_n \leq s < \tau_{n+1}, \tag{3.2}$$
$$X_{\tau_{n+1}} = X_{\tau_n} - \zeta_{n+1}. \tag{3.3}$$

3.1.1   Price impact

For the price impact we let $\alpha(\zeta, p)$ be the post-trade price when the investor trades ζ shares of the asset at a pre-trade price of p. We assume that α is smooth, non-increasing in ζ, and non-decreasing in p. We will also assume that $\alpha(\zeta, p) \leq p$ for $\zeta \geq 0$ and $\alpha(0, p) = p$ for all p. Furthermore, we will also assume that for all $\zeta_1, \zeta_2, p \in \mathbb{R}_+$

$$\alpha(\zeta_1, \alpha(\zeta_2, p)) = \alpha(\zeta_1 + \zeta_2, p). \tag{3.4}$$

This assumption says that the impact in the price of trading twice at the same moment in time is the same as trading the total number of shares once. This assumption will prevent any price manipulation from the investor. Two possible choices for α are:

$$\alpha_1(\zeta, p) = p - \lambda \zeta$$

$$\alpha_2(\zeta, p) = p e^{-\lambda \zeta}$$

where $\lambda > 0$. A linear impact like α_1 has the drawback that the post-trade price can be negative. Given a price impact α and an admissible strategy ν, the price dynamics are given by:

$$dP_s = \mu(P_s)ds + \sigma(P_s)dB_s, \text{ for } \tau_n \leq s < \tau_{n+1}, \tag{3.5}$$
$$P_{\tau_n} = \alpha(\zeta_n, P_{\tau_{n-}}). \tag{3.6}$$

With this general model, we can achieve different types of price impact by choosing the appropriate price process. The first type studied in the literature is the permanent impact. By permanent impact we mean a change in the equilibrium price process due to the trading

itself, as explained in [Almgren and Chriss, 2000]. The first and widely used price process that we can use to model permanent price impact is the geometric Brownian motion. Clearly, the arithmetic Brownian motion also allows for this type of impact.

A second kind of impact present in the references is a temporary impact. We can describe temporary impact as caused by temporary imbalances in supply/demand dynamics. Mean-reverting processes like OrnsteinUhlenbeck process and CIR process allow us to model this temporary impact in the price.

3.2   Infinite Horizon

Given $y = (x, p) \in \bar{\mathcal{O}}$ we define $V(y)$, the value function as the maximum (or supremum), taken over all admissible trading strategies such that $(X_{0-}, P_{0-}) = Y_{0-} = y$. We call $\beta > 0$ the discount factor and $k \geq 0$ the transaction cost. Note that we can always do nothing, in which case the expected revenue is 0. Therefore $V \geq 0$ for all y. Formally, given $y = (x, p) \in \bar{\mathcal{O}}$ the value function V has the form:

$$V(y) = \sup_{\nu} \mathbb{E} \left[\sum_{n=1}^{M} e^{-\beta \tau_n} (\zeta_n P_{\tau_n} - k) \right]. \tag{3.7}$$

As usual, we assume that $e^{-\beta \tau} = 0$ on $\{\tau = \infty\}$.

3.2.1   Hamilton-Jacobi-Bellman equation

In order to characterize the value function we will use the dynamic programming approach. This principle has been proved for several frameworks and types of control. Some of the references that prove it in a fairly general context are [Ishikawa, 2004, Ma and Yong, 1999]. We have that the following Dynamic Programming Principle (DPP) holds: For all $y = (x, p) \in \mathcal{O}$ we have

$$V(y) = \sup_{\nu} \mathbb{E} \left[\sum_{\tau_n \leq \tau} e^{-\beta \tau_n} (\zeta_n P_{\tau_n} - k) + e^{-\beta \tau} V(Y_\tau) \right], \tag{3.8}$$

where τ is any stopping time. Let's define the impulse transaction function as

$$\Gamma(y, \zeta) = (x - \zeta, \alpha(\zeta, p))$$

for all $y \in \bar{\mathcal{O}}$ and $\zeta \in \mathbb{R}$. This corresponds to the change in the state variables when a trade of ζ shares has taken place. We define the intervention operator as

$$\mathcal{M}\varphi(y) = \sup_{0 \leq \zeta \leq x} \varphi(\Gamma(y, \zeta)) + \zeta \alpha(\zeta, p) - k,$$

for any measurable function φ. Also, let's define the infinitesimal generator operator associated with the price process when no trading is done, that is

$$A\varphi = \mu(p)\frac{\partial\varphi}{\partial p} + \frac{1}{2}\sigma(p)^2\frac{\partial^2\varphi}{\partial p^2},$$

for any function $\varphi \in C^2(\mathcal{O})$. The HJB equation that follows from the DPP is then ([Øksendal and Sulem, 2005])

$$\min\{\beta\varphi - A\varphi, \varphi - \mathcal{M}\varphi\} = 0 \text{ in } \mathcal{O}. \tag{3.9}$$

We call the continuation region to

$$\mathcal{C} = \{y \in \mathcal{O} : \mathcal{M}\varphi - \varphi < 0\}$$

and the trade region to

$$\mathcal{T} = \{y \in \mathcal{O} : \mathcal{M}\varphi - \varphi = 0\}.$$

The intuition behind the equation (3.9) is the following: Let $y \in \mathcal{O}$. Clearly $\mathcal{M}V(y) \leq V(y)$. Now, assuming enough regularity for the value function V, by DPP and Dynkin's formula

$$V(y) \geq \mathbb{E}[e^{-\beta t}V(Y_t)] = V(y) + \mathbb{E}\left[\int_0^t e^{-\beta s}(-\beta V + AV)(Y_s)ds\right],$$

for all $t \geq 0$ where Y_s is the process with no intervention. Therefore $(-\beta V + AV)(y) \leq 0$. If the no intervention is optimal, then $(-\beta V + AV)(y) = 0$ and $\mathcal{M}V(y) < V(y)$, that is $y \in \mathcal{C}$. On the other hand if the optimal strategy is to trade, then $\mathcal{M}V(y) = V(y)$ and $(-\beta V + AV)(y) < 0$, so $y \in \mathcal{T}$.

3.2.2 Growth condition

We will define a particular optimal stopping problem and use some of the results in [Dayanik and Karatzas, 2003] to establish an upper bound on the value function V and therefore a growth condition. Consider the case where there is no price impact, that is, $\alpha(\zeta, p) = p$ for all $\zeta \geq 0$. We define

$$V_{NI}(y) = \sup_{\nu} \mathbb{E}\left[\sum_{n=1}^{M} e^{-\beta\tau_n}(\zeta_n P_{\tau_n} - k)\right], \tag{3.10}$$

where P_s follows the unperturbed price process. It is clear that $V \leq V_{NI}$. When there is no price impact, the investor would need to trade only one time.

Proposition 16. *For all $y \in \mathcal{O}$*

$$V_{NI}(x, p) = U(x, p) := \sup_{\tau} \mathbb{E}[e^{-\beta\tau}(xP_\tau - k)^+] \tag{3.11}$$

where the supremum is taken over all stopping times with respect to the filtration $(\mathcal{F}_s)$.

Proof. Since (τ, x) is an admissible strategy for any stopping time τ, then $U \leq V_{NI}$. Now, let Υ_n the set of admissible strategies with at most n interventions. The proof will continue by induction in n to show that for all n

$$\sup_{\nu \in \Upsilon_n} \mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta \tau_i}(\zeta_i P_{\tau_i} - k)\right] \leq U(y). \tag{3.12}$$

Clearly (3.12) is true for $n = 1$. Let $\nu \in \Upsilon_n$. Note that $xp - k \leq U(x, p)$, therefore, conditioning on $\mathcal{F}_{\tau_1}$ we have

$$\mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta \tau_i}(\zeta_i P_{\tau_i} - k)\right] = \mathbb{E}\left[\mathbb{E}\left[[e^{-\beta \tau_1}(\zeta_1 P_{\tau_1} - k)\big| \mathcal{F}_{\tau_1}]\right] + \right.$$

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\beta \tau_1} \sum_{i=2}^{n} e^{-\beta(\tau_i - \tau_1)}(\zeta_i P_{\tau_i} - k)\bigg| \mathcal{F}_{\tau_1}\right]\right]$$

$$\leq \mathbb{E}\left[\mathbb{E}\left[e^{-\beta \tau_1} U(\zeta_1, P_{\tau_1})\big| \mathcal{F}_{\tau_1}\right]\right] +$$

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\beta \tau_1} U(x - \zeta_1, P_{\tau_1})\big| \mathcal{F}_{\tau_1}\right]\right]$$

$$\leq \mathbb{E}\left[e^{-\beta \tau_1} U(x, P_{\tau_1})\right]$$

$$\leq U(x, p),$$

where the last inequality follows from the fact that the process $e^{-\beta s} U(x, P_s)$ is a supermartingale ([Øksendal and Reikvam, 1998]). This proves (3.12). By lemma 10, the left hand side of (3.12) converges to V_{NI} as $n \to \infty$ and the proof is complete. $\square$

From the previous proposition we have the bound

$$0 \leq V(x, p) \leq U(x, p) = \sup_{\tau} \mathbb{E}[e^{-\beta \tau}(xP_{\tau} - k)^+], \tag{3.13}$$

where the supremum is taken over all stopping times with respect to the filtration $(\mathcal{F}_t)$. Following section 5 in [Dayanik and Karatzas, 2003], let ψ and ϕ be the unique, up to multiplication by a positive constant, strictly increasing and strictly decreasing (respectively) solutions of the ordinary differential equation $Au = \beta u$ and such that $0 \leq \psi(0+)$ and $\psi(p) \to \infty$ as $p \to \infty$. For any $x \geq 0$, let

$$\ell_x = \lim_{p \to \infty} \frac{(xp - k)^+}{\psi(p)}. \tag{3.14}$$

Then U is finite in $\mathcal{O}$ if and only if ℓ_x is finite for all $x \geq 0$. Furthermore, when U is finite we also have that for some $C > 0$

$$U(x, p) \leq Cx\psi(p) \tag{3.15}$$

and

$$\lim_{p \to \infty} \frac{U(x, p)}{\psi(p)} = \ell_x. \tag{3.16}$$

We will assume that U is finite for the rest of this section.

3.2.3 Boundary condition

Since the investor is not allowed to purchase shares of the asset we have that $V(0, p) = 0$ for all $p \geq 0$. Also, the price process gets absorbed at 0, therefore $V = 0$ on $\partial \mathcal{O}$. If we assume that U is finite then by (3.15) we have that $V(x, p) \to 0$ as $x \to 0$ for all $p \geq 0$, that is, V is continuous on $\{x = 0\}$. Now we distinguish two cases:

1. 0 is an absorbing boundary for the price process P. This means that for any $p > 0$, $\mathbb{P}(P_t = 0 \text{ for some } t > 0 | P_0 = p) > 0$. A simple example is the arithmetic Brownian motion. Since the process is stopped at 0, we must have that for all $x \geq 0$

$$U(x, 0) = 0.$$

   Also, [Dayanik and Karatzas, 2003] shows that in this case U is continuous at $\{p = 0\}$ whenever U is finite. Therefore the boundary conditions for the value function V are

$$V = 0 \text{ on } \partial \mathcal{O} \text{ and } \lim_{y' \to y} V(y') = 0 \text{ for all } y \in \partial \mathcal{O}. \tag{3.17}$$

2. 0 is a natural boundary for the price process P. This means that for any $p > 0$, $\mathbb{P}(P_t = 0 \text{ for some } t > 0 | P_0 = p) = 0$. For example the geometric Brownian motion. In this case we can have different situations in $V(x, p)$ as p goes to 0 depending on the price process. In particular, we can have the situation where V is discontinuous on the set $\{p = 0\}$.

3.2.4 Viscosity solution

We now are going to prove that the value function is a viscosity solution of the HJB equation (3.9) and find the appropriate conditions that make this value function unique. The appropriate notion of solution of the HJB equation (3.9) is the notion of discontinuous viscosity solution since we cannot know a priori if the value function is continuous in $\mathcal{O}$.

