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Dynamics and Phase Transitions in Interacting Electron Systems

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Abstract

This dissertation studies interacting electron models for quantum matter from two perspectives with different approaches respectively. The first part of this text focuses on rigorously proving Lieb-Robinson bounds for continuous space integer quantum Hall systems perturbed by short range two-body interactions. A Lieb-Robinson bound is an operator norm inequality that implies a finite rate of information propagation in a quantum system, a low energy analogue to the relativistic limit given by the speed of light. Many results of this nature have been obtained for lattice systems, but over the last several years, progress has been made in identifying classes of continuous space models that admit propagation estimates. In this thesis, we prove Lieb-Robinson bounds for two-dimensional continuum fermions in a constant, perpendicular magnetic field using the strategy of Gebert et al. [31], and use this result to prove strong continuity of the Heisenberg dynamics in the limit where inter-particle interactions occur in infinite volume. We then shift to a different framework, namely that of lattice-localized frames, to represent a class of interacting quantum Hall systems in continuous space, as introduced by Bachmann and De Nittis, [12]. With this framework, we define a family of parametrized Hamiltonians and construct a continuum analogue of the Hastings-Wen quasi-adiabatic evolution, i.e. dynamics in parameter space that satisfy Lieb-Robinson bounds themselves. Moreover, we then directly apply this to prove quantization of Hall conductance for these models. The new results of these sections are based on works in preparation, [10, 47].

The second portion of this dissertation focuses on two specific materials and phases: magic angle twisted bilayer graphene (MATBG) and superconducting lanthanum nickel gallium-2 ($LaNiGa_2$). MATBG has garnered a tremendous amount of attention since works such as [18] and [101], which first suggested the existence of flat bands in the electronic band structure of twisted bilayer graphene: bilayer graphene with a relative twist between the layers. Astonishingly, this phenomenon occurs for specific twist angles dubbed magic angles. This quenching of kinetic energy sets the stage for electron-electron interactions to dominate the behavior of the material. The onset of Mott insulating states at several carrier densities as seen through peaks in inverse compressibility [127] drew our interest. In collaboration with Oitmaa and Singh, [48], we study the SU(4) Fermi-Hubbard model

as a candidate effective model for describing the low energy behavior of MATBG, where we uncover the onset of charge incompressibility and Mott gaps, indicating the existence of the insulating phase.

Additionally, we investigate the superconducting pairing mechanism of $LaNiGa_2$, which has been highlighted as a potential time-reversal symmetry breaking (TRSB) material below its critical temperature due to experimental evidence for internal magnetization, [34], [46], [97]. In our recent work, [92], we report the existence of a Hebel-Slichter (HS) peak in the nuclear-spin relaxation rate, an effect attributed to the divergence in the electronic density of states as the superconducting gap opens. Furthermore, we study models with triplet and singlet spin pairing respectively to ascertain a mechanism that reproduces a HS peak. In doing so, we find that two distinct superconducting gaps and TRSB may not coexist, which raises questions about whether $LaNiGa_2$ really does break time reversal symmetry or whether a different model is required to fully characterize this material in its superconducting phase.

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Throughout my doctoral degree, I have been fortunate enough to learn from and collaborate with a number of brilliant faculty members. It would not be an exaggeration to say that my advisor Bruno Nachtergaele is a wizard, possessing a breadth of knowledge in mathematical physics whose vastness never ceases to amaze me. As a mentor, he was kind, patient, and open to discussion, and he allowed me to grow not only as a mathematician/physicist but also as a communicator. He taught me the invaluable lesson that simple and less are always more. It would be an understatement to say that he impacted my growth tremendously. Rajiv Singh thoroughly contributed towards my appreciation and passion for condensed matter physics/materials; his love and interest in the field are contagious, and his insightful questions in our meetings and seminars helped cultivate my physical intuition.

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Table of Contents

1. Introduction 1

1.1. Outline 5

Part 1. Dynamics of Interacting Fermions in the Continuum 9

2. Quantum Lattice Systems and Dynamics 9

2.1. Local Hilbert Spaces and Algebras of Observables 10

2.2. Interactions and Decay Classes 13

2.2.1. Short range interactions 14

2.3. Example Models 15

2.3.1. Heisenberg Model 16

2.3.2. Hubbard Model 17

2.4. Heisenberg Dynamics and Lieb-Robinson Bounds 20

2.4.1. Lattice Fermion Lieb-Robinson Bounds 24

2.5. Thermodynamic Limit and Quasi-locality 27

3. Integer Quantum Hall Effect 29

3.1. Landau Hamiltonian 30

3.2. Magnetic Translation and Frame 37

4. Continuum Fermions 44

4.1. Fock Space and Second Quantization 45

4.2. CAR Algebra and Dynamics 50

4.3. Many-Body Quantum Hall Model 52

4.3.1. UV Regularization 54

4.4. Single-particle Propagation Estimates 58

4.5. Lieb-Robinson Bounds 71

4.6. Thermodynamic Limit 82

5. Spectral Flow 86
5.1. Preliminaries and Notation 86
5.1.1. Localized Frames 86
5.1.2. Fermionic CAR algebra and Conditional Expectation 93
5.2. Hamiltonians and Heisenberg Dynamics 95
5.2.1. Interactions and Decay Assumptions 95
5.2.2. Finite volume Hamiltonians and Heisenberg Dynamics 97
5.3. Hastings Generator and Spectral Flow 107
5.3.1. Finite System Ground States 118
5.3.2. Thermodynamic Limit 119

6. Quantization of Hall Conductance 129
6.1. Continuum Systems via Lattices 129
6.1.1. Dynamics and Lieb-Robinson Bounds 134
6.2. Statement of Results Concerning Hall Conductance 137
6.3. Proofs of Theorems 6.6-6.8 143

Part 2. Studies of Effective Models for Correlated Electron Systems 160

7. Fermi-Hubbard model and Multi-layer Materials 160

8. Unconventional Superconductivity in $LaNiGa_2$ 174

Part 3. Conclusion 190

9. Review and Discussion 190

A. Appendix 197

References 201

1. Introduction

As we go about our day, we interact with and use objects that are nothing more than solids, liquids, or gasses, be it a cup of coffee in the morning, a smartphone, or the very air we breathe. The fact that matter like this is ordinary and stable is something we often take for granted because at the microscopic level, matter is of course made up of order 10^{23} many particles which all interact with each other via electromagnetic forces. Thus, it is quite spectacular to think that matter is actually stable and not highly volatile, considering that its constituent electrons and positive nuclei feel an attractive force. As it turns out, by zooming in and studying the quantum mechanical picture of electrons interacting among each other and with nuclei, the stability of matter has been rigorously proved by showing that the lowest energy state of this system is bounded from below, [66, 67]. However, the behavior of matter is considerably more complex than just a stable existence as a solid, liquid, or gas. In the last hundred years, experimental and theoretical progress in quantum mechanics have uncovered the existence of highly nontrivial electronic phases of matter, shattering our conventional understanding of materials.

One of the most unusual states that was discovered, which is the focus of the first part of this dissertation, occurs when an effectively two-dimensional sample of metal is subject to a constant perpendicular magnetic field. The classical behavior of electrons in this environment was first discovered in 1897 by Edwin Hall [37], who put a gold leaf in a circuit which drove current across the sample and supplied a perpendicular magnetic field at room temperature. What Hall found was the existence of a transverse voltage measured perpendicular to the direction of current driven by the circuit. This phenomenon can be explained by classical electricity and magnetism, where the Lorentz force indicates that a moving charge in a magnetic field experiences a force perpendicular to both the field and its velocity. This net force drives a perpendicular current, and hence voltage, across the sample, which have since been named the Hall current and voltage.

Almost one hundred years later, once liquid helium-esque temperatures and stronger magnetic fields became experimentally accessible, the quantum setting for the Hall effect could be studied, and this led to a remarkable finding. In 1980, Klaus von Klitzing published a seminal paper based on his studies of silicon field-effect transistors, [108], which allowed him to produce a two-dimensional electron gas more faithfully, where he discovered the quantum Hall effect. At temperatures on the order of 1-4 kelvin, he found that in varying the magnetic field strength, the Hall resistance formed quantized plateaus characterized by integer values, and at these same points, the longitudinal resistance effectively drops to zero. In fact, as had been solved in 1930 by Landau [64], the Hamiltonian describing a single charged particle in a magnetic field yields a spectrum of equally spaced energy eigenvalues, known as Landau levels. The integer quantization of Hall resistance is thus intimately linked to the filling of an integer number of Landau levels in the many-body state. Then, two years later came the discovery of a fractional quantization of Hall resistance, characterized by the specific fraction of $\frac{1}{3}$, in Gallium Arsenide based materials by Tsui, Stormer, and Gossard, [106], and more fractions have since been found in numerous studies. Notably, the seminal work of Laughlin the following year, [65], established the pivotal role of electron-electron interactions in this fractional setting, in contrast to the noninteracting integer case.

The theoretical investigation of this quantization of Hall resistance, arising from the so called integer/fractional quantum Hall effect, respectively (I/F QHE), has been the subject of heavy research. In 1982, [103], Thouless, Kohmoto, Nightingale, and den Nijs established that the integer case is characterized by a topological invariant, precisely the Chern number of Landau level bands. To do so, they considered the IQHE with a periodic potential and used the Kubo formula to calculate the Hall conductance, which yields an integer. Then, in 1985, [7], Avron and Seiler established that this topological invariant is robust under more general forms of background potentials, including interactions. Quantum Hall systems are thus widely considered to be the first discovered topological phase of matter, a state characterized by a topological invariant that is robust under deformations and perturbations to the system.

Rigorously proving the stability of a phase under perturbation is a challenging task, especially for systems featuring multi-particle interactions. However, a tool has emerged from studying the theory of operators, which physically translate to quantum mechanical observables. In their seminal 1972 paper, [69], Elliott Lieb and Derek Robinson proved the first instance of what are now called Lieb-Robinson bounds. Concretely, they establish a finite rate of information propagation in lattice spin systems governed by local, finite range interactions, which can be thought of as a non-relativistic analogue to the speed of light. The authors do so by considering two observables A and B that act on distant groups of sites, meaning $[A, B] = 0$. They then show that the size of $[A(t), B]$, with $A(t)$ the time evolved observable in the Heisenberg picture, decays in how far apart A and B initially act but grows in time at a maximal rate v. This result and many more that have been published since establish that for an initially local operator A, the predominant support of $A(t)$ does not suddenly grow to become all of space, but rather it gradually expands.

Due to the work of Bachmann, Michalakis, Nachtergaele, and Sims [11], inspired by the construction of Hastings and Wen [39], it was shown that the same notion of locality can hold for parameter space evolution $U(s)$, which is what maps the ground state of an initial Hamiltonian $H(0)$ to that of a perturbed model $H(s)$. In [58, 59] Tosio Kato rigorously establishes the mechanism by which this parameter space evolution $U(s)$ is generated, via equations of parallel transport. In this formalism, however, there is no reason to believe that $U(s)$ is a locality preserving evolution as described in the previous paragraph. This is precisely where the results of [11] show that there is a quasi-adiabatic evolution, as modeled by Hastings, which satisfies Lieb-Robinson bounds and connects the gapped ground state phases of lattice systems.

In lattice systems, the conditions required for these mechanisms to work naturally arise in physical systems, such as finite/short range interactions that have bounded energy per finite volume/collection of particles. Moreover, in condensed matter physics, the crystalline description of materials lends nicely to lattice-based Hamiltonians to describe electron behavior, such as the Heisenberg spin and Fermi-Hubbard models. However, these electrons ultimately live in continuous space, a realm that has eluded profound study in terms of Lieb-Robinson bounds until a result from 2020 by Gebert et al. [31]. Thus in Part 1 of this dissertation, the objective is to further bridge the gap between continuum space models, Lieb-Robinson bounds, and their applications. We do this by studying the dynamics in time and parameter space of interacting quantum Hall systems in the continuum towards proving quantization of Hall conductance (in finite and infinite volume).

The central point of emphasis until now has been the role of electron interactions in generating phases such as FQHE, propagating information, and supporting stability. Some of the most groundbreaking progress in our understanding of materials has emerged from the discovery of new interaction-driven phases of matter and materials that exhibit them, such as phonon-mediated Cooper pair formation in conventional superconductors (Bardeen-Cooper-Schrieffer BCS) [16]. Recently, multi-layer moiré materials have attracted a lot of research effort due to their ability to host a variety of phases, from superconducting to anomalous Hall insulating phases, but the story formally began with magic-angle twisted bilayer graphene (MATBG). In 2011, Bistritzer and MacDonald theorized that when two layers of graphene are stacked with a relative twist angle between them, specific twist angles give rise to nearly flat bands in the electronic spectrum, [18]. Flat bands are a playground for electron interactions to dominate and pave the way for different phases to emerge, such as Mott insulation as seen by rigidity in the carrier density [127]. Thus, Part 2 begins with a study of the onset of this Mott insulating phase through the SU(4) Hubbard model, which is a canonical starting point for describing electrons that experience Coulomb repulsion at short distances and possess two spin-$\frac{1}{2}$ degrees of freedom.

Returning back to the story of superconductivity, while BCS theory provided the first mechanism that explains the phenomenon, in time it became clear that it could not be the only one. For example, in 1986 Bednorz and Müller observed a critical temperature (T_c) of 35K in lanthanum barium copper oxide [17], which earned them a Nobel prize the following year and galvanized the physics community to pursue the discovery of high-T_c materials and other unconventional superconductors. One that has garnered the efforts of several faculty members at UC Davis is lanthanum nickel gallium-2 ($LaNiGa_2$), which hosts a band structure with five Fermi surfaces that intersect to form Dirac lines and loops, [13]. Moreover, it is believed to be a candidate triplet superconductor that breaks time reversal symmetry below its critical temperature and has two superconducting gaps, as supported by some experimental evidence, [46, 97]. A theoretical explanation of this effect in $LaNiGa_2$ has evaded discovery; however, an effective model does exist which could shed some light, namely the nonunitary triplet model, as defined in [80]. In Part 2 of this thesis, we also explore what lessons can be taken from this model in relation to $LaNiGa_2$'s superconducting state and pairing mechanism.

1.1. **Outline.**

The presentation of results in this dissertation is written in two parts. Part 1 focuses on the use of functional analysis and operator algebra theory to rigorously study the Heisenberg dynamics of interacting electron systems, and Part 2 contains studies of two specific materials based on effective models that aim to describe experimentally observed features.

In Section 2, we begin with an introduction to quantum lattice systems, the setting in which most progress has been achieved in proving properties of the dynamics generated by interacting Hamiltonians. This involves building the local Hilbert spaces and algebras of observables needed to define many-body states and operators that act on these states for spin and fermion systems. We then formally define time evolution in the Heisenberg picture (dynamics) before stating theorems that establish Lieb-Robinson bounds. These have many uses, including establishing continuity of dynamics in the thermodynamic limit, so we conclude this section with discussion of this result.

Section 3 introduces the well known Landau Hamiltonian, the single-particle model that describes a charged particle in two-dimensions in the presence of a uniform, perpendicular magnetic field, the simplest model to begin studying the quantum Hall effect. We take time to work through the elegant solution of this Hamiltonian, stating the unique spectrum of this operator, and we elaborate on different sets of functions which can be used to span the associated eigenspaces.

With an understanding of the environment that underlies the behavior of electrons in our system of interest, barring interactions, Section 4 builds the machinery needed to define a many-body Hamiltonian that features inter-particle interactions. This process begins with a review of the Fock space construction and the requirement of antisymmetrization for fermionic systems to ensure that accurate particle-exchange statistics are enforced. The bounded observables that act on these states belong to a canonical anti-commutation relations (CAR) algebra, which is generated by creation and annihilation operators that create and remove single particle states respectively. These are the building blocks of interactions as we shall see in detail. We use these tools to define a many-body Hamiltonian, inspired by [31], which features a bounded pairwise interaction in finite volume, a requirement of lattice models ahead of proving Lieb-Robinson bounds in the discrete setting. The remainder of this section then proceeds with an argument for how to prove these information propagation bounds and apply these to prove strong continuity of the dynamics in the limit of infinite volume interactions. The contents of this section are based on [47], which is joint work with

Bruno Nachtergaele and Robert Sims (soon to be published).

Towards addressing questions concerning the stability of a phase of matter under perturbation however, we employ Lieb-Robinson bounds as introduced in [12]. Thus to begin Section 5, we note the preliminaries needed to define a lattice-localized frame that can generate interacting fermion systems in continuous space. We then show how to construct a generalized notion of partial trace that can be used to write local approximations of quasi-local operators of the CAR algebra, namely the Heisenberg evolution of a local operator. Following these preliminaries, we then define a family of parametrized Hamiltonians and prove that the standard Hastings generator gives rise to quasi-local dynamics, meaning evolution in parameter space that satisfies Lieb-Robinson bounds.

In Section 6, the results of Section 5 are directly applied towards the objective of proving quantization of Hall conductance for continuum fermions in two dimensions with a constant perpendicular magnetic field. We restrict the analysis to the lowest two Landau levels, which are relevant for describing fractional quantum Hall physics with the possibility for inter-Landau level interactions. The contents of both Sections 5 and 6 are based on joint work in preparation with Sven Bachmann, Giuseppe De Nittis, and Bruno Nachtergaele, [10]. The results of this section conclude the first part focusing on the dynamics of continuum fermion systems.

Part 2 dives into numerical studies of the fascinating materials: magic angle twisted bilayer graphene and $LaNiGa_2$. Section 7 is an article that discusses lessons we can learn about the Mott insulating phase in magic angle graphene systems as seen via inverse electronic compressibility, which indicates a locked in charge carrier density per site. The first portion of the work discusses the strong coupling expansion that we use to study this phenomenon, namely one where kinetic energy is treated perturbatively, and we then use this to generate plots of thermodynamic quantities with the aim of diagnosing Mott insulation. Furthermore, in this insulating regime, we comment on how the expansions at high temperature are related to a generalized SU(4) spin model. The work of this study is based on a publication in collaboration with Rajiv Singh and Jaan Oitmaa, [48].

In the second work of this part of the dissertation, we present novel findings regarding the superconducting state of $LaNiGa_2$. Namely, Section 8 is based on [92], a project spearheaded by Phurba Sherpa from Curro's NMR group at UC Davis. We begin with discussion of the fundamental questions concerning the pairing mechanism and breaking of time reversal symmetry in the superconducting phase. We then transition into an explicit calculation of the excitation spectrum and nuclear-spin relaxation rate (T_1^{-1}), all taken from the supplemental materials of the article. Then we show a plot of experimental data and best fit curves, from spin triplet/singlet Hamiltonians, for T_1^{-1} versus temperature and a table with the superconducting gap values. These are the basis for discussion of future work that can elucidate the competing descriptions of $LaNiGa_2$'s behavior. This study is based on a publication in collaboration with Phurba Sherpa, the Curro group at UC Davis, and the strongly correlated electron group at Los Alamos National Laboratory.

We conclude the dissertation in Part 3, where we review the key results of each section and outline directions for future work.

Part 1. Dynamics of Interacting Fermions in the Continuum

2. Quantum Lattice Systems and Dynamics

In our daily lives, the changes of phase that occur between solid, liquid, and gaseous states are phenomena that feel commonplace due to how established these are in our understanding of day-to-day nature. If one breathes hot air onto a window, the air condenses to form droplets on the glass surface. If left out at room temperature for long enough, the ice cubes in a beverage melt away to become liquid water. Furthermore, it is common knowledge that varying temperature and pressure allow materials to transition between these three phases.

Over the last hundred years since the inception of quantum mechanics, research has enhanced our understanding of materials such as metals, insulators, and semi-conductors and elucidated the existence of a diverse variety of phases such as ferro/antiferro-magnetic, superconducting, and topological insulating states, for example. By modifying some parameter or physical feature, which could be temperature, it is possible for a material to undergo a transition between these exotic phases as well. In addition to discovering such states of matter, uncovering key features of these states at zero/very low temperatures has been the subject of tremendous research in mathematical physics. Real materials themselves are extremely difficult to study analytically due to their complexity, which poses the challenge of how to construct models that can capture the physics of real materials while also being solvable.

The discovery of crystalline structures in materials, in which the atoms in a compound form a periodic arrangement, motivated the use of a lattice-based picture to model the locations and interactions of particles. In such a setting, the nuclei form a periodic lattice, and electrons are free to localize and move between the sites of the lattice. This microscopic description has paved the way for many precise models, with repulsive and attractive interactions between nearest neighbor particles or particles occupying the same site to name examples.

One important feature of all these lattice descriptions is the locality of their constituent interactions. In real materials, where electric and magnetic forces dictate the behaviors of electrons, protons, and other quasiparticles, interactions between particles are strongest when they are close to each other, and the farther apart they grow, the weaker their interaction becomes. Thus, in quantum lattice systems, interactions have either finite or short range, meaning the strength decays sufficiently fast in separation distance. This is seen in the fundamental Coulomb repulsion with r^{-1} decay in the particle separation or the Yukawa potential arising from screening with e^{-ar}/r decay, [118]. The absence of arbitrarily strong, long range interactions allows for a rigorous notion of causality in low energy systems through Lieb-Robinson bounds, which were first introduced by Elliott Lieb and Derek Robinson in their seminal work published in 1972, [69]. If A, B are two distinct measurements/observables that act on distant regions in space, their actions commute with each other. What this result shows is that if A acts some time after B, this will no longer hold; however, the further apart the actions of A and B, the longer it would take for this change to become noticeable.

As shall be made evident through the contents of this dissertation, the finite rate of propagation of information that Lieb-Robinson bounds justify has a number of powerful applications, and one we will concretely review is the extent to which Heisenberg dynamics are continuous in the thermodynamic limit. Thus, in this chapter, we introduce the setting in which these results have been predominantly studied, i.e. lattice systems, before defining Lieb-Robinson bounds and showing how one can prove such an estimate.

2.1. **Local Hilbert Spaces and Algebras of Observables.**

Formally, the setting in which a collection of quantum particles exists is represented by the metric space (Γ, d). Γ is the lattice which serves as a discretization of space; for 1-dimensional systems, $\Gamma = \mathbb{Z}$ or any finite subset of the set of integers. In two dimensions, one could make use of a square lattice $\mathbb{Z}^2$, the more general rectangular one given by $a_1\mathbb{Z} \times a_2\mathbb{Z}$ for some $a_1, a_2 > 0$, or other more complex structures formed by linear combinations of pairs of the so-called lattice vectors in $\mathbb{Z}^2$. In any case, Γ encodes the physical locations of particles, and the metric d, which we can take to be the standard Euclidean distance, provides a measure of the distance between sites of Γ. In general, a distance function d is a suitable metric if it satisfies the following properties for all $x, y, z \in \Gamma$.

$$i)\ d(x,y) \geq 0 \text{ (Non-negativity)}$$

$$ii)\ d(x,y) = 0 \text{ implies } x = y \text{ (Non-degeneracy)}$$

$$iii)\ d(x,y) = d(y,x) \text{ (Symmetry)}$$

$$iv)\ d(x,y) \leq d(x,z) + d(z,y) \text{ (Triangle inequality)}$$

For a detailed discussion of metric spaces and their features, one may refer to Chapter 2 of [87].

To each site x of the lattice, we ascribe a complex Hilbert space $\mathcal{H}_x$, which encodes the possible states ψ of the particle at site x, and an algebra of observables $\mathcal{A}_x := \mathcal{B}(\mathcal{H}_x)$, the group of bounded operators that can act on states $\psi \in \mathcal{H}_x$. By bounded operator on $\mathcal{H}_x$, we mean that for any $A \in \mathcal{A}_x$,

$$\sup_{\substack{\psi \in \mathcal{H}_x: \\ \|\psi\|=1}} \|A\psi\| < \infty$$

I refrain from making these objects more precise now, but when we introduce example models, specific Hilbert spaces and algebras of observables will be required. Denote by $\mathcal{P}_0(\Gamma)$ the collection of finite subsets of Γ. Then for any $\Lambda \in \mathcal{P}_0(\Gamma)$, one forms the multi-site Hilbert space $\mathcal{H}_\Lambda$ and algebra $\mathcal{A}_\Lambda$ via tensor products of the site-wise Hilbert spaces and algebras respectively:

(1)
$$\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{H}_x;\ \mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x = \mathcal{B}(\mathcal{H}_\Lambda).$$

When the particles in question have exchange statistics, these definitions require slight modifications, but we shall return to this subtlety upon encountering the said case. This framework establishes how to express the possible states and observables for finite collections of sites; however, generalizing this notion to the infinite volume system requires caution. For any two $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Gamma)$ such that $\Lambda_1 \subset \Lambda_2$, any operator $A \in \mathcal{A}_{\Lambda_1}$ can be identified with

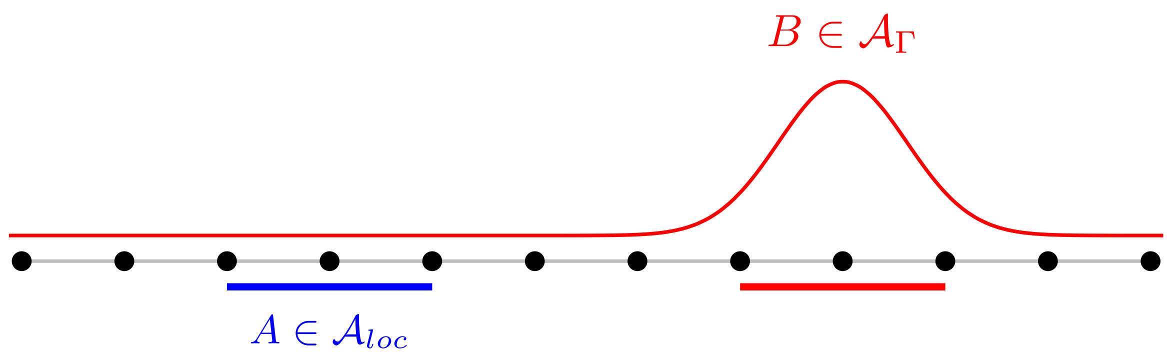

Figure 1. A visual representation of the supports of local versus quasi-local operators. A only acts nontrivially on the three underlined sites in blue, whereas B acts on every site of the lattice. However, the action of B is most pronounced on the three underlined sites in red, suggesting that B can be approximated by a local operator acting on just those sites.

$A \in \mathcal{A}_{\Lambda_2}$ using the tensor product structure of the finite volume spaces in 1. Since A only acts on sites in Λ_1, as an element of $\mathcal{A}_{\Lambda_2}$ it can be written as $A \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1}$ to explicitly encode the absence of action on sites outside of Λ_1. With respect to identifying A with $A \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1}$ for any Λ_2 containing Λ_1, the **support** of A is Λ_1. Thus, the collection of all **local** observables, i.e. those with finite sized support, is given by the following inductive limit:

(2)
$$\mathcal{A}_{loc} = \bigcup_{\Lambda \in \mathcal{P}_0(\Gamma)} \mathcal{A}_\Lambda.$$

There also exists a more general class of observables, named the quasi-local algebra, consisting of those operators that can be written as the limits of sequences of local ones. Such objects are especially pertinent in describing the Heisenberg dynamics of local operators, and we define this general algebra as the completion with respect to the operator norm on $\mathcal{A}_{loc}$.

(3)
$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$$

The one-dimensional example in Figure 1 illustrates the difference between strictly local and quasi-local operators in terms of their supports and norms. We now have the ingredients needed to formally define interactions between sites and build finite volume Hamiltonians, which is the subject of the next section.

2.2. **Interactions and Decay Classes.**

An interaction $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_{loc}$ is a map that takes in a finite subset Λ and outputs a local operator whose support is given by Λ. It is important to add that $\Phi^* = \Phi$ (self-adjoint) and is bounded in norm (i.e. finite energy). Physically, this defines an energy of interaction of all the quantum particles situated in a subset, and in the following sections we will analyze specific examples through models that are paramount to condensed matter physics. In general, these interactions can also be time-dependent; however, for the purpose of constructing quantum lattice models and presenting Lieb-Robinson bounds without additional details, we restrict our present discussion to time-independent interactions.

In terms of these maps Φ, Hamiltonians in finite regions $\Lambda \in \mathcal{P}_0(\Gamma)$ take the general form,

$$H_\Lambda = \sum_{X \in \Lambda} \Phi(X). \tag{4}$$

Since Φ is self-adjoint, the same holds for H_Λ. With our general construction, Φ is a map that can take in any finite region, regardless of size, and output an operator. However, we can be more precise about how the norm of $\Phi(X)$ actually depends on the number of sites in X and its spatial diameter, meaning the largest distance between a pair of sites in X. Thus, we introduce the notion of the range of an interaction.

Definition 2.1. An interaction Φ is said to be **finite range** if there exists a constant $R > 0$ such that $\Phi(X) = 0$ if $\mathrm{diam}(X) > R$.

Thus, such an interaction only provides an energetic contribution to the Hamiltonian if the support X is sufficiently small, meaning the particles are close together. Noting, for example, that the Coulomb potential energy between electrons decays as $\frac{1}{r}$, beyond a certain distance R, this energy becomes arbitrarily small. Hence, establishing a strict cutoff distance between particles after which the interaction yields zero is a natural way to encode the spatial decay of Φ in the diameter of its support. Conversely, an alternate assumption on Φ would be to permit its domain to include subsets of arbitrary size but impose decay conditions on Φ. Coulomb decay in particular is longer range than what we will consider to be short range, but this interaction still captures the feature of decreasing strength with particle separation. Characterizing a useful notion of short range decay is then precisely the subject of the subsequent portion of this section.

2.2.1. *Short range interactions.* When defining finite range interactions, it is straightforward to introduce a cutoff distance R beyond which lattice sites cannot interact with each other. In the case of short range the question becomes one of determining how rapidly the interaction strength decays as the separation between sites increases. In a number of works that followed the first result of a Lieb-Robinson bound in 1972, the decay of lattice-based interactions has been described using the so-called *F-functions*. These have been discussed in detail in multiple works pertaining to locality estimates in quantum lattice systems, e.g. [74].

An *F-function* on the metric space (Γ, d) is a positive, non-increasing function $F : [0, \infty) \to (0, \infty)$ that satisfies the following two properties.

$$(5) \quad i) \ \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) =: \|F\| < \infty$$

$$ii) \text{ There exists a constant } C_F > 0 \text{ such that } \sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) \le C_F F(d(x,y))$$

The first feature is a statement regarding integrability of the function, which necessarily implies that it decays fast enough and does not have any singular behavior. The second establishes that F satisfies a convolution condition. As we shall see in the discussion of dynamics and Lieb-Robinson bounds, these properties play a crucial role in the rigorous proofs.

A standard example of such a function is $F_0(r) = \frac{1}{(1+r)^\nu}$, where ν is a positive constant larger than the lattice dimension, $\nu > 2$ in the quantum Hall setting of Section 3 onward. Its form has polynomial decay, and due to the form of the denominator, there is no $r = 0$ divergence as in the case of the Coulomb potential. More generally, $F_0(r)$ can also serve as the base for a class of F-functions of varying decay. Now, let $g : [0, \infty) \to [0, \infty)$ be any non-decreasing function satisfying subadditivity: $g(r + r') \leq g(r) + g(r')$. Then the family of functions $F_g(r) = e^{-g(r)} F_0(r)$ all satisfy 5. A couple of examples of functions $g(r)$ that will be relevant in future proofs are $g(r) = ar^\theta$ with $a > 0, 0 \leq \theta \leq 1$ and $g(r) = qln(1 + r)$ with $q > 0$. There are of course other candidate functions satisfying these properties, and when we shift to the setting of continuous space as opposed to the lattice, these will be presented. With respect to F-functions, it is possible to introduce a norm on the space of bounded operators that encodes the ability of an interaction Φ to decay sufficiently fast in the size of its support.

The F-norm $\|\cdot\|_F : \mathcal{A}_{loc} \to \mathbb{R}$ is defined by its action as shown,

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{Z \in \mathcal{P}_0(\Gamma): \\ x,y \in Z}} \|\Phi(Z)\|, \tag{6}$$

and the space of observables with finite F-norms is denoted by,

$$\mathcal{B}_F = \{\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma | \Phi \text{ is an interaction and } \|\Phi\|_F < \infty\}. \tag{7}$$

Ultimately for quantum lattice systems, interactions that are either finite range or have bounded F-norm are conducive towards studying how interactions affect the propagation of information over time. Before elaborating on this phenomenon, we present some well-studied Hamiltonians in condensed matter physics that satisfy either the finite range or short range conditions introduced in this section.

2.3. **Example Models.**

In the pursuit of studying systems of interacting electrons at a microscopic level inside various materials, a number of simple yet tremendously influential models have been introduced. Part of what renders them simple is the range of their interactions; the examples we present here are all finite range, with short range extensions feasible for them all. In each case, we construct the local Hilbert spaces and algebras of observables so that the domain and range of the interaction maps are clear. We begin with arguably the most important description of the spin-exchange interaction between electrons localized on atomic sites.

2.3.1. *Heisenberg Model.* Inspired by the progress of Heitler and London in showing the spin dependence of the Pauli-exclusion induced exchange interaction in the Hydrogen atom, [44], Werner Heisenberg realized that lifting such an interaction to a lattice of electron spins could yield the ferromagnetic behavior that Weiss theorized, [43]. Furthermore, with the contributions of Dirac in 1929 which made use of operator algebra theory of the so-called Pauli matrices, [30], the precise form of the Heisenberg spin interaction was deduced in terms of operators acting on spin-$\frac{1}{2}$ electron states. With this in mind, we now lay the foundation for defining the Heisenberg spin model.

In this lattice system, each electron is localized about an atomic lattice site $i \in \Gamma$ and admits an on-site Hilbert space $\mathcal{H}_i = \mathbb{C}^2$. Moreover, the algebra of observables $\mathcal{A}_i = \mathcal{B}(\mathcal{H}_i)$ corresponds to the space of complex Hermitian 2×2 matrices. As defined in 1, the local Hilbert spaces and algebras of observables are formed by taking tensor products over sites of the on-site Hilbert spaces and algebras. For a finite $\Lambda \in \mathcal{P}_0(\Gamma)$,

$$\mathcal{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathbb{C}^2 = \mathbb{C}^{2|\Lambda|}; \ \mathcal{A}_\Lambda = \mathcal{B}\left(\mathcal{H}_\Lambda\right) \ \text{(complex Hermitian } 2^{|\Lambda|} \times 2^{|\Lambda|} \text{ matrices).}$$

The corresponding local and quasi-local algebras $\mathcal{A}_{loc}$ and $\mathcal{A}_\Gamma$ are then constructed as in equations 2-3. We may now introduce the finite volume Hamiltonian that is the Heisenberg spin model.

Fix a finite volume $\Lambda \in \mathcal{P}_0(\Gamma)$ and recall the well-known Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

In terms of these matrices, the vector valued spin operator $\mathbf{S}$ can be defined component-wise as $S^q = \frac{\hbar}{2}\sigma^q$, where $q = x, y, z$. Note that each such spin matrix is a 2×2 complex Hermitian matrix and is thus an element of $\mathcal{A}_i$. As previously explained, such an operator can be lifted to finite volume algebras via tensor products with the identity:

$$S_i^q \sim S_i^q \otimes \left(\bigotimes_{j \in \Lambda \setminus \{i\}} \mathbb{1}_2 \right),$$

where the subscript i on S_i^q indicates that the spin operator S^q acts on the site i. Additionally, with this notation in hand denote by $\mathbf{S}_i$ the vector valued spin operator associated with site i. Then, letting $J > 0$, the Heisenberg interaction $\Phi_H : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma$ is precisely the following.

$$\Phi_H(X) = \begin{cases} -J\, \mathbf{S}_i \cdot \mathbf{S}_j & \text{if } X = \{i,j\} \text{ and } d(i,j) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The Hamiltonian for this model is then given by $H_\Lambda = \sum_{X \in \Lambda} \Phi_H(X)$. Evidently, this interaction is finite range, with $R = 1$, and has bounded norm, given by $J \frac{3\hbar^2}{4}$. Given how influential this spin system has been in condensed matter physics, extensions of spin-spin interacting models exist with longer range interactions.

2.3.2. *Hubbard Model.* Like the previous example, the Hubbard model was also conceived to better understand the onset of ferromagnetism, in this case particularly within transition metals. In the Heisenberg spin system, electrons were treated as particles that localized at sites, and as Heitler and London considered f-electrons, this was an effective approach as Hubbard notes, [52]. However, with the d-electrons of transition metals, dynamical motion is also feasible. Furthermore, as Hubbard, Gutzwiller [36], and Kanamori [56] all remark, a solely free electron based model is insufficient when it comes to describing the correlation effects that lead to ferromagnetism in such metals. Thus, in the Hubbard model, which each of these three authors describe in works published in 1963, electrons are modeled with features that draw from both the tight-binding free electron and atomic pictures.

In order to construct the interactions that constitute the Hamiltonian in question, as in the previous example we must carefully introduce the local algebra of observables. Precisely, the relevant algebra is generated by the so called fermionic creation and annihilation operators derived from second quantization. Towards the discussion of the dynamics generated by the Hubbard Hamiltonian, it is not essential here to define the Hilbert spaces in detail, so we focus on the construction of the algebra of observables. The starting point is the metric space (Γ, d) in two-dimensional space, which one can envision as a square lattice with Euclidean metric, and restrict to a finite volume $\Lambda \in \mathcal{P}_0(\Gamma)$. This model seeks to describe electrons as spin-$\frac{1}{2}$ fermions, where fermions are particles that are antisymmetric under exchange. This means that if, for example, $\psi \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is a two-particle wavefunction describing two identical fermions, it must satisfy antisymmetry in its coordinates: $\psi(x_1, x_2) = -\psi(x_2, x_1)$. Algebraically, this can be encoded for many-fermion models via the canonical anticommutation relations (CAR) algebra.

In the finite sized lattice Λ, each site admits four possible occupation states: 1) no fermion present, 2) one up-spin fermion present, 3) one down-spin fermion present, and 4) two fermions of opposite spin present, with no more allowed states because of the Pauli exclusion principle. The on-site operators that create or remove a spinful fermion from a given lattice site are denoted by $c_{i\sigma}^*$ and $c_{i\sigma}$ respectively for the site $i \in \Gamma$ with z-component of spin either $\sigma =\uparrow, \downarrow$. Importantly, each of these are bounded operators with operator norm of 1. The fermionic CAR algebra, associated with the entire lattice, $\mathcal{A}_\Gamma := \mathcal{A}\left(\ell^2(\Gamma) \otimes \mathbb{C}^2\right)$, is thus generated by $\{c_{i\sigma}, c_{i\sigma}^* : \ i \in \Gamma, \ \sigma =\uparrow, \downarrow\}$. Thus, this algebra includes linear combinations of monomials in the operators $c_{i\sigma}^*$ and $c_{i\sigma}$, with bounded operator norm. These individual operators satisfy the following algebra defining relations, the CAR relations:

$$\{c_{i\sigma}, c_{j\sigma'}\} = \{c_{i\sigma}^*, c_{j\sigma'}^*\} = 0; \ \{c_{i\sigma}, c_{j\sigma'}^*\} = \delta_{ij}\delta_{\sigma\sigma'}\mathbb{1}. \tag{10}$$

Furthermore, for finite subsets $\Lambda \in \mathcal{P}_0(\Gamma)$, local subalgebras of $\mathcal{A}_\Gamma$ are simply generated by those creation and annihilation operators with lattice sites contained in Λ. We refrain from discussing the origins of these operators in too much detail here as this will be done more carefully when discussing fermions in continuous space. Before constructing the Hubbard Hamiltonian, it is important to introduce the local number operators. For a given site $i \in \Gamma$, $n_{i\sigma} := c_{i\sigma}^* c_{i\sigma}$ is the number operator associated with spin σ at site i with spectrum $\{0, 1\}$ denoting the possible number of fermions with spin σ at the site. More generally,

$n_i := \sum_{\sigma\in\{\uparrow,\downarrow\}} n_{i\sigma}$ counts the number of fermions at site i without specifying spin, and $n_\Lambda := \sum_{i\in\Lambda} n_i$ is the local number operator associated with the finite region Λ. We are now equipped to construct the Hubbard model. The interaction $\Phi_{\mathrm{Hub}} : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma$ is defined as follows.

$$\Phi_{\mathrm{Hub}}(X) = \begin{cases} U\left(n_{i\uparrow}n_{i\downarrow}\right) - \mu n_i & X = \{i\} \\ -t\sum_{\sigma\in\{\uparrow,\downarrow\}} c_{i\sigma}^* c_{j\sigma} + h.c. & X = \{i,j\} \text{ and } d(i,j) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The constants U and t are both positive, and $\mu \in \mathbb{R}$ is the chemical potential. As described previously, the Hubbard model is built with the goal of incorporating features of both an atomic limit and a tight-binding free particle setting. This is achieved via an on-site electron-electron repulsion, whereby there is an energy cost of U for double occupancy of a lattice site, and a nearest-neighbor hopping with coefficient $-t$. In some cases, the on-site interaction is written as $U(n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2) - \mu n_i$ to emphasize particle-hole symmetry, which is very useful in numerical studies as discussed in [90]. The chemical potential μ is of course present to set the ground-state fermion occupancy density over the lattice. The Hubbard model is then just a sum over such subsets of the interaction: $H_\Lambda = \sum_{X\subset\Lambda} \Phi_{\mathrm{Hub}}(X)$. Variations of this model do exist, where for example a next-nearest-neighbor hopping term is also included, and it is also possible to define a Hubbard model for lattice bosons as opposed to fermions. The bosonic case differs drastically due to the absence of a bound on the on-site occupancy number due to the symmetric exchange statistics of bosons.

There are plenty of other important lattice-based Hamiltonians that describe the interactions of particles, and these can also be built using the language of local algebras of observables. Towards analyzing the propagation of information in such quantum systems, we proceed with defining the Heisenberg dynamics, which govern time evolution of quantum operators, and considering how the support of local observables changes under these dynamics.

2.4. **Heisenberg Dynamics and Lieb-Robinson Bounds.**

Crucially, Hamiltonians as in equation 4 are the generators of Heisenberg dynamics, $\tau_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, which act as automorphisms over the local algebra $\mathcal{A}_\Lambda$, meaning maps that preserve the additive and multiplicative structure of the algebra. Letting $t_0 = 0$ denote the initial time without loss of generality, this map is implemented via conjugation by a unitary operator $U_\Lambda(t)$ and its adjoint, and it has the precise form $U_\Lambda(t) = \exp\left(-itH_\Lambda\right)$ since the Hamiltonian is time independent. More generally, this family of 1-parameter unitaries gives the solution to the initial value problem:

$$\frac{d}{dt}U_\Lambda(t) = -iH_\Lambda U_\Lambda(t);\ U_\Lambda(0) = \mathbb{1}. \tag{12}$$

For an operator $A \in \mathcal{A}_\Lambda$, its Heisenberg time evolution can be written as,

$$\tau_t^\Lambda(A) = A(t) = U_\Lambda(t)^* A U_\Lambda(t). \tag{13}$$

A motivating question which underpins understanding of the propagation of information in interacting lattice models concerns how the support of a local operator changes under time evolution. Furthermore, how can this spread be concretely detected? Consider the setting of a quantum spin system such as the Heisenberg model, subsection 2.3.1, and take two elements $A, B \in \mathcal{A}_\Lambda$ that act on disjoint clusters of lattice sites separated by a distance r. It necessarily follows that their commutator $[A, B] = 0$; however, will the same hold true for $\left[\tau_t^\Lambda(A), B\right]$? By leaving B unevolved, it serves as a spatial probe for detecting the support of $A(t)$ and determining how rapidly it may grow. A Lieb-Robinson bound is an upper bound on precisely the norm of this commutator: $\left\|\left[\tau_t^\Lambda(A), B\right]\right\|$. Note that this is a sensible procedure for spin systems; however, Lieb-Robinson bounds for lattice fermion models take on a different form due to the canonical anticommutation relations that define the algebras of observables.

In any case, we begin with a statement of a Lieb-Robinson bound for quantum spin systems as presented in the seminal paper by Elliott Lieb and Derek Robinson from 1972, using the machinery we have introduced thus far in this section.

Theorem 2.2 (Lieb-Robinson Bound, [69]). *Let Φ be a finite-range interaction, 2.1, and $\Lambda \in \mathcal{P}_0(\Gamma)$ a finite lattice subset. Let H_Λ be a finite volume Hamiltonian as constructed in equation 4. Then there exist a constant $v_\Phi > 0$ and an increasing function μ such that for any $v > v_\Phi$ and $A, B \in \mathcal{A}_\Lambda$ such that $d(supp(A), supp(B)) = r$, the following equality holds.*

$$\lim_{\substack{|t|\to\infty \\ r>v|t|}} e^{\mu(v)|t|} \left\| \left[\tau_t^\Lambda(A), B \right] \right\| = 0 \tag{14}$$

At the core of this result is the notion that distant spins are communicating via the finite range interactions taking place in between them, and this process occurs nontrivially after finite time, with Lieb-Robinson velocity v_Φ dictating how rapidly this occurs. This first result does not provide an explicit bound on this commutator as a function of r, the distance between supports of A and B. However, the limit suggests that so long as r is outside the effective light cone $v|t|$, then $\left\| \left[\tau_t^\Lambda(A), B \right] \right\|$ decays to zero in the asymptotic limit of large t faster than $e^{\mu(v)|t|}$. Furthermore, the timescale set by $t_r = \frac{r}{v}$ is roughly how long it would take for this commutator to possibly become nontrivial, and by taking this to ∞, the Lieb-Robinson bound establishes that this nontriviality is never achieved. Figure 2 illustrates this phenomenon whereby the local operators A and B have supports separated by a distance r. Additionally, Heisenberg evolution of A effectively grows the support by a distance vt, but so long as this is less than r, then $\left\| \left[\tau_t^\Lambda(A), B \right] \right\|$ remains negligible. The constant v_Φ in the statement of the theorem denotes the so called Lieb-Robinson velocity, which indicates the maximum rate at which information travels, as seen via B as a probe for detecting the support of $A(t)$. Following this first result concerning information propagation bounds, generalizations of Lieb-Robinson bounds have been proved which account for short range interactions. Next, we present a statement of a Lieb-Robinson bound for quantum spin systems as written by Nachtergaele, Sims, and Young in their review from 2019, which makes use of the decaying F-functions. First, we introduce one useful piece of notation. Let the Φ-boundary of a subset X be the following:

$$\partial_\Phi X := \{ x \in X : \exists\, Z \in \mathcal{P}_0(\Gamma) \text{ with } x \in Z,\ Z \cap (\Gamma \backslash X) \neq \emptyset,\ \Phi(Z) \neq 0 \}\,. \tag{15}$$

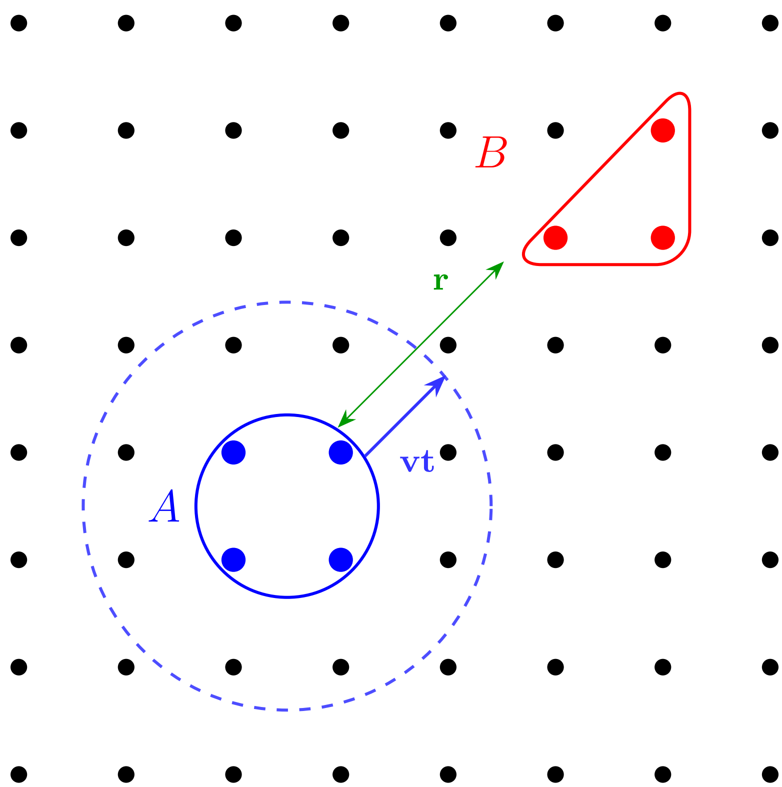

FIGURE 2. A finite lattice subset with two operators A and B that have disjoint supports X and Y, enclosed in blue and red respectively, separated by a distance r. Heisenberg evolution of A causes its support to spread, which is seen most notably within a distance vt of the original support as indicated by the dotted blue boundary line.

This is the set of points in X belonging to the supports of nontrivial interactions supported on $\Gamma \backslash X$. We now state the result.

Theorem 2.3 (Lieb-Robinson Bound, [74]). *Let $\Phi \in \mathcal{B}_F$, i.e. bounded F-norm 6, and $X, Y \in \mathcal{P}_0(\Gamma)$ such that $X \cap Y = \emptyset$. Then, for any $\Lambda \in \mathcal{P}_0(\Gamma)$ such that $X, Y \subset \Lambda$ and $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, we have,*

$$\left\| \left[\tau_t^\Lambda(A), B \right] \right\| \leq \frac{2\|A\|\|B\|}{C_F} \left(e^{2C_F \|\Phi\|_F |t|} - 1 \right) D(X, Y), \tag{16}$$

where $D(X,Y) = \min \left\{ \sum_{x \in X} \sum_{y \in \partial_\Phi Y} F(d(x,y)), \sum_{x \in \partial_\Phi X} \sum_{y \in Y} F(d(x,y)) \right\}$.

Notice two significant properties of this bound. First, it increases exponentially in time from zero, which captures the feature that two initially commuting operators that act on distant spins commute less as time progresses, due to an increase in the effective support of $A(t)$. Additionally, the function $D(X, Y)$ in the bound decays in the distance between the supports of A and B, and the rate of decay is governed by the decay class of the interaction Φ. Therefore, the further apart the initial supports of these operators and the shorter the range of the interaction, the smaller the bound and the longer it takes for it to become significant. Precisely, the Lieb-Robinson velocity in this case is given by $2C_F\|\Phi\|$, meaning that the rate of information propagation as understood by the commutator $[A(t), B]$, increases as the interaction strength $\|\Phi\|_F$ grows stronger. This interpretation further validates the picture shown in Figure 2.

One can also derive a nice interpretation of the bound 2.3 by refining this estimate when the F-function is precisely of the form $F_a(r) = e^{-ar}(1+r)^{-\nu}$. Suppose that the set X is smaller than Y. Then one has,

$$\sum_{x\in\partial_\Phi X}\sum_{y\in Y} F_a(d(x,y)) = \sum_{x\in\partial_\Phi X}\sum_{y\in Y} e^{-ad(x,y)} F_0(d(x,y)) \tag{17}$$

$$\leq e^{-ad(X,Y)} \sum_{x\in\partial_\Phi X}\sum_{y\in Y} F_0(d(x,y)) \tag{18}$$

$$\leq e^{-ad(X,Y)} |\partial_\Phi X| \sup_{x\in\partial_\Phi X} \sum_{y\in Y} F_0(d(x,y)) \tag{19}$$

$$= |\partial_\Phi X| \|F\| e^{-ad(X,Y)} \tag{20}$$

Then, returning to the Lieb-Robinson bound, one obtains,

$$\left\| \left[\tau_t^\Lambda(A), B\right] \right\| \leq \frac{2\|A\|\|B\|}{C_F} |\partial_\Phi X| \|F\| e^{-a(d(X,Y) - v_a|t|)}, \tag{21}$$

where $v_a = \frac{2C_F\|\Phi\|_F}{a}$, and the 1 subtracted from the exponential growth in time is estimated away. Crucially, this bound establishes an effective light cone that indicates for what times the upper bound will be exponentially small. This is illustrated concretely via the graph in Figure 3, where the line $v_a|t|$ is the threshold separation distance between the supports X and Y that enables $\left\| \left[\tau_t^\Lambda(A), B\right] \right\|$ to be small. We now present one more statement of a

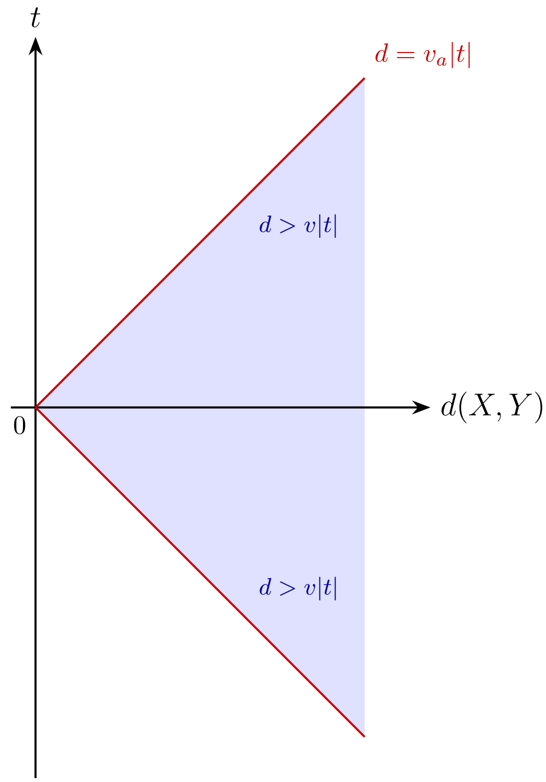

FIGURE 3. This graph features time t plotted against $d(X,Y)$, and the red lines correspond to the function $v_a|t|$, i.e. the light cone. The shaded region in blue defines the region where $d(X,Y) > v_a|t|$, which is precisely the regime where the Lieb-Robinson bound is exponentially small.

Lieb-Robinson bound that will be pertinent to our discussion of interacting fermions as we transition from discretized to continuous space.

2.4.1. *Lattice Fermion Lieb-Robinson Bounds.* As introduced in our discussion of the Fermi-Hubbard model 2.3.2, interactions and bounded observables belong to the fermionic CAR algebra generated by creation and annihilation operators $c_{i\sigma}^{(*)}$ that satisfy the canonical anticommutation relations, 10. Hence, it is not reasonable to consider general commutators of observables when framing Lieb-Robinson bounds. In their 2018 paper, [73], Nachtergaele, Sims, and Young provide a thorough setup to prove Lieb-Robinson bounds for the dynamics generated by interactions such as the Hubbard model, 11. Before stating the bound as a theorem, we discuss operator parity for fermion systems, and for the remainder of this section we suppress spin degrees of freedom without loss of generality.

Recall the local number operators associated with finite regions $X \in \mathcal{P}_0(\Gamma)$: $N_X = \sum_{i \in X} n_i = \sum_{i \in X} c_i^* c_i$. Let Θ_X denote the parity automorphism over $\mathcal{A}_X$, which acts on elements $A \in \mathcal{A}$ as,

$$\Theta_X(A) = (-1)^{N_X} A (-1)^{N_X}. \tag{22}$$

This map can be extended to a unique limit Θ in the limit $X \uparrow \Gamma$ which behaves as Θ_X when restricted to elements of $\mathcal{A}_X$. Note that this map has eigenvalues of ± 1 indicating even $(+1)$ and odd (-1) parity. We shall denote by $\mathcal{A}_\Gamma^\pm$ the respective subalgebras consisting of even and odd elements. Simple examples of odd elements are c_i and c_i^* whereas n_i is an even element. The following proposition proved by Nachtergaele and co. establishes the (anti-)commutator structure needed to study even and odd elements respectively.

Proposition 2.4 (2.1 from [73])**.** *Let $X, Y \in \mathcal{P}_0$ such that $X \cap Y = \emptyset$.*

i) If $A \in \mathcal{A}_X^+$ and $B \in \mathcal{A}_Y$, then $[A, B] = 0$. Additionally, if $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ and $[A, B] = 0$, then either $A \in \mathcal{A}_X^+$ or $B \in \mathcal{A}_Y^+$.

ii) If $A \in \mathcal{A}_X^-$, $B \in \mathcal{A}_Y^-$, then $\{A, B\} = 0$. Further, if $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ and $\{A, B\} = 0$, then either A or B is identically zero or $A \in \mathcal{A}_X^-$ and $B \in \mathcal{B}_Y^-$.

Any $A \in \mathcal{A}_X$ can be written as $A = A^+ + A^-$ where $A^\pm \in \mathcal{A}_X^\pm$ and is a linear combination of even(+) and odd(-) degree monomials in $a_i^{(*)}$ respectively. We do not include a full proof, but remark that these properties can be seen easily with monomials in c_i and c_i^*, from which the general statements can be justified through linearity. Below we review sample calculations justifying these claims with low degree monomials.

$$\{c_i^* c_j, c_k\} = c_i^* \{c_j, c_k\} - \{c_i^*, c_k\} c_j = 0; \; k \neq i, j$$

$$\{c_i^* c_j^* c_k, c_m\} = c_i^* [c_j^* c_k^*, c_m] + \{c_i, c_m\} c_j^* c_k^* = 0; \; m \neq i, j, k.$$

(Anti-)Commutators of higher degree monomials simply reduce to calculations such as these through the properties of (anti-)commutators.

The interactions introduced in this work take on a particular form, which will appear in our continuum analysis later on as well. So we take some time to carefully construct these. As we saw before, an interaction Φ takes a group of sites X and maps it to an operator. In their work, Nachtergaele et al. write their interactions associated with X as sums of operators of the following form.

$$h(Y,Z)\mathbf{c}_Y^*\mathbf{c}_Z + h.c.\,,\ \mathbf{c}_Y^{(*)} = \prod_{y\in Y} c_y^{(*)} \tag{23}$$

The sets Y and Z together constitute X, meaning $X = Y \cup Z$, and a suitable ordering of sites is chosen for the monomials. Additionally, these interactions are assumed to be even parity, meaning that $|Y| + |Z|$ is even, which ensures that electrons can only be created/destroyed in even multiples. This constraint allows them to capture physically relevant interactions, which are present in Hamiltonians such as the Hubbard model, Bogoliubov-de-Gennes model for BCS superconductors, and more. The finite volume Hamiltonians and their dynamics are then written in the usual fashion,

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X),\ \tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} = A(t)\ \forall\ A \in \mathcal{A}_\Lambda. \tag{24}$$

We may now state the lattice fermion Lieb-Robinson bound as in the work of Nachtergaele, Sims, and Young.

Theorem 2.5 (3.1 from [73]). *Fix $\Lambda \in \mathcal{P}_0(\Gamma)$ and let the interaction $\Phi \in \mathcal{A}_\Lambda^+ \cap \mathcal{B}_F$. Denote by τ_t^Λ the dynamics associated with Φ and let $X, Y \subset \Lambda$ such that $X \cap Y = \emptyset$.*
i) If $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $[A, B] = 0$, then,

$$\left\| \left[\tau_t^\Lambda(A), B\right] \right\| \leq 2\|A\|\|B\| \left(e^{2C_F\|\Phi\|_F|t|} - 1\right) \sum_{x\in\partial_\Phi X} \sum_{y\in Y} F(d(x,y)) \tag{25}$$

for all finite t.

ii) If $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $\{A, B\} = 0$, then for any finite t,

$$\left\| \left\{ \tau_t^\Lambda(A), B \right\} \right\| \leq 2\|A\|\|B\| \left(e^{2C_F \|\Phi\|_F |t|} - 1 \right) \sum_{x \in \partial_\Phi X} \sum_{y \in Y} F(d(x, y)). \tag{26}$$

Proving these bounds follows the same structure as the proof of lattice Lieb-Robinson bounds for short range interactions. However, noting that operator parity determines whether two operators with disjoint supports commute or anti-commute, it is important to emphasize the existence of Lieb-Robinson estimates for both the norms of commutators and anti-commutators of Heisenberg evolved observables. Nonetheless, the essence of the bound is just as in the case of spin systems. The failure of $c_i(t)$ and c_j^* to anticommute, for example, notably depends on how far apart the sites i, j are, and there is a finite rate at which the support of $c_i(t)$ grows. Moreover, just as derived in equation 21, the specific choice of F-function $F_a(r) = e^{-ar}F_0(r)$ gives rise to the more explicit bound with exponential factor $e^{-a(d(X,Y) - v_a|t|)}$.

For the complete proofs of lattice system Lieb-Robinson bounds, it is recommended to review the cited sources for each stated theorem. With these results in hand, there are a number of useful applications that can be achieved. In the following subsection, we explore one such case, which involves proving the existence and quasi-locality of Heisenberg dynamics in the infinite volume limit.

2.5. **Thermodynamic Limit and Quasi-locality.**

For both the lattice spin and fermion systems, the Lieb-Robinson bounds stated in Theorems 2.3 and 2.5 carry no dependence on the finite volume Λ. In fact, if we enlarge the spatial region Λ to Λ' such that $\Lambda \subset \Lambda'$, $H_{\Lambda'}$ features additional interaction terms that are either supported near the boundary of Λ' or have supports whose diameters render their strengths negligible. Consequently, this allows one to show that the infinite volume limit of the dynamics exists as an automorphism of $\mathcal{A}_\Gamma$, which means it is a quasi-local map. We state this result as was proved for spin systems.

Theorem 2.6 (3.5 in [74]). *Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be an exhaustive increasing sequence of subsets of Γ. Then under the same assumptions as Theorem 2.3, the thermodynamic limit τ_t of the Heisenberg dynamics exists,*

$$\lim_{n\to\infty} \tau_t^{\Lambda_n}(A) = \tau_t(A) \tag{27}$$

for all $A \in \mathcal{A}_{loc}$ and t in compact intervals.

Moreover, $\{\tau_t\}_{t\in\mathbb{R}}$ forms a strongly continuous 1-parameter group of automorphisms of $\mathcal{A}_\Gamma$. Thus, the infinite volume dynamics exist as a well defined map over the algebra of quasi-local observables and is itself a quasi-local map, as a result of Lieb-Robinson bounds. What this entails is that for finite t and any choice of $\epsilon > 0$, there exist $\Lambda_\epsilon \subset \Lambda$ and $A_\epsilon \in \mathcal{A}_{\Lambda_\epsilon}$ such that the following holds,

$$\left\|\tau_t^{\Lambda}(A) - A_\epsilon\right\| \leq \epsilon, \tag{28}$$

and this holds in both lattice spin and fermion settings. The Lieb-Robinson bound implies that any B supported at least a distance r, larger than $v|t|$, from X satisfies the bound $\left\|\left[\tau_t^{\Lambda}(A), B\right]\right\| \leq Ce^{-a(r-v|t|)}$, with $a, C, v > 0$. Thus, one can construct a local approximation of $\tau_t^{\Lambda}(A)$ supported on $X_r := \{j \in \Gamma : d(j, X) \leq r\}$, and it deviates in norm from $\tau_t^{\Lambda}(A)$ by at most an error proportional to $e^{-a(r-v|t|)}$, which is what we denote by ϵ in equation 28. As we show in Section 5, such approximations can be made for local operators that are Heisenberg evolved in time t or in a perturbation parameter describing a modification of an initial Hamiltonian. In particular, we will discuss how to construct such local approximations of quasi-local observables in the context of continuum fermion systems.

Having introduced quantum lattice systems, it is evident that bounded interactions with finite/short range emerge as a requirement for Lieb-Robinson bounds to hold. As we move into the continuous space setting, it will be important to ensure that this feature of our model remains. However, before considering continuum dynamics, we first discuss in detail the continuous space electron model that will be the focus of Part 1 of this dissertation.

3. Integer Quantum Hall Effect

To establish the physical setting of the quantum Hall effect, we begin with a classical perspective on the behavior of charged particles in the presence of uniform magnetic and electric fields. Physically, we consider the following setup depicted in Figure 4. One has a closed circuit with current I driven by a battery with voltage $\mathcal{E}$, and a two-dimensional sheet of metal is inserted midway with a voltmeter connected to the horizontal sides of the sample as shown. The driven current in the longitudinal direction (red) gives rise to an electric field in the same direction, which results in electrons moving rightwards in the sample. Due to the magnetic field however, the charges feel a Lorentz force: $F_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$, and the electrons exhibit circular "cyclotron" orbits. This phenomenon gives rise to a transverse "Hall" current and voltage across the metal sample and as derived in a number of references, e.g. [35], the Hall voltage and resistivity are proportional to the magnetic field strength B.

Such is the classical description of charged particles moving in two dimensions in the presence of a perpendicular magnetic field, which occurs at room temperature and small magnetic fields. A quantum description of this setting at very low temperatures and higher field strength uncovers a fascinating phenomenon by which the Hall resistivity is quantized at integer/fractional plateaus as the magnetic field strength grows. In particular, the onset of fractional versus integer plateaus is attributed to dominant electron-electron interactions. In order to build the interacting setting, we first take time in this section to study the simplest model that describes a charged particle in two dimensions in the presence of a constant magnetic field. This is the well known Landau Hamiltonian, which encodes the non-interacting behavior of particles in this physical environment. This particular Hamiltonian has a rich spectral structure and different methods for spanning its eigenspaces, which become very useful as we later impose inter-particle interactions and study the dynamics of these systems.

