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M. B. Halpern and P. Senjanović

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SURFACE MEDICATION FOR GAUGE THEORIES IN TWO DIMENSIONS *

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December 14, 1976

ABSTRACT

When surface terms cannot be neglected, the usual canonical formalism is plagued by a host of ambiguities and inconsistencies. We review these problems in the context of the massive Schwinger model, and present a complete cure effected by a well-chosen dynamical surface term. In the presence of the surface term, canonical formulation can be again naive. We also offer an introductory application of the method to the non-Abelian case: In a box, only singlet states are possible. The "new" singlet states involve surface excitation and may not survive the large box limit.

I. INTRODUCTION

Workers in the field^{1,2,3,4} have been generally aware of a number of pseudo-problems which plague the formalism of gauge theories when the fields hold up at large distances. Although first noticed in expanding around classical solutions possessing this feature, the problems arise generally for certain gauge choices. Notably, in axial and axial-like (e.g. light-cone) gauges, fields are automatically forced to hold up at large distances (in certain sectors).

The massive Schwinger model⁵ is an excellent example. There one can see clearly the following problems:

- (a) The Lagrange equations of motion do not follow from the action for interesting variations around the axial-like gauge solutions (except at zero charge and "angle").
- (b) Faddeev-Popov determinants vanish for these gauges.
- (c) Canonical Hamiltonian equations of motion do not agree with Lagrange equations of motion (except at zero charge and angle).

One can guess through these pseudo-problems: The (essentially) "correct" Hamiltonian for the Schwinger model is the stress-tensor energy taken by Coleman,⁵ but, looking back, the whole canonical super-structure lays behind one in tatters.

It is our purpose in this paper to show that all these problems are different faces of the same problem: the historical canonical formalism was derived assuming no contributions from surface terms. Dynamical surface terms of the type proposed by Regge and Teitelboim¹ and Sakita et al.³ solve all these problems cleanly. Action, Lagrange and canonical Hamiltonian formulations are finally in complete accord.

In the process, Coleman's angle is elevated to a dynamical surface variable, but the final physical interpretation is precisely his.

An outline of the paper is as follows. Section II is an overview of the problems in the naive formulation of the massive Schwinger model. Section III introduces our Poincaré invariant dynamical surface term and its corresponding generalized classical gauge-fixing. In Section IV, we introduce a suitable grid, discuss Faddeev-Popov determinants in the action formulation, and make a quick derivation of the Hamiltonian in a generalized light-cone gauge. The less-technically-inclined reader may be satisfied with the results in this form. In Section V, we complete the full Dirac-Faddeev canonical formulation. Section VI is an introductory extension to the non-Abelian case. In the presence of the surface term, and while the system is in a box, all states are color singlet. These include "new" singlet states, involving surface excitation. It is by no means clear these can meaningfully survive the large box limit. Section VII is for remarks, including some possibilities for extension to four dimensions.

II. NAIVE FORMULATION

We begin with the usual action for the massive Schwinger model,

$$S = \int d^2x \left\{ \bar{\psi} (i \not{\partial} - M - e \not{A}) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \quad (II.1)$$

and a totally accepting attitude toward the conventional approach. For reasons which will become clear later, we choose to consider a light-cone quantization of the model (and light-cone gauge). All the remarks of this section will, however, be equally valid for the more usual time-like quantization and axial gauge. Our notation is

standard,

$$S = \int dx^+ dx^- \left\{ \bar{\psi} (i \gamma^+ \partial_+ + i \gamma^- \partial_- - M) \psi + \frac{1}{2} F_{+-}^2 - e A^+ J^- - e A^- J^+ \right\}$$

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1), \quad A^\pm = \frac{1}{\sqrt{2}} (A^0 \pm A^1), \quad F_{+-} = \partial_+ A^- - \partial_- A^+$$

$$\gamma^\pm = \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^1), \quad J^\pm = \bar{\psi} \gamma^\pm \psi. \quad (II.2)$$

The first problem is that noticed by Sakita et al.³ Under variations δA_μ which vanish at initial and final $x^\pm$ (and $\delta \psi$ which vanishes at all boundaries), we find

$$\delta S = - \int dx^+ \left\{ \delta A^- F_{+-} \right\} \Big|_{x^-=-\infty}^{x^-=+\infty} + \dots \quad (II.3)$$

The dots indicate our omission of the usual "equation of motion terms". The reader familiar with light-cone gauge Schwinger model will recognize that for variations around such solutions, this surface term is not zero (unless both charge and Coleman angle are zero). Thus, equations of motion do not follow from variations around their own solutions! Undaunted, we continue to ply the trade.

The equations of motion are

$$\partial_+ F_{+-} = -e J_+, \quad \partial_- F_{+-} = e J_-$$

$$(i \gamma^+ \partial_+ + i \gamma^- \partial_- - M - e A^+ \gamma^- - e A^- \gamma^+) \psi = 0. \quad (II.4)$$

In passing, we also mention that the field equations imply charge conservation ($Q = e \int_{-L}^{+L} dx^- J^+$) only if $J^-(L) - J^-(-L) = 0$. In this approach, we cannot "derive" such, but we are willing to assume it anyway.

In the light-cone gauge ($A^+ = 0$), Eqs. (II.4) reduce to

$$\begin{aligned} \partial_+ \partial_- A^- &= e J^-, \quad \partial_-^2 A^- = -e J^+, \\ (i \gamma^+ \partial_+ + i \gamma^- \partial_- - M - e A^- \gamma^+) \psi &= 0. \end{aligned} \quad (\text{II.5})$$

To help define the theory, we put it in a box $(-L, +L)$. Unfortunately, the Faddeev-Popov determinant is then clearly zero

$$\det(\partial_-) = 0 \quad (\text{II.6})$$

and so the gauge is, strictly speaking, not legitimate. This also we pass by, with hope.

The most general solution to both Maxwell equations in the light-cone gauge is

$$A^-(x^+, x^-) = -\frac{e}{2} \int_{-L}^L dy^- |x^- - y^-| J^+(x^+, y^-) + a(x^+) + bx^-. \quad (\text{II.7})$$

This is the "usual" form, but in fact, we had to assume

$$J^-(L) + J^-(-L) = 0 \text{ to derive it. (Otherwise } b = \frac{e}{2} [J^-(L) + J^-(-L)].)$$

This we are also willing to do; the assumption tally is now

$$J^-(\pm L) = 0.$$

We pass on now to the canonical Hamiltonian. Without any choice of gauge, we find, in the usual manner:

$$\begin{aligned} H &= H_{OF} + \int_{-L}^{+L} dx^- \left[\frac{1}{2} \pi_+^2 + \pi_+ \partial_- A^- + e J^+ A^- + e J^- A^+ \right] \\ H_{OF} &= \int_{-L}^{+L} dx^- \bar{\psi} (-i \gamma^- \partial_- + M) \psi \\ \pi_+ &= \frac{\partial L}{\partial(\partial_+ A^+)} = F_{+-}, \quad \pi_- = \frac{\partial L}{\partial(\partial_+ A^-)} = 0. \end{aligned} \quad (\text{II.8})$$

and we have not yet studied the secondary constraints. There are two commonly used methods to obtain a useful Hamiltonian from this. They are usually assumed to give equivalent results.

