Title

ELECTRON-ENERGY DISTRIBUTIONS AND IONIZATION RATES IN HYDROGEN WITH CROSSED ELECTRIC AND STRONG MAGNETIC FIELDS

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Berkeley, California

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Subject: UCRL-10366, "Electron-Energy Distributions and Ionization Rates In Hydrogen with Crossed Electric and Strong Magnetic Fields," Gary A. Pearson and Wulf B. Kunkel, September 5, 1962.

Please add the following to the last paragraph of page 37:

Recently, Engelhardt and Phelps have obtained results which agree very well with curve (1) of Fig. 6; this indicates a real disagreement between the experimental and calculated values of α/B. Their results also agree fairly well with curve (1) of Figs. 4, 5, and 10.

Research and Development

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ABSTRACT

The calculation of the electron velocity distribution in a slightly ionized gas with $(\omega_b \tau)^2 \gg 1$ is discussed, including a detailed treatment of inelastic processes. The derived equations apply whenever the gas density n_g, the gas-flow velocity, $\vec{E}$, and $\vec{B}$ are spacially uniform and constant in time, the component of $\vec{E}$ parallel to $\vec{B}$ is negligible, and E is much less than B (in Gaussian units). The equations are suitable for numerical computation of the electron energy distribution and drift velocity in the reference frame in which $\vec{E}$ is negligible and the gas flow is perpendicular to $\vec{B}$ so that the electron velocity distribution is nearly isotropic. When the rms molecular speed in this frame is much smaller than the rms electron speed, the equations can be greatly simplified; although the results are not new, this derivation clarifies their physical interpretation and limitations.

With simplified equations, the electron-energy distribution, drift velocity, ionization rate, and diffusion tensor in cold stationary H_2 gas are calculated for cE/B between 4×10^6 and 6×10^7 cm/sec. Whereas the electron-energy distribution depends only on cE/B, the other quantities also have simple dependences upon n_g and ω_b. In view of the large uncertainty in important collision cross sections, the agreement with recent experimental results is reasonably good.

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I. INTRODUCTION

If a sufficiently small density of free electrons is introduced into a cold gas, the motion of the electrons is controlled by the external fields and by collisions with the gas molecules. The electron distribution function can then be determined by solving the Boltzmann equation. The "usual" method of solution consists of expanding the distribution function in spherical harmonics in velocity space and evaluating the collision terms in the reference frame in which the gas is at rest, which we call the gas frame. The method is useful when the drift speed of the electrons relative to the gas is small in comparison with the rms electron speed. An important parameter is the ratio of the electron-cyclotron frequency ω_b to the electron-collision frequency $1/\tau$. [In H_2 gas, $\omega_b\tau \approx 3\ B(kG)/p(mm\ Hg)$.]

When $\omega_b\tau \ll 1$, the effects of the magnetic field can be ignored or treated as a small perturbation. This case has been extensively studied, both experimentally and theoretically. The usual method of solution is adequate until the phenomenon of so-called "runaway" electrons becomes important.[1]

When $\omega_b\tau \gg 1$, the problem is complicated unless the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are mutually perpendicular, spacially uniform, and constant in time with $E \ll B$ (in Gaussian units). Under these conditions the electron motion in the applied fields is simple and only the effects of relatively infrequent collisions need to be evaluated. In this case--which is discussed in this paper--we can consider any nonrelativistic electron drift speed in the gas frame since the problem of runaway electrons cannot appear. The usual method of solution is not adequate when the drift speed is high.

We find it useful to consider the distribution function in a reference frame moving with velocity $\vec{v}_d = c\vec{E}\times\vec{B}/B^2$ relative to the gas frame, which we call the drift frame and in which the electric field vanishes. In the drift frame,

a free electron follows a helical trajectory in which its speed and energy are constant, while in the gas frame, its speed, kinetic energy, and potential energy oscillate in time.

The effects of elastic collisions--which are often the most probable--are most easily discussed in the drift frame, where there is no electric field but there is a wind of gas molecules with relatively high kinetic energy. The collisions produce diffusion of the electrons perpendicular to the magnetic field and reduce the diffusion along the magnetic field. Intuitively, it seems that elastic collisions tend to heat the electrons until their mean energy is near that of the gas molecules and tend to spread the electron-energy distribution until it resembles a Maxwellian distribution. Notice that in the drift frame the electrons gain energy from the molecules, while in the gas frame they gain energy by moving through the electric field.

We use the term inelastic collisions to include all but elastic, electron attachment, and ionizing collisions. Inelastic processes have effects similar to those of elastic collisions, but their most important effect is to increase or decrease the energy of the electrons involved.

Ionization and electron attachment processes also have similar effects, but their primary effect is to alter the number of free electrons.

The usual method of solution is outlined in Sec. II. We then develop a method of solution in the drift frame that is useful whenever all speeds involved are nonrelativistic. Although the resulting equations are relatively complicated, only simple physical concepts--like those discussed above--are used.

The results of numerical computations of the electron-energy distribution, drift speed, diffusion tensor, and ionization rate in hydrogen gas with v_d between 4×10^6 cm/sec and 6×10^7 cm/sec are given in Sec. III.

II. METHODS OF SOLUTION OF THE BOLTZMANN EQUATION

On the basis of the discussion in the introduction we make the following explicit assumptions:

(a) The electric and magnetic fields are mutually perpendicular, spacially uniform, and constant in time.

(b) The distribution function of the gas molecules is spacially uniform and constant in time.

(c) The magnetic field and n_g, the number of gas molecules per unit volume, are such that $\omega_b \tau \gg 1$.

(d) The gas density n_g is so much greater than the electron and ion densities that only collisions of electrons with such molecules are important.

(e) All speeds involved are nonrelativistic.

Because of (d) and (e), the electron-distribution function $F(\vec{r}, \vec{v}, t)$ satisfies the Boltzmann equation

$$\frac{\partial F}{\partial t} + \nabla \cdot \vec{v} F + \nabla_v \cdot \left[-\frac{e}{m} \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) F \right] - \left(\frac{\partial F}{\partial t} \right)_{coll} = 0 \qquad \text{(II-1)}$$

where only collisions with neutral molecules must be included. When ionization of the gas molecules by electron impact is not the only important process producing free electrons, additional source terms must be included in Eq. (II-1). We do not discuss such terms here.

Because of (a) and (b), we can avoid considering the spacial dependence either by assuming that $F(\vec{r}, \vec{v}, t)$ is independent of $\vec{r}$ or by integrating Eq. (II-1) over a volume on whose surface $F(\vec{r}, \vec{v}, t)$ vanishes. The latter interpretation permits the treatment of electron avalanches.

A. The "Usual" Method; Expansion in Spherical Harmonics in the Gas Frame

Equation (II-1) is often solved by expanding $F(\vec{v}, t)$ in spherical harmonics in velocity space.[2, 3] It is usually assumed that the expansion converges rapidly, so that only the first two terms are needed. The expansion then takes the form

$$F(\vec{v}, t) = F_0(v, t) + \frac{\vec{v}}{v} \cdot \vec{F}_1(v, t). \tag{II-2}$$

Substituting Eq. (II-2) into Eq. (II-1) and equating the coefficients of each spherical harmonic to zero, we obtain the two coupled equations

$$\frac{\partial F_0}{\partial t} - \frac{e}{3mv^2} \vec{E} \cdot \frac{\partial (v^2 \vec{F}_1)}{\partial v} - \left[\left(\frac{\partial F}{\partial t}\right)_{coll}\right]_0 = 0, \tag{II-3}$$

and

$$\frac{\partial F_1}{\partial t} - \frac{e}{m} \vec{E} \frac{\partial F_0}{\partial v} - \frac{e}{mc} \vec{B} \times \vec{F}_1 - \left[\left(\frac{\partial F}{\partial t}\right)_{coll}\right]_1 = 0 \tag{II-4}$$

It is found that the spherical harmonic expansion converges well when the electron speed v is high compared with the electron drift speed in the reference frame being used. The collision terms in Eq. (II-3) and Eq. (II-4) are simplest when the gas rest frame is used. Often when elastic collisions are much more probable than any other, the rms electron speed is much higher than the electron drift speed in the gas frame, so the spherical harmonic expansion converges well in the gas frame for all but a negligible fraction of the electrons.

Allis[2] and Holstein[3] have derived the collision terms in Eqs. (II-3) and (II-4) in the gas frame by assuming that elastic collisions are much more probable than any other. They find

$$\left[\left(\frac{\partial F}{\partial t}\right)_{coll}\right]_1 = - \nu_m^{el} \vec{F}_1,$$

where ν_m^{el} is the elastic momentum-transfer collision frequency as defined in Appendix A. If we choose $\vec{E} = E \hat{a}_y$ and $\vec{B} = B \hat{a}_z$, so $\vec{v}_d = (cE/B)\hat{a}_x$, (II-4) becomes

$$\frac{\partial F_{1x}}{\partial t} + \omega_b F_{1y} + \nu_m^{el} F_{1x} = 0 \tag{II-5}$$

$$\frac{\partial F_{1y}}{\partial t} - \frac{eE}{m}\frac{\partial F_0}{\partial v} - \omega_b F_{1x} + \nu_m^{el} F_{1y} = 0 \tag{II-6}$$

$$\frac{\partial F_{1z}}{\partial t} + \nu_m^{el} F_{1z} = 0 \tag{II-7}$$

Notice that Eq. (II-7) simply implies that F_{1z} damps to zero because of collisions.

It is clear that near equilibrium, the time-derivative terms in Eqs. (II-5) and (II-6) can be neglected. Solving for F_{1x} and F_{1y}, we find

$$F_{1x} = -v_d \gamma(v) \frac{\partial F_0}{\partial v}, \text{ where } \gamma(v) = \left[1 + \left(\frac{\nu_m^{el}(v)}{\omega_b}\right)^2\right]^{-1} \tag{II-8}$$

and

$$F_{1y} = -\frac{\nu_m^{el}(v)}{\omega_b} F_{1x}. \tag{II-9}$$

Using these results, we can write Eq. (II-3) as

$$\frac{\partial F_0(v,t)}{\partial t} = \sum_j \left(\frac{\partial F_0}{\partial t}\right)_j, \tag{II-10}$$

where, using the collision terms derived by Allis and Holstein, we have

$$\left(\frac{\partial F_0}{\partial t}\right)_{el} = \frac{v_d^2}{3v^2}\frac{\partial}{\partial v}\left[v^2 \nu_m^{el}(v)\gamma(v)\frac{\partial F_0(v,t)}{\partial v}\right] + \frac{m}{Mv^2}\frac{\partial}{\partial v}\left[v^3 \nu_m^{el}(v) F_0(v,t)\right] + \frac{kT}{Mv^2}\frac{\partial}{\partial v}\left[v^2 \nu_m^{el}(v)\frac{\partial F_0(v,t)}{\partial v}\right] \tag{II-11}$$

for elastic collisions;

$$\left(\frac{\partial F_0}{\partial t}\right)_j = \nu^j\left[(v^2+a_j)^{1/2}\right]\frac{(v^2+a_j)^{1/2}}{v}\,F_0\left[(v^2+a_j)^{1/2},\,t\right] - \nu^j(v)F_0(v,\,t) \qquad \text{(II-12)}$$

for inelastic processes; and

$$\left(\frac{\partial F_0}{\partial t}\right)_j = \frac{n_j}{v^2}\int \nu^j(v')v'^2P^j(v;\,v')F_0(v',\,t)dv' - \nu^j(v)F_0(v,\,t) \qquad \text{(II-13)}$$

for ionization collisions. In these formulas $\nu^j(v)$ is the collision frequency for process j, n_j is the number of electrons leaving each collision, and $P^j(v;\,v')$ is their distribution of speed v for specified incident speed v'. It is clear from Eq. (II-13) that for electron attachment we have

$$\left(\frac{\partial F_0}{\partial t}\right)_j = -\nu^j(v)F_0(v,\,t) \quad . \qquad \text{(II-14)}$$

The physical interpretation of Eqs. (II-12), (II-13), and (II-14) is clear. However, Eq. (II-11) has some interesting features. The first term, which arises from the second term in Eq. (II-3), can be written as

$$\nabla_v\cdot\left[D(v)\nabla_v F_0(v,\,t)\right] \quad , \qquad \text{where} \qquad D(v) = v_d^{\,2}\nu_m^{el}(v)\gamma(v)/3 \quad ,$$

i. e., as a diffusion in velocity space. It is interesting to note that in the large $\omega_b\tau$ limit where $\gamma(v)$ can be replaced by unity, the diffusion coefficient D(v) has the same form as that found in elementary kinetic theory with the mean free path merely replaced by v_d. This result can be qualitatively understood on the basis of a random-walk description of diffusion. Although the above derivation of this term is valid only near the steady state solution, we show in Appendix D that for large $\omega_b\tau$ no such restriction is necessary.

The second term of Eq. (II-11) accounts for the effects of molecular recoil. The third term accounts for the temperature T of the gas; notice that it has the same form as the first term. It is also interesting to note that $(\partial F_0/\partial t)_{el}$ from Eq. (II-11) vanishes when

$$F_0(v) \propto \exp\left[-\frac{3m}{M}\int\frac{vdv}{\gamma(v)v_d^{\,2}+3kT/M}\right] \quad . \qquad \text{(II-15)}$$

When $v_d = 0$, this reduces to the expected result

$$F_0(v) \propto \exp\left[-(mv^2/2)/kT\right].$$

In the large $\omega_b\tau$ limit, Eq. (II-15) becomes

$$F_0(v) \propto \exp\left[-\frac{3}{2}\left(\frac{mv^2}{2}\right)\Big/\left(\frac{Mv_d^2}{2}+\frac{3}{2}kT\right)\right] .$$

Thus, in this limit, elastic collisions always tend to heat the electrons until their mean energy is $Mv_d^2/2 + 3kT/2$--the mean energy of the gas molecules in the drift frame--and to spread their energy distribution until it is a Maxwellian distribution. This result is completely independent of the differential cross section for elastic scattering!

Once $F_0(v,t)$ is known from solving Eq. (II-10), most quantities of interest can be calculated.

The drift velocity of the electrons in the gas frame is given by

$$\vec{v}_D(t) = \frac{1}{n}\int d^3v F(\vec{v},t)\vec{v} \qquad \text{with} \qquad n = \int d^3v F(\vec{v},t) .$$

By introducing Eq. (II-2) we find

$$\vec{v}_D(t) = \frac{4\pi}{3n}\int v^3\vec{F}_1(v,t)dv \quad \text{with} \qquad n = 4\pi\int F_0(v,t)v^2dv .$$

We now introduce Eqs. (II-8) and (II-9) and integrate by parts to get

$$v_{Dx} = \frac{4\pi v_d}{3n}\int F_0(v,t)\frac{\partial\left[v^3\gamma(v)\right]}{\partial v}\,dv , \tag{II-16}$$

and

$$v_{Dy} = \frac{-4\pi v_d}{3n\omega_b}\int F_0(v,t)\frac{\partial\left[v^3\gamma(v)\nu_m^{el}(v)\right]}{\partial v}\,dv . \tag{II-17}$$

In the large $\omega_b\tau$ limit Eq. (II-16) becomes $v_{Dx} = v_d$. If $\nu_m^{el}(v)$ is a constant, $v_{Dy} = -v_d(\nu_m^{el}/\omega_b)$ in the large $\omega_b\tau$ limit.

A consideration of spacial gradients also leads to the diffusion tensor[2]

$$\vec{\vec{D}} = \begin{pmatrix} D_T & D_\perp & 0 \\ -D_\perp & D_T & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} ,$$

where

$$D_T = \frac{4\pi}{3n}\int \frac{\nu_m^{el}(v)}{\omega_b^2}\,\gamma(v)v^4F_0(v)dv ,$$

$$D_\perp = \frac{4\pi}{3n}\int \frac{1}{\omega_b}\,\gamma(v)v^4F_0(v)dv , \qquad \text{(II-18)}$$

$$D_\parallel = \frac{4\pi}{3n}\int \frac{1}{\nu_m^{el}(v)}\,\gamma(v)v^4F_0(v)dv ,$$

which are also simplified in the large $\omega_b\tau$ limit.

