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SPECTRAL DISSECTION OF FINITE RANK PERTURBATIONS OF NORMAL OPERATORS

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To Dan Virgil Voiculescu on the occasion of his seventieth birthday

Abstract. Finite rank perturbations $T = N + K$ of a bounded normal operator N on a separable Hilbert space are studied thanks to a natural functional model of T; in its turn the functional model solely relies on a perturbation matrix/ characteristic function previously defined by the second author. Function theoretic features of this perturbation matrix encode in a closed-form the spectral behavior of T. Under mild geometric conditions on the spectral measure of N and some smoothness constraints on K we show that the operator T admits invariant subspaces, or even it is decomposable.

1. INTRODUCTION

Finite rank perturbations of Hilbert space operators were studied for at least a century, for instance for their far reaching connections with function theory of a complex variable, boundary value problems of mathematical physics or for applications to quantum theory. It is sufficient to mention the case of dissipative operators with rank-one imaginary part or the celebrated phase-shift and related perturbation determinant, see [22] for the golden era references, or [44] for more recent developments. The booming topics of Aleksandrov-Clark measures [38] and the resurrection of Aronszjan-Donoghue theory for matrix valued measures associated to finite rank perturbations of self-adjoint operators [33, 21] are two other notable examples. To name only one less known, additional relevant ramification: an apparently non-related open problem of approximation theory, known as Sendov conjecture, can be translated into spectral estimates of rank-two perturbations of normal matrices, see [28].

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Let $\mathcal{H}$ be a complex, separable Hilbert space. The class of operators under study is

$$T = N + \sum_{k=1}^{m} u_j \otimes v_j = N + \sum_{k=1}^{m} u_j \langle \cdot, v_j \rangle, \tag{1}$$

where N is a normal operator on $\mathcal{H}$ and the finite rank summand is subjected to certain "smoothness" conditions. To fix ideas we assume first that N has spectral multiplicity 1. According to the Spectral Theorem, N is unitarily equivalent to the multiplication operator by the complex variable:

$$Nf(z) = zf(z) \quad \text{on} \quad \mathcal{H} = L^2(\mu).$$

In this functional model setting the announced "smoothness" properties of the functions u_j, v_j will be quite natural, for instance imposing their uniform boundedness. The case of general spectral multiplicity of N is covered by our main results, with notation and conventions described in Section 3.

The second author has already treated smooth trace class perturbations in the case of a measure absolutely continuous with respect to area measure [49]. A natural quotient model of T, defined in terms of certain Sobolev-type spaces was introduced there. The present article builds on [49] by proposing a quotient functional model for a wide class of spectral measures μ. The concrete function theoretic features of the quotient model provide the announced spectral decomposition results. In the present article we only treat finite rank perturbations due to inerent technical complications.

To give a preview of the nature of the functional model we propose in this work, we consider the simplest case

$$T = M_z + u(z)\langle \cdot, v \rangle : L^2(\mu) \longrightarrow L^2(\mu),$$

with bounded measurable functions u, v defined on the support σ of the positive measure μ. Let Ω be a domain with smooth boundary containing σ, so that the perturbation function

$$\psi(z) = 1 + \int \frac{u(w)\overline{v(w)}d\mu(w)}{w - z},$$

does not vanish outside $\bar{\Omega}$. Define the model space $\mathrm{Mod}(\Omega)$ of Cauchy transforms plus analytic functions

$$\int \frac{x(w)\overline{v(w)}d\mu(w)}{w - z} + b(z), \ z \in \Omega,$$

where $x \in L^2(\mu)$ and $b(z)$ is analytic in the associated Hardy space $H^2(\Omega)$. This space is complete with respect to the Sobolev type norm

$$\|x\|_{2,\mu}^2 + \|b\|_{2,\partial\Omega}^2.$$

The reader can immediately verify that the operator

$$x \mapsto \int \frac{x(w)\overline{v(w)}d\mu(w)}{w - z}$$

intertwines T and multiplication by the complex variable on the quotient space $\mathrm{Mod}(\Omega)/\psi\mathrm{Mod}(\Omega)$. This precise similarity transform allows us to infer spectral decomposition properties of T from function and measure theoretic results.

Our inquiry is affiliated to a series of recent developments. For instance, in a series of papers [18]–[20], Foiaş, Jung, Ko and Pearcy considered rank one perturbations of normal operators with a discrete spectral measure μ. Their technique of cutting the spectrum of the perturbation relied on carefully chosen integration contours applied to localized resolvent and producing appropriate Riesz projections. Later, their results were extended by Fang and Xia [16], and by Klaja [29] to finite rank and compact perturbations. Klaja also has some results [29] for a non-discrete measure μ, provided there is a Jordan curve Γ with certain continuity and growth conditions on the local resolvents $(N-\lambda)^{-1}u_j$ and $(N^*-\bar{\lambda})^{-1}v_j$, where $\lambda \in \Gamma$.

An earlier work on the same subject is [26]. And even before, studies of similar flavor go back to [9, 43]; see also the references in [49].

In [30], Klaja proves that if N is diagonalizable and its spectrum is a perfect set, then N possesses a rank one perturbation without eigenvalues. If N is compact and self-adjoint, this is no longer true in general; the criterion for the existence of rank one perturbation of this type was given in [4].

On a related, but totally different line of research, weakly convergent integrals of localized resolvents of operators belonging to a II_1 factor appeared in the works of Haagerup and Schultz [24] and followers [14]. These integrals provided Riesz type projections and ultimately unveiled a rich spectral decomposition behavior.

Rank-one perturbations of normal operators were intensively studied during the last years by Baranov and his collaborators (see [1, 3]), in particular, by Baranov and the second author [4]–[6]. These works make use of a de Branges type functional model and heavily rely on the theory of entire functions.

By far from being exhaustive, more references to old and new analyses of finite rank perturbations of linear transforms are scattered through the present article.

Our aims are:

- To find conditions for possibly "dissecting" the spectrum of T along a curve. Given a domain Ω containing $\sigma(T)$, such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, the domains Ω_1 and Ω_2 have no common points and have piecewise smooth boundaries, our Theorem 3.9 says that under additional conditions, there is a direct sum decomposition $T = T_1 \dotplus T_2$, where $\sigma(T_j) \subset \bar{\Omega}_j$.
- To deduce from this fact sufficient conditions for the existence of invariant subspaces.
- To give sufficient conditions for the decomposability of T in the sense of Foiaş [17].

While we do not seek in the present article the most general conditions imposed on the spectral measure μ and/or the finite rank perturbations, our approach gives rise to challenging questions in geometric function theory. Some of these problems are explicitly formulated throughout the text and in the last section.

The contents is the following. Section 2 contains some preliminaries of geometric measure theory and function theory of a complex variable. There we introduce the concept of dissectible Borel measure to become the key technical tool for the rest of the article. Section 3 is devoted to the construction of the Sobolev space type functional model for the perturbed operator. Our main spectral dissection theorem is stated in Section 3. In section 4 we provide proofs for the geometric measure theory lemmas and in Section 5 we turn to the proofs involving the functional model. Section 6 is focused on the existence criterion of a non-trivial invariant subspace for the perturbed operator. In Section 7 we recall some basic terminology and facts from local spectral theory; there we propose the notion of *dissectible operator*, slightly stronger than decomposable operator. In the same section we provide sufficient criteria for the perturbed operator to be either dissectible, or only to possess Bishop's property (β). In Section 8 we formulate a few open questions.

We dedicate this article to Dan Virgil Voiculescu whose exceptional insight into perturbation theory of linear operators has reformed modern mathematical analysis. His contributions to the subject span more than four decades, to cite only [47, 48].

2. PRELIMINARIES

This section collects a few function and measure theoretic results referred to in the sequel. We also introduce some necessary notation and terminology.

Henceforth we rely on the standard dyadic system $\mathcal{D}$ of squares in $\mathbb{C}$ with sides parallel to coordinate axes. The kth generation $\mathcal{D}_k$ consists of squares of the form $[m2^{-k}, (m+1)2^{-k}) \times [n2^{-k}, (n+1)2^{-k})$; here $k, m, n \in \mathbb{Z}$. Then

$$\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k.$$

A square Q is dyadic if $Q \in \mathcal{D}_k$ for some k; in this case we set $|Q| = 2^{-k}$ to be its side length.

For any dyadic $Q \in \mathcal{D}$ and any $c > 0$, we denote by $\overline{Q}$ the closure of Q, by Q^o its interior and by cQ the square homothetic to Q with the same center, such that $|cQ| = c|Q|$.

We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and by $\mathcal{H}^s_{\text{cont}}$ the s-dimensional Hausdorff content, calculated with respect to the dyadic system $\mathcal{D}$.

Definition 2.1. Throughout this article h stands for a nonnegative increasing function defined on $[0, +\infty)$ satisfying $h(t) > 0$ for $t > 0$, $\lim_\infty h = +\infty$,

and

$$\int_0^1 t^{-2} h(t)\, dt < \infty. \tag{2}$$

Such a function h is called *a scale function*.

The main example is

$$h(t) = t^a,$$

where $1 < a \leq 2$. To capture finer effects (related with measures "close" to the arc length measure in the sense of dimension), one can also consider functions of the form $h(t) = t \cdot \log^{-a}(2 + 1/t)$, $a > 1$.

Definition 2.2. Let ν be a finite positive measure on $\mathbb{C}$ with compact support.

1) We say that a dyadic square Q is *light (w.r. to ν)* if

$$\nu(Q) \leq h(|Q|),$$

and *heavy* in the opposite case.

2) Given a measure ν, consider the increasing family of open sets

$$\mathcal{G}_s(\nu) = \bigcup_{Q \text{ dyadic and heavy, } |Q| \leq s} 10Q^o, \quad \text{where } s > 0. \tag{3}$$

We say that ν is *dissectible* if

$$\lim_{s \to 0^+} \mathcal{H}^1_{\text{cont}}(\mathcal{G}_s(\nu)) = 0.$$

In this case, we will also use the terms *h-dissectible*, or *a-dissectible*, for the case of $h(t) = t^a$.

Notice that condition

$$\sum_{Q \text{ dyadic and heavy}} |Q| < \infty$$

is sufficient for ν to be h-dissectible.

We remark that for any finite measure ν, each sufficiently large square in $\mathcal{D}$ is light. This implies the following observation. Suppose ν is a h-dissectible measure as above. Choose any function h_1 meeting the same conditions as h and such that $h_1(t)/h(t) \to \infty$ as $t \to 0^+$. Then for any nonnegative function $f \in L^\infty(\nu)$, $f\nu$ is h_1-dissectible.

In what follows, we set

$$\nu = \big(\sum_j |u_j|^2 + |v_j|^2\big)\mu. \tag{4}$$

Most of our results require the following condition:

(D) **The measure ν is dissectible.**

Given a complex measure τ of finite total variation, we put

$$C(\tau)(z) := \int \frac{d\tau(t)}{t-z} \tag{5}$$

to be its Cauchy transform. It is known that it is defined at least as a function in $L^1_{\rm loc}(\mathbb{C})$.

As it was shown by Verdera [46] and by Mattila and Melnikov [34], for any measure ν and any Ahlfors-David regular curve γ, the Cauchy integral $C(\nu)$ exists a.e. on γ and satisfies a weak L^1 estimate on γ. In our constructions, we will need to know a better regularity of the Cauchy integrals $C(f\nu)$. In particular, we will use the following lemma, which shows the importance of dissectible measures in our study.

Lemma 2.3. *For any positive measure ν, any $s \in (0,1]$ and any $f \in L^\infty(\nu)$, one has*

$$|C(f\nu)(z)| \le K s^{-1} \|f\|_\infty, \quad z \in \mathbb{C} \setminus \mathcal{G}_s(\nu), \tag{6}$$

where K is a constant only depending on ν. Moreover, $C(f\nu)$ is a continuous function on $\mathbb{C} \setminus \mathcal{G}_s(\nu)$.

In particular, if ν is a dissectible measure, then $C(f\nu)$ is continuous on sets with "small" complement.

If Ω is a Jordan domain with piecewise C^1 smooth boundary, we will use the abbreviation

$$C_{\partial\Omega}(w) := C(w \cdot dz\big|_{\partial\Omega}),$$

(here dz corresponds to the positive orientation of $\partial\Omega$). We adopt the notation $L^p(\partial\Omega) = L^p(\partial\Omega, |dz|)$.

If Ω is a Jordan domain in the complex plane, by its *exhaustion* we will mean an increasing sequence of Jordan domains Ω_n with rectifiable boundaries such that $\cup_n \Omega_n = \Omega$. For $1 \le p < \infty$, $E^p(\Omega)$ will denote the Smirnov space of analytic functions in Ω. We recall that a function f, holomorphic in Ω, belongs to $E^p(\Omega)$ if there is an exhaustion $\{\Omega_n\}$ of Ω such that

$$\sup_n \int_{\partial\Omega_n} |f(z)|^p \, |dz| < \infty. \tag{7}$$

These are Banach spaces for any p as above; for $p = 2$, they are Hilbert spaces. The space $E^\infty(\Omega) = H^\infty(\Omega)$ is just the space of bounded analytic functions on Ω.

