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Author

Nguyen, David Tuan

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David Tuan Nguyen

Committee in charge:

Professor Yitang Zhang, Chair
Professor Jeffrey Stopple
Professor Mihai Putinar

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The Dissertation of David Tuan Nguyen is approved.

Professor Jeffrey Stopple

Professor Mihai Putinar

Professor Yitang Zhang, Committee Chair

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Topics in Multiplicative Number Theory

Dedicated to: all my teachers.

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Isla Vista,
May 2021.

Curriculum Vitæ
David Tuan Nguyen

Education

2016-2021	Ph.D. in Mathematics, University of California, Santa Barbara. Advisor: Yitang Zhang Thesis: Topics in Multiplicative Number Theory
2016-2019	M.A. in Mathematics, University of California, Santa Barbara.
2014-2016	M.S. in Mathematics, Cal Poly Pomona.
2009-2014	B.S. in Mathematics, Cal Poly Pomona.
2007-2009	A.S. in Engineering, Orange Coast College.

Publications

1. Generalized divisor functions in arithmetic progressions: I. The k-fold divisor function in arithmetic progressions to large moduli, *J. Number Theory* **227** (2021), 30-93.

2. Explicit Formulas for Enumeration of Lattice Paths: Basketball and the Kernel Method (with C. Banderier, C. Krattenthaler, A. Krinik, D. Kruchinin, V. Kruchinin, and M. Wallner), *Springer Developments in Mathematics Series*, vol. **58** (2019), 78-118.

Preprints

1. Generalized divisor functions in arithmetic progressions: II. Variance of the ternary divisor function in arithmetic progressions (2021), 22 pages.

2. A note on the zeros of the derivative of the Riemann zeta function near the critical line (2020), 6 pages.

Abstract

Topics in Multiplicative Number Theory

by

David Tuan Nguyen

We investigate several topics in multiplicative number theory on the distributions of the k-fold divisor function in arithmetic progressions, divisor sums, and gaps between zeros of L-functions. Applications of these topics to the distribution of prime numbers, zeros of the derivative of the Riemann zeta function near the critical line, the generalized de Bruijn-Newman constant, and connections to moments of L-functions and random matrix theory are also discussed.

Contents

Chapter 1

Introduction

1.1 Overview and summary of statement of results

We summarize in this first chapter statement of results of this dissertation. The expositions here are brief and we refer to the Introduction sections in subsequent chapters for more background and discussions.

1.1.1 Distribution of the k-fold divisor function in arithmetic progressions

For a fixed natural number k, let $\tau_k(n)$ denote the k-fold divisor function: the number of ways to write a natural number n as an ordered product of k positive integers. The distribution of $\tau_k(n)$

$$\sum_{n \le X} \tau_k(n)$$

and over arithmetic progressions

$$\sum_{\substack{n \le X \\ n \equiv a (\bmod d)}} \tau_k(n) \tag{1.1}$$

are an important topic and has many applications in analytic number theory. In particular, understanding the distribution (1.1) for the 3-fold divisor function $\tau_3(n)$ was one of the crucial steps in the striking proof of bounded gaps between primes of Y. Zhang [87]. Further understanding of the 4-fold divisor function, which is rather poor at the present, will undoubtedly have application in reducing the current gap of 246 between consecutive pairs of primes to a smaller number. In our first result, we prove a distribution estimate for $\tau_k(n)$, $k \geq 4$, in arithmetic progressions to a special set of moduli d that exceeds the square-root of the length X of the sum.

Theorem 1 *Let*

$$\varpi = \frac{1}{1168}$$

and

$$\theta_k = \min\left\{ \frac{1}{12(k+2)}, \varpi^2 \right\}.$$

For $a \neq 0$, let

$$\mathcal{D} = \{d \geq 1 : (d,a) = 1,\ |\mu(d)| = 1,\ (d, \prod_{p \leq X^{\varpi^2}} p) < X^{\varpi},\ \textit{and}\ (d, \prod_{p \leq X^{\varpi}} p) > X^{71/584}\},$$

where μ is the Möbius function. Then for each $k \geq 4$ we have

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{293/584}}} \left| \sum_{\substack{n \leq X \\ n \equiv a (\textit{mod } d)}} \tau_k(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right| \ll X^{1-\theta_k}. \tag{1.2}$$

The implied constant is effective, and depends at most on a and k.

See Remark 2 for a possible application of this theorem to prime numbers and Section 1.2 below for notation. Conditionally, if we assume the Generalized Lindelöf Hypothesis for Dirichlet L-functions we can obtain a stronger result.

Theorem 2 *On the Generalized Lindelöf Hypothesis, the estimate (1.2) holds with the right side replaced by*

$$X^{1-\varpi^2},$$

where the θ_k power saving is replaced by a positive constant independent of k.

In our next result, we prove a distribution estimate for $\tau_k(n)$ similar to that of Barban-Davenport-Halberstam type theorems for the von Mangoldt function $\Lambda(n)$ averaging over both a and d to moduli d as large as X.

Theorem 3 *For $k \geq 4$ we have*

$$\sum_{d\leq D}\sum_{\substack{a=1\\(a,d)=1}}^{d}\left|\sum_{\substack{n\leq X\\ n\equiv a(mod\ d)}}\tau_k(n)-\frac{1}{\varphi(d)}\sum_{\substack{n\leq X\\(n,d)=1}}\tau_k(n)\right|^2 \ll (D+X^{1-1/6(k+2)})X(\log X)^{k^2-1}.$$

Motivated by the recent work [42] of Heath-Brown and Li in 2017, we also prove analogous estimates for pairs of $\tau_k(n)$'s and $\tau_k(n)\Lambda(n)$ to moduli d that can taken to be almost as large as X^2.

Theorem 4 *For $k \geq 4$ and any $\epsilon > 0$ there holds*

$$\sum_{d\leq D}\sum_{\substack{a=1\\(a,d)=1}}^{d}\left(\sum_{\substack{m,n\leq X\\ m\equiv an(mod\ d)}}\tau_k(m)\tau_k(n)-\frac{1}{\varphi(d)}\left(\sum_{\substack{n\leq X\\(n,d)=1}}\tau_k(n)\right)^2\right)^2 \ll X^{4-1/3(k+4)}$$

for any $D \leq X^{2-1/3(k+2)}$.

In particular, the above estimate is valid if one of the τ_k is replaced by the von Mangoldt function Λ. We have

$$\sum_{d\leq D}\sum_{\substack{a=1\\(a,d)=1}}^{d}\left(\sum_{\substack{m,n\leq X\\ m\equiv an(mod\ d)}}\tau_k(m)\Lambda(n)-\frac{X}{\varphi(d)}\sum_{\substack{n\leq X\\(n,d)=1}}\tau_k(n)\right)^2 \ll X^{4-1/3(k+4)}$$

for any $D \leq X^{2-1/3(k+2)}$.

We replace the upper bound in Theorem 3 for the 3-fold divisor function by an asymptotic with the condition $(a,d)=1$ removed.

Theorem 5 *We have the following asymptotic equality, with effectively computable numerical constants*

$$\mathfrak{S}_j, \ (0 \le j \le 8),$$

$$\sum_{1\le \ell \le N} \sum_{1 \le b \le \ell} \left| \sum_{\substack{1\le n\le N \\ n\equiv b (mod\ \ell)}} \tau_3(n) - NP_2(\log N) \right|^2 = N^2 \sum_{j=0}^{8} \mathfrak{S}_{8-j} \log^{8-j} N + O\left(N^{599/300}\right),$$

where

$$P_2(\log N) = \operatorname*{Res}_{s=1} \left\{ \sum_{n\equiv b (mod\ \ell)} \frac{\tau_3(n)}{n^s} \frac{N^{s-1}}{s} \right\}.$$

Motivated by the recent work of Banks, Heath-Brown, and Shparlinski [2] in 2005 on the distribution of the divisor function $\tau(n)$ on average, we prove an analogous result for $\tau_3(n)$, in particular, slightly extending the range of the modulus from the well-known result [31] of Friedlander and Iwaniec in 1985.

Theorem 6 *For any $\epsilon > 0$, we have*

$$\sum_{\substack{1\le a\le q \\ (a,q)=1}} \left| \sum_{\substack{n\le X \\ n\equiv a (mod\ q)}} \tau_3(n) - \frac{X}{\varphi(q)} P_2(\log X) \right| \ll \begin{cases} X^{5/6+\epsilon} q^{1/4}, & if\ X^{1/2} < q < X^{2/3}, \\ X^{7/9+\epsilon} q^{1/2} & if\ X^{1/4} < q < X^{4/9}, \\ X^{11/12+\epsilon}, & if\ 1 \le q \le X^{1/4}, \end{cases}$$

where $P_2(\log X)$ is a polynomial of degree 2 in $\log X$ with coefficients depending on q, and the implied constant depends only on ϵ.

We obtain an asymptotic for a modified additive divisor sum for $\tau_3(n)$ averaged over the shift h up to $X - 1$.

Theorem 7 *There are numerical constants c_0, c_1, c_2, c_3 such that*

$$\sum_{1\le h\le X-1} \sum_{1\le n\le X-h} \tau_3(n)\tau_3(n+h) = X^2 \left(\frac{\log^4 X}{4} + c_3 \log^3 X + c_2 \log^2 X + c_1 \log X + c_0 \right) + O(X^{3/2+\epsilon}),$$

for any $\epsilon > 0$.

This result is closely connected to the sixth power moment of the Riemann zeta function. For any $\epsilon > 0$, it is known that the Lindelöf hypothesis

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{\epsilon}) \iff \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll T^{1+\epsilon} \ \forall k$$

is equivalent to good bounds on moments of zeta. Understanding moments of zeta requires a good handling on the so-called correlation sum

$$\sum_{n\leq X} \tau_k(n)\tau_k(n+h), \ h \leq X.$$

In fact, even an asymptotic formula for

$$\sum_{n\leq X} \tau_3(n)\tau_3(n+1) \tag{1.3}$$

is not known. In Remark 8, we comment on the intractability of obtaining an asymptotic for the correlation (1.3).

1.1.2 Gaps between zeros of L-functions

We give a quantitative lower bound for the positive proportion of zeros of $\zeta'(s)$ near the critical line, complementing the result of Zhang [86, Theorem 1].

Theorem 8 *For* $\nu > 10^{22}$, *we have*

$$m^{-}(\nu) > 10^{-83},$$

where the function $m^{-}(\nu)$ *is defined in* (7.1).

We extend this result to gaps between zeros of quadratic Dirichlet L-functions $L(s,\chi)$.

Theorem 9 *There exist computable constants $c > 0$ and $\lambda > 1$ such that for T large, we have*

$$\sum_{\substack{0<\gamma_n<T \\ \frac{\gamma_{n+1}-\gamma_n}{2\pi/\log T}>\lambda}} 1 \geq cN(T,\chi),$$

where $N(T,\chi)$ is the number of zeros $1/2+i\gamma_n$ of $L(s,\chi)$, χ a quadratic Dirichlet character modulo D, with imaginary parts at most T.

This result, together with the recent work of Rodgers and Tao [72] in 2020 on the Newman conjecture for $\zeta(s)$, has possible application to the generalized Newman conjecture introduced by Stopple in [75]; see Section 9.1 in Chapter 9 Future projects for more discussions.

1.2 Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$.

p–a prime number.

a, b, c–integers.

$d, n, m, k, q, r, s, Q, R$–positive integers.

$\Lambda(q)$–the von Mangoldt function.

$\tau_k(n)$–the k-fold divisor function: the number of ways to write a natural number n as an ordered product of k positive integers.

$\tau(n) = \tau_2(n)$–the usual divisor function.

$\varphi(n)$–the Euler's totient function.

$s = \sigma + it$

X–a large real number.

$\mathcal{L} = \log X$.

$\chi(n)$–a Dirichlet character.

$e(y)$–the additive character $\exp\{2\pi iy\}$.

$e_d(y) := \exp\{2\pi iy/d\}$.

$c_q(b)$: Ramanujan's sum

$$c_q(b) = \sum_{\substack{1\le a\le q\\(a,q)=1}} e_q(ab).$$

$\hat{f}$–the Fourier transform of f, i.e.,

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(y)e(yz)dy.$$

$m \equiv a(q)$ means $m \equiv a(\text{mod } q)$.

$q \sim Q$ means $Q \le q < 2Q$.

ϵ–any sufficiently small, positive constant, not necessarily the same in each occurrence.

B–some positive constant, not necessarily the same in each occurrence.

$\|\alpha\|$–means the L^2 norm of $\alpha = (\alpha(m))$, i.e.,

$$\|\alpha\| = \left(\sum_m |\alpha(m)|^2\right)^{1/2}.$$

χ_N–the characteristic function of the subset $[N, (1+\rho)N) \subset \mathbb{R}$.

$\sideset{}{'}\sum_{\chi(\text{mod } d)}$ –means a summation over nonprincipal characters $\chi(\text{mod } d)$.

$\sideset{}{^*}\sum_{\chi(\text{mod } d)}$ –means a summation over primitive characters $\chi(\text{mod } d)$.

$\sum_{b(\text{mod } q)}$ –means $\sum_{b=1}^{q}$.

$\sideset{}{^*}\sum_{b(\text{mod } q)}$ –means $\sum_{\substack{b=1\\(b,q)=1}}^{q}$.

$\log_+(x) = \log(2+|x|)$.

$\Gamma(s)$: Gamma function.

γ: Euler's constant $= 0.5722\ldots.$

$\gamma_0(\alpha)$: 0-th generalized Stieltjes constant

$$\gamma_0(\alpha) = \lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{1}{k+\alpha} - \log(m+\alpha)\right).$$

(m, n): the greatest common divisor of m and n.

$[m, n]$: the least common divisor of m and n.

$P_r(\log N)$: a polynomial of degree r in $\log N$, not necessarily the same in each occurrence.

We follow standard notations and write $f(X) = O(g(X))$ or $f(X) \ll g(X)$ to mean that $|f(X)| \leq Cg(X)$ for some fixed constant C, and $f(X) = o(g(X))$ if $|f(X)| \leq c(X)g(X)$ for some function $c(X)$ that goes to zero as X goes to infinity. The sequences $\alpha(n)$ and $\beta(n)$ we consider are all real; in particular, the absolute value sign is not needed in several expressions.

1.3 Permissions and Attributions

1. Theorems 1-4 of this dissertation are based on the author's article titled "Generalized divisor functions in arithmetic progressions: I" [68], published in *Journal of Number Theory*, `doi.org/10.1016/j.jnt.2021.03.021`, Copyright © 2021 Elsevier.

Chapter 2

The k-fold divisor function in arithmetic progressions to large moduli

2.1 Introduction and statement of results

Let $n \geq 1$ and $k \geq 1$ be integers. Let $\tau_k(n)$ denote the *k-fold divisor function*

$$\tau_k(n) = \sum_{n_1 n_2 \cdots n_k = n} 1,$$

where the sum runs over ordered k-tuples $(n_1, n_2, \ldots, n_k)$ of positive integers for which $n_1 n_2 \cdots n_k = n$. Thus $\tau_k(n)$ is the coefficient of n^{-s} in the Dirichlet series

$$\zeta(s)^k = \sum_{n=1}^{\infty} \tau_k(n) n^{-s}.$$

We investigate the distribution of $\tau_k(n)$ in arithmetic progressions to moduli d that exceed the square-root of length of the sum, in particular, provides a sharpening of the result in [82]. We begin by giving background of the problem.

2.1.1 Distribution of arithmetic functions

C. F. Gauss, towards the end of the 18th century, conjectured the celebrated Prime Number Theorem concerning the sum

$$\sum_{p\leq X} 1$$

as X approaches infinity, where p denotes a prime. It is more convenience to count primes with weight $\log p$ instead of weight 1, c.f. Chebyshev; this leads to consideration of the sum

$$\sum_{p\leq X} \log p.$$

To access the Riemann zeta function more conveniently we also count powers of primes, leading to the sum

$$\sum_{\substack{p^\alpha\leq X\\ \alpha\geq 1}} \log p,$$

which is equal to the unconstrained sum over n

$$\sum_{n\leq X} \Lambda(n)$$

where $\Lambda(n)$ is the von Mangoldt function–the coefficient of n^{-s} in the series $-\zeta'(s)/\zeta(s)$. In 1837, P. G. L. Dirichlet considered the deep question of primes in arithmetic progression, leading him to consider sums of the form

$$\sum_{\substack{n\leq X\\ n\equiv a (\mathrm{mod}\ d)}} \Lambda(n)$$

for $(d,a)=1$. More generally, the function $\Lambda(n)$ is replaced by an arithmetic function $f(n)$, satisfying certain growth conditions, and we arrive at the study of the congruence sum

$$\sum_{\substack{n\leq X\\ n\equiv a (\mathrm{mod}\ d)}} f(n). \tag{2.1}$$

This sum (2.1) is our main object of study.

For most function f appearing in applications, it is expected that f is distributed equally among the reduced residue classes $a(\text{mod } d)$ with $(a,d)=1$, e.g., that the sum (2.1) is well approximated by the average

$$\frac{1}{\varphi(d)} \sum_{\substack{n \le X \\ (n,d)=1}} f(n) \tag{2.2}$$

since there are $\varphi(d)$ reduced residue classes modulo d, where $\varphi(n)$ is the Euler's totient function. The quantity (2.2) is often thought of as the 'main term'. Different main terms are also considered. Thus, the study of (2.1) is reduced to studying the 'error term'

$$\Delta(f; X, d, a) := \sum_{\substack{n \le X \\ n \equiv a (\text{mod } d)}} f(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \le X \\ (n,d)=1}} f(n), \quad \text{for } (a,d)=1, \tag{2.3}$$

measuring the discrepancy between the the sum (2.1) and the expected value (2.2). If f satisfies

$$f(n) \le C\tau^B(n) \log^B X$$

for some constants $B, C > 0$, which is often the case for most f in applications, then a trivial bound for the discrepancy $\Delta(f; X, d, a)$ is

$$\Delta(f; X, d, a) \le C' X \log^{B'} X,$$

for some constants $B', C' > 0$. The objective is then to obtain an upper bound for $\Delta(f; X, d, a)$ as small as possible for d as large as possible–the smaller the discrepancy and the larger the range of d, the better the distribution estimates for f is.

For us we consider the k-fold divisor function

$$f(n) = \tau_k(n)$$

introduced in the beginning. It is well known that the function τ_k is closely related to prime numbers; see Remark 1 below. Let us next survey known results on the distribution of $\tau_k(n)$.

2.1.2 Individual estimates for each modulus d

For $(a,d)=1$ define

$$T_k(X,d,a) = \sum_{\substack{n\le X\\ n\equiv a (\mathrm{mod}\ d)}} \tau_k(n).$$

For $k=1$ we have

$$T_1(X,d,a) = \sum_{\substack{n\le X\\ n\equiv a (\mathrm{mod}\ d)}} 1 = \frac{X}{d} + O(1),$$

and this is valid for all $d < X$. We wish to find real numbers $\theta_k > 0$, as large as possible, such that the following statement holds.

(S1) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$T_k(X,d,a) - \frac{X}{\varphi(d)} P_k(\log X) \ll \frac{X^{1-\delta}}{\varphi(d)} \tag{2.4}$$

uniformly for all $d \le X^{\theta_k - \epsilon}$.

Here $P_k(\log X)$ is a polynomial in $\log X$ of degree $k-1$ given by Cauchy's theorem as

$$P_k(\log X) = \mathrm{Res}(s^{-1}L^k(s,\chi_0)X^{s-1}; s=1),$$

where χ_0 is the principal character of modulus d; for instance, see [30] and [65]. The number θ_k is called the level of distribution for τ_k. It is widely believed that (S1) is valid for all $\theta_k < 1$ for each $k \ge 1$; though, the only known case is for $k=1$. For ease of referencing, we record this as

Conjecture 1 *For each $k \ge 2$ statement (S1) holds for any $\theta_k < 1$.*

The Generalized Riemann Hypothesis implies that statement (S1) holds for all $\theta_k < 1/2$ for any k. In Table 2.1 we summarize known unconditional results towards Conjecture 1. We now give a brief survey of the known results.

Table 2.1: Known results towards Conjecture 1 for individual estimates and references. Only for $k = 1, 2, 3$ is the exponent of distribution θ_k for τ_k known to hold for a value larger than 1/2.

k	θ_k	**References**
$k = 2$	$\theta_2 = 2/3$	Selberg, Linnik, Hooley (independently, unpublished, 1950's); Heath-Brown (1979) [39, Corollary 1, p. 409].
$k = 3$	$\theta_3 = 1/2 + 1/230$	Friedlander and Iwaniec (1985) [31, Theorem 5, p. 338].
	$\theta_3 = 1/2 + 1/82$	Heath-Brown (1986) [40, Theorem 1, p. 31].
	$\theta_3 = 1/2 + 1/46$	Fouvry, Kowalski, and Michel (2015) [27, Theorem 1.1, p. 122], (for prime moduli, polylog saving).
$k = 4$	$\theta_4 = 1/2$	Linnik (1961) [56, Lemma 5, p. 197].
$k \geq 4$	$\theta_k = 8/(3k + 4)$	Lavrik (1965) [55, Teopema 1, p. 1232].
$k = 5$	$\theta_5 = 9/20$	Friedlander and Iwaniec (1985) [30, Theorem I, p. 273].
$k = 6$	$\theta_6 = 5/12$	Friedlander and Iwaniec (1985) [30, Theorem II, p. 273].
$k \geq 7$	$\theta_k = 8/3k$	Friedlander and Iwaniec (1985) [30, Theorem II, p. 273].
$k \geq 5$	$\theta_k \geq 1/2$	Open.

The classical result for $k = 2$ giving $\theta_2 = 2/3$ depends crucially on the Weil bound for Kloosterman's sum $S(a, b; d)$:

$$S(a, b; d) := \sum_{\substack{n=1 \\ (n,d)=1}}^{d} e_d\left(an + b\overline{n}\right) \ll \tau(d)(a, b, d)^{1/2} d^{1/2}. \tag{2.5}$$

This important result for θ_2 is an unpublished result of Selberg, Hooley, and Linnik obtained independently in the 1950's, though none of them seem to have formally written it down. They discovered that Weil bound (2.5) for Kloosterman sums implies that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$T_2(X, d, a) = \frac{X}{\varphi(d)} P_d(\log X) + O\left(\frac{X^{1-\delta}}{\varphi(d)}\right),$$

uniformly for all $d < X^{2/3-\epsilon}$, where P_d is the linear polynomial given by

$$\begin{aligned} P_d(\log X) &= \operatorname{Res}\left\{\zeta^2(s)\prod_{p|d}(1 - p^{-s}\frac{X^{s-1}}{s}; s = 1\right\} \qquad (2.6) \\ &= \frac{\varphi(d)^2}{d^2}(\log X + 2\gamma - 1) - \frac{2\varphi(d)}{d}\sum_{\delta|d}\frac{\mu(\delta)\log\delta}{\delta}. \end{aligned}$$

(Note that the main term here is differed from that in (2.3) by an admissible quantity

$$\sum_{\substack{n\le X\\(n,d)=1}} \tau(n) - XP_d(\log X) \ll X^{1/2}d^{\epsilon};$$

see, e.g., [2, Lemma 2.5, p. 7].) As noted in [40], the work [43] of Hooley in 1957 essentially gives this result for θ_2. A formal proof of this result can be found in [39, Corollary 1, p. 409], the work of Heath-Brown in 1979.

The *divisor problem for arithmetic progressions* (c.f. [30]) then amounts to extending θ_2 beyond $2/3$. This difficult question has seen no improvement since the 1950's. In one aspect, the problem is asking for a better uniform estimates for Kloosterman sums beyond those that are immediately available from the Weil bound (2.5); see, e.g., the discussion in [2]. The next, and only known, case where $\theta_k > 1/2$ is for $k = 3$.

Friedlander and Iwaniec's spectacular breakthrough work [31] in 1985 yields, in particular, $\theta_3 = 58/115 > 1/2$ for $k = 3$. More precisely, they proved in [31, Theorem 5, p. 338] that, for any $\epsilon > 0$, $X^{92/185} < d < X^{58/115}$, $(a, d) = 1$, we have

$$T_3(X, d, a) = \frac{X}{\varphi(d)}P(\log X) + O(X^{A+\epsilon}d^{-B}),$$

where $A = 271/300$, $B = 97/120$, and P is the quadratic polynomial for which $P(\log X)$ is the residue at $s = 1$ of $(\prod_{p|d}(1-p^{-s}\zeta(s))^3(X^{s-1}/s)$. The proof uses multiple exponential sums which rests on Deligne's deep work on the Riemann Hypothesis over finite fields together with Burgess' bounds [10] on character sums.

Heath-Brown's improvement [40, Theorem 1, p. 31] in 1986 of this exponent of distribution θ_3 to $1/2+1/82$ gives a different proof and removes the condition $(a, d) = 1$. Consequently, he replaced the expected main term in (2.4) by

$$M_k(X, d, a) = \frac{X}{\varphi(d/\delta)}\text{Res}\left\{ (\sum_{\substack{m=1\\(m,d)=\delta}}^{\infty} \tau_k(m)m^{-s})\frac{X^{s-1}}{s}; s = 1\right\}$$

where $\delta = (a, d)$. Heath-Brown showed in [40, Theorem 1, p. 31] that for any $\epsilon > 0$, if $q \leq X^{21/41}$, then

$$T_3(X, d, a) = M_3(X, d, a) + O(X^{86/107+\epsilon} d^{-66/107}).$$

His proof uses deep estimates for multiple Kloosterman sums also powered by Deligne's Riemann Hypothesis.

Most recently, in 2015, Fouvry, Kowalski, and Michel's result [27] for $k = 3$ to prime moduli uses the spectral theory of modular forms and estimates for exponential sums using Frobenius trace functions. They showed in [27, Theorem 1.1, p. 122] that for every non-zero integer a, every $\epsilon, A > 0$, every $X \geq 2$, and every prime q, coprime with a, satisfying

$$q \leq X^{1/2+1/46+\epsilon},$$

we have

$$\Delta(\tau_3; X, q, a) \ll \frac{X}{q \log^A X},$$

where the implied constant depends on ϵ and A and not on a. The estimates for $k = 1, 2$, and 3 are the only known cases where (S1) holds with $\theta_k > 1/2$.

The exponent of distribution $\theta_4 = 1/2$ for $k = 4$ is explicit in Linnik's work [56] of 1961. Lavrik's result [55] in 1965 for $k \geq 4$ uses Burgess' estimates for character sums and the fourth power moment estimate for $L(s, \chi)$ averaged over characters χ modulo q. Friedlander and Iwaniec's improvement [30] in 1985 of Lavrik's result for $k \geq 5$ uses Burgess' estimates for character sums and Heath-Brown and Iwaniec's work [41] on the difference between consecutive primes.

Remark 1 *The k-fold divisor problem for arithmetic progressions, which asks whether the range of d for which statement (S1) holds can be extended beyond $X^{1/2}$ for $k \geq 4$, has important implications to the distribution of prime numbers. For instance, the estimate*

(2.4) *has application to the asymptotic for the divisor problem*

$$\sum_{n \le X} \tau_k(n)\tau_\ell(n+h).$$

See, for instance, the recent works [58, 59], or [85, p. 4] for discussion towards application to power moments of the Riemann zeta function.

More concretely, the distribution of the ternary divisor function $\tau_3(n)$ in arithmetic progression [31] has shown to play an important rôle in the sensational work of Y. Zhang [87] towards the problem of bounded gaps between primes, bringing the gap from infinity to a finite number. The distribution of τ_4 to large moduli will undoubtedly have important consequence to prime numbers. See also [23, Théorème 4] for precise connection between distribution of τ_k and distribution of prime numbers.

2.1.3 Average estimates over d

To obtain further progress on Conjecture 1, a larger range for d can be obtained by averaging over d. This type of result is of Bombieri-Vinogradov type. More precisely, let (S2) be the following statement.

(S2) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{d < X^{\theta_k}} \max_{(a,d)=1} \left| T_k(X,d,a) - \frac{X}{\varphi(d)} P_k(\log X) \right| \ll X^{1-\delta}, \qquad (2.7)$$

provided that $d \le X^{\theta_k - \epsilon}$.

We have an analogous conjecture for θ_k.

Conjecture 2 *For each $k \ge 2$ statement (S2) holds for any $\theta_k < 1$.*

Of course, Conjecture 1 implies Conjecture 2. In Table 2.2 we list known results towards Conjecture 2.

Table 2.2: Known results towards Conjecture 2 for average over all moduli d, and references.

$\boldsymbol{k}$	$\boldsymbol{\theta_k}$	**References**
$k = 2$	$\theta_2 = 1$	Fouvry (1985) [24, Corollaire 5, p. 74] (exponential saving); Fouvry and Iwaniec (1992) [26, Theorem 1, p. 272].
$k = 3$	$\theta_3 = 1/2 + 1/42$	Heath-Brown (1986) [40, Theorem 2, p. 32].
	$\theta_3 = 1/2 + 1/34$	Fouvry, Kowalski, and Michel (2015) [27, Theorem 1.2, p. 123], (for prime moduli, polylog saving).
$k \geq 4$	$\theta_k < 1/2$	Follows from the general version of Bombieri-Vinogradov theorem, see, e.g., [65] or [83], (polylog saving).
$k \geq 4$	$\theta_k \geq 1/2$	Open.

Averaging over d, Fouvry in 1985 was able to break through the $X^{2/3}$-barrier for θ_2 and showed in [24, Corollaire 5, p. 74] that for any $\epsilon > 0$, there exists a constant $c = c(\epsilon) > 0$ such that

$$\sum_{\substack{(q,a)=1 \\ d \leq X^{1-\epsilon} \\ d \notin [X^{2/3-\epsilon}, X^{2/3+\epsilon}]}} |\Delta(\tau_2; X, d, a)| \ll X \exp(-c \log^{1/2} x)$$

uniformly for all $|a| \leq \exp(c \log^{1/2} x)$. His proof uses estimates for Kloosterman sums together with Poisson's formula and the dispersion method.

The gap $X^{2/3-\epsilon} < d < X^{2/3+\epsilon}$ is closed by Fouvry and Iwaniec [26] for the case of squarefree moduli d which have a square-free factor r of a certain size. They showed in [26, Theorem 1, p. 272] that if r is square-free with $(a, r) = 1$, $r \leq X^{3/8}$, then we have

$$\sum_{\substack{rs^2 < X^{1-6\epsilon} \\ (s,ar)=1}} |\Delta(\tau_2; X, d, a)| \ll r^{-1} X^{1-\epsilon},$$

where the implied constant depends on ϵ alone. The proof depends on an estimate for sums of Kloosterman sums [26, Theorem 2, p. 272]. In the proof of Theorem 2, some estimates of sums of Laurent polynomials in five variables over a finite field were required; an estimate for these sums is proved in the appendix of [26] by Katz.

A larger range for θ_3 obtained by Heath-Brown [40] is a result of a sharper average bound for

$$\#\{n_1, n_2, n_3, n_4 : 1 \leq n_i \leq N, (n_i, d) = 1, n_1 + n_2 \equiv n_3 + n_4 \pmod{d}\}.$$

He proved in [40, Theorem 2, p. 32] that for any $\epsilon > 0$, we have

$$\sum_{d \leq D} \max_{x \leq X} \max_{a (\text{mod } d)} |T_3(x, d, a) - M_3(x, d, a)| \ll X^{40/51+\epsilon} D^{7/17},$$

and this is non-trivial for $D \leq X^{11/21-3\epsilon}$.

With a fixed, Fouvry, Kowalski, and Michel's second result in [27] shows that for every non-zero integer a, for every $\epsilon > 0$, and every $A > 0$, we have,

$$\sum_{\substack{q \leq X^{9/17-\epsilon} \\ q \text{ prime}, q \nmid a}} |\Delta(\tau_3; X, d, a)| \ll \frac{X}{\log^A X},$$

where the implied constant depends on ϵ, A, and a. This is the best result on the numerical value of the exponent of distribution θ_k for τ_k presently. Their proof combines estimates for divisor twists of trace functions together with "Kloostermaniac" techniques through the seminal work of Deshouillers and Iwaniec in [20].

Notably, in this type, the statement (S2) holds for any $\theta_k < 1/2$ for all $k \geq 5$. This follows from the general version of the Bombieri-Vinogradov theorem; see, e.g., [65] or [83].

2.1.4 Average estimates over smooth moduli

Recently progress has been made breaking the $X^{2/3}$-barrier towards the divisor problem for arithmetic progression for special moduli. It has been discovered that if, in addition to averaging over moduli d, we also restrict d to those that have good factorization properties, we can obtain results for $\theta_k \geq 1/2$ beyond the 1/2-barrier. A natural number m is called X^δ-smooth if all proper prime divisors of m are less than or equal to X^δ for some $\delta > 0$. Let (S3) be the following statement.

(S3) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{\substack{d < X^{\theta_k} \\ d \text{ is } X^{\eta}\text{-smooth}}} |T_k(X, d, a) - M_k(X, q)| \ll X^{1-\delta}$$

for some $\eta > 0$, provided that $d \leq X^{\theta_k - \epsilon}$.

Conjecture 3 *For each $k \geq 2$ statement (S3) holds for any $\theta_k < 1$.*

Table 2.3: Known results towards Conjecture 3 for average over smooth moduli d, and references.

$\boldsymbol{k}$	$\boldsymbol{\theta_k}$	**References**
$k = 2$	$\theta_2 = 2/3 + 1/246$	Irving (2015) [45, Theorem 1.2, p. 6677] (square-free moduli).
	$\theta_2 = 2/3 + \epsilon$	Khan (2016) [51, Theorem 1.1, p. 899] (prime powers moludi).
$k = 3$	$\theta_3 = 1/2 + 1/34$	Xi (2018) [84, Theorem 1.1, p. 702], (square-free moduli, polylog saving).
$k \geq 4$	$\theta_k = 1/2 + 1/584$	Wei, Xue, and Y. Zhang (2016) [82, Theorem 1.1, p. 1664] (square-free moduli, exponential saving).

In 2015, Irving [45] succeeded in broken through the Weil bound for smooth moduli and obtained sharp individual estimate for each d for d almost as large as $X^{55/82}$, thus showing $\theta_2 > 2/3$. More precisely, Irving proved in [45, Theorem 1.2, p. 6677] that for any $\varpi, \eta > 0$ satisfying $246\varpi + 18\eta < 1$, there exists a $\delta > 0$, depending on ϖ and η, such that for any X^{η}-smooth, squarefree $d \leq X^{2/3+\varpi}$ and any $(a, d) = 1$, we have

$$\Delta(\tau_2; X, d, a) \ll d^{-1} X^{1-\delta}. \tag{2.8}$$

The proof is based on a q-analog of van der Corput's method and bounds on complete Kloosterman sums of Fouvry, Kowalski, and Michel [28, Corollary 3.3].

Khan [51] in 2016 considered an important case of prime power moduli $d = p^m$ and proved in [51, Theorem 1.1, p. 899] that for a fixed integer $m \geq 7$, there exists some constants $\delta > 0$ and $\rho > 0$, depending only on m, such that (2.8) holds uniformly for $X^{2/3-\rho} < d < X^{2/3+\rho}$. Khan's method is different from that of Irving. The proof uses cancellation of a sum of Kloosterman sums to prime power moduli via the method of Weyl differencing. Khan's result is made uniform in m in [57].

For the ternary divisor function, in an extension of the work of Fouvry, Kowalski, and Michel [27] on θ_3 for prime moduli, Xi [84] in 2018 obtained individual estimates for smooth, square-free d to the same level of distribution $\theta_3 = 9/17$. More precisely, for each $\epsilon > 0$ and $A > 0$, there exists a constant $\eta > 0$ such that if $X^{9/17-\epsilon} \ll d \ll X^{9/17-\epsilon}$, d is X^{η}-smooth and square-free, then

$$\Delta(\tau_3; X, d, a) \ll \frac{X}{d}(\log X)^{-A}$$

holds uniformly for $(a, d) = 1$.

In [82], F. Wei, B. Xue, and Y. Zhang showed that the methods of Zhang in [87] in the problem of bounded gaps between primes applies not only to the von Mangoldt function Λ, but also equally to the k-fold divisor function τ_k. They proved in [82, Theorem 1.1, p. 1664] that for any $k \geq 4$ and $a \neq 0$, we have

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{1/2+1/584}}} \mu(d)^2 \left|\Delta(\tau_k; X, d, a)\right| \ll X \exp(-\log^{1/2} X), \tag{2.9}$$

where

$$\mathcal{D} = \{d \geq 1 : (d, a) = 1, (d, \prod_{p < X^{1/1168}} p) > X^{71/584}\}, \tag{2.10}$$

and the implied constant depends on k and a. The condition on the moduli d in (2.10) slightly relaxes the constraint on d being smooth, allowing for d to have some, but not too many, prime factors larger than $X^{1/1168}$. The error term and, more importantly, the exponent of distribution $\theta_k = 293/584 = 1/2 + 1/548$ in (2.9) hold uniformly in k. The proof uses the Cauchy-Schwartz inequality, combinatorial arguments, the dispersion method, the Weil bound on Kloosterman sums, together with an estimate of Birch and Bombieri for a variant of a three-variable Kloosterman sum proved in the Appendix to

[31] powered by Deligne's Riemann Hypothesis, and, crucially, the factorization

$$d = qr$$

to Weyl shift a certain incomplete Kloosterman sum to the modulus r, thus gaining over Deligne's Riemann Hypothesis by a power of r, hence saving a power of d, since d is a multiple of r.

In the main result of this chapter, we provide a sharpening of the error term in (2.9), saving a power of X from the trivial bound, with a constraint on the moduli d not having too many very small prime factors. Actually, our arguments follow closely those of [87] in treating contribution coming from large moduli; see Section 2.2 below for more discussion.

Theorem 1 *Let*

$$\varpi = \frac{1}{1168} \tag{2.11}$$

and

$$\theta_k = \min\left\{\frac{1}{12(k+2)}, \varpi^2\right\}. \tag{2.12}$$

For $a \neq 0$, let

$$\mathcal{D} = \{d \geq 1 : (d,a) = 1,\ |\mu(d)| = 1,\ (d, \prod_{p \leq X^{\varpi^2}} p) < X^{\varpi},\ \textit{and}\ (d, \prod_{p \leq X^{\varpi}} p) > X^{71/584}\},$$

where μ is the Möbius function. Then for each $k \geq 4$ we have

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{293/584}}} \left| \sum_{\substack{n \leq X \\ n \equiv a (mod\ d)}} \tau_k(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right| \ll X^{1-\theta_k}. \tag{2.13}$$

The implied constant is effective, and depends at most on a and k.

Remark 2 *As remarked by M. R. Murty, Theorem 1, in conjunction with a result of Fouvry [23, Théorème 4], can be used to possibly give another proof of bounded gaps between primes.*

Remark 3 *Theorem 1 admits several refinements. The particular choice of* $\varpi = 1/1168$ *comes from the condition (2.4.3) and, while being uniform in* k*, it is not optimal. There are certain ways to improve the numerics in Theorem 1, for instance using the extensive work [70] of the Polymath 8 project. Though we do not focus on this aspect here, it is an open problem to replace* ϖ *and* θ_k *on the right side of (2.13) by values that are as large as possible.*

Remark 4 *Fouvry has informed us of recent paper with Radziwiłł [29], to appear in Annales Scientifiques de l'Ecole Normale Supérieure, in which the authors proved in Corollary 1.2 that for fixed integer* $k \geq 1$ *and* $\epsilon > 0$*, one has*

$$\sum_{\substack{Q \leq q \leq 2Q \\ (q,a)=1}} \left| \sum_{\substack{X<n\leq 2X \\ n\equiv a (mod\ q)}} \tau_k(n) - \frac{1}{\varphi(q)} \sum_{\substack{X<n\leq 2X \\ (n,q)=1}} \tau_k(n) \right| \ll \frac{X}{(\log X)^{1-\epsilon}}$$

uniformly for $X \geq 2$*,* $Q \leq X^{\frac{17}{33}-\epsilon}$ *and* $1 \leq |a| \leq X/12$*.*

Conditionally, if we assume the Generalized Riemann Hypothesis, or the weaker Generalized Lindelöf Hypothesis, for Dirichlet L-functions we can obtain a stronger result than (2.13).

Theorem 2 *On the Generalized Lindelöf Hypothesis, the estimate* (2.13) *holds with the right side replaced by*

$$X^{1-\varpi^2},$$

where the θ_k *power saving is replaced by a positive constant independent of* k*.*

This uniform power saving is the result of sharper estimates of $L(s,\chi)^k$ on the critical line that are independent of k. We next present two results when we are allowed to take an extra averaging over the residue classes $a(\text{mod } d)$.

2.2 Outline of the proof of Theorem 1

Here, and in the rest of the chapter, we fix an integer $k \geq 4$, unless specified otherwise.

To prove (2.13) we follow standard practice and split the summation over moduli d in into two sums: one over $d < X^{\frac{1}{2}-\delta}$ which are called small moduli and the other over $X^{\frac{1}{2}-\delta} \leq d < X^{\frac{1}{2}+2\varpi}$ which are called large moduli. For small moduli, we estimate (2.13) directly using the large sieve inequality together with a direct substitute for the Siegel-Walfisz condition. For the von Mangoldt function $\Lambda(n)$, the Möbius function $\mu(n)$ is involved and, hence, the Siegel-Walfisz theorem is needed to handle very small moduli. For us, fortunately, τ_k is simpler than Λ in that μ is absent–this feature of τ_k allows us to get a sharper bound in place of the Siegel-Walfisz theorem; see Lemma 8 below. The constant here is effective.