We have the following theorem:

Theorem 17. *The value function V defined in (3.7) is a viscosity solution of (3.9) in $\mathcal{O}$.*

Proof. By the bounds given in the section 3.2.2, it is clear that V is locally bounded. Now we show the viscosity solution property.

Subsolution property: Let $y_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ such that y_0 is a maximizer of $V^* - \varphi$ on $\mathcal{O}$ with $V^*(y_0) = \varphi(y_0)$. Now suppose that there exists $\theta > 0$ and $\delta > 0$ such that

$$-\beta \varphi(y) + A\varphi(y) \leq -\theta \tag{3.18}$$

for all $y \in \mathcal{O}$ such that $|y - y_0| < \delta$. Let (y_n) be a sequence in $\mathcal{O}$ such that $y_n \to y_0$ and

$$\lim_{n \to \infty} V(y_n) = V^*(y_0).$$

By the dynamic programming principle (3.8), for all $n \geq 1$ there exist an admissible control $\nu_n = (\tau_m^n, \zeta_m^n)_m$ such that for any stopping time τ we have that

$$V(y_n) \leq \mathbb{E}\left[\sum_{\tau_m^n \leq \tau} e^{-\beta \tau_m^n} (\zeta_m^n P_{\tau_m^n}^n - k) + e^{-\beta \tau} V(Y_\tau^n) \right] + \frac{1}{n}, \tag{3.19}$$

where Y_s^n is the process controlled by ν_n for $s \geq 0$. Now consider the stopping time

$$T_n = \inf\{s \geq 0 : |Y_s^n - y_0| \geq \delta\} \wedge \tau_1^n,$$

where τ_1^n is the first intervention time of the impulse control ν_n. By (3.19) we have that

$$V(y_n) \leq \mathbb{E}\left[e^{-\beta T_n} V(Y_{T_n}^n) 1_{\{T_n < \tau_1^n\}} \right] + \mathbb{E}\left[e^{-\beta T_n} \left(\zeta_1^n P_{\tau_1^n}^n - k + V(Y_{\tau_1^n}^n) \right) 1_{\{T_n = \tau_1^n\}} \right] + \frac{1}{n}$$

$$\leq \mathbb{E}\left[e^{-\beta T_n} V(Y_{T_n-}^n) 1_{\{T_n < \tau_1^n\}} \right] + \mathbb{E}[e^{-\beta T_n} \mathcal{M}V(Y_{\tau_1^n-}^n) 1_{\{T_n = \tau_1^n\}}] + \frac{1}{n} \tag{3.20}$$

$$\leq \mathbb{E}\left[e^{-\beta T_n} V(Y_{T_n-}^n) \right] + \frac{1}{n} \tag{3.21}$$

Now, by Dynkin's formula and (3.18) we have

$$\mathbb{E}[e^{-\beta T_n} \varphi(Y_{T_n-}^n)] = \varphi(y_n) + \mathbb{E}\left[\int_0^{T_n} e^{-\beta s} \left(-\beta \varphi(Y_s^n) + A\varphi(Y_s^n) \right) ds \right]$$

$$\leq \varphi(y_n) - \frac{\theta}{\beta}(1 - \mathbb{E}[e^{-\beta T_n}]).$$

Since $V \leq V^* \leq \varphi$ and $T_n \leq \tau_1^n$, by (3.21)

$$V(y_n) \leq \varphi(y_n) - \frac{\theta}{\beta}(1 - \mathbb{E}[e^{-\beta T_n}]) + \frac{1}{n},$$

for all n. Letting n go to infinity we have that

$$\lim_{n \to \infty} \mathbb{E}[e^{-\beta T_n}] = 1,$$

which implies that

$$\lim_{n \to \infty} \mathbb{P}[\tau_1^n = 0] = 1.$$

Combining the above with (3.20) when we let $n \to \infty$ we get

$$V^*(y_0) \leq \sup_{|y' - y_0| < \delta} \mathcal{M}V(y').$$

Since this is true for all δ small enough, then sending δ to 0 we have

$$V^*(y_0) \leq (\mathcal{M}V)^*(y_0).$$

If we show that $(\mathcal{M}V)^* \leq \mathcal{M}V^*$, then we would have proved that if $-\beta\varphi(y_0) + A\varphi(y_0) < 0$, then $\mathcal{M}V^*(y_0) - V^*(y_0) \geq 0$ and therefore

$$\min\{\beta\varphi(y_0) - A\varphi(y_0), V^*(y_0) - \mathcal{M}V^*(y_0)\} \leq 0.$$

The proof of this last fact is as follows: Let φ be a locally bounded function on $\bar{\mathcal{O}}$. Let (y_n) be a sequence in $\mathcal{O}$ such that $(y_n) \to y_0$ and

$$\lim_{n\to\infty} \mathcal{M}\varphi(y_n) = (\mathcal{M}\varphi)^*(y_0).$$

Since φ^* is usc and Γ is continuous, for each $n \geq 1$ there exists $0 \leq \zeta_n \leq x_n$ such that

$$\mathcal{M}\varphi^*(y_n) = \varphi^*(\Gamma(y_n, \zeta_n)) + \zeta_n\alpha(\zeta_n, p_n) - k.$$

The sequence (ζ_n) is bounded (since $x_n \to x_0$) and therefore converges along a subsequence to $\zeta \in [0, x_0]$. Hence

$$\begin{aligned}
(\mathcal{M}\varphi)^*(y_0) &= \lim_{n\to\infty} \mathcal{M}\varphi(y_n) \\
&\leq \limsup_{n\to\infty} \mathcal{M}\varphi^*(y_n) \\
&= \limsup_{n\to\infty} \varphi^*(\Gamma(y_n, \zeta_n)) + \zeta_n\alpha(\zeta_n, p_n) - k \\
&\leq \varphi^*(\Gamma(y_0, \zeta)) + \zeta\alpha(\zeta, p_0) - k \\
&\leq \mathcal{M}\varphi^*(y_0).
\end{aligned}$$

Supersolution property: Let $y_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ such that y_0 is a minimizer of $V_* - \varphi$ on $\mathcal{O}$ with $V_*(y_0) = \varphi(y_0)$. By definition of V and $\mathcal{M}V$ we have that $\mathcal{M}V \leq V$ on $\mathcal{O}$ and therefore $(\mathcal{M}V)_* \leq V_*$. Let (y_n) be a sequence in $\mathcal{O}$ such that $y_n \to y_0$ and

$$\lim_{n\to\infty} V(y_n) = V_*(y_0).$$

Now, since $V_* \leq V$ is lower semi-continuous and Γ is continuous we have

$$\begin{aligned}
\mathcal{M}V_*(y_0) &= \sup_{0\leq\zeta\leq x_0} V_*(\Gamma(y_0, \zeta)) + \zeta\alpha(\zeta, p_0) - k \\
&\leq \sup_{0\leq\zeta\leq x_0} \liminf_{n\to\infty} V(\Gamma(y_n, \zeta)) + \zeta\alpha(\zeta, p_n) - k \\
&\leq \liminf_{n\to\infty} \sup_{0\leq\zeta\leq x_n} V(\Gamma(y_n, \zeta)) + \zeta\alpha(\zeta, p_n) - k \\
&\leq \lim_{n\to\infty} \mathcal{M}V(y_n) \\
&= (\mathcal{M}V)_*(y_0).
\end{aligned}$$

Hence $\mathcal{M}V_*(y_0) \leq (\mathcal{M}V)_*(y_0) \leq V_*(y_0)$. Now suppose that there exists $\theta > 0$ and $\delta > 0$ such that

$$\beta\varphi(y) - A\varphi(y) \leq -\theta \tag{3.22}$$

for all $y \in \mathcal{O}$ such that $|y - y_0| < \delta$. Fix n large enough such that $|y_n - y_0| < \delta$ and consider the process Y_s^n for $s \geq 0$ with no intervention such that $Y_0^n = y_n$. Let

$$T_n = \inf\{s \geq 0 : |Y_s^n - y_0| \geq \delta\}.$$

Now, by Dynkin's formula and (3.22) we have

$$\mathbb{E}[e^{-\beta T_n}\varphi(Y_{T_n}^n)] = \varphi(y_n) + \mathbb{E}\left[\int_0^{T_n} e^{-\beta s}\left(-\beta\varphi(Y_s^n) + A\varphi(Y_s^n)\right)ds\right]$$

$$\geq \varphi(y_n) + \frac{\theta}{\beta}(1 - \mathbb{E}[e^{-\beta T_n}]).$$

On the other hand, $\varphi \leq V_* \leq V$ and using the dynamic programming principle (3.8) we have

$$\mathbb{E}[e^{-\beta T_n}\varphi(Y_{T_n}^n)] \leq \mathbb{E}[e^{-\beta T_n}V(Y_{T_n}^n)] \leq V(y_n).$$

Notice that $\eta := \lim_{n\to\infty} \mathbb{E}[e^{-\beta T_n}] < 1$ since $T_n > 0$ a.s by a.s continuity of the processes Y_s^n, then by the above two inequalities and taking $n \to \infty$, we have that

$$V_*(y_0) \geq \varphi(y_0) + \frac{\theta}{\beta}(1 - \eta) > \varphi(y_0)$$

contradicting the fact that $V_*(y_0) = \varphi(y_0)$. This establishes the supersolution property. $\quad\square$

3.2.5 Uniqueness

Let ψ be defined as before and assume that the function U defined in (3.13) is finite. Also assume that the transaction cost $k > 0$. We want to prove that V is the unique viscosity solution of the equation (3.9) that is bounded by U. We will need an additional assumption about the function ψ: For all $x \geq 0$

$$\lim_{p\to\infty} \frac{U(x,p)}{\psi(p)} = \ell_x = 0. \tag{3.23}$$

Following the ideas in [Crandall et al., 1992, Ishii, 1993] let u be an upper semi-continuous (usc) viscosity subsolution of the HJB equation (3.9) and v be a lower semi-continuous (lsc) viscosity supersolution of the same equation in $\mathcal{O}$, such that they are bounded by U and

$$\limsup_{y'\to y} u(y') \leq \liminf_{y'\to y} v(y') \text{ for all } y \in \partial\mathcal{O}. \tag{3.24}$$

Define
$$v_m(x, p) = v(x, p) + \frac{1}{m}x^2\psi(p)$$
for all $m \geq 1$. Then v_m is still lsc and clearly $\beta v_m - \mathcal{A}v_m \geq 0$ by definition of ψ. Now,

$$\mathcal{M}v_m(x, p) = \sup_{0 \leq \zeta \leq x} v(x - \zeta, \alpha(\zeta, p)) + \frac{1}{m}(x - \zeta)^2\psi(\alpha(\zeta, p)) + \zeta\alpha(\zeta, p) - k$$

$$\leq \sup_{0 \leq \zeta \leq x} v(x - \zeta, \alpha(\zeta, p)) + \zeta\alpha(\zeta, p) - k + \sup_{0 \leq \zeta \leq x} \frac{1}{m}(x - \zeta)^2\psi(\alpha(\zeta, p))$$

$$= \mathcal{M}v(x, p) + \frac{1}{m}x^2\psi(p)$$

$$\leq v(x, p) + \frac{1}{m}x^2\psi(p) = v_m(x, p).$$

Therefore v_m is supersolution of (3.9). Now, by the growth condition of u and v and equations (3.15) and (3.23) we get
$$\lim_{|y| \to \infty} (u - v_m)(y) = -\infty. \tag{3.25}$$

We will show now that
$$u \leq v \text{ in } \mathcal{O}. \tag{3.26}$$

It is sufficient to show that $\sup\limits_{y \in \bar{\mathcal{O}}}(u - v_m) \leq 0$ for all $m \geq 1$ since the result is obtained by letting $m \to \infty$. Suppose that there exists $m \geq 1$ such that $\eta = \sup\limits_{y \in \bar{\mathcal{O}}}(u - v_m) > 0$. Since $u - v_m$ is usc, by (3.25) and (3.24) there exist $y_0 \in \mathcal{O}$ such that $\eta = (u - v_m)(y_0)$. Let $y_0 = (x_0, p_0)$ be the one with minimum norm over all possible maximizers of $u - v_m$. For $i \geq 1$, define
$$\phi_i(y, y') = \frac{i}{2}|y - y'|^4 + |y - y_0|^4,$$
$$\Phi_i(y, y') = u(y) - v_m(y') - \phi_i(y, y').$$