The first method we might call the pedestrian approach. Here, one makes a gauge choice ($A^+ = 0$) and is willing to use Lagrange equations (with no time derivatives) in H . Since in our gauge, $F_{+-} = -\partial_- A^-$, we reach

$$H_{(1)} = H_{OF} + \int_{-L}^{+L} dx^- \left\{ -\frac{1}{2} (\partial_- A^-)^2 + e J^+ A^- \right\}. \quad (\text{II.9})$$

Inserting Eq. (II.7) and doing a little algebra, we have also

$$\begin{aligned} H_{(1)} &= H_{OF} - \frac{e^2}{4} \iint_{-L}^L dx^- dy^- J^+(x^-) |x^- - y^-| J^+(y^-) \\ &\quad + a Q - \frac{1}{4} L Q^2 - L b^2. \end{aligned} \quad (\text{II.10})$$

This is the Hamiltonian of Ref. (4). b is essentially the Coleman angle, but the b dependence is trivial. Reference (4) concludes the (true) Hamiltonian is not a function of b . Is this a fact?

The truth is that $H_{(1)}$ is an unacceptable Hamiltonian (except at $Q = b = 0$) because it does not reproduce the fermion equation of motion. In fact, it could not, because the equation of motion contains A^- , and hence b . For completeness, we record the (unacceptable) equation of motion implied by $H_{(1)}$:

$$\left\{ i \not{\partial} - M - e \gamma^+ \left[A^- - \left(b + \frac{1}{2} L Q \right) x^- \right] \right\} \psi = 0. \quad (\text{II.11})$$

But now we are really in the soup. We have been good-natured about assumptions, but we cannot tolerate contradictions. Is there any hope in the second method, the Dirac-Faddeev⁶ approach to the Hamiltonian? For this, we seek the secondary constraints by commuting π_- with H . We find

$$\begin{aligned} 0 &= \int_{-L}^{+L} dy^- \pi_+(y^-) \partial_-^y \delta(y^- - x^-) + e J^+(x^-) \\ &= -\partial_- \pi_+ + e J^+ + \pi_+(L) \delta(L - x^-) - \pi_+(-L) \delta(-L - x^-). \end{aligned} \quad (II.12)$$

The only interpretation we can consistently place on this is

$$-\partial_- \pi_+ + e J^+ = 0, \quad \pi_+(\pm L) = 0. \quad (II.13)$$

The constraints can be solved only when $Q = 0$ and then

$$\pi_+ = - \int_{x^-}^L dy^- e J^+(x^+, y^-). \quad (II.14)$$

We also make the usual gauge choices $A^+ = A^- = 0$. (We ignore the question of more gauges for the "extra" $[\pi_+(\pm L) = 0]$ first class constraints--this will not help anyway). Thus, we reach (with $Q = 0$)

$$\begin{aligned} H_{(2)} &= H_{OF} + \frac{e^2}{2} \int_{-L}^L dx^- \left(\int_{x^-}^L dy^- J^+ \right)^2 \\ &= H_{OF} - \frac{e^2}{4} \iint_{-L}^{+L} dx^- dy^- J^+(x^-) |x^- - y^-| J^+(y^-). \end{aligned} \quad (II.15)$$

This agrees with $H_{(1)}$ when $Q = 0$, and is equally unsatisfactory.

What to do? As mentioned in the Introduction, it is not difficult to guess through all these problems, as did Coleman⁵: The (essentially) correct Hamiltonian is the integral of the stress-tensor,

$$\begin{aligned} H_C &= \int_{-L}^{+L} \theta_{00} dx^- = H_{OF} + \int_{-L}^{+L} dx^- \frac{1}{2} F_{+-}^2 \\ &= H_{OF} + \int_{-L}^{+L} dx^- \frac{1}{2} (\partial_- A^-)^2 \\ &= H_{(1)} + \frac{1}{2} \partial_- \left[(A^-)^2 \right] \Big|_{-L}^{+L}. \end{aligned} \quad (II.16)$$

This Hamiltonian differs from $H_{(1)}$ by precisely the integration-by-parts surface term involved in the (doubtful) transition

$$\pi_+ \partial_- A^- \approx -\partial_- \pi_+ A^-. \quad (II.17)$$

With a little algebra and Eq. (II.7), H_C can be expressed,

$$\begin{aligned} H_C &= H_{OF} - \frac{e^2}{4} \iint_{-L}^{+L} dx^- dy^- J^+(x^-) |x^- - y^-| J^+(y^-) \\ &\quad + \frac{1}{4} L Q^2 + b D + b^2 L. \end{aligned} \quad (II.18)$$

Here, D is the dipole moment operator $D \equiv e \int_{-L}^{+L} dx^- x^- J^+(x^-)$.

Using the identity

$$[\partial_- A^-(x^-), \psi(y^-)] = \frac{e}{2} \epsilon(x^- - y^-) \psi(y^-)$$

the implied equation of motion is

$$[i \not{\partial} - M - e \gamma^+ (A^- - a)] \psi = 0. \quad (II.19)$$

This is "almost" correct; it is correct at $a = 0$, but " a " involves no real physics; this slight deviation could also be anticipated, because H_C is a function of $\partial_- A^-$ only.

The final answer is, as promised, easy enough to guess, but what about our canonical formalism? It lies behind us in rags with an ever-accumulating number of assumptions (plus now: drop surface terms during integration by parts $\pi_+ \partial_- A^- \rightarrow -\partial_- \pi_+ A^-$).

When a subject reaches such a state, it is time for a clarifying reformulation. One suspects that all the trouble we encountered (the formalism is consistent only at $b = Q = 0$) is traceable back to the action itself--for, in fact, the bothersome surface terms in δS vanish just when $b = Q = 0$. In the following sections, we show that a modified action, with a dynamical surface term, cures all these problems cleanly.

III. THE SURFACE-MODIFIED ACTION

In the succeeding sections, we will suppress explicit reference to the fermionic variables, dealing with the current as if it were external; at any stage, the reader may append the fermion kinetic energy.