Let us recall some of their basic properties (see [13, 52]). Denote by $K(\partial\Omega)$ *the David constant* of the curve $\partial\Omega$, that is, the least constant such that $\mathcal{H}^1(\partial\Omega \cap B(z,r)) \le Kr$ for all disks $B(z,r)$ in the plane. If $\partial\Omega$ is sufficiently good (say, locally Lipschitz), then one can use the same exhaustion $\{\Omega_n\}$ in (7) for all functions f. Namely, take any exhaustion $\{\Omega_n\}$ of Ω with uniformly bounded David constants $K(\partial\Omega_n)$ (such a sequence of domains always exists). Then a function f, holomorphic in Ω, is in $E^p(\Omega)$ if and only if (7) holds for this sequence of domains.

Let $\Omega^c = \widehat{\mathbb{C}} \setminus \bar{\Omega}$ be the complementary domain, where $\widehat{\mathbb{C}}$ is the Riemann sphere. For a function f in $E^2(\Omega)$ or in $E^2(\Omega^c)$, its boundary values on $\partial\Omega$ are well-defined as a function in $L^2(\partial\Omega) = L^2(\partial\Omega, |dz|)$. In this sense, the spaces $E^2(\Omega)$ and $E^2(\Omega^c)$ can be identified with closed subspaces of $L^2(\partial\Omega)$. The direct sum decomposition

$$L^2(\partial\Omega) = E^2(\Omega) \dotplus E^2_0(\Omega^c) \tag{8}$$

holds; here $E^2_0(\Omega^c) = \{f \in E^2(\Omega^c) : f(\infty) = 0\}$. In general the above direct sum is not orthogonal. However, the parallel projections onto the two direct summands in (8) are given by the linear bounded transformations $f \mapsto C_{\partial\Omega} f\big|_{\Omega}$ and $f \mapsto -C_{\partial\Omega} f\big|_{\Omega^c}$.

Definition 2.4. Given a measure ν on $\mathbb{C}$ and a subset $E \subset \mathbb{C}$, define the corresponding *Carleson constant* of E with respect to ν by

$$\mathrm{Carleson}(E, \nu) := \sup_{z \in E, r > 0} \frac{\nu(B(z, r))}{r},$$

where $B(z, r)$ is the open disc in $\mathbb{C}$ of radius r, centered at z.

Suppose Ω is a domain such that $\nu(\partial\Omega) = 0$. We will use *the Carleson embedding operator*

$$J_\Omega : E^2(\Omega) \to L^2(\nu), \quad J_\Omega f := f. \tag{9}$$

It is known [13, 52] that both operators J_Ω and J_{Ω^c} are bounded if and only if the Carleson constant of the boundary $\partial\Omega$ with respect to ν is bounded.

Definition 2.5. A Jordan curve γ is called *admissible* (with respect to ν) if it satisfies the following conditions:

(1) γ is a piecewise C^1 smooth curve;
(2) Let Ω and Ω^c be the two connected components of $\widehat{\mathbb{C}} \setminus \gamma$. There is an exhaustion $\{\Omega_n\}$ of the domain Ω such that all the curves γ and $\partial\Omega_n$ are contained in $\mathbb{C} \setminus \mathcal{G}_s(\mu)$ for some (fixed) $s > 0$ and the David constants $K(\partial\Omega_n)$ and the Carleson constants $\mathrm{Carleson}(\partial\Omega_n, \nu)$ are uniformly bounded. The same property holds for Ω^c.

If γ is admissible, we simply call Ω and Ω^c admissible domains.

Remark 2.6. It is easy to see that $\nu(\gamma) = 0$ whenever the Jordan curve γ does not intersect $\mathcal{G}_s(\nu)$ for some $s > 0$.

Lemma 2.7. *There are subsets $\mathbb{A}_x, \mathbb{A}_y$ of the real line such that $\mathcal{H}^1(\mathbb{R} \setminus \mathbb{A}_x) = \mathcal{H}^1(\mathbb{R} \setminus \mathbb{A}_y) = 0$ possessing the following properties:*

1) Any Jordan broken line γ is admissible whenever it consists of finitely many intervals parallel to the axes and its vertices belong to $\mathbb{A}_x + i\mathbb{A}_y$.

2) For any $x_0 \in \mathbb{A}_x$, there is an increasing sequence $\{x_n^-\}$ and a decreasing sequence $\{x_n^+\}$ both tending to x_0 and such that for some $s > 0$, all the points x_0, x_n^- and x_n^+ belong to $\mathbb{A}_x \setminus \mathrm{Re}\,\mathcal{G}_s(\mu)$. A similar approximation property hold for the set $\mathbb{A}_y$ (with $\mathbb{A}_y \setminus \mathrm{Im}\,\mathcal{G}_s(\mu)$ replacing $\mathbb{A}_x \setminus \mathrm{Re}\,\mathcal{G}_s(\mu)$).

Examples 2.8. 1) Let μ be the area measure restricted to some bounded Borel subset of $\mathbb{C}$. Let $d\nu = u\,d\mu$, where $u \in L^p(\mu)$. Let $1/p + 1/q = 1$. If $p > 2$, then by Hölder's inequality, any dyadic square is light:

$$\nu(Q) \le \mu(Q)^{1/q}\left(\int_Q |u|^p\,d\mu\right)^{1/p} \le \|u\|_p |Q|^{2/q},$$

and $a = 2/q > 1$. So each measure ν of this form is a-dissectible (as it was mentioned, the constant $\|u\|_p$ does not matter).

2) Similarly, assume that μ satisfies an estimate $\mu(Q) \le C|Q|^\sigma$ for any square Q, where $\sigma \in [1, 2)$ and C are constants.

Choose $q \in (1, \sigma)$ and the corresponding $p \in (1, \infty)$. Put $d\nu = u\,d\mu$, where $u \in L^p(\mu)$, as in the previous example. Then we derive in the same way that ν is a-dissectible for $a = \sigma/q$.

So, if $u \in L^p(\mu)$, we find that ν is a-dissectible for $a = \sigma/q > 1$.

3) On the opposite side, if the closed support $\operatorname{supp}\mu$ has Minkowski dimension less than 1, then it is easy to see that μ is a-dissectible for any $a > 1$ (a square can be heavy only if it touches $\operatorname{supp}\mu$).

4) Of course, the closed support is a very rough characteristic of "the dimension" of μ. Suppose, however, that measures μ_k, $k \ge 0$, are mutually singular and h_k-dissectible, for some scale functions h_k. Suppose $\sum \mu_k(\mathbb{C})$ is finite. Put $\mu = \sum \mu_k$. This measure can fail to be dissectible, but there is an equivalent measure to μ that is h-dissectible for some h. Namely, the following Proposition holds.

Proposition 2.9. *Suppose that measures μ_k are h_k-dissectible for some scale functions h_k, and suppose that the union of the closed supports of these measures is a bounded subset of $\mathbb{C}$. Then there are (small) positive constants c_k such that the measure $\sum_k c_k\mu_k$ is finite and dissectible with respect to some scale function h.*

5) In particular, for any atomic measure $\mu = \sum \delta_{t_k}$ there is an equivalent measure $\nu = \sum c_k\delta_{t_k}$, which is dissectible. This can be related with the techniques proposed by Foiaş and collaborators [18]–[20].

The proofs of Lemmas 2.3, 2.7 and Proposition 2.9 will be given in Section 4.

Remark 2.10. A measure ν is not h-dissectible (for any scale function h) whenever for some Ahlfors-David curve γ, $\nu|\gamma$ has a nontrivial part, absolutely continuous with respect to the arc length. This can be deduced from Lemma 2.3.

One can infer from the above examples that measures of "dimension" 1 are the most difficult case, at least for this approach.

3. THE FUNCTIONAL MODEL

We embark now on the higher spectral multiplicity framework. Specifically, the Spectral Theorem implies that by passing to a unitarily equivalent operator, one can assume that the normal operator N is represented by a von Neumann direct integral:

$$Nf(z) = zf(z), \quad f \in \mathcal{H} = L^2(\mu, H) := \int^{\oplus} H(z)\, d\mu(z), \tag{10}$$

where $\{H(z)\}$ is a measurable family of Hilbert spaces, referred to as fibers.

If the fiber space is constant, that is, $h(z) \equiv L$ μ-a.e., then $\mathcal{H} = L^2(\mu) \otimes L$. This is why we adopt the shorter notation $L^2(\mu, H)$ for the direct integral in (10).

Put $H_0(z) = \operatorname{span}\{u_j(z), v_j(z) : 1 \le j \le m\}$, then $H_0(z) \subset H(z)$ and $\dim H_0(z) \le 2m$ μ-a.e. We make the following second standing assumption:

(K) **For μ-almost every $z \in \mathbb{C}$, the vectors $v_j(z)$, $j = 1, \dots, m$ generate the space $H_0(z)$.**

Any finite rank perturbation can be rewritten in a form that satisfies (K). Indeed, it suffices to add to the expression (1) finitely many new formal terms $\langle \cdot, v_j \rangle\, u_j$ with $u_j = 0$.

From now on, we build the quotient functional model only for the restriction of T to its reducing subspace

$$\mathcal{H}_0 := L^2(\mu, H_0). \tag{11}$$

Notice that the restriction of T to $\mathcal{H} \ominus \mathcal{H}_0$ is normal and is already represented in its von Neumann functional model.

Let $x, y \in \mathcal{H}_0$ be a pair of vectors (so that $x(z), y(z)$ are measurable cross-sections, $x(z), y(z) \in H_0(z)$, and $\|x(\cdot)\|, \|x(\cdot)\| \in L^2(\mu)$).

Their $\odot$ product is defined by:

$$\big(x \odot y\big)(z) := [x(z), y(z)]_{H_0(z)}$$

(so that $x \odot y \in L^1(\mu)$). To simplify notation, here we have to assume that here the scalar products $[x(z), y(z)]_{H_0(z)}$ are bilinear. To be more precise, fix a measurable family of orthonormal bases $e_j(z)$, $1 \le j \le n(z)$ in spaces $H_0(z)$ and define these products in terms of corresponding coordinates:

$$[x(z), y(z)]_{H_0(z)} = \sum_{j=1}^{n(z)} x_j(z) y_j(z).$$

The usual sesquilinear product in a fiber $H_0(z)$ (used in (1)) is given by

$$\langle x(z), y(z) \rangle_{H_0(z)} = [x(z), \bar{y}(z)]_{H_0(z)}.$$

Here $\bar{y}(z) = \sum_j \bar{y}_j(z) e_j(z)$ whenever $y(z) = \sum_j y_j(z) e_j(z) \in H_0(z)$.

In this situation, in the definition (4) of the measure ν, we define $|u_j(z)| := \|u_j\|_{H(z)}$, and the same for $|v_j(z)|$.

If $H_0(z) \equiv L$ μ-a.e., where $\dim L = 1$, then $\mathcal{H}_0$ can be identified with $L^2(\mu)$, and $x \odot y$ is just the pointwise product of L^2 functions x and y.

We will use the formal columns $u = (u_1, \dots, u_m)^t$ and $v = (v_1, \dots, v_m)^t$. Then the perturbation in (1) can be expressed as follows:

$$\sum_{j=1}^{m} \langle \cdot, v_j \rangle \, u_j = u^t \langle \cdot, v \rangle. \tag{12}$$

In what follows, instead of condition (D) we will assume for convenience the following two conditions, that are formally stronger:

(D′) **The measure μ is dissectible**;

(B) **The functions u_j and v_j are bounded**.

If (K) holds, there is no loss of generality in assuming both (D′) and (B): one can achieve it by substituting μ with an equivalent measure. Indeed, the function $\rho = \sum_j |u_j|^2 + |v_j|^2 \in L^1(\mu)$ is positive and, by (K), nonzero μ-a.e. Then the data

$$\widetilde{\mu} := \rho\mu, \ \widetilde{u}_j := \rho^{-1/2} u_j, \ \widetilde{v}_j := \rho^{-1/2} v_j$$

satisfy (D′) and define new operators N and T, which are unitarily equivalent to the original ones.

To construct our model, formally we do not need (D′). However, to get most of its consequences we need to require that the spectral measure is dissectible.

Lemma 3.1. *Let Ω be a Jordan domain with admissible boundary with respect to μ. Then formula*

$$f \mapsto C(f\mu)$$

defines bounded operators from $L^2(\mu|\Omega)$ to $E^2(\Omega^c)$ and from $L^2(\mu|\Omega^c)$ to $E^2(\Omega)$.

Proof. Let $f \in L^2(\mu)$ and assume that the closed support of f does not touch $\partial\Omega$. For any function $g \in E^2(\Omega)$, analytic on a neighbourhood of the closure of Ω, one has

$$\frac{1}{2\pi i} \int_{\partial\Omega} C(f\mu) \cdot g \, dz = \int_{\Omega} f g \, d\mu.$$

Since Ω is a Smirnov domain [13], the functions g as above are dense in $E^2(\Omega)$.

Notice also that $E^2(\Omega)$ is dual to $E_0^2(\Omega^c)$ with respect to the Cauchy duality (defined by the left hand side of the above identity). So the boundedness of the Carleson embedding (9) implies the boundedness of the operator $f \mapsto C(f\mu)$ from $L^2(\chi_\Omega \cdot \mu)$ to $E_0^2(\Omega^c)$. The boundedness of the second operator is checked similarly. □

Definition 3.2. We associate with any domain Ω in $\mathbb{C}$ with admissible boundary the following *model space*

$$\mathrm{Mod}(\Omega) = \big\{g = C(x\odot\bar{v}\cdot\mu)+b : \quad x \in L^2(\mu|\Omega, H_0),\ b \in E^2(\Omega)\otimes\mathbb{C}^m\big\}, \tag{13}$$

where $x \odot \bar{v} = (x \odot \bar{v}_1, \dots, x \odot \bar{v}_m)^t$ is a column function. According to (D′), $x \odot \bar{v} \in L^2(\mu) \otimes \mathbb{C}^m$.