Let
$$\eta_i = \sup_{|y|, |y'| \leq |y_0|} \Phi_i(y, y') = \Phi_i(y_i, y_i').$$

Clearly $\eta_i \geq \eta$. Then, this inequality reads $\frac{i}{2}|y_i - y_i'|^4 + |y_i - y_0|^4 \leq u(y_i) - v_m(y_i') - (u - v_m)(y_0)$. Since $|y_i|, |y_i'| \leq |y_0|$ and u and $-v_m$ are bounded above in that region, this implies that $y_i, y_i' \to y_0$ and $\frac{i}{2}|y_i - y_i'|^4 \to 0$ (along a subsequence) as $i \to \infty$. We also find that $\eta_i \to \eta$, $u(y_i) - v_m(y_i') \to \eta$ and $u(y_i) \to u(y_0), v_m(y_i') \to v(y_0)$. By theorem 3.2 in [Crandall et al., 1992], for all $i \geq 1$, there exist symmetric matrices M_i and M_i' such that $(\frac{\partial\phi_i}{\partial y}(y_i, y_i'), M_i) = (d_i, M_i) \in \bar{J}^{2,+}u(y_i)$, $(-\frac{\partial\phi_i}{\partial y'}(y_i, y_i'), M_i') = (d_i', M_i') \in \bar{J}^{2,-}v_m(y_i')$ and

$$\begin{pmatrix} M_i & 0 \\ 0 & M_i' \end{pmatrix} \leq D^2\phi_i(y_i, y_i') + \frac{1}{i}(D^2\phi_i(y_i, y_i'))^2.$$

Since u is a subsolution of (3.9) and v_m is a supersolution, we have

$$\min\{\beta u(y_i) - \mu(p_i)d_{i,2} - \frac{1}{2}\sigma(p_i)^2 M_{i,22}, u(y_i) - \mathcal{M}u(y_i)\} \le 0,$$

and

$$\min\{\beta v_m(y_i') - \mu(p_i')d_{i,2}' - \frac{1}{2}\sigma(p_i')^2 M_{i,22}', v_m(y_i') - \mathcal{M}v_m(y_i')\} \ge 0.$$

Now, if we show that for infinitely many i's we have that

$$\beta u(y_i) - \mu(p_i)d_{i,2} - \frac{1}{2}\sigma(p_i)^2 M_{i,22} \le 0, \tag{3.27}$$

and since it is always true that

$$\beta v_m(y_i') - \mu(p_i')d_{i,2}' - \frac{1}{2}\sigma(p_i')^2 M_{i,22}' \ge 0,$$

we have that $u \le v_m$ by following the classical comparison proof in [Crandall et al., 1992]. Suppose then, that there exists i_0 such that (3.27) is not true for all $i \ge i_0$, then for $i \ge i_0$

$$u(y_i) - \mathcal{M}u(y_i) \le 0.$$

Since v_m is a supersolution, we must have that

$$v_m(y_i') - \mathcal{M}v_m(y_i') \ge 0.$$

Since u is usc, there exist ζ_i such that $\mathcal{M}u(y_i) = u(x_i - \zeta_i, \alpha(\zeta_i, p_i)) + \zeta_i \alpha(\zeta_i, p_i) - k$. Then

$$u(y_i) \le u(x_i - \zeta_i, \alpha(\zeta_i, p_i)) + \zeta_i \alpha(\zeta_i, p_i) - k.$$

Extracting a subsequence if necessary, we assume that $\zeta_i \to \zeta_0$ as $i \to \infty$. First, consider $\zeta_0 = 0$, then by taking lim sup in the inequality above we get $u(y_0) \le u(y_0) - k$. This is a contradiction since $k > 0$. Now assume that $\zeta_0 \ne 0$. From the above inequalities we have that

$$u(y_i) - v_m(y_i') \le u(x_i - \zeta_i, \alpha(\zeta_i, p_i)) + \zeta_i \alpha(\zeta_i, p_i) - v_m(x_i' - \zeta_i', \alpha(\zeta_i', p_i')) - \zeta_i' \alpha(\zeta_i', p_i'),$$

for any $0 \le \zeta_i' \le p_i'$. Since $p_i' \to p_0$, let $\zeta_i' \to \zeta_0$ and taking lim sup in the above inequality we get that

$$\eta \le (u - v_m)(x_0 - \zeta_0, \alpha(\zeta_0, p_0)).$$

This is a contradiction since y_0 was chosen with minimum norm among maximizers of $u - v_m$ and $\zeta_0 > 0$. Therefore (3.27) must hold for infinitely many i's and (3.26) holds. As usual continuity in $\mathcal{O}$ and uniqueness of V follow from the fact that V is a viscosity solution of (3.9).

We have just proved the following theorem:

Theorem 18. *Assume condition* (3.23) *and that the transaction cost $k > 0$. If W is a viscosity solution of equation* (3.9) *that is bounded by U and satisfies the same boundary conditions as V, then $W = V$. Furthermore, V is continuous in $\mathcal{O}$.*

Remark 19. *Condition* (3.23) *is satisfied by Itô processes like Brownian Motion, Geometric Brownian Motion, Mean Reverting and Cox-Ingersoll-Ross.*

3.3 Finite Horizon

In this case we need to know how much time is left to execute the position. Hence, given $y = (x, p) \in \bar{\mathcal{O}}$ and $0 \leq t \leq T$ the value function V has the form:

$$V(t, y) = \sup_{\nu} \mathbb{E} \left[\sum_{n=1}^{M} e^{-\beta(\tau_n - t)} (\zeta_n \alpha(\zeta_n, P_{\tau_n-}) - k) \right]. \tag{3.28}$$

When T is small the effect of discounting could be negligible, so we allow $\beta = 0$ in this formulation.

In general, the impulse control formulation allows multiple actions at the same moment. Hence, in this case multiple trading is allowed and it could be optimal as well . The presence of transaction cost will off course forbid infinite tradings but do not forbid that for some n we could have optimally $\tau_n = \tau_{n+1}$ and $\zeta_n, \zeta_{n+1} \neq 0$. In fact, sometimes it is optimal as we will show later in a special case. This is an important difference with respect to the formulation in [Ly Vath et al., 2007], since they imply that this degenerate behavior is ruled out because of the presence of transaction cost. This in particular has implications in the terminal condition (see Chapter 1).

3.3.1 Hamilton-Jacobi-Bellman equation

Again, we will use the dynamic programming approach. In this case, we have that the following Dynamic Programming Principle holds: For all $(t, y) \in [0, T) \times \mathcal{O}$ we have

$$V(t, y) = \sup_{\nu} \mathbb{E} \left[\sum_{\tau_n \leq \tau} e^{-\beta(\tau_n - t)} (\zeta_n \alpha(\zeta_n, P_{\tau_n-}) - k) + e^{-\beta(\tau - t)} V(\tau, Y_\tau) \right], \tag{3.29}$$

where $t \leq \tau \leq T$ is any stopping time. The HJB equation that follows from the DPP is then ([Øksendal and Sulem, 2005])

$$\min \left\{ -\frac{\partial \varphi}{\partial t} + \beta \varphi - A\varphi, \varphi - \mathcal{M}\varphi \right\} = 0 \text{ in } \mathcal{O}. \tag{3.30}$$

3.3.2   Growth Condition

As in the infinite horizon case we can use the solution of the optimal stopping problem to bound the value function. Following the same arguments as in proposition 16 we have that

$$V(t, y) \leq U(t, x, p) \leq x \sup_{t \leq \tau \leq T} \mathbb{E}[e^{-\beta(\tau - t)} P_\tau] =: x U_0(t, p). \tag{3.31}$$

3.3.3   Boundary Condition

Since the investor is not allowed to purchase shares of the asset we have that $V(t, 0, p) = 0$ for all $p \geq 0$ and $0 \leq t \leq T$. Also, the price process gets absorbed at 0, therefore $V = 0$ on $[0, T] \times \partial \mathcal{O}$. Since U_0 is finite then by (3.31) we have that $V(t, x, p) \to 0$ as $x \to 0$, that is, V is continuous on $\{x = 0\}$. Now, if U_0 is continuous at $\{p = 0\}$ the boundary conditions for the value function V are

$$V = 0 \text{ on } [0, T] \times \partial \mathcal{O} \text{ and } \lim_{y' \to y} V(t, y') = 0 \text{ for all } (t, y) \in [0, T) \times \partial \mathcal{O}. \tag{3.32}$$

3.3.4   Terminal Condition

Assumption (3.4) guarantees that the price impact does not change by splitting the trades, but the profit obtained by doing so could be greater than with a single transaction. We define the following sequence of functions for $(t, y) \in [0, T) \times \mathcal{O}$:

$$\varphi_0(t, y) = 0$$

and

$$\varphi_n(t, y) = \mathcal{M}\varphi_{n-1}(t, y) = \sup_{0 \leq \zeta \leq x} \varphi_{n-1}(t, \Gamma(y, \zeta)) + \zeta \alpha(\zeta, p) - k \text{ for } n = 1, 2, \ldots$$

So, $\varphi_n(t, y)$ is the best that we can do by trading n times starting at y at time t. Since these functions are constant in t we will consider them as functions in $\bar{\mathcal{O}}$ only for easy notation. When $k > 0$ we cannot trade infinitely many times, but when there is no transaction cost we can actually trade infinitely many times. Hence, let's define the following important function

$$W(y) = \int_0^x \alpha(s, p) ds. \tag{3.33}$$

When $\alpha(\zeta, p) = p e^{-\lambda \zeta}$ for $\lambda > 0$, figure 3.1(a) shows φ_n for various n and $W = \frac{p}{\lambda}(1 - e^{-\lambda x})$ for some values of x and keeping p fixed.

Lemma 20. $\varphi_n(y) \leq W(y)$ *for all* $n \geq 0$ *and all* $y \in \mathcal{O}$.

(a) φ_n for $n = 1, 2, 3, 4, 5$ and W with $p = 2$.

(b) φ_∞ with $p = 0.5$.
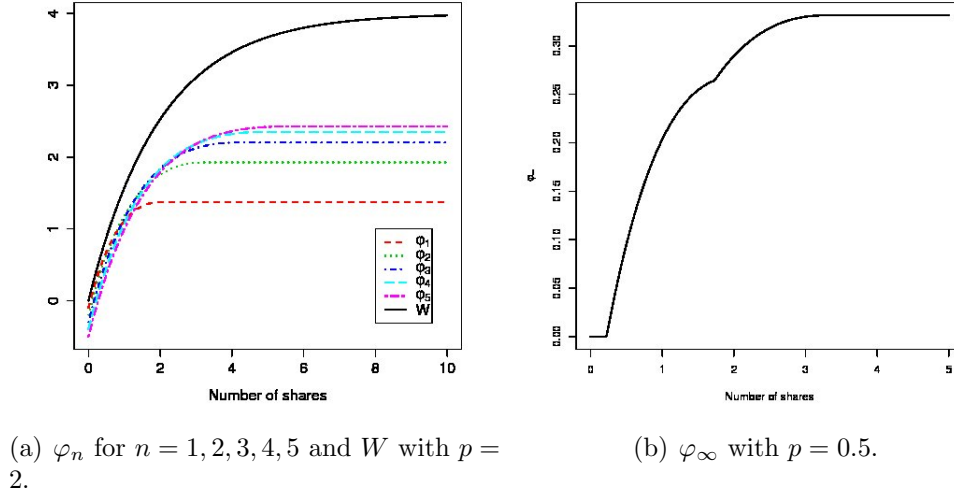
Figure 3.1: $\lambda = 0.5$ and $k = 0.1$.