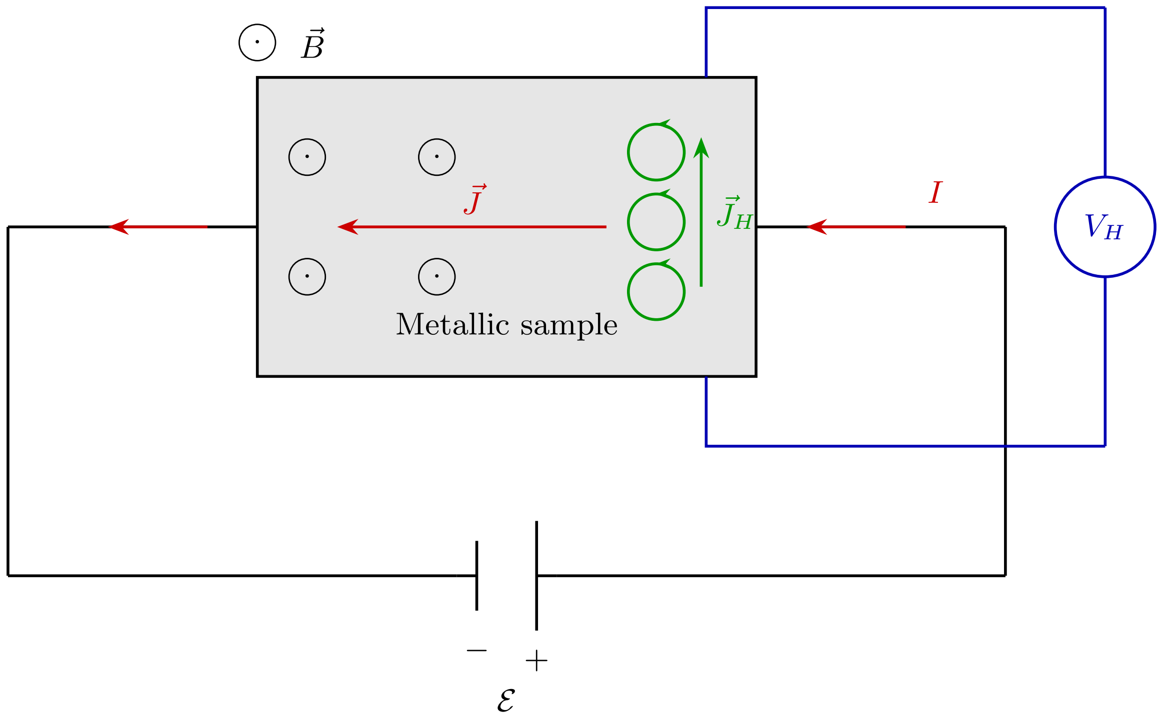

FIGURE 4. Experimental apparatus for detecting the Hall effect via a voltage measured vertically across the metallic sample. The longitudinal and transverse (Hall) currents are portrayed as red and green arrows inside the metallic sample, and the Hall voltage V_H is measured across the two horizontal edges of the metal.

3.1. **Landau Hamiltonian.**

The starting point for understanding the Integer Quantum Hall effect quantum mechanically is to consider the well known Landau Hamiltonian, which describes a single charged particle in two dimensions with a constant perpendicular magnetic field. For a charge q with mass m, this Hamiltonian is given by

$$H_L = \frac{1}{2m}\left(p - q\vec{A}(x)\right)^2, \tag{29}$$

where $x = (x_1, x_2)$ is the vector valued position operator and $p = -i\nabla = -i\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ is the two-dimensional momentum operator. The coordinates x_1 and x_2 refer to Cartesian coordinates. The vector potential $\vec{A}(x)$ is an operator that acts by multiplication as a function of x and encodes the perpendicular magnetic field: $\nabla \times \vec{A}(x) = \vec{B} = B\hat{x}_3$. There is of course freedom in how we choose to define this function $\vec{A}$ due to gauge invariance of H_L. In the analysis of the dynamics of interacting electrons to come, we find it most suitable to work in the symmetric gauge where $\vec{A}(x) = \frac{B}{2}(-x_2, x_1)$, and a quick calculation verifies that $\nabla \times \vec{A}(x) = B\hat{x}_3$. Henceforth, we also set $m = \frac{1}{2}$ and $q < 0$, making the identification $qB \equiv B > 0$ for convenience (convention used in [85]). We will now solve the time-independent Schrödinger equation with H_L using an elegant algebraic approach, which introduces complex valued coordinates, and we also state the eigenfunctions of H_L in terms of two-dimensional polar coordinates. For the former, we follow the solution as presented in the work of Rougerie and Yngvason. [85].

With our choice of units and gauge, the Landau Hamiltonian explicitly takes the form,

$$\begin{aligned} H_L &= (p - \vec{A}(x))^2 \\ &= \left(-i\partial_{x_1} + \frac{B}{2}x_2\right)^2 + \left(-i\partial_{x_2} - \frac{B}{2}x_1\right)^2 \\ &= \Pi_1^2 + \Pi_2^2, \end{aligned} \tag{30}$$

where we denote by $\Pi_j := p_j - A(x)_j$ the gauge-invariant momenta operators. Notably, the components of Π do not commute with one another.

$$[\Pi_1, \Pi_2] = \left[p_1 + \frac{B}{2}x_2, p_2 - \frac{B}{2}x_1\right] \tag{31}$$

$$= \frac{B}{2}\left([p_1, -x_1] + [x_2, p_2]\right) \tag{32}$$

$$= iB \tag{33}$$

In terms of the components Π_j, we introduce the so called ladder operators, akin to those which appear in the context of the quantum harmonic oscillator, which satisfy canonical commutation relations.

$$\mathfrak{a}^* = \frac{1}{\sqrt{2B}}\left(-\Pi_2 - i\Pi_1\right); \ \mathfrak{a} = \frac{1}{\sqrt{2B}}\left(-\Pi_2 + i\Pi_1\right) \tag{34}$$

(35) $$[\mathfrak{a}, \mathfrak{a}^*] = \mathbb{1}$$

The Landau Hamiltonian may then be written in terms of $\mathfrak{a}, \mathfrak{a}^*$ to give,

(36) $$H_L = 2B\left(\mathfrak{a}^*\mathfrak{a} + \frac{1}{2}\mathbb{1}\right).$$

A lowest energy eigenfunction $\psi_0(r)$ of this Hamiltonian is thus one that is annihilated by the operator $\mathfrak{a}$: $\mathfrak{a}\psi_0(x) = 0$, giving rise to an eigenvalue $E_0 = B$. Higher energy states are generated by acting on ψ_0 with powers of $\mathfrak{a}^*$: $\psi_n(r) = \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \psi_0(r)$, with eigenvalues $2B\left(n + \frac{1}{2}\right)$. Below, we demonstrate this feature of the eigenstates of H_L which will require evaluating $[\mathfrak{a}, (\mathfrak{a}^*)^n]$.

$$\begin{aligned}
\left[\mathfrak{a}, (\mathfrak{a}^*)^n\right] &= [\mathfrak{a}, \mathfrak{a}^*](\mathfrak{a}^*)^{n-1} + \mathfrak{a}^*\left[\mathfrak{a}, (\mathfrak{a}^*)^{n-1}\right] \\
&= (\mathfrak{a}^*)^{n-1} + \mathfrak{a}^*[\mathfrak{a}, \mathfrak{a}^*](\mathfrak{a}^*)^{n-2} + (\mathfrak{a}^*)^2\left[\mathfrak{a}, (\mathfrak{a}^*)^{n-2}\right] \\
&= 2(\mathfrak{a}^*)^{n-1} + (\mathfrak{a}^*)^2\left[\mathfrak{a}, (\mathfrak{a}^*)^{n-2}\right] \\
&. \\
&. \\
&. \\
\left[\mathfrak{a}, (\mathfrak{a}^*)^n\right] &= n(\mathfrak{a}^*)^{n-1}
\end{aligned}$$

Thus, $\mathfrak{a}^*\mathfrak{a}$ acting on ψ_n yields,

$$\begin{aligned}
\mathfrak{a}^*\mathfrak{a}\psi_n &= \frac{1}{\sqrt{n!}}\mathfrak{a}^*\mathfrak{a}(\mathfrak{a}^*)^n\psi_0 \\
&= \frac{1}{\sqrt{n!}}\mathfrak{a}^*\left(n(\mathfrak{a}^*)^{n-1} + (\mathfrak{a}^*)^n\mathfrak{a}\right)\psi_0 \\
&= \left(n\frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n + (\mathfrak{a}^*)^n\mathfrak{a}\right)\psi_0 \\
&= n\psi_n.
\end{aligned}$$

In the second equality, we make use of the evaluation of $[\mathfrak{a}, (\mathfrak{a}^*)^n]$, and the final equality holds because $\mathfrak{a}\psi_0 = 0$ by definition of the ground state. Therefore, we explicitly show the eigenvalues of H_L to be of the form,

$$E_n = 2B\left(n + \frac{1}{2}\right). \tag{37}$$

The factor of $2B$ with our choice of units is termed the cyclotron frequency ω_c, and this is actually in agreement with the orbit frequency of the classical cyclotron motion of charged particles moving in a magnetic field. Moreover, by solving the differential equation given by $\mathfrak{a}\psi_0(x) = 0$, the ground state eigenfunction ψ_0 can be written explicitly. We also introduce the length scale $\ell_B = \frac{1}{\sqrt{B}}$ called the magnetic length, and then one has,

$$\psi_0(x) = \frac{1}{\ell_B\sqrt{2\pi}} e^{-|x|^2/4\ell_B^2};\ \psi_n(x) = \frac{1}{\ell_B\sqrt{2\pi}} \frac{(x_1 - ix_2)^n}{\sqrt{n!}} e^{-|x|^2/4\ell_B^2}. \tag{38}$$

Each eigenspace with associated eigenvalue E_n will henceforth be termed a **Landau level** as in literature. The operators $\mathfrak{a}$ and $\mathfrak{a}^*$ thus map a state in the n^{th} Landau level to the $(n-1)^{\text{th}}$ and $(n+1)^{\text{th}}$ levels respectively. Notably, each Landau level is infinitely degenerate, and this can be seen by introducing another pair of ladder operators $\mathfrak{b}$ and $\mathfrak{b}^*$.

$$\mathfrak{b}^* = \frac{1}{\sqrt{2B}}\left(\pi_y - i\pi_x + B(x+iy)\right);\ \mathfrak{b} = \frac{1}{\sqrt{2B}}\left(\pi_y + i\pi_x + B(x-iy)\right) \tag{39}$$

$$[\mathfrak{b}, \mathfrak{b}^*] = \mathbb{1} \tag{40}$$

Crucially, these also satisfy canonical commutation relations just like the Landau level raising and lowering operators $\mathfrak{a}^{(*)}$. Moreover, using equation 31 and $[r_j, p_k] = i\delta_{jk}\mathbb{1}$, one has,

$$\begin{aligned}
[\mathfrak{a}^{(*)}, \mathfrak{b}] &= \frac{1}{\sqrt{2B}}\left[-\pi_y \pm i\pi_x, \pi_y + i\pi_x + B(x-iy)\right] && (41)\\
&= \frac{1}{\sqrt{2B}}\left(-B + B \mp B \pm B\right) && (42)\\
&= 0, && (43)
\end{aligned}$$

and the same holds if one replaces $\mathfrak{b}$ with $\mathfrak{b}^*$. As a result, $\left[H_L, \mathfrak{b}^{(*)}\right] = 0$, meaning that the $\mathfrak{b}$ operators can be used to generate additional eigenstates of H_L with the same energy E_n. In other words, they leave each Landau level invariant. The existence of an operator that commutes with the Hamiltonian suggests that H_L is invariant under some symmetry action. This ends up being exactly the case, and to elucidate the identity of this symmetry, we study the $\mathfrak{b}$ operators more carefully. Acting on ψ_0 with $\mathfrak{b}$ and $\mathfrak{b}^*$, one obtains,

$$\mathfrak{b}\psi_0(x) = \left(-i\partial_{x_2} + \frac{B}{2}x_1 + \partial_{x_1} - i\frac{B}{2}x_2\right)\frac{1}{\ell_B\sqrt{2\pi}}e^{-|x|^2/4\ell_B^2} \tag{44}$$

$$= \left(i\frac{B}{2}x_2 + \frac{B}{2}x_1 - \frac{B}{2}x_1 - i\frac{B}{2}x_2\right)\frac{1}{\ell_B\sqrt{2\pi}}e^{-|x|^2/4\ell_B^2} \tag{45}$$

$$= 0 \tag{46}$$

$$\mathfrak{b}^*\psi_0(x) = \left(-i\partial_{x_2} + \frac{B}{2}x_1 - \partial_{x_1} + i\frac{B}{2}x_2\right)\frac{1}{\ell_B\sqrt{2\pi}}e^{-|x|^2/4\ell_B^2} \tag{47}$$

$$= B(x_1 + ix_2)\psi_0. \tag{48}$$

Thus, $\mathfrak{b}$ annihilates ψ_0 just like $\mathfrak{a}$, whereas $\mathfrak{b}^*$ can be used to generate additional eigenstates with energy E_0. We may now introduce the more general eigenfunction of H_L given by,

$$\psi_{nm}(x) := \frac{1}{\sqrt{n!m!}}\left(\mathfrak{a}^*\right)^n\left(\mathfrak{b}\right)^m\psi_0(x);\ n, m \in \mathbb{N}_0. \tag{49}$$

Now consider the action of $\mathfrak{b}^*\mathfrak{b}$ on ψ_{nm}. Since $[\mathfrak{b}, \mathfrak{b}^*] = 1$, it follows just as in the case of the Landau level operators $\mathfrak{a}^{(*)}$ that $\mathfrak{b}^*\mathfrak{b}\psi_{nm} = m\psi_{nm}$. So for each fixed n, ψ_{nm} is an eigenvector of $\mathfrak{b}^*\mathfrak{b}$ with eigenvalue m, and each Landau level is actually infinitely degenerate, unless we specify finite volume boundary conditions. Furthermore, as mentioned previously, the fact that $\left[H_L, \mathfrak{b}^{(*)}\right] = 0$ hints at the existence of a symmetry whose action leaves H_L invariant. To see this, we expand $\mathfrak{b}^*\mathfrak{b} - \mathfrak{a}^*\mathfrak{a}$ in terms of position and momenta operators.

$$\begin{aligned}(\mathfrak{b}^*\mathfrak{b} - \mathfrak{a}^*\mathfrak{a}) &= \frac{1}{2B}\big(\left(\pi_2 - i\pi_1 + B(x_1 + ix_2)\right)\left(\pi_2 + i\pi_1 + B(x_1 - ix_2)\right) \\ &\quad - \left(-\pi_2 - i\pi_1\right)\left(-\pi_2 + i\pi_1\right)\big) \\ &= \frac{1}{2B}\left(2i\left[\pi_2, \pi_1\right] + B^2(x_1^2 + x_2^2) + 2B\left(x_1\pi_2 - x_2\pi_1\right) + iB\left(\left[x_2, \pi_2\right] + \left[x_1, \pi_1\right]\right)\right)\end{aligned}$$

$$= \frac{1}{2B}\left(2B + B^2(x_1^2 + x_2^2) + 2B(x_1p_2 - \frac{B}{2}x_1^2 - x_2p_1 - \frac{B}{2}x_2^2) - 2B\right)$$

$$= x_1p_2 - x_2p_1$$

$$= L_3$$

The operator $\mathfrak{b}^*\mathfrak{b} - \mathfrak{a}^*\mathfrak{a}$ is precisely the perpendicular component of angular momentum, and $[H_L, L_3] = 0$ in the symmetric gauge. Therefore, the Landau Hamiltonian is invariant under rotations in the $x - y$ plane, a consequence that is visible due to our choice of gauge. Furthermore, it is evident that angular momentum L_3 is quantized with integer-valued spectrum bounded below by $-n$ in the n^{th} Landau level. The operators $\mathfrak{b}$ and $\mathfrak{b}^*$ now carry the physical interpretation of raising and lowering the angular momentum of a wavefunction. Denoting by $\mathcal{H}_n$ the Hilbert space associated with the n^{th} Landau level, $\{\psi_{nm} : m \in \mathbb{N}_0\}$ is then an orthonormal basis for $\mathcal{H}_n$.

Moreover, it is clear that each Landau level is infinitely degenerate, but per unit area, degeneracy is constant and proportional to the magnetic field. Importantly, this means that at large magnetic fields, the lowest Landau level will be able to accommodate more electrons due to an increase in number of states per unit area. Lastly, in the many-body picture, note that with a chemical potential located in any gap between Landau levels, the ground state of the system is precisely that associated with full occupancy of the Landau levels with energy below the chemical potential. This number of filled levels, being an integer, is precisely what sets the stage for the integer quantum Hall effect. Importantly as a many-body state, this is a non-degenerate ground state with gapped particle/hole excitations consisting of particles in higher Landau levels and holes in the occupied levels respectively.

This algebraic decomposition of the Landau Hamiltonian is an elegant procedure; one observes that it simplifies to a quantum harmonic oscillator model, where each band gives rise to infinitely many angular momentum states. For some of the future calculations in this dissertation, we also write the eigenfunctions of H_L in two dimensional polar coordinates. A detailed solution of the time-independent Schrödinger equation $H_L\psi = E\psi$ is presented in [24], and we summarize the key features of the result below, adopting the notation used in this work.

For what follows, we parametrize solutions of the Schrödinger equation by the integers n_r and m_l. In terms of the previously introduced labels n and m, $m_l = m - n$ and $n_r = \frac{m+n-|m-n|}{2}$. The number m_l is precisely the eigenvalue of the z-component of angular momentum, and $n_r \in \mathbb{N}_0$ is an integer that helps to characterize the Landau level index (for all $m \geq 0$, $n_r = n$). One may then write E_n as in equation 37 as

$$E_{n_r m_l} = 2B\left(\frac{1}{2} + n_r + \frac{|m_l| - m_l}{2}\right), \tag{50}$$

and the associated normalized eigenfunctions of H_L with eigenvalue $E_{n_r m_l}$ are given by,

$$\psi_{n_r m_l}(r,\phi) = R_{n_r m_l}(r)\Phi_{m_l}(\phi) \tag{51}$$

$$\Phi_{m_l}(\phi) = \frac{e^{im_l\phi}}{\sqrt{2\pi}} \tag{52}$$

$$R_{n_r m_l}(r) = N_{n_r m_l}\left(\frac{r}{\ell_B}\right)^{|m_l|} e^{-r^2/4\ell_B^2} L_{n_r}^{|m_l|}\left(\frac{r}{2\ell_B^2}\right), \tag{53}$$

where the normalization constant $N_{n_r m_l}$ is,

$$N_{n_r m_l} = \sqrt{\frac{n_r!}{\ell_B^2 2^{|m_l|}(n_r + |m_l|)!}}. \tag{54}$$

For any two integers $a, b \geq 0$, $L_a^b(x)$ are the generalized Laguerre polynomials which are defined as shown in the following equation.

$$L_a^b(x) = \frac{x^{-b}e^x}{a!}\frac{d^a}{dx^a}\left(e^{-x}x^{a+b}\right) \tag{55}$$

We also note the orthogonality relations that the radial and angular functions $R_{n_r m_l}$ and Φ_{m_l} respectively satisfy.

$$\int_0^\infty dr\, R^*_{n_r m_l}(r) R_{n'_r m_l}(r) = \delta_{n_r n'_r} \tag{56}$$

$$\int_0^{2\pi} d\phi\, \Phi_{m_l}(\phi)^* \Phi_{m'_l}(\phi) = \delta_{m_l m'_l} \tag{57}$$

Of particular importance will be the (standard) Laguerre polynomials $L_a(x) \equiv L_a^0(x) = \frac{1}{a!} e^x \frac{d^a}{dx^a}\left(e^{-x} x^a\right)$. One useful relation we will make use of in subsequent locality estimates is the generating function involving these Laguerre polynomials, which one can find in references such as Section 13.2 of [5]. For any $t \in \mathbb{C}$ such that $|t| < 1$, one has,

$$\sum_{j=0}^{\infty} t^j L_j(x) = \frac{1}{1-t} e^{-\frac{tx}{1-t}}. \tag{58}$$

Having discussed the Landau Hamiltonian, its discrete spectrum, and different orthonormal bases that span its eigenspaces, we now introduce an alternate perspective from which one can express the state of a charged particle in a perpendicular magnetic field. Precisely, we introduce an overcomplete coherent state decomposition of the Landau level Hilbert spaces $\mathcal{H}_n$.

3.2. **Magnetic Translation and Frame.**

The domain of the Landau Hamiltonian,

$$\mathcal{D}_L = \{f \in L^2(\mathbb{R}^2) :\ \left(\partial_{x_j} - A_j\right) f \in L^2(\mathbb{R}^2,\ j = 1, 2)\},$$

as written in references such as [68], is one that admits rapidly decaying functions, seen with the Gaussian factor in $\psi_{nm}(x)$. However, the topologically nontrivial nature of the Landau levels prevents each $\mathcal{H}_n$ from possessing an orthonormal basis with each element exponentially localized at discrete points in space. This fact is a crucial feature of electron band structures with topological character that has been explored in works such as [22]. Nonetheless, such results do not prevent Hilbert spaces such as the Landau level spaces $\mathcal{H}_n$

from admitting spanning sets with elements that are exponentially localized at points in $\mathbb{R}^2$. In order to properly construct the functions in this spanning set, we need to introduce the so called magnetic translation operator.

While the physical setting of a charged particle in two dimensions with a uniform perpendicular magnetic field is translation invariant, the Landau Hamiltonian does not possess this feature due to the position dependent vector potential $\vec{A}(x)$. Therefore, the translation operator $\mathcal{S}_a = e^{-ia\cdot p}$ which translates functions in $L^2(\mathbb{R}^2)$ does not commute with H_L. Therefore, if $f \in \mathcal{H}_0$ is a lowest Landau level state, then $\mathcal{S}_a f(x) = f(x-a) \notin \mathcal{H}_0$. There is, however, a way to construct a modified translation operator that commutes with H_L, and this is precisely the magnetic translation.

Definition 3.1. Let $\Pi^{\pm} = p \pm A$; then, $T_a^{\pm} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with $a \in \mathbb{R}^2$ is a magnetic translation operator defined as $T_a^{\pm} = e^{\mp ia\cdot\Pi^{\pm}}$. The action of this operator on any $f \in L^2(\mathbb{R}^2)$ is given by

$$T_a^{\pm} h(x) = e^{\mp ia\cdot A(x)} h(x-a). \tag{59}$$

Recalling our choice of the symmetric gauge to define the vector potential,

$$\begin{aligned} T_a^{\pm} &= \exp\left[-i\left((p_1 \mp \frac{B}{2}x_2)a_1 + (p_2 \pm \frac{B}{2}x_1)a_2\right)\right] \\ &= \exp\left[-i\left(a_1p_1 \pm \frac{a_2B}{2}x_1\right)\right]\exp\left[-i\left(a_2p_2 \mp \frac{a_1B}{2}x_2\right)\right] \end{aligned}$$

where the second line is true as all operators relating to x_1 commute with those relating to x_2. Now, as x_i and p_i have a commutator equal to a constant, we can use the Baker-Campbell-Hausdorff formula to rewrite the terms of $T_a^{\pm}$.

$$\begin{aligned} e^{\mp i\frac{a_2B}{2}x_1}e^{-ia_1p_1} &= \exp\left(-i(a_1p_1 \pm \frac{a_2B}{2}x_1) + \frac{1}{2}\left[\mp i\frac{a_2B}{2}x_1,\, -ia_1p_1\right]\right) \\ &= \exp\left(-i(a_1p_1 \pm \frac{a_2B}{2}x_1)\right)e^{\mp i\frac{Ba_1a_2}{4}} \end{aligned}$$

Similarly,

$$e^{\pm i\frac{a_1 B}{2}x_2}e^{-ia_2p_2} = \exp\left(-i(a_2p_2 \mp \frac{a_1B}{2}x_2)\right)e^{\pm i\frac{Ba_1a_2}{4}}.$$

Therefore, we finish the calculation as follows.

$$\begin{aligned} T_a^{\pm}f(x) &= e^{\mp i\frac{a_2B}{2}x_1}e^{-ia_1p_1}e^{\pm i\frac{a_1B}{2}x_2}e^{-ia_2p_2}f(x) \\ &= e^{\mp i\frac{a_2B}{2}x_1 \pm i\frac{a_1B}{2}x_2}e^{-i(a_1p_1+a_2p_2)}f(x) \\ &= e^{\mp ia\cdot A}f(x-a). \end{aligned}$$

Going from the first to the second line, we utilize the trivial commutation relations between x_1-operators and x_2-operators to rearrange terms. Then upon inspection, the second exponential term simply defines the standard translation operator of an element in $L^2(\mathbb{R}^2)$ by $a \in \mathbb{R}^2$, which yields $f(x-a)$. Operators of this form have been introduced and carefully studied by Joshua Zak, giving rise to the magnetic translation group and the Zak transform, [119]-[120].

For our purposes, $T^-_{(\cdot)}$ appears in some of the technical calculations to come, and $T^+_{(\cdot)}$ will be used to generate the spanning sets of $\mathcal{H}_n$ for each $n \in \mathbb{N}_0$. In particular, $T^+_{(\cdot)}$ is effective towards this end because it commutes with H_L. To show this, we calculate the commutation relations between the components of Π^+ and Π^-.

$$\left[\Pi_1^+, \Pi_1^-\right] = \left[\Pi_2^+, \Pi_2^-\right] = 0 \tag{60}$$

$$\left[\Pi_1^+, \Pi_2^-\right] = \left[p_1 - \frac{B}{2}x_2, p_2 - \frac{B}{2}x_1\right] = 0 \tag{61}$$

$$\left[\Pi_2^+, \Pi_1^-\right] = \left[p_2 + \frac{B}{2}x_1, p_1 + \frac{B}{2}x_2\right] = 0 \tag{62}$$

As a result of these relations, it follows that for any $a \in \mathbb{R}^2$, $[T_a^+, H_L] = 0$ in the symmetric gauge. We will now use $T^+_{(\cdot)}$ to construct an overcomplete frame for $\mathcal{H}_0$ and with the raising operators $\mathfrak{a}^*$, we can lift this frame to any Landau level.

First, we define a frame in a Hilbert space $\mathcal{H}$ and discuss some general properties, as can be found in Chapter 5 of [23]. Let $\{f_i\}_{i\in I} \subset \mathcal{H}$ be a possibly infinite sequence of functions. $\{f_i\}_{i\in I}$ is said to be a frame if there exist constants $0 < c \leq C$ such that for all $\psi \in \mathcal{H}$, the following bound holds,

$$c\|\psi\|^2 \leq \sum_{i\in I} |\langle f_i, \psi\rangle|^2 \leq C\|\psi\|^2 \tag{63}$$

where the constants c, C are not unique and are called frame bounds. Moreover, $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if $\overline{\operatorname{span}\{f_i : \ i \in I\}} = \mathcal{H}$. Let $S : \mathcal{H} \to \mathcal{H}$ denote the frame operator, which acts on $\psi \in \mathcal{H}$ as,

$$S\psi = \sum_{i\in I} \langle f_i, \psi\rangle f_i. \tag{64}$$

The frame operator S is positive, bounded, self-adjoint, and invertible and has an operator norm which is bounded by the frame bounds:

$$c\mathbb{1} \leq S \leq C\mathbb{1}.$$

If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then any $\psi \in \mathcal{H}$ admits the following decomposition in terms of elements f_i.

$$\psi = S \circ S^{-1}\psi = \sum_{i\in I} \langle f_i, S^{-1}\psi\rangle f_i = \sum_{i\in I} \langle S^{-1} f_i, \psi\rangle f_i \tag{65}$$

The inner products $s_i(\psi) := \langle S^{-1} f_i, \psi\rangle$ are termed the frame coefficients. We will be interested in the setting where $\{f_i\}_{i\in I}$ actually forms an overcomplete frame for $\mathcal{H}$, meaning there exists at least one sequence of complex coefficients $\{c_i\}_{i\in I}$, with not all $c_i = 0$, such that $\sum_{i\in I} c_i f_i = 0$. In this case, there does not exist a unique sequence of coefficients $\{a_i\}_{i\in I}$ such that for any $\psi \in \mathcal{H}$, $\psi = \sum_{i\in I} a_i f_i$. However, the frame coefficients have the minimum ℓ^2-norm of all sequences that can be used to express $\psi \in \mathcal{H}$, i.e.

$$\sum_{i\in I} |s_i(\psi)|^2 \leq \sum_{i\in I} |a_i|^2.$$

We are now equipped to construct an overcomplete frame for $\mathcal{H}_0$. Let $\Gamma := \alpha\mathbb{Z} \times \beta\mathbb{Z}$, with $\alpha, \beta > 0$, be a rectangular lattice and let $d : \Gamma \times \Gamma \to \mathbb{R}$ be the metric associated with Euclidean distance. The sites of this lattice will be used to index the elements of the frame in question; such frames are called regular. Denote by $\chi_0(x) := \psi_{00}(x)$, the lowest Landau level state with zero angular momentum, and $\chi_a := T_a^+ \chi_0(x)$, the same function magnetically translated to be centered at a. Since, $\left[T_{(\cdot)}^+, H_L\right] = 0$, $\chi_a(x) \in \mathcal{H}_0 \; \forall a \in \mathbb{R}^2$. In [12], Bachmann and De Nittis prove that under precise conditions, the set $\{\chi_\gamma\}_{\gamma \in \Gamma}$ is a frame for $\mathcal{H}_0$.

Theorem 3.2 (Theorem 5.6 from [12])**.** *Let $\Gamma = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. If $\alpha\beta < 2\pi\ell_B^2$, then $\{\chi_\gamma\}_{\gamma\in\Gamma}$ is an overcomplete frame for $\mathcal{H}_0$.*

The proof of this theorem and the process of deriving frame bounds for the associated frame operator are rather technical, so we refer the reader to Section 5 of [12] for details on these steps.

Importantly, this frame $\{\chi_\gamma\}_{\gamma\in\Gamma}$ satisfies the following two properties that we will use to define a localized subset of $\mathcal{H}_0$.

Definition 3.3. Let (I, d) be a metric space. A sequence of functions $\{\{f_i\}_{i\in I}\}$ in a Hilbert space $\mathcal{H}$ is called a **localized** set if,

$$i) \; \exists\, \sigma_1, G_1 > 0 \text{ such that } \forall\, i \in I,\; |f_i(x)| \leq G_1 e^{-\sigma_1 d(i,x)}.$$

$$ii) \; \exists\, \sigma_2, G_2 > 0 \text{ such that } \forall\, i, j \in I,\; |\langle f_i, f_j\rangle| \leq G_2 e^{-\sigma_2 d(i,j)}.$$

Thus, the set $\{\chi_\gamma\}_{\gamma\in\Gamma}$ is a localized frame for $\mathcal{H}_0$ since $|\chi_\gamma(x)| = \frac{1}{\ell_B\sqrt{2\pi}} e^{-|x-\gamma|^2/4\ell_B^2}$ and $|\langle \chi_\gamma, \chi_{\gamma'}\rangle| = e^{-d(\gamma,\gamma')^2/4\ell_B^2}$, which can be found through direct computation. As mentioned before, we may now lift this frame to arbitrary Landau levels via the raising operator $\mathfrak{a}^*$. Thus, denote by $\chi_{\gamma,n} := \frac{1}{\sqrt{n!}} (\mathfrak{a}^*)^n \chi_\gamma = \frac{1}{\sqrt{n!}} (\mathfrak{a}^*)^n T_\gamma^+ \chi_0 = T_\gamma^+ \frac{1}{\sqrt{n!}} (\mathfrak{a}^*)^n \psi_{00} = T_\gamma^+ \psi_{n0}$. From the form of ψ_{n0} in equation 38, it is clear that each $\chi_{\gamma,n}$ will be exponentially localized about $\gamma \in \Gamma$, with a slower decay rate than χ_γ. Moreover, the following inner product calculation confirms that for each $n \in \mathbb{N}_0$, $\{\chi_{\gamma,n}\}_{\gamma\in\Gamma}$ is a localized frame.

$$\langle \chi_{\gamma,n}, \chi_{\gamma',n}\rangle = \frac{1}{n!}\left\langle T^+_\gamma \left(\mathfrak{a}^*\right)^n \chi_0, T^+_{\gamma'} \left(\mathfrak{a}^*\right)^n \chi_0\right\rangle \tag{66}$$

$$= \frac{1}{n!}\left\langle \left(\mathfrak{a}^*\right)^n T^+_\gamma \chi_0, \left(\mathfrak{a}^*\right)^n T^+_{\gamma'} \chi_0\right\rangle \tag{67}$$

$$= \frac{1}{n!}\left\langle \mathfrak{a}^n \left(\mathfrak{a}^*\right)^n \chi_\gamma, \chi_{\gamma'}\right\rangle \tag{68}$$

$$= \langle \chi_\gamma, \chi_{\gamma'}\rangle \tag{69}$$

The inner product $\langle \chi_{\gamma,n}, \chi_{\gamma',n'}\rangle$ features $\delta_{nn'}$, since functions from different Landau levels are orthogonal. However, this may also be bounded from above by $e^{-|n-n'|^2/4\ell_B^2}$, which then gives the bound,

$$|\langle \chi_{\gamma,n}, \chi_{\gamma',n'}\rangle| \leq \frac{1}{\ell_B\sqrt{2\pi}} e^{-\left(d(\gamma,\gamma')^2+|n-n'|^2\right)/4\ell_B^2} \tag{70}$$

We then consider the superlattice $\Theta := \Gamma \times \mathbb{Z}_N$, where sites $\theta := (\gamma, n) \in \Theta$ correspond to a spatial site γ and Landau level n. Furthermore, let $\rho : \Theta\times\Theta \to \mathbb{R}$ be a metric that is defined as $\rho(\theta,\theta') = d(\gamma,\gamma') + d_0|n-n'|$, where $d_0 > 0$ is a constant with units of distance. (Θ, ρ) is then a metric space. Denote by $\chi_\theta := \chi_{\gamma,n}$. Then, $\{\chi_\theta\}_{\theta\in\Theta}$ is an overcomplete, localized frame for $\mathcal{H}^N := \bigoplus_{n=0}^{N-1} \mathcal{H}_n$, a finite collection of Landau levels.

Before concluding this section, we present one locality estimate involving the frame operator S associated with $\{\chi_\theta\}_{\theta\in\Theta}$.

Lemma 3.4. *[Proposition 2.2 from [12]] Let (Γ, d) be a metric space, with Γ a lattice, and let $\{f_\gamma\}_{\gamma\in\Gamma}$ be an overcomplete, exponentially localized frame for a separable Hilbert space $\mathcal{H}$ with decay rate $\lambda > 0$, meaning $|\langle f_\gamma, f_{\gamma'}\rangle| \leq Ce^{-\lambda d(\gamma,\gamma')}$, with $C > 0$. Denote by S the associated frame operator. Then for every $p \in \mathbb{N}$, there exist constants $0 < \lambda_p < \lambda$ and $a_p > 0$ such that,*

$$\left|\langle f_\gamma, S^{-p} f_{\gamma'}\rangle\right| \leq a_p e^{-\lambda_p d(\gamma,\gamma')}, \tag{71}$$

for all $\gamma, \gamma' \in \Gamma$, where $S^{-p} := (S^{-1})^p$.

Inner products of this form will appear frequently in later calculations, so establishing this locality estimate is important. For the technical proof, we refer the reader to Section 6 of [12].

As we will see in the following few sections, functions of this form will be very useful for constructing the Hamiltonians of many interacting charged particles in a perpendicular magnetic field as we aim to bridge the gap between the integer and fractional quantum Hall effects.

4. Fermions in Continuous Space and Lieb-Robinson Bounds

In Section 2, we introduced the machinery that is required to construct quantum lattice spin systems from an algebraic perspective, define finite/short-range interactions, and build the associated dynamics. Under natural assumptions on these interactions, that we will recall shortly, it is possible to prove Lieb-Robinson bounds, which manifest as estimates on the norm of the (anti-)commutator of Heisenberg evolved local operators with disjoint supports. Notably, these decay in the distance between supports of the two operators in question but grow exponentially in time.

Then in Section 3, we reviewed the model that describes non-interacting electrons in two dimensions with a perpendicular magnetic field, i.e. the Landau Hamiltonian H_L (equation 29). In the symmetric gauge, the eigenspace of H_L is spanned by harmonic oscillator like eigenfunctions that can be parametrized in terms of complex spatial coordinates, as with ψ_{nm} in equation 49, or in terms of polar coordinates, $\psi_{n_r m_l}$ as in equation 51. Moreover, there exists a lattice-based frame for the Landau level eigenspaces consisting of exponentially localized functions.

We now seek to study fermionic systems in $\mathbb{R}^2$ with short range interactions in the presence of a constant magnetic field. We obtain Lieb-Robinson type propagation bounds for such systems and apply them to obtain a strongly continuous dynamics on the CAR algebra of the infinite system, which is based on joint work in preparation with Bruno Nachtergaele and Robert Sims [47]. To be able to do this we introduce a UV cut-off in the interaction by considering smeared creation and annihilation operators. Without such regularization, the dynamics are not expected to be given by a strongly continuous group of automorphisms on the CAR algebra [89].

As we will discuss, proving strong continuity of the dynamics of interacting fermions in the continuum has been achieved via a few different strategies, as in the works of Gebert et al. [31], Hinrichs et al. [49], Siebert [93], and Bachmann and De Nittis [12]. However, these are not the only studies of the dynamics of fermions in the continuum. As far as we are aware, Narnhofer and Thirring were the first to construct infinite volume dynamics over the CAR algebra on $L^2(\mathbb{R}^d)$ with the use of a regularized pair-wise interaction, [77], that was motivated by the desire to preserve Galilean invariance of the dynamics. From our point of view it is more natural to use Gaussian smearing of the point particles in the pair interaction, as is done in [31]. This choice allows us to construct a finite volume interaction that is a bounded operator, a crucial ingredient to proving strong continuity of the dynamics via Lieb-Robinson bounds. In this section, we focus on employing the framework of [31] to prove strong continuity of the Heisenberg dynamics for interacting fermions.

Towards this goal, we begin by constructing the CAR algebra of observables from fermionic Fock space before then formally introducing the Heisenberg dynamics over this algebra. Then, we will introduce the second quantized finite volume Hamiltonians, where the UV regularization procedure for the two-body interaction will be defined using a Gaussian smearing. In the remainder of this section, we prove propagation estimates and bounds of Lieb-Robinson type towards proving strong continuity of the dynamics in the thermodynamic limit.

4.1. **Fock Space and Second Quantization.**

When studying an individual particle whose state is governed by a Schrödinger Hamiltonian of the form $H = \frac{p^2}{2m} + V(x)$, it is insightful to describe this particle with a wavefunction that depends on the position coordinate $x \in \mathbb{R}^2$ for example. This representation determines the probability of the particle being located in some region $\Lambda \subset \mathbb{R}^2$, which can be used to calculate the expectation values of observables such as momentum, energy, etc. Formulating quantum mechanics in this fashion is termed first quantization. In the setting of N indistinguishable particles that may be interacting, the wavefunctions describing the physically relevant states of this collection of particles become cumbersome to work with. Moreover, learning the properties of one specific particle will not generally lend towards understanding the behavior of the entire collection of particles. This is where the language of second quantization becomes very useful in reformulating the description of states and operators, where the number of occupied single particle states determines an N-particle state and operators acting on such states involve combinations of removing or creating particles in these states. We now make this construction very precise, following the general structure of Ola Bratteli and Derek Robinson in Chapter 5 of Volume 2 of their seminal textbook Operator Algebras and Quantum Statistical Mechanics ([21]).

To begin constructing this space of multi-particle states, we first define the single particle Hilbert space of interest. In this dissertation, we will work with $L^2(\mathbb{R}^2)$ and direct sums of the Landau level spaces $\mathcal{H}_n$. Thus, for the purpose of this section we will let $\mathcal{H}$ refer to either $L^2(\mathbb{R}^2)$ or $\bigoplus_{n=0}^{N_L} \mathcal{H}_n$, where N_L is finite but arbitrarily large. Introducing additional particles is achieved via the tensor product, where the general N-particle Hilbert space is given by $\bigotimes_{n=0}^{N} \mathcal{H}$. The vacuum 0-particle space $\mathcal{H}^{(0)}$ is just $\mathbb{C}$. Electrons, which are the particles of interest in this work, are fermions and thus have wavefunctions that are antisymmetric under particle exchange, meaning that for a 2-electron wavefunction ψ, one has $\psi(x_1, x_2) = -\psi(x_2, x_1)$. Such a condition is not enforced by the tensor product, so we use the antisymmetric wedge product $\wedge$ to denote an antisymmetrized tensor product. Explicitly for $f, g \in \mathcal{H}$,

$$(f \wedge g)(x_1, x_2) = f(x_1)g(x_2) - f(x_2)g(x_1);\ x_1, x_2 \in \mathbb{R}^2$$

Hence, the N-particle fermionic Hilbert space is precisely $\mathcal{H}^{(N)} = \bigwedge_{n=0}^{N} \mathcal{H}$. In terms of these multi-particle Hilbert spaces, we now introduce fermionic Fock space, the space of sequences of N-particle fermion states.

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(N)} \tag{72}$$

An element $\psi \in \mathcal{F}$ is thus a sequence $\psi = \{\psi_n\}_{n\in\mathbb{N}_0}$ with each $\psi_n \in \mathcal{H}^{(N)}$. For example, the sequences associated to systems of N-particles will be of the form $f_N = \{0, 0, ..., f, 0, ...\}$ with zeros everywhere except the $(N+1)^{\text{th}}$ entry. We will now introduce a set of operators that add and remove particles, shifting a sequence f_N to $F_{N\pm1}$ for example. These are precisely the so-called fermionic creation and annihilation operators, which we will now take time to carefully define.

For $f \in \mathcal{H}$, denote by $a^*(f)$ and $a(f)$ the creation and annihilation operators associated with the single particle function f. For the particular function $\psi^{(N)} = \psi_1 \wedge \psi_2 ... \wedge \psi_N$, $a(f)$ and $a^*(f)$ act as,

$$a^*(f)\psi^{(N)} = \sqrt{N+1}\,(f \wedge \psi_1 \wedge \psi_2 ... \wedge \psi_N) \tag{73}$$

$$a(f)\psi^{(N)} = \sqrt{N}\langle f, \psi_1\rangle\,(\psi_2 \wedge \,...\, \wedge \psi_N) \tag{74}$$

One clearly sees how these operators carry the physical interpretation of adding a particle, via the wedge product, or removing a particle, with an inner product. More general functions will involve linear combinations of ones such as $\psi^{(N)}$ as above. We thus write the action of $a^{(*)}(f)$ more explicitly:

$$\begin{aligned} (a(f)\phi)^{(n)}(x_1, ..., x_n) &= \sqrt{n+1}\int_{\mathbb{R}^2} dx\,\overline{f(x)}\phi^{(n+1)}(x, x_1, ..., x_n) \\ (a^*(f)\phi)^{(n)}(x_1, ..., x_n) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(-1)^{i-1}f(x_i)\phi^{(n-1)}(x_1, ..., \hat{x}_i, ..., x_n), \end{aligned} \tag{75}$$

where the $\hat{x}_i$ denotes an omitted coordinate. We shall more generally denote the algebra generated by $\{a(f), a^*(f) :\ f \in \mathcal{H}\}$ as $\mathcal{A}(\mathcal{H})$. This is the fermionic algebra of observables associated with the Hilbert space $\mathcal{H}$ known as the Canonical Anticommutation Relations (CAR) algebra. Its properties as a C^*-algebra can be found described in great detail in references such as Chapter 2 of Volume 1 of the textbook by Bratteli and Robinson, [20].

Importantly, we now state the algebra defining canonical anticommutation relations.

$$\{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0;\ \{a(f), a^*(g)\} = \langle f, g\rangle \mathbb{1};\ \forall\, f, g \in \mathcal{H} \tag{76}$$

Additionally, these can be used to show that the operator norm of $a^{(*)}(f)$ is given by $\left\|a^{(*)}(f)\right\| = \|f\|_{L^2}$. From the form of $a(f)$ and $a^*(f)$ in equation 75, there is an alternate way to define these operators in terms of fermionic field operators $c_x^{(*)}$ that are operator valued distributions.

$$a^*(f) = \int_{\mathbb{R}^2} dx\, f(x) c_x^*,\ a(f) = \int_{\mathbb{R}^2} dx\, \overline{f(x)} c_x \tag{77}$$

$$(c_x\phi)^{(n)}(x_1, ..., x_n) = \sqrt{n+1}\phi^{(n+1)}(x, x_1, ..., x_n); \tag{78}$$

$$(c_x^*\phi)^{(n)}(x_1, ..., x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i-1} \delta(x - x_i) \phi^{(n-1)}(x_1, ..., \hat{x}_i, ..., x_n)$$

These field operators can be thought of as creation/annihilation operators that add a fermion at $x \in \mathbb{R}^2$ through a delta function centered at x. Moreover, these field operators satisfy the following anticommutation relations:

$$\{c_x, c_y\} = \{c_x^*, c_y^*\} = 0;\ \{c_x, c_y^*\} = \delta(x-y)\mathbb{1};\ \forall\, x, y \in \mathbb{R}^2. \tag{79}$$

Again note that we are using $\mathbb{R}^2$ in all of the definitions and properties here since this is the space of interest in this work, so the formalism here holds in $\mathbb{R}^d$ more generally. We may also define the total fermion number operator $\hat{N}$ in terms of the field operators as,

$$\hat{N} = \int_{\mathbb{R}^2} dx\, c_x^* c_x, \tag{80}$$

and the local number operator $\hat{N}_X$ associated with $X \subset \mathbb{R}^2$,

$$\hat{N}_X = \int_X dx\, c_x^* c_x. \tag{81}$$

By the anticommutation relations in equation 79, it follows that for all $X, Y \subset \mathbb{R}^2$, $\left[\hat{N}_X, \hat{N}_Y\right] = 0$.

Crucially, these field operators may be used to lift self-adjoint operators acting on $\mathcal{H}^{(N)}$ for a particular N to operators acting on sequences of states in $\mathcal{F}(\mathcal{H})$; this mapping is what is known as second quantization. Of utmost importance for us is the second quantization of single particle operators acting on $\mathcal{H}$ and two-body interactions acting on $\mathcal{H} \wedge \mathcal{H}$. For example, if $H_1 = \frac{p^2}{2m} + V(x)$ is a one-particle Hamiltonian, its second quantization $\mathbb{H}$ is obtained as shown in the subsequent equation.

$$\mathbb{H} = \int_{\mathbb{R}^2} dx\, c_x^* H_1 c_x \tag{82}$$

For example, as we will need later in this section and in future ones, the following procedure demonstrates how to second quantize the Landau Hamiltonian $H_L = (p - A(x))^2$.

$$\mathbb{H}_L = \int_{\mathbb{R}^2} dx\, c_x^* H_L c_x \tag{83}$$

$$= \int_{\mathbb{R}^2} dx\, c_x^* \left(\sum_{n=0}^{\infty} 2B \left(n + \frac{1}{2} \right) \sum_{m=0}^{\infty} \left| \psi_{nm} \right\rangle \left\langle \psi_{nm} \right| \right) c_x \tag{84}$$

$$= 2B \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\, \psi_{nm}(x) c_x^* \overline{\psi_{nm}(y)} c_y \tag{85}$$

$$= 2B \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \sum_{m=0}^{\infty} a^*(\psi_{nm}) a(\psi_{nm}) \tag{86}$$

$$= \sum_{n=0}^{\infty} E_n \hat{N}_n;\ \ \hat{N}_n = \sum_{m=0}^{\infty} a^*(\psi_{nm}) a(\psi_{nm}) \tag{87}$$

In the second equality we make use of a spectral resolution of H_L, writing it as a sum over Landau levels of the associated energy multiplied by the orthogonal projection onto the corresponding subspace. The third equality expresses these projections using an integral kernel, which gives rise to a form where $a^{(*)}(\psi_{nm})$ becomes apparent by equation 77. In the final equality, we introduce the Landau level number operator $\hat{N}_n$ which counts the number of fermions occupying the n^{th} Landau level.

Similarly, if $W(x,y)$ is a two-body interaction operator that acts via multiplication by the function W of the position operators x and y, then its second quantization is as shown in the following equation.

$$\mathbb{W} = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dxdy\, W(x,y)c_x^* c_y^* c_y c_x. \tag{88}$$

We defer stating assumptions on the function $W(x,y)$ until the Hamiltonian of interest is introduced. The second quantization $\mathbb{H}_L$ of the Landau Hamiltonian and more generally $\mathbb{H}$, the second quantization of $H_L + V$, as well as interactions such as 88 will appear in the Hamiltonians generating the dynamics of interacting fermions. In the next subsection, we introduce Heisenberg time evolution as an automorphism, a map from $\mathcal{A}(\mathcal{H})$ to itself that preserves its algebraic structure, namely the canonical anticommutation relations. Specifically, we introduce the free dynamics, which provides an algebraic means of defining the second quantization of a self-adjoint operator such as H_L or $H_L + V$, and then we set up the problem of studying the Heisenberg dynamics of interacting fermions.

4.2. **CAR Algebra and Dynamics.**

As Bratteli and Robinson note in Section 5.2.2 of [21], the preceding introduction of the fermionic creation and annihilation operators involved deriving the space of sequences, i.e. fermionic Fock space $\mathcal{F}(\mathcal{H})$, upon which they act to add or remove single particle states. However, one may also directly consider $\mathcal{A}(\mathcal{H})$, generated by $\{a(f), a^*(f) :\ f \in \mathcal{H}\}$, from an algebraic perspective without having to represent its elements as operators on sequences of the Fock space. More abstract properties of $a^*(f)$ and $a(f)$ can be found in the cited reference.

Towards proving propagation bounds and Lieb-Robinson type estimates, it is important to define how the Heisenberg dynamics act on elements of $\mathcal{A}(\mathcal{H})$, and we begin with a discussion of free dynamics, meaning time evolution generated by the second quantization of single particle Hamiltonians. Denote by $\mathbb{H}_1$ the second quantization of a general Hamiltonian $H_1 = H_L + V$, where the magnetic field in H_L could even be taken to be zero to cover the non-magnetic system. Then, the free dynamics $\tau_t^{\emptyset} : \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ generated by $\mathbb{H}_1$ behave explicitly as shown in the following equation.

$$\tau_t^{\emptyset}(a^{\#}(f)) = a^{\#}\left(e^{-itH_1} f\right). \tag{89}$$

We will henceforth use $a^{\#}(f)$ to denote either $a^*(f)$ or $a(f)$ in cases where either operator can exist. This action of free time evolution can be derived via second quantization in the Fock space picture or algebraically, as is done in 3.1.2 of [12]. Formally, the second quantization of H_1 is precisely the operator $\mathbb{H}_1$ that implements Heisenberg dynamics over $\mathcal{A}(\mathcal{H})$ as,

$$\tau_t^{\emptyset}(a^{\#}(f)) = e^{it\mathbb{H}_1} a^{\#}(f) e^{-it\mathbb{H}_1}.$$

Towards considering the interacting dynamics, we first restrict the interaction $\mathbb{W}$ as in equation 88 to a finite volume $\Lambda \subset \mathbb{R}^2$.

$$\mathbb{W}_\Lambda = \frac{1}{2} \int_\Lambda \int_\Lambda dxdy\, W(x,y) c_x^* c_y^* c_y c_x \tag{90}$$

The finite volume interacting Hamiltonian is then formally written as,

$$\mathbb{H}_\Lambda = \mathbb{H}_1 + \mathbb{W}_\Lambda. \tag{91}$$

Importantly, even with suitable assumptions on the function $W(x,y)$ and the restriction to finite volume, $\mathbb{W}_\Lambda$ is not bounded, in the sense of an operator on sequences of $\mathcal{F}(\mathcal{H})$ as well as in terms of C^* operator norm, due to the fermion field operators $c_x^{\#}$. As emphasized in Section 2, the existence of a bounded interaction in finite volume is necessary to prove a Lieb-Robinson bound. Thus, we will smear the pointwise field operators into creation and annihilation operators that are elements of $\mathcal{A}(\mathcal{H})$. The focus of the next subsection is thus to introduce this regularization procedure and define a class of interacting Hamiltonians with bounded finite volume interactions. Moreover, for the remainder of this section, we concretely take $\mathcal{H}$ to be $L^2(\mathbb{R}^2)$.

4.3. **Many-Body Quantum Hall Model.**

We begin with a statement of the N-particle model of interest in the language of first quantization. The Hamiltonian is precisely the following:

$$H^{(N)} = \sum_{i=1}^{N} H_{1,i} + \sum_{i<j} W(x_i - x_j) \tag{92}$$

$$= \sum_{i=1}^{N} \left((p_i - A(x_i))^2 + V(x_i)\right) + \sum_{i<j} W(x_i - x_j). \tag{93}$$

Note that the first subscript on x and p refers to one of the N particles, meaning that x_{i1} and x_{i2} will refer to the Cartesian components of the i^{th} particle's position for example. We now state general assumptions on the one-body potential V and provide an example of an operator that satisfies the specified conditions.

Assumption 4.1. *(On V) Let $V : \mathbb{R}^2 \to \mathbb{C}$ have the form*

$$V(x) = \int_{\mathbb{R}^2} d\mu(k) e^{ik\cdot x}$$

where μ : Borel$(\mathbb{R}^2) \to \mathbb{R}$ is a Borel measure on $\mathbb{R}^2$ satisfying

(i) μ has support contained in a ball of radius M centered at the origin.

(ii) $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are nonnegative finite measures on Borel$(\mathbb{R}^2)$. Set $|\mu| = \mu^+ + \mu^-$.

(iii) μ is even on Borel$(\mathbb{R}^2)$.

As a consequence, we introduce the constants C_μ and M, which appear frequently in subsequent calculations:

$$\int_{\mathbb{R}^2} d|\mu|(x) \leq C_\mu, \quad \sup\{|k| | k \in \operatorname{supp}(|\mu|)\} \leq M. \tag{94}$$

Ultimately, V as a function of the position operator x is the Fourier transform of the measure μ, where the Fourier transform is evaluated over a finite range of frequencies. Physically, V is therefore a periodic function. Thus, a candidate potential function that is also relevant to condensed matter systems is the following:

$$V(x) = \sum_{i=1}^{J} b_i \cos(a_i \cdot x).$$

Physically, a potential of this form could represent a screened Coulomb interaction of electrons with the periodic arrangement of metallic atoms in the two-dimensional metal. Importantly, we must also require this potential to be bounded in energy. We now state assumptions on the two-body potential function $W(x, y)$.

Assumption 4.2. $W : \mathbb{R}^2 \to \mathbb{R}$ *satisfies,*
(i) W *is even and only depends on inter-particle distance:* $W(x, y) = W(x - y) = W(y - x)$ *for almost every* $x, y \in \mathbb{R}^2$*;*
(ii) W *is short range, meaning there exist* $a > 0, \;\; C_w > 0 \in \mathbb{R}$ *such that*

$$|W(x, y)| \leq C_w e^{-a|x-y|} \text{ for almost every } x, y \in \mathbb{R}^2.$$

This condition establishes that this pointwise interaction is bounded in energy and decays at most exponentially in the distance between interacting fermions. This is somewhat analogous to the construction of short-range lattice interactions where the range of interaction is dictated by an F-function, as in 5. However, as alluded to before, bounding the maximum value and decay of the potential function W is not sufficient in order to guarantee that $\mathbb{W}_\Lambda$ is a bounded operator. Therefore, we implement a UV regularization that resolves this issue, and this is the focus of the upcoming discussion.

4.3.1. *UV Regularization.*

Let $\sigma > 0$ be a fixed parameter and denote by $\varphi_0^\sigma(x) := \frac{1}{2\pi\sigma^2} e^{-|x|^2/2\sigma^2}$ an L^1-normalized Gaussian function over $\mathbb{R}^2$. Moreover, let $\Psi_a^\sigma(x) = T_a^+ \varphi_0^\sigma(x)$ be the L^1-normalized Gaussian magnetically translated to be centered at $a \in \mathbb{R}^2$. In terms of these translates, we introduce the regularized two-body interaction $\mathbb{W}_\Lambda^\sigma$:

$$\mathbb{W}_\Lambda^\sigma = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dxdy \, W_\Lambda(x,y) a^*(\Psi_x^\sigma) a^*(\Psi_y^\sigma) a(\Psi_y^\sigma) a(\Psi_x^\sigma). \tag{95}$$

The fermionic field operators $c_x^\#$ are replaced with bounded operators associated to the Gaussians Ψ_x^σ, indicating that the pointwise interaction occurs in the vicinity of the points $x, y \in \mathbb{R}^2$. The constant σ carries the physical interpretation of describing the effective width of the interacting fermions. Since $\Psi_x^\sigma \in L^2(\mathbb{R}^2)$, each $a^\#(\Psi_x^\sigma)$ has a bounded norm, and $\mathbb{W}_\Lambda^\sigma$ has a finite operator norm for finite Λ.

It is natural to wonder whether the first quantization of $\mathbb{W}_\Lambda^\sigma$ corresponds to a suitable two-body potential. If we restrict $\mathbb{W}_\Lambda^\sigma$ to the n-particle sector of fermionic Fock space, it acts on n-fold antisymmetric tensor products of functions. However in doing so, it acts on every pair of particles as it is a two-body interaction. Hence, to justify (88) as describing a two-body interaction, we prove that in the limit $\sigma \downarrow 0$, $\mathbb{W}_\Lambda^\sigma$ behaves as multiplication by the potential function W restricted to the region Λ when acting on a 2-particle state. For this reason, let $W_\Lambda(x_1, x_2) = \mathbb{1}_{\Lambda\times\Lambda}(x_1, x_2) W(x_1, x_2)$, where $\mathbb{1}_{\Lambda\times\Lambda}(x_1, x_2)$ is the indicator function that evaluates to 1 when $x_1, x_2 \in \Lambda$ and 0 otherwise. We then prove the following limit.

Proposition 4.3. *Let $\phi \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ be a two-particle state and $W_{\Lambda;2}^\sigma$ the two-body interaction which is the restriction of $\mathbb{W}_\Lambda^\sigma$ to the 2-particle sector of $\mathcal{F}(L^2(\mathbb{R}^2))$. Then,*

$$\lim_{\sigma\downarrow 0} (W_{\Lambda;2}^\sigma \phi)(x_1, x_2) = W_\Lambda(x_1, x_2) \phi(x_1, x_2). \tag{96}$$

This is a rather technical proof, so the inclined reader can proceed to view the details, but it would also do no harm to skip over.

proof of Proposition 4.3. We step away from the second quantized notation and express the action of $W^{\sigma}_{\Lambda;2}$ on ϕ as follows.

$$(W^{\sigma}_{\Lambda;2}\phi)(x_1,x_2) = \int_{\mathbb{R}^8} dxdydz_1dz_2\ W_\Lambda(x,y)\phi(z_1,z_2)\overline{\Psi^{\sigma}_{z_1}(x)\Psi^{\sigma}_{z_2}(y)}\Psi^{\sigma}_{x_1}(x)\Psi^{\sigma}_{x_2}(y)$$

A magnetically translated function consists of translation by $j \in \mathbb{R}^2$ followed by multiplication of a complex phase $\mathcal{U}_j(x) = e^{-ij\cdot A(x)}$. Rewriting the expression above in terms of these phases, we obtain

$$(W^{\sigma}_{\Lambda;2}\phi)(x_1,x_2) = \int_{\mathbb{R}^8} dxdydz_1dz_2\ W_\Lambda(x,y)\phi(z_1,z_2)\mathcal{U}_{-z_1}(x)\mathcal{U}_{-z_2}(y)\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)$$
$$\times\, \varphi^{\sigma}_0(x-z_1)\varphi^{\sigma}_0(y-z_2)\varphi^{\sigma}_0(x_1-x)\varphi^{\sigma}_0(x_2-y).$$

This equality utilizes the symmetry of the Gaussian states as well as the fact that the translated Gaussian is equivalent to an origin-centered Gaussian evaluated at shifted coordinates. Before proceeding, introduce the following function:

$$\Phi^{\sigma}(x,y) := \varphi^{\sigma}_0(x)\varphi^{\sigma}_0(y).$$

Thus, evaluating the z_1 and z_2 integrals followed by the x and y integrals, the previous expression can be written in terms of convolutions.

$$\begin{aligned}
(W^{\sigma}_{\Lambda;2}\phi)(x_1,x_2) &= \int_{\mathbb{R}^8} dxdy\ \left(\Phi^{\sigma}(z_1,z_2) * \mathcal{U}_{-z_1}(x)\mathcal{U}_{-z_2}(y)\phi(z_1,z_2)\right)(x,y) \\
&\quad \times W_\Lambda(x,y)\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)\Phi^{\sigma}(x-x_1,y-x_2) \\
&= \big[\Phi^{\sigma}(x,y) * \big(\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\big(\Phi^{\sigma}(z_1,z_2)* \\
&\quad \mathcal{U}_{-z_1}(x)\mathcal{U}_{-z_2}(y)\phi(z_1,z_2)\big)(x,y)\big)\big](x_1,x_2) \\
&= [\Phi^{\sigma}(x,y) * (\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\phi(x,y))] +
\end{aligned}$$

$$[\Phi^\sigma(x,y) * \mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\times$$
$$(\Phi^\sigma(z_1,z_2) * \mathcal{U}_{-z_1}(x)\mathcal{U}_{-z_2}(y)\phi(z_1,z_2) - \phi(x,y))]$$

The last equality arises plainly from adding and subtracting the term in the first line of the last expression. Now consider the limit that $\sigma \downarrow 0$ of each term separately. Start with the first term.

$$\begin{aligned}
&\Phi^\sigma(x,y) * (\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\phi(x,y)) \\
&= \int_{\mathbb{R}^4} dxdy\ \Phi^\sigma(x_1 - x, x_2 - y)\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\phi(x,y) \\
&= \int_{\mathbb{R}^4} dxdy\ \varphi_0^\sigma(x_1 - x)\varphi_0^\sigma(x_2 - y)\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\phi(x,y)
\end{aligned}$$

In the limit $\sigma \downarrow 0$, $\varphi_0(x - j)$ behaves as a Dirac Delta function; hence, integrating this function over the variables x and y sends $x \to x_1$ and $y \to x_2$. Therefore, we obtain a first result.

$$\begin{aligned}
&\lim_{\sigma\downarrow 0}\left[\Phi^\sigma(x,y) * (\mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\phi(x,y))\right] \\
&= \mathcal{U}_{x_1}(x_1)\mathcal{U}_{x_2}(x_2)W_\Lambda(x_1,x_2)\phi(x_1,x_2) \\
&= W_\Lambda(x_1,x_2)\phi(x_1,x_2)
\end{aligned}$$

The last equality follows as each $\mathcal{U}_j(j)$ evaluates to 1 by construction. Next, we analyze the L^2 norm of the second term,

$$\left[\Phi^\sigma(x,y) * \mathcal{U}_{x_1}(x)\mathcal{U}_{x_2}(y)W_\Lambda(x,y)\left(\Phi^\sigma(z_1,z_2) * \mathcal{U}_{-z_1}(x)\mathcal{U}_{-z_2}(y)\phi(z_1,z_2) - \phi(x,y)\right)\right].$$

Further, with the arguments of each function made clear in the previous sentence, for compact notation, we remove these in the subsequent calculation.

$$\|\Phi^\sigma * \mathcal{U}_{x_1}\mathcal{U}_{x_2}W_\Lambda\left(\Phi^\sigma * \mathcal{U}_{-z_1}\mathcal{U}_{-z_2}\phi - \phi\right)\|_2 \le \|\Phi^\sigma\|_1\|\mathcal{U}_{x_1}\mathcal{U}_{x_2}W_\Lambda\|_1\|\Phi^\sigma * \mathcal{U}_{-z_1}\mathcal{U}_{-z_2}\phi - \phi\|_2$$

$$= \|\Phi^\sigma\|_1 \|W_\Lambda\|_1 \|\Phi^\sigma * \mathcal{U}_{-z_1}\mathcal{U}_{-z_2}\phi - \phi\|_2$$

The convolution that remains in the last line behaves just like the x,y variable convolution from before in the limit that $\sigma \downarrow 0$; however here, $z_1 \to x$ and $z_2 \to y$. Thus we finish the calculation by taking this limit.

$$\begin{aligned}\lim_{\sigma\downarrow 0} \|\Phi^\sigma * \mathcal{U}_{x_1}\mathcal{U}_{x_2} W_\Lambda \left(\Phi^\sigma * \mathcal{U}_{-z_1}\mathcal{U}_{-z_2}\phi - \phi\right)\|_2 &\le \|\Phi^\sigma\|_1 \|W_\Lambda\|_1 \|\mathcal{U}_{-x}\mathcal{U}_{-y}\phi - \phi\|_2 \\ &= 0\end{aligned}$$

Consequently, we conclude that $\lim_{\sigma\downarrow 0}(W^\sigma_{\Lambda;2}\phi)(x_1,x_2) = W_\Lambda(x_1,x_2)\phi(x_1,x_2)$, thus the proof is complete.

