Title

On the Brauer groups of fibrations

Author

Qin, Yanshuai

Publication Date

2023

On the Brauer groups of fibrations

by

Yanshuai Qin

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Xinyi Yuan, Co-chair
Professor Martin Olsson, Co-chair
Professor Paul Vojta
Professor Martin White

Fall 2023

On the Brauer groups of fibrations

Abstract

On the Brauer groups of fibrations

by

Yanshuai Qin

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Xinyi Yuan, Co-chair

Professor Martin Olsson, Co-chair

Let C be a smooth projective geometrically connected curve over a finite field with function field K or the spectrum of the ring of integers in a number field K. Let X be a smooth projective geometrically connected curve over K. Let $\pi : \mathcal{X} \longrightarrow C$ be a proper regular model of X/K. Artin and Grothendieck proved that there is an isomorphism $Ш(\mathrm{Pic}^0_{X/K}) \cong \mathrm{Br}(\mathcal{X})$ up to finite groups. As a result, this implies that the BSD conjecture for $\mathrm{Pic}^0_{X/K}$ is equivalent to the Tate conjecture for the surface $\mathcal{X}$ when K is in positive characteristic. In this thesis, we generalize this result to fibrations $\pi : \mathcal{X} \longrightarrow C$ of arbitrary relative dimensions fibered over C, where C is a smooth projective curve over arbitrary finitely generated fields or the spectrum of the ring of integers in a number field. As a consequence, we reprove the reduction theorem of the Tate conjecture for divisors due to André and Ambrosi, and give a simpler proof of a theorem of Geisser who proved it using the étale motivic cohomology theory. We also reduce Artin's question on the finiteness of Brauer groups of proper regular schemes to dimension at most 3.

To my parents

Contents

Contents **ii**

1 Introduction **1**
1.1 Tate conjectures 1
1.2 Brauer groups and Tate-Shafarevich groups 2
1.3 The Artin-Grothendieck Theorem for fibered surfaces 3
1.4 Fibrations of higher relative dimensions 4
1.5 Applications 5
1.6 Outline of Thesis 6

2 Preliminaries **8**
2.1 Notation and Terminology 8
2.2 Cohomological Brauer group 9
2.3 Picard variety and Neron-Severi group 16
2.4 Colliot-Thélène and Skorobogatov's pull-back trick 17
2.5 Tate conjecture and Brauer group 21

3 Geometric case **24**
3.1 The Galois invariant parts of geometric Brauer groups 24
3.2 Geometric Tate-Shafarevich group 27
3.3 Proof of Theorem 1.4.1 30
3.4 Applications of Theorem 1.4.1 37

4 Arithmetic case **39**
4.1 Local results 39
4.2 Proof of Theorem 1.4.5 56
4.3 Applications 67

Bibliography **70**

Acknowledgments

I would like to thank my advisor, Xinyi Yuan. He introduced me to these intriguing problems and provided encouragement during challenging times. I gained substantial insights from our discussions and he generously shared his ideas on the problems. The key idea that led to solving these problems is the pull-back trick, which he shared with me. I am really grateful for his help. I also extend my thanks to Professor Hourong Qin for his encouragement and support. Additionally, I appreciate Professor Martin Olsson for his help with enrollment matters when Xinyi was no longer at Berkeley.

My parents and my girlfriend gave me lots of support during my tough time. I'm grateful for their support.

Lastly, I am grateful for the support provided by the Lehmer Fellowship during Spring 2022.

Chapter 1

Introduction

1.1 Tate conjectures

In [60], Tate formulated the Tate conjecture [60, Conj. 1], and the rank BSD conjecture [60, Conj. B+SD], the strong Tate conjecture [60, Conj. 2] and the B+SD+2 conjecture [60, Conj. B+SD+2] in terms of L-functions. He asked about relations between these conjectures and proved some equivalences between them for smooth projective varieties over finite fields (cf. [61, Thm. 2.9]). Let's first recall Tate's formulations of these conjectures.

Let X be a smooth projective geometrically connected variety over a finitely generated field K (i.e. finitely generated over a finite field or $\mathbb{Q}$). By spreading out, one can then construct a projective and smooth morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of schemes of finite type over $\mathbb{Z}$, with $\mathcal{Y}$ integral and regular, whose generic fiber is X/K. Let $|\mathcal{X}|$ (resp. $|\mathcal{Y}|$) denote the set of closed points of $\mathcal{X}$ (resp. $\mathcal{Y}$). For each $y \in |\mathcal{Y}|$, let $k(y)$ denote the residue field of y and q_y denote the cardinality of $k(y)$. Set

$$P_{y,i}(T) := \det(1 - \sigma_y^{-1} T | H^i_{\text{ét}}(\mathcal{X}_{\bar{y}}, \mathbb{Q}_\ell)),$$

$$\Phi_i(s) := \prod_{y \in |\mathcal{Y}|} \frac{1}{P_{y,i}(q_y^{-s})}. \tag{1.1}$$

Here $\sigma_y \in \mathrm{Gal}(\overline{k(y)}/k(y))$ is the arithmetic Frobenius element and $\ell \neq \mathrm{char}(k(y))$ is a prime. Define the zeta function of $\mathcal{X}$ as

$$\zeta(\mathcal{X}, s) := \prod_{x \in |\mathcal{X}|} \frac{1}{1 - q_x^{-s}},$$

where q_x denotes the cardinality of $k(x)$.

Let $\ell \neq \mathrm{char}(K)$ be a prime number. Let $A^i(X)$ denote the group of classes of algebraic cycles of codimension i on X, with coefficients in $\mathbb{Q}$, for ℓ-adic homological equivalence. Let $N^i(X) \subseteq A^i(X)$ denote the group of classes of cycles that are numerically equivalent to zero. Let d denote the dimension of X. Tate made the following conjectures:

Conjecture 1.1.1 ($T^i(X,\ell)$)**.** *The cycle class map*

$$A^i(X)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \longrightarrow H^{2i}_{\text{ét}}(X_{K^s},\mathbb{Q}_\ell(i))^{\mathrm{Gal}(K^s/K)}$$

is surjective.

Conjecture 1.1.2 ($E^i(X,\ell)$)**.** *$N^i(X)=0$, i.e. numerical equivalence is equal to ℓ-adic homological equivalence for algebraic cycles of codimension i on X.*

Conjecture 1.1.3 (BSD)**.** *The rank of* $\mathrm{Pic}^0_{X/K}(K)$ *is equal to the order of the zero of* $\Phi_1(s)$ *at* $s=\dim(\mathcal{Y})$.

Conjecture 1.1.4 ($T^i(X)$)**.** *The dimension of $A^i(X)/N^i(X)$ is equal to the order of the poles of $\Phi_{2i}(s)$ at $s=\dim\mathcal{Y}+i$.*

Conjectures 1.1.1 and 1.1.2 are called the Tate conjecture for X. Conjecture 1.1.3 is known as the rank part of the BSD conjecture for the Picard variety $\mathrm{Pic}^0_{X/K}$. Conjecture 1.1.4 is called the strong Tate conjecture.

Let $\mathcal{X}$ be a regular scheme of finite type over $\mathbb{Z}$. Tate made the following conjecture about the zeta function of $\mathcal{X}$, which is equivalent to $T^1(\mathcal{X})$ when $\mathcal{X}$ is a smooth projective variety over a finite field.

Conjecture 1.1.5. *If $\mathcal{X}$ is a regular scheme of finite type over $\mathbb{Z}$, then the order of $\zeta(\mathcal{X},s)$ at the point $s=\dim\mathcal{X}-1$ is equal to* $\operatorname{rank}H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})-\operatorname{rank}H^1(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})$.

Tate showed that Conjecture 1.1.5 for an open dense subscheme of $\mathcal{X}$ is equivalent to the conjecture for $\mathcal{X}$ itself. So Conjecture 1.1.5 in positive characteristic can be regarded as the strong Tate conjecture for divisors on non-proper varieties over finite fields. Let $\mathcal{X}\to\mathcal{Y}$ be a morphism as discussed at the beginning, whose generic fiber is a smooth projective variety over a global field K. Tate conjectured a relation between the conjectures for X/K that he made before:

Conjecture 1.1.6. *Conjecture 1.1.5 for $\mathcal{X}$ $\Leftrightarrow$ the BSD conjecture for* $\mathrm{Pic}^0_{X/K}+T^1(X)$.

It is this conjecture of Tate that has motivated our thesis.

1.2 Brauer groups and Tate-Shafarevich groups

In this thesis, we are mainly interested in the Tate conjecture for divisors on X (i.e. $T^i(X,\ell)$ with $i=1$). The conjecture $T^1(X,\ell)$ was proved for abelian varieties by Tate [62] (over finite fields), Zarhin [68, 69] (over positive characteristics) and Faltings [17, 18] (over characteristic zero), for $K3$ surfaces over characteristic zero by André [2] and Tankeev [58, 59], for $K3$

surfaces over positive characteristics by Nygaard [45], Nygaard-Ogus [44], Artin-Swinnerton-Dyer [3], Maulik [34], Charles [7] and Madapusi-Pera [39].

The Tate conjecture for divisors and the BSD conjecture for abelian varieties have the Brauer groups and the Tate-Shafarevich groups as their obstructions respectively. Next, we will introduce these groups of obstructions.

For any noetherian scheme X, the *cohomological Brauer group*

$$\mathrm{Br}(X) := H^2(X, \mathbb{G}_m)_{\mathrm{tor}}$$

is defined to be the torsion part of the étale cohomology group $H^2(X, \mathbb{G}_m)$. Let X be a smooth projective geometrically connected variety over a finitely generated field k and $\ell \neq \mathrm{char}(k)$ be a prime. By Kummer's theory (see Proposition 2.5.2), there is a canonical exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(k^s/k))} \longrightarrow V_\ell \mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)} \longrightarrow 0,$$

where $V_\ell \mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)}$ is the Tate module of $\mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)}$ tensoring with $\mathbb{Q}_\ell$. Since the algebraic equivalence coincides with the numerical equivalence for divisors up to torsions by Matsusaka's theorem, $T^1(X, \ell)$ is equivalent to $V_\ell \mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)} = 0$, which is equivalent to the finiteness of $\mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)}[\ell^\infty]$. If k is finite, the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})^{\mathrm{Gal}(k^s/k)}$ has a finite kernel and a finite cokernel, so $T^1(X, \ell)$ is also equivalent to the finiteness of $\mathrm{Br}(X)[\ell^\infty]$.

For an abelian variety A over a global field K, the *Tate-Shafarevich group* is defined as

$$Ш(A) := \mathrm{Ker}(H^1(K, A) \longrightarrow \prod_v H^1(K_v, A)),$$

where K_v denotes the completion of K at a place v. If $p = \mathrm{char}(K) > 0$, Schneider [51] proved that the BSD conjecture for A is equivalent to the finiteness of $Ш(A)[\ell^\infty]$ for $\ell \neq p$.

1.3 The Artin-Grothendieck Theorem for fibered surfaces

Let C be a smooth projective geometrically connected curve over a finite field with function field K or the spectrum of the ring of integers in a number field K. Let $\mathcal{X}$ be a 2-dimensional regular scheme and $\mathcal{X} \longrightarrow C$ be a proper flat morphism such that the generic fiber X is smooth and geometrically connected over K. Since the Tate conjecture for X is known, Conjecture 1.1.6 suggests that there should be a relation between $\mathrm{Br}(\mathcal{X})$ and $Ш(\mathrm{Pic}^0_{X/K})$. To state Artin-Grothendieck's theorem, we need to define *isomorphic up to finite groups* for two abelian groups.

For two abelian groups M and N, we say that they are *isomorphic up to finite groups* if there exists a sub-quotient M_1/M_0 of M (resp. N_1/N_0 of N) such that M_0 (resp. N_0) and M/M_1 (resp. N/N_1) are finite and $M_1/M_0 \cong N_1/N_0$. Let $f : M \to N$ be a homomorphism with a finite cokernel and a kernel isomorphic to an abelian group H up to finite groups, in this situation, we say that the sequence $0 \to H \to M \to N \to 0$ is *exact up to finite groups.*

Theorem 1.3.1 (Artin-Grothendieck)**.** *There is an isomorphism* $Ш(\mathrm{Pic}^0_{X/K}) \cong \mathrm{Br}(\mathcal{X})$ *up to finite groups.*

Artin-Grothendieck's theorem implies that Conjecture 1.1.6 holds for the case that $\mathcal{X}$ is a smooth surface over a finite field, since $T^1(\bar{\mathcal{X}}) \Leftrightarrow T^1(\bar{\mathcal{X}}, \ell)$ (cf. [61]) for a smooth projective compactification $\bar{\mathcal{X}}$ of $\mathcal{X}$.

1.4 Fibrations of higher relative dimensions

Motivated by the Conjecture 1.1.6 of Tate, we study the relation between these obstructions: the Brauer group $\mathrm{Br}(\mathcal{X})$, the Tate-Shafarevich group $Ш(\mathrm{Pic}^0_{X/K})$ and the geometric Brauer group $\mathrm{Br}(X_{K^s})^{\mathrm{Gal}(K^s/K)}$. We generalize Artin-Grothendieck's theorem to fibrations of arbitrary relative dimensions fibered over smooth projective geometrically connected curves over finitely generated fields (geometrical case) and the spectrum of the ring of integers in number fields (arithmetic case). For a torsion abelian group M and a prime p, denote by $M(\text{non-}p)$ the subgroup of elements of order prime to p in M. For a field k, denote by G_k the absolute Galois group of k.

Theorem 1.4.1 (geometrical case)**.** *Let k be a finitely generated field of characteristic $p \geq 0$. Let $\mathcal{X}$ be a smooth geometrically connected variety over k and C be a smooth projective geometrically connected curve over k with function field K. Let $\pi : \mathcal{X} \longrightarrow C$ be a dominant k-morphism such that the generic fiber X is smooth projective geometrically connected over K. Let K' denote Kk^s and write P for the Picard variety* $\mathrm{Pic}^0_{X/K,\mathrm{red}}$. *Define*

$$Ш_{K'}(P) := \mathrm{Ker}(H^1(K', P) \longrightarrow \prod_{v \in |C_{k^s}|} H^1(K_v^{sh}, P)).$$

Then there is an exact sequence up to finite groups

$$0 \longrightarrow Ш_{K'}(P)^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(\mathcal{X}_{k^s})^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p) \longrightarrow 0.$$

In the case that k is a finite field and $\mathcal{X}$ is projective over k, the above theorem gives the following result, which generalizes Artin-Grothendieck's theorem in positive characteristics to arbitrary relative dimensions.

Corollary 1.4.2. *Let $\pi : \mathcal{X} \longrightarrow C$ be a proper flat k-morphism, where C is a smooth projective geometrically connected curve over a finite field k of characteristic p with function* *field K. Assuming that $\mathcal{X}$ is smooth projective over k and the generic fiber X of π is smooth projective geometrically connected over K, then there is an exact sequence up to finite groups*

$$0 \longrightarrow Ш(\mathrm{Pic}^0_{X/K,\mathrm{red}})(\text{non-}p) \longrightarrow \mathrm{Br}(\mathcal{X})(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p) \longrightarrow 0.$$

In Theorem 1.4.1, when k is finite, the canonical maps $Ш(P)(\text{non-}p) \to Ш_{K'}(P)^{G_k}(\text{non-}p)$ and $\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(\mathcal{X}_{k^s})^{G_k}$ have finite kernels and finite cokernels (cf. Proposition 3.2.2 and [67, Cor. 1.4]). Thus, Corollary 1.4.2 follows directly from Theorem 1.4.1. As a consequence of Corollary 1.4.2, we reprove a theorem of Geisser:

Theorem 1.4.3 ([23]). *$T^1(\mathcal{X}, \ell)$ is equivalent to $T^1(X, \ell)$ + finiteness of $Ш(\mathrm{Pic}^0_{X/K,\mathrm{red}})[\ell^\infty]$.*

Remark 1.4.4. *Geisser's theorem implies that Conjecture 1.1.6 is true in positive characteristics. Geisser's proof used the étale motivic cohomology theory. Comparing with his result, our result gives a quantitative relation between these obstructions without need to assume any of these conjectures.*

Conjecture 1.1.6 in mixed characteristic may not be accessible in general, nevertheless, we prove a relation for their obstructions:

Theorem 1.4.5 (arithmetic case). *Let $\pi : \mathcal{X} \longrightarrow C$ be a proper flat morphism, where C is $\mathrm{Spec}(\mathcal{O}_K)$ for some number field K. Assume that $\mathcal{X}$ is regular and the generic fiber X of π is projective and geometrically connected over K. Then there is an exact sequence up to finite groups*

$$0 \longrightarrow Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow \mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\overline{K}})^{G_K} \longrightarrow 0.$$

Remark 1.4.6. *For arithmetic schemes of dimension ≥ 3, the above question was first studied by Tankeev (cf.[55, 56, 57]), and he proved the above result in some special cases.*

1.5 Applications

Since Brauer groups are the obstructions of the Tate conjecture for divisors, in combination with a spreading out argument, Theorem1.4.1 implies the following reduction theorem of the Tate conjecture due to André [2] and Ambrosi [1]:

Theorem 1.5.1. *Let k be a prime field and K be a finitely generated field over k. Let ℓ be a prime different from char(k). Assuming that $T^1(\mathcal{X}, \ell)$ is true for all smooth projective varieties $\mathcal{X}$ over k, then $T^1(X, \ell)$ holds for all smooth projective varieties X over K.*

Remark 1.5.2. *$T^1(X, \ell)$ for smooth projective varieties over finite fields has been reduced to $T^1(X, \ell)$ for smooth projective surfaces X over finite fields with $H^1(X, \mathcal{O}_X) = 0$ by the work of de Jong [14], Morrow [40, Thm. 4.3] and Yuan [66, Thm. 1.6].*

In [54], Skorobogatov and Zarkhin conjectured that $\mathrm{Br}(X_{k^s})^{G_k}(\text{non-}p)$ is finite for any smooth projective variety X over a finitely generated field k. In [54, 53], they proved the finiteness of $\mathrm{Br}(X_{k^s})^{G_k}(\text{non-}p)$ for abelian varieties and $K3$ surfaces over k with $\mathrm{char}(k) \neq 2$. By using our Theorem 1.4.1, we reduce the question about the finiteness of $\mathrm{Br}(X_{k^s})^{G_k}(\text{non-}p)$ to the question over prime fields:

Theorem 1.5.3. *Let k be a prime field and K be a field finitely generated over k. Then the following statements are true.*

(1) If $k = \mathbb{Q}$ and $\mathrm{Br}(X_{k^s})^{G_k}$ is finite for any smooth projective variety X over k, then $\mathrm{Br}(X_{K^s})^{G_K}$ is finite for any smooth projective variety X over K.

(2) If $k = \mathbb{F}_p$ and the Tate conjecture for divisors holds for any smooth projective variety over k, then $\mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$ is finite for any smooth projective variety X over K.

Artin asked the following question about the finiteness of Brauer groups:

Question 1.5.4 (Artin)**.** *Let $\mathcal{X}$ be proper scheme over $\mathrm{Spec}(\mathbb{Z})$, is the Brauer group $\mathrm{Br}(\mathcal{X})$ finite ?*

For a smooth projective variety $\mathcal{X}$ over a finite field, we have seen that the finiteness of $\mathrm{Br}(\mathcal{X})$ is equivalent to the Tate conjecture for divisors on $\mathcal{X}$ (cf. [61]). For a proper flat regular integral scheme $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$, taking $C = \mathrm{Spec}(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$, there is a proper flat morphism (the Stein factorization) $\pi : \mathcal{X} \longrightarrow C$ with a generic fiber geometrically connected over $K = K(C)$. It follows from our Theorem 1.4.5 that the finiteness of $\mathrm{Br}(\mathcal{X})$ is equivalent to the finiteness of $Ш(\mathrm{Pic}^0_{X/K})$ and the finiteness of $\mathrm{Br}(X_{K^s})^{G_K}$. Orr and Skorobogatov [46, Thm. 5.1] proved that $\mathrm{Br}(X_{K^s})^{G_K}$ is finite under the assumption of the integral Mumford-Tate conjecture. Thus, Theorem 1.4.5 implies the following result on Artin's question for regular schemes.

Corollary 1.5.5. *Assuming the integral Mumford-Tate conjecture and the Tate-Shafarevich conjecture, then $\mathrm{Br}(\mathcal{X})$ is finite for all proper flat regular schemes over $\mathrm{Spec}(\mathbb{Z})$.*

By a theorem of André [2] and a result of Ambrosi [1, Cor. 1.6.2.1], the finiteness of $\mathrm{Br}(X_{\bar{K}})^{G_K}$ has been reduced to smooth projective surfaces over K. Thus, Theorem 1.4.5 reduces Artin's question for regular schemes to arithmetic threefolds:

Theorem 1.5.6. *Assuming that $\mathrm{Br}(\mathcal{X})$ is finite for all 3-dimensional regular proper flat schemes over $\mathbb{Z}$, then $\mathrm{Br}(\mathcal{X})$ is finite for all regular proper flat schemes over $\mathbb{Z}$.*

1.6 Outline of Thesis

In Chapter 2, we introduce notations, basic definitions and facts about Brauer groups, which will be used throughout the article.

In Chapter 3, we prove Theorem 1.4.1 i.e. the geometrical case of our main results and its corollaries.

In Chapter 4, we prove Theorem 1.4.5 i.e. the arithmetic case of our main results and its corollaries. First, we prove a local result for fibrations over a henselian DVR. Then, we use the local result to deduce Theorem 1.4.5. In the proof of the local result, we use the syntomic cohomology to compute the flat cohomology $H^2(\mathcal{X}, \mu_{p^n})$ and then relate the flat cohomology $H^2(\mathcal{X}, \mu_{p^n})$ to the étale cohomology $H^2(X_{\bar{K}}, \mu_{p^n})^{G_K}$ through the Fontaine-Messing period morphism. This gives an isomorphism $\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}$ up to groups of finite exponent.

Chapter 2

Preliminaries

2.1 Notation and Terminology

Fields

By a *finitely generated field*, we mean a field which is finitely generated over a prime field. For any field k, denote by k^s (resp. $\bar{k}$) the separable closure (resp. algebraic closure). Denote by $G_k = \mathrm{Gal}(k^s/k)$ the absolute Galois group of k.

Henselization

Let R be a noetherian local ring, denote by R^h (resp. R^{sh}) the henselization (resp. strict henselization) of R at the maximal ideal. If R is a discrete valuation ring, denote by K^h (resp. K^{sh}) the fraction field of R^h (resp. R^{sh}).

Varieties

By a *variety* over a field k, we mean a scheme which is separated and of finite type over k. For a smooth projective variety X over a field k, we use $\mathrm{Pic}^0_{X/k}$ to denote the identity component of the Picard scheme $\mathrm{Pic}_{X/k}$. Denote by $\mathrm{Pic}^0_{X/k,\mathrm{red}}$ the underlying reduced closed subscheme of $\mathrm{Pic}^0_{X/k}$.

Cohomology

The default sheaves and cohomology over schemes are with respect to the small étale site. So H^i is the abbreviation of $H^i_{\text{ét}}$.

Abelian group

For any abelian group M, integer m and prime ℓ, we set

$$M[m] = \{x \in M | mx = 0\}, \quad M_{\mathrm{tor}} = \bigcup_{m \geq 1} M[m], \quad M[\ell^\infty] = \bigcup_{n \geq 1} M[\ell^n],$$

$$M(\text{non-}\ell) = \bigcup_{m \geq 1, \ell \nmid m} M[m], \; T_\ell M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M) = \varprojlim_n M[\ell^n], \; V_\ell M = T_\ell(M) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

We also set $M(\text{non-}0) = M$. Denote the cardinality of a group M by $|M|$. A torsion abelian group M is *of cofinite type* if $M[\ell]$ is finite for all prime ℓ. For two abelian groups M and N, we say that they are *isomorphic up to finite groups* if there exist filtrations $M_0 \subseteq M_1 \subseteq M$ and $N_0 \subseteq N_1 \subseteq N$ such that M_0 (resp. N_0) and M/M_1 (resp. N/N_1) are finite and M_1/M_0 is isomorphic to N_1/N_0.

2.2 Cohomological Brauer group

In this section, we recall basic definitions and facts about the cohomological Brauer groups. The main reference for this section is the book [12].

Definition 2.2.1. *The* cohomological Brauer group *of a scheme X is*

$$\mathrm{Br}(X) := H^2(X, \mathbb{G}_m)_{\mathrm{tor}}.$$

In the particular case $X = \mathrm{Spec}(k)$, where k is a field, this is same as the classical description of the Brauer group of a field given by the Galois cohomology:

$$\mathrm{Br}(k) = H^2(k, (k^s)^\times).$$

A morphism of schemes $f : X \longrightarrow Y$ gives rise to a morphism:

$$f^* : H^n(Y, \mathbb{G}_m) \longrightarrow H^n(X, \mathbb{G}_m)$$

For $n = 2$ this gives a natural map of Brauer groups $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$, which is sometimes referred to as the *restriction* map.

The Kummer exact sequence

The Brauer group is linked to étale cohomology with finite coefficients by the Kummer exact sequence on X

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \longrightarrow 0.$$

Here ℓ is a prime invertible on X and n is a positive integer. The associated long exact sequence of cohomology gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow H^2(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0.$$

One can drop the restriction that ℓ is invertible on X by using the fppf site X_{fppf} instead of $X_{\text{ét}}$. For any integer $n \geq 1$, the sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 0$$

is exact on X_{fppf}. Since $\mathbb{G}_m$ is a smooth group scheme, there is a canonical isomorphism

$$H^i_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sim} H^i_{\text{fppf}}(X, \mathbb{G}_m),$$

it gives rise to an exact sequence

$$0 \longrightarrow \text{Pic}(X)/n \longrightarrow H^2(X, \mu_n) \longrightarrow \text{Br}(X)[n] \longrightarrow 0.$$

Localization of Brauer groups

Let X be a regular, integral, noetherian scheme. Let $j : \text{Spec}(F) \longrightarrow X$ be the generic point of X. There is a natural exact sequence of sheaves on the small étale site $X_{\text{ét}}$, which describes the embedding of the group of invertible regular functions into the group of non-zero rational functions as the kernel of the divisor map:

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D \in X^1} i_{D*}\mathbb{Z}_{k(D)} \longrightarrow 0.$$

Here X^1 denotes the set of divisors on X and $i_D : \text{Spec}(k(D)) \hookrightarrow X$ is the embedding of the generic point of D.

