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Robust Synchronization of Two Linear Systems over Intermittent Communication Networks

Sean Phillips and Ricardo G. Sanfelice

Abstract— **The property of synchronization between two identical linear time-invariant (LTI) systems connected through a network with stochastically-driven isolated communication events is studied. More precisely, the goal is to design feedback controllers that, using information obtained over such networks, asymptotically drive the values of their state to synchronization and render the synchronization condition Lyapunov stable. To solve this problem, we propose a dynamic controller with hybrid dynamics, namely, the controller exhibits continuous dynamics between communication events while it has variables that jump at such events. Due to the additional continuous and discrete dynamics inherent in the networked systems and communication structure, we utilize a hybrid systems framework to model the closed-loop system. The problem of synchronization is then recast as a set stabilization problem and, by utilizing recent Lyapunov stability tools for hybrid systems, sufficient conditions for asymptotic stability of the synchronization set are provided for two network topologies: a cascade (unidirectional) network and a feedback (bidirectional) communication network with independent transmission instances. Furthermore, we study the robustness of synchronization by considering a class of perturbations on the transmitted data. Numerical examples are provided.**

I. Introduction

The idea of synchronization in networked systems has been extensively studied in the literature and has been approached by various viewpoints and methodologies. Synchronization of coupled identical linear systems has been throughly investigated in both the continuous and discrete time domains [1], [2], [3]. Further studies involving nonlinear systems within complex network structures are also available, [4]. A typical approach is to study the network structure using graph theory, which provides a solid understanding of the connectivity of the network and its effect on the individual dynamics of the systems [5], [6]. The study of the stability of synchronization using systems theory tools, like Lyapunov functions [7], [8], contraction theory [9], and incremental input-to-state stability [10], [11] have also been explored. Synchronization of state information in impulsive networks with sporadic communication between LTI systems naturally leads to a complex communication structure, for which, perhaps due to a theory for synchronization of hybrid systems not being available, there is a surprising lack of solutions. Notably, recent research efforts on sample-data systems [12], event-triggered control [13], [14], and tracking [15] for the stabilization of sets provide results that can become useful for synchronization, though some of the assumptions need to be carefully fit to the synchronization problem under intermittent communication networks; e.g., the common event times conditions in [15] and [14] might require specific structure in the network.

This article deals with the problem of synchronization of the states between two linear continuous-time systems when communication occurs only impulsively, at stochastically determined instants. To solve this problem, we design a dynamic hybrid controller that continuously evolves between communication times and undergoes an instantaneous change in its state when a new measurement is available. Due to the combination of continuous and impulsive dynamics, we use hybrid systems theory to model the systems, the controllers, and two network topologies, and apply tools for the stabilization of sets. More precisely, we recast the synchronization problem as the stabilization of a compact set and apply a Lyapunov theorem for hybrid systems. We show that the resulting closed-loop system has an asymptotic synchronization property that is robust to small perturbations.

The remainder of this paper is organized as follows. In Section II we introduce the impulsive network model, control structure and recast this model into the hybrid systems framework. In Section III, we provide the sufficient conditions that guarantee stability for each topology considered. The results on robustness of synchronization are in Section III-C. Section IV provides examples and numerical simulations. Due to space limitations, the proofs will be published elsewhere.

Notation: The set $\mathbb{R}$ denotes the space of real numbers. The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The set $\mathbb{N}$ denotes the natural numbers including zero, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$. Given two vectors x, y, we denote $(x, y) = [x^\top, y^\top]^\top$. Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. The identity matrix is denoted by I. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A and $|A|$ denotes the induced 2-norm. For two symmetric matrices with same dimensions, A and B, $A > B$ means that $A - B$ is positive definite. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing and is a class $\mathcal{K}_\infty$ function if it belongs to class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if it is nondecreasing in its first argument, non-increasing in its second argument, and is such that $\lim_{s\to 0} \beta(s, t) = \lim_{t\to\infty} \beta(s, t) = 0$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is $|x|_\mathcal{A} := \inf_{z\in\mathcal{A}} |x - z|$.

II. Problem Description and

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Mathematical Modeling

A. Problem Description

We consider the problem of synchronizing the states x_1 and x_2 of two identical LTI systems given by

$$\dot{x}_1 = Ax_1 + Bu_1, \qquad \dot{x}_2 = Ax_2 + Bu_2, \tag{1}$$

exchanging information about their variables intermittently, where, for each $i \in \{1,2\}$, $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^p$ is the input to the i-th system. More precisely, our goal is to design a feedback controller assigning the inputs u_1 and u_2 to drive the solutions to (1) to synchronization between x_1 and x_2 asymptotically and also rendering the set of points where x_1 is equal to x_2 stable. Moreover, the outputs of each system are available to the other system only at isolated time instances.

