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Cluster Structures in Double Canonical Bases

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by

Dane M. Lawhorne

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To Susie.

ABSTRACT OF THE DISSERTATION

Cluster Structures in Double Canonical Bases

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2020
Dr. Jacob Greenstein, Chairperson

We study connections between quantum cluster algebras and the double canonical bases of subalgebras of the Heisenberg and Drinfeld double associated to a quantized Borel subalgebra of $\mathfrak{sl}_3$. We show that the Heisenberg double has a finite type quantum cluster algebra structure for which the set of quantum cluster monomials is equal to the double canonical basis. Furthermore, we identify an affine quantum cluster algebra structure on parabolic subalgebras of the Drinfeld double and prove that all quantum cluster variables belong to the double canonical basis. Finally, we identify an infinite subset of quantum clusters for which the quantum cluster monomials are contained in the double canonical basis.

Contents

Chapter 1

Introduction

The notion of a canonical basis originates with Kazhdan and Lusztig and the construction in [KL79] of their eponymous basis in the Hecke algebra $H(W)$ of a Coxeter group W. This basis, known now as the *Kazhdan-Lusztig basis*, is characterized by the invariance of its elements with respect to a *bar-involution* and by the unitriangularity (with respect to a natural order on W called the Bruhat order) of the transition matrix from the natural basis $\{T_w : w \in W\}$. When W is a Weyl group, the structure constants of this basis (the famous *Kazhdan-Lusztig polynomials*) have nonnegative integer coefficients and contain important geometric and representation theoretic information.

A basis of the positive part of the quantized enveloping algebra $U_q(\mathfrak{n}_+)$ with similar properties was discovered independently by Lusztig [Lus90] (there called the *canonical basis*) and Kashiwara [Kas91] (there called the *global basis*). The canonical basis provides combinatorial tools for studying $U_q(\mathfrak{g})$-modules (through *crystal graphs*) and has many other incredible properties, including nonnegative structure constants when $\mathfrak{g}$ is simply laced. Although not how they were first discovered, the existence of both the Kazhdan-Lusztig basis and the canonical basis is guaranteed by *Lusztig's lemma* (a version of which is given in Thm. 2.10).

The algebra $U_q(\mathfrak{n}_+)$ possesses a well-known nondegenerate bilinear form (see Section 2.5) and thus contains a basis dual to the canonical basis, called the *dual canonical basis* (or in

Kashiwara's terminology, *upper global basis*). The dual canonical basis is naturally viewed as a basis of a deformation of the coordinate ring $\mathbb{C}[N_+]$. To be precise, taking the $q \to 1$ limit of $U_q(\mathfrak{n}_+)$ depends on a choice of integral form. If that form is the span of the canonical basis, $U_q(\mathfrak{n}_+)$ specializes to the enveloping algebra $U(\mathfrak{n}_+)$. If the integral form is the span of the dual canonical basis, the specialization is isomorphic to $\mathbb{C}[N_+]$ (see [Kim12, Thm. 4.35, Prop. 4.36]). Since the dual canonical basis specializes to a commutative algebra, it is reasonable to expect that it has an interesting multiplicative structure.

The multiplicative structure of the dual canonical basis was investigated by Berenstein and Zelevinsky in [BZ93]. They conjectured ([BZ93, Conj. 1.7], known as the *Berenstein-Zelevinsky conjecture*, that when two dual canonical basis elements quasi-commute (that is, commute up to an integer power of q), their product belongs to the dual canonical basis up to rescaling by a power of $q^{\frac{1}{2}}$. They proved the conjecture for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ in [BZ93] (see also the descriptions in [BG17b, Ex. 5.13], [BG17a, Ex. 5.2]). The conjecture also holds for $\mathfrak{sp}_4$ (see, for example, [BG17a, Ex. 5.3]) and $\mathfrak{sl}_5$, but counterexamples were found by Leclerc [Lec03] for all other finite types. However, the Berenstein-Zelevinsky conjecture may be weakened and restated using the language of *quantum cluster algebras.*

Cluster algebras were introduced by Fomin and Zelevinsky in [FZ02] in order to create an "algebraic framework" for the studying (the $q \to 1$ specializations of) dual canonical bases. Cluster algebras are $\mathbb{Q}$-algebras generated by a possibly infinite set of *cluster variables* organized into finite subsets called *clusters.* For any cluster $\mathbf{X}$ and cluster variable $X \in \mathbf{X}$, there is a unique cluster $\mathbf{X}'$ and cluster variable X' such that $\mathbf{X}' = (\mathbf{X} \setminus \{X\}) \cup \{X'\}$. The cluster variables X, X' satisfy an *exchange relation* $XX' = N + M$, where N, M are monomials in $\mathbf{X} \setminus \{X\}$. The cluster variables and exchange relations are determined by the rules of *quiver mutation.*

In [BZ05], Berenstein and Zelevinsky introduced noncommutative deformations of cluster algebras called quantum cluster algebras. These algebras are $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebras similar in structure to commutative cluster algebras except that the cluster variables in any given cluster quasi-commute instead of commute. Motivated by a series of results of Geiss, Leclerc, and Schröer (beginning in [GLS05]) on connections between commutative cluster algebras and Lusztig's dual semicanonical basis, Kimura conjectured in [Kim12] (the *quantization conjecture*) that $U_q(\mathfrak{n}_+)$ has a quantum cluster algebra structure for which the quantum cluster monomials are contained in the dual canonical basis (in fact, Kimura stated the conjecture for all quantum Schubert cells $U_q(w)$ associated to a Weyl group element w). In [GLS13], Geiss, Leclerc, and Schröer confirmed that $U_q(\mathfrak{n}_+)$ is a quantum cluster algebra for symmetric Kac-Moody $\mathfrak{g}$. The quantization conjecture was proven for types A, D, and E by Qin [Qin17], and in the symmetric Kac-Moody case by Kang, Kashiwara, Kim, and Oh [KKKO18]. The existence of quantum cluster structures on $U_q(\mathfrak{n}_+)$ for symmetrizable Kac-Moody $\mathfrak{g}$ follows from the work of Goodearl and Yakimov ([GY16], [GY20]). A proof of the quantization conjecture in the symmetrizable case was recently announced by Qin [Qin20].

In [BG17b], Berenstein and Greenstein introduced a canonical basis for the Drinfeld double $U_q(\tilde{\mathfrak{g}})$ of $U_q(\mathfrak{b}_+)$ (see Section 2.4) called the *double canonical basis.* A key intermediate step in the construction of the double canonical basis of is the construction of another canonical basis in (a subalgebra of) the braided Heisenberg double, also called the *double canonical basis.* Both of these bases contain the dual canonical bases of the positive and negative parts of $U_q(\mathfrak{g})$.

Since the double canonical basis is an extension of the dual canonical basis, it is natural to ask in what cases the adaptions of the Berenstein-Zelevinksy and quantization conjectures to the double canonical basis setting hold. Although the double canonical basis does not have a quantum cluster structure in general (see Ex. 2.15), the conjectures may still hold in certain cases. This dissertation is an investigation of these questions for $\mathfrak{g} = \mathfrak{sl}_3$. In Chapter 3, we show that a version of the Berenstein-Zelevinsky conjecture holds for the Heisenberg double of $\mathfrak{sl}_3$. The main result of this chapter is:

Theorem (Thm. 3.1)**.** *The Heisenberg double $\mathcal{H}_q^+(\mathfrak{sl}_3)$ has a finite type quantum cluster algebra structure for which the quantum cluster monomials coincide with the double canonical* *basis. In particular, if two elements of the double canonical basis quasi-commute, their product belongs to the double canonical basis (up to rescaling by a power of $q^{\frac{1}{2}}$).*

In Chapter 4, we provide evidence that a version of the quantization conjecture holds for parabolic subalgebras of $U_q(\widetilde{\mathfrak{sl}_3})$. The main result of this chapter is:

Theorem (Thm. 4.1)**.** *Parabolic subalgebras of the Drinfeld double $U_q(\widetilde{\mathfrak{sl}_3})$ have an affine type quantum cluster algebra structure for which the quantum cluster variables are contained in the double canonical basis.*

We also show (Prop. 4.27) that quantum cluster monomials from an infinite subset of quantum clusters are contained in the double canonical basis.

Chapter 2

Preliminaries

All material in this chapter is well-known. Our main references are [BG17b], [BZ05] and [Lus93].

2.1 Notation

Let ν be an indeterminate, and for $n \in \mathbb{Z}_{\geq 0}$, set

$$[n]_\nu = \frac{\nu^n - 1}{\nu - 1},$$

$$[n]_\nu! = \prod_{j=1}^{n} [j]_\nu.$$

Note that $[n]_\nu = 1 + \nu + \cdots + \nu^{n-1}$, so $[n]_\nu, [n]_\nu! \in \mathbb{Z}[\nu]$. We define the *Gaussian binomial coefficients* as

$$\begin{bmatrix} n \\ m \end{bmatrix}_\nu = \frac{[n]_\nu!}{[m]_\nu![n-m]_\nu!}, \quad 0 \leq m \leq n.$$

We use the convention that $\begin{bmatrix} n \\ m \end{bmatrix}_\nu = 0$ if $m < 0$ or $m > n$. The Gaussian binomial coefficients are easily seen to lie in $1 + \nu\mathbb{Z}_{\geq 0}[\nu]$ and satisfy the Pascal identities

$$\begin{bmatrix} n \\ m \end{bmatrix}_\nu = \begin{bmatrix} n-1 \\ m \end{bmatrix}_\nu + \nu^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_\nu.$$

Let $\Bbbk$ be a field and let A and B be $\Bbbk$-algebras. When defining an algebra structure on $A \otimes B$ such that that maps

$$a \mapsto a \otimes 1, \quad a \in A,$$

$$b \mapsto 1 \otimes b, \quad b \in B$$

are algebra embeddings, we often omit the tensor product symbol and write

$$ab = (a \otimes 1)(1 \otimes b),$$

$$ba = (1 \otimes b)(a \otimes 1).$$

The multiplicative structure is given by *cross relations*

$$ba = \sum_{j \in J} a_j b_j,$$

where J is a finite index set and $a_j \in A, b_j \in B$.

The symbol $\triangleright$ denotes the action of an algebra on a vector space.

2.2 Nichols algebras and bosonization

Let H be a Hopf algebra with comultiplication Δ and invertible antipode S. Let V be a left H-module which is also a left H-comodule. We call V a *Yetter-Drinfeld module* if

$$\delta(h \triangleright v) = h_{(1)} v^{(-1)} S(h_{(3)}) \otimes (h_{(2)} \triangleright v^{(0)}),$$

where we use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the comultiplication and the similar notation $\delta(v) = v^{(-1)} \otimes v^0$ for the coaction $\delta : V \to H \otimes V$. Yetter-Drinfeld modules, along with H-module H-comodule homomorphisms, form a category denoted ${}^H_H\mathcal{YD}$. Given $V \in {}^H_H\mathcal{YD}$, the formula $c(v \otimes w) = (v^{(-1)} \triangleright w) \otimes v^{(0)}$ defines an automorphism of $V \otimes V$

which satisfies the braid relation

$$(c \otimes 1_V)(1_V \otimes c)(c \otimes 1_V) = (1_V \otimes c)(c \otimes 1_V)(1_V \otimes c)$$

in $V \otimes V \otimes V$. The inverse of the braiding is $c^{-1}(v \otimes w) = v^{(0)} \otimes (S_H^{-1}(v^{(-1)}) \triangleright w)$. If $V, W \in {}^H_H\mathcal{YD}$, then so is $V \otimes W$ with the usual H-module structure and $\delta(v \otimes w) = v^{(-1)}w^{(-1)} \otimes v^{(0)} \otimes w^{(0)}$. Furthermore, if A and B are algebras in ${}^H_H\mathcal{YD}$ (meaning the multiplication and unit are H-module and H-comodule homomorphisms), then so is $A \otimes B$ with the twisted multiplication

$$(a \otimes b)(a' \otimes b') = a(b^{(-1)} \triangleright a') \otimes b^{(0)}b'.$$

We denote this algebra by $A \underline{\otimes} B$. If B is also a coalgebra in ${}^H_H\mathcal{YD}$ and the comultiplication $\underline{\Delta} : B \to B \underline{\otimes} B$ and counit $\underline{\epsilon} : B \to \Bbbk$ are algebra maps, then B is called a *braided bialgebra.* For the comultiplication, we write $\underline{\Delta}(b) = \underline{b}_{(1)} \otimes \underline{b}_{(2)}$ in Sweedler-like notation. If B also has a braided antipode, that is, a map of Yetter-Drinfeld modules $\underline{S} : B \to B$ satisfying

$$\underline{b}_{(1)}\underline{S}(\underline{b}_{(2)}) = \underline{S}(\underline{b}_{(1)})\underline{b}_{(2)} = \underline{\epsilon}(b),$$

B is called a *braided Hopf algebra.* The braided antipode satisfies the braided antimultiplicative property

$$\underline{S} \circ \underline{\mu} = \underline{\mu} \circ c \circ (\underline{S} \otimes \underline{S}),$$

where $\underline{\mu} : B \underline{\otimes} B \to B$ is the multiplication. it is easy to check that the tensor algebra $T(V)$ of any $V \in {}^H_H\mathcal{YD}$ is a braided Hopf algebra with $\underline{\Delta}(v) = 1 \otimes v + v \otimes 1$, $\underline{\epsilon}(v) = 0$, and $\underline{S}(v) = -v$ for all $v \in V$.

Given $V \in {}^H_H\mathcal{YD}$, there is a unique $\mathbb{Z}_{\geq 0}$-graded braided Hopf algebra $\mathcal{B}(V)$ generated by V, called the *Nichols algebra* of V, such that V is equal to both the degree 1 graded component and the set of primitive elements of $\mathcal{B}(V)$. Let J be the largest coideal of $T(V)$ which is contained entirely in the degree ≥ 2 components of $T(V)$. Then J is also an ideal and a Yetter-Drinfeld submodule of $T(V)$, and $T(V)/J$ is the Nichols algebra of V ([AS02, Prop. 2.2]).

Any braided Hopf algebra B in ${}^H_H\mathcal{YD}$ can be embedded, as an algebra, into an honest Hopf algebra called its *bosonization*. As an algebra, the bosonization $B \rtimes H$ of B is $B \otimes H$ with the *smash product* algebra structure

$$hb = (h_{(1)} \triangleright b)h_{(2)}.$$

The coalgebra structure and antipode are defined by

$$\Delta(bh) = \underline{b}_{(1)}(\underline{b}_{(2)})^{(-1)}h_{(1)} \otimes (\underline{b}_{(2)})^{(0)}h_{(2)},$$

$$\epsilon(bh) = \underline{\epsilon}(b)\epsilon_H(h),$$

$$S(bh) = S_H(h)S_H(b^{(-1)}) \otimes \underline{S}(b_{(0)})$$

(the coalgebra sturcture is the *smash coproduct* structure [Maj95, p. 26]).

2.3 The quantized enveloping algebra $U_q(\mathfrak{n}_+)$

Let $\Bbbk = \mathbb{Q}(q^{\frac{1}{2}})$ and let $\mathfrak{g}$ be a semisimple Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $n = \dim \mathfrak{h}$ be the rank of $\mathfrak{g}$ and let $I = \{1, \ldots, n\}$. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ denote the root latice and $C = \{a_{ij}\}_{i,j \in I}$ the Cartan matrix of $\mathfrak{g}$. Let $\{d_1, \ldots, d_n\}$ be positive integers satisfying $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$ and set $q_i = q^{d_i}$. Consider the vector space

$$V^+ = \bigoplus_{i \in I} \Bbbk E_i$$

graded by the free abelian monoid $Q_{\geq 0} = \bigoplus_{i=0}^n \mathbb{Z}_{\geq 0}\alpha_i$ (the $\mathbb{Z}_{\geq 0}$-span of the simple roots). Then V^+ is a Yetter-Drinfeld module over the group algebra $\Bbbk Q$ via

$$K_{+i} \triangleright E_j = q_i^{a_{ij}} E_j,$$

$$\delta(E_i) = K_{+i} \otimes E_i,$$

where K_{+i} denotes the generator of $\Bbbk Q$ corresponding to $\alpha_i \in Q$. Then the braiding on V^+ is given by $c(E_i \otimes E_j) = q_i^{a_{ij}}(E_j \otimes E_i)$.

Definition 2.1. The Nichols algebra of V^+ in ${}^{\Bbbk Q}_{\Bbbk Q}\mathcal{YD}$, denoted $U_q(\mathfrak{n}_+)$, is called the *quantized enveloping algebra* of $\mathfrak{n}_+$.

We often use the abbreviation U_q^+. The braided antipode for U_q^+ is determined by $\underline{S}(E_i) = -E_i$ (extended to all of U_q^+ using the braided antimultiplicative property). As an algebra, U_q^+ is isomorphic to the algebra generated by the $E_i, i \in I$ and the *quantum Serre relations*

$$\sum_{\substack{r,s \in \mathbb{Z}_{\geq 0} \\ r+s=1-a_{ij}}} (-1)^s E_i^{\langle r \rangle} E_j E_i^{\langle s \rangle} = 0, \quad i \neq j \in I$$

where

$$E_i^{\langle t \rangle} = \frac{E_i^t}{\prod_{j=1}^t q_i^j - q_i^{-j}}.$$

The radical of the form in [Lus93, Prop. 1.2.3] is equal to the coideal J in the definition of a Nichols algebra [AS02, Prop. 2.10], so Lustig's algebra $\mathbf{f}$ is isomorphic to U_q^+. The quantum Serre relations hold by [Lus93, Prop. 1.4.3] and are defining relations by [Lus93, Cor. 33.1.5]. The algebra U_q^+ is $Q_{\geq 0}$-graded via $\deg_{Q_{\geq 0}} E_i = \alpha_i$. For a homogeneous element, $x \in U_q^+$ we denote its $Q_{\geq 0}$-degree by $|x|$. The bosonization $U_q^+ \rtimes \Bbbk Q$ has Hopf algebra structure

$$K_{+i}E_j = q_i^{a_{ij}} E_j K_{+i}, \quad i, j \in I$$

$$\Delta(E_i) = E_i \otimes 1 + K_{+i} \otimes E_i, \quad \Delta(K_{+i}) = K_{+i} \otimes K_{+i}$$

$$\epsilon(E_i) = 0, \quad \epsilon(K_{+i}) = 1,$$

$$S(E_i) = -K_{+i}^{-1} E_i, \quad S(K_{+i}) = K_{+i}^{-1}, \quad i \in I.$$

The $Q_{\geq 0}$-grading extends to $U_q^+ \rtimes \Bbbk Q$ via $|K_{+i}| = |K_{+i}^{-1}| = 0$.

2.4 Heisenberg and Drinfeld doubles

Let V^- be another $Q_{\geq 0}$-graded $\mathbb{k}$-vector space with basis $\{F_i : i \in I\}$. We view V^- as a Yetter-Drinfeld module over $\mathbb{k}Q$ via $K_{-i} \triangleright F_i = q_i^{a_{ij}} F_i$, where K_{-i} denotes the generator of $\mathbb{k}Q$ corresponding to $\alpha_i \in Q$ (this notation allows us to distinguish between two copies of $\mathbb{k}Q$ in the Drinfeld and Heisenberg doubles). We denote the Nichols algebra of V^- by U_q^- (as algebras, U_q^- and U_q^+ are isomorphic). The *Drinfeld double* $\mathcal{D}(U_q^- \rtimes \mathbb{k}Q, U_q^+ \rtimes \mathbb{k}Q)$ (see [BG17b, Sections A.7–A.9], [Maj95, p. 26] for the general definition) contains $U_q^- \rtimes \mathbb{k}Q$, and $(U_q^+ \rtimes \mathbb{k}Q)^{\mathrm{cop}}$ as sub-Hopf algebras and has cross relations

$$F_i E_j = E_j F_i + \delta_{ij}(q_i - q_i^{-1})(K_{+i} - K_{-i}),$$

$$K_{+i} F_j = q_i^{-a_{ij}} F_j K_{+i},$$

$$E_i K_{-j} = q_i^{-a_{ij}} K_{-j} E_i.$$

Here $(U_q^+ \rtimes \mathbb{k}Q)^{\mathrm{cop}}$ is isomorphic to $U_q^+ \rtimes \mathbb{k}Q$ as an algebra but has the opposite comultiplication (and thus inverted antipode). The multiplication map is a vector space isomorphism

$$\mu : U_q^- \otimes \mathbb{k}Q^2 \otimes U_q^+ \to \mathcal{D}(U_q^- \rtimes \mathbb{k}Q, U_q^+ \rtimes \mathbb{k}Q),$$

where we identify $\mathbb{k}Q \otimes \mathbb{k}Q$ with $\mathbb{k}Q^2$. The *braided Heisenberg double* $\mathcal{H}(U_q^- \rtimes \mathbb{k}Q, U_q^+ \rtimes \mathbb{k}Q)$ (see [BG17b, Section A.7], [BB09] for the general definition) contains $U_q^- \rtimes \mathbb{k}Q$, and $U_q^+ \rtimes \mathbb{k}Q$ as subalgebras and has cross relations

$$F_i E_j = E_j F_i + \delta_{ij}(q_i - q_i^{-1}) K_{+i}$$

$$K_{+i} F_j = q_i^{-a_{ij}} F_j K_{+i},$$

$$E_i K_{-j} = q_i^{-a_{ij}} K_{-j} E_i.$$

The Heisenberg double is not a Hopf algbra. Again, the multiplication map is an isomorphism of vector spaces

$$\mu : U_q^- \otimes \Bbbk Q^2 \otimes U_q^+ \to \mathcal{H}(U_q^- \rtimes \Bbbk Q, U_q^+ \rtimes \Bbbk Q).$$

We use the abbreviations $\mathcal{H}_q$ and $\mathcal{D}_q$ for $\mathcal{H}(U_q^- \rtimes \Bbbk Q, U_q^+ \rtimes \Bbbk Q)$ and $\mathcal{D}(U_q^- \rtimes \Bbbk Q, U_q^+ \rtimes \Bbbk Q)$, respectively. Both $\mathcal{H}_q$ and $\mathcal{D}_q$ are Q-graded via $\deg_Q E_i = \alpha_i, \deg_Q F_i = -\alpha_i$, and $\deg_Q K_{+i}^{\pm 1} = \deg K_{-i}^{\pm 1} = 0$. We denote the Q-degree of a homogeneous element x by $|x|$.

There are many useful symmetries of $\mathcal{H}_q$ and $\mathcal{D}_q$. First, there exists a $\mathbb{Q}$-linear anti-automorphism of $\mathcal{D}_q$, called the *bar involution*, defined by

$$\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}, \quad \overline{K_{+i}} = K_{+i}, \quad \overline{K_{-i}} = K_{-i}, \quad \overline{E_i} = E_i, \quad \overline{F_i} = F_i.$$

We also have $\Bbbk$-linear involutive antiautomorphisms $* : \mathcal{D}_q \to \mathcal{D}_q$ and $\tau : \mathcal{D}_q \to \mathcal{D}_q$ determined by

$$E_i^* = E_i, \quad F_i^* = F_i, \quad (K_{+i})^* = K_{-i},$$

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K_{+i}) = K_{+i}, \quad \tau(K_{-i}) = K_{-i}.$$

The composite $* \circ \tau$ is an automorphism which restricts to isomorphisms between U_q^+ and U_q^-. Furthermore, $\bar{\cdot} : \mathcal{H}_q \to \mathcal{H}_q$ and $\tau : \mathcal{H}_q \to \mathcal{H}_q$ are defined using the same formulas (there is no analogue of $*$ on $\mathcal{H}_q$). We also remark that the composite $\sigma = \bar{\cdot} \circ * : \mathcal{D}_q \to \mathcal{D}_q$ is a $\mathbb{Q}$-linear involutive automorphism. (The involution σ is often denoted by $\bar{\cdot}$ and called the "bar involution," but we reserve this notation and terminology for the *anti*-involution defined above.)

It is convenient to work with smaller subalgebras of $\mathcal{H}_q$ and $\mathcal{D}_q$.

Definition 2.2. Let $\mathcal{H}_q^+(\mathfrak{g})$ be the subalgebra of $\mathcal{H}_q$ generated by $\{E_i, F_i, K_{+i} : i \in I\}$. Let $U_q(\tilde{\mathfrak{g}})$ be the subalgebra of $\mathcal{D}_q$ generated by $\{E_i, F_i, K_{+i}, K_{-i} : i \in I\}$.

We use the abbreviations $\mathcal{H}_q^+$ and $\widetilde{U}_q$. We also denote by $\widetilde{\mathcal{H}}_q$ the subalgebra of $\mathcal{H}_q$ generated by $\{E_i, F_i, K_{+i}, K_{-i} : i \in I\}$. One advantage of the subalgebras $\mathcal{H}_q^+, \widetilde{\mathcal{H}}_q$, and $\widetilde{U}_q$

is that they are $Q^2_{\geq 0}$-graded via

$$\deg_{Q^2_{\geq 0}} E_i = (0, \alpha_i),$$

$$\deg_{Q^2_{\geq 0}} F_i = (\alpha_i, 0),$$

$$\deg_{Q^2_{\geq 0}} K_{+i} = \deg_{Q^2_{\geq 0}} K_{-i} = (\alpha_i, \alpha_i).$$

Note that $\mathcal{H}_q^+$ is isomorphic to the quotient of $\widetilde{U}_q$ by the ideal generated by the K_{-i}'s. Furthermore, the vector space isomorphism $\iota : \mathcal{H}_q \to \mathcal{D}_q$ restricts to a vector space inclusion $\iota : \mathcal{H}_q^+ \to \widetilde{U}_q$. The symmetries $\bar{\cdot}$ and τ restrict to anti-automorphisms $\mathcal{H}_q^+ \to \mathcal{H}_q^+$, $\widetilde{U}_q \to \widetilde{U}_q$ and $*$ restricts to an anti-automorphism $\widetilde{U}_q \to \widetilde{U}_q$. Although $\widetilde{U}_q$ is a bialgebra and not a Hopf algebra, the quotient $\widetilde{U}_q/\langle K_{-i}K_{+i} - 1 : i \in I\rangle$ is a Hopf algebra via $S(E_i) = -E_iK_i^{-1}, S(F_i) = -K_iF_i$, and $S(K_i) = K_i^{-1}$, where we denote the images of K_{+i}, K_{-i} by K_i, K_i^{-1}. This quotient is called the *quantized enveloping algebra* of $\mathfrak{g}$ and is denoted $U_q(\mathfrak{g})$. Note that the $Q^2_{\geq 0}$ grading does not descend to $U_q(\mathfrak{g})$ since the relation $K_{+i}K_{-i} = 1$ is not homogeneous.