We propose the following modified action for the massive Schwinger model

$$\begin{aligned} S = & \int dx^+ \left\{ \int_{-\infty}^{+\infty} dx^- \left(\frac{1}{2} F_{+-}^2 - e J^+ A^- - e J^- A^+ \right) \right. \\ & + \left. (A^- - x^- \partial_+ \lambda) G \right|_{x^- = -\infty}^{x^- = \infty} \\ & \sim \int dx^+ \left\{ \int_{-L}^{+L} dx^- \left(\frac{1}{2} F_{+-}^2 - e J^+ A^- - e J^- A^+ \right) \right. \\ & + [A^-(L) - L \partial_+ \lambda(L)] G(L) \\ & \left. \left. - [A^-(-L) + L \partial_+ \lambda(-L)] G(-L) \right\} \right. \quad (III.1) \end{aligned}$$

Here $\lambda(x^+, \pm L)$, $G(x^+, \pm L)$ are extra surface variables, which we have chosen to write $\lambda(\pm L)$, $G(\pm L)$, suppressing the time dependence. This surface term locks us into light-cone quantization. It is the only simple surface term we have found which preserves (at least formally, as $L \rightarrow \infty$) Poincaré invariance. We are taking advantage, of course, of the simple Lorentz-transformation properties of light-cone variables in two dimensions. A similar term for time-like quantization $\left\{ \int dx^0 (A^0 - x \partial_0 \lambda) \right|_{-\infty}^{+\infty} \right\}$ is not invariant. Later, however, we will mention the possibility of more general invariant surface terms.

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This action retains a generalized gauge-invariance,

$$\bar{A}^{\pm} = A^{\pm} + \partial_{\mp} \Lambda, \quad \bar{\lambda}(\pm L) = \lambda(\pm L) \pm \frac{1}{L} \Lambda(\pm L). \quad (\text{III.2})$$

In addition it is now true that the equations of motion do follow from the action:

$$\partial_{+} F_{+-} = -e J^{-}, \quad \partial_{-} F_{+-} = e J^{+}$$

$$G(\pm L) = F_{+-}(\pm L)$$

$$A^{-}(\pm L) = \pm L \partial_{+} \lambda(\pm L), \quad L \partial_{+} G(\pm L) = 0. \quad (\text{III.3})$$

It takes but a line of algebra to show that Eqs. (III.3) imply

$$J^{-}(\pm L) = 0. \quad (\text{III.4})$$

Equation (III.4) is no longer a separate assumption. It is a consistency requirement on an external J , but it will, of course, remain true (as a dynamical consequence of the action) when the fermionic degrees of freedom are added. It is not surprising that Eqs. (III.3) are sufficient to guarantee charge conservation

$$[Q = e \int_{-L}^{+L} dx^{-} J^{+}, \quad \dot{Q} = \partial_{+} Q = 0].$$

Notice also that complete gauge-fixing is now possible. As an example, one might choose light-cone gauge ($\bar{A}^{+} = 0$). This determines Λ only up to a function of x^{+} . This, of course, is

a primal source of trouble for the unmodified system. Now we complete the light-cone gauge-fixing by requiring, say $\bar{\lambda}(-L) = 0$. Thus Λ is completely determined,

$$\Lambda(x^{+}, x^{-}) = - \int_{-L}^{x^{-}} dy^{-} A^{+}(x^{+}, y^{-}) + L \lambda(-L). \quad (\text{III.5})$$

We can anticipate that Faddeev-Popov determinants such as

$$\Delta \int \mathcal{D}\Lambda \delta(A^{+} - \partial_{-} \Lambda) \delta[L \lambda(-L) - \Lambda(-L)] = 1 \quad (\text{III.6})$$

do not vanish. Formally, this is immediate because the zero of the combined δ -functions [Eq. (III.6)] is unique, and hence the inversion is possible. To be more precise, it is necessary to define such formal expressions as (III.6) on a grid. This will also be helpful in our phase-space formulation, and is our next topic.

IV. A SUITABLE GRID, FADDEEV-POPOV DETERMINANTS, AND A QUICK DERIVATION OF THE HAMILTONIAN

Our choice of space grid is shown in Fig. 1. We break up the range $-L \leq x^{-} \leq L$ in such a manner that there are $N+1$ A^{+} 's, F_{+-} 's and J^{-} 's, but $N+2$ A^{-} 's, Λ 's, and J^{+} 's. Thus, e.g.,

$$A_N^{+} = A^{+}(L), \quad A_0^{+} = A^{+}(-L), \quad A_{N+1}^{-} = A^{-}(L), \quad A_0^{-} = A^{-}(-L),$$

$$\Lambda_0 = \Lambda(-L) \quad \Lambda_{N+1} = \Lambda(+L) \quad (\text{IV.1})$$

and so on. This unorthodox grid choice is dictated primarily by

invariance under gauge transformations, which now take the form,

$$\bar{A}_\ell^- = A_\ell^- + \partial_+ \Lambda_\ell \quad 0 \leq \ell \leq N+1$$

$$\bar{A}_\ell^+ = A_\ell^+ + \frac{\Lambda_{\ell+1} - \Lambda_\ell}{\epsilon} \quad 0 \leq \ell \leq N$$

$$\bar{\lambda}(L) = \lambda(L) + \frac{1}{L} \Lambda_{N+1}, \quad \bar{\lambda}(-L) = \lambda(-L) - \frac{1}{L} \Lambda_0. \quad (IV.2)$$

The invariant action is $S = \int dx^+ L$,

$$L = \frac{\epsilon}{2} \sum_{\ell=0}^N (F_{+-})_\ell^2 - e \epsilon \sum_{\ell=0}^{N+1} J_\ell^+ A_\ell^- - e \epsilon \sum_{\ell=0}^N J_\ell^- A_\ell^+ + [A_{N+1}^- - L \dot{\lambda}(L)] G(L) - [A_0^- + L \dot{\lambda}(-L)] G(-L) \quad (IV.3)$$

where $(F_{+-})_\ell = \partial_+ A_\ell^+ - \frac{1}{\epsilon} (A_{\ell+1}^- - A_\ell^-)$ $0 \leq \ell \leq N$, and $\dot{\lambda} = \partial_+ \lambda$.