Proof. Since α is non-increasing on x and positive, we have for all $y \in \mathcal{O}$

$$x\alpha(x, p) \leq \int_0^x \alpha(s, p)ds. \tag{3.34}$$

Clearly $\varphi_0(y) \leq W(y)$ for all $y \in \mathcal{O}$. Now assume that $\varphi_n(y) \leq W(y)$ for all $y \in \mathcal{O}$. Hence for all $0 \leq \zeta \leq x$

$$\varphi_n(\Gamma(y, \zeta)) + \zeta\alpha(\zeta, p) - k \leq W(\Gamma(y, \zeta)) + \zeta\alpha(\zeta, p) - k$$

$$= \zeta\alpha(\zeta, p) - k + \int_0^{x-\zeta} \alpha(s, \alpha(\zeta, p))ds$$

$$\leq \zeta\alpha(\zeta, p) + \int_0^x \alpha(s, p)ds - \int_0^\zeta \alpha(s, p)ds$$

$$= W(y) + \zeta\alpha(\zeta, p) - \int_0^\zeta \alpha(s, p)ds$$

$$\leq W(y),$$

where the last inequality follows from (3.34). Therefore

$$\varphi_{n+1}(y) = \sup_{0 \leq \zeta \leq x} \varphi_n(\Gamma(y, \zeta)) + \zeta\alpha(\zeta, p) - k \leq W(y).$$

This completes the proof by induction. $\square$

Let's define
$$\varphi_\infty(y) := \sup_n \varphi_n(y) \le W(y).$$

φ_∞ is the best that we can achieve at any particular moment by just thinking what is best at that moment, without looking into the future of the process. Now, when $k = 0$ consider the strategy that trades $\frac{x}{n}$ number of shares each time for $n \ge 1$. Thus

$$\varphi_n(y) \ge \frac{x}{n} \sum_{i=1}^n \alpha(i\frac{x}{n}, p).$$

Taking $n \to \infty$ we have that
$$\varphi_\infty(y) \ge W(y),$$

and therefore $\varphi_\infty(y) = W(y)$. Note that W is not attainable for any strategy.

Lemma 21. *The function φ_∞ is continuous.*

Proof. When $k = 0$ this is clear since α is continuous. Let $k > 0$, then we cannot trade infinitely many times, hence for all $y \in \mathcal{O}$ there exists some $n \ge 0$ such that $\varphi_m(y) \le \varphi_n(y)$ for all m, that is, φ_∞ is the sup over a finite number of functions. We need to show now that these functions are continuous, i.e, we need to show that for any $\varphi \in \mathcal{C}(\bar{\mathcal{O}})$, $\mathcal{M}\varphi \in \mathcal{C}(\bar{\mathcal{O}})$. Let (y_n) be a sequence in $\mathcal{O}$ such that $y_n \to y = (x, p)$. Since φ, Γ and α are continuous, there exists $0 \le \zeta_n \le x_n$ such that

$$\mathcal{M}\varphi(y_n) = \varphi(\Gamma(y_n, \zeta_n)) + \zeta_n \alpha(\zeta_n, p_n) - k.$$

Since $x_n \to x$, let $0 \le \zeta^* \le x$ a limit point of the sequence (ζ_n). Let $\hat{\zeta}$ such that

$$\mathcal{M}\varphi(y) = \varphi(\Gamma(y, \hat{\zeta})) + \hat{\zeta}\alpha(\hat{\zeta}, p) - k.$$

Hence,

$$\varphi(\Gamma(y_n, \hat{\zeta})) + \hat{\zeta}\alpha(\hat{\zeta}, p_n) - k \le \mathcal{M}\varphi(y_n) = \varphi(\Gamma(y_n, \zeta_n)) + \zeta_n \alpha(\zeta_n, p_n) - k,$$

and taking $n \to \infty$

$$\mathcal{M}\varphi(y) \le \varphi(\Gamma(y, \zeta^*)) + \zeta^* \alpha(\zeta^*, p) - k \le \mathcal{M}\varphi(y).$$

Therefore $\mathcal{M}\varphi(y_n) \to \mathcal{M}\varphi(y)$ and $\mathcal{M}\varphi$ is continuous. $\qquad\square$

For the following lemma we introduce a superscript in the functions φ_∞ and V that indicates the value of the transaction cost. Hence, we showed that $\varphi_\infty^{(0)} = W$. For easy notation we will use this only when necessary.

Lemma 22. *For all $(t, y) \in [0, T] \times \bar{\mathcal{O}}$ we have*

$$\lim_{k \to 0} \varphi_\infty^{(k)}(y) = \varphi_\infty^{(0)}(y) = W(y)$$

and

$$\lim_{k \to 0} V^{(k)}(t, y) = V^{(0)}(t, y).$$

Proof. By lemma 20 the limit on the left is bounded by W. Let $\epsilon > 0$, then there exist $\zeta_1, \ldots, \zeta_m$ such that

$$W(y) \leq \sum_{i=1}^{m} \zeta_i \alpha \left(\sum_{j=1}^{i} \zeta_j, p \right) + \epsilon.$$

For any $k \leq \frac{\epsilon}{m}$ we have that

$$W(y) \leq \sum_{i=1}^{m} \left(\zeta_i \alpha \left(\sum_{j=1}^{i} \zeta_j, p \right) - k \right) \leq \varphi_\infty^{(k)}(y) + 2\epsilon.$$

The proof of the other limit is similar. Again, it is clear that $V^{(0)}$ is an upper bound. Let $\epsilon > 0$, then there is $m \geq 0$ and $\nu \in \Upsilon_m$ (as in the proof of proposition 16) such that

$$V^{(0)}(t, y) \leq \mathbb{E} \left[\sum_{i=1}^{m} e^{-\beta(\tau_i - t)} \zeta_i \alpha(\zeta_i, P_{\tau_i -}) \right] + \epsilon.$$

The rest of the proof follows as in the previous case. $\square$

We can now state the terminal condition:
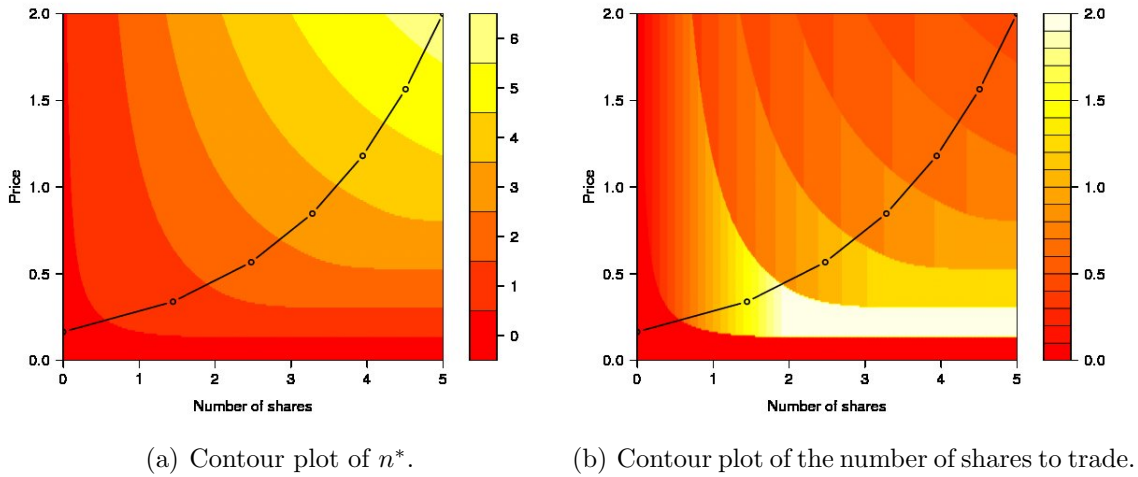
Proposition 23. *We have*

$$V(T, y) = \varphi_\infty(y) \text{ for all } y \in \mathcal{O}.$$

Proof. Let $y \in \mathcal{O}$ and consider a sequence (t_n, y_n) in $[0, T) \times \mathcal{O}$ such that $(t_n, y_n) \to (T, y)$ and

$$\lim_{n \to \infty} V(t_n, y_n) = V_*(T, y).$$

It is clear that $V(t_n, y_n) \geq \varphi_\infty(y_n)$. Taking $n \to \infty$, by lemma 21, $V_*(T, y) \geq \varphi_\infty(y)$. Now, let $y \in \mathcal{O}$ and consider a sequence (t_n, y_n) in $[0, T) \times \mathcal{O}$ such that $(t_n, y_n) \to (T, y)$ and

$$\lim_{n \to \infty} V(t_n, y_n) = V^*(T, y).$$

(a) Contour plot of n^*.

(b) Contour plot of the number of shares to trade.

Figure 3.2: $\lambda = 0.5$ and $k = 0.1$.

For any n there is an admissible control $\nu_n = (\tau_m^n, \zeta_m^n)_m$ such that

$$
V(t_n, y_n) \leq \mathbb{E}\left[\sum_{m=1}^{M^n} \zeta_m^n P_{\tau_m^n}^n - k\right] + \frac{1}{n}
$$

$$
= \mathbb{E}\left[\sum_{m=1}^{M^n} \zeta_m^n \alpha\left(\sum_{i=1}^{m} \zeta_i^n, p_n\right) - k\right] + \mathbb{E}\left[\sum_{m=1}^{M^n} \zeta_m^n \left(P_{\tau_m^n}^n - \alpha\left(\sum_{i=1}^{m} \zeta_i^n, p_n\right)\right)\right] + \frac{1}{n}
$$

$$
\leq \varphi_\infty(y_n) + \mathbb{E}\left[\sum_{m=1}^{M^n} \zeta_m^n \left(P_{\tau_m^n}^n - \alpha\left(\sum_{i=1}^{m} \zeta_i^n, p_n\right)\right)\right] + \frac{1}{n}
$$

by condition 3.4 and definition of φ_∞. Since the process P_s^n is a.s. continuous between intervention times and $t_n \to T$ then $\left|P_{\tau_m^n}^n - \alpha\left(\sum_{i=1}^{m} \zeta_i^n, p_n\right)\right| \to 0$ a.s as $n \to \infty$ for all m. If $k > 0$, the expected number of trades will remain bounded as $y_n \to y$, then taking $n \to \infty$ we get the reverse inequality $V^*(T, y) \leq \varphi_\infty(y)$. When $k = 0$ we use lemma 22 to complete the proof. $\qquad\square$

Figure 3.2(a) shows the contour plot of the optimal number of trades n^* for α as above. Also, figure 3.2(b) shows the contour plot of the number of shares that the investor must trade at each state. Both figures display the path of consecutive trades starting with 5 shares and price 2.

3.3.5 Viscosity Characterization

Here we can follow the same ideas as in the infinite horizon case. Note that we do not have to assume that U is finite since this is true in the finite case. Thus, we have the following theorem.

Theorem 24. *The value function V defined by* (3.28) *is a viscosity solution of* (3.30) *in* $[0, T) \times \mathcal{O}$.

To prove uniqueness before we used the results in [Dayanik and Karatzas, 2003] to guarantee the existence of the function ψ. We do not have those results in this case, so let us assume that there exists $\psi \in \mathcal{C}([0, T) \times [0, \infty))$ such that:

1. $-\frac{\partial \psi(t,p)}{\partial t} = A\psi(t, p) - \beta\psi(t, p)$.

2. ψ is non-decreasing in p.

3. For all $t \in [0, T)$

$$\lim_{p \to \infty} \frac{U_0(t, p)}{\psi(t, p)} = 0 \tag{3.35}$$

Example If the price process is a geometric Brownian motion that satisfies

$$dP_s = \mu P_s ds + \sigma P_s dB_s$$

for $t \le s \le T$ and $P_t = p$, then it is easy to see that $U_0(t, p) = pe^{(\mu-\beta)^+(T-t)}$. Therefore, the function $\psi(t, p) = p^2 e^{(2\mu+\sigma^2-\beta)(T-t)}$ satisfies the above conditions. Also, by the bound (3.31), the value function in this case satisfies the boundary condition (3.32).