We remark that the presentation for $U_q(\mathfrak{g})$ given here is slightly nonstandard. If $\{e_i, f_i, k_i, k_i^{-1}\}$ are the generators of $U_q(\mathfrak{g})$ in the standard presentation, the cross relations are

$$e_if_j = f_je_i + \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}.$$

However, let $\psi : \widetilde{U}_q \to \widetilde{U}_q$ be the automorphism determined by

$$\psi(E_i) = (q_i^{-1} - q_i)^{-1}E_i, \quad \psi(F_i) = (q_i - q_i^{-1})^{-1}F_i, \quad \psi(K_i) = K_i.$$

Then $\{\psi(E_i), \psi(F_i), \psi(K_i), \psi(K_{-i})\}$ satisfy the relations of the standard presentation (with K_i, K_i^{-1} replaced by K_{+i}, K_{-i}).

2.5 Dual canonical bases

Identify the subalgebra of $\widetilde{U}_q$ (or of $\widetilde{\mathcal{H}}_q$) generated by $\{K_{+i}, K_{-i} : i \in I\}$ with the monoidal algebra $\Bbbk Q^2_{\geq 0}$. Let $\alpha_i^\vee : Q^2_{\geq 0} \to \mathbb{Z}, i \in I$ be the unique monoid homomorphism determined by $\alpha_i^\vee(0, \alpha_j) = a_{ij}$ and $\alpha_i^\vee(\alpha_j, 0) = -a_{ij}, j \in I$. There is an action $\diamond$ of $\Bbbk Q^2_{\geq 0}$ on $\widetilde{U}_q$ (or on $\widetilde{\mathcal{H}}_q$) defined by

$$K_{\pm i} \diamond x = q^{\mp \frac{1}{2}\alpha_i^\vee(\deg_{Q^2_{\geq 0}} x)} K_{\pm i} x$$

for homogeneous x which satisfies $\overline{K \diamond x} = \overline{K} \diamond \overline{x}$ for all $K \in \Bbbk Q^2_{\geq 0}, x \in \widetilde{U}_q$.

Let $\partial_i, \partial_i^{\mathrm{op}} : U_q^+ \to U_q^+$ be the linear quasi-derivations defined by

$$[F_i, x] = (q_i - q_i^{-1})(K_{+i} \diamond \partial_i(x) - K_{-i} \diamond \partial_i^{\mathrm{op}}(x))$$

(cf. [BG17b] and [Kas91, Lemma 3.4.1]). Let $(\cdot, \cdot)$ be the unique (nondegenerate) bilinear form on U_q^+ such that $(1, 1) = 1$ and

$$(x, E_i^{\langle n \rangle} y) = (\partial_i^{\mathrm{op}(n)} x, y)$$

(cf. [BG17a, Lemma 2.10] and [Kas91, Prop. 3.4.4]). Recall the automorphism $\psi : \widetilde{U}_q \to \widetilde{U}_q$ which rescales the generators. Let $\mathbf{B}_+^{\mathrm{low}}$ denote the inverse image under ψ of Lusztig's *canonical basis* of U_q^+ [Lus90] (which coincides with Kashiwara's *lower global basis* [Kas91]). Recall that $\mathbf{B}_+^{\mathrm{low}}$ is $Q_{\geq 0}$-homogeneous. For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbf{B}_+^{\mathrm{low}} = \{E_1^{\langle r \rangle} : r \in \mathbb{Z}_{\geq 0}\}$. For $\mathfrak{g} = \mathfrak{sl}_3$,

$$\mathbf{B}_+^{\mathrm{low}} = \{E_1^{\langle a \rangle} E_2^{\langle b \rangle} E_1^{\langle c \rangle} : b \geq a + c\} \cup \{E_2^{\langle a \rangle} E_1^{\langle b \rangle} E_2^{\langle c \rangle} : b \geq a + c\}.$$

Since $(\cdot, \cdot)$ is nondegenerate and remains nondegenerate when restricted to $Q_{\geq 0}$-graded components (which are finite dimensional), for every $b \in \mathbf{B}_+^{\mathrm{low}}$, there exists a unique homogeneous $\delta_b \in U_q^+$ of the same degree such that $(\delta_b, b') = \delta_{b,b'}$ for all $b' \in \mathbf{B}_+^{\mathrm{low}}$.

Definition 2.3. The set $\mathbf{B}_+^{\mathrm{up}} = \{\delta_b : b \in \mathbf{B}_+^{\mathrm{low}}\}$ is a $Q_{\geq 0}$-homogeneous basis of U_q^+ called the *dual canonical basis*.

An alternative description of $\mathbf{B}_+^{\text{up}}$ can be found in [BG17a]. The following example can be found in [BG17b, Example 3.25], [BG17b, Example 5.13]. It is used extensively in the following chapters.

Example 2.4. For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbf{B}_+^{\text{up}} = \{E^r : r \in \mathbb{Z}_{\geq 0}\}$. For $\mathfrak{g} = \mathfrak{sl}_3$,

$$\mathbf{B}_+^{\text{up}} = \{q^{\frac{1}{2}a(b-c)} E_1^a E_{12}^b E_{21}^c : a, b, c \in \mathbb{Z}_{\geq 0}\} \cup \{q^{-\frac{1}{2}a(b-c)} E_2^a E_{12}^b E_{21}^c : a, b, c \in \mathbb{Z}_{\geq 0}\},$$

where

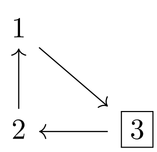
$$E_{12} = (q - q^{-1})^{-1}(q^{\frac{1}{2}} E_2 E_1 - q^{-\frac{1}{2}} E_1 E_2),$$

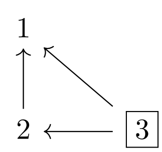
$$E_{21} = (q - q^{-1})^{-1}(q^{\frac{1}{2}} E_1 E_2 - q^{-\frac{1}{2}} E_2 E_1).$$

Note that $E_1 E_2 = q^{-\frac{1}{2}} E_{12} + q^{\frac{1}{2}} E_{21}$ and that E_{12}, E_{21} are bar-invariant.

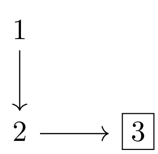
2.6 Quantum cluster algebras

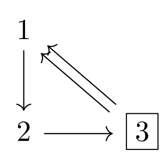
Let Q be a *quiver* (directed graph) with n vertices. We assume that Q has neither loops nor 2-cycles. We allow a subset of the vertices to be designated as *frozen*. Frozen vertices are indicated with rectangular boxes. For example,





are quivers with one frozen vertex. For any non-frozen vertex k, we define the involutive operation of *mutation* of Q at k as follows. First, for any subgraph $i \to k \to j$, add an arrow from $i \to j$. Next, reverse all arrows with source or target k. Finally, remove any 2-cycles produced in the previous two steps. For example, mutating the two quivers above at vertex 2 results in





.

Let $I = \{1, \ldots, n\}$ and let $\Lambda = \{\lambda_{ij}\}_{i,j \in I}$ be a skew-symmetric integer matrix. To Λ we associate the *quantum torus* $\mathcal{T}(\Lambda)$, which is the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra generated by $\{X_1^{\pm 1}, \ldots X_n^{\pm 1}\}$ subject to the relations

$$X_i X_j = q^{\lambda_{ij}} X_j X_i,$$

$$X_i X_i^{-1} = 1.$$

We say $x, y \in \mathcal{T}(\Lambda)$ *quasi-commute* if $xy = q^c yx$ for some $c \in \mathbb{Z}$. We say that an element $x \in A \subseteq \mathcal{T}(\Lambda)$ is *quasi-central* in A if it quasi-commutes with all elements of A. Observe that $\mathcal{T}(\Lambda)$ posses a $\mathbb{Z}$-linear anti-involution, called the *bar involution*, satisfying $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$ and $\overline{X_i} = X_i$ for each $i \in I$. For any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, the element

$$M_\Lambda(\mathbf{a}) = q^{\frac{1}{2}\sum_{i>j} a_i a_j \lambda_{ij}} X_1^{a_1} \cdots X_n^{a_n}$$

is fixed by the bar involution. Observe that the set $\{M_\Lambda(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^n\}$ is a basis of $\mathcal{T}(\Lambda)$. Since $\mathcal{T}(\Lambda)$ is an Ore domain (see [BZ05, Appendix A]), it has a skew-field of fractions which we denote by $\mathcal{F}$.

Let m be the number of frozen vertices of Q and label the vertices such that the first $n - m$ are non-frozen. Let B be the $n \times (n - m)$ submatrix of the adjacency matrix of Q consisting of the first $n - m$ columns. If for all $0 \le i \le n, j \le n - m$,

$$\sum_{k=1}^{n} b_{kj} \lambda_{ki} = \delta_{ij} d_i$$

for some positive integers d_i, then Q is said to be *compatible* with Λ. This is equivalent to requiring that $B^T \Lambda$ consists of an $(n - m) \times (n - m)$ diagonal matrix with positive diagonal entries followed by the $(n - m) \times m$ zero matrix. Let $\mathbf{X} = \{X_1, \ldots, X_n\}$. The elements of $\mathbf{X}$ are called *initial cluster variables* and $\mathbf{X}$ is called the *initial cluster*. We call Q the *initial quiver* and $(\mathbf{X}, \Lambda, Q)$ the *initial quantum seed*. Let $\{\mathbf{e}_i : i \in I\}$ denote the standard basis vectors of $\mathbb{Z}^n$. For any non-frozen vertex k of Q, define

$$\mu_k(X_k) = M_\Lambda(-\mathbf{e}_k + \sum_{b_{ik}>0} b_{ik}\mathbf{e}_i) + M_\Lambda(-\mathbf{e}_k - \sum_{b_{ik}<0} b_{ik}\mathbf{e}_i),$$

or equivalently,

$$\mu_k(X_k) = X_k^{-1}\left(q^{\frac{1}{2}T}\prod_{i\to k} X_i + q^{\frac{1}{2}S}\prod_{k\to i} X_i\right),$$

where the first product is taken over all arrows in Q with target k and the second with source k and S and T are the unique integers such that $\mu_k(X_k)$ is bar-invariant (note that these integers depend on the order in which the product is taken). The compatibility condition on Λ and Q guarantees that the elements of

$$\mu_k(\mathbf{X}) = \{X_1, \ldots, \mu_k(X_k), \ldots, X_n\}$$

quasi-commute with each other. Let $\mu_k(\Lambda)$ be this quasi-commutation matrix. Observe also that $\mu_k(\Lambda)$ is compatible with $\mu_k(Q)$ (using the same integers d_i), where $\mu_k(Q)$ denotes Q mutated at the vertex k. Thus iterated mutations of the initial quantum seed are well-defined. The *quantum seeds* consist of the initial seed and any seed $(\mathbf{X}', \Lambda', Q')$ produced through iterated mutation. The subset $\mathbf{X}'$ of $\mathcal{F}$ is called a *quantum cluster* and its elements *quantum cluster variables*. The m cluster variables corresponding to frozen vertices of Q appear in every quantum cluster and are called *coefficients*. The remaining cluster variables are *exchangeable*. The bar-invariant elements $M_{\Lambda'}(\mathbf{a})$, $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ are called *quantum cluster monomials*.

Definition 2.5. The *quantum cluster algebra* $\mathcal{A}_{q^{1/2}}(Q, \Lambda)$ associated to Q and Λ is the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by all quantum cluster variables.

This definition is slightly different from the original definition in [BZ05] in that the coefficients are not invertible. We say that two quantum seeds are *mutation equivalent* if one can be produced from the other through a sequence of mutations. Two quantum cluster algebras are isomorphic if and only their initial seeds are mutation equivalent. Furthermore, $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$ is contained in $\mathcal{T}(\Lambda)$ (this is called the *quantum Laurent phenomenon*).

Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be the initial cluster of $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$. For $0 \leq i \leq n-m$, let X_i' denote the cluster variable obtained by mutating the initial quiver Q at the vertex i. We denote the exchange relations by $X_i X_i' = P_i(\mathbf{X})$. Let $\widehat{\mathcal{A}}_{q^{1/2}}(Q,\Lambda)$ denote $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$ localized at the multiplicative submonoid $\mathbf{C}$ generated by the coefficients (since the generators of $\mathbf{C}$ quasi-commute with each other, $\mathbf{C}$ satisfies the Ore condition). We call the subgraph of Q obtained by deleting frozen vertices the *principal part* of Q. The following theorem is a corollary of [BZ05, Thm. 7.3] (see also [BFZ05, Cor. 1.21]) and provides a presentation for $\widehat{\mathcal{A}}_{q^{1/2}}(Q,\Lambda)$ when the principal part Q is *acyclic*, that is, has no oriented cycles.

Theorem 2.6 (Berenstein, Fomin, Zelevinsky)**.** *If the principal part Q is acyclic, then $\widehat{\mathcal{A}}_{q^{1/2}}(Q,\Lambda)$ is isomorphic to the algebra generated by*

$$\{X_1, \ldots X_{n-m}, X_{n-m+1}^{\pm 1}, \ldots, X_n^{\pm 1}, X_1', \ldots X_{n-m}'\}$$

subject to the relations $X_i X_i' = P_i(\mathbf{X})$ and the quasi-commutation relations determined by Λ.

The following obvious lemma is used in the proofs of Thm. 3.1 and Thm. 4.1.

Lemma 2.7. *Let A be an algebra and let S be a multiplicative submonoid of A whose elements are quasi-central. Suppose that $A[S^{-1}]$ is isomorphic to $\widehat{\mathcal{A}}_{q^{1/2}}(Q,\Lambda)$ and the image of A is contained in $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$. Then A is isomorphic to $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$*

If the principal part of Q is mutation equivalent to an orientation of a Dynkin diagram, then $\mathcal{A}_{q^{1/2}}(Q,\Lambda)$ is said to be of *finite type*. A quantum cluster algebra has finitely many cluster variables if and only if it is of finite type. The number of non-initial quantum cluster variables is equal to the number of positive roots of the root system corresponding to the Dynkin diagram.

There are several bases for quantum cluster algebras which contain the quantum cluster monomials when the principal part Q is acyclic (e.g., [BZ14]).

Theorem 2.8. *If the principal part of Q is acyclic, the quantum cluster monomials are linearly independent. If the principal part of Q is mutation equivalent to a Dynkin diagram, the quantum cluster monomials span $\mathcal{A}_{q^{1/2}}(Q, \Lambda)$.*

Example 2.9. Let

$$Q = \begin{array}{ccc} 1 & \longleftarrow & \boxed{2} \\ \big\downarrow & & \\ \boxed{3} & & \end{array},$$

and

$$\Lambda = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Mutating at vertex 1, we have

$$\begin{aligned} \mu_1(X_1) &= M_\Lambda(-e_1 + e_2) + M_\Lambda(-e_1 + e_3) \\ &= q^{-\frac{1}{2}} X_1^{-1} X_2 + q^{\frac{1}{2}} X_1^{-1} X_3. \end{aligned}$$

Let $\mathfrak{g} = \mathfrak{sl}_3$. Recall the notation of Example 2.4. Since $E_1E_2 = q^{-\frac{1}{2}} E_{12} + q^{\frac{1}{2}} E_{21}$, the assignments $X_1 \mapsto E_1, X_2 \mapsto E_{21}$, and $X_3 \mapsto E_{12}$ determine a surjective homomorphism of algebras

$$\Bbbk \otimes_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \mathcal{A}_{q^{1/2}}(Q, \Lambda) \to U_q^+.$$

Since this map restricts to a bijection between the quantum cluster monomials (a basis for $\mathcal{A}_{q^{1/2}}(Q, \Lambda)$) and the dual canonical basis, it is an isomorphism.

2.7 Double canonical bases

Let $\bar{\cdot} : \mathbb{Z}[\nu, \nu^{-1}] \to \mathbb{Z}[\nu, \nu^{-1}]$ be the $\mathbb{Z}$-linear ring homomorphism such that $\overline{\nu} = \nu^{-1}$. Let $(\mathcal{I}, \preceq)$ be a partially ordered set such that for all $i \in \mathcal{I}$, the length of chains descending from i is bounded from above. Let M be a free $\mathbb{Z}[\nu, \nu^{-1}]$-module with *initial basis* $\{x_i : i \in \mathcal{I}\}$ indexed by $\mathcal{I}$. Suppose also that $\bar{\cdot} : M \to M$ is a $\mathbb{Z}$-linear ring involution satisfying $\overline{fm} = \overline{f}\,\overline{m}$, for all $f \in \mathbb{Z}[\nu, \nu^{-1}], m \in M$. The following theorem (see, for example, [BZ14, Thm. 1.1]) is a version of *Lusztig's lemma.*

Theorem 2.10. *Suppose that for all* $i \in \mathcal{I}$,

$$\overline{x_i} - x_i \in \sum_{j \prec i} \mathbb{Z}[\nu, \nu^{-1}] x_j.$$

Then for each x_i*, there exists a unique* b_i *such that* b_i *is bar-invariant and*

$$b_i - x_i \in \sum_{j \neq i} \nu \mathbb{Z}[\nu] x_j.$$

The set $\mathbf{B} = \{b_i : i \in \mathcal{I}\}$ *is a basis for* M*.*

Let $\mathbf{K}_+$ (respectively $\mathbf{K}_-$) denote the multiplicative submonoid of $\widetilde{\mathcal{H}}_q$ or $\widetilde{U}_q$ generated by the K_{+i}'s (respectively K_{-i}'s). Let $\mathbf{K} = \mathbf{K}_-\mathbf{K}_+$. Let $\mathbf{B}_-^{\mathrm{up}}$ be the dual canonical basis of U_q^-, that is, the image of $\mathbf{B}_+^{\mathrm{up}}$ under $* \circ \tau$. The double canonical basis of $\widetilde{U}_q$ is constructed through two applications of Lusztig's Lemma. First, applying Lusztig's Lemma to the $\mathbb{Z}[q, q^{-1}]$-submodule M of $\mathcal{H}_q$ spanned by the initial basis

$$\{K \diamond b_- b_+ : K \in \mathbf{K}, (b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}\}$$

with partial order $K \diamond b_- b_+ \preceq K' \diamond b'_- b'_+$ if $\deg_{Q^2_{\geq 0}} b_- b_+ \leq \deg_{Q^2_{\geq 0}} b'_- b'_+$ and

$$\deg_{Q^2_{\geq 0}} K + \deg_{Q^2_{\geq 0}} b_- b_+ = \deg_{Q^2_{\geq 0}} K' + \deg_{Q^2_{\geq 0}} b'_- b'_+,$$

Berenstein and Greenstein constructed a basis for $\mathcal{H}_q^+$ (note that $\mathcal{H}_q^+$ is a proper submodule of M):

Theorem 2.11 ([BG17b], Thm. 1.3)**.** *For any* $(b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}$*, there exists a unique bar-invariant element* $b_- \circ b_+ \in \mathcal{H}_q^+(\mathfrak{g})$ *such that*

$$b_- \circ b_+ - b_- b_+ \in \sum q \mathbb{Z}[q] K_+ \diamond (b'_- b'_+)$$

where the sum is over all $K_+ \in \mathbf{K}_+ \setminus \{1\}$ *and* $(b'_-, b'_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}$ *such that* $\deg_{Q^2_{\geq 0}} b'_- b'_+ + \deg_{Q^2_{\geq 0}} K_+ = \deg_{Q^2_{\geq 0}} b_- b_+$.

Definition 2.12. The set $\mathbf{B}_{\mathfrak{g}}^+ = \{K_+ \diamond (b_- \circ b_+) : (b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}, K_+ \in \mathbf{K}_+\}$ is a bar-invariant basis of $\mathcal{H}_q^+(\mathfrak{g})$ called the *double canonical basis.*

Applying Lusztig's Lemma again to the $\mathbb{Z}[q, q^{-1}]$-submodule of $\widetilde{U}_q$ spanned by the initial basis

$$\{K \diamond \iota(b_- \circ b_+) : K \in \mathbf{K}, (b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}\}$$

and a similar partial order, they constructed a basis for $\widetilde{U}_q$:

Theorem 2.13 ([BG17b], Thm. 1.5)**.** *For any* $(b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}$*, there exists a unique bar-invariant element* $b_- \bullet b_+ \in U_q(\tilde{\mathfrak{g}})$ *such that*

$$b_- \bullet b_+ - \iota(b_- \circ b_+) \in \sum q^{-1}\mathbb{Z}[q^{-1}] K \diamond \iota(b'_- \circ b'_+)$$

where the sum is over all $K \in \mathbf{K} \setminus \mathbf{K}_+$ *and* $(b'_-, b'_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}$ *such that* $\deg_{Q^2_{\geq 0}} b'_- b'_+ + \deg_{Q^2_{\geq 0}} K_+ = \deg_{Q^2_{\geq 0}} b_- b_+$.

Definition 2.14. The set $\mathbf{B}_{\tilde{\mathfrak{g}}} = \{K \diamond (b_- \bullet b_+) : (b_-, b_+) \in \mathbf{B}_-^{\mathrm{up}} \times \mathbf{B}_+^{\mathrm{up}}, K \in \mathbf{K}\}$ is a bar-invariant basis of $U_q(\tilde{\mathfrak{g}})$ called the *double canonical basis.*

If an element of $U_q(\tilde{\mathfrak{g}})$ (resp. $\mathcal{H}_q^+(\mathfrak{g})$) satisfies the unitriangularity condition of Thm. 2.13 (resp. Thm. 2.11), we say it has *correct triangularity.*

The following example can be found in [BG17b, Section 4.1] and is used extensively throughout the rest of this dissertation.

Example 2.15. Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $C_+ = FE - qK_+ \in \mathcal{H}_q^+(\mathfrak{sl}_2)$. Since C_+ is bar-invariant and has correct triangularity, $C_+ = F \circ E$. Furthermore,

$$C_+^k = \sum_{j=0}^{k} (-1)^j q^j \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} K_+^j F^{k-j} E^{k-j}$$

and thus $C_+^k = F^k \circ E^k$. Observe also that C_+ is central in $\mathcal{H}_q^+(\mathfrak{sl}_2)$, and so products of powers of C_+ with either powers of F or E remain bar-invariant. It is straightforward to check that such products belong to the double canonical basis. Thus

$$\mathbf{B}^+_{\mathfrak{sl}_2} = \{K_+^a \diamond F^r C_+^s : a, r, s \in \mathbb{Z}_{\geq 0}\} \cup \{K_+^a \diamond C_+^s E^r : a, r, s \in \mathbb{Z}_{\geq 0}\}.$$

Since $FE = C_+ + qK_+$, $\mathcal{H}_q^+(\mathfrak{sl}_2)$ is a quantum cluster algebra with type A_1 initial quiver

$$\begin{array}{ccc} E & \longrightarrow & \boxed{C_+} \\ \uparrow & & \\ \boxed{K_+} & & \end{array}$$

and the double canonical basis coincides with the quantum cluster monomials.

Let $C = \iota(C_+) - q^{-1}K_- \in U_q(\widetilde{\mathfrak{sl}_2})$. Then C is bar-invariant and central in $U_q(\tilde{\mathfrak{g}})$. Since it has correct triangularity, $C = F_1 \bullet E_1$. But

$$C^2 = \iota(C_+^2) - (q^{-1} + q^{-3})K_-\iota(C_+) + (1 + q^{-2})K_-K_+$$

and thus $F_1^2 \bullet E_1^2 = C^2 - K_-K_+$. So $\mathbf{B}_{\widetilde{\mathfrak{sl}_2}}$ contains *imaginary* elements, that is, elements whose powers do not belong to the double canonical basis. Let $C^{(0)} = 1, C^{(1)} = C$, and for $k \geq 1$ define $C^{(k+1)} = C^{(k)}C - K_-K_+C^{(k-1)}$. Each $C^{(k)}$ is bar-invariant and central, and

$$C^{(k)} = \sum_{\substack{0 \leq i \leq j \\ i+j \leq k}} (-1)^j q^{-j-i^2} \begin{bmatrix} k-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_-^j K_+^i \iota(F^{k-i-j} \circ E^{k-i-j}).$$

Therefore $C^{(k)} = F^k \bullet E^k$. It is straightfoward to show

$$\mathbf{B}_{\widetilde{\mathfrak{sl}_2}} = \{K_-^a K_+^b \diamond F^r C^{(s)} : a, b, r, s \in \mathbb{Z}_{\geq 0}\} \cup \{K_-^a K_+^b \diamond C^{(s)} E^r : a, b, r, s \in \mathbb{Z}_{\geq 0}\}.$$

Although it has a "cluster-like" structure, the double basis of $U_q(\widetilde{\mathfrak{sl}_2})$ does not have a strict quantum cluster structure since $FE = C + qK_+ + q^{-1}K_-$ does not have the proper form of an exchange relation.

We conclude this section by observing that the braid group of $\mathfrak{g}$ acts on the Drinfeld double $\mathcal{D}_q$ by algebra automorphisms via the *modified Lusztig symmetries*

$$T_i(K_{\pm j}) = K_{\pm j}K_{\pm i}^{-a_{ij}},$$

$$T_i(E_j) = \begin{cases} q_i^{-1}K_{+i}^{-1}F_i & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} E_i^{\langle r\rangle} E_j E_i^{\langle s\rangle} & i \neq j \end{cases},$$

$$T_i(F_j) = \begin{cases} q_i^{-1}K_{+i}^{-1}E_i & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} F_i^{\langle r\rangle} F_j F_i^{\langle s\rangle} & i \neq j \end{cases}.$$

Note that these symmetries commute with the bar-involution, that is, $\overline{T_i(x)} = T_i(\overline{x})$ for all $x \in \mathcal{D}_q$. We also remark that $(T_i(x))^* = T_i^{-1}(x^*)$, $\tau(T_i(x)) = T_i^{-1}(\tau(x))$, and $T_i(K \diamond x) = T_i(K) \diamond T_i(X)$ for $x \in D_q, K \in \Bbbk Q^2$ (here we have extended $\diamond$ to an action of $\Bbbk Q^2$ on $\mathcal{D}_q$). The set $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}} = \mathbf{K}^{-1} \diamond \mathbf{B}_{\tilde{\mathfrak{g}}}$ is a bar-invariant basis of $\mathcal{D}_q$ and it is conjectured [BG17b, Conjecture 1.15] (when $\mathfrak{g}$ is semisimple, as assumed here) that $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ is preserved by the braid group action.