□

What this proof establishes is that in first quantization, when the regularization parameter σ is removed, we recover a bona fide two-body interaction. Therefore, our choice of UV regularization is a suitable strategy for producing a bounded, finite volume interaction consisting of elements of $\mathcal{A}(L^2(\mathbb{R}^2))$. We thus make use of the following interacting finite volume Hamiltonians in our study of the dynamics of continuum fermions.

$$\mathbb{H}^\sigma_\Lambda = \mathbb{H}_1 + \mathbb{W}^\sigma_\Lambda \tag{97}$$

As previously noted, the strategy of using Lieb-Robinson bounds to obtain strong continuity of the dynamics with a UV-regularized interaction is inspired by the work of Gebert et al., [31], who consider interacting fermions in arbitrary dimension, $\mathbb{R}^d$, without a magnetic field. Regularization of their pairwise interaction is achieved with plainly translated Gaussians $\varphi_0(x-a)$. As we will see in the upcoming subsection on single-particle propagation estimates, our choice of magnetic translates is essential in showing that the dynamics generated by the interaction $\mathbb{W}^\sigma_\Lambda$ do not result in non-local growth of operator supports. By support, we mean the supports in $\mathbb{R}^2$ of functions $f \in L^2(\mathbb{R}^2)$ in operators involving $a^\#(f)$.

The dynamics of interacting fermions in two dimensions with a constant magnetic field have also been studied before in [12], where the authors prove Lieb-Robinson bounds and strong continuity of the infinite volume dynamics using a localized frame based resolution of a Hamiltonian that is similar to $\mathbb{H}^{\sigma}_{\Lambda}$. The UV regularization is effectively implemented via localized functions, which in the case of fermions in a magnetic field are the functions $\chi_{\gamma,n}$ introduced in 3.2. Here the smearing parameter σ is set to be $\sqrt{2}\ell_B$, and the regularization extends to the kinetic energy as well, i.e. the second quantization of the Landau Hamiltonian, $\mathbb{H}_L$. However, this comes at the cost of requiring $\mathbb{H}_L$ to be restricted to finitely many Landau levels, which is still very useful for many applications. In fact, the lattice localization of the frames for the Landau level Hilbert spaces $\mathcal{H}_n$ provides a natural way to formulate quasi-adiabatic evolution and prove quantization of Hall conductance for continuum fermion systems. This is the subject of Sections 5 and 6.

4.4. **Single-particle Propagation Estimates.**

As in the case of lattice systems, we consider how the interaction between particles drives the propagation of information in a quantum system. In the continuum model, this phenomenon is dictated by the operators $a^{\#}(\Psi^{\sigma}_{x})$; the Gaussians centered at $x, y \in \mathbb{R}^2$ serve as probes of fermion interactions in the vicinities of these points. As we will see in the following subsection, it will be crucial to understand the locality of these states under time evolution governed by H_L and H_1. Towards this end, we begin with a lemma concerning an eigenfunction decomposition of φ^{σ}_0 in the polar coordinate basis as in 51.

Lemma 4.4. *Fix $\sigma > 0$ and let φ^{σ}_0 the L^1 normalized Gaussian centered at the origin. Then, the following eigenfunction decomposition with respect to H_L holds.*

$$\varphi^{\sigma}_0(r) = \sum_{n=0}^{\infty} c_n R_{n0}(r) \tag{98}$$

$$c_n = \frac{\ell_B}{\pi} \frac{1}{2\ell_B^2 + \sigma^2} \left(\frac{2\ell_B^2 - \sigma^2}{2\ell_B^2 + \sigma^2} \right)^n ; \quad \frac{\ell_B^2}{\sigma^2} \neq \frac{1}{2} \tag{99}$$

$$c_n = \begin{cases} \frac{1}{4\pi\ell_B} & n = 0 \\ 0 & n \neq 0 \end{cases} ; \quad \frac{\ell_B^2}{\sigma^2} = \frac{1}{2} \tag{100}$$

Note that in the case where $\frac{\ell_B^2}{\sigma^2} = \frac{1}{2}$, φ_0^σ is exactly the lowest Landau level radial wavefunction, up to normalization. Henceforth we will drop the subscript B in the magnetic length $\ell \equiv \ell_B$ for readability.

Proof. Recalling the orthogonality relation (56), we may express each c_n as shown below.

$$c_n = \int_0^\infty dr\, r\, R_{n0}(r)\varphi_0^\sigma(r) \tag{101}$$

$$= \int_0^\infty dr\, r\, \frac{1}{2\pi\ell\sigma^2} \exp\left(-\left(\frac{1}{4\ell^2} + \frac{1}{2\sigma^2}\right) r^2\right) L_n\left(\frac{r^2}{2\ell^2}\right). \tag{102}$$

One sees that this integral converges by noting that the integrand is the product of a Gaussian and a finite degree polynomial. At this stage, perform a change of variables $u = \frac{r^2}{2\ell^2}$ and note the definition of the standard n^{th} Laguerre polynomial: $L_n(u) = \frac{e^u}{n!}\frac{d^n}{du^n}(e^{-u}u^n)$.

$$c_n = \int_0^\infty du\, \frac{\ell}{2\pi\sigma^2}\, \exp\left(\left(\frac{1}{2} - \frac{\ell^2}{\sigma^2}\right) u\right) \frac{1}{n!}\frac{d^n}{du^n}(e^{-u}u^n) \tag{103}$$

We then use integration by parts n times to eliminate the n-fold differentiation with respect to u in the integrand. The boundary terms vanish, and this is seen by recognizing that with each successive integration of $\frac{d^n}{du^n}(e^{-u}u^n)$, we obtain $\frac{d^m}{du^m}(e^{-u}u^n)$ with $m < n$, meaning that what remains will always be an expression of the form $e^{-u}p_n(u)$, where $p_n(u)$ is a degree-n polynomial of u with no constant term. Letting $\delta = \frac{1}{2} - \frac{\ell^2}{\sigma^2}$ and $\delta' = 1 - \delta$, we obtain the following.

$$c_n = \frac{\ell}{2\pi\sigma^2}\frac{(-\delta^n)}{n!}\int_0^\infty du\, e^{-\delta' u}u^n \tag{104}$$

Another n-fold application of integration by parts will eliminate u^n and leave $e^{-\delta' u}$ in the integrand, which upon evaluation yields,

$$c_n = \frac{\ell}{2\pi\sigma^2}\frac{1}{\delta'}\left(\frac{-\delta}{\delta'}\right)^n. \tag{105}$$

A straightforward rewriting of the above expression in terms of ℓ and σ yields the result of the lemma. $\square$

With this lemma, we are now equipped to prove the following theorem concerning the time evolution governed by H_L.

Theorem 4.5. ***Non-interacting Dynamics*** *Let H_L be the Landau Hamiltonian in the symmetric gauge. Define constants $\eta_\sigma = min(\sqrt{2}\ell,\ \sigma)$ and $\Delta_\sigma = max(\frac{2\ell^2}{\sigma},\ \sigma)$ with $\sigma > 0$. Then the following equality and inequality hold for all $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^2)$.*

(106)

$$\left|e^{-itH_L}\varphi_0^\sigma\right|(x) = \frac{1}{\pi}\frac{1}{\sqrt{8\ell^4\left(1-\cos\left(\frac{2t}{\ell^2}\right)\right)+2\sigma^4\left(1+\cos\left(\frac{2t}{\ell^2}\right)\right)}}e^{-\frac{\sigma^2 x^2}{4\ell^4\left(1-\cos\left(\frac{2t}{\ell^2}\right)\right)+\sigma^4\left(1+\cos\left(\frac{2t}{\ell^2}\right)\right)}}$$

(107)

$$\left|\langle e^{-itH_L}f, \Psi_y^\sigma\rangle\right| \leq \frac{1}{2\pi\eta_\sigma^2}\int_{\mathbb{R}^2} dx\ |f(x)|e^{-\frac{|x-y|^2}{2\Delta_\sigma^2}}$$

Proof. Time evolving φ_0^σ just involves a linear combination of the time evolved radial wavefunctions with associated eigenvalues $E_n = \frac{2}{\ell^2}\left(n+\frac{1}{2}\right)$. We introduce the parameter $\gamma \equiv \frac{2\ell^2-\sigma^2}{2\ell^2+\sigma^2}$. Then, using the expression for the spectrum of H_L given $m_l = 0$, (50),

$$\left(e^{-iH_Lt}\varphi_0^\sigma\right)(x) = \sum_{n=0}^{\infty} c_n e^{-iE_nt}R_{n0}(x) \tag{108}$$

$$= \frac{1}{\pi}\frac{e^{-\frac{it}{\ell^2}}e^{-\frac{x^2}{4\ell^2}}}{2\ell^2+\sigma^2}\sum_{n=0}^{\infty}\gamma^n e^{-\frac{2int}{\ell^2}}L_n\left(\frac{x^2}{2\ell^2}\right) \tag{109}$$

$$= \frac{1}{\pi}\frac{e^{-\frac{it}{\ell^2}}e^{-\frac{x^2}{4\ell^2}}}{2\ell^2+\sigma^2}\sum_{n=0}^{\infty}\tilde{\gamma}^n L_n\left(\frac{x^2}{2\ell^2}\right) \tag{110}$$

where we have now defined the complex parameter $\tilde{\gamma} \equiv \gamma e^{-\frac{2it}{l^2}}$. It is clear from the form of γ that for all $\sigma > 0$, we have $|\tilde{\gamma}| < 1$. Consequently, the sum in the last equality is none other than the generating function involving Laguerre polynomials (58): $\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t}\exp\left(\frac{-tx}{1-t}\right)$ for all t with $|t| < 1$. This leads to a closed form expression for the time evolved

Gaussian.

$$(111)\qquad \left(e^{-iH_Lt}\varphi_0^\sigma\right)(x) = \frac{1}{\pi}\frac{e^{-\frac{it}{\ell^2}}e^{-\frac{x^2}{4\ell^2}}}{2\ell^2+\sigma^2}\frac{1}{1-\tilde{\gamma}}e^{\frac{-\tilde{\gamma}x^2}{2\ell^2(1-\tilde{\gamma})}}$$

$$= \frac{1}{\pi}\frac{e^{-\frac{it}{\ell^2}}e^{-\frac{x^2}{4\ell^2}}}{2\ell^2+\sigma^2}\frac{1}{1-\gamma e^{-\frac{2it}{\ell^2}}}e^{-\frac{x^2}{2\ell^2}\frac{i\gamma\sin\left(\frac{2t}{\ell^2}\right)}{1+\gamma^2-2\gamma\cos\left(\frac{2t}{\ell^2}\right)}}e^{-\frac{x^2}{2\ell^2}\frac{\gamma\cos\left(\frac{2t}{\ell^2}\right)-\gamma^2}{1+\gamma^2-2\gamma\cos\left(\frac{2t}{\ell^2}\right)}}$$

Now we evaluate the absolute value of the last equality.

(112)

$$\left|\left(e^{-iH_Lt}\varphi_0^\sigma\right)\right|(x) = \frac{1}{\pi}\frac{e^{-\frac{x^2}{4\ell^2}}}{2\ell^2+\sigma^2}\frac{1}{\sqrt{1+\gamma^2-2\gamma\cos\left(\frac{2t}{\ell^2}\right)}}e^{-\frac{x^2}{2\ell^2}\frac{\gamma\cos\left(\frac{2t}{\ell^2}\right)-\gamma^2}{1+\gamma^2-2\gamma\cos\left(\frac{2t}{\ell^2}\right)}}$$

$$= \frac{1}{\pi}\frac{1}{\sqrt{8\ell^4\left(1-\cos\left(\frac{2t}{\ell^2}\right)\right)+2\sigma^4\left(1+\cos\left(\frac{2t}{\ell^2}\right)\right)}}e^{-\frac{\sigma^2x^2}{4\ell^4\left(1-\cos\left(\frac{2t}{\ell^2}\right)\right)+\sigma^4\left(1+\cos\left(\frac{2t}{\ell^2}\right)\right)}}$$

This is the first statement of the theorem. To obtain the second result, we establish a time independent inequality on the first result which arises from finding the supremum over $t \in \mathbb{R}$ of the prefactor and the width of the Gaussian term. Let $\eta_\sigma = \min\left\{\sqrt{2}\ell, \sigma\right\}$ and $\Delta_\sigma = \max\left\{\frac{2\ell^2}{\sigma}, \sigma\right\}$. We can then compute a time-independent bound with the following inequality.

$$\left|e^{-iH_Lt}\varphi_0^\sigma\right|(x) \le \frac{1}{2\pi\eta_\sigma^2}e^{-\frac{x^2}{2\Delta_\sigma^2}}$$

Now we are equipped to time evolve the magnetically translated Gaussian Ψ_y^σ. From the previous section, $[T_x^+, H] = 0\ \forall x \in \mathbb{R}^2$. Hence,

$$e^{-iH_Lt}\Psi_y^\sigma(x) = e^{-iH_Lt}T_y^+\varphi_0^\sigma(x) = T_y^+e^{-iH_Lt}\varphi_0^\sigma(x).$$

Then by Definition 3.1,

$$T_y^+e^{-iH_Lt}\varphi_0^\sigma(x) = e^{-iy\cdot A(x)}\left(e^{-iH_Lt}\varphi_0^\sigma\right)(x-y).$$

Taking the absolute value of the previous inequality guides us towards the end of this proof,

$$\left|\left(e^{-iH_L t}\Psi_y^\sigma\right)\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2} e^{-\frac{|x-y|^2}{2\Delta_\sigma^2}},$$

from which the second result of the theorem plainly follows. □

Importantly, these estimates illustrate that the functions Ψ_x^σ remain localized around their magnetic centers $x \in \mathbb{R}^2$ even under time evolution by the Landau Hamiltonian, albeit with a fluctuating width. It is also worth noting that in the zero-field limit, equation 106 is consistent with the analogous result from [31], equation 2.18, where the time evolution of φ_0^σ governed by $H = -\Delta$ is calculated:

$$\lim_{B\to 0} \left|e^{-itH_L}\varphi_0^\sigma\right|(x) = \frac{1}{2\pi} \frac{e^{-\frac{\sigma^2|x|^2}{8t^2+2\sigma^4}}}{\sqrt{4t^2+\sigma^4}}.$$

The next theorem establishes a similar locality estimate when the single particle dynamics are generated by the more general Hamiltonian H_1 with potential V satisfying the previously stated assumptions, Assumption 4.1.

Theorem 4.6. *Let $H_1 = H_L + V$ where V satisfies Assumption 4.1. Then $\forall t \in \mathbb{R}$ and any $\sigma > 0$, the following bounds hold for all finite t and $f \in L^2(\mathbb{R}^2)$.*

(113)

$$\left|e^{-itH_1}\Psi_y^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2}\left(e^{-\frac{|x-y|^2}{2\Delta_\sigma^2}} + e^{-\frac{|x-y|^2}{8\Delta_\sigma^2}}\left(e^{C_\mu|t|}-1\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{B|x-y|}{8M}\left(\ln\frac{B|x-y|}{8MC_\mu|t|}-1\right)} e^{C_\mu|t|}\right)$$

(114)

$$\left|\langle e^{-itH_1}f,\ \Psi_y^\sigma\rangle\right| \leq C_1 e^{C_2|t|\ln|t|}\int_{\mathbb{R}^2} dx\, e^{-\frac{C_3}{C_\mu|t|+1}|x-y|}|f(x)|$$

The constants C_1, C_2, and C_3 are made precise in the proof of this theorem, but note that they in general depend on σ and the magnetic field B. C_μ and M are as in 94. The key features to highlight in the first inequality are the Gaussian and exponential factors, which all enforce decay in the distance between the center y of Ψ_y^σ and the arbitrary point x. Even with a periodic potential V added to the Landau Hamiltonian, the states Ψ_x^σ remain spatially localized. However, in Lieb-Robinson fashion, this upper bound grows exponentially in size as time grows. Furthermore as illustrated by the time dependence in the third term inside parentheses, the effective width of the localized particle grows in time, which is not surprising considering the setting of evolving a Gaussian-type wavepacket in a non-interacting Hamiltonian.

The second inequality effectively measures the amplitude of f in the vicinity of the point y after time evolution governed by H_1. This can be seen by noting that integrating $f(x)$ against a delta function $\delta(x-y)$ yields $f(y)$; smearing out the delta function into a Gaussian converts the integral into a convolution, which carries the weight of f near y. The upper bound consists of a convolution of $|f|$ with an exponentially decaying function, which still outputs the weight of f in a neighborhood of y. Crucially, this neighborhood increases in spatial extent linearly in time as the exponent indicates, and the overall bound itself increases exponentially in time due to the prefactor.

Ultimately, the results of both Theorems 4.5 and 4.6 illustrate that the functions Ψ_x^σ remain spatially localized about x after time evolution generated by either H_L or $H_L + V$. As inequalities such as equation 114 emphasize, the states Ψ_x^σ also serve as effective probes for measuring the spatial spread of other arbitrary single particle states $f \in L^2(\mathbb{R}^2)$ under time evolution. This consequence will play a very important role in characterizing the effect of $\mathbb{W}_\Lambda^\sigma$ on the supports of many-body observables.

The proof of Theorem 4.6 requires a handful of lemmas, so we state and prove a few of these first, delegating the purely technical results to the appendix section A. An early step in this proof requires the Heisenberg time evolution, governed by the Landau Hamiltonian, of the operator $e^{ik\cdot x}$ which acts by multiplication. So the first lemma here states the form of this time evolved operator.

Lemma 4.7. *Let H_L denote the Landau Hamiltonian with magnetic field strength B, where the vector potential is defined according to the symmetric gauge, and $\tau^{\emptyset}_{1,t}$ the single-particle Heisenberg dynamics governed by H_L. Then for all $k \in \mathbb{R}^2$, one has,*

$$\tau^{\emptyset}_{1,t}\left(e^{ik\cdot x}\right) = e^{\frac{i}{2B}k\cdot b_k(t)} e^{ik\cdot x} T^{-}_{\frac{1}{B}b_k(t)}, \tag{115}$$

where $b_k(t) = (-k_1(\sin(2Bt)) + k_2(\cos(2Bt) - 1), k_1(1 - \cos(2Bt)) + k_2 \sin(2Bt))$.

The reader does not lose the meaning of this section by skipping this proof, but the inclined reader will find that this evolved operator illustrates why a plainly translated Gaussian becomes non-local under Landau Hamiltonian time evolution. This is apparent by noting the following attempt at calculating this state.

$$\begin{aligned}
\varphi^{\sigma}_0(x-a) &= e^{ia\cdot A(x)}\Psi^{\sigma}_a(x) \\
&= e^{-iA(a)\cdot x}\Psi^{\sigma}_a(x) \\
e^{-itH_L}\varphi^{\sigma}_0(x-a) &= e^{-itH_L}e^{-iA(a)\cdot x}\Psi^{\sigma}_a(x) \\
&= \tau^{\emptyset}_{1,t}\left(e^{-iA(a)\cdot x}\right)e^{-itH_L}\Psi^{\sigma}_a(x)
\end{aligned}$$

From Theorem 4.5, $e^{-itH_L}\Psi^{\sigma}_a$ remains localized about the point a, but from this lemma, it follows that $\tau^{\emptyset}_{1,t}\left(e^{-iA(a)\cdot x}\right)$ shifts this point of localization in a time dependent manner that depends on how far a is from the origin. We now proceed with a proof of the lemma at hand.

proof of Lemma 4.7. To start, we consider the Heisenberg dynamics of Π^-_1 and Π^-_2. For brevity of notation in the rest of this proof, we drop the $-$ superscript. Towards this end, recall that $H_L = \Pi_1^2 + \Pi_2^2$ and that $[\Pi_1,\, \Pi_2] = iB\mathbb{1}$. Further, define the function $f : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{R}^2)) \times \mathcal{B}(L^2(\mathbb{R}^2))$ by

$$f(t) = \begin{pmatrix} \tau^{\emptyset}_{1,t}(\Pi_1) \\ \tau^{\emptyset}_{1,t}(\Pi_2) \end{pmatrix}.$$

Then, it follows that $f(t)$ satisfies the differential equation,

$$f'(t) = i \begin{pmatrix} \tau^{\emptyset}_{1,t}([H_L, \Pi_1]) \\ \tau^{\emptyset}_{1,t}([H_L, \Pi_2]) \end{pmatrix} = i \begin{pmatrix} -2iB\tau^{\emptyset}_{1,t}(\Pi_2) \\ 2iB\tau^{\emptyset}_{1,t}(\Pi_1) \end{pmatrix} = \beta f(t) \tag{116}$$

where β is the constant matrix given by

$$\begin{pmatrix} 0 & 2B \\ -2B & 0 \end{pmatrix}.$$

The unique solution of (116) is given by

$$f(t) = e^{\beta t} f(0) = \begin{pmatrix} \cos(2Bt) & \sin(2Bt) \\ -\sin(2Bt) & \cos(2Bt) \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} \tag{117}$$

The form of $e^{\beta t}$ in (117) can be obtained by noting that $\beta = 2iB\sigma^y$ where σ^y is a Pauli matrix. Now, define the function $g : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{R}^2)) \times \mathcal{B}(L^2(\mathbb{R}^2))$ and compute its first derivative as shown below.

$$g(t) = \tau^{\emptyset}_{1,t}(k \cdot x) \tag{118}$$

(119)

$$\begin{aligned} g'(t) = i\tau^{\emptyset}_{1,t}\left([H_L, k \cdot x]\right) &= i\left(\tau^{\emptyset}_{1,t}\left([\Pi_1^2, k \cdot x]\right) + \tau^{\emptyset}_{1,t}\left([\Pi_2^2, k \cdot x]\right)\right) \\ &= 2k_1\tau^{\emptyset}_{1,t}(\Pi_1) + 2k_2\tau^{\emptyset}_{1,t}(\Pi_2) \\ &= 2\left(k_1\cos(2Bt) - k_2\sin(2Bt)\right)\Pi_1 + 2\left(k_1\sin(2Bt) + k_2\cos(2Bt)\right)\Pi_2 \end{aligned}$$

The last equality follows from (117), and noting that $g(0) = k \cdot x$ and $b_k(t) = (k_1(\sin(2Bt)) + k_2(\cos(2Bt) - 1), k_1(1 - \cos(2Bt)) + k_2\sin(2Bt))$, one has the solution,

$$g(t) = \tau_{1,t}^{\emptyset}(k \cdot x) = k \cdot x + \frac{1}{B} b_k(t) \cdot \Pi \tag{120}$$

Exponentiating $ig(t)$ yields $\tau_{1,t}^{\emptyset}(e^{ik\cdot x})$ and an application of the Baker-Campbell-Hausdorff formula gives the desired result. □

Remark that the time dependent translation $b_k(t)$ has an orbit frequency equal to the cyclotron frequency we found earlier. The next lemma concerns the commutation of the magnetic translation operators $T_a^{\pm}$ and the operator $e^{ik\cdot x}$

Lemma 4.8. *Let $T_{(\cdot)}^{\pm}$ denote the aforementioned magnetic translation operators. Then for any $a, k \in \mathbb{R}^2$,*

$$T_a^{\pm} e^{ik\cdot x} f = e^{-ik\cdot a} e^{ik\cdot x} T_a^{\pm} f \tag{121}$$

holds for all $f \in L^2(\mathbb{R}^2)$.

This equality follows from the behavior of the magnetic translation operators shown in Definition 3.1. We now proceed with proving the Theorem in question.

proof of Theorem 4.6. Start with a Dyson series expansion for the dynamics generated by $H = H_L + V$.

$$e^{-iHt} = e^{-itH_L} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_n ... \int_0^{t_2} dt_1 e^{-i(t-t_n)H_L} V e^{-i(t_n - t_{n-1})H_L} V \\ ... V e^{-it_1 H_L} \tag{122}$$

$$\left| \left(e^{-itH} - e^{-itH_L} \right) \Psi_y^{\sigma}(x) \right| = \left| \sum_{n=1}^{\infty} \int_0^t dt_n ... \int_0^{t_2} dt_1 \int_{\mathbb{R}^2} d\mu(k_n) ... \int_{\mathbb{R}^2} d\mu(k_1) \mathcal{I}_n \left(\{k_i\}_{i=1}^n, y, x \right) \right| \tag{123}$$

$$\leq \sum_{n=1}^{\infty} \int_0^t dt_n ... \int_0^{t_2} dt_1 \int_{\mathbb{R}^2} d|\mu|(k_n) ... \int_{\mathbb{R}^2} d|\mu|(k_1) |\mathcal{I}_n (\{k_i\}_{i=1}^n, y, x)|$$

$$\mathcal{I}_n (\{k_i\}_{i=1}^n, y, x) = e^{-i(t-t_n)H_L} V_{k_n} e^{-i(t_n - t_{n-1})H_L} V_{k_{n-1}} ... V_{k_1} e^{-it_1 H_L} \Psi_y^\sigma (x) \tag{124}$$

Each V_{k_j} corresponds to the multiplication operator by the plane wave function $e^{ik_j \cdot x}$. We then proceed to evaluate $\mathcal{I}_n$ for each n.

$$\mathcal{I}_n (\{k_i\}_{i=1}^n, y, x) = e^{-itH_L} \prod_{j=1}^{n} \tau_{1,t_j}^{\emptyset} (e^{ik_j \cdot x}) \Psi_y^\sigma (x) \tag{125}$$

From Lemmas 4.7, 4.8, and those proved in Appendix A, we arrive at the following inequality:

$$|\mathcal{I}_n| (\{k_i\}_{i=1}^n, y, x) \leq \frac{1}{2\pi \eta_\sigma^2} e^{-\frac{\left| x - y - \frac{1}{B} \sum_{j=1}^n b_{k_j}(t_j) + b_{k_j}(-t) \right|^2}{2\Delta_\sigma^2}}, \tag{126}$$

where η_σ and Δ_σ are as in equation 107 and $b_k(t)$ is as introduced in 4.7. We now separate the Dyson series, (122), into two sums, and this choice of partition will guide us towards spatial decay in $|x - y|$. Towards this end, remark the following bound on $b_{k_j}(t)$.

$$b_{k_j}(t) = \begin{pmatrix} \sin(2Bt) & \cos(2Bt) - 1 \\ 1 - \cos(2Bt) & \sin(2Bt) \end{pmatrix} \begin{pmatrix} k_{j,1} \\ k_{j,2} \end{pmatrix} = 2\sin(Bt) \begin{pmatrix} \cos(Bt) & -\sin(Bt) \\ \sin(Bt) & \cos(Bt) \end{pmatrix} \begin{pmatrix} k_{j,1} \\ k_{j,2} \end{pmatrix} \tag{127}$$

$$\left| b_{k_j}(t) \right| \leq 2|k_j| \leq 2M \tag{128}$$

The constant M is as in (94). The Gaussian in (126) depends on $x - y$; therefore w.l.o.g set $y = 0$ for the ensuing analysis of the exponent. One then finds,

$$\left| x - \frac{1}{B} \sum_{j=1}^n b_{k_j}(t_j) + b_{k_j}(-t) \right|^2 \geq |x|^2 - \frac{8nM}{B}|x| + \frac{16n^2 M^2}{B^2} \tag{129}$$

Introduce the quantity K_n defined as,

$$K_n = \frac{1}{2\pi \eta_\sigma^2} \int_0^t dt_n ... \int_0^{t_2} dt_1 \int_{\mathbb{R}^2} d|\mu|(k_n) ... \int_{\mathbb{R}^2} d|\mu|(k_1) e^{-\frac{\left| x - y - \frac{1}{B} \sum_{j=1}^n b_{k_j}(t_j) + b_{k_j}(-t) \right|^2}{2\Delta_\sigma^2}}, \tag{130}$$

and following a similar approach as in [31] denote by Σ_0 and Σ_1 the sums,

$$\Sigma_0 = \sum_{n=1}^{N_0} K_n, \quad \Sigma_1 = \sum_{n=N_0+1}^{\infty} K_n \text{ with } N_0 = \left[\frac{B}{8M}|x|\right], \tag{131}$$

where $[a]$ denotes the integer part of a. Now we aim to bound from above Σ_0 and Σ_1 to obtain an overall estimate on $\left|\left(e^{-itH} - e^{-itH_L}\right)\Psi_y^\sigma\right|(x)$, and we continue to set $y = 0$ just for the sake of calculation. Starting with Σ_0, note that $1 \leq n \leq \frac{B}{8M}|x|$; hence, one has the inequality,

$$\left|x - \frac{1}{B}\sum_{j=1}^{n} b_{k_j}(t_j) + b_{k_j}(-t)\right|^2 \geq \frac{|x|^2}{4}. \tag{132}$$

Therefore, we can estimate Σ_0 as follows.

$$\Sigma_0 \leq e^{-\frac{|x|^2}{8\Delta_\sigma^2}} \sum_{n=1}^{N_0} \frac{C_\mu^n |t|^n}{n!} \leq e^{-\frac{|x|^2}{8\Delta_\sigma^2}} \left(e^{C_\mu|t|} - 1\right) \tag{133}$$

Remark that to estimate Σ_1, the Gaussian factor is bounded from above by 1. So we are only concerned with the n-dependence involving $C_\mu|t|$. This calculation is similar to equations 3.17-3.19 of [31], but with N_0 taking on a different value. Hence, we obtain the subsequent bound on Σ_1.

$$\Sigma_1 \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{B|x|}{8M}\left(\ln\frac{B|x|}{8MC_\mu|t|} - 1\right)} e^{C_\mu|t|} \tag{134}$$

Then, combining (112), (133), (134), we obtain (113):

$$\begin{aligned}\left|e^{-itH}\Psi_y^\sigma\right|(x) \leq &\frac{1}{2\pi\eta_\sigma^2}\left(e^{-\frac{|x-y|^2}{2\Delta_\sigma^2}} + e^{-\frac{|x-y|^2}{8\Delta_\sigma^2}}\left(e^{C_\mu|t|} - 1\right)\right) + \\ &\frac{1}{2\pi\eta_\sigma^2}\frac{1}{\sqrt{2\pi}} e^{-\frac{B|x-y|}{8M}\left(\ln\frac{B|x-y|}{8MC_\mu|t|} - 1\right)} e^{C_\mu|t|}.\end{aligned} \tag{135}$$

The proof of 114 starts by estimating away the Gaussian and natural log dependence of the terms in the expression above. First remark that the Gaussian terms can be combined by bounding from above the first Gaussian with a Gaussian with the same width as the second term. As before, we set $y=0$ for the time being.

$$\left|e^{-itH}\Psi_0^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2}\left(e^{-\frac{|x|^2}{8\Delta_\sigma^2}}e^{C_\mu|t|} + \frac{1}{\sqrt{2\pi}}e^{-\frac{B|x|}{8M}\left(\ln\frac{B|x|}{8MC_\mu|t|}-1\right)}e^{C_\mu|t|}\right) \tag{136}$$

From here we begin by estimating the second term, which we recall is an upper bound on Σ_1, by considering first the case where $\frac{B|x|}{8MC_\mu|t|} \geq e^2$. In this regime, one has,

$$\Sigma_1 \leq \frac{1}{\sqrt{2\pi}}e^{-\frac{B|x|}{8M}+C_\mu|t|} \tag{137}$$

Then, we estimate Σ_1 when $\frac{B|x|}{8MC_\mu|t|} < e^2$, which as shown below, makes use of the inequality $e^{-z\ln(z)} < e^{\frac{1}{e}}$ for all $z>0$.

$$\begin{aligned}\Sigma_1 &\leq \frac{1}{\sqrt{2\pi}}e^{-\frac{B|x|}{8M}\ln\frac{B|x|}{8MC_\mu|t|}}e^{\frac{B|x|}{8M}}e^{C_\mu|t|}\\ &\leq \frac{1}{\sqrt{2\pi}}e^{\frac{1}{e}}e^{e^2C_\mu|t|\ln C_\mu|t|}e^{e^2C_\mu|t|}e^{C_\mu|t|}\\ &\leq \frac{1}{\sqrt{2\pi}}e^{\frac{1}{e}}e^{e^2C_\mu|t|(\ln C_\mu|t|+1)+C_\mu|t|}e^{-\frac{B|x|}{8MC_\mu|t|}+e^2}\end{aligned} \tag{138}$$

Note that the last inequality follows from the fact that $e^{-\frac{B|x|}{8MC_\mu|t|}+e^2} \geq 1$ given the initial assumption. Next, we combine (137) and (138) to obtain an upper bound on Σ_1 with an exponential decay in $|x|$ across all terms, which we use to bound $\left|e^{-itH}\Psi_y^\sigma\right|(x)$.

$$\left|e^{-itH}\Psi_0^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2}e^{C_\mu|t|}\left(e^{-\frac{|x|^2}{8\Delta_\sigma^2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{B|x|}{8M}} + \frac{1}{\sqrt{2\pi}}e^{\frac{1}{e}+e^2}e^{e^2C_\mu|t|(\ln C_\mu|t|+1)}e^{-\frac{B|x|}{8MC_\mu|t|}}\right) \tag{139}$$

Then, using the inequality that for all $z\in\mathbb{R}$, $z^2 \geq z-\frac{1}{4}$, the Gaussian dependence can be relaxed to an exponential decay.

$$\left|e^{-itH}\Psi_0^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2}e^{C_\mu|t|}\left(e^{\frac{1}{32\Delta_\sigma^2}}e^{-\frac{|x|}{8\Delta_\sigma^2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{B|x|}{8M}} + \frac{1}{\sqrt{2\pi}}e^{\frac{1}{e}+e^2}e^{e^2C_\mu|t|(\ln C_\mu|t|+1)}e^{-\frac{B|x|}{8MC_\mu|t|}}\right) \tag{140}$$

We then seek to combine the three spatially decaying terms above, which requires finding a rate of decay which is smaller than either of the three exponents above.

$$\min\left(\frac{1}{\Delta_\sigma^2}, \frac{B}{M}, \frac{B}{MC_\mu|t|}\right) \geq \frac{1}{\Delta_\sigma^2 + \frac{M}{B}\left(1 + C_\mu|t|\right)}$$
$$\geq \frac{1}{\Delta_\sigma^2 + \frac{M}{B}} \frac{1}{1 + C_\mu|t|}$$

Thus, we obtain the following.

(141)
$$\left|e^{-itH}\Psi_0^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2} e^{C_\mu|t|}\left(e^{\frac{1}{32\Delta_\sigma^2}} + \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} e^{\frac{1}{e}+e^2} e^{e^2C_\mu|t|(\ln C_\mu|t|+1)}\right) e^{-\frac{1}{8\left(\Delta_\sigma^2+\frac{M}{B}\right)}\frac{|x|}{C_\mu|t|+1}}$$

Lastly, the terms inside parentheses can be estimated by multiplying the first two by $e^{e^2C_\mu|t|(\ln C_\mu|t|+1)}$. The second statement of Theorem 4.6 follows from this inequality, with the constants C_1, C_2, and C_3 explicitly computed and y re-introduced.

$$\left|e^{-itH}\Psi_y^\sigma\right|(x) \leq \frac{1}{2\pi\eta_\sigma^2}\left(e^{\frac{1}{32\Delta_\sigma^2}} + \frac{1}{\sqrt{2\pi}} + \frac{e^{\frac{1}{e}+e^2}}{\sqrt{2\pi}}\right) e^{e^2C_\mu|t|(\ln C_\mu|t|+1+\frac{1}{e^2})} e^{-\frac{1}{8\left(\Delta_\sigma^2+\frac{M}{B}\right)}\frac{|x-y|}{C_\mu|t|+1}} \tag{142}$$

□

Having proved single particle propagation estimates, with Theorems 4.5 and 4.6, we have established that under time evolution by H_1, where V could trivially be zero, the functions Ψ_x^σ remain localized around x. The range of this localization is prone to growing in time, namely if V is a nonzero periodic potential, which is expected of a free Gaussian wavepacket under evolution by a Hamiltonian that is quadratic in x and p.

In the following subsection, we introduce the quantity to be estimated that will yield a Lieb-Robinson type estimate for collections of interacting fermions.

4.5. **Lieb-Robinson Bounds.**

We begin with a definition of the relevant Heisenberg dynamics which will be used in the derivation of a Lieb-Robinson bound. In particular, we comment on the free dynamics, i.e. non-interacting, and then introduce the interacting case as a perturbation of the free dynamics.

Denote by $\tau_t^{\emptyset} : \mathcal{A}(L^2(\mathbb{R}^2)) \to \mathcal{A}(L^2(\mathbb{R}^2))$ the Heisenberg dynamics generated by the second quantization of the single particle Hamiltonian H_1, i.e. $\mathbb{H}_1$. The reason for the empty set notation will become apparent after introducing the interacting analogue. The map $\tau_t^{\emptyset}$ is precisely the map that implements single particle time evolution over the CAR algebra as,

$$\tau_t^{\emptyset}(a(f)) = e^{it\mathbb{H}_1} a(f) e^{-it\mathbb{H}_1} = a(e^{-itH_1} f) \tag{143}$$

This behavior can be derived, as discussed in Bratteli and Robinson's book [21], by considering the second quantization of e^{-itH_1}, which is of course a unitary operator on $L^2(\mathbb{R}^2)$. The interacting dynamics $\tau_t^{\Lambda} : \mathcal{A}(L^2(\mathbb{R}^2)) \to \mathcal{A}(L^2(\mathbb{R}^2))$ act as shown in the next equation.

$$\tau_t^{\Lambda}(a(f)) = e^{itH_{\Lambda}^{\sigma}} a(f) e^{-itH_{\Lambda}^{\sigma}};\ \forall\, f \in L^2(\mathbb{R}^2) \tag{144}$$

Remark that due to the UV regularization of the two-body interaction, the dynamics τ_t^{Λ} must inherit dependence on σ as well. To avoid cluttered notation, σ is not written explicitly in the dynamics.

Recall as discussed in Section 2.4.1, the operator parity determines whether a meaningful Lieb-Robinson bound will consist of a norm estimate of the commutator versus anticommutator of two operators. We work with the individual operators $a^{\#}(f)$, noting that using the properties of the (anti-)commutators of products of operators, one can reduce the computation to a series of anti-commutators of the individual $a^{\#}(f)$ operators. For $f, g \in L^2(\mathbb{R}^2)$, the following is thus the object to be estimated towards a Lieb-Robinson bound:

$$F_t^\Lambda(f,g) = \left\|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\}\right\| + \left\|\{\tau_t^\Lambda(a^*(f)), a^*(g)\}\right\|. \tag{145}$$

This function may appear contrived upon inspection, but there is good reason to work with it. Using the Duhamel identity, one may write τ_t^Λ explicitly as a perturbation of the free dynamics (as in 5.4.1 of [21]).

$$\tau_t^\Lambda(a(f)) = \tau_t^\emptyset(a(f)) + i\int_0^t dt'\ \tau_{t'}^\Lambda\left(\left[\mathbb{W}_\Lambda^\sigma, \tau_{t-t'}^\emptyset(a(f))\right]\right) \tag{146}$$

The state f might not be in the domain of the single particle Hamiltonian H_1, meaning that $(-i\nabla - A(x))f \notin L^2(\mathbb{R}^2)$. As a result, $e^{-itH_1}f$ might not be differentiable in time, leading to highly oscillatory behavior. However, by subtracting away the free dynamics as is done in $F_t^\Lambda(f,g)$, what remains is the dynamics generated by the pairwise interaction. Moreover, the commutator of $\tau_{t-t'}^\emptyset(a(f))$ with $\mathbb{W}_\Lambda^\sigma$ will precisely yield inner products of the form $\langle e^{-itH_1}f, \Psi_x^\sigma\rangle$ due to the canonical anticommutation relations. From Theorem 4.6, this has an upper bound localized around x. The second term in $F_t^\Lambda(f,g)$ does not have the free dynamics subtracted away because $\{\tau_t^\emptyset(a^*(f)), a^*(g)\}$ is already zero.

Moreover, at time $t = 0$, one trivially obtains $F_0^\Lambda(f,g) = 0$. As time grows however, one expects a bound on F_t^Λ to grow at least exponentially in time. Spatial locality enters via the functions f, g. These may be compactly supported or more generally, localized in regions that are very far apart. Thus, what we will show is that the distance between supports of f, g dictates how small the Lieb-Robinson bound will be, analogous to the distance between observables A, B acting on disjoint clusters of lattice sites.

The remainder of this section will thus be devoted to proving an upper bound on $F_t^\Lambda(f,g)$, and this result is summarized in the following theorem.

Theorem 4.9. ***(Lieb-Robinson Bound)*** *Fix $\sigma > 0$. Let $\mathbb{H}_\Lambda^\sigma$ be the Hamiltonian as defined in (97), with H_L the Landau Hamiltonian in the symmetric gauge, V a potential satisfying*

Assumption 4.1, and $\mathbb{W}_\Lambda^\sigma$ as in 95 with $W(x)$ satisfying Assumption 4.2. For all $t \in \mathbb{R}$ and any bounded, measurable set $\Lambda \subset \mathbb{R}^2$, τ_t^Λ describes the Heisenberg dynamics associated with $\mathbb{H}_\Lambda^\sigma$. Then, for all $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $g \in L^2(\mathbb{R}^2)$,

$$F_t^\Lambda(f,g) \le D(t)(e^{P(t)} - 1) \int_{\mathbb{R}^4} dxdy\, |f(x)||g(y)| e^{-\frac{\nu_t |x-y|}{4}} \tag{147}$$

In the case $V = 0$, if we alter Assumption 4.2.ii to let $|W(x-y)| \le C_w e^{-a|x-y|^2}$, we obtain the estimate:

$$F_t^\Lambda(f,g) \le A(e^{\kappa t} - 1) \int_{\mathbb{R}^4} dxdy\, |f(x)||g(y)| e^{-\alpha |x-y|^2} \tag{148}$$

where the constants A, κ, and α and the continuous functions $D(t)$, $P(t)$, and ν_t have dependence on σ, B, and C_μ as well as the parameters a and C_w from Assumption 2.5. Expressions for these constants and functions are derived as we work through the proof of this theorem. As previously discussed, the distance between the supports of f and g now clearly enters the estimate. In either bound, evaluating the integral over x for example yields the convolution of $|f|$ with either an exponential or Gaussian function with decay in the distance to y, i.e. the support of $|g|$. This convolution thus yields the weight of $|f|$ near y. Therefore, the remaining integral over y is effectively the inner product of $|g|$ with a smeared out form of $|f|$, which is induced by the UV regularization.

Note that convergence of the integrals in 147 and 148 is ensured by Young's inequality, which is stated for reference.

Theorem 4.10 (Young's Inequality)**.** *Let $p, q, r \ge 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, and $h \in L^r(\mathbb{R}^n)$. Then there exists a constant C such that the following inequality holds.*

$$\left| \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, f(x) g(y) h(x-y) \right| \le C \|f\|_p \|g\|_q \|h\|_r \tag{149}$$

The precise form of the constant C can be found in Section 4.2 of [68], from which we state this theorem. Thus, letting $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R}^2)$, Young's inequality holds since $e^{-\nu_t |x|/4}$ and $e^{-D_4|x|^2}$ are both in $L^1(\mathbb{R}^2)$.

Notably, this theorem could not be proved in the absence of smeared creation/annihilation operators, i.e. $\sigma = 0$, otherwise the prefactors of the Lieb-Robinson estimate diverge. Furthermore, the bounds in this theorem are uniform in the size of the interaction domain Λ, which allows us to take the thermodynamic limit of the dynamics governed by $\mathbb{H}^\sigma_\Lambda$ as Λ extends to all of $\mathbb{R}^2$, which is an application of the Lieb-Robinson type bound in 147 that we elaborate on in the next subsection.

proof of Theorem 4.9. As we proceed, note that we consider the case of general V taking on the assumed form; however, we shall comment in parallel on how the calculation can be carried out for $V = 0$ with a slightly tighter bound on the range of the interaction. Towards this end, we begin by stating the result of a preliminary bound on $F^\Lambda_t(f, g)$.

Lemma 4.11. *Under the assumptions introduced on V, and W, for any $f \in L^2(\mathbb{R}^2)$ and $t \geq 0$ the following holds.*

$$\begin{aligned}F^\Lambda_t(f, g) \leq C_\sigma \int_0^t ds \int_{\mathbb{R}^2} dx K_{t-s}(f, x) \left|\langle e^{-isH} \Psi^\sigma_x, g\rangle\right| \\ + C_\sigma \int_0^t ds \int_{\mathbb{R}^2} dx K_{t-s}(f, x) F^\Lambda_s(\Psi^\sigma_x, g)\end{aligned} \tag{150}$$

Here, $C_\sigma = \frac{1}{4\pi\sigma^2}$, and the kernel function $K_{t-s}(f, x)$ is given by,

$$K_t(f, x) = \|W\|_1 \left|\langle e^{-itH} f, \Psi^\sigma_x\rangle\right| + 2\left(|W| * \left|\langle e^{-itH} f, \Psi^\sigma_{(\cdot)}\rangle\right|\right)(x), \tag{151}$$

where the symbol $*$ denotes convolution. For the interested reader, refer to the proof of Lemma 4.1 of [31], where an analogous bound is proven with zero magnetic field and UV regularization implemented via plainly translated Gaussian functions. Notice that this is a bound that suggests iteration as a natural next step towards approaching a final bound. In order to carry out this procedure however, we begin by computing an estimate on the kernel function, and for ease of notation we introduce the function $\mathcal{E}_a(x) = e^{-a|x|}$ for all $x \in \mathbb{R}^2$, $a \in \mathbb{R}$.

Lemma 4.12. *Let $t \in \mathbb{R}$, $f \in L^2(\mathbb{R}^2)$, and $x \in \mathbb{R}^2$. We then have the estimate,*

$$K_t(f,x) \le P_1(t) e^{C_2|t|\ln|t|} \left(\mathcal{E}_{a_t} * |f|\right)(x) \tag{152}$$

where

$$4a_t = \frac{aC_3}{a(C_\mu|t|+1)+C_3}; \quad P_1(t) = C_1\left(2C_w\left(\frac{D}{4a_t}\right)^2 + \|W\|_1\right).$$

Note that $P_1(t) : \mathbb{R} \to (0,\infty)$ is a degree-2 polynomial, the constants C_1, C_2, and C_3 are from Theorem 4.6, and $D > 0$.

Proof of Lemma 4.12. We begin by recalling (114) which provides the inequality,

$$\begin{aligned} \left|\langle e^{-itH} f,\ \Psi_x^\sigma\rangle\right| &\le C_1 e^{C_2|t|\ln|t|} \int_{\mathbb{R}^2} dy\, e^{-\frac{C_3}{C_\mu|t|+1}|x-y|}|f(y)| \\ &\le C_1 e^{C_2|t|\ln|t|} \left(\mathcal{E}_{c_t} * |f|\right)(x) \end{aligned} \tag{153}$$

The integral has been rewritten in terms of the convolution operator, and for brevity we have introduced the term,

$$c_t = \frac{C_3}{C_\mu|t|+1}.$$

The bound on $W(x)$ from Assumption 4.2 can also be expressed compactly as $|W(x)| \le C_w\mathcal{E}_a(x)$. Hence, an initial upper bound on the kernel function can be written as follows.

$$K_t(f,x) \le C_1\|W\|_1 e^{C_2|t|\ln|t|} \left(\mathcal{E}_{c_t} * |f|\right)(x) + 2C_1C_w e^{C_2|t|\ln|t|} \left(\mathcal{E}_a * \left(\mathcal{E}_{c_t} * |f|\right)\right)(x) \tag{154}$$

We now want to consider how to estimate the double convolution of functions that occurs in the second term by utilizing the associativity of this operation.

$$\begin{aligned} \left(\mathcal{E}_a * \left(\mathcal{E}_{c_t} * |f|\right)\right)(x) &= \int_{\mathbb{R}^2} dz e^{-a|x-z|} \left(\mathcal{E}_{c_t} * |f|\right)(z) \\ &= \int_{\mathbb{R}^2} dz \int_{\mathbb{R}^2} dy e^{-a|x-z|} e^{-c_t|z-y|} |f|(y) \end{aligned} \tag{155}$$

$$\leq \int_{\mathbb{R}^2} dz \int_{\mathbb{R}^2} dy e^{-4a_t|x-z|} e^{-4a_t|z-y|} |f|(y)$$

The time dependent factor $4a_t$, defined in the statement of the lemma, serves as a common lower bound for both a and c_t. Using Lemma B.1 from [31], one has,

$$\int_{\mathbb{R}^2} dz e^{-4a_t|x-z|} e^{-4a_t|z-y|} \leq \left(\frac{D}{4a_t}\right)^2 e^{-a_t|x-y|} \tag{156}$$

where $D > 0$ is a constant. It is worth noting that in the case $V = 0$ where one has a Gaussian bound, (107), on $\left|\langle e^{-itH_L} f, \Psi_x^\sigma \rangle\right|$, this calculation could have been carried out in identical fashion. However, by using a Gaussian bound on $|W(x)|$, which falls under the exponential bound of Assumption 4.2, the analogue of (156) could have been calculated as an equality as the convolution of two Gaussian functions is a Gaussian. In any case, combining (155) and (156), (154) can be further estimated.

$$K_t(f,x) \leq C_1 e^{c_2|t|\ln|t|} \left(\|W\|_1 \left(\mathcal{E}_{c_t} * |f|\right)(x) + 2C_w \left(\frac{D}{4a_t}\right)^2 \left(\mathcal{E}_{a_t} * |f|\right)(x) \right) \tag{157}$$

Remarking that $a_t \leq c_t$ and introducing the polynomial $P_1(t) = C_1 \left(\|W\|_1 + 2C_w \left(\frac{D}{4a_t}\right)^2 \right)$, the first convolution in (157) can be bounded from above by replacing c_t with a_t, and a common factor of $P_1(t)e^{C_2|t|\ln|t|}$ can be extracted. This yields the expression in the statement of the lemma. □

Setting $V = 0$, the analogue of this calculation with Gaussian bound on the interaction function yields a time independent Gaussian convolved with f rather than $\mathcal{E}_{a_t}$. This occurs as the convolution of two Gaussians is Gaussian as well. Proceeding onward, one additional ingredient we will need to iterate our bound on $F_t^\Lambda(f,g)$ infinitely is an estimate of the form of the lemma we just proved, but with f specifically chosen to be the magnetically translated Gaussian Ψ_x^σ.

Lemma 4.13. *Let $t \in \mathbb{R}$, $\sigma > 0$, and $x, y \in \mathbb{R}^2$. One has,*

$$K_t(\Psi_x^\sigma, y) \leq P_2(t) e^{C_2|t|\ln|t|} \mathcal{E}_{\nu_t}(x-y) \tag{158}$$

where $P_2(t) = \frac{P_1(t)e^{\frac{1}{8\sigma^2}}}{2\pi\sigma^2}\left(\frac{D}{4\nu_t}\right)^2$ *is a degree-4 polynomial and* $4\nu_t = \frac{aC_3}{4a(C_\mu|t|+1)+2C_3(2+a\sigma^2)}$.

Proof. Begin by applying Lemma 4.12 and the inequality $z^2 \geq z - \frac{1}{4}$ which holds for all $z \in \mathbb{R}$ to obtain,

$$K_t(\Psi_x^\sigma, y) \leq P_1(t)e^{C_2|t|\ln|t|}\left(\mathcal{E}_{a_t} * |\Psi_x^\sigma|\right)(y) \tag{159}$$
$$= \frac{P_1(t)e^{\frac{1}{8\sigma^2}}}{2\pi\sigma^2}e^{C_2|t|\ln|t|}\int_{\mathbb{R}^2} dz e^{-a_t|y-z|}e^{-\frac{|z-x|}{2\sigma^2}}$$
$$\leq \frac{P_1(t)e^{\frac{1}{8\sigma^2}}}{2\pi\sigma^2}e^{C_2|t|\ln|t|}\int_{\mathbb{R}^2} dz e^{-a_t|y-z|}e^{-\frac{|z-x|}{2\sigma^2}}$$

The second inequality holds by taking the absolute value of the integrand. Then, we invoke Lemma B.1 from [31] to compute the following estimate, where ν_t which appears is as in the statement of the lemma.

$$\int_{\mathbb{R}^2} dz e^{-a_t|y-z|}e^{-\frac{|z-x|}{2\sigma^2}} \leq \left(\frac{D}{4\nu_t}\right)^2 e^{-\nu_t|x-y|}$$

□

Remark again that if $V = 0$ and our interaction decay is strengthened to be Gaussian, $\mathcal{E}_{\nu_t}$ would be replaced by a Gaussian function. We are now in a position to iterate the bound in Lemma 4.11 infinitely towards a Lieb-Robinson estimate. This results in the subsequent bound with terms defined as follows.

$$F_t^\Lambda(f,g) \leq \sum_{n=1}^N a_n(t,f,g) + R_N(t,f,g) \tag{160}$$

$$a_n(t,f,g) = C_\sigma^n \int_0^t dt_1 \int_{\mathbb{R}^2} dx_1 K_{t-t_1}(f,x_1)\int_0^{t_1} dt_2 \int_{\mathbb{R}^2} dx_2 K_{t_1-t_2}(\Psi_{x_1}^\sigma, x_2)... \tag{161}$$
$$\times \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^2} dx_n K_{t_{n-1}-t_n}(\Psi_{x_{n-1}}, x_n)\left|\langle e^{-it_nH}\Psi_{x_n}^\sigma, g\rangle\right|$$

$$R_N(t,f,g) = C_\sigma^N \int_0^t dt_1 \int_{\mathbb{R}^2} dx_1 K_{t-t_1}(f,x_1)\int_0^{t_1} dt_2 \int_{\mathbb{R}^2} dx_2 K_{t_1-t_2}(\Psi_{x_1}^\sigma, x_2)... \tag{162}$$
$$\times \int_0^{t_{N-1}} dt_N \int_{\mathbb{R}^2} dx_N K_{t_{N-1}-t_N}(\Psi_{x_{N-1}}, x_N)F_{t_N}^\Lambda(\Psi_{x_N}^\sigma, g)$$

Notice that this series consists of a finite series up to an N^{th} iteration along with a remainder term that estimates the tail of the series. Additionally, there is time ordering of the temporal variables of integration, meaning $t_n \leq t_{n-1} \leq ...t_1 \leq t$. Towards proving Theorem 4.9, we will prove lemmas concerning the behavior of $a_n(t, f, g)$ for each $n \geq 1$, the sum of a_n up to N terms, and R_N in the limit that N is infinite.

Lemma 4.14. *Let $t > 0$ and $n \in \mathbb{N}$. Then, under the same assumptions as Lemmas 4.11-4.13, for all $f, g \in L^2(\mathbb{R}^2)$, the following bound holds.*

$$a_n(t, f, g) \leq D_1(t)\frac{P_3(t)^n}{n!}\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy e^{-\frac{-\nu_t|x-y|}{4}}|f(x)||g(y)| \tag{163}$$

$P_3(t) : \mathbb{R} \to \mathbb{R}$ is a degree-7 polynomial given by $P_3(t) = \frac{C_\sigma D t P_2(t)}{\nu_t^2}$, and $D_1(t) : \mathbb{R} \to \mathbb{R}$ is defined as $D_1(t) = C_1 D \frac{P_1(t)}{P_2(t)} e^{t \ln t}$.

Importantly, this estimate introduces the inner product of $|g|$ with the smeared form of $|f|$, i.e. its convolution with an exponentially decaying function, which carries over to the result of Theorem 4.9.

proof of Lemma 4.14. For all $0 \leq t_1 \leq t$, by Lemma 4.12 one has the estimate,

$$\begin{aligned} K_{t-t_1}(f, x_1) &\leq P_1(t - t_1)e^{C_2|t-t_1|\ln|t-t_1|}\left(\mathcal{E}_{a_{t-t_1}} * |f|\right)(x_1) \\ &\leq P_1(t)e^{C_2(t-t_1)\ln t}\left(\mathcal{E}_{\nu_t} * |f|\right)(x_1) \end{aligned} \tag{164}$$

To reach the second inequality, we note that $P_1(t)$ is an increasing function in its argument as well as the fact that the time ordering implies that $t \geq t - t_1 \geq 0$. We also note $a_{t-t_1} \geq \nu_t$. Then, invoking Lemma 4.13, for all $j \in \mathbb{N}$, $0 \leq t_j \leq t_{j-1} \leq t$, and $x_j \in \mathbb{R}^2$ we have the bound,

$$\begin{aligned} K_{t_{j-1}-t_j}(\Psi^\sigma_{x_{j-1}}, x_j) &\leq P_2(t_{j-1} - t_j)e^{C_2(t_{j-1}-t_j)\ln(t_{j-1}-t_j)}\mathcal{E}_{\nu_{t_{j-1}-t_j}}(x_{j-1} - x_j) \\ &\leq P_2(t)e^{C_2(t_{j-1}-t_j)\ln t}\mathcal{E}_{\nu_t}(x_{j-1} - x_j) \end{aligned} \tag{165}$$

This follows from $P_2(t)$ being increasing and that ν_t is decreasing in t. One more necessary ingredient is a bound on $\left|\langle e^{-t_nH}\Psi^\sigma_{x_n}, g\rangle\right|$, for which we call on 114 from Theorem 4.6:

$$
\begin{aligned}
\left|\langle e^{-t_nH}\Psi^\sigma_{x_n}, g\rangle\right| &\leq C_1 e^{C_2|t_n|\ln|t_n|}\left(\mathcal{E}_{c_{t_n}} * |g|\right)(x_n) \\
&\leq C_1 e^{C_2 t_n \ln t}\left(\mathcal{E}_{\nu_t} * |g|\right)(x_n),
\end{aligned}
\tag{166}
$$

where we use that $\nu_t \leq c_t$ to obtain the second inequality. Then using 164-166, letting $t_0 = t$, and $t_{n+1} = 0$, we estimate $a_n(t, f, g)$ as shown in the subsequent calculations.

$$
\begin{aligned}
a_n(t,f,g) \leq{}& C_1 C_\sigma^n P_1(t) P_2(t)^{n-1}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} d_{t_n} e^{\sum_{j=1}^{n+1}(t_{j-1}-t_j)\ln t}\times \\
& \int_{\mathbb{R}^2} dx_1 \ldots \int_{\mathbb{R}^2} dx_n \left(\mathcal{E}_{\nu_t} * |f|\right)(x_1)\mathcal{E}_{\nu_t}(x_1-x_2)\ldots\mathcal{E}_{\nu_t}(x_{n-1}-x_n)\left(\mathcal{E}_{\nu_t} * |g|\right)(x_n) \\
={}& C_1 C_\sigma^n P_1(t) P_2(t)^{n-1} e^{t\ln t}\frac{t^n}{n!}\int_{\mathbb{R}^2} dx_1 \left(\mathcal{E}_{\nu_t} * |f|\right)(x_1)\left(\mathcal{E}_{\nu_t} * \mathcal{E}_{\nu_t} * \ldots * \mathcal{E}_{\nu_t} * |g|\right)(x_1) \\
={}& C_1 \frac{P_1(t)}{P_2(t)} e^{t\ln t}\frac{(C_\sigma P_2(t) t)^n}{n!}\int_{\mathbb{R}^2} dx_1 |f(x_1)| \left(\mathcal{E}_{\nu_t} * \ldots * \mathcal{E}_{\nu_t} * |g|\right)(x_1)
\end{aligned}
\tag{167}
$$

In the second equality, we evaluate the telescopic sum to yield $e^{t\ln t}$ and note that there is an n-fold convolution of $\mathcal{E}_{\nu_t}$ with itself. To obtain the third equality, we use the definition of the convolution to rearrange the functions and obtain an $(n+1)$-fold convolution. Using Lemma B.1 of [31], this series of convolutions can be bounded from above in the same manner as has been done in the proofs of the previous lemmas. With a rewriting of the integration variables, this gives the following bound.

$$
a_n(t,f,g) \leq C_1 D \frac{P_1(t)}{P_2(t)} e^{t\ln t}\left(\frac{D}{\nu_t^2}\right)^n \frac{(C_\sigma P_2(t) t)^n}{n!}\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy e^{-\frac{\nu_t|x-y|}{4}}|f(x)||g(y)|
\tag{168}
$$

Letting $D_1(t) = C_1 D \frac{P_1(t)}{P_2(t)} e^{t\ln t}$ and $P_3(t) = \frac{C_\sigma D t P_2(t)}{\nu_t^2}$, we obtain the statement of the lemma.

□

An important corollary of this lemma is the following, which estimates an upper bound on $\sum_{n=1}^N a_n(t,f,g)$ in the limit that N is infinite.

Corollary 4.15. *Under the same assumptions as Lemma 4.14, for any $t>0$ one has the upper bound,*

$$\lim_{N\to\infty}\sum_{n=1}^{N} a_n(t,f,g) \le D_1(t)\left(e^{P_3(t)}-1\right)\int_{\mathbb{R}^2} dx\int_{\mathbb{R}^2} dy e^{-\frac{\nu_t|x-y|}{4}}|f(x)||g(y)| \tag{169}$$

All that remains towards proving Theorem 4.9 is to prove that $\lim_{N\to\infty} R_N(t,f,g)=0$, and this is indeed true by the subsequent calculations. Fix $T>0$ and let $t\in[-T,T]$. Remark that a_n and R_N only differ in the last term, where $\left|\langle e^{-it_nH}\Psi^{\sigma}_{x_n}, g\rangle\right|$ is replaced by $F^{\Lambda}_{t_N}(\Psi^{\sigma}_{x_N}, g)$. We use the simple estimate that the latter can be bounded from above by $6\left\|\Psi^{\sigma}_{x_N}\right\|_2\|g\|_2 = 6\sqrt{C_\sigma}\|g\|_2$. Then, via the same arguments used to estimate $a_n(t,f,g)$, we arrive at the following inequality.

(170)

$$\begin{aligned}
R_N(t,f,g) &\le 6\sqrt{C_\sigma}\|g\|_2 e^{t\ln t} C_\sigma^N P_1(t)P_2(t)^{N-1}\frac{t^N}{N!}\int_{\mathbb{R}^2} dx_1 \ldots \int_{\mathbb{R}^2} dx_N \left(\mathcal{E}_{\nu_t} * \ldots * \mathcal{E}_{\nu_t} * |f|\right)(x_N)\\
&= 6\sqrt{C_\sigma}\|g\|_2 e^{t\ln t} C_\sigma^N P_1(t)P_2(t)^{N-1}\frac{t^N}{N!}\int_{\mathbb{R}^2} dx_N \int_{\mathbb{R}^2} dy \left(\mathcal{E}_{\nu_t} * \ldots * \mathcal{E}_{\nu_t}\right)(x_n-y)|f(y)|
\end{aligned}$$

From the second equality above, through one more application of Lemma B.1 of [31] and noting that $f\in L^1(\mathbb{R}^2)$ by assumption in Theorem 4.9, one has,

(171)

$$\begin{aligned}
R_N(t,f,g) &\le 6\sqrt{C_\sigma}\|g\|_2 e^{t\ln t} C_\sigma^N P_1(t)P_2(t)^{N-1}\frac{t^N}{N!}\left(\frac{D}{\nu_t^2}\right)^N \nu_t^2\int_{\mathbb{R}^2} dx_N \int_{\mathbb{R}^2} dy e^{-\frac{\nu_t|x_N-y|}{4}}|f(y)|\\
&\le 6\sqrt{C_\sigma}\|g\|_2 e^{t\ln t} C_\sigma^N P_1(t)P_2(t)^{N-1}\frac{t^N}{N!}\left(\frac{D}{\nu_t^2}\right)^N \nu_t^2\left\|\mathcal{E}_{\frac{1}{4}\nu_t}\right\|_1\|f\|_1
\end{aligned}$$

From this last inequality, it is clear that $\lim_{N\to\infty}\sup_{t\in[-T,T]} R_N(t,f,g)=0$. This completes the proof and gives us a Lieb-Robinson type bound:

$$F^{\Lambda}_t(f,g) \le D_1(t)\left(e^{P_3(t)}-1\right)\int_{\mathbb{R}^2} dx\int_{\mathbb{R}^2} dy\, e^{-\frac{\nu_t|x-y|}{4}}|f(x)||g(y)|;\quad V\neq 0 \tag{172}$$

Furthermore in the case that $V = 0$, where $W(x) \leq C_w e^{-ax^2}$ is used, due to the propagation of spatial Gaussian dependence through convolutions, we arrive at the following estimate,

$$F_t^{\Lambda}(f,g) \leq D_2 \left(e^{D_3 t} - 1\right) \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \, e^{-D_4|x-y|^2} |f(x)||g(y)|; \;\; V = 0, \tag{173}$$

where D_2, D_3, and D_4 are all real positive constants. In both of these regimes, we have a bound with no dependence on the $|\Lambda|$. This completes the proof of Theorem 4.9. □

As commented on previously, a Lieb-Robinson estimate in the setting of the continuous fermion CAR algebra is one that should demonstrate an approximate preservation of the canonical anticommutation relations, 76, for short times. If the Gaussian or exponential functions were replaced by $\delta(x-y)$, one would of course directly recover the inner product $\langle |f|, |g| \rangle$, which describes the overlap of the supports of f and g due to the absolute values. However, this delta function is effectively regularized like the interaction to give a smeared inner product of $|f|$ and $|g|$. Furthermore, the prefactor induces exponential growth polynomial/linear in time of the upper bounds, which can be thought of as growing the support of f in $\tau_t^{\Lambda}(a(f))$ to progressively overlap the dominant support of g. These characteristics are all consistent with the propagation of information that a Lieb-Robinson bound describes, as introduced in Section 2 for lattice systems.

A subtle technical point that is worth revisiting is the assumption that $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, as opposed to f simply being in $L^2(\mathbb{R}^2)$. As described towards the end of the previous proof, upon iterating the preliminary estimate in Lemma 4.11 infinitely, i.e. taking N to ∞, f is required to be in $L^1(\mathbb{R}^2)$ to ensure that the remainder term $R_N(t,f,g)$ vanishes. While this fact is necessary to ensure that we can prove a Lieb-Robinson bound with the finite volume dynamics, it will not prevent us from proving existence and strong continuity of the infinite system dynamics over $\mathcal{A}(L^2(\mathbb{R}^2))$. The focus of the final subsection is thus to demonstrate how Theorem 4.9 can be applied to study the thermodynamic limit.