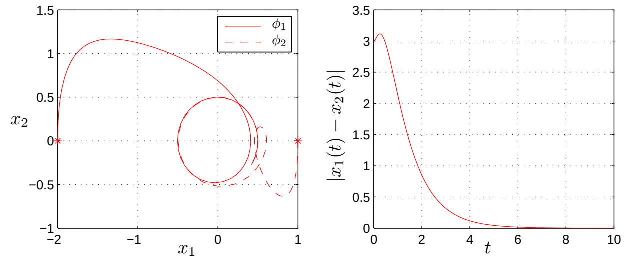
To accommodate many real-world applications, we do not assume that information arrival to each system occurs simultaneously. Specifically, we assume that the state x_ℓ of the ℓ-th system is available to the i-th system ($i, \ell \in \{1,2\}$, $i \neq \ell$) only at impulsive time instances t_k^i, where $k \in \mathbb{N} \setminus \{0\}$ is the communication event index and $i \in \{1,2\}$ is the index denoting the system receiving information at such instants. Given positive scalars $T_2 > T_1$, we assume that the time between these events are governed by a discrete random variable with some bounded probability distribution; i.e., for each $i \in \{1,2\}$, the random variable $\gamma^i \in [T_1, T_2]$ determines the time elapsed between such events, namely,

$$t_{k+1}^i - t_k^i = \gamma^i \qquad \forall k \in \mathbb{N} \setminus \{0\} \tag{2}$$

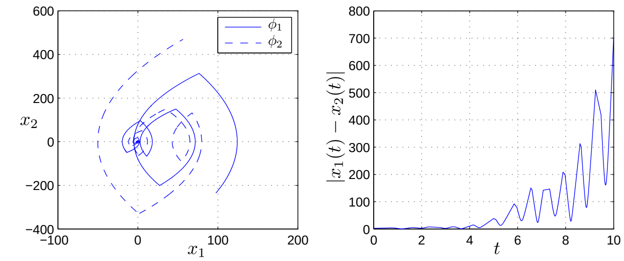
The positive values T_1 and T_2 define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive transmission time instances. In this way, the random variable γ^i may take values only on the bounded interval $[T_1, T_2]$, while the probability density function governing its distribution can be arbitrary on this interval.

Synchronization itself is most generally described as the property that the distance between every pair of solutions, one to each system, converges to zero. At times, stability of the synchronization condition is also required. In this paper, due to the interest in robustness, we consider stability and attractivity of the set defining synchronization, which will be presented in detail for hybrid systems in Definition 2.3.

To illustrate that achieving synchronization with intermittent information is difficult, consider a pair of linear oscillators, each with the dynamics $\dot{x}_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$, with state $x_i \in \mathbb{R}^2$, initial condition $x_i(0)$, and input $u_i \in \mathbb{R}$ for each $i \in \{1,2\}$. It can be shown that for a network transmitting continuously, the static controller given by $u_i = K_i(x_i - x_\ell)$ ($i, \ell \in \{1,2\}$ $i \neq \ell$) asymptotically synchronizes the interconnected system if there exist $K_1, K_2 \in \mathbb{R}^{1\times 2}$ such that the matrix $A + B(K_1 + K_2)$ is Hurwitz. See Figure 1(a) for a numerical solution for $K_1 = K_2 = [-1, -2]$ and initial conditions $x_1(0) = (-2, 0)$ and $x_2(0) = (1, 0)$. The norm of the error between x_1 and x_2 along the solution is also plotted. As Figure 1(b) shows, a sample-and-hold implementation of this controller may not work over a network in which



(a) The distance between the solutions using continuous control decreases to zero (right), implying that solutions synchronize asymptotically (left).



(b) With $t_{k+1}^i - t_k^i \in [0.5, 1]$ for $i \in \{1,2\}$, $k \in \mathbb{N} \setminus \{0\}$, the intermittent communication prevents the controller from driving the distance between solutions to zero (right) and thus preventing synchronization (left).

Fig. 1. A numerical solution to (1) with a controller given by $u_i = K_i(x_i - x_j)$ with $K_1 = K_2 = [-1, -2]$ under (a) continuous communication and (b) intermittent communication from initial conditions $x_1(0) = (-2, 0)$ and $x_2(0) = (1, 0)$.

information cannot be transmitted arbitrarily fast. Since information arrives only at impulsive events (in this simulation, we assume they occur at the same time instances for both systems and $[T_1, T_2] = [0.5, 1]$), the controller is not capable of synchronizing the systems. While very fast transmission would solve the problem, we show in Section IV-A that our controller synchronizes the systems with this "slow" and nonperiodic transmission rate.