This space is contained in $L^1(\Omega, dA) \otimes \mathbb{C}^m$, where dA is the area measure. It follows from (K) that the function x is determined by g uniquely from $x \odot v \cdot \mu = -\pi\bar{\partial}g$ on Ω (in the sense of distributions). Hence b is also determined by g.

Proposition 3.3. *Let Ω be an admissible domain and γ an admissible Jordan curve, which is either contained in Ω or is a subarc of $\partial\Omega$.*

Then any function g in $\mathrm{Mod}(\Omega)$ has a well-defined "trace" $g|\gamma$ on γ, which is an element of $L^2(\gamma) \otimes \mathbb{C}^m$. The map $g \mapsto g|\gamma \in L^2(\gamma) \otimes \mathbb{C}^m$ is bounded.

The exact definition of these traces will be given in Section 5.

Formula

$$\|g\|^2 := \|x\|^2_{L^2(\mu|\Omega,H_0)} + \|b\|^2_{E^2}$$

defines a Hilbert space structure on $\mathrm{Mod}(\Omega)$ (we leave the details to the reader). Proposition 3.3 implies that the maps

$$g \mapsto g|\partial\Omega \in L^2(\partial\Omega) \otimes \mathbb{C}^m, \quad g \mapsto x = x(g) \in L^2(\mu, H_0) \tag{14}$$

are bounded on $\mathrm{Mod}(\Omega)$. Hence formula

$$\|g\|_1^2 := \|x(g)\|^2_{L^2(\mu|\Omega,H_0)} + \|g\|^2_{L^2(\partial\Omega)} \tag{15}$$

defines an equivalent Hilbert space norm on $\mathrm{Mod}(\Omega)$.

For a domain Ω as in the above Definition, set

$$W_{0,\Omega}x(z) = C\big(x \odot \bar{v} \cdot \mu\big)(z), \quad z \in \Omega, \tag{16}$$

so that

$$W_{0,\Omega} : L^2(\mu|\Omega, H_0) \to \mathrm{Mod}(\Omega).$$

The $m \times m$ matrix function Ψ, defined by

$$\Psi = I_m + C(\bar{v} \odot u^t \cdot \mu)$$

is called *the perturbation matrix* and will play a major role in the sequel. It belongs to $L^2_{loc}(\mathbb{C}, dA)$. By Lemma 2.3, this function is defined and continuous on $\mathbb{C} \setminus \mathcal{G}_s(\mu)$, for any $s > 0$.

Notice that Ψ might be not continuous on the complement of the union of the sets $\mathcal{G}_s(\mu)$.

A direct calculation gives the intertwining property

$$W_{0,\Omega}Tx(z) = zW_{0,\Omega}x(z) + \Psi(z)\,\langle x, v\rangle. \tag{17}$$

Put

$$\psi = \det \Psi;$$

this is called *the perturbation determinant*. We observe that Ψ and ψ are holomorphic on $\mathbb{C} \setminus \operatorname{supp} \mu$, $\Psi(\infty) = I_m$ and $\psi(\infty) = 1$.

We denote by $\operatorname{Der} \operatorname{supp} \mu$ the derivative set of the support of μ, that is, the set of all accumulation points of $\operatorname{supp} \mu$.

Proposition 3.4. *Set $T_0 = T|\mathcal{H}_0$.*

(1) *The essential spectrum of T_0 coincides with the set* $\operatorname{Der} \operatorname{supp} \mu$.
(2) *Suppose that $\lambda \in \mathbb{C} \setminus \operatorname{supp} \mu$. Then $\lambda \in \sigma(T_0)$ iff $\psi(\lambda) = 0$. The same criterion applies for λ to belong to the point spectrum of T_0.*

Proposition 3.5. *For any admissible domain Ω such that $\psi \neq 0$ on $\partial\Omega$, the multiplication operator $f(z) \mapsto zf(z)$ is bounded on* $\operatorname{Mod}(\Omega)$ *and the linear manifold*

$$\Psi \cdot \left(E^2(\Omega) \otimes \mathbb{C}^m\right)$$

is a closed subspace of $\operatorname{Mod}(\Omega)$. *Moreover, for any $a \in E^2(\Omega) \otimes \mathbb{C}^m$, the trace of Ψa on $\partial\Omega$ in the sense of Proposition 3.3 equals $\Psi \cdot a|\partial\Omega$ (notice that Ψ is continuous on $\partial\Omega$).*

Lemma 3.6. *Suppose Ω is a domain with admissible boundary. Then the operator*

$$f(z) \mapsto zf(z) \tag{18}$$

is bounded on $\operatorname{Mod}(\Omega)$. *Its spectrum is contained in $\bar{\Omega}$.*

For any domain Ω as in the above Proposition 3.5, we consider the quotient space

$$\mathcal{Q}(\Omega) := \operatorname{Mod}(\Omega) \big/ \Psi \cdot (E^2(\Omega) \otimes \mathbb{C}^m)$$

and call it *the quotient model space corresponding to* Ω. The quotient multiplication operator M_Ω, induced by the above mapping (18), is correctly defined and bounded on $\mathcal{Q}(\Omega)$. We state the following analogue of Theorem 1 in [49].

Theorem 3.7. *Assume conditions (B) and (K) hold true. Let $\mathcal{B}$ be a domain with admissible boundary, such that $\sigma(T) \subset \mathcal{B}$. Then the transform*

$$W_{\mathcal{B}} x := C(x \odot \bar{v} \cdot \mu), \qquad W_{\mathcal{B}} : L^2(\mu, H_0) \to \mathcal{Q}(\mathcal{B}),$$

is an isomorphism. It intertwines the operator T with the quotient multiplication operator $M_{\mathcal{B}}$.

This theorem shows that the perturbation matrix Ψ is a close analogue of the characteristic function of a contractive or dissipative linear operator. This function appeared first in 1946 in the paper by Livsič [31], dedicated to quasi-hermitian operators with defect indices $(1, 1)$.

The following two statements are analogous to [49], Lemmas 2 and 6.

Lemma 3.8 (Glueing lemma)**.** *Suppose Ω_1, Ω_2 and Ω are admissible domains, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and $\overline{\Omega}_1 \cap \overline{\Omega}_2$ is an arc. If $w_j \in \operatorname{Mod}(\Omega_j)$ and $w_1 = w_2$ on $\overline{\Omega}_1 \cap \overline{\Omega}_2$ (in the sense of Proposition 3.3), then $w_j = w|\overline{\Omega}_j$ for a function $w \in \operatorname{Mod}(\Omega)$.*

Theorem 3.9. *Assume conditions (B) and (K) are satisfied. Define $\mathcal{H}_0$ by (11), and let $T_0 = T|\mathcal{H}_0$. Suppose $\mathcal{B}$ is an admissible domain which contains $\sigma(T_0)$, and let $\overline{\mathcal{B}} = \cup_{j=1}^N \overline{\Omega}_j$ be its finite partition, where Ω_j are open and disjoint admissible domains. Assume that the union of boundaries $\partial\Omega_j$ does not intersect $\mathcal{G}_s(\mu)$ for some $s > 0$ and that the perturbation determinant ψ does not vanish on $\cup_{j=1}^N \partial\Omega_j$.*

Then T_0 splits into a direct sum

$$T_0 = \dot{+}_{j=1}^N T_j,$$

where for each j, the spectrum of T_j is contained in the closure of Ω_j.

Formally, the above theorem does not require the dissectability condition (D′). Its main application, however, is for measures μ meeting this condition. In this case, the lengths of the sets $\operatorname{Re}\mathcal{G}_s(\mu)$ and $\operatorname{Im}\mathcal{G}_s(\mu)$ tend to zero as $s \to 0$. Therefore, by Lemmas 2.3 and 2.7, Theorem 3.9 permits one to dissect the spectrum along straight lines that cover densely the whole plane. This property is reflected in the abstract notion of a dissectible Banach space operator, which we introduce in Section 7. Theorem 7.2, which is a consequence of the above Theorem 3.9, gives a sufficient condition for a finite rank perturbation of a normal operator to be dissectible.

4. PROOFS OF LEMMAS FROM GEOMETRIC FUNCTION THEORY

In the present section we prove Lemmas 2.3, 2.7 and Proposition 2.9.

Lemma 4.1. *For any real number x, there is a strictly increasing sequence $\{x_k^- : k \geq 0\}$ of numbers of the form $x_k^- = m_k/2^k$, $m_k \in \mathbb{Z}$, such that*

1) $\lim_k x_k^- = x$;

2) $2^{-k} \leq x - x_k^- \leq 2^{-k+1}$.

Proof. Set s_k to be the truncation of the binary representation of x with k digits after the point. Then $\{s_k\}$ increases, tends to x, and $0 \leq x - s_k \leq 2^{-k}$. Now put $x_k^- = s_k - 2^{-k}$. □

Proof of Lemma 2.3. We can assume that $s = 2^{-k_0}$ for some $k_0 \geq 1$. Take any $f \in L^\infty(\nu)$ with $\|f\|_{L^\infty(\nu)} \leq 1$ and set $\eta = \eta_f = C(f\nu)$. Put $z = x + iy$, and let $\{x_k^-\}_{k\geq 1}$ be the sequence approximating x from below, constructed in Lemma 4.1 and $\{x_k^+\}$ the sequence approximating x from above, with analogous properties. Similarly, take two sequences $\{y_k^-\}, \{y_k^+\}$, approximating y from below and from above with the same properties. We have $x_k^- < x < x_k^+$, $2^{-k} \leq |x - x_k^\pm| \leq 2^{-k+1}$, and the same for y and $\{y_k^\pm\}$. Consider the nested sequence of rectangles

$$R_k(z) = [x_k^-, x_k^+) \times [y_k^-, y_k^+), \quad k \geq k_0 - 1,$$

whose intersection is the point z. For any $k \geq k_0$, $R_{k-1}(z) \setminus R_k(z)$ is a union of no more than 16 dyadic squares from the generation $\mathcal{D}_k$. The (dyadic)

squares Q satisfying

$$Q \in \mathcal{D}_k \quad \text{and} \quad Q \subset R_{k-1}(z) \setminus R_k(z) \quad \text{for some } k \geq k_0$$

are disjoint, and their union over all $k \geq k_0$ equals to $R_{k_0-1}(z) \setminus \{z\}$. We arrange them in a single sequence $\{Q_m\}_{m\geq 1}$.

Notice that any square in the sequence $\{Q_m\}$ is at a moderate distance from z in the sense that

$$|Q_m| \leq \operatorname{dist}(Q_m, z) \leq 2^{3/2}|Q_m|. \tag{19}$$

Since $z \notin \mathcal{G}_s(\nu)$ and $|Q_m| \leq 2^{-k_0}$ for all $m \geq 1$, it follows that none of the squares Q_m is heavy (see (3)). We also observe that $\nu(\{z\}) = 0$ (otherwise any sufficiently small dyadic square containing z would be heavy, which would imply that $z \in \mathcal{G}_s(\nu)$). We get

$$\begin{aligned} |\eta_f(z)| &\leq \int_{\mathbb{C}\setminus R_{k_0-1}(z)} \frac{|d\nu(t)|}{|t-z|} + \sum_m \int_{Q_m} \frac{|d\nu(t)|}{|t-z|} \\ &\leq 2^{-k_0+1}\,\nu(\mathbb{C}) + \sum_m \frac{\nu(Q_m)}{|Q_m|} \\ &\leq 2^{-k_0+1}\,\nu(\mathbb{C}) + \sum_m \frac{h(|Q_m|)}{|Q_m|} \\ &\leq 2^{-k_0+1}\,\nu(\mathbb{C}) + 16 g(2^{-k_0}), \end{aligned} \tag{20}$$

where $g(\tau) = \int_0^\tau \frac{h(t)}{t^2}\,dt < \infty$. This proves the estimate (6).

With a small extra effort, we also derive the continuity of η_f on $\mathbb{C} \setminus \mathcal{G}_s(\nu)$. Fix some $z \in \mathbb{C} \setminus \mathcal{G}_s(\nu)$, and let us prove the continuity of $\eta_f|\mathbb{C} \setminus \mathcal{G}_s(\nu)$ at z. It suffices to consider the case when $z \in (\operatorname{supp}\nu) \setminus \mathcal{G}_s(\nu)$.

First we isolate the following observation. Take some $w \in B(z,r) \setminus \mathcal{G}_s(\nu)$, where the radius r is small. Choose k_1 so that $2^{-k_1-2} \leq r < 2^{-k_1-1}$, and repeat the above construction of squares Q_m and the estimate (20), taking k_1 in place of k_0. We get that for any f as above and any $\varepsilon > 0$, there exists $r > 0$ such that

$$\Big|\int_{B(z,r)} \frac{f(t)d\nu(t)}{t-w}\Big| < \Big|\int_{R_{k_1-1}(z)} \frac{f(t)d\nu(t)}{t-w}\Big| < 16 g(2^{-k_1}) < \frac{\varepsilon}{3}$$

whenever $w, z \in B(z,r) \setminus \mathcal{G}_s(\nu)$.