4.6. **Thermodynamic Limit.**

A sample of metal studied in a laboratory can be well described by quantum and statistical mechanics in the thermodynamic limit because its finite size is still much larger than the microscopic scales of the interacting particles. It is thus important to lift our discussion of the Heisenberg dynamics of interacting fermions in continuous space to the limit where interactions take place across all of space. The finite volume restriction is of course necessary to prove the propagation estimates we presented in the previous subsection; however, the physically relevant model is one in which interactions are not just restricted to a finite subset $\Lambda \subset \mathbb{R}^2$. Towards this end, two questions of significance are the following.

i) For any $f \in L^2(\mathbb{R}^2)$, does $\lim_{\Lambda \uparrow \mathbb{R}^2} \tau_t^\Lambda (a(f)) =: \tau_t (a(f))$ exist as an element of $\mathcal{A}(L^2(\mathbb{R}^2))$?

ii) Is the one-parameter group $\{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathcal{A}(L^2(\mathbb{R}^2))$ strongly continuous, meaning for any $f \in L^2(\mathbb{R}^2)$, does the following limit hold?

$$\lim_{t' \to t} \|\tau_t (a(f)) - \tau_{t'}(a(f))\| = 0 \tag{174}$$

The answer to both of these questions is affirmative and is encapsulated in the following theorem.

Theorem 4.16. *(**Thermodynamic Limit**) Following the same assumptions as Theorem 4.9, for all $f \in L^2(\mathbb{R}^2)$ and any increasing sequence of bounded subsets $(\Lambda_n)_{n\in\mathbb{N}}$ with $\Lambda_n \subset \mathbb{R}^2$ $\forall n \in \mathbb{N}$, i.e. $\Lambda_m \subset \Lambda_n$ for all $m \leq n$ and $\cup_n \Lambda_n = \mathbb{R}^2$, there exists a strongly continuous 1-parameter group of automorphisms $\{\tau_t\}_{t\in\mathbb{R}}$ on the CAR algebra such that*

$$\lim_{\Lambda \uparrow \mathbb{R}^2} \tau_t^\Lambda(a(f)) = \tau_t(a(f)). \tag{175}$$

Consequently, the infinite volume limit as Λ tends to all of $\mathbb{R}^2$ of the Heisenberg dynamics does exist as an automorphism τ_t, and this limiting map is strongly continuous. As an element of $\mathcal{A}(L^2(\mathbb{R}^2))$, $\tau_t(a^\#(f))$ does not undergo any sharp or discontinuous variations in time.

This can equivalently be phrased in terms of Schrödinger-evolved states, whereby expectation values with respect to these states of bounded observables are continuous functions in time. In the Heisenberg picture however, one does not need to make assumptions regarding the initial unevolved state as the dynamics act as a map on bounded operators and not states.

The remainder of this section will be dedicated to proving the existence and strong continuity of the dynamics in the thermodynamic limit, which will follow the same strategy used in [31]. We start with a corollary of Theorem 4.9, where the function g is taken to be Ψ_x^σ and f is taken to be compactly supported.

Corollary 4.17. *Fix $y \in \mathbb{R}^2$ and let $f \in L^2(\mathbb{R}^2)$ have compact support supp$(f) =: X_f$. Under the same assumptions as Theorem 4.9, there exist continuous functions $\tilde{a}(t)$ and $\tilde{c}(t)$ such that the following bound holds for all finite t.*

$$\left\|\left\{\tau_t^\Lambda(a(f)), a^\#(\Psi_y^\sigma)\right\}\right\| \le \|f\|_1 e^{\tilde{c}(t)} e^{-\tilde{a}(t) d(X_f, y)} \tag{176}$$

As before, the # notation indicates either a creation or annihilation operator. This bound follows from applying Theorem 4.9 to $F_t^\Lambda(f, \Psi_y^\sigma)$ and Theorem 4.6 to the term involving the free dynamics:

$$\begin{aligned}\left\|\left\{\tau_t^\Lambda(a(f)), a^\#(\Psi_y^\sigma)\right\}\right\| &\le \left\|\left\{\tau_t^\Lambda(a(f)), a^\#(\Psi_y^\sigma)\right\} - \left\{\tau_t^\emptyset(a(f)), a^\#(\Psi_y^\sigma)\right\}\right\| + \\ &\quad \left\|\left\{\tau_t^\emptyset(a(f)), a^\#(\Psi_y^\sigma)\right\}\right\| \\ &\le F_t^\Lambda(f, \Psi_y^\sigma) + \left\|\left\{\tau_t^\emptyset(a(f)), a^\#(\Psi_y^\sigma)\right\}\right\|.\end{aligned}$$

We will use this estimate to first prove Theorem 4.16 for compactly supported $f \in L^2(\mathbb{R}^2)$. Since the algebra $\mathcal{A}_c(L^2(\mathbb{R}^2))$, generated by

$$\{a(f), a^*(f) :\ f \in L^2(\mathbb{R}^2) \text{ with compact support}\},$$

is dense in $\mathcal{A}(L^2(\mathbb{R}^2))$, the infinite volume dynamics over $\mathcal{A}_c(L^2(\mathbb{R}^2))$ can be lifted to a unique automorphism over $L^2(\mathbb{R}^2)$.

proof of Theorem 4.16. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be an exhaustive sequence of subsets of $\mathbb{R}^2$ such that $\bigcup_{n\in\mathbb{N}}\Lambda_n=\mathbb{R}^2$. Fix $n,m\in\mathbb{N}$ such that $n>m$. Using the Duhamel identity, we expand the difference in dynamics as follows.

$$\begin{aligned}\tau_t^{\Lambda_n}(a(f))-\tau_t^{\Lambda_m}(a(f)) &= \frac{i}{2}\int_0^t dt'\int_{\Lambda_n\setminus\Lambda_m\times\Lambda_n\setminus\Lambda_m} dxdy\, W(x-y)\times \\ &\qquad \tau_t^{\Lambda_n}\left(\left[a^*(\Psi_x^\sigma)a^*(\Psi_y^\sigma)a(\Psi_y^\sigma)a(\Psi_x^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\right]\right)\end{aligned}\tag{177}$$

Then, using the identities $[AB,C]=A[B,C]+[A,C]B$ and $\{AB,C\}=A\{B,C\}-\{A,C\}B$, the Heisenberg evolved commutator of the integrand can be expanded as,

$$a^*(\Psi_x^\sigma)\{a^*(\Psi_y^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\}a(\Psi_y^\sigma)a(\Psi_x^\sigma)-\{a^*(\Psi_x^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\}a^*(\Psi_y^\sigma)a(\Psi_y^\sigma)a(\Psi_x^\sigma)\tag{178}$$

$$+a^*(\Psi_x^\sigma)a^*(\Psi_y^\sigma)a(\Psi_y^\sigma)\{a(\Psi_x^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\}-a^*(\Psi_x^\sigma)a^*(\Psi_y^\sigma)\{a(\Psi_y^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\}.a(\Psi_x^\sigma)\tag{179}$$

To estimate an upper bound on $\left\|\tau_t^{\Lambda_n}(a(f))-\tau_t^{\Lambda_m}(a(f))\right\|$, we bound the norm of $\tau_t^{\Lambda_n}\left(\left[a^*(\Psi_x^\sigma)a^*(\Psi_y^\sigma)a(\Psi_y^\sigma)a(\Psi_x^\sigma),\tau_{t-t'}^{\Lambda_m}(a(f))\right]\right)$, which, by the triangle inequality, yields four terms as in Corollary 4.17. Furthermore, since $W(x-y)$ is an even function, each of the four integrands resulting from the triangle inequality is equivalent. Thus, one obtains,

$$\left\|\tau_t^{\Lambda_n}(a(f))-\tau_t^{\Lambda_m}(a(f))\right\|\le 2\|f\|_1\int_0^t dt'\, e^{\tilde{c}(t')}\int_{\Lambda_n\setminus\Lambda_m\times\Lambda_n\setminus\Lambda_m} dxdy\,|W(x-y)|e^{-\tilde{a}(t')d(X_f,x)}.\tag{180}$$

We then find that for any $t\in[-T,T]$ with T finite, there are constants $C,b>0$ such that,

$$\left\|\tau_t^{\Lambda_n}(a(f))-\tau_t^{\Lambda_m}(a(f))\right\|\le C\|f\|_1\int_{\Lambda_n\setminus\Lambda_m\times\Lambda_n\setminus\Lambda_m} dxdy\,|W(x-y)|e^{-bd(X_f,x)},\tag{181}$$

and since $|W(x-y)|\le C_w e^{-\sigma_w|x-y|}$, it follows that this converges to 0 as $n,m\to\infty$. Thus, $\{\tau_t^{\Lambda_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence and thermodynamic limit of τ_t^{Λ} exists, with convergence uniform for $t\in[-T,T]$. As mentioned before, its action can be uniquely extended to a $*$-homomorphism over all of $\mathcal{A}(L^2(\mathbb{R}^2))$. Furthermore, strong continuity of the limit τ_t follows from this uniform convergence in $t\in[-T,T]$, the already known strong continuity of the free dynamics $\tau_t^{\emptyset}$, and compactly supported functions on $\mathbb{R}^2$ being dense in $L^2(\mathbb{R}^2)$. $\square$

This completes the proof of the existence and strong continuity of the infinite volume dynamics using an approach that relies on Lieb-Robinson bounds. We successfully employed the strategy of Gebert et al. and introduced a class of UV regularized two-body interactions, built from magnetically translated Gaussian functions. This smearing protocol is necessary to produce bounded interactions when restricted to finite volumes, and with suitable decay assumptions on the interaction, namely exponential/Gaussian, one can obtain Lieb-Robinson bounds.

The mechanics of the mathematical arguments are quite similar to the lattice case, as seen in works such as [72] and [74]. However, locality in the continuum setting is related to the supports of single particle wavefunctions $f \in L^2(\mathbb{R}^2)$ effectively spreading as opposed to lattice subsets. In the following section, we introduce the setting of lattice-localized frames which Bachmann and De Nittis used to also prove Lieb-Robinson bounds for interacting fermions in continuous space, [12]. In collaboration with these authors and Nachtergaele, we use this precise framework to construct quasi-adiabatic evolution for these systems. Before doing so and considering the application of proving quantization of Hall conductance, we spend time in the next section introducing the setup and key results from [12] as well as some new machinery that will be necessary to define a spectral flow automorphism.

5. Quasi-Adiabatic Evolution and Spectral Flow Automorphism

The propagation estimates and Lieb-Robinson bounds of Section 4 were conducive towards proving strong continuity of the dynamics in the infinite volume limit of the interaction. Precisely, they illustrate how the dynamics approximately preserve the inner product structure of the canonical anticommutation relations of $\mathcal{A}(L^2(\mathbb{R}^2))$. For $f, g \in L^2(\mathbb{R}^2)$, their inner product $\langle f, g\rangle = \int_{\mathbb{R}^2} dx\, \overline{f(x)} g(x)$ is smeared into $\langle f, g\rangle_t = \int_{\mathbb{R}^2} dx\, f^t(x)|g(x)|$ with $f^t(x) = \int_{\mathbb{R}^2} dy\, |f(y)| e^{-\sigma_t|x-y|}$, i.e. the convolution of $|f|$ with an exponentially decaying function.

The quasi-locality of the interacting dynamics are apparent in that $|f(x)|$ and $f^t(x)$ have predominant support in similar regions of space; however, this locality is with reference to a particular function f. For applications of Lieb-Robinson bounds such as towards proving stability of the ground state energy gap or of a topological invariant under local perturbations, it helps to establish the quasi-locality of the dynamics directly with reference to regions in space, as is the case for lattice systems (discussed in Section 2). In the work of Bachmann and De Nittis, [12], the authors demonstrate a method for achieving this via a lattice-localized frame decomposition of the underlying single particle Hilbert space of interest. So in this section, we recall this framework and use it to construct the Hastings-Wen generator of quasi-adiabatic evolution for continuum fermions, which is based on joint work in preparation with Sven Bachmann, Giuseppe De Nittis, and Bruno Nachtergaele [10].

5.1. **Preliminaries and Notation.**

5.1.1. *Localized Frames.* Let (Γ, d) be a metric space, with $\Gamma := \Gamma_{\alpha,\beta} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and d the standard Euclidean distance. Let $\mathcal{H}$ be a separable Hilbert space that can be written as the direct sum of N mutually orthogonal subspaces,

$$\mathcal{H} = \bigoplus_{n=0}^{N-1} \mathcal{H}_n. \tag{182}$$

In each $\mathcal{H}_n$, assume that there exists a set of L^2-normalized functions $\{\chi_{\gamma,n}\}_{\gamma\in\Gamma}$ that forms an overcomplete frame for $\mathcal{H}_n$. This means that the following is true.

i) $\mathcal{H}_n = \overline{\operatorname{span}\{\chi_{\gamma,n} :\ \gamma \in \Gamma\}}$. This set remains complete even upon removing finitely many elements.

ii) $\exists\, 0 < A < B$ such that for each $n \in \mathbb{Z}_N$ and $\forall\, f \in \mathcal{H}_n$,

$A\|f\|^2 \le \sum_{\gamma\in\Gamma} |\langle \chi_{\gamma,n}, f\rangle|^2 \le B\|f\|^2.$

iii) There exists a positive, self-adjoint, invertible operator $S^{(n)}$ named the frame operator that acts on any $f \in \mathcal{H}_n$ as $S^{(n)}f = \sum_{\gamma\in\Gamma}\langle \chi_{\gamma,n}, f\rangle \chi_{\gamma,n}$. Moreover,

$f = \sum_{\gamma\in\Gamma} \langle \left(S^{(n)}\right)^{-1} \chi_{\gamma,n}, f\rangle \chi_{\gamma,n}.$

Importantly, we assume that for each n, $\{\chi_{\gamma,n}\}_{\gamma\in\Gamma}$ forms a localized Gabor frame. For each n, let $g_{n\ell}(x) \in \mathcal{H}_n$ and define $\chi_{\gamma,n}(x) := T_\gamma g_{n\ell}(x) = e^{-\frac{i}{2\ell^2}\gamma\wedge x} g_{n\ell}(x-\gamma)$, where ℓ is a positive constant related to the decay rate of g_{nl} and $\gamma \wedge x = \gamma_1 x_2 - \gamma_2 x_1$. The operator $T_{\gamma'}$ introduced here is a unitary operator on $L^2(\mathbb{R}^2)$ that translates and frequency-shifts, and for now assume that each $\mathcal{H}_n$ is an invariant subspace of T_γ for all $\gamma \in \Gamma$. Furthermore, the localization of this set of functions is characterized by the following features.

i) $\forall\, n \in \mathbb{Z}_N,\ \exists\, \sigma_1, G_1 > 0$ such that $|\chi_{\gamma,n}(x)| \le G_1 e^{-\sigma_1 d(\gamma,x)}$.

ii) $\forall\, n, n' \in \mathbb{Z}_n,\ \exists\, \sigma_2, G_2 > 0$ such that $|\langle \chi_{\gamma,n}, \chi_{\gamma',n'}\rangle| \le \delta_{nn'} G_2 e^{-\sigma_2 d(\gamma,\gamma')}$.

The constant σ_2 above will be called the frame's localization rate. We will also assume the following two bounds pertaining to lattice regularity of Γ.

$$\exists\, \kappa > 0 \text{ such that } |\{\gamma' \in \Gamma :\ d(\gamma,\gamma') \le r\}| \le \kappa r^2 \tag{183}$$

$$\sup_{\gamma\in\Gamma} \sum_{\gamma'\in\Gamma} e^{-a d(\gamma,\gamma')} =: m_a < \infty \ \forall\, a > 0 \tag{184}$$

For what is to come, it will be important that these frames for $\mathcal{H}_n$ are overcomplete, and as discussed in the example of the Landau level systems, [12], this can be achieved with a lower bound on the unit cell area $\alpha\beta$. We thus assume that $\alpha\beta < M\ell^2$ for a particular $M > 0$, which ensures that for each n the set of functions $\{\chi_{\gamma,n}\}_{\gamma\in\Gamma}$ is overcomplete. We will make M precise when introducing the particular frame relevant to two dimensional fermions in a magnetic field. Before moving forward, we state a useful result from [12] concerning the locality of functions of the form $\left(S^{(n)}\right)^{-p}\chi_{\gamma,n}$.

Proposition 5.1 (2.2 from [12])**.** *Let $\mathcal{H}$ be a separable Hilbert space with an overcomplete localized frame $\{\chi_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{H}$ and localization rate $\sigma > 0$ (note here (Γ, d) is an arbitrary metric space which can include ours). Let S be the associated frame operator. Then for every $p \in \mathbb{N}$ there exist constants $0 < \sigma_p < \sigma$ and $a_p > 0$ such that,*

$$\left|\langle\chi_\gamma, S^{-p}\chi_{\gamma'}\right| \leq a_p e^{-\sigma_p d(\gamma,\gamma')}, \tag{185}$$

for all $\gamma, \gamma' \in \Gamma$, where $S^{-p} = (S^{-1})^p$.

This particular estimate will prove useful in subsequent proofs. We now discuss how we will treat finite volume subsets of the functions $\{\chi_{\gamma,n}\}$, i.e. when the sites γ are restricted to be in a finite subset.

When restricting to a finite open-boundary subset $\Lambda \subset \Gamma$, we simply consider the functions $\{\chi_\gamma\}_{\gamma\in\Lambda}$ and their span. When we instead use a finite subset Λ_L to generate a toroidal lattice, modifications will be needed to account for periodicity. Let $L \in \mathbb{N}$ and at this stage take Γ to be a square lattice $\alpha\mathbb{Z} \times \alpha\mathbb{Z}$. Define the larger lattice $\Gamma_L := L\alpha\mathbb{Z} \times L\alpha\mathbb{Z}$ and denote by $\mathbb{V}_L := \alpha[-\frac{L}{2}, \frac{L}{2}) \times \alpha[-\frac{L}{2}, \frac{L}{2})$ the square centered at the origin in $\mathbb{R}^2$. Denote by $\Lambda_L := (\mathbb{V}_L \cap \Gamma)/\Gamma_L$ the set of sites contained in this fundamental cell and let

$$d_T(\gamma,\gamma') := \min_{\tilde{\gamma}\in\Gamma_L} d(\gamma, \gamma' + \tilde{\gamma}) \tag{186}$$

denote a toroidal metric. We require a few more technical assumptions regarding the lattices due to the fact that for each n, $\{\chi_{n,\gamma}\}_{\gamma\in\Gamma}$ is a Gabor frame. In particular, we require that for any $\gamma,\gamma'\in\Gamma_L$, $\gamma\wedge\gamma' = 4m\pi\ell^2$ with $m\in\mathbb{N}$, and this can be achieved if $\alpha = \frac{2\ell\sqrt{m\pi}}{L}$. There is freedom in how we choose the integers m and L, but L must be large enough that $\alpha^2 < M\ell^2$. Now, for $\gamma\in\Lambda_L$, we introduce the following set of functions.

$$\chi^L_{\gamma,n}(x) = \mathbb{1}_{\mathbb{V}_L}(x)\sum_{\gamma'\in\Gamma_L} e^{-\frac{i}{2\ell^2}\gamma'\wedge x}\chi_{\gamma,n}(x-\gamma) = \mathbb{1}_{\mathbb{V}_L}(x)\sum_{\gamma'\in\Gamma_L} T_{\gamma'}\chi_{\gamma,n}(x) \tag{187}$$

For reasons that will become apparent later, we call this the magnetic periodization of $\chi_{\gamma,n}$ restricted to the fundamental cell $\mathbb{V}_L$ of the torus. Notably, due to the exponential localization of each $\chi_{\gamma,n}$, the series above converges in $L^2(\mathbb{V}_L)$. It is also trivial to see that $\chi^L_{\gamma,n}\to\chi_{\gamma,n}$ pointwise and in L^2 norm as L approaches ∞. In a preliminary estimate, we will show that these restricted, periodized functions have pairwise inner products whose absolute values decay in the distance between associated lattice sites.

Proposition 5.2. *Fix $\gamma,\gamma'\in\Lambda_L$, $n,n'\in\mathbb{Z}_N$, then there exist constants $\sigma_1,\sigma_2,G>0$ such that,*

$$\left|\langle\chi^L_{\gamma,n},\chi^L_{\gamma',n'}\rangle\right| \le \delta_{nn'}\left(Ge^{-\sigma_1 d_T(\gamma,\gamma')} + e^{-\sigma_2 L}\right). \tag{188}$$

The inner product here is over $L^2(\mathbb{V}_L)$.

Proof. We begin with an explicit calculation using the definition of the inner product.

$$\langle\chi^L_{\gamma,n},\,\chi^L_{\gamma',n'}\rangle = \int_{\mathbb{V}_L} dx \sum_{\gamma_1,\gamma_2\in\Gamma_L} e^{\frac{i}{2\ell^2}\gamma_1\wedge x}\overline{\chi_{\gamma,n}(x-\gamma_1)}e^{-\frac{i}{2\ell^2}\gamma_2\wedge x}\chi_{\gamma',n'}(x-\gamma_2) \tag{189}$$

Introduce the change of variable $y = x-\gamma_1$, in which case $\gamma_1\wedge(y+\gamma_1) = \gamma_1\wedge y$ by definition. Then one has,

$$\langle\chi^L_{\gamma,n},\,\chi^L_{\gamma',n'}\rangle = \sum_{\gamma_1,\gamma_2\in\Gamma_L}\int_{\mathbb{V}_L+\gamma_1} dy\, e^{\frac{i}{2\ell^2}\gamma_1\wedge y}\overline{\chi_{\gamma,n}(y)}e^{-\frac{i}{2\ell^2}\gamma_2\wedge(y+\gamma_1)}\chi_{\gamma',n'}(y-(\gamma_2-\gamma_1)) \tag{190}$$

$$= \sum_{\gamma_1,\gamma_2\in\Gamma_L}\int_{\mathbb{V}_L+\gamma_1} dy\, e^{\frac{i}{2\ell^2}\gamma_1\wedge y}\overline{\chi_{\gamma,n}(y)}e^{-\frac{i}{2\ell^2}\gamma_2\wedge y}\chi_{\gamma',n'}(y-(\gamma_2-\gamma_1)) \tag{191}$$

$$= \sum_{\gamma_1,\gamma_2\in\Gamma_L} \int_{\mathbb{V}_L+\gamma_1} dy\, \overline{\chi_{\gamma,n}(y)} e^{-\frac{i}{2\ell^2}(\gamma_2-\gamma_1)\wedge y} \chi_{\gamma',n'}(y-(\gamma_2-\gamma_1)) \tag{192}$$

$$= \sum_{\tilde{\gamma}\in\Gamma_L} \int_{\mathbb{R}^2} dy \overline{\chi_{\gamma,n}(y)} T_{\tilde{\gamma}} \chi_{\gamma',n'}(y) \tag{193}$$

$$= \sum_{\tilde{\gamma}\in\Gamma_L} \langle \chi_{\gamma,n}, T_{\tilde{\gamma}} \chi_{\gamma',n'} \rangle \tag{194}$$

Since $\gamma, \gamma' \in \Lambda_L$, denote by $\gamma_* := \operatorname{argmin}_{\tilde{\gamma}\in\Gamma_L} d(\gamma, \gamma'+\tilde{\gamma})$, i.e. the element of Γ_L that minimizes the distance between γ and γ'. We then split the series:

$$\langle \chi^L_{\gamma,n},\ \chi^L_{\gamma',n'} \rangle = \langle \chi_{\gamma,n}, T_{\gamma_*} \chi_{\gamma',n'} \rangle + \sum_{\substack{\tilde{\gamma}\in\Gamma_L:\\ \tilde{\gamma}\neq\gamma_*}} \langle \chi_{\gamma,n}, T_{\tilde{\gamma}} \chi_{\gamma',n'} \rangle \tag{195}$$

Since $\mathcal{H}_n$ are invariant subspaces of $T_{(\cdot)}$ and from the definition of γ_*, it immediately follows that,

$$|\langle \chi_{\gamma,n}, T_{\gamma_*} \chi_{\gamma',n'} \rangle| \leq \delta_{nn'} G_2 e^{-\sigma_2 d_T(\gamma,\gamma')} \tag{196}$$

To address the remaining sum over $\tilde{\gamma} \neq \gamma_*$, note that since $\tilde{\gamma} \neq \gamma_*$, there exists a constant $c > 0$ such that $d(\gamma, \gamma' + \tilde{\gamma}) \geq c\alpha L$. Let $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = \sigma_2$, with σ_2 the decay rate of the set $\{\chi_{\gamma,n}\}_{\gamma\in\Gamma}$. Then, proceeding with the estimate,

$$\sum_{\substack{\tilde{\gamma}\in\Gamma_L:\\ \tilde{\gamma}\neq\gamma_*}} |\langle \chi_{\gamma,n}, T_{\tilde{\gamma}} \chi_{\gamma',n'} \rangle| \leq \delta_{nn'} G_2 e^{-\lambda_1 c\alpha L} \sum_{\substack{\tilde{\gamma}\in\Gamma_L:\\ \tilde{\gamma}\neq\gamma_*}} e^{-\lambda_2 d(\gamma,\gamma'+\tilde{\gamma})} \tag{197}$$

$$\leq \delta_{nn'} G_2 m_{\lambda_2} e^{-\lambda_1 c\alpha L}, \tag{198}$$

where the factor m_{λ_2} as in 184 arises due to lattice regularity. Combining the bounds in 196 and 198 yields the statement of the proposition, and the proof is complete. □

With this result in hand, we ensure that in applications requiring open and periodic boundaries as well as all of $\mathbb{R}^2$, we have a localized set of functions as in 183, up to vanishing error in L for the periodic case. Henceforth, until we specify the finite volume boundary conditions, we shall denote by χ_γ either its original definition or χ^L_γ, since up to decay in L, $\{\chi_\gamma\}$ is a localized set of functions. Additionally, with the index n effectively adding another dimension to the discrete labeling of these functions, let $\Theta = \Gamma \times \mathbb{Z}_N$ equipped with metric ρ, such that for $\theta, \theta' \in \tilde{\Gamma}$ and $d_0 > 0$, $\rho(\theta, \theta') := d_0|n - n'| + d(\gamma, \gamma')$, be a metric space now embedded in $\mathbb{R}^3$. Noting as well that $|\delta_{nn'}| \leq e^{-a|n-n'|}$ for all $a \geq 0$, we use the following general localization bound,

$$|\langle \chi_\theta, \chi_{\theta'} \rangle| \leq G_1 e^{-\lambda \rho(\theta, \theta')} \tag{199}$$

As first introduced, we have that $\{\chi_\theta\}_{\theta \in \Theta}$ is an overcomplete frame for $\mathcal{H}$, with associated frame operator S.

For notational purposes, we denote by $\mathcal{P}_0(\Theta)$ the collection of finite subsets of Θ. Now, when considering finite collections $\{\chi_\theta\}_{\theta \in \Lambda_1}$ for any $\Lambda_1 \subset \Theta$, the closure of the span of this set is of course a finite dimensional Hilbert space, call this $\mathcal{H}_{\Lambda_1} := \overline{\text{span}\{\chi_\theta : \ \theta \in \Lambda_1\}}$. However, linear independence of $\{\chi_\theta\}_{\theta \in \Lambda_1}$ is not guaranteed, as justified by the open HRT conjecture [42]. Thus, there exists an orthonormal basis $\{f_j\}_{j=1}^{N_1}$ whose span is $\mathcal{H}_{\Lambda_1}$, with $N_1 \leq |\Lambda_1|$. However, it will also be important to have an overcomplete, localized frame for $\mathcal{H}_{\Lambda_1}$. Let $\Lambda_2 \in \mathcal{P}_0(\Theta)$ such that $\Lambda_1 \subset \Lambda_2$. Then, $\{\chi_\theta\}_{\theta \in \Lambda_2}$ is by construction an overcomplete frame for $\mathcal{H}_{\Lambda_1}$, with an associated frame operator S_{Λ_1}. Letting P_{Λ_1} and $P_{\Lambda_1}^{\perp}$ denote the orthogonal projections onto $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_1}^{\perp} \cap \mathcal{H}_{\Lambda_2}$ respectively, it will be useful to have explicit formulas for these in terms of the functions χ_θ. The following proposition grants us the ability to do so.

Proposition 5.3 (5.3.5 in [23]). *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$ and let P denote the orthogonal projection onto a closed subspace V of $\mathcal{H}$. Then the following hold:*

(i) If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then $\{Pf_k\}_{k=1}^{\infty}$ is a frame for V.

(ii) If $\{f_k\}_{k=1}^{N}$ is a frame for V with frame operator S, where N need not be finite, then the orthogonal projection onto V is given by $Pf = \displaystyle\sum_{k=1}^{N} \langle S^{-1} f_k, \ f \rangle f_k$.

Notice that the action of P is just given by the frame expansion in 183.iii. Consequently, for any $f \in \mathcal{H}_{\Lambda_2}$, one has,

$$P_{\Lambda_1} f = \sum_{\theta \in \Lambda_2} \langle S_{\Lambda_1}^{-1} \chi_\theta, f \rangle \chi_\theta \tag{200}$$

$$P_{\Lambda_1}^{\perp} f = (P_{\Lambda_2} - P_{\Lambda_1}) f = f - \sum_{\theta \in \Lambda_2} \langle S_{\Lambda_1}^{-1} \chi_\theta, f \rangle \chi_\theta \tag{201}$$

From Proposition 5.3.i, it follows that $\left\{P_{\Lambda_1}^{\perp} \chi_\theta\right\}_{\theta \in \Lambda_2}$ is an overcomplete frame for $\mathcal{H}_{\Lambda_1}^{\perp}$, but this set necessarily simplifies to become $\left\{P_{\Lambda_1}^{\perp} \chi_\theta\right\}_{\theta \in \Lambda_2 \setminus \Lambda_1}$. The next proposition confirms locality in the pairwise inner products within this projected set.

Proposition 5.4. *Let $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Theta)$ with $\Lambda_1 \subset \Lambda_2$. The set $\left\{P_{\Lambda_1}^{\perp} \chi_\theta\right\}_{\theta \in \Lambda_2 \setminus \tilde{\Lambda}_1}$ satisfies the following pairwise inner product bound:*

$$\forall\, \theta, \theta' \in \Lambda_2 \setminus \Lambda_1,\ \exists\, G, \sigma > 0 \text{ such that } \left|\langle P_{\Lambda_1}^{\perp} \chi_\theta, P_{\Lambda_1}^{\perp} \chi_{\theta'} \rangle\right| \leq G e^{-\sigma \rho(\theta, \theta')}. \tag{202}$$

Proof. As $P_{\Lambda_1}^{\perp}$ is an orthogonal projection, the inner product whose absolute value we estimate is equal to $\left|\langle \chi_\theta,\ P_{\Lambda_1}^{\perp} \chi_{\theta'} \rangle\right|$. Using the explicit formula for the action of $P_{\Lambda_1}^{\perp}$, we have,

$$P_{\Lambda_1}^{\perp} \chi_{\theta'} = \chi_{\theta'} - \sum_{j \in \Lambda_2} \langle S_{\Lambda_1}^{-1} \chi_j, \chi_{\theta'} \rangle \chi_j \tag{203}$$

Then, taking the inner product of χ_θ with 203, we obtain

$$\langle \chi_\theta, P_{\Lambda_1}^{\perp} \chi_{\theta'} \rangle = \langle \chi_\theta, \chi_{\theta'} \rangle - \sum_{j \in \Lambda_2} \langle S_{\Lambda_1}^{-1} \chi_j, \chi_{\theta'} \rangle \langle \chi_\theta, \chi_j \rangle \tag{204}$$

Towards proving the proposition, we now take the absolute value of the previous equation and begin estimating an upper bound.

$$\left|\langle \chi_\theta, P_{\Lambda_1}^{\perp} \chi_{\theta'} \rangle\right| \leq \langle \chi_\theta, \chi_{\theta'} \rangle + \sum_{j \in \Lambda_2} \left|\langle S_{\Lambda_1}^{-1} \chi_j, \chi_{\theta'} \rangle\right| \left|\langle \chi_\theta, \chi_j \rangle\right| \tag{205}$$

The set $\{\chi_\theta\}_{\theta\in\Theta}$ is already a localized set. Moreover, the terms of the form $\left|\langle S_{\Lambda_1}^{-1}\chi_j, \chi_{\theta'}\rangle\right|$ can be bounded using Proposition 5.1. Thus, one obtains,

$$\left|\langle \chi_\theta, P_{\Lambda_1}^{\perp}\chi_{\theta'}\rangle\right| \leq G_1 e^{-\sigma_2\rho(\theta,\theta')} + a_1 G_1 \sum_{j\in\Lambda_2} e^{-\sigma_1\rho(j,\theta')} e^{-\sigma_2\rho(\theta,j)} \tag{206}$$

$$\leq G_1 e^{-\sigma_2\rho(\theta,\theta')} + \tilde{G} e^{-\tilde{\sigma}\rho(\theta,\theta')} \tag{207}$$

$$\leq G e^{-\sigma\rho(\theta,\theta')} \tag{208}$$

Here, the constants $\tilde{G}, \tilde{\sigma} > 0$ arise from estimating the convolution in the first inequality. This thus completes the proof. □

Now, we will lift this framework to the fermionic CAR algebra which will be used to represent bounded observables and interactions to describe interacting continuum fermions.

5.1.2. *Fermionic CAR algebra and Conditional Expectation.*

Let $\mathcal{A}_\Theta$ denote the CAR algebra generated by $\{a(\chi_\theta), a^*(\chi_\theta) : \theta \in \Theta\}$ and henceforth denote by $a_\theta^\# := a^\#(\chi_\theta)$, where the # symbol indicates either a fermionic annihilation or creation operator. Additionally, for finite subsets $\Lambda \in \mathcal{P}_0(\Theta)$, $\mathcal{A}_\Lambda$ will denote the finite volume subalgebras generated by $\{a_\theta, a_\theta^* : \theta \in \Lambda\}$. While the functions χ_θ have support over all of $\mathbb{R}^2$, or $\mathbb{V}_L$ in the periodic boundary case, they are associated with a local subset, namely a singleton $\{\theta\}$, of Θ. We thus introduce the following terminology to establish a recurring notion of locality in this work.

Definition 5.5. An operator $A \in \mathcal{A}_\Theta$ will be called **local** if there exists a finite $\Lambda \subset \Theta$ such that $A \in \mathcal{A}_\Lambda$.

A local operator is one whose creation and annihilation operators are centered in a finite subset of the lattice Θ. Moreover, we similarly term an operator to be quasi-local if it can be expressed as the limit of a sequence of local operators. Moving forward, it will be useful to have a generalized notion of a partial trace that will allow us to generate local approximations of quasi-local operators. This is where we seek inspiration from the works of Araki and Moriya [4] and Nachtergaele, Sims, and Young [73] to introduce a conditional expectation map. Remark however that as pointed out with such a map for fermionic systems, [73], this map is defined in terms of global operators unless we restrict the domain to be generated by even degree monomials in $a_\theta^\#$. For a subalgebra $\mathcal{A}_\Lambda$, denote by $\mathcal{A}_\Lambda^+$ the algebra generated by even degree monomials in a_θ, a_θ^*: $\mathcal{M}_\Lambda := \left\{\mathbb{1}, a_{\theta_1}^\# ... a_{\theta_{2k}}^\# : \ k \in \mathbb{N}, \ \theta_i \in \Lambda\right\}$. Let $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Theta)$ such that $\Lambda_1 \subset \Lambda_2$. The conditional expectation is the map $\mathbb{E}_{\Lambda_1}^{\Lambda_2} : \mathcal{A}_{\Lambda_2} \to \mathcal{A}_{\Lambda_1}$ where $\mathcal{A}_{\Lambda_1}$ is a subalgebra of $\mathcal{A}_{\Lambda_2}$. What follows is the construction of this map.

Since $\mathcal{H}_{\Lambda_1}$ is a closed subspace of the finite-dimensional space $\mathcal{H}_{\Lambda_2}$, we may choose to construct an orthonormal basis for $\mathcal{H}_{\Lambda_1}$ that is a subset of the orthonormal basis for $\mathcal{H}_{\Lambda_2}$. Let $N_1 < N_2$ denote the dimensions of each space respectively and $\{f_j\}_{j=1}^{N_i}$ an orthonormal basis for $\mathcal{H}_{\Lambda_i}$. Furthermore, $\{f_j\}_{j=N_1+1}^{N_2}$ is an orthonormal basis for $\mathcal{H}_{\Lambda_1}^\perp \cap \mathcal{H}_{\Lambda_2}$, and for notational purposes let $I_{12} = \{N_1+1, N_1+2, ..., N_2\}$ be the indexing set for this basis. For each such basis function spanning $\mathcal{H}_{\Lambda_1}^\perp \cap \mathcal{H}_{\Lambda_2}$, let $a_i^\# := a^\#(f_i)$ be the associated fermionic creation and annihilation operators. We introduce the following four unitary operators:

$$u_i^0 = \mathbb{1}; \ u_i^1 = a_i^* + a_i; \ u_i^2 = a_i^* - a_i; \ u_i^3 = \mathbb{1} - 2a_i^* a_i. \tag{209}$$

Let $\mathcal{I}_{12} = \{0,1,2,3\}^{N_2 - N_1}$ and for any $\iota \in \mathcal{I}_{12}$, define the unitary operator $u(\iota)$ as,

$$u(\iota) = u_{N_1+1}^{\iota(N_1+1)} ... u_{N_2}^{\iota(N_2)}. \tag{210}$$

Then for any $A \in \mathcal{A}_{\Lambda_2}^+$, $\mathbb{E}_{\Lambda_1}^{\Lambda_2} : \mathcal{A}_{\Lambda_2}^+ \to \mathcal{A}_{\Lambda_1}^+$ acts as,

$$\mathbb{E}_{\Lambda_1}^{\Lambda_2}(A) = \frac{1}{4^{N_2 - N_1}} \sum_{\iota \in \mathcal{I}_{12}} u(\iota)^* A u(\iota). \tag{211}$$

Notably, this construction is equivalent to the lattice fermion setting with the sites here corresponding to orthonormal basis elements f_i. It is thus trivial to prove the following properties of this map.

Lemma 5.6 (Lemma 4.1, [73]). *For any $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Theta)$ with $\Lambda_1 \subset \Lambda_2$, the map $\mathbb{E}_{\Lambda_1}^{\Lambda_2}$ as in 211 satisfies*

$$i) \left\| \mathbb{E}_{\Lambda_1}^{\Lambda_2} \right\| = 1$$

$$ii) \left(\mathbb{E}_{\Lambda_1}^{\Lambda_2} \right)^2 = \mathbb{E}_{\Lambda_1}^{\Lambda_2}$$

$$iii) \mathbb{E}_{\Lambda_1}^{\Lambda_2} \left(\mathcal{A}_{\Lambda_2}^{+} \right) = \mathcal{A}_{\Lambda_2}^{+}$$

This map will be crucial in constructing local sequences of operators with limit contained in $\mathcal{A}_\Theta$, namely the Heisenberg dynamics of local operators. Thus, we now proceed with an introduction to the class of interactions that will be used to build the finite volume Hamiltonians of interest. Then, we will discuss their Heisenberg dynamics, Lieb-Robinson bounds, and approximations of Heisenberg-evolved operators using the conditional expectation map.

5.2. **Hamiltonians and Heisenberg Dynamics.**

5.2.1. *Interactions and Decay Assumptions.* The class of interactions we consider are constructed from elements of $\mathcal{A}_\Theta^+$ which feature the same number of creation and annihilation operators. We use the following terminology to distinguish between interactions according to how many creation/annihilation operators are present, i.e. how many fermions are involved in an interaction.

Definition 5.7. Let $\Lambda \in \mathcal{P}_0(\Theta)$. An element of $\mathcal{A}_\Lambda$ is said to be **k-body** if $A \in \mathcal{M}_\Lambda$, i.e. it is of the form $A = a_{\theta_{2k}}^{\#} ... a_{\theta_{k+1}}^{\#} a_{\theta_k}^{\#} ... a_{\theta_1}^{\#}$, with each $\theta_i \in \Lambda$.

Furthermore for $\Lambda \subset \Theta$ and $k \in \mathbb{N}$, we will denote by

$$\Lambda^{2k} = \{(\theta_1, ..., \theta_{2k}) : \forall\, i = 1, ..., 2k\ \theta_i \in \Lambda\}$$

the collection of ordered $2k$-tuples of sites in Λ. Then, let $\Lambda^+ := \bigcup_{k=1}^{|\Lambda|} \Lambda^{2k}$ the set of $2k$-tuples of sites in Λ. This definition also works for the entire lattice Θ, where Θ^+ is the collection of all even-sized tuples of sites in Θ. Going forward in this work, we will use the notation $\underline{\theta} := (\theta_1, ..., \theta_{2k})$ to denote $2k$-tuples of sites. For any $\underline{\theta} \in \Theta^+$, one has $\underline{\theta} \in \Lambda^+$ for any $\Lambda \in \mathcal{P}_0(\Theta)$ containing the sites of $\underline{\theta}$. For $\underline{\theta} \in \Theta^+$, denote by $\boldsymbol{\theta} := \mathrm{supp}(\underline{\theta}) \in \mathcal{P}_0(\Theta)$ the collection of sites in the tuple $\underline{\theta}$ and $|\underline{\theta}|$ the number of sites, including repetition, in the tuple. Then, for $\underline{\theta}, \underline{\mu} \in \Theta^+$, one has $\underline{\theta} \subset \underline{\mu}$ if the sites θ_i are each in the tuple $\underline{\mu}$, and $\boldsymbol{\theta} \subset \boldsymbol{\mu}$ if the collection of sites $\{\theta_i\}$ are a subset of $\{\mu_i\}$, disregarding repetition.

Throughout this work, we define $a_{\underline{\theta}} := a^*_{\theta_{2k}}...a^*_{\theta_{k+1}}a_{\theta_k}...a_{\theta_1}$ to be the k-body monomial generated by $\underline{\theta}$. Note that the maximum degree of such a monomial supported in Λ is $2|\Lambda|$ as the CAR relations eliminate any higher degree operators. We then consider interactions $\Phi : \Theta^+ \to \mathcal{A}_\Theta$ with $\Phi^* = \Phi$ of the form,

$$\Phi(\underline{\theta}) = \frac{1}{2}\phi(\underline{\theta})\left(a_{\underline{\theta}} + a^*_{\underline{\theta}}\right). \tag{212}$$

This operator $\Phi(\underline{\theta})$ will be assumed to be short range, and to define this more precisely, we recall F-functions, which have been used to characterize the decay of interactions in quantum lattice systems.

Definition 5.8 (F-Function)**.** An *F-function* on an arbitrary metric space (Γ, d) is a non-increasing function $F : [0, \infty) \to (0, \infty)$ satisfying the next two properties.

$$(i)\|F\| = \sup_{x\in\Gamma}\sum_{y\in\Gamma} F(d(x,y)) < \infty \tag{213}$$

(ii)There exists a constant $C_F > 0$ such that

$$\sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) \le C_F F(d(x,y));\ \forall x,y \in \Gamma.$$

Additionally, let $g(r) : [0,\infty] \to [0,\infty)$ be a non-decreasing, subadditive function and let F_0 be an F-function. Weighted F-functions $F_g(r)$ are then of the form:

$$F_g(r) = e^{-g(r)}F_0(r). \tag{214}$$

A very useful example of such a function g is $g(r) = ar^p$ with $a > 0$ and $0 < p \leq 1$. For the precise case $g(r) = ar$ with $a > 0$, we explicitly denote the corresponding weighted F-function as $F_a(r)$. Interestingly, the function e^{-ar} on its own is not an F-function because it does not exactly satisfy the convolution property. By Lemma B.1 of [31], the absolute value of the convolution of two exponentially decaying functions, $e^{-a|x|} * e^{-a|x|}$ can be bounded by above by $Ce^{-a|x|/4}$, where $C > 0$, so the convolution preserves the decay class at the cost of sacrificing decay rate. Therefore, while not precisely an F-function, we still include functions of the form e^{-ar} as examples of admissible functions for the forthcoming analysis.

The following is an assumption on the decay of the interaction potential function ϕ.

Assumption 5.9.

i) $\phi(\underline{\theta}) \in \mathbb{R} \,\forall\, \underline{\theta} \in \Theta^+$,

ii) $\exists\, C > 0$ *and an* F*-function* F *such that for any* $\theta_1, \theta_2 \in \Theta$,

$$\sum_{\substack{\underline{\theta} \in \Theta^+: \\ x,y \in \boldsymbol{\theta}}} |\phi(\underline{\theta})| \leq CF(\rho(x, y)).$$

This condition implies decay in the interaction in the diameter of its support as well as the number of sites in its support, which will be essential when constructing the dynamics associated with this interaction. Moreover, since the operators $a^{\#}_{\theta}$ are already normalized, this F-function decay fully describes the strength and range of Φ. The next subsection introduces the general class of Hamiltonians that we study in this work.

5.2.2. *Finite volume Hamiltonians and Heisenberg Dynamics.* We are now equipped to construct Hamiltonians using the map Φ defined in equation 212. Fix a finite volume, either open or periodic boundary, $\Lambda \in \mathcal{P}_0(\Theta)$. We then consider the following Hamiltonian,

$$H_\Lambda = \sum_{\underline{\theta} \in \Lambda^+} \Phi(\underline{\theta}) = \sum_{\underline{\theta} \in \Lambda^+} \phi(\underline{\theta}) a_{\underline{\theta}}. \tag{215}$$

H_Λ is clearly self-adjoint since the interaction is summed over all lattice tuples, which includes ordering of sites belonging to a single subset of Θ. The Heisenberg dynamics generated by these Hamiltonians, $\tau_t^\Lambda : \mathcal{A}_\Theta \to \mathcal{A}_\Theta$, are automorphisms of the CAR algebra $\mathcal{A}_\Theta$ and act on elements $A \in \mathcal{A}_\Theta$ as,

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}. \tag{216}$$

Crucially, these dynamics give rise to a Lieb-Robinson bound, which is the subject of the following theorem.

Theorem 5.10. *Let $\Lambda \in \mathcal{P}_0(\Theta)$ and Φ be an interaction as in 212 satisfying Assumption 5.9 with decay governed by a sub-exponential weighted F-function $F_g(r) = e^{-g(r)}F_0(r)$, meaning that for $c > 0$, there exists a constant G_F such that $e^{-cr} \leq G_F F_g(r)$. Then there exist constants $G, v > 0$ such that for all $x, y \in \Theta$ and $t \in \mathbb{R}$, the following bound holds,*

$$\left\| \left\{ \tau_t^\Lambda(a_x^\#), a_y^\# \right\} \right\| \leq G F_0(0) e^{-(g(\rho(x,y)) - v|t|)}, \tag{217}$$

where F_0 is the base F-function of F_g.

Proof. According to the Duhamel identity, we write the anticommutator in question as,

$$\{\tau_t^\Lambda(a_x^\#),\ a_y^\#\} = \left\{a_x^\#, a_y^\#\right\} + i \sum_{\underline{\theta} \in \Lambda^+} \int_0^t ds\ \left\{ \tau_s^\Lambda \left(\left[\Phi(\underline{\theta}), a_x^\# \right] \right), a_y^\# \right\}. \tag{218}$$

Noting the form of $\Phi(\underline{\theta})$, we evaluate the commutator $[a_{\underline{\theta}}, a_x^\#]$. Recalling that $a_{\underline{\theta}}$ is a degree-$|\underline{\theta}|$ monomial, we have

$$[a_{\underline{\theta}},\ a_x^\#] = \Big[\prod_{j=1}^{|\underline{\theta}|} a_{\theta_{2j-1}}^\# a_{\theta_{2j}}^\#,\ a_x^\# \Big] \tag{219}$$

$$= \sum_{j=1}^{|\underline{\theta}|} a_{\theta_{2k}}^\# a_{\theta_{2k-1}}^\# \ldots \big[a_{\theta_{2j}}^\# a_{\theta_{2j-1}}^\#,\ a_x^\# \big] \ldots a_{\theta_2}^\# a_{\theta_1}^\#, \tag{220}$$

where θ_i denotes the i^{th} element of the tuple $\underline{\theta}$. Using the property that $[A_0A_1, A_2] = A_0\{A_1, A_2\} - \{A_0, A_2\}A_1$, we then obtain,

$$[a_{\underline{\theta}},\ a_x^{\#}] = \sum_{j=1}^{|\theta|} a^{\#}_{\theta_{2k}} a^{\#}_{\theta_{2k-1}} ... \left(a^{\#}_{\theta_{2j}} \{a^{\#}_{\theta_{2j-1}}, a_x^{\#}\} - \{a^{\#}_{\theta_{2j}}, a_x^{\#}\} a^{\#}_{\theta_{2j-1}} \right) ... a^{\#}_{\theta_2} a^{\#}_{\theta_1}. \tag{221}$$
$$= \sum_{j=1}^{|\theta|} (-1)^{j-1} f\left(\theta_j, x\right) a^{\#}_{\theta_{2k}} ... \hat{a}^{\#}_{\theta_j} ... a^{\#}_{\theta_1},$$

where $\hat{a}^{\#}_{(\cdot)}$ indicates an absent operator and $f\left(\theta_j, \theta\right) = \langle \chi_{\theta_j}, \chi_x \rangle$ or the complex conjugate of this inner product depending on which fermion operator is creation/annihilation. Noting that there is a positive constant G_F such that $|f(x,y)| \leq G_F F_g(\rho(x,y))$,

$$\{\tau_s^{\Lambda}\left([a_{\underline{\theta}}, a_x^{\#}]\right), a_y^{\#}\} = \sum_{j=1}^{|\theta|} (-1)^{j-1} f\left(\theta_j, x\right) \{\tau_s^{\Lambda}(a^{\#}_{\theta_{2k}} ... \hat{a}^{\#}_{\theta_j} ... a^{\#}_{\theta_1}), a_y^{\#}\} \tag{222}$$

$$\left\|\{\tau_s^{\Lambda}\left([a_{\underline{\theta}}, a_x^{\#}]\right),\ a_y^{\#}\}\right\| \leq G_F \sum_{i \neq j}^{|\theta|} F_g(\rho(\theta_j, x)) \left\|\{\tau_s^{\Lambda}(a^{\#}_{\theta_i}), a_y^{\#}\}\right\| \tag{223}$$

We will now use this to begin estimating $\left\|\{\tau_t^{\Lambda}(a_x^{\#}), a_y^{\#}\}\right\| =: \mathcal{F}_t^{\Lambda}(x,y)$.

$$\mathcal{F}_t^{\Lambda}(x,y) \leq G_F F_g(\rho(x,y)) + G_F \sum_{\underline{\theta} \in \Lambda^+} \phi(\underline{\theta}) \sum_{i \neq j}^{|\theta|} F_g(\rho(x, \theta_j)) \int_0^{|t|} ds\, \mathcal{F}_s^{\Lambda}(\theta_i, y) \tag{224}$$

This preliminary upper bound naturally suggests that we proceed via iteration, which brings us to the following bound as a series.

$$\mathcal{F}_t^{\Lambda}(x,y) \leq G_F F_g(\rho(x,y)) + \sum_{n=1}^{\infty} \frac{|a_n(x,y)||t|^n}{n!} \tag{225}$$

$$a_1(x,y) = G_F \sum_{\underline{\theta} \in \Lambda^+} \phi(\underline{\theta}) \sum_{i \neq j}^{|\theta|} F_g(\rho(x, \theta_j)) F_g(\rho(\theta_i, y)) \tag{226}$$

$$a_n(x,y) = G_F^{n+1} \sum_{\underline{\theta}^n \in \Lambda^+} \sum_{i_n \neq j_n} ... \sum_{\underline{\theta}^1 \in \Lambda^+} \sum_{i_1 \neq j_1} \tag{227}$$
$$F_g(\rho(x, \theta^1_{j_1})) \left(\prod_{m=2}^{n} F_g(\rho(\theta^{m-1}_{i_{m-1}}, \theta^m_{j_m})) \right) F_g(\rho(\theta^n_{i_n}, y)) \prod_{j=1}^{n} \phi(\underline{\theta}^j)$$

To motivate the closed expression for our estimate of this series upper bound, we begin by computing an upper bound on $a_1(x,y)$.

$$a_1(x,y) = G_F \sum_{\underline{\theta}\in\Lambda^+} \phi(\underline{\theta}) \sum_{i\neq j}^{|\underline{\theta}|} F_g(\rho(x,\theta_j))F_g(\rho(\theta_i,y)) \tag{228}$$

$$a_1(x,y) \leq G_F \sum_{\underline{\theta}\in\Lambda^+} \sum_{\{\theta_1,\theta_2\}\subset\underline{\theta}} |\phi(\underline{\theta})| F_g(\rho(x,\theta_1))F_g(\rho(\theta_2,y)) \tag{229}$$

$$\leq G_F \sum_{\theta_1,\theta_2\in\Theta} \sum_{\substack{\underline{\theta}\in\Lambda^+: \\ \theta_1,\theta_2\in\boldsymbol{\theta}}} |\phi(\underline{\theta})| F_g(\rho(x,\theta_1))F_g(\rho(\theta_2,y)) \tag{230}$$

$$\leq G_F C \sum_{\theta_1,\theta_2\in\Theta} F_g(\rho(\theta_1,\theta_2))F_g(\rho(x,\theta_1))F_g(\rho(\theta_2,y)) \tag{231}$$

$$\leq G_F C C_F^2 F_g(\rho(x,y)) \tag{232}$$

Using the same series of arguments with the convolution property of the F-function, we bound the general term a_n:

$$|a_n(x,y)| \leq G_F^{n+1} C^n C_F^{2n} F_g(\rho(x,y)). \tag{233}$$

Evaluating the series over n yields the final bound,

$$\mathcal{F}_t^{\Lambda}(x,y) \leq G_F e^{v|t|} F_g(\rho(x,y)), \tag{234}$$

where $v = G_F C_\Phi C_F^2$. Recalling the definition of the weighted F-function, it necessarily holds that $F_g(r) \leq e^{-g(r)}F_0(0)$, from which the claim of the theorem follows. $\square$

From this result, we may also write a more general Lieb-Robinson bound when the operators in the (anti-)commutator are monomials of degree larger than one.

Corollary 5.11. *Let $\Lambda \in \mathcal{P}_0(\Theta)$ and $\underline{\theta}, \underline{\mu} \in \Lambda^+$. Under the same assumptions as Theorem 5.10, for all $t \in \mathbb{R}$ one has the bound,*

$$\left\| \left[\tau_t^{\Lambda}(a_{\underline{\theta}}^{\#}), a_{\underline{\mu}}^{\#}\right]_{\pm} \right\| \leq G_F \|F_0\| \min\{|\underline{\theta}|, |\underline{\mu}|\} e^{-(g(\rho(\boldsymbol{\theta},\boldsymbol{\mu}))-v|t|)}, \tag{235}$$

where $[\cdot,\cdot]_+ := \{\cdot,\cdot\}$ *if both* $|\underline{\theta}|, |\underline{\mu}|$ *are odd and* $[\cdot,\cdot]_- := [\cdot,\cdot]$ *if at least one of* $|\underline{\theta}|, |\underline{\mu}|$ *is even. Moreover,* $\rho(\boldsymbol{\theta},\boldsymbol{\mu}) := \min_{\theta\in\boldsymbol{\theta}}\min_{\mu\in\boldsymbol{\mu}}\rho(\theta,\mu)$.

Proving this estimate follows the same procedure as Theorem 5.10, but integrability of the function F_0 is exploited to give the factor $\min\{|\underline{\theta}|, |\underline{\mu}|\}\|F_0\|$ instead of $F_0(0)$ as in the single-site case. Furthermore, as pointed out in Proposition 2.1 of [73], the parity of the monomials will dictate whether one should consider the anti-commutator or commutator of the respective monomials. In addition to these Lieb-Robinson bounds featuring elements of $\mathcal{A}_\Theta$, we require one more Lieb-Robinson bound involving elements of $\mathcal{A}(\mathcal{H}_{\Lambda_1}^{\perp}\cap\mathcal{H}_{\Lambda_2})$ with $\Lambda_1\subset\Lambda_2$ two finite subsets of Θ.

Lemma 5.12. *Let* $\Lambda_1,\Lambda_2\in\mathcal{P}_0(\Theta)$ *such that* $\Lambda_1\subset\Lambda_2$ *and assume the same conditions on the Hamiltonian* H_Λ *as Theorem 5.10 with sub-exponential weighted* F*-function* $F_g(r) = e^{-g(r)}F_0(r)$. *For all* $x\in\Lambda_1$ *and any* $f\in\mathcal{H}_{\Lambda_1}^{\perp}\cap\mathcal{H}_{\Lambda_2}$, *there exist constants* $G_\perp, v_p > 0$ *such that the following bound holds,*

$$\left\|\left\{\tau_t^{\Lambda_2}(a_x^{\#}), a^{\#}(f)\right\}\right\| \leq G_\perp\|f\|\|F_0\|e^{-(g(\rho(x,\partial\Lambda_1))-v_p|t|)}, \tag{236}$$

where $\partial\Lambda_1$ *denotes the boundary of* Λ_1.

The proof of this bound follows the same strategy as the proof of Theorem 5.10; however, from the step of writing a preliminary bound, the proof deviates due to the function f. Thus, the proof of this proposition picks up from that initial inequality.

proof of Proposition 5.12. Let $\mathcal{F}_t^{\Lambda_2}(x,f) := \left\|\left\{\tau_t^{\Lambda_2}(a_x^{\#}), a^{\#}(f)\right\}\right\|$. Then as in equation 224, one has,

$$\mathcal{F}_t^{\Lambda_2}(x,f) \leq |\langle\chi_x, f\rangle| + G_F\sum_{\underline{\theta}\in\Lambda^+}\phi(\underline{\theta})\sum_{i\neq j}^{|\underline{\theta}|}F_g(\rho(x,\theta_j))\int_0^{|t|}ds\,\mathcal{F}_s^{\Lambda}(\theta_i,f) \tag{237}$$

Note that the first term on the right hand side of 237 is actually 0, but we will proceed as if x is not necessarily in Λ_1. Then to conclude the calculation we will adjust our estimate to account for $x \in \Lambda_1$. Moving on, upon infinite iteration of, we obtain the series,

$$\mathcal{F}_t^{\Lambda_2}(x, f) \leq |\langle \chi_x,\ f\rangle| + \sum_{n=1}^{\infty} \frac{a_n(x,f)|t|^n}{n!} \tag{238}$$

$$a_1(x,f) = G_F \sum_{\underline{\theta}\in\Lambda^+} \phi(\underline{\theta}) \sum_{i\neq j}^{|\underline{\theta}|} F_g(\rho(x,\theta_j))|\langle \chi_{\theta_i}, f\rangle| \tag{239}$$

$$\begin{aligned} a_n(x,f) = G_F^{n+1} &\sum_{\underline{\theta}^n\in\Lambda^+} \sum_{i_n\neq j_n} \cdots \sum_{\underline{\theta}^1\in\Lambda^+} \sum_{i_1\neq j_1} \\ &F_g(\rho(x,\theta^1_{j_1})) \left(\prod_{m=2}^{n} F_g(\rho(\theta^{m-1}_{i_{m-1}}, \theta^m_{j_m}))\right) \left|\langle \chi_{\theta^n_{i_n}}, f\rangle\right| \prod_{j=1}^{n} \phi(\underline{\theta}^j). \end{aligned} \tag{240}$$

Since $f \in \mathcal{H}_{\Lambda_1}^{\perp} \cap \mathcal{H}_{\Lambda_2}$, by Proposition 5.3, we have the following decomposition of f:

$$f = \sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \langle S^{-1}_{\Lambda_2\backslash\Lambda_1} P^{\perp}_{\Lambda_1}\chi_{\theta'},\ f\rangle P^{\perp}_{\Lambda_1}\chi_{\theta'},$$

where $S^{-1}_{\Lambda_2\backslash\Lambda_1}$ denotes the inverse frame operator associated with $\{P^{\perp}_{\Lambda_1}\chi_{\theta'}\}_{\theta'\in\Lambda_2\backslash\Lambda_1}$. Taking the inner product of χ_{θ_i} with f then gives us the following.

$$|\langle \chi_{\theta_i}, f\rangle| = \left|\langle \chi_{\theta_i}, \sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \langle S^{-1}_{\Lambda_2\backslash\Lambda_1} P^{\perp}_{\Lambda_1}\chi_{\theta'}, f\rangle P^{\perp}_{\Lambda_1}\chi_{\theta'}\rangle\right| \tag{241}$$

$$\leq \sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \left|\langle S^{-1}_{\Lambda_2\backslash\Lambda_1} P^{\perp}_{\Lambda_1}\chi_{\theta'},\ f\rangle\right| \left|\langle \chi_{\theta_i}, P^{\perp}_{\Lambda_1}\chi_{\theta'}\rangle\right| \tag{242}$$

$$= \sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \left|\langle S^{-1}_{\Lambda_2\backslash\Lambda_1} P^{\perp}_{\Lambda_1}\chi_{\theta'},\ f\rangle\right| \left|\langle P^{\perp}_{\Lambda_1}\chi_{\theta_i}, P^{\perp}_{\Lambda_1}\chi_{\theta'}\rangle\right| \tag{243}$$

The first inequality is an application of the triangle inequality and linearity of the inner product, while the equality that follows makes use of the fact that $P^{\perp}_{\Lambda_1}$ is an orthogonal projection. To proceed, we make use of the Cauchy-Schwarz inequality.

$$|\langle \chi_{\theta_i}, f\rangle| \leq \left(\sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \left|\langle S^{-1}_{\Lambda_2\backslash\Lambda_1} P^{\perp}_{\Lambda_1}\chi_{\theta'},\ f\rangle\right|^2\right)^{\frac{1}{2}} \left(\sum_{\theta'\in\Lambda_2\backslash\Lambda_1} \left|\langle P^{\perp}_{\Lambda_1}\chi_{\theta_i}, P^{\perp}_{\Lambda_1}\chi_{\theta'}\rangle\right|^2\right)^{\frac{1}{2}} \tag{244}$$

In the first term, note that $S^{-1}_{\Lambda_2\backslash\Lambda_1}$ inherits the overcompleteness of the original frame for $\mathcal{H}_{\Lambda_1}$, which means that it has a finite operator norm that does not diverge as $\Lambda_2 \to \Theta$, i.e.

the frame bound. Thus there exists a positive constant $C > 0$ such that $\left(\sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} \left|\langle S^{-1}_{\Lambda_2 \setminus \Lambda_1} P^{\perp}_{\Lambda_1} \chi_{\theta'},\ f\rangle\right|^2\right)^{\frac{1}{2}} \leq C\|f\|$. The second term can be bounded as a result of proposition 5.4. So, one has,

$$|\langle \chi_{\theta_i}, f\rangle| \leq CG\|f\| \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} e^{-\sigma\rho(\theta_i,\theta')}. \tag{245}$$

Furthermore, there exists a weighted F-function F_{g_1} that can bound the exponentially decaying term above. Estimating a bound on $a_1(\theta, f)$ then yields,

$$a_1(x,f) \leq CGG_F\|f\| \sum_{\underline{\theta} \in \Lambda^+} |\phi(\underline{\theta})| \sum_{i \neq j}^{|\underline{\theta}|} F_g(\rho(x,\theta_j)) \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_{g_1}(\rho(\theta_i,\theta')) \tag{246}$$

$$\leq CGG_F\|f\| \sum_{\underline{\theta} \in \Lambda^+} |\phi(\underline{\theta})| \sum_{\{\theta_1,\theta_2\} \subset \underline{\theta}} \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_g(\rho(x,\theta_1)) F_{g_1}(\rho(\theta_2,\theta')) \tag{247}$$

$$\leq CGG_F\|f\| \sum_{\theta_1,\theta_2 \in \Theta} \sum_{\substack{\underline{\theta} \in \Lambda^+: \\ \theta_1,\theta_2 \in \boldsymbol{\theta}}} |\phi(\underline{\theta})| \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_g(\rho(x,\theta_1)) F_{g_1}(\rho(\theta_2,\theta')) \tag{248}$$

These steps follow the same trajectory as the proof of Theorem 5.10 with the exception of a new decay function F_{g_1} introduced to handle the element $f \in \mathcal{H}^{\perp}_{\Lambda_1} \cap \mathcal{H}_{\Lambda_2}$. Let F_{g_2} be a weighted F-function with identical base $F_0(r)$ that bounds from above both F_g and F_{g_1}. Then, we may proceed to use Assumption 5.9:

$$a_1(x,f) \leq CGG_F\|f\| \sum_{\theta_1,\theta_2 \in \Theta} \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_{g_2}(\rho(x,\theta_1)) F_{g_2}(\rho(\theta_1,\theta_2)) F_{g_2}(\rho(\theta_2,\theta')) \tag{249}$$

$$\leq CGG_F C_F^2 \|f\| \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_{g_2}(\rho(x,\theta')) \tag{250}$$

$$\leq CGG_F C_F^2 \|f\| e^{-g_2(\rho(x,\partial\Lambda_1))} \sum_{\theta' \in \Lambda_2 \setminus \Lambda_1} F_0(\rho(x,\theta')) \tag{251}$$

$$\leq CGG_F C_F^2 \|f\| \|F_0\| e^{-g_2(\rho(x,\partial\Lambda_1))}. \tag{252}$$

By $\partial\Lambda_1$ we mean the boundary of Λ_1 in Λ_2, which appears since $x \in \Lambda_1$. The inequalities above make use of the convolution properties of the F-function as well as the definition of a weighted F-function, which lets us extract exponential decay in $g_2(r)$. Similarly, one bounds the terms $a_n(x, f)$ with the estimate,

$$CGG_F^{n+1}C_F^{2n}\|f\|\|F_0\|e^{-g_2(\rho(x,\partial\Lambda_1))} \tag{253}$$

Bounding the infinite series of the a_n terms then gives the final bound,

$$\left\|\left\{\tau_t^{\Lambda_2}(a_x^{\#}), a^{\#}(f)\right\}\right\| \leq CGG_F\|f\|\|F_0\|e^{-(g_2(\rho(x,\partial\Lambda_1))-v_p|t|)} \tag{254}$$

□

Since the localization properties of f have not been specified more precisely, this bound captures its general location as being outside of Λ_1, meaning that the distance of its support to $x \in \Lambda_1$ is at least $\rho(x, \partial\Lambda_1)$. With this lemma in hand, we now prove a crucial estimate that involves local approximations of Heisenberg evolved elements of $\mathcal{A}_\Theta$. For notational purposes, if $\Lambda \in \mathcal{P}_0(\Theta)$, denote by $\Lambda_r := \{\theta \in \Theta : \ \rho(\theta, \Lambda) \leq r\}$.