B. The Method of Solving in the Drift Frame

1. Motivation for the Approach

We know that in the large $\omega_b\tau$ limit the drift velocity in the gas frame is approximately $\vec{v}_d$. It is clear that, as v_d is increased, inelastic and ionizing collisions become more important and that they hold the mean energy of the electrons down. Thus, at high v_d, a large fraction of the electrons in the distribution will not satisfy the conditions that make the "usual" method appropriate. However, if we instead expand the electron-velocity distribution in spherical harmonics in the drift frame, where the apparent drift speed is small, the expansion should converge rapidly for most of the electrons in the distribution. This reasoning can be clarified by the following extreme example. Suppose there are no collisions and that all the electrons are at rest in the drift frame. Then the spherical harmonic expansion in the drift frame requires only one term while that in the gas frame converges very slowly. Notice that there is no time independent distribution whose expansion converges well in the gas frame but not in the drift frame. When the Boltzmann equation is applied in the drift frame, $\vec{E} = 0$ so that Eq. (II-3) becomes

$$\frac{\partial f(\epsilon, t)}{\partial t} = \sum_j \left(\frac{\partial f}{\partial t}\right)_j \quad , \tag{II-19}$$

where $f(\epsilon, t)$ is the distribution of electron energy ϵ as measured in the drift frame. With the assumption that $\omega_b \tau \gg 1$ the collision term in Eq. (II-3) has been written as a sum over the various collision processes j between the electrons and the gas molecules.

Notice that Eq. (II-19) is general and need not be based on an expansion in spherical harmonics. In other words, because in the drift frame the energy of each electron remains constant between collisions, the only contribution to $\partial f(\epsilon, t)/\partial t$ comes from the collisions themselves. The problem is thus reduced to evaluating the collision terms in the presence of the gas "wind" of speed v_d. The form of these terms becomes rather complicated, but this method is useful because the velocity distribution at large $\omega_b \tau$ is expected to be nearly isotropic in the drift frame for any nonrelativistic value of v_d.

2. The Collision Terms

We will evaluate the collision terms with the assumption that the molecules are at rest in the gas frame. Throughout this discussion the symbol v will denote the electron speed in the gas frame and $\epsilon = mV^2/2$ will be the electron energy in the drift frame.

The collision terms for each collision process j can be written in the form

$$\left(\frac{\partial f}{\partial t}\right)_j = \int G^j(\epsilon; \epsilon') f(\epsilon', t) d\epsilon' - N^j(\epsilon) f(\epsilon, t) \quad , \tag{II-20}$$

which has the interpretation that electrons are removed from the energy distribution at a rate proportional to the mean-collision-frequency function $N^j(\epsilon)$ and are inserted into the energy distribution by the first term in a manner determined by the energy-scatter function $G^j(\epsilon; \epsilon')$.

The mean-collision-frequency function $N^j(\epsilon)$ is defined as the mean collision frequency for process j by electrons of energy ϵ, and it is given by

$$N^j(\epsilon) = \int \nu^j(v) P(v; \epsilon) dv \tag{II-21}$$

where $P(v;\epsilon)$ is the probability distribution of the speed v of the electrons with energy ϵ, and $\nu^j(v)$ is the collision frequency for process j. (For the notation and nomenclature on cross sections and collision frequencies see Appendix A; for that on probability distributions see the first page of Appendix B.)

The energy-scatter function $G^j(\epsilon;\epsilon')$ is defined such that $G^j(\epsilon;\epsilon')f(\epsilon',t)$ is the rate at which electrons are inserted into the energy distribution at ϵ by collisions of process j in which the incident electrons have energy ϵ'. If n_j electrons leave each collision of process j, the general properties of $G^j(\epsilon;\epsilon')$ are

$$\int G^j(\epsilon;\epsilon')d\epsilon = n_j N^j(\epsilon')$$

and $G^j(\epsilon;\epsilon') \geq 0$. The exact nature of this function depends upon the type of collisions process involved.

For electron attachment processes, we have $n_j = 0$, so $G^j(\epsilon;\epsilon') = 0$.

For elastic and inelastic processes, we have $n_j = 1$, so the effects of collisions are completely determined by a differential cross section $\sigma_\theta^j(\theta;v)$ and by the discrete energy transfer $\epsilon_j = (m/2)\alpha_j$ associated with the change in the internal state of the molecule (which is zero for elastic collisions). We here assume that the differential cross section is independent of the azimuthal angle of scattering, since although the strong fields could introduce a slight asymmetry, the quantitative effect is not known and should be very small. A little thought shows that we may write

$$G^j(\epsilon;\epsilon') = \int dv' \int_0^\pi d\theta\, P(v';\epsilon')\left[2\pi n_g \sin\theta\, v'\, \sigma_\theta^j(\theta;v')\right] \times P^j(\epsilon;v',\theta,\epsilon') \quad , \qquad \text{(II-22)}$$

where $P^j(\epsilon;v',\theta,\epsilon')$ is the probability distribution of the final energy of the electrons with incident speed v' that are deflected by an angle θ, $P(v';\epsilon')$ is the probability distribution of the incident speed v', and $\left[2\pi n_g \sin\theta\, v'\, \sigma_\theta^j(\theta;v')\right]$ is the rate at which electrons of incident speed v' are deflected by θ. The function $P^j(\epsilon;v',\theta,\epsilon')$ depends only on the collision kinematics and is given to lowest order in m/M by Eq. (B-14) of the Appendix B. If the scattering is isotropic, the angular integration in Eq. (II-22) is trivial and yields

$$G^j(\epsilon;\epsilon') = \frac{1}{2mv_d}\int \frac{\nu^j(v)}{(v'^2 - a_j)^{1/2}}\left[1 - \frac{2mv'}{Mv_d}\cos\psi + \left(\frac{mv'}{Mv_d}\right)^2\right]^{-1/2}$$

$$\times P(v';\epsilon')dv' \qquad \text{(II-23)}$$

where $\cos\psi = (v'^2 + v_d^2 - V'^2)/2v_dV'$, $\epsilon' = mV'^2/2$, and the limits are given in Appendix B. The result is particularly simple if molecular recoil is neglected by assuming $m/M = 0$; in this case the limits are given by the "triangle" inequalities $|V' - v_d| \le v' \le V' + v_d$ and $|V - v_d| \le (v'^2 - a_j)^{1/2} \le V + v_d$, where $\epsilon = mV^2/2$. For elastic and inelastic processes, $G^j(\epsilon;\epsilon')$ is nonzero only within a finite range of ϵ about $\epsilon' - \epsilon_j$.

For ionization processes where $n \ge 2$, the effects of collisions are characterized by n_j, the discrete energy transfer ϵ_j, the cross section $\sigma^j(v)$, and the distribution of angle and speed $I^j(\theta, v; v')$ of the resulting electrons as discussed in Appendix A. As above, one can write

$$G^j(\epsilon;\epsilon') = n_j\int dv'\int_0^\pi d\theta\int dv\, P(v';\epsilon')\left[2\pi n_g \sin\theta\, v'\sigma^j(v')I^j(\theta, v; v')\right] \qquad \text{(II-24)}$$

$$\times P(\epsilon; v', v, \theta, \epsilon)$$

where $P(\epsilon; v', v, \theta, \epsilon)$ is the probability distribution of the energy ϵ of the electrons ejected at speed v and angle θ from collisions in which the incident electron speed is v'; it is given by Eq. (B-8) in Appendix B. If the scattering is isotropic, the angular integration of Eq. (II-24) yields

$$G^j(\epsilon;\epsilon') = \frac{n_j}{2mv_d}\int_{|V'-v_d|}^{V'+v_d} dv' \int_{|V-v_d|}^{V+v_d} \frac{\nu^j(v')}{v}\, P^j(v; v')P(v';\epsilon')dv, \qquad \text{(II-25)}$$

where $P^j(v; v') = 2\pi\int_0^\pi \sin\theta\, I^j(\theta, v; v')\, d\theta$. For ionization processes $G^j(\epsilon;\epsilon')$ is nonzero for ϵ from zero up to about $\epsilon' - \epsilon_j$.

3. The Assumed Form of $P(v; \epsilon)$

The theory outlined above is complete except for a specification of the function $P(v; \epsilon)$, which depends only upon the angular distribution of the velocities in the drift frame of the electrons of energy ϵ. In the large $\omega_b \tau$ limit, the most general angular distribution of interest is independent of the azimuthal angle about $\vec{B}$ and is an even function of $\cos \xi$ where ξ is the polar angle measured from $\vec{B}$. This is true because for large $\omega_b \tau$ the electrons follow their trajectories through many revolutions about $\vec{B}$ between collisions and the infrequent collisions tend to produce a random distribution of phases. Such a distribution can be expanded in the even Legendre polynomials of $\cos \xi$, and $P(v; \epsilon)$ is then given by Eq. (B-5) in Appendix B.

We expect the electron velocity distribution to be nearly isotropic in the drift frame. In fact, it is difficult to imagine a process by which the velocity distribution will become very anisotropic. Thus, the expansion in Legendre polynomials will converge quite rapidly. We assume that using

$$P(v; \epsilon) = v/2v_d V \qquad \text{for} \qquad |V - v_d| \leq v \leq V + v_d \ ,$$

$$\text{where} \qquad \epsilon = mV^2/2 \ , \qquad \text{(II-26)}$$

which from Eq. (B-5) is exact for an isotropic velocity distribution in the drift frame, will give a good approximation to the correct physical results. As is illustrated below, this assumption is not always as stringent as assuming that the velocity distribution is isotropic in the drift frame.

Any process--such as ionization--that produces electrons approximately at rest in the lab frame will enhance the angular distribution for ξ near $\pi/2$ and ϵ near $mv_d^2/2$. To illustrate the quantitative effect we can evaluate $G^j(\epsilon; \epsilon')$ explicitly for $\epsilon' = mv_d^2/2$ by neglecting molecular recoil in Eq. (II-23) and assuming either $\sigma^{el}(v)$ or $\nu^{el}(v)$ is equal to a constant. The results for an isotropic velocity distribution, a velocity distribution proportional to $\sin^2 \xi$, and a velocity distribution containing only $\xi = \pi/2$ are compared in Fig. 1. We see that even in the latter case--the largest possible anisotropy--the assumption of using Eq. (II-26) is reasonably good.

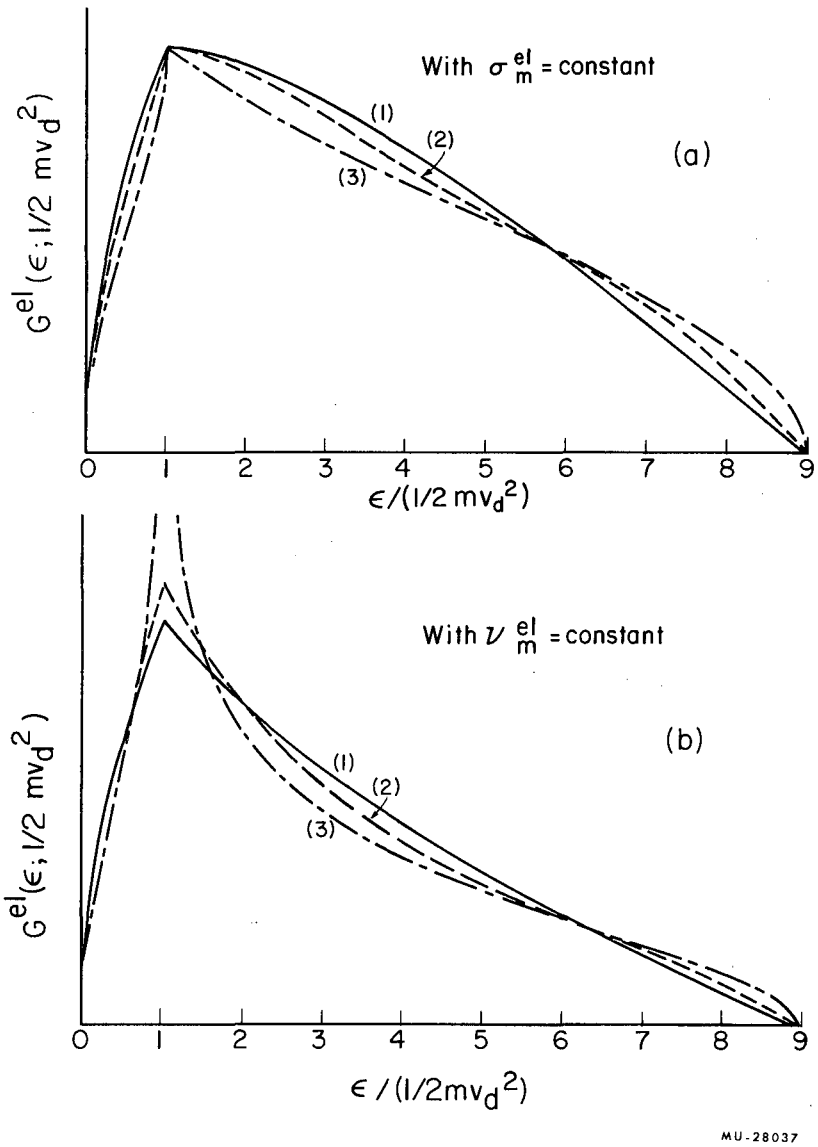

Fig. 1. The elastic energy-scatter function for $\epsilon' = mv_d^2/2$ calculated assuming the electron-velocity distribution in the drift frame
(1) is isotropic,
(2) is weighted by $\sin^2 \xi$, and
(3) contains only $\xi = \pi/2$.

C. Discussion of the Theory

1. General Discussion

Because it is based only on the assumption of using Eq. (II-26), the theory outlined in the previous section is useful at all nonrelativistic values of v_d in the large $\omega_b\tau$ limit. (In Sec. II-A we showed that the "large $\omega_b\tau$" limit is reached when $(\omega_b\tau)^2 \gg 1$.) In Section D of the Appendix we show that for electrons whose energies are large compared with $mv_d^2/2$ and for which elastic collisions are much more probable than any other, the results of this method of solution agree with those of the "usual" method as presented in Sec. II-A. Whenever most of the electrons in the distribution satisfy these conditions, the "usual" method of solution should be used because of its simplicity.

Notice that we have not attempted to include the effect of a component of E parallel to B. The criterion for validity of the theory being discussed is probably that $\left|\vec{E}\cdot\vec{B}\right|\omega_b\tau \ll \left|\vec{E}\times\vec{B}\right|$, in the gas frame.

When v_d is so large that the method outlined in Section II-B must be used, the angular dependences of the scattering processes, which are needed in evaluating Eqs. (II-22) and (II-24), will often be unknown. In this case it is useful to assume the scattering is isotropic and to use Eqs. (II-23) and (II-25), but we must determine the proper collision frequencies to use.

In ionization processes it is clear that the collision frequency should be used in Eq. (II-25), since this will give the correct rate of production of free electrons.