B. Outline of Proposed Solution

We propose a hybrid controller and a design procedure for synchronization over intermittent networks. The proposed controller for each system has state η_i, which assigns the input as $u_i = \eta_i$, where $\eta_i \in \mathbb{R}^p$ has the following dynamics:

$$\begin{aligned} \dot{\eta}_i(t) &= f_{ci}(x_i(t)) & t &\notin \{t_k^i\}_{k=1}^\infty \\ \eta_i(t^+) &\in G_{ci}(x_i(t), x_\ell(t)) & t &\in \{t_k^i\}_{k=1}^\infty \end{aligned} \tag{3}$$

where $\eta_i(t^+)$ denotes the value of η_i after an instantaneous change at time t. The map $f_{ci} : \mathbb{R}^n \to \mathbb{R}^p$ governs the continuous evolution of the controller between time instances t_k^i while $G_{ci} : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a set-valued map updating the state of the controller when new information arrives; the functions f_{ci} and G_{ci} from (3) are to be designed.

Due to the nonlinearities, nonperiodic arrival of information, and impulsive dynamics, classical control theory provides little insight into modeling and design. This motivates us to design the controller in (3) by recasting the closed-loop interconnection as a hybrid system [16].

C. Hybrid Modeling

The closed-loop system obtained when controlling (1) with (3) consists of a continuous-time plant and an impulsive controller that is allowed to evolve both continuously between transmission time instances (i.e., when the continuous time parameter t satisfies $t \in [t_k^i, t_{k+1}^i)$) and discretely when information is received (i.e., when $t = t_k^i$, $k \in \mathbb{N} \setminus \{0\}$). Following [17], for each system $i \in \{1, 2\}$, we model these events using a timer variable τ_i that decreases during flows and, once it reaches zero, is reset to any point in $[T_1, T_2]$.

To model this mechanism and the closed-loop system, we employ the hybrid systems framework in [16], where a hybrid system is given by four objects (C, f, D, G) defining its *data:*

- *Flow set*: a set $C \subset \mathbb{R}^n$ specifying the points where (the continuous evolution or) flows are possible.
- *Flow map*: a single-valued map $f : \mathbb{R}^n \to \mathbb{R}^n$ defining the flows.
- *Jumps set*: a set $D \subset \mathbb{R}^n$ specifying the points where (the discrete evolution or) jumps are possible.
- *Jump map*: a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the value of the state after jumps.

A hybrid system with state $\xi \in \mathbb{R}^n$ is denoted by $\mathcal{H} = (C, f, D, G)$ and can be written in the compact form

$$\mathcal{H}: \qquad \xi \in \mathbb{R}^n \quad \begin{cases} \dot{\xi} = f(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D. \end{cases} \tag{4}$$

Using this framework, the evolution of the timers τ_i modeling the events is given, for each $i \in \{1, 2\}$, by

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2], \\ \tau_i^+ &\in [T_1, T_2] & \tau_i &= 0. \end{aligned} \tag{5}$$

The reset law of the timer state is set valued and it resets to any value within the interval $[T_1, T_2]$. Due to this fact, any sequence of points $\{t_k^i\}_{k=1}^\infty$ that satisfies (2) with γ^i given by any probability density function is captured by (5).

A model of the closed-loop bidirectional system, denoted $\mathcal{H}^b$, has state $\xi = (\tilde{x}_1, \tilde{x}_2)$ where, for each $i \in \{1, 2\}$, $\tilde{x}_i = (x_i, \eta_i, \tau_i) \in \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] =: S$. The flow and the jump sets are defined to constrain the evolution of the timers. From the model in (5), the controller dynamics in (3), and the LTI model given by (1), the flow set is defined by constraining the timer states to the interval $[0, T_2]$. Then, we have that the flow set and flow map are given by

$$C := \{\xi \in S^2 : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\}, \tag{6}$$

$$f(\xi) = (f_1(\xi), f_2(\xi)), \tag{7}$$

respectively, where $f_i(\xi) = (Ax_i + B\eta_i, f_{ci}(x_i), -1)$, for each $i \in \{1, 2\}$. The impulsive events are captured by the jump set D and the jump map G. Since jumps occur when either $\tau_1 = 0$ or $\tau_2 = 0$, the jump set is defined as $D := D_1 \cup D_2$, where

$$D_1 = \{\xi \in S^2 : \tau_1 = 0\}, \quad D_2 = \{\xi \in S^2 : \tau_2 = 0\}. \tag{8}$$