For large moduli, we adapt the methods of Zhang in [87] to bound the error term which goes as follows. After applying suitable combinatorial arguments, we split τ_k into appropriate convolutions as Type I, II, and III, as modeled in [87]. We treat the Type I and II in our Case (b), Type III in our Case (c), and Case (a) corresponds to a trivial case which we treat directly. The main ingredients in Case (b) are the dispersion method and Weil bound on Kloosterman sums. The Case (c) depends crucially on the factorization $d = qr$ of the moduli to Weil shift a certain incomplete Kloosterman sum to the modulus r. The shift modulo this r then induces a Ramanujan sum, which is known to have better than square-root cancellation. This allows for a saving of a power of r, and since d is a multiple of r, and d is less than X, this saves a small power of X from the trivial bound.

Table 2.4: Table of parameters and their first appearance.

Parameters	First apprearance
$\varpi = 1/1168$	(2.11)
$\theta_k = \min\left\{\frac{1}{12(k+2)}, \varpi^2\right\}$	(2.12)
$Q_0 = X^{1/12(k+1)}$	(2.44)
$D_0 = X^{\varpi^{4/3}}$	(2.137)
$D_1 = X^{\varpi}$	(2.41)
$D_2 = X^{1/2-1/12(k+1)}$	(2.45)
$D_3 = X^{1/2+2\varpi}$	(9.18)
$\mathcal{P}_0 = \prod_{p\le D_0} p$	(2.37)
$\mathcal{P}_1 = \prod_{p\le D_1} p$	(2.38)
$\rho = X^{-\varpi}$	(2.54)
$X_1 = X^{3/8+8\varpi}$	(2.105)
$X_2 = X^{1/2-4\varpi}$	(2.105)

2.3 Preliminary lemmas

We collect here lemmas that shall be used to prove our theorems. Some lemmas are standard and we quote directly from the literature.

Lemma 1 *For any* $\epsilon > 0$ *we have*

$$\tau_j(n) \ll n^{\epsilon}. \tag{2.14}$$

Proof: See [47, Equation (1.81)]. ■

Lemma 2 *Let* γ *be an arithmetic function. If* $\chi(\text{mod } d)$ *is nonprincipal, then there exists a unique* $q|d$, $q>1$, *and a unique primitive character* $\chi^*(\text{mod } q)$, *such that, with* $r = d/q$,

$$\sum_n \gamma(n)\chi(n) = \sum_{(n,r)=1} \gamma(n)\chi^*(n).$$

Proof: See, e.g., [18, Section 5] for definition of characters and proofs. ■

In reducing nonprincipal characters, which may have not too small moduli, to primitive characters for the application of large sieve inequality, very small moduli of the primitive characters may occur. We treat contributions from those small moduli via the following lemma.

Lemma 3 *Let χ be a primitive character (mod d). For $d < X^{1/3(k+1)}$ we have*

$$\sum_{n \leq X} \tau_k(n)\chi(n) \ll X^{1-\frac{1}{3(k+2)}}. \tag{2.15}$$

Proof: Decompose the interval $[1, X]$ in to dyadic intervals of the form $[N, 2N)$. Denote by

$$\psi(\chi) = \sum_{n \sim N} \tau_k(n)\chi(n).$$

Let $0 < \eta < 1$ be a parameter to be specified latter (see (2.27) below). Let $f(x)$ be a function of $C^\infty(-\infty, \infty)$ class such that $0 \leq f(y) \leq 1$,

$$f(y) = 1 \quad \text{if} \quad N \leq y \leq 2N,$$

$$f(y) = 0 \quad \text{if} \quad y \notin [N - N^\eta, 2N + N^\eta],$$

and obeying the derivative bound

$$f^{(j)}(y) \ll N^{-j\eta}, \ j \geq 1, \tag{2.16}$$

where the implied constant depends on η and j at most. Let

$$\psi^*(\chi) = \sum_{n=1}^{\infty} \tau_k(n)\chi(n)f(n). \tag{2.17}$$

By (2.14), we have

$$\psi^*(\chi) - \psi(\chi) = \sum_{N - N^\eta \leq n \leq N} \tau_k(n)\chi(n) + \sum_{2N \leq n \leq 2N + N^\eta} \tau_k(n)\chi(n) \ll N^{\eta+\epsilon}$$

for any $\epsilon > 0$. Let

$$F(s) = \int_0^\infty f(x)x^{s-1}dx$$

be the Mellin transform of $f(x)$. The function $F(s)$ is absolutely convergent for $\sigma > 0$ with inverse Mellin transform

$$f(x) = \frac{1}{2\pi i}\int_{(2)} F(s)x^{-s}ds, \tag{2.18}$$

where $\int_{(c)}$ denotes the integration $\int_{c-i\infty}^{c+i\infty}$ over the vertical line $c + it$ where t runs from $-\infty$ to ∞. Substituting (2.18) into (2.17) and changing the order of summation and integration, we get

$$\begin{aligned}\psi^*(\chi) &= \sum_{n=1}^{\infty}\tau_k(n)\chi(n)\left(\frac{1}{2\pi i}\int_{(2)} F(s)n^{-s}ds\right)\\ &= \frac{1}{2\pi i}\int_{(2)} F(s)\left(\sum_{n=1}^{\infty}\tau_k(n)\chi(n)n^{-s}\right)ds\\ &= \frac{1}{2\pi i}\int_{(2)} F(s)L(s,\chi)^k ds, \qquad (2.19)\end{aligned}$$

where $L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}$ is the Dirichlet series for χ. Since the function $L(s,\chi)$, and thus, $F(s)L(s,\chi)^k$ has no poles in $\sigma \geq 0$, we may move the line of integration in (2.19) from $\sigma = 2$ to $\sigma = 1/2$ and obtain

$$\psi^*(\chi) = \frac{1}{2\pi i}\int_{(\frac{1}{2})} F(s)L(s,\chi)^k ds. \tag{2.20}$$

We next estimate this integral by bounding the integrand and splitting the line of integration into two parts, over $|t| < T$ and $|t| \geq T$, then choosing T suitably (see (2.26) below). For $\sigma = 1/2$, we have the convexity bound; see, e.g., [47, Theorem 5.23],

$$|L(s,\chi)|^k \ll d^{k/4}|s|^k. \tag{2.21}$$

We next obtain upper bound for $F(s)$. On the line $\sigma = 1/2$, we have, by definitions of

$F(s)$ and $f(x)$,

$$F(s)=\int_{N-N^{\eta}}^{2N+N^{\eta}} f(x)x^{s-1}dx \leq \int_{N-N^{\eta}}^{2N+N^{\eta}} x^{-1/2}dx \ll N^{1/2}. \tag{2.22}$$

This bound is sufficient for bounding small $|t|$ in (2.20), but too large for $|t|$ large. To bound contribution from large $|t|$ we fix an

$$\ell > k+1 \tag{2.23}$$

and apply integration by parts ℓ times to $F(s)$:

$$F(s)=(-1)^{\ell}\frac{1}{s(s+1)\cdots(s+\ell+1)}\int_0^{\infty} f^{(\ell)}(x)x^{s+\ell-1}dx.$$

Hence, by the derivative bound (2.16), $F(s)$ is bounded by

$$|F(s)| \ll \frac{1}{|s|^{\ell}}N^{-\ell\eta+1/2+\ell} \ll \frac{1}{|s|^{\ell}}N^{(1-\eta)\ell+1/2}. \tag{2.24}$$

This bound allows us to save an arbitrary negative power of $|s|$; we will use this bound for large $|t|$.

We now split the integral in (2.20) into two and estimate each part individually. Let $s=1/2+it$. For $T>2$, we can write $\psi^*(\chi)$ in (2.20) as

$$\psi^*(\chi)=\frac{1}{2\pi i}\int_{|t|<T} F(s)L(s,\chi)^k ds+\frac{1}{2\pi i}\int_{|t|\geq T} F(s)L(s,\chi)^k ds.$$

By (2.21) and (2.22), the first term on the right side is

$$\frac{1}{2\pi i}\int_{|t|<T} F(s)L(s,\chi)^k ds \ll N^{1/2}d^{k/4}\int_{-T}^{T}|1/2+it|^k dt \ll N^{1/2}d^{k/4}T^{k+1}; \tag{2.25}$$

while by (2.21), (2.24), and (2.23), the second term on the right side is

$$\frac{1}{2\pi i}\int_{|t|\geq T} F(s)L(s,\chi)^k ds \ll N^{(1-\eta)\ell+1/2}d^{k/4}\int_T^{\infty} t^{k-\ell}dt \ll N^{(1-\eta)\ell+1/2}d^{k/4}T^{k-\ell+1}.$$

Hence we choose

$$T=N^{1-\eta}, \tag{2.26}$$

so that the contributions in (2.25) and (2.3 are of the same order. With this choice of T, we get

$$\psi^*(\chi) \ll N^{(1-\eta)(k+1)+1/2} d^{k/4}.$$

It remains to specify η. Let

$$\eta = 1 - \frac{1}{3(k+1)}, \tag{2.27}$$

so that $(1-\eta)(k+1) = 1/3$. Thus

$$\psi^*(\chi) \ll N^{5/6} d^{k/4}$$

and

$$\psi(\chi) \ll N^{5/6} d^{k/4} + N^{\eta+\epsilon}.$$

Summing over $N = X2^{-k}$, we find that the left side of (2.15) is bounded by

$$\ll X^{5/6+\epsilon} d^{k/4} + X^{1-\frac{1}{3(k+1)}+\epsilon}.$$

Thus, if $d < X^{1/3(k+1)}$, then the above estimate is

$$\ll X^{1-\frac{1}{3(k+1)}+\epsilon} \ll X^{1-\frac{1}{3(k+2)}}$$

for small enough ϵ. This gives the estimate (2.15). ■

Lemma 4 *Let γ be an arithmetic function. For $(a,d)=1$ we have*

$$\Delta(\gamma; X, d, a) = \frac{1}{\varphi(d)} \sideset{}{'}\sum_{\chi (\mathit{mod}\ d)} \overline{\chi}(a) \left(\sum_{n \le X} \gamma(n)\chi(n) \right). \tag{2.28}$$

Proof: By the orthogonality condition

$$\frac{1}{\varphi(d)} \sum_{\chi (\mathrm{mod}\ d)} \overline{\chi}(a)\chi(n) = \begin{cases} 1, & \text{if } n \equiv a(d), \\ 0, & \text{otherwise,} \end{cases}$$

we may write

$$\sum_{\substack{n\le X\\ n\equiv a(d)}}\gamma(n)=\sum_{n\le X}\gamma(n)\left(\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\ d)}\overline{\chi}(a)\chi(n)\right)=\frac{1}{\varphi(d)}\sum_{\chi(\mathrm{mod}\ d)}\overline{\chi}(a)\left(\sum_{n\le X}\gamma(n)\chi(n)\right). \tag{2.29}$$

If χ (mod d) is principal, then

$$\begin{cases}\overline{\chi}(a)=1,\\ \displaystyle\sum_{n\le X}\gamma(n)\chi(n)=\sum_{\substack{n\le X\\ (n,d)=1}}\gamma(n).\end{cases}$$

Hence the contribution from the principal character gives the main term in (2.29) and the discrepancy $\Delta(\gamma;X,d,a)$ is given by a sum over nonprincipal characters. This gives (2.28). ■

The next lemma is the well-known multiplicative large sieve inequality.

Lemma 5 *Let χ be a primitive character mod q. For $a(n)$ a sequence of complex numbers, we have*

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi(\mathit{mod}\ q)}\left|\sum_{n\le N}a(n)\chi(n)\right|^2\ll(Q^2+N)\sum_{n\le N}|a(n)|^2. \tag{2.30}$$

Proof: See [47, Theorem 7.13]. ■

The next lemma is a truncated Poisson formula.

Lemma 6 *Suppose that $\eta^*>1$ and $X^{1/4}<M<X^{2/3}$. Let f be a function of $C^\infty(-\infty,\infty)$ class such that $0\le f(y)\le 1$,*

$$f(y)=1\quad\text{if}\quad M\le y\le\eta^*M,$$

$$f(y)=0\quad\text{if}\quad y\notin[(1-M^{-\epsilon})M,(1+M^{-\epsilon}\eta^*)M],$$

and

$$f^{(j)}(y)\ll M^{-j(1-\epsilon)},\quad j\ge 1,$$

with the implied constant depending on ϵ and j at most. Then we have

$$\sum_{m\equiv a(d)} f(m) = \frac{1}{d}\sum_{|h|<H} \hat{f}(h/d)e_d(-ah) + O(d^{-1})$$

for any $H \geq dM^{-1+2\epsilon}$, where $\hat{f}$ is the Fourier transform of f.

Proof: See [8, Lemma 2]. ■

Lemma 7 *Suppose that $1 \leq N < N' < 2x$, $N' - N > X^{\epsilon}d$, and $(c,d) = 1$. Then for $j, \nu \geq 1$ we have*

$$\sum_{N\leq n\leq N'} \tau_j(n)^{\nu} \ll (N'-N)\mathcal{L}^{j^{\nu}-1}, \tag{2.31}$$

and

$$\sum_{\substack{N\leq n\leq N'\\ n\equiv c(d)}} \tau_j(n)^{\nu} \ll \frac{N'-N}{\varphi(d)}\mathcal{L}^{j^{\nu}-1}.$$

The implied constants depending on $\epsilon, j,$ and ν at most.

Proof: See [73, Theorem 2]. ■

In the next lemma we verify a substitute for the "Siegel-Walfisz" condition.

Lemma 8 *Let $\beta = \beta_{i_1} * \cdots * \beta_{i_\ell}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_\ell \leq k$, and $\beta_j = \chi_{N_j}$, with $N := N_{i_1}N_{i_2}\cdots N_{i_\ell} \gg X^{\kappa}$ for some constant $\kappa > 0$. For χ a primitive character modulo $r \ll X^{\kappa}$, we have*

$$\sum_{n} \beta(n)\chi(n) \ll X^{-\kappa/12}N. \tag{2.32}$$

Proof: We first verify (2.32) for a single $\beta = \beta_i$. For the general case, it suffices to check that if β_i and β_j satisfy (2.32), then so does their convolution $\beta_i * \beta_j$.

Let $\beta = \chi_{N_i}$ $N = N_i \gg X^{\kappa}$. We proceed analogously as to the proof of Lemma 3. Let $f(x)$ be a function of $C^{\infty}(-\infty,\infty)$ class such that $0 \leq f(y) \leq 1$,

$$f(y) = 1 \quad \text{if} \quad N \leq y \leq (1+\rho)N,$$

$$f(y) = 0 \quad \text{if} \quad y \notin [N - N^{11/12}, (1+\rho)N + N^{11/12}],$$

and obeying the derivative bound

$$f^{(j)}(y) \ll N^{-11j/12}, \ j \geq 1,$$

where the implied constant depends on j. Let

$$F(s) = \int_0^\infty f(x)x^{s-1}dx$$

denote the Mellin transform of $f(x)$. Let

$$\psi(\chi) = \sum_n \beta(n)\chi(n)$$

and

$$\psi^*(\chi) = \sum_n \beta(n)\chi(n)f(n).$$

Analogously, we have

$$\psi^*(\chi) - \psi(\chi) = \sum_{N-N^{11/12} \leq n \leq N} \chi(n) + \sum_{(1+\rho)N \leq n \leq (1+\rho)N + N^{11/12}} \chi(n) \ll N^{11/12}$$

and

$$|L(1/2+it,\chi)| \ll r^{1/4}|s|.$$

Thus,

$$\begin{aligned}
\psi^*(\chi) &= \frac{1}{2\pi i}\int_{(\frac{1}{2})} F(s)L(s,\chi)ds \\
&= \frac{1}{2\pi i}\int_{|t|<N^{1/12}} F(1/2+it)L(1/2+it,\chi)ds \\
&\quad + \frac{1}{2\pi i}\int_{|t|\geq N^{1/12}} F(1/2+it)L(1/2+it,\chi)ds \\
&\ll N^{1/2}r^{1/4}N^{1/6} + N^{1/2}r^{1/4}N^{1/6} \ll N^{2/3}r^{1/4}.
\end{aligned}$$

Assume $r \ll X^{\kappa}$. We deduce

$$\psi(\chi) \ll N^{11/12} + N^{2/3}r^{1/4} \ll N\frac{1}{N^{1/12}} + N\frac{r^{1/4}}{N^{1/3}} \ll NX^{-\kappa/12} + N\frac{X^{\kappa/4}}{X^{\kappa/3}} \ll NX^{-\kappa/12}.$$

Now assume β_i and β_j satisfy (2.32) with $N := N_iN_j \gg X^{\kappa}$. Write $N_i = X^{\kappa_i}$ and $N_j = X^{\kappa_j}$ so that $\kappa_i + \kappa_j \geq \kappa$. Since β_i and β_j satisfy (2.32), we have

$$\sum_n \beta_i(n)\chi(n) \ll N_iX^{-\kappa_i/12}$$

and

$$\sum_n \beta_j(n)\chi(n) \ll N_jX^{-\kappa_j/12}.$$

Thus, writing n as mn and separate variables, we get

$$\sum_n \beta_i * \beta_j(n)\chi(n) = \sum_m \beta_i(m)\chi(m)\sum_n \beta_i(n)\chi(n) \ll N_iX^{-\kappa_i/12}N_jX^{-\kappa_j/12} \ll NX^{-\kappa/12}.$$

This completes the proof of Lemma 8. ■

Lemma 9 *Let β be given as in (2.102), with N given in (2.103) satisfying (2.104). Assume $R \leq X^{-\varpi/6}N$. Then for any $q \geq 1$ and (r, ℓ), we have*

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\ell (mod\ r)} \left(\sum_{\substack{n\equiv \ell(r)\\(n,q)=1}} \beta(n) - \frac{1}{\varphi(r)} \sum_{(n,qr)=1} \beta(n) \right)^2 \ll \tau(q)^B N^2 X^{-\frac{\varpi}{12}}. \tag{2.33}$$

Proof: By Möbius inversion, the condition $(n, q) = 1$ may be removed at the cost of removing the $\tau(q)^B$ factor on the right side of (2.33); see, e.g., [32, p. 21-22]. Thus it suffices to show

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\ell(\mathrm{mod}\ r)} \Delta(\beta; X, r, \ell)^2 \ll N^2X^{-\varpi/12}. \tag{2.34}$$

By (2.28), we have

$$\begin{aligned}\Delta(\beta;X,r,\ell)^2 &= \frac{1}{\varphi(r)^2}\left|\sideset{}{'}\sum_{\chi(\text{mod } r)} \overline{\chi}(a)\left(\sum_{n\le X}\beta(n)\chi(n)\right)\right|^2\\ &= \frac{1}{\varphi(r)^2}\sideset{}{'}\sum_{\chi_1(\text{mod } r)} \overline{\chi_1}(a)\left(\sum_{n\le X}\beta(n)\chi_1(n)\right)\sideset{}{'}\sum_{\chi_2(\text{mod } r)} \chi_2(a)\overline{\left(\sum_{n\le X}\beta(n)\chi_2(n)\right)}.\end{aligned}$$

Summing over primitive $\ell(\text{mod } r)$ and changing the order of summation, we get

$$\sideset{}{^*}\sum_{\ell(\text{mod } r)}\Delta(\beta;X,r,\ell)^2 = \frac{1}{\varphi(r)^2}\sideset{}{'}\sum_{\chi_1(\text{mod } r)}\sideset{}{'}\sum_{\chi_2(\text{mod } r)}\left(\sum_{n\le X}\tau_k(n)\chi_1(n)\right)\overline{\left(\sum_{n\le X}\tau_k(n)\chi_2(n)\right)}\sideset{}{^*}\sum_{a(\text{mod } r)}\overline{\chi_1}(a)\chi_2(a).$$

By the orthogonality relation

$$\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{a(\text{mod } r)}\overline{\chi_1}(a)\chi_2(a) = \begin{cases}1, & \text{if } \chi_1=\chi_2,\\ 0, & \text{if } \chi_1\ne\chi_2,\end{cases} \tag{2.35}$$

this becomes

$$\sideset{}{^*}\sum_{\ell(\text{mod } r)}\Delta(\beta;X,r,\ell)^2 = \frac{1}{\varphi(r)}\left|\sideset{}{'}\sum_{\chi(\text{mod } r)}\left(\sum_{n\le X}\beta(n)\chi(n)\right)\right|^2.$$

We now reduce to primitive characters as in the proof of Proposition 1. By Lemma 2, we have

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\ell(\text{mod } r)}\Delta(\beta;X,r,\ell)^2 \ll \log\mathcal{L}\sum_{s\le R}\frac{1}{s}\left(\sum_{1<q\le R/s}\frac{1}{q}\left|\sideset{}{^*}\sum_{\chi(\text{mod } q)}\left(\sum_{n\le X}\beta(n)\chi(n)\right)\right|^2\right).$$

We apply the Siegel-Walfisz condition to bound contribution coming from tiny moduli. By Lemma 8, we get, for $1<q\le X^{\varpi/6}$,

$$\sum_{1<q\le X^{\varpi/12}}\frac{1}{q}\left|\sideset{}{^*}\sum_{\chi(\text{mod } q)}\left(\sum_{n\le X}\beta(n)\chi(n)\right)\right|^2 = \sum_{1<q\le X^{\varpi/6}}\frac{1}{q}\left(\varphi(q)X^{-3/96-2\varpi/3}N\right)^2 \ll N^2X^{-1/16}.$$

Assume $X^{\varpi/6} \ll Q \ll R$. By the large sieve inequality (2.30) and the bound (2.31), we have

$$\frac{1}{Q}\sum_{q\sim Q}\left|\sum_{\chi(\mathrm{mod}\ q)}^{*}\left(\sum_{n\le X}\beta(n)\chi(n)\right)\right|^2 \ll \frac{1}{Q}(Q^2+N)\left(\sum_{n\le X}\beta(n)\right)^2 \ll \left(Q+\frac{N}{Q}\right)N\mathcal{L}^{k-1}.$$

For $R \le X^{-\varpi/6}N$, this leads to (2.34). ■

The next lemma restricts d to moduli that have 'well-factorable' property.

Lemma 10 (Factorization lemma) *Write*

$$D_3 = X^{1/2+2\varpi}. \tag{2.36}$$

Suppose d is square-free such that

$$D_2 < d < D_3,$$

$$(d,\mathcal{P}_0) < X^{\varpi}, \quad \varpi = 1/1168, \tag{2.37}$$

and

$$(d,\mathcal{P}_1) > X^{1/8-4\varpi} \tag{2.38}$$

Then, for any R^* satisfying

$$X^{2\varpi} \le R^* \le X^{45\varpi} \tag{2.39}$$

or

$$X^{3/8+7\varpi} \le R^* \le X^{1/2-2\varpi}, \tag{2.40}$$

there is a factorization $d = qr$ such that $X^{-\varpi}R^* < r < R^*$ and $(q,\mathcal{P}_0) = 1$.

Proof: Since d is square-free, we may write d as $d = d_0d_1d_2$ with

$$d_0 = (d,\mathcal{P}_0)$$

$$d_1 = \frac{(d,\mathcal{P}_1)}{(d,\mathcal{P}_0)} = \prod_{j=1}^{n} p_j, \quad D_0 < p_1 < p_2 < \cdots < p_n < D_1, \quad n \geq 2, \tag{2.41}$$

and

$$d_2 = \prod_{\substack{p|d \\ p>D_1}} p.$$

We have $d_0 < X^{\varpi}$. By the first inequality in (2.39), $R^* \geq X^{2\varpi}$, and there is an $n' < n$ such that

$$d_0 \prod_{j=1}^{n'} p_j < R^* \quad \text{and} \quad d_0 \prod_{j=1}^{n'+1} p_j \geq R^*.$$

Similarly, by (2.37), we have

$$d_2 = \frac{d}{(d,\mathcal{P}_1)} < X^{3/8+6\varpi}.$$

By the first inequality in (2.40), $R^* \geq X^{3/8+7\varpi}$, and there is an $n'' < n$ such that

$$d_2 \prod_{j=1}^{n''} p_j < R^* \quad \text{and} \quad d_2 \prod_{j=1}^{n''+1} p_j \geq R^*.$$

The assertion follows by choosing

$$r = d_0 r_i, \;\; i = 1, 2,$$

where

$$r_1 = d_0 \prod_{j=1}^{n'} p_j \quad \text{and} \quad r_2 = d_0 d_2 \prod_{j=1}^{n''} p_j$$

and noting that $r_i \geq R^*/p_{n'+1}$. ■

Lemma 11 *([87, Lemma 9]) Suppose that $H, N \geq 2$ and $(c,d) = 1$. Then we have*

$$\sum_{\substack{n\leq N \\ (n,d)=1}} \min\{H, \|c\overline{n}/d\|^{-1}\} \ll (dN)^{\epsilon}(H+N),$$

where $\overline{n}/d$ means $a/d(\text{mod } 1)$ with $an \equiv 1(\text{mod } d)$.

We quote a crucial bound on an incomplete Kloosterman sum obtained in [87, Lemma 11].

Lemma 12 ([87, Lemma 11]) *Suppose that* $N \geq 1$, $d_1d_2 > 10$, *and* $|\mu(d_1)| = |\mu(d_2)| = 1$. *Then for any* c_1, c_2, *and* ℓ, *we have*

$$\sum_{\substack{n\leq N\\(n,d_1)=1\\(n+\ell,d_2)=1}} e\left(\frac{c_1\overline{n}}{d_1}+\frac{c_2\overline{(n+\ell)}}{d_2}\right) \ll (d_1d_2)^{1/2}\tau(d_1d_2)+\frac{(c_1,d_1)(c_2,d_2)(d_1,d_2)^2N}{d_1d_2}. \tag{2.42}$$

In the case $d_2 = 1$, (2.42) *becomes a Ramanujan sum*

$$\sum_{\substack{n\leq N\\(n,d_1)=1}} e_{d_1}(c_1\overline{n}) \ll d_1^{1/2}\tau(d_1)+\frac{(c_1,d_1)N}{d_1}; \tag{2.43}$$

see, e.g., [8, Lemma 6] for a proof.

This next lemma is the Birch-Bombieri bound.

Lemma 13 *([87, Lemma 12]) Let*

$$T(k;m_1,m_2;q) = \sum_{\substack{\ell(q)\\(\ell(\ell+k),q)=1}} \sideset{}{^*}\sum_{t_1(q)} \sideset{}{^*}\sum_{t_2(q)} e_q\left(\overline{\ell}t_1-\overline{(\ell+k)}t_2+m_1\overline{t}_1-m_2\overline{t}_2\right).$$

Suppose that q *is square-free. Then we have*

$$T(k;m_1,m_2;q) \ll (k,q)^{1/2}q^{3/2}\tau(q).$$

2.4 Proof of the main result Theorem 1

We start with the proof of Theorem 1 which is the longest of the four. We begin by making some preliminary reductions. Writing

$$Q_0 = X^{\frac{1}{12(k+1)}} \tag{2.44}$$

and

$$D_2 = \frac{X^{1/2}}{Q_0} = X^{\frac{1}{2}-\frac{1}{12(k+1)}}, \tag{2.45}$$

we first show that contributions coming from moduli $d \le D_2$, which we call small moduli, are acceptable. (See Remark 5 below for a discussion on the dependency of D_2 on k.) The main ingredients in this first step are the multiplicative large sieve inequality (2.30) in conjunction with Lemma 3 to control contributions from primitive characters with very small moduli.

Proposition 1 *Let*

$$\tau_k = \tau_\ell * \tau_s \tag{2.46}$$

with $k = \ell + s$*,* τ_ℓ *supported on* $[M, 2M)$*,* τ_s *supported on* $[N, 2N)$*,* $M, N > X^{1/6(k+1)}$*, and* $MN = X$*. For* $D \le D_2$ *we have*

$$\sum_{d\le D} \max_{(a,d)=1} |\Delta(\tau_k; X, d, a)| \ll X^{1-\frac{1}{12(k+2)}}. \tag{2.47}$$

Proof: As mentioned above, we estimate each $|\Delta(\tau_k; X, d, a)|$ directly using the large sieve inequality (2.30). By Lemma 4, we first reduce the task of estimating $|\Delta(\tau_k; X, d, a)|$ to a sum over nonprincipal characters, then, with Lemmas 2 and 3 and the factorization

$$d = qr,$$

we further reduce this sum to one involving only primitive characters, to which, we apply the large sieve inequality (2.30) to obtain (2.47).

By Lemma 4, the left side of (2.47) is

$$\le \sum_{d\le D} \frac{1}{\varphi(d)} \sideset{}{'}\sum_{\chi \,(\mathrm{mod}\ d)} \left| \sum_{n\le X} \tau_k(n)\chi(n) \right|.$$

By the bound $\frac{1}{\varphi(d)} \ll \frac{\log \mathcal{L}}{d}$ and Lemma 2, this is

$$\begin{aligned} &\ll \log \mathcal{L} \sum_{d \leq D} \frac{1}{d} \sum_{\substack{d=qr \\ q>1}} \sideset{}{^*}\sum_{\chi^*(\bmod q)} \left| \sum_{\substack{n \leq X \\ (n,r)=1}} \tau_k(n) \chi^*(n) \right| \\ &= \log \mathcal{L} \sum_{r \leq D} \frac{1}{r} \left(\sum_{1<q \leq D/r} \frac{1}{q} \sideset{}{^*}\sum_{\chi(\bmod q)} \left| \sum_{\substack{n \leq X \\ (n,r)=1}} \tau_k(n) \chi(n) \right| \right). \end{aligned}$$

The sum $\sum_{r \leq D} \frac{1}{r}$ contributes a factor of $\log D$. Thus, to show (2.47), it suffices to show that, for each fixed $r \leq D$,

$$\sum_{1<q \leq D/r} \frac{1}{q} \sideset{}{^*}\sum_{\chi(\bmod q)} \left| \sum_{\substack{n \leq X \\ (n,r)=1}} \tau_k(n) \chi(n) \right| \ll X^{1-1/12(k+1)}. \tag{2.48}$$

Fix an $r \leq D$. Recall Q_0 given as in (2.44). We split the range of primitive conductors $q \in (1, D/r]$ into two, one over $q < Q_0$ and the other over $Q_0 \leq q \leq D/r$, with the intention of applying Lemma 3 to the former and large sieve inequality (2.30) to the latter.

By Lemma 3, we have, for $q \leq Q_0$,

$$\sum_{n \leq X} \tau_k(n) \chi(n) \ll X^{1-1/3(k+1)}.$$

And since the number of characters with modulus less than Q_0 is at most Q_0^2, we get

$$\sum_{1<q<Q_0} \frac{1}{q} \sideset{}{^*}\sum_{\chi(\bmod q)} \left| \sum_{(n,r)=1} \tau_k(n) \chi(n) \right| \ll Q_0^2 X^{1-1/3(k+1)} \sum_{1<q<D_0} \frac{1}{q} \ll X^{1-1/6(k+2)}.$$

Thus, to prove (2.48) it is sufficient to show

$$\frac{1}{Q} \sum_{Q \leq q \leq 2Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \left| \sum_{(n,r)=1} \tau_k(n) \chi(n) \right| \ll X^{1-1/12(k+1)} \tag{2.49}$$

for any $Q_0 \leq Q \leq D$. By (2.46), we have

$$\sum_{(n,r)=1} \tau_k(n)\chi(n) = \left(\sum_{(m,r)=1} \tau_\ell(m)\chi(m)\right)\left(\sum_{(n,r)=1} \tau_s(n)\chi(n)\right). \tag{2.50}$$

By (2.50) and Cauchy's inequality, the left side of (2.49) is

$$\leq \frac{1}{Q}\left(\sum_{Q\leq q\leq 2Q} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \left|\sum_{(m,r)=1} \tau_\ell(m)\chi(m)\right|^2\right)^{1/2} \left(\sum_{Q\leq q\leq 2Q} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \left|\sum_{(n,r)=1} \tau_s(n)\chi(n)\right|^2\right)^{1/2}.$$

By the large sieve inequality (2.30), the above quantity is

$$\leq \frac{1}{Q}\left((Q^2+M)M^{2\epsilon}\right)^{1/2}\left((Q^2+N)N^{2\epsilon}\right)^{1/2} = \frac{1}{Q}(Q^2+M)^{1/2}(Q^2+N)^{1/2}X^{1/2+\epsilon}.$$

By the inequality $\sqrt{x+y} < \sqrt{x}+\sqrt{y}$, the above is bounded by

$$\begin{aligned} &< \frac{1}{Q}(Q+\sqrt{M})(Q+\sqrt{N})X^{1/2+\epsilon} = \frac{1}{Q}(Q^2+Q\sqrt{M}+Q\sqrt{N}+X^{1/2})X^{1/2+\epsilon} \\ &= \left(Q+\sqrt{M}+\sqrt{N}+\frac{X^{1/2}}{Q}\right)X^{1/2+\epsilon}. \end{aligned} \tag{2.51}$$

Since $Q_0 < Q < D_2$ and $N, M > X^{1/6(k+1)}$, we have

$$\begin{cases} Q < D_2 = X^{1/2-1/12(k+1)}, \\ \sqrt{M} = \sqrt{\dfrac{X}{N}} < \sqrt{\dfrac{X}{X^{1/6(k+1)}}} < X^{1/2-1/12(k+1)}, \\ \sqrt{N} = \sqrt{\dfrac{X}{M}} < \sqrt{\dfrac{X}{X^{1/6(k+1)}}} < X^{1/2-1/12(k+1)}, \\ \dfrac{X^{1/2}}{Q} < \dfrac{X^{1/2}}{Q_0} = X^{1/2-1/12(k+1)}. \end{cases}$$

Combining the above estimates, (2.51) is

$$\ll X^{1-1/12(k+1)+\epsilon} \ll X^{1-1/12(k+2)}.$$

This leads to (2.47). ∎

Thus, by Proposition 1, Theorem 1 holds for $1 \leq d \leq D_2$. From this, to prove (2.13), it suffices to show that

$$\sum_{\substack{d \in \mathcal{D} \\ D_2 < d < D_3}} |\Delta(\tau_k; X, d, a)| \ll X^{1-\theta_k}. \tag{2.52}$$

The rest of this section is devoted to proving the estimate (2.52).

Remark 5 *The cutoff parameter D_2 introduced in (2.45) separating small and large moduli unfortunately has a dependence on k. We are unable to resolve this dependency without appealing to GRH or the Lindelöf Hypothesis. This dependency on k is the result of the convexity bound (2.21) of $L(s,\chi)^k$ on the critical line which depends on k. On GRH, the bound on $L(1/2+it,\chi)^k$ can be made uniform in k; see Section 2.5 below for more.*

2.4.1 Combinatorial argument

The goal of this subsection is to apply combinatorial arguments to reduce the proof of (2.52) to showing that

$$\sum_{\substack{d \in \mathcal{D} \\ D_2 < d < D_3}} |\Delta(\gamma; X, d, a)| \ll X^{1-\theta_k}, \tag{2.53}$$

for suitable γ. Combinatorial arguments amount to estimating $\sum_{\substack{n \leq X \\ n \equiv a (\mathrm{mod}\ d)}} \tau_k(n)$ by a sum of $\sum_{n \equiv a(d)} \gamma(n)$, where each γ is of the form

$$\gamma = \beta_1 * \beta_2 * \cdots * \beta_k,$$

a convolution of simpler arithmetic functions β_j.

Following the fundamental work of Friedlander and Iwaniec in their treatment of the ternary divisor function $\tau_3(n)$ in [31, Section 3], after decomposing the interval $[1, X]$ to $O(\mathcal{L}^B)$ dyadic intervals of the form $[N, 2N)$, we perform a finner-than-dyadic subdivision of the interval $[N, 2N)$ as follows. Let

$$\rho = X^{-\varpi}. \tag{2.54}$$

Let R be the largest positive integer r for which $(1+\rho)^r < 2x$. We have the following bound for R:

$$R \leq \frac{\log 2x}{\log(1+\rho)} \ll \rho^{-1} \log X.$$

For $n \sim N$, we have $\tau_k(n) = T_1(n)$ where

$$T_1(n) = \sum_{\mathcal{N}=(N_1,N_2,\cdots,N_k)} \chi_{N_k} * \chi_{N_{k-1}} * \cdots * \chi_{N_1}. \tag{2.55}$$

Here $N_1, N_2, \ldots, N_k \geq 1$ run over the powers of $1+\rho$ satisfying

$$[N_k \cdots N_1, (1+\rho)^k N_k \cdots N_1) \cap [N, 2N) \neq \emptyset. \tag{2.56}$$

Let T_2 have the same expression as T_1 but with the constraint (2.56) replaced by

$$[N_k \cdots N_1, (1+\rho)^k N_k \cdots N_1) \subset [N, 2N). \tag{2.57}$$

Since $T_1 - T_2$ is supported on $[(1+\rho)^{-k}N, (1+\rho)^k N]$ and $(T_1 - T_2)(n) \ll \tau_k$, by Lemma 7 we have

$$\sum_{\substack{d \in \mathcal{D} \\ D_2 < d < D_3}} |\Delta(T_1 - T_2; X, d, a)| \ll X^{1-\varpi^2}.$$

Let γ be of the form

$$\gamma = \chi_{N_k} * \chi_{N_{k-1}} * \cdots . * \chi_{N_1}$$

with $N_k, \ldots, N_1$ satisfying (2.57) and $N_k \leq \cdots \leq N_1$. Write $N_i = X^{\nu_i}$. We have

$$0 \leq \nu_k \leq \cdots \leq \nu_1, \tag{2.58}$$

and

$$0 \leq \nu_k + \cdots + \nu_1 < 1 - k\frac{\log \rho}{\mathcal{L}}. \tag{2.59}$$

We deduce the proof of (2.52) from the following

Proposition 2 *With the same notation as above, we have*

$$\sum_{\substack{d \in \mathcal{D} \\ D_2 < d < D_3}} |\Delta(\gamma; X, d, a)| \ll X^{1-\varpi^{7/2}}. \tag{2.60}$$

In view of (7), the proof of (2.60) is divided into three cases:

Case (a):

$$\nu_1 \geq \frac{5}{8} - 8\varpi.$$

Case (b):

$$\nu_1 < \frac{5}{8} - 8\varpi \tag{2.61}$$

and

$$\sum_{i \in I} \nu_i \notin \left[\frac{3}{8} + 8\varpi, \frac{5}{8} - 8\varpi\right] \tag{2.62}$$

for any subset I of $\{1, 2, \ldots, k\}$.

Case (c):

$$\nu_1 < \frac{5}{8} - 8\varpi$$

and there is a subset I of $\{1, 2, \ldots, k\}$ such that

$$\sum_{i \in I} \nu_i \in \left[\frac{3}{8} + 8\varpi, \frac{5}{8} - 8\varpi\right].$$

By Lemma 8, all the choices of $\beta_i = \chi_{N_i}$ above satisfy the Siegel-Walfisz condition (2.32). Noting that the sum in (2.55) contains $R \ll \rho^{-1} \log X \ll X^{\varpi^{15/16}}$ terms, by the above discussion, we conclude that (2.60) implies (2.52).

We start with the proof of Case (a), the simplest of the three.

2.4.2 Proof of Case (a)

This is the simplest case of the three. Let

$$\beta = \beta_1, \quad N = N_1,$$

and

$$\alpha = \beta_2 * \cdots * \beta_k, \quad M = N_2 \cdots N_k,$$

so that $\gamma = \alpha * \beta$. Since $\nu_1 \geq 5/8 - 8\varpi$ in this case, by Lemma 8 with $\kappa = X^{5/8-8\varpi}$, we have

$$\Delta(\beta; X, d, a) \ll X^{5/16-4\varpi} N.$$

By definition of $\Delta(\gamma; X, d, a)$ and bounding

$$\sum_{m\sim M} \alpha(m) \ll \sum_{m\sim M} \tau_k(m) \ll M\mathcal{L}^B$$

trivially, we get

$$\Delta(\gamma; X, d, a) \leq \sum_{(m,d)=1} \left| \sum_{m\sim M} \alpha(m) \right| |\Delta(\beta; X, d, a\overline{m})| \ll M\mathcal{L}^B \max_{(a,d)=1} \Delta(\alpha; X, d, a) \ll X^{5/16-5\varpi}.$$

Thus,

$$\sum_{d\leq D_3} \Delta(\gamma; X, d, a) \ll X^{5/16-5\varpi} D_3 \leq X^{13/16-3\varpi}.$$

This leads to (2.53).

2.4.3 Proof of Case (b)

This case corresponds to the Type III estimate in [87, §§13, 14], which we will follow closely. The main tool we need is the Birch-Bombieri bound from Lemma 13. We will reduce $\Delta(\gamma; X, d, a)$ into an exponential sum of that form; see (2.95). Write $\alpha = \beta_4 * \cdots * \beta_k$ so that

$$\gamma = \alpha * \beta_1 * \beta_2 * \beta_3.$$

Let

$$M = N_4 \cdots N_k.$$

Note that $\alpha * \beta_1$ is supported on $[MN_1, 2MN_1)$. We have the following lemma.