Lemma 5.13. *Let $\Lambda \in \mathcal{P}_0(\Theta)$ and let $a_{\underline{\theta}}^{\#} \in \mathcal{M}_{\boldsymbol{\theta}}$. Let $\{\boldsymbol{\theta}_r\}_{r\in\mathbb{N}_0}$ be a sequence of exhaustive subsets of Θ. Under the same assumptions as Lemma 5.12, with weighted F-function $F_a(r)$, for any $r, r' \in \mathbb{N}$ such that $r' > r$, one has the following bounds.*

$$\left\|\left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)\left(\tau_t^{\Lambda}(a_{\underline{\theta}}^{\#})\right)\right\| \leq \min\left\{2, 8\kappa G_\perp\|F_0\||\underline{\theta}|e^{-(ar-v_p|t|)}\sum_{m=1}^{r'-r}\left(\frac{d_{\boldsymbol{\theta}}}{2}+r+m\right)e^{-am}\right\} \tag{255}$$

$$\left\|\left(\mathbb{1} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)\left(\tau_t^{\Lambda}(a_{\underline{\theta}}^{\#})\right)\right\| \leq \min\left\{2, C\|F_0\||\underline{\theta}|\left(\frac{d_{\boldsymbol{\theta}}}{2}+r+c\right)e^{-(ar-v_p|t|)}\right\} \tag{256}$$

where constants $C, c > 0$ arise from evaluating the sum in the first inequality as $r' \to \infty$, and $d_{\boldsymbol{\theta}} := \operatorname{diam}(\boldsymbol{\theta})$ (this notation will be used henceforth).

The constant v_p is as in Proposition 5.12. We now proceed with the proof of this lemma.

proof of Lemma 5.13. Naively for any $A \in \mathcal{A}_\Lambda$, $\left\|\left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)\left(\tau_t^{\Lambda}(A)\right)\right\| \leq 2\|A\|$. To obtain the precise bound, we call on the definition of the conditional expectation map, 211, to write

$\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}}$ and $\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r}}$ acting on an arbitrary k-body operator A.

$$\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}(A) = \frac{1}{4^{N_\Lambda - N_r}} \sum_{\iota \in \mathcal{I}_r} u(\iota)^* A u(\iota) \tag{257}$$

$$\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}}(A) = \frac{1}{4^{N_\Lambda - N_{r'}}} \sum_{\iota \in \mathcal{I}_{r'}} u(\iota)^* A_0 u(\iota) \tag{258}$$

As before, N_Λ and N_r denote the dimensions of the Hilbert spaces $\mathcal{H}_\Lambda$ and $\mathcal{H}_{\boldsymbol{\theta}_r}$ respectively, and $\mathcal{I}_r = \{0,1,2,3\}^{N_\Lambda - N_r}$. The same applies for $\boldsymbol{\theta}_{r'}$. By construction, since $\boldsymbol{\theta}_r \subset \boldsymbol{\theta}_{r'}$, the orthonormal basis partition $\{f_j\}_{j=1}^{N_{r'}} = \{f_j\}_{j=1}^{N_r} \cup \{f_j\}_{j=N_r+1}^{N_{r'}}$ can be chosen to yield sets that span $\mathcal{H}_{\boldsymbol{\theta}_r}$ and $\mathcal{H}_{\boldsymbol{\theta}_{r'}} \cap \mathcal{H}^{\perp}_{\boldsymbol{\theta}_r}$ respectively. This allows us to write $\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}(A)$ as,

$$\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}(A) = \frac{1}{4^{N_\Lambda - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \sum_{\iota_2 \in \mathcal{I}_{r'}} u(\iota_1)^* u(\iota_2)^* A u(\iota_2) u(\iota_1), \tag{259}$$

$$\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}}(A) = \frac{1}{4^{N_\Lambda - N_{r'}}} \frac{1}{4^{N_{r'} - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \sum_{\iota_2 \in \mathcal{I}_{r'}} u(\iota_1)^* u(\iota_1) u(\iota_2)^* A u(\iota_2) \tag{260}$$

$$= \frac{1}{4^{N_\Lambda - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \sum_{\iota_2 \in \mathcal{I}_{r'}} u(\iota_1)^* u(\iota_1) u(\iota_2)^* A u(\iota_2), \tag{261}$$

where $\mathcal{I}_{rr'} := \{0,1,2,3\}^{N_{r'} - N_r}$. The difference of these two conditional expectation maps acting on A is then given by,

$$\left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)(A) = \frac{1}{4^{N_\Lambda - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \sum_{\iota_2 \in \mathcal{I}_{r'}} u(\iota_1)^* \left[u(\iota_1), u(\iota_2)^* A u(\iota_2)\right] \tag{262}$$

$$= \frac{1}{4^{N_\Lambda - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \sum_{\iota_2 \in \mathcal{I}_{r'}} u(\iota_1)^* u^*(\iota_2) \left[u(\iota_1), A\right] u(\iota_2), \tag{263}$$

where the second inequality holds because $[u(\iota_1)^{\#}, u(\iota_2)^{\#}] = 0$ from the orthonormality of the basis $\{f_j\}$. Now, letting $A = \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}})$ and estimating the norm of $\left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)\left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}})\right)$, one has,

$$\left\| \left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r}\right)\left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}})\right) \right\| \leq \frac{4^{N_\Lambda - N_{r'}}}{4^{N_\Lambda - N_r}} \sum_{\iota_1 \in \mathcal{I}_{rr'}} \left\| \left[u(\iota_1), \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}})\right] \right\| \tag{264}$$

$$\leq \sum_{j=N_r+1}^{N_{r'}} \left\| \left[u_j^{\iota(j)}, \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}})\right] \right\| \tag{265}$$

where we use 210 to expand $u(\iota_1)$, and in the second inequality, one specific $\iota \in \mathcal{I}_{rr'}$ is used. From the definition of each $u_{N_r+j}^{\iota(N_r+j)}$, a bound on the last inequality above can be written as,

$$\left\| \left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r} \right) \left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right) \right\| \leq 4 \sum_{j=N_r+1}^{N_{r'}} \left\| \left[a^{\#}\left(f_j\right), \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right] \right\|. \tag{266}$$

Now, the space $\mathcal{H}_{\boldsymbol{\theta}_{r'}} \cap \mathcal{H}^{\perp}_{\boldsymbol{\theta}_r}$ can also be written as,

$$\mathcal{H}_{\boldsymbol{\theta}_{r'}} \cap \mathcal{H}^{\perp}_{\boldsymbol{\theta}_r} = \left(\bigcup_{m=1}^{d_N(\partial\boldsymbol{\theta}_r, \partial\Lambda)} \mathcal{H}_{\boldsymbol{\theta}_{r+m} \setminus \boldsymbol{\theta}_{r+m-1}} \cap \mathcal{H}^{\perp}_{\boldsymbol{\theta}_{r+m-1}} \right),$$

meaning that we divide the region $\Lambda \setminus \boldsymbol{\theta}_r$ into unit-thickness annuli. Thus, the basis $\{f_j\}_{j=N_r+1}^{N_{r'}}$ can also be written as the union of disjoint sets $\{f_{m,j}\}_{j=1}^{N_m}$ which individually span each $\mathcal{H}_{\boldsymbol{\theta}_{r+m} \setminus \boldsymbol{\theta}_{r+m-1}} \cap \mathcal{H}^{\perp}_{\boldsymbol{\theta}_{r+m-1}}$. Thus, proceeding with the estimate, one obtains,

$$\left\| \left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r} \right) \left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right) \right\| \leq 4 \sum_{m=1}^{r'-r} \sum_{j=1}^{N_m} \left\| \left[a^{\#}\left(f_{m,j}\right), \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right] \right\|. \tag{267}$$

Furthermore, each annulus $\boldsymbol{\theta}_{r+m} \setminus \boldsymbol{\theta}_{r+m-1}$ has at most $\kappa\left(\frac{d_{\boldsymbol{\theta}}}{2} + r + m\right)$-many sites. Thus,

$$\left\| \left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r} \right) \left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right) \right\| \leq 4\kappa \sum_{m=1}^{r'-r} \left(\frac{d_{\boldsymbol{\theta}}}{2} + r + m \right) \left\| \left[a^{\#}\left(f_{m,1}\right), \tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right] \right\|, \tag{268}$$

where we now just pick an arbitrary function $f_{m,1}$ from the basis for each annulus. The norm in the inequality above can be estimated using Proposition 5.12, and we may actually improve the precision of the spatial decay by noting that for each m, $f_{m,1}$ is associated to an annulus which is a distance $r+m$ from $\boldsymbol{\theta}$. This insight leads to the following bound.

$$\left\| \left(\mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_{r'}} - \mathbb{E}^{\Lambda}_{\boldsymbol{\theta}_r} \right) \left(\tau_t^{\Lambda}(a^{\#}_{\underline{\theta}}) \right) \right\| \leq 4k\kappa G_{\perp} \|F_0\| |\underline{\theta}| e^{-(ar - v_p|t|)} \sum_{m=1}^{r'-r} \left(\frac{d_{\boldsymbol{\theta}}}{2} + r + m \right) e^{-am} \tag{269}$$

The factor of $|\underline{\theta}|$ appears due to A being a $|\underline{\theta}|$-body operator, which results in the previous commutator splitting into $|\underline{\theta}|$ terms. By our assumption on the growth of g, it follows that the sum above converges to a finite constant, and this in turn becomes 0 as $r \to r'$. □

Having proved these crucial estimates, we are equipped to formally develop quasi-adiabatic evolution for a family of parametrized Hamiltonians, which is the focus of the next section.

5.3. **Hastings Generator and Spectral Flow.** The construction of the unitary operators that implement the spectral flow automorphism begins with the introduction of a function $w_\epsilon \in L^1(\mathbb{R})$ parametrized by $\epsilon > 0$. Our particular choice of function, and its existence in $L^1(\mathbb{R})$, is presented in the following proposition.

Proposition 5.14. *Let $\epsilon > 0$ and define a positive sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n = \frac{a_1}{n\ln^2(n)}$. Here a_1 is chosen in such a way that $\sum_{n=1}^{\infty} a_n = \frac{\epsilon}{2}$. Then,*

$$w_\epsilon(t) = c_\epsilon \prod_{n=1}^{\infty} \left(\frac{\sin(a_n t)}{a_n t} \right)^2 \tag{270}$$

defines an even function in $L^1(\mathbb{R})$. We choose c_ϵ such that $\|w_\epsilon\|_1 = 1$.

This has been proved in a number of prior works, [11],[74], and in terms of w_ϵ, we now define the filter function W_ϵ.

$$W_\epsilon(t) := \begin{cases} \int_t^\infty d\xi\, w_\epsilon(\xi) & \text{if } t \geq 0 \\ -\int_{-\infty}^t d\xi\, w_\epsilon(\xi) & \text{if } t < 0 \end{cases} \tag{271}$$

From this definition, it is clear that W_ϵ is an odd function that is also in $L^1(\mathbb{R})$. The following two lemmas help to establish the decay properties of this function, which have also been proved in the same cited works.

Lemma 5.15 (Lemma 2.5 from [11])**.** *For $a > 0$ define*

$$u_a(\eta) = e^{-a\frac{\eta}{\ln^2 \eta}}, \tag{272}$$

on the domain $\eta > 1$. For all integers $k \geq 0$ and for all $t \geq e^4$ such that also

$$a\frac{t}{\ln^2 t} \geq 2k + 2,$$

we have the bound

$$\int_t^\infty d\eta\, \eta^k u_a(\eta) \leq \frac{2k+3}{a} t^{2k+2} u_a(t) \tag{273}$$

The function $u_a(\eta)$ appears in bounds on W_ϵ and its integral, as shown in the next lemma.

Lemma 5.16 (Lemma 2.6 from [11])**.** *Let $\epsilon > 0$ and $W_\epsilon(t)$ as in 271. Then,*

(i) $|W_\epsilon(t)|$ is continuous and monotone decreasing for $t \geq 0$. In particular,

$$\|W_\epsilon\|_\infty = W_\epsilon(0) = \frac{1}{2}$$

(ii) $|W_\epsilon(t)| \leq G^{(W)}(\epsilon|t|)$, where for $\eta \geq 0$,

$$G^{(W)}(\eta) = \begin{cases} \frac{1}{2} & 0 \leq \eta \leq \eta^* \\ 35e^2\eta^4 u_{2/7}(\eta) & \eta > \eta^* \end{cases}.$$

Additionally, η^* is the largest real solution of $35e^2\eta^4 u_{2/7}(\eta) = \frac{1}{2}$.

(iii) There is a constant K_w such that

$$\|W_\epsilon\|_1 \leq \frac{K_w}{\epsilon}$$

(iv) For $t > 0$, let

$$J_\epsilon(t) = \int_t^\infty d\xi\, W_\epsilon(\xi). \tag{274}$$

Then, $|J_\epsilon(t)| \leq G^{(I)}(\epsilon|t|)$, where for $\eta > 0$,

$$G^{(I)}(\eta) = \frac{1}{\epsilon} \begin{cases} \frac{K_w}{2} & 0 \leq \eta \leq \eta^* \\ 130e^2\eta^{10} u_{2/7}(\eta) & \eta > \eta^* \end{cases},$$

where K_w is as in (iii) and a $\eta^* > 0$.

These properties will be essential in establishing the spectral flow as a quasi-local map. We now introduce a general family of finite volume, parametrized Hamiltonians that will give rise to a notion of quasi-adiabatic evolution (spectral flow).

Generally, under assumptions that will be made precise in this section, the framework we introduce is valid for two forms of perturbation of an initial Hamiltonian as in 215: 1) interaction(s) whose strength is tuned via a perturbation parameter $s \in [0,1]$ and 2) interactions that are transformed via an automorphism $\mathcal{R}_\varphi : \mathcal{A}(L^2(\mathbb{R}^2)) \to \mathcal{A}(L^2(\mathbb{R}^2))$, depending on one or finitely many parameters $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in [0, 2\pi]^n$, that can be implemented via a unitary map $U_\varphi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. Explicit forms of these models are as in the subsequent equations. Let $\Lambda \in \mathcal{P}_0(\Theta)$ and $\Lambda_p \subset \Lambda$ be a perturbation region that can be as large as Λ itself. Denote by H_Λ^0 an initial Hamiltonian as in 215. Then we consider the families of models given by,

$$\begin{aligned} H_\Lambda(s) &= H_\Lambda^0 + V_1(s) \\ &= \sum_{\underline{\theta} \in \Lambda^+} \phi(\underline{\theta}) a_{\underline{\theta}} + \sum_{\substack{\underline{\theta} \in \Lambda^+ : \\ \boldsymbol{\theta} \cap \Lambda_p \neq \emptyset}} h(\underline{\theta}, s) a_{\underline{\theta}}, \end{aligned} \tag{275}$$

$$\begin{aligned} H_\Lambda(\varphi) &= H_\Lambda^0 + V_2(\varphi) \\ &= \sum_{\underline{\theta} \in \Lambda^+} \phi(\underline{\theta}) a_{\underline{\theta}} + \sum_{\substack{\underline{\theta} \in \Lambda^+ : \\ \boldsymbol{\theta} \cap \Lambda_p \neq \emptyset}} \phi(\underline{\theta}) \left(\mathcal{R}_\varphi - \mathbb{1}\right) (a_{\underline{\theta}}) \, . \end{aligned} \tag{276}$$

We now state the necessary assumptions on these perturbations that guarantee we can construct a quasi-local evolution in s or φ space from these models.

Assumption 5.17. *Let $\Lambda \in \mathcal{P}_0(\Theta)$ and $H_\Lambda(s)$, $H_\Lambda(\varphi)$ Hamiltonians as in 275 and 276 respectively, with $s \in [0,1]$ and $\varphi \in [0,2\pi]^n$ for a positive finite n. We then assume the following.*

i) $\forall\, \underline{\theta} \in \Theta^+,\ h(\underline{\theta}, \cdot) \in C^1[0,1]$.

ii) $\forall\, \underline{\theta} \in \Theta^+,\ h(\underline{\theta}, 0) = 0$.

iii) $\exists\, C_h > 0$ *and an* $F-$*function* F *such that for any* $x, y \in \Theta$,

$$\sup_{s\in[0,1]} \sum_{\substack{\underline{\theta}\in\Theta^+: \\ x,y\in\boldsymbol{\theta}}} |h(\underline{\theta}, s)| \leq C_h F(\rho(x,y)).$$

iv) $U_{(0,\ldots,0)} = U_{(2\pi,\ldots,2\pi)} = \mathbb{1}$.

v) $\forall\, \theta \in \Theta,\ \varphi \in [0,2\pi]^n,\ \mathcal{R}_\varphi(a_\theta^\#) = a^\#(U_\varphi \chi_\theta)$.

vi) $\exists\, C_1, C_2, c_1, c_2 > 0$ *such that* $\forall\, \theta, \theta' \in \Theta,\ \varphi \in [0,2\pi]^n,\ j \leq n$,

$$\|\{a_\theta, \mathcal{R}_\varphi(a_{\theta'}^*)\}\| \leq C_1 e^{-c_1\rho(\theta,\theta')}, \left\|\left\{a_\theta, \frac{\partial}{\partial\varphi_j}\mathcal{R}_\varphi(a_{\theta'}^*)\right\}\right\| \leq C_2 e^{-c_2\rho(\theta,\theta')}.$$

These conditions ultimately ensure that the dynamics $\tau_t^{\Lambda,s/\varphi}$, generated by $H_\Lambda(s)$ and $H_\Lambda(\varphi)$ respectively, satisfy Lieb-Robinson bounds as in Theorem 5.10 and Lemma 5.12. Furthermore, the decay of the interactions present in $\frac{\partial}{\partial s}H_\Lambda(s)$ and $\frac{\partial}{\partial\varphi_j}H_\Lambda(\varphi)$ is also ensured by these assumptions. In the case of the twist perturbation 276, it is important to note that $\frac{\partial}{\partial\varphi_j}U_\varphi\chi_\theta$ is not necessarily contained in $\mathcal{H} = \overline{\text{span}\,\{\chi_\theta :\ \theta \in \Theta\}}$. Thus in this setting, we introduce the enlarged Hilbert space

$$\mathcal{H}_\varphi := \overline{\text{span}\left\{\chi_\theta, U_\varphi\chi_\theta, \frac{\partial}{\partial\varphi_j}U_\varphi\chi_\theta :\ \theta \in \Theta,\ \varphi \in [0,2\pi]^n, j = 1,...,n\right\}}$$

and CAR algebra $\mathcal{A}(\mathcal{H}_\varphi)$.

Also note that both of the perturbations we defined can be included simultaneously to obtain a more general family of perturbed models, and under the stated assumptions, the forthcoming analysis of this section would still hold. To avoid cluttered notation, we introduce a general set of functions $\{\psi_\theta\}_{\theta\in\Theta}$, which correspond to either $\{\chi_\theta\}_{\theta\in\Theta}$ or the extended set $\{\chi_\theta, U_\varphi\chi_\theta, \frac{\partial}{\partial\varphi_j}U_\varphi\chi_\theta\}_{\substack{\theta\in\Theta,\\ j\leq n}}$, and fermionic creation/annihilation operators $b_\theta^\# := a^\#(\psi_\theta)$. The functions ψ_θ may of course depend on φ, but we suppress this for compact notation. Additionally, we use the notation $b_{\underline{\theta}}$ to denote the degree-2k monomial in the $b_\theta^\#$ operators analogous to $a_{\underline{\theta}}$. With respect to these monomials, let λ denote the perturbation parameter(s), either s or φ, and K the compact domain, either $[0,1]$ or $[0,2\pi]^n$, in which λ belongs.

We study the general Hamiltonians $H_\Lambda(\lambda)$ and $\frac{d}{d\lambda}H_\Lambda(\lambda)$, which are both of the following form.

$$H_\Lambda(\lambda) = \sum_{\underline{\theta}\in\Lambda^+} \phi(\underline{\theta})a_{\underline{\theta}} + \sum_{\substack{\underline{\theta}\in\Lambda^+:\\ \boldsymbol{\theta}\cap\Lambda_p\neq\emptyset}} h^{(1)}(\underline{\theta},\lambda)b_{\underline{\theta}} \tag{277}$$

$$\frac{\partial}{\partial\lambda}H_\Lambda(\lambda) = \sum_{\substack{\underline{\theta}\in\Lambda^+:\\ \boldsymbol{\theta}\cap\Lambda_p\neq\emptyset}} h^{(2)}(\underline{\theta},\lambda)b_{\underline{\theta}} \tag{278}$$

We will also require the following assumption on the function $h^{(2)}$ as written.

Assumption 5.18. *Assume there exists a weighted function $F_a(r)$ such that for any $x, y \in \Theta$, the following bound holds.*

$$\sup_{\lambda\in K} \sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}}} d_{\boldsymbol{\theta}}|\underline{\theta}|\left|h^{(2)}(\underline{\theta},\lambda)\right| \leq F_a(\rho(x,y)) \tag{279}$$

Before defining the generator of quasi-adiabatic evolution in λ, we state a necessary assumption on the spectra of the Hamiltonians $H_\Lambda(\lambda)$.

Assumption 5.19. *For all $\lambda \in K$, and $\Lambda \in \mathcal{P}_0(\Theta)$, assume that spec$(H_\Lambda(\lambda)) = \Sigma_1(\lambda) \cup \Sigma_2(\lambda)$. Additionally, we assume that there is a uniform gap between Σ_1 and Σ_2:*

$$\inf_{\substack{\lambda\in K\\ E_1\in\Sigma_1(\lambda),\, E_2\in\Sigma_2(\lambda)}} |E_2 - E_1| = \epsilon.$$

This uniform gap will be used to parametrize the filter function defined in equation 271. We now define the Hastings generator.

$$D_\Lambda(\lambda) = \int_{\mathbb{R}} dt\, W_\epsilon(t)\tau_t^{\Lambda,\lambda}\left(\frac{\partial}{\partial\lambda}H_\Lambda(\lambda)\right) \tag{280}$$

$$= \sum_{\substack{\underline{\theta}\in\Lambda^+:\\ \boldsymbol{\theta}\cap\Lambda_p\neq\emptyset}} h^{(2)}(\underline{\theta},\lambda) \int_{\mathbb{R}} dt\, W_\epsilon(t)\tau_t^{\Lambda,\lambda}\left(b_{\underline{\theta}}\right) \tag{281}$$

For brevity of notation, we drop the superscript (2) on the potential function $h_k^{(2)}$ of $\frac{\partial}{\partial\lambda}H_\Lambda(\lambda)$. The goal of this section is to prove that $D_\Lambda(\lambda)$ is of the form of a Hamiltonian as in 215, which would result in the dynamics generated by $D_\Lambda(\lambda)$ satisfying a Lieb-Robinson bound as in Theorem 5.10. Towards this goal, we begin by writing $D_\Lambda(\lambda)$ as a sum of local terms with the help of the conditional expectation map $\mathbb{E}^\Lambda_{(\cdot)}$. Note that in constructing the map $\mathbb{E}^\Lambda_{(\cdot)}$ with the functions ψ_θ instead of χ_θ, its properties that we have established, namely Lemma 5.13, still hold.

Denote the CAR algebra generated by $\{b_\theta, b_\theta^* : \theta \in \Theta\}$ by $\mathcal{B}_\Theta$ and adopt the same notation for finite volume subalgebras. Additionally, let $\mathcal{M}^b_\Lambda := \{\mathbb{1}, b^\#_{\theta_1}...b^\#_{\theta_{2k}} : k \in \mathbb{N},\ \theta_i \in \Lambda\}$ be the k-body monomials in the $b^\#$ operators. For an arbitrary monomial $b_{\underline{\theta}} \in \mathcal{M}^b_\Lambda$, let Δ^0_Λ and Δ^r_Λ be maps from $\mathcal{B}_\Theta \times K \to \mathcal{B}_\Theta$ that act as,

$$\Delta^0_\Lambda(b_{\underline{\theta}}, \lambda) = \int_{\mathbb{R}} dt\, W_\epsilon(t) \mathbb{E}^\Lambda_{\boldsymbol{\theta}} \left(\tau_t^{\Lambda,\lambda}(b_{\underline{\theta}})\right) \tag{282}$$

$$\Delta^r_\Lambda(b_{\underline{\theta}}, \lambda) = \int_{\mathbb{R}} dt\, W_\epsilon(t) \left(\mathbb{E}^\Lambda_{\boldsymbol{\theta}_r} - \mathbb{E}^\Lambda_{\boldsymbol{\theta}_{r-1}}\right) \left(\tau_t^{\Lambda,\lambda}(b_{\underline{\theta}})\right), \ r \geq 1. \tag{283}$$

Due to its recurrence, let $\mathcal{E}^\Lambda_{\boldsymbol{\theta}_r} := \mathbb{E}^\Lambda_{\boldsymbol{\theta}_r} - \mathbb{E}^\Lambda_{\boldsymbol{\theta}_{r-1}}$. As a preliminary estimate, we bound the norm of $\Delta^r_\Lambda(A)$.

Lemma 5.20. *Let $\Lambda \in \mathcal{P}_0(\Theta)$. Under the same conditions as Lemma 5.13, for any $b^\#_{\underline{\theta}} \in \mathcal{M}^b_\Lambda$, the following bound holds.*

$$\left\|\Delta^r_\Lambda(b^\#_{\underline{\theta}}, \lambda)\right\| \leq \min\left\{2\|W_\epsilon\|_1,\ \mathcal{G}_\epsilon(\underline{\theta}, r)\right\} \tag{284}$$

where $\mathcal{G}_\epsilon(\underline{\theta}, r) = \frac{C\|F_0\|}{v_p}|\underline{\theta}|\left(\frac{d_{\boldsymbol{\theta}}}{2} + r + 1\right) e^{-\delta a r} + 4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right)$, with $C, \delta > 0$ and $G^{(I)}$ as in Lemma 5.16.

Proof. A simple estimate of course gives that $\left\|\Delta^r_\Lambda(b^\#_{\underline{\theta}}, \lambda)\right\| \leq 2\|W_\epsilon\|_1$ for $r \geq 0$. To proceed with the precise bound, we split the integral into two pieces with a cutoff $T > 0$.

$$\left\|\Delta^r_\Lambda(b^\#_{\underline{\theta}}, \lambda)\right\| \leq \int_{|t|<T} dt\, |W_\epsilon(t)| \left\|\mathcal{E}^\Lambda_{\boldsymbol{\theta}_r}\left(\tau_t^{\Lambda,\lambda}(b^\#_{\underline{\theta}})\right)\right\| + \int_{|t|>T} dt\, |W_\epsilon(t)| \left\|\mathcal{E}^\Lambda_{\boldsymbol{\theta}_r}\left(\tau_t^{\Lambda,\lambda}(b^\#_{\underline{\theta}})\right)\right\| \tag{285}$$

Starting with the first finite time integral, make use of Lemma 5.13 and Proposition 5.16.i to obtain,

$$\int_{|t|<T} dt\, |W_\epsilon(t)| \left\| \mathcal{E}^{\Lambda}_{\boldsymbol{\theta}_r} \left(\tau_t^{\Lambda,\lambda}(b^{\#}_{\underline{\theta}}) \right) \right\| \leq 4\kappa G_\perp \|F_0\| |\underline{\theta}| e^{-ar} \left(\frac{d_{\boldsymbol{\theta}}}{2} + r + 1 \right) \int_{-T}^{T} dt\, e^{v_p|t|} \tag{286}$$

$$\leq \frac{8\kappa G_\perp}{v_p} \|F_0\| |\underline{\theta}| \left(\frac{d_{\boldsymbol{\theta}}}{2} + r + 1 \right) e^{-(ar - v_p T)} \tag{287}$$

Now, letting $0 < \delta_1 < 1$ and $\delta = 1 - \delta_1$, one may choose $T = \frac{\delta_1 ar}{v_p}$, after which the bound becomes,

$$\int_{|t|<T} dt\, |W_\epsilon(t)| \left\| \mathcal{E}^{\Lambda}_{\boldsymbol{\theta}_r} \left(\tau_t^{\Lambda,\lambda}(b^{\#}_{\underline{\theta}}) \right) \right\| \leq \frac{8\kappa G_\perp}{v_p} \|F_0\| |\underline{\theta}| \left(\frac{d_{\boldsymbol{\theta}}}{2} + r + 1 \right) e^{-\delta ar}. \tag{288}$$

Moreover, with this choice of T, the integral over $|t| > T$ is estimated using Proposition 5.16.

$$\int_{|t|>T} dt\, |W_\epsilon(t)| \left\| \mathcal{E}^{\Lambda}_{\boldsymbol{\theta}_r} \left(\tau_t^{\Lambda,\lambda}(b^{\#}_{\underline{\theta}}) \right) \right\| \leq 4 \int_T^\infty dt\, |W_\epsilon(t)| \tag{289}$$

$$\leq 4G^{(I)} \left(\epsilon(1-\delta)ar/v_p \right) \tag{290}$$

Putting the two integral estimates together completes the proof. □

This lemma is crucial as it establishes decay in r of each interaction supported in an r-fattening of $\boldsymbol{\theta}$, and coupled with the already assumed decay in $h(\underline{\theta}, \lambda)$, we have the ingredients to show that $D_\Lambda(\lambda)$ is in fact an operator as in equation 215. First, we rewrite the Hastings generator by re-organizing the sum over lattice tuples.

$$D_\Lambda(s) = \sum_{\substack{\underline{\theta} \in \Lambda^+: \\ \boldsymbol{\theta} \cap \Lambda^p \neq \emptyset}} h(\underline{\theta}, \lambda) \sum_{r=0}^{\infty} \Delta^r_\Lambda(b_{\underline{\theta}}) \tag{291}$$

$$= \sum_{\underline{\mu} \in \Lambda^+} \sum_{r=0}^{\infty} \sum_{\substack{\underline{\theta} \in \Lambda^+: \\ \boldsymbol{\theta} \cap \Lambda_p \neq \emptyset, \\ \boldsymbol{\theta}_r = \boldsymbol{\mu}}} h(\underline{\theta}, \lambda) \Delta^r_\Lambda \left(b_{\underline{\theta}}, \lambda \right) \tag{292}$$

$$= \sum_{\underline{\mu} \in \Lambda^+} \beta_\Lambda(\underline{\mu}, \lambda) b_{\underline{\mu}} \tag{293}$$

This new sum is now entirely over local elements of $\mathcal{B}_\Theta$, namely monomials in $\mathcal{M}^b_\Lambda$. Our goal is now to prove that this particular interaction satisfies Assumption 5.9 towards proving a Lieb-Robinson bound for the dynamics generated by $D_\Lambda(\lambda)$, i.e. the finite volume spectral flow automorphism.

Lemma 5.21. *Let $\Lambda \in \mathcal{P}_0(\Theta)$. Under the same assumptions as Lemmas 5.13 and 5.20, there exists an F-function F^β such that for any $x, y \in \Lambda$, the following bound holds.*

$$\sup_{\lambda\in K} \sum_{\substack{\underline{\mu}\in\Lambda^+: \\ x,y\in\boldsymbol{\mu}}} \left|\beta_\Lambda(\underline{\mu},\lambda)\right| \leq F^\beta\left(\rho(x,y)\right) \tag{294}$$

To complete the proof of this lemma, we need to identity the interaction terms whose local supports contain $\{x, y\}$. There are two distinct types of interaction that could achieve this.

Case 1: Those interactions with local supports that are r-fattenings $\boldsymbol{\theta}_r$ with $x, y \in \boldsymbol{\theta}$, i.e. terms of the form $h(\underline{\theta},\lambda)\Delta^r_\Lambda\left(b_{\underline{\theta}},\lambda\right)$ with $x, y \in \boldsymbol{\theta}$.

Case 2: Those interactions of the form $h(\underline{\theta},\lambda)\Delta^r_\Lambda\left(b_{\underline{\theta}},\lambda\right)$ where $x, y \in B_r(\boldsymbol{\theta})$ and $\{x, y\} \cap \Lambda\backslash\boldsymbol{\theta} \neq \emptyset$.

To prove the claim of this lemma, we will show that by summing over the norms of these two forms of interaction, there exists an F-function F^β such that equation 294 is satisfied.

proof of Lemma 5.21. We proceed by proving 294 for each of the two types of interaction.

Case 1

The precise bound we seek to prove here is one on the sum,

$$\sup_{\lambda\in K} \sum_{r=0}^{\infty} \sum_{\substack{\underline{\theta}\in\Lambda^+: \\ x,y\in\boldsymbol{\theta}}} \left\|h(\underline{\theta},\lambda)\Delta^r_\Lambda\left(b_{\underline{\theta}},\lambda\right)\right\|. \tag{295}$$

To begin estimating a bound, we refer to the result of Lemma 5.20.

(296)

$$\sup_{\lambda\in K}\sum_{r=0}^{\infty}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}}}\|h(\underline{\theta},\lambda)\Delta_{\Lambda}^{r}\left(b_{\underline{\theta}},\lambda\right)\| \leq$$

$$\sum_{r=0}^{\infty}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}}}|h(\underline{\theta},\lambda)|\left[\frac{C\|F_0\|}{v_p}|\underline{\theta}|\left(\frac{d_{\boldsymbol{\theta}}}{2}+r+1\right)e^{-\delta ar}+4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right)\right]$$

$$\leq\frac{C\|F_0\|}{v_p}\sum_{r=0}^{\infty}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}}}\left(\frac{d_{\boldsymbol{\theta}}}{2}+2\right)|\underline{\theta}||h(\underline{\theta},\lambda)|\left(re^{-\delta ar}+4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right)\right) \tag{297}$$

$$\leq\frac{C\|F_0\|}{v_p}F_a(\rho(x,y))\sum_{r=0}^{\infty}\mathcal{J}_{\epsilon}(r) \tag{298}$$

In the second to last inequality, we separate the summand into pieces depending only on $\underline{\theta}$ and r respectively. Then to obtain the final bound, we introduce the function $\mathcal{J}_{\epsilon}(r):=re^{-\delta ar}+4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right)$ and use Assumption 5.18. Furthermore, from Lemma 5.16, it follows that $\mathcal{J}_{\epsilon}$ is summable. Thus, the interactions of the Case 1 category satisfy the claim of this lemma.

Case 2

We now consider those interactions generated by tuples supported on the r-fattenings $\boldsymbol{\theta}_r$ containing x and y. In particular, we require $\{x,y\}\cap\boldsymbol{\theta}_r\backslash\boldsymbol{\theta}\neq\emptyset$, which ensures that the interaction does not fall into the Case 1 category. Towards this end, we seek to bound the quantity,

$$S_2:=\sup_{\lambda\in K}\sum_{m=1}^{\infty}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}_r}}|h(\underline{\theta},\lambda)|\mathrm{Ind}\left[\{\theta_1,\theta_2\}\cap(\boldsymbol{\theta})_{m-1}^{c}\neq\emptyset\right]\sum_{r\geq m}\|\Delta_{\Lambda}^{r}(b_{\underline{\theta}},\lambda)\|, \tag{299}$$

where the index m that is summed over represents the minimal fattening of $\boldsymbol{\theta}$ required to include x and y, and the superscript c denotes complement set. We then use the overcounting estimate,

$$\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ x,y\in\boldsymbol{\theta}_m}} \operatorname{Ind}\left[\{x,y\}\cap(\boldsymbol{\theta})^c_{m-1}\neq\emptyset\right] \leq \sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ y_1,y_2\in\boldsymbol{\theta}}} . \tag{300}$$

We then continue the overall estimate on 299.

$$S_2 \leq \sup_{\lambda\in K}\sum_{m=1}^{\infty}\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)}\sum_{\substack{\underline{\theta}\in\Lambda^+:\\ y_1,y_2\in\boldsymbol{\theta}}} |h(\underline{\theta},\lambda)|\sum_{r\geq m}\|\Delta^r_\Lambda(b_{\underline{\theta}},\lambda)\| \tag{301}$$

$$\leq \sum_{m=1}^{\infty}\hat{\mathcal{J}}_\epsilon(m)\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)} F_a(\rho(y_1,y_2)) \tag{302}$$

In the last inequality, we estimate as we did for the Case 1 interactions and introduce the function $\hat{\mathcal{J}}_\epsilon(m) := \sum_{r\geq m}\mathcal{J}_\epsilon(r)$. Note the following estimate due to the triangle inequality.

$$\rho(x,y)\leq\rho(y_1,y_2)+2m$$

Then, setting $m_* = m_*(\delta_1)$ to be the largest integer smaller than $(1-\delta_1)\rho(x,y)/2$. For $m\leq m_*$, one has,

$$\rho(x,y)\leq\rho(y_1,y_2)+2m_*$$

$$\delta_1\rho(x,y)\leq\rho(y_1,y_2).$$

With this bound, we split the sum over m in 302 into a first sum up to m_* and a second series consisting of the remaining terms. Beginning with the former, the estimate proceeds as follows.

$$\sum_{m=1}^{m_*}\hat{\mathcal{J}}_\epsilon(m)\sum_{y_1\in B_m(x)}\sum_{y_2\in B_m(y)} F_a(\rho(y_1,y_2)) \leq \hat{\mathcal{J}}_\epsilon(1)F_a(\delta\rho(x,y))\sum_{m=1}^{m_*}|B_m(x)||B_m(y)| \tag{303}$$

$$\leq \kappa^2\hat{\mathcal{J}}_\epsilon(1)F_a(\delta\rho(x,y))\sum_{m=1}^{m_*}m^4 \tag{304}$$

Noting that $\hat{\mathcal{J}}_\epsilon$ is a decreasing function and using lattice regularity, we arrive at the second inequality. Using the latter, we can also begin to bound the remaining interactions corresponding to $m > m_*$.

$$\sum_{m\geq m_*+1} \hat{\mathcal{J}}_\epsilon(m) \sum_{y_1\in B_m(x)} \sum_{y_2\in B_m(y)} F_a(\rho(y_1,y_2)) \leq \|F_a\| \sum_{m\geq m_*+1} |B_m(x)| \hat{\mathcal{J}}_\epsilon(m) \tag{305}$$

$$\leq \kappa \|F_a\| \sum_{m\geq m_*+1} m^2 \hat{\mathcal{J}}_\epsilon(m) \tag{306}$$

$$= \kappa \|F_a\| \sum_{m\geq m_*+1} m^2 \tag{307}$$

$$\times \sum_{r\geq m} \left(re^{-\delta ar} + 4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right)\right)$$

Starting with the term with exponential decay in r, one then obtains,

$$\sum_{m\geq m_*+1} m^2 \sum_{r\geq m} re^{-\delta ar} = \sum_{n\geq 0} (n+m_*+1)e^{-\delta an} \sum_{m\geq m_*+1} m^2 e^{-\delta am}, \tag{308}$$

which for large m_* will be dominated by decay of the form $m_*^2 e^{-\delta a m_*}$. Lastly, we bound the portion of the series involving $G^{(I)}$ using Lemma 5.16.

(309)

$$\sum_{m\geq m_*+1} m^2 \sum_{r\geq m} 4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right) \leq \sum_{m\geq m_*+1} m^2 \sum_{r\geq m} \frac{C}{\epsilon} \left(\epsilon(1-\delta)r/v_p\right)^{10} u_{2/7}\left(\epsilon(1-\delta)r/v_p\right)$$

The constant C is used to encompass constant prefactors, and in the subsequent bounds, it will absorb any numerical factors. Then, by Lemma 5.15,

(310)

$$\sum_{m\geq m_*+1} m^2 \sum_{r\geq m} 4G^{(I)}\left(\epsilon(1-\delta)ar/v_p\right) \leq \frac{Cv_p}{\epsilon^2(1-\delta)} \sum_{m\geq m_*+1} m^2 \int_{\epsilon(1-\delta)m/v_p}^{\infty} dR\, R^{10} u_{2/7}(R)$$

$$\leq \frac{Cv_p}{\epsilon^2(1-\delta)} \sum_{m\geq m_*+1} m^2 \left(\frac{\epsilon(1-\delta)m}{v_p}\right)^{22} \times \tag{311}$$

$$u_{2/7}\left(\frac{\epsilon(1-\delta)m}{v_p}\right)$$

$$\leq \frac{Cv_p^2}{\epsilon^3(1-\delta)^2} \int_{\epsilon(1-\delta)(m_*+1)/v_p}^{\infty} dR \left(\frac{v_pR}{\epsilon(1-\delta)}\right)^2 R^{22} u_{2/7}(R) \tag{312}$$

$$\leq \frac{Cv_p^4}{\epsilon^5(1-\delta)^4} \left(\frac{\epsilon(1-\delta)(m_*+1)}{v_p}\right)^{50} \times \tag{313}$$

$$u_{2/7}\left(\frac{\epsilon(1-\delta)(m_*+1)}{v_p}\right)$$

Consolidating our estimates for the case 1 and case 2 interactions, we have the following estimate, where any C_i or c_i is a positive constant.

$$\sup_{\lambda\in K}\sum_{\substack{\underline{\mu}\in\Lambda^+:\\ x,y\in\boldsymbol{\mu}}}\left|\beta_\Lambda(\underline{\mu},\lambda)\right| \leq C_1F_a(\rho(x,y)) + C_2(m_*+c_1)\sum_{m\geq m_*+1} m^2e^{-\delta am}$$
$$+ C_3\left(\frac{\epsilon(1-\delta)(m_*+1)}{v_p}\right)^{c_2} u_{2/7}\left(\frac{\epsilon(1-\delta)(m_*+1)}{v_p}\right) \tag{314}$$

This bound is of course similar to the analogous step of the proof in the work of Bachmann et al. in [11]; thus, it holds that the third term decays the slowest in m_*, and consequently $\rho(x,y)$ by the definition of m_*. Therefore, there exists an F-function,

$$F^\beta(r) = u_\nu\left(c\frac{\epsilon}{v_p}r\right)F\left(c\frac{\epsilon}{v_p}r\right);\ 0<\nu<\frac{2}{7}, \tag{315}$$

such that,

$$\sup_{\lambda\in K}\sum_{\substack{\underline{\mu}\in\Lambda^+:\\ x,y\in\boldsymbol{\mu}}}\left|\beta_\Lambda(\underline{\mu},\lambda)\right| \leq F^\beta(\rho(x,y)). \tag{316}$$

Importantly, the base function F in 315 must satisfy the following condition to ensure suitable decay of F^β.

$$\sup_{r\geq 1}\frac{u_a(r)}{F(r)} < \infty,\ 0<a<\frac{2}{7}$$

One example of a base function that satisfies this condition is $F(r) = \frac{1}{(1+r)^\zeta}$ with $\zeta > 2$. The proof of the lemma is now complete. □

5.3.1. *Finite System Ground States.* As emphasized previously, for a finite $\Lambda\subset\Theta$, $\mathcal{H}_\Lambda$ is finite-dimensional and has an orthonormal basis with N_Λ elements. Thus, the algebra $\mathcal{A}_\Lambda$ is isomorphic to the matrix algebra $\mathcal{M}_{2^{N_\Lambda}}$. Denote by $[A]$ the associated finite dimensional matrix for an element $A\in\mathcal{A}_\Lambda$. In this finite dimensional setting, $H_\Lambda(\lambda)$ is isomorphic to a $2^{N_\Lambda}\times 2^{N_\Lambda}$ matrix $[H_\Lambda(\lambda)]$, which can be diagonalized. Then, let $P_\Lambda(\lambda)$ denote the orthogonal projection onto the lowest eigenvalue eigenspace of $H_\Lambda(\lambda)$, separated from the rest of the spectrum by at least $\epsilon > 0$. Additionally, let

$$\mathbf{\Omega} = \{\omega_{\Lambda,\lambda} : \mathcal{A}_\Lambda \to [0,\infty) \mid \omega_{\Lambda,\lambda}(A^*[H_\Lambda(\lambda), A]) \geq 0 \, \forall \, A \in \mathcal{A}_\Lambda\}$$

denote the set of normalized ground states, as defined in terms of the derivation governed by $H_\Lambda(\lambda)$. For these states, $\omega_{\Lambda,\lambda}(P_{\Lambda,\lambda}) = 1$. Then, if $A \in \mathcal{A}_\Lambda$, one has

$$\omega_{\Lambda,\lambda}(A) = \mathrm{Tr}([P_\Lambda(\lambda)][A]). \tag{317}$$

As a consequence of the properties we proved regarding $D_\Lambda(\lambda)$ and $\alpha_{\Lambda,\lambda}$, the following equalities necessarily hold, as demonstrated in works such as [11] and [74].

$$\omega_{\Lambda,\lambda}(A) = \mathrm{Tr}([U_{\Lambda,\lambda}][P_\Lambda(0)][U^*_{\Lambda,\lambda}][A]) = \mathrm{Tr}([P_\Lambda(0)][U^*_{\Lambda,\lambda}][A][U_{\Lambda,\lambda}]) = (\omega_{\Lambda,0} \circ \alpha_{\Lambda,\lambda})(A) \tag{318}$$

This relation establishes the automorphic equivalence of $\omega_{\Lambda,0}$ and $\omega_{\Lambda,\lambda}$.

5.3.2. *Thermodynamic Limit.* Letting α^Λ_λ denote the dynamics generated by $D_\Lambda(\lambda)$, Lemma 5.21 establishes that the local interactions making up the Hastings generator satisfy the conditions of Assumption 5.9. Therefore, α^Λ_λ satisfies a Lieb-Robinson bound as in Theorem 5.10. To complete this section, we use this fact to prove that $\lim_{\Lambda\uparrow\Theta} \alpha^\Lambda_\lambda$ exists and forms a strongly continuous 1-parameter group of automorphisms on $\mathcal{B}_\Theta$. This is encapsulated by a theorem, whose proof concludes this section.

Theorem 5.22. *Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be an increasing and exhaustive sequence of Θ, meaning $\Lambda_m \subset \Lambda_n$ for all $m \leq n$ and $\bigcup_{n\in\mathbb{N}} \Lambda_n = \Theta$. For $\lambda \in K$, denote by $\alpha^{\Lambda_n}_\lambda : \mathcal{B}_\Theta \to \mathcal{B}_\Theta$ the dynamics generated by $D_\Lambda(\lambda)$, as defined in equation 280. Under the same assumptions as Lemma 5.21, for all $\theta \in \Theta$,*

$$\alpha_\lambda(b^\#_\theta) := \lim_{n\to\infty} \alpha^{\Lambda_n}_\lambda(b^\#_\theta) \tag{319}$$

exists in $\mathcal{B}_\Theta$ and $\{\alpha_\lambda\}_{\lambda\in K}$ forms a strongly continuous 1-parameter group of automorphisms of the algebra.

proof of Theorem 5.22. We start by obtaining a bound concerning the difference of two dynamics, namely the finite volume dynamics $\tau_t^{\Lambda,\lambda}$ associated with two different finite subsets.

Lemma 5.23. *Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be an increasing and absorbing sequence of subsets in Θ. Let $\tau_t^{\Lambda_n,\lambda}$ be the dynamics generated by $H_{\Lambda_n}(\lambda)$, satisfying Assumptions 5.9 with weighted F-function F_a. For any $\theta \in \Theta$, $\lambda \in K$, and finite $t \in \mathbb{R}$, the sequence $\{\tau_t^{\Lambda_n,\lambda}(b_\theta^\#)\}_{n\in\mathbb{N}} \subset \mathcal{B}_\Theta$ is Cauchy.*

Proof. For brevity, let $\phi(\underline{\theta},\lambda)$ denote either $\phi(\underline{\theta})$ or $h^{(1)}(\underline{\theta},\lambda)$ as in the definition of $H_\Lambda(\lambda)$, 277. Let $m, n \in \mathbb{N}$ with $m < n$ and let $\theta \in \Lambda_m$. Then, using the Duhamel identity for each respective dynamics,

$$\left\|\tau_t^{\Lambda_n,\lambda}(b_\theta^\#) - \tau_t^{\Lambda_m,\lambda}(b_\theta^\#)\right\| \leq \sum_{\substack{\underline{\theta}\in\Lambda_n^+: \\ \boldsymbol{\theta}\cap\Lambda_n\setminus\Lambda_m\neq\emptyset}} |\phi(\underline{\theta},\lambda)| \int_0^{|t|} dt' \left\|\left[b_{\underline{\theta}}, \tau_{t-t'}^{\Lambda,\lambda}(b_\theta^\#)\right]\right\| \tag{320}$$

$$\leq \frac{G_F}{v}(e^{v|t|}-1) \sum_{\substack{\underline{\theta}\in\Lambda_n^+: \\ \boldsymbol{\theta}\cap\Lambda_n\setminus\Lambda_m\neq\emptyset}} |\phi(\underline{\theta},\lambda)| \sum_{i=1}^{|\underline{\theta}|} F_a(\rho(\theta,\theta_i)) \tag{321}$$

$$\leq \frac{G_F}{v}(e^{v|t|}-1) \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{y\in\Lambda_n} \sum_{\substack{\underline{\theta}\in\Lambda_n^+: \\ x,y\in\boldsymbol{\theta}}} |\phi(\underline{\theta},\lambda)| e^{-a\rho(\theta,y)} \tag{322}$$

$$\leq \frac{G_F}{v}(e^{v|t|}-1) \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{y\in\Lambda_n} F_a(\rho(x,y))F_a(\rho(\theta,y)) \tag{323}$$

$$\leq \frac{G_F C_F}{v}(e^{v|t|}-1) \sum_{x\in\Lambda_n\setminus\Lambda_m} F_a(\rho(x,\theta)) \tag{324}$$

Since F_a is uniformly integrable by construction, it is clear that $\{\tau_t^{\Lambda_n,\lambda}(b_\theta^\#)\}_{n\in\mathbb{N}}$ is Cauchy. $\square$

As an immediate consequence, the infinite lattice limit $\tau_t^\lambda(b_\theta^\#) := \tau_t^{\Theta,\lambda}(b_\theta^\#)$ exists in $\mathcal{B}_\Theta$ for all $\theta \in \Theta$ and $\lambda \in K$. Furthermore, the following bound holds for any $\Lambda \in \mathcal{P}_0(\Theta)$ uniformly in $\lambda \in K$.

$$\left\|\tau_t^\lambda(b_\theta^\#) - \tau_t^{\Lambda,\lambda}(b_\theta^\#)\right\| \leq \frac{G_F C_F}{v}(e^{v|t|}-1) \sum_{x\in\Theta\setminus\Lambda} F_a(\rho(x,\theta)) \tag{325}$$

The next lemma is a useful result that will help compare the infinite volume limit $\Delta_{\Theta}^{r}(b_{\underline{\theta}},\lambda)$ with its finite volume counterpart $\Delta_{\Lambda}^{r}(b_{\underline{\theta}},\lambda)$.

Lemma 5.24. *Let $\{\Lambda_n\}_{n\in\mathbb{N}_0}$ be an increasing and exhaustive sequence of subsets in Θ and $\underline{\theta}\in\Lambda_0^{+}$. Let $m,n\in\mathbb{N}_0$ such that $m<n$. Then for all $\lambda\in K$, the following bound holds.*

$$\left\|\Delta_{\Lambda_n}^{r}(b_{\underline{\theta}},\lambda)-\Delta_{\Lambda_m}^{r}(b_{\underline{\theta}},\lambda)\right\|\le 2\left(\sqrt{\mathcal{G}_{\epsilon}(\underline{\theta},r)\mathcal{K}_{\epsilon}^{1}(\underline{\theta},\Lambda_m)}+\mathcal{K}_{\epsilon}^{2}(\underline{\theta},r,\Lambda_m)\right)\tag{326}$$

The function $\mathcal{G}_{\epsilon}$ is as introduced in Lemma 5.20, while the functions $\mathcal{K}_{\epsilon}^{1}$ and $\mathcal{K}_{\epsilon}^{2}$ are defined to be,

$$\mathcal{K}_{\epsilon}^{1}(\underline{\theta},\Lambda_m):=\frac{G_F C_F\|F_0\|}{v^2}|\underline{\theta}|e^{-a\zeta\rho(\boldsymbol{\theta},\Lambda_m^c)}+4G^{(I)}(\epsilon(1-\zeta)a\rho(\boldsymbol{\theta},\Lambda_m^c)/v),\tag{327}$$

$$\mathcal{K}_{\epsilon}^{2}(\underline{\theta},r,\Lambda_m):=C_1|\underline{\theta}|\left(d_{\Lambda_m}+C_2\right)e^{-a\zeta(r+\rho(\boldsymbol{\theta},\Lambda_m^c))}+4G^{(I)}(\epsilon(1-\zeta)a(r+\rho(\boldsymbol{\theta},\Lambda_m^c))/v_p),\tag{328}$$

with the C_i being constant prefactors.

We may not directly write the operator $\Delta_{\Theta}^{r}(b_{\underline{\theta}},\lambda)$ because the conditional expectation map is not well defined on the infinite volume system. However, by first proving that $\{\Delta_{\Lambda_n}^{r}(b_{\underline{\theta}},\lambda)\}_{n\in\mathbb{N}}$ is Cauchy, we can define $\Delta_{\Theta}^{r}(b_{\underline{\theta}},\lambda):=\lim_{n\to\infty}\Delta_{\Lambda_n}^{r}(b_{\underline{\theta}},\lambda)$ and obtain the desired comparison in norm between this limit and any finite volume Δ_{Λ}^{r}.

Proof. Explicitly, one has,

$$\Delta_{\Lambda_n}^{r}(b_{\underline{\theta}},\lambda)-\Delta_{\Lambda_m}^{r}(b_{\underline{\theta}},\lambda)=\int_{\mathbb{R}}dt\,W_{\epsilon}(t)\left(\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}))-\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right)\tag{329}$$

$$\begin{aligned}&=\int_{\mathbb{R}}dt\,W_{\epsilon}(t)\big(\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}\left(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}})-\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})\right)\\&\quad-\left(\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}-\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}\right)\left(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})\right)\big)\end{aligned}\tag{330}$$

$$\begin{aligned}\left\|\Delta_{\Lambda_n}^{r}(b_{\underline{\theta}},\lambda)-\Delta_{\Lambda_m}^{r}(b_{\underline{\theta}},\lambda)\right\|&\le\int_{\mathbb{R}}dt\,|W_{\epsilon}(t)|\left\|\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}\left(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}})-\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})\right)\right\|\\&\quad+\int_{\mathbb{R}}dt\,|W_{\epsilon}(t)|\left\|\left(\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}-\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}\right)\left(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})\right)\right\|\end{aligned}\tag{331}$$

To estimate the first norm of the last inequality, we can make use of the proof strategy of Lemma 5.20 to obtain,

$$\int_{\mathbb{R}} dt\, |W_\epsilon(t)| \left\| \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n} \left(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right) \right\| \leq 2\mathcal{G}_\epsilon(\underline{\theta}, r). \tag{332}$$

However, we may also invoke Lemma 5.23 to obtain an alternate bound.

$$\int_{\mathbb{R}} dt\, |W_\epsilon(t)| \left\| \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n} \left(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right) \right\| \leq 2 \int_{\mathbb{R}} dt\, |W_\epsilon(t)| \left\| \tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right\| \tag{333}$$

As we have done previously, we split this integral into a finite time piece and a remainder that takes care of large times. Letting $0 < T < \infty$, one has,

$$2\int_{|t|<T} dt\, |W_\epsilon(t)| \left\| \tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right\| \leq \frac{2G_F C_F}{v^2}\left(e^{vT} - 1\right) \sum_{\theta \in \Lambda_n \setminus \Lambda_m} \sum_{i=1}^{|\underline{\theta}|} F_a(\rho(\theta, \theta_i)) \tag{334}$$

$$\leq \frac{2G_F C_F \|F_0\|}{v^2} |\underline{\theta}| e^{-(a\rho(\boldsymbol{\theta}, \Lambda_m^c) - vT)} \tag{335}$$

Crucially, the first inequality above supports $\{\Delta_{\Lambda_n}^r(b_{\underline{\theta}}, \lambda)\}_{n\in\mathbb{N}}$ being a Cauchy sequence; this needs to be done for the second half of 331. For the remainder integral, one has,

$$2\int_{|t|>T} dt\, |W_\epsilon(t)| \left\| \tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right\| \leq 8G^{(I)}(\epsilon T) \tag{336}$$

Letting $0 < \zeta < 1$ and $T = (1-\zeta)a\rho(\boldsymbol{\theta}, \Lambda_m^c)/v$, we obtain the combined estimate,

$$2\int_{\mathbb{R}} dt\, |W_\epsilon(t)| \left\| \tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right\| \leq \frac{2G_F C_F \|F_0\|}{v^2} |\underline{\theta}| e^{-a\zeta\rho(\boldsymbol{\theta}, \Lambda_m^c)} + 8G^{(I)}(\epsilon(1-\zeta)a\rho(\boldsymbol{\theta}, \Lambda_m^c)/v) \tag{337}$$

$$=: 2\mathcal{K}_\epsilon^1(\underline{\theta}, \Lambda_m) \tag{338}$$

With both bounds in 332 and 337, we elect to use the following bound.

$$\int_{\mathbb{R}} dt\, |W_\epsilon(t)| \left\| \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n} \left(\tau_t^{\Lambda_n,\lambda}(b_{\underline{\theta}}) - \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}) \right) \right\| \leq 2\sqrt{\mathcal{G}_\epsilon(\underline{\theta}, r)\mathcal{K}_\epsilon^1(\underline{\theta}, \Lambda_m)} \tag{339}$$

To estimate a bound on the second term in 331, we revisit the expression for the conditional expectation map $\mathbb{E}$. Recall equation 211,

$$\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(A) = \frac{1}{4^{N_n - N_r}} \sum_{\iota \in \mathcal{I}_{rn}} u(\iota)^* A u(\iota), \tag{340}$$

with N_n and N_r denoting the dimensions of $\overline{\operatorname{span}\{\psi_\theta :\ \theta \in \Lambda_n\}}$ and $\overline{\operatorname{span}\{\psi_\theta :\ \theta \in \boldsymbol{\theta}_r\}}$ respectively. We also have the collection of tuples used to identify the fermionic unitary operators, $\mathcal{I}_{rn} = \{0,1,2,3\}^{N_n - N_r}$. Additionally, note as before that this map is built using orthonormal basis functions f_j which span $\mathcal{H}_{\boldsymbol{\theta}_r}^{\perp} \cap \mathcal{H}_{\Lambda_n}$, and this holds in the case of Λ_m as well. Thus, we can choose to decompose $\mathcal{H}_{\boldsymbol{\theta}_r}^{\perp} \cap \mathcal{H}_{\Lambda_n}$ into $\mathcal{H}_{\boldsymbol{\theta}_r}^{\perp} \cap \mathcal{H}_{\Lambda_m} \oplus \mathcal{H}_{\Lambda_m}^{\perp} \cap \mathcal{H}_{\Lambda_n}$. With this in hand, let $\mathcal{I}_{mn} = \{0,1,2,3\}^{N_n - N_m}$. Then we have the following expressions for $\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(A)$ and $\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(A)$ respectively.

$$\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(A) = \frac{1}{4^{N_m - N_r}} \sum_{\iota \in \mathcal{I}_{rm}} u(\iota)^* A u(\iota), \tag{341}$$

$$= \frac{1}{4^{N_n - N_r}} \sum_{\iota_1 \in \mathcal{I}_{mn}} \sum_{\iota_2 \in \mathcal{I}_{rm}} u(\iota_1)^* u(\iota_1) u(\iota_2)^* A u(\iota_2) \tag{342}$$

$$\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(A) = \frac{1}{4^{N_n - N_r}} \sum_{\iota_1 \in \mathcal{I}_{mn}} \sum_{\iota_2 \in \mathcal{I}_{rm}} u(\iota_1)^* u(\iota_2)^* A u(\iota_2) u(\iota_1) \tag{343}$$

At this stage, recognize that we have used an identical rewriting of $\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(A)$ as in the proof of Lemma 5.13. Thus, estimating the difference in these two conditional expectation maps yields,

$$\left\|\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(A) - \mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(A)\right\| \leq \frac{1}{4^{N_n - N_r}} \sum_{\iota_1 \in \mathcal{I}_{mn}} \sum_{\iota_2 \in \mathcal{I}_{rm}} \|u^*(\iota_1) \left[u(\iota_2)^* A u(\iota_2), u(\iota_1)\right]\| \tag{344}$$

$$\leq \sum_{\iota \in \mathcal{I}_{mn}} \|[A, u(\iota)]\|, \tag{345}$$

and letting $A = \tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})$ yields exactly the sum of terms as in Lemma 5.13. Thus, we can directly write a bound on $\left\|\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\|$.

(346)
$$\left\|\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\| \leq 8\kappa G_\perp \|F_0\| |\underline{\theta}| e^{-(a(r+\rho(\boldsymbol{\theta}_r,\Lambda_m^c))-v_p|t|)} \sum_{k=1}^{n-m} \left(\frac{d_{\Lambda_m}}{2} + k\right) e^{-ak}$$

An identical bound holds when $\boldsymbol{\theta}_r$ is replaced with $\boldsymbol{\theta}_{r-1}$, as is needed to consider $\mathcal{E}_r^{\Lambda}$. Importantly, this estimate confirms that the second half of 331 supports $\{\Delta_{\Lambda_n}^r(b_{\underline{\theta}},\lambda)\}_{n\in\mathbb{N}}$ being Cauchy. So this aspect of the proof is complete.