We check the continuity at z just by definition. Fix $\varepsilon > 0$. Let $w \in \mathbb{C} \setminus \mathcal{G}_s(\nu)$, $|z-w| < \delta$, where δ is to be determined. Choose r as above, and assume $\delta < r$. Then

$$\begin{aligned} |\eta_f(z) - \eta_f(w)| \leq \Big|\int_{B(z,r)} \frac{f(t)d\nu(t)}{t-z}\Big| + \Big|\int_{B(z,r)} \frac{f(t)d\nu(t)}{t-w}\Big| \\ + \Big|\int_{\mathbb{C}\setminus B(z,r)} f(t)d\nu(t)[(t-z)^{-1} - (t-w)^{-1}]\Big|. \end{aligned} \tag{21}$$

Since the last integral is continuous as a function of w on the open disk $B(z,r)$, it follows that $|\eta_f(z)-\eta_f(w)|<\varepsilon$ whenever $w\in\mathbb{C}\setminus\mathcal{G}_s(\nu)$, $|z-w|<\delta$ and δ is sufficiently small. That is, η_f is continuous on $\mathbb{C}\setminus\mathcal{G}_s(\nu)$. □

Now we pass to the proof of Lemma 2.7. We recall that a point x_0 is called a (Lebesgue) density point of a measurable set $A\subset\mathbb{R}$ if $x_0\in A$ and $\mathcal{H}^1([x_0-\varepsilon,x_0+\varepsilon])/(2\varepsilon)\to 1$ as $\varepsilon\to 0^+$. By the Lebesgue density theorem, for any subset A of $\mathbb{R}$ of positive measure, its almost every point is a density point.

Proof of Lemma 2.7. It is easy to see that any scale function h satisfies $h(t)\le Ct$ for $t\in[0,1]$. Hence the Carleson constant of each set $\mathbb{C}\setminus\mathcal{G}_s(\mu)$ with respect to μ is bounded. Indeed, choose $k\ge 0$ with $2^{-k}<s$. Let $z\in\mathbb{C}\setminus\mathcal{G}_s(\mu)$. The closest point p to z in the discrete grid $2^{-k}\mathbb{Z}\times 2^{-k}\mathbb{Z}$ is the vertex of four dyadic squares from the generation $\mathcal{D}_k$. If Q is any of these squares, then $|Q|\le s$ and $10Q^o$ contains z, which implies that $\mu(Q)\le h(|Q|)\le C|Q|$. This shows that the constant $\mathrm{Carleson}(\mathbb{C}\setminus\mathcal{G}_s(\mu),\mu)$ is bounded for any s. Denote by X_s and Y_s the x- and y- projections of the set $\mathcal{G}_s(\mu)$. They are open and their lengths tend to zero as $s\to 0$. Note also that $\{X_s\}$ and $\{Y_s\}$ are increasing families of sets. Define $\mathbb{A}_x$ to be the set of points in $\mathbb{R}$ that are density points of at least one of the sets $\mathbb{R}\setminus X_s$. Define $\mathbb{A}_y$ similarly, using Y_s in place of X_s. Then $\mathbb{A}_x$ and $\mathbb{A}_y$ are measurable, and their complements in $\mathbb{R}$ have Lebesgue measure zero. For any point x_0 in $\mathbb{A}_x$, there is some $s>0$ such that $x_0\in\mathbb{R}\setminus X_s$ and x_0 can be approximated both from above and from below by a sequence of points in the same set $\mathbb{R}\setminus X_s$. The points of $\mathbb{A}_y$ have a similar property.

This gives assertion 2).

Any broken line γ as in Lemma is contained in $\mathbb{C}\setminus\mathcal{G}_s(\mu)$ for some sufficiently small s. It follows that the assertion 1) holds. □

Proof of Proposition 2.9. Case 1: Assume first that the hypotheses hold, and all the scale functions coincide: $h_k=h$ for all k. We will normalize the measures μ_k, assuming that $\mu_k(\mathbb{C})=1$ for all k. We will show that there exist constants $c_k>0$ such that $\sum_k c_k\mu_k$ is a finite and h-dissectible measure. Put $c_k=\alpha_k^2$, where $\alpha_k>0$.

First choose positive numbers $s_k\to 0$ such that

$$\mathcal{H}^1_{\mathrm{cont}}\big(\mathcal{G}_{s_k}(\mu_k)\big)\le 2^{-k}. \tag{22}$$

Note that if a square Q is $\alpha_k\mu_k$-heavy, then

$$h(|Q|)<\alpha_k\mu_k(|Q|)\le\alpha_k.$$

We choose $\alpha_k\in(0,1)$ so that this implies that $|Q|\le s_k$. We also assume that

$$\gamma:=\sum_k\alpha_k\le 1.$$

With this choice, the measure $\nu:=\sum_k c_k\mu_k$ will be finite and h-dissectible. Indeed, fix any $\varepsilon>0$. Find some N such that $2^{-N}<\varepsilon/2$. Since each μ_k

is h-dissectible, we can choose $q > 0$ so that $\mathcal{H}^1_{\rm cont}\big(\mathcal{G}_q(\mu_k)\big) < \varepsilon/(2N)$ for $k = 1, 2, \dots, N$. We assert that whenever $0 < s < q$,

$$\mathcal{G}_s(\nu) \subset \bigcup_{k=1}^{N} \mathcal{G}_q(\mu_k) \cup \bigcup_{k=N+1}^{\infty} \mathcal{G}_{s_k}(\mu_k). \tag{23}$$

By (22), this implies that

$$\mathcal{H}^1_{\rm cont}\big(\mathcal{G}_s(\nu)\big) < N \cdot \frac{\varepsilon}{2N} + \sum_{k=N+1}^{\infty} 2^{-k} = \frac{\varepsilon}{2} + 2^{-N} < \varepsilon,$$

and therefore, ν is dissectible, because ε was arbitrary.

So it remains to check (23). In order to do so, take any dyadic square Q, which is ν-heavy and satisfies $|Q| < s < q$. Then $\gamma\nu(Q) \geq \gamma h(|Q|)$, which we rewrite as

$$\gamma \sum_k \alpha_k^2 \mu_k(Q) \geq \sum_k \alpha_k h(|Q|).$$

Hence for some k,

$$\mu_k(Q) \geq \alpha_k \mu_k(Q) \geq \frac{1}{\gamma}\, h(|Q|) \geq h(|Q|).$$

So Q is $\alpha_k\mu_k$-heavy (and μ_k-heavy). By the choice of α_k, this implies $|Q| \leq s_k$. Hence

$$10Q^o \subset \mathcal{G}_t(\mu_k),$$

where $t = \min(s, s_k) < q$. This implies (23).

Case 2 (the general case): Notice first that whenever h, h_1 are scale functions with $h_1 \leq h$, then Q is h-heavy implies that Q is h_1-heavy. So if a measure ν is h_1-dissectible, then it is also h-dissectible with respect to any scale function h such that $h_1 \leq h$ on an interval $[0, \varepsilon]$, $\varepsilon > 0$.

As a consequence, we will get the general case from the above Case 1 once we check the following: *For any sequence of scale functions* $\{h_k : k \geq 1\}$, *there exists a scale function* h *such that for any* k, $h \geq h_k$ *on some (nonempty) interval* $(0, \varepsilon_k]$.

This is simple. Namely, let $\{\beta_k\}$ be a sequence of positive numbers such that $\sum_k \beta_k < \infty$. Put

$$h(t) = h_1(t) + \sum_{k \geq 2} g_k(t),$$

where $g_k(t) = \min(h_k(t), \beta_k)$. Then $h(t)$ is finite for any $t \geq 0$. If the numbers $\beta_k > 0$ go sufficiently rapidly to zero, then h satisfies the integral condition (2) and so is a desired scale function. $\square$

5. PROOFS PERTAINING TO THE FUNCTIONAL MODEL

First we mention the Cauchy-Pompeiu formula for functions in $\mathrm{Mod}(\Omega)$: *If Ω is admissible and $w \in \mathrm{Mod}(\Omega)$, then*

$$(2\pi i)^{-1} C_{\partial\Omega}(w)(z) - \pi^{-1} C(\bar\partial w)(z) = \begin{cases} w(z), & \text{if } z \in \Omega \\ 0, & \text{if } z \notin \overline{\Omega} \end{cases} \tag{24}$$

(notice that $w|\partial\Omega \in L^2(\partial\Omega, |dz|)$).

Indeed, by the definition of $\mathrm{Mod}(\Omega)$, it suffices to check this formula for functions $w = b \in E^2(\Omega) \otimes \mathbb{C}^m$ and for functions of the form $w = C(x \odot \bar v \cdot \mu)$, where $v \in L^2(\mu, H_0)$. In the first case, it is just the Cauchy formula. In the second case, it is obvious because $\bar\partial w = -\pi x \odot \bar v \cdot \mu$.

Proof of Proposition 3.3. Let $g = C(x \odot \bar v \cdot \mu) + b$ be a function in $\mathrm{Mod}(\Omega)$, as in the Definition 3.2 of the model space and let γ be a curve meeting the hypotheses. Suppose first that $\gamma \subset \Omega$.

If x is bounded, then by (D′) and Lemma 2.3, one can define g on the set $\Omega \setminus \cap_s \mathcal{G}_s(\mu)$; it will be continuous on $\Omega \setminus \mathcal{G}_s(\mu)$ for any $s > 0$. Since γ does not touch some set $\mathcal{G}_{s_0}(\mu)$, this permits one to define $g|\gamma$ as a restriction.

Now Lemma 3.1 implies that this map $g \to g|\gamma$ extends by continuity to functions $g \in \mathrm{Mod}(\Omega)$ as above corresponding to arbitrary $x \in L^2(\mu, H_0)$.

We take this extension as a definition of the trace.

By the same Lemma 3.1, the map $g \to g|\gamma$ is also well-defined and bounded if γ is a subarc of $\partial\Omega$. □

Proof of Proposition 3.4. (1) The definition of $\mathcal{H}_0$ implies that the essential spectrum $\sigma_{ess}(N|\mathcal{H}_0)$, and hence also $\sigma_{ess}(T_0)$, coincide with $\mathrm{Der}\,\mathrm{supp}\,\mu$.

(2) Fix a point λ, which is not in the support of μ. By the above, $\lambda \in \sigma(T_0)$ if and only if λ is an eigenvalue of T_0. This rewrites as $(z - \lambda) f(z) = -u^t c$, where $f \in \mathcal{H}_0$ ($f \neq 0$) is the corresponding eigenfunction and $c = \langle f, v\rangle$. It is easy to see that these two equations are solvable if and only if $\Psi(\lambda)c = 0$ has a nonzero solution. □

Proof of Proposition 3.5. Suppose $\partial\Omega$ admissible.

(a) First we check that $\Psi\big(E^2(\Omega) \otimes \mathbb{C}^m\big)$ is contained in $\mathrm{Mod}(\Omega)$. Take any $a \in E^2(\Omega) \otimes \mathbb{C}^m$. We need to check that $g := \Psi a \in \mathrm{Mod}(\Omega)$. We have

$$\bar\partial g = -\pi(\bar v \odot u^t) \cdot a = -\pi \bar v \odot (u^t \cdot a) \quad \text{in } \Omega. \tag{25}$$

Since the Carleson embedding J_Ω is bounded on $E^2(\Omega)$ and u is a bounded function, $x := -(u^t \cdot a)\chi_\Omega$ is in $L^2(\mu)$. By (25), the function

$$b := g - C(\bar v \odot x \cdot \mu)$$

is analytic in Ω. Let Ω_n be the domains that correspond to Ω, whose existence is asserted in the Definition 2.5 of an admissible Jordan curve. Since the Carleson constants of $\partial\Omega_n$ with respect to μ are uniformly bounded and

Ψ is bounded on the union of these curves (see Lemma 2.3), we get that

$$\sup_n \|C(\bar{v} \odot x \cdot \mu)\|_{L^2(\partial\Omega_n)} < \infty \quad \text{and} \quad \sup_n \|g\|_{L^2(\partial\Omega_n)} < \infty.$$

Therefore, the integrals $\int_{\partial\Omega_n} |b|^2 \, |dz|$ are uniformly bounded, which means that b is in $E^2(\Omega)$. Hence $g = b + C(\bar{v} \odot x \cdot \mu)$ is in $\mathrm{Mod}(\Omega)$.

The same arguments show that the map $a \to \Psi a$ is bounded from $E^2(\Omega) \otimes \mathbb{C}^m$ to $\mathrm{Mod}(\Omega)$.

(b) Let us check the last statement about the trace of Ψa on $\partial\Omega$. Let $C_a(\bar{\Omega})$ stand for the space of functions, analytic in Ω which can be extended continuously to the boundary. Consider first the case when $a \in C_a(\bar{\Omega}) \otimes \mathbb{C}^m$ and define x, b as above. Then $x \in L^\infty(\mu)$, which implies that $C(v \odot x \cdot \mu)$ is well-defined at any point of $\partial\Omega$ and is continuous on this curve. Hence b is also continuous on $\partial\Omega$ and so $b \in C_a(\bar{\Omega}) \otimes \mathbb{C}^m$. Hence the equality $\Psi a = C(\bar{v} \odot x \cdot \mu) + b$ holds pointwise on $\partial\Omega$, which implies that the trace of Ψa on $\partial\Omega$ equals to $\Psi \cdot a | \partial\Omega$.