Proposition 2.2.2. *Let X be a regular, integral, noetherian scheme. Then the groups $H^n(X, \mathbb{G}_m)$ are torsion for $n \geq 2$. In particular, the Brauer group $\text{Br}(X)$ is a torsion group.*

Proof. This follows from the long exact sequence associated to the short exact sequence above and the lemma below. □

Lemma 2.2.3. *Let X be a scheme. Let L be a field and let $f : \text{Spec}(L) \to X$ be a morphism. We have the following properties:*

(i) $H^1(X, f_*\mathbb{Z}) = 0$;

(ii) $H^1(X, f_*\mathbb{G}_m) = 0$;

(iii) $R^1 f_*\mathbb{Z} = 0$;

(iv) $R^1 f_*\mathbb{G}_m = 0$.

If F is an étale sheaf on $\text{Spec}(L)$, then, for any $i \geq 1$,

(v) the sheaf $R^i f_* F$ is a torsion sheaf;

(vi) *if, in addition, X is quasi-compact and quasi-separated, then the group $H^i(X, f_*F)$ is a torsion group.*

Lemma 2.2.4. *Let X be a regular, integral, noetherian scheme. Let $j : \mathrm{Spec}(F) \longrightarrow X$ be the generic point of X. There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow H^2(X, j_*\mathbb{G}_m) \longrightarrow \bigoplus_{D \in X^1} H^1(k(D), \mathbb{Q}/\mathbb{Z}).$$

Proof. This follows from the long exact sequence associated to the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D \in X^1} i_{D_*}\mathbb{Z}_{k(D)} \longrightarrow 0$$

and the lemma above. See [12, Lemma. 3.5.4] for the detailed proof.

□

Proposition 2.2.5. *Let X be a regular, integral, noetherian scheme. Let $j : \mathrm{Spec}(F) \longrightarrow X$ be the generic point of X. the natural map $\mathrm{Br}(X) \to \mathrm{Br}(F)$ is injective. For any non-empty open subset $U \subset X$, the natural map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is also injective.*

Proof. By the lemma above, it suffices to show that the natural map induced by the isomorphism $j^*j_*\mathbb{G}_{m,F} \cong \mathbb{G}_{m,F}$

$$H^2(X, j_*\mathbb{G}_{m,F}) \longrightarrow H^2(\mathrm{Spec}(F), \mathbb{G}_m)$$

is injective. The Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q j_*\mathbb{G}_{m,F}) \Rightarrow H^{p+q}(\mathrm{Spec}(F), \mathbb{G}_m)$$

gives an exact sequence

$$H^0(X, R^1 j_*\mathbb{G}_{m,F}) \longrightarrow H^2(X, j_*\mathbb{G}_{m,F}) \longrightarrow \mathrm{Ker}(H^2(\mathrm{Spec}(F), \mathbb{G}_m) \to H^0(X, R^2 j_*\mathbb{G}_{m,F})).$$

By Lemma 2.2.3, $H^0(X, R^1 j_*\mathbb{G}_{m,F}) = 0$. This shows the injectivity. □

Purity for the Brauer group

In this section, we recall the purity for Brauer group(cf. [12, §3.7] for proofs).

Theorem 2.2.6. *Let X be an excellent, regular, integral, noetherian scheme, let $Z \subset X$ be a closed subset of codimension c. Let $U \subset X$ be the open set $X - Z$. Let ℓ be a prime invertible on X*

(i) *If $c \geq 2$, then the restriction map $\mathrm{Br}(X)[\ell^\infty] \to \mathrm{Br}(U)[\ell^\infty]$ is an isomorphism.*

(ii) If $c = 1$ and $D_1, ..., D_m$ are the connected components of the regular locus of Z of codimension 1, then the Gysin exact sequences

$$H^2(X, \mu_{\ell^n}) \longrightarrow H^2(U, \mu_{\ell^n}) \longrightarrow \bigoplus_{i=1}^{m} H^1(D_i, \mathbb{Z}/\ell^n)$$

for all $n \geq 1$ give rise to exact sequences

$$0 \longrightarrow \mathrm{Br}(X)[\ell^\infty] \longrightarrow \mathrm{Br}(U)[\ell^\infty] \longrightarrow \bigoplus_{i=1}^{m} H^1(D_i, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

$$0 \longrightarrow \mathrm{Br}(X)[\ell^\infty] \longrightarrow \mathrm{Br}(U)[\ell^\infty] \longrightarrow \bigoplus_{i=1}^{m} H^1(k(D_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Theorem 2.2.7 ([6])**.** *Let X be a regular, integral, noetherian scheme. Let $U \subset X$ be an open subset whose complement is of codimension at least 2. Then the restriction map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(U)$$

is an isomorphism

Theorem 2.2.8 ([6])**.** *Let X be a regular, integral, noetherian scheme with function field F. Then $\mathrm{Br}(X) \subset \mathrm{Br}(F)$ is the subgroup*

$$\bigcap_{x \in X^1} \mathrm{Br}(\mathcal{O}_{X,x}),$$

where X^1 denotes the set of generic points of divisors in X.

The restriction and corestriction maps

Let $f : Y \to X$ be a finite locally free morphism of constant rank d of schemes. This means that locally for the Zariski topology on X the morphism is of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, where B a free A-module of finite rank d.

The norm of an element $b \in B$ is the determinant of the matrix that gives the multiplication by b on B with respect to some A-basis of B. It does not depend on the basis. The norm is multiplicative. We obtain a map of quasi-coherent sheaves $f_*\mathcal{O}_Y \to \mathcal{O}_X$. The composition of the canonical map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ with $f_*\mathcal{O}_Y \to \mathcal{O}_X$ sends u to u^d. Recall that the étale sheaf $\mathbb{G}_{m,X}$ is defined by setting $\mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U)^*$ for any étale morphism $U \to X$, and similarly for $\mathbb{G}_{m,Y}$. Thus, we obtain natural morphisms of étale sheaves

$$\mathbb{G}_{m,X} \longrightarrow f_*\mathbb{G}_{m,Y} \longrightarrow \mathbb{G}_{m,X},$$

whose composition sends u to u^d. By the finiteness of f, the functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact (cf.[36, Cor. II.3.6]). Thus the Leray spectral sequence gives an isomorphism $H^n(X, f_*\mathbb{G}_{m,Y}) \xrightarrow{\sim} H^n(Y, \mathbb{G}_{m,Y})$ which identifies the canonical map f^* with $H^n(X, \mathbb{G}_{m,X}) \to H^n(X, f_*\mathbb{G}_{m,Y})$. We thus obtain the *restriction* and *corestriction* maps

$$H^n(X, \mathbb{G}_{m,X}) \xrightarrow{\mathrm{Res}_{Y/X}} H^n(Y, \mathbb{G}_{m,Y}) \xrightarrow{\mathrm{Cor}_{Y/X}} H^n(X, \mathbb{G}_{m,X})$$

whose composition is multiplication by d. Here the restriction $\mathrm{Res}_{Y/X}$ is the canonical map f^*. For $n = 2$, we obtain the restriction and corestriction maps of Brauer groups

$$\mathrm{Res}_{Y/X} : \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y), \qquad \mathrm{Cor}_{Y/X} : \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X)$$

Proposition 2.2.9. *Let $f : Y \to X$ be a finite locally free morphism of constant rank of schemes. Let $i : X' \to X$ be a morphism and let $Y' = X' \times_X Y$. Let $j : Y' \to Y$ be the natural projection. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Br}(Y) & \xrightarrow{j^*} & \mathrm{Br}(Y') \\ \downarrow{\scriptstyle \mathrm{Cor}_{Y/X}} & & \downarrow{\scriptstyle \mathrm{Cor}_{Y'/X'}} \\ \mathrm{Br}(X) & \xrightarrow{i^*} & \mathrm{Br}(X') \end{array}$$

Proof. See [12, Propostion. 3.8.1]. □

Brauer groups of Dedekind schemes

In this section, we give some examples about Brauer groups of regular schemes of dimension at most 1.

Proposition 2.2.10 (Azumaya)**.** *Let R be a henselian local ring with residue field k.*

(i) The embedding of the closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ *induces an isomorphism* $\mathrm{Br}(R) \xrightarrow{\sim} \mathrm{Br}(k)$.

(ii) If R is a strictly henselian local ring, that is, if k is separably closed, then $\mathrm{Br}(R) = 0$.

Proof. See [12, Thm. 3.4.2]. □

Corollary 2.2.11. *Let R be a henselian noetherian local ring with maximal ideal m. Let $\hat{R}$ be the m-adic completion of R. Then the natural map* $\mathrm{Br}(R) \to \mathrm{Br}(\hat{R})$ *is an isomorphism.*

Proposition 2.2.12. *Let R be an excellent discrete valuation ring with field of fractions K and residue field k of characteristic p.*

(i) If k is separably closed, then $\mathrm{Br}(K)(\text{non-}p) = 0$. *If k is algebraically closed, then* $\mathrm{Br}(K) = 0$.

(ii) If k is finite and $\mathrm{char}(K) = 0$, *then* $\mathrm{Br}(R) = 0$ *and* $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$.

Proof. By Theorem 2.2.6 (ii), $\mathrm{Br}(R)[\ell^\infty] \xrightarrow{\sim} \mathrm{Br}(K)[\ell^\infty]$ for any $\ell \neq p$. Since $\mathrm{Br}(R) \cong \mathrm{Br}(k) = 0$, $\mathrm{Br}(K)[\ell^\infty] = 0$. This proves $\mathrm{Br}(K)(\text{non-}p) = 0$. For the case that k is algebraically closed, see [12][Thm. 1.2.15] for the proof.

If k is finite, $\mathrm{Br}(R) \cong \mathrm{Br}(k) = 0$ by [12][Thm. 1.2.13]. There is a Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(k, H^j(K^{sh}, \mathbb{G}_m)) \Longrightarrow H^{i+j}(K, \mathbb{G}_m).$$

It gives a long exact sequence

$$0 \to H^1(k, (K^{sh})^\times) \to H^1(K, \mathbb{G}_m) \to H^0(k, H^1(K^{sh}, \mathbb{G}_m))$$

$$\to H^2(k, (K^{sh})^\times) \to \ker(\mathrm{Br}(K) \to \mathrm{Br}(K^{sh})) \to H^1(k, H^1(K^{sh}, \mathbb{G}_m))$$

Since $H^1(K, \mathbb{G}_m) = 0$ and $\mathrm{Br}(K^{sh}) = 0$,

$$\mathrm{Br}(K) \cong H^2(k, (K^{sh})^\times).$$

By $(K^{sh})^\times \cong (R^{sh})^\times \times \mathbb{Z}$, it suffices to show that $H^2(k, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ and $H^2(k, (R^{sh})^\times)$=0. Taking Galois cohomology for the exact sequence of G_k-modules with trivial actions

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

we get $H^2(k, \mathbb{Z}) \cong H^1(k, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. Let π be a generator of the maximal ideal in R. The exact sequence

$$0 \longrightarrow (1 + \pi R^{sh}) \longrightarrow (R^{sh})^\times \longrightarrow (k^s)^\times \longrightarrow 0$$

induces a long exact sequence of Galois cohomology. Since $H^i(k, (k^s)^\times) = 0$ for $i = 1, 2$, the induced morphism

$$H^2(k, (1 + \pi R^{sh})) \longrightarrow H^2(k, (R^{sh})^\times)$$

is an isomorphism. Since $(1 + \pi^n R^{sh})/(1 + \pi^{n+1} R^{sh}) \cong k^s$ and $H^i(k, k^s) = 0$ for $i > 0$, the natural map

$$H^2(k, (1 + \pi^m R^{sh})) \longrightarrow H^2(k, (1 + \pi^n R^{sh}))$$

is an isomorphism for any $m > n \geq 1$. Let M_n denote $1 + \pi^n R^{sh}$. If $p = \pi^e u$ for some $u \in (R^{sh})^\times$, it follows from Hensel's lemma that for any $y \in M_{n+e}$, the equation $x^p = y$ has a unique solution x in M_n if n is sufficiently large. So $pM_n = M_{n+e}$ for some $n > 1$. It follows that

$$H^2(k, M_n) \xrightarrow{p} H^2(k, M_n)$$

is an isomorphism. So $H^2(k, M_n)[p] = 0$. Since $M_n \xrightarrow{\ell} M_n$ is an isomorphism for any $\ell \neq p$, $H^2(k, M_n)[\ell] = 0$. This proves $H^2(k, M_n) = 0$ for some $n > 0$. So $H^2(k, (R^{sh})^\times) = 0$. □

Proposition 2.2.13. *Let X be a smooth geometrically connected curve over a separably closed field k of characteristic p. Then* $\mathrm{Br}(X)(\text{non-}p) = 0$. *Moreover, if k is algebraically closed, then* $\mathrm{Br}(X) = 0$.

Proof. Let $\ell \neq p$ be a prime, the Kummer exact sequence gives

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow H^2(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0.$$

If X is projective, the second arrow is an isomorphism, so $\mathrm{Br}(X)[\ell^n] = 0$. If X is not projective, $H^2(X, \mu_{\ell^n}) = 0$ by the Poincaré duality. This also gives $\mathrm{Br}(X)[\ell^n] = 0$. So $\mathrm{Br}(X)(\text{non-}p) = 0$. If k is algebraically closed, then the function field $K(X)$ is a C_1-field (c.f. [12][Thm. 1.2.14]) whose Brauer group vanishes. By Proposition 2.2.5, $\mathrm{Br}(X)$ also vanishes. □

Proposition 2.2.14. *Let $\mathcal{O}_K$ be the ring of integers in a number field K, there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v \in M_K} \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Here M_K denotes the set of all places of K and K_v is the completion of K at v. Let X denote $\mathrm{Spec}(\mathcal{O}_K)$. *As a result,* $\mathrm{Br}(X)$ *is finite.*

Proof. The first claim is a consequence of the global class field theory (cf. [35, Chap. II, Prop. 2.1]). By [12, Thm. 3.6.1(ii)], the natural map

$$\mathrm{Br}(K)/\mathrm{Br}(\mathcal{O}_{X,v}) \longrightarrow \mathrm{Br}(K_v)/\mathrm{Br}(\mathcal{O}_v)$$

is injective. By Theorem 2.2.8, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v \in M_K^f} \mathrm{Br}(K)/\mathrm{Br}(\mathcal{O}_{X,v}),$$

where M_K^f is the set of finite places. It gives

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v \in M_K^f} \mathrm{Br}(K_v)/\mathrm{Br}(\mathcal{O}_v).$$

By Proposition 2.2.12, $\mathrm{Br}(\mathcal{O}_v) = 0$ for all $v \in M_K^f$. This gives

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v \in M_K^f} \mathrm{Br}(K_v).$$

Thus $\mathrm{Br}(X) \hookrightarrow \prod_v \mathrm{Br}(K_v)$, where the product is taken for all infinite places. Since $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$, so $\mathrm{Br}(X)$ is finite. □

2.3 Picard variety and Neron-Severi group

Here we review some definitions and basic properties about the Picard group, the Neron–Severi group, and the Picard functor.

Let X be a projective scheme over a field k. Denote by $\mathrm{Pic}^0(X)$ the subgroup of $\mathrm{Pic}(X)$ of algebraically trivial line bundles (cf. [31, Definition 9.5.9]).

Definition 2.3.1. *The* Neron-Severi group $\mathrm{NS}(X)$ *of* X *is the quotient* $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

Note that a line bundle over X is algebraically trivial if and only if it is algebraically trivial over X_{k^s}. Therefore,

$$\mathrm{NS}(X) = \mathrm{Im}(\mathrm{Pic}(X) \to \mathrm{NS}(X_{k^s})).$$

By [19, n°232, §6] or [4, §8.2, Thm. 3], the Picard functor $\underline{\mathrm{Pic}}_{X/k}$ is represented by a group scheme, locally of finite type over k. Denote by $\underline{\mathrm{Pic}}^0_{X/k}$ the identity component of (the group scheme representing) $\underline{\mathrm{Pic}}_{X/k}$. By [31, Lem 9.5.1], $\underline{\mathrm{Pic}}^0_{X/k}$ is a group scheme of finite type over k, open and closed in $\underline{\mathrm{Pic}}_{X/k}$. If X is geometrically normal, by [31, Prop. 9.5.3, Thm. 9.5.4], $\underline{\mathrm{Pic}}^0_{X/k}$ is actually projective over k. In this case, by [19, n°236-16, Cor. 3.2], the reduced structure $(\underline{\mathrm{Pic}}^0_{X/k})_{\mathrm{red}}$ of $\underline{\mathrm{Pic}}^0_{X/k}$ is an abelian variety over k.

Definition 2.3.2. *Let X be a smooth projective variety over k. The* Picard variety $\mathrm{Pic}^0_{X/k,\mathrm{red}}$ *of X/k is the reduced scheme* $(\underline{\mathrm{Pic}}^0_{X/k})_{\mathrm{red}}$.

There are canonical injections

$$\mathrm{Pic}(X) \longrightarrow \underline{\mathrm{Pic}}_{X/k}(k), \quad \mathrm{Pic}^0(X) \longrightarrow \underline{\mathrm{Pic}}^0_{X/k}(k).$$

They are isomorphisms if $X(k)$ is non-empty or k is separably closed. See [4, §8.1, Prop. 4] and [31, Prop 9.5.10, Thm 9.2.5].

By [50, Exp. XIII, Theorem 5.1], $\mathrm{NS}(X^s)$ is a finitely generated abelian group. Thus, we have

Proposition 2.3.3. *The Neron-Severi group* $\mathrm{NS}(X)$ *is finitely generated.*

Poincaré reducibility theorem

Theorem 2.3.4 ([41])**.** *If A is an abelian variety and Y is an abelian subvariety of A. Then, there is an abelian subvariety Z of A such that $Y \cap Z$ is finite and $Y + Z = A$. In other words, X is isogenous to $Y \times Z$.*

Proof. See [41, §19, Thm. 1]. □

Lemma 2.3.5. *Let X be a smooth projective geometrically connected variety over a infinite field k. Let $C \subset X$ be a smooth projective geometrically connected curve obtained by taking hyperplane sections repeatedly. Then there exists an abelian subvariety B of $\mathrm{Pic}^0_{C/k}$ such that the induced morphism*

$$\mathrm{Pic}^0_{X/k,\mathrm{red}} \times B \longrightarrow \mathrm{Pic}^0_{C/k}$$

is an isogeny.

Proof. By the Lefschetz hyperplane theorem, the induced map

$$H^1(X_{k^s}, \mathbb{Q}_\ell(1)) \longrightarrow H^1(C_{k^s}, \mathbb{Q}_\ell(1))$$

is injective. Since $H^1(X_{k^s}, \mathbb{Q}_\ell(1)) \cong V_\ell \mathrm{Pic}^0_{X/k,\mathrm{red}}$ and $H^1(C_{k^s}, \mathbb{Q}_\ell(1)) \cong V_\ell \mathrm{Pic}^0_{C/k}$, the induced map

$$V_\ell \mathrm{Pic}^0_{X/k,\mathrm{red}} \longrightarrow V_\ell \mathrm{Pic}^0_{C/k}$$

is injective. It follows that the kernel of $\mathrm{Pic}^0_{X/k,\mathrm{red}} \longrightarrow \mathrm{Pic}^0_{C/k}$ is finite. Let Y be the image of $\mathrm{Pic}^0_{X/k,\mathrm{red}} \longrightarrow \mathrm{Pic}^0_{C/k}$. Set $A = \mathrm{Pic}^0_{C/k}$. Y is an abelian subvariety of A, by the theorem above, there exists $Z \subset A$ such that $Y \times Z \to A$ is an isogeny. Taking $B = Z$, $\mathrm{Pic}^0_{X/k,\mathrm{red}} \times B \to \mathrm{Pic}^0_{C/k}$ is an isogeny.

□

2.4 Colliot-Thélène and Skorobogatov's pull-back trick

In this section, we will recall a pull-back trick developed by Colliot-Thélène and Skorobogatov in their paper [11]. This pull-back trick play the essential role in the proof our main theorems.

Lemma 2.4.1. *Let X be a smooth projective geometrically connected variety over a field k.*

(i) There exist a finite separable extension k'/k and smooth projective geometrically connected curves $C_1, ..., C_m \subset X_{k'}$ and an abelian subvariety $B \subset \prod_{i=1}^m \mathrm{Pic}^0_{C_i/k'}$ such that the induced morphism of k^s-points

$$\mathrm{Pic}(X_{k^s}) \times B(k^s) \longrightarrow \bigoplus_{i=1}^m \mathrm{Pic}(C_{i,k^s})$$

has a kernel and a cokernel of finite exponent.

(ii) There exist smooth projective integral curves $C_1, ..., C_m \subset X$ over k and a G_k-module B with a G_k-equivariant map $B \to \oplus_{i=1}^m \mathrm{Pic}(C_{i,k^s})$ such that the induced G_k-morphism

$$\mathrm{Pic}(X_{k^s}) \times B \longrightarrow \bigoplus_{i=1}^m \mathrm{Pic}(C_{i,k^s})$$

has a kernel and a cokernel of finite exponent.

Proof. For the proof of (i), see [67, p17]. Let $C_1, ..., C_m$ be curves as in (i). As a k-scheme, C_i is a smooth projective integral curve over k. The induced morphism

$$\mathrm{Pic}^0_{X/k,\mathrm{red}} \longrightarrow \prod_{i=1}^{m} \mathrm{Pic}^0_{C_i/k}$$

has a finite kernel. By the argument in the proof of Lemma 2.3.5, there exists a morphism of abelian varieties $B' \longrightarrow \prod_{i=1}^{m} \mathrm{Pic}^0_{C_i/k}$ such that the induced morphism

$$\mathrm{Pic}^0_{X/k,\mathrm{red}} \times B' \longrightarrow \prod_{i=1}^{m} \mathrm{Pic}^0_{C_i/k}$$

is an isogeny. It follows that the natural map

$$\mathrm{Pic}(X_{k^s}) \times B'(k^s) \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$$

has a kernel of finite exponent and a cokernel M of finite dimension after tensor with $\mathbb{Q}$. There exists a G_k-submodule $N \subset \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$ such that $N_{\mathbb{Q}} \xrightarrow{\sim} M_{\mathbb{Q}}$. We may assume that N is finitely generated over $\mathbb{Z}$. It suffices to show that

$$\mathrm{Pic}(X_{k^s}) \oplus B'(k^s) \oplus N \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$$

has a kernel and a cokernel of finite exponent. It is easy to see that the induced map

$$\mathrm{NS}(X_{k^s})_{\mathbb{Q}} \times N_{\mathbb{Q}} \longrightarrow (\bigoplus_{i=1}^{m} \mathrm{NS}(C_{i,k^s}))_{\mathbb{Q}}$$

is surjective. So

$$\mathrm{NS}(X_{k^s}) \times N \longrightarrow \bigoplus_{i=1}^{m} \mathrm{NS}(C_{i,k^s})$$

has a finite cokernel. This implies that

$$\mathrm{Pic}(X_{k^s}) \oplus B'(k^s) \oplus N \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$$

has a cokernel of finite exponent. Let (a, b, c) be an element in the kernel. Since $N_{\mathbb{Q}} \xrightarrow{\sim} M_{\mathbb{Q}}$, $c \in N_{\mathrm{tor}}$. Since $|N_{\mathrm{tor}}|$ is finite, $|N_{\mathrm{tor}}|(a, b, c)$ lies in the kernel of

$$\mathrm{Pic}(X_{k^s}) \times B'(k^s) \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$$

which is of finite exponent. □

Next, we use the pull-back trick above to prove Colliot-Thélène and Skorobogatov's theorem. In Chapter 3 and 4, we will extend this technique to prove our main theorems.

Theorem 2.4.2 ([11])**.** *Let X be a smooth, projective, geometrically integral variety over a field k. Then the cokernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})^{G_k}$ has a cokernel of finite exponent.*

Lemma 2.4.3. *Let $L \subset k^s$ be a finite separable extension of a field k of degree d. Let X be a k-scheme. The following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Br}(X) & \xrightarrow{\mathrm{Res}_{X_L/X}} & \mathrm{Br}(X_L) & \xrightarrow{\mathrm{Cor}_{X_L/X}} & \mathrm{Br}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Br}(X_{k^s})^{G_k} & \xrightarrow{i} & \mathrm{Br}(X_{k^s})^{G_L} & \xrightarrow{\sigma} & \mathrm{Br}(X_{k^s})^{G_k} \end{array}$$

Here i is the inclusion map and $\sigma(x) = \sum_i \sigma_i(x)$, where $\sigma_i \in G_k$ are coset representatives of G_k/G_L. The composition of maps in each row of the diagram is the multiplication by d.