Consider the case of $\xi \in D_1 \setminus D_2$ (i.e., $\tau_1 = 0$ and $\tau_2 > 0$). Then, a jump is triggered such that $\tilde{x}_1$ is updated via G_1 and $\tilde{x}_2$ is mapped to itself. Likewise, when $\xi \in D_2 \setminus D_1$, $\tilde{x}_2$ is updated via G_2 and $\tilde{x}_1$ is mapped to itself. Such dynamics are captured by the jump map given by

$$G(x) := \begin{cases} \begin{bmatrix} G_1(\xi) \\ \tilde{x}_2 \end{bmatrix} & \text{if } \xi \in D_1 \setminus D_2 \\ \left\{ \begin{bmatrix} \tilde{x}_1 \\ G_2(\xi) \end{bmatrix}, \begin{bmatrix} G_1(\xi) \\ \tilde{x}_2 \end{bmatrix} \right\} & \text{if } \xi \in D_1 \cap D_2 \\ \begin{bmatrix} \tilde{x}_1 \\ G_2(\xi) \end{bmatrix} & \text{if } \xi \in D_2 \setminus D_1 \end{cases} \tag{9}$$

where $G_i(\xi) = (x_i, G_{ci}(x_i, x_\ell), [T_1, T_2])$, for each $i, \ell \in \{1, 2\}$, $i \neq \ell$. If both τ_1 and τ_2 expire simultaneously, then the second piece in the definition of the jump map indicates that either $\tilde{x}_1$ is reset or $\tilde{x}_2$ is reset. Due to the properties of the data of $\mathcal{H}^b$, we have the following result.

Lemma 2.1: If, for each $i \in \{1, 2\}$, $f_{ci} : \mathbb{R}^n \to \mathbb{R}^p$ is continuous for all $\xi \in C$ and $G_{ci} : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is outer semicontinuous, bounded and nonempty for every $\xi \in D$, then the hybrid system $\mathcal{H}$ with data given in (6) – (9) *satisfies the hybrid basic conditions.*[1]

A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if, for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as $\cup_{j=0}^J (I_j \times \{j\})$, where $I_j = [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_J \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps.

With a hybrid system having the data above, in particular the jumps triggered by the timer in (5), we immediately have the following result on solutions to $\mathcal{H}$.

Lemma 2.2: Given two positive scalars $0 < T_1 \leq T_2$, any maximal solution[2] *ϕ to $\mathcal{H}$ in* (4) *is such that for every $T > 0$, $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq T$, imply $j \geq \frac{T - T_2}{T_2 + 1}$.*

We recast asymptotic synchronization as a set stabilization problem, where we determine the stability of a set, which we label $\mathcal{A}$, enforcing that the appropriate state components of the solutions to the resulting closed-loop hybrid system are equal. For this purpose, we employ the following notion of asymptotic stability for general hybrid systems.

Definition 2.3: [16, Definition 3.6] Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be

- *uniformly globally stable (UGA)* for $\mathcal{H}$ if there exists a class-$\mathcal{K}_\infty$ function α such that any solution ϕ to $\mathcal{H}$ satisfies $|\phi(t, j)|_\mathcal{A} \leq \alpha(|\phi(0, 0)|_\mathcal{A})$ for all $(t, j) \in \operatorname{dom} \phi$;
- *uniformly globally attractive (UGS)* for $\mathcal{H}$ if every maximal solution to $\mathcal{H}$ is complete and for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_\mathcal{A} \leq r$, $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq T$ imply $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$;
- *uniformly globally asymptotically stable (UGAS)* for $\mathcal{H}$ if it is both UGA and UGS.

[1] See [16] for more information.

[2] A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded.

With an appropriate definition of the set $\mathcal{A}$, the notion in Definition 2.3 corresponds to a synchronization property for $\mathcal{H}$. At times, when such a property holds, we say that the hybrid system $\mathcal{H}$ asymptotically synchronizes.

III. Main Results

A. Cascade Topology

As a special case of the network structure considered in Section II-C, in this section, we address the case of a cascade topology of two systems. In this setting, information is transmitted only from system 1 to system 2, where system 1 is autonomous. Since the network is unidirectional, we use a single timer, denoted by τ, satisfying the dynamics in (5) to trigger the update of information. To adapt the hybrid system model in Section II-C to this setting, we remove the timer and the input to system 1. Then, the cascade topology is reduced to a hybrid system $\mathcal{H}^c = (C, f, D, G)$ with state $\xi = (x_1, x_2, \eta_2, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] =: S_1$ and the following data:

$$\begin{aligned} f(\xi) &:= \begin{bmatrix} Ax_1 \\ Ax_2 + B\eta_2 \\ f_{c2}(x_2) \\ -1 \end{bmatrix} \quad C := \{\xi \in S_1 : \tau \in [0, T_2]\} \\ G(\xi) &:= \begin{bmatrix} x_1 \\ x_2 \\ G_{c2}(x_2, x_1) \\ [T_1, T_2] \end{bmatrix} \quad D := \{\xi \in S_1 : \tau = 0\}. \end{aligned} \tag{10}$$

with f_{c2} and G_{c2} defining the hybrid controller during flows and jumps, respectively.