It is not difficult to repeat our continuum manipulations in this language. The Lagrange equations of motion which follow from the action are

$$\partial_+ (F_{+-})_\ell = -e J_\ell^- \quad (0 \leq \ell \leq N)$$

$$(F_{+-})_0 = e \epsilon J_0^+ + G(-L), \quad (F_{+-})_N = -e \epsilon J_{N+1}^+ + G(L)$$

$$\frac{1}{\epsilon} [(F_{+-})_\ell - (F_{+-})_{\ell-1}] = e J_\ell^+ \quad 1 \leq \ell \leq N$$

(Equation IV.4 continued)

$$-L \partial_+ G(\pm L) = 0, \quad A_{N+1}^- - L \dot{\lambda}(L) = 0, \quad A_0^- + L \dot{\lambda}(-L) = 0. \quad (IV.4)$$

It is easy to see that these reduce to the continuum Eq's. (III.3) in the limit $\epsilon \rightarrow 0$. It is also not hard to see that the equations of motion imply

$$\partial_+ J_0^+ = \frac{1}{\epsilon} J_0^-, \quad \partial_+ J_{N+1}^+ = \frac{1}{\epsilon} J_N^-$$

$$\partial_+ J_\ell^+ + \frac{1}{\epsilon} (J_\ell^- - J_{\ell-1}^-) = 0, \quad 1 \leq \ell \leq N \quad (IV.5)$$

which is sufficient to guarantee charge conservation

$\left\{ Q = e \epsilon \sum_{\ell=0}^{N+1} J_\ell^+, \quad \dot{Q} = 0 \right\}$. Again, we emphasize that these equations remain as dynamical implications of the action when the fermions are added explicitly.

We now have our gauge degrees of freedom in plain sight. A complete gauge-fixing means determining all $N+2$ Λ 's. Thus, we could specify a light-cone gauge as

$$\bar{A}_\ell^+ = A_\ell^+ + \frac{1}{\epsilon} (\Lambda_{\ell+1} - \Lambda_\ell) = 0 \quad 0 \leq \ell \leq N$$

$$\bar{\lambda}(-L) = \lambda(-L) - \frac{1}{L} \Lambda_0 = 0. \quad (IV.6)$$

This fixes all $N+2$ gauge functions. Similarly, we could specify an axial gauge as

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$$\bar{A}_\ell^+ - \bar{A}_\ell^- = 0 = A_\ell^+ - A_\ell^- + \frac{1}{\epsilon} (\Lambda_{\ell+1} - \Lambda_\ell) - \partial_+ \Lambda_\ell,$$

$$0 \leq \ell \leq N$$

$$\bar{\lambda}(-L) = \lambda(-L) - \frac{1}{L} \Lambda_0.$$

(IV.7)

ACTION FADDEEV-POPOV FORMALISM

Up to a measure, the effective action is the classical action. We shall show in the next section that the measure is trivial, and concentrate here on the "results". In general, one can use the ordinary Faddeev-Popov formalism to change among completely-fixed gauges. As an example, for our light-cone gauge, we define the Faddeev-Popov determinant Δ as

$$1 = \Delta \int \prod_{r=0}^{N+1} \mathcal{D}_{\Lambda_r} \prod_{\ell=0}^N \delta \left[A_\ell^+ + \frac{1}{\epsilon} (\Lambda_{\ell+1} - \Lambda_\ell) \right] \delta \left[\lambda(-L) - \frac{1}{L} \Lambda_0 \right].$$

(IV.8)

The integrations are functional integrations over the (suppressed) time dependence of each spatial variable. Equation (IV.8) is the precise form of the formal statement Eq. (III.6).

A more convenient form is obtained by arranging the A^+ 's and $\lambda(-L)$ into an $(N+1)$ -column vector, called a^+ ($a_{N+1}^+ = \lambda(-L)$). Thus

$$1 = \Delta \int \mathcal{D}_\Lambda \delta [a^+ + \partial_- \Lambda],$$

(IV.9)

where

$$(\partial_-)_{lm} = \begin{array}{c} \begin{array}{ccccc} & 0 & & & N+1 \\ 0 & \begin{bmatrix} -\frac{1}{\epsilon} & \frac{1}{\epsilon} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{\epsilon} \\ -\frac{1}{L} & 0 & 0 & \cdots & 0 \end{bmatrix} & \end{array} \end{array}$$

(IV.10)

The integration is completed via the usual shift $\Lambda = \bar{\Lambda} - (\partial_-)^{-1} a^+$, and hence

$$\Delta = \det \partial_- = \prod_{x^+} \frac{1}{L} \left(\frac{1}{\epsilon} \right)^{N+1}.$$

(IV.11)

The determinant is quite finite, and may be ignored on the grid. Note however that, if we had forgotten to do the last gauge-fixing (with $\lambda(-L)$), we would obtain Δ^{-1} infinite--because one of the Λ integrations covers its entire range. This, of course, is what happens in the naive approach. The important lesson is that vanishing Faddeev-Popov determinants signal incomplete gauge-fixing. More generally, vanishing Jacobians signal non-unique inversions.

With the surface term, all this is cured.

Accepting that the measure is trivial, we may make a quick transition to a useful Hamiltonian (for our light-cone gauge). We write the generating functional as

$$Z = \Delta \int \mathcal{D}A^- \mathcal{D}A^+ \mathcal{D}G(L) \mathcal{D}G(-L) \mathcal{D}\lambda(L) \mathcal{D}\lambda(-L) \delta(A^+) \delta(\lambda(-L))$$

$$\otimes \exp i \left\{ \int dx^+ \int_{-L}^L dx^- \left(\frac{1}{2} F_{+-}^2 - e J^+ A^- - e J^- A^+ \right) \right\}$$

$$\otimes \exp i \left\{ \int dx^+ \left[(A^-(L) - L \dot{\lambda}(L)) G(L) - (A^-(-L) + L \dot{\lambda}(-L)) G(-L) \right] \right\}.$$

(IV.12)

Doing first the A^+ and $\lambda(-L)$ integrations, and then the $G(-L)$ integration, we obtain

$$Z = \Delta \int \mathcal{D}A^- \mathcal{D}G(L) \mathcal{D}\lambda(L) \delta[A^-(-L)]$$

$$\otimes \exp i \left\{ \int dx^+ \int_{-L}^L dx^- \left(\frac{1}{2} (\partial_- A^-)^2 - e J^+ A^- \right) \right\}$$

$$\otimes \exp i \left\{ \int dx^+ \left[(A^-(L) - L \dot{\lambda}(L)) G(L) \right] \right\}.$$

(IV.13)

Now consider the change of variables $\partial_- A^- = B$. Again, the inversion is unique: Because $A^-(-L) = 0$,

$$A^- = \int_{-L}^{x^-} B(y^-) dy^-.$$

(IV.14)