The case of a general a in $E^2(\Omega) \otimes \mathbb{C}^m$, is obtained by approximating a in the norm of $E^2(\Omega) \otimes \mathbb{C}^m$ by a sequence of functions a_n in $C_a(\bar{\Omega}) \otimes \mathbb{C}^m$ and applying the boundedness of the trace mapping $w \mapsto w|\partial\Omega$, $w \in \mathrm{Mod}(\Omega)$.

(c) To prove that $\Psi\big(E^2(\Omega) \otimes \mathbb{C}^m\big)$ is closed in $\mathrm{Mod}(\Omega)$, apply the expression (15) for an equivalent norm in $\mathrm{Mod}(\Omega)$. Since $\|\Psi^{-1}\|$ is uniformly bounded on $\partial\Omega$, it follows that $\|\Psi a\|_{\mathrm{Mod}(\Omega)} \geq \delta \|a\|_{E^2}$ for any $a \in E^2(\Omega) \otimes \mathbb{C}^m$, where $\delta > 0$. Hence the image in $\mathrm{Mod}(\Omega)$ of the map $a \in E^2(\Omega) \otimes \mathbb{C}^m \mapsto \Psi a$ is closed. □

Proof of Lemma 3.6. Let $g \mapsto x(g)$ be the map defined in (14). Since $x(g)$ is obtained by taking $\bar{\partial} g$, it follows that $x(ag) = a x(g)$ for $g \in \mathrm{Mod}(\Omega)$ and any a analytic in a neighborhood of $\bar{\Omega}$. Applying this to $a(z) = z$ and using the equivalent norm (15) on the model space, we get that the operator $g \mapsto zg$ is bounded on $\mathrm{Mod}(\Omega)$. By the same reasons, for any $\lambda \in \Omega^c$, the multiplicaton operator $g \mapsto (z - \lambda)^{-1} g$ is a bounded inverse to the operator $g \mapsto (z - \lambda) g$. □

Proof of Theorem 3.7. We can repeat the arguments of the proof of Theorem 1 in [49] (see [49], p. 66). The intertwining property follows from (17). It remains to check that, given w_0 in Mod_B, there exist a unique x in $L^2(\mu, H_0)$ and a unique $a \in E^2(\mathcal{B}) \otimes \mathbb{C}^m$ such that

$$(W_{0,\mathcal{B}} x)(z) = w_0(z) + \Psi(z) a(z), \quad z \in \mathcal{B}. \tag{26}$$

We appeal to a class of Toeplitz type operators. If F is matrix-valued function in $L^\infty(\partial\mathcal{B}, L(\mathbb{C}^m))$, then the Toeplitz operator τ_F acts on $E^2(\mathcal{B}) \otimes \mathbb{C}^m$ by

$$\tau_F f(z) := C_{\partial\mathcal{B}}(Ff)(z), \quad z \in \mathcal{B}.$$

It is bounded. It is easy to verify that $\tau_G \tau_F = \tau_{GF}$ whenever $F, G \in H^\infty(\mathcal{B}^c, L(\mathbb{C}^m))$. In particular, τ_Ψ is invertible, and $\tau_\Psi^{-1} = \tau_{\Psi^{-1}}$.

Notice that, for a function g in $\mathrm{Mod}(\mathcal{B})$, one has

$$g \in W_{0,\mathcal{B}}L^2(\mu|\mathcal{B}, H_0) \Leftrightarrow C_{\partial\mathcal{B}}(g) = 0 \quad \text{in } \mathcal{B}. \tag{27}$$

Therefore (26) is solvable with respect to x if and only if

$$a = -\tau_{\Psi^{-1}}\big(C_{\partial\mathcal{B}}(w_0)\big)\big|_{\mathcal{B}}.$$

If we define a by this formula, by Proposition 3.5, Ψa and hence $w := w_0 + \Psi a$ belong to $\mathrm{Mod}(\mathcal{B})$. Hence a and x are determined by (26). □

Proof of the Glueing lemma 3.8. Suppose w_1, w_2 satisfy the hypotheses and define a function w on $\Omega_1 \cup \Omega_2$ by $w\big|_{\Omega_j} = w_j$. By summing formulas (24) applied to Ω_1, Ω_2, we infer

$$w = (2\pi i)^{-1}C_{\partial\Omega}(w) - \pi^{-1}C\big((\bar\partial w_1)\chi_{\Omega_1} + (\bar\partial w_2)\chi_{\Omega_2}\big)$$

on $\Omega_1 \cup \Omega_2$. The right hand part is a function in Mod_Ω. □

We remark that the same lemma holds for finitely many bordering domains, instead of two of them.

Proof of Theorem 3.9. Suppose first that (K) holds. Then we can follow the lines of the proof of Lemma 6 in [49]. Consider the restriction maps $\mathcal{J}_k w = w|\Omega_k \in \mathrm{Mod}(\Omega_k)$, $w \in \mathrm{Mod}(\Omega)$. It follows from Proposition 3.3 and the expression (15) for the norm in the model spaces that these are bounded linear operators. They induce operators $\widehat{\mathcal{J}}_k : \mathcal{Q}(\Omega) \to \mathcal{Q}(\Omega_k)$, which are well-defined and bounded. We assert that $\widehat{\mathcal{J}} = (\widehat{\mathcal{J}}_1, \dots, \widehat{\mathcal{J}}_N) : \mathcal{Q}(\Omega) \to \oplus_k \mathcal{Q}(\Omega_k)$ is an isomorphism. Let $w \in \mathrm{Mod}(\Omega)$, and let $\hat{w}$ be the corresponding element (class of equivalence) in $\mathcal{Q}(\Omega)$. If $\widehat{\mathcal{J}}\hat{w} = 0$, then $w|\Omega_k = \Psi a_k$ for some functions $a_k \in E^2(\Omega_k)$, $k = 1, \dots, N$. It is easy to check that for any $g \in \mathrm{Mod}(\Omega)$ and any domains Ω_j, Ω_k, bordering by an arc γ, the trace of $g|\Omega_j$ on γ equals to the trace of g on γ. Hence, by the last statement of Proposition 3.5, we get that $a_k = a_j$ on $\partial\Omega_k \cap \partial\Omega_j$, $1 \le k < j \le N$. By applying the Cauchy integral representation of E^2 functions, one finds there is some $a \in E^2(\Omega) \otimes \mathbb{C}^m$ such that $a_k = a|_{\Omega_k}$ for all k. This shows that $\ker \widehat{\mathcal{J}} = 0$.

The fact that $\widehat{\mathcal{J}}$ is onto also is shown in the same way as in the proof Lemma 6 in [49], and we leave the details to the reader.

In the general case, we use the reducing space $\mathcal{H}_0$ of T, defined by (11). Since $T|\mathcal{H}_0$ satisfies (K) and (D) and has the same perturbation determinant as T and $T|\mathcal{H} \ominus \mathcal{H}_0$ is normal, our assertion follows. □

6. THE EXISTENCE OF INVARIANT SUBSPACES

The aim of this section is to prove the following result.

Theorem 6.1. *Suppose T is given by* (1) *and satisfies (D) and (K). If either* $\mathrm{Der}\,\mathrm{supp}\,\mu$ *is not connected or there exists a domain G, whose boundary is a Lipschitz Jordan curve, such that $G \cap \mathrm{Der}\,\mathrm{supp}\,\mu = \emptyset$ and the intersection*

of $\operatorname{supp}\mu$ *with the boundary of* G *contains an arc, then* T *has a nontrivial invariant subspace.*

The proof of this result will rely on the following known fact.

Theorem A (Privalov's uniqueness theorem, see [39], Ch. IV, §2.6)**.** *If a function* $f(z)$*, meromorphic on a domain, bounded by a rectifiable Jordan curve* Γ*, has angular limit values equal to zero on a subset* E *of* Γ *with* $\mathcal{H}^1(E) > 0$*, then* f *is identically zero.*

Recall also F. and M. Riesz theorem [42, 40]: if Ω is any Jordan domain, bounded by a rectifiable curve, then the Hausdorff measure $\mathcal{H}^1$ and the harmonic measure are mutually absolutely continuous on $\partial\Omega$.

Proof of Theorem 6.1. We may assume (D′), (B) and also that $\mathcal{H} = \mathcal{H}_0$. First let us make the following reduction. Consider the open set

$$\Omega := \mathbb{C} \setminus \operatorname{Der}\operatorname{supp}\mu.$$

Note that Ψ and ψ are meromorphic functions on Ω. If ψ vanishes on one of its connected components, then, by part (2) of Proposition 3.4, T has an eigenvalue, and therefore has an invariant subspace. So, from now on, we will assume that ψ does not vanish identically on any of the connected components of Ω.

Case 1: The (compact) set $\operatorname{Der}\operatorname{supp}\mu$ is disconnected. Then it decomposes into a disjoint union of two non-empty closed and relatively open subsets, say, F_1 and F_2. Since the zeros and the poles of ψ form discrete subsets of the complement of $F_1 \cup F_2$, one can find open disjoint sets $\mathcal{O}_1 \supset F_1$ and $\mathcal{O}_2 \supset F_2$, whose boundaries are finite unions of rectifiable Jordan curves, do not intersect with the set $F_1 \cup F_2$ and do not contain neither zeros nor poles of ψ. Note that $\partial\mathcal{O}_1$ and $\partial\mathcal{O}_2$ are contained in $\mathbb{C} \setminus \sigma(T)$, whereas F_1 and F_2 are contained in $\sigma(T)$. It follows that the corresponding Riesz projection $P_{\mathcal{O}_1}$ is a non-trivial idempotent, commuting with T. The range of $P_{\mathcal{O}_1}$ is a nontrivial invariant subspace of T.

Case 2: there exists a domain G, meeting the hypotheses of the theorem. By passing to a smaller domain, we may assume that $\operatorname{supp}\mu \cap \partial G$ has the form

$$\gamma = \{x + if(x) : \ x \in J\},$$

where $J \subset \mathbb{R}$ is a finite closed interval and f is a Lipschitz function. Since ψ is meromorphic in G and is not identically zero in this domain, Privalov's uniqueness theorem implies that $\psi(\lambda) \neq 0$ for a.e $\lambda \in \gamma$. We can also assume that f is non-constant on J.

There is a compact subset F of $f(J) \cap \mathbb{A}_y$ of positive measure. Since f maps sets of zero measure to sets of zero measure and $f(f^{-1}(F)) = F$, the preimage $f^{-1}(F)$ has positive measure. Hence one can choose $z_0 = x_0 + iy_0 \in \gamma$ so that $x_0 \in J^o \cap \mathbb{A}_x \cap f^{-1}(\mathbb{A}_y)$ and $\psi(z_0) \neq 0$. Then $y_0 \in \mathbb{A}_y$.

Part 2) of Lemma 2.7 implies that there exist sequences $\{x_n^-\}, \{x_n^+\}$ tending to x_0 and $\{y_n^-\}, \{y_n^+\}$ tending to y_0 such that $x_n^- < x_0 < x_n^+$ and

$y_n^- < y_0 < y_n^+$ for all n. Moreover, there is $s > 0$ such that for all n, $x_n^-, x_n^+ \in \mathbb{R} \setminus \operatorname{Re} \mathcal{G}_s(\mu)$ and $y_n^-, y_n^+ \in \mathbb{R} \setminus \operatorname{Im} \mathcal{G}_s(\mu)$.

Consider a rectangle $R = [x_n^-, x_n^+] \times [y_n^-, y_n^+]$. If n is sufficiently large, then $|\psi| > \delta > 0$ on $R \setminus \mathcal{G}_s(\mu)$ and γ is not contained in R. Notice also that the boundary of R is contained in $\mathbb{C} \setminus \mathcal{G}_s(\mu)$. Therefore R^o is an admissible domain.

By Theorem 3.9, T has invariant subspaces L and M such that $\mathcal{H}_0 = L \dotplus M$, $\sigma(T|L) \subset \sigma(T) \cap R$ and $\sigma(T|M) \subset \sigma(T) \setminus R^o$. Notice that $\sigma(T) = \sigma(T|L) \cup \sigma(T|M)$. Therefore each one of the spectra $\sigma(T|L)$ and $\sigma(T|M)$ contains a nontrivial subarc of γ. Hence $L \neq 0$ and $M \neq 0$, so that L is a nontrivial invariant subspace of T. □

7. BISHOP PROPERTIES ON THE MODEL SPACE AND DECOMPOSABILITY

A landmark contribution to axiomatic spectral theory is Bishop's 1959 article [8]. Inspired by generalized spectral decompositions of linear and bounded Banach space operators lacking a spectral measure, he has identified four different behaviors of the resolvent of the dual which imply the existence of invariant subspaces localizing the spectrum. Soon afterwards Foiaş has isolated in 1963 the concept of decomposable operator [17]; this class of linear operators and Bishop's properties have simplified and unified conceptually many lines of research in spectral analysis and produced over the years far reaching applications. For an early account of the theory we refer to [10, 12] as for recent developments, including multivariate generalizations, see [15].

A linear and bounded operator T acting on a Banach space X is called *decomposable*, if for every finite open cover of its spectrum

$$\sigma(T) \subset \cup_{j=1}^n U_j, \tag{28}$$

there are T-invariant subspaces $X_j \subset X$, $1 \leq j \leq n$, with the properties

$$X = X_1 + X_2 + \ldots + X_n, \tag{29}$$

and

$$\sigma(T|_{X_j}) \subset U_j, \quad 1 \leq j \leq n.$$

Note that the above decomposition is not a direct sum, nor there are bounded linear projections onto its terms. Later on it was proved that only open covers with two sets suffice for the decomposability condition. Examples are all classes of operators possessing a spectral measure, and beyond, for instance operators admitting a functional calculus with smooth functions.