We consider the case of a sample-and-hold state-feedback controller. In this control structure, the controller state does not change during flows, i.e., $f_{c2}(x_2) = 0$, but is updated at jumps by the mapping $G_{c2}(x_2, x_1) = -K(x_2 - x_1)$. Following [17], we write the system in error coordinates $\chi = (\varepsilon, \eta_2, \tau) \in \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] =: \widetilde{S}_1$, where $\varepsilon = x_1 - x_2$. The resulting hybrid system is denoted as $\mathcal{H}^c_\varepsilon$ with data $(\widetilde{C}, \tilde{f}, \widetilde{D}, \widetilde{G})$. During flows, it follows that the error dynamics are given by $\dot{\varepsilon} = A\varepsilon - B\eta_2$. Note that since the dynamics of the timer state did not change from that of (5), the flow set is given by $\widetilde{C} := \{\chi \in \widetilde{S}_1 : \tau \in [0, T_2]\}$. If $\tau = 0$, then a jump is triggered, which, according to the error dynamics, leads to $\varepsilon^+ = \varepsilon$. Furthermore, due to the fact that x_1 and x_2 appear in G_{c2} as a function of $x_2 - x_1$ only, we have that $\eta_2^+ = K\varepsilon$. The jump set is $\widetilde{D} := \{\chi \in \widetilde{S}_1 : \tau = 0\}$. This change of coordinates leads to

$$\mathcal{H}^c_\varepsilon : \begin{cases} \dot{\chi} = \begin{bmatrix} A\varepsilon - B\eta_2 \\ 0 \\ -1 \end{bmatrix} =: \tilde{f}(\chi) & \chi \in \widetilde{C}, \\ \chi^+ \in \begin{bmatrix} \varepsilon \\ K\varepsilon \\ [T_1, T_2] \end{bmatrix} =: \widetilde{G}(\chi) & \chi \in \widetilde{D} \end{cases} \tag{11}$$

Note that this system is independent of the dynamics of the hybrid system $\mathcal{H}^c$ defined by the data in (10). Due to the change of variables, we study synchronization by determining the stability of the set given by

$$\mathcal{A} := \{\chi \in \widetilde{S}_1 : \varepsilon = 0, \tau \in [0, T_2]\} \tag{12}$$

for the hybrid system $\mathcal{H}^c_\varepsilon$ in (11).

Following [16, Example 3.21], we partition the state as $\chi = (z, \tau)$, where $z = (\varepsilon, \eta_2)$. Then, the flow map and jump map of $\mathcal{H}_\varepsilon$ are defined by

$$\tilde{f}(\chi) = \begin{bmatrix} A_f z \\ -1 \end{bmatrix} \qquad \widetilde{G}(\chi) = \begin{bmatrix} A_g z \\ [T_1, T_2] \end{bmatrix}, \tag{13}$$

respectively, where

$$A_f = \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} \qquad A_g = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}. \tag{14}$$

Due to the definition of the controller, in fact, we want to determine the stability of the set $\{\chi \in \widetilde{S}_1 : z = 0, \tau \in [0, T_2]\}$, which is equivalent to $\mathcal{A}$ in (12). The following result employs sufficient conditions for stability of hybrid systems in [16] to determine the stability of $\mathcal{A}$ for $\mathcal{H}_\varepsilon$. Moreover, it characterizes the rate of convergence via a class-$\mathcal{KL}$ function.

Theorem 3.1: Given two positive scalars $T_1 \leq T_2$, if there exists a positive definite symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and a matrix $K \in \mathbb{R}^{p\times n}$ such that

$$A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0 \qquad \forall \nu \in [T_1, T_2], \tag{15}$$

then the set $\mathcal{A}$ is UGAS for the hybrid system (11). Furthermore, every maximal solution ϕ to $\mathcal{H}_\varepsilon$ satisfies

$$|\phi(t,j)|_\mathcal{A} \leq \sqrt{\frac{\overline{c}}{\underline{c}}} e^{\theta j/2} |\phi(0,0)|_\mathcal{A}$$

for every $(t, j) \in \operatorname{dom} \phi$, where $\theta = \ln(1 - \beta/\underline{c})$,

$$\beta \leq \min_{\tau \in [T_1, T_2]} |A_g^\top e^{A_f^\top \tau} P e^{A_f \tau} A_g - P|,$$

$$\underline{c} = \min_{\tau \in [0, T_2]} \lambda_{\min}(e^{A_f^\top \tau} P e^{A_f \tau}), \overline{c} = \max_{\tau \in [0, T_2]} \lambda_{\max}(e^{A_f^\top \tau} P e^{A_f \tau}).$$

Remark 3.2: Note that condition (15) is akin to the discrete Lyapunov equation with system matrix $H(\nu) = e^{A_f \nu} A_g$. Furthermore, condition (15) is satisfied if the eigenvalues of $H(\nu)$ are within the unit circle for all $\nu \in [T_1, T_2]$.