Theorem 25. *Let ψ as above and assume that the transaction cost $k > 0$. If W is a viscosity solution of equation* (3.30) *that is bounded by U and satisfies the same boundary conditions as V, then $W = V$. Furthermore, V is continuous in $\mathcal{O}$.*

3.3.6 Special Case

A few different price impacts have been proposed in the literature. [Subramanian and Jarrow, 2001] considers impact functions of the form $\alpha(x, p) = pc(x)$, where $0 \le c \le 1$ is nonincreasing. In our case, by condition (3.4), c must satisfy $c(x_1)c(x_2) = c(x_1 + x_2)$ and therefore we end up with the following price impact function and its corresponding W:

$$\alpha(x, p) = pe^{-\lambda x}$$
$$W(x, p) = \frac{p}{\lambda}(1 - e^{-\lambda x})$$

with $\lambda > 0$. This function was proposed also in [He and Mamaysky, 2005] and [Ly Vath et al., 2007]. Let's consider this price impact function from now on. The advantage of it is that the impact is linear in p, which is very useful as we will see.

Theorem 26. *Suppose that* $U(t,x,p) = e^{-\beta(T-t)}\mathbb{E}[(xP_T - k)^+]$ *for all* $(t,y) \in [0,T] \times \bar{\mathcal{O}}$. *Then* $V(t,x,p) = e^{-\beta(T-t)}\mathbb{E}[\varphi_\infty(x, P_T)]$, *where* P_s *is the unperturbed process.*

Proof. We will follow the same idea as in the proof of proposition 16, that is, induction in the number of trades. Note that the function $\zeta \mapsto \zeta e^{-\lambda\zeta}$ in $[0,x]$ attains its maximum at $\hat{x} = \min\{x, \frac{1}{\lambda}\}$. Then,

$$\sup_{\nu \in \Upsilon_1} \mathbb{E}[e^{-\beta(\tau_1-t)}(\zeta_1 P_{\tau_1-}e^{-\lambda\zeta_1} - k)] \leq U(t,\hat{x},p)$$
$$= e^{-\beta(T-t)}\mathbb{E}[(\hat{x}P_T - k)^+]$$
$$\leq e^{-\beta(T-t)}\mathbb{E}[\varphi_\infty(x, P_T)].$$

Now, let $\nu \in \Upsilon_n$ and P_s^ν controlled by ν. Hence,

$$\mathbb{E}[e^{-\beta(\tau_1-t)}(\zeta_1 P_{\tau_1}^\nu - k)] = \mathbb{E}\left[\mathbb{E}[e^{-\beta(\tau_1-t)}(\zeta_1 P_{\tau_1-}e^{-\lambda\zeta_1} - k)|\mathcal{F}_{\tau_1}]\right]$$
$$\leq \mathbb{E}\left[e^{-\beta(\tau_1-t)}\mathbb{E}[U(\tau_1, \zeta_1 e^{-\lambda\zeta_1}, P_{\tau_1-})|\mathcal{F}_{\tau_1}]\right]$$
$$\leq \mathbb{E}\left[e^{-\beta(\tau_1-t)}\mathbb{E}[e^{\beta(T-\tau_1)}\mathbb{E}[(\zeta_1 e^{-\lambda\zeta_1}P_T - k)^+]|\mathcal{F}_{\tau_1}]\right]$$
$$= e^{-\beta(T-t)}\mathbb{E}[(\zeta_1 e^{-\lambda\zeta_1}P_T - k)^+].$$

On the other hand, by induction hypothesis we have

$$\mathbb{E}\left[e^{-\beta(\tau_1-t)}\sum_{i=2}^{n}e^{-\beta(\tau_i-\tau_1)}(\zeta_i P_{\tau_i}^\nu - k)\right]$$
$$= \mathbb{E}\left[e^{-\beta(\tau_1-t)}\mathbb{E}\left[\sum_{i=2}^{n}e^{-\beta(\tau_i-\tau_1)}(\zeta_i P_{\tau_i}^\nu - k)\bigg|\mathcal{F}_{\tau_1}\right]\right]$$
$$\leq \mathbb{E}\left[e^{-\beta(\tau_1-t)}\mathbb{E}[V(\tau_1, x - \zeta_1, e^{-\lambda\zeta_1}P_{\tau_1})|\mathcal{F}_{\tau_1}]\right]$$
$$= \mathbb{E}\left[e^{-\beta(\tau_1-t)}\mathbb{E}[e^{\beta(T-\tau_1)}\mathbb{E}[\varphi_\infty(x - \zeta_1, e^{-\lambda\zeta_1}P_T)]|\mathcal{F}_{\tau_1}]\right]$$
$$= e^{-\beta(T-t)}\mathbb{E}[\varphi_\infty(x - \zeta_1, e^{-\lambda\zeta_1}P_T)].$$

Combining both inequalities above we have

$$\sup_{\nu \in \Upsilon_n} \mathbb{E}\left[\sum_{i=1}^{n}e^{-\beta(\tau_i-t)}(\zeta_i P_{\tau_i}^\nu - k)\right] \leq e^{-\beta(T-t)}\mathbb{E}[\varphi_\infty(x, P_T)].$$

By lemma 10 the left hand side converges to V as $n \to \infty$. Clearly the other inequality holds and the proof is complete. $\square$

Corollary 27. *Let the unperturbed price process be such that* $e^{-\beta(s-t)}P_s$ *is a submartingale. Then for each* $x \geq 0$, $V(t,x,p)$ *solves the problem*

$$\begin{cases} -\frac{\partial\varphi}{\partial t}(t,p) + \beta\varphi(t,p) - A\varphi(t,p) = 0 & in \ [0,T) \times (0,\infty) \\ \varphi(T,p) = \varphi_\infty(x,p) & on \ [0,\infty). \end{cases}$$

Proof. Since the map $p \mapsto (xp - k)^+$ is convex, increasing and 0 at $p = 0$ for all $x \geq 0$, then the process $e^{-\beta(s-t)}(xP_s - k)^+$ is a submartingale. Hence, the theorem holds and by Feynman-Kac theorem we get the result. $\qquad\square$

Corollary 28. *Let the unperturbed price process be such that $e^{-\beta(s-t)}P_s$ is a submartingale. Then*

$$V^{(0)}(t, x, p) = e^{-\beta(T-t)}\frac{1 - e^{-\lambda x}}{\lambda}\mathbb{E}[P_T].$$

Chapter 4

No Transaction Cost: Infinite Horizon Case

From the proof of the uniqueness result in the previous chapter we can see that the result depends on the assumption that the transaction cost $k > 0$. In this chapter we analyze the case where there is no transaction cost. Let's start by pointing out that in this case the intervention operator becomes

$$\mathcal{M}\varphi(y) = \sup_{0 \leq \zeta \leq x} \varphi(\Gamma(y, \zeta)) + \zeta\alpha(\zeta, p) \geq \varphi(\Gamma(y, 0)) = \varphi(y),$$

for any measurable function φ. This implies in particular that any measurable function is a viscosity subsolution of (3.9). On the other hand, $V \geq \mathcal{M}V$ for the value function. Then we have that

$$V \geq \mathcal{M}V \geq V.$$

Assume now that $V \in C^1(\mathcal{O})$. Since $\zeta = 0$ is a maximum for $\zeta \mapsto V(\Gamma(y, \zeta)) + \zeta\alpha(\zeta, p)$, then for all $y \in \mathcal{O}$:

$$0 \geq \frac{\partial\alpha}{\partial\zeta}(\zeta, p)\frac{\partial V}{\partial p}(y) - \frac{\partial V}{\partial x}(y) + \alpha(\zeta, p) + \zeta\frac{\partial\alpha}{\partial\zeta}(\zeta, p)\bigg|_{\zeta=0}$$

$$= \frac{\partial\alpha}{\partial\zeta}(0, p)\frac{\partial V}{\partial p}(y) - \frac{\partial V}{\partial x}(y) + p.$$

Recall that α is non-increasing in ζ, so we define

$$\gamma(p) = -\frac{\partial\alpha}{\partial\zeta}(0, p), \tag{4.1}$$

for all $p \geq 0$. Hence, we get the following condition for V:

$$-\gamma(p)\frac{\partial V}{\partial p}(y) - \frac{\partial V}{\partial x}(y) + p \leq 0. \tag{4.2}$$

This suggests that if we assume no fixed transaction cost we should look at a different HJB equation, that is

$$\min\left\{\beta\varphi - A\varphi, \gamma(p)\frac{\partial\varphi}{\partial p} + \frac{\partial\varphi}{\partial x} - p\right\} = 0. \tag{4.3}$$

In fact, equation (4.3) is the associated HJB equation of a singular control problem. We describe this model and the price impact function in the first section. Sections 2 and 3 characterize the value function in terms of the HJB equation. In section 4 we consider a special case and find the explicit value function. The last section includes some numerical examples for different type of price processes.

4.1 Singular Control Model

In this case our control satisfies

$$dX_t = -d\xi_t,$$

where $\xi_0 = 0$, ξ is an adapted, càdlàg non-decreasing and non-negative process. The price process in this case follows the dynamics

$$dP_t = \mu(P_{t-})dt + \sigma(P_{t-})dB_t - \gamma(P_{t-})d\xi_t,$$

where γ (see (4.1)) is a non-negative smooth function that accounts for the price impact. The main concern here is the existence and uniqueness of the process P_t. Fortunately we have a result similar to 4.

Theorem 29 ([Protter, 1990]). *Let $f_i : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, d$ be Lipschitz functions. Let Z^i be semimartingales with $Z_0^i = 0$ for $i = 1, \ldots, d$. Then the equation*

$$X_t = x + \sum_{i=1}^{d} \int_0^t f_i(s, X_{s-})dZ_s^i$$

admits an unique solution. This solution is also a semimartingale.

There is no need to give a full definition of semimartingale but we must say that any finite variation process and the Brownian motion are semimartingales. Hence, we need γ to be Lipschitz. Now, the form of the value function V_0 changes to

$$V_0(y) = \sup_\xi \mathbb{E}\left[\int_0^\infty e^{-\beta t}P_t d\xi_t\right], \tag{4.4}$$

for all $y \in \bar{\mathcal{O}}$. In this case the appropriate form of the DPP is

$$V_0(y) = \sup_\xi \mathbb{E}\left[\int_0^\tau e^{-\beta s}P_s d\xi_s + e^{-\beta\tau}V_0(Y_\tau)\right], \tag{4.5}$$

for any stopping time τ. As before, we can define the continuation region as

$$\mathcal{C} = \{y \in \mathcal{O} : \gamma(p)\frac{\partial\varphi}{\partial p} + \frac{\partial\varphi}{\partial x} - p > 0\}$$

and the trade region as

$$\mathcal{T} = \{y \in \mathcal{O} : \gamma(p)\frac{\partial\varphi}{\partial p} + \frac{\partial\varphi}{\partial x} - p = 0\}.$$

The intuition behind the equation (4.3) is the following: Let $y \in \mathcal{O}$ and assume enough regularity for the value function V_0. By DPP and Dynkin's formula, and considering the process with no intervention we can show that $(-\beta V_0 + AV_0)(y) \leq 0$, as before. Also, if the no intervention strategy is optimal, then $(-\beta V + AV)(y) = 0$. Now consider any admissible strategy ξ, by Dynkin's formula for semimartingales (see [Protter, 1990])

$$\begin{aligned}
V_0(y) \geq &\mathbb{E}\left[\int_0^t e^{-\beta s}P_s d\xi_s + e^{-\beta t}V_0(Y_t)\right] \\
= &V_0(y) + \mathbb{E}\left[\int_0^t e^{-\beta s}(-\beta V_0 + AV_0)(Y_s)ds\right] \\
&+ \mathbb{E}\left[\int_0^t e^{-\beta s}\left(P_s - \gamma(P_s)\frac{\partial V_0}{\partial p}(Y_s) - \frac{\partial V_0}{\partial x}(Y_s)\right)d\xi_s\right] \\
&+ \mathbb{E}\left[\sum_{s \leq t}e^{-\beta s}\left(P_s - \gamma(P_s)\frac{\partial V_0}{\partial p}(Y_s) - \frac{\partial V_0}{\partial x}(Y_s)\right)\Delta\xi_s\right]
\end{aligned}$$

for all $t \geq 0$. Since this is true for any ξ, we must have that $-\gamma(p)\frac{\partial V_0}{\partial p}(y) - \frac{\partial V_0}{\partial x}(y) + p \leq 0$. If taking an action is optimal, $-\gamma(p)\frac{\partial V}{\partial p}(y) - \frac{\partial V}{\partial x}(y) + p = 0$.