Example 2.16. For $\mathfrak{sl}_2$, we have $T(C^{(k)}) = K_-^{-k}K_+^{-k}C^{(k)}$. For $\mathfrak{sl}_3$, we have $T_i(E_j) = E_{ji}$ and $T_i(E_{ji}) = E_j$ $(i \neq j)$.

The double canonical basis of both $\mathcal{H}_q^+$ and $\widetilde{U}_q$ is preserved by τ. In fact, $\tau(b_- \circ b_+) = \tau(b_+) \circ \tau(b_-)$ and $\tau(b_- \bullet b_+) = \tau(b_+) \bullet \tau(b_-)$ [BG17b, Thm. 1.10]. It is conjectured that the $\mathbf{B}_{\tilde{\mathfrak{g}}}$ is preserved by $*$ [BG17b, Conjecture 1.11].

Chapter 3

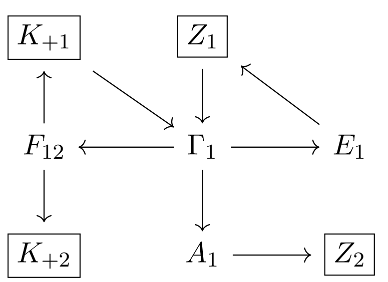
The double canonical basis of $\mathcal{H}_q^+(\mathfrak{sl}_3)$

In this chapter we investigate the structure of the double canonical basis of $\mathcal{H}_q^+(\mathfrak{sl}_3)$. For $i, j \in \{1, 2\}$, $i \neq j$, let $A_i = F_i \circ E_{ij}$, $H_i = F_{ij} \circ E_j$, $\Gamma_i = F_{ij} \circ E_{ij}$, $C_{+i} = F_i \circ E_i$, and $Z_i = F_{ij} \circ E_{ji}$. Our main result is the following theorem:

Theorem 3.1. *The algebra $\mathcal{H}_q^+(\mathfrak{sl}_3)$ has a type D_4 quantum cluster structure. The exchangeable cluster variables are*

$$\{E_1, E_2, E_{12}, E_{21}, F_1, F_2, F_{12}, F_{21}, C_{+1}, C_{+2}, A_1, A_2, H_1, H_2, \Gamma_1, \Gamma_2\}$$

and the coefficients are $\{K_{+1}, K_{+2}, Z_1, Z_2\}$. An initial cluster is $\{F_{12}, \Gamma_1, E_1, A_1\}$ with initial quiver



The double canonical basis is equal to the set of quantum cluster monomials.

In Section 3.1, we determine the quasi-commutation relations between the quantum cluster variables listed in Thm. 3.1. In Section 3.2, we show that monomials in a subset of quasi-commuting variables belong to the double canonical basis (up to a power of $q^{\frac{1}{2}}$). We complete the proof of Thm. 3.1 in Section 3.3.

3.1 Quasi-commutation relations

We begin by computing the quasi-commutation relations between the sixteen cluster variables listed in Thm. 3.1. We also show that the cluster variables are *real*, that is, their powers belong to the double canonical basis.

First, we have

$$F_1E_{21} = E_{21}F_1 = F_1 \circ E_{21}.$$

Furthermore, $E_{12}F_1 = F_1E_{12} + (q^{-1} - q)K_{+1} \diamond E_2$ and thus

$$F_1 \circ E_{12} = F_1E_{12} - qK_{+1} \diamond E_2.$$

The following lemma is immediate:

Lemma 3.2. *The assignments $F \mapsto F_1, E \mapsto E_{12}, K \mapsto q^{\frac{1}{2}}K_{+1}E_2$ extend to a homomorphism of algebras $\mathcal{H}_q^+(\mathfrak{sl}_2) \to \mathcal{H}_q^+(\mathfrak{sl}_3)$. In particular, $F_1 \circ E_{12}$ commutes with F_1 and E_{12}, and*

$$\begin{aligned}(F_1 \circ E_{12})^k &= \sum_{j=0}^{k}(-1)^j q^j \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} K_{+1}^j \diamond F_1^{k-j} q^{-\frac{1}{2}j(k-j)} E_2^j E_{12}^{k-j} \\ &= F_1^k \circ E_{12}^k.\end{aligned}$$

Lemma 3.3. *We have the following quasi-commutation relations:*

$$(F_1 \circ E_{12})E_{21} = E_{21}(F_1 \circ E_{12}), \tag{3.1}$$

$$(F_1 \circ E_{12})E_2 = q^{-1}E_2(F_1 \circ E_{12}), \tag{3.2}$$

$$(F_1 \circ E_{12})E_1 = qE_1(F_1 \circ E_{12}), \tag{3.3}$$

and

$$(F_1 \circ E_{12})(F_1 \circ E_1) = q(F_1 \circ E_1)(F_1 \circ E_{12}). \tag{3.4}$$

Proof. Equations (3.1) and (3.2) are immediate. For (3.3), we have

$$\begin{aligned}
(F_1 \circ E_{12})E_1 &= F_1E_{12}E_1 - qK_{+1}q^{\frac{1}{2}}E_2E_1 \\
&= qF_1E_1E_{12} - qK_{+1}(qE_{12} + E_{21}) \\
&= q(E_1F_1 + (q - q^{-1})K_{+1})E_{12} - q^2K_{+1}E_{12} - qK_{+1}E_{21} \\
&= qE_1F_1E_{12} - K_{+1}(q^{-1}E_{12} + qE_{21}) \\
&= qE_1F_1E_{12} - K_{+1}q^{\frac{1}{2}}E_1E_2 \\
&= qE_1(F_1 \circ E_{12}).
\end{aligned}$$

Equation (3.4) follows immediately from (3.3) and Lemma 3.2. □

By symmetry of the defining relations, we have $F_2 \circ E_{21} = F_2E_{21} - qK_{+2} \diamond E_1$. Applying τ, we obtain

$$F_{12} \circ E_2 = \tau(F_2 \circ E_{21}) = F_{12}E_2 - qK_{+2} \diamond F_1,$$

as well as

$$\begin{aligned}
(F_{12} \circ E_2)^k &= \sum_{j=0}^{k} (-1)^j q^j \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} K_{+2}^j \diamond q^{\frac{1}{2}j(k-j)} F_1^j F_{12}^{k-j} E_2^{k-j} \\
&= F_{12}^k \circ E_2^k.
\end{aligned}$$

Corollary 3.4. *The element $F_{12} \circ E_2$ commutes with F_{12} and E_2,*

$$(F_{12} \circ E_2)F_{21} = F_{21}(F_{12} \circ E_2),$$

$$(F_{12} \circ E_2)F_1 = qF_1(F_{12} \circ E_2),$$

$$(F_{12} \circ E_2)F_2 = q^{-1}F_2(F_{12} \circ E_2),$$

$$(F_{12} \circ E_2)(F_2 \circ E_2) = q^{-1}(F_2 \circ E_2)(F_{12} \circ E_2).$$

From the formulas for F_{12} and E_{12}, we compute

$$E_{12}F_{12} = F_{12}E_{12} + (q^{-1} - q)K_{+1}K_{+2},$$

and thus

$$F_{12} \circ E_{12} = F_{12}E_{12} - qK_{+1}K_{+2}.$$

The following lemma is immediate:

Lemma 3.5. *The assignments* $F \mapsto F_{12}, E \mapsto E_{12}$, *and* $K \mapsto K_{+1}K_{+2}$ *determine an algebra homomorphism* $\mathcal{H}_q^+(\mathfrak{sl}_2) \to \mathcal{H}_q^+(\mathfrak{sl}_3)$. *In particular,* $F_{12} \circ E_{12}$ *commutes with* F_{12} *and* E_{12} *and*

$$\begin{aligned}(F_{12} \circ E_{12})^k &= \sum_{j=0}^{k} (-1)^j q^j \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} K_{+1}^j K_{+2}^j F_{12}^{k-j} E_{12}^{k-j} \\ &= F_{12}^k \circ E_{12}^k.\end{aligned}$$

Lemma 3.6. *We have*

$$(F_{12} \circ E_{12})E_1 = qE_1(F_{12} \circ E_{12}), \tag{3.5}$$

$$(F_{12} \circ E_{12})F_1 = q^{-1}F_2(F_{12} \circ E_{12}), \tag{3.6}$$

$$(F_{12} \circ E_2)E_{12} = qE_{12}(F_{12} \circ E_2), \tag{3.7}$$

$$(F_1 \circ E_{12})F_{12} = q^{-1}F_{12}(F_1 \circ E_{12}). \tag{3.8}$$

Proof. Applying τ to the equation $F_1E_{21} = E_{21}F_1$ gives $F_{12}E_1 = E_1F_{12}$, from which (3.5) follows. Similarly, (3.6) holds since F_2 commutes with E_{12}. For (3.7), we have,

$$\begin{aligned}(F_{12} \circ E_2)E_{12} &= (F_{12}E_2 - qK_{+2} \diamond F_1)E_{12} \\ &= q(E_{12}F_{12} + (q - q^{-1})K_{+1}K_{+2})E_2 - q^{\frac{1}{2}}K_{+2}(E_{12}F_1 + (q - q^{-1})K_{+1} \diamond E_2) \\ &= qE_{12}F_{12}E_2 - q^2E_{12}K_{+2} \diamond F_1 \\ &= qE_{12}(F_{12} \circ E_2).\end{aligned}$$

Applying τ to $(F_{21} \circ E_1)E_{21} = qE_{21}(F_{21} \circ E_1)$ gives (3.8). □

From Lemma 3.3, Cor. 3.4 and Lemma 3.6, we obtain:

Corollary 3.7. *The following quasi-commutation relations hold:*

$$(F_{12} \circ E_2)(F_{12} \circ E_{12}) = q(F_{12} \circ E_{12})(F_{12} \circ E_2),$$

$$(F_1 \circ E_{12})(F_{12} \circ E_2) = q^{-2}(F_{12} \circ E_2)(F_1 \circ E_{12}),$$

$$(F_1 \circ E_{12})(F_{12} \circ E_{12}) = q^{-1}(F_{12} \circ E_{12})(F_1 \circ E_{12}).$$

Next, we have

$$\begin{aligned}E_{21}F_{12} &= F_{12}E_{21} + (q^{-1} + q)K_{+2}F_1E_1 - (1 + q^2)K_{+1}K_{+2} \\ &= F_{12}E_{21} + (q^{-1} - q)K_{+2}(F_1 \circ E_1),\end{aligned}$$

and thus

$$\begin{aligned}F_{12} \circ E_{21} &= F_{12}E_{21} - qK_{+2}(F_1 \circ E_1) \\ &= F_{12}E_{21} - qK_{+2}F_1E_1 + q^2K_{+1}K_{+2}.\end{aligned}$$

Lemma 3.8. *The assignments* $F \mapsto F_{12}, E \mapsto E_{21}, K_+ \mapsto K_{+2}(F_1 \circ E_1)$ *determine another algebra homomorphism* $\mathcal{H}_q^+(\mathfrak{sl}_2) \to \mathcal{H}_q^+(\mathfrak{sl}_3)$, *and thus* $(F_{12} \circ E_{21})^k = F_{12}^k \circ E_{21}^k$.

Proof. We have

$$\begin{aligned}(F_{12} \circ E_{21})^k &= \sum_{j=0}^{k} (-1)^j q^j \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} K_{+2}^j (F_1 \circ E_1)^j F_{12}^{k-j} E_{21}^{k-j} \\ &= \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^{j+l} q^{j+l} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} \begin{bmatrix} j \\ l \end{bmatrix}_{q^2} K_{+1}^l K_{+2}^j q^{\frac{1}{2}(j-l)(k-j)} F_1^{j-l} F_{12}^{k-j} \\ &\qquad\qquad \times q^{-\frac{1}{2}(j-l)(k-j)} E_1^{j-l} E_{21}^{k-j} \\ &= F_{12}^k \circ E_{21}^k.\end{aligned}$$

□

Lemma 3.9. *The element* $F_{12} \circ E_{21}$ *is quasi-central in* $\mathcal{H}_q^+(\mathfrak{sl}_3)$.

Proof. It is straightforward that $(F_{12} \circ E_{21})E_1 = q^{-1}E_1(F_1 \circ E_{21})$ and we also have

$$\begin{aligned}(F_{12} \circ E_{21})E_2 &= q(E_2 F_{12} + (q - q^{-1})K_{+2} \diamond F_1)E_{21} - qK_{+2}(q^{-\frac{1}{2}}(F_1 \circ E_{12}) + q^{\frac{1}{2}} F_1 E_{21}) \\ &= qE_2 F_{12} E_{21} - K_{+2} q^{-\frac{1}{2}} F_1 E_{21} - K_{+2} q^{\frac{1}{2}} (F_1 \circ E_{12}) \\ &= qE_2 F_{12} E_{21} - K_{+2} E_2 (F_1 \circ E_1) \\ &= qE_2(F_{12} \circ E_{21}).\end{aligned}$$

Applying τ to the previous two computations gives

$$(F_{12} \circ E_{21})F_1 = qF_1(F_{12} \circ E_{21}),$$

$$(F_{12} \circ E_{21})F_2 = q^{-1}F_2(F_{12} \circ E_{21}).$$

□

Using symmetry, we now have a complete set of quasi-commutation relations. We summarize these using the notation of Thm. 3.1:

- A_i quasi-commutes with $F_i, F_{ij}, E_1, E_2, H_i$, Γ_i, and C_{+i},
- H_i quasi-commutes with $F_1, F_2, E_j, E_{ij}, A_i$, Γ_i, and C_{+j},
- Γ_i quasi-commutes with $F_j, F_{ij}, E_i, E_{ij}, A_i$, and H_i,
- C_{+i} quasi-commutes with $F_i, F_{ij}, E_i, E_j, A_i$ and H_j,
- Z_i is quasi-central,

where $\{i, j\} = \{1, 2\}$.

3.2 Computation of the double canonical basis

From the computations in Section 3.1, we conclude that any maximal set of mutually quasi-commuting elements from

$$\mathbf{X} = \{E_1, E_2, E_{12}, E_{21}, F_1, F_2, F_{12}, F_{21}, C_{+1}, C_{+2}, A_1, A_2, H_1, H_2, \Gamma_1, \Gamma_2\}$$

has size four. There are fifty such sets. In no particular order, they are:

$\{A_i, H_i, \Gamma_i, E_{ij}\}, \{A_i, H_i, \Gamma_i, F_{ij}\}, \{A_i, H_i, E_j, E_{ij}\}, \{A_i, H_i, E_j, F_i\}, \{A_i, H_i, F_{ij}, F_i\},$

$\{A_i, \Gamma_i, E_{ij}, E_i\}, \{A_i, \Gamma_i, F_{ij}, E_i\}, \{A_i, E_{ji}, E_j, F_i\}, \{A_i, E_{ij}, E_i, E_{ji}\}, \{A_i, E_{ij}, E_{ji}, E_j\},$

$\{A_i, E_{ji}, C_{+i}, E_i\}, \{A_i, E_{ji}, C_{+i}, F_i\}, \{A_i, C_{+i}, F_i, F_{ij}\}, \{A_i, C_{+i}, E_i, F_{ij}\}, \{H_i, \Gamma_i, E_{ij}, F_j\},$

$\{H_i, \Gamma_i, F_{ij}, F_j\}, \{H_i, E_j, F_i, F_{ji}\}, \{H_i, F_{ij}, F_i, F_{ji}\}, \{H_i, F_{ij}, F_j, F_{ji}\}, \{H_i, E_j, F_{ji}, C_{+j}\},$

$\{H_i, F_j, F_{ji}, C_{+j}\}, \{H_i, F_j, E_{ij}, C_{+j}\}, \{H_i, E_j, E_{ij}, C_{+j}\}, \{\Gamma_i, E_{ij}, F_j, E_i\}, \{\Gamma_i, F_{ij}, F_j, E_i\},$

where $\{i, j\} = \{1, 2\}$. Let S be any subset of $\mathbf{X}$ whose elements quasi-commute. In this section, we prove that quantum cluster monomials in $S \cup \{K_{+1}, K_{+2}, Z_1, Z_2\}$ are double basis elements, and conversely, that every double basis element is such a monomial. Here we use "quantum cluster monomial" informally to mean the unique bar-invariant multiple of $q^{\frac{1}{2}}$ with a monomial in quasi-commuting elements. In Section 3.3, we show these elements are honest quantum cluster monomials for a quantum cluster structure on $\mathcal{H}_q^+(\mathfrak{sl}_3)$.

Observe that the monoid $\mathbf{K}_+$ identifies with $Q_{\geq 0}$. For $\lambda = r\alpha_1 + s\alpha_2 \in \mathbb{Q}_{\geq 0}$, write $K_\lambda = K_{+1}^r K_{+2}^s$. With this notation, $K_\lambda u = q^{(\lambda,|u|)} u K_\lambda$ where $(\cdot,\cdot)$ is the bilinear form on Q^2 such that $(\alpha_i, \alpha_i) = 2$ (the Killing form) and $K_\lambda \diamond u = q^{-\frac{1}{2}(\lambda,|u|)} K_\lambda u$.

For $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $ab = 0$, let

$$b_+(a,b,c,d) = q^{\frac{1}{2}(a-b)(c-d)} E_1^a E_2^b E_{12}^c E_{21}^d \in \mathbf{B}_+^{\mathrm{up}}$$

and

$$b_-(a,b,c,d) = q^{\frac{1}{2}(a-b)(c-d)} F_1^a F_2^b F_{12}^c F_{21}^d \in \mathbf{B}_-^{\mathrm{up}}.$$

The following lemma and its corollaries provide sufficient conditions for when quantum cluster monomials in quasi-commuting double canonical basis elements belong to the double basis.

Lemma 3.10. *Let $B_- \circ B_+$ be a double basis element and write*

$$B_- \circ B_+ = B_- B_+ + \sum_{j \in J} p_j(q) K_{\lambda_j} \diamond b_{-j} b_{+j}$$

where J is a finite index set, $p_j(q) \in q\mathbb{Z}[q]$, and

$$\deg_{Q^2_{\geq 0}} b_{-j} b_{+j} + \deg_{Q^2_{\geq 0}} K_{\lambda_j} = \deg_{Q^2_{\geq 0}} B_- B_+.$$

Let B'_- be a dual canonical basis element in U_q^- satisfying $(B_- \circ B_+)B'_- = q^c B'_-(B_- \circ B_+)$ for some $c \in \mathbb{Z}$ and, for all $j \in J$, $b_{-j}B'_- = q^{c_j} B'_- b_{-j}$ for some $c_j \in \mathbb{Z}$. Then $q^{\frac{1}{2}c} B'_- B_-$ and $q^{\frac{1}{2}c_j} B'_- b_{-j}$ are dual canonical basis elements. If $q^{\frac{1}{2}c_j} B'_- b_{-j}$ is a quantum cluster monomial *in* $\{F_1, F_{12}, F_{21}\}$*, suppose also that*

$$x - y_j + n_j - m_j \geq 0,$$

where $B_- = b_-(a, 0, x, x')$*,* $b_{-j} = b_-(a_j, 0, y_j, y_j')$*, and* $\lambda_j = n_j\alpha_1 + m_j\alpha_2$*. If* $q^{\frac{1}{2}c_j}B'_-b_{-j}$ *is a quantum cluster monomial in* $\{F_2, F_{12}, F_{21}\}$*, suppose that*

$$y_j - x + m_j \geq 0,$$

where $B_- = b_-(0, a, x, x')$*,* $b_{-j} = b_-(0, a_j, y_j, y_j')$*, and* $\lambda_j = n_j\alpha_1 + m_j\alpha_2$*. Then* $q^{\frac{1}{2}c}B'_-(B_- \circ B_+)$ *is in the double canonical basis.*

Observe that if a dual basis element commutes with each b_{-j} and with B_-, then for each j, B_- and b_{-j} are quantum cluster monomials from the same cluster. Therefore, if there exist $j, l \in J$ such that $q^{\frac{1}{2}c_j}B'_-b_{-j}$ and $q^{\frac{1}{2}c_l}B'_-b_{-l}$ are quantum cluster monomials from different clusters, then $a = 0$.

Proof. We know that $q^{\frac{1}{2}c}B'_-(B_- \circ B_+)$ is bar-invariant and

$$q^{\frac{1}{2}c}B'_-(B_- \circ B_+) = q^{\frac{1}{2}c}B'_-B_-B_+ + \sum_{j\in J} q^{-\frac{1}{2}(\lambda_j, |B'_-|)}q^{\frac{1}{2}(c-c_j)}p_j(q)K_{\lambda_j} \diamond q^{\frac{1}{2}c_j}B'_-b_{-j}b_{+j}.$$

If $|B_-| = -r\alpha_1 - s\alpha_2$ and $\lambda_j = n_j\alpha_1 + m_j\alpha_2$, our $Q^2_{\geq 0}$-degree assumptions also imply that

$$|b_{-j}| = (-r + n_j)\alpha_1 + (-s + m_j)\alpha_2.$$

Since B'_- quasi-commutes with B_- and with each b_{-j}, $q^{\frac{1}{2}c}B'_-B_-$ and each $q^{\frac{1}{2}c_j}B'_-b_{-j}b_{+j}$ are quantum cluster monomials (and therefore dual canonical basis elements).

Suppose first that $q^{\frac{1}{2}c}B'_-B_-$, $q^{\frac{1}{2}c_j}B'_-b_{-j}b_{+j}$ are cluster monomials in $\{F_1, F_{12}, F_{21}\}$. Then $B_- = b_-(r - s, 0, x, s - x)$ and $b_{-j} = b_-(r - n_j - s + m_j, 0, y_j, s - m_j - y_j)$ for some

$x, y_j \geq 0$. If $B'_- = b_-(\alpha, 0, \beta, \gamma)$, then

$$c = -\alpha(s - x) + \alpha x + (\gamma - \beta)(r - s),$$

$$c_j = -\alpha(s - m_j - y_j) + \alpha y_j + (\gamma - \beta)(r - n_j - s + m_j),$$

and so

$$c - c_j = 2\alpha(x - y_j) - \alpha m_j + (\beta - \gamma)(m_j - n_j).$$

Furthermore,

$$(\lambda_j, |B'_-|) = (-2\alpha - \beta - \gamma)n_j + (\alpha - \beta - \gamma)m_j$$

and thus

$$c - c_j - (\lambda_j, |B'_-|) = 2\alpha(x - y_j + n_j - m_j) + 2(\beta m_j + \gamma n_j) \in 2\mathbb{Z}.$$

If $x - y_j + n_j - m_j \geq 0$, we have $q^{-\frac{1}{2}(\lambda_j, |B'_-|)} q^{\frac{1}{2}(c - c_j)} \in \mathbb{Z}[q]$.

Next, suppose that $q^{\frac{1}{2}c} B'_- B_-$, $q^{\frac{1}{2}c_j} B'_- b_{-j} b_{+j}$ are cluster monomials in $\{F_2, F_{12}, F_{21}\}$. Then $B_- = b_-(0, s - r, x, r - x)$ and $b_{-j} = b_-(0, s - m_j - r + n_j, y_j, r - n_j - y_j)$ for some $x, y_j \geq 0$. If $B'_- = b_-(0, \alpha, \beta, \gamma)$, then

$$c = \alpha(r - x) - \alpha x + (\beta - \gamma)(s - r),$$

$$c_j = \alpha(r - n_j - y_j) - \alpha y_j + (\beta - \gamma)(s - m_j - r + n_j),$$

and so

$$c - c_j = 2\alpha(y_j - x) + \alpha n_j + (\gamma - \beta)(n_j - m_j).$$

Since

$$(\lambda_j, |B'_-|) = (\alpha - \beta - \gamma)n_j + (-2\alpha - \beta - \gamma)m_j,$$

we have

$$c - c_j - (\lambda_j, |B'_-|) = 2\alpha(y_j - x + m_j) + 2(\beta m_j + \gamma n_j) \in 2\mathbb{Z}.$$

If $y_j - x + m_j \geq 0$, then $q^{-\frac{1}{2}(\lambda_j, |B'_-|)} q^{\frac{1}{2}(c-c_j)} \in \mathbb{Z}[q]$.

By the assumptions of the lemma, $q^{\frac{1}{2}c} B'_-(B_- \circ B_+)$ has correct triangularity and is therefore a double basis element. □

In particular, the assumption of Lemma 3.10 is true if B'_- and each b_{-j} is a monomial in only F_1 and F_{12} (so that $x - y_j = m_j$).

Corollary 3.11. *Let $B_- \circ B_+$ be as in Lemma 3.10. Suppose that B'_+ is a dual canonical basis element in U_q^+ which satisfies $(B_- \circ B_+)B'_+ = q^c B'_+(B_- \circ B_+)$ for some $c \in \mathbb{Z}$ and, for all $j \in J$, $b_{+j}B'_+ = q^{c_j} B'_+ b_{+j}$ for some $c_j \in \mathbb{Z}$. Then $q^{-\frac{1}{2}c} B_+ B'_+$ and $q^{-\frac{1}{2}c_j} b_{+j} B'_+$ belong to dual canonical basis. If $q^{-\frac{1}{2}c_j} b_{+j} B'_+$ is a quantum cluster monomial in $\{E_1, E_{12}, E_{21}\}$, suppose also that*

$$x - y_j + n_j - m_j \geq 0,$$

where $B_+ = b_+(a, 0, x', x)$, $b_{+j} = b_+(a_j, 0, y'_j, y_j)$, and $\lambda_j = n_j\alpha_1 + m_j\alpha_2$. If $q^{-\frac{1}{2}c_j} b_{+j} B'_+$ is a quantum cluster monomial in $\{E_2, E_{12}, E_{21}\}$, suppose that

$$y_j - x + m_j \geq 0,$$

where $B_+ = b_+(0, a, x', x)$, $b_{+j} = b_+(0, a_j, y'_j, y_j)$, and $\lambda_j = n_j\alpha_1 + m_j\alpha_2$. Then $q^{-\frac{1}{2}c}(B_- \circ B'_+)B'_+$ is in the double canonical basis.