Proof. See [12, Lemma 5.4.13]. □

The Hochschild–Serre spectral sequence

Let X be a smooth projective geometrically connected variety over a field k. There is a Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(k, H^j(X^s, \mathbb{G}_m)) \Longrightarrow H^{i+j}(X, \mathbb{G}_m).$$

It gives a long exact sequence of seven terms

$$0 \to H^1(k, (k^s)^\times) \to \mathrm{Pic}(X) \to \underline{\mathrm{Pic}}_{X/k}(k) \to \mathrm{Br}(k)$$

$$\to \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})^{G_k}) \to H^1(k, \mathrm{Pic}(X_{k^s})) \to H^3(k, \mathbb{G}_m).$$

By Hilbert's theorem 90, $H^1(k, (k^s)^\times) = 0$. Assuming that $X(k) \neq \varnothing$, let $Y = \mathrm{Spec}(k) \to X$ be a k-point of X, there is a commuative diagram

$$\begin{array}{ccc} H^1(k, \mathrm{Pic}(X_{k^s})) & \longrightarrow & H^3(k, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^1(k, \mathrm{Pic}(Y_{k^s})) & \longrightarrow & H^3(k, \mathbb{G}_m) \end{array}$$

The first vetrical map is 0 and the second vertical map is an isomorphism, so the first row vanishes. This gives

Proposition 2.4.4. *Let X be a smooth projective geometrically connected variety over a field k. Assuming that $X(k) \neq \varnothing$, there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})^{G_k}) \longrightarrow H^1(k, \mathrm{Pic}(X_{k^s})) \longrightarrow 0.$$

Consider the following canonical maps and exact sequences induced by the Hochschild–Serre spectral sequence,

$$d_2^{1,1} : E_2^{1,1} \longrightarrow E_2^{3,0},$$
$$d_3^{0,2} : E_3^{0,2} \longrightarrow E_3^{3,0},$$
$$d_2^{0,2} : E_2^{0,2} \longrightarrow E_2^{2,1}.$$
$$E^2 \longrightarrow E_4^{0,2} \longrightarrow 0,$$
$$0 \longrightarrow E_4^{0,2} \longrightarrow E_3^{0,2} \xrightarrow{d_3^{0,2}} E_3^{3,0},$$
$$0 \longrightarrow E_3^{0,2} \longrightarrow E_2^{0,2} \xrightarrow{d_2^{0,2}} E_2^{2,1}.$$

Assuming that $X(k) \neq \varnothing$, $d_2^{1,1} = 0$ and $d_2^{3,0} = 0$. Thus, $E_3^{3,0} = E_2^{2,0}$. By the similar pull-back argument, $d_3^{0,2} = 0$. So, $E_4^{0,2} = E_3^{0,2}$. This gives

Lemma 2.4.5. *Let X be a smooth projective geometrically connected variety over a field k. Assuming that $X(k) \neq \varnothing$, there is an exact sequence*

$$\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})^{G_k} \longrightarrow H^2(k, \mathrm{Pic}(X_{k^s})).$$

Proof of Theorem 2.4.2

Proof. By Lemma 2.4.3 and Lemma 2.4.1, without loss of generality, we can assume that $X(k) \neq \varnothing$ and there exists $C_1, ..., C_m \subset X$ defined over k satisfying the condition in Lemma 2.4.1. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(X_{k^s})^{G_k} & \longrightarrow & H^2(k, \mathrm{Pic}(X_{k^s})) \\ \downarrow & & \downarrow \\ \oplus_i \mathrm{Br}(C_{i,k^s})^{G_k} & \longrightarrow & \oplus_i H^2(k, \mathrm{Pic}(C_{i,k^s})) \end{array}$$

By the lemma above, it suffices to show that the image of the first row is of finite exponent. Since

$$\mathrm{Pic}(X_{k^s}) \times B(k^s) \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Pic}(C_{i,k^s})$$

has a finite kernel and a finite cokernel, the natural map

$$H^2(k, \mathrm{Pic}(X_{k^s}) \times B(k^s)) \longrightarrow \bigoplus_{i=1}^{m} H^2(k, \mathrm{Pic}(C_{i,k^s}))$$

has a kernel of finite exponent. This implies that the second column in the diagram has a kernel of finite exponent. Since $\mathrm{Br}(C_{i,k^s}) = 0$ for all i, a diagram chasing implies that the image of the first row is of finite exponent. □

2.5 Tate conjecture and Brauer group

In this section, we will recall some facts about the Tate conjecture for divisors which already appeared in the literature.

Lemma 2.5.1. *Let X be a smooth projective geometrically connected variety over a field k. The natural map*

$$\mathrm{NS}(X) \longrightarrow \mathrm{NS}(X_{k^s})^{G_k}$$

has a finite kernel and a finite cokernel.

Proof. The map

$$\mathrm{NS}(X) \longrightarrow \mathrm{NS}(X_{k^s})^{G_k}$$

is injective by definition. For its cokernel, take G_k-invariants of the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X_{k^s}) \longrightarrow \mathrm{Pic}(X_{k^s}) \longrightarrow \mathrm{NS}(X_{k^s}) \longrightarrow 0.$$

We have an exact sequence

$$\mathrm{Pic}(X_{k^s})^{G_k} \longrightarrow \mathrm{NS}(X_{k^s})^{G_k} \longrightarrow H^1(k, \mathrm{Pic}^0(X_{k^s})).$$

The last arrow has a finite image, since $\mathrm{NS}(X_{k^s})$ is finitely generated and $H^1(k, \mathrm{Pic}^0(X_{k^s}))$ is torsion. Then it suffices to prove that $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^{G_k}$ has a torsion cokernel. The Hochschild–Serre spectral sequence

$$H^i(k, H^j(X_{k^s}, \mathbb{G}_m)) \Longrightarrow H^{i+j}(X, \mathbb{G}_m)$$

induces an exact sequence

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_{k^s})^{G_k} \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X).$$

The cokernel of the first map is torsion, since $\mathrm{Br}(k)$ is torsion. □

Proposition 2.5.2. *Let X be a smooth projective geometrically connected variety over a field k, then we have*

(a) For any prime $\ell \neq \mathrm{char}(k)$, the exact sequence of G_k-representations

$$0 \longrightarrow \mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell \mathrm{Br}(X_{k^s}) \longrightarrow 0$$

is split. Taking G_k-invariant, there is an exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_k} \longrightarrow V_\ell \mathrm{Br}(X_{k^s})^{G_k} \longrightarrow 0.$$

(b) For sufficiently large prime $\ell \neq \mathrm{char}(k)$, the exact sequence of G_k-modules

$$0 \longrightarrow \mathrm{NS}(X_{k^s})/\ell^n \longrightarrow H^2(X_{k^s}, \mathbb{Z}/\ell^n(1)) \longrightarrow \mathrm{Br}(X_{k^s})[\ell^n] \longrightarrow 0$$

is split for any $n \geq 1$.

(c) For all but finitely many ℓ, there is an exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_k} \longrightarrow \mathrm{Br}(X_{k^s})^{G_k}[\ell^\infty] \longrightarrow 0.$$

Proof. Let d denote the dimension of X. Let $Z_1(X_{k^s})$ denote the group of 1-cycles on X_{k^s}, it admits a natural G_k action. Since τ-equivalence is same as the numerical equivalence for divisors (cf., e.g. SGA 6 XIII, Theorem 4.6), thus the intersection pairing

$$\mathrm{NS}(X_{k^s})_{\mathbb{Q}} \times Z_1(X_{k^s})_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

is left non-degenerate. Since $\mathrm{NS}(X_{k^s})$ is finitely generated, so there exists a finite dimensional G_k-invariant subspace W of $Z^1(X_{k^s})_{\mathbb{Q}}$ such that the restriction of the intersection pairing to $\mathrm{NS}(X_{k^s})_{\mathbb{Q}} \times W$ is left non-degenerate. Since G_k-actions factor through a finite quotient of G_k, we can choose W such that the pairing is actually perfect. Let $W_{\mathbb{Q}_\ell}$ denote the subspace of $H^{2d-2}(X_{k^s}, \mathbb{Q}_\ell(d-1))$ generated by cycle classes of W. Then the restriction of

$$H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \times H^{2d-2}(X_{k^s}, \mathbb{Q}_\ell(d-1)) \longrightarrow H^{2d}(X_{k^s}, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell$$

to $\mathrm{NS}(X_{k^s})_{\mathbb{Q}_\ell} \times W_{\mathbb{Q}_\ell}$ is also perfect. So we have

$$H^2(X_{k^s}, \mathbb{Q}_\ell(1)) = \mathrm{NS}(X_{k^s})_{\mathbb{Q}_\ell} \oplus W_{\mathbb{Q}_\ell}^{\perp}.$$

This proves (a).

There exists a finitely generated G_k-invariant subgroup W_0 of $Z^1(X_{k^s})$ such that $W = W_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, there exists a positive integer N such that the base change of the pairing $\mathrm{NS}(X_{k^s}) \times W_0 \longrightarrow \mathbb{Z}$ to $\mathbb{Z}[N^{-1}]$ is perfect. So for any $\ell \nmid N$, the intersection pairing

$$\mathrm{NS}(X_{k^s})/\ell^n \times W_0/\ell^n \longrightarrow \mathbb{Z}/\ell^n$$

is perfect. Since it is compatible with

$$H^2(X_{k^s}, \mathbb{Z}/\ell^n(1)) \times H^{2d-2}(X_{k^s}, \mathbb{Z}/\ell^n(d-1)) \longrightarrow H^{2d}(X_{k^s}, \mathbb{Z}/\ell^n(d)) \cong \mathbb{Z}/\ell^n.$$

Thus we have

$$H^2(X_{k^s}, \mathbb{Z}/\ell^n(1)) = \mathrm{NS}(X_{k^s})/\ell^n \oplus (W_0/\ell^n)^{\perp}.$$

This proves (b).

Taking G_k-invariants, we get an exact sequence for all but finitely many ℓ

$$0 \longrightarrow (\mathrm{NS}(X_{k^s})/\ell^n)^{G_k} \longrightarrow H^2(X_{k^s}, \mathbb{Z}/\ell^n(1))^{G_k} \longrightarrow \mathrm{Br}(X_{k^s})^{G_k}[\ell^n] \longrightarrow 0.$$

Taking direct limit, we get

$$0 \longrightarrow (\mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k} \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_k} \longrightarrow \mathrm{Br}(X_{k^s})^{G_k}[\ell^\infty] \longrightarrow 0.$$

To prove (c), it suffices to show that the natural map

$$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow (\mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$$

is an isomorphism for all but finitely many ℓ. Consider the exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{NS}(X_{k^s})/\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow 0.$$

For ℓ sufficiently large, the last group will be a free $\mathbb{Z}_\ell$-module. Thus, it is split for all but finitely many ℓ. So tensoring $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, the sequence is still exact. This proves injectivity. Since the G_k action on $\mathrm{NS}(X_{k^s})$ factors though a finite group G, thus, for ℓ prime to $|G|$, any element in the $(\mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$ can be written as $\sum_{g\in G} gx$. This implies that the natural map

$$\mathrm{NS}(X_{k^s})^{G_k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow (\mathrm{NS}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$$

is surjective. Thus, to prove the surjectivity, it suffices to show that

$$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow \mathrm{NS}(X_{k^s})^{G_k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$$

is surjective. This follows from Lemma 2.5.1. □

Corollary 2.5.3. *Let X be a smooth projective geometrically connected variety over a finitely generated field k. Let $\ell \neq \mathrm{char}(k)$ be a prime. Then $T^1(X,\ell)$ is equivalent to the finiteness of $\mathrm{Br}(X_{k^s})^{G_k}[\ell^\infty]$. If k is finite, it's also equivalent to the finiteness of $\mathrm{Br}(X)[\ell^\infty]$.*

Proof. The first claim follows from Proposition 2.5.2. If k is finite, there is an exact sequence

$$H^1(X, \mathrm{Pic}^0_{X/k}) \longrightarrow H^1(X, \mathrm{Pic}(X_{k^s}) \longrightarrow H^1(k, \mathrm{NS}(X_{k^s}))$$

By Lang's theorem, $H^1(X, \mathrm{Pic}^0_{X/k}) = 0$. $H^1(k, \mathrm{NS}(X_{k^s}))$ is finite since $NS(X_{k^s})$ is finitely generated. This shows that $H^1(X, \mathrm{Pic}(X_{k^s})$ is finite. By Wedderburn's theorem $\mathrm{Br}(k) = 0$, it follows from Proposition 2.4.4, the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})$ has a finite kernel. By Theorem 2.4.2,

$$\mathrm{Br}(X)[\ell^\infty] \longrightarrow \mathrm{Br}(X_{k^s})^{G_k}[\ell^\infty]$$

has a finite kernel and a finite cokernel. The second claim follows from the first one. □

Chapter 3

Geometric case

In this chapter, we will prove Theorem 1.4.1 and its corollaries Corollary 1.4.2, Theorem 1.5.1 and Theorem1.5.3.

3.1 The Galois invariant parts of geometric Brauer groups

In this section, we will show that the Galois invariant part of the geometric Brauer group of a smooth variety is a birational invariant up to finite groups. This will allow us to shrink the base of a fibration without changing the question.

The following results in the case of characteristic 0 were proved in [11] (cf. [11, Prop. 6.1 and Thm. 6.2 (iii)]).

Proposition 3.1.1. *Let X be a smooth geometrically connected variety over a finitely generated field k of characteristic $p \geq 0$. Let $U \subseteq X$ be an open dense subset. Then the natural map*

$$\mathrm{Br}(X_{k^s})^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(U_{k^s})^{G_k}(\text{non-}p)$$

is injectie and has a finite cokernel.

Proof. The injectivity follows from that X_{k^s} is regular and irreducible. To show that the cokernel is finite, we need the lemma below. Let Y denote $X - U$ with reduced scheme structure. By the purity for Brauer groups(Theorem 2.2.7), removing a close subset of codimension ≥ 2 will not change the Brauer group. Thus, by shrinking X, we may assume that Y is regular and of codimension 1 in X. Let $\ell \neq p$ be a prime. By Theorem 2.2.6, there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Br}(X_{k^s})[\ell^\infty] \longrightarrow \mathrm{Br}(U_{k^s})[\ell^\infty] \longrightarrow H^1(Y_{k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

It suffices to show that the group

$$H^1(Y_{k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$$

is finite and vanishes for all but finitely many ℓ. This follows from the lemma below. □

Lemma 3.1.2. *Let U be a regular variety over a finitely generated field k. Let $\ell \neq \mathrm{char}(k)$ be a prime. Then*

$$H^1(U_{k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$$

is finite and vanishes for all but finitely many ℓ.

Proof. Firstly, we will show that it suffices to prove the claim for smooth varieties over k. Notice that we can replace k by a finite separable extension. Thus, we may assume that U_{k^s} is irreducible. Let V be an open dense subset of U_{k^s}. By semi-purity (cf. [22, §8]),

$$H^1(U_{k^s}, \mathbb{Z}/\ell^n) \longrightarrow H^1(V, \mathbb{Z}/\ell^n)$$

is injective. We may assume that V is defined over k. Therefore, it suffices to prove the claim for an open dense subset. Since U_{k^s} and $U_{\bar{k}}$ have the same underlying toplological space, by shrinking U_{k^s}, we may assume that $(U_{\bar{k}})_{\mathrm{red}}$ is irreducible and smooth over $\bar{k}$. So there exists a finite extension l/k such that $(U_l)_{\mathrm{red}}$ is irreducible and smooth over l. We may assume that l/k is purely inseparable. Then $l^s = l \otimes_k k^s$ and $G_l = G_k$. Let V denote $(U_l)_{\mathrm{red}}$. Thus $V_{l^s} = V \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^s)$. The k-morphism $V \longrightarrow U$ induces a G_k-equivariant isomorphism

$$H^1(U_{k^s}, \mathbb{Z}/\ell^n) \cong H^1(V_{l^s}, \mathbb{Z}/\ell^n).$$

It suffices to prove the claim for V. Thus we may assume that U is smooth and geometrically connected over k.

By above arguments, we can always replace k by a finite extension and shrink U. By de Jong's alteration theorem, we may assume that there is a finite flat morphism $f : V \longrightarrow U$ such that V admits a smooth projective geometrically connected compactification over k. Since the kernel of

$$H^1(U_{k^s}, \mathbb{Z}/\ell^n) \longrightarrow H^1(V_{k^s}, \mathbb{Z}/\ell^n)$$

is killed by the degree of f (cf.[12, Prop. 3.8.4]), it suffices to prove the claim for V. Thus we may assume that U is an open subvariety of a smooth projective geometrically connected variety X over k. Let Y_i be irreducible components of $X - U$ of codimension 1. Let D_i be the regular locus of Y_i. By extending k to a finite separable extension, we may assume that D_i is geometrically irreducible. By purity theorem(cf. [22, §8]), there is a canonical exact sequence

$$0 \longrightarrow H^1(X_{k^s}, \mathbb{Z}/\ell^n) \longrightarrow H^1(U_{k^s}, \mathbb{Z}/\ell^n) \longrightarrow \bigoplus_i H^0(D_{i,k^s}, \mathbb{Z}/\ell^n(-1)).$$

Taking G_k-invariants, we get an exact sequence

$$0 \longrightarrow H^1(X_{k^s}, \mathbb{Z}/\ell^n)^{G_k} \longrightarrow H^1(U_{k^s}, \mathbb{Z}/\ell^n)^{G_k} \longrightarrow \bigoplus_i H^0(D_{i,k^s}, \mathbb{Z}/\ell^n(-1))^{G_k}.$$

It suffices to show that the size of the first and third group are bounded independent of n and is equal to 1 for all but finitely many ℓ. Since D_{i,k^s} is connected, we have

$$H^0(D_{i,k^s}, \mathbb{Z}/\ell^n(-1)) = \mathrm{Hom}(\mu_{\ell^n}, \mathbb{Z}/\ell^n).$$

By the lemma below, the size of the third group is bounded independent of n and is equal to 1 for all but finitely many ℓ. Since

$$H^1(X_{k^s}, \mathbb{Z}/\ell^n) \cong \mathrm{Pic}_{X/K}[\ell^n](-1)$$

and

$$0 \longrightarrow \mathrm{Pic}^0_{X/K,\mathrm{red}}[\ell^n] \longrightarrow \mathrm{Pic}_{X/K}[\ell^n] \longrightarrow \mathrm{NS}(X_{k^s})[\ell^n]$$

is exact, it suffices to show that the size of $(\mathrm{Pic}^0_{X/K,\mathrm{red}}[\ell^n](-1))^{G_k}$ is bounded independent of n and is equal to 1 for all but finitely many ℓ. Let A be the dual of $\mathrm{Pic}^0_{X/K,\mathrm{red}}$. By the Weil pairing,

$$\mathrm{Pic}^0_{X/K,\mathrm{red}}[\ell^n](-1) \cong \mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n).$$

Taking G_k-invariants, we get

$$(\mathrm{Pic}^0_{X/K,\mathrm{red}}[\ell^n](-1))^{G_k} \cong \mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n)^{G_k}.$$

By the lemma below, the claim follows. □

Lemma 3.1.3. *Let A be an abelian variety over a finitely generated field K of characteristic $p \geq 0$. Let $\ell \neq p$ be a prime. Then the sizes of*

$$\mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n)^{G_K} \quad \text{and} \quad \mathrm{Hom}(\mu_{\ell^n}, \mathbb{Z}/\ell^n)^{G_K}$$

are bounded independent of n. Moreover, for all but finitely many ℓ, these two groups vanish for any n.

Proof. We will only prove the claim for the first group, since the second one follows from the same arguments. Firstly, we assume that K is finite, then we will use a specialization technique to reduce the general case to the finite field case.

If K is finite, then we have

$$\mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n)^{G_K} = \mathrm{Hom}(A[\ell^n]_{G_K}, \mathbb{Z}/\ell^n),$$

which has the same size as $A[\ell^n]^{G_K}$. Then the claim follows from the finiteness of $A(K)$.

In general, choose an integral regular scheme S of finite type over $\mathrm{Spec}(\mathbb{Z})$ with function field K. By Shrinking S, we may assume A/K extends to an abelian scheme $\mathscr{A}/S$. Fix a closed point $s \in S$. For any $\ell \neq \mathrm{char}(k(s))$, we can shrink S such that ℓ is invertible on S and $s \in S$. Then the étale sheaf $\mathscr{A}[\ell^n]$ is a locally constant sheaf of $\mathbb{Z}/\ell^n$-module since $\mathscr{A}[\ell^n]$ is finite étale over S. Thus

$$\mathscr{H}om(\mathscr{A}[\ell^n], \mathbb{Z}/\ell^n)$$

is also a locally constant sheaf of $\mathbb{Z}/\ell^n$-module and its stalk at the generic point can be identified with

$$\mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n).$$

Set $\mathscr{F} = \mathscr{H}om(\mathscr{A}[\ell^n], \mathbb{Z}/\ell^n)$, we have

$$\mathrm{Hom}(A[\ell^n], \mathbb{Z}/\ell^n)^{G_K} = H^0(S, \mathscr{F}) \hookrightarrow H^0(s, \mathscr{F}).$$

Since $H^0(s, \mathscr{F}) = \mathrm{Hom}(\mathscr{A}_s[\ell^n], \mathbb{Z}/\ell^n)^{G_{k(s)}}$ and $k(s)$ is finite, thus the claim holds for $\ell \neq \mathrm{char}(k(s))$. In the case $\mathrm{char}(K) = 0$, we can choose another closed point s' such that $\mathrm{char}(k(s')) \neq \mathrm{char}(k(s))$, then by the same argument, the claim also holds for $\ell = \mathrm{char}(k(s))$. This completes the proof. □

3.2 Geometric Tate-Shafarevich group

In this section, first, we will study a geometric version of the Tate-Shafarevich group for an abelian variety over a function field with a base field k. Then, we will prove that the Galois fixed part of the geometric Tate-Shafarevich group is canonically isomorphic to Tate-Shafarevich group up to finite groups in the case that k is a finite field. The idea is reducing the question to the relation between arithmetic Brauer groups and geometric Brauer groups which was studied in [11](cf. Theorem 2.4.2).

Proposition 3.2.1. *Let C be a smooth projective geometrically connected curve defined over a finitely generated field k of characteristic $p \geq 0$. Let K be the function field of C and A be an abelian variety over K. Denote Kk^s by K'. Let $U \subseteq C$ be an open dense subscheme. Define*

$$Ш_{U_{k^s}}(A) := \mathrm{Ker}(H^1(K', A) \longrightarrow \prod_{v \in |U_{k^s}|} H^1(K_v^{sh}, A))$$

Then the natural map

$$Ш_{C_{k^s}}(A)^{G_k}(\text{non-}p) \longrightarrow Ш_{U_{k^s}}(A)^{G_k}(\text{non-}p)$$

is injective and has a cokernel of finite exponent.

Proof. By definitions, the injectivity is obvious. It suffices to show that the cokernel is of finite exponent. By Lemma 2.3.5, there exists an abelian variety B/K such that $A \times B$ is isogenous to $\mathrm{Pic}^0_{X/K}$ for some smooth projective geometrically connected curve X over K. It suffices to prove the claim for $A = \mathrm{Pic}^0_{X/K}$. Without loss of generality, we may replace k by a finite extension l/k. This is obvious if l/k is separable. For l/k a purely inseparable extension of degree p^n, the quotient $A(K^s l^s)/A(K^s)$ is killed by p^n. Then $H^1(K', A) \longrightarrow H^1(Kl^s, A)$ has a kernel and a cokernel killed by p^n. It follows that $Ш_{U_{k^s}}(A) \longrightarrow Ш_{U_{l^s}}(A)$ has a kernel and a cokernel killed by some power of p. By the resolution of singularity of surfaces, $X \longrightarrow \mathrm{Spec}(K)$ admits a proper regular model $\pi : \mathcal{X} \longrightarrow C$. By extending k, we may assume that $\mathcal{X}$ is smooth proper geometrically connected over k. Write V for $\pi^{-1}(U)$. The Leray spectral sequence

$$E_2^{p,q} = H^p(U_{k^s}, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(V_{k^s}, \mathbb{G}_m)$$

gives a long exact sequence

$$H^2(U_{k^s}, \mathbb{G}_m) \longrightarrow \mathrm{Ker}(H^2(V_{k^s}, \mathbb{G}_m) \longrightarrow H^0(U_{k^s}, R^2\pi_*\mathbb{G}_m))$$
$$\longrightarrow H^1(U_{k^s}, R^1\pi_*\mathbb{G}_m) \longrightarrow H^3(U_{k^s}, \mathbb{G}_m).$$

By [24, Cor. 3.2 and Lem. 3.2.1], $R^2\pi_*\mathbb{G}_m = 0$ and $H^i(U_{k^s}, \mathbb{G}_m)(\text{non-}p) = 0$ for $i \geq 2$. Thus, there is a canonical isomorphism

$$H^2(V_{k^s}, \mathbb{G}_m)(\text{non-}p) \cong H^1(U_{k^s}, R^1\pi_*\mathbb{G}_m)(\text{non-}p).$$

Let $j : \mathrm{Spec}(K') \longrightarrow C_{k^s}$ be the generic point. By the spectral sequence

$$H^p(U_{k^s}, R^q j_*(j^*R^1\pi_*\mathbb{G}_m)) \Rightarrow H^{p+q}(K', \mathrm{Pic}(X_{K^s})),$$

we have

$$H^1(U_{k^s}, j_*j^*R^1\pi_*\mathbb{G}_m) = \mathrm{Ker}(H^1(K', \mathrm{Pic}(X_{K^s})) \longrightarrow \prod_{v\in|U_{k^s}|} H^1(K_v^{sh}, \mathrm{Pic}(X_{K^s}))).$$

Since $\mathrm{Pic}(X_{K^s}) = \mathrm{Pic}^0_{X/K}(K^s) \oplus \mathbb{Z}$ as G_K-modules, there is a natural isomorphism

$$H^1(U_{k^s}, j_*j^*R^1\pi_*\mathbb{G}_m) \cong Ш_{U_{k^s}}(\mathrm{Pic}^0_{X/K}).$$

Without loss of generality, we may shrink U such that π is smooth on $\pi^{-1}(U)$. By Lemma 3.3.1, the natural map

$$R^1\pi_*\mathbb{G}_m \longrightarrow j_*j^*R^1\pi_*\mathbb{G}_m$$

is an isomorphism on U. It follows that

$$H^2(V_{k^s}, \mathbb{G}_m)(\text{non-}p) \cong H^1(U_{k^s}, R^1\pi_*\mathbb{G}_m)(\text{non-}p)$$
$$\cong H^1(U_{k^s}, j_*j^*R^1\pi_*\mathbb{G}_m)(\text{non-}p) \cong Ш_{U_{k^s}}(\mathrm{Pic}^0_{X/K})(\text{non-}p).$$

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Br}(\mathcal{X}_{k^s})(\text{non-}p) & \longrightarrow & H^1(C_{k^s}, j_*j^*R^1\pi_*\mathbb{G}_m)(\text{non-}p) & \longrightarrow & Ш_{C_{k^s}}(\mathrm{Pic}^0_{X/K})(\text{non-}p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Br}(V_{k^s})(\text{non-}p) & \longrightarrow & H^1(U_{k^s}, j_*j^*R^1\pi_*\mathbb{G}_m)(\text{non-}p) & \longrightarrow & Ш_{U_{k^s}}(\mathrm{Pic}^0_{X/K})(\text{non-}p) \end{array}$$

Since maps on the bottom are isomorphisms, it suffices to show that the natural map

$$\mathrm{Br}(\mathcal{X}_{k^s})^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(V_{k^s})^{G_k}(\text{non-}p)$$

has a finite cokernel. This follows from Proposition 3.1.1. □

Proposition 3.2.2. *Notations as in the above proposition. Assuming that k is a finite field, define*

$$Ш_{K'}(A) := \mathrm{Ker}(H^1(K', A) \longrightarrow \prod_{v \in |C_{k^s}|} H^1(K_v^{sh}, A)).$$

Then the natural map

$$Ш(A)(\text{non-}p) \longrightarrow Ш_{K'}(A)^{G_k}(\text{non-}p)$$

has a kernel and a cokernel of finite exponent.