Remark 3.3: The form of condition (15) may be difficult to satisfy numerically. In fact, this condition is not convex in K and P, and needs to be verified for infinitely many values of ν. In [17], the authors use a polytopic embedding strategy, in which, one needs to find some matrices X_i such that the exponential matrix is an element in the convex hull of the X_i matrices to solve an linear matrix inequality. These results can be adapted to our setting.

Remark 3.4: When full state measurements are not available, but rather, the output $y_i = Mx_i$ is measured, the jump map would have a second component given by $\widetilde{G}_{c2}(\varepsilon) = KM\varepsilon$. Following Lemma 3.1, given $T_1 \leq T_2$, if there exist K and $P = P^\top > 0$ of appropriate dimensions such that (15) holds with $A_g = \begin{bmatrix} I & 0 \\ KM & 0 \end{bmatrix}$, then $\mathcal{A}$ is UGAS.

B. Bidirectional communication with a sample-and-hold controller

Following the construction in Section II-C, we design a sample-and-hold state-feedback controller with $f_{ci}(x_i) = 0$ and $G_{ci}(x_i, x_\ell) = K(x_i - x_\ell)$ for each $i, \ell \in \{1, 2\}, i \neq \ell$. We rewrite the closed-loop hybrid system, denoted by $\mathcal{H}^b_\varepsilon = (\widetilde{C}, \tilde{f}, \widetilde{D}, \widetilde{G})$, in the coordinates $\chi = (\chi_1, \chi_2)$, where $\chi_i = (z_i, \tau_i) \in \mathbb{R}^{n+p} \times [0, T_2]$ and $z_i = (\varepsilon_{i\ell}, \eta_i)$ with $\varepsilon_{i\ell}$ being the error quantity $\varepsilon_{i\ell} := x_i - x_\ell$ for each $i, \ell \in \{1, 2\}, i \neq \ell$. Then, for each $i \in \{1, 2\}$, the continuous dynamics of each z_i are given by

$$\dot{z}_i = A_f z_i + B_f z_\ell \tag{16}$$

for all χ in $C := \{\chi \in S : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\}$, while at jumps (i.e., $\chi \in D$), z_i is updated to $A_g z_i$ when $\tau_i = 0$ and is updated to z_i otherwise, where $A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, A_g = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}$ and $B_f = \begin{bmatrix} 0 & -B \\ 0 & 0 \end{bmatrix}$; this update law is properly implemented by $\widetilde{G}$ below. From the above continuous and discrete dynamics, the closed-loop hybrid system, which we denote $\mathcal{H}^b_\varepsilon$, has the following data:

$$\begin{aligned} \tilde{f}(\chi) &= (f_1(\chi), f_2(\chi)) \\ \widetilde{C} &:= \{\chi \in S : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\} \end{aligned} \tag{17}$$

where, for each $i, \ell \in \{1, 2\}, i \neq \ell$, the local flow map is given by $f_i(\chi) = (A_f z_i + B_f z_\ell, -1)$. The jump map is

$$\widetilde{G}(\chi) = \begin{cases} \begin{bmatrix} \widetilde{G}_1(\chi) \\ \chi_2 \end{bmatrix} & \text{if } \chi \in \widetilde{D}_1 \setminus \widetilde{D}_2 \\ \left\{ \begin{bmatrix} \widetilde{G}_1(\chi) \\ \chi_2 \end{bmatrix}, \begin{bmatrix} \chi_1 \\ \widetilde{G}_2(\chi) \end{bmatrix} \right\} & \text{if } \chi \in \widetilde{D}_1 \cap \widetilde{D}_2 \\ \begin{bmatrix} \chi_1 \\ \widetilde{G}_2(\chi) \end{bmatrix} & \text{if } \chi \in \widetilde{D}_2 \setminus \widetilde{D}_1 \end{cases} \tag{18}$$
$$\widetilde{D} := \widetilde{D}_1 \cup \widetilde{D}_2, \qquad \widetilde{D}_i := \{\chi \in S : \tau_i = 0\}$$

with $\widetilde{G}_i(\chi) = (A_g z_i, [T_1, T_2])$ for each $i \in \{1, 2\}$. Then, we have the following result.