The Jacobian of the transformation is thus a non-vanishing number. Although the matrix involved is slightly different from " ∂_- ", it turns out to have the same determinant Δ (up to factors of L). Thus

$$Z = \int \mathcal{D}B \mathcal{D}G(L) \mathcal{D}\lambda(L)$$

$$\otimes \exp \left[i \int dx^+ \left\{ \int_{-L}^{+L} dx^- \left[\frac{1}{2} B^2 - e B \int_{x^-}^L dy^- J^+(x^+, y^-) + B G(L) \right] \right\} \right]$$

$$\otimes \exp \left\{ i \int dx^+ \left[-L \dot{\lambda}(L) G(L) \right] \right\}.$$

(IV.15)

The B integration is elementary, with the result

$$Z = \int \mathcal{D}G(L) \mathcal{D}\lambda(L) \exp \left\{ -\frac{i}{2} \int dx^+ \int_{-L}^{+L} dx^- \left[G(L) - e \int_{x^-}^L dy^- J^+(y^-) \right]^2 \right\}$$

$$\otimes \exp \left(-i \int dx^+ L \dot{\lambda}(L) G(L) \right).$$

(IV.16)

After a simple change of variables $\bar{\lambda} = -L \dot{\lambda}(-L)$ the functional integral is in canonical form; the canonical Hamiltonian is

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$$H = \int_{-L}^{+L} dx^- \frac{1}{2} \left[G(L) - e \int_{x^-}^L dy^- J^+(y^-) \right]^2 + H_{OF}$$

$$[\bar{\lambda}(x^+, L), G(x^+, L)] = i.$$

(IV.17)

Here we have reinstated the fermionic variables for completeness.

We shall do a more careful derivation of this result in the following section, but the answer is quite correct. With a little algebra,

H can be put in the following alternate form,

$$\begin{aligned} H &= H_{OF} - \frac{1}{4} e^2 \iint dx^- dy^- J^+(x^-) |x^- - y^-| J^+(y^-) \\ &\quad + \int dx^- \left[\frac{1}{2} G^2(L) - e x^- J^+(x^-) G(L) \right] \\ &\quad + \int dx^- \left[-\frac{1}{2} Q G(L) + \frac{1}{4} Q^2 + \frac{1}{2} Q x^- J^+ \right] \\ &= H_{OF} - \frac{1}{4} e^2 \iint dx^- dy^- J^+(x^-) |x^- - y^-| J^+(y^-) \\ &\quad + L G^2(L) - G(L) D + Q \left[-L G(L) + \frac{1}{2} L Q + \frac{1}{2} D \right]. \end{aligned} \quad (IV.18)$$

Here, D is the dipole moment operator $D = e \int_{-L}^L dx^- x^- J^+(x^-)$, and all space integrations run from -L to +L.

Q and G(L) are constants of the motion. If we start with states at large negative times $|Q = 0, G(L) \text{ fixed}\rangle_{in}$, then, in Eq.(IV.18), Q may be dropped, and G(L) taken as a

number. It is clear that we have precisely Coleman's picture, with $\theta = \frac{2\pi}{e} G(L)$ (or, in the large L limit, $\theta = \frac{2\pi}{e} G(\infty)$). We have also checked that our Hamiltonian reproduces all Lagrange equations of motion (for all values of G, Q). It may be interesting to study the large L spectrum of Eq. (IV.18) in sectors other than $Q = 0$.

V. PHASE-SPACE FORMULATION

We return to the action Eq. (IV.3) on the grid. The canonical variables are

$$\begin{aligned} \frac{\partial L}{\partial (\partial_+ A_t^+)} &= e (F_{+-})_t = p_+^t & 0 \leq t \leq N \\ \frac{\partial L}{\partial (\partial_+ A_t^-)} &= 0 = p_-^t & 0 \leq t \leq N+1 \\ \frac{\partial L}{\partial \lambda(\pm L)} &= -L G(\pm L) = \pi_\lambda(\pm L), & \frac{\partial L}{\partial \dot{G}(\pm L)} = 0 = \pi_G(\pm L). \end{aligned} \quad (V.1)$$

The last two lines provide $N + 6$ primary constraints. We construct the first-stage Hamiltonian in the usual manner

$$\begin{aligned} H_0 &= \frac{1}{2e} \sum_{t=0}^N p_+^t p_+^t + \sum_{t=0}^N p_+^t \left(\frac{A_{t+1}^- - A_t^-}{e} \right) + e \sum_{t=0}^{N+1} J_t^+ A_t^- \\ &\quad + e \sum_{t=0}^N J_t^- A_t^+ + \frac{1}{L} \left[\pi_\lambda(L) A_{N+1}^- - \pi_\lambda(-L) A_0^- \right]. \end{aligned} \quad (V.2)$$

Here we have freely used the primary constraints and the definition of $(F_{+-})_t$ in terms of p_+^t . The reader is urged to think also in terms of the continuum limit, the appropriate variables for which are the usual

$$\pi_+^t = \frac{p_+^t}{\epsilon} \quad 0 \leq t \leq N.$$

$$\begin{aligned} H_0 = & \frac{1}{2} \int_{-L}^L dx^- (\pi_+^2 + \pi_+ \partial_- A^- + e J^+ A^- + e J^- A^+) \\ & + \frac{1}{L} [\pi_\lambda(L) A^-(L) - \pi_\lambda(-L) A^-(-L)]. \end{aligned} \quad (V.3)$$

The second term in H_0 is worth focus. The correct integration by parts

$$\begin{aligned} \int_{-L}^{+L} dx^- \pi_+ \partial_- A^- &= - \int_{-L}^{+L} dx^- A^- \partial_- \pi_+ + A^-(L) \pi_+(L) \\ &- A^-(-L) \pi_+(-L), \end{aligned} \quad (V.4)$$

is as easily done on the grid:

$$\begin{aligned} \frac{1}{\epsilon} \sum_{t=0}^N p_+^t (A_{t+1}^- - A_t^-) &= - \frac{1}{\epsilon} \sum_{t=1}^N A_t^- (p_+^t - p_+^{t-1}) + \frac{1}{\epsilon} A_{N+1}^- p_+^N \\ &- \frac{1}{\epsilon} A_0^- p_+^0. \end{aligned} \quad (V.5)$$

Thus

$$\begin{aligned} H_0 = & \frac{1}{2\epsilon} \sum_{t=0}^N p_+^t p_+^t - \frac{1}{\epsilon} \sum_{t=1}^N A_t^- (p_+^t - p_+^{t-1}) + e \epsilon \sum_{t=0}^{N+1} J_t^+ A_t^- \\ & + e \epsilon \sum_{t=0}^N J_t^- A_t^+ + A_{N+1}^- \left[\frac{1}{L} \pi_\lambda(L) + \frac{1}{\epsilon} p_+^N \right] \\ & - A_0^- \left[\frac{1}{L} \pi_\lambda(-L) + \frac{1}{\epsilon} p_+^0 \right]. \end{aligned} \quad (V.6)$$