From Eq. (C-2) in Appendix C the rate of energy gain per electron of energy ϵ by elastic or inelastic process j is

$$W^j(\epsilon) = N^j_m(\epsilon)mv_d^2\left[K^j_m(\epsilon) - \frac{\epsilon}{Mv_d^2/2}\right] - N^j(\epsilon)\epsilon_j \ , \qquad \text{(II-27)}$$

where the subscript m denotes momentum transfer and where $K^j_m(\epsilon)$ is a factor of order unity. For elastic collisions, this result shows that the elastic momentum-transfer collision frequency should be used in evaluating $G^j(\epsilon;\epsilon')$ from Eq. (II-23) and $N^j(\epsilon)$ from Eq. (II-21). For an inelastic process where $|\epsilon_j| \gg mv_d^2/2$ the corresponding collision frequency should be used, and if $|\epsilon_j| \ll mv_d^2/2$ the corresponding momentum-transfer collision frequency should be used. When $|\epsilon_j| \approx mv_d^2/2$ and the collision frequency differs considerably

from the momentum-transfer collision frequency, the choice is not so easy. Often when $|\epsilon_j| \approx mv_d^2/2$ or $|\epsilon_j| \ll mv_d^2$, the collision process can be ignored in comparison with elastic collisions.

We notice also from Eq. (II-27) that whenever the "usual" approach cannot be used, the term involving $\epsilon/(Mv_d^2/2)$ can be ignored. This implies that the effects of molecular recoil can be neglected in Eq. (II-23) by setting $m/M = 0$; the result is very much simpler than the general equation (II-22).

Consideration of differential cross sections for elastic scattering with various angular dependences shows that the use of the elastic momentum-transfer collision frequency in Eqs. (II-21) and (II-23) gives approximately the correct rate of energy "spreading" by elastic collisions. Since in the lab frame this energy spreading corresponds to diffusion along $\vec{E}$, this conclusion is also suggested by the fact that in the usual method the diffusion tensor depends only upon the elastic momentum-transfer collision frequency.

2. The Most Useful Solution

Our basic equation (II-5) can be written symbolically as

$$\frac{\partial f(\epsilon, t)}{\partial t} = n_g \int \chi(\epsilon, \epsilon'; v_d) f(\epsilon', t) d\epsilon' \quad , \tag{II-28}$$

where, for a particular gas, $\chi(\epsilon, \epsilon'; v_d)$ is a kernel that depends only on v_d. We know physically that any initial electron-energy distribution will evolve until it can be written as

$$f(\epsilon, t) = Ce^{\beta t} f_0(\epsilon) \quad , \tag{II-29}$$

where $f_0(\epsilon)$ is normalized as $\int f_0(\epsilon) d\epsilon = 1$, and where β and C are constants. Substitution above yields the eigenvalue equation

$$(\beta/n_g) f_0(\epsilon) = \int \chi(\epsilon, \epsilon'; v_d) f_0(\epsilon) d\epsilon \quad , \tag{II-30}$$

which determines the eigenvalue β/n_g and the energy distribution $f_0(\epsilon)$, both of which clearly depend only upon v_d. The eigenfunction $f_0(\epsilon)$ of interest is everywhere nonnegative. Once it is known, most quantities of interest can be calculated.

The constant β gives the rate of production of free electrons. The quantity that is usually measured experimentally is the Townsend coefficient α, which in the absence of electron attachment is defined as $\alpha = \beta/v_E$, where v_E

is the drift speed of the electrons in the direction of $-\vec{E}$ in the gas frame. This drift speed is

$$v_E = \int f_0(\epsilon) v_E(\epsilon) d\epsilon \quad , \tag{II-31}$$

where $v_E(\epsilon)$ is obtained for large $\omega_b\tau$ by summing Eq. (C-4) over all processes j, and where $v_E(\epsilon)$ depends upon the collision frequencies for ionization and attachment processes and upon the total momentum-transfer collision frequency. for large $\omega_b\tau$ the order of magnitude of v_E is $v_d/(\omega_b\tau)$, and $v_E B/n_g$ depends only on v_d, so we see that α/B depends only upon v_d. (For small $\omega_b\tau$ it is well known that the electron energy distribution and α/n_g depend only on E/n_g.)

3. Extended Applications

From the method of solution we have developed, it is clear that only the speed of the gas molecules in the drift frame is important in determining the electron energy distribution. Because of this, we can also use our equations when the gas molecules have a distribution of velocities in the gas frame. If the distribution of molecular speed U in the drift frame is H(U), where $\int_0^\infty H(U)dU = 1$, then our basic equation (II-28) is simply generalized to

$$\frac{\partial f(\epsilon, t)}{\partial t} = n_g \int f(\epsilon', t) d\epsilon' \int_0^\infty \chi(\epsilon, \epsilon'; U) H(U) dU \quad . \tag{II-32}$$

For example, if the molecular velocity distribution is isotropic in the gas frame, where the distribution of molecular speeds is h(u) with $\int_0^\infty h(u)du = 1$, we have

$$H(U) \equiv \int P(U; u) h(u) du = \frac{U}{2v_d} \int_{|U-v_d|}^{U+v_d} \frac{h(u)}{u} \, du \tag{II-33}$$

as follows from changing the notation in Eq. (II-26). Notice that these formulas can also be used when $v_d = 0$; in this case the presence of the magnetic field is not important.

At least to a good approximation, the drift speed in the drift frame is given by Eq. (II-31), even when the gas has a finite "temperature."

When $\epsilon \gg mv_d^2$, so that the simplified form of $\chi(\epsilon, \epsilon'; v_d)$ as derived in Appendix D can be used, the entire dependence of $\chi(\epsilon, \epsilon'; v_d)$ upon v_d appears

in the factor v_d^2 in Eq. (D-6) for elastic collisions. In this case the integral over U in Eq. (II-32) simply replaces v_d^2 in Eq. (D-6) by the mean square speed of the molecules in the drift frame. From Eq. (II-33) this is $v_d^2 + \langle u^2 \rangle$, which slightly generalizes the result of the "usual" method as given in Eq. (II-11).

Since, if the gas is to remain "slightly" ionized, $(M/2)\langle u^2 \rangle$ must be small in comparison with the ionization energy of the gas, there actually is no case of physical interest where the simplified form of $\chi(\epsilon, \epsilon'; v_d)$ cannot be used and where $\langle u^2 \rangle$ cannot be neglected.

III. NUMERICAL COMPUTATION OF THE IONIZATION RATE IN HYDROGEN GAS

At the time these computations were done, the detailed theory as presented in Part II was not complete. The computational procedure, which was designed only to give an estimate of the ionization rate, will appear crude in comparison with this theory. However, as will become evident, the cross sections are so poorly known that a more detailed numerical computation would be unwarranted at the present time anyway. While the theory is useful at all nonrelativistic values of v_d, we use simplified forms for the collision terms that are only valid for v_d below 6×10^7 cm/sec or, equivalently, for $mv_d^2/2$ below 1 eV.

As discussed in the introduction, each collision process has one or two primary effects and several secondary effects upon the electron energy distribution. We will treat only the primary effects, which are as follows: elastic collisions cause heating of the electrons and spreading of the electron-energy distribution, inelastic collisions remove energy from the electrons, and ionization removes energy from the electrons and increases the total number of free electrons. In hydrogen, electron attachment can be neglected. Among the inelastic collision processes we will include electronic excitation, vibrational excitation, and dissociation processes, but we will consider values of v_d large enough so that the effects of rotational excitation are unimportant (see Sec. III-A). Notice that the "spreading" effect of elastic collisions is very important in filling out the "tail" of the energy distribution and thus in determining the ionization rate.

A. The Collision Terms Used

We will now discuss the cross sections and the approximations used in evaluating the collision terms of our basic equation

$$\frac{\partial f(\epsilon, t)}{\partial t} = \sum_j \left(\frac{\partial f}{\partial t}\right)_j \quad . \tag{III-1}$$

1. For Inelastic Collisions

The collision frequencies $\nu^j(\epsilon)$ and energy transfers ϵ_j given in Fig. 2 are used in the formula

$$\left(\frac{\partial f}{\partial t}\right)_j = \nu^j(\epsilon + \epsilon_j) f(\epsilon + \epsilon_j, t) - \nu^j(\epsilon) f(\epsilon, t) \quad . \tag{III-2}$$

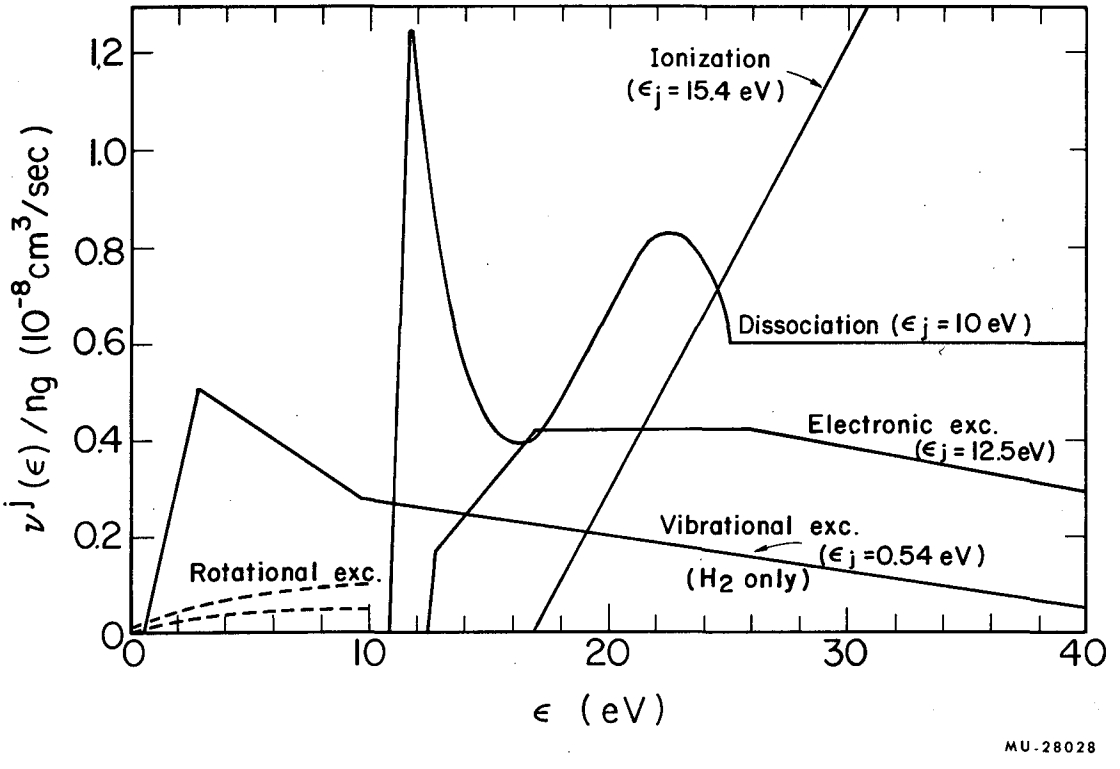

Fig. 2. Collision frequencies ν^j and energy transfers ϵ_j used in the numerical computations.

This corresponds to the first term of an expansion in powers of v_d as given by Eq. (D-5) in Appendix D, and it has a clear physical interpretation. This approximation is good when $\epsilon \gg (m/2)v_d^2$, so for $1/2\ mv_d^2$ below 1 eV, it is good for most of the electrons, particularly those that contribute to ionization.

The collision frequencies for rotational excitation shown in Fig. 2 correspond to the cross sections determined from swarm data by Frost and Phelps.[4] The collision frequencies and energy transfers are so small that the effects of rotational excitation can be neglected for values of v_d above 3×10^6 cm/sec.

Figure 2 shows the collision frequency for excitation of the first excited vibrational state of H_2 by electron impact, essentially as found by Frost and Phelps.[4] It is consistent with the experimental results of Ramien[5] and Schultz,[6] but it disagrees with all theoretical computations, including that of Chen and Magee.[7] The experimental results by Schultz indicate that the excitation of higher vibrational states can be ignored. The vibrational excitation of D_2 has not been investigated.

The collision frequency for molecular dissociation shown in Fig. 2 was determined from the theoretical results of Edelstein.[8] We must recognize that using this collision frequency with $\epsilon_j = 10$ eV cannot account for the effects of molecular dissociation exactly, since the energy loss in exciting the $1^3\Sigma_\mu^+$ state can vary from about 8.8 eV to about 15 eV. An energy loss of about 10 eV is the most probable, and Edelstein assumed that $\epsilon_j = 10$ eV in calculating the given cross section. This cross section exhibits the "anomalous" threshold behavior observed in some beam experiments,[9] and its magnitude is consistent with swarm experiments by Poole,[9] and by Corrigan and von Engel.[10] The odd peaked behavior also agrees roughly with observations by Schultz[11] and by Kruithof and Ornstein.[12] Although the excitation of higher triplet states also leads to dissociation, for our purpose we can consider this simply as electronic excitation.

The cross sections for the excitation of various electronic states are almost completely unknown. For this reason, the effects of all but the $1^3\Sigma_\mu^+$ state have been combined into the collision frequency shown in Fig. 2. The shape of this collision frequency was modelled after that of helium, and the magnitude was adjusted to agree roughly with the swarm data of Corrigan and von Engel.[10] The computational procedure suggested by Gryzinski,[13] which has had considerable success in predicting other cross sections, gives

a collision frequency that is an order of magnitude larger than the one shown in Fig. 2. Because this collision frequency is so uncertain, a variable multiplicative factor is introduced.

2. For Ionizing Collisions

Only single-ionization collisions to the ground state of H_2^+ are included. The collision frequency $\nu^{ion}(\epsilon)$ given by Figs. 2 and 3 is used with $\epsilon_j = 15.4$ eV in the formula

$$\left(\frac{\partial f}{\partial t}\right)_{ion} = 2\int \nu^{ion}(\epsilon')f(\epsilon', t)P^{ion}(\epsilon;\epsilon')d\epsilon' - \nu^{ion}(\epsilon)f(\epsilon, t) \quad , \qquad \text{(III-3)}$$

where $P^{ion}(\epsilon;\epsilon')$ is the probability distribution of energy ϵ of the electrons leaving ionizing collisions in which the incident electrons have energy ϵ'. This formula is the first term of an expansion in powers of v_d as given by Eq. (D-7) in Appendix D; thus the discussion of the validity of Eq. (III-2) also applies to Eq. (III-3).