Lemma 14 *Suppose that*

$$\nu_1 < \frac{5}{8} - 8\varpi \tag{2.63}$$

and, for any subset $I \subseteq \{1, 2, \cdots, k\}$*,*

$$\sum_{i \in I} \nu_i \notin \left[\frac{3}{8} + 8\varpi, \frac{5}{8} - 8\varpi\right]. \tag{2.64}$$

Then we have

$$\nu_3 + \nu_2 > \frac{5}{8} - 8\varpi.$$

Proof: By virtue of (2.64) with $I = \{2, 3\}$, it suffices to show that

$$\nu_3 + \nu_2 > \frac{3}{8} + 8\varpi.$$

By (2.63) and (2.64) we have

$$\nu_1 < \frac{3}{8} + 8\varpi.$$

By (7),

$$\nu_2 \leq \nu_1 < \frac{3}{8} + 8\varpi.$$

By the first inequality in (8),

$$\nu_2 + \cdots + \nu_k = (\nu_k + \cdots + \nu_1) - \nu_1 > 1 - \left(\frac{3}{8} + 8\varpi\right) = \frac{5}{8} - 8\varpi.$$

Hence, by (8), there is an $3 \leq \ell \leq k$ such that

$$\nu_2 + \cdots + \nu_{\ell-1} < \frac{3}{8} + 8\varpi$$

and

$$\nu_2 + \cdots + \nu_\ell > \frac{5}{8} - 8\varpi,$$

so that

$$\nu_\ell > \frac{1}{4} - 16\varpi$$

Thus, by (7) and (2.64), we conclude

$$\nu_3 + \nu_2 \geq 2\nu_\ell > \frac{1}{2} - 32\varpi > \frac{3}{8} + 8\varpi.$$

■

We have

$$N_1 \geq N_2 \geq \left(\frac{X}{MN_1}\right)^{1/2} \geq X^{5/16-4\varpi}, \tag{2.65}$$

and

$$N_3 \geq \frac{X}{MN_1N_2} \geq X^{1/4-16\varpi}. \tag{2.66}$$

By Lemma 14 we have

$$\nu_3 + \nu_2 > \frac{5}{8} - 8\varpi.$$

Hence,

$$MN_1 \ll \frac{X}{N_2N_3} \ll X^{3/8+8\varpi}. \tag{2.67}$$

Let f be as in Lemma 6 with $\eta^* = \eta$, $\epsilon = \varpi$, and with N_1 in place of M. Note that the function $\beta_1 - f$ is supported on $[N_1^-, N_1] \cup [\eta N_1, \eta N_1^+]$ with $N_1^{\pm} = (1 \pm N_1^{-\varpi})N_1$. Letting

$$\gamma^* = \alpha * \beta_2 * \beta_3 * f,$$

we have

$$\begin{aligned}
\sum_{(n,d)=1} (\gamma - \gamma^*)(n) &\ll \sum_{(mn,d)=1} \alpha * \beta_2 * \beta_3(n)\beta_1(m) - \alpha * \beta_2 * \beta_3(n) f(m) \\
&= \sum_{(n,d)=1} \alpha * \beta_2 * \beta_3(n) \sum_{(m,d)=1} (\beta_1(m) - f(m)) \\
&\ll \frac{\varphi(d)}{d} MN_2N_3M^{\epsilon}(N_1^{1-\varpi} + \eta N_1^{1-\varpi}) \\
&\ll \frac{\varphi(d)}{d} X^{1-\varpi/2}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\equiv a(d)} (\gamma - \gamma^*)(n) &\ll \sum_{\substack{N_1^- \leq q \leq N_1 \\ (q,d)=1}} \sum_{\substack{1 \leq \ell < 3x/q \\ \ell q \equiv a(d)}} \tau_k(\ell) + \sum_{\substack{\eta N_1 \leq q \leq \eta N_1^+ \\ (q,d)=1}} \sum_{\substack{1 \leq \ell < 3x/q \\ \ell q \equiv a(d)}} \tau_k(\ell) \\
&\ll \frac{\varphi(d)}{d^2} MN_2N_3N_1^{1-\varpi}\mathcal{L}^B \ll \frac{X^{1-\varpi/2}}{d}.
\end{aligned}$$

It therefore suffices to prove (2.53) with γ replaced by γ^*. We shall prove the bound

$$\Delta(\gamma^*; d, c) \ll \frac{X^{1-\varpi/3}}{d}. \tag{2.68}$$

We remove the condition $(n, d) = 1$ by means of Möbius inversion formula

$$\sum_{\delta|(n,d)} \mu(\delta) = \begin{cases} 1, & \text{if } (n, d) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.69}$$

By (2.69) we have

$$\sum_{(n,d)=1} f(n) = \sum_{n} f(n) \sum_{\substack{\delta|n \\ \delta|d}} \mu(\delta) = \sum_{\delta|d} \mu(\delta) \sum_{n\equiv 0(\delta)} f(n).$$

For $\delta > N^{1-2\epsilon}$, the inner sum on the right side is

$$\sum_{n\equiv 0(\delta)} f(n) \ll \frac{N}{N^{1-2\epsilon}} \ll N^{2\epsilon}.$$

This yields

$$\sum_{(n,d)=1} f(n) = \sum_{\substack{\delta|d \\ \delta\le N^{1-2\epsilon}}} \mu(\delta) \sum_{n\equiv 0(\delta)} f(n) + O(N^\epsilon) = \sum_{\substack{\delta|d \\ \delta\le N^{1-2\epsilon}}} \frac{\mu(\delta)}{\delta} \hat{f}(0) + O(N^\epsilon).$$

Since

$$\sum_{\delta|d} \frac{\mu(\delta)}{\delta} = \frac{\varphi(d)}{d},$$

the above becomes

$$\sum_{(n,d)=1} f(n) = \frac{\varphi(d)}{d} \hat{f}(0) + O(N^\epsilon). \tag{2.70}$$

By (2.65), this yields

$$\frac{1}{\varphi(d)} \sum_{(n,d)=1} \gamma^*(n) = \frac{\hat{f}(0)}{d} \sum_{(m,d)=1} \sum_{\substack{n_3\simeq N_3 \\ (n_3,d)=1}} \sum_{\substack{n_2\simeq N_2 \\ (n_2,d)=1}} \alpha(m) + O(d^{-1}X^{3/4}).$$

Here $n \simeq N$ stands for $N \le n \le \eta N$. On the other hand, we have

$$\sum_{n \equiv a(d)} \gamma^*(n) = \sum_{(m,d)=1} \sum_{\substack{n_3 \simeq N_3 \\ (n_3,d)=1}} \sum_{\substack{n_2 \simeq N_2 \\ (n_2,d)=1}} \alpha(m) \sum_{mn_3n_2n_1 \equiv a(d)} f(n_1).$$

The innermost sum is, by Lemma 6, equal to

$$\frac{1}{d} \sum_{|h|<H^*} \hat{f}(h/d) e_d(-ah\overline{mn_3n_2}) + O(d^{-1})$$

where

$$H^* = dN_1^{-1+2\epsilon}.$$

It follows that the left side of (2.68) is

$$= \frac{1}{d} \sum_{\substack{m \simeq M \\ (m,d)=1}} \sum_{\substack{n_3 \simeq N_3 \\ (n_3,d)=1}} \sum_{\substack{n_2 \simeq N_2 \\ (n_2,d)=1}} \alpha(m) \sum_{1 \le |h| < H^*} \hat{f}(h/d) e_d(-ah\overline{mn_3n_2}) + O(d^{-1}X^{3/4}).$$

Hence the proof of (2.68) is reduced to showing that

$$\sum_{1 \le h < H^*} \sum_{\substack{n_3 \simeq N_3 \\ (n_3,d)=1}} \sum_{\substack{n_2 \simeq N_2 \\ (n_2,d)=1}} \hat{f}(h/d) e_d(ah\overline{n_3n_2}) \ll X^{1-\varpi/2+2\epsilon} M^{-1} \tag{2.71}$$

for any a with $(a,d)=1$. Substituting $d_1 = d/(h,d)$ and applying Möbius inversion, the left side of (2.71) can be written as

$$\begin{aligned} &\sum_{d_1 \mid d} \sum_{\substack{1 \le h < H \\ (h,d_1)=1}} \sum_{\substack{n_3 \simeq N_3 \\ (n_3,d)=1}} \sum_{\substack{n_2 \simeq N_2 \\ (n_2,d)=1}} \hat{f}(h/d_1) e_{d_1}(ah\overline{n_3n_2}) \\ &= \sum_{d_1 d_2 = d} \sum_{b_3 \mid d_2} \sum_{b_2 \mid d_2} \mu(b_3)\mu(b_2) \sum_{\substack{1 \le h < H \\ (h,d_1)=1}} \sum_{\substack{n_3 \simeq N_3/b_3 \\ (n_3,d_1)=1}} \sum_{\substack{n_2 \simeq N_2/b_2 \\ (n_2,d_1)=1}} \hat{f}(h/d_1) e_{d_1}(ah\overline{b_3b_2n_3n_2}), \end{aligned}$$

where

$$H = d_1 N_1^{-1+2\epsilon}. \tag{2.72}$$

It therefore suffices to show that

$$\sum_{\substack{1 \le h < H \\ (h,d_1)=1}} \sum_{\substack{n_3 \simeq N_3' \\ (n_3,d_1)=1}} \sum_{\substack{n_2 \simeq N_2' \\ (n_2,d_1)=1}} \hat{f}(h/d_1) e_{d_1}(bh\overline{n_3n_2}) \ll X^{1-\varpi/2+\epsilon} M^{-1} \tag{2.73}$$

for any d_1, b, N_3', and N_2' satisfying

$$d_1 | d, \quad (b, d_1) = 1, \quad \frac{d_1 N_3}{d} \le N_3' \le N_3, \quad \frac{d_1 N_2}{d} \le N_2' \le N_2, \tag{2.74}$$

which are henceforth assumed. By the lower bound (2.65) we have

$$H \ll X^{3/16+6\varpi+\epsilon}. \tag{2.75}$$

By (2.72), the left side of (2.73) is void if $d_1 \le N_1^{1-2\epsilon}$, so we may assume that $d_1 > N_1^{1-2\epsilon}$. By the trivial bound

$$\hat{f}(z) \ll N_1 \tag{2.76}$$

and the bound (2.43), we find that the left side of (2.73) is

$$\ll H N_3 N_1 (d_1^{1/2+\epsilon} + d_1^{-1} N_2) \ll d_1^{3/2+\epsilon} N_1^{2\epsilon} N_3.$$

If $d_1 \le X^{5/12-6\varpi}$, then the right side of the above is $\ll X^{1-\varpi+3\epsilon}$ by (2.67) and (2.105), and this leads to (2.73). Thus we may further assume that

$$d_1 > X^{5/12-6\varpi}. \tag{2.77}$$

We appeal to the Weyl shift and the factorization $d_2 = rq$. By Lemma 10, with d_1 in place of d, we can choose a factor r of d_1 such that

$$X^{44\varpi} < r < X^{45\varpi}. \tag{2.78}$$

Write

$$\mathcal{N}(d_1, k) = \sum_{\substack{1 \le h < H \\ (h, d_1) = 1}} \sum_{\substack{n_3 \simeq N_3' \\ (n_3, d_1) = 1}} \sum_{\substack{n_2 \simeq N_2' \\ (n_2 + hkr, d_1) = 1}} \hat{f}(h/d_1) e_{d_1}(bh\overline{(n_2 + hkr) n_3}),$$

so that $\mathcal{N}(d_1, 0)$ corresponds to the left side of (2.73). Assume $k > 0$. We have

$$\mathcal{N}(d_1, k) - \mathcal{N}(d_1, 0) = \mathcal{Q}_1(d_1, k) - \mathcal{Q}_2(d_1, k), \tag{2.79}$$

where

$$\mathcal{Q}_i(d_1,k) = \sum_{\substack{1\le h<H\\(h,d_1)=1}} \sum_{\substack{n_3\simeq N_3'\\(n_3,d_1)=1}} \sum_{\substack{\ell\in\mathcal{I}_i(h)\\(\ell,d_1)=1}} \hat{f}(h/d_1)e_{d_1}(bh\overline{\ell n_3}), \quad i=1,2,$$

with

$$\mathcal{I}_1(h) = [\eta N_2', \eta N_2' + hkr), \quad \mathcal{I}_2(h) = [N_2', N_2' + hkr).$$

By Möbius inversion, we have

$$\mathcal{Q}_i(d_1,k) = \sum_{st=d_1} \mu(s) \sum_{1\le h<H/s} \sum_{\substack{n_3\simeq N_3'\\(n_3,d_1)=1}} \sum_{\substack{\ell\in\mathcal{I}_i(sh)\\(\ell,d_1)=1}} \hat{f}(h/t)e_t(bh\overline{\ell n_3}).$$

The h sum is empty unless $s<H$. Since $H^2 = o(d_1)$ by (2.75) and (2.77), it follows, by changing the order of summation, that

$$|\mathcal{Q}_i(d_1,k)| \le \sum_{\substack{st=d_1\\t>H}} \sum_{\substack{n_3\simeq N_3'\\(n_3,d_1)=1}} \sum_{\substack{\ell\in\mathcal{I}_i(H)\\(\ell,d_1)=1}} \left| \sum_{h\in J_i(s,\ell)} \hat{f}(h/t)e_t(bh\overline{\ell n_3}) \right|,$$

where $J_i(s,\ell)$ is some interval of length at most H depending on s and ℓ. By integration by parts, we have

$$\frac{d}{dz}\hat{f}(z) \ll \min\{N_1^2, |z|^{-2}N_1^{\epsilon}\},$$

and by partial summation and (2.76) we obtain

$$\sum_{h\in J_i(s,\ell)} \hat{f}(h/t)e_t(bh\overline{\ell n_3}) \ll N_1^{1+\epsilon}\min\{H, \|b\overline{\ell n_3}/t\|^{-1}\}.$$

Thus,

$$\mathcal{Q}_i(d_1,k) \ll N_1^{1+\epsilon} \sum_{\substack{t|d_1\\t>H}} \sum_{\substack{\ell\in\mathcal{I}_i(H)\\(\ell,d_1)=1}} \sum_{\substack{n_3<2N_3\\(n_3,d_1)=1}} \min\{H, \|b\overline{\ell n_3}/t\|^{-1}\}.$$

Since $H = o(N_3)$ by (2.66) and (2.75), the inner most sum is $\ll N_3^{1+\epsilon}$, by Lemma 11. By (2.72), this leads to

$$\mathcal{Q}_i(d_1,k) \ll d_1^{1+\epsilon}krN_3. \tag{2.80}$$

We now introduce the parameter

$$K = [X^{-1/2-48\varpi} N_1 N_2],$$

which is $\gg X^{1/8-56\varpi}$ by (2.65). By the second inequality in (2.78), the right side of (2.80) is $\ll X^{1-\varpi+\epsilon} M^{-1}$ if $k < 2K$. Hence, by (2.79), the proof of (2.73) is reduced to showing that

$$\frac{1}{K} \sum_{k \sim K} \mathcal{N}(d_1, k) \ll X^{1-\varpi/2+\epsilon} M^{-1}. \tag{2.81}$$

We now prove (2.81). By the relation

$$h\overline{(n_2 + hkr)} \equiv \overline{\ell + kr} \pmod{d_1}$$

for $(h, d_1) = (n_2 + hkr, d_1) = 1$, where $\ell \equiv \overline{h} n_2 \pmod{d_1}$, we may rewrite $\mathcal{N}(d_1, k)$ as

$$\mathcal{N}(d_1, k) = \sum_{\substack{\ell \ (\mathrm{mod}\ d_1) \\ (\ell+kr, d_1)=1}} \nu(\ell; d_1) \sum_{\substack{n_3 \simeq N_3' \\ (n_3, d_1)=1}} e_{d_1}(b\overline{(\ell + kr)} n_3),$$

where

$$\nu(\ell; d_1) = \sideset{}{'}\sum_{\overline{h} n_2 \equiv \ell (d_1)} \hat{f}(h/d_1).$$

Here $\sum'$ is the restriction to $1 \le h < H$, $(h, d_1) = 1$, and $n_2 \simeq N_2'$. It follows by Cauchy's inequality that

$$\left| \sum_{k \sim K} \mathcal{N}(d_1, k) \right|^2 \le P_1 P_2, \tag{2.82}$$

where

$$P_1 = \sum_{\ell \ (\mathrm{mod}\ d_1)} |\nu(\ell; d_1)|^2 \quad \text{and} \quad P_2 = \sum_{\ell \ (\mathrm{mod}\ d_1)} \left| \sum_{\substack{k \sim K \\ (\ell+kr, d_1)=1}} \sum_{\substack{n \simeq N_3' \\ (n, d_1)=1}} e_{d_1}(b\overline{(\ell + kr)} n) \right|^2.$$

By (2.76) we have

$$P_1 \ll N_1^2 N_4,$$

where

$$N_4 = \#\{(h_1, h_2; n_1, n_2) : h_2 n_1 \equiv h_1 n_2 \pmod{d_1}, 1 \le h_i < H, n_i \simeq N_2'\}$$
$$\ll \sum_{\ell \pmod{d_1}} \left(\sum_{\substack{1 \le m < 2HN_2 \\ m \equiv \ell (d_1)}} \tau(m) \right)^2 .$$

Since $HN_2 \ll d_1^{1+\epsilon}$ by (2.72), we get

$$P_1 \ll d_1^{1+\epsilon} N_1^2. \tag{2.83}$$

We claim that

$$P_2 \ll d_1 X^{3/16+52\varpi+\epsilon} K^2. \tag{2.84}$$

With this, the estimate (2.81) follows from (2.82), (2.83), and (2.84) immediately since

$$N_1 \le X^{3/8+8\varpi} M^{-1}, \quad d_1 < X^{1/2+2\varpi},$$

and

$$\frac{31}{32} + 36\varpi = 1 - \frac{\varpi}{2}.$$

It remains to prove (2.84). Write $d_1 = rq$. Note that

$$\frac{N_3'}{r} \gg X^{1/6-69\varpi} \tag{2.85}$$

by (2.74), (2.77), (2.66), and the second inequality in (2.78). Since

$$\sum_{\substack{n \simeq N_3' \\ (n, d_1)=1}} e_{d_1}(b\overline{(\ell + kr)n}) = \sum_{\substack{0 \le s < r \\ (s,r)=1}} \sum_{\substack{n \simeq N_3'/r \\ (nr+s, q)=1}} e_{d_1}(b\overline{(\ell + kr)(nr + s)}) + O(r),$$

we get

$$\sum_{\substack{k \sim K \\ (\ell + kr, d_1)=1}} \sum_{\substack{n \simeq N_3' \\ (n, d_1)=1}} e_{d_1}(b\overline{(\ell + kr)n}) = U(\ell) + O(Kr),$$

where

$$U(\ell)=\sum_{\substack{0\le s<r\\(s,r)=1}}\ \sum_{\substack{k\sim K\\(\ell+kr,d_1)=1}}\ \sum_{\substack{n\simeq N_3'/r\\(nr+s,q)=1}} e_{d_1}\big(b\overline{(\ell+kr)(rn+s)}\big).$$

Hence,

$$P_2 \ll \sum_{\ell \pmod{d_1}} |U(\ell)|^2 + d_1(Kr)^2.$$

By the second inequality in (2.78), the second term on the right side of the above is admissible for (2.84). On the other hand, we have

$$\sum_{\ell \pmod{d_1}} |U(\ell)|^2 = \sum_{k_1\sim K}\sum_{k_2\sim K}\sum_{\substack{0\le s_1<r\\(s_1,r)=1}}\sum_{\substack{0\le s_2<r\\(s_2,r)=1}} V(k_2-k_1;s_1,s_2), \tag{2.86}$$

where

$$V(k;s_1,s_2)=\sum_{\substack{n_1\simeq N_3'/r\\(n_1r+s_1,q)=1}}\ \sum_{\substack{n_2\simeq N_3'/r\\(n_2r+s_2,q)=1}}\ \sideset{}{'}\sum_{\ell \pmod{d_1}} e_{d_1}\big(b\overline{\ell(n_1r+s_1)}-b\overline{(\ell+kr)(n_2r+s_2)}\big). \tag{2.87}$$

Here $\sum'$ is the restriction to $(\ell,d_1)=(\ell+kr,d_1)=1$. Note that if $\ell\equiv \ell_1r+\ell_2q \pmod{d_1}$, then the condition $(\ell(\ell+kr),d_1)=1$ is equivalent to $(\ell_1(\ell_1+k),q)=(\ell_2,r)=1$. In this case, by the relation

$$\frac{1}{d_1}\equiv\frac{\overline{r}}{q}+\frac{\overline{q}}{r} \pmod 1,$$

we have

$$\begin{aligned}&\frac{\overline{\ell(n_1r+s_1)}-\overline{(\ell+kr)(n_2r+s_2)}}{d_1}\\ &\equiv \frac{\overline{r^2\ell_1(n_1r+s_1)}-\overline{r^2(\ell_1+k)(n_2r+s_2)}}{q}+\frac{\overline{q^2s_1s_2\ell_2}(s_2-s_1)}{r} \pmod 1.\end{aligned}$$

Thus, by the Chinese remainder theorem, the innermost sum in (2.87) is equal to

$$C_r(s_2-s_1)\sum_{\substack{\ell \pmod q\\(\ell(\ell+k),q)=1}} e_q\big(b\overline{r^2\ell(n_1r+s_1)}-b\overline{r^2(\ell+k)(n_2r+s_2)}\big),$$

and, thus,

$$V(k;s_1,s_2)=W(k;s_1,s_2)C_r(s_2-s_1), \tag{2.88}$$

where

$$W(k,s_1,s_2)=\sum_{\substack{n_1\simeq N_3'/r\\(n_1r+s_1,q)=1}}\sum_{\substack{n_2\simeq N_3'/r\\(n_2r+s_2,q)=1}}\sideset{}{'}\sum_{\ell \pmod q} e_q(\overline{br^2\ell(n_1r+s_1)}-\overline{br^2(\ell+k)(n_2r+s_2)}).$$

Here $\sum'$ is the restriction to $(\ell(\ell+k),q)=1$.

We first estimate the contribution from terms with $k_1=k_2$ on the right side of (2.86) as follows. For $(n_1r+s_1,q)=(n_2r+s_2,q)=1$, we have

$$\sideset{}{^*}\sum_{\ell \pmod q} e_q((\overline{br^2\ell(n_1r+s_1)}-\overline{br^2\ell(n_2r+s_2)})=C_q((n_1-n_2)r+s_1-s_2).$$

Since $N_3'\ll X^{1/3}$, by (2.77) and the second inequality in (2.78), we have

$$\frac{N_3'}{d_1}\ll X^{-1/12+6\varpi}\ll r^{-1}. \tag{2.89}$$

This implies $N_3'/r=o(q)$, giving

$$\sum_{n\simeq N_3'/r}|C_q(nr+m)|\ll q^{1+\epsilon}$$

for any m. Thus

$$W(0;s_1,s_2)\ll q^{1+\epsilon}r^{-1}N_3'.$$

Substituting the above into (2.88) and using the simple estimate

$$\sum_{0\le s_1<r}\sum_{0\le s_2<r}|C_r(s_2-s_1)|\ll r^{2+\epsilon},$$

we get that

$$\sum_{\substack{0\le s_1<r\\(s_1,r)=1}}\sum_{\substack{0\le s_2<r\\(s_2,r)=1}}V(0;s_1,s_2)\ll d_1^{1+\epsilon}N_3.$$

Hence the contribution from the terms with $k_1 = k_2$ on the right side of (2.86) is $\ll d_1^{1+\epsilon} K N_3$; this reduces to showing that

$$\sum_{k_1 \sim K} \sum_{\substack{k_2 \sim K \\ k_2 \neq k_1}} \sum_{\substack{0 \leq s_1 < r \\ (s_1, r)=1}} \sum_{\substack{0 \leq s_2 < r \\ (s_2, r)=1}} V(k_2 - k_1; s_1, s_2) \ll d_1 X^{3/16+52\varpi+\epsilon} K^2. \tag{2.90}$$

By (2.85) and (2.89), letting

$$n' = \min\{n : n \simeq N_3'/r\}, \quad n'' = \max\{n : n \simeq N_3'/r\},$$

we may rewrite $W(k; s_1, s_2)$ as

$$\sum_{\substack{n_1 \leq q \\ (n_1 r + s_1, q)=1}} \sum_{\substack{n_2 \leq q \\ (n_2 r + s_2, q)=1}} \sideset{}{'}\sum_{\ell \pmod q} F\left(\frac{n_1}{q}\right) F\left(\frac{n_2}{q}\right) e_q\left(b\overline{r^2 \ell (n_1 r + s_1)} - b\overline{r^2 (\ell + k)(n_2 r + s_2)}\right),$$

where $F(y)$ is a function of $C^2[0,1]$ class such that

$$0 \leq F(y) \leq 1,$$

$$F(y) = \begin{cases} 1, & \text{if} \quad y \in \left[\frac{n'}{q}, \frac{n''}{q}\right], \\ 0, & \text{if} \quad y \notin \left[\frac{n'}{q} - \frac{1}{2q}, \frac{n''}{q} + \frac{1}{2q}\right], \end{cases}$$

for which the Fourier coefficient

$$\kappa(m) = \int_0^1 F(y) e(-my) dy$$

satisfies

$$\kappa(m) \ll \kappa^*(m) := \min\left\{\frac{1}{r}, \frac{1}{|m|}, \frac{q}{m^2}\right\}. \tag{2.91}$$

Here we have used (2.89). Fourier expand $F(y)$ we obtain

$$W(k; s_1, s_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \kappa(m_1)\kappa(m_2) Y(k; m_1, m_2; s_1, s_2), \tag{2.92}$$

where

$$Y(k;m_1,m_2;s_1,s_2) = \sum_{\substack{n_1\le q\\(n_1r+s_1,q)=1}} \sum_{\substack{n_2\le q\\(n_2r+s_2,q)=1}} \sideset{}{'}\sum_{\ell \pmod q} e_q(\delta(\ell,k;m_1,m_2;n_1,n_2;s_1,s_2)),$$

with

$$\delta(\ell,k;m_1,m_2;n_1,n_2;s_1,s_2) = b\overline{r^2\ell(n_1r+s_1)} - b\overline{r^2(\ell+k)(n_2r+s_2)} + m_1n_1 + m_2n_2.$$

Moreover, if $n_jr+s_j \equiv t_j \pmod q$, then $n_j \equiv \overline{r}(t_j-s_j) \pmod q$ so that

$$m_1n_1+m_2n_2 \equiv \overline{r}(m_1t_1+m_2t_2) - \overline{r}(m_1s_1+m_2s_2) \pmod q.$$

Hence, on substituting $n_jr+s_j = t_j$, we may write $Y(k;m_1,m_2;s_1,s_2)$ as

$$Y(k;m_1,m_2;s_1,s_2) = Z(k;m_1,m_2)e_q\left(-\overline{r}(m_1s_1+m_2s_2)\right), \tag{2.93}$$

where

$$Z(k;m_1,m_2) = \sideset{}{^*}\sum_{t_1 \pmod q} \sideset{}{^*}\sum_{t_2 \pmod q} \sideset{}{'}\sum_{\ell \pmod q} e_q(b\overline{r^2\ell t_1} - b\overline{(r^2(\ell+k)t_2)} + \overline{r}(m_1t_1+m_2t_2)).$$

By (2.88), (2.92), and (2.93) we get

$$\sum_{\substack{0\le s_1<r\\(s_1,r)=1}} \sum_{\substack{0\le s_2<r\\(s_2,r)=1}} V(k;s_1,s_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \kappa(m_1)\kappa(m_2)Z(k;m_1,m_2)J(m_1,m_2), \tag{2.94}$$

where

$$J(m_1,m_2) = \sum_{\substack{0\le s_1<r\\(s_1,r)=1}} \sum_{\substack{0\le s_2<r\\(s_2,r)=1}} e_q(-\overline{r}(m_1s_1+m_2s_2))C_r(s_2-s_1).$$

We now appeal to Lemma 13.

By simple substitution we have

$$Z(k;m_1,m_2) = T(k, bm_1\overline{r}^3, -bm_2\overline{r}^3; q),$$

so Lemma 13 gives

$$Z(k;m_1,m_2) \ll (k,q)^{1/2}q^{3/2+\epsilon}, \tag{2.95}$$

the right side does not depend on m_1 and m_2. We claim the following estimate

$$\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\kappa^*(m_1)\kappa^*(m_2)|J(m_1,m_2)| \ll r^{1+\epsilon}. \tag{2.96}$$

Combining the above two estimates together with (2.94) we obtain

$$\sum_{\substack{0\le s_1<r\\(s_1,r)=1}}\sum_{\substack{0\le s_2<r\\(s_2,r)=1}} V(k;s_1,s_2) \ll (k,q)^{1/2}q^{3/2+\epsilon}r^{1+\epsilon}.$$

This leads to (2.90), since

$$q^{1/2}=(d_1/r)^{1/2}<X^{1/4-21\varpi}=X^{3/16+52\varpi}$$

by the first inequality in (2.78) and

$$\sum_{k_1\sim K}\sum_{\substack{k_2\sim K\\k_2\neq k_1}}(k_1-k_1,q)^{1/2}\ll q^\epsilon K^2,$$

whence (2.84) follows.

We now prove (2.96). We rewrite the left side of (2.96) as

$$\frac{1}{r}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\sum_{0\le k<r}\kappa^*(m_1)\kappa^*(m_2+k)|J(m_1,m_2+k)|.$$

By (2.91), we have

$$\sum_{m=-\infty}^{\infty}\kappa^*(m)\ll \mathcal{L},$$

and $\kappa^*(m+k)\ll\kappa^*(m)$ for $0\le k<r$, since $r<q$ by (2.77) and the second inequality in (2.78). Thus, to prove (2.96), it suffices to show that

$$\sum_{0\le k<r}|J(m_1,m_2+k)|\ll r^{2+\epsilon} \tag{2.97}$$

for any m_1 and m_2. Substituting $s_2-s_1=t$ and applying Möbius inversion we obtain

$$J(m_1,m_2)=\sum_{|t|<r}C_r(t)\sum_{\substack{s\in I_t\\(s(s+t),r)=1}}e_q\left(-\overline{r}(m_2t+(m_1+m_2)s)\right) \tag{2.98}$$

$$\ll\sum_{|t|<r}|C_r(t)|\left|\sum_{\substack{s\in I_t\\s(s+t)\equiv 0(r_1)}}e_q(\overline{r}(m_1+m_2)s)\right|,$$

where I_t is some interval of length less than r depending on t. For any fixed t and any square-free r_1, there are exactly $\tau(r_1/(t,r_1))$ distinct residue classes modulo r_1 such that

$$s(s+t) \equiv 0 \pmod{r_1}$$

if and only if s belongs to one of these classes. On the other hand, if $r = r_1 r_2$, then

$$\sum_{\substack{s \in I_t \\ s \equiv a (r_1)}} e_q\left(\overline{r}(m_1+m_2)s\right) \ll \min\{r_2, \|\overline{r}_2(m_1+m_2)/q\|^{-1}\}$$

for any a. Hence the inner sum on the right side of (2.98) is

$$\ll \tau(r) \sum_{r_2 | r} \min\{r_2, \|\overline{r}_2(m_1+m_2)/q\|^{-1}\},$$

which does not depend on t. This, together with the simple estimate

$$\sum_{|t|<r} |C_r(t)| \ll \tau(r) r,$$

yields

$$J(m_1, m_2) \ll \tau(r)^2 r \sum_{r_2|r} \min\{r_2, \|\overline{r}_2(m_1+m_2)/q\|^{-1}\}.$$

Thus the left side of (2.97) is

$$\ll \tau(r)^2 r \sum_{r_1 r_2 = r} \sum_{0 \le k_1 < r_1} \sum_{0 \le k_2 < r_2} \min\{r_2, \|\overline{r}_2(m_1+m_2+k_1r_2+k_2)/q\|^{-1}\}. \tag{2.99}$$

Assume $r_2 | r$. By the relation

$$\frac{\overline{r_2}}{q} \equiv -\frac{\overline{q}}{r_2} + \frac{1}{qr_2} \pmod 1,$$

we have

$$\sum_{0 \le k < r_2} \min\{r_2, \|\overline{r_2}(m+k)/q\|^{-1}\} \ll r_2 \mathcal{L} \tag{2.100}$$

for any m. Hence the estimate (2.97) follows at once from (2.99) and (2.100).

2.4.4 Proof of Case (c)

We finish the proof of Theorem 1 with the last and most involved case. This case corresponds to the Type I and II estimates in [87, §§7-12]. Without loss of generality, assume there is a subset I of $\{1, 2, \ldots, k\}$ such that

$$\frac{3}{8} + 8\varpi < \sum_{i \in I} \nu_i < \frac{1}{2} + \frac{\log 2}{2\mathcal{L}}. \tag{2.101}$$

Let J be the complement of I in $\{1, 2, \ldots, k\}$. Write

$$\alpha = \beta_{j_1} * \beta_{j_2} * \cdots * \beta_{j_m}, \quad J = \{j_1, j_2, \ldots, j_m\},$$

and

$$\beta = \beta_{i_1} * \beta_{i_2} * \cdots * \beta_{i_\ell}, \quad I = \{i_1, i_2, \ldots, i_\ell\}, \tag{2.102}$$

so that $\gamma = \alpha * \beta$. We have α supported on $[M, 2M)$ and β supported on $[N, 2N)$, where

$$M = \prod_{j \in J} N_j, \quad N = \prod_{i \in I} N_i. \tag{2.103}$$

By (2.101), we have

$$X^{3/8+8\varpi} < N \ll X^{1/2}. \tag{2.104}$$

We now treat (2.107) via the methods in [31] and [8, §§3-7], following [87]. Write

$$X_1 = X^{3/8+8\varpi} \quad \text{and} \quad X_2 = X^{1/2-4\varpi}. \tag{2.105}$$

We apply Lemma 10 with

$$R^* = \begin{cases} X^{-\varpi/6} N, & \text{if } X_1 < N \leq X_2, \\ X^{-3\varpi} N, & \text{if } X_2 < N \leq 2X^{1/2}. \end{cases} \tag{2.106}$$

Hence, by Lemma 10, the proof of Case (c) is reduced to showing that

$$\sum_{q \sim Q} \sum_{\substack{r \sim R \\ (r,a)=1 \\ (q, r\mathcal{P}_0)=1}} \mu(qr)^2 |\Delta(\gamma; X, qr, a)| \ll X^{1-\varpi^{5/3}} \tag{2.107}$$

subject to the conditions

$$X^{-\varpi}R^* < R < R^* \quad \text{and} \quad \frac{1}{2}D_2 < QR < X^{1/2+2\varpi}.$$

Therefore, it suffices to prove that

$$\mathcal{B}(\gamma; Q, R) := \sum_{\substack{r\sim R \\ (r,a)=1}} |\mu(r)| \sum_{\substack{q\sim Q \\ q|\mathcal{P} \\ (q,r\mathcal{P}_0)=1}} |\Delta(\gamma; X, qr, a)| \ll X^{1-\varpi^{5/3}} \tag{2.108}$$

subject to the constraints

$$X^{-\varpi}R^* < R < R^* \tag{2.109}$$

and

$$D_2 \ll QR \ll X^{1/2+2\varpi}, \tag{2.110}$$

which are henceforth assumed.

In what follows we assume that

$$r \sim R, \quad |\mu(r)| = 1, \quad \text{and} \quad (r,a) = 1. \tag{2.111}$$

Let $c(q,r)$ denote

$$c(q,r) = \begin{cases} \operatorname{sign}\Delta(\gamma; X, qr, a), & \text{if } q \sim Q, q|\mathcal{P}, \text{and } (q, r\mathcal{P}_0) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Splitting $\gamma = \alpha * \beta$, writing n as mn, and changing the order of summation, the inner sum over q in (2.108) becomes

$$\sum_{\substack{q\sim Q \\ q|\mathcal{P} \\ (q,r\mathcal{P}_0)=1}} |\Delta(\gamma; X, q, a)| = \sum_{(m,r)=1} \alpha(m)\mathcal{D}(r,m), \tag{2.112}$$

where

$$\mathcal{D}(r,m) := \sum_{(q,m)=1} c(q,r) \left(\sum_{mn\equiv a(qr)} \beta(n) - \frac{1}{\varphi(qr)} \sum_{(n,qr)=1} \beta(n) \right).$$

Substituting (2.112) into (2.108) and applying Cauchy's inequality to the m variable, we get

$$\mathcal{B}(\gamma; Q, R)^2 \ll MR\mathcal{L}^B \sum_{r\sim R} |\mu(r)| \sum_{(m,r)=1} f(m)\mathcal{D}(r,m)^2, \tag{2.113}$$

where $f(y)$ is as in Lemma 6. Squaring out $\mathcal{D}(r,m)$ and summing over m, we have

$$\sum_{(m,r)=1} f(m)\mathcal{D}(r,m)^2 = \mathcal{S}_1(r) - 2\mathcal{S}_2(r) + \mathcal{S}_3(r), \tag{2.114}$$

where $\mathcal{S}_j(r,a)$, $j = 1,2,3$, are defined by

$$\mathcal{S}_1(r) = \sum_{(m,r)=1} f(m) \left(\sum_{(q,m)=1} c(q,r) \sum_{mn\equiv a(qr)} \beta(n) \right)^2,$$

$$\mathcal{S}_2(r) = \sum_{q_1}\sum_{q_2} \frac{c(q_1,r)c(q_2,r)}{\varphi(q_2 r)} \sum_{n_1} \sum_{(n_2,q_2r)=1} \beta(n_1)\beta(n_2) \sum_{\substack{mn_1\equiv a(q_1r)\\(m,q_2)=1}} f(m),$$

$$\mathcal{S}_3(r) = \sum_{q_1}\sum_{q_2} \frac{c(q_1,r)c(q_2,r)}{\varphi(q_1r)\varphi(q_2r)} \sum_{(n_1,q_1r)=1} \sum_{(n_2,q_2r)=1} \beta(n_1)\beta(n_2) \sum_{(m,q_1q_2r)=1} f(m).$$

By (2.113) and (2.114), the proof of (2.108) is reduced to showing that

$$\sum_r (\mathcal{S}_1(r) - 2\mathcal{S}_2(r) + \mathcal{S}_3(r)) \ll NR^{-1}X^{1-\varpi^{5/3}}, \tag{2.115}$$

where r is constrained as in (2.111).

We begin with the evaluation of $\mathcal{S}_3(r)$ which is the simplest of the three sums. We make frequent use of the trivial bound

$$\hat{f}(z) \ll M. \tag{2.116}$$

Similar to the proof of (2.70), we have, for $q_j \sim Q$, $j = 1,2$,

$$\sum_{(m,q_1q_2r)=1} f(m) = \frac{\varphi(q_1q_2r)}{q_1q_2r}\hat{f}(0) + O(X^\epsilon).$$

This yields

$$\mathcal{S}_3(r) = \hat{f}(0) \sum_{q_1}\sum_{q_2} \frac{c(q_1,r)c(q_2,r)}{\varphi(q_1r)\varphi(q_2r)} \frac{\varphi(q_1q_2r)}{q_1q_2r} \sum_{(n_1,q_1r)=1} \sum_{(n_2,q_2r)=1} \beta(n_1)\beta(n_2) + O(X^\epsilon N^2 R^{-2}).$$

If $(q_1q_2, \mathcal{P}_0) = 1$, then either $(q_1, q_2) = 1$ or $(q_1, q_2) > D_0$. Thus, on the right side of the above, the contribution from terms with $(q_1, q_2) > 1$ is, by (2.116) and trivial estimation,

$$\ll XND_0^{-1}R^{-2}\mathcal{L}^B.$$

It follows that

$$\mathcal{S}_3(r) = \hat{f}(0)X(r) + O(XND_0^{-1}R^{-2}\mathcal{L}^B), \tag{2.117}$$

where

$$X(r) = \sum_{q_1} \sum_{(q_2,q_1)=1} \frac{c(q_1, r)c(q_2, r)}{q_1q_2r\varphi(r)} \sum_{(n_1,q_1r)=1} \sum_{(n_2,q_2r)=1} \beta(n_1)\beta(n_2). \tag{2.118}$$

The value of $X(r)$ is not essential, since it will cancel out, with acceptable error (see (2.142) below), when we insert it back into the left side of (2.115). We next evaluate $\mathcal{S}_2(r)$.