We cycle the primary constraints through H_0 to find the secondary constraints. The only non-trivial result comes from $[H_0, p_-^t] = 0$:

$$\begin{aligned} e \epsilon J_0^+ - \left[\frac{1}{L} \pi_\lambda(-L) + \frac{1}{\epsilon} p_+^0 \right] &= 0 \\ e \epsilon J_{N+1}^+ + \left[\frac{1}{L} \pi_\lambda(+L) + \frac{1}{\epsilon} p_+^N \right] &= 0 \\ - \frac{1}{\epsilon} (p_+^t - p_+^{t-1}) + e \epsilon J_t^+ &= 0, \quad 1 \leq t \leq N. \end{aligned} \quad (V.7)$$

These $N + 2$ secondary constraints have an obvious continuum limit:

$$\begin{aligned} - \partial_- \pi_+ + e J^+ &= 0 \\ \frac{1}{L} \pi_\lambda(L) + \pi_+(L) &= \frac{1}{L} \pi_\lambda(-L) + \pi_+(-L) = 0. \end{aligned} \quad (V.8)$$

This is in agreement with the direct continuum calculation of $[H_0, \pi^-] = 0$, with the understanding $\int_{-L}^{+L} dx^- \delta(L - x^-) = \frac{1}{2}$:

$$0 = -\partial_- \pi^+ + e J^+ + 2 \delta(L - x^-) \left[\frac{1}{L} \pi_\lambda(L) + \pi_+(L) \right] \\ - 2 \delta(-L - x^-) \left[\frac{1}{L} \pi_\lambda(-L) + \pi_+(-L) \right]. \quad (V.9)$$

What is important about the grid is that we know precisely how many constraints are present. In the subspace of the secondary constraints, we can write

$$H_0' = \frac{1}{2\epsilon} \sum_{\ell=0}^N p_+^\ell p_+^\ell + e \epsilon \sum_{\ell=0}^N J_\ell^- A_\ell^+ \quad (V.10)$$

Collecting all the constraints, we can write the correct phase-space functional integral as

$$Z = \int \prod_{r=0}^N \mathcal{D}p_+^r \prod_{s=0}^{N+1} \mathcal{D}p_-^s \prod_{t=0}^N \mathcal{D}A_t^+ \prod_{\ell=0}^{N+1} \mathcal{D}A_\ell^- \\ \otimes \int \mathcal{D}\lambda_{(+L)} \mathcal{D}\lambda_{(-L)} \mathcal{D}G(L) \mathcal{D}G(-L) \\ \otimes \int \mathcal{D}\pi_\lambda_{(+L)} \mathcal{D}\pi_\lambda_{(-L)} \mathcal{D}\pi_G(L) \mathcal{D}\pi_G(-L)$$

(Equation V.11 continued)

$$\otimes \left\{ \prod_{m=0}^{N+1} \delta[p_-^m] \delta \left[e \in J_0^+ - \left(\frac{1}{L} \pi_\lambda(-L) + \frac{1}{\epsilon} p_+^0 \right) \right] \right. \\ \otimes \prod_{n=1}^N \delta \left[-\frac{1}{\epsilon} (p_+^n - p_+^{n-1}) + e \in J_n^+ \right] \\ \otimes \left. \delta \left[e \in J_{N+1}^+ + \frac{1}{L} \pi_\lambda(L) + \frac{1}{\epsilon} p_N^+ \right] \right\} \\ \otimes \delta \left[\pi_\lambda(+L) + L G(L) \right] \delta \left[\pi_\lambda(-L) + L G(-L) \right] \prod_{\pm} \delta \left[\pi_G(\pm L) \right] \prod_{x^+} (L^2) \\ \otimes \delta^{2N+4}(\text{gauges}) \otimes (F - P \text{ determinant}) \\ \otimes \exp \left[i \int dx^+ \left\{ \sum_{\ell=0}^N p_+^\ell \dot{A}_\ell^+ + \sum_{\pm} \pi_G(\pm L) \dot{G}(\pm L) \right. \right. \\ \left. \left. + \sum_{\pm} \pi_\lambda(\pm L) \dot{\lambda}(\pm L) - H_0' \right\} \right]. \quad (V.11)$$

Here we have collected the first-class constraints in curly brackets. There are $2(N+2)$ of these. Hence, we must choose $2(N+2)$ gauges. The second-class constraints are in the seventh line, together with their proper subdeterminant.⁸

It is conventional to choose half the gauges as $\prod_{\ell=0}^{N+1} \delta(A_{\ell}^-)$, leaving $N+2$ to be specified. A non-singular choice is our generalized light-cone gauge $\prod_{\ell=0}^N \delta(A_{\ell}^+) \delta(\lambda(-L))$. The reader is invited to verify that the Faddeev-Popov determinant for this choice is precisely $\Delta = \det " \partial_- "$, as discussed in Section IV. We record for the "light-cone" gauge

$$\Delta \delta^{2(N+2)}(\text{gauges}) = \prod_{\ell=0}^{N+1} \delta(A_{\ell}^-) \prod_{s=0}^N \delta(A_s^+) \delta[\lambda(-L)] \det " \partial_- " \quad (V.12)$$

With this counting in hand, and the grid for difficult questions, there can be no objection to writing the functional integral in the continuum limit,

$$Z = \int \prod_{\lambda} \pi_{\lambda} \prod_{\lambda} A^+ \prod_{\lambda} A^- \prod_{\lambda(+L)} \lambda(-L) \prod_{\lambda(-L)} \lambda(+L) \prod_{\lambda} G(L) \prod_{\lambda} G(-L)$$

$$\delta \pi_{\lambda}(L) \delta \pi_{\lambda}(-L) \delta \pi_G(L) \delta \pi_G(-L) \delta(\pi_-)$$

$$\delta \left[+ \frac{1}{L} \pi_{\lambda}(-L) + \pi_+(-L) \right] \delta (-\partial_- \pi_+ + e J^+)$$

$$\delta \left[\frac{1}{L} \pi_{\lambda}(L) + \pi_+(L) \right]$$

$$\delta [\pi_{\lambda}(L) + L G(L)] \delta [\pi_{\lambda}(-L) + L G(-L)] \delta [\pi_G(L)]$$

$$\delta [\pi_G(-L)] \prod_{x^+} (L^2)$$

(Equation V.13 continued)