Proof. Under these assumptions, $\tau(B_- \circ B_+) = \tau(B_-) \circ \tau(B_+)$ satisfies the assumptions of Lemma 3.10 and thus $q^{-\frac{1}{2}c}\tau(B'_+)(\tau(B_-) \circ \tau(B_+))$ is a double basis element. Applying τ once more completes the proof. □

Corollary 3.12. *Let $B_- \circ B_+$ and $B'_- \circ B'_+$ be double basis elements. Write*

$$B_- \circ B_+ = B_- B_+ + \sum_{j \in J} p_j(q) K_{\lambda_j} \diamond b_{-j} b_{+j}$$

and

$$B'_- \circ B'_+ = B'_- B'_+ + \sum_{l \in L} p_l(q) K_{\lambda_l} \diamond b_{-l} b_{+l}$$

where J and L are finite index sets, $p_j(q), p_l(q) \in q\mathbb{Z}[q]$, and the necessary $Q^2_{\geq 0}$-degree assumptions are satisfied. Suppose that $B_- B'_- = q^{c_-} B'_- B_-$, $B_+ B'_+ = q^{c_+} B'_+ B_+$, and $(B_- \circ B_+)(B'_- \circ B'_+) = q^{c_- + c_+}(B'_- \circ B'_+)(B_- \circ B_+)$ for some $c_-, c+ \in \mathbb{Z}$. Suppose also that for all l, $B_- b_{-l} = q^{c_l^-} b_{-l} B_-$ and $B_+ b_{+l} = q^{c_l^+} b_{+l} B_+$ some $c_l^-, c_l^+ \in \mathbb{Z}$. Finally, assume that $(B_- \circ B_+) b_{-l} = q^{c_l^-} b_{-l}(B_- \circ B_+)$ for all l and that each b_{-l} quasi-commutes with each b_{-j}. If $B_- \circ B_+$ satisfies the assumptions of Lemma 3.10/Cor. 3.11, then $q^{-\frac{1}{2}(c_- + c_+)}(B_- \circ B_+)(B'_- \circ B'_+)$ is in the double canonical basis if

$$q^{-\frac{1}{2}(\lambda_l, |B_- \circ B_+|)(c_- - c_l^- + c_+ - c_l^+)} \in \mathbb{Z}[q]$$

for all l

Proof. We have

$$\begin{aligned} &q^{-\frac{1}{2}(c_- + c_+)}(B_- \circ B_+)(B'_- \circ B'_+) \\ &= q^{-\frac{1}{2}(c_+ - c_-)} B'_-(B_- \circ B_+) B'_+ + \sum_{l \in L} p_l(q) q^{-\frac{1}{2}(\lambda_l, |B_- \circ B_+|)(c_- - c_l^- + c_+ - c_l^+)} \\ &\quad \times K_{\lambda_l} \diamond q^{\frac{1}{2}(c_l^- - c_l^+)} b_{-l}(B_- \circ B_+) b_{+l}. \end{aligned}$$

The proofs of Lemma 3.10 and Cor. 3.11 imply that $q^{-\frac{1}{2}(c_+ - c_-)} B'_-(B_- \circ B_+) B'_+$ and $q^{\frac{1}{2}(c_l^- - c_l^+)} b_{-l}(B_- \circ B_+) b_{+l}$ have correct triangularity. □

We now apply Lemma 3.10 and its corollaries to compute the double basis of a parabolic subalgebra of $\mathcal{H}_q^+$ (that is, all basis elements of the form $F_i \circ b_+, b_+ \in \mathbf{B}_+^{\mathrm{up}}$). We require one additional computation (Lemma 3.15). First, we must generalize the exchange relation $E_2 E_1 = q^{\frac{1}{2}} E_{12} + q^{-\frac{1}{2}} E_{21}$ to rewrite products of the form $E_2^k E_1^k$. The following lemma is a straightforward consequence of the q-binomial theorem:

Lemma 3.13. *If X and Y are commuting elements in a $\mathbb{Q}(q)$-algebra, then*

$$\prod_{j=0}^{k-1}(q^{2j+1}X+Y)=\sum_{j=0}^{k} q^{j^2}\begin{bmatrix}k\\j\end{bmatrix}_{q^2} X^jY^{k-j}$$

for $k \geq 1$.

Corollary 3.14. *For $k \geq 1$,*

$$E_2^kE_1^k = q^{-\frac{1}{2}k^2}\sum_{j=0}^{k} q^{j^2}\begin{bmatrix}k\\j\end{bmatrix}_{q^2} E_{12}^jE_{21}^{k-j}, \tag{3.9}$$

$$E_2^k(F_1\circ E_1)^k = q^{-\frac{1}{2}k^2}\sum_{j=0}^{k} q^{j^2}\begin{bmatrix}k\\j\end{bmatrix}_{q^2} (F_1^k\circ E_{12}^jE_{21}^{k-j}). \tag{3.10}$$

Proof. Note that $(q^{\frac{1}{2}}E_{12}+q^{-\frac{1}{2}}E_{21})E_1 = E_1(q^{\frac{3}{2}}E_{12}+q^{-\frac{3}{2}}E_{21})$. Since $E_2E_1 = q^{\frac{1}{2}}E_{12}+q^{-\frac{1}{2}}E_{21}$, it follows that

$$E_2^kE_1^k = \prod_{j=0}^{k-1}(q^{j+\frac{1}{2}}E_{12}+q^{-j-\frac{1}{2}}E_{21}) = q^{-\frac{1}{2}k^2}\prod_{j=0}^{k-1}q^{2j+1}E_{12}+E_{21},$$

since

$$\prod_{j=0}^{k-1}q^{-j-\frac{1}{2}} = q^{\sum_{j=0}^{k-1}-j-\frac{1}{2}} = q^{-\frac{1}{2}(k-1)k-\frac{1}{2}k} = q^{-\frac{1}{2}k^2}.$$

So (3.9) follows from Lemma 3.13 since E_{12} commutes with E_{21}. The proof of (3.10) proceeds by a similar argument since

$$\begin{aligned}E_2(F_1\circ E_1) &= F_1(q^{\frac{1}{2}}E_{12}+q^{-\frac{1}{2}}E_{21}) - q^{\frac{3}{2}}K_{+1}\diamond E_2\\ &= q^{\frac{1}{2}}(F_1\circ E_{12}) + q^{-\frac{1}{2}}F_1E_{21}\end{aligned}$$

(and since $F_1\circ E_{12}$ commutes with $F_1E_{21} = F_1\circ E_{21}$). □

Lemma 3.15. *For all $a, b, c \in \mathbb{Z}_{\geq 0}$,*

$$q^{-\frac{1}{2}a(b+c)}(F_1 \circ E_{12})^a (F_1 \circ E_1)^b E_1^c$$

is in the double canonical basis.

Proof. Bar-invariance follows from Lemma 3.3. We have

$$\begin{aligned}
&q^{-\frac{1}{2}a(b+c)}(F_1 \circ E_{12})^a (F_1 \circ E_1)^b E_1^c \\
&= q^{-\frac{1}{2}a(b+c)} \sum_{0 \leq j \leq a, 0 \leq l \leq b} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \\
&\qquad \times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} q^{-\frac{1}{2}j(a-j)} E_2^j E_1^{b+c-l} E_{12}^{a-j} \\
&= q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \leq j \leq a, 0 \leq l \leq b \\ j \leq b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \\
&\qquad \times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} (E_2^j E_1^j) E_1^{b+c-l-j} E_{12}^{a-j} \\
&\quad + q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \leq j \leq a, 0 \leq l \leq b \\ j > b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \\
&\qquad \times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} E_2^{j-b-c+l} (E_2^{b+c-l} E_1^{b+c-l}) E_{12}^{a-j}.
\end{aligned}$$

Using Cor. 3.13, we expand the first sum to get

$$\begin{aligned}
&q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \leq j \leq a, 0 \leq l \leq b \\ j \leq b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \\
&\qquad \times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} \times (E_2^j E_1^j) E_1^{b+c-l-j} E_{12}^{a-j} \\
&= q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \leq j \leq a, 0 \leq l \leq b \\ j \leq b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \\
&\qquad \times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} q^{-\frac{1}{2}j^2} \sum_{r=0}^{j} q^{r^2} \begin{bmatrix} j \\ r \end{bmatrix}_{q^2} E_{12}^r E_{21}^{j-r} E_1^{b+c-l-j} E_{12}^{a-j} \\
&= \sum_{\substack{0 \leq r \leq j \leq a, 0 \leq l \leq b \\ j \leq b+c-l}} (-1)^{j+l} q^{j+l} q^{j(b+c-l)} q^{r(b+c-j-l+r)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \begin{bmatrix} j \\ r \end{bmatrix}_{q^2}
\end{aligned}$$

$$\times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} q^{\frac{1}{2}(b+c-l-j)(a-2j+2r)} E_1^{b+c-l-j} E_{12}^{a-j+r} E_{21}^{j-r}.$$

Similarly, the second sum becomes

$$q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \le j \le a, 0 \le l \le b \\ j > b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2}$$

$$\times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} E_2^{j-b-c+l} (E_2^{b+c-l} E_1^{b+c-l}) E_{12}^{a-j}$$

$$= q^{-\frac{1}{2}a(b+c)} \sum_{\substack{0 \le j \le a, 0 \le l \le b \\ j > b+c-l}} (-1)^{j+l} q^{j+l} q^{\frac{1}{2}al} q^{(j+l)c} q^{(a+j)(b+c-l)} q^{-\frac{1}{2}j(a-j)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2}$$

$$\times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} E_2^{j-b-c+l} q^{-\frac{1}{2}(b+c-l)^2} \sum_{r=0}^{b+c-l} q^{r^2} \begin{bmatrix} b+c-l \\ r \end{bmatrix}_{q^2} E_{12}^{a-j+r} E_{21}^{b+c-l-r}$$

$$= \sum_{\substack{0 \le j \le a, 0 \le l \le b \\ 0 \le r \le b+c-l < j}} (-1)^{j+l} q^{j+l} q^{(j+l)c} q^{j(b+c-l)} q^{r(j-b-c+l+r)} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} \begin{bmatrix} b \\ l \end{bmatrix}_{q^2} \begin{bmatrix} b+c-l \\ r \end{bmatrix}_{q^2}$$

$$\times K_{+1}^{j+l} \diamond F_1^{a+b-j-l} q^{-\frac{1}{2}(j-b-c+l)(a-j+2r-b-c+l)} E_2^{j-b-c+l} E_{12}^{a-j+r} E_{21}^{b+c-l-r}.$$

Thus $q^{-\frac{1}{2}a(b+c)}(F_1 \circ E_{12})^a (F_1 \circ E_1)^b E_1^c$ has correct triangularity. $\square$

Corollary 3.16. *Let S be one of the following sets:*

$$\{F_1, F_1 \circ E_{12}, F_1 \circ E_1, E_{21}\},$$

$$\{F_1 \circ E_{12}, F_1 \circ E_1, E_1, E_{21}\},$$

$$\{F_1 \circ E_{12}, E_1, E_{12}, E_{21}\},$$

$$\{F_1 \circ E_{12}, E_2, E_{12}, E_{21}\},$$

$$\{F_1, F_1 \circ E_{12}, E_2, E_{21}\},$$

$$\{F_1, F_1 \circ E_{12}, F_1 \circ E_1, F_{12}\},$$

$$\{F_1 \circ E_{12}, F_1 \circ E_1, E_1, F_{12}\}.$$

Bar-invariant products of monomials in S with a power of $q^{\frac{1}{2}}$ belong to the double canonical basis.

Proof. From Lemma 3.15, we see that $q^{-\frac{1}{2}a(b+c)}(F_1 \circ E_{12})^a (F_1 \circ E_1)^b E_1^c$ satisfies the assumptions of Lemma 3.10/Cor. 3.11. □

The following lemmata establish that Z_1 and Z_2 play the role of coefficients (this statement is made precise in Section 3.3).

Lemma 3.17. *For all $a, b, x, y \in \mathbb{Z}_{\geq 0}$, $q^{-\frac{1}{2}(a-b)(x-y)} q^{\frac{1}{2}ab} F_2^a F_1^b F_{12}^x F_{21}^y$ is in the $\mathbb{Z}[q]$-span of the dual canonical basis of U_q^-.*

Proof. If $a \geq b$, we have

$$\begin{aligned}
F_2^a F_1^b F_{12}^x F_{21}^y &= F_2^{a-b} q^{-\frac{1}{2}b^2} \sum_{j=0}^{b} q^{j^2} \begin{bmatrix} b \\ j \end{bmatrix}_{q^2} F_{12}^{x+j} F_{21}^{y+b-j} \\
&= \sum_{j=0}^{b} q^{\frac{1}{2}(a-b)(x-y)} q^{-\frac{1}{2}ab} q^{(a-b)j+j^2} \begin{bmatrix} b \\ j \end{bmatrix}_{q^2} q^{-\frac{1}{2}(a-b)(x+2j-y-b)} F_2^{a-b} F_{12}^{x+j} F_{21}^{y+b-j} \\
&= \sum_{j=0}^{b} q^{\frac{1}{2}(a-b)(x-y)} q^{-\frac{1}{2}ab} q^{(a-b)j+j^2} \begin{bmatrix} b \\ j \end{bmatrix}_{q^2} b_-(0, a-j, x+j, y+b-j).
\end{aligned}$$

If $a < b$, we have

$$\begin{aligned}
F_2^a F_1^b F_{12}^x F_{21}^y &= q^{-\frac{1}{2}a^2} \sum_{j=0}^{a} q^{j^2} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} F_{12}^{j} F_{21}^{a-j} F_1^{b-a} F_{12}^x F_{21}^y \\
&= q^{(b-a)(2j-a)} q^{-\frac{1}{2}a^2} \sum_{j=0}^{a} q^{j^2} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} F_1^{b-a} F_{12}^{x+j} F_{21}^{y+a-j} \\
&= \sum_{j=0}^{a} q^{\frac{1}{2}(a-b)(x-y)} q^{\frac{1}{2}ab} q^{(b-a)j+j^2} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} q^{\frac{1}{2}(b-a)(x+2j-y-a)} F_1^{b-a} F_{12}^{x+j} F_{21}^{y+a-j} \\
&= \sum_{j=0}^{a} q^{\frac{1}{2}(a-b)(x-y)} q^{\frac{1}{2}ab} q^{(b-a)j+j^2} \begin{bmatrix} a \\ j \end{bmatrix}_{q^2} b_-(b-a, 0, x+j, y+a-j).
\end{aligned}$$

In either case, these computations show that $q^{-\frac{1}{2}(a-b)(x-y)} q^{\frac{1}{2}ab} F_2^a F_1^b F_{12}^x F_{21}^y$ is a $\mathbb{Z}[q]$-linear combination of dual basis elements. □

For the proof of Corollary 3.18, observe that

$$\tau(q^{-\frac{1}{2}(a-b)(x-y)}q^{\frac{1}{2}ab}F_2^aF_1^bF_{12}^xF_{21}^y) = q^{-\frac{1}{2}(a-b)(x-y)}q^{\frac{1}{2}ab}E_{12}^yE_{21}^xE_1^bE_2^a$$
$$= q^{\frac{1}{2}(a-b)(x-y)}q^{\frac{1}{2}ab}E_1^bE_2^aE_{12}^yE_{21}^x$$

is also in the the $\mathbb{Z}[q]$-span of the dual canonical basis.

Corollary 3.18. *For all $n, m \in \mathbb{Z}_{\geq 0}$, $(F_{12} \circ E_{21})^m(F_{21} \circ E_{12})^n$ is in the double canonical basis.*

Proof. We have

$$\begin{aligned}
&(F_{12} \circ E_{21})^n(F_{21} \circ E_{12})^m \\
&= \sum_{j,l} p_{jl}(q)K_{+1}^lK_{+2}^jF_1^{j-l}F_{12}^{m-j}E_1^{j-l}E_{21}^{m-j} \\
&\qquad \times \sum_{r,s} p_{rs}(q)K_{+1}^rK_{+2}^sF_2^{r-s}F_{21}^{n-r}E_2^{r-s}E_{12}^{n-r} \\
&= \sum_{j,l,r,s} p_{jl}(q)p_{rs}(q)q^{-m(r-s)}q^{(2j-l)(r-s)}q^{(l+j)(n-r)}q^{(l-j)(n-r)}q^{(r-s)(m-j)}K_{+1}^{l+r}K_{+2}^{j+s} \\
&\qquad \times F_2^{r-s}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j} \\
&= \sum_{j,l,r,s} p_{jl}(q)p_{rs}(q)q^{2l(n-r)}K_{+1}^{l+r}K_{+2}^{j+s}q^{(r-s)(j-l)}F_2^{r-s}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j}
\end{aligned}$$

where $0 \leq l \leq j \leq m, 0 \leq s \leq r \leq n$, and

$$p_{jl}(q) = (-1)^{j+l}q^{j+l}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\begin{bmatrix} j \\ l \end{bmatrix}_{q^2}, \quad p_{rs}(q) = (-1)^{r+s}q^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}_{q^2}\begin{bmatrix} r \\ s \end{bmatrix}_{q^2} \in q\mathbb{Z}[q].$$

By rewriting

$$\begin{aligned}
&q^{(r-s)(j-l)}F_2^{r-s}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j} \\
&= q^{-\frac{1}{2}(r-s-j+l)(m-j-n+r)}q^{\frac{1}{2}(r-s)(j-l)}F_2^{r-s}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r} \\
&\qquad \times q^{\frac{1}{2}(r-s-j+l)(m-j-n+r)}q^{\frac{1}{2}(r-s)(j-l)}E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j},
\end{aligned}$$

we conclude from Lemma 3.17 that $(F_{12}\circ E_{21})^m(F_{21}\circ E_{12})^n$ is in the double canonical basis. □

Lemma 3.19. *For all $m,n\in\mathbb{Z}_{\geq 0}$, products of $(F_{12}\circ E_{21})^m(F_{21}\circ E_{12})^n$ with dual canonical basis elements are (up to a power of $q^{\frac{1}{2}}$) in the double canonical basis.*

Proof. From Lemma 3.18, we have

$$\begin{aligned}
&q^{\frac{1}{2}a(n-m)}F_2^a(F_{12}\circ E_{21})^m(F_{21}\circ E_{12})^n\\
&= q^{\frac{1}{2}a(n-m)}\sum_{j,l,r,s} p_{jlrs}(q)q^{a(j+s)}q^{-\frac{1}{2}a(l+r)}K_{+1}^{l+r}K_{+2}^{j+s}\diamond q^{(r-s)(j-l)}F_2^{r-s+a}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}\\
&\qquad\times E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j}\\
&= \sum_{j,l,r,s} p_{jlrs}(q)q^{as}q^{-\frac{1}{2}a(m-j-n+r)}K_{+1}^{l+r}K_{+2}^{j+s}\diamond q^{\frac{1}{2}a(j-l)}q^{(r-s)(j-l)}F_2^{r-s+a}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}\\
&\qquad\times E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j}
\end{aligned}$$

where $0\leq l\leq j\leq m, 0\leq s\leq r\leq n$, and

$$p_{jlrs}(q)=(-1)^{j+l+r+s}q^{j+l+r+s}\begin{bmatrix} n\\ j\end{bmatrix}_{q^2}\begin{bmatrix} j\\ l\end{bmatrix}_{q^2}\begin{bmatrix} m\\ r\end{bmatrix}_{q^2}\begin{bmatrix} r\\ s\end{bmatrix}_{q^2}\in q\mathbb{Z}[q].$$

But we know from (the proof of) Lemma 3.17 that

$$q^{\frac{1}{2}a(j-l)}q^{(r-s)(j-l)}F_2^{r-s+a}F_1^{j-l}F_{12}^{m-j}F_{21}^{n-r}E_1^{j-l}E_2^{r-s}E_{12}^{n-r}E_{21}^{m-j}$$

is in the $q^{\frac{1}{2}a(m-j-n+r)}\mathbb{Z}[q]$-span of products of dual canonical basis elements. So

$$q^{\frac{1}{2}a(n-m)}F_2^a(F_{12}\circ E_{21})^m(F_{21}\circ E_{12})^n$$

is in the double canonical basis.

From the above computation and the proof of Cor. 3.18, we know that

$$q^{\frac{1}{2}a(n-m)}F_2^a(F_{12}\circ E_{21})^m(F_{21}\circ E_{12})^n$$

is in the $\mathbb{Z}[q]$-span of elements which look like $q^{as}K_{+1}^{l+r}K_{+2}^{j+s}b_{jlrs}^{-}b_{jlrs}^{+}$ where

$$b_{jlrs}^{-} = b_{-}(0, r-s+a-j+l, m-j+i, n-r-i)$$

or

$$b_{jlrs}^{-} = b_{-}(j-l-r+s-a, 0, m-j+i, n-r-i)$$

where $0 \le l \le j \le m, 0 \le s \le r \le n$, and $i \le \min(j-l, r-s+a)$. Note that dual canonical basis elements of the form $b_{-}(0,0,a,b)$ quasi-commute with each summand. Since

$$\begin{aligned} as + m - (m-j+i) + l + r - (j+s) &= as + l + r - s - i \\ &= l + r - s + a - i + a(s-1) \in \mathbb{Z}_{\ge 0} \end{aligned}$$

and

$$as + (m-j+i) - m + j + s = as + i + s \in \mathbb{Z}_{\ge 0},$$

we conclude from Lemma 3.10 that $(F_{12} \circ E_{21})^m (F_{21} \circ E_{12})^n b_{-}(0,0,a,b)$ is (up to a power of $q^{\frac{1}{2}}$) a double basis element. The remaining cases are proved using symmetry and τ. $\square$

Corollary 3.20. *For all $m, n \in \mathbb{Z}_{\ge 0}$, products of $(F_{12} \circ E_{21})^m (F_{21} \circ E_{12})^n$ with double canonical basis elements are (up to a power of $q^{\frac{1}{2}}$) in the double canonical basis.*

Proof. Let $B_{-} \circ B_{+}$ be a double basis element and write

$$B_{-} \circ B_{+} = B_{-}B_{+} + \sum_{j \in J} p_j(q) K_{\lambda_j} \diamond b_{-j} b_{+j}$$

where J is a finite index set, $p_j(q) \in q\mathbb{Z}[q]$, and

$$\deg_{Q^2_{\ge 0}} b_{-j} b_{+j} + \deg_{Q^2_{\ge 0}} K_{\lambda_j} = \deg_{Q^2_{\ge 0}} B_{-}B_{+}.$$

Since $Z_1^m Z_2^n$ has Q-degree zero,

$$(B_- \circ B_+) Z_1^m Z_2^n = B_- Z_1^n Z_2^m B_+ + \sum_{j \in J} p_j(q) K_{\lambda_j} \diamond b_{-j} Z_1^m Z_2^n b_{+j}.$$

The result follows immediately from Lemma 3.19. □

Proposition 3.21. *Let S be any subset of*

$$\mathbf{X} = \{E_1, E_2, E_{12}, E_{21}, F_1, F_2, F_{12}, F_{21}, C_{+1}, C_{+2}, A_1, A_2, H_1, H_2, \Gamma_1, \Gamma_2\}$$

whose elements quasi-commute. Quantum cluster monomials in $S \cup \{K_{+1}, K_{+2}, Z_1, Z_2\}$ are double basis elements (and conversely, every double basis element is such a monomial).