Remark 30. *Note that any impulse control is also a singular control. However, the revenue obtained with an impulse control in the first formulation is different to the one obtain with the same control for the singular formulation.*

4.2 Viscosity Solution

When the singular control is discontinuous the stochastic integral may not be properly defined. To avoid this we will require the control to be continuous to be sure that the price process has càglàd paths (see [Protter, 1990]). Nevertheless, the value function is still a viscosity solution of equation (4.3) (which definition is similar to 13).

Theorem 31. *The value function V_0 defined in (4.4) is a viscosity solution of (4.3) in $\mathcal{O}$.*

Proof. Since the process P_t is continuous, we can approach the integral with respect to a finite variation process by simple functions. Then by proposition 16 we have that

$$V_0 \leq U. \tag{4.6}$$

Therefore, V_0 is locally bounded.

Subsolution property: Let $y_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ such that y_0 is a maximizer of $V_0^* - \varphi$ on $\mathcal{O}$ with $V_0^*(y_0) = \varphi(y_0)$. Now suppose that there exists $\kappa > 0$ and $\delta > 0$ such that

$$-\beta\varphi(y) + A\varphi(y) \leq -\kappa \text{ and } p - \gamma(p)\frac{\partial\varphi}{\partial p}(y) - \frac{\partial\varphi}{\partial x}(y) \leq -\kappa \tag{4.7}$$

for all $y \in \mathcal{O}$ such that $|y - y_0| < \delta$. Let (y_n) be a sequence in $\mathcal{O}$ such that $y_n \to y_0$ and

$$\lim_{n\to\infty} V_0(y_n) = V_0^*(y_0).$$

Given any stopping time τ, by (4.5), for all $n \geq 1$ there exists an admissible control ξ^n such that

$$V_0(y_n) \leq \mathbb{E}\left[\int_0^\tau e^{-\beta s} P_s^n d\xi_s^n + e^{-\beta\tau}V_0(Y_\tau^n)\right] + \frac{1}{n},$$

where Y_s^n is the process controlled by ξ^n for $s \geq 0$ starting at y_n. Since $V_0 \leq V_0^* \leq \varphi$, using Dynkin's formula for semimartingales we have that

$$V_0(y_n) \leq \mathbb{E}\left[\int_0^\tau e^{-\beta s} P_s^n d\xi_s^n\right] + \varphi(y_n) + \mathbb{E}\left[\int_0^\tau e^{-\beta s}\left(-\beta\varphi(Y_s^n) + A\varphi(Y_s^n)\right)ds\right]$$
$$- \mathbb{E}\left[\int_0^\tau e^{-\beta s}\left(\gamma(P_s^n)\frac{\partial\varphi}{\partial p}(Y_s^n) + \frac{\partial\varphi}{\partial x}(Y_s^n)\right)d\xi_s^n\right] + \frac{1}{n}.$$

Consider again the stopping time

$$\tau_n = \inf\{s \geq 0 : |Y_s^n - y_0| \geq \delta\},$$

then by (4.7)

$$V_0(y_n) \leq -\kappa\mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}(ds + d\xi_s^n)\right] + \varphi(y_n) + \frac{1}{n}.$$

Taking $n \to \infty$ we obtain a contradiction since the integral inside the expectation is bounded away from 0 for any admissible control ξ by the a.s continuity of the process Y_s^n. Hence at least one of the inequalities in (4.7) is not possible and this establishes the subsolution property.

Supersolution property: Let $y_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ such that y_0 is a minimizer of $V_{0*} - \varphi$ on $\mathcal{O}$ with $V_{0*}(y_0) = \varphi(y_0)$. Let (y_n) be a sequence in $\mathcal{O}$ such that $y_n \to y_0$ and

$$\lim_{n\to\infty} V_0(y_n) = V_{0*}(y_0).$$

First, suppose that there exists $\theta > 0$ and $\delta > 0$ such that

$$\beta\varphi(y) - A\varphi(y) \leq -\theta \tag{4.8}$$

for all $y \in \mathcal{O}$ such that $|y - y_0| < \delta$. Fix n large enough such that $|y_n - y_0| < \delta$ and consider the process Y_s^n for $s \geq 0$ with no intervention, i.e. $\xi = 0$, such that $Y_0^n = y_n$. Let

$$\tau_n = \inf\{s \geq 0 : |Y_s^n - y_0| \geq \delta\}.$$

Now, by Dynkin's formula for semimartingales and (4.8) we have

$$
\begin{aligned}
\mathbb{E}[e^{-\beta\tau_n}\varphi(Y_{\tau_n}^n)] &= \varphi(y_n) + \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}\left(-\beta\varphi(Y_s^n) + A\varphi(Y_s^n)\right) ds\right] \\
&\quad - \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}\left(\gamma(P_s^n)\frac{\partial\varphi}{\partial p}(Y_s^n) + \frac{\partial\varphi}{\partial x}(Y_s^n)\right) d\xi_s\right] \\
&= \varphi(y_n) + \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}\left(-\beta\varphi(Y_s^n) + A\varphi(Y_s^n)\right) ds\right] \\
&\geq \varphi(y_n) - \theta\mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} ds\right].
\end{aligned}
$$

As before, from here we can draw a contradiction with $V_{0*}(y_0) = \varphi(y_0)$ by the a.s. continuity of the process Y_s^n. Now, take $h > 0$ and consider the process Y_t with control process $d\xi_t = \frac{1}{h}\mathbb{1}_{[0,h]}(t)dt$ and $Y_0 = y$ for given $y \in \mathcal{O}$. Using (4.5) we can show that

$$
\begin{aligned}
V_0(y) &\geq \mathbb{E}\left[\int_0^h e^{-\beta s}P_s d\xi_s + e^{-\beta h}V(Y_h)\right] \\
&\geq \mathbb{E}\left[\int_0^h e^{-\beta s}P_s d\xi_s + e^{-\beta h}\varphi(Y_h)\right] \\
&= \mathbb{E}\left[\frac{1}{h}\int_0^h e^{-\beta s}P_s ds + e^{-\beta h}\varphi(Y_h)\right].
\end{aligned}
$$

By Dynkin's formula again,

$$
\begin{aligned}
\mathbb{E}[e^{-\beta h}\varphi(Y_h)] &= \varphi(y) + \mathbb{E}\left[\int_0^h e^{-\beta s}\left(-\beta\varphi(Y_s) + A\varphi(Y_s)\right) ds\right] \\
&\quad - \mathbb{E}\left[\int_0^h e^{-\beta s}\left(\gamma(P_s)\frac{\partial\varphi}{\partial p}(Y_s) + \frac{\partial\varphi}{\partial x}(Y_s)\right) d\xi_s\right] \\
&= \varphi(y) + \mathbb{E}\left[\int_0^h e^{-\beta s}\left(-\beta\varphi(Y_s) + A\varphi(Y_s)\right) ds\right] \\
&\quad - \frac{1}{h}\mathbb{E}\left[\int_0^h e^{-\beta s}\left(\gamma(P_s)\frac{\partial\varphi}{\partial p}(Y_s) + \frac{\partial\varphi}{\partial x}(Y_s)\right) ds\right].
\end{aligned}
$$

Letting $h \to 0$, we have

$$V_0(y) \geq \varphi(y) + p - \gamma(p)\frac{\partial \varphi}{\partial p}(y) - \frac{\partial \varphi}{\partial x}(y).$$

Therefore, for all $n \geq 1$ we have

$$V_0(y_n) \geq \varphi(y_n) + p_n - \gamma(p_n)\frac{\partial \varphi}{\partial p}(y_n) - \frac{\partial \varphi}{\partial x}(y_n).$$

Since γ is continuous, letting $n \to \infty$ we get

$$\varphi(y_0) = V_{0*}(y_0) \geq \varphi(y_0) + p_0 - \gamma(p_0)\frac{\partial \varphi}{\partial p}(y_0) - \frac{\partial \varphi}{\partial x}(y_0)$$

as desired. This establishes the supersolution property. $\qquad\square$

4.3 Uniqueness

Recall that with the impulse formulation we do not have uniqueness in absence of transaction cost. This is not the case with the singular control formulation.

Theorem 32. *Assume that* (3.35) *is satisfied. If W is a viscosity solution of equation* (4.3) *that is bounded by U and satisfies the same boundary conditions as V_0, then $W = V_0$. Furthermore, V_0 is continuous in $\mathcal{O}$.*

Proof. The proof follows the same strategy as in the impulse control case. Let u be an upper semi-continuous (usc) viscosity subsolution of the HJB equation (4.3) and v be a lower semi-continuous (lsc) viscosity supersolution of the same equation in $\mathcal{O}$, such that they are bounded by U and condition (3.24) holds. Define

$$v_m(x,p) = \left(1 - \frac{1}{m}\right)v(x,p) + \frac{1}{m}\left(C(x+1)^2\psi(p) + 1\right)$$

for all $m \geq 1$ and C as in (3.15). Recall that γ is non-negative and ψ is an increasing function, then (3.15) implies that

$$-p + \frac{\partial v_m}{\partial x} + \gamma(p)\frac{\partial v_m}{\partial p} \geq -p + \left(1 - \frac{1}{m}\right)p + \frac{\partial}{\partial x}\frac{1}{m}C(x+1)^2\psi(p) + \gamma(p)\frac{\partial}{\partial p}\frac{1}{m}C(x+1)^2\psi(p)$$

$$= -\frac{1}{m}p + \frac{1}{m}2C(x+1)\psi(p) + \gamma(p)\frac{1}{m}C(x+1)^2\psi'(p)$$

$$\geq -\frac{1}{m}p + \frac{2}{m}p(x+1) + \gamma(p)\frac{1}{m}C(x+1)^2\psi'(p)$$

$$\geq \frac{1}{m}p.$$

Also $(\beta I - A)\left(\frac{1}{m}\right) = \frac{\beta}{m} > 0$, where I is the identity operator. Therefore v_m is a strict supersolution of (4.3) in $\mathcal{O}$. Following the same lines and definitions as in the previous proof we have

$$\min\{\beta u(y_i) - \mu(p_i)d_{i,2} - \frac{1}{2}\sigma(p_i)^2 M_{i,22}, -p_i + d_{i,1} + \gamma(p_i)d_{i,2}\} \leq 0,$$

and

$$\min\{\beta v_m(y_i') - \mu(p_i')d_{i,2}' - \frac{1}{2}\sigma(p_i')^2 M_{i,22}', -p_i' + d_{i,1}' + \gamma(p_i')d_{i,2}'\} \geq \delta_i,$$

where $\delta_i = \min\left\{\frac{p_i'}{m}, \frac{\beta}{m}\right\}$. Since $p_i' \to p_0$ and $y_0 \in \mathcal{O}$, $\delta_i > 0$ for large enough i. We need to show now that for infinitely many i's we have that

$$\beta u(y_i) - \mu(p_i)d_{i,2} - \frac{1}{2}\sigma(p_i)^2 M_{i,22} \leq 0. \tag{4.9}$$

Suppose then, that there exists i_0 such that (4.9) is not true for all $i \geq i_0$, then for $i \geq i_0$

$$-p_i + d_{i,1} + \gamma(p_i)d_{i,2} \leq 0.$$

Since v_m is a supersolution, we must have that

$$-p_i' + d_{i,1}' + \gamma(p_i')d_{i,2}' \geq \delta_i.$$

Hence,

$$p_i - p_i' - (d_{i,1} - d_{i,1}') - (\gamma(p_i)d_{i,2} - \gamma(p_i')d_{i,2}') \geq \delta_i.$$

Since d_i, d_i' goes to 0 as i goes to ∞, we get the contradiction $0 \geq \delta_0 = \min\left\{\frac{p_0}{m}, \frac{\beta}{m}\right\} > 0$. Therefore (4.9) must hold for infinitely many i's and the comparison result holds. Everything follows now as before. $\square$