The collision frequency for ionization shown in Figs. 2 and 3 is essentially that of Tate and Smith.[14]

The distribution $P^{ion}(\epsilon;\epsilon')$ has not been studied in hydrogen. For a single-ionization process we know that $P^{ion}(\epsilon;\epsilon')$ is zero unless $0 \leq \epsilon \leq \epsilon' - \epsilon_j$ and that neglecting recoil, $P^{ion}(\epsilon;\epsilon') = P^{ion}(\epsilon' - \epsilon_j - \epsilon;\epsilon')$. In using Eq. (III-3) we assume that

$$P^{ion}(\epsilon;\epsilon') = \frac{1}{\epsilon' - \epsilon_j} \qquad \text{for} \quad 0 \leq \epsilon \leq \epsilon' - \epsilon_j \quad ,$$

i.e., that the distribution of ϵ is uniform. Theoretical results of applying the Born approximation to helium and to atomic hydrogen indicate that this assumption is quite reasonable for ϵ' below about 40 to 50 eV. With this assumption Eq. (III-3) becomes

$$\left(\frac{\partial f}{\partial t}\right)_{ion} = 2\int_{\epsilon+\epsilon_j}^{\infty} \frac{\nu^{ion}(\epsilon')f(\epsilon';t)}{\epsilon' - \epsilon_j}\, d\epsilon' - \nu^{ion}(\epsilon)f(\epsilon, t) \quad . \qquad \text{(III-4)}$$

3. For Elastic Collisions

Because the angular distribution of the scattered electrons is poorly known, we use

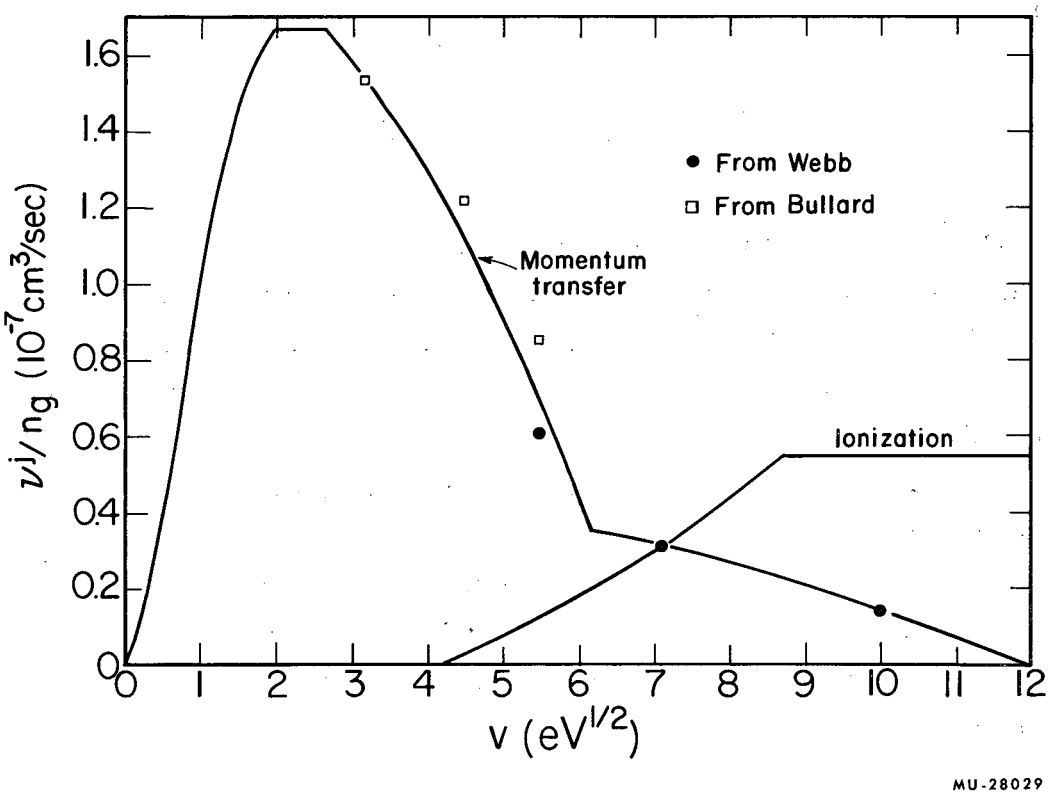

Fig. 3. Collision frequencies used in the numerical computations.

$$\left(\frac{\partial f}{\partial t}\right)_{el} = \int G^{el}(\epsilon;\epsilon')f(\epsilon', t)d\epsilon' - N_m^{el}(\epsilon)f(\epsilon, t) \quad , \tag{III-5}$$

where

$$G^{el}(\epsilon;\epsilon') = \frac{1}{4mv_d^2V'}\int \nu_m^{el}(v')\left[1 - \frac{m}{Mv_d^2}(v'^2 + v_d^2 - V'^2)\right.$$

$$\left. + \left(\frac{mv'}{Mv_d}\right)^2\right]^{-1/2} dv' \tag{III-6}$$

and

$$N_m^{el}(\epsilon) = \frac{1}{2v_dV}\int_{|V-v_d|}^{V+v_d} \nu_m^{el}(v)v\,dv \quad , \tag{III-7}$$

as was suggested in Section II-C. In Eq. (III-6), $\epsilon' = (m/2)V'^2$, and the limits are given by $|V' - v_d| \leq v' \leq V' + v_d$ and by Eq. (B-18).

The elastic momentum-transfer collision frequency used is given in Fig. 3. Below 7 eV, the results of Frost and Phelps[4] are used. At higher energies the cross section for elastic scattering was obtained by subtracting the inelastic and ionization cross sections from the total cross section found by Normand.[15] The values of the elastic momentum transfer cross section at 30, 50, and 100 eV we calculated using angular distributions measured by Webb,[16] and the values at 10, 20, and 30 eV we calculated from the results in helium at 20, 30, and 50 eV as suggested by Bullard and Massey.[17] It is clear that the elastic momentum-transfer collision frequency shown is uncertain at high energies. This collision frequency was taken as zero above 143 eV as a convenient way of limiting the range of energy over which the energy distribution must be calculated.

To simplify the equations even further, the integration in Eq. (III-6) was done explicitly by assuming that, over each individual region of integration, σ_m^{el} is effectively constant if $\epsilon' < 3$ eV, or ν_m^{el} is effectively constant if $\epsilon' > 3$ eV. The same procedure was used in explicitly evaluating Eq. (III-7). It is easy to verify that this gives about the correct rate of energy spreading, but, in retrospect, it turns out that this assumption does not give the correct rate of heating of the electrons.

For energies below about 11 eV, this procedure gives the rate of energy gain from elastic collisions to within about 15% of that given by Eq. (II-27). The error changes sign at several energies so the error introduced in the overall heating of the electrons by elastic collisions is much less than 15%.

For energies above 11 eV, this procedure gives a rate of energy gain from elastic collisions that is considerably higher than that given by Eq. (II-27), but for these energies the energy gain from elastic collisions is unimportant in comparison with the energy loss from inelastic collisions and ionization unless $1/2\ mv_d^2$ is large in the sense of Eq. (C-3).[18] If $1/2\ mv_d^2$ is large, our treatment of inelastic collisions and ionization gives too large an energy loss since in Eq. (C-2) we have accounted only for the term involving ϵ_j. Thus we have two errors which tend to cancel. To investigate how well they cancel, it is necessary to estimate the momentum-transfer collision frequencies for inelastic collisions and ionization. These estimates indicate that our procedure gives an overall rate of energy gain that is too high in comparison with that given by Eq. (C-3). The largest fractional error occurs at about 25 eV and is of the order of $mv_d^2/3eV$, but the error introduced into the overall energy balance is much smaller.

We conclude that while our treatment of the collision terms is somewhat crude, the errors introduced appear to be smaller than the uncertainty in the collision frequencies. While all of the collision frequencies are somewhat in doubt, the collision frequency for electronic excitation is most uncertain.

B. The Computational Procedure

We are interested in finding the electron-energy distribution $f_0(\epsilon)$ and the ionization rate β as determined by the eigenvalue equation (II-30). Holstein[3] has used physical arguments to show that the eigenfunction $f_0(\epsilon)$ of interest has the largest eigenvalue β/n_g. This suggests that the Rayleigh-Ritz procedure could be used, particularly to find the eigenvalue β/n_g, but we will use a more physical approach.

It is clear that Eq. (II-30) can also be solved by putting an electron-energy distribution into Eq. (II-28) and calculating its time evolution until it approaches the form of Eq. (II-29). The resulting iteration procedure consists of calculating a change in $f(\epsilon, t)$ from

$$\Delta f(\epsilon, t) = (n_g \Delta t) \int \chi(\epsilon, \epsilon'; v_d) f(\epsilon', t) d\epsilon \qquad \text{(III-8)}$$

and adding this to $f(\epsilon, t)$ to obtain $f(\epsilon, t + \Delta t)$. The eigenfunction $f_0(\epsilon)$ is obtained by normalizing $f(\epsilon, t)$ after enough iterations have been made so that $\Delta f(\epsilon, t)$ and $f(\epsilon, t)$ have the same dependence on ϵ. All quantities of interest, including the eigenvalue β/n_g, can be calculated from $f_0(\epsilon)$. This iteration procedure was used in the numerical computations.

It is clear physically that if Δt is chosen small enough, the above procedure will converge to the correct eigenfunction. However, to minimize the number of iterations needed, Δt must be chosen as large as possible. It was sometimes found that if Δt was chosen larger than τ, the collision time for an electron, the iteration method converged to the wrong eigenfunction. This problem could probably have been eliminated by restricting $f(\epsilon, t)$ to be nonnegative.

To estimate the time required for the energy distribution to approach the solution (II-29), we notice from Eq. (II-27) that the average energy gain per elastic collision is of the order of mv_d^2. The time required is thus of the order of $(\bar{\epsilon}/mv_d^2)\tau$, where $\bar{\epsilon}$ is the final mean energy of the electrons. Because the electron drift speed along $-\vec{E}$ in the lab frame is of the order of $v_d/(\omega_b \tau)$, this time corresponds to the electrons drifting along $-\vec{E}$ by a distance of the order of $\bar{\epsilon}/eE$, as also follows from energy considerations alone. These estimates were verified by the numerical computations.

The procedure outlined above was used to compute $f_0(\epsilon)$ numerically for various values of v_d in hydrogen gas. In evaluating the collision terms found in the previous section, only the integral in Eq. (III-5) for elastic collisions was difficult. The region of ϵ' for which $G^{el}(\epsilon; \epsilon')$ exists for a given value of ϵ was divided into ten equal intervals, and the integral was evaluated by using a straight-line approximation to $G^{el}(\epsilon; \epsilon') f(\epsilon', t)$ in each of these intervals. The energy distribution $f_0(\epsilon)$ was calculated at 204 values of ϵ between 0 and 170 eV; the spacing of these points varied from 0.005 eV for $\epsilon < 0.1$ eV to 2.5 eV for $\epsilon > 70$ eV. On an IBM 7090 computer, each iteration took less than 2 sec, and the number of iterations required for adequate convergence varied from 40 for the highest value of v_d to 650 for the lowest.

Once $f_0(\epsilon)$ is known, the other quantities of interest are easily calculated. The eigenvalue β/n_g is given by

$$\beta/n_g = \int \left(\frac{\nu^{ion}(\epsilon)}{n_g} \right) f_0(\epsilon) d\epsilon \quad . \tag{III-9}$$

The mean energy of the electrons in the drift frame is

$$\bar{\epsilon} = \int \epsilon f_0(\epsilon) d\epsilon \quad , \tag{III-10}$$

and from Eq. (B-2), the mean kinetic energy of the electrons in the lab frame is $\bar{\epsilon} + 1/2\ m v_d^2$. The components of the diffusion tensor are calculated from

$$D_T = \frac{2}{3m} \int \left(\frac{\nu_m^{el}(\epsilon)}{\omega_b^2} \right) \epsilon f_0(\epsilon) d\epsilon \quad ,$$

$$D_\perp = \frac{2}{3m} \int \frac{1}{\omega_b} \epsilon f_0(\epsilon) d\epsilon = \frac{2\bar{\epsilon}}{3m\omega_b} \quad , \tag{III-11}$$

and

$$D_{||} = \frac{2}{3m} \int \frac{1}{\nu_m^{el}(\epsilon)} \epsilon f_0(\epsilon) d\epsilon \quad ,$$

which just correspond to Eq. (II-13) and are thus only valid when the "usual" method of solving the Boltzmann equation is valid.

To calculate the Townsend coefficient α we must evaluate v_E from Eq. (II-31). However, the momentum-transfer collision frequencies for the inelastic and ionization processes are not known. For this reason Eq. (II-31) was evaluated first by assuming that the momentum-transfer collision frequencies equal the corresponding collision frequencies and then by assuming that only forward scattering occurs so that these momentum-transfer collision frequencies have their minimum possible values.

C. Results and Comparison with Experiment

Some of the computed results can be compared directly with recent experimental results by Bernstein.[19, 20]

The calculated values of β/n_g are shown by curve (1) of Fig. 4, and the calculated values of $Bv_E/n_g v_d$ are shown by curve (1) of Fig. 5 in comparison with the experimental results. The cross-hatched area of Fig. 5

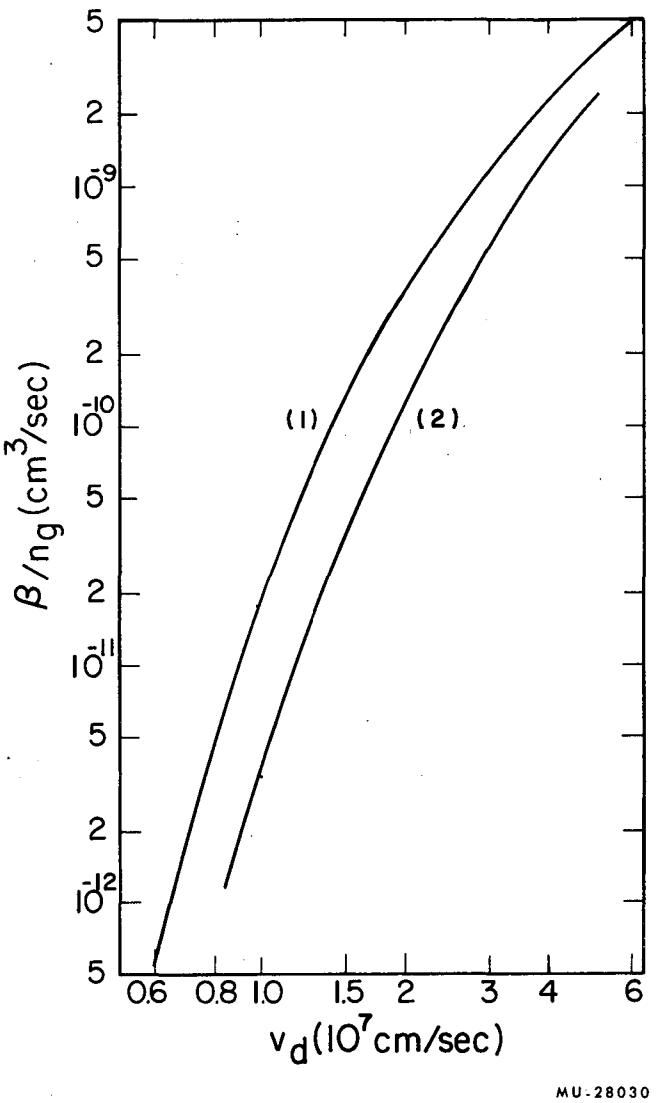

Fig. 4. Calculated values of the ionization rate β in H_2 gas using the collision frequency for electronic excitation shown in Fig. 2 multiplied by unity for curve (1) and by 4.7 for curve (2).

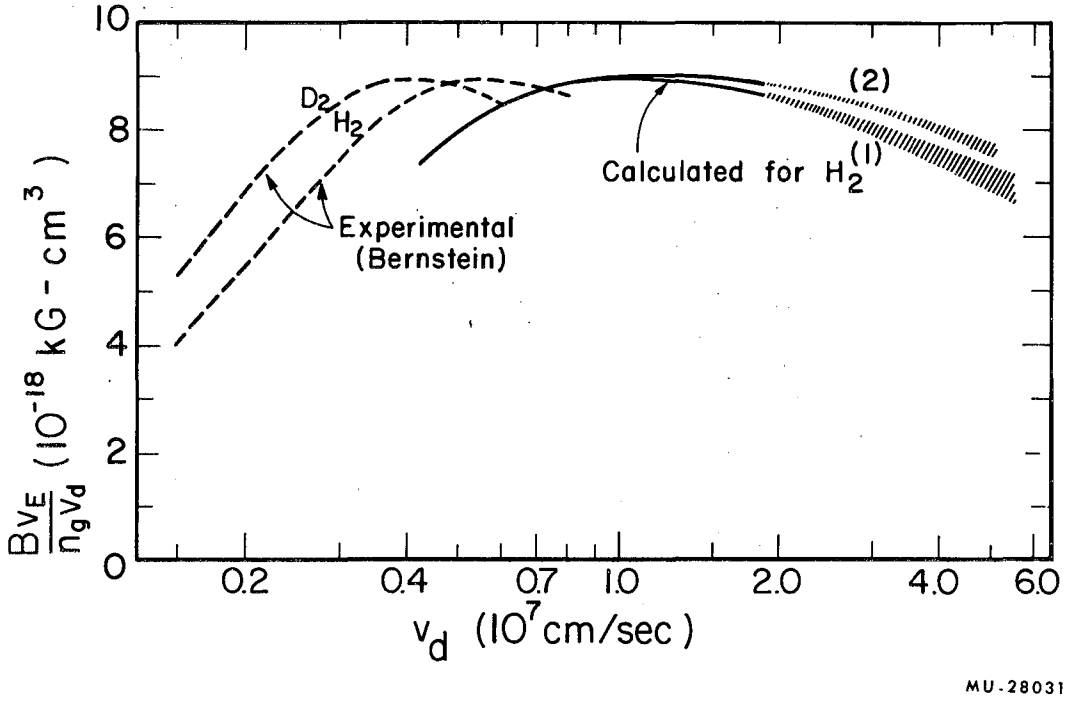

Fig. 5. The electron drift speed v_E in the drift frame. The collision frequency for electronic excitation shown in Fig. 2 was multiplied by unity for curve (1) and by 4.7 for curve (2).

indicates the uncertainty introduced by our lack of knowledge of the momentum-transfer collision frequencies for inelastic and ionization processes. The results in Fig. 5 depend mostly on the elastic momentum-transfer collision frequency, which we have noted is somewhat uncertain.