A decomposable operator $T \in L(X)$ has the *single valued extension property*: for every open set $U \subset \mathbb{C}$ and any vector valued analytic function $f \in \mathcal{O}(U, X)$, $(zI - T)f(z) = 0$, $z \in U$, implies $f = 0$. As soon as an operator $T \in L(X)$ has the single valued extension property, one can speak without ambiguity about the localized spectrum $\sigma_x(T)$ of T with respect to a vector $x \in X$, defined as the smallest closed subset of the complex plane

allowing the localized resolvent $(zI-T)^{-1}x$ to have an analytic continuation on its complement [10].

One step further, the operator $T \in L(X)$ satisfies *Bishop's property* (β) if the map

$$(zI - T) : \mathcal{O}(U, X) \longrightarrow \mathcal{O}(U, X)$$

is one to one with closed range for every open set $U \subset \mathbb{C}$. Here $\mathcal{O}(U, X) = \mathcal{O}(U) \otimes X$ stands for the Fréchet space of all analytic X-valued functions on U. Obviously it is sufficient to check this condition on open disks U.

A decomposable operator possesses property (β) [8, 10]. A more recent theorem due to Albrecht and Eschmeier [2] completes Bishop's visionary program by stating that T is decomposable if and only if T and its dual T^* both have property (β). All these results have an analog in the case of commuting tuples of operators. In that context the analytic sheaf model

$$\mathcal{F}(U) = \mathcal{O}(U, X)/(zI - T)\mathcal{O}(U, X)$$

is prevalent, opening the gate to homological algebra techniques [15].

Prompted by the spectral behavior our main theorem reveals, we propose the following more restrictive variation of the decomposability property. If $A \subset B \subset \mathbb{C}$, we denote by $\partial(A; B)$ the relative boundary of A in B.

Definition 7.1. Let X be a Banach space. We will say that an operator $T \in L(X)$ is *dissectible* if for any open cover

$$\sigma(T) \subset \cup_{j=1}^{n} U_j$$

there are closed sets $F_j \subset U_j$ such that $\sigma(T) = \cup_{j=1}^{n} F_j$, $F_j \cap F_k = \partial(F_j; \sigma(T)) \cap \partial(F_k; \sigma(T))$ for all $j \neq k$, and there are T-invariant subspaces $X_j \subset X$, $1 \leq j \leq n$, such that

$$\sigma(T|_{X_j}) \subset F_j, \quad 1 \leq j \leq n$$

and a direct sum decomposition holds:

$$X = X_1 \dotplus X_2 \dotplus \ldots \dotplus X_n. \tag{30}$$

It is clear that this notion is stronger than decomposability. Any normal operator and any compact operator are dissectible. On the other hand, plenty of decomposable operators are not dissectible.

To fix ideas we discuss a simple case. Let $Y \subset \mathbb{C}$ be any connected compact set, which has more than one point. Let $X = C(Y)$ be the space of continuous functions on Y. The multiplication operator $Tf(z) = zf(z)$ is decomposable but not dissectible.

Indeed, it is easy to see that $\sigma(T) = Y$. Take any open cover $Y \subset U_1 \cup U_2$ of $\sigma(T)$ such that neither U_1 nor U_2 covers Y. Notice that if F_1, F_2 correspond to this open cover, then $f|\partial(F_j; Y) = 0$ for all f in X_j. Since $Y = F_1 \cup F_2$ is connected and $F_j \neq \emptyset$, there is a point w in $F_1 \cap F_2 = \partial(F_1; Y) \cup \partial(F_2; Y)$. By (30), any function in X vanishes at w, a contradiction.

To increase generality, take now X to be a continuously embedded Banach space into $C(Y)$, so that $T = M_z$ on X is decomposable and $\sigma(T) = Y$ (such

as a Sobolev space). The argument above adapts and implies that T is not dissectible.

We observe that Theorem 3.9 permits us to prove the following fact.

Theorem 7.2. *Assume that conditions (D) and (K) are satisfied. If, moreover, for any $s > 0$, the one-dimensional Hausdorff measure of the set*

$$\{z \in \mathbb{C} \setminus \mathcal{G}_s(\mu) : \psi(z) = 0\}$$

equals to zero, then T is dissectible.

We recall that, by Lemma 2.3, ψ is continuous on $\mathbb{C} \setminus \mathcal{G}_s(\mu)$ for any $s > 0$; the sets $\mathcal{G}_s(\mu)$ have been defined in (3).

In the proof, we use the following notation. Given some $z = x + iy$ and some radius $r > 0$, we set

$$B_{x-}(z,r) = \{w \in B(z,r) : \operatorname{Re} w < x\}, \quad B_{x+}(z,r) = \{w \in B(z,r) : \operatorname{Re} w > x\};$$

these are the left and the right half of the disc $B(z,r)$. We will also use the lower half $B_{y-}(z,r)$ and the upper half $B_{y+}(z,r)$, that have similar definitions. The four sets are open.

Let $Y \subset \mathbb{C}$, and let $z \in Y$. We will say that *z is r-accessible from the right in Y* if the intersection $Y \cap B_{x+}(z,r)$ is not empty. We say that a point *z is accessible from the right in Y* if it r-accessible from the right in Y for any positive r. Equivalently, there should exist a sequence $\{w_k\}$ of points of Y that tends to z and satisfies $\operatorname{Re} w_k > \operatorname{Re} z$ for all k.

Note that z is inaccessible from the right whenever it not r-accessible from the right for some $r > 0$. We define similarly the accessibility from the left, from above and from below.

Lemma 7.3. *For any bounded subset Y of the complex plane, the set of its points, inaccessible from the right in Y, is contained in a countable union of vertical (straight) lines.*

Proof. A point of Y is inaccessible from the right (in Y) if and only if it is $1/k$-inaccessible from the right for some $k \in \mathbb{N}$. So it will be enough to check that, say, for any $r \in (0,2)$, the set of points of Y, r-inaccessible from the right, can be covered by finitely many vertical lines.

Fix some $r \in (0,2)$ and assume that $Y \subset B(z_0, R)$ for some z_0 and some $R > 0$. We will prove that the above set of points can be covered by no more than $[4(R+1)^2/r^2]$ vertical lines, where $[t]$ stands for the entire part of t.

Suppose it is false. Then there exist $N > 4(R+1)^2/r^2$ points $z_j \in Y$ that are r-inaccessible from the right and all have distinct real parts. By comparing areas, we get that for some $k \neq \ell$, the discs $B(z_k, r/2)$ and $B(z_\ell, r/2)$ have to intersect. Hence $|z_k - z_\ell| < r$. Let, for instance, $\operatorname{Re} z_k < \operatorname{Re} z_\ell$. Then $z_\ell \in B_{x+}(z_k, r) \cap Y$. Therefore z_k is r-accessible from the right, a contradiction. This finishes the proof. $\square$

Proof of Theorem 7.2. By applying last Lemma four times to the set $Y = \sigma(T)$, we get that in Lemma 2.7, the sets $\mathbb{A}_x, \mathbb{A}_y \subset \mathbb{R}$ can be chosen so that they satisfy additionally the following properties:

(1) Any point $z \in \sigma(T)$ such that $\operatorname{Re} z \in \mathbb{A}_x$ is accessible from the right and from the left in $\sigma(T)$;
(2) Any point $z \in \sigma(T)$ such that $\operatorname{Im} z \in \mathbb{A}_y$ is accessible from above and from below in $\sigma(T)$.
(3) The lines $\{\operatorname{Re} z = x_0\}$, where $x_0 \in \mathbb{A}_x$, and $\{\operatorname{Im} z = y_0\}$, where $y_0 \in \mathbb{A}_y$, are contained in $\{z \in \mathcal{G}_s : \psi(z) \neq 0\}$ for some positive s.

We will also assume that whenever a is an isolated point of $\sigma(T)$, $\operatorname{Re} a \notin \mathbb{A}_x$ and $\operatorname{Im} a \notin \mathbb{A}_y$. This is achieved by quitting some countable subsets from $\mathbb{A}_x$ and $\mathbb{A}_y$, once again.

Assume we have an open cover (28) of $\sigma(T)$. Draw finitely many vertical lines $\{\operatorname{Re} z = x_j\}$ $(1 \leq x_j \leq N)$, where $x_j \in \mathbb{A}_x$, and finitely many horizontal lines $\{\operatorname{Im} z = y_j\}$ $(1 \leq y_j \leq M)$, where $y_j \in \mathbb{A}_y$. Let us assume that the finite sequences $\{x_j\}$, $\{y_j\}$ are increasing and that the open rectangle $(x_1, x_N) \times (y_1, y_M)$ contains $\sigma(T)$. Put $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$. By the Lebesgue lemma, we can also assume that the lines that were drawn are so close to each other that for each pair (j,k), there is an index $\hat{m}(j,k)$ such that $R_{jk} \subset U_{\hat{m}(j,k)}$. Fix these numbers $\hat{m}(j,k)$, and set

$$\widetilde{R}_m = \cup\{R_{jk} : \ \hat{m}(j,k) = m\}.$$

Then $\cup_m \widetilde{R}_m$ contains a neighbourhood of $\sigma(T)$.

The desired sets F_m will be defined as

$$F_m = \sigma(T) \cap R_m, \tag{31}$$

where R_m are certain modifications of $\widetilde{R}_m$. These modifications are performed as follows. By a vertex, we mean a point of the form (x_j, y_k), which is on the boundary of one of polygonal sets $\widetilde{R}_m$. We say that a vertex p is special if it is a limit point of $\sigma(T) \cap \widetilde{R}_m$ only for one index $m = m(p)$. (For any vertex $p \in \sigma(T)$, such index m should exist.)

For any special vertex p, choose a small closed rectangle $\rho = \rho(p)$, whose vertices lie on $\mathbb{A}_x + iA_y$, that contains p in its interior and does not touch any drawn line, except those that pass through p. We also require that $\rho \subset U_{m(p)}$ and that $\rho \cap \sigma(T) = \widetilde{R}_{m(p)} \cap \sigma(T)$.

Notice that p cannot be a limit point of $\sigma(T) \cap \partial \widetilde{R}_m$ (otherwise, as it is easy to see from (1) and (2) above, p would be also a limit point of $\sigma(T) \cap \widetilde{R}_t$ for some $t \neq m$). Therefore ρ can be chosen so that

$$\rho \cap (\sigma(T) \setminus \{p\}) = \widetilde{R}^o_m \cap (\sigma(T) \setminus \{p\}). \tag{32}$$

We make replacements

$$\widetilde{R}_{m(p)} \mapsto \widetilde{R}_{m(p)} \cup \rho(p)$$

and

$$\widetilde{R}_t \mapsto \widetilde{R}_t \setminus \rho(p)^o$$

whenever $t \neq m$ and p lies on the boundary of $\widetilde{R}_t$. After making these replacements for all special vertex points p, we will get modified sets R_m, such that special vertex points are no longer their vertices. (We may assume that the rectangles $\rho(p)$, corresponding to different special vertex points p, are disjoint, so that the result does not depend on the order of these replacements.)

Define the sets F_m by (31). By (32), all vertices of the sets R_m that were not among the vertices of $\widetilde{R}_m$ do not belong to $\sigma(T)$.

By (3) and Theorem 3.9, there is a decomposition $T = T_1 \dotplus \cdots \dotplus T_N$, where $\sigma(T_m) \subset F_m$.

It remains to check the condition concerning the boundaries of F_m. Let $w \in F_m \cap F_t$, with $m \neq t$. Then w belongs to the boundaries of R_m and of R_t. If w is not a vertex point of R_m, then it follows from (1) and (2) that w is a limit of points in $\sigma(T) \setminus F_m$, and so $w \in \partial(F_m; \sigma(T))$. Similarly, $w \in \partial(F_t; \sigma(T))$.

Finally, let w be a vertex point. Since $w \in \sigma(T)$, it is not a special point. Therefore there are indices $k \neq \ell$ such that w is a limit point of both sets F_k and F_ℓ. Either $k \neq m$ or $\ell \neq m$; assume for instance that $k \neq m$. Then w is a limit point of the set $\sigma(T) \cap R_k^o$, which does not intersect F_m, and therefore $w \in \partial(F_m; \sigma(T))$. Similarly, $w \in \partial(F_t; \sigma(T))$.

This shows that $F_m \cap F_t = \partial(F_m; \sigma(T)) \cap \partial(F_t; \sigma(T))$. □

It turns out that a sufficient condition for decomposability of other sort can be proved.

Set

$$\tilde{\psi}(z) = \|\Psi(z)^{-1}\|^{-1}$$

(the right hand part is understood as zero if the matrix $\Psi(z)$ is not invertible). Notice that for a rank one perturbation, when $m = 1$, $\tilde{\psi} = \psi = \Psi$. We also remark that $|\tilde{\psi}(z)| \leq \|\Psi(z)\|$ for a.e. z, so that $\tilde{\psi}$ is a nonnegative locally L^1 function. The set of zeros of ψ and of $\tilde{\psi}$ on each set $\mathbb{C} \setminus \mathcal{G}_s(\mu)$ coincide.