Proof. First, we apply Cor. 3.12 to $B_- \circ B_+ = A_i^r$ and $B'_- \circ B'_+ = H_i^s$. Since A_i satisfies the requirements of Lemma 3.10/Cor. 3.11 and

$$-\frac{1}{2}(K_{+j}^l, |A_i^r|) - \frac{1}{2}(-rs + r(s-l) - rs + r(s-l)) = 0$$

the proposition holds for $S = \{A_i, H_i\}$. Now we apply Cor. 3.12 to $B_- \circ B_+ = \Gamma_i$ and $B'_- \circ B'_+ = q^{rs} A_i^r H_i^s$. Since Γ_i^n satisfies the requirements of Lemma 3.10/Cor. 3.11 and

$$q^{-\frac{1}{2}(K_{+i}^j K_{+j}^l, |\Gamma_i^n|)} q^{-\frac{1}{2}(rn - (r-j+l)n - sn - (s-l+j)n)} = q^{sn},$$

the proposition holds for $S = \{A_i, H_i, \Gamma_i\}$. Observe also that quantum cluster monomials in $\{A_i, H_i, \Gamma_i\}$ satisfy the requirement of Lemma 3.10/Cor. 3.11 (see the remark following the proof of that lemma). We conclude that the proposition holds for any S consisting of quasi-commuting elements from

$$\{A_i, H_i, \Gamma_i\} \cup \mathbf{B}_-^{\mathrm{up}} \cup \mathbf{B}_+^{\mathrm{up}}.$$

The remaining cases follow from Cor. 3.16 using symmetry and τ. □

3.3 Proof of Theorem 3.1

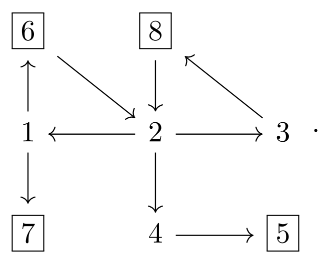
Let

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -2 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that B is the first four columns of the adjacency matrix of the quiver in Thm. 3.1 and Λ is the quasi-commutation matrix of $\{F_{12}, \Gamma, E_1, A_1, Z_2, K_{+1}, K_{+2}, Z_1\}$ (ordered from left to right). These matrices are compatible since $B^T\Lambda$ is a block matrix consisting of a 4×4 diagonal matrix with 2's on the diagonal followed by the 4×4 zero matrix. Label the quiver Q as follows:



In the notation of Theorem 2.6, the defining relations of $\widehat{\mathcal{A}}_{q^{1/2}}(Q, \Lambda)$ are

$$X_1X_1' = X_2 + qX_6X_7,$$

$$X_2X_2' = X_6X_8 + qX_1X_3X_4,$$

$$X_3X_3' = q^{-\frac{1}{2}}X_2 + q^{\frac{1}{2}}X_8,$$

$$X_4X_4' = q^{-\frac{1}{2}}X_2 + q^{\frac{1}{2}}X_5$$

since the principal part of Q is acyclic (it is an orientation of the Dynkin diagram D_4). Observe that

$$F_{12}E_{12} = \Gamma_1 + qK_{+1}K_{+2},$$

$$\begin{aligned}
\Gamma_1 C_{+1} &= (F_{12}E_{12} - qK_{+1}K_{+2})(F_1E_1 - qK_{+1}) \\
&= F_{12}(A_1 + q^{-1}K_{+1} \diamond E_2)E_1 - qK_{+1}K_{+2}F_1E_1 - qK_{+1}F_{12}E_{12} + q^2K_{+1}^2K_{+2} \\
&= F_{12}A_1E_1 + K_{+1}F_{12}(qE_{12} + E_{21}) - qK_{+1}K_{+2}F_1E_1 - qK_{+1}F_{12}E_{12} + q^2K_{+1}^2K_{+2} \\
&= qF_{12}E_1A_1 + K_{+1}(F_{12}E_{21} - qK_{+2}F_1E_1 + q^2K_{+1}K_{+2}) \\
&= qF_{12}E_1A_1 + K_{+1}Z_1,
\end{aligned}$$

$$\begin{aligned}
E_1H_1 &= E_1(F_{12}E_2 - qK_{+2} \diamond F_1) \\
&= F_{12}(q^{-\frac{1}{2}}E_{12} + q^{\frac{1}{2}}E_{21}) - q^{\frac{3}{2}}K_{+2}(F_1E_1 + (q^{-1} - q)K_{+1}) \\
&= q^{-\frac{1}{2}}\Gamma_1 + q^{\frac{1}{2}}Z_1.
\end{aligned}$$

By symmetry, $E_2H_2 = q^{-\frac{1}{2}}\Gamma_2 + q^{\frac{1}{2}}Z_2$. Applying τ gives

$$A_1F_2 = q^{-\frac{1}{2}}\Gamma + q^{\frac{1}{2}}Z_2.$$

Therefore the assignments $X_1 \mapsto F_{12}, X_2 \mapsto \Gamma, X_3 \mapsto E_1, X_4 \mapsto A_1, X_1' \mapsto E_{12}, X_2' \mapsto C_{+1}, X_3' \mapsto H_1, X_4' \mapsto F_2$ determine a surjective algebra homomorphism (after extending scalars)

$$\phi : \widehat{\mathcal{A}}_{q^{1/2}}(Q, \Lambda) \to \widehat{\mathcal{H}}_q^+(\mathfrak{sl}_3),$$

where $\widehat{\mathcal{H}}_q^+(\mathfrak{sl}_3)$ denotes $\mathcal{H}_q^+$ localized at the multiplicative submonoid generated by Z_1 and Z_2. The isomorphism ϕ restricts to a bijection between the quantum cluster variables and the subset $\mathbf{X}$ of double canonical basis elements listed in Thm. 3.1. For example, mutating Q repeatedly at vertices 1, 3, 4, and 2 (in that order) produces the non-initial cluster variables $E_{12}, H_1, F_2, C_{+2}, F_{21}, A_2, E_2, \Gamma_2, E_{21}, F_1, H_2$, and C_{+1}. Furthermore, ϕ determines a bijection between the quantum clusters and the maximal quasi-commuting subsets of $\mathbf{X}$ listed at the beginning of Section 3.2. So ϕ takes quantum cluster monomials (a basis for $\mathcal{A}_{q^{1/2}}(Q, \Lambda)$) to double canonical basis elements and is therefore an isomorphism.

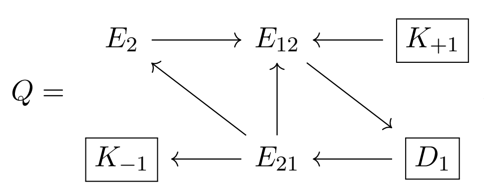
Chapter 4

The double canonical basis of $U_q(\tilde{\mathfrak{p}})$

Let $U_q(\tilde{\mathfrak{p}})$ be the subalgebra of $U_q(\widetilde{\mathfrak{sl}_3})$ generated by $F_1, K_{-1}, K_{+1}, K_{-2}, K_{+2}$, and U_q^+. Note that the modified Lusztig symmetry T_1 restricts to an automorphism $T_1 : U_q(\tilde{\mathfrak{p}}) \to U_q(\tilde{\mathfrak{p}})$. Through symmetry and the use of τ, the computations in this chapter apply to any of the four parabolic subalgebras of $U_q(\widetilde{\mathfrak{sl}_3})$.

In this chapter, we investigate the structure of the double canonical basis of $U_q(\tilde{\mathfrak{p}})$. For $l \geq -1$, we define elements A_l and B_l of $U_q(\tilde{\mathfrak{p}})$ called A-elements and B-elements, respectively. The main result of this section is the following theorem:

Theorem 4.1. *The A-elements and B-elements belong to the double canonical basis. Together with F_1 and E_1, they are the exchangeable quantum cluster variables for a quantum cluster algebra structure on $U_q(\tilde{\mathfrak{p}})$ with type $\widetilde{A_2}$ initial quiver*



where $K_{-1}, K_{+1}, K_{-2}, K_{+2}$, and $D_1 = F_1 \bullet E_{12}E_{21}$ are coefficients. The set of quantum clusters is preserved by the automorphism $T_1 : U_q(\tilde{\mathfrak{p}}) \to U_q(\tilde{\mathfrak{p}})$. For $m \geq 0$, quantum cluster *monomials from* $\{F_1, A_{2m-1}, A_{2m+1}, D_1, K_{\pm 1}, K_{\pm 2}\}$ *and* $\{F_1, A_{2m-1}, A_{2m+1}, D_1, K_{\pm 1}, K_{\pm 2}\}$ *are contained in the double canonical basis.*

The coefficients K_{-2} and K_{+2} never appear in exchange relations (that is, there are no arrows in the initial quiver Q whose source or target is a vertex corresponding to K_{-2} or K_{+2}). Therefore we omit these vertices from Q and ignore K_{-2} and K_{+2} in the proofs that follow. Thm. 4.1 is an immediate corollary of Prop.'s 4.21, 4.22, and 4.27. Although $C_1 = F_1 \bullet E_1$ is not a quantum cluster variable, we also show (Cor. 4.6) that *modified* quantum cluster monomials in $\{F_1, D_1, C_1, K_{\pm 1}, K_{\pm 2}\}$ and $\{D_1, C_1, E_1, K_{\pm 1}, K_{\pm 2}\}$ belong to the double canonical basis. Here we use "modified quantum cluster monomials" to mean the unique bar-invariant multiple of a power of $q^{1/2}$ with a monomial but with powers C_1^k replaced by the Chebyshev polynomials $C_1^{(k)}$ (see Ex. 2.15).

4.1 The image of ι and the element $F_1 \bullet E_{21}$

We begin with some observations about the linear inclusion $\iota : \mathcal{H}_q^+(\widetilde{\mathfrak{sl}_3}) \to U_q(\widetilde{\mathfrak{sl}_3})$. Observe that $\iota(E_{12}F_1) = E_{12}F_1$, and thus

$$\iota(F_1^r \circ E_{12}^r) = \iota((F_1 \circ E_{12})^r) = (F_1 \bullet E_{12})^r = F_1^r \bullet E_{12}^r.$$

Note also that that $F_1 \bullet E_{12}$ commutes with F_1 and E_{12}, and $(F_1 \bullet E_{12})E_2 = q^{-1}E_2(F_1 \bullet E_{12})$. We can now describe ι applied to all elements $F_1^r \circ b_+$ computed in the previous chapter:

$$\iota(q^{\frac{1}{2}s(a+c)}F_1^r(F_1 \circ E_{12})^a E_2^s E_{21}^c) = q^{\frac{1}{2}s(a+c)}F_1^r(F_1 \bullet E_{12})^a E_2^s E_{21}^c,$$

$$\iota(q^{\frac{1}{2}s(a+b+c)}(F_1 \circ E_{12})^a E_{12}^b E_2^s E_{21}^c) = q^{\frac{1}{2}s(a+b+c)}(F_1 \bullet E_{12})^a E_{12}^b E_2^s E_{21}^c,$$

$$\iota(q^{-\frac{1}{2}(a+b-c)s}(F_1 \circ E_{12})^a E_{12}^b E_1^s E_{21}^c) = q^{-\frac{1}{2}(a+b-c)s}(F_1 \bullet E_{12})^a E_{12}^b E_1^s E_{21}^c,$$

$$\iota(q^{-\frac{1}{2}(a+c)b}F_1^r(F_1 \circ E_{12})^a C_{+1}^b E_{21}^c) = q^{-\frac{1}{2}(a+c)b}F_1^r(F_1 \bullet E_{12})^a \iota(C_{+1}^b) E_{21}^c,$$

$$\iota(q^{-\frac{1}{2}(a+c)(b+s)}(F_1 \circ E_{12})^a C_{+1}^b E_1^s E_{21}^c) = q^{-\frac{1}{2}(a+c)(b+s)}(F_1 \bullet E_{12})^a \iota(C_{+1}^b) E_1^s E_{21}^c.$$

Observe also that F_1 and E_{21} do not commute in $U_q(\widetilde{\mathfrak{sl}_3})$. Applying σ to $E_{12}F_1 = F_1E_{12} + (q^{-1} - q)K_{+1} \diamond E_2$, we have

$$E_{21}F_1 = F_1E_{21} + (q - q^{-1})K_{-1} \diamond E_2$$

and thus

$$F_1 \bullet E_{21} = F_1E_{21} - q^{-1}K_{-1} \diamond E_2 = \iota(F_1 \circ E_{21}) - q^{-1}K_{-1} \diamond E_2.$$

Observe that $F_1 \bullet E_{21} = \sigma(F_1 \bullet E_{12})$. Applying σ to $(F_1 \bullet E_{12})^k$, we have

$$\begin{aligned}(F_1 \bullet E_{21})^k &= \sum_{j=0}^{k} (-1)^j q^{-j} \begin{bmatrix} k \\ j \end{bmatrix}_{q^{-2}} K_{-1}^j \diamond q^{\frac{1}{2}j(k-j)} E_2^j E_{21}^{k-j} \\ &= F_1^k \bullet E_{21}^k.\end{aligned}$$

It also follows that $F_1 \bullet E_{21}$ quasi-commutes with F_1, E_2, and E_{21}.

4.2 The central element D_1

In this section we show that the element $D_1 = F_1 \bullet E_{12}E_{21}$ is central in $U_q(\tilde{\mathfrak{p}})$ and that its powers belong to the double canonical basis. We also show that dual basis elements from U_q^+, as well as $C_1^{(k)}$, remain in the double canonical basis when multiplied by powers of D_1.

Although $F_1 \circ E_{12}$ and E_{21} commute in $\mathcal{H}_q^+(\mathfrak{sl}_3)$, $F_1 \bullet E_{12} = \iota(F_1 \circ E_{12})$ and E_{21} do not commute in $U_q(\widetilde{\mathfrak{sl}_3})$. Instead, we have

$$(F_1 \bullet E_{12})E_{21} = E_{21}(F_1 \bullet E_{12}) + (q^{-1} - q)K_{-1}q^{-\frac{1}{2}}E_2E_{12}$$

and thus

$$(F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12}$$

is bar-invariant. Since $\iota(F_1 \circ E_{12}E_{21}) = (F_1 \bullet E_{12})E_{21}$, we have

$$F_1 \bullet E_{12}E_{21} = (F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12}. \tag{4.1}$$

Lemma 4.2. *The element $F_1 \bullet E_{12}E_{21}$ is central in $U_q(\tilde{\mathfrak{p}})$.*

Proof. Commutation with $E_2, K_{\pm 1}$, and $K_{\pm 2}$ is immediate. For F_1, we have

$$\begin{aligned}
(F_1 \bullet E_{12}E_{21})F_1 &= ((F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12})F_1 \\
&= (F_1 \bullet E_{12})(F_1E_{21} + (q - q^{-1})K_{-1} \diamond E_2) - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2 \\
&\qquad \times (F_1E_{12} + (q^{-1} - q)K_{+1} \diamond E_2) \\
&= F_1(F_1 \bullet E_{12})E_{21} + (1 - q^{-2})K_{-1}q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_2 \\
&\qquad - K_{-1}q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_2 - q^{-1}K_{-1}K_{+1}E_2^2 \\
&= F_1(F_1 \bullet E_{12})E_{21} - q^{-2}K_{-1}q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_2 - q^{-1}K_{-1}K_{+1}E_2^2 \\
&= F_1(F_1 \bullet E_{12})E_{21} - q^{-3}K_{-1}(F_1q^{-\frac{1}{2}}E_2E_{12} - q^2K_{+1}E_2^2) - q^{-1}K_{-1}K_{+1}E_2^2 \\
&= F_1(F_1 \bullet E_{12}E_{21}).
\end{aligned}$$

For the E_1 computation, we first observe that

$$\begin{aligned}
E_1(F_1 \bullet E_{12}) &= (F_1E_1 + (q^{-1} - q)K_{+1} + (q - q^{-1})K_{-1})E_{12} - q^{-1}K_{+1}(E_{12} + qE_{21}) \\
&= F_1E_1E_{12} - qK_{+1}E_{12} - K_{+1}E_{21} + (q - q^{-1})K_{-1}E_{12} \\
&= q^{-1}F_1E_{12}E_1 - qK_{+1}(E_{12} + q^{-1}E_{21}) + (q - q^{-1})K_{-1}E_{12} \\
&= q^{-1}F_1E_{12}E_1 - qK_{+1}q^{-\frac{1}{2}}E_2E_1 + (q - q^{-1})K_{-1}E_{12} \\
&= q^{-1}(F_1 \bullet E_{12})E_1 + (q - q^{-1})K_{-1}E_{12}
\end{aligned}$$

and thus

$$E_1(F_1 \bullet E_{12}E_{21}) = E_1((F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12})$$

$$= q^{-1}(F_1 \bullet E_{12})E_1E_{21} + (q-q^{-1})K_{-1}E_{12}E_{21}$$
$$- qK_{-1}(q^{-1}E_{12} + E_{21})E_{12}$$
$$= (F_1 \bullet E_{12})E_{21}E_1 + (q-q^{-1})K_{-1}E_{12}E_{21} - q^{-1}K_{-1} \diamond E_{12}^2 - qK_{-1}E_{12}E_{21}$$
$$= (F_1 \bullet E_{12})E_{21}E_1 - q^{-1}K_{-1}(qE_{12} + E_{21})E_{12}$$
$$= (F_1 \bullet E_{12})E_{21}E_1 - q^{-1}K_{-1}q^{\frac{1}{2}}E_2E_1E_{12}$$
$$= (F_1 \bullet E_{12}E_{21})E_1.$$

□

Lemma 4.3. *For $r \geq 0$,*

$$(F_1 \bullet E_{12}E_{21})^r = \sum_{j=0}^{r}(-1)^j q^{-j} \begin{bmatrix} r \\ j \end{bmatrix}_{q^{-2}} K_{-1}^j \iota(F_1^{r-j} \circ q^{-\frac{1}{2}j^2}E_2^jE_{12}^rE_{21}^{r-j}) = F_1^r \bullet E_{12}^rE_{21}^r.$$

Proof. We use induction on r. The base case holds by (4.1). Suppose the claim holds for $r-1$. Since $(F_1 \bullet E_{12}E_{12})^{r-1}$ is central in $U_q(\tilde{\mathfrak{p}})$, we have

$$\begin{aligned}(F_1 \bullet E_{12}E_{21})^r &= ((F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12})(F_1 \bullet E_{12}E_{12})^{r-1} \\ &= (F_1 \bullet E_{12})(F_1 \bullet E_{12}E_{12})^{r-1}E_{21} - q^{-1}K_{-1}(F_1 \bullet E_{12}E_{12})^{r-1}q^{-\frac{1}{2}}E_2E_{12}.\end{aligned}$$

Using $\iota(F_1^{r-1-j} \circ q^{-\frac{1}{2}j^2}E_2^jE_{12}^{r-1}E_{21}^{r-1-j}) = q^{-\frac{3}{2}j^2}q^{j(r-1)}(F_1 \bullet E_{12})^{r-j-1}E_2^jE_{12}^jE_{21}^{r-1-j}$, our assumption gives

$$\begin{aligned}(F_1 \bullet E_{12}E_{21})^r &= \sum_{j=0}^{r-1}(-1)^j q^{-j}q^{-2j} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^{-2}} K_{-1}^j q^{-\frac{3}{2}j^2}q^{jr}(F_1 \bullet E_{12})^{r-j}E_2^jE_{12}^jE_{21}^{r-j} \\ &\quad + \sum_{j=0}^{r-1}(-1)^{j+1}q^{-j-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^{-2}} K_{-1}^{j+1}q^{-\frac{3}{2}(j+1)^2}q^{(j+1)r}(F_1 \bullet E_{12})^{r-1-j} \\ &\qquad \times E_2^{j+1}E_{12}^{j+1}E_{21}^{r-1-j} \\ &= \sum_{j=0}^{r}(-1)^j q^{-j}\left(q^{-2j}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^{-2}} + \begin{bmatrix} r-1 \\ j-1 \end{bmatrix}_{q^{-2}}\right) K_{-1}^j q^{-\frac{3}{2}j^2}q^{jr}(F_1 \bullet E_{12})^{r-j}\end{aligned}$$

$$\times E_2^j E_{12}^j E_{21}^{r-j}$$

$$= \sum_{j=0}^{r} (-1)^j q^{-j} \begin{bmatrix} r \\ j \end{bmatrix}_{q^{-2}} K_{-1}^j \iota(F_1^{r-j} \circ q^{-\frac{1}{2}j^2} E_2^j E_{12}^r E_{21}^{r-j}),$$

which completes the proof. ☐

Using the formula in Lemma 4.3, a straightforward computation shows:

Corollary 4.4. *Quantum cluster monomials in* $\{E_2, E_{12}, E_{21}, D_1\}$ *and* $\{E_1, E_{12}, E_{21}, D_1\}$ *belong to the double canonical basis.*

Lemma 4.5. *For* $r, k \in \mathbb{Z}_{\geq 0}$, $D_1^r C_1^{(k)}$ *is in the double canonical basis.*

Proof. We fix r and show by induction on s that $D_1^r \iota(C_{+1}^s)$ has correct triangularity. We claim that

$$D_1^r \iota(C_{+1}^s) = A_1^r \iota(C_{+1}^s) E_{21}^r + \sum_{j \in J} K_{-1}^{m_j} K_{+1}^{n_j} c_j(q^{-1}) b_j$$

where J is a finite index set, $c_j(q^{-1}) \in q^{-1}\mathbb{Z}[q^{-1}]$, and b_j has one of the following four forms:

$$(1) \quad b_j = q^{-\frac{1}{2}\alpha_j(2\beta_j - \alpha_j)} E_2^{\alpha_j} A_1^{r-\alpha_j} E_{12}^{\beta_j} E_{21}^{s-\beta_j},$$

where $\alpha_j \geq 1, \beta_j \geq 1$ and $-m_j + n_j + \beta_j \leq 0$,

$$(2) \quad b_j = q^{\frac{1}{2}\beta_j^2} F_1^{\alpha_j} E_2^{\beta_j} A_1^{r-\alpha_j-\beta_j} E_{21}^{r+\alpha_j},$$

where $\alpha_j \geq 0, \beta_j \geq 1$ and $-m_j + n_j \leq 0$,

$$(3) \quad b_j = q^{-\alpha_j(\alpha_j+\beta_j)} A_1^{r+\alpha_j} E_{12}^{\beta_j} E_{21}^{r-\beta_j-\alpha_j} E_1^{\alpha_j},$$

where $\alpha_j \geq 1, \beta_j \geq 1$ and $-m_j + n_j + \alpha_j + \beta_j \leq 0$, or

$$(4) \quad b_j = q^{-(\alpha_j^2+\beta_j r)} A_1^{r+\alpha_j} \iota(C_{+1}^{\beta_j}) E_{21}^{r-\alpha_j} E_1^{\alpha_j},$$

where $\alpha_j \geq 0, \beta_j \geq 0$ and $-m_j + n_j + \alpha_j \leq 0$. The claim for $s = 0$ is established in Lemma 4.3. For the inductive step, observe that

$$D_1^r \iota(C_{+1}^{s+1}) = F_1 D_1^r \iota(C_{+1}^s) E_1 - qK_{+1} D_1^r \iota(C_{+1}^s).$$

We begin by computing the $F_1 K_{-1}^{m_j} K_{+1}^{n_j} b_j E_1$ for each b_j. For notational simplicity, we write α in computations instead of α_j, etc. First, we have for type (1) terms

$$\begin{aligned}
&F_1 K_{-1}^m K_{+1}^n q^{-\frac{1}{2}\alpha(2\beta-\alpha)} E_2^\alpha A_1^{r-\alpha} E_{12}^\beta E_{21}^{r-\beta} E_1 \\
&= q^{-2m+2n} K_{-1}^m K_{+1}^n q^{-\frac{1}{2}\alpha(2\beta-\alpha)} E_2^\alpha A_1^{r-\alpha} (F_1 E_{12} - qK_{+1} \diamond E_2) E_{12}^{\beta-1} E_{21}^{r-\beta} E_1 \\
&\qquad + q^{-2m+2n} K_{-1}^m K_{+1}^{n+1} q^{-\frac{1}{2}\alpha(2\beta-\alpha)} q^{2\beta} E_2^\alpha A_1^{r-\alpha} E_{12}^{\beta-1} E_{21}^{r-\beta} q^{\frac{1}{2}} E_2 E_1 \\
&= q^{-2m+2n} K_{-1}^m K_{+1}^n q^{-\frac{1}{2}\alpha(2\beta-\alpha)} q^{2\beta-\alpha} E_2^{\alpha-1} A_1^{r-\alpha+1} E_{12}^{\beta-1} E_{21}^{r-\beta} (q^{\frac{1}{2}} E_{12} + q^{-\frac{1}{2}} E_{21}) \\
&\qquad + q^{-2m+2n} K_{-1}^m K_{+1}^{n+1} q^{-\frac{1}{2}\alpha(2\beta-\alpha)} q^{2\beta} E_2^\alpha A_1^{r-\alpha} E_{12}^{\beta-1} E_{21}^{r-\beta} (qE_{12} + E_{21}) \\
&= q^{-2m+2n+\beta} K_{-1}^m K_{+1}^n q^{-\frac{1}{2}(\alpha-1)(2\beta-\alpha+1)} E_2^{\alpha-1} A_1^{r-\alpha+1} E_{12}^\beta E_{21}^{r-\beta} \\
&\qquad + q^{-2m+2n+\beta-\alpha} K_{-1}^m K_{+1}^n q^{-\frac{1}{2}(\alpha-1)(2\beta-\alpha-1)} E_2^{\alpha-1} A_1^{r-\alpha+1} E_{12}^{\beta-1} E_{21}^{r-\beta+1} \\
&\qquad + q^{-2m+2n+2\beta+1} K_{-1}^m K_{+1}^{n+1} q^{-\frac{1}{2}\alpha(2\beta-\alpha)} E_2^\alpha A_1^{r-\alpha} E_{12}^\beta E_{21}^{r-\beta} \\
&\qquad + q^{-2m+2n+2\beta-\alpha} K_{-1}^m K_{+1}^{n+1} q^{-\frac{1}{2}\alpha(2\beta-\alpha-2)} E_2^\alpha A_1^{r-\alpha} E_{12}^{\beta-1} E_{21}^{r-\beta+1}.
\end{aligned}$$