Our next goal is to show that

$$\mathcal{S}_2(r) = \hat{f}(0)X(r) + O(XND_0^{-1}R^{-2}\mathcal{L}^B) \tag{2.119}$$

with $X(r)$ given as in (2.118). The main tool we need is the Ramanujan bound (2.43). Assume $c(q_1, r)c(q_2, r) \neq 0$. On substituting $mn_1 = n$ and applying Lemma 7 we have

$$\sum_{n_1} \beta(n_1) \sum_{\substack{mn_1 \equiv a(q_1r) \\ (m,q_2)=1}} f(m) \ll \sum_{\substack{n<2X \\ n\equiv a(q_1r)}} \tau_k(n) \ll \frac{X\mathcal{L}^B}{q_1r}.$$

It follows that contributions from terms with $(q_1, q_2) > 1$ in $\mathcal{S}_2(r)$ is

$$\ll XND_0^{-1}R^{-2}\mathcal{L}^B,$$

so that

$$\mathcal{S}_2(r) = \sum_{q_1} \sum_{(q_2,q_1)=1} \frac{c(q_1, r)c(q_2, r)}{\varphi(q_2r)} \sum_{n_1} \sum_{(n_2,q_2r)=1} \beta(n_1)\beta(n_2) \sum_{\substack{mn_1\equiv a(q_1r) \\ (m,q_2)=1}} f(m) + O(XND_0^{-1}R^{-2}\mathcal{L}^B). \tag{2.120}$$

Note that the innermost sum over m in (2.120) is empty unless $(n_1, q_1 r) = 1$. For $|\mu(q_1 q_2 r)| = 1$ and $(q_2, \mathcal{P}_0) = 1$ we have

$$\frac{q_2}{\varphi(q_2)} = 1 + O(\tau(q_2) D_0^{-1})$$

and, by Lemma 7,

$$\sum_{(n_1, q_1 r)=1} \beta(n_1) \sum_{\substack{mn_1 \equiv a(q_1 r) \\ (m, q_2) > 1}} f(m) \ll \sum_{\substack{n < 2X \\ n \equiv a(q_1 r) \\ (n, q_2) > 1}} \tau_k(n) \ll \frac{\tau_k(q_2) X \mathcal{L}^B}{q_1 r D_0}.$$

Thus (2.120) still holds with the constraint $(m, q_2) = 1$ removed and with $\varphi(q_2 r)$ replaced by $q_2 \varphi(r)$. That is, we have

$$\mathcal{S}_2(r) = \sum_{q_1} \sum_{(q_2, q_1)=1} \frac{c(q_1, r) c(q_2, r)}{q_2 \varphi(r)} \sum_{n_1} \sum_{(n_2, q_2 r)=1} \beta(n_1) \beta(n_2) \sum_{mn_1 \equiv a(q_1 r)} f(m) + O(XND_0^{-1} R^{-2} \mathcal{L}^B). \tag{2.121}$$

By Lemma 6, for $(n_1, q_1 r) = 1$, we have

$$\sum_{mn_1 \equiv a(q_1 r)} f(m) = \frac{1}{q_1 r} \sum_{|h| < H_2} \hat{f}\left(\frac{h}{q_1 r}\right) e_{q_1 r}(-hm) + O(d^{-1}),$$

where

$$H_2 = 4QRM^{-1+2\epsilon}.$$

Substituting this into (2.121) we deduce that

$$\mathcal{S}_2(r) = \hat{f}(0) X(r) + \mathcal{R}_2(r) + O(XND_0^{-1} R^{-2} \mathcal{L}^B), \tag{2.122}$$

where

$$\begin{aligned} \mathcal{R}_2(r) = &\sum_{q_1} \sum_{(q_2, q_1)=1} \frac{c(q_1, r) c(q_2, r)}{q_1 q_2 \varphi(r)} \left(\sum_{(n_2, q_2 r)=1} \beta(n_2) \right) \\ &\times \sum_{(n_1, q_1 r)=1} \beta(n_1) \sum_{1 \le |h| < H_2} \hat{f}\left(\frac{h}{q_1 r}\right) e_{q_1 r}(-hm). \end{aligned}$$

The proof of (2.119) is now reduced to estimating $\mathcal{R}_2(r)$. Note that, by the second inequality in (2.110), we have

$$H_2 \ll X^{-1/2+2\varpi+2\epsilon} N \tag{2.123}$$

since $M^{-1} \ll X^{-1}N$. This implies that $\mathcal{R}_2(r) = 0$ if $X_1 < N < X_2$, since $H_2 < 1$ in this case.

Now assume that $X_2 < N < 2X^{1/2}$. By the 'reciprocity' relation

$$\frac{m}{q_1 r} \equiv \frac{a\overline{q_1 n_1}}{r} + \frac{a\overline{r n_1}}{q_1} \pmod 1,$$

we get

$$\mathcal{R}_2(r) \ll N^{1+\epsilon} R^{-2} \sum_{\substack{n \sim N \\ (n,r)=1}} |\mathcal{R}^*(r,n)|, \tag{2.124}$$

where

$$\mathcal{R}^*(r,n) = \sum_{(q,n)=1} \frac{c(q,r)}{q} \sum_{1 \le |h| < H_2} \hat{f}\left(\frac{h}{qr}\right) e\left(\frac{-ah\overline{qn}}{r} - \frac{ah\overline{rn}}{q}\right).$$

Expanding $|\mathcal{R}^*(r,n)|^2$ out, we have

$$\begin{aligned} |\mathcal{R}^*(r,n)|^2 = &\sum_{(q,n)=1} \sum_{(q',n)=1} \frac{c(q,r)c(q',r)}{qq'} \\ &\times \sum_{1 \le |h| < H_2} \sum_{1 \le |h'| < H_2} \hat{f}\left(\frac{h}{qr}\right) \overline{\hat{f}\left(\frac{h'}{q'r}\right)} e\left(\frac{a(h'\overline{q'} - h\overline{qn}}{r} - \frac{ah\overline{rn}}{q} + \frac{ah'\overline{rn}}{q'}\right). \end{aligned}$$

Changing the order of summation and applying (2.116), we obtain

$$M^{-2} \sum_{\substack{n \sim N \\ (n,r)=1}} |\mathcal{R}^*(r,n)|^2 \ll \sum_{q} \sum_{q'} \frac{|c(q,r)c(q',r)|}{qq'} \sum_{1 \le |h| < H_2} \sum_{1 \le |h'| < H_2} |\mathcal{W}(q,r;q',h')| \tag{2.125}$$

where

$$\mathcal{W}(q,r;q',h') = \sum_{\substack{n \sim N \\ (n,qq'r)=1}} e\left(\frac{a(h'\overline{q}' - h\overline{qn})}{r} - \frac{ah\overline{rn}}{q} + \frac{ah'\overline{rn}}{q'}\right).$$

Since $M^{-1} \ll N^{-1}$, by the second inequality in (2.109) we have

$$H_2 Q^{-1} \ll X^{-3\varpi+\epsilon}. \tag{2.126}$$

It follows that, on the right side of (2.125), the contribution from terms with $h'q = hq'$ is

$$\ll NQ^{-2} \sum_{1 \le |h| < H_2} \sum_{q < 2Q} \tau(hq) \ll X^{-3\varpi+\epsilon} N. \tag{2.127}$$

Now assume that $c(q,r)c(q',r) \neq 0$, $1 \le |h| < H_2$, $1 \le |h'| < H_2$, and $h'q \neq hq'$. Letting $d = [q,q']r$, we have

$$\frac{a(h'\overline{q}' - h\overline{q})}{r} - \frac{ah\overline{r}}{q} + \frac{ah'\overline{r}}{q'} \equiv \frac{c}{d} \pmod 1$$

for some c with

$$(c,r) = (h'\overline{q}' - h\overline{q}, r).$$

By the estimate (2.43) it follows that

$$\mathcal{W}(q,r;q',h') \ll d^{1/2+\epsilon} + \frac{(c,d)N}{d}. \tag{2.128}$$

Since $N > X_2$, by the first inequality in (2.109), (2.106), and (2.105), we have

$$R^{-1} < X^{4\varpi} N^{-1} < X^{-1/2+8\varpi}. \tag{2.129}$$

This and the second inequality in (2.110) imply that

$$Q \ll X^{10\varpi}. \tag{2.130}$$

Thus we have

$$d^{1/2} \ll (Q^2R)^{1/2} \ll X^{1/4+6\varpi}.$$

Note that

$$h'\overline{q}' - h\overline{q} \equiv (h'q - hq')\overline{qq'} \pmod r.$$

This implies

$$(c,d) \le (c,r)[q,q'] \ll [q,q']H_2Q. \tag{2.131}$$

This together with (2.123), (2.129), and (2.130), give

$$\frac{(c,d)N}{d} \ll H_2NQR^{-1} \ll X^{16\varpi+\epsilon}.$$

Combining these estimates with (2.128) we deduce that

$$\mathcal{W}(q,r;q',h') \ll X^{1/4+7\varpi}.$$

This together with (2.123) imply that the contribution from terms with $h'q \neq hq'$ on the right side of (2.125) is $\ll X^{1/4+12\varpi}$, which is sharper than the right side of (2.127). Combining these estimates with (2.125) we conclude that

$$\sum_{\substack{n\sim N\\(n,r)=1}} |\mathcal{R}^*(r,n)| \ll X^{1-3\varpi/2+\epsilon}.$$

Substituting this into (2.124) we obtain

$$\mathcal{R}_2(r) \ll NR^{-2}X^{1-\varpi}, \tag{2.132}$$

which is sharper than the big-Oh term in (2.122).

The relation (2.119) follows from the bounds (2.122) and (2.132) immediately. It remains to deal with $\mathcal{S}_1(r)$. The evaluation of the last sum $\mathcal{S}_1(r)$ is the most difficult. The main tool we need is Lemma 12.

We shall instead establish an averaged bound on $\mathcal{S}_1(r)$ of the form

$$\sum_r \mathcal{S}_1(r) = \sum_r (\hat{f}(0)X(r) + \mathcal{R}_1(r)) + O(XNR^{-1}X^{-\varpi^{5/3}}) \tag{2.133}$$

with $\mathcal{R}_1(r)$ to be specified below in (2.139). By the estimates (2.119) for $\mathcal{S}_2(r)$ and (2.117) for $\mathcal{S}_3(r)$, the proof of (2.115) will be reduced to estimating $\mathcal{R}_1(r)$.

By definition of $\mathcal{S}_1(r)$, expanding out the square, we have

$$\mathcal{S}_1(r) = \sum_{q_1}\sum_{q_2} c(q_1,r)c(q_2,r)\sum_{n_1}\sum_{n_2\equiv n_1(r)} \beta(n_1)\beta(n_2) \sum_{\substack{mn_1\equiv a(q_1r)\\ mn_2\equiv a(q_2)}} f(m). \tag{2.134}$$

Let $\mathcal{U}(r,q_0)$ denote the sum of terms on the right side of the above with $(q_1,q_2)=q_0$. The sum $\mathcal{U}(r,q_0)$ vanishes unless

$$q_0 < 2Q, \quad q_0|\mathcal{P}, \quad \text{and} \quad (q_0, r\mathcal{P}_0) = 1,$$

which we will assume. We first show that the contribution

$$\sum_{r}\sum_{q_0>1}\mathcal{U}(r,q_0) \ll XN(D_0R)^{-1}\mathcal{L}^B \tag{2.135}$$

coming from terms with $q_0 > 1$ is admissible. Assume that, for $j = 1, 2$,

$$q_j \sim Q, \quad q_j|\mathcal{P}, \quad (q_j, r\mathcal{P}_0) = 1, \quad \text{and} \quad (q_1, q_2) = q_0.$$

Writing $q_1' = q_1/q_0$, $q_2' = q_2/q_0$, we get

$$\sum_{r}\sum_{q_0>1}\mathcal{U}(r,q_0) = \sum_{r}\sum_{q_0>1}\sum_{\substack{(q',q'')=1\\(a,q'q'')=1}}\sum c(q_0q',r)c(q_0q'',r)\sum_{n_1}\sum_{n_2\equiv n_1(r)}\beta(n_1)\beta(n_2)\sum_{m\equiv\mu(q_1q_2r)}f(m)$$

where $\mu(\text{mod } q_1q_2r)$ is a common solution to

$$\mu n_1 \equiv a(\text{mod } q_1r) \quad \text{and} \quad \mu n_2 \equiv a(\text{mod } q_2r). \tag{2.136}$$

Since q_1 and q_2 have no prime factor less than D_0, we have either $q_0 = 1$ or $q_0 \geq D_0$. By Cauchy's inequality we have

$$\sum_{r}\sum_{q_0>1}\mathcal{U}(r,q_0) \ll \sum_{D_0<q_0\leq 2Q}\sum_{r}\sum_{m}f(m)\sum_{q'}\sum_{n_1\equiv a\overline{m}(q'r)}|\beta(n_1)|^2\sum_{q_2}\sum_{n_2\equiv a\overline{m}(q_2r)}1.$$

By Lemma 7 the two inner sums of the above is

$$\ll \sum_{n\equiv a\overline{m}(q_0r)}\tau^B(mn-a)\tau^B(q_0) \ll N(D_0R)^{-1}\mathcal{L}^B.$$

This and Lemma 7 imply

$$\sum_{r}\sum_{q_0>1}\mathcal{U}(r,q_0) \ll N(D_0R)^{-1}\mathcal{L}^B\sum_{m}\sum_{n}\tau^B(mn-a) \ll NR^{-1}\mathcal{L}^BXD_0^{-1}.$$

This bound is admissible provided

$$D_0 = X^{\varpi^{4/3}} \tag{2.137}$$

which we henceforth assume.

We now turn to $\mathcal{U}(r,1)$. Assume $|\mu(q_1q_2r)|=1$. In the case $(n_1,q_1r)=(n_2,q_2r)=1$, the innermost sum in (2.134) is, by Lemma 6, equal to

$$\frac{1}{q_1q_2r}\sum_{|h|<H_1}\hat{f}\left(\frac{h}{q_1q_2r}\right)e_{q_1q_2r}(-\mu h)+O(d^{-1}),$$

where

$$H_1=8Q^2RM^{-1+2\epsilon}.$$

It follows that

$$\mathcal{U}(r,1)=\hat{f}(0)X^*(r)+\mathcal{R}_1(r)+O(1), \tag{2.138}$$

where

$$X^*(r)=\sum_{q_1}\sum_{(q_2,q_1)=1}\frac{c(q_1,r)c(q_2,r)}{q_1q_2r}\sum_{(n_1,q_1r)=1}\sum_{\substack{n_2\equiv n_1(r)\\(n_2,q_2)=1}}\beta(n_1)\beta(n_2)$$

and

$$\mathcal{R}_1(r)=\sum_{q_1}\sum_{(q_2,q_1)=1}\frac{c(q_1,r)c(q_2,r)}{q_1q_2r}\sum_{\substack{n_2\equiv n_1(r)\\(n_2,q_2)=1}}\beta(n_1)\beta(n_2)\sum_{1\le|h|<H}\hat{f}\left(\frac{h}{q_1q_2r}\right)e_{q_1q_2r}(-\mu h). \tag{2.139}$$

By (2.134), (2.135), and (2.138) we conclude that

$$\sum_r\mathcal{S}_1(r)=\sum_r(\hat{f}(0)X^*(r)+\mathcal{R}_1(r))+O(XN(D_0R)^{-1}\mathcal{L}^B).$$

In view of (2.116), the proof of (2.133) is thus reduced to showing that

$$\sum_r(X^*(r)-X(r))\ll N^2R^{-1}X^{-\varpi^{5/3}}. \tag{2.140}$$

We have

$$X^*(r)-X(r)=\sum_{q_1}\sum_{(q_2,q_1)=1}\frac{c(q_1,r)c(q_2,r)}{q_1q_2r}\mathcal{V}(r;q_1,q_2),$$

with

$$\mathcal{V}(r;q_1,q_2)=\sum_{(n_1,q_1r)=1}\sum_{\substack{n_2\equiv n_1(r)\\(n_2,q_2)=1}}\beta(n_1)\beta(n_2)-\frac{1}{\varphi(r)}\sum_{(n_1,q_1r)=1}\sum_{(n_2,q_2r)=1}\beta(n_1)\beta(n_2).$$

It follows that

$$\sum_r (X^*(r) - X(r)) \ll \frac{1}{R} \sum_{q_1 \sim Q} \sum_{q_2 \sim Q} \frac{1}{q_1 q_2} \sum_{\substack{r \sim R \\ (r, q_1 q_2) = 1}} |\mathcal{V}(r; q_1, q_2)|. \tag{2.141}$$

Noting that

$$\mathcal{V}(r; q_1, q_2) = \sideset{}{^*}\sum_{\ell (\text{mod } r)} \left(\sum_{\substack{n \equiv \ell(r) \\ (n, q_1) = 1}} \beta(n) - \frac{1}{\varphi(r)} \sum_{(n, q_1 r) = 1} \beta(n) \right) \left(\sum_{\substack{n \equiv \ell(r) \\ (n, q_2) = 1}} \beta(n) - \frac{1}{\varphi(r)} \sum_{(n, q_2 r) = 1} \beta(n) \right),$$

and by Cauchy's inequality and Lemma 9, we find that the innermost sum in (2.141) is

$$\ll \tau(q_1 q_2)^B N^2 X^{-\varpi/12},$$

which leads to (2.140).

Combining (2.117), (2.119), and (2.133) leads to

$$\sum_r (\mathcal{S}_1(r) - 2\mathcal{S}_2(r) + \mathcal{S}_3(r)) = \sum_r \mathcal{R}_1(r) + O(XNR^{-1} X^{-\varpi^{5/3}}).$$

Note that

$$\frac{\mu}{q_1 q_2 r} \equiv \frac{a \overline{q_1 q_2 n_1}}{r} + \frac{a \overline{q_2 r n_1}}{q_1} + \frac{a \overline{q_1 r n_2}}{q_2} \pmod 1$$

by (2.136). Hence, on substituting $n_2 = n_1 + kr$, we may write $\mathcal{R}_1(r)$ as

$$\mathcal{R}_1(r) = \frac{1}{r} \sum_{|k| < N/R} \mathcal{R}_1(r, k),$$

where

$$\begin{aligned} \mathcal{R}_1(r, k) = & \sum_{q_1} \sum_{(q_2, q_1) = 1} \frac{c(q_1, r) c(q_2, r)}{q_1 q_2} \sum_{1 \leq |h| < H_1} \hat{f}\left(\frac{h}{q_1 q_2 r} \right) \\ & \times \sum_{\substack{(n, q_1 r) = 1 \\ (n + kr, q_2) = 1}} \beta(n) \beta(n + kr) e(-h \xi(r; q_1, q_2; n, k)), \end{aligned}$$

with

$$\xi(r;q_1,q_2;n,k) = \frac{a\overline{q_1q_2n}}{r} + \frac{a\overline{q_2rn}}{q_1} + \frac{a\overline{qr(n+kr)}}{q_2}.$$

Thus, the proof of (2.115) will follow from the bound

$$\mathcal{R}_1(r,k) \ll X^{1-\varpi/2} \tag{2.142}$$

which we shall prove in the next two subsections. We note that the bound (2.142) amounts to saving a power of X from the trivial estimate; indeed, it trivially follows from (2.116) that

$$\mathcal{R}_1(r,k) \ll X^{1+\epsilon}H_1.$$

On the other hand, in view of (2.105), since

$$H_1 \ll X^{\epsilon}(QR)^2(MN)^{-1}NR^{-1},$$

and, by the first inequality in (2.109) and (2.106),

$$NR^{-1} < \begin{cases} X^{\varpi+\epsilon}, & \text{if } X_1 < N \le X_2, \\ X^{4\varpi}, & \text{if } X_2 < N < 2X^{1/2}, \end{cases} \tag{2.143}$$

it follows from the second inequality in (2.110) that

$$H_1 \ll \begin{cases} X^{5\varpi+2\epsilon} & \text{if } X_1 < N \le X_2, \\ X^{8\varpi+\epsilon} & \text{if } X_2 < N \le 2X^{1/2} \end{cases} \tag{2.144}$$

is bounded by a small power of X.

Estimation of $\mathcal{R}_1(r,k)$: The Type I case

In this and the next subsection we assume that $|k| < NR^{-1}$, and we write

$$\mathcal{R}_1, \quad c(q_1), \quad c(q_2), \quad \text{and} \quad \xi(q_1,q_2;n)$$

for

$$\mathcal{R}_1(r,k), \quad c(q_1,r), \quad c(q_2,r), \quad \text{and} \quad \xi(r;q_1,q_2;n,k),$$

respectively, with the goal of proving (2.142). The variables r and k may also be omitted for notational simplicity. The proof is analogous to the estimation of $\mathcal{R}_2(r)$; the main ingredient is Lemma 12.

Assume that $X_1 < N \leq X_2$ and R^* is as in (2.106). We have

$$\mathcal{R}_1 \ll N^\epsilon \sum_{q_1} \frac{c(q_1)}{q_1} \sum_{\substack{n\sim N \\ (n,q_1r)=1}} |\mathcal{F}(q_1,n)|, \tag{2.145}$$

where

$$\mathcal{F}(q_1,n) = \sum_{0<|h|<H_1} \sum_{(q_2,q_1(n+kr))=1} \frac{c(q_2)}{q_2} \hat{f}\left(\frac{h}{q_1q_2r}\right) e(-h\xi(q_1,q_2;n)).$$

We assume $c(q_1)\neq 0$. To bound the sum of $|\mathcal{F}(q_1,n)|$ we observe that, similar to (2.125),

$$M^{-2} \sum_{\substack{n\sim N \\ (n,q_1r)=1}} |\mathcal{F}(q_1,n)|^2 \ll \sum_{(q_2,q_1)=1} \sum_{(q_2',q_1)=1} \frac{|c(q_2)c(q_2')|}{q_2q_2'} \sum_{0<|h|<H_1} \sum_{0<|h'|<H_1} |\mathcal{G}(h,h';q_1,q_2;q_2')|, \tag{2.146}$$

where

$$\mathcal{G}(h,h';q_1,q_2;q_2') = \sum_{\substack{n\sim N \\ (n,q_1r)=1 \\ (n+kr,q_2q_2')=1}} e(h'\xi(q_1,q_2';n) - h\xi(q_1,q_2;n)).$$

The condition $N \leq X_2$ is essential for bounding the diagonal terms $h'q_2 = hq_2'$ in (2.146). By (2.110) we have

$$H_1Q^{-1} \ll X^\epsilon (QR)(MN)^{-1}N \ll X^{-2\varpi+\epsilon}.$$

It follows that, on the right side of (2.146), the contribution from terms with $h'q_2 = hq_2'$ is

$$\ll NQ^{-2} \sum_{1\leq|h|<H_1} \sum_{q\sim Q} \tau(hq)^B \ll X^{-2\varpi+\epsilon}N. \tag{2.147}$$

Now assume that $c(q_2)c(q_2') \neq 0$, $(q_2q_2', q_1) = 1$, and $h'q_2 \neq hq_2'$. We have

$$h'\xi(q_1, q_2'; n) - h\xi(q_1, q_2; n) \equiv \frac{h'\overline{q_2'} - h\overline{q_2}a\overline{q_1 n}}{r} + \frac{h'\overline{q_2'} - h\overline{q_2}a\overline{rn}}{q_1} + \frac{h'a\overline{q_1 r(n+kr)}}{q_2'} - \frac{ha\overline{q_1 r(n_k r)}}{q_2} \pmod 1.$$

Letting $d_1 = q_1 r$ and $d_2 = [q_2, q_2']$, we may write

$$\frac{h'\overline{q_2'} - h\overline{q_2}a\overline{q_1}}{r} + \frac{h'\overline{q_2'} - h\overline{q_2}a\overline{r}}{q_1} \equiv \frac{c_1}{d_1} \pmod 1$$

for some c_1 with

$$(c_1, r) = (h'\overline{q_2'} - h\overline{q_2}, r),$$

and

$$\frac{h'a\overline{q_1 r}}{q_2'} - \frac{ha\overline{q_1 r}}{q_2} \equiv \frac{c_2}{d_2} \pmod 1$$

for some c_2, so that

$$h'\xi(q_1, q_2'; n) - h\xi(q_1, q_2; n) \equiv \frac{c_1\overline{n}}{d_1} + \frac{c_2\overline{(n+kr)}}{d_2} \pmod 1.$$

Since $(d_1, d_2) = 1$, it follows by Lemma 12 that

$$\mathcal{G}(h, h'; q_1, q_2; q_2') \ll (d_1d_2)^{1/2+\epsilon} + \frac{(c_1, d_1)N}{d_1}. \tag{2.148}$$

By the condition $N > X_1$, this gives, by (2.143),

$$R^{-1} < X^{\varpi+\epsilon}N^{-1} < X^{-3/4-15\varpi+\epsilon}N. \tag{2.149}$$

Together with (2.110), this yields

$$(d_1d_2)^{1/2} \ll (Q^3R)^{1/2} \ll X^{3/4+3\varpi}R^{-1} \ll X^{-12\varpi+\epsilon}N.$$

A sharper bound for the second term on the right side of (2.148) can be obtained as follows. In a way similar to the proof of (2.131), we find that

$$(c_1, d_1) \leq (c_1, r)q_1 \ll H_1Q^2.$$

It follows by (2.144), (2.110), and the first inequality in (2.149) that

$$\frac{(c_1,d_1)}{d_1} \ll H_1(QR)R^{-2} \ll X^{1/2+9\varpi+4\epsilon}N^{-2} \ll X^{-1/4-6\varpi}.$$

Here the condition $N > X_1$ is used again. Combining these estimates with (2.148) we deduce that

$$\mathcal{G}(h,h';q_1,q_2;q_2') \ll X^{-12\varpi+\epsilon}N.$$

Together with (2.144), this implies that, on the right side of (2.146), the contribution from terms with $h'q_2 \neq hq_2'$ is

$$\ll X^{-12\varpi+\epsilon}H_1^2 N \ll X^{-2\varpi+5\epsilon}N,$$

which has the same order of magnitude as the right side of (2.147) essentially. Combining these estimates with (2.146) we obtain

$$\sum_{\substack{n\sim N\\(n,q_1r)=1}} |\mathcal{F}(q_1,n)|^2 \ll X^{1-2\varpi+5\epsilon}M.$$

This yields, by Cauchy's inequality,

$$\begin{aligned}\sum_{\substack{n\sim N\\(n,q_1r)=1}} |\mathcal{F}(q_1,n)| &\ll \Bigg(\sum_{\substack{n\sim N\\(n,q_1r)=1}} 1\Bigg)^{1/2}\Bigg(\sum_{\substack{n\sim N\\(n,q_1r)=1}} |\mathcal{F}(q_1,n)|^2\Bigg)^{1/2} \\ &\ll N^{1/2}(X^{1-2\varpi+5\epsilon}M)^{1/2} \ll X^{1-\varpi+3\epsilon}.\end{aligned} \tag{2.150}$$

The estimate (2.142) follows from (2.145) and (2.150) immediately.

Estimation of $\mathcal{R}_1(r,k)$: The Type II case

We now assume that $X_2 < N < 2X^{1/2}$ and R^* is as in (2.106). We have

$$\mathcal{R}_1 \ll N^{\epsilon} \sum_{\substack{n\sim N\\(n,r)=1}} |\mathcal{K}(n)|, \tag{2.151}$$

where

$$\mathcal{K}(n) = \sum_{(q_1,n)=1} \sum_{(q_2,q_1(n+kr))=1} \frac{c(q_1)c(q_2)}{q_1q_2} \sum_{1\leq|h|<H_1} \hat{f}\left(\frac{h}{q_1q_2r}\right) e(-h\xi(q_1,q_2;n)).$$

Let $\sum^{\#}$ stands for a summation over the tuples $(q_1,q_2;q_1',q_2')$ with

$$(q_1,q_2)=(q_1',q_2')=1.$$

To estimate the sum of $\mathcal{K}(n)$ we observe that, similar to (2.125),

$$M^{-2}\sum_{\substack{n\sim N\\(n,r)=1}} |\mathcal{K}(n)|^2 \ll \sum\nolimits^{\#} \frac{c(q_1)c(q_2)c(q_1')c(q_2')}{q_1q_2q_1'q_2'} \sum_{1\leq|h|<H_1}\sum_{1\leq|h'|<H_1} |\mathcal{M}(h,h';q_1,q_2;q_1',q_2')|, \tag{2.152}$$

where

$$\mathcal{M}(h,h';q_1,q_2;q_1',q_2') = \sideset{}{'}\sum_{n\sim N} e(h'\xi(q_1,q_2;n)-h\xi(q_1,q_2;n)).$$

Here $\sum'$ denote a sum is restricted to $(n,q_1q_1'r)=(n+kr,q_2q_2')=1$.

Similar to (2.126), we have

$$H_1Q^{-2} \ll X^{-3\varpi+\epsilon}.$$

Hence, on the right side of (2.152), the contribution from terms with $h'q_1q_2 = hq_1'q_2'$ is

$$\ll NQ^{-4}\sum_{1\leq|h|<H_1}\sum_{q\sim Q}\sum_{q'\sim Q}\tau(hqq')^B \ll X^{-3\varpi+\epsilon}N. \tag{2.153}$$

Note that the bounds (2.129) and (2.130) are valid in this present situation. Since R is slightly smaller than $X^{1/2}$ and Q is small, by Lemma 12, contribution from terms with $h'q_1q_2 \neq hq_1'q_2'$ on the right side of (2.152) is small compare to (2.153). Assume that

$$c(q_1)c(q_2)c(q_1')c(q_2')\neq 0, \quad (q_1,q_2)=(q_1',q_2')=1, \quad \text{and} \quad h'q_1q_2\neq hq_1'q_2'.$$

We have

$$h'\xi(q_1',q_2';n)-h\xi(q_1,q_2;n) \equiv \frac{s\overline{n}}{r}+\frac{t_1\overline{n}}{q_1}+\frac{t_1'\overline{n}}{q_1'}+\frac{t_2\overline{(n+kr)}}{q_2}+\frac{t_2'\overline{(n+kr)}}{q_2'} \pmod 1 \tag{2.154}$$

with

$$\begin{aligned} s &\equiv a(h'\overline{q_1'q_2'} - h\overline{q_1q_2}) \pmod r, \\ t_1 &\equiv -ah\overline{q_2r} \pmod{q_1}, \\ t_1' &\equiv ah'\overline{q_2'r} \pmod{q_1'}, \\ t_2 &\equiv -ah\overline{q_1r} \pmod{q_2}, \\ t_2' &\equiv ah'\overline{q_1'r} \pmod{q_2'}. \end{aligned}$$

Letting $d_1 = [q_1, q_1']r$ and $d_2 = [q_2, q_2']$, we may rewrite (2.154) as

$$h'\xi(q_1', q_2'; n) - h\xi(q_1, q_2; n) \equiv \frac{c_1\overline{n}}{d_1} + \frac{c_2\overline{(n_kr)}}{d_2} \pmod 1$$

for some c_1 and c_2 with

$$(c_1, r) = (h'\overline{q_1'q_2'} - h\overline{q_1q_2}, r).$$

It follows by Lemma 12 that

$$\mathcal{M} \ll (d_1d_2)^{1/2+\epsilon} + \frac{(c_1, d_1)(d_1, d_2)^2N}{d_1}. \tag{2.155}$$

By (2.110) and (2.130) we have

$$(d_1d_2)^{1/2} \ll (Q^4R)^{1/2} \ll X^{1/4+16\varpi}.$$

On the other hand, we have $(d_1, d_2) \le (q_1q_1', q_2q_2') \ll Q^2$, since $(q_2q_2', r) = 1$, and, similar to (2.131),

$$(c_1, d_1) \le (c_1, r)[q_1, q_1'] \ll [q_1, q_1']H_1Q^2.$$

It follows by (2.143), (2.130), and the first inequality in (2.129) that

$$\frac{(c_1, d_1)(d_1, d_2)^2N}{d_1} \ll H_1NQ^6R^{-1} \ll X^{72\varpi}.$$

Combining these estimates with (2.155) we deduce that

$$\mathcal{M}(h, h'; q_1, q_2; q_1', q_2') \ll X^{1/4+16\varpi+\epsilon}.$$

Together with (2.143), this implies that, on the right side of (2.152), the contribution from terms with $h'q_1q_2 \neq hq_1'q_2'$ is

$$X^{1/4+16\varpi+\epsilon}H_1^2 \ll X^{1/4+33\varpi},$$

which is sharper than the right side of (2.153). Combining these estimates with (2.152) we obtain

$$\sum_{\substack{n\sim N\\(n,r)=1}} |\mathcal{K}(n)| \ll X^{1-\varpi}. \tag{2.156}$$

The estimate (2.142) follows from (2.151) and (2.156) immediately. This completes the proof of Theorem 1.

2.5 Proof of uniform power savings Theorem 2

Let χ be a primitive character (mod d) and $L(s,\chi)$ denote its Dirichlet L-function. On the Generalized Lindelöf Hypothesis, we have, for $\sigma \geq 1/2$,

$$L(s,\chi)^k \ll (d|s|)^{\epsilon k} \tag{2.157}$$

for any $\epsilon > 0$; see, e.g., [12]. This bound will allow us to significantly improve the estimate in (2.15).

This Lemma is a truncated Perron's formula.

Lemma 15 *Let*

$$\delta(X) = \begin{cases} 0, & \textit{if } 0 < X < 1,\\ 1/2, & \textit{if } X = 1,\\ 1, & \textit{if } X > 1 \end{cases}$$

and

$$I(X,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{X^s}{s}ds.$$

Then

$$\delta(X) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{X^s}{s} ds,$$

and, for $X > 0$, $c > 0$, and $R > 0$, we have

$$|I(X,T) - \delta(X)| < \begin{cases} X^c \min\{1, T^{-1}|\log X|^{-1}\}, & \textit{if } X \neq 1, \\ c/T, & \textit{if } X = 1. \end{cases}$$

Proof: See [66, Theorem 4.1.4]. ■

With (2.157) we can strengthen Lemma 3, which was used in the proof of Theorem 1, to

Proposition 3 *Assume the Generalized Lindelöf Hypothesis. For χ a primitive character (mod d) we have*

$$\sum_{n\leq X} \tau_k(n)\chi(n) \ll X^{7/8} d^{1/2}. \tag{2.158}$$

Proof: The proof of this proposition is in principal very similar to that of Lemma 3. Indedd, we estimate directly the left side of (2.158) using the truncated Perron's formula, getting

$$\sum_{n\leq X} \tau_k(n)\chi(n) = \frac{1}{2\pi i}\int_{9/8-iT}^{9/8+iT} L(s,\chi)^k \frac{X^s}{s} ds + O\left(\frac{X^{9/8}}{T}\right). \tag{2.159}$$

Since χ is nonprincipal, the function $L(s,\chi)^k$ is analytic and has no poles in $\sigma \geq 1/2$. We move the line of integration to $\sigma = 1/2$ and apply Cauchy's theorem. On the generalized Lindelöf Hypothesis, we apply (2.157) with $\epsilon = 1/2k$ giving

$$L(s,\chi)^k \ll (d|s|)^{1/2}, \quad \sigma \geq 1/2.$$

The contribution from horizontal segments is

$$\left|\frac{1}{2\pi i}\left(\int_{9/8+iT}^{1/2+iT} + \int_{9/8-iT}^{1/2-iT}\right) L(s,\chi)^k \frac{X^s}{s} ds\right| \ll d^{1/2}\frac{X^{9/8}}{T^{1/2}},$$

and contribution from the vertical segment $\sigma = 1/2$ is

$$\left| \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} L(s,\chi)^k \frac{X^s}{s} ds \right| \ll d^{1/2} X^{1/2} T^{1/2}.$$

Hence, by Cauchy's theorem, (2.159) becomes

$$\sum_{n \le X} \tau_k(n)\chi(n) \ll d^{1/2} \frac{X^{9/8}}{T^{1/2}} + d^{1/2} X^{1/2} T^{1/2} + \frac{X^{9/8}}{T}.$$

We choose $T = X^{1/2}$. Thus, the error term of the above is

$$\ll d^{1/2} X^{7/8}.$$

This leads to the right side of (2.158). ■

Proof of Theorem 2. By Proposition 2, we have

$$\sum_{\substack{d \in \mathcal{D} \\ D_2 < d < D_3}} |\Delta(\tau_k; X, d, a)| \ll X^{1-\varpi^2}. \tag{2.160}$$

By Proposition 3 above together with the large sieve inequality (2.30), in a way similar to the proof of Proposition 1, for $D \le D_2$, we get

$$\sum_{d \le D} \max_{(a,d)=1} |\Delta(\tau_k; X, d, a)| \ll X^{1-\varpi^2}.$$

This, together with (2.160), give the desired estimate

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{1/2+1/584}}} |\Delta(\tau_k; X, d, a)| \ll X^{1-\varpi^2}.$$

Chapter 3

Upper bounds for the variance and bilinear estimates for $\tau_k(n)$ in arithmetic progressions

3.1 Introduction

3.1.1 Average estimates over a

Instead of averaging over the modulus d, a different average over the reduced residue classes $a (\mathrm{mod}\ d)$ has also been considered by several authors. For instance, in 2005, Banks, Heath-Brown, and Shparlinski showed in [2, Theorem 3.1, p. 9] that for every $\epsilon > 0$, we have the mean value estimate

$$\sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left| T_2(X,d,a) - \frac{X}{\varphi(d)} P_d(\log X) \right| \ll \begin{cases} d^{1/5} X^{4/5+\epsilon}, & \text{if } d > X^{1/2}, \\ d^{2/5} X^{7/10+\epsilon}, & \text{if } X^{1/3} < d \leq X^{1/2}, \\ d^{1/2} X^{2/3+\epsilon}, & \text{if } X^{1/6} < d \leq X^{1/3}, \\ X^{3/4+\epsilon}, & \text{if } d \leq X^{1/6}, \end{cases} \tag{3.1}$$

where $P_d(\log X)$ is given as in (2.6), and the implied constant depends only on ϵ. In particular, their result shows that the left side of the above is $\ll X^{1-\epsilon/6}$ uniformly for all $d < X^{1-\epsilon}$. Their method relies on average bounds for incomplete Kloosterman sums–they showed and crucially used that the Weil-type bound for certain incomplete Kloosterman sums can be sharpened when averaged over all reduced residue classes modulo d.

The upper bounds in (3.1) are substantially sharpened by Blomer [6] in 2008 where the condition $(a,d)=1$ is removed and the summand is squared instead. More precisely, letting

$$\tilde{P}_d(\log X) = \sum_{\delta \mid d} \frac{r_\delta(a)}{\delta}(P(\log X) + 2\gamma - 1 - 2\log\delta), \tag{3.2}$$

Blomer showed in [6, Theorem 1.1, p. 277] that for X a large real number and $d \le X$ a positive integer, then for any $\epsilon > 0$, we have

$$\sum_{a=1}^{d} \left(T_2(X,d,a) - \frac{X}{d}\tilde{P}_d(\log X) \right)^2 \ll X^{1+\epsilon}, \tag{3.3}$$

where $r_c(h)$ is the Ramanujan's sum. When $(a,d)=1$, the above main term matches that of (3.1). In particular, by Cauchy's inequality, (3.3) gives, for all $d > X^{1/3}$

$$\sum_{a=1}^{d} \left| T_2(X,d,a) - \frac{X}{d}\tilde{P}_d(\log X) \right| \ll (dX)^{1/2+\epsilon}.$$

The proof of (3.3) uses Voronoi summation (the functional equation of twists of the corresponding L-function). Blomer's proof also works for other arithmetic functions such as Fourier coefficients of primitive (eigenform of all Hecke operators) holomorphic cusp forms [6, Theorem 1.2] and the square-free numbers μ^2 [6, Theorem 1.3]. Even though $\tau(n) > 0$ for all n while Hecke eigenvalues can be negative, it is well known that their mean values share analogous oscillatory properties; see, e.g., the detailed exposition [48] of Jutila.

Lau and Zhao [53] in 2012 obtained asymptotic results for the variance of $\tau(n)$ for d and X going to infinity at different rates. Fouvry, Ganguly, Kowalski, and Michel [25] in 2014 show, in a restricted range, that the divisor function $\tau(n)$ in residue classes modulo a prime follows a Gaussian distribution, and a similar result for Hecke eigenvalues of classical holomorphic cusp forms.

There is also a conjecture for general k. In a function field variant, the work of Keating, Rodgers, Roditty-Gershon, and Rudnick in [50] leads to the following conjecture over the integers for the variance of τ_k (c.f. [71, Conjecture 1]).

Conjecture 4 (Keating–Rodgers–Roditty-Gershon–Rudnick [50]) *For* $X, d \to \infty$ *such that* $\log X/\log d \to c \in (0,k)$*, we have*

$$\sum_{\substack{a=1 \\ (a,d)=1}}^{d} \Delta(\tau_k; X, d, a)^2 \sim a_k(d)\gamma_k(c)X(\log d)^{k^2-1}, \tag{3.4}$$

where $a_k(d)$ *is the arithmetic constant*

$$a_k(d) = \lim_{s\to 1^+}(s-1)^{k^2} \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} \frac{\tau_k(n)^2}{n^s},$$

and $\gamma_k(c)$ *is a piecewise polynomial of degree* k^2-1 *defined by*

$$\gamma_k(c) = \frac{1}{k!G(k+1)^2} \int_{[0,1]^k} \delta_c(w_1 + \cdots w_k)\Delta(w)^2 d^k w,$$

where $\delta_c(x) = \delta(x-c)$ *is a Dirac delta function centered at* c*,* $\Delta(w) = \prod_{i<j}(w_i - w_j)$ *is a Vandermonde determinant, and* G *is the Barnes G-function, so that in particular* $G(k+1) = (k-1)!(k-2)!\cdots 1!$*.*

This conjecture is closely related to the problem of moments of Dirichlet L-functions; see, e.g, the works [14, 15, 16, 17] of Conrey and Keating on moments of the Riemann zeta function and correlations of divisor sums.

3.1.2 Average estimates over both a and d

The estimate of the form (2.7) in (S2) is of Bombieri-Vinogradov type where an average over moduli d is taken. In addition to this average, taking an additional average over all primitive residue classes a in each modulus d yields better result for the range for d. These results are of Barban-Davenport-Halberstam type (see, e.g., [18, §29] and have a long history, starting from the initial works [4, 5] of Barban in 1963 on primes in arithmetic progressions, and of Davenport and Halberstam [19] a few years latter. Hooley has written a series of papers on the Barban-Davenport-Halberstam sums, dating from 1975, which numbers nineteen, as of currently.