Definition 7.4. We say that $\tilde{\psi}$ *has no deep zeros* if for any bounded domain $\mathcal{D}$ and any its compact subset K, there is a constant $C(\mathcal{D}, K, \tilde{\psi})$ such that the estimate

$$\sup_K |f| \leq C(\mathcal{D}, K, \tilde{\psi}) \int_{\mathcal{D}} |\tilde{\psi}||f| \tag{33}$$

holds for any function f, holomorphic in $\mathcal{D}$.

Theorem 7.5. *If the function $\tilde{\psi}$ has no deep zeros, then T has property (β).*

By means of a result by Domar [11], we derive a more tangible criterion.

First we define an auxiliary function F^*. Choose a disc $B(0, R)$, containing the spectrum of T, and let F^* be the decreasing rearrangement of

$\log\log(e+\tilde{\psi}^{-1})|B(0,R)$. That is, F^* is a decreasing non-negative function on $[0,\pi R^2]$ such that, for any $s>0$, the length of the interval $\{t : F^*(t)\geq s\}$ is equal to the area measure of the set $\{z\in B(0,R) : \log\log(e+\tilde{\psi}(z)^{-1})\geq s\}$.

Theorem 7.6. *As usual, assume (D′), (B) and (K). If Ψ is a Hölder-α function for some $\alpha\in(0,1]$ and*

$$\int_0^{\varepsilon}(t^{-1}F^*(t))^{1/2}<\infty \tag{34}$$

for some positive ε, then T has property (β).

Theorem 7.6 has a simple corollary about decomposability.

Corollary 7.7. *Suppose the conditions (D′) and (B). Suppose that for μ-a.e. z, the linear span of the vectors $u_1(z),\dots,u_m(z)$ coincides with the linear span of the vectors $v_1(z),\dots,v_m(z)$. If Ψ is a Hölder-α function for some $\alpha\in(0,1]$ and (34) holds, then T is decomposable.*

Indeed, one obtains the representation of T^* as a perturbation of N^* by passing to conjugates in (1). The hypotheses on u_j and v_j imply that both representations of T and of T^* satisfy condition (K). Moreover, in this case the corresponding perturbation matrices Ψ_T and Ψ_{T^*} are of size m and satisfy $\Psi_{T^*}(z)=\Psi_T(\bar{z})^*$. (If only (K) is assumed, one has to add new "fake" terms to the representation of T^*, which makes the size of Ψ_{T^*} greater than m.) Hence $\tilde{\psi}_{T^*}(z)=\tilde{\psi}_T(\bar{z})$. So we get (β) for both T and T^*, and therefore T is decomposable.

Example 7.8. The decomposability can fail if ψ is a smooth function that vanishes on an smooth arc δ and decays very rapidly when approaching to this arc. This was (rather briefly) explained in [49] at the end of Section 6. We reproduce this argument here, supplying more details. Assume that μ is absolutely continuous with respect to the area measure and $m=1$. Assume also that $\mathcal{H}=\mathcal{H}_0$. Notice the following simple fact: if $S\in L(X)$ is a Hilbert space operator and $W:K\to C(\delta)$ is its diagonalization, that is, $(WSx)(z)=z(Wx)(z)$ for all $x\in X$, then $(Wx)|\delta\setminus\sigma(S)=0$, $x\in X$. Indeed, it follows that the function $\lambda\mapsto(\cdot-\lambda)^{-1}Wx\in C(\delta)$ extends analytically from $\mathbb{C}\setminus(\sigma(S)\cup\delta)$ to $\mathbb{C}\setminus\sigma(S)$.

Fix a domain $\mathcal{B}\supset\sigma(T)$. If $\delta\subset\sigma(T)$ and ψ and $\bar{\partial}\psi$ decay sufficiently fast when approaching δ, then, as explained in [49], the operator $Wf:=f|\delta$ is a well-defined diagonalization operator from the model space $\mathcal{Q}(\mathcal{B})$ to a space of quasianalytic functions on δ. Take now any open cover $\sigma(T)\subset U_1\cup U_2$ such that each of the sets $\delta\setminus U_1$ and $\delta\setminus U_2$ contains a subarc of δ. If there exists the corresponding decomposition $L^2(\mu,H_0)=X_1+X_2$, as in (29), then $(Wx)|\delta\setminus U_j=0$ for $x\in X_j$. By quasianalyticity, $(Wx)|\delta=0$ for all $x\in X_j$ and hence for all $x\in L^2(\mu,H_0)$, which is obviously false. This shows that T cannot be decomposable.

Remark 7.9. Notice also that Corollary 7.7 ensures decomposability in many cases when the zero set of ψ is larger than it is allowed by Theorem 7.2

(which assumes it to be of zero length). The arguments similar to those in the above example show that, whenever the zero set of ψ on some set $\mathbb{C} \setminus \mathcal{G}_s(\mu)$ is connected, operator T will not be dissectible, in general.

However, Corollary 7.7 says nothing if the zero set of ψ contains an open set U. If T is a rank one perturbation of N, then by applying the model Theorem 3.7 and the diagonalization map $w \in \mathcal{Q}(\mathcal{B}) \mapsto w|U$ (which is well-defined, because $\Psi|U = \psi|U = 0$), it is easy to see that T is not decomposable. We do not know whether this fact extends to higher rank perturbations.

We remark that condition (34) resembles the well-known criterion for decomposability by Lyubich and Macaev [32].

Diagonalization operators as above are certainly important in the spectral study of perturbations of normal operators and have been used intensively in [49]. Some additional material can be found in unpublished preprint [50] by the second author, where completeness of generalized eigenvectors of T of several kinds has been discussed. Later, the local spectral multiplicity and completeness of "systems of generalized eigenvectors" have been defined and studied in [51] for a general Banach space operator.

We turn now to the proofs. We return to the functional model described in the previous sections. Henceforth we adopt the notation $W_0 x = W_{0,\widehat{\mathbb{C}}}\, x$, where $W_{0,\widehat{\mathbb{C}}}$ is the transform defined in (16) for the case of $\Omega = \widehat{\mathbb{C}}$. We set

$$\mathrm{Mod}(\widehat{\mathbb{C}}) = W_0 L^2(\mu, H_0).$$

Notice that for any x in $L^2(\mu, H_0)$,

$$\langle x, v\rangle = \big(zW_0 x(z)\big)\big|_{z=\infty}.$$

Moreover, by (17),

$$W_0 Tx(z) = zW_0 Tx(z) + \Psi(z)\,\langle x, v\rangle, \quad x \in L^2(\mu, H_0). \tag{35}$$

We will use the space $\mathcal{O}(\mathcal{D}, \mathrm{Mod}(\widehat{\mathbb{C}})) = \mathcal{O}(\mathcal{D}) \otimes \mathrm{Mod}(\widehat{\mathbb{C}})$. For a function $f(\lambda, z)$ in this space, $f(\lambda, \cdot)$ is an element of $\mathrm{Mod}(\widehat{\mathbb{C}})$ for any $\lambda \in \mathcal{D}$ and the map $\lambda \mapsto f(\lambda, \cdot) \in \mathrm{Mod}(\widehat{\mathbb{C}})$ is analytic. We start by the following observation.

Lemma 7.10. *Let $\mathcal{D}$ be any domain in $\mathbb{C}$ and let $\mathcal{D}_0$ be its subdomain with admissible boundary such that $\overline{\mathcal{D}}_0 \subset \mathcal{D}$. Then the restriction-to-diagonal map*

$$f(\lambda, z) \in \mathcal{O}(\mathcal{D}, \mathrm{Mod}(\widehat{\mathbb{C}})) \mapsto f(z, z) \in \mathrm{Mod}(\mathcal{D}_0)$$

is well defined and continuous. It sends to 0 any function of the form $(z - \lambda) f(\lambda, z)$, where $f(\lambda, z) \in \mathcal{O}(\mathcal{D}, \mathrm{Mod}(\widehat{\mathbb{C}}))$.

Proof. Let $\mathcal{D}_1$ be a domain with smooth boundary, relatively compact in $\mathcal{D}$ and containing the closure of $\mathcal{D}_0$. Due to the nuclearity of $\mathcal{O}(U)$, for

any open set U, the restriction map factors through the complete projective tensor product

$$\mathcal{O}(\mathcal{D}) \otimes \mathrm{Mod}(\widehat{\mathbb{C}}) \longrightarrow L^2_a(\mathcal{D}_1)\hat{\otimes}_\pi \mathrm{Mod}(\widehat{\mathbb{C}}) \longrightarrow \mathcal{O}(\mathcal{D}_0) \otimes \mathrm{Mod}(\widehat{\mathbb{C}}).$$

Above $L^2_a(\mathcal{D}_1)$ stands for the Bergman space.

Choose an orthonormal basis $\{f_n(\lambda)\}$ of $L^2_a(\mathcal{D}_1)$ such that any $f \in \mathcal{O}(\mathcal{D})$ expands when restricted to $\mathcal{D}_1$ into a convergent series $\sum c_n f_n(\lambda)$. Thus, after taking restrictions to $\mathcal{D}_1$ of the elements of $\mathcal{O}(\mathcal{D}) \otimes \mathrm{Mod}(\widehat{\mathbb{C}})$ we can work with convergent series with respect to the projective norm:

$$f(\lambda, z) = \sum f_n(\lambda) g_n(z) \tag{36}$$

where $g_n(z) \in \mathrm{Mod}(\widehat{\mathbb{C}})$. Moreover, the maps $f \in L^2_a(\mathcal{D}_1, \mathrm{Mod}(\widehat{\mathbb{C}})) \mapsto g_n|\mathcal{D}_0$ are bounded and their norms decay exponentially as $n \to \infty$. This implies that the above restriction-to-diagonal operator is bounded. If $f(\lambda, z)$ is expressed in a series as above, then the restriction-to-diagonal operator sends both functions $zf(\lambda, z)$ and $\lambda f(\lambda, z)$ to $\sum z f_n(z) g_n(z)$, which implies the last statement. □

Proof of Theorem 7.5. We have to prove that for every domain $\mathcal{D}$ in $\mathbb{C}$ and every sequence of $L^2(\mu, H_0)$-valued analytic functions $x_n(\lambda)$ defined for $\lambda \in \mathcal{D}$, subject to

$$\lim_n (T - \lambda) x_n(\lambda) = 0 \tag{37}$$

with respect to the topology of $\mathcal{O}(\mathcal{D}, L^2(\mu, H_0))$ satisfies

$$\lim_n x_n = 0 \quad \text{in } \mathcal{O}(\mathcal{D}, L^2(\mu, H_0))).$$

Notice that (37) rewrites as

$$\lim_n \left[(N - \lambda) x_n(\lambda) + u^t \langle x_n(\lambda), v\rangle\right] = 0 \quad \text{in } \mathcal{O}(\mathcal{D}, L^2(\mu, H_0)). \tag{38}$$

We already know that normal operators (and more general, all decomposable operators) have property (β). Hence it is sufficient to prove that

$$\lim_n \langle x_n, v\rangle = 0 \quad \text{in } \mathcal{O}(\mathcal{D}, \mathbb{C}^m). \tag{39}$$

Set $f_n(\lambda, \cdot) = W_0 x_n(\lambda)$.

By applying the isomorphism W_0 to (37) and using (35), we get

$$(z - \lambda) f_n(z, \lambda) + \Psi(z) \langle x_n(\lambda), v\rangle \to 0 \quad \text{in } \mathcal{O}(\mathcal{D}, \mathrm{Mod}(\widehat{\mathbb{C}})).$$

Put $a_n(\lambda) = \langle x_n(\lambda), v\rangle$. Take any compact subset K of $\mathcal{D}$. There is a domain $\mathcal{D}_0$ with admissible boundary such that $K \subset \mathcal{D}_0 \subset \overline{\mathcal{D}_0} \subset \mathcal{D}$. By virtue of the above Lemma, we get

$$\Psi(z) a_n(z) \to 0 \quad \text{in } \mathrm{Mod}(\mathcal{D}_0).$$

Then

$$\max_{z \in K} \|a_n(z)\| \le C(\mathcal{D}_0, K) \|\tilde{\psi} a_n\|_{L^1(\mathcal{D}_0)} \le C(K, \mathcal{D}_0) \|\Psi a_n\|_{L^1(\mathcal{D}_0)} \to 0$$

as $n \to \infty$. This implies that $a_n \to 0$ in $C(K) \otimes \mathbb{C}^m$. We conclude that (39) holds. □

Proof of Theorem 7.6. We prove that the hypotheses imply $\tilde{\psi}$ has no deep zeros. First we observe that for any two $m \times m$ invertible matrices A and B, the formula $B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1}$ implies that

$$\big| \|A^{-1}\|^{-1} - \|B^{-1}\|^{-1} \big| \le \|A - B\|.$$

By passing to a limit, we get that this inequality holds for arbitrary $m \times m$ matrices A and B. This implies that $\tilde{\psi}$ is Hölder-α whenever the matrix-valued function Ψ is.