4.4 Special Case

Recall the function W defined in (3.33):

$$W(y) = \int_0^x \alpha(s, p)ds \text{ for } y \in \mathcal{O}.$$

This function actually satisfies (4.2) with equality. Indeed, by the condition (3.4) we have that for any ζ_1, ζ_2 and p

$$\frac{\partial \alpha}{\partial \zeta}(\zeta_1 + \zeta_2, p) = \frac{\partial \alpha}{\partial p}(\zeta_1, \alpha(\zeta_2, p))\frac{\partial \alpha}{\partial \zeta}(\zeta_2, p),$$

and taking $\zeta_2 = 0$ we obtain

$$\frac{\partial \alpha}{\partial \zeta}(\zeta_1, p) = \frac{\partial \alpha}{\partial p}(\zeta_1, p)\frac{\partial \alpha}{\partial \zeta}(0, p) = -\gamma(p)\frac{\partial \alpha}{\partial p}(\zeta_1, p).$$

Now, since α is smooth we find

$$-\gamma(p)\frac{\partial W}{\partial p}(y) - \frac{\partial W}{\partial x}(y) + p = -\gamma(p)\int_0^x \frac{\partial \alpha}{\partial p}(s,p)ds - \frac{\partial}{\partial x}\int_0^x \alpha(s,p)ds + p$$

$$= \int_0^x \frac{\partial \alpha}{\partial \zeta}(s,p)ds - \alpha(x,p) + p$$

$$= \alpha(x,p) - \alpha(0,p) - \alpha(x,p) + p = 0.$$

Let us consider the price impact function used in section 3.3.6, that is:

$$\alpha(x,p) = pe^{-\lambda x} \tag{4.10}$$

$$\gamma(p) = \lambda p \tag{4.11}$$

$$W(x,p) = \frac{p}{\lambda}(1 - e^{-\lambda x}) \tag{4.12}$$

with $\lambda > 0$. Clearly γ is Lipschitz for this price impact. In this case we have the following:

Theorem 33. $V_0 = W = V$ if and only if $U(x,p) = xp$.

Proof. If $V_0 = W$ then $\beta W - AW \geq 0$ and therefore $\beta \varphi - A\varphi \geq 0$ for $\varphi(p) = p$. By the uniqueness result for optimal stopping problems (see Theorem 3.1 in [Øksendal and Reikvam, 1998])

$$p = \sup_\tau \mathbb{E}[e^{-\beta \tau} P_\tau],$$

that is $U(x,p) = xp$. Suppose that

$$U(x,p) = x \sup_\tau \mathbb{E}[e^{-\beta \tau} P_\tau] = xp,$$

for $y \in \mathcal{O}$. This means that $\beta \varphi - A\varphi \geq 0$ for $\phi(p) = p$. Therefore $\beta W - AW \geq 0$ and W satisfies the HJB equation (4.3) with $\mathcal{T} = \mathcal{O}$. Also, W satisfies the growth condition and has the same boundary conditions as V_0 by (3.13). By Theorem 32, we have that $W = V_0$. To prove the second equality we will do induction in the number of trades. Note that the function $\zeta \mapsto \zeta e^{-\lambda \zeta}$ in $[0,x]$ attains its maximum at $\hat{x} = \min\{x, \frac{1}{\lambda}\}$. Then,

$$\sup_{\nu \in \Upsilon_1} \mathbb{E}[e^{-\beta \tau_1} \zeta_1 P_{\tau_1-} e^{-\lambda \zeta_1}] \leq U(\hat{x},p) = \hat{x}p \leq W(x,p).$$

Now, let $\nu \in \Upsilon_n$. Hence,

$$\mathbb{E}[e^{-\beta \tau_1} \zeta_1 P_{\tau_1}] = \mathbb{E}\left[e^{-\beta \tau_1} \mathbb{E}[\zeta_1 P_{\tau_1-} e^{-\lambda \zeta_1}|\mathcal{F}_{\tau_1}]\right].$$

On the other hand, by induction hypothesis we have

$$\mathbb{E}\left[e^{-\beta \tau_1}\sum_{i=2}^n e^{-\beta(\tau_i - \tau_1)}\zeta_i P_{\tau_i}\right] = \mathbb{E}\left[e^{-\beta \tau_1}\mathbb{E}\left[\sum_{i=2}^n e^{-\beta(\tau_i - \tau_1)}\zeta_i P_{\tau_i}\bigg|\mathcal{F}_{\tau_1}\right]\right]$$

$$\leq \mathbb{E}\left[e^{-\beta \tau_1}\mathbb{E}[V(x - \zeta_1, e^{-\lambda \zeta_1}P_{\tau_1-})|\mathcal{F}_{\tau_1}]\right]$$

$$\leq \mathbb{E}\left[e^{-\beta \tau_1}\mathbb{E}[W(x - \zeta_1, e^{-\lambda \zeta_1}P_{\tau_1-})|\mathcal{F}_{\tau_1}]\right]$$

Combining both inequalities above we have

$$\mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta \tau_i} \zeta_i P_{\tau_i}^{\nu}\right] \leq \mathbb{E}[e^{-\beta \tau_1} W(x, P_{\tau_1-})] \leq W(x, p).$$

By lemma 10, the left hand side converges to V as $n \to \infty$. Clearly the other inequality holds and the proof is complete. $\square$

If the price process is a geometric Brownian motion the unperturbed price process is

$$dP_t = \mu P_t dt + \sigma P_t dB_t,$$

with $\sigma > 0$. It is easy to see that the value function U is finite if and only if $\beta > \mu$. In this case the function ψ takes the form

$$\psi(p) = p^a,$$

where $a > 1$, therefore condition (3.35) holds. Now, the condition (3.13) reads

$$0 \leq V(x, p) \leq U(x, p) = xp.$$

This implies that $V_0 = V = W$.

4.4.1 Regular control

Since we are considering continuous trading strategies when there is no transaction cost, another possibility would be to consider a regular control formulation. In this case the control has to be absolutely continuous (with respect to Lebesgue measure), therefore we replace $d\xi_t$ by $u_t dt$ where u is a non-negative adapted process. Hence, the dynamics and value function become

$$dX_t = -u_t dt,$$

$$dP_t = \mu(P_t)dt + \sigma(P_t)dB_t - \gamma(P_t)u_t dt,$$

$$V(y) = \sup_u \mathbb{E}\left[\int_0^\infty e^{-\beta t} P_t u_t dt\right].$$

The corresponding HJB equation for this formulation is:

$$\inf_{u \geq 0} \left\{\beta \varphi - A\varphi - pu + u\frac{\partial \varphi}{\partial x} + \gamma(p)u\frac{\partial \varphi}{\partial p}\right\} = 0. \tag{4.13}$$

As in the case of impulse and singular control, similar heuristic arguments based on DPP and Dynkin's formula can be used to write this equation. Now, observe that equality above holds if and only if

$$\beta \varphi - A\varphi = 0$$

and

$$-\gamma(p)\frac{\partial\varphi}{\partial p} - \frac{\partial\varphi}{\partial x} + p \leq 0.$$

Let $u > 0$ and consider the strategy $d\xi_t = udt$, that is, selling shares at a constant speed u until the investor executes the position. Then,

$$P_t = p\exp\{(\mu - \lambda u - \frac{1}{2}\sigma^2)t + \sigma B_t\}$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}P_t d\xi_t\right] = u\mathbb{E}\left[\int_0^{x/u} e^{-\beta t}P_t dt\right]$$

$$= u\int_0^{x/u} e^{-\beta t}\mathbb{E}[P_t]dt$$

$$= up\int_0^{x/u} e^{(\mu-\lambda u-\beta)t}dt$$

$$= \frac{pu}{\mu - \lambda u - \beta}\left(e^{(\mu-\lambda u-\beta)x/u} - 1\right)$$

by using Fubini's theorem since the integrand is positive. Taking $u \to \infty$ this expression converges to W. Note that the class of singular controls contains the class of regular controls. Thus, W is an upper bound for the value function obtained with a regular control formulation. On the other hand, the calculation above shows that we can approach to W with regular-type controls. This means that W is the value function in this formulation. However, W does not satisfy the equation (4.13). This means that it is not possible to prove theorems like 17 and 31 in this context. Here we find a difference with the work in [Schied and Schöneborn, 2009], where the formulation allows to show the characterization of the value function as the solution of a HJB equation (see Theorem 1). The primary reason for this difference is the inclusion of both permanent and temporary impact in the price dynamics.

4.5 Numerical Examples

We are now going to present different choices of price processes. Throughout this section we will continue considering the price impact function:

$$\gamma(p) = \lambda p. \tag{4.14}$$

4.5.1 Permanent impact

By permanent impact we mean a change in the equilibrium price process due to the trading itself, as explained in [Almgren and Chriss, 2000]. The first price process that we can use
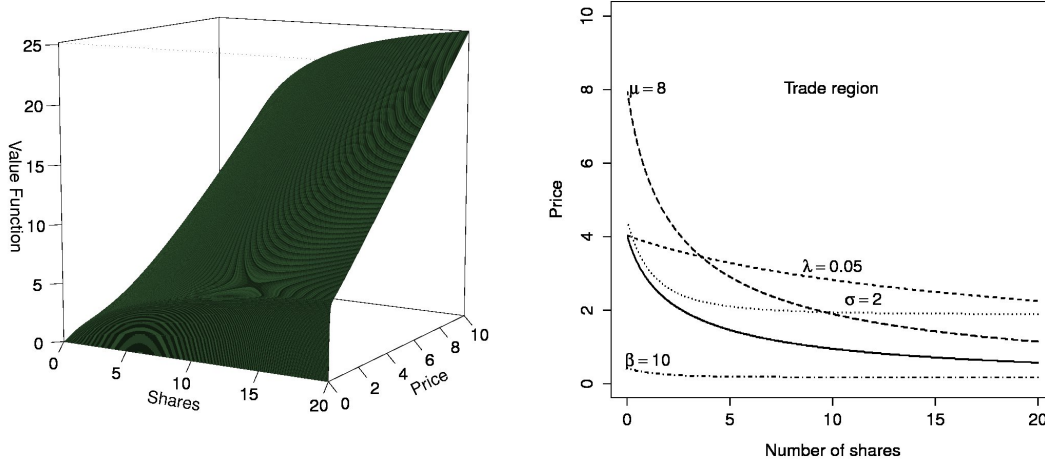
(a) Value function in the BM case with parameters $\lambda = 0.5$, $\mu = 4$, $\sigma = 0.5$ and $\beta = 1$.

(b) Continuation-trade region in the BM case. The solid line shows the contour with parameters $\lambda = 0.5$, $\mu = 4$, $\sigma = 0.5$ and $\beta = 1$. In the other lines only the indicated parameter has been changed.

Figure 4.1: Value function and continuation-trade region in the BM case.

to model permanent price impact was already discussed in detail, that is the geometric Brownian motion. The next easy process that allows a permanent impact is the arithmetic Brownian motion. The price process becomes

$$dP_t = \mu dt + \sigma dB_t - \lambda P_t d\xi_t,$$

with $\sigma > 0$. In this case the value function is always finite, regardless of μ, due to the exponential decay of the discount factor. Since 0 is an absorbing boundary for this process the boundary conditions are given by (3.17). An analytic solution for V does not seem easy to find here, so we used an implicit numerical scheme following chapter 6 in [Kushner and Dupuis, 1992]. In particular, we used the Gauss-Seidel iteration method for approximation in the value space. Figure 4.1(a) shows the value function obtained by this scheme.