In 1976 Motohashi [65] obtained an asymptotic formula for $\tau(n)$ averaged over both a and d. More specifically, there are explicit numerical constants $\mathfrak{S}_j$, $0 \le j \le 3$, such that

$$\begin{aligned}&\sum_{d<X}\sum_{a=1}^{d}\Big(T_2(X,d,a) - \tilde{P}_d(\log X)\Big)^2\\ &\qquad = X^2(\mathfrak{S}_3\log^3 X + \mathfrak{S}_2\log^2 X + \mathfrak{S}_1\log X + \mathfrak{S}_0) + O(X^{15/8}\log^2 X),\end{aligned}$$

where $\tilde{P}_d(\log X)$ is as in (3.2)

Very recently, Rodgers and Soundararajan [71] in 2018 consider smoothed sums of τ_k averaging over both a and d, and confirm an averaged version of Conjecture 5 for a restricted range. More precisely, for $k \ge 2$, let

$$\Delta_k(D;X) = \sum_{d} V_k(d;X)\Phi\left(\frac{d}{D}\right),$$

where

$$V_k(d;X) = \sum_{\substack{a=1\\(a,d)=1}}^{d}\left(\sum_{n\equiv a(d)}\tau_k(n)\Psi\left(\frac{n}{N}\right) - \frac{1}{\varphi(d)}\sum_{(n,d)=1}\tau_k(n)\Psi\left(\frac{n}{N}\right)\right)^2,$$

with Φ and Ψ fixed smooth non-negative functions compactly supported in the positive reals normalized so that $\int\Phi = 1$ and $\int\Psi^2 = 1$. Then letting $c = \log x/\log D$, Rodgers and Soundararajan in [71, Theorem 1] obtained asymptotics for $\Delta_k(D;X)$ as $X, D \to \infty$ uniformly in c for all $\delta \le c \le (k+2)/k - \delta$ for some $\delta > 0$ sufficiently small. Under GRH, a larger range $\delta \le c \le 2-\delta$ independent of k is obtained in [71, Theorem 2].

Their results are closely related to moments of Dirichlet L-functions as discussed above, and their proof relies on the asymptotic large sieve.

The latter work of de la Bretèche and Fiorilli in [9] considers a related variance in the range $1 \leq c < 1 + O(1/k)$ using an arithmetic approximation motivated by work of Vaughan. Interestingly, the asymptotic for their arithmetic variance in [9, Theorem 1.2] matches those in [71, Theorem 1].

In our first result, we prove a distribution estimate for τ_k averaging over both a and d to moduli d as large as X, c.f. Lemma 9, similar to that of Barban-Davenport-Halberstam type theorems for Λ.

Theorem 3 *For $k \geq 4$ we have*

$$\sum_{d \leq D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \Delta(\tau_k; X, d, a)^2 \ll (D + X^{1-1/6(k+2)}) X (\log X)^{k^2-1}. \tag{3.5}$$

Motivated by the recent work [42] of Heath-Brown and Li in 2017, we also prove analogous estimates for pairs of $\tau_k(n)$'s and $\tau_k(n)\Lambda(n)$ to moduli d that can taken to be almost as large as X^2.

Theorem 4 *For $k \geq 4$ and any $\epsilon > 0$ there holds*

$$\sum_{d \leq D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left(\sum_{\substack{m,n \leq X \\ m \equiv an (mod\ d)}} \tau_k(m)\tau_k(n) - \frac{1}{\varphi(d)} \left(\sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right)^2 \right)^2 \ll X^{4-1/3(k+4)} \tag{3.6}$$

for any $D \leq X^{2-1/3(k+2)}$.

In particular, the above estimate is valid if one of the τ_k is replaced by the von Mangoldt function Λ. We have

$$\sum_{d \leq D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left(\sum_{\substack{m,n \leq X \\ m \equiv an (mod\ d)}} \tau_k(m)\Lambda(n) - \frac{X}{\varphi(d)} \sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right)^2 \ll X^{4-1/3(k+4)} \tag{3.7}$$

for any $D \leq X^{2-1/3(k+2)}$.

It might look surprising at first that the moduli in Theorems 4 can be taken almost as large as X^2, but proof is in fact rather simple; the proof of Theorem 4 follows essentially from the multiplicative large sieve inequality.

3.2 Proof of Theorem 3

We proceed analogously as in the proof of Lemma 9 and Proposition 1. By (2.28), we have

$$\Delta(\tau_k; X, d, a)^2 = \frac{1}{\varphi(d)^2} \sideset{}{'}\sum_{\chi_1 (\mathrm{mod}\ d)} \overline{\chi_1}(a) \left(\sum_{n \le X} \tau_k(n) \chi_1(n) \right) \times \sideset{}{'}\sum_{\chi_2 (\mathrm{mod}\ d)} \chi_2(a) \overline{\left(\sum_{n \le X} \tau_k(n) \chi_2(n) \right)}.$$

Summing over primitive $a(\mathrm{mod}\ d)$ and changing the order of summation, we get

$$\sideset{}{^*}\sum_{a (\mathrm{mod}\ d)} \Delta(\tau_k; X, d, a)^2 = \frac{1}{\varphi(d)^2} \sideset{}{'}\sum_{\chi_1 (\mathrm{mod}\ d)} \sideset{}{'}\sum_{\chi_2 (\mathrm{mod}\ d)} \left(\sum_{n \le X} \tau_k(n) \chi_1(n) \right) \overline{\left(\sum_{n \le X} \tau_k(n) \chi_2(n) \right)} \times \sideset{}{^*}\sum_{a (\mathrm{mod}\ d)} \overline{\chi_1}(a) \chi_2(a).$$

By the orthogonality relation (2.35) this becomes

$$\sideset{}{^*}\sum_{a (\mathrm{mod}\ d)} \Delta(\tau_k; X, d, a)^2 = \frac{1}{\varphi(d)} \left| \sideset{}{'}\sum_{\chi (\mathrm{mod}\ d)} \left(\sum_{n \le X} \tau_k(n) \chi(n) \right) \right|^2.$$

We now reduce to primitive characters. By Lemma 2, we have

$$\sum_{d \le D} \sideset{}{^*}\sum_{a (\mathrm{mod}\ d)} \Delta(\tau_k; X, d, a)^2 \ll \log \mathcal{L} \sum_{r \le D} \frac{1}{r} \left(\sum_{1 < q \le D/r} \frac{1}{q} \left| \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \left(\sum_{n \le X} \tau_k(n) \chi(n) \right) \right|^2 \right).$$

By Lemma 3, we get, for $1 < q \le X^{1/3(k+2)}$,

$$\sum_{1 < q \le X^{1/3(k+2)}} \frac{1}{q} \left| \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \left(\sum_{n \le X} \tau_k(n) \chi(n) \right) \right|^2 \ll \sum_{1 < q \le X^{1/3(k+2)}} \frac{1}{q} \left(\varphi(q) X^{2 - \frac{2}{3(k+2)}} \right) \ll X^{2 - \frac{1}{3(k+2)}}.$$

Assume $X^{1/3(k+2)} \ll Q \ll D$. By the large sieve inequality (2.30) and the bound (2.31), we have

$$\frac{1}{Q}\sum_{q\sim Q}\left|\sideset{}{^*}\sum_{\chi(\text{mod } q)}\left(\sum_{n\leq X}\tau_k(n)\chi(n)\right)\right|^2 \ll \frac{1}{Q}(Q^2+X)\left(\sum_{n\leq X}\tau_k(n)\right)^2 \ll \left(Q+\frac{X}{Q}\right)X\mathcal{L}^{k-1}.$$

This leads to (3.5).

3.3 Proof of Theorem 4

Denote

$$E(f_1,f_2;X,d,a)=\sum_{\substack{m,n\leq X\\ m\equiv an(d)}}f_1(m)f_2(n)-\frac{1}{\varphi(d)}\left(\sum_{\substack{m\leq X\\ (m,d)=1}}f_1(m)\right)\left(\sum_{\substack{n\leq X\\ (n,d)=1}}f_2(n)\right).$$

We start by writing

$$\psi_1(\chi)=\sum_{m\leq X}f_1(m)\chi(m) \quad\text{and}\quad \psi_2(\chi)=\sum_{n\leq X}f_2(n)\chi(n).$$

We first prove the estimate (3.6). Let $f_1=f_2=\tau_k$. By (2.28), we have

$$E(\tau_k,\tau_k;X,d,a)=\frac{1}{\varphi(d)}\sideset{}{'}\sum_{\chi(\text{mod } d)}\overline{\chi}(a)\psi_1(\chi)\overline{\psi_2(\chi)}.$$

Taking the square of the modulus of both sides yields

$$E(\tau_k,\tau_k;X,d,a)^2=\frac{1}{\varphi(d)^2}\sideset{}{'}\sum_{\chi_1(\text{mod } d)}\overline{\chi_1}(a)\psi_1(\chi_1)\overline{\psi_2(\chi_1)}\sideset{}{'}\sum_{\chi_2(\text{mod } d)}\chi_2(a)\overline{\psi_1(\chi_2)}\psi_2(\chi_2).$$

Summing over primitive $a(\text{mod } d)$ and changing the order of summation, we get

$$\sideset{}{^*}\sum_{a(\text{mod } d)}E(\tau_k,\tau_k;X,d,a)^2=\frac{1}{\varphi(d)^2}\sideset{}{'}\sum_{\chi_1(\text{mod } d)}\sideset{}{'}\sum_{\chi_2(\text{mod } d)}\psi_1(\chi_1)\overline{\psi_2(\chi_1)\psi_1(\chi_2)}\psi_2(\chi_2)$$
$$\times\sideset{}{^*}\sum_{a(\text{mod } d)}\overline{\chi_1}(a)\chi_2(a).$$

By the orthogonality relation (2.35) we get

$$\sum_{a(\bmod d)}{}^{*} E(\tau_k, \tau_k; X, d, a)^2 = \frac{1}{\varphi(d)} \left| \sum_{\chi(\bmod d)}{}' \psi_1(\chi)\overline{\psi_2(\chi)} \right|^2 .$$

We now reduce to primitive characters. By Lemma 2, we have

$$\sum_{d\le D} \sum_{a(\bmod d)}{}^{*} E(\tau_k, \tau_k; X, d, a)^2 \ll \log \mathcal{L} \sum_{r\le D} \frac{1}{r} \left(\sum_{1<q\le D/r} \frac{1}{q} \left| \sum_{\chi(\bmod q)}{}^{*} \psi_1(\chi)\overline{\psi_2(\chi)} \right|^2 \right) .$$

By Lemma 3, we get, for $1 < q \le X^{1/3(k+2)}$,

$$\begin{aligned} \sum_{1<q\le X^{1/3(k+2)}} \frac{1}{q} \left| \sum_{\chi(\bmod q)}{}^{*} \psi_1(\chi)\overline{\psi_2(\chi)} \right|^2 &\ll \sum_{1<q\le X^{1/3(k+2)}} \frac{1}{q} \left(\varphi(q) X^{2-\frac{2}{3(k+2)}}\right)^2 \\ &\ll X^{4-\frac{2}{3(k+2)}}. \end{aligned} \tag{3.8}$$

Assume $X^{1/3(k+2)} \ll Q \ll D$. It suffices then to show, for each fixed $r \le D$,

$$\frac{1}{Q} \sum_{q\sim Q} \left| \sum_{\chi(\bmod q)}{}^{*} \psi_1(\chi)\overline{\psi_2(\chi)} \right|^2 \ll X^{4-1/3(k+3)}.$$

By the large sieve inequality (2.30) and the bound (2.31), the left-side of the above is

$$\le \frac{1}{Q} \sum_{q\sim Q} \sum_{\chi(\bmod q)}{}^{*} |\psi_1(\chi)\overline{\psi_2(\chi)}|^2 \ll \frac{1}{Q}(Q^2 + X^2) \left(\sum_{n\le X} \tau_k(n) \right)^2 \ll \left(Q + \frac{X^2}{Q} \right) X^2 \mathcal{L}^{2k-2}. \tag{3.9}$$

For $D \le X^{2-1/3(k+2)}$, the above is

$$\ll X^{2-1/3(k+2)} + X^{2-1/3(k+2)} X^2 \mathcal{L}^{2k} \ll X^{4-1/3(k+3)}.$$

This, together with (3.8), lead to the right side of (3.6). Note also that, for $X^{2-1/3(k+2)} < D \le X^2$, (3.9) becomes

$$\left(D + \frac{X^2}{X^{2-1/3(k+2)}} \right) X^2 \mathcal{L}^{2k-2} \ll D X^2 \mathcal{L}^{2k-2}.$$

This gives an estimate for (3.6) in this range.

We next prove (3.7). Let $f_1 = \tau_k$ and $f_2 = \Lambda$. The proof of (3.7) is analogous to that of (3.6), except that in (3.8) and (3.9) we estimate the sum over Λ by

$$\psi_2(\chi) = \sum_{n \le X} \Lambda(n)\chi(n) \ll X \quad \text{and} \quad \sum_{n \le X} \Lambda(n) \ll X.$$

Thus (3.8) becomes

$$\sum_{1<q\le X^{1/3(k+2)}} \frac{1}{q} \left| \sum_{\chi (\text{mod } q)}^{*} \psi_1(\chi)\overline{\psi_2(\chi)} \right|^2 \ll X^{4-\frac{1}{3(k+2)}},$$

and (3.9) becomes

$$\begin{aligned} \frac{1}{Q}\sum_{q\sim Q} \sum_{\chi (\text{mod } q)}^{*} |\psi_1(\chi)\overline{\psi_2(\chi)}|^2 &\ll \frac{1}{Q}(Q^2+X^2)\left(\sum_{n\le X}\tau_k(n)\right)\left(\sum_{n\le X}\Lambda(n)\right) \\ &= \left(Q + \frac{X^2}{Q}\right) X^2\mathcal{L}^{k-1}, \end{aligned}$$

both of which are admissible for (3.7). This concludes the proof of Theorem 4.

Chapter 4

Variance of the ternary divisor function in arithmetic progressions

4.1 Introduction

One form of the celebrated Bombieri-Vinogradov Theorem [7] [79] (1965) asserts that

$$\sum_{1\le q\le N^{1/2}(\log N)^{-B}} \max_{y\le N} \max_{(a,q)=1} \left| \sum_{\substack{1\le n\le y\\ n\equiv a(q)}} \Lambda(n) - \frac{y}{\varphi(y)} \right| \ll N(\log N)^{-A} \tag{4.1}$$

where $\Lambda(n)$ is the von Mangoldt function and $B = 4A + 40$ with $A > 0$ arbitrary. Analogues of (4.1) has been found for all $\tau_k(n)$ and $\tau_2(n)^2$ [63, Lemma 8], where $\tau_k(n)$ is the k-fold divisor function: $\sum_{n=1}^{\infty} \tau_k(n)n^{-s} = \zeta^k(s)$.

Around the same time, Barban [3] [4] (1963-1964), Davenport-Halberstam [19] (1966), and Gallagher [36] (1967) found the following related inequality in which the absolute value is being squared:

$$\sum_{1\le q\le N(\log N)^{-B}} \sum_{\substack{1\le a\le q\\ (a,q)=1}} \left| \sum_{\substack{1\le n\le N\\ n\equiv a(q)}} \Lambda(n) - \frac{N}{\varphi(q)} \right|^2 \ll N^2(\log N)^{-A}, \tag{4.2}$$

giving a much wider range for q. In fact, Davenport and Halberstam proved a slightly stronger result than Barban's, while Gallagher gave a simplified elegant proof. For this reason, this type of inequalities are often referred to as Barban-Davenport-Halberstram type inequalities.

Barban-Davenport-Halberstram type inequalities have many applications in number theory. For instance, a version of this inequality (with $\Lambda(n)$ replaced by related convolutions over primes) was skillfully used by Zhang [87, Lemma 10] (2014) in his spectacular work on bounded gaps between primes.

Shortly after, in 1970 Montgomery [60] succeeded in replacing the inequality in (4.2) by an asymptotic equality. One of his results is

$$\sum_{1\le q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\left|\sum_{\substack{1\le n\le N\\ n\equiv a(q)}}\Lambda(n)-\frac{N}{\varphi(q)}\right|^2=QN\log N+O(QN\log(2N/Q))+O\left(N^2(\log N)^{-A}\right) \tag{4.3}$$

for $Q\le N$ and $A>0$ arbitrary. A few years latter, Hooley [44] (1975), by introducing new ideas in treatment of the off-diagonal terms specific to primes, sharpened the right side of (4.3) to

$$QN\log N+O(QN)+O(N^2(\log N)^{-A}))$$

with $\Lambda(n)$ replaced by the Chebyshev function $\theta(n)$.

Motohashi [64] (1973), by using an approach similar to Montgomery, elaborately established a more precise asymptotic with lower order and power saving error terms for the divisor function $\tau(n)$. Recently, by function field analogues, Rodgers and Soundararajan [71] (2018) were led to the following conjecture for the variance of the k-fold divisor function τ_k over the integers.

Conjecture 5 *For $X,d\to\infty$ such that $\log X/\log d\to c\in(0,k)$, we have*

$$\sum_{\substack{a=1\\(a,d)=1}}^{d}\left|\sum_{\substack{1\le n\le X\\ n\equiv a(mod\ d)}}\tau_k(n)-\frac{1}{\varphi(d)}\sum_{\substack{1\le n\le X\\(n,d)=1}}\tau_k(n)\right|^2\sim a_k(d)\gamma_k(c)X(\log d)^{k^2-1}, \tag{4.4}$$

where $a_k(d)$ *is the arithmetic constant*

$$a_k(d) = \lim_{s\to 1^+} (s-1)^{k^2} \sum_{\substack{n=1\\(n,d)=1}}^{\infty} \frac{\tau_k(n)^2}{n^s},$$

and $\gamma_k(c)$ *is a piecewise polynomial of degree* k^2-1 *defined by*

$$\gamma_k(c) = \frac{1}{k!G(k+1)^2} \int_{[0,1]^k} \delta_c(w_1 + \cdots w_k)\Delta(w)^2 d^k w,$$

where $\delta_c(x) = \delta(x-c)$ *is a Dirac delta function centered at* c*,* $\Delta(w) = \prod_{i<j}(w_i - w_j)$ *is a Vandermonde determinant, and* G *is the Barnes* G*-function, so that in particular* $G(k+1) = (k-1)!(k-2)!\cdots 1!$.

This conjecture is closely related to the problem of moments of Dirichlet L-functions [13] and correlations of divisor sums [14]. In the same paper [71], Rodgers and Soundararajan confirmed an averaged version of this conjecture in a restricted range over smooth cutoffs. Harper and Soundararajan [38] obtained a lower bound of the right order of magnitude for the average of this variance. By using the large sieve inequality, we obtained an upper bound of the same order of magnitude for this averaged variance in Theorem 3.

In this chapter, we replace the upper bound in Theorem 3 by an asymptotic equality for the ternary divisor function $\tau_3(n)$ averaged over moduli up to the length of the sum. Our approach is based on Motohashi's treatment for the case of τ_2, with appropriate modifications; see Sections 4.2.1 and 4.2.2 below for discussions and possible extensions.

4.2 Statement of result

Our main result is the following

Theorem 5 *We have the following asymptotic equality, with effectively computable numerical constants*

$$\mathfrak{S}_j, \ (0 \leq j \leq 8),$$

$$\sum_{1\leq \ell \leq N} \sum_{1 \leq b \leq \ell} \left| \sum_{\substack{1 \leq n \leq N \\ n \equiv b (mod\ \ell)}} \tau_3(n) - N P_2(\log N) \right|^2 = N^2 \sum_{j=0}^{8} \mathfrak{S}_{8-j} \log^{8-j} N + O\left(N^{599/300}\right), \tag{4.5}$$

where

$$P_2(\log N) = \operatorname*{Res}_{s=1} \left\{ \sum_{n \equiv b (mod\ \ell)} \frac{\tau_3(n)}{n^s} \frac{N^{s-1}}{s} \right\}. \tag{4.6}$$

4.2.1 Remarks

The function $P_2(\log N)$, which can be thought of as the expected main term (EMT), is a certain second degree polynomial in $\log N$ whose coefficients can be determined explicitly. In fact, one has (c.f. Lemma 21)

$$P_2(\log N) = \frac{1}{2} \tilde{A} \log^2 N - (\tilde{A} - \tilde{B}) \log N + (\tilde{A} - \tilde{B} + \tilde{C}),$$

where

$$\tilde{A} = \tilde{A}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma),$$

$$\tilde{B} = \tilde{B}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q) - 3\log q),$$

$$\tilde{C} = \tilde{C}(\ell, b) = \ell^{-1} \sum_{q|\ell} q^{-3} c_q(b) \sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q)\gamma_0(\beta/q) - 9\gamma_0(\alpha/q)\log q + \frac{9}{2}\log^2 q),$$

γ is Euler's constant, $\gamma_0(\alpha)$ is the 0-th Stieltjes constant, and $c_q(b)$ is the Ramanujan sum. Different main terms are also considered by other authors. Our choice of EMT (4.6) differs from that of (4.4) by an admissible amount which can be shown to be at most $O(X^{2/3+\epsilon})$. This has the harmless effect of changing lower order terms coefficients

$\mathfrak{S}_j$ in the asymptotic; see the discussion proceeding Lemma 32 below. Since our average over $b(\text{mod } \ell)$ is over all residue classes not necessarily coprime to ℓ, the expression (4.6) is the natural EMT to consider, as can readily be seen from its shape. When coprime condition is imposed on $b(\text{mod } \ell)$, the corresponding EMT is the one appearing in (5); this EMT comes from the contribution of the principal character χ mod ℓ.

The constants $\mathfrak{S}_j, 0 \leq j \leq 8$, have complicated expressions but can be explicitly determined from our proof. We give here the value of the leading constant $\mathfrak{S}_8$:

$$\mathfrak{S}_8 = \frac{1}{8!} \prod_p \left(1 - 9p^{-2} + 16p^{-3} - 9p^{-4} + p^{-6}\right) = 1.223 \cdots \times 10^{-6}.$$

We note that it is possible to obtain asymptotics for (4.5) with ℓ averaged over the range $1 \leq \ell \leq L$ for $L < N$ and $L > N$. A phase transition in the coefficient of the leading asymptotic might begin to occur. It is also plausible to use this method, in conjunction with subconvexity bounds for $\zeta(s)$, to treat the variance of $\tau_4(n)$. On the generalized Riemann hypothesis, it might also be possible to treat all $\tau_k(n)$. We hope to return to these ideas in a future article.

4.2.2 Outline of the proof

We follow the approach of Motohashi [64] in his treatment of the divisor function $\tau(n)$, which in turn was based on Montgomery's adaptation [60] of a result of Lavrik [54] on twin primes on average.

To control the error term, we prove an analog of Lavrik's result for τ_3, using a simpler version of Vinogradov's method of trigonometric sums, as in Motohashi. The standard convexity bound for $\zeta(s)$ in the critical strip suffices for our purpose. We remark here that our analogue of Lavrik's result can be seen as an average result concerning the mean square error of the following modified additive divisor sum

$$\sum_{1\le n\le N-h} \tau_3(n)\tau_3(n+h) \tag{4.7}$$

of length $N-h$ averaged over h up to $h \le N-1$. The advantage of considering (4.7) is that the length of this sum becomes shorter the larger the shift h is, making contribution from large shifts small, so a power saving is possible when an average over h is taken. This idea might also have application to the sixth power moment of $\zeta(s)$, which we plan to revisit in the near future.

To evaluate the main term, we proceed slightly different from Motohashi due to some complications involving an exponential sum in three variables. We show that the resulting sum can be evaluated, on average, thanks to the orthogonality property of the Ramanujan's sum.

4.3 Preparatory lemmas

For $\sigma > 1$ and $(a,q)=1$, let

$$E\left(s;\frac{a}{q}\right) = E_3\left(s;\frac{a}{q}\right) = \sum_{n=1}^{\infty} \tau_3(n)e_q(an)n^{-s}. \tag{4.8}$$

The case for the usual divisor function $\tau(n)$ was considered by Estermann (1930) who obtained analytic continuation and the functional equation for the corresponding generating function. Smith (1982) extended the result to all τ_k. We specialize to a special case his results.

Lemma 16 *[74, Theorem 1, pg. 258] The function $E_3(s;a/q)$ has a meromorphic continuation to the whole complex plane where it is everywhere holomorphic except for a pole of order 3 at $s=1$. Moreover, $E(s;a/q)$ satisfies the functional equation*

$$E(s;a/q) = \left(\frac{q}{\pi}\right)^{-\frac{3}{2}(2s-1)} \frac{\Gamma^3\left(\frac{1-s}{2}\right)}{\Gamma^3\left(\frac{s}{2}\right)} E^{+}(1-s;a/q) + i\left(\frac{q}{\pi}\right)^{\frac{3}{2}(2s-1)} \frac{\Gamma^3\left(\frac{2-s}{2}\right)}{\Gamma^3\left(\frac{1+s}{2}\right)} E^{-}(1-s;a/q), \tag{4.9}$$

where

$$E^{\pm}(s;a/q)=\sum_{m_1,m_2,m_3\geq 1}G^{\pm}(m_1,m_2,m_3;a/q)(m_1m_2m_3)^{-s},\quad (\sigma>1),$$

$$G^{\pm}(m_1,m_2,m_3;a/q)=\frac{1}{2q^{3/2}}\{G(m_1,m_2,m_3;a/q)\pm G(m_1,m_2,m_3;-a/q)\},$$

and

$$G(m_1,m_2,m_3;a/q)=\sum_{x_1,x_2,x_3(q)}e_q(am_1m_2m_3+m_1x_1+m_2x_2+m_3x_3).$$

We rewrite the functional equation (4.9) as follows (c.f. Ivic [46]). Let

$$A^{\pm}(n,a/q)=\sum_{n_1n_2n_3=n}\sum_{x_1,x_2,x_3=1}^{q}\frac{1}{2}\big(e_q(ax_1x_2x_3+n_1x_1+n_2x_2+n_3x_3)$$
$$\pm e_q(-ax_1x_2x_3+n_1x_1+n_2x_2+n_3x_3)\big).$$

We have that

$$|A^{\pm}(n,a/q)|\leq q^3\tau_3(n).$$

Then from Lemma 16 we obtain the following form of the functional equation.

Lemma 17 *[46, Lemma 2, pg. 1007] For $\sigma<0$ and $(a,q)=1$, we have*

$$E(s;a/q)=\left(\frac{q}{\pi}\right)^{-\frac{3}{2}(2s-1)}\left\{\frac{\Gamma^3\left(\frac{1-s}{2}\right)}{\Gamma^3\left(\frac{s}{2}\right)}\sum_{n=1}^{\infty}A^{+}(n,a/q)n^{s-1}+i\frac{\Gamma^3\left(\frac{2-s}{2}\right)}{\Gamma^3\left(\frac{1+s}{2}\right)}\sum_{n=1}^{\infty}A^{-}(n,a/q)n^{s-1}\right\}, \tag{4.10}$$

where the two series on the right-side are absolutely convergent.

We also need the Laurent expansion of $E(s;a/q)$ at $s=1$ for residue calculations. We first recall a lemma from Motohashi [64].

Lemma 18 *We have, uniformly for any integer d,*

$$\sum_{1\leq m\leq y}\tau(dm)=y\sum_{q|d}\frac{\varphi(q)}{q}(\log dy+2\gamma-1-2\log q)+O(\tau^2(d)y^{1/2}\log^2 y).$$

Proof: See [64, Lemma 4.6.1, p. 193]. ■

Lemma 19 *For $(a,q)=1$, we have*

$$E(s;a/q)=\frac{1}{q}\left(\frac{A}{(s-1)^3}+\frac{B}{(s-1)^2}+\frac{C}{s-1}\right)+\sum_{n=0}^{\infty}c_n(a,q)(s-1)^n, \tag{4.11}$$

where

$$A=A(q)=q^{-2}\sum_{\alpha,\beta,\gamma=1}^{q}e_q(a\alpha\beta\gamma),$$

$$B=B(q)=q^{-2}\sum_{\alpha,\beta,\gamma=1}^{q}e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q)-3\log q),$$

$$C=C(q)=q^{-2}\sum_{\alpha,\beta,\gamma=1}^{q}e_q(a\alpha\beta\gamma)(3\gamma_0(\alpha/q)\gamma_0(\beta/q)-9\gamma_0(\alpha/q)\log q+\frac{9}{2}\log^2 q),$$

with

$$\gamma_0(\alpha)=\lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{1}{k+\alpha}-\log(m+\alpha)\right). \tag{4.12}$$

The coefficients A,B,C are independent of a and satisfy

$$A(q)\asymp\log^2 q,\quad B(q)\asymp\log^3 q,\quad C(q)\asymp\log^4 q, \tag{4.13}$$

uniformly in a.

Proof: See Ivić [22, pp. 1007-1008] for the Laurent expansion (4.11). In fact, Ivić also gave upper bound of the form q^{ϵ} for A,B, and C which came from the bound

$$\sum_{\alpha,\beta,\gamma=1}^{q}e_q(a\alpha\beta\gamma)\le q\sum_{1\le\alpha\le q}\tau(\alpha q)\ll q^{2+\epsilon}.$$

We can sharpened this upper bound by applying Lemma 18 and bounding $\sum_{q|d}\frac{\varphi(q)}{q}$ by $\log(q)$, giving

$$\sum_{\alpha,\beta,\gamma=1}^{q}e_q(a\alpha\beta\gamma)\le q\sum_{1\le\alpha\le q}\tau(\alpha q)\ll q^2\log^2 q.$$

On the other hand, we have

$$\sum_{\alpha,\beta,\gamma=1}^{q} e_q(a\alpha\beta\gamma) \geq q \sum_{1\leq\alpha\leq q} \tau_3(\alpha) = q^2\log^2 q + O(q^2\log q).$$

Thus, noting that $\gamma_0(\alpha) \ll 1$ by (4.12), the bounds (4.13) follow. ■

Lemma 20 *For $n \geq 1$ and $(a,q)=1$, we have*

$$\operatorname*{Res}_{s=1} E\left(s;\frac{a}{q}\right)\frac{n^s}{s} = q^{-1}n\left(\frac{A}{2}\log^2 n - (A-B)\log n + (A-B+C)\right), \tag{4.14}$$

where A, B, C are given in Lemma 19.

Proof: We have, by (4.11),

$$\begin{aligned}\operatorname*{Res}_{s=1} E\left(s;\frac{a}{q}\right)\frac{n^s}{s} &= \frac{1}{2}\lim_{s\to1}\frac{d^2}{ds^2}\left((s-1)^3 E\left(s;\frac{a}{q}\right)\frac{n^s}{s}\right)\\ &= \frac{1}{2q}\lim_{s\to1}\frac{d^2}{ds^2}\left((A+B(s-1)+C(s-1)^2+O((s-1)^3))\frac{n^s}{s}\right)\\ &= q^{-1}n\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right).\end{aligned}$$

■

Lemma 21 *For $\sigma > 1$, let*

$$R(s;\ell,b) = \sum_{n\equiv b(mod\ \ell)} \tau_3(n)n^{-s}.$$

We have

$$\operatorname*{Res}_{s=1} R(s;\ell,b)\frac{N^s}{s} = N\left(\frac{\tilde{A}}{2}\log^2 N - (\tilde{A}-\tilde{B})\log N + (\tilde{A}-\tilde{B}+\tilde{C})\right),$$

where

$$\tilde{A} = \tilde{A}(\ell,b) = \ell^{-1}\sum_{q|\ell} q^{-1}c_q(b)A(q),$$

$$\tilde{B} = \tilde{B}(\ell,b) = \ell^{-1}\sum_{q|\ell} q^{-1}c_q(b)B(q),$$

$$\tilde{C} = \tilde{C}(\ell,b) = \ell^{-1}\sum_{q|\ell} q^{-1}c_q(b)C(q),$$

with $A(q), B(q), C(q)$ given in Lemma 19.

Proof: We can write $R(s;\ell,b)$ as

$$\begin{aligned}R(s;\ell,b) &= \frac{1}{\ell}\sum_{q|\ell}\sum_{\substack{1\le a\le q\\(a,q)=1}} e_q(-ab)E\left(s;\frac{a}{q}\right)\\ &= \frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)\left(\frac{A(q)}{(s-1)^3}+\frac{B(q)}{(s-1)^2}+\frac{C(q)}{s-1}\right)+\sum_{n=0}^{\infty}\frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)c_n(a,q)(s-1)^n\\ &= \frac{\tilde{A}(\ell,b)}{(s-1)^3}+\frac{\tilde{B}(\ell,b)}{(s-1)^2}+\frac{\tilde{C}(\ell,b)}{s-1}+\sum_{n=0}^{\infty}\frac{1}{\ell}\sum_{q|\ell}\frac{1}{q}c_q(b)c_n(a,q)(s-1)^n.\end{aligned}$$

The lemma follows as in the previous one. ■

For $\alpha\in\mathbb{R}$, let

$$D(\alpha,N)=\sum_{1\le n\le N}\tau_3(n)e(\alpha n). \tag{4.15}$$

Using (4.8) we first estimate $D(\alpha,N)$ for $\alpha=a/q$ with $(a,q)=1$.

Lemma 22 *For $(a,q)=1$, we have*

$$D\left(\frac{a}{q},n\right)=\frac{n}{q}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)+O\left\{(nq+q^2)^{3/5+\epsilon}\right\},$$

with A,B,C given in Lemma 19.

Proof: We have

$$\begin{aligned}D\left(\frac{a}{q},n\right) &= \operatorname*{Res}_{s=1} E\left(s;\frac{a}{q}\right)\frac{n^s}{s}+\operatorname*{Res}_{s=0} E\left(s;\frac{a}{q}\right)\frac{n^s}{s}+\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT}E\left(s;\frac{a}{q}\right)\frac{n^s}{s}ds\\ &\quad+O\left\{\frac{n^{1+\epsilon}}{T}+n^{\epsilon}+\frac{1}{T}\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT;\frac{a}{q}\right)\right|n^{\sigma}d\sigma\right\},\end{aligned}\tag{4.16}$$

where $\delta=(\log(nq+1))^{-1}$ and T is to be determined latter. By expressing the residue as an integral around the origin,

$$\left|\operatorname*{Res}_{s=0} E\left(s;\frac{a}{q}\right)\frac{n^s}{s}\right|\ll(\log(qn+1))^3. \tag{4.17}$$

By the functional equation (7.12) and the convexity argument,

$$\left|E\left(\sigma+iT;\frac{a}{q}\right)\right| \ll (qT)^{\frac{3}{2}(1-\sigma)}(\log qT)^6$$

uniformly for $-\delta \le \sigma \le 1+\delta$. Hence we get

$$\left|\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT} E\left(s;\frac{a}{q}\right)\frac{n^s}{s}ds\right| \ll (Tq)^{\frac{3}{2}}(\log qT)^7 \tag{4.18}$$

and

$$\frac{1}{T}\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT;\frac{a}{q}\right)\right|n^\sigma d\sigma \ll \frac{n}{T}(\log qT)^6\int_{-\delta}^{1+\delta}\left(\frac{Tq}{n^{2/3}}\right)^{\frac{3}{2}(1-\sigma)}d\sigma. \tag{4.19}$$

Taking

$$T = q^{-1}(nq+q^2)^{2/5}$$

it follows from (4.14), (4.16), (4.17), (4.18) and (4.19) that

$$D\left(\frac{a}{q},n\right) = \frac{n}{q}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)+O\left\{(nq+q^2)^{3/5+\epsilon}\right\}.$$

■

Lemma 23 *For $\alpha\in\mathbb{R}$, we have*

$$D(\alpha,N) = \frac{1}{q}\sum_{1\le n\le N}\left(\frac{A}{2}\log^2 n-(A-B)\log n+(A-B+C)\right)e\left(\left(\alpha-\frac{a}{q}\right)n\right) \tag{4.20}$$
$$+O\left\{(Nq+q^2)^{3/5+\epsilon}\left(1+\left|\alpha-\frac{a}{q}\right|N\right)\right\},$$

with A,B,C given in Lemma 19.

Proof: We have

$$D(\alpha,N)=\sum_{1\le n\le N}\left\{D(a/q,n)-D(a/q,n-1)\right\}e((\alpha-a/q)n).$$

This, together with Lemma 22 and partial summation, gives (4.20). ■

Let

$$F\left(\alpha, \frac{a}{q}, N\right) = \frac{1}{q} \sum_{1 \le n \le N} \left(\frac{A}{2}\log^2 n - (A-B)\log n + (A-B+C)\right) e\left(\left(\alpha - \frac{a}{q}\right) n\right) \tag{4.21}$$

and

$$G_\Delta(\alpha, N) = \sum_{1 \le q \le \Delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^2, \tag{4.22}$$

where Δ satisfies

$$4\Delta \le \Omega \tag{4.23}$$

and Δ is to be determined more precisely latter; see (4.37) below. By Lemma 23 and equation (9.11),

$$|D(\alpha, N) - F(\alpha, a/q, N)| \ll (Nq + q^2)^{3/5+\epsilon} \left(1 + \left|\alpha - \frac{a}{q}\right| N\right). \tag{4.24}$$

Now, by (9.11) and (4.22),

$$G_\Delta(\alpha, N) = \sum_{|k| \le N-1} e(\alpha k) \left(\sum_{1 \le q \le \Delta} \frac{1}{q^2} W_q(k, N) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e_q(-ak) \right), \tag{4.25}$$

where

$$\begin{aligned}W_q(k,N) &= \frac{1}{4}A^2\sum_{1\le n\le N-|k|}\log^2 n\log^2(n+|k|) \\ &- \frac{1}{2}A(A-B)\sum_{1\le n\le N-|k|}\log n\log(n+|k|)\log n(n+|k|) \\ &+ (A-B)^2\sum_{1\le n\le N-|k|}\log n\log(n+|k|) \\ &- \frac{1}{2}A(A-B+C)\sum_{1\le n\le N-|k|}(\log^2 n+\log^2(n+|k|)) \\ &- (A-B)(A-B+C)\sum_{1\le n\le N-|k|}\log n(n+|k|) \\ &+ (A-B+C)^2(N-|k|) \\ &= w_1(q)T_1(k,N)+\cdots+w_6(q)T_6(k,N),\end{aligned} \tag{4.26}$$

say. For the innermost sum in (4.25) we have

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} e_q(-ak) = \mu\left(\frac{q}{(q,|k|)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,|k|)}\right)} = c_q(|k|).$$

Thus we write (4.25) as

$$G_\Delta(\alpha,N) = \sum_{|k|\le N-1}\left(\sum_{1\le q\le\Delta} q^{-2}c_q(|k|)W_q(k,N)\right)e(\alpha k) = \sum_{|k|\le N-1} S_\Delta(k,N)e(\alpha k), \tag{4.27}$$

say. Now, by (4.15), we have

$$|D(\alpha,N)|^2 = \sum_{|k|\le N-1} V(k,N)e(\alpha k),$$

where

$$V(k,N) = \sum_{1\le n\le N-|k|}\tau_3(n)\tau_3(n+|k|). \tag{4.28}$$

Thus,

$$|D(\alpha,N)|^2 - G_\Delta(\alpha,N) = \sum_{|k|\le N-1}(V(k,N)-S_\Delta(k,N))e(\alpha k).$$

and we obtain

Lemma 24

$$\sum_{|k|\leq N-1} (V(k,N) - S_\Delta(k,N))^2 = \int_0^1 \left||D(\alpha,N)|^2 - G_\Delta(\alpha,N)\right|^2 d\alpha, \tag{4.29}$$

with $D(\alpha,N)$*,* $G_\Delta(\alpha,N)$*,* $V(k,N)$*, and* $S_\Delta(k,N)$ *given by* (4.15)*,* (4.22)*,* (4.28)*, and* (4.27)*, respectively.*

This integral will be estimated in Section 4.4 below.

Lemma 25 *With*

$$\begin{aligned} T_1(k,N) &= \sum_{1\leq n\leq N-|k|} \log^2 n \log^2(n+|k|), \qquad (4.30)\\ T_2(k,N) &= \sum_{1\leq n\leq N-|k|} \log n \log(n+|k|)\log(n(n+|k|)),\\ T_3(k,N) &= \sum_{1\leq n\leq N-|k|} \log n \log(n+|k|),\\ T_4(k,N) &= \sum_{1\leq n\leq N-|k|} (\log^2 n + \log^2(n+|k|)),\\ T_5(k,N) &= \sum_{1\leq n\leq N-|k|} \log(n(n+|k|)). \end{aligned}$$

given from (4.26)*, we have*

$$\begin{aligned} T_1(k,N) &= (N-|k|)\log^2 N \log^2(N-|k|) + O(N\log^3 N),\\ T_2(k,N) &= (N-|k|)(\log^2 N\log(N-|k|) + \log N \log^2(N-|k|)) + O(N\log^2 N),\\ T_3(k,N) &= (N-|k|)\log N\log(N-|k|) + O(N\log N),\\ T_4(k,N) &= (N-|k|)(\log^2 N + \log^2(N-|k|)) + O(N\log N),\\ T_5(k,N) &= (N-|k|)(\log N + \log(N-|k|)) + O(N). \end{aligned}$$

Proof: For $k>0$, by partial summation, we have

$$T_5(k,N) = (N-k)\log(N-k) + N\log N - k\log k - 2(N-k) + O(\log N).$$

Similarly, we obtain the other T_j's. ■

Lemma 26 *For positive integer δ and $q > 1$, we have*

$$\sum_{1\le m\le X} c_q(\delta m) = O(q\delta^{-1/2}X^{1/2}\log^2 X).$$

Proof: Let us consider the function

$$f(s) = \sum_{m=1}^{\infty} \frac{c_q(\delta m)}{m^s}.$$

We have

$$f(s) = \sum_{m=1}^{\infty} \sum_{\substack{d\mid\delta m\\ dd'=q}} \mu(d')dm^{-s} = \delta^{-s}\sum_{m=1}^{\infty} m^{-s}\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-s} = \delta^{-s}\zeta(s)\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-s}.$$

Hence we have

$$\sum_{1\le m\le X} c_q(\delta m) = \operatorname*{Res}_{s=1}\left\{\delta^{-s}\zeta(s)\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-s}\frac{X^s}{s}\right\} + \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\delta^{-s}\zeta(s)\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-s}\frac{X^s}{s}ds$$

$$+ O\left\{\frac{X^{1+\epsilon}}{T} + X^{\epsilon} + \frac{X^{\epsilon}}{T}\int_{1/2}^{1}|\zeta(\sigma+iT)|\left|\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-\sigma-iT}\right|X^{\sigma}d\sigma\right\}.$$

Since we have

$$\int_{-T}^{T}|\zeta(1/2+it)|\frac{dt}{|t|+1} \ll \log^2 T,$$

$$\left|\sum_{d\mid q}\mu\left(\frac{q}{d}\right)d^{1-s}\right| \le \sum_{d\mid q} d^{1/2} \ll q,$$

and

$$|\zeta(\sigma+iT)| \ll T^{\frac{1}{3}(1-\sigma)}\log^5 T;$$

taking $T = X$ we complete the proof. ■

We will apply Perron's formula in the following form.