The values of α/B calculated from curves (1) of Figs. 4 and 5 are shown by curve (1) of Fig. 6 in comparison with Bernstein's experimental results. The values agree within a factor of 2 to 6, which is not bad for an initial attempt. The experimental values are considered to be more accurate than the calculated ones, because of the rather large uncertainties in the cross sections used.

To obtain better agreement with the experimental values of α/B, the collision frequency for electronic excitation was increased to 4.7 times that shown in Fig. 2. The results are shown by curves (2) in Figs. 4, 5, and 6. We see that the calculated results are useful in extrapolating the experimental values of α/B to higher and lower values of v_d and in extrapolating the experimental values of $v_E B/v_d n_g$ to higher v_d.

Some typical examples of the calculated energy distributions are shown in Figs. 7, 8, and 9 in comparison with Maxwellian distributions with the same mean energy. We see that the calculated distributions resemble the Maxwellian distributions fairly closely except far out in the "tail" where the calculated distributions are much smaller than the Maxwellian distributions.

The calculated and experimental values of $n_g D_{||}$ are compared in Fig. 10. Using this experimental data and that shown in Fig. 5 and using an analysis based on Maxwellian energy distributions, Bernstein[19] deduced that the elastic momentum-transfer collision frequency is 10 to 20% higher than that shown in Fig. 3 for energies of 0.3 to 4.0 eV. Such an assumption would improve the agreement between the calculated and experimental results in Figs. 5 and 10. From the same analysis, Bernstein deduced the mean energy of the electrons, which is compared with the calculated values in Fig. 11.

From Eq. (III-11) we see that $BD_{||}$ is proportional to the mean energy as given in Fig. 11. The calculated values of $B^2 D_T/n_g$ are shown in Fig. 12.

The experimental results for D_2 gas shown in Figs. 5, 10, and 11 differ considerably from those for H_2. As has been pointed out, the vibrational excitation of D_2 by electron impact has not been investigated. To account for the magnitude of the difference in the experimental results for D_2 and H_2, one

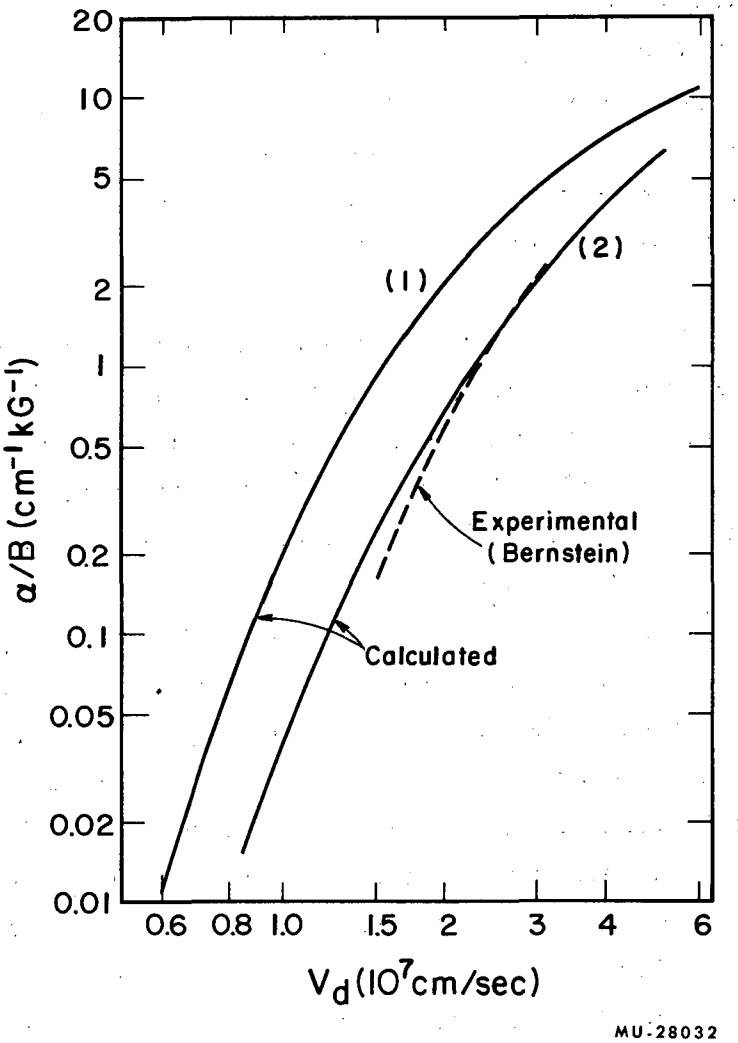

Fig. 6. The ratio of the Townsend coefficient α to the magnetic field B in H_2 gas. The collision frequency for electronic excitation shown in Fig. 2 was multiplied by unity for curve (1) and by 4.7 for curve (2).

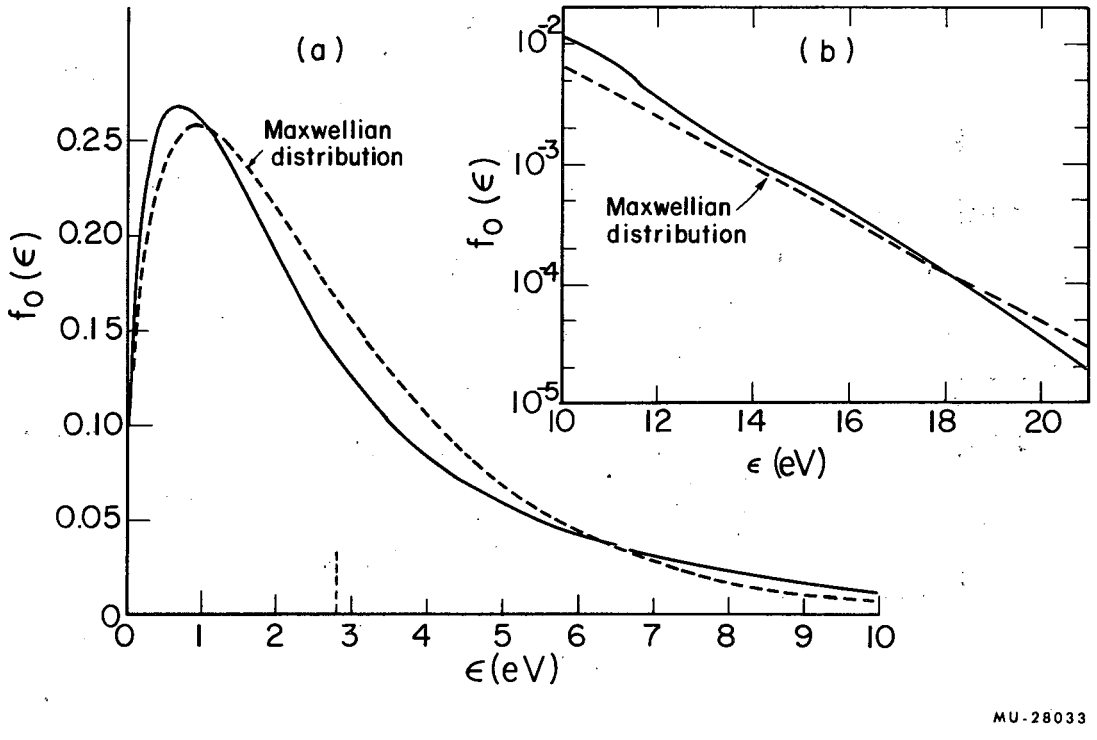

Fig. 7. The electron-energy distribution in H_2 calculated with $v_d = 5.93 \times 10^6$ cm/sec (so $mv_d^2/2 = 0.01$ eV) and using the collision frequency for electronic excitation, shown in Fig. 2, multiplied by 4.7. The Maxwellian distribution shown has the same mean energy, which is indicated by the vertical dashed line. The inserted drawing (b) shows the "tail" of the distribution on a logarithmic scale.

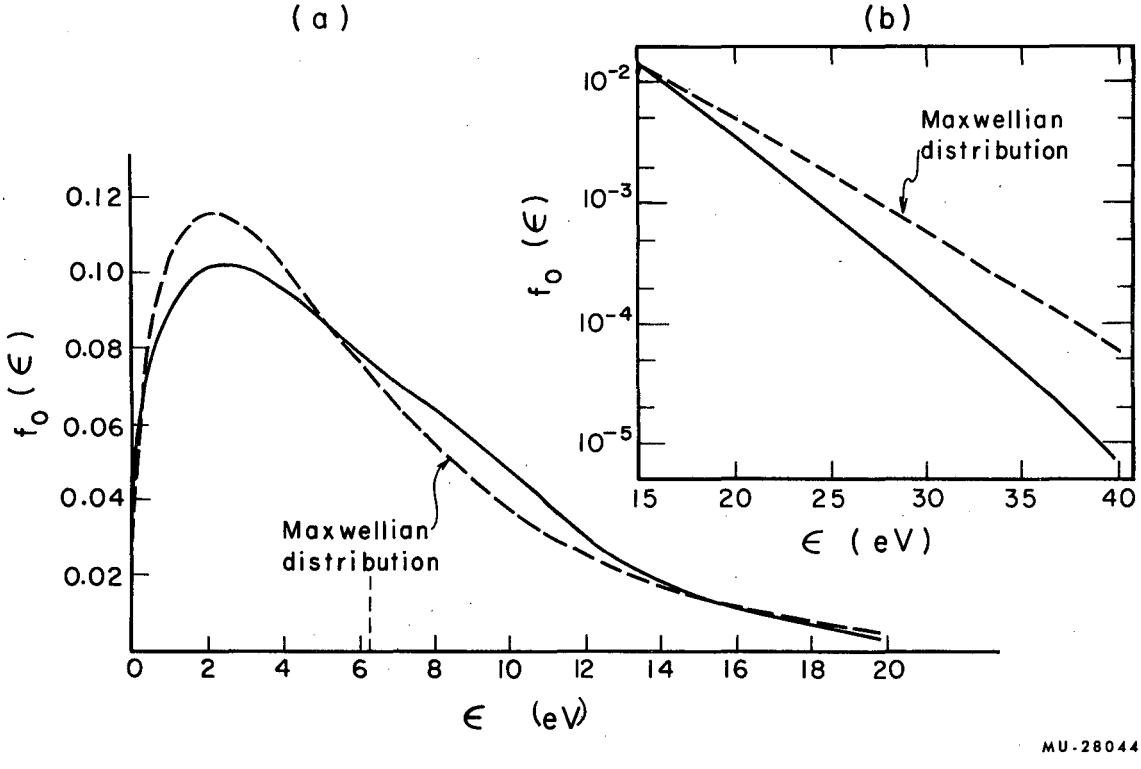

Fig. 8. The electron-energy distribution in H_2 calculated with $v_d = 1.88 \times 10^7$ cm/sec (so $mv_d^2/2 = 0.1$ eV) and using the collision frequency for electronic excitation, shown in Fig. 2, multiplied by 4.7. The Maxwellian distribution shown has the same mean energy, which is indicated by the vertical dashed line. The inserted drawing (b) shows the "tail" of the distribution on a logarithmic scale.

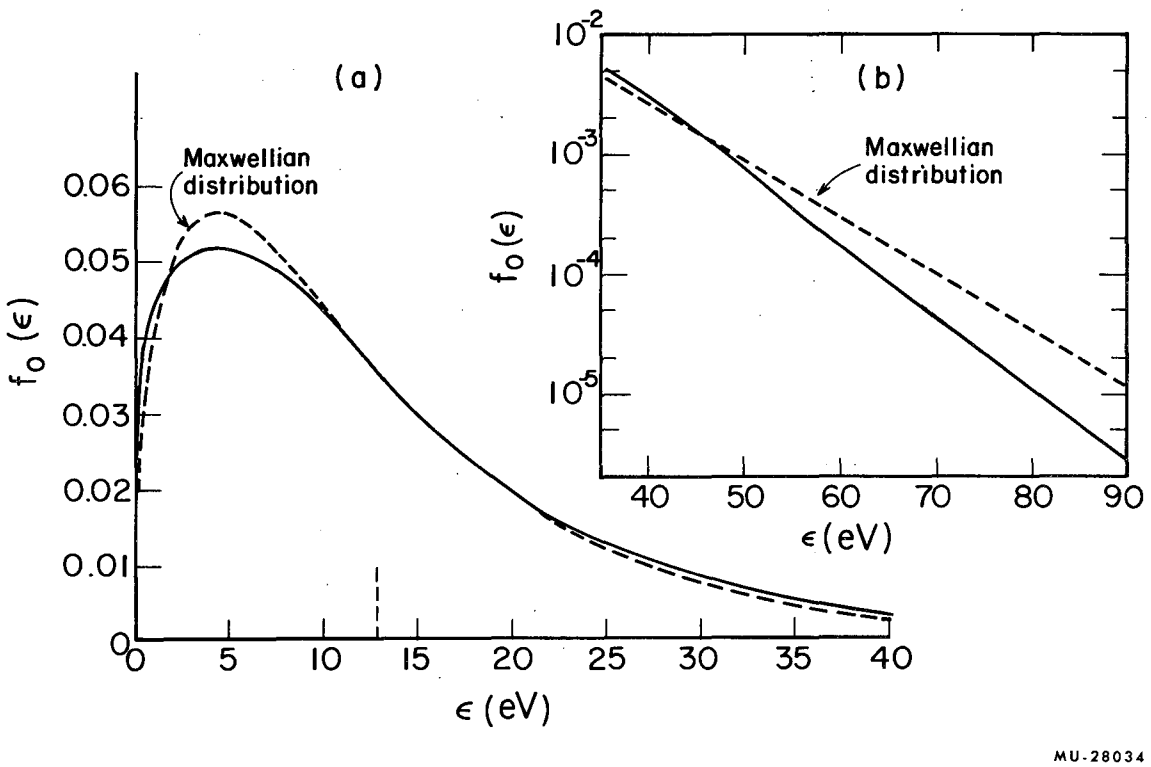

Fig. 9. The electron-energy distribution in H_2 calculated with $v_d = 5.14 \times 10^7$ cm/sec (so $mv_d^2/2 = 0.75$ eV) and using the collision frequency for electronic excitation, shown in Fig. 2, multiplied by 4.7. The Maxwellian distribution shown has the same mean energy, which is indicated by the vertical dashed line. The inserted drawing (b) shows the "tail" of the distribution on a logarithmic scale.