If $-m_j + n_j + \beta < 0$, then each summand has the desired form. That is, either the terms remain of type (1) (when $\alpha_j, \beta_j > 1$) and $-m_{j+1} + n_{j+1} + \beta_{j+1} \leq 0$, or they are of types (2)-(4) and satisfy the corresponding assumptions there. If $-m_j + n_j + \beta = 0$, then every summand has the desired form except the third. Such terms, however, also appear in $qK_{+1} D_1^r \iota(C_{+1}^s)$ and are therefore cancelled in $D_1^r \iota(C_{+1}^{s+1})$. For types (2)-(4), we compute

$$\begin{aligned}
&F_1 K_{-1}^m K_{+1}^n F_1^\alpha q^{\frac{1}{2}\beta(2\alpha+\beta)} E_2^\beta A_1^{r-\alpha-\beta} E_{21}^{r+\alpha} E_1 \\
&= q^{-2m+2n-\beta-2\alpha} K_{-1}^m K_{+n}^n F_1^{\alpha+1} q^{\frac{1}{2}\beta(2\alpha+\beta)} E_2^{\beta-1} A_1^{r-\alpha-\beta} (q^{\frac{1}{2}} E_{12} + q^{-\frac{1}{2}} E_{21}) E_{21}^{r+\alpha} \\
&= q^{-2m+2n-\beta-2\alpha} K_{-1}^m K_{+1}^n F_1^\alpha q^{\frac{1}{2}\beta(2\alpha+\beta)} E_2^{\beta-1} A_1^{r-\alpha-\beta} q^{\frac{1}{2}} (F_1 E_{12} - qK_{+1} \diamond E_2) E_{21}^{r+\alpha}
\end{aligned}$$

$$
\begin{aligned}
&q^{-2m+2n+1}K_{-1}^{m}K_{+1}^{n+1}F_1^{\alpha}q^{\frac{1}{2}\beta(2\alpha+\beta)}E_2^{\beta}A_1^{r-\alpha-\beta}E_{21}^{r+\alpha}\\
&q^{-2m+2n-\beta-2\alpha}K_{-1}^{m}K_{+1}^{n}F_1^{\alpha+1}q^{\frac{1}{2}\beta(2\alpha+\beta)}q^{-\frac{1}{2}}E_2^{\beta-1}A_1^{r-\alpha-\beta}E_{21}^{r+\alpha+1}\\
=\ &q^{-2m+2n-\alpha}K_{-1}^{m}K_{+1}^{n}F_1^{\alpha}q^{\frac{1}{2}(\beta-1)(2\alpha+\beta-1)}E_2^{\beta-1}A_1^{r-\alpha-\beta+1}E_{21}^{r+\alpha}\\
&q^{-2m+2n+1}K_{-1}^{m}K_{+1}^{n+1}F_1^{\alpha}q^{\frac{1}{2}\beta(2\alpha+\beta)}E_2^{\beta}A_1^{r-\alpha-\beta}E_{21}^{r+\alpha}\\
&q^{-2m+2n-\beta-\alpha}K_{-1}^{m}K_{+1}^{n}F_1^{\alpha+1}q^{\frac{1}{2}(\beta-1)(2\alpha+\beta+1)}E_2^{\beta-1}A_1^{r-\alpha-\beta}E_{21}^{r+\alpha+1},
\end{aligned}
$$

$$
\begin{aligned}
&F_1K_{-1}^{m}K_{+1}^{n}q^{-\alpha(\alpha+\beta)}A_1^{r+\alpha}E_{12}^{\beta}E_{21}^{r-\beta-\alpha}E_1^{\alpha}E_1\\
&=q^{-2m+2n}K_{-1}^{m}K_{+1}^{n}q^{-\alpha(\alpha+\beta)}A_1^{r+\alpha}(F_1E_{12}-qK_{+1}\diamond E_2)E_{12}^{\beta-1}E_{21}^{r-\beta-\alpha}E_1^{\alpha+1}\\
&\qquad+q^{-2m+2n+2\alpha+2\beta}K_{-1}^{m}K_{+1}^{n+1}q^{-\alpha(\alpha+\beta)}A_1^{r+\alpha}E_{12}^{\beta-1}E_{21}^{r-\beta-\alpha}(qE_{12}+E_{21})E_1^{\alpha}\\
&=q^{-2m+2n}K_{-1}^{m}K_{+1}^{n}q^{-(\alpha+1)(\alpha+\beta)}A_1^{r+\alpha+1}E_{12}^{\beta-1}E_{21}^{r-\beta-\alpha}E_1^{\alpha+1}\\
&\qquad+q^{-2m+2n+2\alpha+2\beta+1}K_{-1}^{m}K_{+1}^{n+1}q^{-\alpha(\alpha+\beta)}A_1^{r+\alpha}E_{12}^{\beta}E_{21}^{r-\beta-\alpha}E_1^{\alpha}\\
&\qquad+q^{-2m+2n+\alpha+2\beta}K_{-1}^{m}K_{+1}^{n+1}q^{-\alpha(\alpha+\beta-1)}A_1^{r+\alpha}E_{12}^{\beta-1}E_{21}^{r-\beta-\alpha+1}E_1^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
&F_1K_{-1}^{m}K_{+1}^{n}q^{-(\alpha^2+\beta r)}A_1^{r+\alpha}\iota(C_{+1}^{\beta})E_{21}^{r-\alpha}E_1^{\alpha+1}\\
&=q^{-2m+2n-r+\alpha}K_{-1}^{m}K_{+1}^{n}q^{-(\alpha^2+\beta r)}A_1^{r+\alpha}F_1\iota(C_{+1}^{\beta})E_1E_{21}^{r-\alpha}E_1^{\alpha}\\
&=q^{-2m+2n-r+\alpha}K_{-1}^{m}K_{+1}^{n}q^{-(\alpha^2+\beta r)}A_1^{r+\alpha}(F_1\iota(C_{+1}^{\beta})E_1-qK_{+1}\iota(C_{+1}^{\beta}))E_{21}^{r-\alpha}E_1^{\alpha}\\
&\qquad+q^{-2m+2n+2\alpha+1}K_{-1}^{m}K_{+1}^{n+1}q^{-(\alpha^2+\beta r)}A_1^{r+\alpha}\iota(C_{+1}^{\beta})E_{21}^{r-\alpha}E_1^{\alpha}.
\end{aligned}
$$

In each case, as in type (1), any terms which do not have the desired form are cancelled in $D_1^r\iota(C_{+1}^{s+1})$ by terms in $qK_{+1}D_1^r\iota(C_{+1}^{s})$ (and thus all of $qK_{+1}D_1^r\iota(C_{+1}^{s})$ is cancelled). The remaining terms satisfy the requirements on $m_{j+1}, n_{j+1}, \alpha_{j+1}$, and β_{j+1}. □

Corollary 4.6. *Modified quantum cluster monomials in $\{F_1, C_1, D_1\}$ and $\{E_1, C_1, D_1\}$ belong to the double canonical basis.*

Proof. The proof for $\{F_1, C_1, D_1\}$ proceeds as for Lemma 4.5. The proof for $\{E_1, C_1, D_1\}$ is a routine computation using Lemma 3.13. □

Generalizing the proof of Lemma 4.5, we also conclude:

Lemma 4.7. *Quantum cluster monomials in* $\{F_1, E_2, A_1, D_1\}$ *are contained in the double canonical basis. Furthermore, although not bar-invariant,*

$$F_1^a q^{-\frac{1}{2}s(u+v)} q^{\frac{1}{2}uv} A_1^u E_2^v D_1^r C_1^{(s)}$$

and

$$F_1^a q^{-\frac{1}{2}sk} A_1^k D_1^r C_1^{(s)}$$

have correct triangularity.

4.3 A-elements and B-elements

In this section we introduce the A-elements and B-elements and prove that they are double canonical basis elements.

Recall that $F_1 \bullet E_{12} = F_1 E_{12} - qK_{+1} \diamond E_2$ and $F_1 \bullet E_{21} = F_1 E_{21} - qK_{-1} \diamond E_2$. Furthermore, $E_1(F_1 \bullet E_{12}) = q^{-1}(F_1 \bullet E_{12})E_1 + (q - q^{-1})K_{-1}E_{12}$ and thus

$$F_1 \bullet q^{\frac{1}{2}} E_1 E_{12} = q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_1 - q^{-1}K_{-1} \diamond E_2.$$

Definition 4.8. Let $A_{-1} = E_2, A_0 = E_{12}$ and $A_1 = F_1 \bullet E_{12}$. For $m \in \mathbb{Z}_{>0}$, define the *A-elements* by

$$A_{2m} = q^{-\frac{1}{2}} A_{2m-1} E_1 - q^{-1} K_{-1} \diamond A_{2m-2}$$

and let

$$A_{2m+1} = F_1 A_{2m} - qK_{+1} \diamond A_{2m-1}.$$

Definition 4.9. Let $B_{-1} = E_2, B_0 = E_{21}$ and $B_1 = F_1 \bullet E_{21}$. For $m \in \mathbb{Z}_{>0}$, let

$$B_{2m} = q^{\frac{1}{2}} B_{2m-1} E_1 - qK_{-1} \diamond A_{2m-2}$$

and let

$$B_{2m+1} = F_1 B_{2m} - q^{-1} K_{-1} \diamond B_{2m-1}.$$

We call these *B-elements.*

Note that for all $l \geq 0$, $\sigma(A_l) = B_l$. We show in Prop. 4.19/4.20 that each A_l and B_l belongs to the double canonical basis. First, we use the braid group action to establish bar-invariance.

Lemma 4.10. *For* $m \in \mathbb{Z}_{\geq 0}$,

$$T_1(A_{2m-1}) = K_{-1}^{-m} K_{+1}^{-m} A_{2m}$$

and

$$T_1(A_{2m}) = K_{-1}^{-m} K_{+1}^{-(m+1)} \diamond A_{2m+1}.$$

In particular, each A-element is preserved by the bar-involution.

Proof. We use induction on m. First, we have

$$\begin{aligned}
T_1(A_0) &= (q - q^{-1})^{-1} T_1(q^{\frac{1}{2}} E_2 E_1 - q^{-\frac{1}{2}} E_1 E_2) \\
&= (q - q^{-1})^{-1} (q^{-\frac{1}{2}} E_{12} K_{+1}^{-1} F_1 - q^{-\frac{3}{2}} K_{+1}^{-1} F_1 E_{12}) \\
&= (q - q^{-1})^{-1} K_{+1}^{-1} \diamond (qE_{12} F_1 - q^{-1} F_1 E_{12}) \\
&= (q - q^{-1})^{-1} K_{+1}^{-1} \diamond ((q - q^{-1}) F_1 E_{12} + (1 - q^2) K_{+1} \diamond E_2) \\
&= K_{+1}^{-1} \diamond A_1.
\end{aligned}$$

Next, we have

$$\begin{aligned}
T_1(A_1) &= T_1(F_1E_{12} - qK_{+1} \diamond E_2) \\
&= q^{-1}K_{-1}^{-1}E_1K_{+1}^{-1} \diamond (F_1 \bullet E_{12}) - qK_{+1}^{-1} \diamond E_{12} \\
&= K_{-1}^{-1}K_{+1}^{-1}(q^{\frac{1}{2}}E_1(F_1 \bullet E_{12}) - qK_{-1} \diamond E_{12}) \\
&= K_{-1}^{-1}K_{+1}^{-1}A_2,
\end{aligned}$$

which completes the base case. Suppose the lemma holds for all $l < m$. Then A_{2m-1} is bar-invariant since $T_1(A_{2(m-1)}) = K_{-1}^{-(m-1)}K_{+1}^{-m} \diamond A_{2m-1}$. Therefore

$$\begin{aligned}
T_1(A_{2m-1}) &= T_1(F_1A_{2m-2} - qK_{+1} \diamond A_{2m-3}) \\
&= q^{-1}K_{-1}^{-1}E_1T_1(A_{2m-2}) - qK_{+1}^{-1} \diamond T_1(A_{2m-3}) \\
&= q^{-1}K_{-1}^{-1}E_1(K_{-1}^{-(m-1)}K_{+1}^{-m} \diamond A_{2m-1}) - qK_{+1}^{-1} \diamond K_{-1}^{-(m-1)}K_{+1}^{-(m-1)}A_{2m-2} \\
&= K_{-1}^{-m}K_{+1}^{-m}q^{\frac{1}{2}}E_1A_{2m-1} - qK_{-1}^{-(m-1)}K_{+1}^{-m} \diamond A_{2m-2} \\
&= K_{-1}^{-m}K_{+1}^{-m}(q^{\frac{1}{2}}E_1A_{2m-1} - qK_{-1} \diamond A_{2m-2}) \\
&= K_{-1}^{-m}K_{+1}^{-m}(q^{-\frac{1}{2}}A_{2m-1}E_1 - q^{-1}K_{-1} \diamond A_{2m-2}) \\
&= K_{-1}^{-m}K_{+1}^{-m}A_{2m}
\end{aligned}$$

(in the second to last step we apply $\bar{\cdot}$, using the bar-invariance of $T_1(A_{2m-1})$ and the fact that $K_{-1}K_{+1}$ is central). So the first half of the lemma holds and A_{2m} is bar-invariant. Therefore

$$\begin{aligned}
T_1(A_{2m}) &= T_1(q^{-\frac{1}{2}}A_{2m-1}E_1 - q^{-1}K_{-1} \diamond A_{2m-2}) \\
&= q^{-\frac{1}{2}}T_1(A_{2m-1})q^{-1}K_{+1}^{-1}F_1 - q^{-1}K_{-1}^{-1} \diamond T_1(A_{2m-2}) \\
&= q^{-\frac{1}{2}}K_{-1}^{-m}K_{+1}^{-m}A_{2m}q^{-1}K_{+1}^{-1}F_1 - q^{-1}K_{-1}^{-1} \diamond (K_{-1}^{-(m-1)}K_{+1}^{-m} \diamond A_{2m-1}) \\
&= q^{-\frac{1}{2}}K_{-1}^{-m}K_{+1}^{-(m+1)}A_{2m}F_1 - q^{-1}K_{-1}^{-m}K_{+1}^{-m}A_{2m-1} \\
&= K_{-1}^{-m}K_{+1}^{-(m+1)} \diamond (A_{2m}F_1 - q^{-1}K_{+1} \diamond A_{2m-1})
\end{aligned}$$

$$= K_{-1}^{-m} K_{+1}^{-(m+1)} \diamond (F_1 A_{2m} - qK_{+1} \diamond A_{2m-1})$$
$$= K_{-1}^{-m} K_{+1}^{-(m+1)} \diamond A_{2m+1},$$

which completes the proof. □

Applying $\sigma \circ T_1 = T_1^{-1}$ to Lemma 4.10 immediately yields:

Corollary 4.11. *For all $m \in \mathbb{Z}_{\geq 0}$,*

$$T_1^{-1}(B_{2m-1}) = K_{-1}^{-m} K_{+1}^{-m} B_{2m}$$

and

$$T_1^{-1}(B_{2m}) = K_{-1}^{-(m+1)} K_{+1}^{-m} \diamond B_{2m+1}.$$

In particular, each B-element is preserved by the bar anti-involution.

The following lemmata establish quasi-commutation relations between the A-elements. The statements for B-elements follow by applying σ or $\bar{\cdot} \circ \sigma = *$.

Lemma 4.12. *For all $m \in \mathbb{Z}_{\geq 0}$, $A_{2m} E_1 = qE_1 A_{2m}$ and $F_1 A_{2m+1} = A_{2m+1} F_1$.*

Proof. We use induction on m. The base case holds since $E_{12}E_1 = qE_1E_{12}$ and $F_1(F_1 \bullet E_{12}) = (F_1 \bullet E_{12})F_1$. Assuming the lemma holds for $m-1$, we have

$$\begin{aligned} A_{2m}E_1 &= (q^{-\frac{1}{2}} A_{2m-1} E_1 - q^{-1} K_{-1} \diamond A_{2m-2}) E_1 \\ &= (q^{\frac{1}{2}} E_1 A_{2m-1} - qK_{-1} \diamond A_{2m-2}) E_1 \\ &= q^{\frac{3}{2}} E_1^2 A_{2m-1} - E_1 K_{-1} \diamond A_{2m-2} \\ &= qE_1 A_{2m}, \end{aligned}$$

$$\begin{aligned} F_1 A_{2m+1} &= F_1(F_1 A_{2m} - qK_{+1} \diamond A_{2m-1}) \\ &= F_1(A_{2m} F_1 - q^{-1} K_{+1} \diamond A_{2m-1}) \end{aligned}$$

$$= F_1 A_{2m} F_1 - q(K_{+1} \diamond A_{2m-1}) F_1$$
$$= A_{2m+1} F_1,$$

completing the proof. □

Corollary 4.13. *For all $m \in \mathbb{Z}_{\geq 0}$, $B_{2m}E_1 = q^{-1}E_1B_{2m}$ and $F_1B_{2m+1} = B_{2m+1}F_1$.*

Lemma 4.14. *For $m \in \mathbb{Z}_{\geq 0}$, we have the following quasi-commutation relations:*

$$A_{2m}A_{2m+1} = A_{2m+1}A_{2m},$$

$$A_{2m+1}A_{2m+3} = qA_{2m+3}A_{2m+1},$$

$$A_{2m+1}A_{2m+2} = qA_{2m+2}A_{2m+1},$$

$$A_{2m}A_{2m+2} = qA_{2m+2}A_{2m}.$$

Proof. Applying T_1 to $E_{12}(F_1 \bullet E_{12}) = (F_1 \bullet E_{12})E_{12}$ gives

$$(K_{+1}^{-1} \diamond A_1)K_{-1}^{-1}K_{+1}^{-1}A_2 = K_{-1}^{-1}K_{+1}^{-1}A_2(K_{+1}^{-1} \diamond A_1)$$
$$q^{-\frac{1}{2}}K_{-1}^{-1}K_{+1}^{-2}A_1A_2 = q^{\frac{1}{2}}K_{-1}^{-1}K_{+1}^{-2}A_2A_1$$
$$A_1A_2 = qA_2A_1.$$

Similarly, $E_2A_0 = qA_0E_2$, so

$$A_0K_{-1}^{-1}K_{+1}^{-1}A_2 = qK_{-1}^{-1}K_{+1}^{-1}A_2A_0$$
$$A_0A_2 = qA_2A_0.$$

Applying T_1 again, we have

$$(K_{+1}^{-1} \diamond A_1)(K_{-1}^{-1}K_{+1}^{-2} \diamond A_3) = q(K_{+1}^{-1}K_{+1}^{-2} \diamond A_3)(K_{+1}^{-1} \diamond A_1)$$
$$q^{-2}K_{-1}^{-1}K_{+1}^{-3}A_1A_3 = q^{-1}K_{-1}^{-1}K_{+1}^{-3}A_3A_1$$

$$A_1A_3 = qA_3A_1.$$

So the lemma holds for $m = 1$. The inductive step is performed similarly. □

Corollary 4.15. *For all $m \in \mathbb{Z}_{\geq 0}$, we have the following quasi-commutation relations:*

$$B_{2m}B_{2m+1} = B_{2m+1}B_{2m},$$

$$B_{2m+1}B_{2m+3} = q^{-1}B_{2m+3}B_{2m+1},$$

$$B_{2m+1}B_{2m+2} = q^{-1}B_{2m+2}B_{2m+1},$$

$$B_{2m}B_{2m+2} = q^{-1}B_{2m+2}B_{2m}.$$

To prove Prop. 4.19/4.20, we need closed forms for the A-elements and B-elements.

Lemma 4.16. *For all $m \in \mathbb{Z}_{>0}$,*

$$A_{2m} = q^{\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}E_{12} - q^m K_{+1} \diamond q^{-\frac{1}{2}(m-1)}(F_1 \bullet E_1)^{(m-1)}E_{21},$$

$$A_{2m+1} = q^{-\frac{1}{2}m}(F_1 \bullet E_{12})(F_1 \bullet E_1)^{(m)} - q^{-\frac{1}{2}(m+1)}K_{-1}K_{+1}E_2(F_1 \bullet E_1)^{(m-1)}.$$

Proof. We have

$$\begin{aligned} A_2 &= q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_1 - q^{-1}K_{-1} \diamond E_{12} \\ &= q^{-\frac{1}{2}}F_1E_{12}E_1 - qK_{+1}(q^{\frac{1}{2}}E_{12} + q^{-\frac{1}{2}}E_{21}) - q^{-\frac{1}{2}}K_{-1}E_{12} \\ &= q^{\frac{1}{2}}(F_1 \bullet E_1)E_{12} - qK_{+1} \diamond E_{21}. \end{aligned}$$

For A_3, we have

$$\begin{aligned} A_3 &= F_1A_2 - qK_{+1} \diamond A_1 \\ &= q^{\frac{1}{2}}(F_1 \bullet E_1)F_1E_{12} - q^2K_{+1} \diamond F_1E_{21} - qK_{+1} \diamond (F_1 \bullet E_{12}) \end{aligned}$$

$$= q^{\frac{1}{2}}(F_1 \bullet E_1)F_1E_{12} - qK_{+1} \diamond (F_1 \bullet E_{12} + qF_1E_{21})$$
$$= q^{\frac{1}{2}}(F_1 \bullet E_1)F_1E_{12} - qK_{+1} \diamond (q^{\frac{1}{2}}(F_1 \bullet E_1)E_2 + K_{-1} \diamond E_2)$$
$$= q^{\frac{1}{2}}(F_1 \bullet E_1)(F_1 \bullet E_{12}) - qK_{-1}K_{+1}E_2,$$

where in the second to last step we use the "exchange relation"

$$(F_1 \bullet E_1)E_2 = q^{-\frac{1}{2}}(F_1 \bullet E_{12}) + q^{\frac{1}{2}}(F_1 \bullet E_{21}).$$

(This is not a true exchange relation since neither $F_1 \bullet E_{12}$ nor $F_1 \bullet E_{21}$ quasi-commute with $F_1 \bullet E_1$.) Applying the bar involution completes the base case. Assuming the lemma holds for m, we have

$$\begin{aligned} T_1(A_{2m}) &= T_1(q^{\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}E_{12} - q^m K_{+1} \diamond q^{-\frac{1}{2}(m-1)}(F_1 \bullet E_1)^{(m-1)}E_{21}) \\ &= q^{\frac{1}{2}m}K_{+1}^{-m}K_{-1}^{-m}(F_1 \bullet E_1)^{(m)}K_{+1}^{-1} \diamond (F_1 \bullet E_{12}) \\ &\qquad - q^m K_{+1}^{-1} \diamond q^{-\frac{1}{2}(m-1)}K_{-1}^{-m+1}K_{+1}^{-m+1}(F_1 \bullet E_1)^{(m-1)}E_2 \\ &= K_{-1}^{-m}K_{+1}^{-m-1} \diamond (q^{\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}(F_1 \bullet E_{12}) - q^{\frac{1}{2}(m+1)}K_{-1}K_{+1}(F_1 \bullet E_1)^{(m-1)}E_2), \end{aligned}$$

$$\begin{aligned} T_1(A_{2m+1}) &= T_1(q^{\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}(F_1 \bullet E_{12}) - q^{\frac{1}{2}(m+1)}K_{-1}K_{+1}(F_1 \bullet E_1)^{(m-1)}E_2) \\ &= q^{\frac{1}{2}m}K_{-1}^{-m}K_{+1}^{-m}(F_1 \bullet E_1)^{(m)}K_{-1}^{-1}K_{+1}^{-1}(q^{\frac{1}{2}}(F_1 \bullet E_1)E_{12} - qK_{+1} \diamond E_{21}) \\ &\qquad - q^{\frac{1}{2}(m+1)}K_{-1}^{-1}K_{+1}^{-1}K_{-1}^{-m+1}K_{+1}^{-m+1}(F_1 \bullet E_1)^{(m-1)}E_{12} \\ &= K_{-1}^{-m-1}K_{+1}^{-m-1}(q^{\frac{1}{2}(m+1)}(F_1 \bullet E_1)^{(m)}(F_1 \bullet E_1)E_{12} - q^{\frac{1}{2}m+1}K_{+1} \diamond (F_1 \bullet E_1)E_{21} \\ &\qquad - q^{\frac{1}{2}(m+1)}K_{-1}K_{+1}(F_1 \bullet E_1)^{(m-1)}E_{12} \\ &= K_{-1}^{-m-1}K_{+1}^{-m-1}(q^{\frac{1}{2}(m+1)}(F_1 \bullet E_1)^{(m+1)}E_{12} - q^{\frac{1}{2}m+1}K_{+1} \diamond (F_1 \bullet E_1)E_{21}), \end{aligned}$$

which completes the proof. □

Corollary 4.17. *For all $m \in \mathbb{Z}_{>0}$,*

$$B_{2m} = q^{-\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}E_{21} - q^{-m}K_{-1} \diamond q^{\frac{1}{2}(m-1)}(F_1 \bullet E_1)^{(m-1)}E_{12},$$

$$B_{2m+1} = q^{-\frac{1}{2}m}(F_1 \bullet E_1)^{(m)}(F_1 \bullet E_{21}) - q^{-\frac{1}{2}(m+1)}K_{-1}K_{+1}(F_1 \bullet E_1)^{(m-1)}E_2.$$

Lemma 4.18. *For $r \in \mathbb{Z}_{>0}$,*

$$q^{\frac{1}{2}r}\iota(F_1^r \circ E_1^r)E_{12} = \iota(F_1^r \circ q^{\frac{1}{2}r}E_1^rE_{12}) + q^rK_{+1} \diamond \iota(F_1^{r-1} \circ q^{-\frac{1}{2}(r-1)}E_1^{r-1}E_{21}),$$

$$q^{-\frac{1}{2}r}E_2\iota(F_1^r \circ E_1^r) = \iota(F_1^r \circ q^{\frac{1}{2}(r-1)}E_1^{r-1}E_{12}) + q^{-r}\iota(F_1^r \circ q^{-\frac{1}{2}(r-1)}E_1^{r-1}E_{21}).$$

Proof. For the first equation, we compute

$$\begin{aligned}
&\iota(F_1^r \circ q^{\frac{1}{2}r}E_1^rE_{12}) \\
&= q^{-\frac{1}{2}r}(F_1 \bullet E_{12})\iota(F_1^{r-1} \circ E_1^{r-1})E_1 \\
&= q^{-\frac{1}{2}r}\sum_{j=0}^{r-1}(-1)^jq^{2j}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^jF_1^{r-1-j}(F_1E_{12} - qK_{+1} \diamond E_2)E_1^{r-j} \\
&= q^{-\frac{1}{2}r}\sum_{j=0}^{r-1}\Bigg((-1)^jq^{r+j}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^jF_1^{r-j}E_1^{r-j}E_{12} \\
&\qquad +(-1)^{j+1}q^{2j+1}q^{2(r-1-j)}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^{j+1}F_1^{r-1-j}(qE_{12} + E_{21})E_1^{r-j-1}\Bigg) \\
&= q^{\frac{1}{2}r}\sum_{j=0}^{r-1}\Bigg((-1)^jq^j\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^jF_1^{r-j}E_1^{r-j}E_{12} \\
&\qquad +(-1)^{j+1}q^{2(r-j-1)}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^{j+1}F_1^{r-1-j}E_1^{r-1-j}E_{12}\Bigg) \\
&\qquad - q^{-\frac{1}{2}r}\sum_{j=0}^{r-1}(-1)^jq^{r+j}\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^{j+1}F_1^{r-1-j}E_1^{r-1-j}E_{21} \\
&= q^{\frac{1}{2}r}\sum_{j=0}^{r-1}(-1)^jq^j\left(\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} + q^{2(r-j)}\begin{bmatrix} r-1 \\ j-1 \end{bmatrix}_{q^2}\right)K_{+1}^jF_1^{r-j}E_1^{r-j}E_{12} \\
&\qquad - q^rK_{+1} \diamond q^{-\frac{1}{2}(r-1)}\sum_{j=0}^{r-1}(-1)^jq^j\begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} K_{+1}^jF_1^{r-1-j}E_1^{r-1-j}E_{21}
\end{aligned}$$

$$= q^{\frac{1}{2}r} \sum_{j=0}^{r-1} (-1)^j q^j \begin{bmatrix} r \\ j \end{bmatrix}_{q^2} K_{+1}^j F_1^{r-j} E_1^{r-j} E_{12}$$

$$- q^r K_{+1} \diamond \iota(F_1^{r-1} \circ q^{-\frac{1}{2}(r-1)} E_1^{r-1} E_{21})$$

$$= q^{\frac{1}{2}r} \iota(F_1^r \circ E_1^r) E_{12} - q^r K_{+1} \diamond \iota(F_1^{r-1} \circ q^{-\frac{1}{2}(r-1)} E_1^{r-1} E_{21}).$$