So, we assume $|\tilde{\psi}(z) - \tilde{\psi}(w)| \le C_0 |z - w|^\alpha$ for all $z, w \in \mathbb{C}$. Fix a bounded domain $\mathcal{D}$ and its compact subset K and take an arbitrary function f, analytic in $\mathcal{D}$; we have to check (33). Since $|\tilde{\psi}| > \delta > 0$ on the complement of $B(0, R)$, we may assume that $\mathcal{D} \subset B(0, R)$. Choose a compact set K_0 such that $K \subset K_0^o \subset K_0 \subset \mathcal{D}$. The distance $d_0 := \operatorname{dist}(K_0, \partial\mathcal{D})$ is positive. Put

$$r_0(z) = \min\big(d_0,\, (2C_0)^{-1/\alpha} |\tilde{\psi}(z)|^{1/\alpha}\big).$$

Let $z \in K_0$. Then any point $w \in B(z, r_0(z))$ is in $\mathcal{D}$ and

$$|\tilde{\psi}(z) - \tilde{\psi}(w)| \le C_0 |z - w|^\alpha \le |\tilde{\psi}(z)|/2.$$

Hence $|\tilde{\psi}(w)| \ge |\tilde{\psi}(z)|/2$ for all $w \in B(z, r_0(z))$. We get

$$|f(z)| \le \frac{1}{\pi r_0(z)^2} \int_{B(z, r_0(z))} |f| \, dA$$
$$\le \frac{2}{|\tilde{\psi}(z)| \, \pi r_0(z)^2} \int_{B(z, r_0(z))} |\tilde{\psi}(w) f(w)| \, dA(w) \le C |\tilde{\psi}(z)|^{-1-2/\alpha} \int_{\mathcal{D}} |\tilde{\psi} f| \, dA.$$

Hence the subharmonic function $\log|f|$ satisfies $\log|f| \le C' \log(e + |\tilde{\psi}|^{-1})$ on K_0. By applying Theorem 1 from Domar's work [11] (with $\alpha = n = 2$), we infer that condition (34) ensures that f is locally bounded. □

If (D′) holds and the measure μ satisfies an estimate $\mu(Q) \le C|Q|^\sigma$ for all squares Q, where $\sigma \in (1, 2]$ and C are constants, then Ψ is a Hölder-α function for some positive α. This is shown along the lines of the last part of the proof of Lemma 2.3, using the estimate (21) with some minor modifications. We leave the details to the reader.

Remark 7.11. As we mentioned above, the case when the "dimension" of μ is one is the most difficult for our approach. However, if the spectrum of N lies on a smooth curve and $T - N$ is compact, belongs to the Matsaev class and the spectrum of T does not "fill in" the interior of the curve, then, by a result by Radjabalipour and Radjavi [41], T is decomposable. This fails for larger compact perturbations, see [25]. We also refer to [35] for a study of the property (β) and decomposability of unilateral and bilateral weighted shifts; the later can be viewed as perturbations of the unweighted bilateral shift, which is unitary. We also can mention that functional models

for perturbations of normal operators with spectrum on a straight line or on a curve have been devised by Naboko [36] and by Tikhonov [45].

8. FINAL REMARKS

The present article leaves open a few strings.

Question 8.1. Is it possible to give sufficient conditions for dissectability of T along a curve γ, which can intersect the spectrum of T in a set of positive length, which would be applicable to an arbitrary measure μ, at least for sufficiently smooth perturbations, in a sense to be made precise?

Our construction does not apply to non-dissectible measures μ. Notice, however, that a simplest and most representative measure μ of this type is one that is absolutely continuous with respect to $\mathcal{H}^1$, restricted to a smooth curve. For this case, any finite rank perturbation of N is decomposable, due to the above-cited paper [41]. This is, in fact, much better behavior than in our case, because no conditions on the finite rank perturbation are necessary. So we ask whether it is possible to merge these two cases and to design a technique which would work for arbitrary measures.

Question 8.2. Let N be a normal operator as above and let μ be its scalar spectral measure. For which measures μ, can one prove that *any* finite rank perturbation of N has a nontrivial invariant subspace?

Question 8.3. Give necessary conditions and sufficient conditions for the perturbation T to be *similar* to a normal operator.

Notice that Theorems 7.5 and 7.6 imply a sort of necessary condition (any operator similar to normal is decomposable). However, as examples show, what we obtain is very far from optimal.

For the case studied in [49], it has been proved there that T is similar to N if and only if $\psi \neq 0$ everywhere on $\mathbb{C}$. However, this cannot be true, for instance, if μ is a discrete measure.

Question 8.4. Given an operator N, the corresponding measure μ and one more measure ν on the plane, when is it possible to find a finite rank perturbation T of N, similar to a normal operator N_1, whose scalar spectral measure is ν?

We refer to [37] for answers in the case of rank one perturbations and selfadjoint operators N and N_1. Notice that compact perturbations of normal operators that are normal themselves is a particular case of Voiculescu's study [47] of the scattering theory for commuting tuples of selfadjoint operators.

References

[1] E. Abakumov, A. Baranov, Yu. Belov, Krein-type theorems and ordered structure for Cauchy - de Branges spaces. J. Funct. Anal. 277 (2019), no. 1, 200-226.

[2] E. Albrecht, J. Eschmeier, Analytic functional models and local spectral theory, Proc. London Math. Soc. 75 (1997), no. 2, 323–348.
[3] A. Baranov, Spectral theory of rank one perturbations of normal compact operators. Algebra i Analiz 30(2018), no. 5, 1-56.
[4] A. Baranov, D. Yakubovich, One-dimensional perturbations of unbounded selfadjoint operators with empty spectrum, J. Math. Anal. Appl. 424 (2015), 1404-1424.
[5] A. Baranov, D. Yakubovich, Completeness and spectral synthesis of nonselfadjoint one dimensional perturbations of selfadjoint operators, Adv. Math. 302 (2016), 740-798.
[6] A. Baranov, D. Yakubovich, Completeness of rank one perturbations of normal operators with lacunary spectrum, J. Spectral Theory 8 (2018), 1-32.
[7] S. R. Bell, The Cauchy transform, Potential Theory, and Conformal Mapping, CRC Press, Boca Raton, 2015.
[8] E. Bishop, A duality theorem for an arbitrary operator. Pacific J. Math 9(1959), 379-397.
[9] S. E. Cheremshantsev, Theory of scattering by a Brownian particle. (Russian) Translated in Proc. Steklov Inst. Math. 1991, no. 2, 1-115.
[10] I. Colojoara, C. Foiaş, Theory of Generalized Spectral Operators, Gordon and Breach, New York, 1968.
[11] Y. Domar, Uniform boundedness in families related to subharmonic functions, J Lond Math Soc (2) 38 (1988), 485-491.
[12] N.Dunford and J. L.Schwartz, Linear operators III. Spectral Operators, Wiley Interscience, New York, 1971.
[13] P. Duren, Theory of H^p Spaces. Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London, 1970.
[14] K. Dykema, F. Sukochev and D. Zanin, A decomposition theorem in II_1-factors, Journal für die Reine und Angewandte Mathematik 708 (2015), 97-114.
[15] J. Eschmeier, M. Putinar, Spectral Decompositions and Analytic Sheaves, London Math. Monographs Vol. 10, Clarendon Press, Oxford, 1996.
[16] Q. Fang, J. Xia, Invariant subspaces for certain finite-rank perturbations of diagonal operators, J. Funct. Analysis 263 (2012), 1356-1377.
[17] C. Foiaş, Spectral maximal spaces and decomposable operators in Banach space, Archiv Mat. 14(1963), 341-349.
[18] C. Foiaş, E. B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators. J. Funct. Anal. 253 (2007), no. 2, 628-646.
[19] C. Foiaş, E. B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators. II. Indiana Univ. Math. J. 57 (2008), no. 6, 2745-2760.
[20] C. Foiaş, E. B. Jung, E. Ko, C. Pearcy, Spectral decomposability of rank-one perturbations of normal operators, J. Funct. Analysis 375(2011), 602-609.
[21] D. Frymark, C. Liaw, Spectral Analysis, Model Theory and Applications of Finite-Rank Perturbations, arXiv:1904.09833.
[22] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Amer. Math. Soc., Providence, R.I., 1969.
[23] S. Grivaux, A hypercyclic rank one perturbation of a unitary operator, Math. Nachr., 285(2012), 533-544.
[24] U. Haagerup, H. Schultz, Invariant subspaces for operators in a general II_1-factor, Publ. Math. Inst. Hautes Etudes Sci. 109 (2009), 19-111.
[25] D.A. Herrero, Indecomposable compact perturbations of the bilateral shift, Proc. Amer. Math. Soc. 62 (1977), no. 2, 254-258.
[26] E. Ionascu, Rank-one perturbations of diagonal operators, Integral Equations Oper. Theory 39 (2001), no. 4, 421-440.
[27] N. Kerzman, E.M. Stein, The Cauchy kernel, the Szegö kernel, and the Riemann mapping function. Math. Ann., 236 (1) (1978), pp. 85–93.

[28] D. Khavinson, R. Pereira, M. Putinar, E.B. Saff, S. Shimorin, Borcea's variance conjectures on the critical points of polynomials, in vol. Notions of Positivity and the Geometry of Polynomials (P. Brändén et al., eds.), Trends in Mathematics 2011, Birkhäuser, Basel, pp 283-309.

[29] H. Klaja, Hyperinvariant subspaces for some compact perturbations of multiplication operators, J. Operator Theory 73 (2015), no. 1, 127-142.

[30] H. Klaja, Rank one perturbations of diagonal operators without eigenvalues, Integral Equations Operator Theory 83 (2015), no. 3, 429-445.

[31] M.S. Livšic, On a class of linear operators in Hilbert space. Mat. Sbornik 19 (61), no. 2 (1946), 239–262 (Russian); English translation in Amer. Math. Soc. Transl. (2) 13 (1960), 61–83.

[32] Ju. I. Lyubič, V. I. Macaev, Operators with separable spectrum, Mat. Sb. (N.S.) 56 (98)(1962), 433-468. (in Russian).

[33] C. Liaw, S. R. Treil, Matrix Measures and Finite Rank Perturbations of Self-adjoint Operators, arXiv:1806.08856.

[34] P. Mattila, M. S. Melnikov, Existence and weak-type inequalities for Cauchy integrals of general measures on rectifiable curves and sets, Proc. Amer. Math. Soc. 120(1994), 143-149.

[35] T. L. Miller, V. G. Miller, M. M. Neumann, Local spectral properties of weighted shifts. J. Operator Theory 51 (2004), no. 1, 71–88.

[36] S. N. Naboko, Functional model of perturbation theory and its applications to scattering theory. Boundary value problems of mathematical physics, 10. Trudy Mat. Inst. Steklov. 147 (1980), 86–114, 203 (Russian); English transl. in Proc. Steklov Inst. Math., 147 (1981), 85–116.

[37] A. Poltoratski, Kreĭn's spectral shift and perturbations of spectra of rank one, Algebra and Analysis 10, no. 5 (1998), 143-183, English transl. in St. Petersburg Math. J., 10, no. 5 (1999), 833-859.

[38] A. Poltoratski, D. Sarason, Aleksandrov-Clark measures, Contemp. Math. 393 (2006), 1-14.

[39] I.I. [Privalov] Priwalow, Randeigenschaften analytischer Funktionen , Deutsch. Verlag Wissenschaft, Berlin, 1956 (Translated from the 1950 Russian edition).

[40] Privalov, I. I., The Cauchy integral (in Russian), Saratov, 1919.

[41] M. Radjabalipour, H. Radjavi, On invariant subspaces of compact perturbations of operators. Rev. Roumaine Math. Pures Appl. 21 (1976), no. 9, 1247-1260.

[42] F. and M. Riesz, Über die Randwerte einer analytischen Funktion, Quatrième Congrès des Mathématiciens Scandinaves, Stockholm, (1916), 27–44.

[43] J. Schwartz, Some non-selfadjoint operators. II. A family of operators yielding to Friedrichs' method. Comm. Pure Appl. Math. 14(1961), 619-626.

[44] B. Simon, Spectral Analysis of Rank One Perturbations and Applications, Centre Recherches Math. Proc. vol. 8(1995), 109-142.

[45] A. S. Tikhonov, A functional model and duality of spectral components for operators with a continuous spectrum on a curve. (Russian) Algebra i Analiz 14 (2002), no. 4, 158–195; English transl. in St. Petersburg Math. J. 14 (2003), no. 4, 655–682.

[46] J. Verdera, A weak type inequality for Cauchy transforms of finite measures, Publ. Mat. 36(1992), 1029-1034.

[47] D. Voiculescu, Some results on norm-ideal perturbations of Hilbert space operators. I, J. Operator Theory 2 (1979), 3-37; part II, J. Operator Theory 5(1981), 77-100.

[48] D.Voiculescu, K-theory and perturbations of absolutely continuous spectra, Comm. Math. Phys. 365 (2019), 363-373.

[49] D. V. Yakubovich, Spectral Properties of Smooth Perturbations of Normal Operators with Planar Lebesgue Spectrum, Indiana U. Math J. 42(1993), 55-83.

[50] D. V. Yakubovich, Spectral Properties of Smooth Finite Rank Perturbations of a Planar Lebesgue Spectrum Cyclic Normal Operator, Preprint Institut Mittag-Leffler 15, 1990-1991, 1–37.

[51] Yakubovich, D. V. Local spectral multiplicity of a linear operator with respect to measure. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222 (1995), Issled. po Linein. Oper. i Teor. Funktsii. 23, 293–306, 311; translation in J. Math. Sci. (New York) 87 (1997), no. 5, 3971–3979.

[52] M. Zinsmeister, Les domaines de Carleson. Michigan Math. J. 36 (1989), no. 2, 213–220.

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