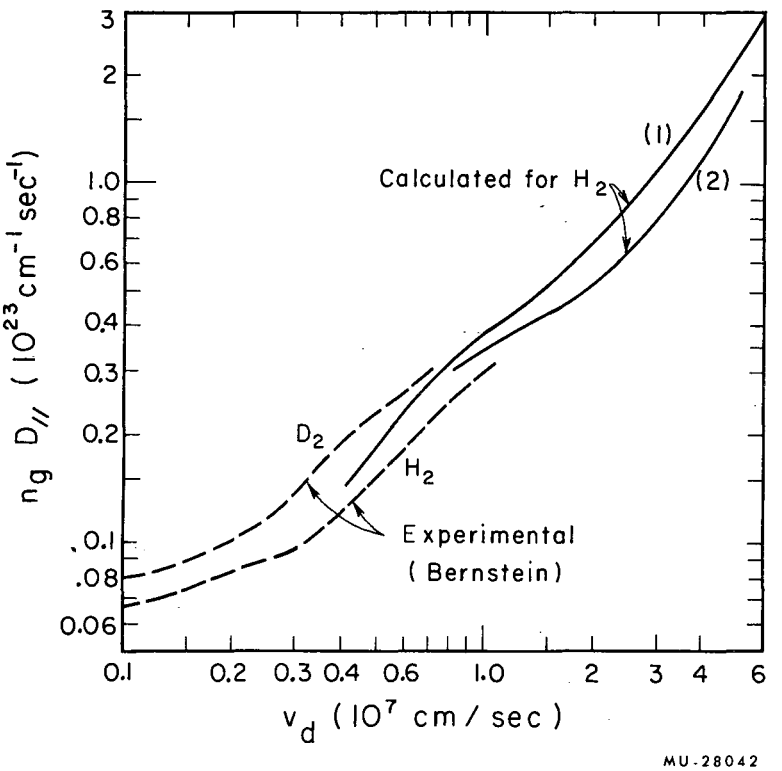

Fig. 10. The coefficient of diffusion along the magnetic field. The collision frequency for electronic excitation as shown in Fig. 2 was multiplied by unity for curve (1) and by 4.7 for curve (2).

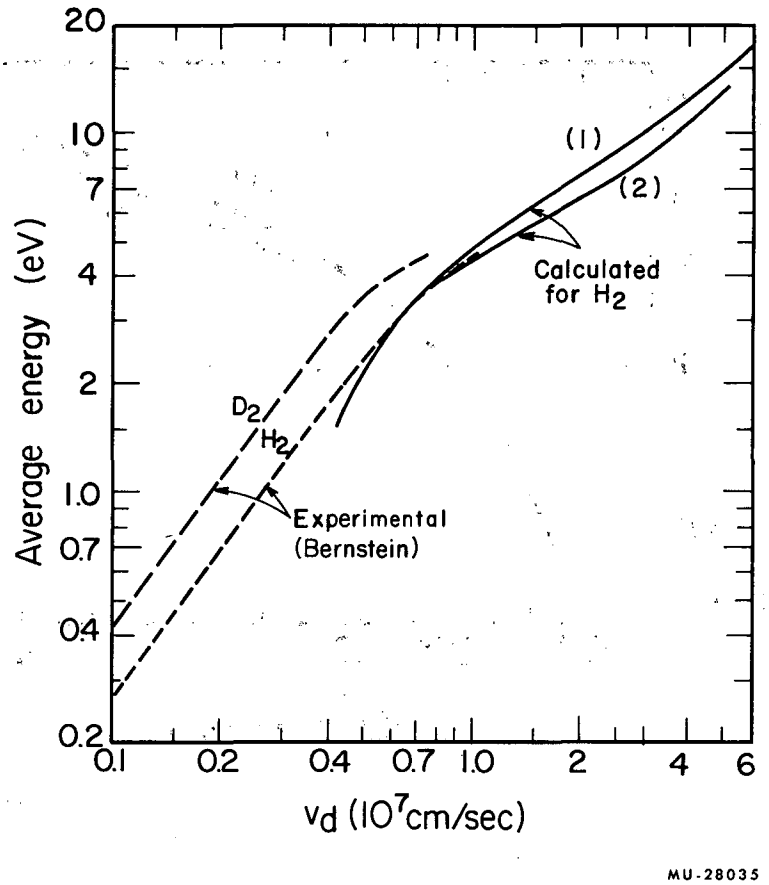

Fig. 11. The mean energy of the electrons in the gas frame. The collision frequency for electronic excitation shown in Fig. 2 was multiplied by unity for curve (1) and by 4.7 for curve (2).

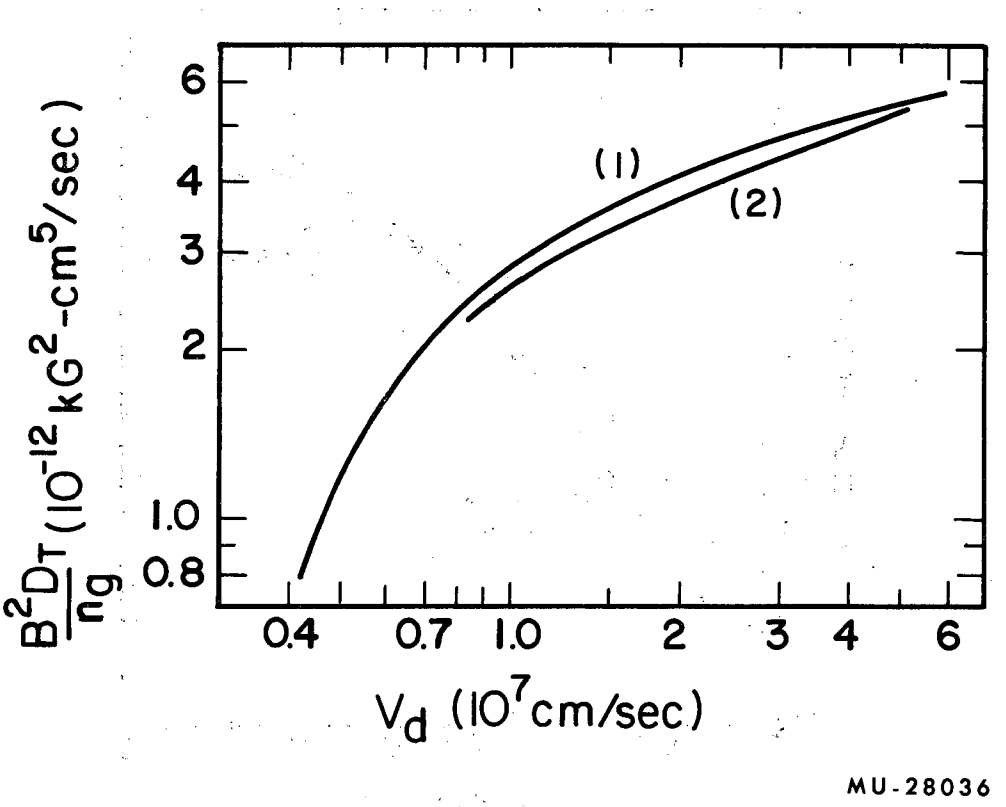

Fig. 12. The coefficient for diffusion across the magnetic field. The collision frequency for electronic excitation shown in Fig. 2 was multiplied by unity for curve (1) and by 4.7 for curve (2).

must assume that the energy loss to vibrational excitation in D_2 is only about 1/4 of that in H_2. Above about $v_d = 10^7$ = cm/sec, the results in H_2 and D_2 should be about the same.

Engelhardt and Phelps[21] are currently doing numerical computations to find cross sections for electronic excitation and dissociation that will fit the swarm data. Using these results and those of Frost and Phelps, they are also doing calculations to compare with Bernstein's experimental results. Their results should be more accurate than ours, mainly because of the cross sections used.

IV. CONCLUSION

The equations we have derived are suitable for calculating the electron energy distribution in the drift frame, where the electron velocity distribution is expected to be nearly isotropic. The other quantities of interest can also be calculated. When the rms molecular speed in this frame is very small in comparison with that of the electrons, the complicated equations can be greatly simplified; the resulting equations agree with those of the "usual" method of solving the Boltzmann equation by expanding the electron-velocity distribution in spherical harmonics in the rest frame of the gas.

From this approach we have gained insight into the physics and mathematics of the problem, particularly in regard to elastic collisions. Specifically, we have shown that in the drift frame:

(1) Elastic collisions always tend to make the electron-energy distribution a Maxwellian distribution with mean energy equal to that of the gas molecules, which we denote by $\mathcal{E}$.

(2) For electrons with energies far below $\mathcal{E}$, the average energy gain per elastic collision is of order $2m\mathcal{E}/M$.

(3) Under certain conditions the effect of elastic collisions upon the electron-velocity distribution has the mathematical form of a diffusion in velocity space.

(4) The mean guiding-center shift per collision in the direction of $-\vec{E}$ in the gas frame is of the order of v_d/ω_b.

Although the numerical computations for H_2 gas were somewhat crude, a more accurate analysis would have been unwarranted because of the large uncertainty in some important cross sections. In view of this, we consider the agreement of the calculated results with recent experimental results to be reasonably good.

ACKNOWLEDGMENTS

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APPENDICES

A. Cross Sections of Interest and Their Properties

In this appendix we establish the nomenclature and notation for the various cross sections of interest in the scattering of electrons from neutral molecules and for the corresponding collision frequencies. We are primarily interested in the effects of such collisions upon the electron energy distribution.

Consider a general collision in which an electron with velocity $\vec{v}'$ collides with a molecule at rest and n_j electrons leave the collision with velocities $\vec{v}_i$, where $i = 1, 2, \ldots n_j$. For each collision process j there is a discrete energy transfer $\epsilon_j = (m/2)a_j$ associated with the change in the internal state of the molecule. We assume that on the average the distribution of the electrons leaving the collision is symmetric about the direction of $\vec{v}'$, and we define $\cos\theta_i = (\vec{v}' \cdot \vec{v}_i)/v'v_i$.

In each such process the cross section $\sigma^j(v')$ and the collision frequency $\nu^j(v') = n_g v' \sigma^j(v')$ have a clear meaning, except perhaps in elastic scattering where forward scattering is identical to no scattering. The rate of transfer of momentum from the electrons to the molecules is determined by

$$\nu^j(v')\left\langle v' - \Sigma v_i \cos\theta_i \right\rangle_j = v'\nu_m^j(v') \quad , \tag{A-1}$$

which serves to define the momentum-transfer collision frequency

$$\nu_m^j(v') = n_g v' \sigma_m^j(v')$$

for process j, where $\sigma_m^j(v')$ is called the momentum-transfer cross section for the process. Notice that $\sigma_m^j(v')$ is well defined even for elastic collisions since forward scattering does not contribute.

To illustrate the importance of these cross sections, we consider the rate of energy transfer from the electrons by process j. By eliminating the final velocity of the molecule from the equations of conservation of energy and momentum, we find

$$v'^2 = \Sigma v_i^2 + a_j + m(\Sigma \vec{v}_i - \vec{v}')^2/M$$

or by expanding the last term and substituting $\Sigma v_i^2 = v'^2 - a_j$ because m/M is

small, we get

$$v'^2 - \Sigma v_i^2 = a_j + \frac{2mv'}{M}(v' - \Sigma v_i \cos\theta_i) + \frac{2m}{M}\sum_{i>\ell}\sum_{\ell}\vec{v}_i \cdot \vec{v}_\ell \qquad \text{(A-2)}$$

to within terms of higher order in m/M. Using this result the rate of energy transfer from the electrons by process j is determined by

$$\nu^j(v')\left\langle \frac{m}{2}(v'^2 - \Sigma v_i^2)\right\rangle_j = \nu^j(v')\epsilon_j + \nu_m^j(v')\frac{2m}{M}\left(\frac{1}{2}mv'^2\right)$$

$$+ \nu^j(v')\frac{m^2}{M}\left\langle \sum_{i>\ell}\sum_{\ell}\vec{v}_i \cdot \vec{v}_\ell \right\rangle_j \qquad \text{(A-3)}$$

and is thus determined completely by the collision frequency and the momentum-transfer collision frequency for the process, except in ionizing collisions where the last term contributes. Notice that the last term involves quantities which are not easily measurable.

In an electron-attachment process, $n_j = 0$, so the summations in the above formulas vanish. Thus Eq. (A-1) implies $\nu^j(v') = \nu_m^j(v')$, Eq. (A-2) implies $a_j = v'^2$, and the rate of energy transfer from the electrons given by Eq. (A-3) reduces to $\nu^j(v')(mv'^2/2)$.

In elastic or inelastic processes, $n_j = 1$. These two-body collisions are characterized completely by a discrete energy transfer ϵ_j (which is zero for elastic collisions) and a differential cross section $\sigma_\theta^j(\theta; v')$. The cross section is

$$\sigma^j(v') = 2\pi\int_0^\pi \sigma_\theta^j(\theta; v')\sin\theta\, d\theta \ . \qquad \text{(A-4)}$$

The momentum-transfer cross section can be determined from Eq. (A-1) by using $\sigma_\theta^j(\theta; v')$ and $v = (v'^2 - a_j)^{1/2}$ from Eq. (A-2); the result is

$$\sigma_m^j(v') = 2\pi\int_0^\pi \sigma_\theta^j(\theta; v')\sin\theta\left[1 - \left(1 - \frac{a_j}{v'^2}\right)^{1/2}\cos\theta\right]d\theta \ . \qquad \text{(A-5)}$$

Notice that the last term in Eq. (A-3) vanishes for both elastic and inelastic

processes and that the first term vanishes for elastic collisions. Both Eqs. (A-3) and (A-5) reduce to the well-known results for elastic collisions.

For ionization processes, $n_j \geq 2$, and again there is a discrete energy transfer ϵ_j. The two quantities that are the most easily observed experimentally are the cross section $\sigma^j(v')$ and the distribution in angle and speed of the electrons resulting, which we denote $I^j(\theta, v; v')$ and which is normalized as

$$2\pi \int_0^\pi d\theta \sin\theta \int dv\, I^j(\theta, v; v') = 1 \quad . \tag{A-6}$$

Using these quantities, the momentum-transfer cross section evaluated from Eq. (A-1) is

$$\sigma_m^j(v') = \sigma^j(v')\, 2\pi \int_0^\pi d\theta \sin\theta \int dv\, I^j(\theta, v; v')(1 - n_j v \cos\theta/v') \quad .$$

Notice that in principle the rate of energy transfer from the electrons could be evaluated by using these same quantities, although $I^j(\theta, v; v')$ will usually not be known sufficiently accurately. Usually Eq. (A-3) will give a more accurate result even though the last term is not known. This is because the last two terms of Eq. (A-3) are usually small compared with the first term since in ionization processes α_j is relatively large. At high energies where these terms could contribute, there is experimental evidence that the angular distribution of the "scattered" electrons has a pronounced forward peak while that of the "ejected" electrons is more nearly isotropic. This suggests that the third term of (A-3) is small compared with the second whenever either is important.

By summing Eqs. (A-1) and (A-3) over all collision processes, we find that the rate of energy transfer from the electrons is determined by

$$\nu_m(v') \frac{2m}{M} \left(\frac{1}{2} m v'^2 \right) + \sum_j \nu^j(v') \epsilon_j$$

where the last term in Eq. (A-3) has been omitted, and that the rate of momentum transfer from the electrons is determined by $v' \nu_m(v')$. The total momentum transfer collision frequency $\nu_m(v')$ is just the sum of $\nu_m^j(v')$ over all collision processes j. Notice that the total cross section, which is often measured in beam experiments,[15] has not appeared in our formulas.

B. Kinematic Functions in the Collision Terms

In the derivations we make extensive use of probability distributions. The probability distribution of a real quantity α is given by

$$P(\alpha) = \left. \frac{\partial P(\alpha < \alpha_0)}{\partial \alpha_0} \right|_{\alpha = \alpha_0}$$

where $P(\alpha < \alpha_0)$ is the probability that α is less than α_0. With this definition, $P(\alpha)$ is normalized such that $\int P(\alpha)\, d\alpha = 1$; in such integrals the integration is over all values of α, although usually the integrand is nonzero only over a finite range of α.