Proof. We use the same arguments as in the proof of the previous lemma. It suffices to prove the claim for $A = \mathrm{Pic}^0_{Y/K}$ where Y is a smooth projective geometrically connected curve over k. Y admits a projective regular model $\mathcal{Y} \longrightarrow C$. Then we have

$$\mathrm{Br}(\mathcal{Y}) \cong Ш(\mathrm{Pic}^0_{Y/K})$$

and

$$\mathrm{Br}(\mathcal{Y}_{k^s}) \cong Ш_{K'}(\mathrm{Pic}^0_{Y/K})$$

up to finite groups. Thus the question is reduced to show that

$$\mathrm{Br}(\mathcal{Y})(\text{non-}p) \longrightarrow \mathrm{Br}(\mathcal{Y}_{k^s})^{G_k}(\text{non-}p)$$

has a finite kernel and a finite cokernel. This follows from Theorem 2.4.2. □

Cofiniteness of Brauer groups and Tate-Shafarevich groups

Let ℓ be a prime number. Recall that a ℓ-torsion abelian group M is of *cofinite type* if $M[\ell]$ is finite. This is also equivalent to that M can be written as $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus M_0$ for some integer $r \geq 0$ and some finite group M_0. We also say that a torsion abelian group M is of cofinite type if $M[\ell^\infty]$ is of cofinite type for all primes ℓ. It is easy to see that a morphism $M \to N$ between abelian groups of cofinite type has a kernel and a cokernel of finite exponent if and only if the kernel and cokernel are finite.

In the following, we will show the cofinitness of geometric Tate-Shafarevich groups and geometric Brauer groups defined in previous sections.

Lemma 3.2.3. *The group $Ш_{U_{k^s}}(A)[\ell^\infty]$ defined in the Proposition 3.2.1 is of cofinite type for any $\ell \neq p$.*

Proof. We may shrink U such that A extends to an abelian scheme $\mathscr{A}$ over U. By the exact sequence

$$0 \longrightarrow \mathscr{A}[\ell] \longrightarrow \mathscr{A} \xrightarrow{\ell} \mathscr{A} \longrightarrow 0$$

we get a surjection

$$H^1(U_{k^s}, \mathscr{A}[\ell]) \longrightarrow H^1(U_{k^s}, \mathscr{A})[\ell].$$

Since $H^1(U_{k^s}, \mathscr{A}[\ell])$ is finite, $H^1(U_{k^s}, \mathscr{A})[\ell]$ is also finite. Thus, it suffices to show

$$H^1(U_{k^s}, \mathscr{A}) \cong Ш_{U_{k^s}}(A).$$

This follows from $\mathscr{A} \cong j_* A$, since $\mathscr{A}$ is a Néron model of A over U. Here $j : \mathrm{Spec}(K) \longrightarrow U$ is the generic point.

□

Lemma 3.2.4. *Let X be a smooth variety over a separable closed field k of characteristic $p \geq 0$. Let $\ell \neq p$ be a prime. Then $\mathrm{Br}(X)[\ell^\infty]$ is of cofinite type.*

Proof. The Kummer exact sequence

$$0 \longrightarrow \mu_\ell \longrightarrow \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \longrightarrow 0$$

induces a surjection

$$H^2(X, \mu_\ell) \longrightarrow \mathrm{Br}(X)[\ell].$$

Then the claim follows from the finiteness of $H^2(X, \mu_\ell)$.

□

3.3 Proof of Theorem 1.4.1

The left exactness in Theorem 1.4.1

In this section, we will prove the left exactness of the sequence in Theorem 1.4.1 by following Grothendieck's arguments in [24, §4].

Lemma 3.3.1. *Let U be an irreducible regular scheme of dimension 1 with function field K. Let $\pi : \mathcal{X} \longrightarrow U$ be a smooth proper morphism with a generic fiber geometrically connected over K. Let $j : \mathrm{Spec}(K) \longrightarrow U$ be the generic point of U. Then we have*

(a) the natural map

$$R^1\pi_*\mathbb{G}_m \longrightarrow j_*j^*R^1\pi_*\mathbb{G}_m$$

is an isomorphism,

(b) the natural map

$$R^2\pi_*\mathbb{G}_m[\ell^\infty] \longrightarrow j_*j^*R^2\pi_*\mathbb{G}_m[\ell^\infty]$$

is an isomorphism for any prime ℓ invertible on U.

Proof. It suffices to show that the induced maps on stalks are isomorphism. Thus, we may assume that $U = \mathrm{Spec}(R)$ where R is a strictly henselian DVR.

Let X denote the generic fiber. Let $s \in U$ be the closed point. Then we have

$$(R^1\pi_*\mathbb{G}_m)_{\bar{s}} = \mathrm{Pic}(\mathcal{X}) \quad \text{and} \quad (j_*j^*R^1\pi_*\mathbb{G}_m)_{\bar{s}} = \mathrm{Pic}_{X/K}(K).$$

Since $\mathcal{X}_s$ admits a section $s \longrightarrow \mathcal{X}_s$ and π is smooth, the section can be extended to a section $U \longrightarrow \mathcal{X}$. Thus $X(K)$ is not empty. So

$$\underline{\mathrm{Pic}}_{X/K}(K) = \mathrm{Pic}(X).$$

Since $\mathcal{X}$ is regular, the natural map

$$\mathrm{Pic}(\mathcal{X}) \longrightarrow \mathrm{Pic}(X)$$

is surjective and has a kernel generated by vertical divisors. It suffices to show that $\mathcal{X}_s$ is connected. This actually follows from $\pi_*\mathscr{O}_{\mathcal{X}} = \mathscr{O}_U$ (cf. [25, Chap. III, Cor. 11.3]). This proves (a).

Let I denote G_K. For (b), the induced map on the stalk at s is

$$\mathrm{Br}(\mathcal{X})[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^s})^I[\ell^\infty].$$

Since π is smooth and proper, we have

$$H^2(\mathcal{X}, \mu_{\ell^\infty}) \cong H^2(X_{K^s}, \mu_{\ell^\infty}) = H^2(X_{K^s}, \mu_{\ell^\infty})^I.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^2(\mathcal{X}, \mu_{\ell^\infty}) & \longrightarrow & \mathrm{Br}(\mathcal{X})[\ell^\infty] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^2(X_{K^s}, \mu_{\ell^\infty}) & \longrightarrow & \mathrm{Br}(X_{K^s})[\ell^\infty] & \longrightarrow & 0 \end{array}$$

Since $\mathrm{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is I-invariant and $\mathrm{Pic}(\mathcal{X}) = \mathrm{Pic}(X)$. It suffices to show that

$$\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow (\mathrm{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^I$$

is surjective. Write $\mathrm{NS}(X_{K^s})_{\mathrm{free}}$ for $\mathrm{NS}(X_{K^s})/\mathrm{NS}(X_{K^s})_{\mathrm{tor}}$. The action of I on $\mathrm{NS}(X_{K^s})$ factors through a finite quotient I'. Consider the exact sequence

$$0 \longrightarrow (\mathrm{NS}(X_{K^s})_{\mathrm{free}})^{I'} \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{NS}(X_{K^s})^{I'} \otimes \mathbb{Q}_\ell \longrightarrow (\mathrm{NS}(X_{\bar{K}}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^I$$
$$\longrightarrow H^1(I', \mathrm{NS}(X_{K^s})_{\mathrm{free}} \otimes \mathbb{Z}_\ell).$$

$H^1(I', \mathrm{NS}(X_{K^s})_{\mathrm{free}} \otimes \mathbb{Z}_\ell)$ is killed by the order of I'. Since $(\mathrm{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^I = \mathrm{NS}(X_{K^s}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible, so the image of the last map is zero. By Lemma 2.5.1,

$$\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{NS}(X_{K^s})^I \otimes \mathbb{Q}_\ell$$

is surjective, the claim follows. By the Snake Lemma, the natural map

$$\mathrm{Br}(\mathcal{X})[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^s})^I[\ell^\infty]$$

is an isomorphism.

□

Lemma 3.3.2. *Let U be a smooth geometrically connected curve over a field k of characteristic $p \geq 0$ with function field K. Let $\pi : \mathcal{X} \longrightarrow U$ be a smooth proper morphism with a generic fiber X geometrically connected over K. Let K' denote Kk^s. Define*

$$Ш_{U_{k^s}}(\mathrm{Pic}_{X/K}) := \mathrm{Ker}(H^1(K', \mathrm{Pic}(X_{K^s})) \longrightarrow \prod_{v \in |U_{k^s}|} H^1(K_v^{sh}, \mathrm{Pic}(X_{K^s}))).$$

Then there is a canonical exact sequence

$$0 \to (Ш_{U_{k^s}}(\mathrm{Pic}_{X/K}))^{G_k}(\text{non-}p) \to \mathrm{Br}(\mathcal{X}_{k^s})^{G_k}(\text{non-}p) \to \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p).$$

Moreover, the natural map

$$(Ш_{U_{k^s}}(\mathrm{Pic}^0_{X/K,\mathrm{red}}))^{G_k} \longrightarrow (Ш_{U_{k^s}}(\mathrm{Pic}_{X/K}))^{G_k}$$

has a kernel and cokernel of finite exponent.

Proof. We may assume that $k = k^s$. The Leray spectral sequence

$$E_2^{p,q} = H^p(U, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m)$$

gives a long exact sequence

$$\begin{aligned} H^2(U, \mathbb{G}_m) \longrightarrow \mathrm{Ker}(H^2(\mathcal{X}, \mathbb{G}_m) &\longrightarrow H^0(U, R^2\pi_*\mathbb{G}_m)) \\ &\longrightarrow H^1(U, R^1\pi_*\mathbb{G}_m) \longrightarrow H^3(U, \mathbb{G}_m). \end{aligned}$$

By [24, Lem. 3.2.1], $H^i(U, \mathbb{G}_m)(\text{non-}p) = 0$ for $i \geq 2$. Thus, there is a canonical exact sequence

$$0 \to H^1(U, R^1\pi_*\mathbb{G}_m)(\text{non-}p) \to H^2(\mathcal{X}, \mathbb{G}_m)(\text{non-}p) \to H^0(U, R^2\pi_*\mathbb{G}_m)(\text{non-}p).$$

Let $j : \mathrm{Spec}(K) \longrightarrow U$ be the generic point. By Lemma 3.3.1, there are canonical isomorphisms

$$H^1(U, R^1\pi_*\mathbb{G}_m)(\text{non-}p) \cong H^1(U, j_*j^*R^1\pi_*\mathbb{G}_m)(\text{non-}p)$$

and

$$H^0(U, R^2\pi_*\mathbb{G}_m)(\text{non-}p) \cong H^0(U, j_*j^*R^2\pi_*\mathbb{G}_m)(\text{non-}p).$$

Since $j^*R^i\pi_*\mathbb{G}_m$ corresponds to the G_K-module $H^i(X_{K^s}, \mathbb{G}_m)$, we get

$$H^1(U, j_*j^*R^1\pi_*\mathbb{G}_m) = Ш_U(\mathrm{Pic}_{X/K}) \quad \text{and} \quad H^0(U, j_*j^*R^2\pi_*\mathbb{G}_m) = \mathrm{Br}(X_{K^s})^{G_K}.$$

This proves the first claim.

For the second claim, consider the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X_{K^s}) \longrightarrow \mathrm{Pic}(X_{K^s}) \longrightarrow \mathrm{NS}(X_{K^s}) \longrightarrow 0.$$

By Proposition 2.3.3, $\mathrm{NS}(X_{K^s})$ is finitely generated. So there exists a finite Galois extension L/K such that $\mathrm{Pic}(X_L) \longrightarrow \mathrm{NS}(X_{K^s})$ is surjective. Taking Galois cohomoloy, we get a long exact sequence

$$0 \longrightarrow H^0(K, \mathrm{Pic}^0(X_{K^s})) \longrightarrow H^0(K, \mathrm{Pic}(X_{K^s})) \longrightarrow H^0(K, \mathrm{NS}(X_{K^s}))$$

$$\xrightarrow{a_K} H^1(K, \mathrm{Pic}^0(X_{K^s})) \longrightarrow H^1(K, \mathrm{Pic}(X_{K^s})) \longrightarrow H^1(K, \mathrm{NS}(X_{K^s})).$$

We have a similar long exact sequence for $H^i(L, -)$. Since $\mathrm{Pic}(X_L) \to \mathrm{NS}(X_{K^s})$ is surjective, we have $a_L = 0$ and

$$H^1(L, \mathrm{NS}(X_{K^s})) = \mathrm{Hom}(G_L, \mathrm{NS}(X_{K^s})) = \mathrm{Hom}(G_L, \mathrm{NS}(X_{K^s})_{\mathrm{tor}}).$$

$a_L = 0$ implies that the image of a_K is contained in

$$\mathrm{Ker}(H^1(K, \mathrm{Pic}^0(X_{K^s})) \longrightarrow H^1(L, \mathrm{Pic}^0(X_{K^s})))$$

which is killed by $[L : K]$. Similarly, one can show that $H^1(K, \mathrm{NS}(X_{K^s}))$ is killed by $[L : K]|\mathrm{NS}(X_{K^s})_{\mathrm{tor}}|$. Therefore, the kernel and cokernel of

$$H^1(K, \mathrm{Pic}^0(X_{K^s})) \longrightarrow H^1(K, \mathrm{Pic}(X_{K^s}))$$

are killed by $[L : K]|\mathrm{NS}(X_{K^s})_{\mathrm{tor}}|$. The claim also holds for

$$H^1(K_v^{sh}, \mathrm{Pic}^0(X_{K^s})) \longrightarrow H^1(K_v^{sh}, \mathrm{Pic}(X_{K^s})).$$

By diagram chasings, the kernel and cokernel of

$$Ш_U(\mathrm{Pic}^0_{X/K,\mathrm{red}}) \longrightarrow Ш_U(\mathrm{Pic}_{X/K})$$

is of finite exponent. This completes the proof. □

The pull-back trick

In this section, we will use Colliot-Thélène and Skorobogatov's pull-back trick(cf. Chapter 2.4) to reduce the question to cases of relative dimension 1. The approach is similar to the proof of Theorem 2.4.2.

Let U be a regular integral excellent scheme of dimension 1 with function field K. Let $\pi : \mathcal{X} \longrightarrow U$ be a smooth projective morphism with the generic fiber X geometrically connected over K. The Leray spectral sequence

$$E_2^{p,q} = H^p(U, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m)$$

induces canonical maps

$$d_2^{1,1} : E_2^{1,1} \longrightarrow E_2^{3,0},$$
$$d_3^{0,2} : E_3^{0,2} \longrightarrow E_3^{3,0},$$
$$d_2^{0,2} : E_2^{0,2} \longrightarrow E_2^{2,1}.$$

Lemma 3.3.3. *Assuming that $X(K)$ is not empty and the natural map $\mathrm{Pic}(X) \to \mathrm{NS}(X_{K^s})$ is surjective, then the canonical maps $d_2^{1,1}$ and $d_3^{0,2}$ vanish and $E_3^{3,0} = E_2^{3,0}$. Moreover, there exists an open dense subscheme $V \subseteq U$ such that the canonical map $d_2^{0,2}$ has an image of finite exponent when replacing U by V. As a result, the natural map*

$$H^2(\pi^{-1}(V), \mathbb{G}_m) \longrightarrow H^0(V, R^2\pi_*\mathbb{G}_m)$$

has a cokernel of finite exponent.

Proof. Let $s \in X(K)$. Since π is proper, it extends to a section $s : U \longrightarrow \mathcal{X}$. Let $\tilde{E}_2^{p,q}$ denote the Leray spectral sequence for the identy map $U \longrightarrow U$. Then s induces a commutative diagram

$$\begin{array}{ccc} E_2^{1,1} & \xrightarrow{d_2^{1,1}} & E_2^{3,0} \\ \downarrow & & \downarrow \\ \tilde{E}_2^{1,1} & \longrightarrow & \tilde{E}_2^{3,0} \end{array}$$

The second column is an isomorphism since $E_2^{3,0} = H^3(U, \mathbb{G}_m) = \tilde{E}_2^{3,0}$. Since $\tilde{E}_2^{1,1} = 0$, thus $d_2^{1,1} = 0$. By definition, $d_2^{3,0} = 0$, it follows that $E_3^{3,0} = E_2^{3,0}$. By the same arguments, we have $d_3^{0,2} = 0$.

For the proof of the second claim, we will use the pullback method for finitely many U-morphisms $\mathcal{Y}_i \longrightarrow \mathcal{X}$ where $\mathcal{Y}_i$ is of relative dimension 1 over U. Let ${}^iE_2^{p,q}$ denote the Leray spectral sequence for $\pi_i : \mathcal{Y}_i \longrightarrow U$. There is a commutative diagram

$$\begin{array}{ccc} E_2^{0,2} & \xrightarrow{d_2^{0,2}} & E_2^{2,1} \\ \downarrow & & \downarrow \\ \oplus_i {}^iE_2^{0,2} & \longrightarrow & \oplus_i {}^iE_2^{2,1} \end{array}$$

By shrinking U, we can assume that π_i is smooth and projective for all i. By Artin's theorem [24, Cor. 3.2], $R^2\pi_{i,*}\mathbb{G}_m = 0$. Thus, to show that $d_2^{0,2}$ has an image of finite exponent, it suffices to show that the second column has a kernel of finite exponent. The second column is the map

$$H^2(U, R^1\pi_*\mathbb{G}_m) \longrightarrow \bigoplus_i H^2(U, R^1\pi_{i,*}\mathbb{G}_m).$$

By Lemma 3.3.1(a), there is a canonical isomorphism

$$R^1\pi_{i,*}\mathbb{G}_m \cong j_*j^*R^1\pi_{i,*}\mathbb{G}_m,$$

where $j : \mathrm{Spec}(K) \longrightarrow U$ is the generic point. It suffices to show that there exists an étale sheaf $\mathscr{F}$ on $\mathrm{Spec}(K)$ and a morphism $\mathscr{F} \longrightarrow \oplus_i j^*R^1\pi_{i,*}\mathbb{G}_m$ such that the induced map

$$j^*R^1\pi_*\mathbb{G}_m \oplus \mathscr{F} \longrightarrow \bigoplus_i j^*R^1\pi_{i,*}\mathbb{G}_m \tag{3.1}$$

has a kernel and a cokernel killed by some positive integer. This will imply that the induced map

$$H^2(U, j_*j^*R^1\pi_*\mathbb{G}_m) \oplus H^2(U, j_*\mathscr{F}) \longrightarrow \bigoplus_i H^2(U, j_*j^*R^1\pi_{i,*}\mathbb{G}_m)$$

has a kernel and a cokernel of finite exponent. Since $j^*R^1\pi_*\mathbb{G}_m$ corresponds to the G_K-module $\mathrm{Pic}(X_{K^s})$, the claim about the map (3.1) can be interpreted as that there is a G_K-module M and a G_K-equivariant map $M \longrightarrow \oplus_i \mathrm{Pic}(Y_{i,K^s})$ such that

$$\mathrm{Pic}(X_{K^s}) \oplus M \longrightarrow \bigoplus_i \mathrm{Pic}(Y_{i,K^s})$$

has a kernel and a cokernel of finite exponent. By Lemma 2.4.1, there exist smooth projective curves Y_i in X and an abelian subvariety A of $\prod_i \mathrm{Pic}^0_{Y_i/K}$ such that the induced morphism

$$\mathrm{Pic}(X_{K^s}) \times A(K^s) \longrightarrow \prod_i \mathrm{Pic}(Y_{i,K^s})$$

has a finite kernel and a finite cokernel. Taking $\mathcal{Y}_i$ to be the Zariski closure of Y_i in $\mathcal{X}$ and then shrinking U, we get smooth and proper morphisms $\pi : \mathcal{Y}_i \longrightarrow U$. This proves the second claim.

For the last claim, consider the following canonical exact sequences induced by the Leray spectral sequence,

$$E^2 \longrightarrow E_4^{0,2} \longrightarrow 0,$$

$$0 \longrightarrow E_4^{0,2} \longrightarrow E_3^{0,2} \xrightarrow{d_3^{0,2}} E_3^{3,0},$$

$$0 \longrightarrow E_3^{0,2} \longrightarrow E_2^{0,2} \xrightarrow{d_2^{0,2}} E_2^{2,1}.$$

Since $d_3^{0,2}$ vanishes and $d_2^{0,2}$ has an image of finite exponent, it follows that

$$E^2 \longrightarrow E_2^{0,2}$$

has a cokernel of finite exponent. This completes the proof.

□

Proof of Theorem 1.4.1

Now we prove Theorem 1.4.1. Let U be an open dense subscheme of C such that π is smooth and projective over U. By Lemma 3.3.2, the natural map

$$(\text{Ш}_{U_{k^s}}(\mathrm{Pic}^0_{X/K,\mathrm{red}}))^{G_k}(\text{non-}p) \to \mathrm{Ker}(\mathrm{Br}(\pi^{-1}(U_{k^s}))^{G_k}(\text{non-}p) \to \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p))$$

has a kernel of finite exponent. By Lemma 3.2.3 and 3.2.4, all groups here are of cofinite type. Thus the kernel is actually finite. By Proposition 3.1.1 and 3.2.1, it suffices to show that the natural map

$$\mathrm{Br}(\pi^{-1}(U_{k^s}))^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$$

has a cokernel of finite exponent.

Firstly, we will prove this under the assumption that $X(K)$ is not empty and the natural map $\mathrm{Pic}(X) \to \mathrm{NS}(X_{K^s})$ is surjective. We still use $\mathcal{X}$ to denote $\pi^{-1}(U)$. By Proposition 3.1.1, without loss of generality, we can always replace U by an open dense subset. By Lemma 3.3.1(b), we have

$$H^0(U, R^2\pi_*\mathbb{G}_m)(\text{non-}p) \cong H^0(U, j_*j^*R^2\pi_*\mathbb{G}_m)(\text{non-}p) = \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p).$$

It suffices to show that the natural map induced by the Leray spectral sequence for π

$$H^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow H^0(U, R^2\pi_*\mathbb{G}_m)$$

has a cokernel of finite exponent. By shrinking U, the claim follows from Lemma 3.3.3.

Secondly, we will show that the question can be reduced to the case that $X(K)$ is not empty and the natural map $\mathrm{Pic}(X) \to \mathrm{NS}(X_{K^s})$ is surjective. There exists a finite Galois extenion L/K such that $X(L)$ is not empty and the natural map $\mathrm{Pic}(X_L) \to \mathrm{NS}(X_{K^s})$ is surjective. Let W be a smooth curve over k with function field L. By shrinking U and W, the map $\mathrm{Spec}(L) \longrightarrow \mathrm{Spec}(K)$ extends to a finite étale Galois covering map $W \longrightarrow U$. Let $\mathcal{X}_W \longrightarrow W$ be the base change of $\mathcal{X} \longrightarrow U$ to W. By the arguments above, the natural map

$$\mathrm{Br}(\mathcal{X}_W)(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_L}(\text{non-}p)$$

has a cokernel of finite exponent. Let G denote $\mathrm{Gal}(L/K)$. The above map is compatible with the G-action. Taking G-invariant, the natural map

$$\mathrm{Br}(\mathcal{X}_W)^G(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$$

has a cokernel of finite exponent. Then the question is reduced to show that the natural map

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(\mathcal{X}_W)^G$$

has a cokernel of finite exponent. Consider the spectral sequence

$$H^p(G, H^q(\mathcal{X}_W, \mathbb{G}_m)) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m).$$

Since $H^p(G, -)$ is killed by the order of G, by similar arguments as in Lemma 3.3.3, we can conclude that the cokernel of

$$H^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow H^2(\mathcal{X}_W, \mathbb{G}_m)^G$$

is of finite exponent. This completes the proof of the theorem.

3.4 Applications of Theorem 1.4.1

Proof of Theorem 1.5.3

Firstly, note that if k is finite, by [33, Thm. 2.3], the finiteness of $\mathrm{Br}(\mathcal{X}_{k^s})^{G_k}(\text{non-}p)$ is equivalent to the Tate conjecture for divisors on $\mathcal{X}$.

We will prove the finiteness of $\mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$ by the induction on the transcendence degree of K/k. If K/k is of transcendence degree 0, a smooth projective variety over K can be view as a smooth projective variety over k. So the claim is true by assumptions.

Assume that the claim is true for all extensions l/k of transcendence degree n. Let K/k be a finitely generated extension of transcendence degree $n+1$ and X/K be a smooth projective connected variety over K. By extending K, we may assume that X/K is geometrically connected. K can be regarded as the function field of a smooth projective geometrically connected curve C over a field l of transcendence degree n over k. The structure map $X \longrightarrow \mathrm{Spec}(K)$ can be extended to a smooth morphism $\pi : \mathcal{X} \longrightarrow C$. By Theorem 1.4.1,

$$\mathrm{Br}(\mathcal{X}_{l^s})^{G_k}(\text{non-}p) \longrightarrow \mathrm{Br}(\mathcal{X}_{K^s})^{G_K}(\text{non-}p)$$

has a finite cokernel. It suffices to show that $\mathrm{Br}(\mathcal{X}_{l^s})^{G_l}(\text{non-}p)$ is finite. By Proposition 3.1.1, we may shrink $\mathcal{X}$ such that there is an alteration $f : \mathcal{X}' \longrightarrow \mathcal{X}$ such that $\mathcal{X}'$ admits a smooth projective compactification over a finite extension of l. Replacing l by a finite extension will not change the question, we may assume that $\mathcal{X}'$ admits a smooth projective compactification over l. By shrinking $\mathcal{X}$, we may assume that f is finite flat. Then the kernel of

$$\mathrm{Br}(\mathcal{X}_{l^s})(\text{non-}p) \longrightarrow \mathrm{Br}(\mathcal{X}'_{l^s})(\text{non-}p)$$

is killed by the degree of f and therefore is finite. Since $\mathrm{Br}(\mathcal{X}'_{l^s})^{G_l}(\text{non-}p)$ is finite by induction, so $\mathrm{Br}(\mathcal{X}_{l^s})^{G_l}(\text{non-}p)$ is finite. This completes the proof.

Proof of Theorem 1.5.1

Let $\ell \neq p$ be a prime, the the claim in Theorem 1.5.3 also holds when replacing the prime to p part by the ℓ primary part in the statement. So by the same argument as in the proof of Theorem 1.5.3, Theorem 1.5.1 is true.