Returning back to the integral at hand, starting with the finite time piece, one has,

(347)
$$\int_{|t|<T} |W_\epsilon(t)| \left\|\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\| \leq C_1 |\underline{\theta}| \left(d_{\Lambda_m} + C_2\right) e^{-(a(r+\rho(\boldsymbol{\theta},\Lambda_m^c))-v_pT)}.$$

For larger times, we of course have the trivial bound

$$\left\|\mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathbb{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\| \leq 2, \tag{348}$$

so for the large time remainder,

$$\int_{|t|>T} |W_\epsilon(t)| \left\|\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\| \leq 8G^{(I)}(\epsilon T). \tag{349}$$

Then, let $T = (1-\zeta)a(r+\rho(\boldsymbol{\theta},\Lambda_m^c))/v_p$. We then obtain an overall bound:

$$\begin{aligned}\int_{\mathbb{R}} |W_\epsilon(t)| \left\|\mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_n}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}})) - \mathcal{E}_{\boldsymbol{\theta}_r}^{\Lambda_m}(\tau_t^{\Lambda_m,\lambda}(b_{\underline{\theta}}))\right\| &\leq 2C_1|\underline{\theta}| \left(d_{\Lambda_m} + C_2\right) e^{-a\zeta(r+\rho(\boldsymbol{\theta},\Lambda_m^c))} \\ &\quad + 8G^{(I)}(\epsilon(1-\zeta)a(r+\rho(\boldsymbol{\theta},\Lambda_m^c))/v_p) \qquad (350) \\ &= 2\mathcal{K}^2(\underline{\theta}, r, \Lambda_m) \qquad (351)\end{aligned}$$

Putting 339 and 350 together gives the final upper bound, which is the statement of this lemma.

$$\left\|\Delta^r_{\Lambda_n}(b_{\underline{\theta}},\lambda)-\Delta^r_{\Lambda_m}(b_{\underline{\theta}},\lambda)\right\|\leq 2\left(\sqrt{\mathcal{G}_\epsilon(\underline{\theta},r)\mathcal{K}^1_\epsilon(\underline{\theta},\Lambda_m)}+\mathcal{K}^2_\epsilon(\underline{\theta},r,\Lambda_m)\right) \tag{352}$$

□

With no dependence on $n\in\mathbb{N}$, the same estimate serves as an upper bound on $\left\|\Delta^r_{\Theta}(b_{\underline{\theta}},\lambda)-\Delta^r_{\Lambda_m}(b_{\underline{\theta}},\lambda)\right\|$, where we may safely define $\Delta^r_\Theta(b_{\underline{\theta}},\lambda):=\lim_{n\to\infty}\Delta^r_{\Lambda_n}(b_{\underline{\theta}},\lambda)$. Moreover, we can now introduce an infinite volume version of the local interactions constituting the Hastings generator.

$$\beta(\underline{\mu},\lambda)\mathfrak{b}_{\underline{\mu}}=\sum_{r=0}^{\infty}\sum_{\substack{\underline{\theta}\in\Theta^+:\\ \boldsymbol{\theta}\cap\Lambda_p\neq\emptyset,\\ \boldsymbol{\theta}_r=\boldsymbol{\mu}}} h(\underline{\theta},\lambda)\Delta^r_\Theta(b_{\underline{\theta}},\lambda) \tag{353}$$

Noting that convergence of $\{\alpha^{\Lambda_n}_\lambda(b^{\#}_\theta)\}_{n\in\mathbb{N}}$ is equivalent to convergence of $\{\left(\alpha^{\Lambda_n}_\lambda\right)^{-1}(b^{\#}_\theta)\}_{n\in\mathbb{N}}$, we elect to proceed by proving that $\{\left(\alpha^{\Lambda_n}_\lambda\right)^{-1}(b^{\#}_\theta)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for any $\theta\in\Theta$ and $\lambda\in K$. For brevity of notation, we will denote by $\tilde{\alpha}^\Lambda_\lambda:=\left(\alpha^\Lambda_\lambda\right)^{-1}$. Let $n>m$ and assume that $K=[0,\lambda_M]$, then we have the preliminary estimate,

$$\left\|\tilde{\alpha}^{\Lambda_n}_\lambda(b^{\#}_\theta)-\tilde{\alpha}^{\Lambda_m}_\lambda(b^{\#}_\theta)\right\|\leq\int_0^\lambda ds\left\|\left[D_{\Lambda_n}(s)-D_{\Lambda_m}(s),\tilde{\alpha}^{\Lambda_m}_s(b^{\#}_\theta)\right]\right\| \tag{354}$$

Now, we decompose the difference $D_{\Lambda_n}(s)-D_{\Lambda_m}(s)$ according to two different kinds of lattice subsets.

$$D_{\Lambda_n}(s)-D_{\Lambda_m}(s)=\sum_{\substack{\underline{\mu}\in\Lambda_n^+:\\ \boldsymbol{\mu}\cap\Lambda_n\setminus\Lambda_m\neq\emptyset}}\beta_{\Lambda_n}(\underline{\mu},s)b_{\underline{\mu}}+\sum_{\underline{\mu}\in\Lambda_m^+}\left(\beta_{\Lambda_n}(\underline{\mu},s)-\beta_{\Lambda_m}(\underline{\mu},s)\right)b_{\underline{\mu}} \tag{355}$$

The first sum directly gives rise to the application of a Lieb-Robinson bound as in Theorem 5.10.

$$\left\|\left[b_{\underline{\mu}},\tilde{\alpha}^{\Lambda_m}_s(b^{\#}_\theta)\right]\right\|\leq G_F e^{vs}\sum_{i=1}^{|\underline{\mu}|}F^\beta(\rho(\mu_i,\theta)) \tag{356}$$

Then, summing over all $\underline{\mu}$ with support intersecting $\Lambda_n \backslash \Lambda_m$ gives,

$$G_F e^{vs} \sum_{\substack{\underline{\mu}\in\Lambda_n^+: \\ \boldsymbol{\mu}\cap\Lambda_n\backslash\Lambda_m\neq\emptyset}} \left|\beta_{\Lambda_n}(\underline{\mu}, s)\right| \sum_{i=1}^{|\underline{\mu}|} F^\beta(\rho(\mu_i,\theta)) \leq G_F \sum_{x\in\Lambda_n\backslash\Lambda_m} \sum_{y\in\Lambda_n} \sum_{\substack{\underline{\mu}\in\Lambda_n^+: \\ x,y\in\boldsymbol{\mu}}} \sup_{s\in K} e^{vs}\left|\beta_{\Lambda_n}(\underline{\mu}, s)\right| F^\beta(\rho(y,\theta)) \tag{357}$$

$$\leq G_F C_\beta e^{v\lambda_M} \sum_{x\in\Lambda_n\backslash\Lambda_m} F^\beta(\rho(x,\theta)) \tag{358}$$

This vanishes as $m, n \to \infty$ due to uniform integrability of F^β. To complete the proof, we estimate a bound for those terms supported in Λ_m, and we proceed with inspiration from the analogous proof from [11]. Let $\mathcal{P}_0(\Lambda_m) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ constitute a partition of the finite subsets of Λ_m:

$$\mathcal{P}_1 = \{X \in \mathcal{P}_0(\Lambda_m) :\ X \subset \Lambda_{m/3}^c\},\ \mathcal{P}_2 = \{X \in \mathcal{P}_0(\Lambda_m) :\ X \subset \Lambda_{2m/3}\},$$
$$\mathcal{P}_3 = \{X \in \mathcal{P}_0(\Lambda_m) :\ X \cap \Lambda_{m/3} \neq \emptyset \text{ and } X \cap \Lambda_{2m/3}^c \neq \emptyset\}.$$

We will show that $\left\|\left[\left(\beta_{\Lambda_n}(\underline{\mu}, s) - \beta_{\Lambda_m}(\mu, s)\right) b_{\underline{\mu}}, \tilde{\alpha}_s^{\Lambda_m}(b_\theta^\#)\right]\right\|$ vanishes as $m, n \to \infty$ as we sum over subsets $\boldsymbol{\mu}$ belonging to each of the three partition elements. Starting with $\mathcal{P}_1$, and letting $\beta_M \in \mathbb{R}$ be such that $\left|\beta_\Lambda(\underline{\mu}, \lambda)\right| \leq \beta_M$ for all $\Lambda \in \Theta^+$, $\underline{\mu} \in \Lambda^+$, and $\lambda \in K$, we have,

$$\sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_1}} \left\|\left[\left(\beta_{\Lambda_n}(\underline{\mu}, s) - \beta_{\Lambda_m}(\underline{\mu}, s)\right) b_{\underline{\mu}}, \tilde{\alpha}_s^{\Lambda_m}(b_\theta^\#)\right]\right\| \leq 2\beta_M G_F e^{vs} \sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_1}} \sum_{i=1}^{|\underline{\mu}|} F^\beta(\rho(\mu_i,\theta)) \tag{359}$$

$$\leq 2\beta_M G_F C_\beta e^{v\lambda_M} \sum_{x\in\Lambda_{m/3}^c} F^\beta(\rho(x,\theta)). \tag{360}$$

This bound decays to 0 as $m \to \infty$. Moving onto those sets in $\mathcal{P}_2$, we proceed to estimate.

$$\left\|\sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_2}} \left[\left(\beta_{\Lambda_n}(\underline{\mu}, s) - \beta_{\Lambda_m}(\underline{\mu}, s)\right) b_{\underline{\mu}}, \tilde{\alpha}_s^{\Lambda_m}(b_\theta^\#)\right]\right\| \leq 2\left\|\sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_2}} \left(\beta_{\Lambda_n}(\underline{\mu}, s) - \beta(\underline{\mu}, s)\right) b_{\underline{\mu}}\right\| \tag{361}$$

$$+ 2\left\| \sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_2}} \left(\beta(\underline{\mu},s)-\beta_{\Lambda_m}(\underline{\mu},s)\right) b_{\underline{\mu}} \right\|$$

These two terms behave identically and yield similar bounds. To obtain these, we make use of Lemma 5.24. By construction,

$$\sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_2}} \left(\beta(\underline{\mu},s)-\beta_{\Lambda_m}(\underline{\mu},s)\right) b_{\underline{\mu}} = \sum_{\underline{\mu}\in\Lambda_{2m/3}^+} \sum_{r=0} \sum_{\substack{\underline{\theta}\in\Theta^+: \\ \boldsymbol{\theta}_r=\boldsymbol{\mu}}} h(\underline{\theta},s)\left(\Delta_\Theta^r(b_{\underline{\theta}},s)-\Delta_{\Lambda_m}^r(b_{\underline{\theta}},s)\right) \tag{362}$$

$$\left\| \sum_{\substack{\underline{\mu}\in\Lambda_m^+: \\ \boldsymbol{\mu}\in\mathcal{P}_2}} \left(\beta(\underline{\mu},s)-\beta_{\Lambda_m}(\underline{\mu},s)\right) b_{\underline{\mu}} \right\| \leq 2\sum_{r\geq 0} \sum_{\substack{\underline{\theta}\in\Theta^+: \\ \boldsymbol{\theta}_r\subset\Lambda_{2m/3}}} |h(\underline{\theta},s)|\Big(\sqrt{\mathcal{G}_\epsilon(\underline{\theta},r)\mathcal{K}_\epsilon^1(\underline{\theta},\Lambda_m)} + \mathcal{K}_\epsilon^2(\underline{\theta},r,\Lambda_m)\Big) \tag{363}$$

To simplify the subsequent expressions, we note the general form of each function that appears inside the parentheses.

$$\mathcal{G}_\epsilon(\underline{\theta},r) = C_0|\underline{\theta}|\left(\frac{d_{\boldsymbol{\theta}}}{2}+r+1\right)e^{-\delta a r} + 4G^{(I)}(c_0 r) \tag{364}$$

$$\mathcal{K}_\epsilon^1(\underline{\theta},\Lambda_m) = C_1|\underline{\theta}|e^{-a\zeta\rho(\boldsymbol{\theta},\Lambda_m^c)} + 4G(I)(c_1\rho(\boldsymbol{\theta},\Lambda_m^c)) \tag{365}$$

$$\mathcal{K}_\epsilon^2(\underline{\theta},r,\Lambda_m) = C_2|\underline{\theta}|(d_{\Lambda_m}+c_2)e^{-a\zeta(r+\rho(\boldsymbol{\theta},\Lambda_m^c))} + 4G^{(I)}(c_3(r+\rho(\boldsymbol{\theta},\Lambda_m^c))) \tag{366}$$

Remark as well that for $\boldsymbol{\theta}\subset\Lambda_{2m/3}$, $\rho(\boldsymbol{\theta},\Lambda_m^c)\leq\rho(\Lambda_{2m/3},\Lambda_m^c)$, and up to a constant κ_1, $\rho(\Lambda_{2m/3},\Lambda_m^c)\leq\kappa_1\frac{m}{3}$. Thus both terms in the sum vanish as $m\to\infty$, and the same holds at least as fast for the analogous terms involving Λ_n. Now, we just need to prove that the sums over r and $\underline{\theta}$ yield finite bounds. Starting with the term involving $\mathcal{K}_\epsilon^2$, one has,

(367)

$$2\sum_{r\geq 0} \sum_{\substack{\underline{\theta}\in\Theta^+: \\ \boldsymbol{\theta}_r\subset\Lambda_{2m/3}}} |h(\underline{\theta},s)|\mathcal{K}_\epsilon^2(\underline{\theta},r,\Lambda_m) \leq 8C_2\sum_{r\geq 0}\sum_{x\in\Lambda_{2m/3}} \sum_{\substack{\underline{\theta}\in\Theta^+: \\ x\in\boldsymbol{\theta}}} |h(\underline{\theta},s)||\underline{\theta}|\,(d_{\Lambda_m}+c_2)\times \left(e^{-a\zeta(r+\kappa_1 m/3)}+G^{(I)}(c_3(r+\kappa_1 m/3))\right)$$

$$\leq 8C_2 F_a(0)\,(d_{\Lambda_m} + c_2) \times \left(e^{-a\zeta\kappa_1 m/3} \sum_{r \geq 0} e^{-a\zeta r} + G^{(I)}(c_3\kappa_1 m/3) \sum_{r\geq 0} G^{(I)}(c_3 r) \right), \tag{368}$$

where we have used that the exponentially decaying function and $G^{(I)}(x)$ are submultiplicative, for large x in the case of $G^{(I)}$. By Lemma 5.16, we know that both the sums over r are finite, which is exactly what we sought, and, noting lattice regularity, the decay as $m \to \infty$ is evident as well. Moving on to those Hastings generator terms supported in $\mathcal{P}_3$, we have,

$$\sum_{\substack{\underline{\mu} \in \Lambda_m^+: \\ \boldsymbol{\mu} \in \mathcal{P}_3}} \left\| \left[\left(\beta_{\Lambda_n}(\underline{\mu}, s) - \beta_{\Lambda_m}(\underline{\mu}, s) \right) b_{\underline{\mu}}, \tilde{\alpha}_s^{\Lambda_m}(b_\theta^{\#}) \right] \right\| \leq 2 \sum_{x \in \Lambda_m} \sum_{y \in \Lambda_{2m/3}^c} \sum_{\substack{\underline{\mu} \in \Theta^+: \\ x,y \in \boldsymbol{\mu}}} \Big(\sup_{s\in K} \left|\beta_{\Lambda_n}(\underline{\mu}, s)\right| + \sup_{s \in K} \left| \beta_{\Lambda_m}(\underline{\mu}, s) \right| \Big) \tag{369}$$

$$\leq 2 \sum_{x \in \Lambda_{m/3}} \sum_{y \in \Lambda_{2m/3}^c} F^{\beta}(\rho(x,y)) \tag{370}$$

$$\leq 2\|F\| \left|\Lambda_{m/3}\right| u_\mu(\kappa_1 m/3) \tag{371}$$

The second inequality follows from Lemma 5.21, and the final inequality holds due to the particular weighted F-function whose existence we prove in the same lemma. Additionally, this last bound goes to zero as $m \to \infty$; thus, we have shown that $\{\alpha_\lambda^{\Lambda_n}(b_\theta^{\#})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_\Theta$ for all $\theta \in \Theta$, which proves the existence of the thermodynamic limit of the spectral flow automorphism, α_λ. ☐

6. Quantization of Hall Conductance

6.1. **Continuum Systems via Lattices.**

Having constructed and validated the machinery needed to implement quasi-adiabatic evolution for interacting fermions in continuous space, we now turn to a concrete physical application: quantization of Hall conductance for a two-dimensional continuum electron gas in the presence of a uniform, perpendicular magnetic field (quantum Hall setting). Proving this statement, which is the subject of this section, is based on joint work with Sven Bachmann, Giuseppe De Nittis, and Bruno Nachtergaele [10], where we employ the Hastings-Michalakis strategy used in [38] and [8].

In the absence of inter-particle interactions, a single charged particle in this environment can be described via the well known Landau Hamiltonian, which admits harmonic oscillator-like energy levels and eigenspaces. In their work, [12], Bachmann and De Nittis introduce lattice-localized frames as a tool for studying the local properties of dynamics of continuum fermion systems, and in this article they prove that each eigenspace of the Landau Hamiltonian admits an overcomplete localized frame.

As we have seen thus far, models for systems of many interacting fermions are conveniently written using creation and annihilation operators tied to single particle states. Formally, the procedure named second quantization, which maps operators acting on $(L^2(\mathbb{R}^2))^{\wedge n}$ to operators on Fock space which alter and/or count the number of occupied single particle states, makes use of an orthonormal basis that spans the domain $\mathcal{D} \subset L^2(\mathbb{R}^2)$ of a Hamiltonian. This process was outlined in Section 4 equation 83. Since the set $\{\chi_\theta\}_{\theta\in\Theta}$ is a dense subset of $\mathcal{H} := \bigoplus_{n=0}^{N-1} \mathcal{H}_n$ that spans the entire space, it is reasonable to propose a second quantization of a Hamiltonian acting on $\mathcal{H}$ generated by $\{\chi_\theta\}_{\theta\in\Theta}$. Using a frame-based expansion, we rewrite the second quantization of the Landau Hamiltonian restricted to the first N Landau levels. Recalling equation 83, one has,

$$\mathbb{H}_L = \sum_{n=0}^{N-1} E_n \sum_{m=0}^{\infty} a^*(\psi_{nm}) a(\psi_{nm}). \tag{372}$$

Each ψ_{nm} admits the following reconstruction formula:

$$\psi_{nm} = \sum_{\theta \in \Gamma_n} \langle \chi_\theta, S^{-1}\psi_{nm} \rangle \chi_\theta, \tag{373}$$

where $\Theta_n := \{\theta \in \Theta : \theta = (\gamma, n) \text{ with } \gamma \in \Gamma\}$, meaning that the Landau level index of the extended lattice is held fixed while the spatial lattice index is variable. Consequently, the creation/annihilation operators associated with each ψ_{nm} can also be expanded as a series. For brevity of notation, let $a_\theta^\# := a^\#(\chi_\theta)$ and as before let $s_\theta(f) := \langle \chi_\theta, S^{-1}\psi_{nm} \rangle$ denote the frame expansion coefficients for $f \in \mathcal{H}$.

$$a^*(\psi_{nm}) = \sum_{\theta \in \Theta_n} s_\theta(\psi_{nm}) a_\theta^*; \; a(\psi_{nm}) = \sum_{\theta \in \Theta_n} \overline{s_\theta(\psi_{nm})} a_\theta. \tag{374}$$

Thus, the second quantization of H_L restricted to the first N Landau levels can be formally rewritten as,

$$\mathbb{H}_L = \sum_{n=0}^{N-1} E_n \sum_{\theta, \theta' \in \Theta_n} \left(\sum_{m=0}^{\infty} \langle \chi_\theta, S^{-1}\psi_{nm} \rangle \langle S^{-1}\psi_{nm}, \chi_{\theta'} \rangle \right) a_\theta^* a_{\theta'} \tag{375}$$

$$= \sum_{n=0}^{N-1} E_n \sum_{\theta, \theta' \in \Theta_n} \left(\sum_{m=0}^{\infty} \langle S^{-1}\chi_\theta, \psi_{nm} \rangle \langle \psi_{nm}, S^{-1}\chi_{\theta'} \rangle \right) a_\theta^* a_{\theta'} \tag{376}$$

$$= \sum_{n=0}^{N-1} E_n \sum_{\theta, \theta' \in \Theta_n} \langle S^{-1}\chi_\theta, \mathbb{1}_n S^{-1}\chi_{\theta'} \rangle a_\theta^* a_{\theta'} \tag{377}$$

$$= \sum_{n=0}^{N-1} E_n \sum_{\theta, \theta' \in \Theta_n} \langle \chi_\theta, S^{-2}\chi_{\theta'} \rangle a_\theta^* a_{\theta'} \tag{378}$$

$$=: \sum_{n=0}^{N-1} E_n \sum_{\theta, \theta' \in \Theta_n} t(\theta, \theta') a_\theta^* a_{\theta'} \tag{379}$$

In the third equality above, $\mathbb{1}_n = \sum_{m=0}^{\infty} |\psi_{nm}\rangle \langle \psi_{nm}|$ is the orthogonal projection onto $\mathcal{H}_n$. The hopping element $t(\theta, \theta')$ can be further simplified by noting that under action by the frame operator, a function cannot jump to another Landau level. Thus, explicitly consider the frame operator S_n associated with elements of the n^{th} Landau level acting on an element $f_n \in \mathcal{H}_n$, where $f_n = \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n f_0$ with $f_0 \in \mathcal{H}_0$.

$$S_n f = \sum_{\gamma \in \Gamma} \langle \chi_{\gamma,n}, f \rangle \chi_{\gamma,n} \tag{380}$$

$$= \sum_{\gamma \in \Gamma} \left\langle \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_\gamma, \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n f_0 \right\rangle \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_\gamma \tag{381}$$

$$= \sum_{\gamma \in \Gamma} \frac{1}{n!} \langle \mathfrak{a}^n (\mathfrak{a}^*)^n \chi_\gamma, f_0 \rangle \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_\gamma \tag{382}$$

$$= \sum_{\gamma \in \Gamma} \langle \chi_\gamma, f_0 \rangle \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_\gamma \tag{383}$$

$$= \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n S_0 f_0 \tag{384}$$

For ease of notation, we let $\chi_\gamma := \chi_{\gamma,0}$. The cancellation of the factor of $n!$ arises due to the canonical commutation relations that the Landau level ladder operators satisfy, as discussed in Section 3. Importantly, this reveals that one can write $S_n = \frac{1}{n!}(\mathfrak{a}^*)^n S_0 \mathfrak{a}^n$, and the same formula extends to S_n^{-1}. Therefore, we can write,

$$\langle \chi_{\gamma,n}, S_n^{-2} \chi_{\gamma',n} \rangle = \left(\frac{1}{n!}\right)^2 \left\langle \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_\gamma, (\mathfrak{a}^*)^n S_0^{-1} \mathfrak{a}^n (\mathfrak{a}^*)^n S_0^{-1} \mathfrak{a}^n \frac{1}{\sqrt{n!}}(\mathfrak{a}^*)^n \chi_{\gamma'} \right\rangle \tag{385}$$

$$= \left(\frac{1}{n!}\right)^3 \left\langle (\mathfrak{a}^*)^n \chi_\gamma, (\mathfrak{a}^*)^n S_0^{-1} \mathfrak{a}^n (\mathfrak{a}^*)^n S_0^{-1} \mathfrak{a}^n (\mathfrak{a}^*)^n \chi_{\gamma'} \right\rangle \tag{386}$$

$$= \left(\frac{1}{n!}\right)^3 \left\langle \mathfrak{a}^n (\mathfrak{a}^*)^n \chi_\gamma, S_0^{-1} \mathfrak{a}^n (\mathfrak{a}^*)^n S_0^{-1} \mathfrak{a}^n (\mathfrak{a}^*)^n \chi_{\gamma'} \right\rangle \tag{387}$$

$$= \langle \chi_\gamma, S_0^{-2} \chi_{\gamma'} \rangle \tag{388}$$

$$=: t(\gamma, \gamma') \tag{389}$$

These equalities again follow from the commutation relations of $\mathfrak{a}$ and $\mathfrak{a}^*$, and they show that the hopping matrix elements are uniform across Landau levels. Recall as well that by

Lemma 3.4, there exist constants $C, \sigma > 0$ such that,

$$|t(\gamma,\gamma')| \leq Ce^{-\sigma d(\gamma,\gamma')},\ \forall\, \gamma,\gamma' \in \Gamma. \tag{390}$$

For the guiding objective of this section, which is ultimately to prove quantization of Hall conductance, we consider the restriction of $\mathbb{H}_L$ to just the lowest two levels. So, this yields the non-interacting model,

$$\mathbb{H}_0 = 2B\sum_{n=0}^{1}(n+\frac{1}{2})\sum_{\gamma,\gamma'\in\Gamma} t(\gamma,\gamma')a^*_{\gamma n}a_{\gamma' n}, \tag{391}$$

To denote the algebra where these operators live, let $\mathcal{A}_{01}$ be the CAR algebra generated by $\{a^{\#}_{\gamma,n} :\ \gamma \in \Gamma,\ n \in \{0,1\}\}$. Ultimately, towards probing candidate systems that could describe the fractional quantum Hall effect, where electron-electron interactions must be considered, we also want to include a two-body interaction. The form of this operator that we consider is inspired by the UV regularization implemented in the work of Gebert et al. [31] and discussed in [12]. Precisely, we consider the general interaction,

$$I_V = \frac{1}{2}\sum_{n_1,\dots,n_4\in\{0,1\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dxdy\, \tilde{V}(x,y,\underline{n})a^*_{(x,n_4)}a^*_{(y,n_3)}a_{(y,n_2)}a_{(x,n_1)}, \tag{392}$$

We now state assumptions on the function $\tilde{V}(x,y,\underline{n})$, where $\underline{n} := \{n_1,...,n_4\}$.

Assumption 6.1. *Let $\tilde{V} : \mathbb{R}^2\times\mathbb{R}^2\times\{0,1\}^4 \to \mathbb{R}$ satisfy:*

i) $\tilde{V}(x,y,\underline{n}) = \delta_{n_1+n_2,n_3+n_4}\left(1-b_0\delta_{1,|n_2-n_1|}\right)\tilde{V}(x,y)$ *with* $0<b_0<1$,

ii) $\tilde{V}(x,y) = \tilde{V}(x-y)$

iii) $\left|\tilde{V}(x,y)\right| \leq C_v e^{-\sigma_v\rho(x,y)}$ *where* $C_v,\sigma_v > 0$.

These conditions establish V as a symmetric and exponentially short range potential function giving rise to both intra and inter-Landau level interactions, with weaker strength given to the inter-level case. Using a frame expansion for each $\chi_{(x,n)}$, we rewrite I_V to obtain,

$$I_V = \sum_{\{\gamma_1,...,\gamma_4\}\in\Gamma^+} V(\{\gamma_1,...,\gamma_4\},\underline{n}) a^*_{\gamma_4 n_4} a^*_{\gamma_3 n_3} a_{\gamma_2 n_2} a_{\gamma_1 n_1}. \tag{393}$$

The effective potential $V(\underline{\gamma},\underline{n})$ is expressed as,

$$V(\underline{\gamma},\underline{n}) = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dxdy\, \tilde{V}(x,y,\underline{n}) s_{\gamma_4}(x,n_4) s_{\gamma_3}(y,n_3) \overline{s_{\gamma_2}(y,n_2) s_{\gamma_1}(x,n_1)} \tag{394}$$

$$s_\gamma(x,n) := \langle S^{-1}\chi_{\gamma,n}, \chi_{x,n}\rangle \tag{395}$$

We now state a proposition which establishes the range of V, which was proved in Proposition 3.7 of [12].

Proposition 6.2. *Assume that there exist constants $C, c > 0$ such that $\left|S^{-1}\chi_{(0,n)}\right| \le Ce^{-c|x|}$. Under the same conditions as Assumption 6.1, there exist constants $C_V, \zeta > 0$ such that for all $\{\gamma_1,...,\gamma_4\} \in \Gamma^+$ and $\underline{n} \in \{0,1\}^4$,*

$$\left|V(\underline{\gamma},\underline{n})\right| \le C_V e^{-\zeta \operatorname{diam}(\{\gamma_1,...,\gamma_4\})}, \tag{396}$$

where $\operatorname{diam}(\{\gamma_1,...,\gamma_4\}) := \max_{\gamma,\gamma'\in\underline{\gamma}}\{d(\gamma,\gamma')\}$.

As shown in Section 3 of the work of Del Prete, [26], if the function $g \in L^2(\mathbb{R})$ used to generate a frame is exponentially localized, then its dual $S^{-1}g$ is also exponentially localized. We believe this should hold in our setting over $\mathbb{R}^2$, but we defer proving this rigorously to a future work.

The general (unperturbed) Hamiltonian we consider is given by $\mathbb{H} = \mathbb{H}_L + I_V$, and we may write this in terms of a single self-adjoint interaction $\Phi : \Gamma^+ \to \mathcal{A}_{01}$ as

$$\mathbb{H} = \sum_{\underline{\gamma}\in\Gamma^+} \Phi(\underline{\gamma}). \tag{397}$$

Precisely,

(398)
$$\Phi(\underline{\gamma}) = \begin{cases} B\sum_{n=0}^{1}(n+\frac{1}{2})\left(t(\gamma_2,\gamma_1)a_{\underline{\gamma,n}} + t(\gamma_1,\gamma_2)a^*_{\underline{\gamma,n}}\right) & \text{if } \underline{\gamma,n} = \{(\gamma,n),(\gamma',n)\} \\ \frac{1}{2}\sum_{\underline{n}\in\{0,1\}^4}\left(V(\underline{\gamma,n})a_{\underline{\gamma,n}} + \overline{V(\underline{\gamma,n})}a^*_{\underline{\gamma,n}}\right) & \text{if } \underline{\gamma,n} = \{(\gamma_1,n_1),...,(\gamma_4,n_4)\} \end{cases}$$

The finite volume restriction of this model occurs with respect to the lattice Θ and not continuous space $\mathbb{R}^2$. Let $\Lambda \in \mathcal{P}_0(\Gamma)$. This lends to the finite volume Hamiltonian,

$$\mathbb{H}_\Lambda = \sum_{\underline{\gamma}\in\Lambda^+} \Phi(\underline{\gamma}) \tag{399}$$

6.1.1. *Dynamics and Lieb-Robinson Bounds.* In Assumption 5.9, we write a necessary condition on the interaction Φ that ensures the dynamics generated by a finite volume Hamiltonian satisfy a Lieb-Robinson bound as in Theorem 5.10. The following proposition establishes that the interaction Φ as in equation 398 does indeed satisfy this assumption.

Proposition 6.3. *Let Φ be the map as defined in equation 398. Then, Φ satisfies Assumption 5.9.*

Proof. Let $x, y \in \Gamma$ be two specified lattice sites. We now want to show that there exists an F-function F^ϕ such that,

$$\sum_{\substack{\underline{\gamma}\in\Gamma^+: \\ x,y\in\boldsymbol{\gamma}}} \left\|\Phi(\underline{\gamma})\right\| \leq F^\phi(d(x,y)) \tag{400}$$

When $\underline{\gamma} = \{\gamma_1, \gamma_2\}$, the only way $\boldsymbol{\mu} \subset \boldsymbol{\gamma}$ is possible is if $\underline{\gamma} = \{\mu_1, \mu_2\}$ or $\underline{\gamma} = \{\mu_2, \mu_1\}$. Therefore in this case, by Proposition 5.1, there exist constants $C, c > 0$ such that $\left\|\Phi(\underline{\gamma})\right\| \leq Ce^{-cd(x,y)}$.

When summing over interactions generated by 4-tuples of sites, whose support contains $\{x, y\}$, more detail is required to prove the claim. In particular, we split the sum over 4-tuples containing $\{x, y\}$ into one sum over tuples whose diameter is $d(x, y)$ and another over tuples with diameter larger than $d(x,y)$. Starting with the sum over tuples with diameter equal to $d(x,y)$, one has,

$$\sum_{\substack{\underline{\gamma}\in\Gamma^+: \\ x,y\in\boldsymbol{\gamma}, \\ |\underline{\gamma}|=4, \\ d_{\boldsymbol{\gamma}}=d(x,y)}} \left\|\Phi(\underline{\gamma})\right\| \leq \kappa C_V e^{-\zeta d(x,y)} \left(\frac{d(x,y)}{2}\right)^4. \tag{401}$$

To obtain this bound, we make use of Proposition 6.2 to obtain exponential decay in the diameter of $\underline{\gamma}$, which is $d(x,y)$ in this case. Moreover by lattice regularity, there are at most $\kappa\left(\frac{d(x,y)}{2}\right)^2$ sites of Γ in the ball with extremal points x and y whose distance gives the ball's diameter.

Let $B(x,y)$ denote the ball with diameter $d(x,y)$ that contains these two sites. Counting the number of 4-tuples containing $\{x,y\}$ with diameter larger than $d(x,y)$ is done by decomposing $\Gamma\backslash B(x,y)$ into annuli of unit thickness and summing over all tuples containing at least one site in each of these annuli.

$$\sum_{\substack{\underline{\gamma}\in\Gamma^+: \\ x,y\in\boldsymbol{\gamma}, \\ |\underline{\gamma}|=4, \\ d_{\boldsymbol{\gamma}}>d(x,y)}} \left\|\Phi(\underline{\gamma})\right\| \leq \kappa^2 C e^{-\zeta d(x,y)} \sum_{r=1}^{\infty} \left(\frac{d(x,y)}{2}+r\right)^3 e^{-\zeta r} \tag{402}$$

$$\leq \kappa^2 C p_3(d(x,y)) e^{-\zeta d(x,y)} \tag{403}$$

In the first inequality, we use lattice regularity to bound the number of sites contained in the unit-thickness annulus of outer radius $\frac{c(x,y)}{2}+r$ and the number of sites contained in the ball of equal radius. The sum over r certainly converges, giving rise to a degree-3 polynomial in $c(x,y)$. Thus, combining the bounds in 401, 403, and the simple estimate involving 2-tuples, one has,

$$\sum_{\substack{\underline{\gamma}\in\Gamma^+: \\ x,y\in\boldsymbol{\gamma}}} \left\|\Phi(\underline{\gamma})\right\| \leq Ce^{-cd(x,y)} + \kappa C_V e^{-\zeta d(x,y)} \left(\frac{d(x,y)}{2}\right)^4 + \kappa^2 C p_3(d(x,y)) e^{-\zeta d(x,y)} \tag{404}$$

$$\sum_{\substack{\underline{\gamma}\in\Gamma^+:\\ x,y\in\gamma}} \left\|\Phi(\underline{\gamma})\right\| \le C_1 e^{-\xi d(x,y)} \tag{405}$$

The second two terms in the first inequality, being finite degree powers of $d(x,y)$ multiplied by exponential decay in $d(x,y)$, can be bounded by a constant multiplied by an exponential function with decay rate ξ that is necessarily smaller than c, ζ. As mentioned before, functions of the form $F(r) = e^{-\xi r}$ are not F-functions because they do not exactly satisfy the convolution property (by Lemma B.1 of [31], the decay rate following convolution reduces to $\xi/4$). This is sufficient for Lieb-Robinson bounds to hold with exponential decay spatially, but one can still choose an F-function with sub-exponential decay, and the claim is satisfied. □

Having proved this proposition, the dynamics generated by $\mathbb{H}_\Lambda$ thus yield a Lieb-Robinson bound and strongly continuous dynamics in the thermodynamic limit by Theorem 5.10.

Corollary 6.4. *Let τ_t^Λ denote the dynamics generated by $\mathbb{H}_\Lambda$, and denote by σ the decay rate of the $n = 0, 1$ Landau level functions $\{\chi_{\theta,n}\}_{\theta\in\Theta}$, i.e. the rate of decay in distance between $\gamma, \gamma' \in \Gamma$ of $|\langle \chi_{\gamma,n}, \chi_{\gamma',n'}\rangle|$. Then, there exist constants $G, v, \xi > 0$ such that for all $t \in \mathbb{R}$, $\gamma, \gamma' \in \Gamma$, and $n, n' \in \{0, 1\}$, one has the bound,*

$$\left\|\left\{\tau_t^\Lambda(a^\#_{\gamma n}), a^\#_{\gamma' n'}\right\}\right\| \le G e^{-\xi(d(\gamma,\gamma')-v|t|)}. \tag{406}$$

More precisely, $\xi = \frac{1}{4}\min\{\sigma, c, \zeta\}$, where c, ζ are the decay rates of the one and two-body interactions respectively. The factor of $\frac{1}{4}$ is as presented in Lemma B.1 of [31].

We now look to prove quantization of Hall conductance for interacting electrons in a magnetic field, whose bulk in the thermodynamic limit is described by $\mathbb{H}$. To do so, we shift to a toroidal geometry where we can use Laughlin's flux threading argument as has been done for interacting lattice systems, [38], [8]. So in the following subsection, we introduce a family of parametrized Hamiltonians on the torus, which we will then use to state results regarding Hall conductance.

6.2. **Statement of Results Concerning Hall Conductance.** As we introduced in the preliminaries, in finite volume we work in a toroidal setting to model the bulk of a two dimensional system. Recalling the spatial subsets of interest, $\mathbb{V}_L := \alpha[-\frac{L}{2}, \frac{L}{2}] \times \alpha[-\frac{L}{2}, \frac{L}{2}] \subset \mathbb{R}^2$ is the fundamental cell of the torus, in continuous space, and $\Gamma_L := L\alpha\mathbb{Z} \times L\alpha\mathbb{Z}$ is the sublattice of the spatial lattice Γ denoting the periodicity of the torus. Lastly, $\Lambda_L := (\mathbb{V}_L \cap \Gamma)/\Gamma_L$ will denote the torus lattice. We equip Λ_L with a metric that encodes periodicity in elements of Γ_L, i.e. d_T as introduced in equation 186. Moreover, recall the magnetically-periodized functions:

$$\chi^L_{\gamma,n}(x) = \mathbb{1}_{\mathbb{V}_L}(x) \sum_{\gamma' \in \Gamma_L} T_{\gamma'} \chi_{\gamma,n}(x). \tag{407}$$

Since $[T_a, H_L] = 0$ for all $a \in \mathbb{R}^2$, none of the translates in the formula above jump to different Landau levels. Furthermore, as proved in Proposition 5.2, there exist constants $G, \sigma_1, \sigma_2 > 0$ such that for all $\gamma, \gamma' \in \Theta_L$,

$$\left|\langle \chi^L_{\gamma,n}, \chi^L_{\gamma',n'} \rangle\right| \leq G\delta_{nn'} \left(e^{-\sigma_1 d_T(\gamma,\gamma')} + e^{-\sigma_2 L}\right). \tag{408}$$

Thus, the set of functions $\{\chi_{\gamma,0}, \chi_{\gamma,1}\}_{\theta \in \Theta_L}$ is a localized set up to exponentially decaying error in the torus length L. Towards computing the Hall conductance in this setting, we will require an automorphism over the fermionic CAR algebra $\mathcal{A}(L^2(\mathbb{R}^2))$ that threads flux according to the charge in a region. So we now introduce a local charge (number) operator, which makes use of fermionic field operators $c_x^{\#}$. These are operator-valued distributions and not elements of $\mathcal{A}_\Theta$. They satisfy the anticommutation relations,

$$\{c_x, c_y\} = 0; \ \{c_x, c_y^*\} = \delta(x - y)\mathbb{1}. \tag{409}$$

In terms of these distributions, the total fermion number operator and local number operator for the compact region $Y \subset \mathbb{R}^2$ are given by,

$$\hat{N} = \int_{\mathbb{R}^2} dx \, c_x^* c_x \tag{410}$$

$$\hat{N}_Y = \int_Y dx\, c_x^* c_x. \tag{411}$$

It then follows that for any two regions $Y, Z \subset \mathbb{R}^2$, $[\hat{N}_Y, \hat{N}_Z] = 0$.

$$\begin{aligned}
[\hat{N}_Y, \hat{N}_Z] &= \int_Y \int_Z dydz\, [c_y^* c_y, c_z^* c_z] && (412)\\
&= \int_Y \int_Z dydz\, c_y^* \{c_y, c_z^*\} c_z - c_z^* \{c_y^*, c_z\} c_y && (413)\\
&= \int_Y \int_Z dydz\, \delta(y-z) \left(c_y^* c_z - c_z^* c_y\right) && (414)\\
&= 0 && (415)
\end{aligned}$$

The final equality follows straightforwardly because if $Y \cap Z = \emptyset$, y can never equal z. If Y and Z do overlap, then the integrand becomes 0 after integration of one of the variables. Thus, for a region $Y \subset \mathbb{V}_L$, we define the local charge operator to be $Q_Y = q\hat{N}_Y$, where q is the charge of a single electron. Additionally, let $\mathcal{A}\left(L^2(\mathbb{V}_L)\right)$ denote the CAR algebra associated with the Hilbert space $L^2(\mathbb{V}_L)$. For any $f \in L^2(\mathbb{V}_L)$,

$$a^*(f) = \int_{\mathbb{V}_L} dx\, f(x) c_x^*. \tag{416}$$

We will use this to evaluate $[Q_Y, a^*(f)]$ for some $Y \subset \mathcal{L}$.

$$\begin{aligned}
[Q_Y, a^*(f)] &= q \int_Y \int_{\mathcal{L}} dydx\, f(x) [c_y^* c_y, c_x^*] && (417)\\
&= q \int_Y \int_{\mathcal{L}} dydx\, f(x) \delta(x-y) c_y^* && (418)\\
&= q \int_Y dy\, f(y) c_y^* && (419)\\
&= q a^*(\mathbb{1}_Y f) && (420)
\end{aligned}$$

In the last equality, $\mathbb{1}_Y(x)$ denotes the indicator function on Y. Similarly, $[Q_Y, a(f)] = -qa(\mathbb{1}_Y f)$. Let $\delta_Y : \mathcal{A}\left(L^2(\mathbb{V}_L)\right) \to \mathcal{A}\left(L^2(\mathbb{V}_L)\right)$ be the derivation acting on elements of the CAR algebra associated with the Hilbert space $L^2(\mathbb{V}_L)$ whose action on $A \in \mathcal{A}\left(L^2(\mathbb{V}_L)\right)$ is given by $[Q_Y, A]$. For $\varphi_1 \in [0, 2\pi]$, define $\mathcal{R}_{Y,\varphi_1} : \mathcal{A}\left(L^2(\mathbb{V}_L)\right) \to \mathcal{A}\left(L^2(\mathbb{V}_L)\right)$ be the automorphism whose action is given by $\mathcal{R}_{Y,\varphi_1} = \exp\left(i\varphi_1\delta_Y\right)$. Thus for any $f \in L^2(\mathbb{V}_L)$,

$$\mathcal{R}_{Y,\varphi_1}\left(a^*(f)\right) = e^{iq\varphi_1}a^*(\mathbb{1}_Y f) + a^*(\mathbb{1}_{\mathbb{V}_L\setminus Y} f) \tag{421}$$
$$\mathcal{R}_{Y,\varphi_1}\left(a(f)\right) = e^{-iq\varphi_1}a(\mathbb{1}_Y f) + a(\mathbb{1}_{\mathbb{V}_L\setminus Y} f).$$

Now, let $X_1 = \{(x_1, x_2) \in \mathbb{V}_L :\ x1 \leq 0\}$ and $X_2 = \{(x_1, x_2) \in \mathbb{V}_L :\ x_2 \leq 0\}$ denote the left and bottom halves of the fundamental cell $\mathbb{V}_L$ and Q_1, Q_2 the charge operators associated with each respective region. We define a specific automorphism implementing flux insertion with respect to the charges in X_1 and X_2.

$$\mathcal{R}_\varphi := \exp\left(i\left(\varphi_1 Q_1 + \varphi_2 Q_2\right)\right) = \mathcal{R}_{X_1,\varphi_1} \circ \mathcal{R}_{X_2,\varphi_2}. \tag{422}$$

In the definition above, the parameter $\varphi = (\varphi_1, \varphi_2) \in [0, 2\pi]^2 =: \mathbb{T}^2$. Remark that $\mathcal{R}_\varphi$ is precisely an automorphism of the class introduced in equation 276. We now define the subset of Θ_L that will be the support of interactions that are acted on by $\mathcal{R}_\varphi$. Let $0 < R < \frac{L}{2}$ and $Y_R = \{\gamma \in Y_L : |\gamma_x| \leq R\} \cup \{\gamma \in Y_L : |\gamma_y| \leq R\}$. First, define the finite volume Hamiltonians on the torus:

$$\mathbb{H}_{\Lambda_L} = \sum_{\underline{\gamma} \in \Lambda_L^+} \Phi(\underline{\gamma}), \tag{423}$$

with interaction $\Phi : \Lambda_L^+ \to \mathcal{A}_{01}^L$ as defined in equation 398, where we let $\mathcal{A}_{01}^L$ be the algebra generated by $\{a^\#(\chi_{\gamma,n}^L) : \gamma \in \Lambda_L, n \in \{0, 1\}\}$. Note that for any $Z \subset \Lambda_L$, we can write

$$\mathbb{H}_{\Lambda_L} := \mathbb{H}_{\Lambda_L}^{Z,1} + \mathbb{H}_{\Lambda_L}^{Z,2} \tag{424}$$
$$= \sum_{\substack{\underline{\gamma} \in \Lambda_L^+ : \\ \boldsymbol{\gamma} \cap Z \neq \emptyset}} \Phi(\underline{\gamma}) + \sum_{\substack{\underline{\gamma} \in \Lambda_L^+ : \\ \boldsymbol{\gamma} \subset \Lambda_L \setminus Z}} \Phi(\underline{\gamma}) \tag{425}$$

Adopting the nomenclature used in [8], we now introduce the families of twist and twist-antitwist Hamiltonians. Respectively, these are defined to be,

$$\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) := \mathcal{R}_\varphi\left(\mathbb{H}_{\Lambda_L}^{Y_R,1}\right) + \mathbb{H}_{\Lambda_L}^{Y_R,2} \tag{426}$$

$$\mathbb{H}_{\Lambda_L}(\varphi^+,\varphi^-) := \mathcal{R}_{-\varphi^-}\left(\tilde{\mathbb{H}}_{\Lambda_L}(\varphi^+ + \varphi^-)\right) \tag{427}$$

The twist Hamiltonians in 426 have interactions that undergo flux insertion only if their supports intersect the strips Y_R. Conversely, the twist-antitwist Hamiltonians in 427 undergo flux insertion globally, with different fluxes threaded depending on the support of the interaction. We also introduce notation for the Hamiltonian that is globally transformed in a uniform manner:

$$\mathbb{H}_{\Lambda_L}(\varphi) := \mathbb{H}_{\Lambda_L}(\varphi,-\varphi) = \mathcal{R}_{\varphi}\left(\mathbb{H}_{\Lambda_L}\right). \tag{428}$$

We now state assumptions regarding the spectra of these operators.

Assumption 6.5. *$\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ has a non-degenerate ground state with a spectral gap $\epsilon_L(\varphi)$ between its lowest two eigenvalues that is bounded from below uniformly in L and $\varphi \in \mathbb{T}^2$, i.e. there exists $\epsilon > 0$ such that $\inf_{L>0}\inf_{\varphi\in\mathbb{T}^2}\epsilon_L(\phi) \geq \epsilon$.*

By unitary equivalence, this necessarily implies that the twist-antitwist Hamiltonians $\mathbb{H}_{\Lambda_L}(\varphi^+,\varphi^-)$ also have a non-degenerate ground state with energy gap to the first excited energy that is bounded from below uniformly in L and $\varphi^+,\varphi^- \in \mathbb{T}^2$. Now, we denote by $\tilde{P}_\varphi$ the ground state projection associated with $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$, where this projection is as introduced in 317 for the finite dimensional setting. Then, for $A \in \mathcal{A}_{\Lambda_L}$, we have the following evaluation of expectation values in terms of the positive linear functional $\tilde{\omega}_{\varphi,L}$.

$$\tilde{\omega}_{\varphi,L}(A) = \mathrm{Tr}(\tilde{P}_\varphi A) \tag{429}$$

Note here that the bracket notation to indicate matrices and L dependence are dropped, and we bear in mind the isomorphism between A and a finite dimensional matrix in $\mathcal{M}_{2^{N_L}}$, where N_L is $\dim \mathcal{H}_{\Lambda_L}$. The Hall adiabatic curvature $\kappa_L(\varphi)$ is then defined as follows.

$$\kappa_L(\varphi) = i\tilde{\omega}_{\varphi,L}\left(\left[\partial_1\tilde{P}_\varphi, \partial_2\tilde{P}_\varphi\right]\right) \tag{430}$$

We may now state the three main results of this work. The first theorem concerns the φ-independence of Hall adiabatic curvature.

Theorem 6.6. *The Hall adiabatic curvature is exponentially φ-independent, meaning there exist constants $C, \sigma > 0$ not depending on L such that,*

$$\sup_{\varphi,\varphi' \in \mathbb{T}^2} |\kappa_L(\varphi) - \kappa_L(\varphi')| \leq Ce^{-\sigma L}. \tag{431}$$

Due to results such as the seminal work of Avron and Seiler, [7], it follows that the integral of κ is an integer multiple of 2π, so asymptotic convergence to $2\pi n$ is an immediate consequence of Theorem 6.6, with $n \in \mathbb{Z}$.

Theorem 6.7. *For any $\varphi \in \mathbb{T}^2$, there exist constants $C, \sigma > 0$ not depending on L such that,*

$$\inf_{n \in \mathbb{Z}} |\kappa_L(\varphi) - 2\pi n| \leq Ce^{-\sigma L}, \tag{432}$$

with the minimizing $n_{min} \in \mathbb{Z}$ being φ-independent as well.

Furthermore, due to our proof of the existence of the spectral flow automorphism $\alpha_{(\cdot)}$ in the thermodynamic limit, it also holds that the Hall adiabatic curvature has a well defined limit as $L \to \infty$.

Theorem 6.8. *Assume that for all $A \in \mathcal{A}_\Theta$, $\lim_{L\to\infty} \tilde{\omega}_{0,L}$ exists. Then, the thermodynamic limit of κ_L exists.*

$$\lim_{L\to\infty} \kappa_L(0) \in 2\pi\mathbb{Z} \tag{433}$$

The following section is devoted to proving these three theorems, where we present several intermediate results that hinge on our proven Lieb-Robinson bounds and quasi-locality of the spectral flow automorphism. First, we briefly comment on the extension of these results to fractional quantization of Hall conductance.

Fractional Quantization. Our proof of integer quantization of Hall conductance following the strategy of Hastings and Michalakis relies on the crucial assumption of a non-degenerate ground state, as in Assumption 6.5. However, the construction of quasi-adiabatic evolution, which supports the validity of Theorems 6.6-6.8, does not only function with one-dimensional ground state spaces. It is exactly the consideration of degenerate ground state spaces that gives rise to fractional quantization using the methods of this section. We comment on this extension as noted in Section 9 of [38] and 3.1 of [8].

More generally, one can let each $\tilde{H}_{\Lambda_L}(\varphi)$ have a p-fold degenerate ground state space and denote by $\tilde{Q}_\varphi = \frac{1}{p}\tilde{P}_\varphi$ the orthogonal projection onto the equally weighted superposition of degenerate ground states, referred to as the chaotic ground state projection. Let $\omega_{\varphi,c}$ denote the ground state function associated with $\tilde{Q}_\varphi$. With respect to $\tilde{P}_\varphi$, the integer quantization still arises from Theorems 6.6-6.8 due to the gapped spectrum; however, the factor of p^{-1} carried through in $\tilde{Q}_\varphi$ results in fractional quantization of Hall conductance given by $\frac{n}{p}$ with $n \in \mathbb{Z}$.

To prove fractional quantization for any individual ground state, we require one more condition, which takes the general name of topological order. This notion has been described in the cited references and made further precise in the work of Nachtergaele, Sims, and Young [75]. We state a more general form, which would be sufficient to establish the claim. Let $\omega_{\varphi,p}$ be any pure ground state belonging to the degenerate ground state space, then for any local $A \in \mathcal{B}_X$ with $X \subset \Lambda_L$, one has,

$$\omega_{\varphi,p}(A) = \omega_{\varphi,c}(A) + \mathcal{O}(L^{-a}), \tag{434}$$

for some $a > 0$. Here $\mathcal{O}(L^{-a})$ indicates decay of order no slower than L^{-a}. Since we ultimately show that the observable, whose expectation value gives Hall adiabatic curvature, is quasi-local, this suggests that the respective Hall adiabatic curvatures $\kappa_{L,p}$ and $\kappa_{L,c}$ are asymptotically close:

$$|\kappa_{L,p}(\varphi) - \kappa_{L,c}| \leq C_a \mathcal{O}(L^{-a}). \tag{435}$$

This thus establishes fractional quantization of Hall conductance for individual ground states as well. We now proceed with the general proof of integer quantization, which implies such extensions.

6.3. **Proofs of Theorems 6.6-6.8.**

Before presenting the general strategy for proving the three theorems in question, we first introduce some notation to build the tools we will use in our proofs. Moving forward, let $\mathbb{1}_{ij}(x)$ denote the indicator function associated with the four quadrants of $\mathbb{V}_L$:

$$\mathbb{1}_{jk}(x) = \begin{cases} \mathbb{1}_{\mathbb{V}_L \setminus (X_1 \cup X_2)}(x) & j,k = 0 \\ \mathbb{1}_{X_1 \setminus (X_1 \cap X_2)}(x) & j = 1, k = 0 \\ \mathbb{1}_{X_2 \setminus (X_1 \cap X_2)}(x) & j = 0, k = 1 \\ \mathbb{1}_{X_1 \cap X_2}(x) & j,k = 1 \end{cases} \tag{436}$$

For abbreviated notation, let $X_{jk} \subset \mathbb{V}_L$ denote the quadrants associated with each configuration of j,k. With our periodic setting of sites $\gamma \in \Lambda_L$ understood, we suppress L-dependence in the fermionic operator abbreviations for the remainder of this section. We can then write the action of $\mathcal{R}_\varphi$ on $a^*_{\gamma,n} := a^*(\chi^L_{\gamma,n})$ as,

$$\mathcal{R}_\varphi\left(a^*_{\gamma,n}\right) = \sum_{j,k=0}^{1} e^{iq(j\varphi_1 + k\varphi_2)} a^*(\mathbb{1}_{jk}\chi^L_{\gamma,n}) = a^*(U_\varphi \chi^L_{\gamma,n}), \tag{437}$$

where we have introduced the operator U_φ which is unitary as a map from $L^2(\mathbb{V}_L)$ to itself. Furthermore, one can also write the operators,

$$\frac{\partial}{\partial \varphi_1}\mathcal{R}_\varphi(a^*_{\gamma,n}) = iq \sum_{j,k=0}^{1} j e^{iq(j\varphi_1 + k\varphi_2)} a^*(\mathbb{1}_{jk}\chi^L_{\gamma,n}) \tag{438}$$

(439)
$$\frac{\partial}{\partial \varphi_2}\mathcal{R}_\varphi(a^*_{\gamma,n}) = iq\sum_{j,k=0}^{1} k e^{iq(j\varphi_1+k\varphi_2)} a^*(\mathbb{1}_{jk}\chi^L_{\gamma,n})$$

By construction, equations 437 and 438 can be written for $a_{\gamma,n}$ too, via complex conjugation and taking the adjoint $(a^*)^* = a$. For brevity, let $\partial_{1(2)} := \frac{\partial}{\partial\varphi_{1(2)}}$. One can also explicitly write the action of $\mathcal{R}_\varphi$ and $\partial_{1(2)}\mathcal{R}_\varphi$ acting on $a^\#_{\underline{\gamma,n}}$ for some m-body monomial.

(440)
$$\mathcal{R}_\varphi(a_{\underline{\gamma,n}}) = \sum_{\substack{j_1,\ldots,j_{2m}\\ k_1,\ldots,k_{2m}}} \exp\left(iq\left[\sum_{n=1}^{m}(j_{m+n}-j_n)\varphi_1 + \sum_{n=1}^{m}(k_{m+n}-k_n)\varphi_2\right]\right)$$
$$\times \qquad a^*(\mathbb{1}_{j_{2m}k_{2m}}\chi^L_{\gamma_{2m},n_{2m}})\ldots a(\mathbb{1}_{j_1k_1}\chi^L_{\gamma_1,n_1})$$

(441)
$$\partial_1\mathcal{R}_\varphi(a_{\underline{\gamma,n}}) = iq\sum_{\substack{j_1,\ldots,j_{2m}\\ k_1,\ldots,k_{2m}}}\sum_{n=1}^{m}(j_{m+n}-j_n)\exp\left(iq\left[\sum_{n=1}^{m}(j_{m+n}-j_n)\varphi_1 + \sum_{n=1}^{m}(k_{m+n}-k_n)\varphi_2\right]\right)$$
$$\times\, a^*(\mathbb{1}_{j_{2m}k_{2m}}\chi^L_{\gamma_{2m},n_{2m}})\ldots a(\mathbb{1}_{j_1k_1}\chi^L_{\gamma_1,n_2})$$

Evident by these explicit expressions, the action of flux insertion (composed with partial differentiation with respect to a flux component) results in fermionic operators that no longer belong to $\mathcal{A}^L_{01}$. In fact, the larger CAR algebra in question is associated with the extended Hilbert space that we shall define as,

(442)
$$\mathcal{H}_L(\varphi) := \overline{\text{span}\left\{\chi^L_{\gamma,n}, U_\varphi\chi^L_{\gamma,n}, \partial_j U_\varphi \chi^L_{\gamma,n} :\ \gamma\in\Lambda_L, n\in\{0,1\}, \varphi\in\mathbb{T}^2, j=1,2\right\}}.$$

Let $\mathcal{A}^L_{01}(\varphi)$ be the enlarged CAR algebra generated by $\{a(f), a^*(f) :\ f\in\mathcal{H}_L(\varphi)\}$. By the exponential localization of the functions $\{\chi^L_{\gamma,n}\}$ and their inner products, up to exponentially small corrections in L, the same follows for both $\{U_\varphi\chi^L_{\gamma,n}\}$ and $\{\partial_j U_\varphi\chi^L_{\gamma,n}\}$. However, this localization is purely with respect to the spatial lattice Λ_L. The precise form of U_φ results in $U_\varphi\chi^L_{\gamma,n}$ losing its identification with the n^{th} Landau level, so naturally, there is no exact or approximate sense in which $\chi^L_{\gamma,0}$ can be orthogonal to $U_\varphi\chi^L_{\gamma',1}$. Thus, the extended set of functions that span $\mathcal{H}_L(\varphi)$ do not entirely satisfy Assumption 5.17, due to localization in sites of Γ and not Θ in general. In the present setting with the two lowest Landau levels, the localization still gives rise to Lieb-Robinson bounds of the form of Corollary 406, with the caveat that $v \to 2v$ due to the loss of orthogonality of different Landau level labeled functions.

We now show an important calculation concerning the L^2 norm of $\mathbb{1}_{jk}(x)U_\varphi \chi^L_{\gamma,n}$. Without loss of generality, we show the computation for the $n = 0$ function.

$$\left\|\mathbb{1}_{jk}U_\varphi \chi^L_{\gamma,n}\right\| \leq \int_{X_{jk}} dx\, \left|\chi^L_{\gamma,0}(x)\right|^2 \tag{443}$$

$$\leq \int_{X_{jk}} dx \left|\sum_{\gamma' \in \Gamma_L} T_{\gamma'}\chi_\gamma(x)\right|^2 \tag{444}$$

$$\leq \int_{X_{jk}} dx\, |\chi_\gamma(x)|^2 + \mathcal{O}(e^{-L}) \tag{445}$$

$$\leq \frac{1}{2\pi\ell^2}\int_{X_{jk}} dx\, e^{-|x-\gamma|^2/4\ell^2} + \mathcal{O}(e^{-L}) \tag{446}$$

$$\leq Ce^{-\zeta d(\gamma, X_{jk})} + \mathcal{O}(e^{-L}) \tag{447}$$

In the third inequality, we use the fact that the nontrivial translates result in terms decaying exponentially in L, which we separate as $\mathcal{O}(e^{-L})$. Then to reach the final inequality, we split up the Gaussian into two pieces, meaning the positive constant $\zeta < 1$. We bound one of these terms by noting that the shortest distance between x and γ is given by the distance between γ and the domain of integration X_{jk}. What remains can be trivially estimated by integrating over all of $\mathbb{R}^2$. We now introduce the relevant Hastings generators that we will use throughout the proofs of Theorems 6.6-6.8. Let $\tilde{\tau}^\varphi_t$ and τ^φ_t be the Heisenberg dynamics generated by the twist Hamiltonians $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ and twist-antitwist Hamiltonians $\mathbb{H}_{\Lambda_L}(\varphi)$ respectively. Then, we have,

$$\tilde{K}_j(\varphi) = \int_{\mathbb{R}} dt\, W_\epsilon(t)\tilde{\tau}^\varphi_t\left(\partial_j \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right), \tag{448}$$

$$K_j(\varphi) = \int_{\mathbb{R}} dt\, W_\epsilon(t)\tau^\varphi_t\left(\partial_j \mathbb{H}_{\Lambda_L}(\varphi)\right). \tag{449}$$

Remark that since $\mathbb{H}_{\Lambda_L}(\varphi) = \mathcal{R}_\varphi(\mathbb{H}_{\Lambda_L})$, it follows that the Hastings generators are also related in this way, meaning $K_j(\varphi) = \mathcal{R}_\varphi(K_j)$ where $K_j := K_j(0)$. Additionally, letting $s \in [0,1]$, there exists a path of Hamiltonians $\mathbb{H}^\varphi_{\Lambda_L}(s) = \mathbb{H}_{\Lambda_L}(\varphi, (s-1)\varphi)$ that allows us to smoothly deform $\mathbb{H}_{\Lambda_L}(\varphi)$ into $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) = \mathbb{H}_{\Lambda_L}(\varphi, 0)$. This parametrization leads to the s-dependent generators:

$$D(s) = \int_{\mathbb{R}} dt\, W_\epsilon(t)\tau^s_t\left(\partial_s \mathbb{H}^\varphi_{\Lambda_L}(s)\right), \tag{450}$$

where τ^s_t denotes Heisenberg time evolution generated by $\mathbb{H}^\varphi_{\Lambda_L}(s)$. Crucially, $\partial_s\mathbb{H}^\varphi_{\Lambda_L}(s)$ is localized away from $(0,0)$, which will help us show that the ground states $\tilde{\omega}_{\varphi,L}$ of $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ and $\omega_{\varphi,L}$ of $\mathbb{H}_{\Lambda_L}(\varphi)$ agree on observables supported near $(0,0)$.

Carrying on, let $Z_R := [-R,R]\times[-R,R]$ and $\Xi_R := Z_R \cap \Lambda_L$. These two subsets denote the continuous space and sites of Λ_L restricted to the $2R\times 2R$ square centered at $(0,0)$. Due to the short range of the interaction and the fact that $\partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ enforces scattering between X_{0k} and $X_{1k'}$ or X_{j0} and $X_{j'1}$, it follows that $\left[\tilde{K}_1(\varphi), \tilde{K}_2(\varphi)\right]$ is most strongly supported by lattice sites in a neighborhood of Ξ_R. When we restrict $[K_1(\varphi), K_2(\varphi)]$ to this neighborhood, we may obtain a good approximation of the former. This is because in this neighborhood, flux insertion affects both $\tilde{\mathbb{H}}_{\Lambda_L}$ and $\mathbb{H}_{\Lambda_L}$ most strongly close to Y_R.

proof of Theorem 6.6. Towards proving an error estimate on this approximation, we will need a series of lemmas. The first one concerns the difference in dynamics generated by $\tilde{\mathbb{H}}_{\Lambda_L}$ versus $\mathbb{H}_{\Lambda_L}$. As we will also need to keep track of quadrant indices j,k in X_{jk} at times, we re-introduce the θ site index so that we may write $a^\#_{\theta,jk} := a^\#(\mathbb{1}_{jk}\chi^L_{\gamma,n})$.

Lemma 6.9 (Difference of Dynamics)**.** *Fix $\varphi = (\varphi_1,\varphi_2)\in\mathbb{T}^2$, and denote by $\Xi_{R,r}$ an r-sized fattening of Ξ_R. Let $\underline{\gamma}\in\Xi^+_{R,r}$ and $\underline{n}\in\{0,1\}^{|\underline{\gamma}|}$. Then, there exist positive constants ζ, v, and c such that the following bound holds,*

$$\left\|\tau^\varphi_t(a_{\underline{\gamma,n}}) - \tilde{\tau}^\varphi_t(a_{\underline{\gamma,n}})\right\| \le P_1(L,R)|t|e^{-cR} + |\boldsymbol{\gamma}|P_2(L,R)e^{-\zeta\left(\frac{L}{2}-(2R+r)\right)}\left(e^{\zeta v|t|}-1\right), \tag{451}$$

where $P_1(L,R)$ and $P_2(L,R)$ are finite degree polynomials in L and R.

Proof.

$$\tau^{\varphi}_{-t}\tilde{\tau}^{\varphi}_{t}(a_{\underline{\gamma,n}}) - a_{\underline{\gamma,n}} = -i\int_0^t dt'\tau^{\phi}_{-t'}\left(\left[\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi), \tilde{\tau}^{\varphi}_{t'}(a_{\underline{\gamma,n}})\right]\right) \tag{452}$$

Note that explicitly,

(453)

$$\mathbb{H}_{\Lambda}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) = \sum_{\substack{\underline{\gamma'}\in\Lambda_L^+:\\ \boldsymbol{\gamma'}\subset\Lambda_L\backslash Y_R}}\sum_{\substack{j_1,\dots,j_{2p}\\ k_1,\dots,k_{2p}}}\left(1-\exp\left(iq\left[\sum_{m=1}^{p}\left(j_{p+m}-j_m\right)\varphi_1 + \sum_{m=1}^{p}\left(k_{p+m}-k_m\right)\varphi_2\right]\right)\right)$$
$$\times\sum_{\underline{n}\in\{0,1\}^{2p}}\phi(\underline{\gamma,n})a^*_{\theta_{2p},j_{2p}k_{2p}}\dots a_{\theta_1,j_1k_1}$$

To avoid writing $\left|\underline{\gamma}'\right|$ in subscripts, we introduce the index $p := |\gamma|'/2$, where p can be either 1 or 2. In this expression, we make use of the interaction construction from 398 and re-introduce the θ site index so that we may write $a^{\#}_{\theta,jk} := a^{\#}(\mathbb{1}_{jk}\chi^{L}_{\gamma,n})$. Thus, the interaction terms in this difference possess quadrant-based scattering across at least one of the edge boundaries. This is imposed by the flux dependent prefactor of the summand. From Corollaries 5.11 and 406, there exist constants $\zeta, v > 0$ such that,

$$\left\|\left[a^{\#}_{\theta_{2p},j_{2p}k_{2p}}\dots a^{\#}_{\theta_1,j_1k_1}, \tilde{\tau}^{\varphi}_{t}(a_{\underline{\gamma,n}})\right]\right\| \le 4Ce^{-\zeta(d_T(\underline{\gamma}',\underline{\gamma})-v|t|)}. \tag{454}$$

To further refine our estimate and make use of the fact that $a_{\underline{\gamma,n}}$ is localized in a neighborhood of Ξ_R, we will more carefully partition the interactions with lattice support contained in $\Lambda_L\backslash Y_R$. To do so, we introduce another region:

$$\mathcal{B}_R := \left\{\gamma = (\gamma_x,\gamma_y)\in\Lambda_L : \left|\frac{L}{2}-\gamma_x\right| \le R \text{ or } \left|\frac{L}{2}-\gamma_y\right| \le R\right\}.$$

We will then expand the difference interactions as,

$$\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) = \left(\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)^{\mathcal{B}_R,1} + \left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)^{\mathcal{B}_R,2} \tag{455}$$

Those interactions with frame lattice support in $\Lambda_L \setminus (Y_R \cup \mathcal{B}_R)$ are all supported at least a distance R from the boundaries between quadrants or have diameters of at least $2R$, meaning that,

$$\left\| \left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) \right)^{\mathcal{B}_R,2} \right\| \leq P(L) e^{-cR}, \tag{456}$$

where $P(L)$ is a finite degree polynomial in L. The monomial $a_{\underline{\gamma,n}}$ can have support overlapping with this region, so the Lieb-Robinson bound does not give us any useful spatial decay. Moving on to those terms with support intersecting $\mathcal{B}_R$, we can further divide these as follows.

$$\begin{aligned} \left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) \right)^{\mathcal{B}_R,1} = &\left(\left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) \right)^{\mathcal{B}_R,1} \right)^{\mathcal{B}_R^c,1} + \\ &\left(\left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) \right)^{\mathcal{B}_R,1} \right)^{\mathcal{B}_R^c,2} \end{aligned} \tag{457}$$

This distinguishes between interactions fully frame-supported inside $\mathcal{B}_R$ and those that also have supports outside this strip. Interactions in the second grouping above are fully inside the strip $\mathcal{B}_R$ and have support that is at least a distance $\frac{L}{2} - (2R + r)$ from $\boldsymbol{\gamma}$. Thus, for an interaction of this form, equation 454 can be made more precise:

$$\left\| \left[a^{\#}_{\theta_{2p}, j_{2p} k_{2p}} \ldots a^{\#}_{\theta_1, j_1 k_1}, \tilde{\tau}^{\varphi}_t (a_{\underline{\gamma,n}}) \right] \right\| \leq 4 C e^{-\zeta(\frac{L}{2} - (2R+r) - v|t|)}. \tag{458}$$

Summing over all such interactions adds a polynomial prefactor $P(L, R)$ from volume estimates and lattice regularity. Lastly, we consider those interactions that have at least one, but not all, site in $\mathcal{B}_R$. The norm of these terms can be estimated as follows.

$$\begin{aligned} \left\| \left(\left(\mathbb{H}_{\Lambda_L}(\varphi) - \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) \right)^{\mathcal{B}_R,1} \right)^{\mathcal{B}_R^c,1} \right\| &\leq 2LRe^{-cR} \sum_{\tilde{r}=1}^{\infty} P(\tilde{r}) e^{-\sigma \tilde{r}} \\ &\leq CLRe^{-cR} \end{aligned} \tag{459}$$

Putting all of these bounds together and integrating over time, we obtain the final bound as in the statement of the lemma.

$$\left\|\tau_t^{\varphi}(a_{\underline{\gamma,n}}) - \tilde{\tau}_t^{\varphi}(a_{\underline{\gamma,n}})\right\| \leq P_1(L,R)|t|e^{-cR} + P_2(L,R)e^{-\zeta\left(\frac{L}{2}-(2R+r)\right)}\left(e^{\zeta v|t|}-1\right) \tag{460}$$

□

Next, we define a finite subset of Λ_L that will be used to construct an approximation of the Hall adiabatic curvature. Let $r := r(L) > 0$ be a parameter used to describe a fattening $\Xi_{R,r}$ of Ξ_R by a distance r, shown in Figure 5. A preliminary feature of this approximation involves replacing $\partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ by $\partial_j\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$, i.e. the interactions of $\partial_j\mathbb{H}_{\Lambda_L}(\varphi)$ with lattice support strictly contained in $\Xi_{R,r}$. The next lemma provides an error estimate on this approximation of $\partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$.