Theorem 3.5: Let T_1 and T_2 be two positive scalars such that $T_1 \leq T_2$. Suppose there exist $\sigma, \epsilon, \beta > 0$, a matrix $K \in \mathbb{R}^{n \times p}$, and a positive symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ satisfying $\beta < \underline{c}$,

$$1 - \frac{\beta}{\overline{c}} < \exp\left(-T_2\left(\frac{\rho\epsilon}{\underline{c}} - \sigma + \frac{1}{\epsilon}\right)\right),$$

and

$$e^{\sigma\nu} A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0$$

for each $\nu \in [T_1, T_2]$, where

$$\begin{aligned} \beta &\leq \min_{\nu \in [T_1, T_2]} |e^{\sigma\nu} A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P| \\ \overline{c} &= e^{\sigma T_2} \max_{\tau \in [0, T_2]} \lambda_{\max}(e^{A_f^\top \tau} P e^{A_f \tau}) \\ \underline{c} &= \min_{\tau \in [0, T_2]} \lambda_{\min}(e^{A_f^\top \tau} P e^{A_f \tau}) \\ \rho &= \exp(\sigma T_2) \max_{\tau \in [0, T_2]} \left| B^\top e^{A^\top \tau} P_{11} e^{A\tau} B \right| \end{aligned}$$

Then, the set $\mathcal{A} = \{z \in S : z_i = 0, \tau_i \in [0, T_2], i \in \{1, 2\}\}$ is UGAS for the hybrid system $\mathcal{H}^b_\varepsilon$ with data given by (17) *and* (18).

Sketch of Proof: Consider the function $V_i(\tilde{z}_i) := e^{\sigma\tau_i} z_i^\top e^{A_f^\top \tau_i} P e^{A_f \tau_i} z_i$, $\sigma > 0$. Due to the construction of the set $\mathcal{A}$, we have that $|\tilde{z}_i|_\mathcal{A}$ is equivalent to $|z_i|$ and $\underline{c}|\tilde{z}_i|_\mathcal{A} \leq V_i(\tilde{z}_i) \leq \overline{c}|\tilde{z}_i|_\mathcal{A}$, where $\underline{c}$ is defined above and $\overline{c} = e^{\sigma T_2} \max_{\tau \in [0, T_2]} \lambda_{\max}(e^{A_f^\top} P e^{A_f \tau})$. During flows, the change in V_i is $\langle \nabla V_i(\tilde{z}_i), f_i(\chi) \rangle \leq -(\sigma - \frac{1}{\epsilon}) V_i(\tilde{z}_i) + \frac{\rho}{\underline{c}} V_\ell(\tilde{z}_\ell)$ ($\ell \neq i$), which may not be negative for some $\chi \in \widetilde{C}$. During jumps we have that $\chi \in \widetilde{D}$ is such that $\tau_i = 0$, and after the jump we have that for each $(g_1, g_2) \in \widetilde{G}(z)$, $V_i(g_i) - V_i(\tilde{z}_i) \leq -\frac{\beta}{\overline{c}} V_i(\tilde{z}_i) < 0$. Then, we define $V(\chi) = V_1(\tilde{z}_1) + V_2(\tilde{z}_2)$ and study the change of V over the worst case solution that allows a maximum amount of flow time between jumps. Due to the assumptions and by using standard bounding techniques, it can be shown that V decreases after T_2 units of flow time, which corresponds to the worst case. ■

C. Robustness properties of the closed-loop systems

In a realistic setting, the information transmitted would be subjected to some amount of noise. In this section, we consider system in Section III-A under the effect of measurement noise m_i induced by perturbations in the communication network between system 1 and system 2. Specifically, the j-th system receives the state of the i-th system perturbed by $m_i \in \mathbb{R}^n$, i.e., consider the information provided by the i-th system to be given by $y_i = x_i + m_i$. In such a case, the update law of the controller state in (11) becomes $\eta_2^+ = K\varepsilon + K(m_1 - m_2)$. Note that the larger the controller gain the higher the amplification of the noise. Inspired by [17], we show that the hybrid system in Section III-A is input-to-state (ISS) stable with respect to noise measurements m_i (a similar result holds for $\mathcal{H}^b_\varepsilon$ in Section III-B). For a notion of ISS for hybrid systems, see [19].

Theorem 3.6: Given two positive scalars $T_1 \leq T_2$, if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and a matrix $K \in \mathbb{R}^{p \times n}$ satisfying (15)*, then the hybrid system $\mathcal{H}^c_\varepsilon$ in* (11) *with $\eta_2^+ = K\varepsilon + K(m_1 - m_2)$ is ISS with respect to $m_1 - m_2$ relative to the set $\mathcal{A}$ in* (12)*.*

Remark 3.7: Since the hybrid system $\mathcal{H}^b_\varepsilon$ in Section III-A and $\mathcal{H}^c_\varepsilon$ in Section III-B satisfy the hybrid basic conditions and have a compact set $\mathcal{A}$ UGAS, the stability of $\mathcal{A}$ is robust to general perturbations. See [16] for more information.