Brauer groups of integral models of abelian varieties and $K3$ surfaces

Proposition 3.4.1. *Let C be a smooth projective geometrically connected curve over a finite field k of characteristic $p > 2$. Let $\pi : \mathcal{X} \longrightarrow C$ be a projective flat morphism. Let X denote the generic fiber of π. Assuming that $\mathcal{X}$ is regular and X_{K^s} is an abelian variety over K^s, then there is an isomorphism up to finite groups*

$$Ш(\mathrm{Pic}^0_{X/K,\mathrm{red}})(\text{non-}p) \cong \mathrm{Br}(\mathcal{X})(\text{non-}p).$$

Proof. By [54, Thm. 1.1], $\mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$ is finite, the claim follows directly from Corollary 1.4.5. $\square$

Proposition 3.4.2. *Let C be a smooth projective geometrically connected curve over a finite field k of characteristic p. Let $\pi : \mathcal{X} \longrightarrow C$ be a projective flat morphism. Assuming that $\mathcal{X}$ is regular and the generic fiber X of π is a smooth projective geometrically connected surface over K with $H^1(X_{K^s}, \mathbb{Q}_\ell) = 0$ for some prime $\ell \neq p$, then the natural map*

$$\mathrm{Br}(\mathcal{X})(\text{non-}p) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$$

has a finite kernel and cokernel. As a result, $\mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$ *is finite if and only if* $\mathrm{Br}(X_{K^s})^{G_K}(\ell)$ *is finite for some prime $\ell \neq p$.*

Proof. $\mathrm{Pic}^0_{X/K,\mathrm{red}} = 0$ since $H^1(X_{K^s}, \mathbb{Q}_\ell) = 0$. It follows that

$$Ш(\mathrm{Pic}^0_{X/K,\mathrm{red}}) = 0.$$

Then the claim follows from Corollary 1.4.2. $\square$

Corollary 3.4.3. *Notations as above, if X/K is a $K3$ surface and $p > 2$, then* $\mathrm{Br}(\mathcal{X})$ *is finite.*

Proof. Since Tate conjecture for X is known, $\mathrm{Br}(X_{K^s})^{G_K}(\ell)$ is finite for some prime $\ell \neq p$. By the above proposition, $\mathrm{Br}(\mathcal{X})[\ell^\infty]$ is also finite. By [38], $\mathrm{Br}(\mathcal{X})$ is finite. $\square$

Proposition 3.4.4. *Let X be a $K3$ surface over a global field K of characteristic $p > 2$. Then* $\mathrm{Br}(X_{K^s})^{G_K}(\text{non-}p)$ *is finite.*

Proof. By previous results, it suffices to show that X admits a projective regular model. By resolution of singularity of threefolds(cf. [9]), X admits a projective regular model. $\square$

Chapter 4

Arithmetic case

In this chapter, we will first prove a local version of Theorem 1.4.5(i.e., Proposition 4.1.1) for fibrations over a hensenlian DVR. This is the main goal of Section 4.1. Then, we use this local result to prove Theorem 1.4.5. In section 4.3, we prove Theorem 1.5.6.

4.1 Local results

Introduction

Let R denote a henselian DVR of characteristic 0 with a perfect residue field k of characteristic $p > 0$. Let ℓ denote a prime number. Let I denote $\mathrm{Gal}(\bar{K}/K^{sh})$ the inertia group for $K = \mathrm{Frac}(R)$. The goal of this section is to prove the following proposition:

Proposition 4.1.1. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism, where R is a henselian DVR of characteristic 0 with a finite residue field. Assuming that $\mathcal{X}$ is regular and the generic fiber X is geometrically connected over $K = \mathrm{Frac}(R)$, then the natural map*

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}$$

has finite kernel and a finite cokernel.

To prove this proposition, we first prove a p-adic analogue of the local invariant cycle theorem for H^2:

Lemma 4.1.2. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism with $\mathcal{X}$ regular. Assume that the residue field of R is finite. Let X denote the generic fiber of π. Define:*

$$H^2_{\mathrm{fppf}}(\mathcal{X}_{R^{sh}}, \mathbb{Q}_p(1)) := \varprojlim_n H^2_{\mathrm{fppf}}(\mathcal{X}_{R^{sh}}, \mu_{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

then the natural map

$$H^2_{\mathrm{fppf}}(\mathcal{X}_{R^{sh}}, \mathbb{Q}_p(1)) \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^I$$

is surjective.

As a direct result of the lemma above, we get the local result for fibrations over a strictly henselian DVR:

Corollary 4.1.3. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism with $\mathcal{X}$ regular. Assuming that the residue field of R is finite, then the natural map*

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})[p^\infty] \longrightarrow \mathrm{Br}(X_{\bar{K}})^I[p^\infty]$$

has a kernel of finite exponent and a finite cokernel.

Taking G_k-invariant, we get the p-primary part of the Proposition 4.1.1. The prime-to-p part of Proposition 4.1.1 can be deduced from the proposition below:

Proposition 4.1.4. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism where R is a henselian DVR with a finite residue field k and a quotient field K. Assume that $\mathcal{X}$ is regular and the generic fiber X of π is smooth projective geometrically connected over K. Let $\ell \neq \mathrm{char}(k)$ be a prime. Then the natural map*

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k}[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}[\ell^\infty]$$

has a kernel and a cokernel of finite exponent and is an isomorphism for all but finitely many ℓ. Moreover, if π is smooth, the map is an isomorphism for all $\ell \neq \mathrm{char}(k)$.

Proof of Lemma 4.1.2

We will prove Lemma4.1.2 under the assumption that π is a semi-stable projective morphism. The general case can be reduced to this case by de Jong's alteration theorem [13, Cor. 5.1](cf. Lemma 4.1.22).

Recall the definition of semi-stable morphisms.

Definition 4.1.5. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}R$ be a flat separated morphism of finite type. Assume that $\mathcal{X}$ is regular and irreducible. Let Y be the special fiber of π. Let $Y_i, i \in I$ be the irreducible components of Y. Put $Y_J = \cap_{j\in J} Y_j$ (scheme-theoretic intersection) for a non empty subset J of I. We say $\mathcal{X}$ is* strictly semi-stable *over S if the following properties hold:*

a) the generic fiber X of π is smooth over $K = \mathrm{Frac}(R)$,

b) Y is a reduced scheme, and

c) for each nonempty $J \subseteq I$, Y_J is smooth over k and has codimension $\#J$ in $\mathcal{X}$.

We say $\mathcal{X}$ is semi-stable *over S (or with* semi-stable reduction*) if the situation etale locally looks as described above.*

Proposition 4.1.6. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a semi-stable flat projective morphsim, where R is a henselian DVR of characteristic 0 with a perfect residue field k of characteristic $p > 0$. Let $i : Y \longrightarrow \mathcal{X}$ be the special fiber and $j : U = X_K \longrightarrow \mathcal{X}$ be the generic fiber. Then the natural map*

$$\tau_{\leq 1} Rj_*\mathbb{Z}/p^n(1) \longrightarrow Rj_*\mathbb{Z}/p^n(1)$$

induces a surjection:

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1} Rj_*\mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}.$$

To prove the statement above, we need the following two lemmas:

Lemma 4.1.7. ([8, Cor. 5.2(ii) and Thm. 5.4]) *Notation as above, assume that R is complete. Equips $\mathcal{X}$ with the log-structure defined by the special fiber. Let $\mathscr{S}_n(r)_{\mathcal{X}}$ be the (log) syntomic sheaf modulo p^n on $Y_{\text{ét}}$, there exist period morphisms*

$$\alpha^{FM}_{r,n} : \mathscr{S}_n(r)_{\mathcal{X}} \longrightarrow i^* Rj_*\mathbb{Z}/p^n(r)'_{X_K}$$

from logarithmic syntomic cohomology to logarithmic p-adic nearby cycles. Here we set $\mathbb{Z}/p^n(r)' := \frac{1}{p^{a(r)}}\mathbb{Z}_p(r) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ where $a(r) = [r/(p-1)]$. Then we have

(i) for $i \geq r+1$, $\mathscr{H}^i(\mathscr{S}_n(r))$ is annihilated by $p^{N(r)}$.

(ii) for $0 \leq i \leq r$, the kernel and cokernel of the period map

$$\alpha^{FM}_{r,n} : \mathscr{H}^i(\mathscr{S}_n(r)_{\mathcal{X}}) \longrightarrow i^* R^i j_*\mathbb{Z}/p^n(r)'_{X_K}$$

is annihilated by p^N for an integer $N = N(K, p, r)$, which only depends on K, p, r.

Remark 4.1.8. *For $0 \leq r \leq p-2$ and $n \geq 1$, it is known that*

$$\alpha^{FM}_{r,n} : \mathscr{S}_n(r)_{\mathcal{X}} \longrightarrow \tau_{\leq r} i^* Rj_*\mathbb{Z}/p^n(r)_{X_K}$$

is an isomorphism for $\mathcal{X}$ a log-scheme log-smooth over a henselian discrete valuation ring R of mixed characteristic. This was proved by Kato [29, 30], Kurihara [32], and Tsuji [63, 64]. If $p > 2$, we can take $r = 1$, it is enough for our application. To deal with the case $p = 2$, we need the above general result of Colmez and Niziol.

Lemma 4.1.9 ([15, 43])**.** *Notations as above, define $R\Gamma_{\text{syn}}(\mathcal{X}, r)_n := R\Gamma(\mathcal{X}_{\text{ét}}, \mathscr{S}_n(r))$ and $R\Gamma_{\text{syn}}(\mathcal{X}, r) := \mathrm{homlim}_n R\Gamma_{\text{syn}}(\mathcal{X}, r)_n$. Define $H^i_{\text{syn}}(\mathcal{X}, \mathbb{Q}_p(r)) := H^i(R\Gamma_{\text{syn}}(\mathcal{X}, r)) \otimes \mathbb{Q}$. Then there is a spectral sequence*

$${}^{\text{syn}}\mathrm{E}_2^{i,j} = H^i_{st}(G_K, H^j_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) \Rightarrow H^{i+j}_{\text{syn}}(\mathcal{X}, \mathbb{Q}_p(r)),$$

which is compatible with the Hochschild-Serre spectral sequence for etale cohomology

$${}^{\text{ét}}\mathrm{E}_2^{i,j} = H^i(G_K, H^j_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{Q}_p(r)).$$

It degenerates at E_2 for X_K projective and smooth. In particular, the edge map

$$H^2_{\mathrm{syn}}(\mathcal{X}, \mathbb{Q}_p(1)) \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective.

Proof of Proposition 4.1.6.

Proof. Replacing R by its completion, we can assume that R is complete. By Lemma 4.1.7, the maps

$$\mathscr{H}^i(\tau_{\leq 1}\mathscr{S}_n(1)_{\mathcal{X}}) \longrightarrow \mathscr{H}^i(\mathscr{S}_n(1)_{\mathcal{X}})$$

and

$$\mathscr{H}^i(\tau_{\leq 1}\mathscr{S}_n(1)_{\mathcal{X}}) \longrightarrow \mathscr{H}^i(i^*\tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K})$$

have kernel and cokernel annihilated by some p^N, where N only depends on $\pi : \mathcal{X} \longrightarrow S$ and is independent of n. So do

$$H^2_{\text{ét}}(Y, \tau_{\leq 1}\mathscr{S}_n(1)_{\mathcal{X}}) \longrightarrow H^2_{\text{ét}}(Y, \mathscr{S}_n(1)_{\mathcal{X}})$$

and

$$H^2_{\text{ét}}(Y, \tau_{\leq 1}\mathscr{S}_n(1)_{\mathcal{X}}) \longrightarrow H^2_{\text{ét}}(Y, i^*\tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}).$$

Taking limit, we get an isomorphism

$$\varprojlim_n H^2_{\text{ét}}(Y, \mathscr{S}_n(1)_{\mathcal{X}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \varprojlim_n H^2_{\text{ét}}(Y, i^*\tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By the proper base change theorem, we have

$$H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}) \cong H^2_{\text{ét}}(Y, i^*\tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}).$$

There is a natural map

$$H^2_{\mathrm{syn}}(\mathcal{X}, \mathbb{Q}_p(1)) \longrightarrow \varprojlim_n H^2_{\text{ét}}(Y, \mathscr{S}_n(1)_{\mathcal{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By composition, we get a natural map

$$H^2_{\mathrm{syn}}(\mathcal{X}, \mathbb{Q}_p(1)) \longrightarrow \varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Lemma 4.1.9, the natural map

$$H^2_{\mathrm{syn}}(\mathcal{X}, \mathbb{Q}_p(1)) \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective. By the following commutative diagram

$$\begin{array}{ccc} H^2_{\mathrm{syn}}(\mathcal{X}, \mathbb{Q}_p(1)) & \longrightarrow & \varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)'_{X_K}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ & \searrow & \downarrow \\ & & H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K} \end{array}$$

the map

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1} Rj_*\mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective. □

Lemma 4.1.10. *Assuming that R has a finite residue field, let X be a smooth proper variety over K. Assuming the conjecture of purity of the weight filtration* ([26, Conj. 2.6.5)])*(which is known for* $\dim(X) \leq 2$*), then the eigenvalues of the geometric Frobenius action on $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(m))^I$ are Weil numbers of weight $\leq i - 2m$.*

Proof. By de Jong's alteration theorem [13, Cor. 5.1], there exists an alteration of X with semi-stable reduction over R. Without loss of generality, we can assume that there exists a proper semi-stable morphism $\pi : \mathcal{X} \longrightarrow \text{Spec}(R)$ whose generic fiber is identified with X/K.

Replace K by its completion. Let K_0 (resp. K_1) denote the fraction field of $W(k)$(resp. $W(\bar{k})$). By the semi-stable comparison theorem, there is a canonical isomorphism

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \cong H^i_0(\mathcal{X}) \otimes_{K_0} B_{st}$$

compatible with actions of G_K, Frobenius ϕ and N and filtrations after tensor K on both sides, where the ϕ action on right side is given by $\phi_{st} \otimes \phi_B$ and N acts by $N \otimes 1 + 1 \otimes N$ on the right side. Set $V = H^i(X_{\bar{K}}, \mathbb{Q}_p)$. We also write $D_{st}(V)$ for $H^i_0(\mathcal{X})$. Note $D_{st}(V(m)) = D_{st}(V)(m)$. And we have

$$V(m) = \{x \in D_{st}(V(m)) \otimes_{K_0} B_{st} | \phi x = x, Nx = 0, 1 \otimes_{K_0} x \in Fil^0\}.$$

Since $B^I_{st} = K_1$ and $B_{cris} = \text{Ker}(N : B_{st} \longrightarrow B_{st})$, we have

$$V(m)^I = \{x \in D_{st}(V(m)) \otimes_{K_0} K_1 | \phi x = x, Nx = 0\} \cap Fil^0.$$

Since $N\phi = p\phi N$ on $D_{st}(V(m))$, so ϕ keeps $(D_{st}(V(m))^{N=0}$. N acts as zero on K_1, therefore

$$V(m)^I \subset \{x \in D_{st}(V(m))^{N=0} \otimes_{K_0} K_1 | \phi x = x\}.$$

Assuming $k = \mathbb{F}_{p^n}$, there is a ϕ^n-equivariant filtration M_i on $D_{st}(V)$ determined by N. Since N acts on $D_{st}(V(m))$ same as action on $D_{st}(V)$ (identifying the underlying space) and $\phi_{D_{st}(m)} = p^{-m}\phi_{D_{st}}$, we have

$$\{x \in D_{st}(V(m))^{N=0} \otimes_{K_0} K_1 | \phi x = x\} = \{x \in D_{st}(V)^{N=0} \otimes_{K_0} K_1 | \phi^n(x) = p^{mn}x\}.$$

ϕ^n can be written as $(\phi^n_{st} \otimes 1) \circ (1 \otimes \phi^n_B)$ as K_0-linear maps on $D_{st}(V)^{N=0} \otimes_{K_0} K_1$. Set $A = \phi^n_{st} \otimes 1$ and $B = 1 \otimes \phi^n_B$, then $\phi^n = AB = BA$. Set

$$U = \{x \in D_{st}(V)^{N=0} \otimes_{K_0} K_1 | \phi^n(x) = p^{mn}x\}.$$

It is a A and B invariant $\mathbb{Q}_p$-linear subspace. Since $D_{st}(V)^{N=0} \subseteq M_0$, by [26, Conj. 2.6.6(a)], all eigenvalues of ϕ^n on $D_{st}(V)^{N=0}$ are Weil numbers of weight $\leq i$. So there is polynomial

$P(X)$ (the product of Galois conjugates of the characteristic polynomial of ϕ^n) in $\mathbb{Q}[X]$ such that $P(\phi^n) = 0$ on $D_{st}(V)^{N=0}$. So $P(B^{-1}p^{mn}) = 0$ on U. Set $G(X) = X^{\deg(P)}P(p^{mn}X^{-1})$, we have $G \in \mathbb{Q}[X]$ and $G(B) = 0$ on U. Actually, the B action on U is same as the arithmetic Frobenius $F \in \mathrm{Gal}(K^{sh}/K)$ action on U. Therefore, $H^i(X_{\bar{K}}, \mathbb{Q}_p(m))^I$ is a B-invariant subspace of U. Therefore, $G(F) = 0$ on $H^i(X_{\bar{K}}, \mathbb{Q}_p(m))^I$. Since all roots of $P(X)$ are Weil numbers of weight $\leq i$, so all roots of $G(X)$ are Weil numbers of weight $\geq 2m-i$. □

Remark 4.1.11. *The conjecture of purity of the weight filtration is known for the case dim $X \leq 2$. For $i \leq 2$, by a Lefschetz hyperplane argument, the statement in the theorem can be reduced to the case dim $X \leq 2$, thus it holds for $i \leq 2$ unconditionally. In the case that X has a good reduction, the eigenvalues of the geometric Frobenius action on $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(m))^I$ are Weil numbers of pure weight $i - 2m$. The above proof is sketched in [27, §5].*

Lemma 4.1.12. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a semi-stable flat projective morphism. Assume that it is a base change of some $\mathcal{X}_0 \longrightarrow S_0 = \mathrm{Spec}(R_0)$, where R_0 is a henselian DVR with a finite residue field k_0 and $R = R_0^{sh}$. Let $\mathfrak{T}_n(1) \in D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ be the p-adic etale Tate twist defined by Kanetomo Sato in* [49, §1.3], *which is naturally isomorphic to $\mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}[-1]$. Then the natural map*

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective.

Proof. By the definition of $\mathfrak{T}_n(m)$, there is a distinguished triangle in $D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$

$$i_*\nu^{m-1}_{Y,n}[-m-1] \longrightarrow \mathfrak{T}_n(m) \longrightarrow \tau_{\leq m}Rj_*\mathbb{Z}/p^n(m) \longrightarrow i_*\nu^{m-1}_{Y,n}[-m], \tag{4.1}$$

where $\nu^m_{Y,n}$ are generalized Logarithmic Hodge–Witt sheaves which agree with $W_n\Omega^m_{Y,\log}$ if Y is smooth (cf. [49, Lem. 1.3.1]). $\nu^0_{Y,n}$ can be identified with

$$\bigoplus_{y \in Y^0} i_{y*}\mathbb{Z}/p^n\mathbb{Z}$$

where Y^0 is the set of generic points of Y. Taking $m = 1$ in (4.1) and taking cohomology, we get an exact sequence

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1}Rj_*\mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \varprojlim_n H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

G_{k_0} acts on the above sequence, and we will show the third term is of pure weight 1. Let $\widetilde{Y} \longrightarrow Y$be an alteration such that $\widetilde{Y}$ is smooth. It induces a map

$$H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \longrightarrow H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n}).$$

Since $H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \hookrightarrow H^1_{\text{ét}}(Y^0, \mathbb{Z}/p^n\mathbb{Z})$ and $H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n}) \hookrightarrow H^1_{\text{ét}}(\widetilde{Y}^0, \mathbb{Z}/p^n\mathbb{Z})$, and the kernel of $H^1_{\text{ét}}(Y^0, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow H^1_{\text{ét}}(\widetilde{Y}^0, \mathbb{Z}/p^n\mathbb{Z})$ is killed by some positive integer independent of n, we have an injection

$$\varprojlim_n H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \varprojlim_n H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Therefore, the natural map

$$\varprojlim_n H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \varprojlim_n H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. We might assume that the irreducible components of Y and $\widetilde{Y}$ are defined over k_0, therefore $H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n})$ admits a $\text{Gal}(k/k_0)$ action. Since $\widetilde{Y}$ is smooth over k, we have $\nu^0_{\widetilde{Y},n} = \mathbb{Z}/p^n\mathbb{Z}$. Thus

$$\varprojlim_n H^1_{\text{ét}}(\widetilde{Y}, \nu^0_{\widetilde{Y},n}) \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\text{ét}}(\widetilde{Y}, \mathbb{Q}_p).$$

Since $H^1_{\text{ét}}(\widetilde{Y}, \mathbb{Q}_p)$ is of pure weight 1 (cf. [27, §3]), thus $\varprojlim_n H^1_{\text{ét}}(Y, \nu^0_{Y,n}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is also of pure weight 1.

By Proposition 4.1.6,

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \tau_{\leq 1} Rj_* \mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective. By Lemma 4.1.10, $H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$ is of mixed weight ≤ 0. Therefore

$$\varprojlim_n H^2_{\text{ét}}(\mathcal{X}, \mathfrak{T}_n(1)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective. □

Proof of Lemma 4.1.2.

Proof. Let $f : \mathcal{X}_{\text{fppf}} \longrightarrow \mathcal{X}_{\text{ét}}$ be the morphism between Grothendieck topologies. By [36, Thm. 3.9], $R^i f_* \mathbb{G}_m = 0$ for $i > 0$. By the Kummer exact sequence of fppf-sheaves

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 0,$$

the complex $\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m$ can be viewed as an acyclic resolution of μ_{p^n}. It gives

$$Rf_* \mu_{p^n} \cong \mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}[-1].$$

Since $\mathfrak{T}_n(1) \cong \mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}[-1]$ (cf. [49, Prop. 4.5.1]), we have

$$H^2_{\text{fppf}}(\mathcal{X}_{R^{sh}}, \mu_{p^n}) = H^2_{\text{ét}}(\mathcal{X}_{R^{sh}}, Rf_* \mu_{p^n}) \cong H^2_{\text{ét}}(\mathcal{X}_{R^{sh}}, \mathfrak{T}_n(1)).$$

Then the claim follows from Lemma 4.1.12. □

Proof of Corollary 4.1.3

Lemma 4.1.13. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism, where R is an excellent strictly henselian DVR with an algebraically closed residue field. Let K denote the quotient field of R. Assuming that $\mathcal{X}$ is regular and the generic fiber X is smooth projective geometrically connected over K, then the map*

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}$$

has kernel of finite exponent. The claim still holds for the prime to p part if the residue field of R is only separable closed.

Proof. If $\dim(\mathcal{X}) \leq 2$, Artin's theorem [24, Thm.3.1] implies that both groups vanish. In general, we will use Colliot-Thélène and Skorobogatov's pull-back trick to reduce it to cases of relative dimension 1. By Lemma 2.4.1, there are smooth projective geometrically connected curves $Z_i \subset X$, an abelian variety A/K and a G_K-equivariant map

$$\mathrm{Pic}(X_{K^s}) \times A(K^s) \longrightarrow \bigoplus_i \mathrm{Pic}(Z_{i,K^s})$$

with finite kernel and cokernel. It follows that the natural map

$$H^1(K, \mathrm{Pic}_{X/K}) \longrightarrow \bigoplus_i H^1(K, \mathrm{Pic}_{Z_i/K})$$

has a kernel of finite exponent. By taking the Zariski closures of Z_i in $\mathcal{X}$ and then desingularizing it, we may assume that there are $\pi_i : \mathcal{Z}_i \longrightarrow S$ with generic fiber Z_i satisfying same conditions as π and S-morphisms $\mathcal{Z} \longrightarrow \mathcal{X}$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(K, \mathrm{Pic}_{X/K}) & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X_{K^s})^{G_K} \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \bigoplus_i H^1(K, \mathrm{Pic}_{Z_i/K}) & \longrightarrow & \bigoplus_i \mathrm{Br}(Z_i) & \longrightarrow & \bigoplus_i \mathrm{Br}(Z_{i,K^s})^{G_K} \end{array}$$

The exact rows come from the spectral sequences

$$H^p(K, H^q(X_{K^s}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

and the fact $H^q(K, \mathbb{G}_m) = 0, q > 0$. Since $\mathrm{Br}(\mathcal{Z}_i) = 0$, $\mathrm{Br}(\mathcal{X}) \cap H^1(K, \mathrm{Pic}_{X/K})$ will be mapped to zero in $\mathrm{Br}(Z_i)$. Thus, $\mathrm{Br}(\mathcal{X}) \cap H^1(K, \mathrm{Pic}_{X/K})$ is contained in the kernel of the first column which is of finite exponent. ☐

Proof of Corollary 4.1.3.

Proof. It follows directly from the above lemma that the natural map

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})[p^\infty] \longrightarrow \mathrm{Br}(X_{\bar{K}})^I[p^\infty]$$

has a kernel of finite exponent. By Proposition 2.5.2, the natural map

$$H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(1))^I \longrightarrow V_p\text{Br}(X_{\bar{K}})^I$$

is surjective. By Lemma 4.1.2, the natural map

$$V_p\text{Br}(\mathcal{X}_{R^{sh}}) \longrightarrow V_p\text{Br}(X_{\bar{K}})^I$$

is surjective. Since $\text{Br}(X_{\bar{K}})^I$ is of cofinite type, thus

$$\text{Br}(\mathcal{X}_{R^{sh}})[p^\infty] \longrightarrow \text{Br}(X_{\bar{K}})^I[p^\infty]$$

has a finite cokernel. □

Proof of Proposition 4.1.1

We will deduce the finiteness of the kernel and the p-primary part of the cokernel from Corollary 4.1.3. The prime to p-parts of the kernel and the cokernel was proved by Colliot-Thélène and Saito in [10, Cor. 2.6]. To prove the finiteness of the cokernel, we will use a Bertini theorem for strictly semi-stable morphisms over DVRs developed in [28] and Colliot-Thélène and Skorobogatov's pull-back trick to reduce it to cases of relative dimension 1.

Lemma 4.1.14. *Let $\pi : \mathcal{X} \longrightarrow S = \text{Spec}(R)$ be a proper flat map, where R is an excellent henselian DVR with a fraction field K and a finite residue field k of characteristic p. Assuming that $\mathcal{X}$ is regular and the generic fiber X of π is geometrically connected over K, then there is an exact sequence*

$$\text{Br}(\mathcal{X}) \longrightarrow \text{Br}(\mathcal{X}_{R^{sh}})^{G_k} \longrightarrow H^2(S, R^1\pi_*\mathbb{G}_m),$$

and the first map of the above sequence has a finite kernel.