Lemma 27 *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series converges absolutely for $\sigma > 1$. Suppose $a_n = O(n^{\epsilon})$ for any $\epsilon > 0$ and $f(s) = \zeta(s)^{\ell}F(s)$ for some natural number ℓ*

and some Dirichlet series $F(s)$ absolutely converges in $\Re(s) > 1/2$. Then for X not an integer, we have

$$\sum_{n\leq X} a_n = \frac{F(1)}{(\ell-1)!} X P_{\ell-1}(\log X) + O_\epsilon\left(X^{1-\frac{1}{\ell+2}}\right),$$

where $P_{\ell-1}(\log X)$ is the polynomial in $\log X$ of degree $\ell-1$ with leading coefficient 1 given explicitly by

$$P_{\ell-1}(\log X) = (\ell-1)! Res_{s=1}\zeta(s)^\ell F(s)\frac{X^{s-1}}{s}.$$

Proof: See [66, Problems 4.4.16, 4.4.17]. ■

Lemma 28 *We have*

$$\sum_{n\leq X} \tau_3^2(n) = \frac{a_3}{8!} X P_8(\log X) + O\left(X^{10/11}\right),$$

where

$$a_3 = \prod_p \left(1 - 9p^{-2} + 16p^{-3} - 9p^{-4} + p^{-6}\right) = 0.04932\ldots$$

and $P_8(\log X)$ is a polynomial of degree 8 in $\log X$ and leading coefficient 1.

Proof: We have

$$\sum_{n=1}^{\infty} \tau_3^2(n) n^{-s} = \prod_p \left\{1 + \sum_{\nu=1}^{\infty} \binom{\nu+2}{2}^2 p^{-\nu s}\right\},$$

where both members of this equation are absolutely convergent if $\sigma > 1$. Hence, if $\sigma > 1$,

$$\begin{aligned}\{\zeta(s)\}^{-9}\left\{\sum_{n=1}^{\infty} \tau_3^2(n) n^{-s}\right\} &= \prod_p \left\{(1-p^{-s})^9(1+9p^{-s}+36p^{-2s}+\cdots)\right\} \\ &= \prod_p \left\{1 + a_2 p^{-2s} + a_3 p^{-3s} + \cdots\right\} = F(s),\end{aligned}$$

say, where

$$a_\nu = \sum_{r=0}^{\nu} (-1)^r \binom{9}{r} \binom{\nu - r + 2}{2}^2 .$$

We adpot the convention for the binomial coefficients that $\binom{n}{m} = 0$ if $m > n$. The coefficient a_ν satisfies

$$|a_\nu| \le K\nu^2,$$

where K is independent of ν. Hence

$$\sum_{\nu=2}^{\infty} |a_\nu| p^{-\nu s} \le K' p^{-2s},$$

where K' is independent of p. Hence, if $\sigma > 1/2$, then $\sum_p p^{-2s}$ is absolutely convergent, and thus is also

$$F(s) = \prod_p \left\{ 1 + \sum_{\nu=2}^{\infty} a_\nu p^{-\nu s} \right\} .$$

Hence we obtain that

$$\sum_{n=1}^{\infty} \tau_3^2(n) n^{-s} = \{\zeta(s)\}^9 F(s),$$

where $F(s)$ is absolutely convergent for $\sigma > 1/2$. It follows at once, by Lemma 27, that

$$\sum_{n \le X} \tau_3^2(n) = \frac{a_3}{8!} X P_8(\log X) + O\left(X^{10/11}\right),$$

where

$$a_3 = F(1) = \prod_p \left(1 - 9p^{-2} + 16p^{-3} - 9p^{-4} + p^{-6}\right) .$$

■

Lemma 29 *We have*

$$\int_1^{N-1} \frac{t \log t}{N - t} dt = N \left(\log^2 N - \log N - \frac{\pi^2}{6} + 1 \right) + O(\log N)$$

and

$$\int_1^{N-1} \frac{t \log^2 t}{N - t} dt = N \left(\log^3 N - 2\log^2 N - \left(\frac{\pi^2}{3} - 2 \right) \log N + 2\zeta(3) - 2 \right) + O(\log^2 N).$$

Proof: Expanding into a geometric series and integrate by parts, we have

$$\begin{aligned}\int_1^{N-1}\frac{t\log t}{N-t}dt &= \sum_{m=1}^{\infty}\frac{1}{N^m}\int_1^{N-1}t^m\log t dt\\ &= N\log(N-1)\sum_{m=1}^{\infty}\frac{1}{m+1}\left(\frac{N-1}{N}\right)^{m+1}-N\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}\left(\frac{N-1}{N}\right)^{m+1}+O(1)\\ &= N\left(\log^2 N-\log N-\frac{\pi^2}{6}+1\right)+O(\log N).\end{aligned}$$

This gives the first integral. The second integral is computed in a similar way. ■

4.4 An analogue to a result of Lavrik

In this section we estimate the integral in (4.29) by trigonometric method of I.M. Vinogradov along the line of Lavrik, following Motohashi (section 3).

Let a/q be a term of the Farey series of order Ω, which is to be determined latter. Let

$$\frac{a'}{q'},\frac{a}{q},\frac{a''}{q''}$$

be consecutive terms of the Farey series and let $C(a/q)$ be the interval $\left[\frac{a'+a}{q'+q},\frac{a+a''}{q+q''}\right]$. The interval $C(a/q)$ contains the fraction a/q with length bounded by

$$\left|C\left(\frac{a}{q}\right)\right|\le\frac{2}{q\Omega}. \tag{4.31}$$

Let

$$U(N)=\int_0^1\left||D(\alpha,N)|^2-G_\Delta(\alpha,N)\right|^2d\alpha$$

denote the integral in (4.29). We proceed to estimate $U(N)$. We have

$$\begin{aligned} U(N) &= \sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left||D(\alpha,N)|^2-G_{\Delta}(\alpha,N)\right|^2 d\alpha \qquad (4.32)\\ &\le 2\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left||D(\alpha,N)|^2-\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2\right|^2 d\alpha\\ &+2\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left|G_{\Delta}(\alpha,N)-\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2\right|^2 d\alpha\\ &= 2U_1(N)+U_2(N),\text{`} \end{aligned}$$

say. For $U_1(N)$, we have, from (4.24),

$$\left||D(\alpha,N)|^2-\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2\right|^2 \ll (Nq+q^2)^{\frac{6}{5}+2\epsilon}\left(1+\left|\alpha-\frac{a}{q}\right|^2N^2\right)\left(|D(\alpha,N)|^2+\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2\right).$$

Thus, for $\alpha\in C(a/q)$, we have, by (4.31), that the above is bounded by

$$\left((N\Omega)^{\frac{6}{5}+2\epsilon}+\Omega^{\frac{12}{5}+4\epsilon}+\frac{N^{\frac{16}{5}+2\epsilon}}{\Omega^2}\right)\left(|D(\alpha,N)|^2+\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2\right),$$

and we get

$$U_1(N) \ll \left((N\Omega)^{\frac{6}{5}+2\epsilon}+\Omega^{\frac{12}{5}+4\epsilon}+\frac{N^{\frac{16}{5}+2\epsilon}}{\Omega^2}\right)\left\{\int_0^1|D(\alpha,N)|^2d\alpha+\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_0^1\left|F\left(\alpha,\frac{a}{q},N\right)\right|^2d\alpha\right\} \qquad (4.33)$$

$$\ll \left((N\Omega)^{\frac{6}{5}+2\epsilon}+\Omega^{\frac{12}{5}+4\epsilon}+\frac{N^{\frac{16}{5}+2\epsilon}}{\Omega^2}\right)N\log^8 N.$$

For $U_2(N)$, we have, by (4.22),

$$U_2(N) \ll \sum_{1\le q\le\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left| \sum_{1\le q'\le\Delta} \sum_{\substack{a'=1\\(a',q')=1\\a'q\neq aq'}}^{q'} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^2 \right|^2 d\alpha \tag{4.34}$$

$$+ \sum_{\Delta<q\le\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \left| F\left(\alpha, \frac{a}{q}, N\right)\right|^4 d\alpha$$

$$= U_3(N) + U_4(N),$$

say. By (9.11), we have

$$U_4(N) \ll \frac{(N\log^2 N)^4}{\Omega} \sum_{\Delta<q\le\Omega} \frac{1}{q^4} \ll \frac{N^4 \log^8 N}{\Omega\Delta^3}. \tag{4.35}$$

It remains to estimate $U_3(N)$. By partial summation, we can write $F\left(\alpha, a'/q', N\right)$ as

$$\frac{1}{q'}(A(a',q')\log^2 N + (B(a',q') - 2A(a',q'))\log N + 2A(a',q') - B(a',q')$$
$$+ C(a',q')) \sum_{1\le n\le N} e\left(\left(\alpha - \frac{a'}{q'}\right)n\right) - \frac{1}{q'}\int_1^N \left(\frac{2A\log\xi}{\xi} + \frac{B-2A}{\xi}\right) \sum_{1\le n\le\xi} e\left(\left(\alpha - \frac{a'}{q'}\right)n\right) d\xi.$$

Thus,

$$\left| F\left(\alpha, \frac{a'}{q'}, N\right)\right| \ll \frac{q'^{\epsilon}\log^3 N}{q'\left|\sin\pi\left(\alpha - \frac{a'}{q'}\right)\right|}.$$

The function $F\left(\alpha, a'/q', N\right)$ has period 1 in α, and $|a/q - (a'/q' \pm 1)| \le 1/2$. Thus, $U_3(N)$ is at most

$$\ll \Delta^2 \log^{12} N \sum_{1\le q\le\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{C(a/q)} \sum_{1\le q'\le\Delta} \sum_{\substack{a'=-q'\\(a',q')=1\\0<\left|\frac{a'}{q'}-\frac{a}{q}\right|\le\frac{1}{2}}}^{2q'} \frac{q'^{\epsilon}}{q'^4\left|\sin\pi\left(\alpha - \frac{a'}{q'}\right)\right|^4} d\alpha.$$

By (4.23), we have, for $\alpha \in C(a/q)$,

$$\frac{1}{2}\left|\frac{a}{q} - \frac{a'}{q'}\right| \le \left|\alpha - \frac{a'}{q'}\right| \le \frac{3}{4}$$

for N sufficiently large. Hence,

$$U_3(N) \ll \Omega^2\Delta^2 \log^{12} N \sum_{1\le q\le\Omega} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{1\le q'\le\Delta} q'^{\epsilon} \sum_{\substack{a'=-q'\\(a',q')=1\\a'q\ne aq'}}^{2q'} \frac{1}{|aq'-qa'|^4} \ll \Omega^{2+\epsilon}\Delta^2 \log^{12} N \sum_{u=1}^{\infty} \frac{t(u)}{u^4},$$

where $t(u)$ is the number of integer solutions to $|aq'-qa'| = u$ in the range of summation. We have

$$t(u) \ll \Delta^2\Omega$$

which yields

$$U_3(N) \ll \Omega^{3+\epsilon}\Delta^4 \log^{12} N. \tag{4.36}$$

From (4.29), (4.32), (4.33), (4.34), (4.35) and (4.36), we get the inequality

$$\sum_{1\le k\le N-1} (V(k,N) - S_\Delta(k,N))^2 \ll N^{\epsilon}\left(N^{11/5}\Omega^{6/5} + \Omega^{12/5}N + \frac{N^{21/5}}{\Omega^2} + \Omega^3\Delta^4 + \frac{N^4}{\Omega\Delta^3}\right).$$

We now take, for example,

$$\Omega = N^{25/38} \quad \text{and} \quad \Delta = N^{4/19}. \tag{4.37}$$

Then the requirement (4.23) is satisfied, and we have proved

Lemma 30 *The inequality*

$$\sum_{1\le k\le N-1} (V(k,N) - S_\Delta(k,N))^2 \ll N^{299/100}$$

holds for sufficiently large N.

4.5 Proof of Theorem 5

Let $Q(N)$ denote the sum on the left side of (4.5). We have

$$
\begin{aligned}
Q(N) &= \sum_{1\le \ell\le N} \sum_{\substack{1\le n_1,n_2\le N\\ n_1\equiv n_2 (\mathrm{mod}\ \ell)}} \tau_3(n_1)\tau_3(n_2) \qquad\qquad (4.38)\\
&+ \frac{1}{4}N^2\log^4 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} \tilde{A}(\ell,b)^2\\
&- N^2\log^3 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - \tilde{A}(\ell,b)\tilde{B}(\ell,b))\\
&+ N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B} + \tilde{B}(\ell,b)^2)\\
&+ N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 - \tilde{A}(\ell,b)\tilde{B}(\ell,b) + \tilde{A}(\ell,b)\tilde{C}(\ell,b))\\
&+ 2N^2\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 + \tilde{B}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B}(\ell,b) - \tilde{B}(\ell,b)\tilde{C}(\ell,b) + \tilde{A}(\ell,b)\tilde{C}(\ell,b))\\
&+ N^2 \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b)^2 + \tilde{B}(\ell,b)^2 + \tilde{C}(\ell,b)^2 - 2\tilde{A}(\ell,b)\tilde{B}(\ell,b) + 2\tilde{A}(\ell,b)\tilde{C}(\ell,b) - 2\tilde{B}(\ell,b)\tilde{C}(\ell,b\\
&- N\log^2 N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} \tilde{A}(\ell,b) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&+ 2N\log N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b) - \tilde{B}(\ell,b)) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&+ 2N \sum_{1\le \ell\le N}\sum_{1\le b\le \ell} (\tilde{A}(\ell,b) - \tilde{B}(\ell,b) + \tilde{C}(\ell,b)) \sum_{\substack{1\le n\le N\\ n\equiv b(\ell)}} \tau_3(n)\\
&= Q_1(N) + \cdots + Q_{10}(N),
\end{aligned}
$$

say. We start with evaluating $Q_1(N)$, which is the longest of the ten. We have

$$\begin{aligned} Q_1(N) &= N \sum_{1\le n\le N} \tau_3^2(n) + 2 \sum_{1\le \ell\le N-1} \sum_{1\le u\le (N-1)/\ell} \sum_{1\le n\le N-u\ell} \tau_3(n)\tau_3(n+u\ell) \quad (4.39)\\ &= N \sum_{1\le n\le N} \tau_3^2(n) + 2 \sum_{1\le k\le N-1} V(k,N)\tau(k), \end{aligned}$$

where $V(k,N)$ is given by (4.28). Here we have, by Lemma 53,

$$\sum_{n\le N} \tau_3^2(n) = \frac{a_3}{8!} N P_8(\log N) + O\left(N^{10/11}\right) \quad (4.40)$$

with a_3 and $P_8(\log N)$ given in that lemma. Now, by Lemma 30,

$$\begin{aligned} \sum_{1\le k\le N-1} V(k,N)\tau(k) &= \sum_{1\le k\le N-1} S_\Delta(k,N)\tau(k) \quad (4.41)\\ &\quad + O\left\{\left(\sum_{1\le k\le N-1} \tau^2(k)\right)^{1/2} \left(\sum_{1\le k\le N-1} (V(k,N)-S_\Delta(k,N))^2\right)^{1/2}\right\}\\ &= \sum_{1\le k\le N-1} S_\Delta(k,N)\tau(k) + O\left(N^{599/300}\right) = Q_{11}(N) + O\left(N^{599/300}\right), \end{aligned}$$

say. We now calculate $Q_{11}(N)$. By (4.27), (4.26), and (4.30), we have

$$Q_{11}(N) = \sum_{j=1}^{6} \sum_{1\le q\le \Delta} q^{-2} w_j(q) \sum_{1\le k\le N-1} \tau(k) c_q(k) T_j(k,N).$$

If $q=1$, then

$$c_1(k) = 1,\ A(1) = 1,\ B(1) = 3\gamma,\ C(1) = 3\gamma^2,$$

and, hence,

$$\begin{aligned} w_1(1) &= \frac{1}{4}, \\ w_2(1) &= \frac{1}{2}(3\gamma - 1), \\ w_3(1) &= (1-3\gamma)^2, \\ w_4(1) &= -\frac{1}{2}(1-3\gamma+3\gamma^2), \\ w_5(1) &= (3\gamma-1)(1-3\gamma-3\gamma^2), \\ w_6(1) &= (1-3\gamma+3\gamma^2)^2. \end{aligned} \tag{4.42}$$

Thus,

$$\begin{aligned} Q_{11}(N) = &\sum_{j=1}^{6} w_j(1) \sum_{1\le k\le N-1} \tau(k) T_j(k,N) \\ &+ \sum_{j=1}^{6} \sum_{1<q\le\Delta} q^{-2} w_j(q) \sum_{1\le k\le N-1} \tau(k) c_q(k) T_j(k,N). \end{aligned} \tag{4.43}$$

To calculate the k-summations, we need to compute the following sums.

$$\begin{aligned}
H_1(N) &= \sum_{1\le k\le N-1} \tau(k)\log(N-k), \qquad (4.44)\\
H_2(N) &= \sum_{1\le k\le N-1} \tau(k)\log^2(N-k),\\
H_3(X) &= \sum_{1\le k\le X} \tau(k)c_q(k),\\
H_4(N) &= \sum_{1\le k\le N-1} \tau(k)c_q(k)\log(N-k),\\
H_5(N) &= \sum_{1\le k\le N-1} \tau(k)c_q(k)\log^2(N-k),\\
H_6(N) &= \sum_{1\le k\le N-1} k\tau(k)\log(N-k),\\
H_7(N) &= \sum_{1\le k\le N-1} k\tau(k)\log^2(N-k),\\
H_8(X) &= \sum_{1\le k\le X} k\tau(k)c_q(k),\\
H_9(N) &= \sum_{1\le k\le N-1} k\tau(k)c_q(k)\log(N-k),\\
H_{10}(N) &= \sum_{1\le k\le N-1} k\tau(k)c_q(k)\log^2(N-k).
\end{aligned}$$

Assume $q>1$. We now compute the first sum in (4.44). By partial summation, we have

$$H_1(N) = \int_1^{N-1} \frac{t}{N-t}\log t dt + (2\gamma-1)\int_1^{N-1} \frac{t}{N-t}dt + O(N^{1/2}\log N).$$

By the first part of Lemma 29, this is equal to

$$N\left(\log^2 N - \log N - \frac{\pi^2}{6} + 1\right) + (2\gamma-1)(N\log N - N) + O(N^{1/2}\log N).$$

Thus,

$$H_1(N) = N\log^2 N + (2\gamma-2)N\log N + \left(\frac{\pi^2}{6} - 2\gamma\right)N + O(N^{1/2}\log N).$$

Similar, by both parts of Lemma 29, we get

$$\begin{aligned} H_2(N) &= \int_1^{N-1} \frac{1}{N-t}\left(t\log^2 t + (2\gamma-2)t\log t + \left(\frac{\pi^2}{6}-2\gamma\right)t + O(t^{1/2}\log t)\right)dt \\ &= N\log^3 N + (2\gamma-4)N\log^2 N + \left(4-4\gamma-\frac{\pi^2}{6}\right)N\log N \\ &+ \left(2\zeta(3) - 2 - (2\gamma-2)\left(\frac{\pi^2}{6}-1\right) - \frac{\pi^2}{6} + 2\gamma\right)N + O\left(N^{1/2}\log^2 N\right). \end{aligned}$$

We now estimate $H_3(X)$. We have

$$H_3(X) = \sum_{1\le k\le X}\sum_{d|k} c_q(k) = \sum_{1\le d\le X}\sum_{1\le m\le X/d} c_q(dm).$$

By Lemma 26, the inner sum is $\ll qd^{-1}X^{1/2}\log^2 X$. Thus,

$$H_3(X) \ll qX^{1/2}\log^2 X \sum_{1\le d\le X} d^{-1} \ll qX^{1/2}\log^3 X.$$

Using this we get, by partial summation,

$$H_4(N) = O(qN^{1/2}\log^4 N)$$

and

$$H_5(N) = O(qN^{1/2}\log^5 N).$$

By partial summation we can easily obtain

Lemma 31

$$H_6(N) = \frac{1}{2}(N-1)^2\log^2(N-1) + \lambda_1(N-1)^2\log(N-1) + \lambda_2(N-1)^2 + O(N^{3/2}\log N),$$

$$\begin{aligned} H_7(N) &= \frac{1}{2}(N-1)^2\log^3(N-1) + \lambda_3(N-1)^2\log^2(N-1) + \lambda_4(N-1)^2\log(N-1) \\ &+ \lambda_5(N-1)^2 + O(N^{3/2}\log^3 N), \\ H_8(N) &= O(N^{3/2}\log^3 N), \\ H_9(N) &= O(N^{3/2}\log^4 N), \\ H_{10}(N) &= O(N^{3/2}\log^4 N), \end{aligned}$$

with numerical constants λ_j's.

Here we have

$$\lambda_1 = \gamma - 1/2,$$
$$\lambda_2 = \frac{\pi^2}{12} - \frac{1}{2}\gamma - \frac{3}{4},$$
$$\lambda_3 = \gamma - 5/4,$$

etc.

Collecting the $w_j(1)$'s from (4.42) and the H_j's above, we deduce the following

Lemma 32 *There is an explicit polynomial $P_5(\log N)$ of degree 5 in $\log N$ such that the $q = 1$ contribution in $Q_{11}(N)$ from (4.43) is given by*

$$\sum_{j=1}^{6} w_j(1) \sum_{1\le k\le N-1} \tau(k)T_j(k,N) = N^2 P_5(\log X) + O(N^{3/2}\log^3 N).$$

Moreover, the $q > 1$ contributions in $Q_{11}(N)$ from (4.43) is at most $O(N^{3/2})$, and, consequently, from (4.39), (4.40), (4.41), and (4.43), we obtain that

$$Q_1(N) = N^2 P_8(\log N) + O(N^{599/300}).$$

With more effort, though tedious in details, one can calculate similar asymptotic expansions for $Q_2(N)$ to $Q_{10}(N)$ in (4.38). However, for our purpose, it suffices to bound the sums Q_2-Q_{10} and show that they are smaller than the leading term $N^2\log^8 N$. Indeed, by (4.13) and orthogonality of the Ramanujan sum $c_q(b)$, we have that

$$Q_2(N), \cdots, Q_{10}(N) \ll N^2 \log^6 N. \tag{4.45}$$

We demonstrate one such bound for $Q_2(N)$–the other bounds can be obtained similarly. Suppose first that $q = 1$. We have, in this case, $\tilde{A}(\ell, b) = \ell^{-1}$ for any b, and hence

$$\sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \tilde{A}(\ell,b)^2 = \sum_{1\le\ell\le N}\sum_{1\le b\le\ell} \ell^{-2} \ll \log N. \tag{4.46}$$

Assume next $q_1, q_2 > 1$. Suppose $(q_1, q_2) = 1$. Then

$$\sum_{1\le b\le \ell} c_{q_1}(b)c_{q_2}(b) = \sum_{1\le b\le \ell} c_{q_1q_2}(b) \ll q_1q_2.$$

From this and (4.13), we get

$$\begin{aligned}\sum_{1\le b\le \ell} \tilde{A}^2(\ell,b) &= \ell^{-2}\sum_{q_1|\ell}\sum_{q_2|\ell} q_1^{-1}q_2^{-1}\sum_{1\le b\le \ell} c_{q_1}(b)c_{q_2}(b)\log^2 q_1\log^2 q_2\\ &= \ell^{-2}\sum_{q_1|\ell}\log^2 q_1\sum_{q_2|\ell}\log^2 q_2 \ll \ell^{-2}\log^4\ell\end{aligned}$$

and, hence,

$$\sum_{1\le \ell\le N}\sum_{1\le b\le \ell}\tilde{A}(\ell,b)^2 \ll \sum_{1\le \ell\le N}\ell^{-2}\log^4\ell \ll \log N. \tag{4.47}$$

It remains to consider the case where $(q_1, q_2) > 1$. Let $q_0 = [q_1, q_2]$. By orthogonality of $c_q(b)$ we have that

$$\sum_{1\le b\le q_0} c_{q_1}(b)c_{q_2}(b) = \begin{cases} q_0\varphi(q_0), & \text{if } q_1 = q_2,\\ 0, & \text{otherwise.}\end{cases}$$

Thus,

$$\sum_{1\le b\le \ell} c_{q_1}(b)c_{q_2}(b) \ll \begin{cases} \ell\varphi(q_0), & \text{if } q_1 = q_2,\\ q_1q_2, & \text{otherwise,}\end{cases}$$

which gives

$$\sum_{1\le \ell\le N}\sum_{1\le b\le \ell}\tilde{A}^2(\ell,b) \ll \begin{cases} \sum_{1\le \ell\le N}\ell^{-1}\tau(\ell) \ll \log^2 N, & \text{if } q_1 = q_2,\\ \sum_{1\le \ell\le N}\ell^{-1} \ll \log N, & \text{if } q_1 \ne q_2.\end{cases}$$

This, together with (4.46) and (4.47), give that $Q_2(N)$ is at most $O(N^2\log^6 N)$, verifying (4.45) for $Q_2(N)$.

As mentioned before, the estimates in (4.45) are crude simply for the purpose of showing they do not contribute to the leading term. It is possible, by procedures analogous to the computations for $Q_1(N)$ and $\sum_k W_q(k,N)$ demonstrated in the proof, to compute explicitly a polynomial $P_6(\log N)$ of degree 6 in $\log N$ such that

$$Q_2(N) + \cdots + Q_{10}(N) = N^2 P_6(\log N) + O(N^{599/300}).$$

We conclude, therefore, that $Q(N)$, which is the left-hand side of (4.5), is given by

$$N^2 P_8(\log N) + O(N^{2-1/300}),$$

which gives the right-hand side of (4.5). This completes the proof of the theorem.

Chapter 5

Distribution of the ternary divisor function in arithmetic progressions on average

5.1 Introduction and statement of result

Theorem 6 *For any $\epsilon > 0$, we have*

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} \left| \sum_{\substack{n\le X\\ n\equiv a (mod\ q)}} \tau_3(n) - \frac{X}{\varphi(q)} P_2(\log X) \right| \ll \begin{cases} X^{5/6+\epsilon}q^{1/4}, & \text{if } X^{1/2} < q < X^{2/3}, \\ X^{7/9+\epsilon}q^{1/2} & \text{if } X^{1/4} < q < X^{4/9}, \\ X^{11/12+\epsilon}, & \text{if } 1 \le q \le X^{1/4}, \end{cases} \tag{5.1}$$

where $P_2(\log X)$ is a polynomial of degree 2 in $\log X$ with coefficients depending on q, and the implied constant depends only on ϵ.

5.2 Lemmas

Lemma 33 *([2, Lemma 2.1]) For any $Y < Z$ and $\beta \in \mathbb{Z}$,*

$$\sum_{\substack{Y<m\le Z\\ m\equiv\beta (mod\ q)}} 1 = \frac{Z-Y}{q} + \sum_{1\le|h|\le H} c_{Y,Z}(h) e\left(-\frac{h\beta}{q}\right) + O\left(\theta_H\left(\frac{Y-\beta}{q}\right) + \theta_H\left(\frac{Z-\beta}{q}\right)\right),$$

where

$$c_{Y,Z}(\xi) = \frac{1}{2\pi i\xi}\left(e\left(\frac{Z\xi}{q}\right) - e\left(\frac{Y\xi}{q}\right)\right) = \frac{1}{q}\int_Y^Z e\left(\frac{y\xi}{q}\right)dy.$$

Lemma 34 *([2, Lemma 2.2]) Let* $(r,q)=1$, $0<H<q$, *and* $z\in\mathbb{R}$. *Then*

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} \theta_H\left(\frac{z-ra}{q}\right) \ll H^{-1}q^{1+\epsilon} + Hq^{-1}.$$

We use the following result of Cochrane and Shi [11].

Lemma 35 *[11, Theorem 2] Let* $\mathcal{H}_i$ *be intervals of length* H_i, $i=1,2,3,4$. *Then we have, for any* $q\ge 1$,

$$\begin{aligned}&\#\{(x_1,x_2,x_3,x_4)\in\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3\times\mathcal{H}_4, x_1x_2x_3x_4\equiv 0(mod\ q) : x_1x_2\equiv x_3x_4(mod\ q)\}\\ &\quad= \frac{H_1H_2H_3H_4}{q} + O\left((H_1H_2H_3H_4)^{1/2}q^{\epsilon}\right)\end{aligned}$$

for any $\epsilon>0$.

We apply Lemma 35 above to generalize [2, Lemma 2.3].

Lemma 36 *Let*

$$Q = q\prod_{p|q} p^{-1}.$$

Then for all $Y_1<Z_1$, $Y_2<Z_2$, *and* $d,h\ge 1$ *with* $d|q$ *and* $(h,q)=1$, *we have*

$$\sum_{\substack{1\le a\le q\\(a,q)=1}}\left|\sum_{\substack{Y_1<b\le Z_1\\Y_2<c\le Z_2\\(bc,q)=1}} e\left(\frac{adh\overline{bc}}{q}\right)\right| \ll \begin{cases} V_1V_2d + L_1L_2q^{1/2}d^{1/2} + (L_1L_2)^{1/2}q + q, & \text{if } Q|d\\ L_1L_2q^{1/2}d^{1/2} + (L_1L_2)^{1/2}q + q, & \text{otherwise,}\end{cases}$$

where $V_1 = q[(Z_1-Y_1)/q]$, $V_2 = q[(Z_2-Y_2)/q]$, $L_1 = q\{(Z_1-Y_1)/q\}$, *and* $L_2 = q\{(Z_2-Y_2)/q\}$.

Proof: The proof of this lemma follows the proof of Lemma 2.2 in [2], except that in [2, (2.17) pg. 6] we apply Lemma 35 to estimate the sum

$$\sum_{\substack{Y_1+V_1<b_1,c_1\leq Z_1\\ Y_2+V_2<b_2,c_2\leq Z_2\\ (b_1b_2c_1c_2,q)=1}} e\left(\frac{ah(\overline{b_1b_2}-\overline{c_1c_2})}{f}\right)$$

by

$$\sum_{\substack{Y_1+V_1<b_1,c_1\leq Z_1\\ Y_2+V_2<b_2,c_2\leq Z_2\\ b_1b_2\equiv c_1c_2 (\mathrm{mod}\ f)}} 1 = \frac{L_1L_2}{f} + O((L_1L_2)^{1/2}f^{\epsilon}).$$

■

5.3 Proof of Theorem 6

We follow the proof of Banks, Heath-Brown, and Shparlinski [2, Theorem 3.1]. Put

$$T(X,q) = \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} \left| \sum_{\substack{n\leq X\\ n\equiv a (\mathrm{mod}\ q)}} \tau_3(n) - \frac{S^*(X,q)}{\varphi(q)}, \right|$$

where

$$S^*(X,q) = \sum_{\substack{n\leq X\\ (n,q)=1}} \tau_3(n).$$

For arbitrary Δ in the range $0 < \Delta < 1/2$, let

$$M = \left\{\frac{1}{2}(1+\Delta)^j | 0 \leq j \leq R\right\},$$

where

$$R = \left[\frac{\log(2X)}{\log(1+\Delta)}\right] \ll \Delta^{-1}\log X.$$

Then

$$\sum_{\substack{n\leq X\\ n\equiv a (\mathrm{mod}\ q)}} \tau_3(n) = \sum_{\substack{abc\leq X\\ \alpha bc\equiv a (\mathrm{mod}\ q)}} 1 = \sum_{A,B,C\in\mathcal{M}} D_a(A,B,C),$$

where

$$D_a(A,B,C)=\sum_{\substack{A<\alpha\le A(1+\Delta)\\ B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ \alpha bc\le X\\ \alpha bc\equiv a(\mathrm{mod}\ q)}}1. \tag{5.2}$$

Similarly,

$$S^*(X,q)=\sum_{A,B,C\in\mathcal{M}}D^*(A,B,C)$$

where

$$D^*(A,B,C)=\sum_{\substack{A<\alpha\le A(1+\Delta)\\ B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ \alpha bc\le X\\ (\alpha bc,q)=1}}1. \tag{5.3}$$

Following [31], we call a triple (A,B,C) *good* if $ABC\le X(1+\Delta)^{-3}$, and *bad* otherwise. Note that, for good triples (A,B,C), the condition $\alpha bc\le X$ in (5.2) and (5.3) are redundant. We have

$$\sum_{(A,B,C)\ \text{bad}}D_a(A,B,C)\ll(\Delta Xq^{-1}+1)X^{\epsilon},$$

$$\sum_{(A,B,C)\ \text{bad}}D^*(A,B,C)\ll(\Delta X+1)X^{\epsilon}.$$

Consequently,

$$T(X,q)\ll(\Delta X+q)X^{\epsilon}+\sum_{\substack{(A,B,C)\ \text{good}\\ A\ge B\ge C}}\ \sum_{\substack{1\le a\le q\\ (a,q)=1}}\left|D_a(A,B,C)-\frac{D^*(A,B,C)}{\varphi(q)}\right|. \tag{5.4}$$

For any good triple (A,B,C) with $A\ge B\ge C$, we have, by Lemma 33 implies that

$$D_a(A,B,C)-\sum_{\substack{B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ (bc,q)=1}}\Delta Aq^{-1}\ll E_1(A,B,C,a)+E_2(A,B,C,a),$$

where

$$E_1(A,B,C,a) = \left| \sum_{\substack{B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ (bc,q)=1}} \sum_{1\le|h|\le H} c_A(h) e\left(-\frac{h\overline{bc}a}{q}\right)\right|,$$

$$E_2(A,B,C,a) = \sum_{\substack{B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ (bc,q)=1}} \left(\theta_H\left(\frac{A-\overline{bc}a}{q}\right) + \theta_H\left(\frac{A(1+\Delta)-\overline{bc}a}{q}\right)\right),$$

and

$$c_A(\xi) = \frac{1}{2\pi i\xi}\left(e\left(\frac{A(1+\Delta)\xi}{q}\right) - e\left(\frac{A\xi}{q}\right)\right) = q^{-1}\int_A^{A(1+\Delta)} e\left(\frac{y\xi}{q}\right) dy.$$

We have the trivial bound

$$c_A(\xi) \ll \min\{|\xi|^{-1}, \Delta A q^{-1}\}. \tag{5.5}$$

Put $H = q-1$ and $J = [\log(\Delta A)] - 1$. We have

$$E_1(A,B,C,a) \le \sum_{j=0}^{J+1} E_{1,j}(A,B,C,a),$$

where

$$E_{1,j}(A,B,C,a) = \left| \sum_{\substack{B<b\le B(1+\Delta)\\ C<c\le C(1+\Delta)\\ (bc,q)=1}} \sum_{h\in\mathcal{H}_j} c_A(h) e\left(-\frac{h\overline{bc}a}{q}\right)\right|,$$

and the h summation is over the sets

$$\mathcal{H}_j = \begin{cases} \left\{h \middle| 1\le|h|\le \frac{q}{\Delta A}\right\}, & \text{if } j=0,\\ \left\{h \middle| e^j\frac{q}{\Delta A} < |h| \le e^{j+1}\frac{q}{\Delta A}\right\}, & \text{if } j=1,\dots,J,\\ \left\{h \middle| e^{J+1}\frac{q}{\Delta A} < |h| \le q-1\right\} & \text{if } j=J+1. \end{cases}$$

By Cauchy's inequality, we have

$$\left(\sum_{\substack{1\le a\le q\\ (a,q)=1}} E_1(A,B,C,a)\right)^2 \ll q\log X \sum_{j=0}^{J+1} \sum_{\substack{1\le a\le q\\ (a,q)=1}} E_{1,j}(A,B,C,a)^2.$$

For each j, we have

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_{1,j}(A,B,C,a)^2 = \sum_{\substack{1\le a\le q\\(a,q)=1}} \sum_{\substack{B<b_1,b_2\le B(1+\Delta)\\C<c_1,c_2\le C(1+\Delta)\\(b_1b_2c_1c_2,q)=1}} \sum_{h_1,h_2\in\mathcal{H}_j} c_A(h_1)c_A(h_2)e\left(\frac{a(h_1\overline{b_1c_1}-h_2\overline{b_2h_2})}{q}\right)$$

$$\le q \sum_{\substack{B<b_1,b_2\le B(1+\Delta)\\C<c_1,c_2\le C(1+\Delta)\\(b_1b_2c_1c_2,q)=1}} \sum_{\substack{h_1,h_2\in\mathcal{H}_j\\h_1\overline{b_1c_1}\equiv h_2\overline{b_2c_2}(\text{mod } q)}} c_A(h_1)c_A(h_2).$$

By (5.5), $c_A(h) \ll e^{-j}\Delta A/q$ for each $h\in\mathcal{H}_j$. Therefore

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_{1,j}(A,B,C,a)^2 \ll e^{-2j}\frac{\Delta^2A^2}{q}T_j(B,C),$$

where $T_j(B,C)$ is the number of solutions $(h_1,h_2,b_1,b_2,c_1,c_2)$ of the congruence

$$h_1\overline{b_1c_1}\equiv h_2\overline{b_2c_2}(\text{mod } q)$$

with $B<b_1,b_2\le B(1+\Delta)$, $C<c_1,c_2\le C(1+\Delta)$, $(b_1b_2c_1c_2,q)=1$, and $h_1,h_2\in\mathcal{H}_j$. We have that

$$T_j(B,C) \ll \left(1+\frac{e^jq}{\Delta A}\right)(1+\Delta B)(1+\Delta C)\left(1+\frac{e^jBC}{\Delta A}\right)X^{\epsilon}$$

and we derive that

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_{1,j}(A,B,C,a)^2 \ll q^{-1}(1+\Delta B)(1+\Delta C)(e^{-j}\Delta A+q)(e^{-j}\Delta A+BC)X^{\epsilon}$$

$$\ll q^{-1}(1+\Delta B)(1+\Delta C)(\Delta A+q)(\Delta A+BC)X^{\epsilon}.$$

Hence,

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_1(A,B,C,a) \ll (1+\Delta B)^{1/2}(1+\Delta C)^{1/2}(\Delta A+q)^{1/2}(\Delta A+BC)^{1/2}X^{\epsilon}. \quad (5.6)$$

On the other hand, with $H = q-1$, we have

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_1(A,B,C,a) \le \sum_{\substack{d|q\\d<q}} \sum_{\substack{1\le|h|<q/d\\(h,q)=1}} |c_A(dh)| \sum_{\substack{1\le a\le q\\(a,q)=1}} \left| \sum_{\substack{B<b\le B(1+\Delta)\\C<c\le C(1+\Delta)\\(bc,q)=1)}} e\left(-\frac{dha\overline{bc}}{q}\right) \right|,$$

thus, by Lemma 36 and the bound (5.5), we derive that

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_1(A,B,C,a) \ll \begin{cases} (\Delta B\Delta C q^{1/2} + \Delta B^{1/2}C^{1/2}q + q)X^{\epsilon}, & \text{if } \Delta B, \Delta C < q,\\ (\Delta B\Delta C + \Delta C q^{3/2} + \Delta^{1/2}C^{1/2}q^{3/2} + q)X^{\epsilon}, & \text{if } \Delta B \ge q > \Delta C,\\ (\Delta B\Delta C + \Delta B q + q^{5/2})X^{\epsilon}, & \text{if } \Delta B, \Delta C \ge q. \end{cases} \tag{5.7}$$

We now estimate contributions from $E_2(A,B,C,a)$. By Lemma 34 with $H = q-1$, we get

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} E_2(A,B,C,a) \ll (1+\Delta B\Delta C)X^{\epsilon}.$$

Suppose first that $q > X^{1/3}$. We have, by (5.6) and (5.7),

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} \left| D_a(A,B,C) - \sum_{\substack{B<b\le B(1+\Delta)\\C<c\le C(1+\Delta)\\(bc,q)=1}} \Delta A q^{-1} \right| \ll E(A,B,C)X^{\epsilon}, \tag{5.8}$$

where

$$E(A,B,C) \ll \begin{cases} (1+\Delta^{1/2}B^{1/2})(1+\Delta^{1/2}C^{1/2})(\Delta^{1/2}A^{1/2}+B^{1/2}C^{1/2})q^{1/2}, & \text{if } \max\{\Delta A, C\} < q,\\ \Delta B\Delta C q^{1/2} + \Delta^{1/2}B^{1/2}\Delta^{1/2}C^{1/2}q + q, & \text{if } \Delta B < q,\\ \Delta B\Delta C + \Delta C q^{3/2} + \Delta^{1/2}C^{1/2}q^{3/2} + q, & \text{otherwise.} \end{cases}$$

We also have that

$$\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\left|\frac{1}{\varphi(q)}D^*(A,B,C)-\sum_{\substack{B<b\leq B(1+\Delta)\\C<c\leq C(1+\Delta)\\(bc,q)=1}}\Delta Aq^{-1}\right|\ll E(A,B,C)X^\epsilon.$$

This, together with (5.8), substituted into (5.4) yields

$$T(X,q)\ll(\Delta X+q)X^\epsilon+\sum_{\substack{(A,B,C)\text{ good}\\A\geq B\geq C}}E(A,B,C)X^\epsilon. \tag{5.9}$$

We have

$$\sum_{\substack{(A,B,C)\text{ good}\\A\geq B\geq C}}E(A,B,C)\leq U_1+U_2+U_3+U_4,$$

where

$$\begin{aligned}
U_1&=\sum_{BC\leq\Delta X/q}\sum_{A<\Delta^{-1}q}(1+\Delta^{1/2}B^{1/2}\Delta^{1/2}C^{1/2})(\Delta^{1/2}A^{1/2}+B^{1/2}C^{1/2})q^{1/2},\\
U_2&=\sum_{BC\leq\Delta X/q}\sum_{\Delta^{-1}q\leq A\leq X/BC}(\Delta B\Delta C+\Delta B\Delta C)q^{1/2},\\
U_3&=\sum_{\Delta X/q<BC\leq X^{2/3}}\sum_{A\leq X/BC}(1+\Delta B^{1/2}C^{1/2})(\Delta^{1/2}A^{1/2}+B^{1/2}C^{1/2})q^{1/2},\\
U_4&=\sum_{C<X^{1/3}}\sum_{X^{1/3}<B<X^{1/2}}\sum_{A\leq X/BC}\Delta B\Delta Cq^{1/2}.
\end{aligned}$$

Using that M is a geometric series, we have the trivial estimates

$$\sum_{\substack{C\in M\\C\leq Y}}C\ll\Delta^{-1}Y,\quad\sum_{\substack{C\in M\\C\leq Y}}C^{1/2}\ll\Delta^{-1}Y^{1/2},\quad\sum_{\substack{C\in M\\C\leq Y}}1\ll\Delta^{-1}X^\epsilon.$$

Thus, we derive that

$$\begin{aligned} U_1 &\ll (\Delta^{-1/2}q^{-1/2}X + \Delta^{-3/2}X^{1/2})X^{\epsilon}, \\ U_2 &\ll (\Delta q^{-1/2}X + \Delta q^{-1/2}X)X^{\epsilon}, \\ U_3 &\ll \Delta^{-1}q^{1/2}X^{2/3+\epsilon}, \\ U_4 &\ll \Delta^{-1}qX^{1/3}X^{\epsilon}. \end{aligned}$$

Choosing

$$\Delta = \begin{cases} q^{1/4}X^{-1/6}, & \text{if } X^{1/2} < q < X^{2/3}, \\ q^{1/2}X^{-2/9}, & \text{if } X^{1/4} < q < X^{4/9}, \end{cases}$$

we obtain, via (5.9), the estimate

$$T(X,q) \ll \begin{cases} X^{5/6+\epsilon}q^{1/4}, & \text{if } X^{1/2} < q < X^{2/3}, \\ q^{1/2}X^{7/9+\epsilon}, & \text{if } X^{1/4} < q \leq X^{4/9}. \end{cases} \tag{5.10}$$

Similarly, we derive, for $q \leq X^{1/3}$, that

$$\sum_{\substack{(A,B,C) \text{ good} \\ A \geq B \geq C}} E(A,B,C)$$

$$\ll \begin{cases} X^{\epsilon}(X^{5/6} + \Delta^{-1}q^{1/2}X^{5/6} + \Delta^{-3/2}q^{3/2}X^{1/6} + \Delta^{-2}qX^{5/12} + \Delta^{-3}q), & \text{if } q \geq \Delta X^{1/2}, \\ X^{\epsilon}(\Delta^{-1}X^{5/6} + \Delta^{-2}q^{3/2}X^{1/3} + \Delta^{-5/2}q^{3/2}X^{1/6} + \Delta^{-3}q), & \text{if } \Delta X^{1/3} < q < \Delta X^{1/2}, \\ X^{\epsilon}(\Delta^{-1}X^{5/6} + \Delta^{-2}qX^{1/2} + \Delta^{-3}q^{5/6}), & \text{if } q \leq \Delta X^{1/3}. \end{cases}$$

Thus, choosing $\Delta = X^{-1/12}$, we obtain

$$T(X,q) \ll X^{11/12+\epsilon}$$

valid for $q \leq X^{1/4}$.