Lemma 6.10 (Partial Derivative Approximation)**.** *Let $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ denote the family of twist Hamiltonians as previously defined. For $X \subset \Lambda_L$, let $\partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_X$ denote the partial derivative with respect to the j^{th} component of φ with support restricted to X. Then, there exist positive constants c and C as well as a finite degree polynomial $P(r)$ such that the following bound holds.*

$$\left\|\partial_j\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}} - \partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right\| \leq CqP(r)e^{-cR} \tag{461}$$

Proof. We begin by explicitly analyzing the norm of the interaction terms specified by this difference in the case that the derivative is with respect to φ_1. The same arguments will hold when we consider ∂_2 instead.

(462)

$$\left\|\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}} - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right\| = \left\|\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}\setminus\Xi_R} - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}\setminus\Xi_R}\right\|$$

$$\leq \left\|\sum_{\substack{\underline{\gamma}\in\Lambda_L^+:\\ \gamma\subset\Xi_{R,r}\setminus\Xi_R}} \partial_1\mathcal{R}_{\varphi}(\Phi(\underline{\gamma}))\right\| \tag{463}$$

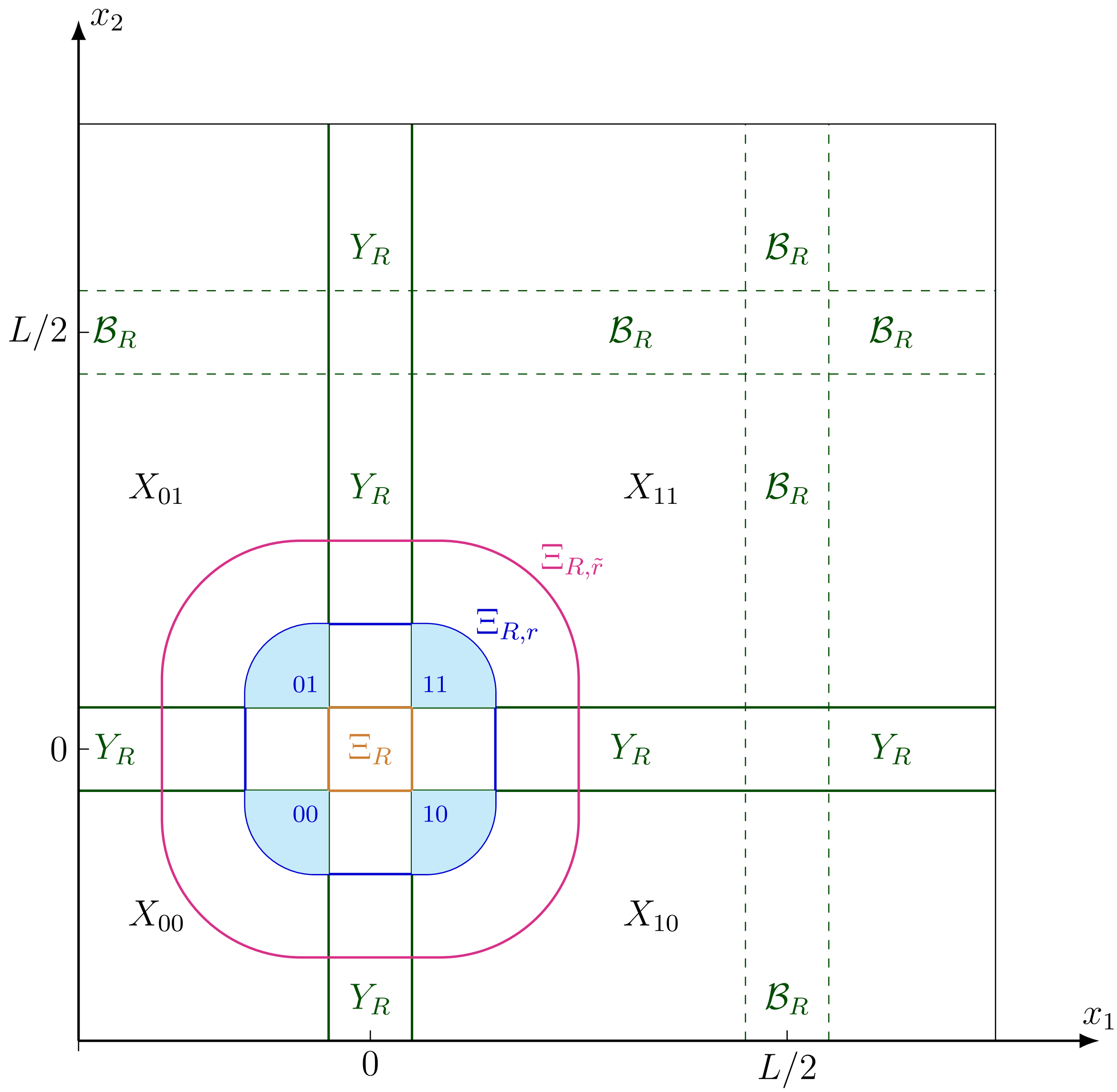

FIGURE 5. Diagram illustrating the relevant regions. $\left[\tilde{K}_1(\varphi), \tilde{K}_2(\varphi)\right]$ is supported in a neighborhood of the central square Ξ. For the purposes of approximation, we approximate $\partial_j \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ by $\partial_j \mathbb{H}_{\Lambda_L}(\varphi)$ within the first circle, i.e. $\Xi_{R,r}$. Then, we use the larger ball with fattening $\tilde{r}$ of Ξ_R as a region where Hall adiabatic curvature will be approximated.

$$\leq q \sum_{\substack{j_1,\ldots,j_{2p} \\ k_1,\ldots,k_{2p}}} \sum_{\substack{\underline{\gamma}\in\Lambda_L^+: \\ \gamma\subset\Xi_{R,r}\setminus\Xi_R}} \tag{464}$$

$$\left\| \left(\sum_{m=1}^{p} j_{p+m} - \sum_{m=1}^{p} j_m \right) \sum_{\underline{n}\in\{0,1\}^{2p}} \phi(\underline{\gamma, n}) a^*_{\gamma_{2p},j_{2p}k_{2p}} \ldots a_{\gamma_1,j_1k_1} \right\| \tag{465}$$

$$\leq qC \sum_{\substack{j_1,\ldots,j_{2p}\\ k_1,\ldots,k_{2p}}} \sum_{\substack{\underline{\gamma}\in\Lambda_L^+:\\ \underline{\gamma}\subset\Xi_{R,r}\setminus\Xi_R}} e^{-\sigma d_{\underline{\gamma}}} \left| \left(\sum_{m=1}^{p} j_{p+m} - \sum_{m=1}^{p} j_m \right) \right|$$

$$\exp\left(-\zeta \sum_{i=1}^{2p} d_T(\gamma_i, X_{j_i k_i}) \right)$$

Thus far, we have only used assumptions and the explicit forms of the twist/anti-twist Hamiltonians to generate these inequalities. The final inequality features competing decay between the diameter of the sets $\underline{\gamma}$ and the distance of sites in these sets from the left/right halves respectively. The reason being that ∂_1 identifies those interactions whose supports change from right to left half and vice versa. With reference to Figure 5, the support of $\chi^L_{\gamma',n'}$ needs to hop from one of X_{10} or X_{11} to the support of $\chi^L_{\gamma,n}$ being one of X_{01} or X_{00} and vice versa. On the other hand, γ and γ' are situated in one of the shaded blue regions.

To proceed with our estimate, we will rewrite the sum over $\underline{\gamma}$ supported in $\Xi_{R,r}\setminus\Xi_R$ as follows. This particular region, the shaded lobes in Figure 5, can be divided in such a way that each lobe is the union of unit width vertical strips of height bounded from above by r. In the case that we consider ∂_2 acting on either flux Hamiltonian, we would use unit height horizontal strips, but the subsequent strategy would be the same. Now, we sum over all $\underline{\gamma}$ contained in the strips furthest from Ξ_R, where either diam$(\underline{\gamma} \geq 2(R+r)$ or for at least one site $\gamma \in \underline{\gamma}$, $d_T(\gamma, X_j k) = R + r$.

Additionally, note that with $\underline{\gamma}$ being supported in up to four lobes, there are at most $4r$ sites that could be involved in either the 1 or 2 body interactions. Then, we sum over all $\underline{\gamma}$ contained in the furthest two strips, where the distances mentioned just now decrease to $(R + r - 1)$, up to constant factor, and the number of sites to choose from increases to $8r$. For notation, denote by $Z^i_{r,r'} := \{\gamma = (\gamma_x, \gamma_y) \in \Xi_{R,r} : |\gamma_i| \geq r - r',\ i = x, y\}$. Noting that

$p \leq 2$, $|(\sum_{m=1}^{p} j_{p+m} - \sum_{m=1}^{p} j_m)| \leq 2$,we continue the estimate as follows.

$$\left\|\partial_1 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}} - \partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right\| \leq 2qC \sum_{\substack{j_1,\dots,j_{2p} \\ k_1,\dots,k_{2p}}} \sum_{r'=1}^{r} \sum_{\substack{\underline{\gamma}\in\Lambda_L^+: \\ \gamma\subset Z^1_{r,r'}}} e^{-c(R+r-r')} \tag{466}$$

$$= 2qCe^{-cR} \sum_{\substack{j_1,\dots,j_{2p} \\ k_1,\dots,k_{2p}}} \sum_{r'=1}^{r} \sum_{\substack{\underline{\gamma}\in\Lambda_L^+: \\ \gamma\subset Z^1_{r,r'}}} e^{-c(r-r')} \tag{467}$$

$$\leq Cqe^{-cR} r^5 \int_0^r dr'(r')^3 e^{-c(r-r')} \tag{468}$$

$$\leq CqP(r)e^{-cR} \tag{469}$$

In the last inequality, $P(r)$ is a polynomial in r which makes use of a worst case counting estimate of the interactions due to the 2-body terms. By the same reasoning, we obtain an identical upper bound when ∂_1 is replaced by ∂_2. □

The final prerequisite lemma we need before constructing the full approximation of the Hall curvature concerns the operator norm of $\partial_j \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$ as well as how much $\partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$ commutes with $\partial_2 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$.

Lemma 6.11. *Fix $r > 0$ and let $\Xi_{R,r}$ be the fattening of Ξ_R by a distance r. Denote by $\mathfrak{r} := \min\{R, r\}$. Then there exist positive constants ζ, v and a finite degree polynomial P such that the following bound holds, where the partial differentiation with respect to φ_1 and φ_2 can be switched.*

$$\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right), \partial_2 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right]\right\| \leq CP(L,R,r)e^{-\zeta(\mathfrak{r}-v|t|)} \tag{470}$$

Using the same proof strategy that will be written below, a similar estimate can be obtained in the case that $\partial_2 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ has its frame lattice support restricted to $\Xi_{R,r}$.

Proof. By definition, the difference $\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$ can be expanded as shown in the subsequent equation.

$$\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}} = iq \sum_{\substack{\underline{\gamma}\in(\Lambda_L\setminus\Xi^+_{R,r}:\\ \boldsymbol{\gamma}\cap Y_R\neq\emptyset}} \sum_{\substack{j_1,\ldots,j_{2p}\\ k_1,\ldots,k_{2p}}} \left(\sum_{m=1}^{p} j_{p+m} - \sum_{m=1}^{p} j_m\right) \times \tag{471}$$

$$e^{\left(iq\left(\sum_m j_{p+m}-\sum_m j_m\right)\varphi_1+\left(\sum_m k_{p+m}-\sum_m k_m\right)\varphi_2\right)} \sum_{\underline{n}\in\{0,1\}^{2p}} \phi(\underline{\gamma,n}) a^*_{\theta_{2p},j_{2p}k_{2p}} \ldots a_{\theta_1,j_1k_1}$$

$\partial_2\tilde{\mathbb{H}}_{\Lambda_L}$ is the same series of interactions except with $\Xi_{R,r}$ included in the support and the outer sums over $j_{(\cdot)}$ replaced by outer sums over $k_{(\cdot)}$. With these lattice restrictions on the supports of each set of interactions, we do not have control on the minimal distance between supports of interactions in $\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$ and in $\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$.

However, these interactions do need to have at least one point in Y_R, which implies one of the following facts about their commutator. 1) The sites are in vertical and horizontal strips respectively and are thus separated by a minimum distance of r. 2) The sites are in the same strip, meaning that either one of these sites or another from their interaction supports is at least a distance R from its required quadrant. If $\underline{\gamma}$ and $\underline{\tilde{\gamma}}$ are the two respective supports of two interaction terms with points γ^* and $\tilde{\gamma}^*$ respectively being their sites contained in Y_R, this implies that diam($\underline{\gamma}$)+ diam($\underline{\tilde{\gamma}}$) $+ d_{\mathbb{T}}(\gamma^*, \tilde{\gamma}^*) \geq \mathcal{C}\min\{R, r\} =: \mathcal{C}\mathfrak{r}$.

For demonstration in the subsequent calculation, let $\underline{\gamma}$ and $\underline{\tilde{\gamma}}$ denote tuples associated with interactions from $\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi) - \partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}$ and $\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ respectively. Note that we will not require quadrant dependence here, so in the first inequality itself, we estimate away this dependence to recover the original operators $a^{\#}_{\gamma,n}$. One term contributing towards the quantity of interest in the statement of the lemma is the following,

$$\left\| \left[\tilde{\tau}^{\varphi}_t \left(\phi(\underline{\gamma,n}) a_{\underline{\gamma,n}} \right), \phi(\underline{\tilde{\gamma},n}) a_{\underline{\tilde{\gamma}}} \right] \right\| \leq Ce^{-\sigma(d_{\boldsymbol{\gamma}}+d_{\tilde{\boldsymbol{\gamma}}})} e^{-\zeta(d_T(\underline{\gamma},\underline{\tilde{\gamma}})-v|t|)} \tag{472}$$

$$\leq Ce^{-\zeta(\mathfrak{r}-v|t|)}, \tag{473}$$

where the constant $\mathcal{C}$ is absorbed through ζ and v. If we then sum over all the possible interactions, one obtains a polynomial prefactor $P(L,R,r)$, which completes the proof. $\square$

With these bounds calculated, we are in a position to justify the following as a suitable approximation of the observable used to define Hall curvature. Let,

$$G^{\Xi}(\varphi) = \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt'\, W_\epsilon(t) W_\epsilon(t') \left[\tau_t^{\varphi}\left(\partial_1 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right), \tau_{t'}^{\varphi}\left(\partial_2 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right] \tag{474}$$

The following lemma establishes that $\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)$ is a good approximation of $\left[\tilde{K}_1(\varphi), \tilde{K}_2(\varphi)\right]$.

Lemma 6.12. *Let $R = r = \frac{L}{16}$ and $\tilde{r} = \frac{L}{4}$. Then, the following upper bound holds,*

$$\left\| \left[\tilde{K}_1(\varphi), \tilde{K}_2(\varphi))\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right\| \leq \mathcal{O}(L^{-\infty}), \tag{475}$$

where $\mathcal{O}(L^{-\infty})$ is a function with decay to zero faster than any rational function in L.

Proof. Explicitly, we consider the quantity,

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}} dt'\, W_\epsilon(t) W_\epsilon(t') \Big(\left[\tilde{\tau}_t^{\varphi}\left(\partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right), \tilde{\tau}_{t'}^{\varphi}\left(\partial_2 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right] - \tag{476}$$

$$\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right), \tau_{t'}^{\varphi}\left(\partial_2 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\Big). \tag{477}$$

To proceed, we will prove the lemma by integrating over a compact set of times and estimating away the tails of integration using the properties of W_ϵ, as proven in [11],

$$\begin{aligned} &\left\| \left[\tilde{K}_1(\varphi), \tilde{K}_2(\varphi))\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right\| \leq \int_{-T}^{T} dt \int_{-T}^{T} dt'\, |W_\epsilon(t)||W_\epsilon(t')| \\ &\left\| \Big(\left[\tilde{\tau}_t^{\varphi}\left(\partial_1 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right), \tilde{\tau}_{t'}^{\varphi}\left(\partial_2 \tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right), \tau_{t'}^{\varphi}\left(\partial_2 \mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\Big)\right\| \\ &+ \mathcal{O}(T^{-\infty}), \end{aligned} \tag{478}$$

where $\mathcal{O}(T^{-\infty})$ decays faster than any rational function in T. What remains now is to estimate a bound on the norm of the integrand and compute its integral. First, we use the triangle inequality to bound the commutator expression.

(479)

$$\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right]-\mathbb{E}^{Y_{\Lambda_L}}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\right\|$$
$$\leq\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right]-\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\right\|$$

(480)

$$\leq\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right]-\left[\tau_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tau_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\right\|$$

(481)

$$\leq\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right]-\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\left[\tau_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tau_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right\|+$$
$$\left\|\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]-\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\right\|$$

The previous three lemmas as well as Proposition 5.13 allow us to bound each of the four norms in the final inequality. Putting all of these results together, we obtain the bound,

(482)

$$\left\|\left[\tilde{\tau}_t^{\varphi}\left(\partial_1\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right),\tilde{\tau}_{t'}^{\varphi}\left(\partial_2\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)\right)\right]-\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tau_t^{\varphi}\left(\partial_1\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right),\tau_{t'}^{\varphi}\left(\partial_2\mathbb{H}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right)\right]\right)\right\|$$
$$\leq CP_3(L,R,r)e^{-\zeta(\mathfrak{r}-v|t-t'|)}+\left\|\partial_j\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right\|P_1(L,R)|t-t'|e^{-cR}+$$
$$|\Xi_R^r|P_2(L,R)e^{-\zeta\left(\frac{L}{2}-(2R+r)\right)}\left(e^{\zeta v|t-t'|}-1\right)+C\left\|\partial_l\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)_{\Xi_{R,r}}\right\|P(r)e^{-cR}+$$
$$C|\Xi_R^r|\left(\frac{\sqrt{2}R-r}{2}+\tilde{r}+c\right)e^{-\zeta(\tilde{r}-r-v_p|t-t'|)}$$

(483)

$$\leq CP_1(L,R,r)e^{-\zeta(\mathfrak{r}-v|t-t'|)} + P_2(L,R,r)|t-t'|^2e^{-cR} + P_3(L,R,r)e^{-\zeta\left(\frac{L}{2}-(2R+r)\right)}\left(e^{\zeta v|t-t'|}-1\right)$$

$$+ P_4(L,R,r)e^{-cR} + P_5(L,R,r)e^{-\zeta(\tilde{r}-r-v_p|t-t'|)},$$

where in the last inequality, the P polynomials have absorbed the additional volume estimates from bounding $\left\|\partial_j\tilde{\mathbb{H}}_{\Lambda_L}\right\|$ and $|\Xi_{R,r}|$. All that remains to complete the proof is to integrate the last inequality above weighted by $W_\epsilon(t)W_\epsilon(t')$ with respect to times t,t' up to T. Doing so yields the bound,

(484)

$$\|W_\epsilon\|_\infty^2P_1(L,R,r)e^{-\zeta(\mathfrak{r}-vT)} + 4\|W_\epsilon\|_1^2T^2P_2(L,R,r)e^{-cR} + \|W_\epsilon\|_\infty^2P_3(L,R,r)e^{-\zeta\left(\frac{L}{2}-(2R+r)-vT\right)}$$

$$+ \|W_\epsilon\|_\infty^2P_4(L,R,r)e^{-cR} + \|W_\epsilon\|_\infty^2P_5(L,R,r)e^{-\zeta(\tilde{r}-r-v_pT)}.$$

To conclude, we introduce the L-dependence of the parameters $R, r, \tilde{r}$ and T. Let $R = r = vT = \frac{L}{16}$, $vT = v_pT = \frac{L}{32}$, and $\tilde{r} = \frac{L}{4}$. Then, letting any monomials in L be absorbed into pre-existing polynomial prefactors, we obtain,

$$\begin{aligned}
&\left\|\left[\tilde{K}_1(\varphi),\tilde{K}_2(\varphi)\right)\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right\| \leq \|W_\epsilon\|_\infty^2P_1(L)e^{-\zeta_1L/32} + 4\|W_\epsilon\|_1^2P_2(L)e^{-cL/16} && (485)\\
&\qquad + \|W_\epsilon\|_\infty^2P_3(L)e^{-\zeta L/4} + \|W_\epsilon\|_\infty^2P_4(L)e^{-cL/16}\\
&\qquad + \|W_\epsilon\|_\infty^2P_5(L)e^{-\zeta L/8} + \mathcal{O}(L^{-\infty})\\
&\qquad \leq \mathcal{O}(L^{-\infty}) && (486)
\end{aligned}$$

□

We are now in position to use this approximation to show that our definition of Hall adiabatic curvature is φ-independent in the thermodynamic limit. Recalling 430, we may now write,

$$\kappa_L(\varphi) = i\tilde{\omega}_{\varphi,L}\left(\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right) + \mathcal{O}(L^{-\infty}), \tag{487}$$

and make use of the family of Hamiltonians $\mathbb{H}^{\varphi}_{\Lambda_L}(s) = \mathbb{H}_{\Lambda_L}(\varphi, (s-1)\varphi)$ and their associated quasi-adiabatic generators $D(s)$. As a first step, we want to verify where the predominant support of $\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s)$ is located in Λ_L, and specifically with reference to $\Xi_{R,\tilde{r}}$. By construction,

(488)

$$\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s) = -iq \sum_{\substack{\underline{\gamma}\in\Lambda_L^+: \\ \gamma\subset\Lambda_L\setminus Y_R}} \sum_{\substack{j_1,\ldots,j_{2p} \\ k_1,\ldots,k_{2p}}} \left(\left(\sum_{m=1}^{p} j_{p+m} - \sum_{m=1}^{p} j_m \right) + \left(\sum_{m=1}^{p} k_{p+m} - \sum_{m=1}^{p} k_m \right) \right) \times$$
$$\exp\left\{ \left(iq \left(\sum_m j_{p+m} - \sum_m j_m \right) (1-s)\varphi_1 + \left(\sum_m k_{p+m} - \sum_m k_m \right) (1-s)\varphi_2 \right) \right\} \times$$
$$\sum_{\underline{n}\in\{0,1\}^{|\underline{\gamma}|}} \phi(\underline{\gamma}, \underline{n}) a^*_{\theta_{2p}, j_{2p}k_{2p}} \ldots a_{\theta_1, j_1 k_1}.$$

As done previously, we may divide these constituent interactions using the bulk strips $\mathcal{B}_R$.

(489) $$\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s) = \left(\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s)\right)^{\mathcal{B}^c_R,1} + \left(\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s)\right)^{\mathcal{B}^c_R,2}$$

The interactions in the second term above have a minimum distance of $\frac{L}{2} - 2R - r$ from those in $\Xi_{R,r}$, and from the assigned L-dependence of these parameters, this distance is linear in L. Additionally, as previously described, the interactions in the first term either have diameters larger than $2R$ or have at least one site that is a distance R away from a quadrant boundary. Then, letting $V(s)$ denote the unitaries of quasi-adiabatic evolution generated by $D(s)$. For an arbitrary operator $A \in \mathcal{A}^L_{01}$, we may write,

(490) $$V^*(1)AV(1) = A + i\int_0^1 ds\, V^*(s)\,[D(s), A]\, V(s)$$

(491) $$= A + i\int_0^1 ds \int_{\mathbb{R}} dt\, W_\epsilon(t) V^*(s) \left[\tau^s_t\left(\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s)\right), A\right] V(s).$$

Focusing on the commutator in the integrand, letting $A = \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)$, we can use the remarks above with Lieb-Robinson bounds to obtain,

$$\left\| \left[\tau_t^s\left(\partial_s \mathbb{H}^{\varphi}_{\Lambda_L}(s)\right), \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right] \right\| \le P_1(L,R,r)e^{-cR} + P_2(L,R,r)e^{-\zeta(\frac{L}{2}-(2R+r+v|t|))} \tag{492}$$

$$= P_1(L)e^{-cL/16} + P_2(L)e^{-\zeta(5L/16 - v|t|)}. \tag{493}$$

Then, we bound the norm of the entire integral over t by splitting the integral as before, where one integral is over $|t| < T = \frac{L}{16v}$. This then results in the following crucial estimate,

$$\left\| i\int_0^1 ds\, V^*(s)\left[D(s), \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right] V(s) \right\| \le \mathcal{O}(L^{-\infty}). \tag{494}$$

Then, noting that the action of the ground states $\tilde{\omega}_{\varphi,L}$ and $\omega_{\varphi,L}$ of $\tilde{\mathbb{H}}_{\Lambda_L}(\varphi)$ and $\mathbb{H}_{\Lambda_L}(\varphi,1)$ respectively on A are related by $\tilde{\omega}_{\varphi,L}(A) = \omega_{\varphi,L}(V^*(1)AV(1))$,

$$\tilde{\omega}_{\varphi,L}\left(\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right) = \omega_{\varphi,L}\left(V^*(1)\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)V(1)\right) \tag{495}$$

$$= \omega_{\varphi,L}\left(\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right) \tag{496}$$

$$+ \omega_{\varphi,L}\left(i\int_0^1 ds\, V^*(s)\left[D(s), \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right] V(s)\right).$$

Thus,

$$\left| \tilde{\omega}_{\varphi,L}\left(\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right) - \omega_{\varphi,L}\left(\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(G^{\Xi}(\varphi)\right)\right)\right| \le \mathcal{O}(L^{-\infty}), \tag{497}$$

and the covariance of $\omega_{\varphi,L}$ and of $G^{\Xi}(\varphi)$ guarantee that this inequality is independent of φ. This completes the proof of Theorem 6.6. □

proof of Theorems 6.7 and 6.8. Now, moving on to the statement of quantization of Hall conductance in the thermodynamic limit, the operator given by $\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}}\left(\left[\tilde{K}_1, \tilde{K}_2\right]\right)$ is an element of $\mathcal{A}_{\Xi_{R,\tilde{r}}}$, which is a subalgebra of $\mathcal{A}^L_{01}$ by construction. Therefore, in the thermodynamic limit, i.e. $L \to \infty$, this operator still remains in $\mathcal{A}_{\Xi_{R,\tilde{r}}}$ because the functions $\chi_{\gamma,L}$ with $\gamma \in Y_L$ converge in L^2 norm to χ_γ; the arguments used to prove Lemma 5.24 also establish that this limit certainly exists within the CAR algebra. Moreover, the preceding proofs establish that

$\left[\tilde{K}_1, \tilde{K}_2\right]$ is a quasi-local operator that can be written as the limit of strictly frame-local operators localized around the box Ξ_R. Thus, we have the following limits,

$$\left[\tilde{K}_1, \tilde{K}_2\right] = \lim_{\tilde{r}\to\infty} \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \tag{498}$$

$$\Upsilon^{\tilde{r}} := \lim_{L\to\infty} \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \left(\left[\tilde{K}_1, \tilde{K}_2\right]\right). \tag{499}$$

Consequently, similar to the lattice analogue ([8])

$$\left[\tilde{K}_1, \tilde{K}_2\right] - \Upsilon^{\tilde{r}} = \left[\tilde{K}_1, \tilde{K}_2\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \left(\left[\tilde{K}_1, \tilde{K}_2\right]\right) + \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \left(\left[\tilde{K}_1, \tilde{K}_2\right]\right) - \Upsilon^{\tilde{r}} \tag{500}$$

$$\left\|\left[\tilde{K}_1, \tilde{K}_2\right] - \Upsilon^{\tilde{r}}\right\| \leq \left\|\left[\tilde{K}_1, \tilde{K}_2\right] - \mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \left(\left[\tilde{K}_1, \tilde{K}_2\right]\right)\right\| + \left\|\mathbb{E}^{\Lambda_L}_{\Xi_{R,\tilde{r}}} \left(\left[\tilde{K}_1, \tilde{K}_2\right]\right) - \Upsilon^{\tilde{r}}\right\| \tag{501}$$

$$\leq \mathcal{O}\left(\tilde{r}^{-\infty}\right) + \mathcal{O}\left(L^{-\infty}\right), \tag{502}$$

and this allows us to conclude the proof:

$$\lim_{L\to\infty} \left|\kappa_L(\varphi) - i\tilde{\omega}_{\varphi,L}\left(\Upsilon^{\tilde{r}}\right)\right| \leq \mathcal{O}\left(\tilde{r}^{-\infty}\right). \tag{503}$$

□

Part 2. Studies of Effective Models for Correlated Electron Systems

7. Fermi-Hubbard model and Multi-layer Materials

Onset of charge incompressibility and Mott gaps in the Honeycomb-Lattice SU(4) Hubbard Model: Lessons for Twisted Bilayer Graphene systems [1]

Introduction: Recent years have seen increasing interest in the Fermionic Hubbard model with more than two spin species [27, 50, 51, 53, 98, 107, 115, 125]. In cold atom platforms, atoms can have internal degrees of freedom such as those associated with hyperfine states. Contact interactions between atoms provide excellent realizations of local Hubbard interaction U with SU(N) symmetry. Even in the solid state context, SU(4) symmetry can arise for multi-orbital systems although full SU(4) symmetry generically requires fine-tuning of parameters [63, 112]. A recent realization of systems with potential SU(4) symmetry are Twisted Multilayer systems at magic angles with extremely narrow or nearly flat bands [18, 57, 61, 114, 117, 121, 122, 123, 124, 126]. Flat bands can provide both strong correlations and enhanced internal symmetry. While the low temperature phase behavior of these systems clearly depend on the complex band structure and details which break SU(N) symmetry [121], a question of interest in this work is: What lessons can be drawn by comparison with high temperature thermodynamics of a simple SU(N) Hubbard model?

Strong coupling expansions around the atomic limit provide a powerful formalism to calculate temperature dependent properties of Hubbard models in the thermodynamic limit [45, 78, 94, 95, 102]. These expansions can be developed at inverse temperature β and chemical potential μ in the grand canonical ensemble and for any set of hopping parameters. Each term in the expansion depends on βt, the fugacity $\zeta = \exp\{\beta\mu\}$, and the Hubbard U which enters the expansions both in terms of $w = \exp\{-\beta U\}$ and $1/\beta U$.

[1]Copyright 2022 American Physical Society. Based on the publication in Physical Review B: `https://doi.org/10.1103/PhysRevB.105.L241410` [48]

Here, we focus on the temperature dependence of the electronic compressibility and entropy. As the temperature is lowered below U, at most densities the compressibility becomes large while at integer densities, the system enters an incompressible Mott regime and the compressibility goes to zero. At the same time, the entropy develops sharp cusps as a function of density. The Mott gap can be obtained from studying the compressibility as a function of temperature.

In the incompressible Mott regime, the strong coupling expansions can be recast as a high temperature expansion for a spin model with SU(4) symmetry. The spin models at $\rho = 1$ and $\rho = 3$ belong to the fundamental representation of SU(4) symmetry with four states per site. The model at $\rho = 2$ has six states per site and can be mapped to one with SO(6) symmetry [121]. The thermodynamic properties such as entropy and specific heat can be arranged in terms of two dimensionless parameters: $\beta t^2/U$ and t/U. At large U/t the system turns into a Heisenberg model with J set by t^2/U whereas additional t/U dependence reflects the presence of various multi-spin and multi-site interactions [28, 70, 113]. Antiferromagnetic ordering and correlations are captured by the $\beta t^2/U$ dependence whereas the t/U terms can be used to study the breakdown of strong coupling expansions.

The manner in which incompressibility sets in with increase in β at various densities, and the shapes of chemical potential, inverse compressibility and entropy as a function of particle density carry potential lessons for magic angle graphene systems. While some features resemble those observed experimentally [86, 88, 105, 111, 127], there are important differences, especially with respect to persistence of band features with temperature, which we will discuss in this work.

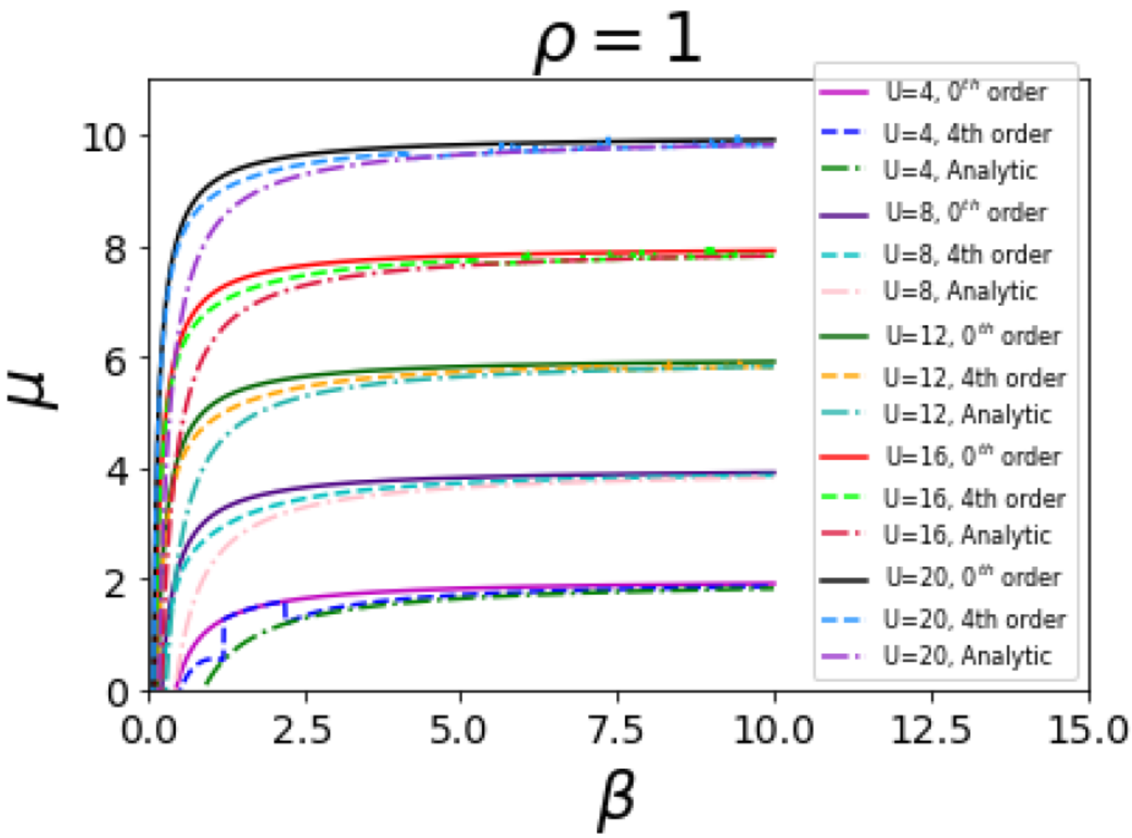

FIGURE 6. Chemical potential μ versus inverse temperature β for $\rho = 1$ and several values of U. Zeroth order and fourth order calculations are shown along with an analytical asymptotic low temperature formula.

Model and Methods: The SU(4) Hubbard model is defined by a Hamiltonian $H = H_0 + V$, where the unperturbed part H_0 is the on-site term:

$$H_0 = U \sum_i \frac{n_i(n_i - 1)}{2} - \mu \sum_i n_i, \tag{504}$$

with μ the chemical potential and n_i the total number operator on site i. The perturbation V is the hopping term:

$$V = -\sum_{i,j} t_{ij} \sum_{\alpha=1}^{4} (C^\dagger_{i,\alpha} C_{j,\alpha} + h.c.), \tag{505}$$

where the sum i, j runs over pairs of sites of a lattice and the sum over α runs over the 4 species of Fermions. In this work, we will consider the nearest-neighbor hopping model on honeycomb lattice. The unperturbed Hamiltonian has a particle-hole symmetry $n_i \to 4-n_i$, $\mu \to 3U - \mu$. The hopping Hamiltonian changes sign under particle-hole symmetry. Thus on a bipartite lattice where thermodynamic properties depend only on the absolute value of the hopping parameter t, the system has particle hole symmetry.

Using the formalism of thermodynamic perturbation theory [45, 78, 94, 95, 102], the logarithm of the grand partition function, per site, can be expanded as

$$\frac{1}{N_s}\ln Z = \ln z_0 + \sum_G L_G z_0^{-s}(\beta t)^r X_G(\zeta, \beta U), \tag{506}$$

where N_s is number of sites in a large lattice, z_0 is the single-site partition function

$$z_0 = 1 + 4\zeta + 6\zeta^2 w + 4\zeta^3 w^3 + \zeta^4 w^6, \tag{507}$$

with $\zeta = e^{\beta\mu}$, and the sum in Eq. 506 is over graphs denoted G. The graph G has s sites and r bonds, and has Lattice Constant L_G. The weight-factor $X_G(\zeta, \beta U)$ is the reduced weight of the graph G for $\ln Z$ [78].

The particle density (per site) can be obtained via the relation

$$\rho = \frac{\zeta}{N_s}\frac{\partial}{\partial\zeta}\ln Z. \tag{508}$$

This relation can be inverted to obtain ζ or μ as a function of ρ and β, which then allows one to obtain various properties at fixed particle density.

In Fig. 6 we show the chemical potential μ as a function of temperature at particle density $\rho = 1$ for different values of U. The solid lines are the calculation in the atomic limit (zeroth order), the dashed line in the fourth order, and the dashed-dotted lines is the asymptotic low temperature formula $\zeta^2 = \frac{1}{6w}$, discussed more in a later section. We can see that the fourth order calculation lies between the zeroth order and the asymptotic expression. At low temperatures it becomes difficult to precisely locate the chemical potential numerically because the Mott plateaus have almost chemical potential independent density. Fortunately,

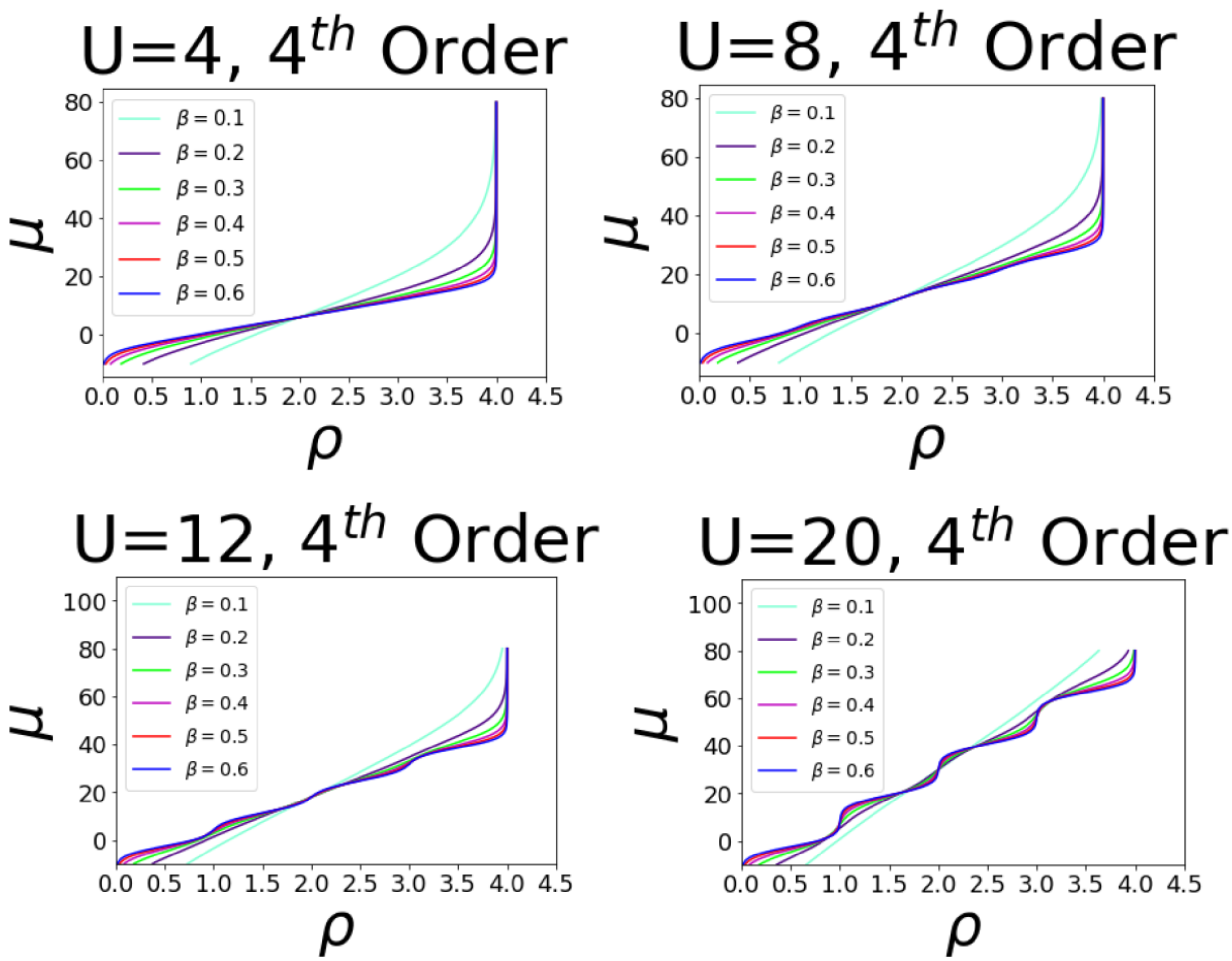

FIGURE 7. Chemical potential μ versus density ρ for several U and β values. As the temperature is lowered, the development of Mott behavior is indicated by rapid changes in chemical potential at integer densities.

the analytical expression can be used there. The compressibility K, entropy S and specific heat C are obtained from the relations

$$K = \left(\frac{\partial \rho}{\partial \mu}\right)_{\beta}, \tag{509}$$

$$S = -\beta\left(\frac{\partial}{\partial \beta} \ln Z\right)_{\zeta} - \rho \ln \zeta + \ln Z, \tag{510}$$

and,

$$C = T\left(\frac{\partial S}{\partial T}\right)_{\rho}. \tag{511}$$

We have carried out the perturbation theory to eighth order. Up to fourth order, we evaluate the full traces and our perturbation theory is complete. For much of the temperature range studied, fourth order perturbation suffices and properties can be calculated accurately for arbitrary densities. Starting with sixth order the number of trace terms becomes too large to evaluate fully. In sixth order, we restrict trace calculations to at most 3 particles per site and in eighth order we restrict to at most two-particles per site. This restricted calculation is sufficient at lower temperatures ($w \to 0$ limit), where we particularly need higher orders, as discussed later.

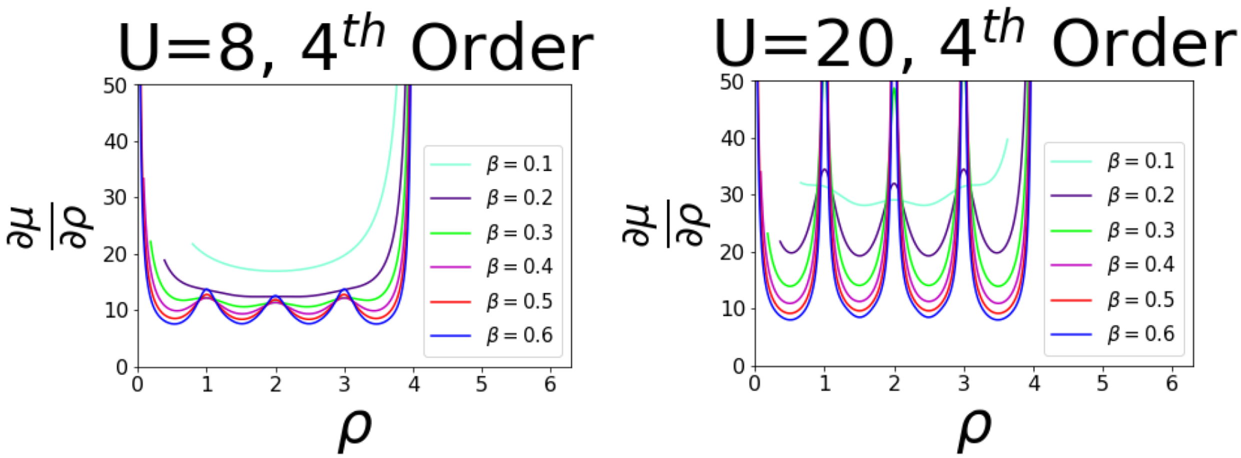

FIGURE 8. Inverse Compressibility versus ρ for $U = 8$ and 20 and several β values. Mott behavior is characterized by a sharp increase in inverse compressibility.

Chemical potential, Compressibility and Entropy: Numerical results for chemical potential μ as a function of particle density ρ for various temperatures and U values are shown in Fig. 7. We find that the results at all density are very well converged down to a temperature of approximately $T/t = 1.5$. Below that temperature, the convergence away from integer densities starts to break down. Hence, the plots are shown up to $\beta = 0.6$. At and near integer densities, the convergence of the expansion is set by t^2/U and hence they remain convergent down to lower temperatures. The striking feature of the plot is the onset of Mott behavior around $T = U/2$ characterized by rapid rise in the chemical potential at integer densities. The larger U system is deeper into the Mott phase and hence the changes in chemical potential are much sharper. Mott behavior can be seen even more clearly in Fig. 8, where we show the inverse compressibility vs density. The Mott phase is characterized by its incompressibility and hence the inverse compressibility shoots up and shows sharp spikes. Note that the spikes are symmetric around the peak.

In Fig. 9, we show the entropy as a function of density. The onset of Mott behavior is characterized by the development of a sharp cusp in the entropy with a minima at integer densities. This is because at small deviation from integer densities and at temperatures much larger than the exchange constant, the system maps on to the high temperature limit of the $t-J$ model [79, 81, 84, 95, 109]. At these temperatures, the motion of charge degrees of freedom is incoherent and corresponds to a dilute gas with an entropy that varies as $-\delta \ln \delta$, where δ is the deviation in density from integer filling.

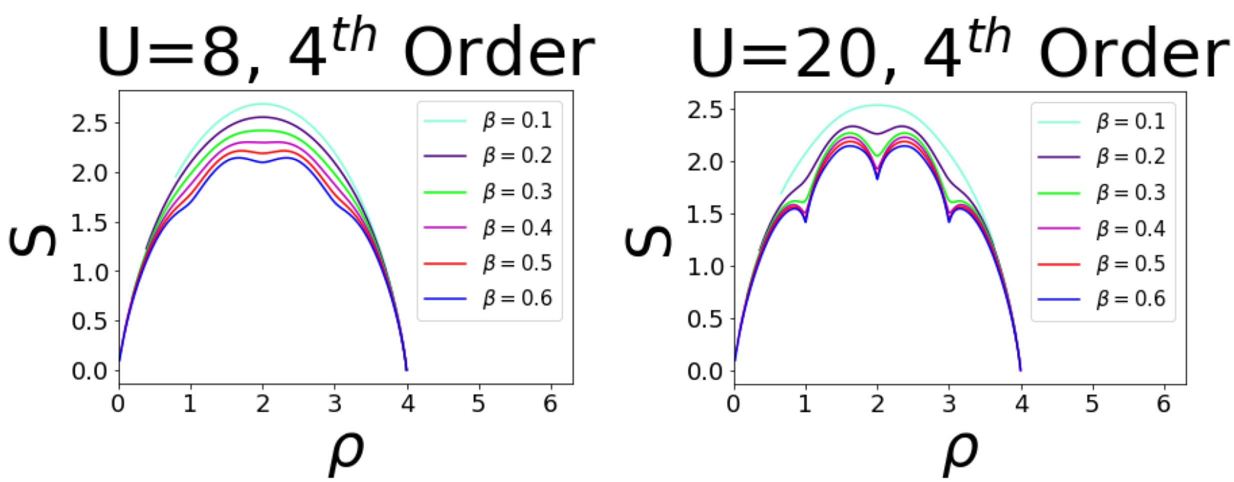

FIGURE 9. Entropy as a function of density for several values of β and $U = 8$ and 20. As the system enters the strongly correlated regime sharp cusps develop at integer densities.

The $w \to 0$ limit: At low temperatures, $w = \exp\{-\beta U\}$ becomes exponentially small and some terms dominate the expansion. At integer densities, the system is dominated by a single occupancy value at each site. Away from integer densities, the system is dominated by only two occupancy values. The remaining terms become exponentially small. To see this, and the range of ζ or chemical potential μ in each case, we consider the unperturbed atomic limit. Particle density in the atomic limit is:

$$\rho_0 = \frac{1}{z_0}(4\zeta + 12\zeta^2 w + 12\zeta^3 w^3 + 4\zeta^4 w^6) \tag{512}$$

The cases $\rho = 0$ and $\rho = 4$ are trivial. We focus on remaining integer densities. Setting $\rho = 1$ and dropping exponentially small terms one obtains:

$$\zeta^2 = \frac{1}{6w} + \mathcal{O}(w). \tag{513}$$

This relation is exact at low temperatures for $\rho = 1$ provided the system remains in the Mott phase, that is, U is not so small that there is a transition away from the insulating phase. The single site partition function is also dominated by a single term

$$z_0 = 4\zeta + \mathcal{O}(\sqrt{w}). \tag{514}$$

Or,

$$\frac{\zeta}{z_0} = \frac{1}{4} + \mathcal{O}(\sqrt{w}) \tag{515}$$

For each graph, in the $w \to 0$ limit, X_G is also dominated by only select terms. For a graph with s sites, there is a z_0^s factor in the denominator and the numerator has a maximum power of ζ^s without any double occupancy. Thus, only the ζ^s terms survive in this limit. In other words, graph by graph, only those terms survive which have exactly one particle at every site in the unperturbed limit. Setting $\rho = 2$ in Eq. 512 we get

$$\zeta^2 = \frac{1}{w^3}. \tag{516}$$

This relation, which implies $\mu = 3U/2$, is exact by particle hole symmetry. The partition function, up to terms which are relatively exponentially small, becomes:

$$z_0 = 6\zeta^2 w. \tag{517}$$

Or,

$$\frac{\zeta^2}{z_0} = \frac{1}{6w}. \tag{518}$$

Once again, contribution from each graph is dominated by terms that have exactly two particles on every site. Other terms are relatively exponentially small.

Setting $\rho = 3$ in Eq. 512 and keeping only the exponentially largest terms we get $\zeta^2 = \frac{1}{6w^5}$. The partition function, up to terms which are relatively exponentially small, becomes $z_0 = 4\zeta^3 w^3$. Or,

$$\frac{\zeta^3}{z_0} = \frac{1}{4w^3}. \tag{519}$$

Once again, contribution from each graph is dominated by terms that have exactly three particles on every site. Other terms are relatively exponentially small. At densities between two integer densities, the fugacity takes values between two commensurate ones and each site can have only one of two occupations. Thus, restricting trace calculations to terms with up to 2 particles per site suffices to get the exponentially largest terms as long as particle density is less than or equal to two. Furthermore, using particle-hole symmetry one can also obtain properties at densities larger than two, so that all densities can still be accurately obtained.

As $w \to 0$, the expansions at integer densities turn into high temperature expansions for a generalized spin model. We can rearrange this expansion in powers of $x = \beta t^2/U$ and $y = (t/U)^2$. At $\rho = 1$, $\ln(Z/4)$ has expansion

$$\begin{aligned} &2.25x(1 - \frac{4}{3}y + 15.81887y^2 - 429.6101y^3 + \ldots) \\ &+ 2.8125x^2(1 - \frac{16}{3}y + 66.49491y^2 + \ldots) \\ &+ 0.9375x^3(1 - 0.8y + \ldots) - 0.00234375x^4(1 + \ldots) \end{aligned} \tag{520}$$

While at $\rho = 2$, $\ln(Z/6)$ has expansion

$$
\begin{aligned}
&3x(1 - 1.833333y + 34.18148y^2 - 1227.889y^3 + \dots) \\
&+ 5x^2(1 - 9.516667y + 182.3093y^2 + \dots) \\
&+ \frac{20}{3}x^3(1 - 9.866667y + \dots) + \frac{26}{9}x^4(1 + \dots)
\end{aligned} \tag{521}
$$

The $y \to 0$ limit corresponds to the Heisenberg model and the properties only depend on $\beta t^2/U$. As long as the expansions in t/U converge, the system remains an incompressible Mott Insulator.

Series in y are too short to determine the location of metal-insulator transition. But, taking the nth root of the absolute value of the coefficients of various y^n terms suggest a convergence radius in $(t/U)^2$ of approximately 0.1 or $U/t \approx 3-4$. Note that the critical U/t need not be the same at different densities. Also, because the series in y are alternating, we cannot rule out a much smaller critical U/t on the real axis. These results are consistent with a previous study [126], which reported a critical U/t in the range of 2.5 to 3.

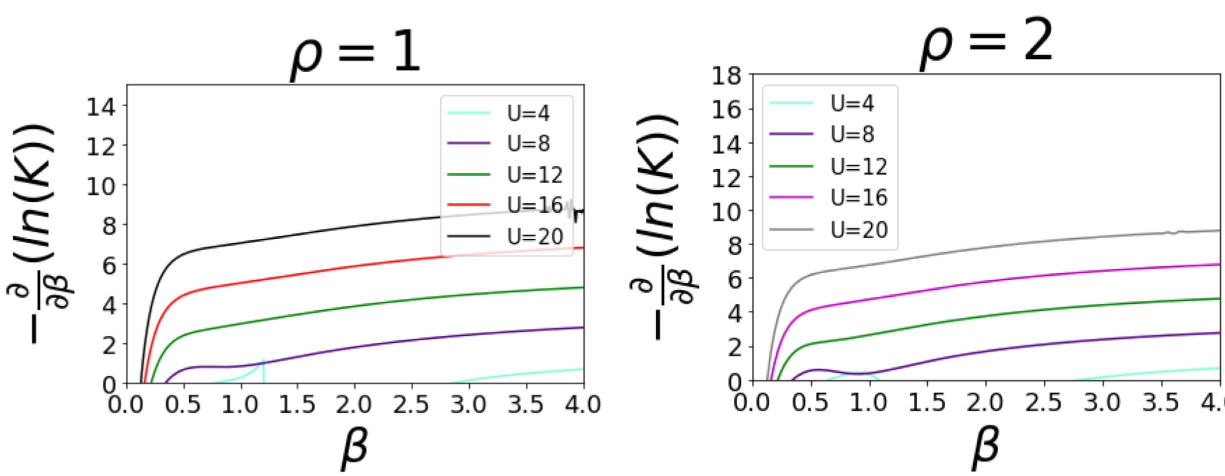

FIGURE 10. Effective Mott Gap defined as $-\frac{d\ln K}{d\beta}$, versus temperature for $\rho = 1$ and $\rho = 2$ for several values of U.

Determining the Mott gap: We define the effective Mott Gap from the behavior of compressibility as $K \sim \exp\{-\Delta/T\}$. The effective Mott gap at inverse temperature β is defined as:

$$\Delta = -\frac{d \ln K}{d\beta} \tag{522}$$

Fig. 10 shows the effective Mott gap as a function of β for several U values. In finite order of perturbation theory, at asymptotically low temperatures the Mott gap thus defined can be shown to go back to $U/2$, the unperturbed value. The reason is simply that $\exp\{(-\beta U/2)\}(\beta t)^n$ goes to zero as $n \to \infty$. Hence, keeping n finite, as $\beta \to \infty$ all perturbative terms vanish. In nth order, convergence should extend up to a β value that increases as n. For our calculation, we thus need to stay at a low but finite temperature. We expect the Mott gap to plateau close to the true answer before drifting back to the unperturbed value as $T \to 0$. It is, however, difficult to precisely pin down the gap from the short series. It seems apparent that the gap becomes small by $U/t = 4$. At smaller U, U-dependence of the gap may take a concave shape so that a small gap may persist to much smaller U values.

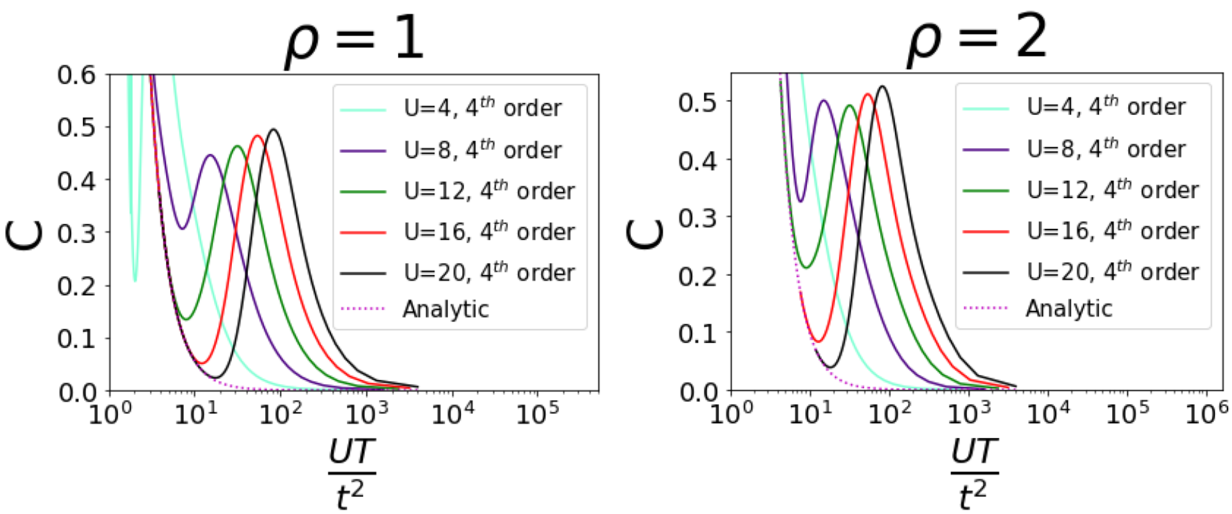

FIGURE 11. Specific Heat (C) as a function of temperature for $\rho = 1$ and $\rho = 2$ for several values of U. The leading order Heisenberg model results for specific heat are shown as dotted lines.

Specific heat in the insulating phase: In Fig. 11 specific heat plots for $\rho = 1$ and $\rho = 2$ at various U values as a function of temperature are shown. The temperature axis has been scaled by t^2/U to match with the large-U exchange constant. The high temperature peak in specific heat corresponds to the transition from temperatures of order U to a strongly correlated regime at $T << U$. For $U \geq 8$, the entropy first plateaus at $ln4$ and $\ln 6$ for $\rho = 1$ and $\rho = 2$ respectively, corresponding to the high temperature limit of the spin model [95]. Subsequently, the entropy decreases with lowering of temperature showing the development of spin correlations. The lower temperature rise of the specific heat is related to the development of spin correlations in the system.

We have also shown in Fig. 11, the high temperature large-U limit of the spin models for the specific heat. It is clear that the spin model captures the numerical variations in the quantities for $U \geq 8$ quite well. But, there are significant corrections for $U/t = 4$.

Magic angle systems: In the Twisted Bilayer Graphene and similar systems near magic angles the bandwidth and effective U have both been estimated to be of order 10 meV [111]. That means the hopping parameter t is of order 10K and U is of order 100K. Thus, our calculations could be relevant above a temperature of a few K at integer densities and above about 10K at all densities.

There have been several measurements of chemical potential, compressibility and entropy as a function of density in these systems some going to temperatures up to 50K or higher [86, 88, 105, 111, 127]. Some features of the experiments are clearly captured by the simple Hubbard model. As temperature is lowered, there occurs a rapid rise in chemical potential at characteristic densities and correspondingly the inverse compressibility shows sharp spikes. However, there are key differences:

1. Mott behavior at $T > 1.5t$ is only seen in our study at integer densities per site not at integer densities per translational unit cell as seen in experiments. The latter would include half integer densities per site, implying twice as many Mott plateaus.

2. The entropy is a minima at integer densities in our studies with sharp cusps, whereas it is a maxima in experiments.

3. No saw-tooth like asymmetry is seen in the inverse compressibility spikes [127], though this maybe related to the particle hole symmetry in the nearest-neighbor hopping model.

Although our model has only nearest-neighbor hopping, results 1 and 2 above should be valid even with more complex hoppings. At temperatures above t and U not too small, incompressibility in the Hubbard model only arises when it can be traced back to the atomic limit, i.e., at integer density per site. Adding further neighbor Coulomb repulsion can cause additional Mott plateaus at half-integer densities, but that would be related to charge density order for which there is no experimental evidence. In fact, there is a band-based argument for insulators at half integer densities. The band structure of the honeycomb system can be regarded as two bands, one below the other in energy, joined together at the Dirac points [127]. Thus, additional incompressibility at half integer density per site is related to integer filling of one of the two bands.

Similarly, hopping of carriers will normally be incoherent at these high temperatures in the Hubbard model and the mobile particle entropy will be that of an ideal gas, regardless of hopping details. This is what causes the entropy function to be a minima at integer densities and have sharp cusps. On the other hand, the kinetic entropy is presumably already quenched even at these temperatures in experiments and the system has turned into a fermi liquid.

We believe these differences point to an important property of magic angle flat-band systems, namely that these flat bands are derived from a much wider band and hence have a much larger energy scale behind them and thus can persist over a larger temperature scale. Persistence of low energy Dirac features to high temperatures has also been emphasized by Zondiner et al [127].

Discussions and Conclusions: We have studied finite temperature properties of the SU(4) Hubbard model on the honeycomb lattice using strong coupling expansions. At integer densities, when U is not too small, the system becomes an incompressible Mott Insulator. This can be seen by examining the density versus chemical potential, which shows the development of sharp plateaus. When this happens, the compressibility becomes exponentially small and the entropy develops cusps at integer densities. The system can be mapped to a spin model with only virtual charge fluctuations. These expansions converge extremely well for $U/t > 8$ and possibly down to $U/t \approx 3-4$. Strong coupling expansion is particularly useful in studying the temperature dependence of properties like compressibility and entropy.

For a more quantitative comparison with experiments on magic angle flat-band materials, it may be useful to extend these studies to include more realistic band structures with many hopping parameters. Furthermore, including smaller terms that break SU(N) symmetry would allow one to study various symmetry breaking transitions in these systems. However, our study suggests that some aspects of the physics of magic angle systems may not be captured by considering a lattice Hubbard model of just the flat bands.