The probability distribution of α for a specified value of another parameter β is denoted $P(\alpha; \beta)$. In this case

$$P(\alpha) = \int P(\alpha; \beta)\, P(\beta)\, d\beta$$

where the above normalization is automatically obtained.

If β is a monotonically increasing function of α, then we have

$$P(\alpha < \alpha_0) = P\left(\beta < \beta(\alpha_0)\right)$$

and by differentiation we have

$$P(\alpha) = P(\beta) \left| \frac{\partial \beta}{\partial \alpha} \right|_{\beta = \beta(\alpha)}$$

where again $P(\alpha)$ is properly normalized. The same formula is valid if β is a monotonically decreasing function of α.

We adopt the convention that a probability distribution is nonzero only where the formula gives a positive, real result. For example,

$$P(\alpha) = 2(\pi)^{-1}(1 - \alpha^2)^{-1/2} \quad \text{for } \alpha \geq 0$$

implies that $P(\alpha)$ vanishes except in the region $0 \leq \alpha < 1$.

1. Derivation of $P(v; \epsilon)$

The equations of motion of a nonrelativistic electron in the gas frame with a homogeneous electric field $\vec{E} = E\, \hat{a}_y$ and a homogeneous magnetic

field $\vec{B} = B\hat{a}_z$ can be written in cartesian components as

$$v_x(t) = v_d - V \sin\xi \sin(\omega_b t + \gamma) \ ,$$

$$v_y(t) = V \sin\xi \cos(\omega_b t + \gamma) \ , \qquad \text{(B-1)}$$

and $$v_z(t) = V \cos\xi \ , \qquad \text{with } 0 \leq \xi \leq \pi \ ,$$

where V, ξ, and γ are constants determined by the initial conditions, $v_d = cE/B$ is the $\vec{E}\times\vec{B}$ drift speed, and $\omega_b = eB/mc$ is the electron-cyclotron frequency. The electron energy in the drift frame is

$$\epsilon = mV^2/2 = \left[m/2\right]\left[v^2(t) - 2v_d v_x(t) + v_d^2\right] \ , \qquad \text{(B-2)}$$

and the electron speed in the gas frame is given by

$$v^2(t) = V^2 + v_d^2 - 2v_d V \sin\xi \sin(\omega_b t + \gamma) \ . \qquad \text{(B-3)}$$

Consider a group of electrons with energy ϵ. According to the discussion in Sec. II-B-3, the most general such group of interest has random phases and a distribution of ξ which is symmetric about $\pi/2$. For the purpose of finding the probability distribution of $\mathcal{Z} = \sin(\omega_b t + \gamma)$, the distribution of phases may be normalized as

$$P(\omega_b t + \gamma) = 1/\pi \qquad \text{for} \qquad -\pi/2 \leq (\omega_b t + \gamma) \leq \pi/2 \ .$$

Since in this interval $\mathcal{Z}$ is a monotonically increasing function of $\omega_b t + \gamma$, we find

$$P(\mathcal{Z}) = (1 - \mathcal{Z}^2)^{-1/2}/\pi \ . \qquad \text{(B-4)}$$

From Eq. (B-3) we see that for specified values of $\epsilon = mV^2/2$ and ξ, v is a monotonic function of $\mathcal{Z}$. Thus we find

$$P(v;\, \xi, \epsilon) = v\left[v_d^2 V^2 \sin^2\xi - \frac{1}{4}(v_d^2 + V^2 - v^2)^2\right]^{-1/2}/\pi \ .$$

The most general distribution of ξ that is symmetric about $\pi/2$ can be written as

$$P(\xi;\epsilon) = \frac{1}{2}\sin\xi\left[1 + \sum_{s=1}^{\infty} A_s(\epsilon) P_{2s}(q)\right]$$

where $q = \cos\xi$ and $P_{2s}(q)$ are the even Legendre polynomials. This form is properly normalized since

$$\int_0^{\pi} \sin\xi \, d\xi = 2 \ ,$$

and

$$\int_0^{\pi} \sin\xi \, P_{2s}(\cos\xi)\, d\xi = 0 \ .$$

Using these results the desired function $P(v;\epsilon)$ is given by

$$P(v;\epsilon) = \int P(v;\xi,\epsilon)\, P(\xi;\epsilon)\, d\xi$$

$$= \frac{v}{2\pi v_d V} \int_{-q_0}^{q_0} \frac{dq}{(q_0^2 - q^2)^{1/2}} \left[1 + \sum_{s=1}^{\infty} A_s(\epsilon) P_{2s}(q)\right]$$

$$= \frac{v}{2v_d V}\left[1 + A_1(\epsilon)\left(\frac{3}{4}q_0^2 - \frac{1}{2}\right) + \cdots\right] \qquad \text{(B-5)}$$

$$\text{for } |V - v_d| \leq v \leq V + v_d$$

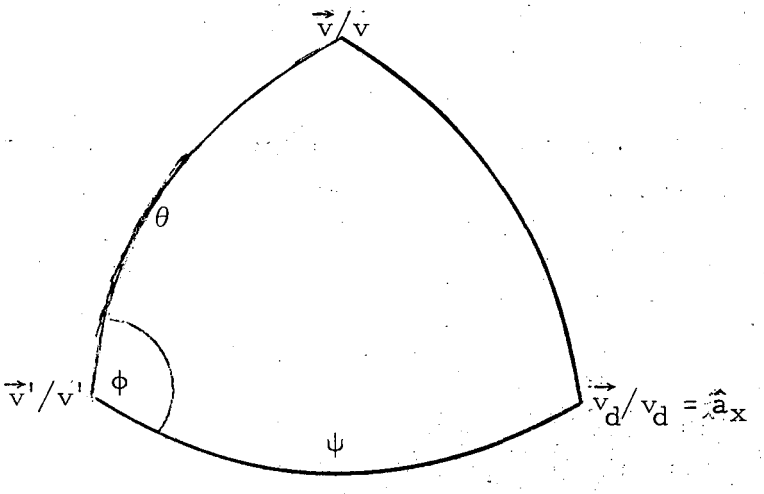
where $q_0^2 = 1 - (V^2 + v_d^2 - v^2)^2/4v_d^2V^2$.

2. Derivation of $P(\epsilon; v', v, \theta, \epsilon')$

Denoting the velocity of the incident electron in the gas frame by $\vec{v}'$ and the velocity of a resulting electron by $\vec{v}$, we define an angle such that $\cos\psi = v_x'/v'$ with $0 \leq \psi \leq \pi$. From Eq. (B-2) we see that

$$\cos\psi = (v'^2 + v_d^2 - V'^2)/2v_dv' \qquad \text{(B-6)}$$

where the energy of the incident electron is $\epsilon' = mV'^2/2$. Referring to the spherical triangle, shown in the following sketch, and using a formula from spherical



trigonometry, we find $v_x = v(\cos\theta\cos\psi - \sin\theta\sin\psi\cos\phi)$. By using this result, Eq. (B-2) becomes

$$\epsilon = mV^2/2 = (m/2)(v^2 + v_d^2) - mv_dv(\cos\theta\cos\psi - \sin\theta\sin\psi\cos\phi). \qquad \text{(B-7)}$$

Since ϵ is a monotonic function of $\cos\phi$, we have

$$P(\epsilon;\, v',\, v,\, \theta,\, \epsilon') = P(\cos\phi;\, v',\, v,\, \theta,\, \epsilon')/(mvv_d \sin\theta \sin\psi)$$

where, because we have assumed the scattering is independent of ϕ, we argue as in deriving Eq. (B-4) and find

$$P(\cos\phi;\, v',\, v, \theta,\, \epsilon') = \pi^{-1}(1 - \cos^2\phi)^{-1/2} .$$

By solving Eq. (B-7) for $\cos\phi$ and substituting above, we get

$$P(\epsilon;\, v',\, v,\, \theta,\, \epsilon') = 2R^{-1/2}/m\pi ,$$

where

$$R = 4v_d^2v^2 \sin^2\theta \sin^2\psi - (V^2 - v^2 - v_d^2 + 2v_dv \cos\theta \cos\psi)^2 , \qquad \text{(B-8)}$$

and where $P(\epsilon; v', v, \theta, \epsilon')$ is nonzero only when $R \geq 0$. To investigate this condition we write

$$R = a\cos^2\theta + 2b\cos\theta + c \ ,$$

where $$a = -4v_d^2 v^2 \ ,$$

$$b = -2v_d v\cos\psi\,(V^2 - v^2 - v_d^2) \ , \qquad \text{(B-9)}$$

and $$c = 4v_d^2 v^2 \sin^2\psi - (V^2 - v^2 - v_d^2)^2 \ .$$

Since $a \leq 0$, we have $R \geq 0$ only when

$$\left[-b + (b^2 - ac)^{1/2}\right]/a \leq \cos\theta \leq \left[-b - (b^2 - ac)^{1/2}\right]/a \ , \qquad \text{(B-10)}$$

and $b^2 - ac \geq 0$. The latter condition is easily shown to reduce to the "triangle" inequality

$$\left|V - v_d\right| \leq v \leq V + v_d \qquad \text{(B-11)}$$

which has an obvious kinematic interpretation. The limits on $\cos\theta$ from Eq. (B-10) are also kinematic and are easily shown to be between -1 and +1.

3. Derivation of $P^j(\epsilon; v', \theta, \epsilon')$ for Elastic and Inelastic Processes

For elastic and inelastic collisions, the quantities v', v, and θ are not independent but are related through the conservation of energy and momentum. From (A-2) with $n_j = 1$, this relationship is

$$v'^2 - v^2 = a_j + 2mv'(v' - v\cos\theta)/M \qquad \text{(B-12)}$$

to within terms of higher order in m/M. The above results are thus not in the most useful form for elastic or inelastic collisions. Using Eq. (B-2), we have $\epsilon' = mV'^2/2 = (m/2)(v'^2 - 2v_d v'\cos\psi + v_d^2)$, and by subtracting this from Eq. (B-7) we find

$$\epsilon - \epsilon' = (m/2)(V^2 - V'^2) = (m/2)(v^2 - v'^2)$$

$$+ mv_d\cos\psi\,(v' - v\cos\theta)$$

$$+ mv_d v\sin\theta\sin\psi\cos\phi \qquad \text{(B-13)}$$

In using this formula we will neglect m/M (but not mv'/Mv_d) compared with unity by using Eq. (B-12) in the first term and then using $v = (v'^2 - a_j)^{1/2}$ throughout. From this point on we will treat Eq. (B-13) with these substitutions as an exact formula, keeping all terms involving m/M, and we will use $v = (v'^2 - a_j)^{1/2}$ simply as notation. Since this formula is as accurate as we desire, the resulting physics will be correspondingly accurate, although as will be pointed out, some of the intermediate formulas are not.

With these substitutions, Eq. (B-13) can be used instead of Eq. (B-7). Proceeding as before, we find that Eq. (B-8) is replaced by

$$P^j(\epsilon; v', \theta, \epsilon') = 2R_j^{-1/2}/m\pi \quad , \tag{B-14}$$

where

$$R_j = 4v_d^2 v^2 \sin^2\psi \sin^2\theta - \left[V^2 - V'^2 + a_j - 2(v_d \cos\psi - mv'/M)(v' - v\cos\theta)\right]^2 ,$$

and where $P^j(\epsilon; v', \theta, \epsilon')$ is nonzero only when $R_j \geq 0$. As before, we write

$$R_j = a' \cos^2\theta + 2b' \cos\theta + c'$$

where

$$\begin{aligned} a' &= -4v^2(v_d^2 \sin^2\psi + A^2) \\ b' &= -2AvB \\ c' &= 4v_d^2 v^2 \sin^2\psi - B^2 \end{aligned} \tag{B-15}$$

and where $A = v_d \cos\psi - mv'/M$ and $B = V^2 - V'^2 + a_j - 2v'A$. As before, since $a' \leq 0$, we have $R_j \geq 0$ only when

$$\left[-b' + (b'^2 - a'c')^{1/2}\right]/a' \leq \cos\theta \leq \left[-b' - (b'^2 - a'c')^{1/2}\right]/a' \tag{B-16}$$

and when $b'^2 - a'c' \geq 0$. The latter condition implies that

$$4v_d^2 v^2 \sin^2\psi - B^2 + 4v^2A^2 \geq 0. \tag{B-17}$$

Addition of $4v_d^2v^2 \sin^2\psi + (B - 2vA)^2 \geq 0$ to this shows that

$$\left.\frac{\partial R_j}{\partial(\cos\theta)}\right|_{\cos\theta = -1} = -2a' + 2b' \geq 0$$

whenever $b'^2 - a'c' \geq 0$.

Similarly, the addition of $4v_d^2v^2 \sin^2\psi + (B + 2vA)^2 \geq 0$ shows that

$$\left.\frac{\partial R_j}{\partial(\cos\theta)}\right|_{\cos\theta = 1} = 2a' + 2b' \leq 0$$

whenever $b'^2 - a'c' \geq 0$.

These results show that the limits on $\cos\theta$ given by Eq. (B-16) are between -1 and +1 whenever $b'^2 - a'c' \geq 0$, since we see from Eq. (B-14) that $R_j \leq 0$ when $\cos\theta = \pm 1$.

Substitution of B, A, $\cos\psi$, and v into Eq. (B-17) yields

$$v'^4 - 2v'^2\left[v_d^2 + V^2 + \alpha_j + 2m(V'^2 - V^2)/M\right] + (V^2 - v_d^2 + \alpha_j)^2 + 4\alpha_j v_d^2 - 4m\alpha_j(v_d^2 - V'^2)/M \leq 0 . \quad \text{(B-18)}$$

This quadratic inequality in v'^2 can be solved for the limits on v'. The results are just kinematic; for example, if molecular recoil is neglected, the limits are determined from the "triangle" inequality

$$\left|V - v_d\right| \leq (v'^2 - \alpha_j)^{1/2} \leq (V + v_d) . \quad \text{(B-19)}$$

However, since for $V \gg v_d$ these limits are quite narrow, the recoil terms must be retained, even though they are not correct because of the approximations made in using Eq. (B-13). As discussed previously, the physical results will still be accurate.

C. Rate of Energy Gain and Drift Speed of the Electrons

The derivations in this section will proceed in the spirit of Appendix A, and the results found in that section will be used.

1. The Rate of Energy Gain

From Eq. (B-2) we see that the increase in the electron energy in the drift frame by a collision is

$$\Sigma\epsilon_i - \epsilon' = (m/2)(\Sigma v_i^2 - v'^2) - mv_d(\Sigma v_{ix} - v_x') + (n_j - 1)(mv_d^2/2) \quad \text{(C-1)}$$

where quantities referring to the incident electron are primed and the sums are over the n_j outgoing electrons. We now average Eq. (C-1) over everything but v' for a particular collision process by using Eq. (A-3) to evaluate the first term and using Eq. (A-1) to write the average of $v_x' - \Sigma v_{ix}$ as $v_x' \nu_m^j(v')/\nu^j(v')$; the resulting energy gain per collision is

$$\left\langle \Sigma\epsilon_i - \epsilon'; v' \right\rangle_j = (n_j - 1)(mv_d^2/2) + \left[\nu_m^j(v')/\nu^j(v')\right]\left[mv_d v_x' - m^2 v'^2/M\right] - \epsilon_j$$

[The last term in Eq. (A-3) was dropped; this means that the term involving m/M is somewhat incorrect for ionization processes.] We now obtain the rate of energy gain per electron of energy ϵ' by process j by multiplying with $\nu^j(v')$ and then averaging over v' by multiplying by $P(v';\epsilon')$ and integrating over v'. The result is

$$W^j(\epsilon') = N^j(\epsilon')\left[(n_j - 1)(mv_d^2/2) - \epsilon_j\right] - N_m^j(\epsilon')2\epsilon' m/M + mv_d^2 \int \nu_m^j(v')P(v';\epsilon')(v_x'/v_d)\,dv' \quad \text{(C-2)}$$

where N^j and N_m^j are mean collision frequency functions as defined in Sec. II-C-2. Since the integral in the last term is of order N_m^j, the second term is significant only for energies far greater than mv_d^2 and ϵ_j. This term was thus evaluated by replacing the mean value of $mv'^2/2$ by ϵ'. We also see that this term will almost never be significant for ionization processes, since these processes have relatively large values of ϵ_j.