The first thing that we notice in this case is that $\mathcal{T} \neq \mathcal{O}$, as shown in figure 4.1(b). The figure also shows how the different parameters affect the continuation/trade regions. Now, let's see how the change in the parameters of the model affect the value function V. Figure 4.2(a) shows that the value function is very sensitive to changes in the parameter λ for small values but not so much for large values. This behavior is common to both processes GBM and BM. This means that the bigger the investor (i.e. the larger the price impact) the less sensitive to small changes in the value of λ. Clearly the value function decreases as the impact increases.
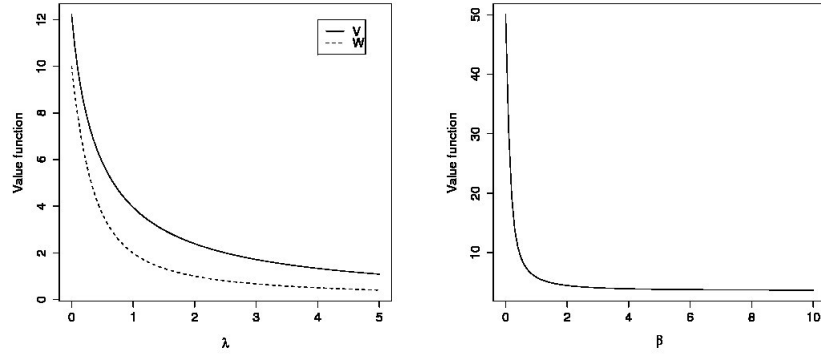
(a) Change in $V(5, 2)$ as λ varies and $\mu = 4$, $\sigma = 0.5$ and $\beta = 1$.

(b) Change in $V(5, 2)$ as β varies and $\mu = 4$, $\sigma = 0.5$ and $\lambda = 0.5$.
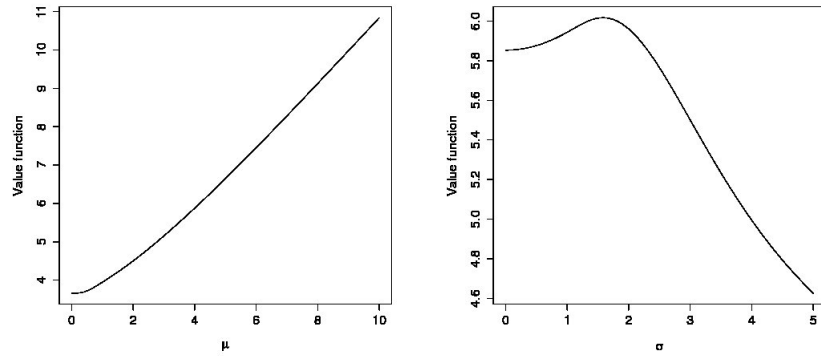
(c) Change in $V(5, 2)$ as μ varies and $\lambda = 0.5$, $\sigma = 0.5$ and $\beta = 1$.

(d) Change in $V(5, 2)$ as σ varies and $\mu = 4$, $\lambda = 0.5$ and $\beta = 1$.

Figure 4.2: Change in the parameters of the model BM.

If $\beta = 0$, the value function would not be finite for any $\mu > 0$, so small values of β yield a very large value of V. As β increases the effect in V is diminishing. Also, the investor has to act greedily and therefore the trade region approaches to $\mathcal{O}$ and V approaches to W.

For $\mu \leq 0$ it is not optimal to wait at all, so $V = W$, but as μ increases clearly the value function increases in an almost linear fashion.

The effect of σ in the value function is probably the most interesting one. In figure 4.2(d) we see that it is beneficial for the investor to have some variance in the asset but not too much. An explanation for this is that when the variance increases it is more likely for the price process to enter the trading region. On the other hand, if the variance is too big, the process can hit 0 too fast. Clearly the variance of the revenue increases with σ, thus as part of future research it would be interesting to consider the risk aversion of the investor.
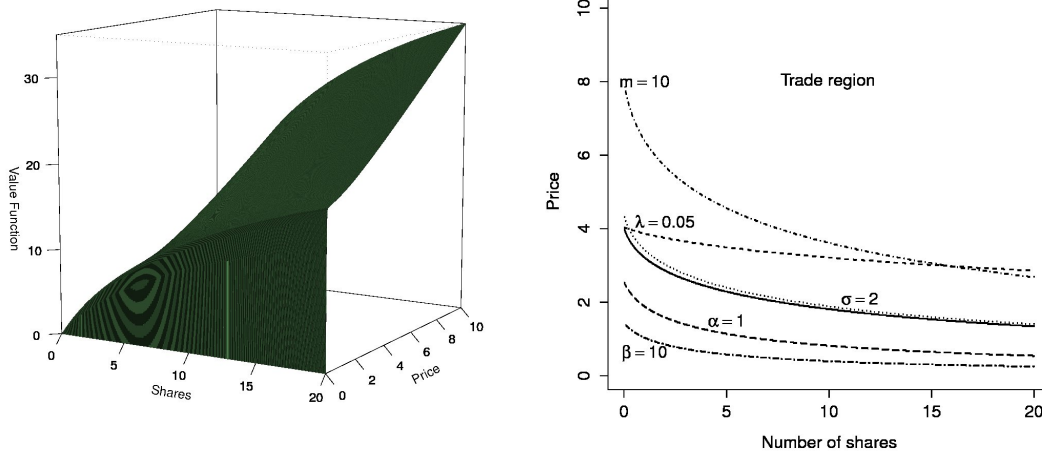
(a) Value function in the OU case with parameters $\lambda = 0.5$, $\alpha = 4$, $\sigma = 0.5$, $m = 5$ and $\beta = 1$.

(b) Continuation-trade region in the OU case. The solid line shows the contour with parameters $\lambda = 0.5$, $\alpha = 4$, $\sigma = 0.5$, $m = 5$ and $\beta = 1$. In the other lines only the indicated parameter has been changed.

Figure 4.3: Value function and continuation-trade region in the mean-reverting case.

4.5.2 Temporary impact

We can describe temporary impact as caused by temporary imbalances in supply/demand dynamics. The OrnsteinUhlenbeck process, also known as the mean-reverting process, allows us to model the temporary impact in the price. The price process becomes

$$dP_t = \alpha(m - P_t)dt + \sigma dB_t - \lambda P_t d\xi_t,$$

with $\sigma, \alpha > 0$. As in the case of arithmetic Brownian motion, the boundary conditions are given by (3.17), since 0 is an absorbing boundary for this process. Figure 4.3 shows the value function and the continuation-trade region. In general, the sensitivity of the function to the parameters is similar to the previous case. The only parameter that is exclusive to the mean-reverting case is the resilience factor α. As we increase α the value function increases (Figure 4.4(d)) and the continuation region grows (Figure 4.3(b)).
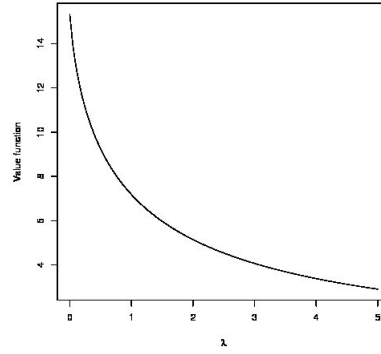
(a) Change in $V(5,2)$ as λ varies and $m = 5$, $\sigma = 0.5$, $\alpha = 4$ and $\beta = 1$.
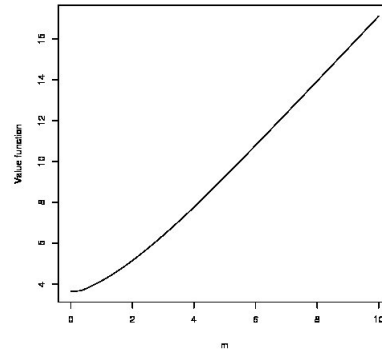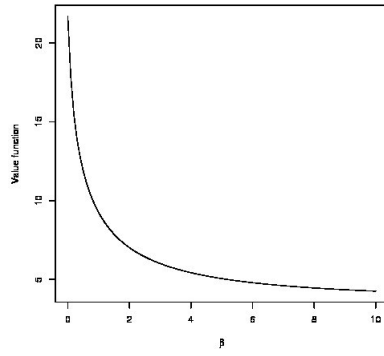
(b) Change in $V(5,2)$ as β varies and $m = 5$, $\sigma = 0.5$, $\alpha = 4$ and $\lambda = 0.5$.

(c) Change in $V(5,2)$ as m varies and $\alpha = 4$, $\sigma = 0.5$, $\beta = 1$ and $\lambda = 0.5$.
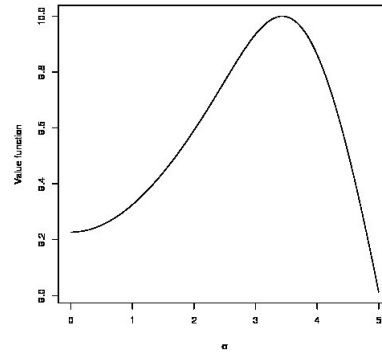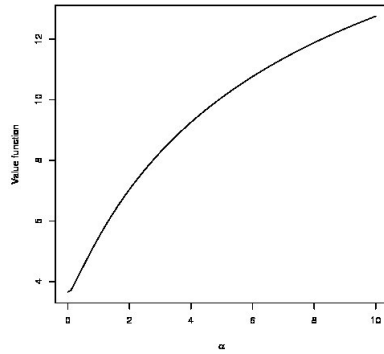
(d) Change in $V(5,2)$ as α varies and $\lambda = 0.5$, $\sigma = 0.5$, $m = 5$ and $\beta = 1$.

(e) Change in $V(5,2)$ as σ varies and $\alpha = 4$, $\lambda = 0.5$, $m = 5$ and $\beta = 1$.

Figure 4.4: Change in the parameters of the model OU.

Chapter 5

Conclusions and Future Work

In this dissertation we analyze two new models for the Optimal Execution Problem in the presence of price impact. The first model is an impulse control formulation that also includes fix transaction cost. We considered both infinite and finite horizon cases as well. In this model we showed that the value function is the unique continuous viscosity solution of the Hamilton-Jacobi-Bellman equation associated to the problem whenever the transaction cost is strictly positive. In the finite horizon situation we considered a particular class of price processes (that includes, for example, the geometric Brownian motion) for which we were able to calculate the value function as the expected value of certain measurable function of the price process at the expiration date. The second model is a singular control formulation (no transaction cost). In this case we also proved continuity and uniqueness of the value function under the viscosity framework. Although any impulse control is a singular control, in general the expected revenue obtained when applying the same impulse control in both formulations is different. Since the singular control formulation was derived naturally from the impulse control formulation we could expect that both value functions are equal. In fact, we were able to show that this is the case for a special type of price impact and we provided the explicit solution. This is particularly challenging since the subsolution property for the HJB equation (3.9), when there is no transaction cost, has no information at all. However, the figure 5.1 shows a numerical exercise, with Brownian motion as price process, that reinforces this conjecture. Notice that the singular control model was formulated only in the infinite horizon case, that is because the terminal condition for the finite horizon case is not easy to establish. However, we also conjecture that both terminal conditions, impulse with $k = 0$ and singular models, coincide. Both conjecture are part of future work.

Another direction for future research would be to find the regularity of the value functions and the free boundaries between trade and continuation regions. Numerical results provided in this work, at least for the second formulation, suggest that the function is more than just continuous and that its regularity is related with the regularity of the function U. Now, from an economic viewpoint, it would be important to include utility functions to account for risk aversion of the investor. Also, consider a stochastic process to describe the price
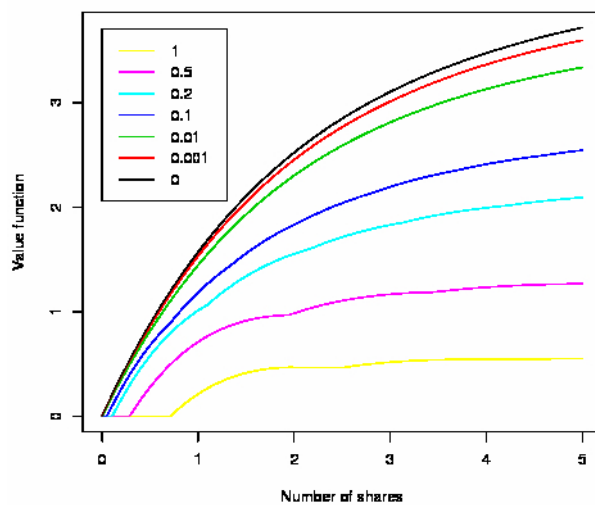
Figure 5.1: Value function for different values of k with $p = 2$ and $\lambda = 0.5$

impact function at different times.

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