Proof. By the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{G}_m),$$

we get a long exact sequence

$$H^2(S, \mathbb{G}_m) \longrightarrow \text{Ker}(H^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow H^0(S, R^2\pi_*\mathbb{G}_m)) \longrightarrow H^1(S, R^1\pi_*\mathbb{G}_m) \longrightarrow H^3(S, \mathbb{G}_m)$$

By [35, Chap. II, Prop. 1.5], $H^i(S, \mathbb{G}_m) = 0$ for all $i \geq 1$. Since $H^0(C, R^2\pi_*\mathbb{G}_m) = \text{Br}(\mathcal{X}_{R^{sh}})^{G_k}$, we get the desired exact sequence. To show the kernel of the first map is finite, it suffices to show that $H^1(S, R^1\pi_*\mathbb{G}_m)$ is finite. We have

$$H^1(S, R^1\pi_*\mathbb{G}_m) = H^1(G_k, \text{Pic}(\mathcal{X}_{R^{sh}})).$$

Since the natural map $\mathrm{Pic}(\mathcal{X}_{R^{sh}}) \to \mathrm{Pic}(X_{K^{sh}})$ is surjective and has a finitely generated cokernel, it suffices to show that $H^1(G_k, \mathrm{Pic}(X_{K^{sh}}))$ is finite. Since $\mathrm{Br}(K^{sh}) = 0$, we have

$$\mathrm{Pic}(X_{K^{sh}}) = \mathrm{Pic}_{X/K}(K^{sh}).$$

Since the cokernel of $\mathrm{Pic}^0_{X/K}(K^{sh}) \to \mathrm{Pic}_{X/K}(K^{sh})$ is finitely generated, it suffices to show that $H^1(G_k, \mathrm{Pic}(X_{K^{sh}}))$ is finite. This follows from [35, Chap. I, Prop.3.8]. □

By Lemma 4.1.13, the kernel of the natural map $\mathrm{Br}(\mathcal{X}_{R^{sh}}) \to \mathrm{Br}(X_{\bar{K}})$ is of finite exponent. By the above lemma, the kernel of the natural map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(X_{\bar{K}})^{G_K}$ is also of finite exponent and is actually finite since $\mathrm{Br}(X)$ is of cofinite type. To show the natural map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(X_{\bar{K}})^{G_K}$ has a finite cokernel, by Corollary 4.1.3, it suffices to show that the natural map

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k} \longrightarrow H^2(S, R^1\pi_*\mathbb{G}_m)$$

has an image of finite exponent. In the case of $\dim(\mathcal{X}) = 2$, this is obvious since $\mathrm{Br}(\mathcal{X}_{R^{sh}}) = 0$. In general, we will need to use a Lefschetz hyperplane argument to reduce this to the case of relative dimension 1.

Lemma 4.1.15. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a projective flat morphism, where R is a strictly henselian DVR of characteristic 0 with an algebraically closed residue field k. Assume $\mathcal{X}$ is strictly quasi-semi-stable over S (cf.[28, Def. 1.1]) and its geometric fiber is geometrically connected over K. Let Y be the special fiber of π. Write $Y_{\mathrm{red}} = \cup_{i=1}^n Y_i$ as the union of irreducible components. By definition, Y_i are smooth projective connected varieties over k. Assume $m = \dim(Y) \geq 1$. Fix an embedding $\mathcal{X} \hookrightarrow \mathbb{P}^n_S$, then there exist hyperplanes $H_1, ..., H_{m-1}$ defined over R such that the scheme-theoretic intersections $\mathcal{Z} = (\cap_{i=1}^{m-1} H_i) \cap \mathcal{X}$ satisfies the same assumption as $\mathcal{X} \longrightarrow S$(strictly quasi-semi-stable and has geometrically connected generic fiber) and $C_j = Y_j \cap (\cap_{i=1}^{m-1} H_i)$ are distinct smooth projective connected curves.*

Proof. If $\dim(Y) = 1$, the claim is trivial. Assume $m = \dim(Y) > 1$, by Bertini's theorem for quasi-stable-schemes over a DVR [28, Thm. 1.2], there is a hyperplane H over S such that H intersects Y_j and $Y_i \cap Y_j$ transversely and $H \cap \mathcal{X}$ is quasi-semi-stable over S. Since $\dim(Y) > 1$, we have scheme-theoretic intersections $H \cap Y_j$ and $H_{\bar{K}} \cap X_{\bar{K}}$ are smooth and connected. Since H intersects $Y_i \cap Y_j$ transversely, $H \cap Y_i$ are distinct irreducible components of the special fiber of $\mathcal{X} \cap H$. By induction, the claim is true for $\mathcal{X} \cap H$, so there exist $H_2, ..., H_{m-1}$ satisfying conditions in the claim for $\mathcal{X} \cap H$. Then $H_1 = H, H_2, ..., H_{m-1}$ are desired hyperplanes for $\mathcal{X}$. □

Lemma 4.1.16. *Let $\pi : \mathcal{Z} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism where R is a strictly henselian DVR with an algebraically closed residue field and $\mathcal{Z}$ is regular and is of dimension 2. Assume that the generic fiber of π is smooth and connected. Let C denote the special fiber and C_i denote irreducible components of C. Then the intersection pairing*

$$\bigoplus_i \mathbb{Q} \cdot [C_i] \times \bigoplus_i \mathbb{Q} \cdot [C_i] \longrightarrow \mathbb{Q}$$

has the left and right kernel spanned by $[C]$.

Proof. Since $[C] = 0$, so $C.C_j = 0$ for any j. Let $D = \sum x_i[C_i]$ such that $D.C_j = 0$ for all j. So $\sum x_i C_i.C_j = 0$ for all j. We have $D.D = 0$, i.e $\sum x_i x_j C_i.C_j = 0$. We may assume $[C] = \sum[C_i]$, since $C.C_i = 0$. It follows $C_i.C_i = -\sum_{j\neq i} C_i.C_j$. Therefore $0 = \sum x_i x_j C_i.C_j = \sum_{i<j}(2x_ix_j - x_i^2 - x_j^2)C_i.C_j$. It follows $x_i = x_j$ if $C_i.C_j > 0$. Since C is connected, so all x_i are same, we have $D = x[C]$. This proves the claim. $\square$

Lemma 4.1.17. *Let G be a group and M, L are G-modules. We say a G-map $f : M \longrightarrow L$ is almost split if there exists a G-map $N \longrightarrow L$ such that $M \oplus N \longrightarrow L$ has kernel and cokernel of finite exponent. Assume that there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

such that the first column, the third column and $A_2 \to B_2$ are almost split. Then the middle column is almost split.

Proof. Let $A_1' \longrightarrow A_2$, $A_2' \longrightarrow B_2$ and $C_1' \longrightarrow C_2$ be G-maps such that $A_1' \oplus A_1 \longrightarrow A_2$, $A_2 \oplus A_2' \longrightarrow B_2$ and $C_1 \oplus C_1' \longrightarrow C_2$ have kernels and cokernels of finite exponent. We may assume $A_2' \subseteq B_2$ and $C_1' \subseteq C_2$. Then $A_2' \longrightarrow C_2$ has a kernel and a cokernel of finite exponent. Let B' denote the preimage of C_1' under this map. One can show that $B' \longrightarrow C_1'$ has a kernel and a cokernel of finite exponent. Replace C_1' by the image of this map, we may assume that the map $B' \longrightarrow C_1'$ is surjective. Let K denote its kernel, so $K \subseteq A_2$ and is of finite exponent. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1' \oplus A_1 \oplus K & \longrightarrow & A_1' \oplus B_1 \oplus B' & \longrightarrow & C_1 \oplus C_1' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

where the first and the third column has kernels and cokernels of finite exponent. By the snake lemma, the second column has kernel and cokernel of finite exponent.

$\square$

Proposition 4.1.18. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a projective flat morphism with a smooth and geometrically connected generic fiber X, where R is a henselian DVR of characteristic 0 with a finite residue field k. Assuming that the base change of π to R^{sh} is strictly quasi-semi-stable, then*

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k}$$

has a kernel and a cokernel of finite exponent.

Proof. By Lemma 4.1.14, it suffices to show that the cokernel is of finite exponent. If π is of relative dimension 1, by Artin's theorem [24, Thm.3.1], $\mathrm{Br}(\mathcal{X}_{R^{sh}}) = 0$, so the claim is true in this case. For the general case, we will use Colliot-Thélène and Skorobogatov's pull-back trick to reduce the problem to the case of relative dimension 1. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{id} & S \end{array}$$

where π' is proper flat and of relative dimenson 1. By Lemma 4.1.14, there is commutative diagram with exact rows

$$\begin{array}{ccccc} \mathrm{Br}(\mathcal{X}) & \longrightarrow & \mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k} & \xrightarrow{\partial} & H^2(S, R^1\pi_*\mathbb{G}_m) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\ \mathrm{Br}(\mathcal{Z}) & \longrightarrow & \mathrm{Br}(\mathcal{Z}_{R^{sh}})^{G_k} & \longrightarrow & H^2(S, R^1\pi'_*\mathbb{G}_m). \end{array}$$

Since $\mathrm{Br}(\mathcal{Z}_{R^{sh}})^{G_k} = 0$, we have $\mathrm{Im}(\partial) \subseteq \mathrm{Ker}(a)$. The idea is to show that $\mathrm{Ker}(a)$ is of finite exponent. Instead of working on one $\mathcal{Z}$, we will find finitely many $\mathcal{Z}_i$ and show that the kernel of

$$H^2(S, R^1\pi_*\mathbb{G}_m) \xrightarrow{\sum a_i} \bigoplus_i H^2(S, R^1\pi'_*\mathbb{G}_m)$$

is of finite exponent. By the Hochschild–Serre spectral sequence

$$H^p(G_k, H^q(\mathrm{Spec}R^{sh}, R^1\pi_*\mathbb{G}_m)) \Rightarrow H^{p+q}(S, R^1\pi_*\mathbb{G}_m)),$$

we get

$$H^2(S, R^1\pi_*\mathbb{G}_m) = H^2(G_k, H^0(\mathrm{Spec}R^{sh}, R^1\pi_*\mathbb{G}_m)) = H^2(G_k, \mathrm{Pic}(\mathcal{X}_{R^{sh}})).$$

Thus, the map a can be identified with

$$H^2(G_k, \mathrm{Pic}(\mathcal{X}_{R^{sh}})) \longrightarrow H^2(G_k, \mathrm{Pic}(\mathcal{Z}_{R^{sh}})).$$

Let Y denote the special fiber of π and Y_i denote its irreducible components. Then there is an exact sequence

$$\bigoplus_i \mathbb{Z}\cdot[Y_i] \longrightarrow \mathrm{Pic}(\mathcal{X}_{R^{sh}}) \longrightarrow \mathrm{Pic}(X_{K^{sh}}) \longrightarrow 0$$

The kernel of the first arrow is generated by $[Y]$, this actually follows from the following long exact sequence (cf.[24, §6])

$$H^0(\mathcal{X}_{R^{sh}}, \mathbb{G}_m) \longrightarrow H^0(X_{K^{sh}}, \mathbb{G}_m) \longrightarrow H^1_Y(\mathcal{X}_{R^{sh}}, \mathbb{G}_m) \longrightarrow H^1(\mathcal{X}_{R^{sh}}, \mathbb{G}_m) \longrightarrow H^1(X_{K^{sh}}, \mathbb{G}_m),$$

and the facts

$$H^0(\mathcal{X}_{R^{sh}}, \mathbb{G}_m) = (R^{sh})^*,\ H^0(X_{K^{sh}}, \mathbb{G}_m) = (K^{sh})^*$$

and

$$H^1_Y(\mathcal{X}_{R^{sh}}, \mathbb{G}_m) = \bigoplus_i \mathbb{Z} \cdot [Y_i].$$

Thus, we have

$$0 \longrightarrow (\bigoplus_i \mathbb{Z} \cdot [Y_i])/\mathbb{Z} \cdot [Y] \longrightarrow \mathrm{Pic}(\mathcal{X}_{R^{sh}}) \longrightarrow \mathrm{Pic}(X_{K^{sh}}) \longrightarrow 0.$$

We will denote $(\bigoplus_i \mathbb{Z} \cdot [Y_i])/\mathbb{Z} \cdot [Y]$ by $D_{\mathcal{X}}$. Let $\mathcal{Z} \longrightarrow S$ be a proper flat morphism with $\mathcal{Z}$ regular and of dimension 2. Let C denote its special fiber and C_i denote irreducible components of C. Consider the G_k-equivariant intersect pairing

$$\mathrm{Pic}(\mathcal{Z}_{R^{sh}}) \times D_{\mathcal{Z}} \longrightarrow \mathbb{Z}, \tag{4.2}$$

by Lemma 4.1.16, the restriction of the pairing to $D_{\mathcal{Z}} \times D_{\mathcal{Z}}$ is perfect after tensor $\mathbb{Q}$. Let $D_{\mathcal{Z}}^{\perp}$ denote the left kernel of the pairing (4.2). Then the map

$$D_{\mathcal{Z}} \oplus D_{\mathcal{Z}}^{\perp} \longrightarrow \mathrm{Pic}(\mathcal{Z}_{R^{sh}})$$

has a kernel and a cokernel of finite exponent. The kernel can be identified with the left kernel of the paring $D_{\mathcal{Z}} \times D_{\mathcal{Z}} \longrightarrow \mathbb{Z}$ which is finite. Since $D_{\mathcal{Z}} \longrightarrow \mathrm{Hom}(D_{\mathcal{Z}}, \mathbb{Z})$ has a cokernel killed by some positive integer m, for any element $v \in \mathrm{Pic}(\mathcal{Z}_{R^{sh}})$, v defines an element v^* in $\mathrm{Hom}(D_{\mathcal{Z}}, \mathbb{Z})$, so there exists an element $u \in D_{\mathcal{Z}}$ such that $mv^* = u^*$. Thus $mv - u \in D_{\mathcal{Z}}^{\perp}$. This proves that the cokernel is killed by m. Thus, $D_{\mathcal{Z}} \longrightarrow \mathrm{Pic}(\mathcal{Z}_{R^{sh}})$ is almost split.

By Lemma 4.1.15, we can choose $\mathcal{Z}_0$ by taking hyperplane sections repeatedly such that $D_{\mathcal{X}} \cong D_{\mathcal{Z}_0}$. By Lemma 2.4.1, we can choose smooth projective integral curves $Z_i \subseteq X$ for $i = 1, ..., n$ such that

$$\mathrm{Pic}(X_{K^s}) \longrightarrow \bigoplus_{i=0}^{n} \mathrm{Pic}(Z_{i,K^s})$$

is almost split as G_K-modules. Taking $G_{K^{sh}}$ invariant, we have

$$\mathrm{Pic}(X_{K^{sh}}) \longrightarrow \bigoplus_{i=0}^{n} \mathrm{Pic}(Z_{i,K^{sh}})$$

is almost split as G_k-modules.

Taking the Zariski closure of Z_i in $\mathcal{X}$, then desingularizes it, we get a S-morphsim $\mathcal{Z}_i \longrightarrow \mathcal{X}$ such that $\mathcal{Z}_i \longrightarrow S$ is proper flat with generic fiber Z_i. Thus, we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_{\mathcal{X}} & \longrightarrow & \mathrm{Pic}(\mathcal{X}_{R^{sh}}) & \longrightarrow & \mathrm{Pic}(X_{K^{sh}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{i=0}^{n} D_{\mathcal{Z}_i} & \longrightarrow & \bigoplus_{i=0}^{n} \mathrm{Pic}(\mathcal{Z}_{i,R^{sh}}) & \longrightarrow & \bigoplus_{i=0}^{n} \mathrm{Pic}(Z_{i,K^{sh}}) & \longrightarrow & 0 \end{array}$$

By the constructions, the first column, the third column and the second row are almost split, by Lemma 4.1.17, the second column is also almost split. Therefore,

$$H^2(G_k, \mathrm{Pic}(\mathcal{X}_{R^{sh}})) \longrightarrow \bigoplus_{i=0}^{n} H^2(G_k, \mathrm{Pic}(\mathcal{Z}_{i,R^{sh}}))$$

has a kernel of finite exponent. This completes the proof. □

Proposition 4.1.19. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ as before. Assuming that π is strictly semi-stable, then the natural map*

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}$$

has a finite kernel and a finite cokernel.

Proof. By Corollary 4.1.3,

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})[p^\infty] \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_{K^{sh}}}[p^\infty]$$

has a kernel and a cokernel of finite exponent. It follows that the map

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k}[p^\infty] \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}[p^\infty]$$

also has a kernel and a cokernel of finite exponent. By the lemma below, the claim also holds for prime-to-p part. Thus, the natural map

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k} \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}$$

has a kernel and a cokernel of finite exponent. By Proposition 4.1.18,

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}$$

has a kernel and a cokernel of finite exponent. Since both groups are of cofinite type, so the kernel and cokernel are actually finite. □

To complete the proof of Proposition 4.1.1, we need to remove the strictly semi-stable assumption. Before removing the strictly semi-stable assumption, we prove Proposition 4.1.4.

Lemma 4.1.20. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism where R is a henselian DVR with a finite residue field k and a quotient field K. Assume that $\mathcal{X}$ is regular and the generic fiber X of π is smooth projective geometrically connected over K. Let $\ell \neq \mathrm{char}(k)$ be a prime. Then*

$$\mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k}[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}[\ell^\infty]$$

has a kernel and a cokernel of finite exponent and is an isomorphism for all but finitely many ℓ.

Proof. Let Y be the special fiber of π and Let Y_i be its irreducible components. We may assume that Y_i is geometrically irreducible. By Theorem 2.2.6 there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathcal{X}_{R^{sh}})[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^{sh}})[\ell^\infty] \longrightarrow \bigoplus_i H^1(D_{i,k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where D_i is the smooth locus of Y_i. It gives an exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathcal{X}_{R^{sh}})^{G_k}[\ell^\infty] \longrightarrow \mathrm{Br}(X_{K^{sh}})^{G_k}[\ell^\infty] \longrightarrow \bigoplus_i H^1(D_{i,k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}.$$

By Lemma 3.1.2, $H^1(D_{i,k^s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_k}$ is finite and vanishes for all but finitely many ℓ. Consider the spectral sequence

$$H^p(K^{sh}, H^q(X_{K^s}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_{K^{sh}}, \mathbb{G}_m).$$

Since $H^p(K^{sh}, \mathbb{G}_m) = 0$ for $p \geq 1$, we have

$$0 \longrightarrow H^1(K^{sh}, \mathrm{Pic}_{X/K}) \longrightarrow \mathrm{Br}(X_{K^{sh}}) \longrightarrow \mathrm{Br}(X_{K^s})^{G_{K^{sh}}}.$$

Taking G_k invariants, we get

$$0 \longrightarrow H^1(K^{sh}, \mathrm{Pic}_{X/K})^{G_k} \longrightarrow \mathrm{Br}(X_{K^{sh}})^{G_k} \longrightarrow \mathrm{Br}(X_{K^s})^{G_K}.$$

By Theorem 2.4.2, the last map has a cokernel of finite exponent. By [47, Lem. 5.4], the group $H^1(K^{sh}, \mathrm{Pic}_{X/K})^{G_k}[\ell^\infty]$ is finite and vanishes for all but finitely many ℓ. This proves the claim. □

Proof of Proposition 4.1.4

Proof. The first claim follows from the lemma above directly. The second claim follows from Lemma 3.3.1 (b). □

Removing the semi-stable assumption

We need the following two technical Lemmas 4.1.21 and 4.1.22 to remove the semi-stable assumption.

Lemma 4.1.21. *Let $f : Y \longrightarrow X$ be a proper morphism between regular noetherian schemes. Assume that X is irreducible and each irreducible component of Y dominates X and there is an open dense subset U of X such that $f|_{f^{-1}(U)}$ is a finite etale Galois covering over U. Let $K(X)$ denote the functions field of X and assume* $\mathrm{char}(K(X)) = 0$. *Set* $G = \mathrm{Aut}(f^{-1}(U)/U) = \mathrm{Aut}(K(Y)/K(X))^{\mathrm{op}}$. *Then G acts on* $\mathrm{Br}(Y)$ *and the natural map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y)^G$$

has a kernel and a cokernel of finite exponent.

Proof. Let $g \in G$, g induces a X-rational map $Y \dashrightarrow Y$. Since $Y \longrightarrow X$ is proper, by the valuation criterion of properness, g can extend to an open dense subset $V \subset Y$ with $\mathrm{codim}(Y - V) \geq 2$.Thus, g induces a map $\mathrm{Br}(Y) \longrightarrow \mathrm{Br}(V)$. By Theorem 2.2.7, $\mathrm{Br}(Y) = \mathrm{Br}(V)$, so G acts on $\mathrm{Br}(Y)$. By Theorem 2.2.8, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(K(Y)) \longrightarrow \prod_{y \in Y^1} \mathrm{Br}(K(Y))/\mathrm{Br}(\mathcal{O}_{Y,y}),$$

where Y^1 denotes the set of point of codimension 1. We get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(K(X)) & \longrightarrow & \prod_{x \in X^1} \mathrm{Br}(K(X))/\mathrm{Br}(\mathcal{O}_{X,x}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathrm{Br}(K(Y))^G & \longrightarrow & \prod_{y \in f^{-1}(X^1)} \mathrm{Br}(K(Y))/\mathrm{Br}(\mathcal{O}_{Y,y})
\end{array}$$

Note all points in $f^{-1}(X^1)$ has codimension 1. Since $\mathrm{Br}(Y)^G \subset \mathcal{K}$, it suffices to show that the first column has a kernel and a cokernel of finite exponent. By the Snake lemma, it is enough to show that the kernel of the third row is killed by some positive integer. Set $B = \mathcal{O}_{Y,y}, x = f(y), A = \mathcal{O}_{X,x}, K = \mathrm{Frac}(A), L = \mathrm{Frac}(B)$. It suffices to show that the kernel of $\mathrm{Br}(K)/\mathrm{Br}(A) \longrightarrow \mathrm{Br}(L)/\mathrm{Br}(B)$ is killed by some positive integer that only depends on $[L : K]$. We may replace L by its normal closure in $\bar{K}$, therefore we can assume that L/K is Galois. By definition, there is an injection

$$\mathrm{Br}(K)/\mathrm{Br}(A) \hookrightarrow H^3_x(\mathrm{Spec}A, \mathbb{G}_m) = H^3_x(\mathrm{Spec}A^h, \mathbb{G}_m).$$

Thus, we have

$$\mathrm{Br}(K)/\mathrm{Br}(A) \hookrightarrow \mathrm{Br}(K^h)/\mathrm{Br}(A^h).$$

Since $\mathrm{char}(K) = 0$, $\mathrm{Br}(K^h) = \mathrm{Br}(\hat{K})$ by the Kummer exact sequence. By Corollary 2.2.11, $\mathrm{Br}(A^h) = \mathrm{Br}(\hat{A})$. Thus, we can assume that A and B are complete DVRs. It suffices to show that the cokernel of $\mathrm{Br}(A) \longrightarrow \mathrm{Br}(B)^{\mathrm{Gal}(L/K)}$ is killed by some positive integer that only depends on $[L : K]$. By Proposition 2.2.10, $\mathrm{Br}(A) = \mathrm{Br}(k)$ and $\mathrm{Br}(B) = \mathrm{Br}(l)$ where k (resp. l) denotes the residue field of A (resp. B). It suffices to show that the natural map

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(l)^{\mathrm{Aut}(l/k)}$$

has a cokernel killed by some fixed power of $[l : k]$. If l/k is separable, it follows from the Hochschild–Serre spectral sequence that the cokernel is killed by $[l : k]^2$. Thus, we may assume that l/k is purely inseparable, $\mathrm{char}(k) = p$ and $[l : k] = p$. By assumptions, $(l^s)^p \subset k^s$. Consider the exact sequence

$$0 \longrightarrow (k^s)^\times \longrightarrow (l^s)^\times \longrightarrow (l^s)^\times/(k^s)^\times \longrightarrow 0,$$

the last group is killed by p. Since $G_k = G_l$, taking Galois cohomology, it follows from the long exact sequence of Galois cohomology that the cokernel of $\mathrm{Br}(k) \to \mathrm{Br}(l)$ is killed by $[l : k]$. This completes the proof.

□

Lemma 4.1.22. (Gabber) *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a proper flat morphism, where R is a henselian DVR of characteristic 0 with a perfect residue field. Let $K = \mathrm{Frac}(R)$ and X denote the generic fiber of π. Assuming that $\mathcal{X}$ is regular and X is geometrically connected over K, then there exists a strictly semi-stable projective morphism $\pi_1 : \mathcal{X}_1 \longrightarrow S_1 = \mathrm{Spec} R_1$, where R_1 is the ring of integer of some finite Galois extension K_1 of K, and an alteration $f : \mathcal{X}_1 \longrightarrow \mathcal{X} \times_S S_1$ over S_1 such that $\mathcal{X}_1$ is irreducible and regular and $K(\mathcal{X}_1)/K(\mathcal{X})$ is a finite Galois extension.*

Proof. It follows from [65, Prop. 4.4.1]. □

Proof of Proposition 4.1.1

Proof. By the previous lemma, there is an alternation $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ satisfying the assumptions in the Lemma 4.1.21. Obviously, $X_1 \otimes_K \bar{K} \longrightarrow X_{\bar{K}}$ also satisfies the assumptions in the Lemma 4.1.21. Set $G = \mathrm{Gal}(K(\mathcal{X}_1)/K(\mathcal{X}))$. By Lemma 4.1.21, the maps $\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(\mathcal{X}_1)^G$ and $\mathrm{Br}(X_{\bar{K}}) \longrightarrow \mathrm{Br}(X_1 \otimes_K \bar{K})^G$ have a kernel and a cokernel of finite exponent. Thus, the map

$$\mathrm{Br}(X_{\bar{K}})^{G_K} \longrightarrow (\mathrm{Br}(X_1 \otimes_K \bar{K})^{G_K})^G$$

also has a kernel and a cokernel of finite exponent. Since

$$X_1 \otimes_K \bar{K} = \bigsqcup_{\sigma : K_1 \hookrightarrow \bar{K}} (X_1)_\sigma,$$

so we have

$$\mathrm{Br}(X_1 \otimes_K \bar{K})^{G_K} = \mathrm{Br}((X_1)_{\bar{K}})^{G_{K_1}}.$$

Since $\mathcal{X}_1 \longrightarrow S_1$ is strictly semi-stable, by Proposition 4.1.19, the natural map

$$\mathrm{Br}(\mathcal{X}_1) \longrightarrow \mathrm{Br}((X_1)_{\bar{K}})^{G_{K_1}}$$

has a kernel and a cokernel of finite exponent. It follows that

$$\mathrm{Br}(\mathcal{X}_1)^G \longrightarrow (\mathrm{Br}((X_1)_{\bar{K}})^{G_{K_1}})^G$$

also has a kernel and a cokernel of finite exponent. Then the claim follows from the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathcal{X}) & \longrightarrow & \mathrm{Br}(X_{\bar{K}})^{G_K} \\ \downarrow & & \downarrow \\ \mathrm{Br}(\mathcal{X}_1)^G & \longrightarrow & (\mathrm{Br}((X_1)_{\bar{K}})^{G_{K_1}})^G \end{array}$$

This completes the proof Proposition 4.1.1. □

Remark 4.1.23. *The strictly semi-stale assumption in the proof of Corollary 4.1.3 can be removed by the same argument as above.*

4.2 Proof of Theorem 1.4.5

Cofiniteness

Proposition 4.2.1. *Let $\mathcal{X}$ be an integral regular scheme flat and of finite type over* $\mathrm{Spec}(\mathbb{Z})$*. Then* $\mathrm{Br}(\mathcal{X})$ *is of cofinite type.*

Proof. Let ℓ be a prime. By shrinking $\mathcal{X}$, we may assume that $\mathcal{X}$ is a $\mathbb{Z}[1/\ell]$-scheme. Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$ the structure morphism. By the Kummer exact sequence, there is a surjective map

$$H^2(\mathcal{X}, \mu_\ell) \longrightarrow \mathrm{Br}(\mathcal{X})[\ell].$$

Thus, it suffices to show that $H^2(\mathcal{X}, \mu_\ell)$ is finite. By the Leray spectral sequence

$$H^p(S, R^q\pi_*\mu_\ell) \Rightarrow H^{p+q}(\mathcal{X}, \mu_\ell),$$

it suffices to show that $H^p(S, R^q\pi_*\mu_\ell)$ is finite. By [21, Thm. 9.5.1], $R^q\pi_*\mu_\ell$ is a constructible ℓ-torsion sheaf. By [35, Chap. II, Thm. 2.13], $H^p(S, R^q\pi_*\mu_\ell)$ is finite. □

Proposition 4.2.2. *Let X be a smooth projective geometrically connected variety over a number field K. Define*

$$Ш(\mathrm{Pic}_{X/K}) := \mathrm{Ker}(H^1(K, \mathrm{Pic}_{X/K}) \longrightarrow \prod_v H^1(K_v^h, \mathrm{Pic}_{X/K})).$$

Then the natural map

$$Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow Ш(\mathrm{Pic}_{X/K})$$

has a finite kernel and a cokernel of finite exponent.