To obtain the rate of energy gain per electron of energy ϵ' by all collision processes, we sum Eq. (C-2) over j and find

$$W(\epsilon') = mv_d^2 \int \nu_m(v') P(v'; \epsilon')(v_x'/v_d)dv' - N_m(\epsilon')2\epsilon' m/M + \sum_j N^j(\epsilon') \left[(n_j - 1)(mv_d^2/2) - \epsilon_j \right] . \quad (C-3)$$

Notice that only the collision frequencies for all but elastic processes and the total momentum-transfer collision frequency are important.

2. The Drift Speed in the $-\vec{E}$ Direction

From the equations of motion (B-1) the y position of an electron relative to that of its guiding center Y is given by $y - Y = (v_d - v_x)/\omega_b$. Thus after a collision at y, the sum of the displacements of the guiding centers of the outgoing electrons relative to the position of the guiding center of the incident electron is

$$\sum_i (Y_i - Y') = (n_j v_x' - \sum_i v_{ix})/\omega_b$$

where we have used the notation of the last section. The drift speed in the drift frame of the electrons of energy ϵ' due to collision process j is now found by using the negative of this expression instead of Eq. (C-1) and proceeding as in the previous section. The result, which is exact in this case, is

$$v_E^j(\epsilon') = \frac{v_d}{\omega_b} \int \left[\nu_m^j(v') + (n_j - 1)\nu^j(v') \right] P(v'; \epsilon')(v_x'/v_d)dv' . \quad (C-4)$$

Electron-attachment collisions make no contribution since $n_j = 0$ and $\nu_m^j = \nu^j$, as was shown in Sec. A. For elastic and inelastic collisions, only the momentum-transfer collision frequency is important since $n_j = 1$.

The drift speed produced by all collision processes is obtained by summing Eq. (C-4) over j. Notice that only the collision frequencies for ionization and electron-attachment processes and the total momentum-transfer collision frequency are important.

3. Properties of the Integral $\int \nu(v) P(v; \epsilon)(v_x/v_d)dv$

Integrals of this form appear in Eqs. (C-2), (C-3), and (C-4). We now write the value of this integral as $N(\epsilon) K(\epsilon)$, where $N(\epsilon)$ is the mean collision

frequency function corresponding to the collision frequency $\nu(v)$ and $K(\epsilon)$ is a factor of order unity.

To investigate the nature of $K(\epsilon)$, we may use $P(v; \epsilon)$ as given by Eq. (II-12), and we may use Eq. (B-2) to write v_x in terms of v. The resulting formula for $K(\epsilon)$ can be evaluated if $\nu(v)$ is known. For example, if $\nu(v)$ is a constant, then $K(\epsilon) = 1$, and if $\nu(v)$ is proportional to v, then

$$K(\epsilon) = 1 + (V^2 - v_d^2/5)/(3V^2 + v_d^2) \qquad \text{if } V \geq v_d ,$$

and

$$K(\epsilon) = 1 + (V^2/v_d^2)(v_d^2 - V^2/5)/(3v_d^2 + V^2) \qquad \text{if } V \leq v_d .$$

More useful formulas are found by carrying out the integrations by expanding the integrands in Taylor series about the larger of V or v_d. The results are

$$K(\epsilon) = 1 + \frac{V}{3\nu(V)} \frac{\partial \nu(V)}{\partial V} + O\left(\frac{v_d^2}{V^2}\right)$$

$$= \frac{1}{3V^2\nu(V)} \frac{\partial}{\partial V} \left[V^3 \nu(V)\right] + O\left(\frac{v_d^2}{V^2}\right) \qquad \text{if } V > v_d , \qquad \text{(C-5)}$$

and

$$K(\epsilon) = 1 + \frac{V^2}{3v_d \nu(v_d)} \frac{\partial \nu(v_d)}{\partial v_d} + O\left(\frac{V^4}{v_d^4}\right) \qquad \text{if } v_d > V ,$$

where the symbol O denotes "order of magnitude." In general, $K(\epsilon)$ is greater than 1 when $\nu(v)$ increases with v and is less than 1 when $\nu(v)$ decreases as v increases.

D. Comparison of the Two Methods of Solution

We will now show that, with the assumptions that the electron speed is much greater than v_d and that elastic collisions are much more probable than any other, the results of our method of solution of the Boltzmann equation agree with those of the "usual" method as given in Sec. II-A. To facilitate the comparison, we will introduce a distribution of speed V in the drift frame by defining $f(\epsilon, t) = 4\pi V g(V, t)/m$, where $\epsilon = mV^2/2$. Our basic equations (II-19) and (II-20) then become

$$\frac{\partial g(V, t)}{\partial t} = \sum_j \left(\frac{\partial g}{\partial t}\right)_j , \qquad \text{(D-1)}$$

and

$$\left(\frac{\partial g}{\partial t}\right)_j = \frac{m}{V}\int G^j(\epsilon;\epsilon')V'^2 g(V', t)dV' - N^j(\epsilon)g(V, t) \quad . \tag{D-2}$$

Because we are considering $V \gg v_d$, $N^j(\epsilon)$ can be evaluated by expanding the integrand of Eq. (II-21) in a Taylor series about V. The result is

$$N^j(\epsilon) = \nu^j(V) + \left(\frac{v_d}{V}\right)^2 \frac{V}{6}\frac{\partial^2}{\partial V^2}\left[V\nu^j(V)\right] + \cdots \quad , \tag{D-3}$$

which is just an expansion in the even powers of v_d/V. The first term of Eq. (D-2) will now be evaluated in a similar manner for each type of collision.

For electron attachment collisions, the first term of Eq. (D-2) is zero, so to the lowest order in v_d/V

$$\left(\frac{\partial g}{\partial t}\right)_j = -\nu^j(V)g(V, t) \quad . \tag{D-4}$$

For elastic and inelastic processes, we use Eq. (II-23) and neglect molecular recoil. By interchanging the order of integration we then find

$$\frac{m}{V}\int G^j(\epsilon;\epsilon')V'^2 g(V', t)dV' = \frac{1}{4v_d^2 V}\int_{\left[(V-v_d)^2+a_j\right]^{1/2}}^{\left[(V+v_d)^2+a_j\right]^{1/2}} \frac{dv'v'\nu^j(v')}{(v'^2-a_j)^{1/2}} \int_{|v'-v_d|}^{v'+v_d} V'g(V', t)dV'$$

where the limits were found by rearranging the "triangle" inequalities $|V' - v_d| \le v' \le V' + v_d$ and $|V - v_d| \le (v'^2 - a_j)^{1/2} \le V + v_d$. The integral over V' can be carried out as above by expanding the integrand in a Taylor series about v'. The result is

$$\int_{|v'-v_d|}^{v'+v_d} V'g(V', t)dV' = 2v_d\left\{v'g(v', t) + \frac{v_d^2}{6}\frac{\partial^2}{\partial v'^2}\left[v'g(v', t)\right] + \cdots\right\}$$

We can now do the remaining integral by expanding the integrand about $(V^2 + a_j)^{1/2}$. For inelastic collisions we keep only the term of lowest order in v_d/V and find

$$\left(\frac{\partial g}{\partial t}\right)_j = \frac{(V^2 + a_j)^{1/2}}{V} \nu^j\left[(V^2 + a_j)^{1/2}\right] g\left[(V^2 + a_j)^{1/2}, t\right] - \nu^j(V) g(V, t) \quad . \qquad \text{(D-5)}$$

For elastic collisions, $a_j = 0$ so these terms cancel; the terms of order v_d/V also vanish so the terms of order v_d^2/V^2 must be retained. After a considerable amount of algebra, the result may be written as

$$\left(\frac{\partial g}{\partial t}\right)_{el} = \frac{v_d^2}{3V^2} \frac{\partial}{\partial V}\left[V^2 \nu_m^{el}(V) \frac{\partial g(V, t)}{\partial V}\right] \quad . \qquad \text{(D-6)}$$

For ionization processes we use Eq. (II-11) and interchange the order of integration as above to find

$$\frac{m}{V}\int G^j(\epsilon; \epsilon') V'^2 g(V', t) dV' = \frac{n_j}{4v_d^2 V} \int_{V - v_d}^{V + v_d} \frac{dv}{v} \int dv' v' \nu^j(v') P^j(v'; v) \int_{|v' - v_d|}^{v' + v_d} V' g(V', t) dV' \quad .$$

As above, we carry out the integrals over V' and v, keeping only the lowest order of v_d/V. The result is

$$\left(\frac{\partial g}{\partial t}\right)_j = \frac{n_j}{V^2} \int dv' v'^2 \nu^j(v') P^j(V; v') g(v', t) - \nu^j(V) g(V, t) \quad . \qquad \text{(D-7)}$$

As we expected, Eqs. (D-1), (D-4), (D-5), and (D-7) agree with Eqs. (II-10), (II-14), (II-12), and (II-13) respectively of Sec. II-A. The results for elastic collisions in Eqs. (D-6) and (II-11) also agree in the large $\omega_b \tau$ limit, except that in Eq. (D-6) molecular recoil and the finite temperature of the gas were neglected. The algebra of deriving the recoil term directly from this approach has not been carried out. The finite-temperature term is discussed in Sec. II-C-3 and the form of the recoil term can then be deduced from the fact that when $v_d = 0$, $(\partial g/\partial t)_{el}$ must vanish when

$$g(V) \propto \exp\left[-(mv^2/2)/kT\right] .$$

We now consider the drift speed of the electrons. From Eq. (II-31) we have, since we assume elastic collisions are the most probable,

$$v_{Dy} = -\frac{1}{n}\int f(\epsilon)v_E(\epsilon)d\epsilon = -\frac{v_d}{\omega_b n}\int f(\epsilon)N_m^{el}(\epsilon)K_m^{el}(\epsilon)d\epsilon \quad ,$$

where $n = \int f(\epsilon)d\epsilon$. Changing to a distribution of speed as before and using $N_m^{el}(\epsilon)$ and $K_m^{el}(\epsilon)$ to the lowest order in v_d/V from Eqs. (D-3) and (C-5), these equations become

$$v_{Dy} = \frac{-4\pi v_d}{3\omega_b n}\int g(V)\frac{\partial}{\partial V}\left[V^3\nu_m^{el}(V)\right]dV \quad , \text{ with } n = 4\pi\int V^2 g(V)dV \quad . \tag{D-8}$$

We see that this agrees with Eq. (II-17) of Sec. II-A.

We conclude that, in the large $\omega_b\tau$ limit, our approach contains all of the results of the "usual" method of solving the Boltzmann equation, although we have not discussed the diffusion tensor here. From our approach we see that the results given by the "usual" method are valid whenever the electron energy is high compared with $mv_d^2/2$ (since only terms of order v_d^2/V^2 have been neglected) and whenever all other collision frequencies can be neglected in comparison with the elastic momentum-transfer collision frequency. Thus, in the large $\omega_b\tau$ limit, it does not appear necessary to assume the energy distribution is near the equilibrium distribution.

E. Partial List of Functions and Symbols Appearing Frequently in the Text

$\hat{a}_x$	Unit vector in the x direction.
$f(\epsilon, t)$	Electron energy distribution in the drift frame.
$f_0(\epsilon)$	Normalized "steady-state" electron energy distribution [Eq. (II-29)].
$F(\vec{r}, \vec{v}, t)$	Electron distribution function in the gas frame.
$g(V, t)$	Distribution of electron speed in the drift frame [Appendix D].
$G^j(\epsilon; \epsilon')$	Energy-scatter function for process j [Eq. (II-20)].
$I^j(\theta, v; v')$	See Appendix A [Eq. (A-6)].
$K(\epsilon)$	See Sec. 3 of Appendix C
n_g	Gas density.
n_j	Number of electrons leaving a collision of process j [Appendix A].
$N^j(\epsilon)$	Mean-collision-frequency function for process j [Eq. (II-21)].
$P(\alpha; \beta, \cdots)$	Probability distribution of α for specified $\beta, \cdots$ [Appendix B].
$\vec{v}$	Electron velocity in the gas frame.
$\vec{v}_d$	$c\vec{E} \times \vec{B}/B^2$
v_E	Drift speed along $-\vec{E}$ in the gas frame.
V	Electron speed in the drift frame
$W^j(\epsilon)$	Rate of electron energy gain by process j [Sec. 1 of Appendix C].
α_j	$2\epsilon_j/m$
β	Rate coefficient for production of free electrons [Eq. (II-29)].
$\gamma(v)$	$[1 + (\nu_m^{el}/\omega_b)^2]^{-1}$
ϵ	Electron energy in the drift frame
ϵ_j	Inelastic energy transfer per collision of process j [Appendix A].
θ	Angle of deflection.
ν^j	Collision frequency for process j [Appendix A].
ν_m^j	Momentum-transfer collision frequency for process j [Appendix A].
ξ	Angle between $\vec{B}$ and the electron velocity in the drift frame.

σ^j	Cross section for collision process j [Appendix A].
$\sigma_\theta^j(\theta, v)$	Differential cross section for process j [Appendix A].
τ	Mean free time between momentum-transfer collisions.
ϕ	Azimuthal angle of scattering.
ψ	$\cos^{-1}(v_x/v)$ with $-\pi/2 \leq \psi \leq \pi/2$ [Eq. (B-6)].

References and Footnotes

1. G. Ecker and K. G. Müller, Z. Naturforsch 16A, 246 (1961); also see H. Dreicer, Phys. Rev. 117, 329 (1960).
2. W. P Allis, Handbuch der Physik (Springer-Verlag, Berlin 1956), Vol. XXI, p. 404.
3. T. Holstein, Phys. Rev. 70, 367 (1946).
4. L. S. Frost and A. V. Phelps, Westinghouse Research Laboratories Scientific Paper 62-908-113-P1 (1962).
5. Hans Ramien, Z. Physik 70, 353 (1931).
6. G. J. Schultz, Westinghouse Research Laboratories, Pittsburgh, Pa., private communication.
7. J. C. Y. Chen and J. L. Magee, J. Chem. Phys. 36, 1407 (1962).
8. L. A. Edelstein, Nature 182, 932 (1958).
9. H. S. W. Massey and E. H. S. Burhop, Electronic and Ionic Impact Phenomena (Oxford University Press, 1952), p. 233.
10. S. J B. Corrigan and A. von Engel, Proc. Roy. Soc. (London) A245, 335 (1958).
11. G. J. Schultz, Phys. Rev. 112, 150 (1958).
12. A. A. Kruithof and L. S. Ornstein, Physica 2, 611 (1935).
13. Michael Gryzinski, Phys. Rev. 115, 374 (1959).
14. John T. Tate and P. T. Smith, Phys. Rev. 39, 270 (1932).
15. C. E. Normand, Phys. Rev. 35, 1217 (1930).
16. Glenn M. Webb, Phys. Rev. 47, 384 (1935)
17. E. C. Bullard and H. S. W. Massey, Proc. Roy. Soc. (London) A133, 637 (1931).
18. This implies that electrons reach energies above 11 eV only by the energy "spreading" effect of collisions, not by heating on the average.
19. Melvin J Bernstein, Phys. Rev. 127, 335 (1962).
20. Melvin J. Bernstein, Phys. Rev. 127, 342 (1962).
21. Arthur V. Phelps (Westinghouse Research Laboratories, Pittsburgh, Pa.) private communication.