$$\delta(\text{gauges}) \Delta$$

$$\exp \left\{ i \int dx^+ \int_{-L}^{+L} dx^- \pi_+ \dot{A}^+ + i \int dx^+ \left[\sum_{\pm} \pi_G(\pm L) \dot{G}(\pm L) + \sum_{\pm} \pi_{\lambda}(\pm L) \dot{\lambda}(\pm L) \right] \right\} \exp \left\{ -i \int dx^+ \int_{-L}^L dx^- \left(\frac{1}{2} \pi_+^2 + e J^- A^+ \right) \right\} \quad (V.13)$$

and fourth

The third line can even be replaced by the quite formal expression

$$\delta \left\{ -\partial_- \pi_+ + e J^+ + 2 \delta(L - x^-) \left[\frac{1}{L} \pi_{\lambda}(L) + \pi_+(L) \right] - 2 \delta(-L - x^-) \left[\frac{1}{L} \pi_{\lambda}(-L) + \pi_+(-L) \right] \right\} \quad (V.14)$$

which is the formal requirement of Eq. (V.9). One must only remember, in the continuum limit, that a gauge choice such as

$$\delta(\text{gauges}) \Delta = \delta(A^-) \delta(A^+) \delta[\lambda(-L)] \det " \partial_- " \quad (V.15)$$

involves the same number of gauges as there are first-class constraints (in $\delta(\pi^-)$ times Eq. (V.14)). Without the grid analysis, such statements are by no means obvious.

RETURN TO THE ACTION

This is easiest to work out for the class of gauges

$$\delta(A^-) \delta^{N+2} \Delta \quad (V.16)$$

where the other $N + 2$ gauges are a function of A^+ and variables other than A^- or the π 's, and such that Δ is non-vanishing. Starting from Eq. (V.11) or Eq. (V.13), it is easy to integrate out A^- and all the π 's but π_+ . For π_+ it is convenient to exponentiate the remaining δ -functions that were the secondary constraints, via a (new) A^- variable. There are $N + 2$ of these constraints, and A^- has $N + 2$ space values; in fact the grid reproduces the following formal continuum manipulation

$$\begin{aligned} & \delta \left\{ -\partial_- \pi_+ + e J^+ + 2 \delta(L - x^-) [-G(L) + \pi_+(L)] \right. \\ & \quad \left. - 2 \delta(-L - x^-) [\pi_+(-L) - G(-L)] \right\} \\ &= \iint \mathcal{D}A^- \exp \left[-i \int dx^+ \int_{-L}^{+L} dx^- \left\{ -\partial_- \pi_+ + e J^+ \right. \right. \\ & \quad \left. \left. + 2 \delta(L - x^-) (-G(L) + \pi_+(L)) \right. \right. \\ & \quad \left. \left. - 2 \delta(-L - x^-) (\pi_+(-L) - G(-L)) \right\} A^- \right] \\ &= \iint \mathcal{D}A^- \exp \left\{ -i \int dx^+ \int_{-L}^{+L} dx^- (-\partial_- \pi_+ + e J^+) A^- \right\} \end{aligned}$$

(Equation V.17 continued)

$$\begin{aligned} & \exp \left[-i \int dx^+ \left\{ (-G(L) + \pi_+(L)) A^-(L) - (\pi_+(-L) \right. \right. \\ & \quad \left. \left. - G(-L)) A^-(-L) \right\} \right]. \end{aligned} \quad (V.17)$$

Now the term $-\partial_- \pi_+ A^-$ may be integrated by parts, as in Eq. (V.4), keeping the surface terms. The final result is

$$\begin{aligned} Z &= \iint \mathcal{D}\lambda(L) \mathcal{D}\lambda(-L) \mathcal{D}G(L) \mathcal{D}G(-L) \mathcal{D}A^+ \mathcal{D}A^- \\ & \quad \delta^{N+2} \Delta \prod_{x^+} (L^2) e^{iS} (\sqrt{\epsilon})^{N+1}. \end{aligned} \quad (V.18)$$

Here $\mathcal{D}A^+ \mathcal{D}A^-$ is written only with the usual time grid measure; $(\sqrt{\epsilon})^{N+1}$ is the resulting space grid measure. S is the classical action with which we began. This precise measure is c-number, and can be neglected in applications. This justifies the discussion of Section IV.

A USEFUL HAMILTONIAN

We present this calculation on the grid, but the reader can easily follow the steps in the continuum. We begin with Eq. (V.11), choosing our "light-cone" gauge

$$\prod_{l=0}^{N+1} \delta(A_l^-) \prod_{k=0}^N \delta(A_k^+) \delta[\lambda(-L)] \det " \partial_- ". \quad (V.19)$$

It is easy to use the δ -functions to integrate out

$$A^+, A^-, p_+, \pi_G(\pm L), \lambda(-L), G(-L), \pi_\lambda(-L)$$

thus arriving at

$$Z = (\det " \partial_- ") \prod_{\ell=0}^N \int \mathcal{D}p_+^\ell \mathcal{D}\lambda(L) \mathcal{D}G(L) \prod_{x^+} (L^2) \\ \otimes \prod_{k=1}^N \delta \left[-\frac{1}{\epsilon} (p_+^\ell - p_+^{\ell-1}) + e \in J_\ell^+ \right] \\ \otimes \delta \left[e \in J_{N+1}^+ - G(L) + \frac{1}{\epsilon} p_+^N \right] \\ \otimes \exp \left\{ -i \int L G(L) \dot{\lambda}(L) dx^+ \right\} \\ \otimes \exp \left\{ -i \frac{1}{2\epsilon} \int \sum_{m=0}^N p_+^m p_+^m dx^+ \right\}. \quad (V.20)$$

The constraint equations are easily solved

$$p_+^\ell = \epsilon G(L) - e \epsilon^2 \sum_{i=\ell+1}^{N+1} J_i^+ \quad (V.21)$$

and the Jacobian is $\prod_{x^+} \epsilon^{N+1}$. Since $\det " \partial_- " = \prod_{x^+} \frac{1}{L} \epsilon^{-N-1}$, we have

$$Z = \int \mathcal{D}\lambda(L) \mathcal{D}G(L) \left(\prod_{x^+} L \right) \exp \left\{ -i \int L G(L) \dot{\lambda}(L) dx^+ \right\} \\ \otimes \exp \left\{ -i \frac{\epsilon}{2} \int dx^+ \sum_{\ell=0}^N \left[G(L) - e \epsilon \sum_{i=\ell+1}^{N+1} J_i^+ \right]^2 \right\} \quad (V.22)$$

Finally, we change variables to $\bar{\lambda}(L) = -L \lambda(L)$ at every point, and go to the continuum limit; thus

$$Z = \int \mathcal{D}\bar{\lambda} \mathcal{D}G(L) \exp \left\{ i \int dx^+ G(L) \dot{\bar{\lambda}}(L) \right\} \exp \left\{ -i \int dx^+ H \right\} \\ H = H_{OF} + \frac{1}{2} \int_{-L}^{+L} dx^- \left[G(L) - e \int_{x^-}^L J^+(y^-) dy^- \right]^2. \quad (V.23)$$

Here we have re-inserted the suppressed fermion kinetic energy. This is the rigorous counterpart of the quick derivation in Section IV.