Proof. Since $Ш(\mathrm{Pic}^0_{X/K})$ is of cofinite type (cf. [35, Chap. II, Cor. 5.3 and Lem. 5.5]), it suffices to show that the map has a kernel and a cokernel of finite exponent. By the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_{X/K} \longrightarrow \mathrm{Pic}_{X/K} \longrightarrow \mathrm{NS}(X_{\bar{K}}) \longrightarrow 0$$

we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & H^1(K, \mathrm{Pic}^0_{X/K}) & \longrightarrow & H^1(K, \mathrm{Pic}_{X/K}) & \longrightarrow & H^1(K, \mathrm{NS}(X_{\bar{K}})) \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ \prod_v H^0(K_v^h, \mathrm{NS}(X_{\bar{K}})) & \xrightarrow{\partial} & \prod_v H^1(K_v^h, \mathrm{Pic}^0_{X/K}) & \longrightarrow & \prod_v H^1(K_v^h, \mathrm{Pic}_{X/K}) & \longrightarrow & \prod_v H^1(K_v^h, \mathrm{NS}(X_{\bar{K}})) \end{array}$$

Let $\bar{a}$ denote the map

$$H^1(K, \mathrm{Pic}^0_{X/K}) \longrightarrow \prod_v H^1(K_v^h, \mathrm{Pic}^0_{X/K})/\mathrm{Im}(\partial)$$

By the Snake Lemma, there is an exact sequence

$$\mathrm{Ker}(\bar{a}) \longrightarrow \mathrm{Ker}(b) \longrightarrow \mathrm{Ker}(c).$$

We also have

$$0 \longrightarrow \mathrm{Ker}(a) \longrightarrow \mathrm{Ker}(\bar{a}) \longrightarrow \mathrm{Im}(\partial).$$

By definition, $\mathrm{Ker}(a) = Ш(\mathrm{Pic}^0_{X/K})$ and $\mathrm{Ker}(b) = Ш(\mathrm{Pic}_{X/K})$. Thus, it suffices to show that $\mathrm{Im}(\partial)$ and $\mathrm{Ker}(c)$ are of finite exponent. Since $\mathrm{NS}(X_{\bar{K}})$ is finitely generated, there exists a finite Galois extension L of K such that $\mathrm{Pic}(X_L)$ maps onto $\mathrm{NS}(X_{\bar{K}})$. By the inflation-restriction exact sequence

$$0 \longrightarrow H^1(\mathrm{Gal}(L/K), \mathrm{NS}(X_{\bar{K}})) \longrightarrow H^1(K, \mathrm{NS}(X_{\bar{K}})) \longrightarrow H^1(L, \mathrm{NS}(X_{\bar{K}}))$$

and the fact that $H^1(\mathrm{Gal}(L/K), \mathrm{NS}(X_{\bar{K}}))$ is killed by $[L : K]$ and $H^1(L, \mathrm{NS}(X_{\bar{K}})) = \mathrm{Hom}(G_L, \mathrm{NS}(X_{\bar{K}})_{\mathrm{tor}})$ is killed by $|\mathrm{NS}(X_{\bar{K}})_{\mathrm{tor}}|$, we have that $H^1(K, \mathrm{NS}(X_{\bar{K}}))$ is also of finite exponent. It follows that $\mathrm{Ker}(c)$ is of finite exponent. To show that $\mathrm{Im}(\partial)$ is of finite exponent, it suffices to show that the map

$$H^0(K_v^h, \mathrm{NS}(X_{\bar{K}})) \xrightarrow{\partial_v} H^1(K_v^h, \mathrm{Pic}^0_{X/K})$$

has an image killed by $[L:K]$. Let w be a place of L over v. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^0(K_v^h, \mathrm{Pic}_{X/K}) & \longrightarrow & H^0(K_v^h, \mathrm{NS}(X_{\bar{K}})) & \xrightarrow{\partial_v} & H^1(K_v^h, \mathrm{Pic}^0_{X/K}) \\ \downarrow & & \downarrow & & \downarrow g \\ H^0(L_w^h, \mathrm{Pic}_{X/K}) & \xrightarrow{f} & H^0(L_w^h, \mathrm{NS}(X_{\bar{K}})) & \xrightarrow{\partial_w} & H^1(L_w^h, \mathrm{Pic}^0_{X/K}) \end{array}$$

By the choice of L, f is surjective. Thus, $\partial_w = 0$. So $\mathrm{Im}(\partial_v) \subseteq \mathrm{Ker}(g)$. By the inflation-restriction exact sequence, we have that $\mathrm{Ker}(g) = H^1(\mathrm{Gal}(L_w^h/K_v^h), \mathrm{Pic}^0_{X/K}(L_w^h))$ is killed by $[L_w^h : K_v^h]$. Since L/K is finite Galois, so $[L_w^h : K_v^h] | [L:K]$. This proves that $\mathrm{Im}(\partial_v)$ is killed by $[L:K]$. □

Lemma 4.2.3. *Let $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(R)$ be a flat morphism with $\mathcal{X}$ integral and regular, where R is a DVR of characteristic 0 with a fraction field K. Let X denote the generic fiber of K. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_{K^h})/\mathrm{Br}(\mathcal{X}_{R^h}).$$

Proof. Let Y denote the special fiber of π. It suffices to show

$$H^3_Y(\mathcal{X}, \mathbb{G}_m) \cong H^3_Y(\mathcal{X}_{R^h}, \mathbb{G}_m).$$

This follows from the Excision theorem (cf.[36, Chap III, Prop. 1.27 and Cor. 1.28]). □

Remark 4.2.4. *In fact, the exact sequence above still holds if we replace R^h by its completion. This actually follows from purity of Brauer groups. We may assume that Y is of codimension 1 with a generic point y. Then there is an injection $\mathrm{Br}(X)/\mathrm{Br}(\mathcal{X}) \hookrightarrow \mathrm{Br}(K(X)^h_y)/\mathrm{Br}(\mathcal{O}^h_{\mathcal{X},y})$. Since $\mathrm{char}(K(X)) = 0$, $\mathrm{Br}(K(X)^h_y) = \mathrm{Br}(\widehat{K(X)}_y)$. By Corollary 2.2.11, $\mathrm{Br}(\mathcal{O}^h_{\mathcal{X},y}) = \mathrm{Br}(\widehat{\mathcal{O}_{\mathcal{X},y}})$. It is easy to see that the completion does not change if we replace R by $\hat{R}$.*

Lemma 4.2.5. *Let $\pi : \mathcal{X} \longrightarrow C = \mathrm{Spec}(\mathcal{O}_K)$ be proper flat morphism with $\mathcal{X}$ regular, where K is a number field. Assuming that the generic fiber X of π is geometrically connected over K, then the natural map*

$$\mathrm{Br}(X)/(\mathrm{Br}(\mathcal{X}) + \mathrm{Br}(K)) \longrightarrow \prod_{v \in C^\circ} \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{O_v^h}) + \mathrm{Br}(K_v^h))$$

has a finite kernel and the group $\mathrm{Br}(\mathcal{X}) \cap \overline{\mathrm{Br}(K)}$ is finite, where C° denotes the set of closed point of C, $\mathcal{O}_v^h$ denotes the henselian local ring of C at v and $\overline{\mathrm{Br}(K)}$ denotes the image of the pullback map $\mathrm{Br}(K) \longrightarrow \mathrm{Br}(X)$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Br}(K) & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X)/\mathrm{Br}(K) & \longrightarrow & 0 \\ \downarrow & & \downarrow b & & \downarrow c & & \\ \oplus_v \mathrm{Br}(K_v^h) & \xrightarrow{f} & \oplus_v \mathrm{Br}(X_{K_v^h})/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) & \longrightarrow & \oplus_v \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)) & & \end{array}$$

Let a denote the map

$$\mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v^h)/\mathrm{Ker}(f).$$

By the Snake Lemma, there is a long exact sequence

$$\mathrm{Ker}(a) \longrightarrow \mathrm{Ker}(b) \longrightarrow \mathrm{Ker}(c) \longrightarrow \mathrm{Coker}(a) \xrightarrow{\bar{f}} \mathrm{Coker}(b).$$

By Theorem 2.2.7 and Lemma 4.2.3, $\mathrm{Ker}(b) = \mathrm{Br}(\mathcal{X})$. Thus, it suffices to show that $\mathrm{Ker}(a)$ and $\mathrm{Ker}(\bar{f})$ are finite groups. Since

$$\mathrm{Ker}(\mathrm{Ker}(a) \longrightarrow \bigoplus_{v|\infty} \mathrm{Br}(K_v) \oplus \mathrm{Ker}(f)) = \mathrm{Ker}(\mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v)),$$

by Proposition 2.2.14, the map

$$\mathrm{Ker}(a) \longrightarrow \bigoplus_{v|\infty} \mathrm{Br}(K_v) \oplus \mathrm{Ker}(f)$$

is injective and $\mathrm{Coker}(a)$ is of cofinite type. Thus, it suffices to show that $\mathrm{Ker}(f)$ is finite and $\mathrm{Coker}(\bar{f})$ is of finite exponent. Let L/K be finite Galois extension such that $X(L)$ is not empty. Let $P \in X(L)$. Then P defines a K-morphism $g : Z := \mathrm{Spec} L \to X$ and the morphism can extend to a C-morphsim $g : \mathcal{Z} := \mathrm{Spec}\mathcal{O}_L \longrightarrow \mathcal{X}$ since π is proper. Consider the composition

$$\bigoplus_v \mathrm{Br}(K_v^h) \xrightarrow{f} \bigoplus_v \mathrm{Br}(X_{K_v^h})/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) \longrightarrow \bigoplus_v \mathrm{Br}(Z_{K_v^h})/\mathrm{Br}(\mathcal{Z}_{\mathcal{O}_v^h}).$$

Obviously, $\mathrm{Ker}(f)$ is a subgroup of the kernel of the composition. We want to show that the kernel of the composition is killed by $[L : K]$. Then, it will imply that $\mathrm{Ker}(f)$ is killed by $[L : K]$. The natural map $\mathcal{Z} \longrightarrow C$ is finite flat of degree $[L : K]$. Its base change $Z_{K_v^h} \longrightarrow \mathrm{Spec} K_v^h$ is also finite flat of degree $[L : K]$. They induce restriction map $\mathrm{res}_v : \mathrm{Br}(K_v^h) \to \mathrm{Br}(Z_{K_v^h})$ and corestiction map $\mathrm{cores}_v : \mathrm{Br}(Z_{K_v^h}) \to \mathrm{Br}(K_v^h)$. By Proposition 2.2.12, $\mathrm{Br}(\mathcal{O}_v^h) = 0$ for all $v \in C^\circ$. The composition map above can be identified with the restriction map

$$\bigoplus_v \mathrm{Br}(K_v^h)/\mathrm{Br}(\mathcal{O}_v^h) \xrightarrow{\oplus_v \mathrm{res}_v} \bigoplus_v \mathrm{Br}(Z_{K_v^h})/\mathrm{Br}(\mathcal{Z}_{\mathcal{O}_v^h}).$$

Since the composition $(\oplus_v \mathrm{cores}_v) \circ (\oplus_v \mathrm{res}_v)$ is equal to the multiplication by $[L : K]$, thus, the kernel of $\oplus_v \mathrm{res}_v$ is killed by $[L : K]$. This proves that $\mathrm{Ker}(f)$ is killed by $[L : K]$. By functoriality, the same argument can imply that $\mathrm{Ker}(\bar{f})$ is also killed by $[L : K]$. To show that $\mathrm{Ker}(f)$ is finite, it suffices to show that for all but finitely many $v \in C^\circ$, the map

$$\mathrm{Br}(K_v^h) \longrightarrow \mathrm{Br}(X_{K_v^h})/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h})$$

is injective. If $X(K_v^h)$ is not empty, the restiction-corestriction argument above will imply that the kernel is killed by 1 and therefore is trivial. Thus, it remains to show that $X(K_v^h)$ is not empty for all but finitely many v. Note that if $\mathcal{X}_{\mathcal{O}_v^h} \longrightarrow \mathrm{Spec}\mathcal{O}_v^h$ is smooth and the special fiber over v admits a rational point, by hensel's lemma, $\mathcal{X}_{\mathcal{O}_v^h}$ admits a section. This will imply that $X(K_v^h)$ is not empty. Thus, it suffices to show that for all but finitely many v, the special fiber over v is smooth and has a rational point. For the proof of this, we refer it to [52, Lem. 2.1]. □

Remark 4.2.6. *In fact, the above proof shows that the kernel is killed by* $[L : K]$ *and* $\mathrm{Br}(\mathcal{X}) \cap \overline{\mathrm{Br}(K)}$ *is killed by* $2[L : K]$ *for any finite extension* L/K *such that* $X(L) \neq \emptyset$*.*

Theorem 4.2.7. *Let* $\pi : \mathcal{X} \longrightarrow C$ *be a proper flat morphism, where* C *is* $\mathrm{Spec}(\mathcal{O}_K)$ *for some number field* K*. Assume that* $\mathcal{X}$ *is regular and the generic fiber* X *of* π *is projective and geometrically connected over* K*. Let* ℓ *be a prime number. Then there are exact sequences up to finite groups*

$$0 \longrightarrow Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow \mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X_{\overline{K}})^{G_K}$$

and

$$0 \longrightarrow Ш(\mathrm{Pic}^0_{X/K})[\ell^\infty] \longrightarrow \mathrm{Br}(\mathcal{X})[\ell^\infty] \longrightarrow \mathrm{Br}(X_{\overline{K}})^{G_K}[\ell^\infty] \longrightarrow 0.$$

Proof. By the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(K, H^q(X_{\overline{K}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

and the fact $H^3(K, \mathbb{G}_m) = 0$ (cf. [37, Prop. 2.7]), we get a long exact sequence

$$\mathrm{Br}(K) \longrightarrow \mathrm{Ker}(\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_{\overline{K}})^{G_K}) \longrightarrow H^1(K, \mathrm{Pic}_{X/K}) \longrightarrow 0.$$

It gives an injective natural map

$$H^1(K, \mathrm{Pic}_{X/K}) \longrightarrow \mathrm{Br}(X)/\mathrm{Br}(K).$$

Similarly, we get

$$H^1(K_v^h, \mathrm{Pic}_{X/K}) \longrightarrow \mathrm{Br}(X_{K_v^h})/\mathrm{Br}(K_v^h).$$

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow H^1(K, \mathrm{Pic}_{X/K}) & \longrightarrow & \mathrm{Br}(X)/\mathrm{Br}(K) & \xrightarrow{b} & \mathrm{Br}(X_{\overline{K}})^{G_K} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_v H^1(K_v^h, \mathrm{Pic}_{X/K}) & \xrightarrow{a} & \prod_v \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)) & \longrightarrow & \prod_v \mathrm{Br}(X_{\overline{K}})^{G_v}/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h})
\end{array}$$

where G_v denotes $\mathrm{Gal}(\overline{K}/K_v^h)$. The above diagram gives the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(K, \mathrm{Pic}_{X/K}) & \longrightarrow & \mathrm{Br}(X)/\mathrm{Br}(K) & \longrightarrow & \mathrm{Im}(b) \\
& \downarrow f & & \downarrow g & & \downarrow h \\
0 \rightarrow & \prod_v H^1(K_v^h, \mathrm{Pic}_{X/K})/\mathrm{Ker}(a) & \rightarrow & \prod_v \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)) & \rightarrow & \prod_v \mathrm{Br}(X_{\bar{K}})^{G_v}/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h})
\end{array}$$

By the Snake Lemma, we get a long exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(g) \longrightarrow \mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f).$$

By Lemma 4.2.5, the natural map

$$\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Ker}(g)$$

has a finite kernel and a finite cokernel. Thus, to prove the theorem, it suffices to show that the natural maps

$$Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow \mathrm{Ker}(f)$$

and

$$\mathrm{Ker}(h)[\ell^\infty] \longrightarrow \mathrm{Br}(X_{\bar{K}})^{G_K}[\ell^\infty]$$

have finite kernels and cokernels and the natural map

$$\mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f)$$

has a finite image. By Lemma 4.2.9 below, $\mathrm{Ker}(a)$ is of finite exponent. Since there is an exact sequence

$$0 \longrightarrow Ш(\mathrm{Pic}_{X/K}) \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(a),$$

and $\mathrm{Ker}(f)$ is of cofinite type (this follows from that $\mathrm{Ker}(g)$ is of cofinite type since $\mathrm{Br}(\mathcal{X})$ is of cofinite type), we have that the injective map

$$Ш(\mathrm{Pic}_{X/K}) \longrightarrow \mathrm{Ker}(f)$$

has a finite cokernel. By Lemma 4.2.2, the natural map

$$Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow Ш(\mathrm{Pic}_{X/K})$$

has a finite kernel and finite cokernel. Thus, the map

$$Ш(\mathrm{Pic}^0_{X/K}) \longrightarrow \mathrm{Ker}(f)$$

also has a finite kernel and finite cokernel. Since $\mathrm{Im}(b)$ is of cofinite type and by Proposition 4.1.1, the target of h is a product of finite groups, thus h will map the maximal divisible subgroup of $\mathrm{Im}(b)$ to zero. Thus, the divisible part of $\mathrm{Im}(b)$ is contained in $\mathrm{Ker}(h)$. By

Theorem 2.4.2, Im(b) has a finite index in $\mathrm{Br}(X_{\overline{K}})^{G_K}$. Thus, they have the same maximal divisible subgroup. So we have

$$\mathrm{Ker}(h)_{\mathrm{div}} = (\mathrm{Br}(X_{\overline{K}})^{G_K})_{\mathrm{div}}.$$

It follows that the inclusion

$$\mathrm{Ker}(h)[\ell^\infty] \hookrightarrow \mathrm{Br}(X_{\overline{K}})^{G_K}[\ell^\infty]$$

has a finite cokernel for any prime ℓ. It remains to be shown that the natural map

$$\mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f)$$

has an image of finite exponent. To prove this, we will use Colliot-Thélène and Skorobogatov's pull-back trick. The idea is that finding finitely many $\pi_i : \mathcal{Z}_i \longrightarrow C$ of relative dimenion 1 which satisfies the same condition as π and admits a C-morphism $\mathcal{Z}_i \longrightarrow \mathcal{X}$, then use the functoriality of the map $\mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f)$ to get a commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}(h) & \longrightarrow & \mathrm{Coker}(f) \\ \downarrow & & \downarrow \\ \bigoplus_i \mathrm{Ker}(h_i) & \longrightarrow & \bigoplus_i \mathrm{Coker}(f_i) \end{array}$$

Since $\mathrm{Ker}(h_i) = 0$ ($\mathrm{Br}(Z_{\overline{K}})^{G_K} = 0$), to prove the claim, it is enough to find $\mathcal{Z}_i$ such that the second column has a kernel of finite exponent. Since $\mathrm{Ker}(a_i)$ is of finite exponent, by the Snake Lemma, it suffices to to find $\mathcal{Z}_i$ such that the natural map

$$\mathrm{Coker}(H^1(K, \mathrm{Pic}_{X/K}) \to \textstyle\prod_v H^1(K_v^h, \mathrm{Pic}_{X/K})) \to \bigoplus_i \mathrm{Coker}(H^1(K, \mathrm{Pic}_{Z_i/K}) \to \prod_v H^1(K_v^h, \mathrm{Pic}_{Z_i/K}))$$

has a kernel of finite exponent. By Lemma 4.2.10 below, there exist smooth projetive geometrically connected curves $Z_i \subset X$ satisfying the above condition. To get $\mathcal{Z}_i$, we can take the Zariski closure of Z_i in $\mathcal{X}$ first and then desingularize it. This completes the proof of the theorem.

□

Remark 4.2.8. *The above proof actually shows the cokernel of* $\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Ker}(h)$ *has a cokernel of finite exponent. Since* $\mathrm{Im}(b)$ *has a finite index in* $\mathrm{Br}(X_{\overline{K}})^{G_K}$*, thus the natural map*

$$\mathrm{Br}(X_{\overline{K}})^{G_K}/\mathrm{Br}(\mathcal{X}) \longrightarrow \prod_v \mathrm{Br}(X_{\overline{K}})^{G_v}/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h})$$

has a finite kernel. To finish the proof of Theorem 1.4.5, it suffices to show that

$$(\mathrm{Br}(X_{\overline{K}})^{G_v}/\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}))[\ell^\infty] = 0$$

for all v *when* $\ell \gg 0$.

Lemma 4.2.9. *Let a denote the map*

$$\prod_v H^1(K_v^h, \mathrm{Pic}_{X/K}) \longrightarrow \prod_v \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)),$$

then $\ker(a)$ *is of finite exponent.*

Proof. By Lemma 2.4.1, there exist smooth projective geometrically connected curves $Z_i \subset X$ over K and an abelian variety A/K with a morphism $A \longrightarrow \oplus_i \mathrm{Pic}^0_{Z_i/K}$ such that the induced G_K-equivariant map

$$\mathrm{Pic}(X_{\overline{K}}) \times A(\overline{K}) \longrightarrow \bigoplus_i \mathrm{Pic}(Z_{i,\overline{K}})$$

has a kernel and a cokernel killed by some positive integer N. Taking the Zariski closure of Z_i in $\mathcal{X}$ and then desingularizing it, we get a C-morphism $\mathcal{Z}_i \longrightarrow \mathcal{X}$. This gives a commutative diagram

$$\begin{array}{ccc} H^1(K_v^h, \mathrm{Pic}_{X/K}) & \xrightarrow{a_v} & \mathrm{Br}(X_{K_v^h})/(\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)) \\ \downarrow & & \downarrow \\ \bigoplus_i H^1(K_v^h, \mathrm{Pic}_{Z_i/K}) & \longrightarrow & \bigoplus_v \mathrm{Br}(Z_{i,K_v^h})/(\mathrm{Br}(\mathcal{Z}_{i,\mathcal{O}_v^h}) + \mathrm{Br}(K_v^h)) \end{array}$$

Since $\mathrm{Br}(\mathcal{Z}_{i,\mathcal{O}_v^h}) = 0$ (cf. [37, Lem. 2.6]), by definition, the second row is injective. It follows that $\ker(a_v)$ is contained in the kernel of the first column, which is a subgroup of the kernel of the map

$$H^1(K_v^h, \mathrm{Pic}_{X/K}) \oplus H^1(K, A) \longrightarrow \bigoplus_i H^1(K_v^h, \mathrm{Pic}_{Z_i/K}).$$

By the long exact sequence of Galois cohomology, it is easy to see the kernel of this map is killed by N^2. This proves $N^2\mathrm{Ker}(a) = 0$. □

Lemma 4.2.10. *Let X be a smooth projective geometrically connected variety over a number field K. Then, there exist smooth projetive geometrically connected curves $Z_i \subset X$ such that the induced map*

$$\mathrm{Coker}(H^1(K, \mathrm{Pic}_{X/K}) \to \textstyle\prod_v H^1(K_v^h, \mathrm{Pic}_{X/K})) \to \bigoplus_i \mathrm{Coker}(H^1(K, \mathrm{Pic}_{Z_i/K}) \to \prod_v H^1(K_v^h, \mathrm{Pic}_{Z_i/K}))$$

has a kernel of finite exponent.