For the second equation, it follows from Lemma 3.13 that in $\mathcal{H}_q^+(\mathfrak{sl}_3)$,

$$\begin{aligned} q^{-\frac{1}{2}r} E_2(F_1^r \circ E_1^r) &= q^{-\frac{1}{2}r} E_2(F_1 \circ E_1)(F_1^{r-1} \circ E_1^{r-1}) \\ &= q^{-\frac{1}{2}r}(q^{\frac{1}{2}}(F_1 \circ E_{12}) + q^{-\frac{1}{2}} F_1 E_{21})(F_1^{r-1} \circ E_1^{r-1}) \\ &= q^{-\frac{1}{2}(r-1)}(F_1 \circ E_{12})(F_1^{r-1} \circ E_1^{r-1}) + q^{-r} q^{-\frac{1}{2}(r-1)} F_1(F_1^{r-1} \circ E_1^{r-1}) E_{21} \\ &= (F_1^r \circ q^{\frac{1}{2}(r-1)} E_1^{r-1} E_{12}) + q^{-r}(F_1^r \circ q^{-\frac{1}{2}(r-1)} E_1^{r-1} E_{21}). \end{aligned}$$

The result follows since $\iota(E_2(F_1^r \circ E_1^r)) = E_2 \iota(F_1^r \circ E_1^r)$. □

Proposition 4.19. *The A-elements belong to the double canonical basis. Specifically,*

$$A_{2m} = F_1^m \bullet b_+(m, 0, 1, 0)$$

and

$$A_{2m+1} = F_1^{m+1} \bullet b_+(m, 0, 1, 0).$$

Proof. We have already established that the A-elements are bar-invariant. It remains to show that they have correct triangularity. Using Lemma 4.18, we compute

$$q^{\frac{1}{2}m}(F_1 \bullet E_1)^{(m)} E_{12}$$

$$= q^{\frac{1}{2}m} \sum_{\substack{0 \le i \le j \\ i+j \le m}} (-1)^j q^{-j-i^2} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} \left(K_{-1}^j K_{+1}^i \diamond \iota(F_1^{m-i-j} \circ E_1^{m-i-j}) \right) E_{12}$$

$$= \sum_{\substack{0 \le i \le j \\ i+j \le m}} (-1)^j q^{-j-i^2+i} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^j K_{+1}^i \diamond q^{\frac{1}{2}(m-i-j)} \iota(F_1^{m-i-j} \circ E_1^{m-i-j}) E_{12}$$

$$= \sum_{\substack{0\le i\le j\\ i+j\le m}} (-1)^j q^{-j-i^2+i} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^j K_{+1}^i \diamond \iota(F_1^{m-i-j} \circ q^{\frac{1}{2}(m-i-j)} E_1^{m-i-j} E_{12})$$

$$+ \sum_{\substack{0\le i\le j\\ i+j\le m-1}} (-1)^j q^{m-2j-i^2} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}}$$

$$\times K_{-1}^j K_{+1}^{i+1} \diamond \iota(F_1^{m-i-j-1} \circ q^{-\frac{1}{2}(m-i-j-1)} E_1^{m-i-j-1} E_{21}).$$

We also have

$$q^m K_{+1} \diamond q^{-\frac{1}{2}(m-1)} (F_1 \bullet E_1)^{(m-1)} E_{21}$$

$$= \sum_{\substack{0\le i\le j\\ i+j\le m-1}} (-1)^j q^{m-2j-i^2} \begin{bmatrix} m-1-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}}$$

$$\times K_{-1}^j K_{+1}^{i+1} \diamond \iota(F_1^{m-1-i-j} \circ q^{-\frac{1}{2}(m-1-i-j)} E_1^{m-1-i-j} E_{21}).$$

Combining these expressions using Lemma 4.16, we have

$$A_{2m} = q^{\frac{1}{2}m} (F_1 \bullet E_1)^{(m)} E_{12} - q^m K_{+1} \diamond q^{-\frac{1}{2}(m-1)} (F_1 \bullet E_1)^{(m-1)} E_{21}$$

$$= \sum_{\substack{0\le i\le j\\ i+j\le m}} (-1)^j q^{-j-i^2+i} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^j K_{+1}^i \diamond \iota(F_1^{m-i-j} \circ q^{\frac{1}{2}(m-i-j)} E_1^{m-i-j} E_{12})$$

$$+ \sum_{\substack{0\le i\le j\\ i+j\le m-1}} (-1)^j q^{m-2j-i^2} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} \left(\begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} - \begin{bmatrix} m-1-i \\ j \end{bmatrix}_{q^{-2}} \right)$$

$$\times K_{-1}^j K_{+1}^{i+1} \diamond \iota(F_1^{m-1-i-j} \circ q^{-\frac{1}{2}(m-1-i-j)} E_1^{m-1-i-j} E_{21})$$

$$= \sum_{\substack{0\le i\le j\\ i+j\le m}} (-1)^j q^{-j-i^2+i} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^j K_{+1}^i \diamond \iota(F_1^{m-i-j} \circ q^{\frac{1}{2}(m-i-j)} E_1^{m-i-j} E_{12})$$

$$+ \sum_{\substack{0\le i\le j\\ i+j\le m-1}} (-1)^j q^{-m-i^2+2i} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} \begin{bmatrix} m-1-i \\ j-1 \end{bmatrix}_{q^{-2}}$$

$$\times K_{-1}^j K_{+1}^{i+1} \diamond \iota(F_1^{m-1-i-j} \circ q^{-\frac{1}{2}(m-1-i-j)} E_1^{m-1-i-j} E_{21}),$$

and thus A_{2m} has correct triangularity. Here we used the Gaussian binomial coefficient identity

$$\begin{bmatrix} r \\ s \end{bmatrix}_\nu - \begin{bmatrix} r-1 \\ s \end{bmatrix}_\nu = \nu^{r-s} \begin{bmatrix} r-1 \\ s-1 \end{bmatrix}_\nu .$$

For the A_{2m+1} computation, we have

$$\begin{aligned}
&q^{-\frac{1}{2}m}(F_1 \bullet E_{12})(F_1 \bullet E_1)^{(m)} \\
&= q^{-\frac{1}{2}m} \sum_{\substack{0\le i\le j \\ i+j\le m}} (-1)^j q^{-j-i^2} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} (F_1 \bullet E_{12}) K_{-1}^i K_{+1}^j \iota(F_1^{m-i-j} \circ E_1^{m-i-j}) \\
&= \sum_{\substack{0\le i\le j \\ i+j\le m}} (-1)^j q^{-2j-i^2} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^i K_{+1}^j \diamond q^{-\frac{1}{2}(m-i-j)} (F_1 \bullet E_{12}) \iota(F_1^{m-i-j} \circ E_1^{m-i-j}) \\
&= \sum_{\substack{0\le i\le j \\ i+j\le m}} (-1)^j q^{-2j-i^2} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^i K_{+1}^j \diamond \iota(F_1^{m+1-i-j} \circ q^{\frac{1}{2}(m-i-j)} E_1^{m-i-j} E_{12})
\end{aligned}$$

and using Lemma 4.18,

$$\begin{aligned}
&q^{-\frac{1}{2}(m+1)} K_{-1} K_{+1} E_2 (F_1 \bullet E_1)^{m-1} \\
&= \sum_{\substack{0\le i\le j \\ i+j\le m-1}} (-1)^j q^{-2j-i^2-1} \begin{bmatrix} m-1-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^{j+1} K_{+1}^{i+1} \diamond q^{-\frac{1}{2}(m-1-i-j)} E_2 \\
&\qquad \times \iota(F_1^{m-1-i-j} \circ E_1^{m-1-i-j}) \\
&= \sum_{\substack{0\le i\le j \\ i+j\le m-1}} (-1)^j q^{-2j-i^2-1} \begin{bmatrix} m-1-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_{-1}^{j+1} K_{+1}^{i+1} \\
&\qquad \diamond \Big(\iota(F_1^{m-1-i-j} \circ q^{\frac{1}{2}(m-2-i-j)} E_1^{m-2-i-j} E_{12}) \\
&\qquad\qquad + q^{-(m-1-i-j)} \iota(F_1^{m-1-i-j} \circ q^{-\frac{1}{2}(m-2-i-j)} E_1^{m-2-i-j} E_{21}) \Big) .
\end{aligned}$$

Combining these expressions using Lemma 4.16 shows that A_{2m+1} has correct triangularity.

□

Proposition 4.20. *The B-elements belong to the double canonical basis. Specifically,*

$$B_{2m} = F_1^m \bullet b_+(m,0,0,1)$$

and

$$B_{2m+1} = F_1^{m+1} \bullet b_+(m,0,0,1).$$

Proof. As for A_{2m}, the claim for B_{2m} follows from Lemma 4.18. For B_{2m+1}, we first note that

$$\begin{aligned} B_{2m+1} = q^{-\frac{1}{2}m} F_1 (F_1 \bullet E_1)^{(m)} E_{21} - q^{-1} K_{-1} \diamond q^{-\frac{1}{2}m} (F_1 \bullet E_1)^{(m)} E_2 \\ - q^{-\frac{1}{2}(m+1)} K_{-1} K_{+1} (F_1 \bullet E_1)^{(m-1)} E_2. \end{aligned}$$

An argument similar to the second part of the proof of Lemma 4.18 shows

$$q^{-\frac{1}{2}r} \iota(F_1^r \circ E_1^r) E_2 = q^{-\frac{1}{2}(r-1)-r} (F_1 \bullet E_{12}) \iota(F_1^{r-1} \circ E_1^{r-1}) + q^{-\frac{1}{2}(r-1)} F_1 (F_1^{r-1} \circ E_1^{r-1}) E_{21},$$

from which the claim for B_{2m+1} follows. □

4.4 The quantum cluster structure

From the quasi-commutation results of the previous section, we conclude that the A and B-elements may grouped into sets $\mathbf{X}_i, \widehat{\mathbf{X}}_i, i \in \mathbb{Z}$ consisting of mutually quasi-commuting elements as follows:

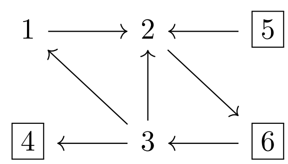
$$\mathbf{X}_i = \begin{cases} \{E_{12}, E_2, E_{21}\} & i = 0 \\ \{A_i, A_{i-1}, A_{i-2}\} & i \in \mathbb{Z}_{>0} \\ \{B_i, B_{i+1}, B_{i+2}\} & i \in \mathbb{Z}_{<0} \end{cases},$$

$$\widehat{\mathbf{X}}_i = \begin{cases} \{E_{12}, E_{21}, E_1\} & i = 0 \\ \{A_i, A_{i-2}, E_1\} & i \in 2\mathbb{Z}_{>0} \\ \{A_i, A_{i-2}, F_1\} & i \in 2\mathbb{Z}_{>0} + 1 \\ \{B_i, B_{i+2}, E_1\} & i \in 2\mathbb{Z}_{<0} \\ \{B_i, B_{i+2}, F_1\} & i \in 2\mathbb{Z}_{<0} + 1 \end{cases} .$$

In this section, we prove that $U_q(\tilde{\mathfrak{p}})$ is isomorphic to the quantum cluster algebra described in Thm. 4.1 and that the sets $\mathbf{X}_i, \widehat{\mathbf{X}}_i, i \in \mathbb{Z}$ are the quantum clusters.

Proposition 4.21. *The algebra $U_q(\tilde{\mathfrak{p}})$ is isomorphic to the quantum cluster algebra with initial quiver*

$Q =$



.

and quasi-commutation matrix

$$\Lambda = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. We have

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

and thus $B^T \Lambda = \begin{bmatrix} 2I_3 & 0_{3,3} \end{bmatrix}$, so the compatibility condition is satisfied. In the notation of Thm. 2.6, $\widehat{\mathcal{A}}_{q^{1/2}}(Q, \Lambda)$ is generated as an algebra by

$$\{X_1, X_2, X_3, X_4^{\pm 1}, X_5^{\pm 1}, X_6^{\pm}, X_1', X_2', X_3'\}$$

subject to the exchange relations

$$X_1X_1' = q^{\frac{1}{2}}X_2 + q^{-\frac{1}{2}}X_3,$$

$$X_2X_2' = X_6 + q^{-\frac{1}{2}}X_1X_3X_5,$$

$$X_3X_3' = X_6 + q^{\frac{1}{2}}X_1X_2X_4.$$

Recall that

$$D_1 = A_1E_{21} - q^{-1}K_{-1}q^{-\frac{1}{2}}E_2E_{12} = E_{21}A_1 - q^{\frac{1}{2}}E_2E_{12}K_{-}.$$

Applying σ gives $D_1 = E_{12}B_1 - q^{-\frac{1}{2}}E_2E_{21}K_{+1}$. Therefore the assignments

$$X_1 \mapsto E_2, \quad X_2 \mapsto E_{12}, \quad X_3 \mapsto E_{21},$$

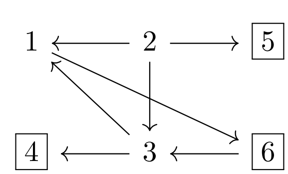
$$X_1' \mapsto E_1, \quad X_2' \mapsto B_1, \quad X_3' \mapsto A_1,$$

$$X_4 \mapsto K_{-1}, \quad X_5 \mapsto K_{+1}, \quad X_6 \mapsto D_1$$

determine an algebra homomorphism (after extending scalars)

$$\phi : \widehat{\mathcal{A}}_{q^{1/2}}(Q, \Lambda) \to \widehat{U}_q(\tilde{\mathfrak{p}}),$$

where $\widehat{U}_q(\tilde{\mathfrak{p}})$ denotes $U_q(\tilde{\mathfrak{p}})$ localized at the multiplicative submonoid generated by K_{-1}, K_{+1}, and D_1. Let $X^{2,3}$ represent the quantum cluster variable obtained by mutating Q at vertex 2 and then again at vertex 3. Mutation at vertex 2 produces



and thus

$$X^{2,3} = X_3^{-1}(X_2' + q^{\frac{1}{2}} X_4 X_1)$$
$$= X_3^{-1} X_2^{-1}(X_6 + q^{-\frac{1}{2}} X_1 X_3 X_5) + X_3^{-1} q^{\frac{1}{2}} X_4 X_1.$$

We immediately have $X_1 X^{2,3} = X^{2,3} X_1$. Furthermore,

$$X_1' X^{2,3} = X_1' X_3^{-1} X_2^{-1} X_6 + q^{-1} X_3^{-1} X_2^{-1}(X_2 + qX_3) X_3 X_5 + q X_3^{-1} X_4 (X_2 + qX_3),$$

$$X^{2,3} X_1' = X_1' X_3^{-1} X_2^{-1} X_6 + X_3^{-1} X_2^{-1}(qX_2 + X_3) X_3 X_5 + X_3^{-1} X_4 (qX_2 + X_3),$$

and thus

$$X_1' X^{2,3} - X^{2,3} X_1' = (q^{-1} - q) X_5 + (q - q^{-1}) X_4.$$

Therefore the assignments

$$E_1 \mapsto X_1', \quad E_2 \mapsto X_1, \quad K_{-1} \mapsto X_4, \quad K_{+1} \mapsto X_5, \quad F_1 \mapsto X^{2,3}$$

determine an algebra homomorphism $\phi' : \widehat{U}_q(\tilde{\mathfrak{p}}) \to \widehat{\mathcal{A}}_{q^{1/2}}(Q, \Lambda)$. Since

$$X_3 X^{2,3} = X_2' + q^{\frac{1}{2}} X_4 X_1,$$

we have

$$E_{21} \phi(X^{2,3}) = B_1 + q^{\frac{1}{2}} K_{-1} E_2 = E_{21} F_1,$$

so $\phi(X^{2,3}) = F_1$ (since $\mathcal{A}_{q^{1/2}}(Q, \Lambda)$ is a domain). Therefore $\phi = \phi^{-1}$ and ϕ is an isomorphism which restricts to an isomorphism $\mathcal{A}_{q^{1/2}}(Q, \Lambda) \to U_q(\tilde{\mathfrak{p}})$. □

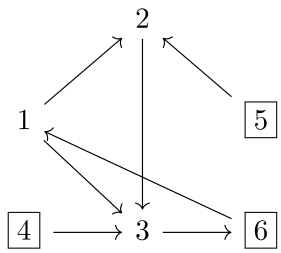
Proposition 4.22. *The A-elements and B-elements are quantum cluster variables.*

Proof. Idenify $\mathcal{A}_{q^{1/2}}(Q, \Lambda) \simeq U_q(\tilde{\mathfrak{p}})$ via the isomorphism in the proof of Prop. 4.21. Let μ_1, μ_2, and μ_3 denote the operation of mutation at vertices 1, 2, and 3, respectively, thought of as functions on the set of quantum seeds. Consider the mutations

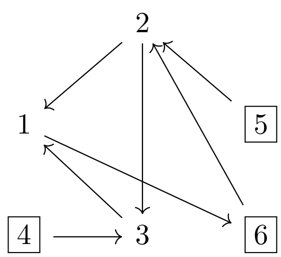
$$S = \{\mu_3, \mu_1\mu_3, \mu_2\mu_1\mu_3, \mu_3\mu_2\mu_1\mu_3, \mu_1\mu_3\mu_2\mu_1\mu_3, (\mu_2\mu_1\mu_3)^2, \dots\}$$

and let Y_l denote the l-th quantum cluster variable produced from the initial quantum seed in Prop. 4.21 by the l-th element of S.

The mutations μ_3 and $\mu_1\mu_3$ produce the quivers/quasi-commutation matrices



$$, \quad \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$



$$, \quad \begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively, and produce the variables Y_1, Y_2. We know from the proof of 4.21 that $Y_1 = A_1$. From these quivers/quasi-commutation matrices, we determine the exchange relations

$$E_2Y_2 = D_1 + qE_{12}A_1,$$

$$E_{12}Y_3 = q^{\frac{1}{2}}K_{+1}D_1 + qY_2A_1$$

Recall that $A_1E_{21} = D_1 + q^{-\frac{3}{2}}K_{-1}E_2E_{12}$. Applying T_1 and using Lemma 4.10, we have

$$\begin{aligned} T_1(A_1E_{21}) &= T_1(D_1 + q^{-\frac{3}{2}}K_-1E_2E_{12}) \\ K_{-1}^{-1}K_{+1}^{-1}A_2E_2 &= K_{-1}^{-1}K_{+1}^{-1}D_1 + q^{-\frac{3}{2}}K_{-1}^{-1}E_{12}(K_{+1}^{-1} \diamond A_1) \end{aligned}$$

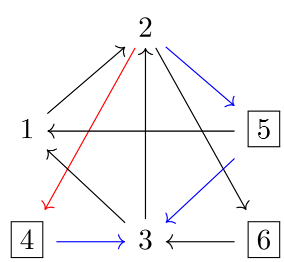
$$K_{-1}^{-1}K_{+1}^{-1}A_2E_2 = K_{-}^{-1}K_{+1}^{-1}(D_1 + q^{-1}E_{12}A_1),$$

and thus $A_2E_2 = D_1 + q^{-1}E_{12}A_1$. Applying the bar involution, we see that this is the first exchange relation above. Hence $Y_2 = A_2$. Through another application of T_1, we have

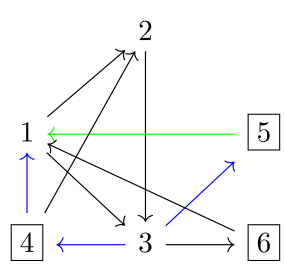
$$\begin{aligned} T_1(A_2E_2) &= T_1(D_1 + q^{-1}E_{12}A_1) \\ (K_{-1}^{-1}K_{+2}^{-2} \diamond A_3)E_{12} &= K_{-1}^{-1}K_{+1}^{-1}D_1 + q^{-1}(K_{+1}^{-1} \diamond A_1)K_{-1}^{-1}K_{+1}^{-1}(A_2) \\ K_{-1}^{-1}K_{+2}^{-2}A_3E_{12} &= K_{-1}^{-1}K_{+1}^{-2}(q^{\frac{1}{2}}K_{+1}D_1 + q^{-1}A_1A_2), \end{aligned}$$

and thus $A_3E_{12} = q^{\frac{1}{2}}K_{+1}D_1 + q^{-1}A_1A_2$. Applying bar, this is equivalent to the second exchange relation above. Hence $Y_3 = A_3$.

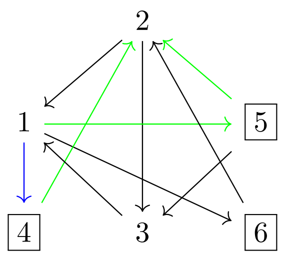
Through induction, we conclude that for $k \in \mathbb{Z}_{\geq 0}$, applying the mutations $(\mu_2\mu_1\mu_3)^{2k+1}$, $\mu_3(\mu_2\mu_1\mu_3)^{2k+1}$, and $\mu_1\mu_3(\mu_2\mu_1\mu_3)^{2k+1}$ to the initial quiver produces the quivers and quasi-commutation matrices



$$, \quad \begin{bmatrix} 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$



$$, \quad \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$



$$\begin{bmatrix} 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

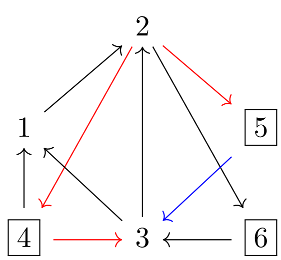
respectively, where a red arrow has multiplicity $3k$, a blue arrow has multiplicity $3k+1$, and a green arrow has multiplicity $3k+2$. The cluster variables produced are Y_{6k+3}, Y_{6k+4}, and Y_{6k+5}, respectively. The corresponding exchange relations are

$$Y_{6k+1}Y_{6k+4} = K_{-1}^{k+1}K_{+1}^{k+1}D_1 + qY_{6k+2}Y_{6k+3},$$

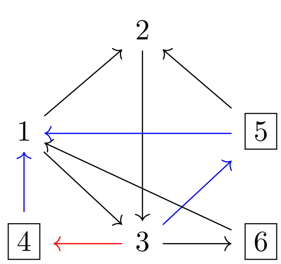
$$Y_{6k+2}Y_{6k+5} = q^{-\frac{1}{2}}K_{-1}^{k+1}K_{+1}^{k+2}D_1 + Y_{6k+3}Y_{6k+4},$$

$$Y_{6k+3}Y_{6(k+1)} = K_{-1}^{k+2}K_{+1}^{k+2}D_1 + qY_{6k+5}Y_{6k+4}.$$

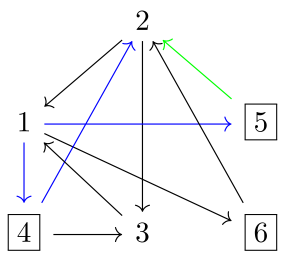
Similarly, for $k \in \mathbb{Z}_{>0}$, applying the mutations $(\mu_2\mu_1\mu_3)^{2k}$, $\mu_3(\mu_2\mu_1\mu_3)^{2k}$, and $\mu_1\mu_3(\mu_2\mu_1\mu_3)^{2k}$ to the initial quiver in Prop. 4.21 produces the quivers/quasi-commutation matrices



$$\begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$



$$\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$



$$\begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively, where a red arrow has multiplicity $3k-1$, a blue arrow has multiplicity $3k$, and a green arrow has multiplicity $3k+1$. The cluster variables produced are Y_{6k}, Y_{6k+1}, and Y_{6k+2}, respectively. The corresponding exchange relations are

$$Y_{6k-2}Y_{6k+1} = q^{-\frac{1}{2}}K_{-1}^{3k-1}K_{+1}^{3k}D_1 + Y_{6k-1}Y_{6k},$$

$$Y_{6k-1}Y_{6k+2} = K_{-1}^{3k}K_{+1}^{3k}D_1 + qY_{6k}Y_{6k+1},$$

$$Y_{6k}Y_{6k+3} = q^{-\frac{1}{2}}K_{-1}^{3k}K_{+1}^{3k+1}D_1 + qY_{6k+2}Y_{6k+1}.$$

We claim that for all $l \in \mathbb{Z}_{>1}, Y_l = A_l$. Applying T_1 to $A_3E_{12} = q^{\frac{1}{2}}K_{+1}D_1 + q^{-1}A_1A_2$,, we have

$$\begin{aligned} T_1(A_3E_{12}) &= T_1(q^{\frac{1}{2}}K_{+1}D_1 + q^{-1}A_1A_2) \\ K_{-1}^{-2}K_{+1}^{-2}A_4(K_{+1}^{-1} \diamond A_1) &= K_{+1}^{-1}D_1 + q^{-1}K_{-1}^{-1}K_{+1}^{-1}A_2(K_{-1}^{-1}K_{+1}^{-2} \diamond A_3) \\ K_{-1}^{-2}K_{+1}^{-3}A_4A_1 &= K_{-1}^{-2}K_{+1}^{-3}(K_{-1}K_{+1}D_1 + q^{-1}A_2A_3), \end{aligned}$$

and thus

$$A_4A_1 = K_{-1}K_{+1}D_1 + q^{-1}A_2A_3.$$

Using the same technique, an inductive argument shows that for all $m \geq 0$,

$$A_{2m+3}A_{2m} = q^{\frac{1}{2}}K_{-1}^{m}K_{+1}^{m+1}D_1 + q^{-1}A_{2m+1}A_{2m+2}$$

$$A_{2m+4}A_{2m+1} = K_{-1}^{m}K_{+1}^{m}D_1 + q^{-1}A_{2m+2}A_{2m+3}.$$

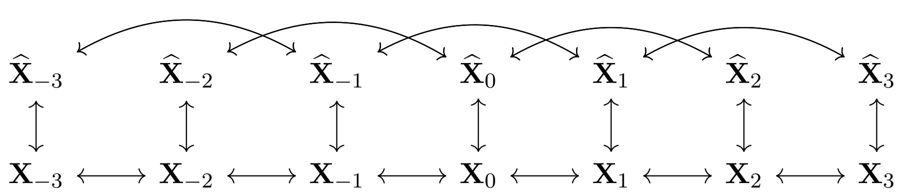
Applying the bar involution, we see that these are the exchange relations of the Y_l elements above. Hence $Y_l = A_l$ for all $l \in \mathbb{Z}_{>0}$. A similar argument using the mutations

$$S' = \{\mu_2, \mu_1\mu_2, \mu_3\mu_1\mu_2, \mu_2\mu_3\mu_1\mu_2, \mu_1\mu_2\mu_3\mu_1\mu_2, (\mu_3\mu_1\mu_2)^2, \dots\}$$

shows that each B_l is a quantum cluster variable.