IV. Examples

A. Linear oscillator over a feedback topology with bidirectional communication

Now, we revisit the harmonic oscillator example in Section II-A with the same time interval $T_1 = 0.5$ and $T_2 = 1$. From Theorem 3.5, we generate a controller matrix $K = [-0.5, -0.7]$, a positive definite symmetric matrix $P \approx \begin{bmatrix} 20.4 & 2.02 & 3.56 \\ \star & 7.11 & -0.38 \\ \star & \star & 3.56 \end{bmatrix}$, $\beta = 2.6965$,

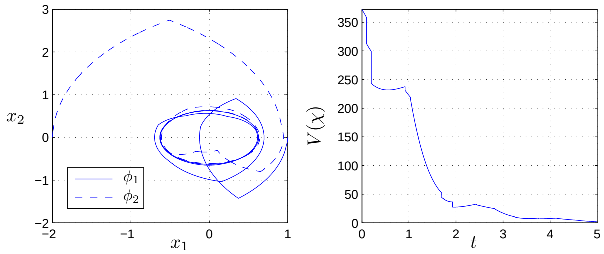


Fig. 2. Solutions to a coupled harmonic oscillator system on the bidirectional topology from $(x_1(0,0), \eta_1(0,0), \tau_1(0,0)) = (1,0,0,0.2)$ $(x_2(0,0), \eta_2(0,0), \tau_2(0,0)) = (-2,0,0,0.1)$.

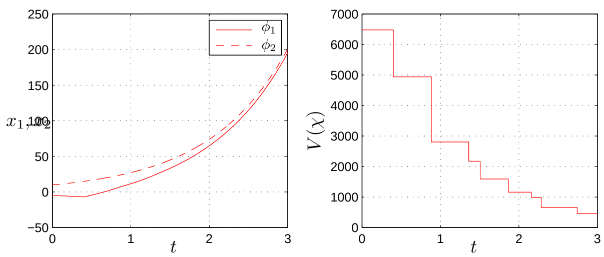


Fig. 3. Solutions of an unstable first order system on the cascade network topology from $x_1(0,0) = -5$, $(x_2(0,0), \eta_2(0,0)) = (10,0)$ and $\tau(0,0) = 0.4$.

$\epsilon = 3$, $\sigma = 0.4$. Figure 2 shows a numerical solution of the hybrid system $\mathcal{H}^b$ in Section II-C from initial conditions $(x_1(0,0), \eta_1(0,0), \tau_1(0,0)) = (1,0,0,0.2)$, $(x_2(0,0), \eta_2(0,0), \tau_2(0,0)) = (-2,0,0,0.1)$. In this figure, we have (left) an $x_1 - x_2$ planar view of the original states x_1 and x_2 and (right) the Lyapunov function V along the solution (based on the error coordinates in Section III-B), where $V(\chi) = \sum_{i=1}^{2} e^{\sigma\tau_i} z_i^\top e^{A_f^\top \tau_i} P e^{A_f \tau_i} z_i$. Notice, while at times, that the Lyapunov function increases due to the injection term in (16), the controller designed for this system asymptotically synchronizes the systems unlike the controller used in Section II-A (see Figure 1(b)).

B. Unstable first order system on the cascade topology

Consider the unstable LTI system given by $\dot{x}_i = x_i + \eta_i$ operating on a cascade network topology with $T_1 = 0.1$, $T_2 = 0.5$. By Theorem 3.1, we find $K = -1.4$ and the positive definite symmetric matrix given by $P \approx \begin{bmatrix} 13.60 & 0.93 \\ \star & 2.35 \end{bmatrix}$. Figure 3 shows a numerical simulation of the hybrid system $\mathcal{H}^c$ given by the data in (10) with state $\xi = (x_1, x_2, \eta_2, \tau)$ from $\xi(0,0) = (-5, 10, 0, 0.4)$. In this figure, we have (left) a plot of each x_i versus the hybrid flow time parameter t. The right figure shows the Lyapunov function V evaluated along the solution, which is constant during flows and strictly decreases at jumps.

V. Conclusion

In this paper, we showed that hybrid sample-and-hold state-feedback controllers are viable algorithms for synchronizing two LTI systems with stochastic transmission events. The communication network between the systems was modeled by a decreasing timer that is reset to some point in a bounded interval, which allowed us to allow for arbitrary probability distributions triggering the transmission events. Recasting synchronization as the stabilization of a compact set, Lyapunov functions were constructed to certify asymptotic stability of this set, implying that the networked systems synchronize asymptotically.

The results in this paper can be used to design large-scale networked systems that communicate at stochastically instants. Future directions of research include the study of such systems with networks defined by general graphs and nonlinear dynamics.

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