8. Unconventional Superconductivity in $LaNiGa_2$

Observation of a pronounced Hebel-Slichter peak in the spin-lattice relaxation rate and implications for gap and pairing symmetry in $LaNiGa_2$ [2]

Superconductors that exhibit spontaneous magnetic fields are unusual because the most energetically favorable state generally involves spin-singlet pairing between Cooper pairs of opposite momentum electrons. Spin triplet pairing can emerge in rare cases, such as in the well-known candidates UPt_3 and UTe_2, where triplet pairing has been observed, and, in case of UPt_3 there is further evidence of a finite magnetization and time-reversal symmetry breaking (TRSB) [83, 91]. TRSB has been recently observed via muon spin resonance (μSR) experiments in the superconducting state of the topological crystalline superconductor $LaNiGa_2$ with $T_c = 2.1$ K [34, 46]. This material has a non-symmorphic orthorhombic crystal structure (C_{mcm}) that gives rise to a non-trivial band topology with band-degeneracies at the Fermi level [14, 80]. This degeneracy can give rise to multi-band superconductivity involving both intra- and inter-band pairing.

It has been proposed that the condensate wavefunction may be described as an internally antisymmetric, non-unitary, triplet (INT) state that is fully gapped with s-wave symmetry in momentum space, antisymmetric in the band channel, and symmetric in spin space with equal-spin triplet ($S = 1$) pairing [33]. In this case, there are two components to a superconducting pair wavefunction corresponding to either both spins up or both spins down [33, 110]. If the order parameter is non-unitary, there will be two distinct superconducting gaps with different magnitudes, and the corresponding superfluid densities will also differ leading to a finite internal magnetization. Support for this interpretation has emerged from specific heat [110], penetration depth [32], and μSR [97] measurements in $LaNiGa_2$ that revealed the presence of two superconducting gaps. However, such measurements are mostly sensitive to

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the density of states and cannot probe the spin structure of the superconducting condensate.

Zero field nuclear quadrupolar resonance (NQR) can shed light on the spins of Cooper pairs because the spin lattice relaxation rate, T_1^{-1}, of nuclear spins is driven by spin-flip scattering with the Bogoliubov excitations. The Hebel-Slichter coherence peak is an increase in T_1^{-1} below T_c that arises due to a temperature-dependent build up of a sharp peak in the density of states and the particular quantum mechanical superposition of holes and electrons describing the excitations [6, 40, 41]. The coherence peak generally emerges in conventional superconductors [60, 62, 71], but is suppressed, apart from extrinsic impurity scattering, for several intrinsic reasons in unconventional superconductors. These include an anisotropic gap that can suppress the peak in the density of states or a change in the symmetry of the pair-wave functions and excitations that can alter coherence factors [19, 96].

Here we find that in an equal-spin pairing triplet state the peak can also be suppressed by having two different gaps for different spin components. At lower temperatures, the temperature dependence of T_1^{-1} reflects the superconducting gap, exhibiting either activated behavior in superconductors with a full gap [29, 76], or power law behavior in unconventional superconductors with nodes in the gap [25, 54]. Less is known about the behavior of T_1^{-1} for the INT state. In this case, there are four types of Bogoliubov excitations involving superpositions of electrons and holes with parallel spins in different bands. Because they involve parallel spins, one might expect spin-flip scattering by nuclear spins would be suppressed, and that T_1^{-1} may depend on the orientation of the **d**-vector that describes the triplet order parameter [99, 100].

In this study we present ^{139}La ($I = 7/2$) NQR T_1^{-1} data in the superconducting state of $LaNiGa_2$ that exhibits a Hebel-Slichter coherence peak as well as activated behavior at low temperatures. We show that a robust coherence peak can only emerge for the INT state if the two gaps are equal in magnitude, with no TRSB. When the equal spin pairing axis is different from the nuclear quantization axis, a weak HS peak arises with TRSB with two distinct gaps, but it is too weak to fit the data. Our experimental results are well described by a two-band singlet model with different gap magnitudes consistent with the previously reported heat capacity and penetration depth experiments.

To compute the superconducting gap and excitations, we ignore details of the band structure and consider two different Bogoliubov-de-Gennes (BdG) Hamiltonians of similar essence with an isotropic gap, inspired by the INT model introduced in [34, 80]. These two models differ in the pairing mechanism: the INT state (a) has equal-spin triplet pairing, whereas the two-gap BCS model (b) has singlet pairing. Both have two bands with inter-band pairing. The INT case (a) is antisymmetric in the band index with parallel spin, but with an admixture of up-up and down-down which allows TRSB and two gaps. The BCS case (b) is symmetric in the band index but is spin singlet. Both models still allow two gaps, and in either case, the gap is assumed to be isotropic (i.e., the spatial wave-function is s-wave). We are, thus, primarily exploring two gaps with the associated build-up of density of states and the nature of coherence factors that come from different pairing symmetries.

We note that the triplet states of the INT model are very different from the BW (Balian-Werthamer) [15] and ABM (Anderson-Brinkman-Morel) [2] states proposed for triplet superfluids and realized in the B and A phases of ^{3}He. In these cases there is no band structure, and the anti-symmetry comes from real space wave-functions, which are $\mathbf{k}$-dependent and odd in $\mathbf{k}$. In $LaNiGa_2$ with D_{2h} point group symmetry, the allowed irreducible representations for a single-band spin triplet model with weak spin orbit coupling all show line or surface nodes in the gap for spherical or cylindrical Fermi surfaces [3].

In order to model the spin lattice relaxation in $LaNiGa_2$, we assume that the La nuclear spins interact with the electron spins in two different bands via a contact hyperfine interaction: $\mathcal{H}_{hyp} = A\mathbf{I} \cdot \mathbf{S}(\phi)$, where A is a hyperfine constant, $\mathbf{I}$ is the nuclear spin,

$\mathbf{S}(\phi) = \frac{1}{2}\sum_{ss'} c^\dagger_{\mathbf{k}s}(\phi)\sigma_{ss'}c_{\mathbf{k}s'}(\phi)$ is the spin of an electron, and $c^\dagger_{\mathbf{k}s}(\phi) = \cos(\phi)a^\dagger_{\mathbf{k}s} + \sin(\phi)b^\dagger_{\mathbf{k}s}$ is a linear combination of creation operators for the two bands ($a^\dagger_{\mathbf{k}s}$ and $b^\dagger_{\mathbf{k}s}$) with wavevector $\mathbf{k}$ and spin s. This hyperfine interaction gives rise to scattering with more or less emphasis on electrons from either band, relative to hyperfine coupling to each band. It is not possible to determine these couplings *a priori* via band structure or experimental observations, thus ϕ is treated as a floating fit parameter.

We now introduce the two Bogoliubov-de-Gennes Hamiltonians inspired by the INT model, which enforce symmetric and antisymmetric spin pairing respectively, before deriving the solutions to these models, i.e. the low-energy band structure and quasi-particle excitations.

(a) INT Model: This particular model is defined by the BdG Hamiltonian: $H_{BdG} = \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}} H(\mathbf{k})\Psi_{\mathbf{k}}$, where

$$H(\mathbf{k}) = \begin{bmatrix} h_0(\mathbf{k}) & \hat{\Delta} \\ \hat{\Delta}^\dagger & -h_0(\mathbf{k}) \end{bmatrix}, \tag{523}$$

$h_0(\mathbf{k}) = \mathbb{1}_2 \otimes (w_{\mathbf{k}} \cdot \mathbf{k}\mathbb{1}_2 + \gamma_{\mathbf{k}} \cdot \mathbf{k}\tau_x)$ exhibits a Dirac dispersion with velocity parameters $w_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ as introduced in [80], the pairing potential $\Delta = (\mathbf{d}\cdot\sigma)i\sigma_y \otimes i\tau_y$ where σ_i and τ_i are the Pauli matrices associated with the electron spins and the a/b bands, respectively, $\mathbf{d}$ describes the triplet gap parameter, and $\Psi_{\mathbf{k}} = \left(a_{\mathbf{k}\uparrow}, b_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, b_{\mathbf{k}\downarrow}, a^\dagger_{-\mathbf{k}\uparrow}, b^\dagger_{-\mathbf{k}\uparrow}, a^\dagger_{-\mathbf{k}\downarrow}, b^\dagger_{-\mathbf{k}\downarrow}\right)$. In this model, the dependence on $\mathbf{k}$ simplifies to dependence only on the component of $\mathbf{k}$ perpendicular to the Fermi surface, as seen in the dispersion relation and subsequent coherence factor expressions. It is also important to note that the analogous model in [34] features an additional τ_z term in $h_0(\mathbf{k})$ that introduces an inherent energy difference in the a and b bands. This has been set to 0, and the τ_x term, constant in [34], has been endowed with linear $\mathbf{k}$ dependence to yield the aforementioned Dirac dispersion.

Furthermore, the order parameter is non-unitary and the condensate has a finite spin polarization $\mathbf{q} = 2i\mathbf{d}\times\mathbf{d}^*$ [82]. We consider a few different choices of pairing vector $\mathbf{d} = \Delta_0\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a complex unit vector. These are (I) $\mathbf{d}_r = \frac{\Delta_0}{\sqrt{1+r^2}}(r, i, 0)$ where r is a real number and $\mathbf{q} \sim \frac{r}{1+r^2}\hat{z}$, (II) $\mathbf{d}_z = (0, 0, 1)$ with $\mathbf{q} = 0$, (III) $\mathbf{d}_\theta^{xy} = \Delta_0(1 + i\cos\theta_\eta, i\sin\theta_\eta, 0)/\sqrt{2}$ with $\mathbf{q} \sim \sin\theta_\eta\hat{z}$, and (IV) $\mathbf{d}_\theta^{yz} = \Delta_0(0, i\sin\theta_\eta, 1 + i\cos\theta_\eta)/\sqrt{2}$ with $\mathbf{q} \sim \sin\theta_\eta\hat{x}$. θ_η characterizes the relative angle between the real and imaginary parts of $\mathbf{d}$ and the degree of non-unitarity. In particular, this parameter gives rise to two possibly distinct gaps:

$$\Delta = \Delta_0\sqrt{1 - \left(\frac{\gamma_k}{w_k}\right)^2}\sqrt{1 \pm |\sin\theta_\eta|} \tag{524}$$

The first two cases correspond to the symmetry allowed irreducible representations for a single-band with weak spin orbit coupling [3]. Here, by increasing r away from 0, the effect is to introduce TRSB with a nonzero $\boldsymbol{\omega} = i(\boldsymbol{\eta}\times\boldsymbol{\eta}*)$. We will now work through the derivation of the superconducting gap(s) and the eigenvectors of this Bogoliubov-de-Gennes Hamiltonian.

Since the pairing vectors $\mathbf{d}$ of interest are in the $x-y$ or $y-z$ planes in $\mathbb{C}^3$, we parametrize them more generally as $\mathbf{d} = (d_{x,r} + id_{x,i}, d_{y,r} + id_{y,i}, 0)$ or $\mathbf{d} = (0, d_{y,r} + id_{y,i}, d_{z,r} + id_{z,i})$. The r and i indices denote real and imaginary parts. With this choice, H_{BdG} can be diagonal in the spin (σ) subspace with respect to either the σ_z or the σ_x eigenbasis. Additionally, note that with this choice, the pairing potential $\hat{\Delta}$ can be written as $(\Delta_r(\sigma) - i\Delta_i(\sigma))\otimes\tau_y$, where:

$$\hat{\Delta}_r(\sigma) = c_j d_{j,i}\sigma_p - d_{y,r}\mathbb{1}_2;\ \hat{\Delta}_i(\sigma) = c_j d_{j,r}\sigma_p + d_{y,i}\mathbb{1}_2, \tag{525}$$

and $j = x$ or z with $c_j = \begin{cases} 1 & \text{if } j = x,\ p = z \\ -1 & \text{if } j = z,\ p = x \end{cases}$. Evidently, this pairing potential is diagonal. From here, we work in the $\lambda\otimes\tau$ subspace, which gives us an effective 4×4 model. We replace the operators $\Delta_r(\sigma)$ and $\Delta_i(\sigma)$ with their eigenvalues which we shall denote by

$\delta_{jr,\pm} = \pm c_j d_{j,i} - d_{y,r}$ and $\delta_{ji,\pm} = \pm c_j d_{j,r} + d_{y,i}$, leading to:

$$H_{\text{eff}}(k) = w_k k \lambda \otimes \mathbb{1}_2 + \gamma_k k \lambda_z \otimes \tau_x + \delta_{jr,\pm} \lambda_x \otimes \tau_y + \delta_{ji,\pm} \lambda_y \otimes \tau_y. \tag{526}$$

Here, one can check that $\lambda_z \otimes \tau_x$ commutes with $H_{\text{eff}}(k)$, so we now change basis from the standard basis to one consisting of eigenvectors of $\lambda_z \otimes \tau_x$, which are of the form $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\\pm1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\\pm1\end{pmatrix}$. In changing basis, we obtain the following Hamiltonian, which we shall denote by $\tilde{H}(k)$, where the Pauli matrices also carry a tilde to emphasize the basis change.

$$\tilde{H}(k) = -\gamma_k k \tilde{\lambda}_z \otimes \mathbb{1}_2 + w_k k \mathbb{1}_2 \otimes \tilde{\tau}_z + \tilde{\lambda}_z \otimes (\delta_{jr,\pm}\tilde{\tau}_y - \delta_{ji,\pm}\tilde{\tau}_x) \tag{527}$$

$$= \mp\gamma_k k \mathbb{1}_2 + w_k k \tilde{\tau}_z \pm \delta_{jr,\pm}\tilde{\tau}_y \mp \delta_{ji,\pm}\tilde{\tau}_x \tag{528}$$

Equation 528 reveals that our transformed Hamiltonian is now diagonal in the $\tilde{\lambda}$ subspace, which means that we can reduce to an effective 2×2 model in $\tilde{\tau}$ space. We now rewrite 528 in a manner that suggests how to diagonalize the 2×2 system.

$$\tilde{H}(k) \pm \gamma_k k \mathbb{1}_2 = V_{\pm}\left(\nu_{\pm\pm} \cdot \tilde{\tau}\right) \tag{529}$$

In the line above, we introduce the constant $V_{\pm} = \sqrt{(w_k k)^2 + \Delta^2(1 \pm |\boldsymbol{\omega}|)}$. The $\pm$ here distinguishes between spin up($+$) and spin down($-$), and $\boldsymbol{\omega} = i\boldsymbol{\eta} \times \boldsymbol{\eta}^*$. This precise dependence of $\Delta^2(1\pm|\boldsymbol{\omega}|)$ can be obtained by explicitly calculating the normalization factor $V_{\pm}$ from equation 529. The unit vector $\nu_{\pm\pm}$ takes the form $\nu_{\pm\pm} = \frac{1}{V_{\pm}}(\mp\delta_{ji,\pm}, \pm\delta_{jr,\pm}, w_k k)$. We now define spherical coordinates $\theta_{\pm} = \arccos\left(\frac{w_k k}{V_{\pm}}\right)$ and $\varphi_{\pm\pm} = \operatorname{sgn}(\pm\delta_{jr,\pm})\arccos\left(\frac{\mp\delta_{ji,\pm}}{\Delta}\right)$. In terms of these angular parameters, the eigenvectors of 529 with energies $E_{\mp\pm\pm}(k) = \mp\gamma_k k \pm V_{\pm}$ are as written below.

$$|t_{\pm\pm-}\rangle = \begin{pmatrix} -e^{-i\varphi_{\pm\pm}}\sin\left(\frac{\theta_{\pm}}{2}\right) \\ \cos\left(\frac{\theta_{\pm}}{2}\right) \end{pmatrix} \tag{530}$$

$$|t_{\pm\pm+}\rangle = \begin{pmatrix} \cos\left(\frac{\theta_{\pm}}{2}\right) \\ e^{i\varphi_{\pm\pm}} \sin\left(\frac{\theta_{\pm}}{2}\right) \end{pmatrix}. \tag{531}$$

To generate the eigenvectors of the initial 8×8 model, we first let $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the color will denote these as eigenvectors of the $\tilde{\lambda}$ subspace (black) or the σ subspace (cyan).

$$|T_{\pm\pm\pm}\rangle_j = |\pm\rangle \otimes |\pm\rangle_j \otimes |t_{\pm\pm\pm}\rangle \tag{532}$$

We use the notation here that the subscript j denotes $|\pm\rangle_j$ to be the eigenvectors of σ_j with $j = x$ or z. We will now use these eigenvectors $|T_{\pm\pm\pm}\rangle_j$ to express the Bogoliubov quasi-particle creation and annihilation operators in terms of a and b electron creation and annihilation operators, and we will drop the j subscript for brevity of notation. Letting $u_{k\pm} = -e^{-i\varphi_{+\pm}} \sin\left(\frac{\theta_{\pm}}{2}\right)$ and $v_{k\pm} = \cos\left(\frac{\theta_{\pm}}{2}\right)$ be the so-called coherence factors (note the switched convention compared to standard BCS Theory), we are able to derive the following expressions:

$$\alpha_{\mathbf{k}\pm} = v_{\mathbf{k}\pm}(a_{\mathbf{k}\pm} - b_{\mathbf{k}\pm}) - u^*_{\mathbf{k}\pm}(a^\dagger_{-\mathbf{k}\pm} + b^\dagger_{-\mathbf{k}\pm}) \tag{533}$$

$$\beta^\dagger_{-\mathbf{k}\pm} = u_{\mathbf{k}\pm}(a_{\mathbf{k}\pm} - b\mathbf{k}\pm) + v_{\mathbf{k}\pm}(a^\dagger_{-\mathbf{k}\pm} + b^\dagger_{-\mathbf{k}\pm}) \tag{534}$$

One can check that these operators satisfy the fermionic anticommutation relations. Note that the general form of these quasi-particle operators is close to the standard BCS form, barring the presence of two bands A and B. However, the basis change we implemented causes a and b electrons to mix, a consequence of the $\gamma_k k$ term in the Hamiltonian. The electron operators may now be expressed in terms of the Bogoliubov operators:

$$a_{\mathbf{k}\pm} = u^*_{\mathbf{k}\pm}\beta^\dagger_{-\mathbf{k}\pm} + v_{\mathbf{k}\pm}\alpha_{\mathbf{k}\pm} - u^*_{-\mathbf{k}\pm}\alpha^\dagger_{-\mathbf{k}\pm} + v_{-\mathbf{k}\pm}\beta_{\mathbf{k}\pm} \tag{535}$$

$$b_{\mathbf{k}\pm} = -u^*_{-\mathbf{k}\pm}\alpha^\dagger_{-\mathbf{k}\pm} + v_{-\mathbf{k}\pm}\beta_{\mathbf{k}\pm} - u^*_{\mathbf{k}\pm}\beta^\dagger_{-\mathbf{k}\pm} - v_{\mathbf{k}\pm}\alpha_{\mathbf{k}\pm} \tag{536}$$

With these operator expressions, we now move on to computing the nuclear relaxation rate, T_1^{-1}, with reference to the S^+ term of the Hyperfine Hamiltonian. In this calculation, the nuclear wavefunction is approximated as constant, so calculating T_1^{-1} effectively involves using Fermi's Golden Rule to calculate scattering amplitudes associated with electron states. Towards this end, we introduce the operator,

$$S^+_{\mathbf{k},\mathbf{k'}}(\phi) = (\cos(\phi)a^\dagger_{\mathbf{k}\downarrow} + \sin(\phi)b^\dagger_{\mathbf{k}\downarrow})(\cos(\phi)a_{\mathbf{k'}\uparrow} + \sin(\phi)b_{\mathbf{k'}\uparrow}). \tag{537}$$

This operator describes electron scattering from momentum **k'** to **k** where spin is flipped, and the parameter ϕ is introduced to allow us to vary the relative weight of a versus b electrons in the scattering process. This angle is given by $\tan\phi = 2A_aA_b/(A_a^2 + A_b^2)$, where $A_{a,b}$ are the hyperfine couplings to the two bands. As mentioned before, we invoke Fermi's Golden Rule and T_1^{-1} involves calculating $\sum_{i,f} \left|\langle f|S^+_{\mathbf{k},\mathbf{k'}}(\phi)|i\rangle\right|^2$, where i and f are the initial and final states of the scattering process which both correspond to states with only one Bogoliubov quasi-particle present. Thus, we sum over initial states featuring an $\alpha_{\mathbf{k'}\pm}$ or a $\beta_{\mathbf{k'}\pm}$ excitation and final states featuring an $\alpha_{\mathbf{k}\pm}$ or a $\beta_{\mathbf{k}\pm}$ excitation. Before proceeding however, we now need to acknowledge the role played by the choice of $j = x$ or z. In the case that $j = z$, we may directly write (537) in terms of 533 and 534. This lets us calculate the scattering amplitudes directly, and this is presented in the next section. Remark again that we will express the dependence of these expressions on **k** as dependence on its component perpendicular to the Fermi surface, denoted as just k.

j = z:

By direct computation, express the electron creation and annihilation operators in (537), and then read off the scattering amplitudes to obtain,

$$A_{\alpha\uparrow\to\alpha\downarrow}(k,k') = v_{k\downarrow}v_{k'\uparrow}\,(1 - \sin(2\phi)) \tag{538}$$

$$A_{\alpha\downarrow\to\alpha\uparrow}(k,k') = u^*_{k\uparrow}u_{k'\downarrow}\,(1 + \sin(2\phi))$$

$$A_{\beta\uparrow\to\beta\downarrow}(k,k') = v_{-k\downarrow}v_{-k'\uparrow}\left(1+\sin(2\phi)\right)$$

$$A_{\beta\downarrow\to\beta\uparrow}(k,k') = u^*_{-k\uparrow}u_{-k'\downarrow}\left(1-\sin(2\phi)\right)$$

$$A_{\alpha\uparrow\to\beta\downarrow}(k,k') = v_{-k\downarrow}v_{k'\uparrow}\cos(2\phi)$$

$$A_{\alpha\downarrow\to\beta\uparrow}(k,k') = u^*_{-k\uparrow}u_{k'\downarrow}\cos(2\phi)$$

$$A_{\beta\uparrow\to\alpha\downarrow}(k,k') = v_{k\downarrow}v_{-k'\uparrow}\cos(2\phi)$$

$$A_{\beta\downarrow\to\alpha\uparrow}(k,k') = u^*_{k\uparrow}u_{-k'\downarrow}\cos(2\phi)$$

The next section outlines the differences imposed by the condition $j = x$.

j=x:

The reason that the calculation of these scattering elements becomes trickier now is that they involve transitions from up spin to down spin particles while our Bogoliubov quasiparticles are labeled by S_x eigenvalues. We make use of the relation:

$$c_{k,\uparrow} = \frac{1}{\sqrt{2}}\left(c_{k,+}+c_{k,-}\right) \tag{539}$$

$$c_{k,\downarrow} = \frac{1}{\sqrt{2}}\left(c_{k,+}-c_{k,-}\right), \tag{540}$$

where the c_k operator here can be either a_k or b_k. With this transformation, we can write $S^+_{k,k'}(\phi)$, but first note the forms of the operators $S^-_{k,k'}$ and $S^z_{k,k'}$:

$$\begin{aligned} S^-_{k,k'}(\phi) &= (\cos(\phi)a^\dagger_{k\uparrow}+\sin(\phi)b^\dagger_{k\uparrow})(\cos(\phi)a_{k'\downarrow}+\sin(\phi)b_{k'\downarrow}), \\ S^z_{k,k'}(\phi) &= \frac{1}{2}\sum_s s(\cos(\phi)a^\dagger_{ks}+\sin(\phi)b^\dagger_{ks})(\cos(\phi)a_{k's}+\sin(\phi)b_{k's}), \end{aligned} \tag{541}$$

where $s=\pm1$ denotes up/down spin. In this case where our natural spin basis is that of σ_x, we can express $S^+_{k,k'}(\phi)$ in the following way.

$$S^{+,\uparrow\backslash\downarrow}_{k,k'}(\phi) = S^{z,\pm}_{k,k'}(\phi)+\frac{1}{2}\left(S^{-,\pm}_{k,k'}(\phi)-S^{+,\pm}_{k,k'}(\phi)\right) \tag{542}$$

$$= S^{z,\pm}_{k,k'}(\phi) - iS^{y,\pm}_{k,k'}(\phi)$$

The notation for the S operators in the equation above is as follows. Those operators with $\uparrow \setminus \downarrow$ as a superscript involve spin matrices written in the usual z-basis. Those with $\pm$ in the superscript involve the exact same matrices, element-wise, except the "z-basis" here actually corresponds to the x-basis. Consequently, the scattering amplitudes associated with $S^{+,\uparrow\setminus\downarrow}_{k,k'}(\phi)$ can be calculated by considering scattering between $\alpha_\pm$ particles and $\beta_\pm$ particles. Furthermore, we can use the same scattering amplitudes found in the previous section, except the $\uparrow\rightarrow\downarrow$ scattering and vice versa will refer to $+\rightarrow-$ scattering and vice versa. With this in mind, we have the following amplitudes that will contribute to our calculation of T_1^{-1}.

(543)
$$\begin{aligned}
A_{\alpha\pm\rightarrow\alpha\pm}(k,k') &= \frac{1}{2}\left(v_{k\pm}v_{k'\pm}(1-\sin(2\phi)) - u^*_{k\pm}u_{k'\pm}(1+\sin(2\phi))\right)\\
A_{\beta\pm\rightarrow\beta\pm}(k,k') &= \frac{1}{2}\left(v_{-k\pm}v_{-k'\pm}(1+\sin(2\phi)) - u^*_{-k\pm}u_{-k'\pm}(1-\sin(2\phi))\right)\\
A_{\alpha\pm\rightarrow\beta\pm}(k,k') &= \frac{1}{2}\cos(2\phi)\left(u^*_{-k\pm}u_{k'\pm} - v_{-k\pm}v_{k'\pm}\right)\\
A_{\beta\pm\rightarrow\alpha\pm}(k,k') &= \frac{1}{2}\cos(2\phi)\left(u^*_{k\pm}u_{-k'\pm} - v_{k\pm}v_{-k'\pm}\right)\\
A_{\alpha+\rightarrow\alpha-}(k,k') &= \frac{1}{2}\left(u^*_{k-}u_{k'+}(1+\sin(2\phi)) - v_{k-}v_{k'+}(1-\sin(2\phi))\right)\\
A_{\alpha-\rightarrow\alpha+}(k,k') &= \frac{1}{2}\left(v_{k+}v_{k'-}(1-\sin(2\phi)) - u^*_{k+}u_{k'-}(1+\sin(2\phi))\right)\\
A_{\beta+\rightarrow\beta-}(k,k') &= \frac{1}{2}\left(u^*_{-k-}u_{-k'+}(1-\sin(2\phi)) - v_{-k-}v_{-k'+}(1+\sin(2\phi))\right)\\
A_{\beta-\rightarrow\beta+}(k,k') &= \frac{1}{2}\left(v_{-k+}v_{-k'-}(1+\sin(2\phi)) - u^*_{-k+}u_{-k'-}(1-\sin(2\phi))\right)\\
A_{\alpha+\rightarrow\beta-}(k,k') &= \frac{1}{2}\cos(2\phi)\left(u^*_{-k-}u_{k'+} - v_{-k-}v_{-k'+}\right)\\
A_{\alpha-\rightarrow\beta+}(k,k') &= \frac{1}{2}\cos(2\phi)\left(v_{-k+}v_{k'-} - u^*_{-k+}u_{k'-}\right)\\
A_{\beta+\rightarrow\alpha-}(k,k') &= \frac{1}{2}\cos(2\phi)\left(u^*_{k-}u_{-k'+} - v_{k-}v_{-k'+}\right)\\
A_{\beta-\rightarrow\alpha+}(k,k') &= \frac{1}{2}\cos(2\phi)\left(v_{k+}v_{-k'-} - u^*_{k+}u_{-k'-}\right)
\end{aligned}$$

These amplitudes can be decomposed into two subsets of scattering processes: those which preserve the x component of spin and those which flip this. The first four equations involve the processes that preserve spin, and these are in fact the same amplitudes that appear in the calculation of ultrasonic attenuation for either orientation of $\mathbf{d}$ that we have considered. The existence of spin-preserving, i.e. gap preserving, scattering is what distinguishes the behavior of quasi-particle scattering when $\mathbf{d}$ has a component along the z-axis.

2-Gap Spin Singlet Model

In addition to explicitly solving the INT model, writing the Bogoliubov quasi-particle operators, and calculating the scattering amplitudes required to obtain nuclear-spin relaxation rate, we also consider a spin-singlet model in addition to the INT model, which promotes triplet pairing. The pivotal question at the heart of this work is whether $LaNiGa_2$ breaks time reversal symmetry below its critical temperature, and one way for us to probe this is by studying another effective 8×8 model, inspired in form by the INT model, 523, where singlet spin pairing occurs, which preserves time reversal symmetry. Explicitly, the effective Bogoliubov-de-Gennes Hamiltonian is identical to equation 523 except for the off-diagonal pairing potential which for this model we call $\hat{\Delta}_s$. We define this to be,

$$\hat{\Delta} = -\sigma_y \otimes \Delta_0 \left(\cos(\theta_b)\mathbb{1}_2 + \sin(\theta_b)\tau_x\right). \tag{544}$$

Here the parameter θ_b is responsible for the existence of two distinct superconducting gaps $\Delta_\pm$ given by

$$\Delta_\pm = \Delta_0\sqrt{1 \pm \sin(2\theta_b)}. \tag{545}$$

Similarly, this model can be solved to yield two species of spinful Bogoliubov excitations:

$$\begin{aligned} \alpha(\beta)_{\mathbf{k}\uparrow} &= u_{\mathbf{k}\alpha(\beta)}\left(a_{\mathbf{k}\uparrow} \pm b_{\mathbf{k}\uparrow}\right) + v_{\mathbf{k}\alpha(\beta)}(a^\dagger_{-\mathbf{k}\downarrow} \pm b^\dagger_{-\mathbf{k}\downarrow}) \\ \alpha(\beta)^\dagger_{-\mathbf{k}\downarrow} &= -u^*_{\mathbf{k}\alpha(\beta)}(a^\dagger_{-\mathbf{k}\downarrow} \pm b^\dagger_{-\mathbf{k}\downarrow}) + v_{\mathbf{k}\alpha(\beta)}(a_{\mathbf{k}\uparrow} \pm b_{\mathbf{k}\uparrow}), \end{aligned} \tag{546}$$

where $u_{\mathbf{k}\gamma}$ and $v_{\mathbf{k}\gamma}$ are determined from the diagonalization of the Hamiltonian. Aside from the difference in spin-pairing, which is singlet in this case, the superconducting state will feature inter-band pairing.

Nuclear-Spin Relaxation Rate

With these amplitudes in hand, we may now proceed to calculate the nuclear relaxation rate using Fermi's Golden Rule. Following the notes of Arovas and Wu, we sum over **k**, **k'**, and spins $s, s' = \pm 1$ of the following (where s, s' can correspond to either the z or x bases):

$$\sum_{\mathbf{k},\mathbf{k'},s,s'} |A_{\alpha s' \to \alpha s}(\mathbf{k}, \mathbf{k'})|^2 f_{\alpha,\mathbf{k},s}(1 - f_{\alpha,\mathbf{k'},s'})\delta(E_{\alpha,\mathbf{k'},s'} - E_{\alpha,\mathbf{k},s} - \hbar\omega)+ \\ |A_{\beta s' \to \beta s}(\mathbf{k}, \mathbf{k'})|^2 f_{\beta,\mathbf{k},s}(1 - f_{\beta,\mathbf{k'},s'})\delta(E_{\beta,\mathbf{k'},s'} - E_{\beta,\mathbf{k},s} - \hbar\omega) \tag{547}$$

where $f_{\alpha,\mathbf{k},s}$ and $f_{\beta,\mathbf{k},s}$ denote the Fermi-Dirac functions corresponding to alpha and beta particles respectively with momentum $\mathbf{k}$ and spin s. This labeling convention is also used to index the energies in the argument of the delta function. Note that the strongest enhancement of relaxation rate will occur due to processes that do not require quasiparticles to hop across the two gaps, i.e. the spin component is preserved. These are more energetically favorable and are only possible if $\mathbf{d}$ is perpendicular to the $\mathbf{x}$ direction (see the first four equations of Eq. 543). To introduce temperature dependence into the formation of a superconducting gap, we let $\Delta(T) = \Delta_0(T)\sqrt{1 - \left(\frac{\gamma_k}{w_k}\right)^2}\sqrt{1 \pm |\sin\theta_\eta|}$. $\Delta_0(T)$ has temperature dependence given by the solution to the self-consistency equation for the singlet BCS Theory gap,

$$\int_0^\infty d\epsilon \left[\frac{\tanh\left(\sqrt{\epsilon^2 + \Delta^2}/2T\right)}{\sqrt{\epsilon^2 + \Delta^2}} - \frac{1}{\sqrt{\epsilon^2 + \Delta(0)^2}}\right], \tag{548}$$

where $\Delta(0)$ is the $T = 0$ value of the parameter such that $T_c = 1$. $\Delta_0(T)$ is then obtained by scaling the solution to the equation above such that $\Delta_0(0)$ takes on any given value.

We are now equipped to review the numerically calculated scattering amplitudes that yield nuclear-spin relaxation rate and comment on their implications for the breaking of time reversal symmetry of superconducting $LaNiGa_2$.

We first consider fits to the INT model, shown in Fig. 12. In this case, the scattering rates between the quasiparticles are dramatically suppressed if the two gaps have different values because energy cannot be conserved in the process. The nuclear resonance frequency is set to $\hbar\omega_N/k_BT_c = 10^{-4}$ and 10^{-6} in the cases of $\mathbf{d}_r$ and $\mathbf{d}_\theta$, respectively, with no additional broadening included. When the two gaps are equal, the model describes the peak very well. However, this case is unitary and has no TRSB. When TRSB is introduced and the two gaps are different, the results depend on whether the nuclear spin quantization axis is the same as the equal-spin pairing axis. In the limit of weak spin-orbit coupling, the spin pairing axis is unrelated to the crystallographic axes. If $\mathbf{d}$ lies in the xy plane, parallel to the nuclear spin quantization axis, the spin-flip process has an immediate downturn in T_1^{-1} and there is no possibility of an HS peak. When the axis for equal spin pairing is away from the nuclear quantization axis, that is, $\mathbf{d}$ lies in the yz plane, a weak HS peak arises due to certain scattering processes that do not require crossing from one gap to the other; see [1] for further discussion. By making the nuclear frequency anomalously small $\hbar\omega_N/k_BT_c = 10^{-12}$ (0.04 Hz), the peak can be enhanced to fit the data well, as shown in the supplemental materials [1]. However, this artificial enhancement of the peak is not credible as one expects other forms of damping to reduce, rather than enhance the peak.

TABLE 1. Superconducting gap values measured in $LaNiGa_2$.

source	Δ_1/k_BT_c	Δ_2/k_BT_c
NQR singlet fit	1.41	1.95
NQR triplet fit ($\mathbf{d}_\theta$)	1.96	1.96
NQR triplet fit ($\mathbf{d}_r$)	1.8599	1.8601
NQR triplet fit ($\mathbf{d}_z$)	1.81	1.81
heat capacity [110]	1.29	2.04
susceptibility [32]	1.57	1.89
μSR [97]	1.3	2.7

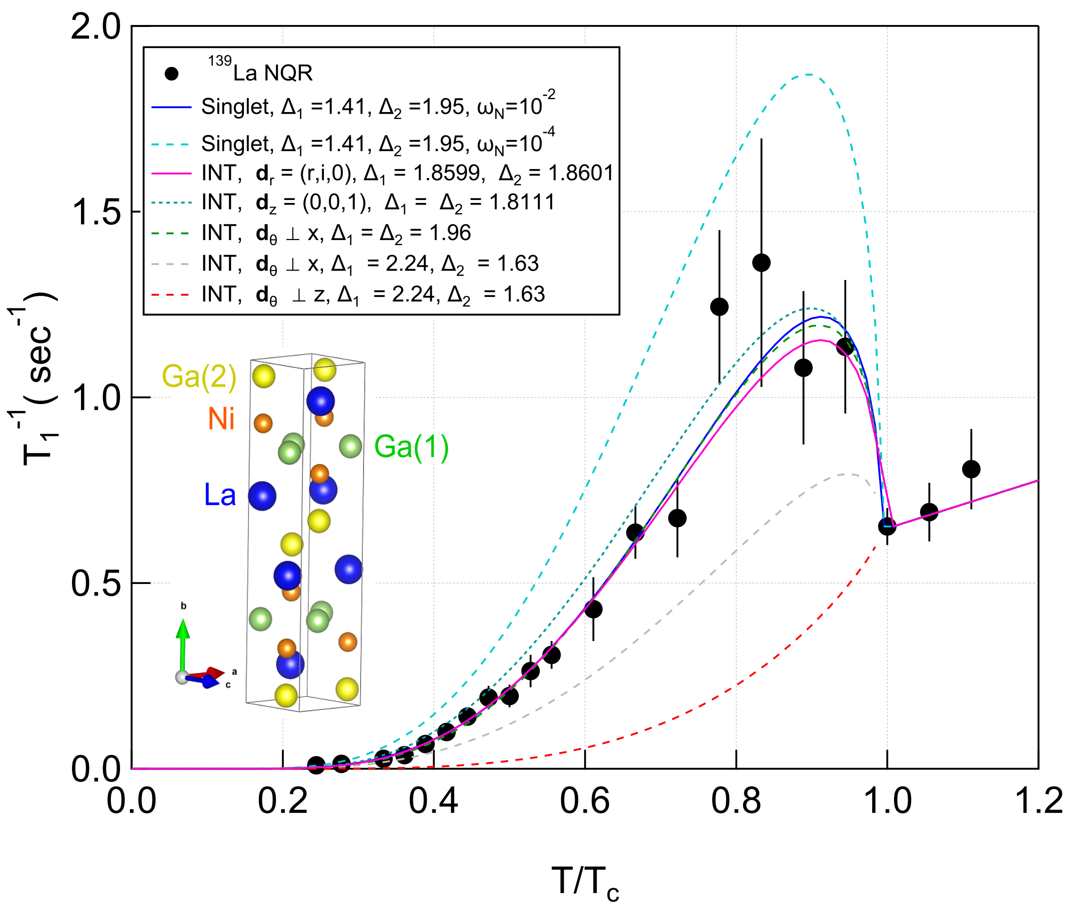

FIGURE 12. ^{139}La T_1^{-1} versus inverse temperature in the superconducting state of $LaNiGa_2$ (with $T_c = 1.83$ K). The solid and dashed lines are calculations as discussed in the text, and the gap values, (Δ_1, Δ_2) and nuclear resonance frequency, ω_N, are given in units of k_BT_c. The data are well fit by a two-gap singlet-pairing model as well as by an INT triplet-pairing model with two equal gaps, but not with the INT model with two different gaps. The inset shows the unit cell of $LaNiGa_2$.

Figure 12 also shows fits to the singlet model, assuming that $\hbar\omega_N/k_BT_c = 10^{-4}$ and 10^{-2} respectively. The quasi-particle scattering must conserve energy, which includes the Zeeman (or quadrupolar) energy of the nuclear spin flip. The peak will diverge if this energy is neglected and there is no impurity broadening, but will be gradually suppressed as this energy difference between the initial and final quasi-particle energies is increased. There is a clear coherence peak below T_c, but it is larger than the experimental data for $\hbar\omega_N/k_BT_c = 10^{-4}$ (dashed cyan line). The coherence peak is often modeled by introducing a broadening term that tends to suppress the divergence [71]. A similar suppression can arise by assuming a larger nuclear frequency, and the solid blue line is calculated assuming $\hbar\omega_N/k_BT_c = 10^{-2}$, two orders of magnitude larger. This assumption suppresses the peak and the BCS case clearly fits the data well. The fitted gap values at $T = 0$, given in Table 1, are comparable to those reported by previous heat capacity [110] and magnetic susceptibility measurements [32], but somewhat closer to one another than those reported by recent μSR experiments [97]. The saturation of the data at the lowest temperatures suggests small residual impurity scattering. The parameter $\phi = 0.157$ for the best fit indicates that the nuclear spins couple to both bands, but one coupling is stronger than the other.

Overall, the robustness of the Hebel-Slichter peak in $LaNiGa_2$ compared to other materials suggests a conventional pairing. Note that the T_c of our powder sample is reduced from that of a single crystal, suggesting the presence of some non-magnetic disorder and/or distributed strain induced by grinding. Although disorder generally suppresses the height of a HS peak, it is unclear whether it can suppress the stability of an INT state in favor of conventional s-wave pairing. However, the initial μSR study of TRSB was reported in a polycrystalline sample, where T_c was similarly suppressed.

In summary, the zero field NQR spin-lattice-relaxation rate data for $LaNiGa_2$ are inconsistent with the INT model unless the two gaps are identical, in which case the state is unitary and there is no TRSB. On the other hand, a two band BCS state with singlet pairing describes the data well. This raises some doubts on the identification of triplet-pairing and time-reversal symmetry breaking in this material. Neither scenario is consistent with μSR measurements of TRSB. However, μSR measurements are close to the lower limit of experimental detection and were conducted only on polycrystalline material. Further measurements should be performed on single crystals, and other measurements, such as optical Kerr rotation [91] should be done to confirm the presence of TRSB. Measurements of the ultrasonic attenuation rate would provide further constraints on the pairing symmetry in $LaNiGa_2$ as that process involves a spin-independent relaxation and is complimentary to our NQR. NQR measurements of the Ga(2) site may also provide information about spin fluctuation asymmetry in the superconducting state.

Part 3. Conclusion

9. Review and Discussion

Our exploration of interacting electron systems using both mathematically rigorous methods and numerical studies of effective models allowed us to probe fundamental features that characterize three distinct electronic phases: interacting quantum Hall states, Mott insulators, and unconventional superconductors.

The integer and fractional quantum Hall effect have shown the existence of one of the most important examples of a topological phase of matter, one which is characterized by an invariant quantity that is rigid under small enough perturbations. In condensed matter physics, a very useful tool for proving stability is the existence of a Lieb-Robinson bound, which implies a finite rate of information propagation where the rate can be with respect to time or a tunable parameter in the system. Initial progress in computing these estimates take place in quantum lattice systems, so in Section 2, we formally build the local Hilbert spaces and algebras of observables and construct general Hamiltonians from bounded, local interactions. Under precise conditions, for two operators $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X \cap Y = \emptyset$ one can then bound $\left\|[A(t), B]_{\pm}\right\|$ by a function that decays in $d(X,Y) - vt$, where v is the so called Lieb-Robinson velocity. This constant v is a limit on how fast information can travel in the system as seen by the failure of $A(t)$ and B to (anti-)commute.

After reviewing the integer quantum Hall setting in Section 3, we second quantize the associated single-particle Landau Hamiltonian with periodic potential and introduce a UV-regularized pairwise interaction in Section 4, with inspiration drawn from works such as [31] and [12]. This construction converts pointwise fermionic field operators to bounded, localized elements of the fermionic CAR algebra, and crucially, the resulting interaction is bounded in finite volume restrictions. Consequently in [47], we are able to prove Lieb-Robinson bounds that imply quasi-locality via the inner product structure of the CAR algebra. In the absence of time evolution, if $f, g \in L^2(\mathbb{R}^2)$, then $\{a(f), a^*(g)\} = \langle f, g\rangle \mathbb{1}$. With interacting finite volume dynamics τ_t^Λ, one obtains bounds of the form $\left\|\{\tau_t^\Lambda(a(f)), a^*(g)\}\right\| \leq G(t)\langle |f^t|, |g|\rangle$, where $f^t(x) = \int_{\mathbb{R}^2} dy\, |f(x)| e^{-|x-y|/\sigma_t}$, where σ_t grows linearly in time. The inner product $\langle |f^t|, |g|\rangle$ can thus be thought of as a "smeared" inner product, where f^t still carries most of its predominant weight in the same regions as f. Following this estimate, we then prove strong continuity of the Heisenberg dynamics in the limit of infinite volume interactions, a powerful result that affirms that the norms and supports of operators do not change drastically in very short time intervals.

In fact, such results can also be proven for dynamics where the time variable is a tunable parameter that lets one interpolate along a curve of Hamiltonians. For a family of finite volume Hamiltonians $H_\Lambda(s)$ with a sufficiently smooth dependence on s and assumed gapped ground state projections $P_\Lambda(s)$, the dynamics that map $P_\Lambda(s)$ to $P_\Lambda(0)$ is generally non-local. However, due to progress made by Hastings and Wen, [39], and Bachmann et al. [11], it has been shown that there exists a procedure by which one can construct the spectral flow dynamics α_s^Λ that satisfy the mapping of ground states, $\alpha_s^\Lambda(P_\Lambda(s)) = P_\Lambda(0)$, and act as quasi-local maps on the algebras of observables. In other words, they satisfy Lieb-Robinson bounds, which in this setting imply a rigidity of local structure as a perturbation becomes more noticeable. For quantum lattice systems with short range interactions, this locality has been rigorously proved, and in Section 5, we overcome the challenges presented by the continuum setting to construct these quasi-local dynamics α_s^Λ for interacting fermion models using the framework of lattice-localized frames, [10]. Crucially, we also prove the existence of the thermodynamic limit of these dynamics, with respect to volume in the frame lattice Θ.

Having explicitly constructed and verified the properties of the spectral flow in continuous space, we put these dynamics to use in Section 6, where we prove quantization of Hall conductance for interacting fermions in a perpendicular magnetic field, both in finite and infinite volume limits. Results of this form have been proved for gapped lattice systems,

[38], [8], [9]. In this thesis, we adopt the strategy of Hastings and Michalakis in a toroidal geometry, as done in [38] and [8], to achieve the objective. In particular, spectral flow provides a quasi-local dynamics for implementing flux insertion along intersecting edges of the torus (where the torus is with respect to the frame lattice Θ). This enables us to calculate a non-trivial Hall curvature and prove its independence in fluxes φ along the torus and existence in the thermodynamic limit, in which the bulk is exactly represented by the Landau Hamiltonian and a short-range pairwise interaction, [10]. In the setting of continuous space interacting models, this result is a first of its kind and opens the door for more applications of the quasi-adiabatic evolution introduced in Section 5.

The work up to this stage made use of rigorous mathematics to prove and make use of the quasi-locality of Heisenberg dynamics, and the solubility of the Landau Hamiltonian is conducive to the concrete results we were able to prove. For general models that aim to describe real complex materials, exact solutions are extremely rare, which means that alternate methods are required to probe the behaviors of such materials. In this case, we seek approximations and effective models that can be studied numerically. The Fermi-Hubbard model, while only solvable in one dimension, has been the focus of numerical studies in higher dimensions through Monte-Carlo simulations, series expansion approximations, and more. In our work outlined in Section 7 based on [48], we make use of a strong coupling expansion, where the kinetic energy hopping is treated perturbatively, to study the SU(4) Hubbard model and probe the onset of Mott insulation in magic angle twisted bilayer graphene.

Notably, we observe peaks in inverse compressibility and cusps in entropy as functions of carrier density at different temperatures and Hubbard constant, U. Our findings did uncover limits to how well the Hubbard model could reproduce experimental data in graphene systems. For example, for sufficiently large temperatures, Mott behavior occurs at half integer carrier densities per site in addition to integer densities, whereas our work only displayed the integer densities. Furthermore, the inverse compressibility data we calculated did not possess asymmetry in the peaks as seen in measurements, e.g. [127]. Thus, while the strong coupling expansion of the Hubbard model paved the way to uncover Mott insulation as seen in a real material, to exactly match the more quantitative features of experimental measurements, a more complex and refined model would be required.

In Section 8, this thesis studies the superconducting phase of the compound $LaNiGa_2$. Its electron pairing mechanism is purported to be triplet in spin space based on experimental work that shows the existence of a non-trivial internal magnetization, e.g. as presented by Hillier et al. [46], meaning a preference in spin orientation by the electron pairs. However, in collaboration with Sherpa et al. [92], we present the existence of a Hebel-Slichter peak: a sudden rise in nuclear-spin relaxation rate below the critical temperature. This can be seen as en enhancement in spin-flip scattering of electrons due to a divergence in the electronic density of states, which occurs as the superconducting gap opens. Importantly, this phenomenon is predicted by BCS theory, where opposite spins form singlet Cooper pairs. Therefore, with competing evidence supporting triplet and singlet pairing, we investigate two effective Bogoliubov-de-Gennes Hamiltonians featuring triplet/singlet spin pairing and singlet/triplet band pairing respectively to preserve overall antisymmetry required by the Cooper pair wavefunction. From these models, we compute the Bogoliubov quasi-particle operators and coherence factors and use this to calculate the scattering amplitude associated with S^+-mediated spin flip of an electron with momentum k' to momentum k. Our theoretical fits reveal that a singlet pairing mechanism best fits the observation of a HS peak coupled with two distinct superconducting gaps, in opposition with certain experimental findings.

Moving forward, there are a number of open questions that the results of this dissertation pose ahead of future work. The continuum construction of the spectral flow made use of a discretization of the space of functions used to generate fermionic observables in $\mathcal{A}(\mathcal{H}_n)$ or the union over finitely many Landau levels of these algebras. This discretization is intrinsically linked to real space and we can therefore define locality via this lattice. Proving quantization of Hall conductance is one application that was made feasible, and one can wonder what more can be done?

In the detailed work of Nachtergaele, Sims and Young, [75], the authors rigorously show how the spectral flow dynamics play a pivotal role in proving stability of the spectral gap for a family of Hamiltonians. In particular, if an initial Hamiltonian $H(s = 0)$ has a gap ϵ between its ground state and first excited state, can one prove that there exists an upper bound s_ϵ on how large the perturbation parameter s can become before the ground state gap closes. For two variants of the AKLT spin chain, this procedure has been used to prove gap stability, [116].

The models we consider in sections 3-6 start from the Landau Hamiltonian in the single-particle picture, which inherently has gapped eigenvalues. Concretely, one can consider a system consisting of a filled lowest Landau level, i.e. chemical potential in the gap between the lowest two Landau levels. From this ground state, excitations take the form of particle-like excitations in higher Landau levels or hole-like excitations in the lowest level. Since these carry nonzero energy, the ground state is necessarily gapped, which is representative of the integer quantum Hall setting. Two body interactions are introduced as a perturbation of the initial model, lending to an intermediate regime between integer and fractional quantum Hall states. Thus, a worthy goal would be to prove that the gap in the spectrum away from the occupied lowest Landau level state is stable under perturbation by the pairwise interaction.

As a more technical point, the charge operators used to build the automorphisms of flux insertion over $\mathcal{A}(L^2(\mathbb{V}_L))$ were defined by spatial number operators $\hat{N}_X$ associated with subsets $X \subset \mathbb{V}_L$ and not $X \subset \Theta_L$, i.e. the lattice of centers of localized functions. The interactions transformed by these maps, while still bounded elements of $\mathcal{A}(L^2(\mathbb{R}^2))$ are no longer tied to the original Landau level subspaces $\mathcal{H}_n$. This is not surprising because whether we consider functions χ_θ, with $\theta \in \Theta$ or some finite open boundary subset, or the magnetically periodized χ_θ^L, continuity and differentiability within $\mathbb{R}^2$ or $\mathbb{V}_L$ is inherent to elements in the spans of $\chi_\theta^{(L)}$. Thus, while the charge operators $Q_X = q\hat{N}_X$ count fermion occupation in the region X and mutually commute, meaning $[Q_X, Q_Y] = 0 \,\forall\, X, Y \subset \mathbb{R}^2$, this counting is not local to an individual Landau level. This obstruction is a key consequence of Landau level physics, where the existence of an exponentially localized orthonormal basis is forbidden, [22]. Therefore, there may not exist a number operator that respects both locality and the spectral subspaces of the underlying Hamiltonian; nonetheless, it would be worth exploring the conditions under which such an operator is admissible.

Additionally, strongly correlated electron systems continue to be the subject of significant research as physicists seek to better understand the conditions that give rise to phases such as Mott insulation, unconventional superconductivity, heavy fermion Kondo behavior, and more. The Fermi-Hubbard model has yielded significant progress in understanding behaviors such as high temperature superconductivity in the family of cuprate materials, [90]. Thus, it will be worthwhile to continue studying this model and its variants and develop more effective methods for doing so, such as series expansions and Monte-Carlo algorithms.

Lastly, our experimental observation of a Hebel-Slichter peak in conjunction with theoretical predictions of this phenomenon raise doubts about the nature of time reversal symmetry breaking/preservation in superconducting $LaNiGa_2$. Consequently, there is lots of room to pursue further experimental and theoretical investigations of the superconducting state of this material. Most importantly, additional measurements of the existence of an internal magnetization would contribute substantial evidence towards either triplet (nonzero magnetization) or singlet (zero magnetization) spin pairing.

Moreover, in [92], we discuss nuclear-spin relaxation rate, but there are other measurements that could shed light on the superconducting mechanism. Ultrasonic attenuation is a quantity that can be calculated using the same general procedure as relaxation rate, as detailed in [104] in the context of BCS theory. For the time-reversal symmetric Uranium ditelluride, considered to be a spin-triplet superconductor, an enhancement in ultrasonic attenuation below critical temperature has been observed, [55], in contrast with a continuous decrease as seen for BCS materials. Therefore, exploring the behavior of ultrasonic attenuation in $LaNiGa_2$ could provide some additional answers regarding the pairing mechanism and presence or lack thereof of time reversal symmetry below the critical temperature.

From a theoretical standpoint, the INT model is elementary in certain aspects as it does not fully capture the full topological landscape of the five intersecting Fermi surfaces that have been detected in $LaNiGa_2$, [13]. Constructing a more complex model might uncover certain features undetectable by the INT model, so long as it is not too contrived or unsolvable by analytic and numerical methods.

A. Appendix

What follows are some initial steps in the proof of Theorem 4.6. This calculation picks up from equation (125).

$$\mathcal{I}_n\left(\{k_i\}_{i=1}^n, y, x\right) = e^{-itH_L} \prod_{j=1}^{n} \tau_{1,t_j}^{\emptyset}(e^{ik_j\cdot x})\Psi_y^\sigma(x) \tag{549}$$

By Lemma 4.7, this yields,

$$\mathcal{I}_n\left(\{k_i\}_{i=1}^n, y, x\right) = e^{-itH_L} \prod_{j=1}^{n} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} \Psi_y^\sigma \tag{550}$$

To proceed, we invoke Lemma 4.8 with the aim of moving all of the T^- operators rightwards so that they act consecutively on $\Psi_y^\sigma(x)$. The result of moving $n-1$ magnetic translation operators past operators of the form $e^{ik_j\cdot x}$ is summarized in the next lemma.

Lemma A.1. *The following equality holds.*

$$\prod_{j=1}^{n} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} = e^{\frac{i}{2B}\sum_{j=1}^{n} k_j\cdot b_{k_j}(t_j)} e^{-\frac{i}{B}\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} k_j\cdot b_{k_l}(t_l)} e^{i\sum_{j=1}^{n} k_j\cdot x} \prod_{j=1}^{n} T^-_{\frac{1}{B}b_{k_j}(t_j)}$$

Proof. This proof shall be done via induction. As the $n = 1$ iteration of the left side of the equality above is just the term itself, we begin with the $n = 2$ form of the expression and show that the right side is correct.

$$\begin{aligned}\prod_{j=1}^{2} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} &= e^{\frac{i}{2B}k_2\cdot b_{k_2}(t_2)} e^{ik_2\cdot x} T^-_{\frac{1}{B}b_{k_2}(t_2)} e^{\frac{i}{2B}k_1\cdot b_{k_1}(t_1)} e^{ik_1\cdot x} T^-_{\frac{1}{B}b_{k_1}(t_1)} \\ &= e^{\frac{i}{2B}(k_2\cdot b_{k_2}(t_2)+k_1\cdot b_{k_1}(t_1))} e^{ik_2\cdot x} T^-_{\frac{1}{B}b_{k_2}(t_2)} e^{ik_1\cdot x} T^-_{\frac{1}{B}b_{k_1}(t_1)}\end{aligned}$$

By Lemma 4.8, one may swap $T^-_{\frac{1}{B}b_{k_2}(t_2)}$ and $e^{ik_1\cdot x}$ at the cost of the additional phase $e^{-\frac{i}{B}k_1\cdot b_{k_2}(t_2)}$. Therefore, the right side can be written as follows.

$$\begin{aligned}\prod_{j=1}^{2} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} &= e^{\frac{i}{2B}(k_2\cdot b_{k_2}(t_2)+k_1\cdot b_{k_1}(t_1))} e^{i(k_1+k_2)\cdot x} e^{-\frac{i}{B}k_1\cdot b_{k_2}(t_2)} T^-_{\frac{1}{B}b_{k2}(t_2)} T^-_{\frac{1}{B}b_{k_1}(t_1)} \\ &= e^{\frac{i}{2B}\sum_{j=1}^{2} k_j\cdot b_{k_j}(t_j)} e^{-\frac{i}{B}\sum_{j=1}^{1}\sum_{l=j+1}^{2} k_j\cdot b_{k_l}(t_l)} e^{i\sum_{j=1}^{2} k_j\cdot x} \prod_{j=1}^{2} T^-_{\frac{1}{B}b_{k_j}(t_j)}\end{aligned}$$

Thus the first step of the proof is complete. Now, we assume that the statement of the lemma is true for some n. Consider the $(n+1)$-fold product of these phases and factors.

$$\prod_{j=1}^{n+1} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} = e^{\frac{i}{2B}k_{n+1}\cdot b_{k_{n+1}}(t_{n+1})} e^{ik_{n+1}\cdot x} T^-_{\frac{1}{B}b_{k_{n+1}}(t_{n+1})} \prod_{j=1}^{n} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)}$$

Now we use our assumption that the n-fold product satisfies the lemma.

$$\begin{aligned}\prod_{j=1}^{n+1} e^{\frac{i}{2B}k_j\cdot b_{k_j}(t_j)} e^{ik_j\cdot x} T^-_{\frac{1}{B}b_{k_j}(t_j)} &= e^{\frac{i}{2B}k_{n+1}\cdot b_{k_{n+1}}(t_{n+1})} e^{ik_{n+1}\cdot x} T^-_{\frac{1}{B}b_{k_{n+1}}(t_{n+1})} e^{\frac{i}{2B}\sum_{j=1}^{n} k_j\cdot b_{k_j}(t_j)} \times \\ &\quad e^{-\frac{i}{B}\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} k_j\cdot b_{k_l}(t_l)} e^{i\sum_{j=1}^{n} k_j\cdot x} \prod_{j=1}^{n} T^-_{\frac{1}{B}b_{k_j}(t_j)} \\ &= e^{\frac{i}{2B}\sum_{j=1}^{n+1} k_j\cdot b_{k_j}(t_j)} e^{-\frac{i}{B}\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} k_j\cdot b_{k_l}(t_l)} e^{ik_{n+1}\cdot x} \\ &\quad T^-_{\frac{1}{B}b_{k_{n+1}}(t_{n+1})} e^{i\sum_{j=1}^{n} k_j\cdot x} \prod_{j=1}^{n} T^-_{\frac{1}{B}b_{k_j}(t_j)}\end{aligned}$$

As before, using Lemma 4.8 one finds that the lone magnetic translation can be swapped with $e^{i\sum_{j=1}^{n} k_j\cdot x}$ at the cost of the phase $e^{-\frac{i}{B}\sum_{j=1}^{n} k_j\cdot b_{k_{n+1}}(t_{n+1})} = e^{-\frac{i}{B}\left(k_n+\sum_{j=1}^{n-1} k_j\right)\cdot b_{k_{n+1}}(t_{n+1})}$. If we combine this factor with the second phase in the equality above, we obtain,

$$\begin{aligned}e^{-\frac{i}{B}\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} k_j\cdot b_{k_l}(t_l)} e^{-\frac{i}{B}\left(k_n+\sum_{j=1}^{n-1} k_j\right)\cdot b_{k_{n+1}}(t_{n+1})} &= e^{-\frac{i}{B}\left(\sum_{j=1}^{n-1}\sum_{l=j+1}^{n+1} k_j\cdot b_{k_l}(t_l)+k_n\cdot b_{k_{n+1}}(t_{n+1})\right)} \\ &= e^{-\frac{i}{B}\sum_{j=1}^{n}\sum_{l=j+1}^{n+1} k_j\cdot b_{k_l}(t_l)}\end{aligned}$$

Hence, by combining $e^{ik_{n+1}\cdot x}$ with $e^{i\sum_{j=1}^{n} k_j\cdot x}$ and absorbing $T^-_{\frac{1}{B}b_{k_{n+1}}(t_{n+1})}$ into the product of magnetic translations, we prove the desired result. □

We then continue by collecting all the complex phases with no position dependence and inserting the identity before the product of translations using $e^{itH_L}e^{-itH_L}$ to obtain the next equality.

$$\mathcal{I}_n = e^{\frac{i}{2B}\sum_{j=1}^{n} k_j\cdot b_{k_j}(t_j)} e^{-\frac{i}{B}\sum_{j=1}^{n}\sum_{l=j}^{n} k_j\cdot b_{k_l}(t_l)} \tau^{\emptyset}_{1,-t}\big(e^{i\sum_{j=1}^{n} k_j\cdot x}\big) e^{-itH_L} \prod_{j=1}^{n} T^-_{\frac{1}{B}b_{k_j}(t_j)} \Psi^{\sigma}_{y}$$

$$= e^{\frac{i}{2B}\sum_{j=1}^{n} k_j\cdot b_{k_j}(t_j)} e^{\frac{i}{2B}\sum_{j,l}^{n} k_j\cdot b_{k_l}(-t)} e^{-\frac{i}{B}\sum_{j=1}^{n}\sum_{l=j}^{n} k_j\cdot b_{k_l}(t_l)} e^{i\sum_{j=1}^{n} k_j\cdot x} T^{-}_{\frac{1}{B}b_{\sum_{j=1}^{n} k_j}(-t)} \prod_{j=1}^{n} T^{-}_{\frac{1}{B}b_{k_j}(t_j)} e^{-itH_L}\Psi_y^{\sigma}$$

$$= e^{\frac{i}{2B}\sum_{j,l}^{n} k_j\cdot\left(b_{k_j}(t_j)+b_{k_l}(-t)\right)} e^{-\frac{i}{B}\sum_{j=1}^{n}\sum_{l=j}^{n} k_j\cdot b_{k_l}(t_l)} e^{i\sum_{j=1}^{n} k_j\cdot x} T^{-}_{\frac{1}{B}\sum_{j=1}^{n} b_{k_j}(-t)} \prod_{j=1}^{n} T^{-}_{\frac{1}{B}b_{k_j}(t_j)} e^{-itH_L}\Psi_y^{\sigma}$$

The second and third equalities involve the application of lemmas 4.7 and 4.8 respectively. To take the absolute value of $\mathcal{I}_n$, we prove one more lemma that relates magnetic translations and standard translations. Denote by S_x a the unitary operator which translates a function in $L^2(\mathbb{R}^2)$ by $x \in \mathbb{R}^2$.

Lemma A.2. *For all finite sets $\{a_1, a_2, ...a_N\} \in \mathbb{R}^2$, the following equality holds for any $\Psi \in L^2(\mathbb{R}^2)$,*

$$\prod_{i=1}^{n} T^{\pm}_{a_i}\Psi(z) = \mathcal{C}_n(z) S_{\sum_{i=1}^{n} a_i}\Psi(z),$$

where $\mathcal{C}_n(z) \in \mathbb{C}$ and $|\mathcal{C}_n| = 1$.

Proof. We complete this proof via induction. In the case $n = 1$, the statement directly follows from the definition of $T^{\pm}_{a_1}$. Hence, assume the equality is true for some $n \leq N-1$. Then,

$$\begin{aligned}
\prod_{i=1}^{n+1} T^{\pm}_{a_i}\Psi(z) &= T^{\pm}_{a_{n+1}} \prod_{i=1}^{n} T^{\pm}_{a_i}\Psi(z) \\
&= T^{\pm}_{a_{n+1}} \mathcal{C}_n(z) S_{\sum_{i=1}^{n} a_i}\Psi(z) \\
&= e^{\pm i a_{n+1}\cdot A(z)} S_{a_{n+1}} \mathcal{C}_n(z) S_{\sum_{i=1}^{n} a_i}\Psi(z) \\
&= e^{\pm i a_{n+1}\cdot A(z)} \mathcal{C}_n(z - a_{n+1}) S_{\sum_{i=1}^{n+1} a_i}\Psi(z) \\
&= \mathcal{C}_{n+1}(z) S_{\sum_{i=1}^{n+1} a_i}\Psi(z)
\end{aligned}$$

In the second equality, we use the assumption on the n-fold product of magnetic translations to simplify that term. Then using the definition of $T^{\pm}_{(\cdot)}$ in terms of $S_{(\cdot)}$, we obtain the desired result noting that a translated complex phase remains a complex number with absolute value equal to 1, where $\mathcal{C}_{n+1}(z)$ is introduced to absorb any additional phases picked up in the calculation. $\square$

Using this lemma, we now take the absolute value of $\mathcal{I}_n$, which gives,

$$
\begin{aligned}
|\mathcal{I}_n|\left(\{k_i\}_{i=1}^n, y, x\right) &= \left| e^{i\sum_{j=1}^n k_j\cdot x} T^-_{\frac{1}{B}\sum_{j=1}^n b_{k_j}(-t)} \prod_{j=1}^n T^-_{\frac{1}{B}b_{k_j}(t_j)} T^+_y e^{-itH_L}\varphi_0^\sigma \right| \\
&= S_{y+\frac{1}{B}\sum_{j=1}^n b_{k_j}(t_j)+b_{k_j}(-t)} \left| e^{-itH_L}\varphi_0^\sigma \right|
\end{aligned} \tag{551}
$$

The translation T^+_y is moved past e^{-itH_L} due to these operators commuting. Using our calculation of $\left|e^{-itH_L}\varphi_0^\sigma\right|$, we arrive at equation (126), where the proof of Theorem 4.6 resumes.

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