Hence, recalling Lemma 51, with this estimate and (5.10), the desired estimate (5.1) follows immediately.

Chapter 6

On correlations of divisor sums

6.1 Averaged correlations

Let

$$D_3(X,h) = \sum_{1\le n\le X-h} \tau_3(n)\tau_3(n+h).$$

In this chapter, we derive an asymptotic for the average correlation sum

$$\sum_{h\le X-1} D_3(X,h).$$

Theorem 7 *There are numerical constants c_0, c_1, c_2, c_3 such that*

$$\begin{aligned}\sum_{1\le h\le X-1}\ \sum_{1\le n\le X-h} &\tau_3(n)\tau_3(n+h)\\ &= X^2\left(\frac{\log^4 X}{4} + c_3\log^3 X + c_2\log^2 X + c_1\log X + c_0\right) + O(X^{3/2+\epsilon}),\end{aligned}$$

for any $\epsilon > 0$.

6.2 Lemmas

We start with some lemmas.

Lemma 37 (Voronoi) *We have*

$$\sum_{n \leq X} \tau_3(n) = X \left(\frac{\log^2 X}{2} + a \log X + b \right) + O(X^{1/2+\epsilon}),$$

where

$$a = 3\gamma - 1$$

and

$$b = 3\gamma^2 - 3\gamma + 3\gamma_1 + 1.$$

By partial summations, we can derive the following variants.

Lemma 38 *We have*

$$\sum_{n \leq X} \tau_3(n) n = X^2 \left(\frac{1}{4} \log^2 X + \tilde{a} \log X + \tilde{b} \right) + O(X^{3/2+\epsilon}),$$

where

$$\tilde{a} = \frac{a}{2} + \frac{1}{4}$$

and

$$\tilde{b} = \frac{a}{4} + \frac{b}{2} - \frac{1}{8}.$$

Lemma 39 *We have*

$$\sum_{n \leq X} \tau_3(n) n \log n = X^2 \left(\frac{\log^3}{4} + \left(\tilde{a} - \frac{1}{8} \right) \log^2 X + \left(\tilde{b} - \frac{\tilde{a}}{2} + \frac{1}{8} \right) \log X + \frac{\tilde{a}}{4} - \frac{\tilde{b}}{2} - \frac{1}{16} \right)$$
$$+ O\left(X^{3/2+\epsilon} \right).$$

Lemma 40 *We have*

$$\sum_{n \leq X} \tau_3(n) n \log^2 n = X^2 \left(\frac{\log^4 X}{4} + \left(\tilde{a} - \frac{1}{4} \right) \log^3 X + \left(\tilde{b} - \tilde{a} + \frac{3}{8} \right) \log^2 X \right)$$
$$+ X^2 \left(\left(\tilde{a} - \tilde{b} - \frac{3}{8} \right) \log X + \frac{\tilde{b}}{2} - \frac{\tilde{a}}{2} + \frac{3}{16} \right) + O(X^{3/2+\epsilon}).$$

6.3 Proof of Theorem 7

Proof: We have

$$\sum_{1\le h\le X-1}\ \sum_{1\le n\le X-h}\tau_3(n)\tau_3(n+h)=\sum_{1\le n\le X-1}\tau_3(n)\sum_{n+1\le h\le X}\tau_3(h).$$

By Lemma 37, the most inner sum of the above is equal to

$$\begin{aligned}\sum_{h\le X}\tau_3(h)-\sum_{h\le n}\tau_3(h)&=X\left(\frac{\log^2 X}{2}+a\log X+b\right)+O(X^{1/2+\epsilon})\\&-n\left(\frac{\log^2 n}{2}+a\log n+b\right)+O(n^{1/2+\epsilon}).\end{aligned}$$

Plugging this in, we see that we need estimates for the following sums

$$\sum_{n\le X-1}\tau_3(n)n\log^2 n,$$

$$\sum_{n\le X-1}\tau_3(n)n\log n,$$

and

$$\sum_{n\le X-1}\tau_3(n)n.$$

These can be derived straightfowardly using partial summations and Lemma 37. This gives the main term, after some algebra. The error term is bounded by

$$O\left(X^{1/2+\epsilon}\sum_{n\le X-1}\tau_3(n)\right)=O(X^{3/2+\epsilon}).$$

The ϵ in the error term can be made more precise if needed, and X^ϵ is probably of the form $\log^C X$ for some C. ■

Remark 6 *If we only average h up to $H\le X-1$, then the appropriate modification is*

$$\begin{aligned}\sum_{1\le h\le H}\ \sum_{1\le n\le X-h}\tau_3(n)\tau_3(n+h)&=\sum_{1\le n\le X-1}\tau_3(n)\sum_{n+1\le h\le \min\{H+n,X\}}\tau_3(h)\\&=\sum_{1\le n\le X-H}\tau_3(n)\sum_{n+1\le h\le H+n}\tau_3(h)+\sum_{X-H+1\le n\le X-1}\tau_3(n)\sum_{n+1\le h\le X}\tau_3(h),\end{aligned}$$

which can be treated similarly as with the case $H=X-1$.

Chapter 7

On the zeros of the derivative of the Riemann zeta function near the critical line

7.1 Introduction

Let $\rho' = \beta' + i\gamma'$ denote the zeros of $\zeta'(s)$, $s = \sigma + it$. The distribution of zeros of $\zeta'(s)$ is intimately related to those of $\zeta(s)$. For $\nu \in \mathbb{R}$, Soundararajan defined

$$m^{-}(\nu) = \liminf_{T\to\infty} \frac{1}{N_1(T)} \sum_{\substack{\beta' \leq 1/2+\nu/\log T \\ 0<\gamma'\leq T}} 1, \tag{7.1}$$

where $N_1(T)$ is the number of zeros of zeta prime with $0 < \gamma' \leq T$. The behavior of the function $m^{-}(\nu)$ determines the horizontal distribution of the zeros of zeta prime near the critical line. On RH, Soundararajan showed that $m^{-}(\nu) > 0$ for $\nu > 2.6$. In 2001 Zhang [86] showed, unconditionally, that $m^{-}(\nu) > 0$ for ν sufficiently large. In this chapter, we give a quantitative lower bound, complementing the result in [86, Theorem 1].

Theorem 8 *For $\nu > 10^{22}$, we have*

$$m^{-}(\nu) > 10^{-83}.$$

7.2 Proof of Theorem 8

We give a sketch of proof. We follow Heath-Brown's note 9.26 in Titchmarsh [76]. Let $0 < h < 1$, $X = T^\epsilon$, $\epsilon > 0$, $k \in \mathbb{N}$,

$$Q(t) = S(t) - \frac{1}{\pi}\sum_{p \le X} \frac{\sin(t\log p)}{\sqrt{p}}, \tag{7.2}$$

$$P(t) = \frac{1}{\pi}\Im \sum_{p \le X} p^{-\frac{1}{2}-it}(p^{-ih} - 1), \tag{7.3}$$

and

$$U(t) = Q(t+h) - Q(t),$$

so that $S(t+h) - S(t) = U(t) + P(t)$.

By Montgomery, Vaughan, and Vaaler's generalization [61, Corollary, page 36] of Hilbert's inequality, we deduce that

$$\int_T^{T+H} \left| \sum_r a_r r^{-it} \right|^2 dt \le H \sum_r |a_r|^2 + 336\theta \sum_r r|a_r|^2,$$

for some θ, $-1 \le \theta \le 1$. It follows from this that

$$\left| \int_T^{T+H} \left(\sum_n a_n n^{-it} \right) \overline{\left(\sum_n b_n n^{-it} \right)} - H \sum_n a_n \overline{b_n} \right| \le 336 \left(\sum_n n|a_n|^2 \right)^{1/2} \left(\sum_n n|b_n|^2 \right)^{1/2}.$$

From the above inequality, it can be shown that

$$\left| \int_T^{T+H} \{f(t)\}^2 dt - \frac{1}{2} H \sum_p |a_p|^2 \right| \le 672 \sum_p p|a_p|^2 \tag{7.4}$$

and

$$\left| \int_T^{T+H} \{f(t)\}^4 dt - \frac{3}{2} H \sum_{p,q} |a_{pq}|^2 \right| \le 4032 \left(\sum_p p|a_p|^2 \right)^2, \tag{7.5}$$

where $f(t)$ denotes either the real or imaginary part of $\sum_p a_p p^{-it}$. For the derivation, see the proofs of Lemmas 1 and 2 of Tsang [78].

In a way analogous to the proof of Theorem 4 of Tsang [78], we get by applying (7.4) and (7.5) to $f(t) = P(t)$ that

$$\left| \int_T^{T+H} \{P(t)\}^2 dt \right| \leq \pi^{-2} H \log(2 + h \log T) \tag{7.6}$$

and

$$\left| \int_t^{T+H} \{P(t)\}^4 dt - \frac{3}{\pi^4} H \log^2(2 + h \log T) \right| \leq 672 H \log(2 + h \log T). \tag{7.7}$$

From Lemma 7 on page 440 of Karatsuba and Korolëv [49], we deduce that

$$\int_T^{T+H} |Q(t)|^2 dt \leq \left(\frac{e^{37}}{10^{-12}\pi^2} \right) H < 10^{29} H \tag{7.8}$$

and

$$\int_T^{T+H} |Q(t)|^4 dt \leq \left(\frac{e^{37}4}{10^{-12}\pi^2} \right)^2 H < 10^{58} H. \tag{7.9}$$

Hence, by (7.6) and (7.8),

$$\begin{aligned}
\int_T^{T+H} \{S(t+h) - S(t)\}^2 dt &= \int_T^{T+H} \{U(t) + P(t)\}^2 dt \\
&\leq \int_T^{T+H} \{U(t)\}^2 dt + \int_T^{T+H} \{P(t)\}^2 dt + 2 \left(\int_T^{T+H} \{P(t)\}^2 dt \right)^{1/2} \left(\int_T^{T+H} \{U(t)\}^2 dt \right)^{1/2} \\
&< 10^{30} H + \pi^{-2} H \log(2 + h \log T) + 2 \left(\pi^{-2} H \log(2 + h \log T) \right)^{1/2} \left(10^{30} H \right)^{1/2} \\
&< H(10^{30} + \pi^{-2} \log(2 + h \log T)) + 10^{15} H \log^{1/2}(2 + h \log T).
\end{aligned}$$

Similarly, by (7.7) and (7.9) and Hölder inequality, we get

$$\int_T^{T+H}\{S(t+h)-S(t)\}^4dt = \int_T^{T+H}\{U(t)+P(t)\}^4dt$$

$$\leq \int_T^{T+H}\{U(t)\}^4dt + \int_T^{T+H}\{P(t)\}^4dt + 4\left(\int_T^{T+H}\{U(t)\}^4\right)^{1/4}\left(\int_T^{T+H}\{P(t)\}^4\right)^{3/4}$$

$$+4\left(\int_T^{T+H}\{U(t)\}^4\right)^{3/4}\left(\int_T^{T+H}\{P(t)\}^4\right)^{1/4} + 6\left(\int_T^{T+H}\{U(t)\}^4\right)^{1/2}\left(\int_T^{T+H}\{P(t)\}^4\right)^{1/2}$$

$$< H\left(10^{60}+\frac{3}{\pi^4}\log^2(2+h\log T)\right) + 10^{16}H\log^{3/2}(2+h\log T).$$

For $2\pi M \leq h\log T \leq 4\pi M$, the inequalities

$$\left|\int_T^{T+H}\{S(t+h)-S(t)\}^2dt - H(10^{30}+\pi^{-2}\log(2+h\log T))\right| \leq 10^{15}H\log^{1/2}(2+h\log T) \tag{7.10}$$

and

$$\left|\int_T^{T+H}\{S(t+h)-S(t)\}^4dt - H\left(10^{60}+\frac{3}{\pi^4}\log^2(2+h\log T)\right)\right| \leq 10^{16}H\log^{3/2}(2+h\log T) \tag{7.11}$$

hold if

$$M > e^{10^{36}}.$$

which we henceforth assume. With this value of M and taking $H = T$, (7.10) and (7.11) yield

$$\int_T^{2T}|S(t+h)-S(t)|^2dt \geq T$$

and

$$\int_T^{2T}|S(t+h)-S(t)|^4dt \leq 10^{36}T$$

for T large. By Hölder inequality we have

$$\int_T^{2T}|S(t+h)-S(t)|^2dt \leq \left(\int_T^{2T}|S(t+h)-S(t)|dt\right)^{\frac{2}{3}}\left(\int_T^{2T}|S(t+h)-S(t)|^4dt\right)^{\frac{1}{3}},$$

so that

$$\int_T^{2T} |S(t+h)-S(t)|dt \geq \frac{\left(\int_T^{2T} |S(t+h)-S(t)|^2 dt\right)^{3/2}}{\left(\int_T^{2T} |S(t+h)-S(t)|^4 dt\right)^{1/2}} \geq \frac{T^{3/2}}{\left(10^{36}\right)^{1/2} T^{1/2}} = 10^{-18}T.$$

We have that

$$N(t) = L(t) + S(t) \quad (t > t_0)$$

with

$$L(t) = \frac{t}{2\pi}\log t - \frac{1+\log 2\pi}{2\pi}t + \frac{7}{8} + O\left(\frac{1}{t}\right),$$

we obtain

$$L(t+h)-L(t) = \frac{1}{2\pi}\left(t\log\frac{t+h}{t} + h\log(t+h)\right) - \frac{1+\log 2\pi}{2\pi}h + O(1/t) = \frac{h\log T}{2\pi} + O\left(\frac{1}{\log T}\right),$$

and hence

$$S(t+h)-S(t) = N(t+h)-N(t) - \frac{h\log T}{2\pi} + O\left(\frac{1}{\log T}\right), \tag{7.12}$$

for $T \leq t \leq 2T$, whence

$$\int_T^{2T} \left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right| dt = \int_T^{2T} |S(t+h)-S(t)|dt + O\left(\frac{T}{\log T}\right) \geq 10^{-19}T.$$

We proceed to write $h = 2\pi\lambda/\log T$, $1 \leq \lambda \leq 2$, and

$$\delta(t,\lambda) = N\left(t + \frac{2\pi\lambda}{\log T}\right) - N(t) - \lambda,$$

so that

$$N(t+h)-N(t)-\frac{h\log T}{2\pi} = \sum_{m=0}^{M-1} \delta\left(t + \frac{2\pi m\lambda}{\log T}, \lambda\right).$$

Thus,

$$\begin{aligned} 10^{-19}T &\leq \int_T^{2T} \left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right| dt \\ &= \sum_{m=0}^{M-1} \int_T^{2T} \left|\delta\left(t+\frac{2\pi m\lambda}{\log T},\lambda\right)\right| = \sum_{m=0}^{M-1} \int_{T+\frac{2\pi m\lambda}{\log T}}^{2T+\frac{2\pi m\lambda}{\log T}} |\delta(t,\lambda)|dt \\ &= \int_T^{2T} |\delta(t,\lambda)|dt, \end{aligned}$$

and hence

$$\int_T^{2T} |\delta(t,\lambda)|dt \geq 10^{-19}T \tag{7.13}$$

uniformly for $1 \leq \lambda \leq 2$. Let I denote the subset of $[T, 2T]$ on which $N\left(t + \frac{2\pi\lambda}{\log T}\right) = N(t)$, then

$$|\delta(t,\lambda)| \leq \begin{cases} \delta(t,\lambda) + 2\lambda, & (t \in I), \\ \delta(t,\lambda) + 2\lambda - 2, & (t \in [T,2T] - I), \end{cases}$$

so that (8.7) yields

$$10^{-19}T \leq \int_T^{2T} |\delta(t,\lambda)|dt = \int_T^{2T} \delta(t,\lambda)dt + (2\lambda - 2)T + 2m(I),$$

where $m(I)$ is the measure of I. However,

$$\delta(t,\lambda) = N\left(t + \frac{2\pi\lambda}{\log T}\right) - N(t) - \lambda = S\left(t + \frac{2\pi\lambda}{\log T}\right) - S(t) + O\left(\frac{1}{\log T}\right)$$

and hence

$$\int_T^{2T} \delta(t,\lambda)dt = \int_T^{2T} \left(S\left(t + \frac{2\pi\lambda}{\log T}\right) - S(t)\right)dt + O\left(\frac{T}{\log T}\right) = O(\log T),$$

whence

$$10^{-19}T \leq (2\lambda - 2)T + 2m(I)$$

and

$$m(I) \geq \frac{1}{2}\left(10^{-19} - 2\lambda + 2\right)T > 10^{-20}T,$$

if we choose $\lambda = 1 + 10^{-20} > 1$. Thus, if

$$S = S_1 = \left\{n : T \leq \gamma_n \leq 2T, \gamma_{n+1} - \gamma_n \geq \frac{2\pi\lambda}{\log T}\right\},$$

then

$$10^{-20}T \leq m(I) \leq \sum_t \frac{2\pi\lambda}{\log T} \leq \sum_{n\in S}(\gamma_{n+1} - \gamma_n) + O(1),$$

so that

$$\left(10^{-20}\right)^2 T^2 \le \left\{\sum_{n\in S}(\gamma_{n+1}-\gamma_n)\right\}^2 \le (\#S)\left(\sum_{n\in S}(\gamma_{n+1}-\gamma_n)^2\right). \tag{7.14}$$

By Theorem 1 of Fujii [35], we have

$$\sum_{\gamma_n\le T}(\gamma_{n+1}-\gamma_n)^2 \le 9\frac{2\pi T}{\log T/2\pi}.$$

Inserting this into (7.14), we get

$$\#S > 10^{-39}N(T)$$

showing that

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \ge 1+10^{-20}$$

holds for a positive proportion of n.

Let $c_1 = 10^{-39}$, $c_2 = 10^{-21}$ and $\nu = 10^{22}$. Then $c_1 > 0$ and c_2 and ν satisfy

$$0 < c_2 < \lambda - 1$$

and

$$\left(\frac{\nu}{\nu+2\pi\lambda}\right)^2 > \frac{\lambda+1}{2\lambda}.$$

We choose

$$c = 5\cdot 10^{-82}$$

so that

$$0 < c < \left(1+\frac{\nu}{\pi\lambda}\right)^{-1}\left(c_1 - 4c - \frac{4c}{c_2}\right)$$

in [86, equation (3.15)] is satisfied. By the proof of Theorem 1 in [86], we conclude that

$$\sum_{\substack{\beta'<1/2+\nu/\log T\\ 0<\gamma'<T}} 1 > cT\log T$$

holds for sufficiently large T, and, consequently,

$$m^{-}(\nu) > 10^{-83}$$

for all $\nu > 10^{22}$.

Chapter 8

Large gaps between critical zeros of $L(s, \chi)$

8.1 Introduction and lemmas

Let

$$S(t, \chi) = \frac{1}{\pi} \arg L(1/2 + it, \chi),$$

where the argument is determined by continuous variation on the half-line $\sigma + it$, $\sigma \geq 1/2$. If t is the ordinate of a zero, then we put

$$S(t, \chi) = \frac{1}{2} \left(S(t + 0, \chi) + S(t - 0, \chi) \right).$$

We start our investigation of large gaps of critical zeros of $L(s, \chi)$ by citing a special case of Fujii's Main Theorem in [33].

Lemma 41 *[33, Main Theorem] Let $\chi(\mathrm{mod}\ D)$ be a primitive character of a fixed modulus D. Suppose $h > 0$ is bounded. Then we have*

$$\int_T^{2T} |S(t + h, \chi) - S(t, \chi)|^{2k} \, dt = \frac{(2k)!}{(2\pi)^{2k} k!} T (2 \log(2 + h \log T))^k + O(T \log(2 + h \log T)^{k-1/2}). \tag{8.1}$$

The following is an analogous result to Littlewood [76, Theorem 9.12].

Lemma 42 *For every large positive T, $L(s,\chi)$ has a zero $\beta+i\gamma$ satisfying*

$$|\gamma - T| \ll (\log\log\log T)^{-1}.$$

Abbreviate $\gamma_n = \gamma_n(\chi)$. Using (8.1), we shall prove

Lemma 43 *Suppose that $T \geq T_0$, $h \gg (\log T)^{-1}$. We have*

$$\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}} (\gamma_{n+1}-\gamma_n)^2 \ll \frac{T\log(2+h\log T)}{\log T}. \tag{8.2}$$

Proof: We follow the proof of Fujii [34, Theorem 2]. By Lemma 42, for every large positive T ($T > e^{e^2}$ is enough), $L(s,\chi)$ has a zero $\beta+i\gamma$ satisfying

$$|\gamma - T| < \frac{A}{\log\log\log T} \tag{8.3}$$

and $\frac{1}{2} \leq \beta < 1$. Hence,

$$\gamma_{n+1} - \gamma_n < \frac{A}{\log\log\log T} \tag{8.4}$$

for $T^{1/2} \leq \gamma_n \leq T$. Between $\gamma_n < t < \gamma_{n+1}$, $L(1/2+it,\chi)$ has no zero and $N(\gamma_n+t,\chi) = N(\gamma_{n+1},\chi)$; hence,

$$|S(t+h,\chi) - S(t,\chi)| > \frac{h}{2\pi}\log\frac{t}{2\pi}.$$

Assume $\gamma_{n+1}-\gamma_n \geq h$. Then

$$\int_{\gamma_n}^{\gamma_{n+1}-\frac{1}{2}h} (S(t+h/2,\chi)-S(t,\chi))^2 dt \geq \frac{1}{(2\pi)^2}(\gamma_{n+1}-\gamma_n)(h\log T)^2.$$

This, together with (8.5), give

$$\begin{aligned}\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}(\gamma_{n+1}-\gamma_n) &\leq \frac{(2\pi)^2}{(h\log T)^2}\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}\int_{\gamma_n}^{\gamma_{n+1}-\frac{1}{2}h}(S(t+h/2)-S(t))^2dt\\ &\leq \frac{(2\pi)^2}{(h\log T)^2}\int_0^T(S(t+h/2)-S(t))^2dt\\ &\ll \frac{T}{(h\log T)^2}.\end{aligned}$$

Using $\gamma_{n+1}-\gamma_n\geq h$ and rewriting $\gamma_{n+1}-\gamma_n-h/2$ as $\int_{\frac{1}{2}h}^{\gamma_{n+1}-\gamma_n}du$, we have

$$\begin{aligned}\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}(\gamma_{n+1}-\gamma_n)^2 &\leq \sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}(\gamma_{n+1}-\gamma_n)(2(\gamma_{n+1}-\gamma_n-h/2))\\ &\leq 2\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}(\gamma_{n+1}-\gamma_n)\int_{\frac{1}{2}h}^{\gamma_{n+1}-\gamma_n}du\\ &\leq 2\int_{h/2}^{16(\log\log\log T)^{-1}}\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq u}}(\gamma_{n+1}-\gamma_n)du\\ &\ll T\int_{h/2}^{16(\log\log\log T)^{-1}}\frac{\log(2+u\log T)}{(u\log T)^2}du,\end{aligned}$$

by (8.4). The integrand is monotone decreasing, and we obtain the bound

$$\sum_{\substack{0<\gamma_n\leq T\\ \gamma_{n+1}-\gamma_n\geq h}}(\gamma_{n+1}-\gamma_n)^2 \ll \frac{T\log(2+h\log T)}{\log T}.$$

This gives (8.2). ■

Using Lemma 43 we shall prove the following corollary with the condition $\gamma_{n+1}-\gamma_n\geq h$ removed.

Lemma 44 *We have*

$$\sum_{0<\gamma_n\leq T}(\gamma_{n+1}-\gamma_n)^2 \ll \frac{N(T,\chi)}{\log^2 T}.$$

Proof: We follow the proof of Fujii [34, Corolary 1]. Choosing $h = 2\pi(\log T)^{-1}$, we get, by Lemma 43,

$$\begin{aligned}\sum_{0<\gamma_n\le T}(\gamma_{n+1}-\gamma_n)^2 &= \sum_{\substack{0<\gamma_n\le T\\ \gamma_{n+1}-\gamma_n\ge 2\pi(\log T)^{-1}}}(\gamma_{n+1}-\gamma_n)^2 + \sum_{\substack{0<\gamma_n\le T\\ \gamma_{n+1}-\gamma_n\ge 2\pi(\log T)^{-1}}}(\gamma_{n+1}-\gamma_n)^2\\ &\ll \frac{T\log(2+2\pi)}{\log T} + N(T,\chi)\left(\frac{2\pi}{\log T}\right)^2\\ &\ll \frac{N(T,\chi)}{(\log T)^2}.\end{aligned}$$

This gives the lemma. ■

We now adapt Heath-Brown's note 9.26 in Titchmarsh [76] as in Chapter 7 to derive the following large gaps result.

8.2 Statement and proof of Theorem 9

Theorem 9 *There exist computable constants $c > 0$ and $\lambda > 1$ such that for T large, we have*

$$\sum_{\substack{0<\gamma_n<T\\ \frac{\gamma_{n+1}-\gamma_n}{2\pi/\log T}>\lambda}} 1 \ge cN(T,\chi).$$

Remark 7 *As in Theorem 8, the constants c and λ can be made explicit by replacing $Q(t)$ and $P(t)$ in (7.2) and (7.3) by*

$$Q(t,\chi) = S(t,\chi) - \frac{1}{\pi}\sum_{p\le X}\frac{\sin(t\log p)\chi(p)}{\sqrt{p}}$$

and

$$P(t,\chi) = \frac{1}{\pi}\Im\sum_{p\le X}p^{-\frac{1}{2}-it}(p^{-ih}-1)\chi(p),$$

for some suitable X.

Proof: By Lemma 41, we deduce

$$\int_T^{2T} (S(t+h,\chi) - S(t,\chi))^2 dt \gg T \tag{8.5}$$

and

$$\int_T^{2T} (S(t+h,\chi) - S(t,\chi))^4 dt \ll T$$

uniformly for

$$2\pi M \le h \log T \le 4\pi M,$$

for a sufficiently large integer M. Hence, by Hölder inequality, we have

$$\int_T^{2T} |S(t+h,\chi) - S(t,\chi)| dt \ge \frac{\left(\int_T^{2T} |S(t+h,\chi) - S(t,\chi)|^2 dt\right)^{1/2}}{\left(\int_T^{2T} |S(t+h,\chi) - S(t,\chi)|^4 dt\right)^{1/2}} \gg T. \tag{8.6}$$

By Riemann-von Mangoldt formula,

$$S(t+h,\chi) - S(t,\chi) = N(t+h,\chi) - N(t,\chi) - \frac{h \log DT}{2\pi} + O\left(\frac{1}{\log DT}\right)$$

for $T \le t \le 2T$, whence

$$\int_T^{2T} \left| N(t+h,\chi) - N(t,\chi) - \frac{h \log DT}{2\pi} \right| dt = \int_T^{2T} |S(t+h,\chi) - S(t,\chi)| dt + O\left(\frac{T}{\log DT}\right) \gg T.$$

We proceed to write $h = 2\pi M\lambda / \log DT$, $1 \le \lambda \le 2$ and

$$\delta(t,\lambda) = N\left(t + \frac{2\pi\lambda}{\log DT}, \chi\right) - N(T,\chi) - \lambda,$$

so that

$$N(t+h,\chi) - N(t,\chi) - \frac{h \log DT}{2\pi} = \sum_{m=0}^{M-1} \delta\left(t + \frac{2\pi m\lambda}{\log DT}, \lambda\right).$$

It follows that

$$\int_T^{2T} |\delta(t,\lambda)| dt \gg T \tag{8.7}$$

uniformly for $1 \le \lambda \le 2$.

Let I denote the subset of $[T,2T]$ on which $N\left(t+\frac{2\pi\lambda}{\log DT},\chi\right) = N(t,\chi)$, then

$$|\delta(t,\lambda)| \le \begin{cases} \delta(t,\lambda) + 2\lambda, & (t \in I), \\ \delta(t,\lambda) + 2\lambda - 2, & (t \in [T,2T] - I), \end{cases}$$

so that (8.7) yields

$$T \ll \int_T^{2T} |\delta(t,\lambda)| dt = \int_T^{2T} \delta(t,\lambda) dt + (2\lambda - 2)T + 2m(I),$$

where $m(I)$ is the measure of I. However,

$$\delta(t,\lambda) = N\left(t + \frac{2\pi\lambda}{\log DT},\chi\right) - N(t,\chi) - \lambda = S\left(t + \frac{2\pi\lambda}{\log DT},\chi\right) - S(t,\chi) + O\left(\frac{1}{\log DT}\right)$$

and hence

$$\int_T^{2T} \delta(t,\lambda) dt = \int_T^{2T} \left(S\left(t + \frac{2\pi\lambda}{\log DT},\chi\right) - S(t,\chi)\right) dt + O\left(\frac{T}{\log DT}\right) = O(\log DT),$$

whence

$$T \ll (2\lambda - 2)T + 2m(I)$$

and

$$m(I) \gg T,$$

if λ is chosen sufficiently close to 1. Thus, if

$$S = \left\{ n : T \le \gamma_n \le 2T, \gamma_{n+1} - \gamma_n \ge \frac{2\pi\lambda}{\log DT} \right\},$$

then

$$T \ll m(I) \le \sum_t \frac{2\pi\lambda}{\log DT} \le \sum_{n \in S} (\gamma_{n+1} - \gamma_n) + O(1),$$

so that

$$T^2 \le \left\{ \sum_{n \in S} (\gamma_{n+1} - \gamma_n) \right\}^2 \le (\#S) \sum_{n \in S} (\gamma_{n+1} - \gamma_n)^2.$$

From this and Lemma 44, it follows that

$$\#S \gg N(T,\chi),$$

showing that

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \geq \lambda$$

holds for a positive proportion of n.

■

Chapter 9

Future projects

9.1 On the generalized de Bruijn-Newman constant for quadratic Dirichlet L-functions

9.1.1 Introduction

Let $-D < 0$ be a fixed fundamental discriminant. Let χ be the Kronecker symbol and $L(s, \chi)$ be its Dirichlet L-function. Define

$$\Xi_t(z, \chi) = \int_0^\infty \exp(tu^2)\Phi(u, \chi)\cos(uz)du,$$

where

$$\Phi(u, \chi) = 4\sum_{n=1}^\infty \chi(n)n\exp\left(\frac{3u}{2} - \frac{n^2\pi\exp(2u)}{D}\right).$$

The function $\Xi_t(z, \chi)$ is entire, even, and satisfies the backward heat equation

$$\frac{\partial\Xi}{\partial t} + \frac{\partial^2\Xi}{\partial z^2} = 0.$$

For $t = 0$, the function $\Xi_t(z, \chi)$ reduces to

$$\Xi_0(z, \chi) = \Xi(z, \chi) := \left(\frac{D}{\pi}\right)^{(s+1)/2}\Gamma((s+1)/2)L(s, \chi). \tag{9.1}$$

Analogous to Newman's theorem [67, Theorem 3] for the Riemann zeta function, there is a real constant $\Lambda_{-D} \in (-\infty, 1/2]$ such that (i) $\Xi_t(z, \chi)$ has only real zeros if and only if $t \geq \Lambda_{-D}$ and (ii) $\Xi_t(z, \chi)$ has some complex zeros if $t < \Lambda_{-D}$. In [75], Stopple defined

$$\Lambda_{\mathrm{Kr}} = \{\sup \Lambda_{-D} | -D \text{ fundamental}\},$$

and conjectured, analogous to Newman's conjecture for $\zeta(s)$, that $\Lambda_{\mathrm{Kr}} \geq 0$. The recent work of Dobner [21] implies, in particular, that $\Lambda_{-D} \geq 0$. In this section we outline a different proof of this pointwise lower bound: One has

$$\Lambda_{-D} \geq 0. \tag{9.2}$$

Our approach is closer to the original proof of Newman's conjecture of Rodgers and Tao [72] and makes use of arithmetic information on the distribution of gaps of zeros of $L(s, \chi)$ on the critical line from Theorem 9.

9.1.2 Outline of the proof of (9.2)

As in [72] we suppose for contradiction that $\Lambda_{-D} < 0$. Since the function $\Xi_t(z, \chi)$ solves the backwards heat equation, this gives the same dynamics on the zeros x_j of $\Xi_t(z, \chi)$ as in [72, Theorem 4.1]. Consequently, c.f. [72, equation (93)],

$$x_{j+1}(0) - x_j(0) = \frac{4\pi + o_{T\to\infty}(1)}{\log T} \tag{9.3}$$

holds for a fraction $1 - o_{T\to\infty}(1)$ of $j \in [T \log T, 2T \log T]$, where the points $x_j(0)$ are twice the imaginary part of the nontrivial zeros of $L(x, \chi)$. This implies that almost all zeros of $L(x, \chi)$ are uniformly distributed close to the average spacing. But, by the result of Theorem 9, there is $\lambda > 1$ and a constant $c(\lambda) > 0$ such that

$$x_{j+1}(0) - x_j(0) \geq \lambda \frac{4\pi}{\log T} \tag{9.4}$$

for a positive proportion $c(\lambda)$ of $j \leq T \log T$. This contradicts (9.3) and therefore the assumption that $\Lambda_{-D} < 0$.

In the next subsection, we summarize the main modifications from $\zeta(s)$ to $L(s,\chi)$ in order to apply Rodgers-Tao's method of convergence to local equilibrium [72, Sections 4-8] to derive (9.3). Then, in the section 8, we prove the large gaps estimate (9.4) leading to the desired contradiction.

9.1.3 Applying Rodgers-Tao's method of convergence to local equilibrium

Since $D > 0$ is fixed, we allow all $O()$ terms and $\ll$ symbols to depend on D. To lighten up the notation, we omit this dependency in the notation and write $O()$ for $O_D()$, and $\ll$ for $\ll_D$.

Asymptotic of $\Xi_t(z,\chi)$

We establish some upper and lower bounds on the function $\Xi_t(z) = \Xi_t(z,\chi)$ and its logarithmic derivative $\Xi_t'(z)/\Xi_t(z)$.

Lemma 45 *Let $z = x - i\kappa \log_+ x$ for some $x \geq 0$ and $0 \leq \kappa \leq C$, and let $\Lambda < t \leq 0$. then one has*

$$\Xi_t(z) \ll \exp\left(-\frac{\pi x}{8} + O_C(\log_+^2 x)\right). \tag{7}$$

Furthermore, there is an absolute constant $C' > 0$ such that if $\kappa \geq C'$, then on has the refinement

$$\Xi_t(z) = \exp\left(-\frac{\pi x}{8} + O_C(\log_+^2 x)\right), \tag{8}$$

as well as the additional estimate

$$\frac{\Xi_t'}{\Xi}(z) = \frac{i}{4}\log\left(\frac{iz}{4\pi}\right) + O_C\left(\frac{\log_+ x}{x}\right), \tag{9}$$

using the standard branch of the complex log.

Proof: We begin by treating the easy case $t = 0$. We have the bound [18, (12.14), page 82]

$$L(s, \chi) \leq 2D|s| = O(|s|) \tag{10}$$

whenever $\sigma \geq 1/2$. In this region, we also have the Stirling approximation

$$\Gamma(\sigma + i\tau) = \exp\left(\left(\sigma + i\tau - \frac{1}{2}\right)\log(\sigma + i\tau) - (\sigma + i\tau) + \log\sqrt{2\pi} + O\left(\frac{1}{|\sigma + i\tau|}\right)\right),$$

where we use the standard branch of the logarithm; in particular

$$\Gamma(\sigma + i\tau) \ll \exp\left((\sigma - \frac{1}{2})\log|\sigma + i\tau| - \tau\arctan\frac{\tau}{\sigma} - \sigma\right).$$

As $\arctan\frac{\tau}{\sigma} = \frac{\pi}{2}sgn(\tau) + O\left(\frac{\sigma}{\sigma+|\tau|}\right)$, we have in particular that

$$\Gamma(\sigma + i\tau) \ll \exp\left(-\frac{\pi}{2}|\tau| + O(\sigma\log_+(|\sigma| + |\tau|))\right).$$

By these bounds we obtain the crude upper bound

$$\Xi(x - iy) \ll \exp\left(-\frac{\pi|x|}{8} + O((1 + y)\log_+(|x| + y))\right) \tag{13}$$

for $x \in \mathbb{R}$ and $y \geq 0$. This gives the $t = 0$ case of (7).

When $\sigma \geq 2$, say, we can improve the bound (10) to

$$L(s, \chi) \ll 1$$

and so we obtain the improvement

$$\Xi(x - iy) = \exp\left(-\frac{\pi|x|}{8} + O((1 + y)\log_+(|x| + y))\right)$$

when $y \geq C'\log_+ x$. This gives the $t = 0$ case of (8).

Finally, from taking logarithmic derivatives of (9.1) one has

$$\frac{\Xi'}{\Xi}(z) = \frac{i}{2}\left(-\frac{1}{2}\log\frac{\pi}{D} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+1)\right) + \frac{L'}{L}(s,\chi)\right)$$

where $s := \frac{1}{2} + \frac{iz}{2}$. One has the well known asymptotic

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \log\frac{s}{2} + O\left(\frac{1}{|s|}\right)$$

for the digamma function $\frac{\Gamma'}{\Gamma}$. From the Dirichlet series expansion $\frac{L'}{L}(s) = -\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^s} \ll \sum_{n=2}^{\infty}\frac{\log n}{n^{\Re s}}$, one can establish the bound

$$\frac{L'}{L}(s) \ll \frac{1}{|s|}$$

in the regime $C'\log_+ x \leq y \leq C\log x$. Putting all this together, one obtains (9) in this case.