Proof. We will use the same argument as in the proof of the above lemma. Let Z_i and A as in the above lemma. So we have a G_K-equivariant map

$$\mathrm{Pic}(X_{\overline{K}}) \times A(\overline{K}) \longrightarrow \bigoplus_i \mathrm{Pic}(Z_{i,\overline{K}})$$

with a kernel and a cokernel killed by some positive integer N. If we think

$$\mathrm{Coker}(H^1(K, -) \longrightarrow \prod_v H^1(K_v^h, -))$$

as a functor, obviously, it commutes with direct sum. Thus, it suffices to show that for any morphism of G_K-modules $P_1 \longrightarrow P_2$ with a kernel and cokernel killed by N, the induced map

$$\mathrm{Coker}(H^1(K,P_1) \to \prod_v H^1(K_v^h,P_1)) \to \mathrm{Coker}(H^1(K,P_2) \to \prod_v H^1(K_v^h,P_2))$$

has a kernel of finite exponent. Consider the following commutative diagram

$$\begin{array}{ccccccc} & & H^1(K,P_1) & \xrightarrow{a} & H^1(K,P_2) & \longrightarrow & \mathrm{Coker}(a) \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \bar{g} \\ \mathrm{Ker}(b) & \longrightarrow & \prod_v H^1(K_v^h,P_1) & \xrightarrow{b} & \prod_v H^1(K_v^h,P_2) & \longrightarrow & \mathrm{Coker}(b) \end{array}$$

By the assumption, we have that $\mathrm{Ker}(b)$ and $\mathrm{Coker}(a)$ are killed by N^2. By the Snake Lemma, there is an exact sequence

$$\mathrm{Ker}(\bar{g}) \longrightarrow \mathrm{Coker}(f)/\mathrm{Ker}(b) \longrightarrow \mathrm{Coker}(g).$$

It follows that the kernel of $\mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(g)$ has a kernel killed by N^4. □

Completion of the proof of Theorem 1.4.5

By Remark 4.2.4, we can replace the henselain local ring by the completed local ring in the proof of the above theorem. If we exclude finitely many places, the map $\mathcal{X}_{\mathcal{O}_v} \longrightarrow \mathrm{Spec}(\mathcal{O}_v)$ will be a smooth projective morphism. By the pull-back trick, one can show that

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v}) \longrightarrow \mathrm{Br}(\mathcal{X}_{\mathcal{O}_v^{sh}})^{G_{k(v)}}$$

has a cokernel killed by some positive integer independent of v(cf. the proof of Proposition 4.1.18). In conjunction with Proposition 4.1.4, this will imply that for all but finitely many v, the natural map

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v})(\text{non-}p_v) \longrightarrow \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}(\text{non-}p_v)$$

has a cokernel killed by some positive integer independent of v. Thus, to prove Theorem 1.4.5, by Remark 4.2.8, it suffices to show that the natural map

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v})(p_v) \longrightarrow \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}(p_v)$$

is surjective for all but finitely many v, where p_v is the characteristic of the residue field at v. We will prove this in the Corollary 4.2.12 below.

Proposition 4.2.11. *Let $K/\mathbb{Q}_p$ be an unramified finite extension and $\pi : \mathcal{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ be a smooth projective morphism with generic fiber X. Assuming $p \geq 5$, then the natural map*

$$H^2_{\mathrm{fppf}}(\mathcal{X}, \mu_{p^n}) \longrightarrow H^2(X_{\overline{K}}, \mu_{p^n})^{G_K}$$

is surjective for any $n \geq 1$

Proof. Let $\mathcal{X}_n$ denote the $\mathcal{O}_K/p^n$-scheme $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. Fontaine and Messing [20, III. 3.1] defined a sheaf $\mathcal{S}_n(r)$ on the small syntomic site $(\mathcal{X}_n)_{syn}$ of $\mathcal{X}_n$. Following definitions and notations in [16, §2.1], for $0 \leq r \leq p-1$, define

$$\mathcal{S}_n(r) := \mathrm{Ker}(\mathcal{J}_n^{[r]} \xrightarrow{1-\varphi_r} \mathcal{O}_n^{cr}).$$

In fact, there is an exact sequence of sheaves on $(\mathcal{X}_n)_{syn}$

$$0 \longrightarrow \mathcal{S}_n(r) \longrightarrow \mathcal{J}_n^{[r]} \xrightarrow{1-\varphi_r} \mathcal{O}_n^{cr} \longrightarrow 0.$$

For our purpose, we only consider the case with $r = 1$. The syntomic cohomology

$$H^i((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1))$$

computes the flat cohomology $H^i_{\mathrm{fppf}}(\mathcal{X}, \mu_{p^n})$ (cf. [32] and [49, §1.4 and §4.5]) and the natural map

$$H^i_{\mathrm{fppf}}(\mathcal{X}, \mu_{p^n}) \longrightarrow H^i(X, \mu_{p^n})$$

is compatible with the Fontaine-Messing map

$$H^i((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) \longrightarrow H^i(X, \mu_{p^n}).$$

Thus, it suffices to show that the following map induced by the Fontaine-Messing map

$$H^2((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) \longrightarrow H^2(X_{\overline{K}}, \mu_{p^n})^{G_K}$$

is surjective for any $n \geq 1$.

Let Y denote the special fiber $\mathcal{X}_1$. Following notations in [42, III. 4.10], denote

$$M_n^2 := H^2((\mathcal{X}_n)_{syn}, \mathcal{O}_n^{cr}) = H^2((Y/W_n)_{\mathrm{cris}}, \mathcal{O}_{Y/W_n}) = H^2_{dR}(\mathcal{X}_n/W_n),$$

$$F^r M_n^2 := H^2((\mathcal{X}_n)_{syn}, \mathcal{J}_n^{[r]}) = H^2((\mathcal{X}_n)_{\mathrm{Zar}}, \sigma_{\geq r}\Omega^{\bullet}_{\mathcal{X}_n/W_n}), \quad T_n^2 := H^2(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}).$$

By [20, II. 2.7], $(M_n^2, F^r M_n^2, \varphi_r)$ defined an object of $MF^{[0,2]}_{W,tors} \subseteq MF^{[0,p-1[}_{W,tors}$ (cf. [42, III. Prop. 4.11] or [5, Thm. 3.2.3]). For $0 \leq r < p-1$, by [42, III. 4.8], the functor T (cf. [42, III. 4.6] for the definition) induces an isomorphism

$$\alpha_{r,M_n^2} : (F^r M_n^2)^{\varphi_r = 1} \longrightarrow (T_n^2(r))^{G_K}.$$

By the exact sequence of sheaves on $(\mathcal{X}_n)_{syn}$

$$0 \longrightarrow \mathcal{S}_n(1) \longrightarrow \mathcal{J}_n^{[1]} \xrightarrow{1-\varphi_1} \mathcal{O}_n^{cr} \longrightarrow 0,$$

taking cohomology, we get a long exact sequence

$$H^2((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) \longrightarrow H^2((\mathcal{X}_n)_{syn}, \mathcal{J}_n^{[1]}) \xrightarrow{1-\varphi_1} H^2((\mathcal{X}_n)_{syn}, \mathcal{O}_n^{cr}) \longrightarrow H^3((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)).$$

Thus, there is a surjective natural map

$$H^2((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) \longrightarrow H^2((\mathcal{X}_n)_{syn}, \mathcal{J}_n^{[1]})^{\varphi_1=1}.$$

By [42, III. Thm. 5.2], there is a commutative diagram

$$\begin{array}{ccc} H^2((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) & \longrightarrow & H^2((\mathcal{X}_n)_{syn}, \mathcal{J}_n^{[1]})^{\varphi_1=1} \\ \downarrow \nu & & \downarrow \alpha_{1,M_n^2} \\ H^2(X, \mathbb{Z}/p^n\mathbb{Z}(1)) & \longrightarrow & H^2(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(1))^{G_K} \end{array}$$

where ν denotes the Fontaine-Messing map. Since the first row is surjective and α_{1,M_n^2} is an isomorphism, it follows that the natural map

$$H^2((\mathcal{X}_n)_{syn}, \mathcal{S}_n(1)) \longrightarrow H^2(X_{\overline{K}}, \mu_{p^n})^{G_K}$$

is surjective. This completes the proof. □

Corollary 4.2.12. *Let $\pi : \mathcal{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ be a proper flat morphism, where K is a number field. Assume that $\mathcal{X}$ is regular and the generic fiber X of π is projective and geometrically connected over K. For any finite place v, let p_v denote the characteristic of the residue field of $\mathcal{O}_K$ at v. Then, for all but finitely many places v, the natural map*

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v})[p_v^\infty]) \longrightarrow \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}[p_v^\infty]$$

is surjective.

Proof. $K/\mathbb{Q}$ only ramifies at finitely many places. Thus, by the proposition above, for all but finitely many v, the natural map

$$H^2_{\mathrm{fppf}}(\mathcal{X}_{\mathcal{O}_v}, \mu_{p_v^n}) \longrightarrow H^2(X_{\overline{K_v}}, \mu_{p_v^n})^{G_{K_v}}$$

is surjective for any $n \geq 1$. By Proposition 2.5.2, for all but finitely many v, the natural map

$$H^2(X_{\overline{K_v}}, \mu_{p_v^n})^{G_{K_v}} \longrightarrow \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}[p_v^n]$$

is surjective for any $n \geq 1$. By the following commutative diagram

$$\begin{array}{ccc} H^2_{\mathrm{fppf}}(\mathcal{X}_{\mathcal{O}_v}, \mu_{p_v^n}) & \longrightarrow & \mathrm{Br}(\mathcal{X}_{\mathcal{O}_v})[p_v^n] \\ \downarrow & & \downarrow \\ H^2(X_{\overline{K_v}}, \mu_{p_v^n})^{G_{K_v}} & \longrightarrow & \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}[p_v^n], \end{array}$$

for all but finitely many v, the natural map

$$\mathrm{Br}(\mathcal{X}_{\mathcal{O}_v})[p_v^n] \longrightarrow \mathrm{Br}(X_{\overline{K_v}})^{G_{K_v}}[p_v^n]$$

is surjective for any $n \geq 1$.

□

Remark 4.2.13. *By [48, Thm. 1.10], the corollary implies that for all but finitely many places v, the p_v-primary part of the kernel of the natural map induced by the Brauer-Manin pairing*

$$\mathrm{Hom}(\mathrm{Br}(X_{K_v})/\mathrm{Br}(K) + \mathrm{Br}(\mathcal{X}_{\mathcal{O}_v}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Alb}_X(K_v)$$

is trivial.

4.3 Applications

Reduction of Artin's question

Lemma 4.3.1. *Assuming that $\mathrm{Br}(X_{\overline{K}})^{G_K}$ is finite for all smooth projective geometrically connected surfaces over a number field K, then $\mathrm{Br}(X_{\overline{K}})^{G_K}$ is finite for all smooth projective geometrically connected varieties over K.*

Proof. Let X be a smooth projective geometrically connected variety over a number field K. Assuming that $\dim(X) > 2$, by a theorem of Ambrosi [1, Cor. 1.6.2.1], there exists a smooth projective geometrically connected hyperplane section D of X such that the induced map

$$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. By Proposition 2.5.2, for any prime ℓ, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \longrightarrow & H^2(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K} & \longrightarrow & V_\ell \mathrm{Br}(X_{\overline{K}})^{G_K} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{NS}(D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \longrightarrow & H^2(D_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K} & \longrightarrow & V_\ell \mathrm{Br}(D_{\overline{K}})^{G_K} & \longrightarrow & 0 \end{array}$$

Since the first column is an isomorphism and the second column is injective (the weak Lefschetz theorem), by the Snake Lemma, the third column is injective. This implies that for any ℓ, the ℓ-primary part of the kernel of the map

$$\mathrm{Br}(X_{\overline{K}})^{G_K} \longrightarrow \mathrm{Br}(D_{\overline{K}})^{G_K}$$

is finite. Since $\mathrm{NS}(X)$ and $\mathrm{NS}(D)$ are finite groups, thus for sufficiently large ℓ, the induced map

$$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

is an isomorphism. By the weak Lefschetz theorem for torsion locally constant sheaves, the induced map

$$H^2(X_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow H^2(D_{\overline{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$$

is injective. By Proposition 2.5.2, we have a similar diagram as before for torsion coefficients when ℓ is sufficently large. Thus, natural map

$$\mathrm{Br}(X_{\overline{K}})^{G_K}[\ell^\infty] \longrightarrow \mathrm{Br}(D_{\overline{K}})^{G_K}[\ell^\infty]$$

is injective for all but finitely many ℓ. This proves that the kernel of

$$\mathrm{Br}(X_{\overline{K}})^{G_K} \longrightarrow \mathrm{Br}(D_{\overline{K}})^{G_K}$$

is finite. By induction of the dimension of X, the claim follows. □

Theorem 4.3.2. *Assuming that* $\mathrm{Br}(X_{\bar{K}})^{G_K}$ *is finite for all smooth projective surfaces and* $Ш(\mathrm{Pic}^0_{X/K})$ *is finite for all smooth projective geometrically connected curves, then* $\mathrm{Br}(\mathcal{X})$ *is finite for all regular proper flat schemes* $\mathcal{X}$ *over* $\mathbb{Z}$*.*

Proof. Let X be a smooth proper geometrically connected variety over a number field K. By resolution of singularity in charateristic 0, there is a smooth projective variety X' and a birational morphism $X' \longrightarrow X$. Since Brauer group is a birational inviants and $\mathrm{Pic}^0_{X/K}$ and $\mathrm{Pic}^0_{X'/K}$ are isogenous to each other, we may assume that X is projective. If $\dim(X) \geq 2$, by Lemma 2.3.5, there is a smooth projective geometrically connected curve $Y \subset X$ and an abelian variety A/K such that $\mathrm{Pic}^0_{X/K} \times A$ is isogenous to $\mathrm{Pic}^0_{C/K}$. Thus, the finiteness of $Ш(\mathrm{Pic}^0_{X/K})$ can be reduced to curves. By the above lemma, the finiteness of $\mathrm{Br}(X_{\overline{K}})^{G_K}$ can be reduced to surfaces. Let X denote the generic fiber of $\mathcal{X}$. By the assumption, $Ш(\mathrm{Pic}^0_{X/K})$ and $\mathrm{Br}(X_{\overline{K}})^{G_K}$ are finite. By Theorem 1.4.5, $\mathrm{Br}(\mathcal{X})$ is finite.

□

Proof of Theorem 1.5.6

Proof. By the theorem above, it suffices to show that $\mathrm{Br}(X_{\overline{K}})^{G_K}$ and $Ш(\mathrm{Pic}^0_{X/K})$ are finite for all smooth projective geometrically connected surfaces over a number field K under the assumption. By de Jong's theorem [13, Cor. 5.1], there is an alteration $X' \longrightarrow X$ such that

X' admits a flat proper regular integral model over the ring of integers of a number field. By extending K, we may assume that X' is smooth and geometrically connected over K. Then we have that $\mathrm{Pic}^0_{X/K} \longrightarrow \mathrm{Pic}^0_{X'/K}$ has a finite kernel since $H^1(X_{\overline{K}}, \mathbb{Q}_\ell) \longrightarrow H^1(X'_{\overline{K}}, \mathbb{Q}_\ell)$ is injective. Thus, the finiteness of $\text{Ш}(\mathrm{Pic}^0_{X'/K})$ can imply the finiteness of $\text{Ш}(\mathrm{Pic}^0_{X/K})$. Since $K(X'_{\overline{K}})$ is a finite extension over $K(X_{\overline{K}})$, by the restriction-corestriction argument, the induced map $\mathrm{Br}(K(X_{\overline{K}})) \longrightarrow \mathrm{Br}(K(X'_{\overline{K}}))$ has a kernel of finite exponent. It follows that

$$\mathrm{Br}(X_{\overline{K}})^{G_K} \longrightarrow \mathrm{Br}(X'_{\overline{K}})^{G_K}$$

has a finite kernel. By the assumption and Theorem 1.4.5, $\mathrm{Br}(X'_{\overline{K}})^{G_K}$ and $\text{Ш}(\mathrm{Pic}^0_{X'/K})$ are finite. Thus, $\mathrm{Br}(X_{\overline{K}})^{G_K}$ and $\text{Ш}(\mathrm{Pic}^0_{X/K})$ are also finite. This completes the proof. □

Examples

Proposition 4.3.3. *Let X be a principal homogeneous space of an abelian variety over a number field K. Assuming that X/K admits a proper regular model $\pi : \mathcal{X} \longrightarrow C$ as in Theorem 1.4.5, then there is an isomorphism up to finite groups*

$$\text{Ш}(\mathrm{Pic}^0_{X/K}) \cong \mathrm{Br}(\mathcal{X}).$$

Proof. By [54, Thm. 1.1], $\mathrm{Br}(X_{K^s})^{G_K}$ is finite. Then the claim follows directly from Theorem 1.4.5. □

Proposition 4.3.4. *(Tankeev) Let $\pi : \mathcal{X} \longrightarrow C$ a proper flat morphism as in Theorem 1.4.5. Assuming that the generic fiber X is a $K3$ surface, then* $\mathrm{Br}(\mathcal{X})$ *is finite.*

Proof. Since $H^1(X, \mathscr{O}_X) = 0$, we have $\mathrm{Pic}^0_{X/K} = 0$. By [54, Thm. 1.2], $\mathrm{Br}(X_{\overline{K}})^{G_K}$ is finite. Then the claim follows directly from Theorem 1.4.5. □

Bibliography

[1] E. Ambrosi, Specialization of Néron-Severi groups in positive characteristic, arXiv preprint arXiv:1810.06481.

[2] Y. André, Pour une théorie inconditionnelle des motifs, Publ. Math. IHES 83, p. 5-49, 1996.

[3] M. Artin, H. P. F. Swinnerton-Dyer, The Shafarevich-Tate conjecture for pencils of elliptic curves on K3 surfaces, Invent. Math. 20 (1973), 249–266.

[4] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 21. Springer-Verlag, Berlin, 1990.

[5] C. Breuil, W. Messing, Torsion étale and crystalline cohomologies. Astérisque 279 (2002): 81-124.

[6] K. Česnavičius, Purity for the Brauer group. Duke Mathematical Journal, 168(8)(2019), pp.1461-1486.

[7] F. Charles, The Tate conjecture for K3 surfaces over finite fields, Invent. Math. 194, 119–145 (2013).

[8] P. Colmez, W. Nizioł, Syntomic complexes and p-adic nearby cycles, Inventiones mathematicae 208, no. 1 (2017): 1-108.

[9] V. Cossart, O. Piltant, Resolution of singularities of threefolds in positive characteristic II. Journal of Algebra, 321(7)(2009), 1836-1976.

[10] J.-L. Colliot-Thélène, S. Saito, Zéro-cycles sur les variétés p-adiques et groupe de Brauer, International Mathematics Research Notices 4 (1996): 151-160.

[11] J.-L. Colliot-Thélène, A.N. Skorobogatov, Descente galoisienne sur le groupe de Brauer. J. reine angew. Math. 682 (2013) 141-165.

[12] J.-L. Colliot-Thélène, A.N. Skorobogatov, The Brauer-Grothendieck group, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 71, Springer, Cham, 2021.

[13] A. J. de Jong, Families of curves and alterations. In Annals of the Fourier Institute (Vol. 47, No. 2, 1997, pp. 599-621).

[14] A. J. de Jong, Tate conjecture for divisors. Unpublished note.

[15] F. Déglise, W. Nizioł, On p-adic absolute Hodge cohomology and syntomic coefficients, I. Commentarii Mathematici Helvetici. 2018.

[16] V. Ertl, W. Niziol, Syntomic cohomology and p-adic motivic cohomology. ALGEBRAIC GEOMETRY 6, no. 1 (2019): 100-131.

[17] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. Invent. Math. 73 (1983), no. 3, 349-366.

[18] G. Faltings, Complements to Mordell. In: G. Faltings, G. Wüstholz et al., Rational points, 2nd edition, F. Viehweg & Sohn, 1986.

[19] A. Grothendieck, "Fondements de la Géométrie Algébrique" [Extraits du Séminaire Bourbaki 1957–1962], Secrétariat mathématique, 11 rue Pierre Curie, Paris 5e, 1962.

[20] J.-M. Fontaine, W. Messing, p-adic periods and p-adic étale cohomology, Contemporary Math. 67, 1987, 179-207.

[21] L. Fu, Etale cohomology theory (Vol. 13). World Scientific, 2011.

[22] K. Fujiwara, A proof of the absolute purity conjecture (after Gabber). In Algebraic geometry 2000, Azumino (pp. 153-183). Mathematical Society of Japan.

[23] T.H. Geisser, Tate's conjecture and the Tate–Shafarevich group over global function-fields, J. Inst. Math. Jussieu 20(3) (2021) 1001–1022.

[24] A. Grothendieck, Le groupe de Brauer. III. Exemples et compléments, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188.

[25] R. Hartshorne, Algebraic geometry. Vol. 52. Springer Science & Business Media, 2013.

[26] L. Illusie, "Crystalline cohomology." Motives (Seattle, WA, 1991) 55 (1994): 43-70.

[27] U. Jannsen, Weights in arithmetic geometry. Japanese Journal of Mathematics, 5(1)(2010), pp.73-102.

[28] U. Jannsen, S. Saito, Bertini theorems and Lefschetz pencils over discrete valuation rings, with applications to higher class field theory. Journal of Algebraic Geometry, 21(4)(2012), 683-705.

[29] K. Kato, On p-adic vanishing cycles (application of ideas of Fontaine-Messing). In: Algebraic Geometry, Sendai, 1985. Adv. Stud. Pure Math., vol. 10, pp. 207–251. North-Holland,Amsterdam (1987)

[30] K. Kato, Semistable reduction and p-adic étale cohomology. Astérisque 223, 269–293(1994)

[31] S. L. Kleiman, The Picard scheme. Fundamental algebraic geometry: Grothendieck's FGA Explained, 235–321, Math. Surveys Monogr., 123, Amer. Math. Soc., Providence, RI, 2005.

[32] M. Kurihara, A note on p-adic etale cohomology. Proceedings of the Japan Academy, Series A, Mathematical Sciences 63, no. 7 (1987): 275-278.

[33] S. Lichtenbaum, Zeta-functions of varieties over finite fields at $s = 1$, in Arithmetic and geometry, pp. 173-194. Birkhäuser, Boston, MA, 1983.

[34] D. Maulik, Supersingular K3 surfaces for large primes, With an appendix by Andrew Snowden, Duke Math. J. 163, 2357–2425 (2014).

[35] J.S. Milne, Arithmetic duality theorems, Vol. 20. Charleston, SC: BookSurge, 2006.

[36] J.S. Milne, Etale cohomology (PMS-33)(1980) (Vol. 5657). Princeton university press.

[37] J.S. Milne, Comparison of the Brauer Group with the Tate-Safarevic group, J. Fac. Science Univ. Tokyo, Sec. IA 28 (1981), 735–743.

[38] J.S. Milne, Values of zeta functions of varieties over finite fields. American Journal of Mathematics (1986), 108(2), 297-360.

[39] K. Madapusi Pera, The Tate conjecture for K3 surfaces in odd characteristic, Invent. Math. 201, 625-668 (2015).

[40] M. Morrow, A Variational Tate Conjecture in crystalline cohomology. Journal of the European Mathematical Society, 21(11)(2019), 3467-3511.

[41] D. Mumford. Abelian varieties. 2nd edition. Oxford University Press, 1974.

[42] J. Nekovář, Syntomic cohomology and p-adic regulators, preprint (1998). https://webusers.imj-prg.fr/ jan.nekovar/pu/syn.pdf

[43] J. Nekovář, W. Nizioł, Syntomic cohomology and p-adic regulators for varieties over p-adic fields, Algebra & Number Theory. 2016 Oct 7;10(8):1695-790.

[44] N. Nygaard, A. Ogus, Tate's conjecture for K3 surfaces of finite height, Ann. of Math. 122, 461-507 (1985).

[45] N. O. Nygaard, The Tate conjecture for ordinary K3 surfaces over finite fields, Invent. Math. 74, 213-237 (1983).

[46] M. Orr, A.N. Skorobogatov, Finiteness theorems for K3 surfaces and abelian varieties of CM type. Compositio Mathematica, 2018, 154(8): 1571-1592.

[47] Y. Qin, On geometric Brauer groups and Tate-Shafarevich groups. arXiv:2012.01681.

[48] Y. Qin, A p-adic local invariant cycle theorem with applications to Brauer groups. arXiv:2103.04945.

[49] K. Sato, p-adic étale Tate twists and arithmetic duality (with an appendix by Hagihara, K.). Ann. Sci. École Norm. Sup. (4) 40, 519–588 (2007)

[50] P. Berthelot, A. Grothendieck, and L. Illusie (eds), Séminaire de Géométrie Algébrique du Bois Marie (1966-67) Théorie des intersections et théorème de Riemann-Roch, (Lecture notes in mathematics 225). Berlin; New York: Springer-Verlag.

[51] Schneider, P. (1982). On the conjecture of Birch and Swinnerton-Dyer about global function fields. Mathematische Annalen , 260 (4), 495-510.

[52] P. Swinnerton-Dyer, Topics in Diophantine equations, Arithmetic geometry, Lecture Notes in Math., vol. 2009, Springer, Berlin, 2011, pp. 45–110

[53] A.N. Skorobogatov, Y.G. Zarkhin, A finiteness theorem for the Brauer group of K3 surfaces in odd characteristic. International Mathematics Research Notices 2015.21 (2015): 11404-11418.

[54] A.N. Skorobogatov, Y.G. Zarkhin, A finiteness theorem for the Brauer group of Abelian varieties and KS surfaces. Journal of Algebraic Geometry, 17(3)(2008), 481-502.

[55] S.G. Tankeev, On the Brauer group of an arithmetic scheme. Izvestiya: Mathematics, 65(2)(2001), 357.

[56] S.G. Tankeev, On the Brauer group of an arithmetic scheme. II. Izvestiya: Mathematics, 67(5)(2003), 1007.

[57] S.G. Tankeev, On the finiteness of the Brauer group of an arithmetic scheme. Mathematical Notes, 95(1-2)(2014), 121-132.

[58] S. G. Tankeev, Surfaces of K3 type over number fields and the Mumford-Tate conjecture. Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 4, 846-861; translation in Math. USSR-Izv. 37 (1991), no. 1, 191- 208.

[59] S. G. Tankeev, Surfaces of K3 type over number fields and the Mumford-Tate conjecture. II. Izv. Ross. Akad. Nauk Ser. Mat. 59 (1995), no. 3, 179-206; translation in Izv. Math. 59 (1995), no. 3, 619-646.

[60] J. Tate, Algebraic cycles and poles of zeta functions. Arithmetical algebraic geometry. 1965:93-110.

[61] J. Tate, Conjectures on algebraic cycles in l-adic cohomology. Motives (Seattle, WA, 1991) 55 (1994): 71-83.

[62] J. Tate, Endomorphisms of abelian varieties over finite fields, Invent. Math. 2, 134-144 (1966).

[63] T. Tsuji, p-adic étale and crystalline cohomology in the semistable reduction case. Invent. math. 137, 233–411 (1999).

[64] T. Tsuji, On p-adic nearby cycles of log smooth families. Bull. SMF 128, 529–575 (2000).

[65] I. Vidal, Théorie de Brauer et conducteur de Swan. Journal of Algebraic Geometry 13.2 (2004): 349-391.

[66] X. Yuan, Positivity of Hodge bundles of abelian varieties over some function fields. Compositio Mathematica. 2021 Sep;157(9):1964-2000

[67] X. Yuan, Comparison of arithmetic Brauer groups with geometric Brauer group. arXiv:2011.12897.

[68] Y. G. Zarhin, Endomorphisms of Abelian varieties over fields of finite characteristic. Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 272-277; Math. USSR Izv. 9 (1975), 255-260.

[69] Y. G. Zarhin, Abelian varieties in characteristic p. Mat. Zametki 19 (1976), 393-400; Math. Notes 19 (1976), 240-244.