□

We now describe all quantum clusters. For $i \in \mathbb{Z}_{>0}$, let $\mathbf{X}_i$ denote the i-th cluster produced from the initial quantum seed by the i-th element of S and let $\mathbf{X}_{-i}$ denote the i-th cluster produced by the i-th element of S'. From the quivers listed in the proof of Prop. 4.22, we can verify that when applied to the initial quantum seed, the mutations $\mu_1(\mu_2\mu_1\mu_3)^r$, $\mu_2\mu_3(\mu_2\mu_1\mu_3)^r$, $\mu_3\mu_1\mu_3(\mu_2\mu_1\mu_3)^r, r \in \mathbb{Z}_{\geq 0}$ respectively produce the cluster variables E_1, F_1, E_1 or F_1, E_1, F_1, depending on the parity of r (one verifies that exchange relations for these variables are the recurrence relations from Def. 4.8). Similarly for the mutations $\mu_1(\mu_3\mu_1\mu_2)^r$, $\mu_3\mu_2(\mu_3\mu_1\mu_2)^r$, $\mu_2\mu_1\mu_2(\mu_3\mu_1\mu_2)^r, r \in \mathbb{Z}_{\geq 0}$. So for each $\mathbf{X}_i$, there is a unique cluster $\widehat{\mathbf{X}}_i$ produced from $\mathbf{X}_i$ by a single mutation and which contains either F_1 or E_1. Mutation of any quantum seed correspond to $\widehat{\mathbf{X}}_i$ produces $\mathbf{X}_i$, $\widehat{\mathbf{X}}_{i-2}$, or $\widehat{\mathbf{X}}_{i+2}$. (A portion of) the exchange graph is



4.5 Quantum cluster monomials

In this section we conclude the proof of Thm. 4.1 by showing (Prop. 4.27) that quantum cluster monomials from $\{F_1, A_{2m-1}, A_{2m+1}, D_1\}$ are contained in the double canonical basis for all $m \in \mathbb{Z}_{>0}$. We begin with the $m = 1$ case. First, we need formulas for powers of A_3.

Lemma 4.23. *For $k \in \mathbb{Z}_{\geq 0}$,*

$$A_3^k = q^{-\frac{1}{2}k^2} A_1^k C_1^{(k)} + \sum_{j \in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j} F_1^{\beta_j} b_j$$

where J is a finite index set, $c(q^{-1}) \in q^{-1}\mathbb{Z}[q^{-1}]$, and

$$b_j = q^{-\frac{1}{2}(k-2\alpha_j)(k-2\beta_j)} A_1^{k-2\beta_j} D_1^{\beta_j} C_1^{(k-2\alpha_j)}, \quad 0 \leq 2\beta_j \leq 2\alpha_j \leq k$$

or

$$b_j = q^{(2\alpha_j - k)(k-\alpha_j-\beta_j)} A_1^{2(k-\alpha_j-\beta_j)} D_1^{\beta_j} E_2^{2\alpha_j - k}, \quad 0 \leq \alpha_j + \beta_j \leq k < 2\alpha_j.$$

Proof. We use induction on k. From Lemma 4.16 we have $A_3 = q^{-\frac{1}{2}} A_1 C_1 - q^{-1} K_{-1} K_{+1} E_2$, so the base case holds. Suppose the lemma holds for all $l \leq k$. We have $A_3 E_2 = F_1 D_1 + q^{-1} A_1^2$, and thus

$$\begin{aligned}
A_3^{k+1} &= A_3^k (q^{-\frac{1}{2}} A_1 C_1 - q^{-1} K_{-1} K_{+1} E_2) \\
&= q^{-k} q^{-\frac{1}{2}} A_1 A_3^k C_1 - q^{-1} K_{-1} K_{+1} A_3^{k-1} (F_1 D_1 + q^{-1} A_1^2) \\
&= q^{-k} q^{-\frac{1}{2}} A_1 A_3^k C_1 - q^{-1} K_{-1} K_{+1} F_1 D_1 A_3^{k-1} - q^{-2k-1} K_{-1} K_{+1} A_1^2 A_3^{k-1}.
\end{aligned}$$

Since the lemma holds for k, we can write $q^{-k} q^{-\frac{1}{2}} A_1 A_3^k C_1$ as

$$q^{-\frac{1}{2}(k+1)^2} A_1^{k+1} (C_1^{(k+1)} - K_{-1} K_{+1} C_1^{(k-1)}) + \sum_{j \in J} q^{-k} q^{-\frac{1}{2}} c_j(q^{-1}) (K_{-1} K_{+1})^{\alpha_j} F_1^{\beta_j} A_1 b_j C_1$$

where J is a finite index set, $c(q^{-1}) \in q^{-1}\mathbb{Z}[q^{-1}]$, and $b_j = q^{-\frac{1}{2}(k-2\alpha_j)(k-2\beta_j)} A_1^{k-2\beta_j} D_1^{\beta_j} C_1^{(k-2\alpha_j)}$ or $b_j = q^{(2\alpha_j-k)(k-\alpha_j-\beta_j)} A_1^{2(k-\alpha_j-\beta_j)} D_1^{\beta_j} E_2^{2\alpha_j-k}$. For the first case, we have

$$\begin{aligned}
&q^{-k} q^{-\frac{1}{2}} A_1 b_j C_1 \\
&= q^{-k} q^{-\frac{1}{2}} q^{-\frac{1}{2}(k-2\alpha)(k-2\beta)} A_1^{k-2\beta+1} D_1^{\beta} (C_1^{(k-2\alpha+1)} - K_{-1} K_{+1} C_1^{(k-2\alpha-1)}) \\
&= q^{-\alpha-\beta} q^{-\frac{1}{2}(k+1-2\beta)(k+1-2\alpha)} A_1^{k-2\beta+1} D_1^{\beta} C_1^{(k+1-2\alpha)}
\end{aligned}$$

$$- q^{-k+\beta-\alpha}K_{-1}K_{+1}q^{-\frac{1}{2}(k+1-2(\alpha+1))(k+1-2\beta)}A_1^{k-2\beta+1}D_1^{\beta}C_1^{(k+1-2(\alpha+1))}.$$

For the second case, using $AB = F_1D_1 + q^{-1}K_{-1}K_{+1}E_2^2$, we have

$$\begin{aligned}
&q^{-k}q^{-\frac{1}{2}}A_1b_jC_1 \\
&= q^{-k}A_1q^{(2\alpha-k)(k-\alpha-\beta)}A_1^{2(k-\alpha-\beta)}D_1^{\beta}E_2^{2\alpha-k-1}(A_1 + q^{-1}B_1) \\
&= q^{-k}q^{(2\alpha-k)(k-\alpha-\beta)}q^{2\alpha-k-1}A_1^{2(k-\alpha-\beta+1)}D_1^{\beta}E_2^{2\alpha-k-1} \\
&\qquad + q^{-(2\alpha-k-1)}q^{-k-1}q^{(2\alpha-k)(k-\alpha-\beta)}A_1^{2(k-\alpha-\beta)}(F_1D_1 + q^{-1}K_{-1}K_{+1}E_2^2)D_1^{\beta}E_2^{2\alpha-k-1} \\
&= q^{-\alpha-\beta}q^{(2\alpha-k-1)(k+1-\alpha-\beta)}A_1^{2(k-\alpha-\beta+1)}D_1^{\beta}E_2^{2\alpha-k-1} \\
&\qquad + q^{-k}F_1q^{(2\alpha-k-1)(k-\alpha-\beta-1)}A_1^{2(k-\alpha-\beta)}D_1^{\beta+1}E_2^{2\alpha-k-1} \\
&\qquad + q^{-2k+2\beta-1}K_{-1}K_{+1}q^{(2(\alpha+1)-k-1)(k-\alpha-1-\beta)}A_1^{2(k-\alpha-\beta)}D_1^{\beta}E_2^{2(\alpha+1)-k-1}.
\end{aligned}$$

Thus $q^{-k}q^{-\frac{1}{2}}A_1A_3^kC_1$ has the desired form. Using the fact that the lemma holds for $k-1$, it is straightforward that $q^{-1}K_{-1}K_{+1}F_1D_1A_3^{k-1}$ and $q^{-2k-1}K_{-1}K_{+1}A_1^2A_3^{k-1}$ also have the desired form. □

Corollary 4.24. *Quantum cluster monomials in $\{F_1, A_1, A_3, D_1\}$ are contained in the double canonical basis.*

Proof. Using Lemma 4.23, we write $q^{-\frac{1}{2}uv}F_1^aA_1^uA_3^vD_1^r$ as

$$q^{-\frac{1}{2}(u+v)v}A_1^{u+v}C_1^{(v)} + \sum_{j\in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j}F_1^{\beta_j+a}b_j$$

where

$$b_j = q^{-\frac{1}{2}(v-2\alpha_j+u)(v-2\beta_j)}A_1^{v-2\beta_j+u}D_1^{\beta_j+r}C_1^{(v-2\alpha_j)}$$

or

$$b_j = q^{-\frac{1}{2}(2\alpha_j-v)(2(v-\alpha_j-\beta_j)+u)}A_1^{2(v-\alpha_j-\beta_j)}D_1^{\beta_j+r}E_2^{2\alpha_j-v}.$$

If b_j has the second form, then $F_1^{\beta_j+a}b_j$ is a double basis element by Lemma 4.7. If b_j has the first form, then $F_1^{\beta_j+a}b_j$ has correct triangularity by Lemma 4.7. □

To generalize Lemma 4.23, we use the fact that $T_1^2(D_1) = D_1$, which is an immediate consequence of the following lemma:

Lemma 4.25. *The element $T_1(D_1)$ belongs to $\widehat{\mathbf{B}}_{\widetilde{\mathfrak{sl}_3}}$. In particular, $T_1(D_1) = K_{-1}^{-1}K_{+1}^{-1}D_1$.*

Proof. We have

$$\begin{aligned}
T_1(D_1) &= T_1((F_1 \bullet E_{12})E_{21} - q^{-1}K_{-1} \diamond q^{-\frac{1}{2}}E_2E_{12} \\
&= (K_{-1}^{-1}K_{+1}^{-1}(q^{-\frac{1}{2}}(F_1 \bullet E_{12})E_1 - q^{-1}K_{-1} \diamond E_{12})E_2 \\
&\qquad - q^{-1}K_{-1}^{-1} \diamond q^{-\frac{1}{2}}E_{12}(K_{+1}^{-1} \diamond (F_1 \bullet E_{12})) \\
&= K_{-1}^{-1}K_{+1}^{-1}((F_1 \bullet E_{12})(q^{-1}E_{12} + E_{21}) - q^{-1}K_{-1} \diamond q^{-\frac{1}{2}}E_2E_{12}) \\
&\qquad - q^{-1}K_{-1}^{-1}K_{+1}^{-1}E_{12}(F_1 \bullet E_{12}) \\
&= K_{-1}^{-1}K_{+1}^{-1}D_1.
\end{aligned}$$

□

Lemma 4.26. *For all $m \in \mathbb{Z}_{>0}$,*

$$A_{2m+1}^k = q^{-\frac{1}{2}k^2}A_{2m-1}^kC_1^{(k)} + \sum_{j \in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j+(m-1)\beta_j}F_1^{\beta_j}b_j$$

where J is a finite index set, $c_j(q^{-1}) \in q^{-1}\mathbb{Z}[q^{-1}]$, and

$$b_j = q^{-\frac{1}{2}(k-2\alpha_j)(k-2\beta_j)}A_{2m-1}^{k-2\beta_j}D_1^{\beta_j}C_1^{(k-2\alpha_j)}, \quad 0 \le 2\beta_j \le 2\alpha_j \le k$$

or

$$b_j = q^{(2\alpha_j-k)(k-\alpha_j-\beta_j)}A_{2m-1}^{2(k-\alpha_j-\beta_j)}A_{2m-3}^{2\alpha_j-k}D_1^{\beta_j}, \quad 0 \le \alpha_j + \beta_j \le k < 2\alpha_j.$$

Proof. We use induction on m. The $m = 1$ case is established in Lemma 4.23. Suppose the lemma holds for $m - 1$. Since T_1^2 fixes D_1 and C_1, $T_1^2(A_{2m-1}^r) = K_{+1}^{-r} \diamond A_{2m+1}^r$, and

$T_1^2(F_1^r) = K_{-1}^r K_{+1}^{-r} \diamond F_1^r$. Applying T_1^2, we have

$$\begin{aligned}
&K_{+1}^{-k} \diamond A_{2m+1}^k \\
&= q^{-\frac{1}{2}k^2} K_{+1}^{-k} \diamond A_{2m-1}^k C_1^{(k)} + \sum_{j \in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j+(m-1)\beta_j}(K_{-1}^{\beta_j}K_{+1}^{-\beta_j} \diamond F_1^{\beta_j})T_1^2(b_j)
\end{aligned}$$

and thus

$$A_{2m+1}^k = q^{-\frac{1}{2}k^2} A_{2m-1}^k C_1^{(k)} + \sum_{j \in J} c_j(q^{-1}) K_{-1}^{\alpha_j+(m-1)\beta_j} K_{+1}^{\alpha+(m-2)\beta_j+k} q^{\frac{1}{2}k^2-2\beta_j^2} F_1^{\beta_j} T_1^2(b_j)$$

where b_j has one of the two forms described above. For the first case, we have

$$\begin{aligned}
&q^{\frac{1}{2}k^2-2\beta^2} F_1^{\beta} T_1^2(b_j) \\
&= q^{\frac{1}{2}k^2-2\beta^2} F_1^{\beta} q^{-\frac{1}{2}(k-2\alpha)(k-2\beta)} (K_{+1}^{-(k-2\beta)} \diamond A_{2m-1}^{k-2\beta}) D_1^{\beta} C_1^{(k-2\alpha)} \\
&= K_{+1}^{-(k-2\beta)} q^{\frac{1}{2}k^2-2\beta_j^2} q^{-\frac{1}{2}(k-2\beta)^2} q^{-2(k-2\beta)\beta} F_1^{\beta} q^{-\frac{1}{2}(k-2\alpha)(k-2\beta)} A_{2m-1}^{k-2\beta} D_1^{\beta} C_1^{(k-2\alpha)} \\
&= K_{+1}^{-(k-2\beta)} F_1^{\beta} q^{-\frac{1}{2}(k-2\alpha)(k-2\beta)} A_{2m-1}^{k-2\beta} D_1^{\beta} C_1^{(k-2\alpha)}.
\end{aligned}$$

In the second case,

$$\begin{aligned}
&q^{\frac{1}{2}k^2-2\beta^2} F_1^{\beta} T_1^2(b_j) \\
&= q^{\frac{1}{2}k^2-2\beta^2} F_1^{\beta} q^{(2\alpha-k)(k-\alpha-\beta)} (K_{+1}^{-2(k-\alpha-\beta)} \diamond A_{2m-1}^{2(k-\alpha-\beta)}) D_1^{\beta} (K_{+1}^{-(2\alpha-k)} \diamond A_{2m-3}^{2\alpha-k}) \\
&= q^{\frac{1}{2}k^2-2\beta^2} q^{-2(k-\alpha-\beta)^2} q^{-\frac{1}{2}(2\alpha-k)^2} q^{-2(k-\alpha-\beta)(2\alpha-k)} q^{-2(k-2\beta)\beta} K_{+1}^{-k+2\beta} F_1^{\beta} \\
&\qquad \times q^{(2\alpha-k)(k-\alpha-\beta)} A_{2m-1}^{2(k-\alpha-\beta)} D_1^{\beta} A_{2m-3}^{2\alpha-k} \\
&= K_{+1}^{-k+2\beta} F_1^{\beta} q^{(2\alpha-k)(k-\alpha-\beta)} A_{2m-1}^{2(k-\alpha-\beta)} D_1^{\beta} A_{2m-3}^{2\alpha-k}.
\end{aligned}$$

□

Proposition 4.27. *Quantum cluster monomials in* $\{F_1, A_{2m-1}, A_{2m+1}, D_1\}$ *are contained in the double canonical basis.*

Proof. Let S_m denote the statement given in the proposition, let T_m denote the statement that $F_1^a q^{-\frac{1}{2}sk} A_{2m-1}^k D_1^r C_1^{(s)}$ has correct triangularity for all $a, k, r, s \in \mathbb{Z}_{\geq 0}$, and let R_m denote the statement that $F_1^a q^{-\frac{1}{2}s(u+v)} q^{\frac{1}{2}uv} A_{2m-1}^u A_{2m-3}^v D_1^r C_1^{(s)}$ has correct triangularity for all $a, u, v, r, s \in \mathbb{Z}_{\geq 0}$. All three statements hold for $m = 0$ by Lemmata 4.7 and 4.24. We claim that

$$R_m \wedge T_m \implies R_{m+1} \wedge T_{m+1}, \tag{$*$}$$

$$T_{m+1} \wedge S_m \implies S_{m+1}. \tag{$**$}$$

Suppose that R_m and T_m hold. For T_{m+1}, we know that

$$\begin{aligned} F_1^a q^{-\frac{1}{2}sk} A_{2m+1}^k D_1^r C_1^{(s)} &= F_1^a q^{-\frac{1}{2}k^2} q^{-\frac{1}{2}sk} A_{2m-1}^k D_1^r C_1^{(k)} C_1^{(s)} \\ &\quad + \sum_{j \in J} c_j(q^{-1}) (K_{-1}K_{+1})^{\alpha_j + (m-1)\beta_j} F_1^{\beta_j + a} q^{-\frac{1}{2}sk} b_j D_1^r C_1^{(s)}, \end{aligned}$$

where b_j has one of the two forms listed in Lemma 4.26. We have (see [BG17b, Eq. (4.7)])

$$F_1^a q^{-\frac{1}{2}k^2} q^{-\frac{1}{2}sk} A_{2m-1}^k D_1^r C_1^{(k)} C_1^{(s)} = \sum_{j=0}^{\min(k,s)} q^{-kj} (K_{-1}K_{+1})^j F_1^a q^{-\frac{1}{2}k(s+k-2j)} A_{2m-1}^k D_1^r C_1^{(k+s-2j)}$$

and for b_j of the first type,

$$\begin{aligned} &F_1^{\beta_j + a} q^{-\frac{1}{2}sk} b_j D_1^r C_1^{(s)} \\ &= F_1^{\beta_j + a} q^{-\frac{1}{2}sk} q^{-\frac{1}{2}(k-2\alpha_j)(k-2\beta_j)} A_{2m-1}^{k-2\beta_j} D_1^{\beta_j + r} C_1^{(k-2\alpha_j)} C_1^{(s)} \\ &= \sum_{l=0}^{\min(k-2\alpha_j, s)} q^{-l(k-2\beta_j) - s\beta_j} (K_{-1}K_{+1})^l F_1^{\beta_j + a} q^{-\frac{1}{2}(k-2\alpha_j+s-2l)(k-2\beta_j)} A_{2m-1}^{k-2\beta_j} D_1^{\beta_j + r} C_1^{(k-2\alpha_j+s-2l)}, \end{aligned}$$

which has correct triangularity since T_m holds. For b_j of the second type,

$$F_1^{\beta_j + a} q^{-\frac{1}{2}sk} b_j D_1^r C_1^{(s)} = F_1^{\beta_j + a} q^{-\frac{1}{2}sk} q^{(2\alpha_j - k)(k - \alpha_j - \beta_j)} A_{2m-1}^{2(k-\alpha_j-\beta_j)} A_{2m-3}^{2\alpha_j - k} D_1^{\beta_j + r} C_1^{(s)},$$

which has correct triangularity since R_m holds. Thus $R_m \wedge T_m \implies T_{m+1}$. Next, to show that R_{m+1} holds, we have

$$
\begin{aligned}
&F_1^a q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} A_{2m-1}^v A_{2m+1}^u D_1^r C_1^{(s)} \\
&= F_1^a q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} A_{2m-1}^v (q^{-\frac{1}{2}u^2} A_{2m-1}^u C_1^{(u)} + \sum_{j\in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j+(m-1)\beta_j} F_1^{\beta_j} b_j) D_1^r C_1^{(s)} \\
&= F_1^a q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} q^{-\frac{1}{2}u^2} A_{2m-1}^{u+v} C_1^{(u)} \\
&\qquad + \sum_{j\in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j+(m-1)\beta_j} q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta_j} A_{2m-1}^v b_j D_1^r C_1^{(s)},
\end{aligned}
$$

where b_j has one of the two forms listed in Lemma 4.26. In the first case,

$$
\begin{aligned}
&q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta} A_{2m-1}^v b_j D_1^r C_1^{(s)} \\
&= q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta} q^{(2\alpha-u)(u-\alpha-\beta)} A_{2m-1}^{2(u-\alpha-\beta)+v} A_{2m-3}^{2\alpha-u} D_1^{\beta+r} C_1^{(s)} \\
&= q^{-\alpha v-s\beta} q^{-\frac{1}{2}s(u+v-2\beta)} F_1^{a+\beta} q^{\frac{1}{2}(2(u-\alpha-\beta)+v)(2\alpha-u)} A_{2m-1}^{2(u-\alpha-\beta)+v} A_{2m-3}^{2\alpha-u} D_1^{\beta+r} C_1^{(s)}
\end{aligned}
$$

which has correct triangularity since R_m holds. In the second case,

$$
\begin{aligned}
&q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta} A_{2m-1}^v b_j D_1^r C_1^{(s)} \\
&= q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta} q^{-\frac{1}{2}(u-2\alpha)(u-2\beta)} A_{2m-1}^{u-2\beta+v} D_1^{\beta+r} C_1^{(u-2\alpha)} C_1^{(s)} \\
&= \sum_{l=0}^{\min(u-2\alpha,s)} (K_{-1}K_{+1})^l q^{-\frac{1}{2}s(u+v)} q^{-\frac{1}{2}uv} F_1^{a+\beta} q^{-\frac{1}{2}(u-2\alpha)(u-2\beta)} A_{2m-1}^{u-2\beta+v} D_1^{\beta+r} C_1^{(u-2\alpha+s-2l)} \\
&= \sum_{l=0}^{\min(u-2\alpha,s)} (K_{-1}K_{+1})^l q^{-l(u-\beta)-\beta s-\alpha v-lv} F_1^{a+\beta} \\
&\qquad \times q^{-\frac{1}{2}(u-2\alpha+s-2l)(u-2\beta+v)} A_{2m-1}^{u-2\beta+v} D_1^{\beta+r} C_1^{(u-2\alpha+s-2l)},
\end{aligned}
$$

which has correct triangularity since T_m holds. Thus $R_m \wedge T_m \implies R_{m+1}$ and $(*)$ is proven. Now suppose that S_m and T_{m+1} hold. Then

$$
q^{-\frac{1}{2}uv} F_1^a A_{2m+1}^u A_{2m+3}^v D_1^r
$$

$$= q^{-\frac{1}{2}(uv+v^2)} A_{2m+1}^{u+v} C_1^{(v)} D_1^r + \sum_{j\in J} c_j(q^{-1})(K_{-1}K_{+1})^{\alpha_j+m\beta_j} F_1^{a+\beta_j} q^{-\frac{1}{2}uv} A_{2m+1}^u b_j D_1^r$$

where b_j has one of the two forms listed in Lemma 4.26. If b_j has the first form, then

$$\begin{aligned} F_1^{a+\beta_j} q^{-\frac{1}{2}uv} A_{2m+1}^u b_j D_1^r &= F_1^{a+\beta_j} q^{-\frac{1}{2}uv} q^{-\frac{1}{2}(v-2\alpha_j)(v-2\beta_j)} A_{2m+1}^{u+v-2\beta_j} D_1^{\beta_j+r} C_1^{(v-2\alpha_j)} \\ &= F_1^{a+\beta_j} q^{-u\alpha_j} q^{-\frac{1}{2}(v-2\alpha_j)(u+v-2\beta_j)} A_{2m+1}^{u+v-2\beta_j} D_1^{\beta_j+r} C_1^{(v-2\alpha_j)}, \end{aligned}$$

which has correct triangularity since T_{m+1} holds. If b_j has the second form, then

$$\begin{aligned} F_1^{a+\beta_j} q^{-\frac{1}{2}uv} A_{2m+1}^u b_j D_1^r &= F_1^{a+\beta_j} q^{-\frac{1}{2}uv} q^{(2\alpha_j-v)(v-\alpha_j-\beta_j)} A_{2m+1}^{2(v-\alpha_j-\beta_j)+u} A_{2m-1}^{2\alpha_j-v} D_1^{\beta_j+r} \\ &= F_1^{a+\beta_j} q^{-u\alpha_j} q^{\frac{1}{2}(2\alpha_j-v)(2(v-\alpha_j-\beta_j)+u)} A_{2m+1}^{2(v-\alpha_j-\beta_j)+u} A_{2m-1}^{2\alpha_j-v} D_1^{\beta_j+r}, \end{aligned}$$

which has correct triangularity since S_m holds. Therefore S_{m+1} holds and $(**)$ is proven. □

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