We now address the $t < 0$ case. We begin with the proof of (7). For any $t < 0$ we have the classical heat equation identity

$$e^{tu^2}\exp(izu) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-r^2/4}\exp(i(z + r|t|^{1/2})u)\,dr$$

for any complex numbers z, u; replacing z, r by $-z, -r$ and averaging we conclude that

$$e^{tu^2}\cos(zu) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{-r^2/4}\cos((z + r|t|^{1/2})u)\,dr.$$

Multiplying by $\Phi(u)$, integrating u from 0 to infinity, and using Fubini's theorem, we conclude that

$$\Xi_t(z) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-r^2/4}\Xi(z + r|t|^{1/2})\,dr.$$

Applying (13), the triangle inequality, and the hypothesis $\Lambda < t \leq 0$, we conclude that

$$\Xi_t(x - iy) \ll \exp\left(-\frac{\pi|x|}{8} + O((1+y)\log_+(|x|+y))\right)\int_{\mathbb{R}}\exp\left(-\frac{r^2}{4} + O((1 + y + |r|\log_+ r))\right)dr.$$

Using $(1+|r|)\log_+ r \leq \epsilon r^2 + O_\epsilon(1)$ and $y\log_+ r \ll \epsilon r^2 + O_\epsilon(y^2)$ for any absolute constant $\epsilon > 0$, we have

$$-\frac{r^2}{4} + O(|r|) + O((1+y+|r|)(1+\log_+ r)) \leq -\frac{r^2}{8} + O((1+y)^2),$$

thus arriving at the bound

$$\Xi_t(x-iy) \ll \exp\left(-\frac{\pi|x|}{8} + O((1+y)\log_+|x| + (1+y)^2)\right).$$

Since $y = O_C(\log_+ x)$, this gives (7).

To prove the remaining two bounds (8) and (9), it is reduced to showing that

$$\frac{\Xi_t}{\Xi_0}(z) = \exp\left(O_C(\log_+^2 x)\right)$$

and

$$\frac{\Xi_t'}{\Xi_t}(z) - \frac{\Xi_0'}{\Xi_0}(z) \ll_C \frac{\log_+ x}{x}$$

when $\kappa \geq C'$. This is accomplished in the next subsection, via the oscillatory integral $I_t(b,\zeta)$ and the method of steepest descent. ■

Modification of the oscillatory integral $I_t(b,\zeta)$

As the function

$$\begin{aligned}\Phi(u,\chi) &= 4\sum_{n=1}^{\infty}\chi(n)n\exp(3u/2 - n^2\pi\exp(2u)/D)\\ &= \sum_{n=1}^{\infty}(4n\chi(n)e^{3u/2})\exp\left(-\frac{\pi n^2}{D}e^{2u}\right)\end{aligned}$$

is even, we may write the function $\Xi_t(z,\chi)$ as

$$\Xi_t(z,\chi) = \frac{1}{2}\int_{\mathbb{R}} e^{tu^2}\Phi(u,\chi)e^{iuz}du.$$

Hence, by Fubini's theorem, we get that

$$\Xi_t(z,\chi) = \frac{1}{2}\sum_{n=1}^{\infty} 4n\chi(n) I_t(\pi n^2/D, 3/2+y+ix), \tag{9.5}$$

where $I_t(b,\zeta)$ denotes the oscillatory integral

$$I_t(b,\zeta) = \int_{\mathbb{R}} \exp(tw^2 - be^{2w} + \zeta w)dw \tag{9.6}$$

which is an absolutely convergent integral whenever $\Re b, \Re\zeta > 0$. Comparing (9.6) with the corresponding oscillatory integral for $\zeta(s)$ (c.f. [72, (19)]), the factor $4w$ is replaced by $2w$, and b now takes the value $b = \pi n^2/D$ which makes the $O()$ term in [72, Lemma 7] depends on D. More specifically, in [72, Lemma 7], the condition $b \geq 1$, which is in the hypothesis of that lemma, is used thrice in pages 13 and 14 between equations (26) and (29). Replacing $b \geq 1$ by $b \geq 1/D$ produces extra term $\log_+ D$ which can be absorbed into the $O(1)$ term. Moreover, the factor $\chi(n)$ in (9.5) will be estimated trivially by the bound $|\chi(n)| \leq 1$. Thus, repeating the calculations of [72, Section 2], we obtain the following analogous asymptotic to [72, Lemma 7].

Lemma 46 *Let b be a complex number with $\Re b \geq 1/D$. Then*

$$I_t(b,2b) = \sqrt{\frac{\pi}{8}}\exp(-b)\left(\frac{1}{\sqrt{b}} + O\left(\frac{1}{|b|^{3/2}}\right)\right)$$

using the standard branch of the square root.

9.1.4 Riemann-von Mangoldt type formulae

For any $\Lambda_{-D} < t \leq 0$, the zeros of $\Xi_t(z,\chi)$ are real and simple. For any interval $I \subset \mathbb{R}$, let $N_t(I)$ denote the number of zeros of $\Xi_t(z,\chi)$ in I. The classical Riemann-von Mangoldt formula for $L(s,\chi)$ (see e.g., [18, (16.1), pg. 101]), together with (9.1), gives the asymptotic

$$N_0([-T,T]) = 2\psi(T) + O(\log_+ T),$$

for $T \geq D$, where

$$\psi(T) = \frac{T}{4\pi}\log\frac{T}{4\pi} + (\log D - 1)\frac{T}{4\pi}.$$

Using Lemma 45, we can derive by analogous calculations as in [72, Section 3] the following.

Proposition 4 (Riemann-von Mangoldt type formulae) *Let* $\Lambda_{-D} < t \leq 0$*,* $T > D$*, and let* $0 \leq \alpha \leq C$ *for some* $C > 0$*. Then one has*

$$N_t([0,T]) = \psi(T) + O(\log_+^2 T)$$

and

$$N_t([T, T + \alpha\log_+ T]) = \frac{\alpha\log_+^2 T}{4\pi} + o(\log_+^2 T).$$

9.2 Variance of the ternary divisor function in arithmetic progressions: II. $Q \leq X$

In Theorem 6 an analogous asymptotic was worked out for $\tau_3(n)$ but only for the case $D = X$ with a different expected main term and without the condition $(a,d) = 1$. The proof of that estimate is based on an averaged additive correlation sum estimate for τ_3 using a simpler version of the circle method. This approach does not work well when the a average is over $(a,d) = 1$ since we are unable to treat correlation sums with coprime conditions at the moment. Also, motivated by certain applications, we consider the expected main term

$$\frac{1}{\varphi(q)}\sum_{\substack{n\leq X\\(n,q)=1}} \tau_3(n) \tag{9.7}$$

which is slightly different from that of (4.5). Hence, we use a different approach, that of Hooley from [44] which was used to treat an analogous sum to (9.7) with $\tau_3(n)$ replaced by the Chebyshev function $\theta(n)$. This approach makes use of the estimate (3.5) which uses the multiplicative large sieve inequality.

9.2.1 Lemmas

We begin our lemmas by noting the following identity.

Lemma 47 *For any $n \leq X$, we have*

$$\tau_3(n) = 3 \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1\ell_2 \leq X^{2/3};\ \ell_1 \leq X^{1/3}}} 1 - 3 \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1\ell_2 \leq X^{2/3};\ \ell_1,\ell_3 \leq X^{1/3}}} 1 + \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1,\ell_2,\ell_3 \leq X^{1/3}}} 1. \tag{9.8}$$

Proof: The identity follows from inclusion-exclusion; see, e.g. Hooley [43, p. 406]. ■

Lemma 48 *We have*

$$\sum_{n \leq X} \frac{1}{n} = \log X + \gamma + \frac{1}{2X} + \frac{1}{12X^2} + O\left(\frac{1}{X^3}\right).$$

Proof: This is a standard estimate and can be derived, for example, from Euler-Maclaurin summation; see, e.g., [66, Example 2.1.10, p. 21]. ■

Lemma 49 *For any $k \geq 1$, we have*

$$\sum_{n \leq X} \frac{\log^k n}{n} = \frac{\log^{k+1} X}{k+1} + (-1)^k k! \gamma_k + O\left(\frac{\log^k X}{X}\right), \tag{9.9}$$

where γ_k is the Stieltjes constant

$$\gamma_n = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1} \right\}.$$

In particular, for $k = 1$,

$$\sum_{n \leq X} \frac{\log n}{n} = \frac{1}{2} \log^2 X - \gamma_1 + O\left(\frac{\log X}{X}\right),$$

with $\gamma_1 = -0.072815 \cdots$.

Proof: Equation (9.9) is equation (1.68) on page 20 of [47]. ■

Lemma 50 *We have*

$$\sum_{n\le X}\tau_3(n)=X\left(\frac{1}{2}\log^2 X+(3\gamma-1)\log X+3(\gamma^2-\gamma+\gamma_1)+1\right)+O(X^{2/3}\log X). \tag{9.10}$$

Proof: This is also a standard result. We apply the identity in Lemma 59 to give another proof. By Lemma 59, we have

$$\sum_{n\le X}\tau_3(n)=3\sum_{\ell_1\le X^{1/3}}\sum_{\ell_2\le\frac{X^{2/3}}{\ell_1}}\sum_{\ell_3\le\frac{X}{\ell_1\ell_2}}1-3\sum_{\ell_1\le X^{1/3}}\sum_{\ell_2\le\frac{X^{2/3}}{\ell_1}}1\sum_{\ell_3\le X^{1/3}}1+\left(\sum_{\ell\le X^{1/3}}1\right)^3=3\Sigma_1-3\Sigma_2+\Sigma_3,$$

say. We first have

$$\Sigma_3=\left(X^{1/3}+O(1)\right)^3=X+O(X^{2/3}).$$

Note that the $O(1)$ can be made more precise by replacing it with the fractional part $\{X^{1/3}\}$, but we will not be needing that for our purpose. Next, we have, by Lemma 48

$$\Sigma_2=\left(\sum_{\ell_1\le X^{1/3}}\frac{X^{2/3}}{\ell_1}+O(X^{1/3})\right)\left(X^{1/3}+O(1)\right)=X\left(\frac{1}{3}\log X+\gamma\right)+O(X^{2/3}\log X).$$

Similarly, we obtain

$$\Sigma_1=X\left(\frac{1}{6}\log^2 X+\gamma\log X+\gamma^2+\gamma_1\right)+O(X^{2/3}\log X).$$

Combining the above three estimates leads to (9.10). ■

Lemma 51 *For any $q \geq 1$ we have*

$$\sum_{\substack{n\leq X\\(n,q)=1}} \tau_3(n)$$

$$= \frac{\varphi(q)^3}{q^3}X\left(\frac{1}{2}\log^2 X + \left(3\gamma - \frac{7}{6} + 2\sum_{p|q}\frac{\log p}{p-1}\right)\log X + 3\gamma^2 + 3\gamma_1 - 3\gamma + 3(\gamma-1)\sum_{p|q}\frac{\log p}{p-1}\right)$$

$$+ \frac{\varphi(q)^3}{q^3}X\sum_{p|q}\frac{\log p}{p-1}\left(\frac{1}{3}\log X + \gamma + \sum_{p|q}\frac{\log p}{p-1}\right) + O\left(\tau(q)X^{2/3}\log X\right).$$

Proof: In what follows we write $\sum_{n}^{*}$ to mean a sum over those n that are coprime to q. By Mobius inversion, we can readily verify the following:

$$\sum_{n\leq X}^{*} 1 = \frac{\varphi(q)}{q}X + O(\tau(q)),$$

$$\sum_{n\leq X}^{*} n = \frac{1}{2}\frac{\varphi(q)}{q}X^2 + O(\tau(q)X),$$

$$\sum_{n\leq X}^{*}\frac{1}{n} = \frac{\varphi(q)}{q}\left(\log X + \gamma + \sum_{p|q}\frac{\log p}{p-1}\right) + O\left(\frac{2^{\omega(q)}}{X}\right),$$

$$\sum_{n\leq X}^{*}\frac{\log n}{n} = \frac{\varphi(q)}{q}\left(\frac{1}{2}\log^2 X - \gamma_1\right) - \frac{1}{2}\sum_{d|q}\frac{\mu(d)\log^2 d}{d} + O\left(\frac{\log X}{X}\right),$$

where $\omega(q) = \sum_{p|q} 1$. Thus, we have, by (9.16),

$$\sum_{n\leq X}^{*}\tau_3(n) = 3\sum_{\ell_1\leq X^{1/3}}^{*}\sum_{\ell_2\leq \frac{X^{2/3}}{\ell_1}}^{*}\sum_{\ell_3\leq\frac{X}{\ell_1\ell_2}}^{*}1 - 3\sum_{\ell_1\leq X^{1/3}}^{*}\sum_{\ell_2\leq\frac{X^{2/3}}{\ell_1}}^{*}1\sum_{\ell_3\leq X^{1/3}}^{*}1 + \left(\sum_{\ell\leq X^{1/3}}^{*}1\right)^3 = 3\Sigma_1 - 3\Sigma_2 + \Sigma_3,$$

say. We treat Σ_3 first. We have

$$\Sigma_3 = \left(\frac{\varphi(q)}{q}X^{1/3} + O(\tau(q))\right)^3 = \frac{\varphi(q)^3}{q^3}X + O(\tau(q)X^{2/3}).$$

Next, we have

$$\Sigma_2 = \left(\sum_{\ell_1\leq X^{1/3}}^{*}\frac{\varphi(q)}{q}\frac{X^{2/3}}{\ell_1} + O(\tau(q)X^{1/3})\right)\left(\frac{\varphi(q)}{q}X^{1/3} + O(\tau(q))\right),$$

which yields

$$\Sigma_2 = \frac{\varphi(q)^3}{q^3} X \left(\frac{1}{3}\log X + \gamma + \sum_{p|q} \frac{\log p}{p-1} \right) + O(\tau(q) X^{2/3} \log X).$$

Similarly, we obtain that

$$\begin{aligned}\Sigma_1 = \frac{\varphi(q)^3}{q^3} X &\left(\frac{1}{6}\log^2 X + \left(\gamma - \frac{1}{18} + \frac{2}{3}\sum_{p|q} \frac{\log p}{p-1} \right) \log X + \gamma^2 + \gamma \sum_{p|q} \frac{\log p}{p-1} + \gamma_1 \right) \\ &+ \frac{\varphi(q)^3}{q^3} X \sum_{p|q} \frac{\log p}{p-1} \left(\frac{1}{3}\log X + \gamma + \sum_{p|q} \frac{\log p}{p-1} \right) + O\left(\tau(q) X^{2/3} \log X\right).\end{aligned}$$

Combining the above three estimates yields the estimate in the lemma. ■

Lemma 52 *For $\ell \ll X^{1/12}$ and $(a, \ell) = 1$, we have*

$$\sum_{\substack{n \le X \\ n \equiv a(\ell)}} \tau_3(n) = \frac{1}{\varphi(\ell)} \sum_{\substack{n \le X \\ (n,\ell)=1}} \tau_3(n) + O\left(X^{14/15}\right).$$

Proof: Instead of working with the generating function for $\tau_k(n)$ in arithmetic progressions directly, we will use the orthogonality of Dirichlet characters to decompose the condition $n \equiv a(d)$. By the orthogonality condition

$$\frac{1}{\varphi(d)} \sum_{\chi(\text{mod } d)} \overline{\chi}(a)\chi(n) = \begin{cases} 1, & \text{if } n \equiv a(d), \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{\substack{n \le X \\ n \equiv a(\ell)}} \tau_k(n) = \sum_{n \le X} \tau_k(n) \frac{1}{\varphi(d)} \sum_{\chi(\text{mod } d)} \overline{\chi}(a)\chi(n) = \frac{1}{\varphi(d)} \sum_{\chi(\text{mod } d)} \overline{\chi}(a) \sum_{n \le X} \tau_k(n)\chi(n).$$

If χ (mod d) is principal, then

$$\begin{cases} \overline{\chi}(a) = 1, \\ \displaystyle\sum_{n \le X} \tau_k(n)\chi(n) = \sum_{\substack{n \le X \\ (n,d)=1}} \tau_k(n). \end{cases}$$

Thus,

$$\sum_{\substack{n\le X\\ n\equiv a(\ell)}}\tau_k(n)=\frac{1}{\varphi(d)}\sum_{\substack{n\le X\\ (n,\ell)=1}}\tau_k(n)+\frac{1}{\varphi(d)}\sideset{}{'}\sum_{\chi(\text{mod }\ell)}\overline{\chi}(a)\sum_{n\le X}\tau_k(n)\chi(n),$$

where $\sideset{}{'}\sum_{\chi(\text{mod }d)}$ restricts to a sum over non-principal characters $\chi(\text{mod }d)$. Decomposing into primitive characters and applying Lemma 3, the second term on the right side of the above is at most $\ll X^{14/15}$. The lemma then follows. ■

Lemma 53 *We have*

$$\sum_{n\le X}\tau_3^2(n)=a_3X\frac{P_8(\log X)}{8!}+O\left(X^{10/11}\right),$$

where

$$a_3=\prod_p(1-p^{-1})^4(1+4p^{-1}+p^{-2})=0.04932\cdots$$

and

$$P_8(\log X)=\log^8 X+c_7\log^7 X+\cdots+c_0,$$

with

$$c_7=8\left(9\gamma-1+6\sum_p\frac{(1+3p)\log p}{(p-1)(1+4p+p^2)}\right)=86.6\cdots,$$

etc.

Proof: We have

$$\sum_{n=1}^{\infty}\tau_3^2(n)n^{-s}=\prod_p\left\{1+\sum_{\nu=1}^{\infty}\binom{\nu+2}{2}^2p^{-\nu s}\right\},$$

where both members of this equation are absolutely convergent if $\sigma>1$. Hence, if $\sigma>1$,

$$\begin{aligned}\{\zeta(s)\}^{-9}\left\{\sum_{n=1}^{\infty}\tau_3^2(n)n^{-s}\right\}&=\prod_p\left\{(1-p^{-s})^9(1+9p^{-s}+36p^{-2s}+\cdots)\right\}\\&=\prod_p\left\{1+a_2p^{-2s}+a_3p^{-3s}+\cdots\right\}=F(s),\end{aligned}$$

say, where

$$a_\nu = \sum_{r=0}^{\nu} (-1)^r \binom{9}{r} \binom{\nu - r + 2}{2}^2 .$$

We adpot the convention for the binomial coefficients that $\binom{n}{m} = 0$ if $m > n$. The coefficient a_ν satisfies

$$|a_\nu| \leq K\nu^2,$$

where K is independent of ν. Hence

$$\sum_{\nu=2}^{\infty} |a_\nu| p^{-\nu s} \leq K' p^{-2s},$$

where K' is independent of p. Hence, if $\sigma > 1/2$, then $\sum_p p^{-2s}$ is absolutely convergent, and thus is also

$$F(s) = \prod_p \left\{ 1 + \sum_{\nu=2}^{\infty} a_\nu p^{-\nu s} \right\} = \prod_p \left(1 - 9p^{-2s} + 16p^{-3s} - 9p^{-4s} + p^{-6s} \right). \qquad (9.11)$$

Hence we obtain that

$$\sum_{n=1}^{\infty} \tau_3^2(n) n^{-s} = \{\zeta(s)\}^9 F(s),$$

where $F(s)$ is absolutely convergent for $\sigma > 1/2$. It follows, from Perron's formula, that

$$\sum_{n \leq X} \tau_3^2(n) = \frac{a_3}{8!} X P_8(\log X) + O\left(X^{10/11}\right),$$

where

$$a_3 = F(1) = \prod_p (1 - p^{-1})^4 (1 + 4p^{-1} + p^{-2})$$

and

$$P_8(\log X) = \frac{8!}{F(1)} \operatorname*{Res}_{s=1} \left(\zeta^9(s) F(s) \frac{X^{s-1}}{s} \right) = \log^8 X + 8 \left(9\gamma - 1 + \frac{F'(1)}{F(1)} \right) \log^7 X + c_6 \log^6 X + \cdots + c_0.$$

Taking the logarithmic derivative of (9.11) yields

$$\frac{F'(1)}{F(1)} = 6 \sum_p \frac{(1+3p)\log p}{(p-1)(1+4p+p^2)} = 6.63\cdots.$$

The other coefficients $c_0, \ldots, c_6$ can be computed similarly. This proves the lemma. ■

Lemma 54 *We have*

$$\sum_{\substack{n\le X\\(n,q)=1}} \tau_3(n)^2 =$$

Lemma 55 *For any real number $\kappa>0$, we have*

$$\sum_{q\le X}\left(\frac{\varphi(q)}{q}\right)^{\kappa} = c_2X + O(\log X),$$

where

$$c_\kappa = \prod_p\left(1-\frac{1}{p}\left(1-\left(1-p^{-1}\right)^{\kappa}\right)\right). \tag{9.12}$$

Thus, by partial summation,

$$\sum_{q\le X}\frac{\varphi(q)^\kappa}{q^{\kappa+1}} = c_\kappa(\log X+1) + O\left(\frac{\log X}{X}\right),$$

Proof: See, e.g., [62, Exercise 2.1.1.14, pg. 42]. ■

Lemma 56 *We have*

$$\sum_{\ell\le\xi}\left(1-\frac{\ell}{\xi}\right)^2\frac{\varphi(\ell)^2}{\ell^3} = c_2\log\xi + C_1 + C_2\frac{\log X}{X} + \frac{C_3}{X} + O\left(\frac{1}{X^{5/4}}\right),$$

where c_2 is the constant given by (9.12) with $\kappa=2$, and C_1, C_2, C_3 are other numerical constants.

Proof: If we let

$$f(s) = \sum_{\ell=1}^{\infty}\frac{\varphi(\ell)^2}{\ell^{s+3}}$$

then we have

$$\begin{aligned} f(s) &= \prod_p\left\{1+\frac{1}{p^s}\left(\frac{(p-1)^2}{p^3}\right)\left(1-\frac{1}{p^s}\right)^{-1}\right\}\\ &= \zeta(s+1)\zeta(s+2)\prod_p\left\{1+\frac{1}{p^{s+2}}\left(1-\frac{1}{p^s}\right)\left(\frac{(p-1)^2}{p^3}\right)\right\} = \zeta(s+1)\zeta(s+2)h(s), \end{aligned}$$

say, where $h(s)$ is regular and bounded for $\sigma > -2$. Thus, for $c > 0$, we have

$$\frac{1}{2}\sum_{\ell\le\xi}\left(1-\frac{\ell}{\xi}\right)^2\frac{\varphi(\ell)^2}{\ell^3} = \frac{1}{2\pi i}\int_{(c)} f(s)\frac{\xi^s}{s(s+1)(s+2)}ds = R_0 + R_{-1} + O(\xi^{-5/4}),$$

where R_0 and R_{-1} are the residues of the integrand at $s=0$ and $s=-1$, respectively. Both are double poles and we find

$$R_0 = \frac{1}{2}\zeta(2)h(0)\log\xi + \frac{1}{2}\left(\zeta'(0)h(0)+\zeta(0)h'(0)-\frac{3}{2}\zeta(0)h(0)\right) = \frac{1}{2}c_2\log\xi + \frac{1}{2}C_1$$

and

$$R_{-1} = -\frac{\zeta(0)h(-1)\log\xi}{\xi} + \frac{\zeta'(0)h(-1)+\zeta(0)h'(0)}{\xi} = -C_2\frac{\log\xi}{2\xi} + \frac{C_3}{2\xi},$$

where C_1, C_2, C_3 are constants. ■

Lemma 57 *We have, uniformly for any integer d,*

$$\sum_{n\le X}\tau(dn) = X\sum_{q|d}\frac{\varphi(d)}{d}(\log dX + 2\gamma - 1 - 2\log q) + O(\tau^2(d)X^{1/2}\log^2 X).$$

Proof: See [64, Lemma 4.6.1, pg. 193]. ■

By Dirichlet's hyperbola method and Lemma 57, we deduce

Lemma 58 *We have, uniformly for an integer d,*

$$\sum_{n\le X}\tau_3(dn) = X\sum_{q|d}\frac{\varphi(q)}{q}\left(\frac{\log^2 X}{2} + \frac{3}{2}\log X\log d + (2\gamma-1-2\log q)\log X + \log^2 d + \frac{1}{2}\log d\right)$$

$$+ O(\tau^2(d)X^{3/4}\log^2 X).$$

Proof: We have

$$\sum_{n\le X}\tau_3(dn) = \sum_{n\le X^{1/2}}\tau(dn)\sum_{m\le X/n}1 + \sum_{n\le X^{1/2}}\sum_{m\le X/n}\tau(dm) - \sum_{n\le X^{1/2}}\tau(dn)\sum_{n\le X^{1/2}}1.$$

By Lemma 57 and after some calculations, we obtain the lemma. ■

9.2.2 Outline of proof

We adapt Hooley's method in [44] to prove (9.7). Recall that

$$D_1 = X^{11/12}.$$

By the large sieve inequality, we have, by (3.5), for $1 \le D \le D_1$,

$$\sum_{d \le D} \sum_{a=1}^{d} \left| \sum_{\substack{n \le X \\ n \equiv a(d)}} \tau_3(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \le X \\ (n,d)=1}} \tau_3(n) \right|^2 \ll X^{2-1/31}.$$

Thus, it suffices to treat the case $D_1 \le D \le X$, which we henceforth assume.

Expanding out the square, we have

$$\sum_{D_1 \le d \le D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left| \sum_{\substack{n \le X \\ n \equiv a(d)}} \tau_3(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \le X \\ (n,d)=1}} \tau_3(n) \right|^2 = \mathcal{A}(D;X) - \mathcal{B}(D;X),$$

say, where

$$\mathcal{A}(D;X) = \sum_{D_1 \le d \le D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left(\sum_{\substack{n \le X \\ n \equiv a(d)}} \tau_3(n) \right)^2$$

and

$$\mathcal{B}(D;X) = \sum_{D_1 \le d \le D} \frac{1}{\varphi(d)} \left(\sum_{\substack{n \le X \\ (n,d)=1}} \tau_3(n) \right)^2.$$

By Lemma 51, we can write

$$\sum_{\substack{n \le X \\ (n,d)=1}} \tau_3(n) = \frac{\varphi(d)^3}{d^3} X \left(\frac{1}{2} \log^2 X + a(d) \log X + b(d) \right) + O\left(\tau(d) X^{2/3} \log X \right), \tag{9.13}$$

with

$$a(d) = 3\gamma - \frac{7}{6} + 2 \sum_{p|d} \frac{\log p}{p-1}$$

and

$$b(d) = 3\gamma^2 + 3\gamma_1 - 3\gamma + \sum_{p|d} \frac{\log p}{p-1}\left(\frac{1}{3}\log X + 4\gamma - 3 + \sum_{p|q}\frac{\log p}{p-1}\right).$$

Hence, we can write $\mathcal{B}(D;X)$ as

$$\mathcal{B}(D;X) = \sum_{D_1 \le d \le D} \frac{1}{\varphi(d)}\left(\frac{\varphi(d)^3}{d^3}X\left(\frac{1}{2}\log^2 X + a(d)\log X + b(d)\right) + O\left(\tau(d)X^{2/3}\log X\right)\right)^2$$

$$\begin{aligned}
&= \frac{1}{4}X^2\log^4 X \sum_{D_1\le d\le D}\frac{\varphi(d)^5}{d^6} + X^2\log^3 X\sum_{D_1\le d\le D}\frac{\varphi(d)^5}{d^6}a_d \\
&+ X^2\log^2 X\sum_{D_1\le d\le D}\frac{\varphi(d)^5}{d^6}(a_d^2+b_d) + X^2\log X\sum_{D_1\le d\le D}\frac{\varphi(d)^5}{d^6}a_d b_d + O(X^{4/3+\epsilon}).
\end{aligned}$$

Next, we we decompose $\mathcal{A}(D,X)$ into a diagonal and non-diagonal sums as

$$\mathcal{A}(D,X) = \sum_{D_1\le d\le D}\sum_{\substack{n\le X\\(n,d)=1}}\tau_3(n)^2 + \sum_{D_1\le d\le D}\sum_{\substack{n,n'\le X\\ n\ne n'\\ n\equiv n'(d)\\ (nn',d)=1}}\tau_3(n)\tau_3(n'). \tag{9.14}$$

We use Lemma 54 for the first sum on the right side. For the second sum, let

$$J(X,D) = \sum_{D<d\le X}\sum_{\substack{n,n'\le X\\ n\ne n'\\ n\equiv n'(d)\\ (nn',d)=1}}\tau_3(n)\tau_3(n'),$$

so that the second term on the right-side of (9.14) is

$$J(X,D_1) - J(X,D).$$

We now treat $J(X,D)$. We have,

$$J(X,D) = 2\sum_{\substack{n-n'=\ell d\\ n'<n\le X\\ D<d\le X\\ (nn',d)=1}}\tau_3(n)\tau_3(n'),$$

where $\ell < X/D \leq X^{1/12}$. Hence,

$$J(X,D) = 2\sum_{\ell<\frac{X}{D}} \sum_{\substack{n\equiv n'(\ell)\\ n\leq X\\ n-n'>\ell D\\ (nn',\ell)=1}} \tau_3(n)\tau_3(n') = 2\sum_{\ell<\frac{X}{D}} \sum_{\substack{a=1\\ (a,\ell)=1}}^{\ell} \sum_{\substack{n'<X-\ell D\\ n'\equiv a(\ell)}} \tau_3(n') \sum_{\substack{\ell D+n'<n\leq X\\ n\equiv a(\ell)}} \tau_3(n) \tag{9.15}$$

By Lemmas 52 and 51, we have, for $d \leq X^{1/12}$,

$$\sum_{\substack{n\leq X\\ n\equiv a(d)}} \tau_3(n) = \frac{\varphi(d)^2}{d^3} X \left(\frac{1}{2}\log^2 X + a(d)\log X + b(d)\right) + O\left(X^{14/15}\right).$$

Substituting this into $J(X,D)$, the most inner sum in (9.15) is equal to

$$\begin{aligned}&\frac{\varphi(\ell)^2}{\ell^3} X \left(\frac{1}{2}\log^2 X + a(\ell)\log X + b(\ell)\right)\\ &- \frac{\varphi(\ell)^2}{\ell^3}(\ell D + n') \left(\frac{1}{2}\log^2(\ell D + n') + a(\ell)\log(\ell D + n') + b(\ell)\right) + O(X^{14/15}).\end{aligned}$$

Thus,

$$\begin{aligned} J(X,D) = {}& X\log^2 X \sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')\\ &+ 2X\log X \sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} a(\ell) \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')\\ &+ 2X \sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} b(\ell) \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')\\ &- \sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')(\ell D + n')\log^2(\ell D + n')\\ &- 2\sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')(\ell D + n')\log(\ell D + n')a(\ell)\\ &- 2\sum_{\ell<\frac{X}{D}} \frac{\varphi(\ell)^2}{\ell^3} \sum_{\substack{n'<X-\ell D\\ (n',\ell)=1}} \tau_3(n')(\ell D + n')b(\ell) + O(X^{29/30}). \end{aligned}$$

The first three $(n', \ell) = 1$ sums can be treated as before by (9.13). For the last three $(n', \ell) = 1$ sums, we remove the condition $(n', \ell) = 1$ by Mobius inversion. We have

$$\sum_{\substack{n' < X - \ell D \\ (n', \ell) = 1}} \tau_3(n')(\ell D + n') \log^2(\ell D + n') = \sum_{n' < X - \ell D} \tau_3(n')(\ell D + n') \log^2(\ell D + n') \sum_{\substack{d | n' \\ d | \ell}} \mu(d)$$

$$= \sum_{d | \ell} \mu(d) \sum_{\substack{n' \leq X - \ell D \\ d | n'}} \tau_3(n')(\ell D + n') \log^2(\ell D + n')$$

$$= \sum_{d | \ell} \mu(d) \sum_{n' \leq \frac{X - \ell D}{d}} \tau_3(dn')(\ell D + dn') \log^2(\ell D + dn').$$

By partial summation, the inner sum of the above is reduced to estimating the sum

$$\sum_{n \leq X} \tau_3(dn)$$

to which we appeal to Lemma 58.

9.3 Zeros of the derivative of the Riemann zeta function

9.3.1 Distribution of the difference $\beta' - 1/2$

It might be possible to work out an asymptotic for the distribution for the difference $\beta' - 1/2$. More precisely, let

$$F(\alpha) = N_1(T)^{-1} \sum_{0 < \gamma' \leq T} T^{\alpha(\beta' - 1/2)} w(\beta' - 1/2).$$

Subtracting

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

from

$$-\frac{\zeta''}{\zeta'}(s) = \frac{2}{s-1} + O(1)$$

gives

$$\frac{\zeta''}{\zeta'}(s) = \frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} + O(1).$$

From this, an application of Perron's formula should give an explicit formula for sum over zeros of $\zeta'(s)$ in terms of sum over primes of the form $L(T) = R(T)$, say. Then evaluating each side directly, it is plausible that

$$F(\alpha) = 1 + o(1)$$

for $0 < \alpha < 2/3$, as in Özlük and Snyder [69].

9.3.2 Zeros of $L'(s,\chi)$ near the critical line

It is natural to generalize Zhang's argument [86] for $\zeta'(s)$ to zeros of $L'(s,\chi)$ by applying the result on large gaps between critical zeros of $L(s,\chi)$ from Theorem 9. The proof should go through without major modifications. More specifically, define

$$m^{-}(\nu) = m^{-}(\nu,\chi) = \liminf_{T\to\infty} N_1(T,\chi)^{-1} \sum_{\substack{\beta' \le 1/2+\nu/\log T \\ 0<\gamma'\le T}} 1.$$

Suppose χ is primitive with $\chi(-1) = 1$. Then $L'(s,\chi)$ is regular with no poles. Let $\eta(s) = h(s)L'(s,\chi)$ and $\xi(s) = h(s)L(s,\chi)$, where $h(s) = (\pi/q)^{-s/2}\Gamma(s/2)$. Define, as in [86],

$$F(t) = \begin{cases} -\Re\frac{\eta'}{\eta}\left(\frac{1}{2}+it\right), & \text{if } \eta(1/2+it) \neq 0, \\ \lim_{v\to t} F(v), & \text{if } \eta(1/2+it) = 0, \end{cases}$$

$$F_1(t) = -\sum_{\beta'>1/2} \Re\frac{1}{1/2+it-\rho'}, \quad \text{and} \quad F_2(t) = \sum_{0<\beta'<1/2} \Re\frac{1}{1/2+it-\rho'}.$$

Then $F(t) = F_1(t) - F_2(t) + O(1)$ and $\int_{\gamma_n}^{\gamma_{n+1}} F(t) \equiv 0 (mod \pi)$ between simple critical zeros of $L(s,\chi)$. By the argument in the proof of [86, Theorem 1], $m^{-}(\nu) > 0$ for sufficiently large ν.

9.3.3 Bounds on the function $m^+(\nu)$

The function $m^+(\nu)$ is defined as

$$m^+(\nu) = \limsup_{T\to\infty} N_1(T)^{-1} \sum_{\substack{\beta' \le 1/2+\nu/L \\ 0<\gamma'\le T}} 1.$$

Can we get bounds on the proportion of simple zeros of $\zeta'(s)$ on the critical line by mollifying $\zeta''(s)$? Let $m(\rho)$ denote the multiplicity of zero ρ of $\zeta(s)$. Then

$$\sum_{\gamma\le T} B\zeta''(\rho) = \sum_{\substack{\gamma\le T \\ m(\rho)=1,2}} B\zeta''(\rho)$$

and, by Cauchy's inequality,

$$N_{m(\rho)=1}(T) + N_{m(\rho)=2}(T) \ge \frac{\sum_{\gamma\le T} |B\zeta''(\rho)|^2}{\left|\sum_{\gamma\le T} B\zeta''(\rho)\right|^2}.$$

9.4 Distribution of arithmetic functions in arithmetic progressions to large smooth moduli

9.4.1 Removing the co-prime condition $(a,d)=1$

In [40] Heath-Brown removed the corpime condition $(a,q)=1$ by defining the function, c.f. [40, equation (8.3)],

$$F(n,\delta) = \sum_{\alpha_1\alpha_2\alpha_3\beta=n} \mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)d_3(\beta\delta)$$

with the property that, c.f. [40, equation (8.5)],

$$\sum_{n|m} F(n,\delta)d_3(m/n) = d_3(m\delta).$$

This implies

$$\sum_{\substack{n\le X \\ n\equiv a(q)}} d_3(n) = \sum_{\substack{n|\delta^2 \\ (n,q_1)=1}} F(n,\delta)D_3(X(n\delta)^{-1}, q_1, a_1\overline{n}),$$

where $(a,q)=\delta$, $q=\delta q_1$, and $a=\delta a_1$.

So to remove the coprime condition $(a,d)=1$, following Heath-Brown, define, e.g. for $k=4$,

$$F(n,\delta)=\sum_{\alpha_1\alpha_2\alpha_3\alpha_4\beta=n}\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\mu(\alpha_4)d_3(\beta\delta)$$

and show that

$$\sum_{n|m}F(n,\delta)\tau_4(m/n)=\tau_4(m\delta).$$

9.4.2 Relaxing the condition $(d,\prod_{p\leq X^{\varpi^2}}p)<X^{\varpi}$

For any $X^{1/2-\epsilon}\leq D\leq X^{1/2+2\varpi}$, let

$$\mathcal{E}(D)=\{d\sim D|\prod_{\substack{p|d\\ p\leq X^{\varpi^2}}}p>X^{\varpi}\}.$$

Trivially these exceptional moduli contribute $X(\log X)^A$ to the sum (2.13). But since the prime factors of d are not too large, the “Siegel-Walfitz” substitute can be applied to save a power of X from the trivial bound. Thus the condition $(d,\prod_{p\leq X^{\varpi^2}}p)<X^{\varpi}$ in (2.13) can probably be relaxed.

9.4.3 Sieving sparse sets

Can we design a sieve that takes in smooth numbers, e.g. analogous to Zhang's sieve, and prove a distribution result to large smooth moduli? This is really vague at the moment, but the idea is that the extra level of distribution might help capture more primes in some sequences.

9.5 Quantitative future works

9.5.1 Proportion of zeros of $L(s,\chi)$ near the critical line

Analogous to Chapter 7, it might be possible to compute an explicit lower bound for the positive proportion of zeros of $L(s,\chi)$ near the critical line.

9.5.2 Gaps between zeros of $\zeta(s)$

Work out explicit bounds for $D^{\pm}(\alpha)$ in [**?**] by working out the asymptotics for

$$h_2(F,\eta,\alpha,T) = \frac{\int_{-\alpha/2L}^{\alpha/2L} \sum_{T<\gamma\le 2T} |F(\gamma+t;T^{\eta})|^4 dt}{\int_T^{2T} |F(t;T^{\eta})|^4 dt}$$

and

$$\int_T^{2T} |F(t;T^{\eta})|^4 dt,$$

where $F(t;T^{\eta}) = \sum_{n\le T^{\eta}} f(n)n^{it}$ and $L = 1/2\pi \log T$.

9.6 Explicit main term in divisor sums

Using the following combinatorial identity found in Hooley [43, p. 406]:

Lemma 59 *[43, p. 406] For any $n \le X$, we have*

$$\tau_3(n) = 3 \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1\ell_2\le X^{2/3};\ \ell_1\le X^{1/3}}} 1 - 3 \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1\ell_2\le X^{2/3};\ \ell_1,\ell_3\le X^{1/3}}} 1 + \sum_{\substack{\ell_1\ell_2\ell_3=n \\ \ell_1,\ell_2,\ell_3\le X^{1/3}}} 1, \tag{9.16}$$

we can derive a (conditional) explicit expression for the main term of the divisor sum

$$D_3(X) = \sum_{n\le X} \tau_3(n)\tau_3(n+h) \tag{9.17}$$

as follows. Fix an h, say $h=1$. Substituting (9.16) in for $\tau_3(n)$ we can write

$$D_3(X) = 3\Sigma_1 - 3\Sigma_2 + \Sigma_3, \tag{9.18}$$

where

$$\Sigma_1 = \sum_{\ell \le X^{1/3}} \sum_{\substack{q \le X^{2/3} \\ q \equiv 0(\ell)}} D_3(X; q, h) + O(kX^{\epsilon}),$$

$$\Sigma_2 = X^{1/3} \sum_{\ell \le X^{1/3}} \sum_{\substack{q \le X^{2/3} \\ q \equiv 0(\ell)}} D_3(X; \frac{q}{\ell}, h) + O(kX^{\epsilon}),$$

and

$$\Sigma_3 = O(\Sigma_2),$$

with

$$D_3(X; q, a) = \sum_{\substack{n \le X \\ n \equiv a(q)}} \tau_3(n).$$

The main term for $D_3(X; q, a)$ is explicit:

$$D_3(X; q, a) = M_3(X; q) + O(X^{\delta_1} q^{-\delta_2}) \tag{9.19}$$

for some $\delta_1, \delta_2 > 0$, and where $M_3(X; q)$ is given explicitly, e.g., by equation (9.2) (pg. 52) in Heath-Brown [40]. Equation (9.19) is known to hold uniformly for all $q \le X^{21/41}$ [40, Theorem 1]. We need (9.19) to hold, on average, for $q \le X^{2/3}$. So if we assume that (9.19) holds on average for $q \le X^{2/3}$, by plugging $M_3(X; q)$ into (9.18), we can derive another explicit main term for the divisor sum $D_3(X)$, ignoring all error terms.

Remark 8 *In a sense, by (9.18), the problem of obtaining an asymptotic for the correlation sum $D_3(X)$ (9.17) is more or less equivalent to showing that $\tau_3(n)$ has level of distribution $2/3$. In other words, if we could obtain an asymptotic for $D_3(X)$, we would likely also obtain a level of distribution $2/3$, on average, for $\tau_3(n)$.*

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