VI. INTRODUCTORY EXTENSION TO NON-ABELIAN GAUGE THEORIES

In the naive formulation,

$$S = \int d^2x \left\{ -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) - 2g \text{Tr} (J \cdot A) + \mathcal{L}_F \right\} \\ F_{\mu\nu} = F_{\mu\nu}^\alpha \frac{\lambda_\alpha}{2}, \quad A_\mu = A_\mu^\alpha \frac{\lambda_\alpha}{2}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu]$$

(Equation VI.1 continued)

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$$J_\mu = \bar{\psi} \gamma^\mu \frac{\lambda_\alpha}{2} \psi \frac{\lambda_\alpha}{2}, \quad \text{Tr} (\lambda^\alpha \lambda^\beta) = 2 \delta^{\alpha\beta}, \quad \mathcal{L}_F = \bar{\psi} (i \not{\partial} - M) \psi, \quad (\text{VI.1})$$

the gluon equations of motion are

$$\partial^\mu F_{\mu\nu} = -i g [A^\mu, F_{\mu\nu}] + g J_\nu, \quad (\text{VI.2})$$

or

$$\begin{aligned} \partial_+ F_{+-} &= -i g [A^-, F_{+-}] - g J^-; \\ \partial_- F_{+-} &= -i g [A^+, F_{+-}] + g J^+. \end{aligned} \quad (\text{VI.3})$$

The usual total color

$$Q = \int_{-L}^{+L} \left\{ g J^+ - i g [A^+, F_{+-}] \right\} dx^-. \quad (\text{VI.4})$$

is conserved only if

$$\left\{ -i g [A^-, F_{+-}] - g J^- \right\} \Big|_{-L}^{+L} = 0. \quad (\text{VI.5})$$

In the light-cone gauge ($A^+ = 0$), the most general solution to the equations of motion is

$$A^-(x^+, x^-) = -\frac{g}{2} \int_{-L}^{+L} dy^- |x^- - y^-| J^+(x^+, y^-) + a(x^+) + b(x^+) x^-. \quad (\text{VI.6a})$$

$$\partial_+ b = \dot{b} = \frac{g}{2} [J^-(L) + J^-(-L)] + \frac{ig}{4} [D, Q] - i g [a, b]. \quad (\text{VI.6b})$$

$$\dot{Q} = 0 = i g \left\{ [D, b] - [a, Q] \right\} - g J^-(L) + g J^-(-L), \quad (\text{VI.6c})$$

$$\text{where, as above, } Q \equiv g \int_{-L}^{+L} dx^- J^+, \quad D \equiv g \int_{-L}^{+L} dx^- x^- J^+.$$

One usually argues that the b 's are zero (or decouple):

It is reasonable (we agree) to take $J^\pm(\pm L) = 0$; then on a color singlet state, charge conservation (Eq. (VI.6c)) forces $[D, b] = 0$. It is not reasonable to have $[D, b]$ follow each other in this manner, so we must take either D or $b = 0$. Although it follows from the algebra of Q and D that D is zero between two singlet states, it is by no means clear that it is zero on a singlet state. We thus prefer to require $b = 0$. In any case, the question of D or b zero is somewhat academic because b will at most decouple. In this case, just as for $\Theta = 0$ in the abelian case, all naive canonical formulations agree. This, of course, is the usual conclusion.

Curiously, this is not the conclusion when a surface term is added. Something that will be useful to have for that discussion is the following (unmotivated) fact about the naive approach.

Suppose (for no good reason), we relaxed our $\dot{Q} = 0$ requirement,

and instead required $A^-(-L) = 0$ $\left(a - \frac{1}{2} D = L \left(b + \frac{1}{2} Q \right) \right)$. Then it is only a line of algebra to show that $Q + 2b$ is conserved.

From the naive approach, this is a meaningless curio, but from the point of view of the surface-term theory, it turns out to be interesting.

SURFACE TERM

To formulate the surface-term theory, we add to the action,

$$\Delta S = \int dx^+ \int_{-L}^{+L} dx^- \text{Tr} \left[2 A^- G - \frac{1}{ig} (M \vec{\sigma}_+ M^\dagger) G \right] \Big|_{-L}^{+L}, \quad (\text{VI.7})$$

in analogy with the abelian case. Here M is a unitary matrix field, transforming as a color triplet and $G = G^\alpha \frac{\lambda_\alpha}{2}$. Gauge and Lorentz invariance are preserved. We shall discuss only this model here, but we mention in passing that similar surface terms, with different representations for the scalar, are also possible. The action now implies the gluonic equations of motion, Eq. (V.3), and

$$\dot{G}(\pm L) + [M(\pm L) \dot{M}^\dagger(\pm L), G(\pm L)] = 0.$$

$$G(\pm L) = F_{+-}(\pm L), \quad A^-(\pm L) = \frac{1}{ig} M(\pm L) \dot{M}^\dagger(\pm L).$$

(VI.8)

As in the abelian case, it is not hard to show that these equations imply

$$J^-(\pm L) = 0.$$

(VI.9)

Noether's theorem on the action gives the conserved total (color) charge

$$Q_T = \int_{-L}^{+L} dx^- \left\{ g J^+ - i g [A^+, F_{+-}] \right\} - (G(L) - G(-L)). \quad (\text{VI.10})$$

Under gauge transformations which satisfy $S(L) = S(-L)$, it can be shown, with the help of the equations of motion, that

$$Q_T \rightarrow S(L) Q_T S^\dagger(L), \quad (\text{VI.11})$$

and hence $\text{Tr}[Q_T^2]$ is invariant as expected. Surprisingly enough it also follows easily from the equations of motion that $Q_T = 0$! (Integrate the $\partial_- F_{+-}$ equation and use Eqs. (VI.8).) This effect is similar to phenomena in the colored "bag".⁹ If one puts a quark in our "bag" (our "bag" is from $-L$ to $+L$), the surface variables apparently form an inverted image, and the total charge is zero. With this extended notion of color, there is no chance of ever finding a non-singlet state.

In a general gauge, we have not found any other conserved charge. There are however two gauge-invariant conserved quadratic forms

$$\frac{d}{dt} \text{Tr} [G^2(\pm L)] = 0 = \frac{d}{dt} \left\{ \frac{1}{2} (\pm L) \right\}. \quad (\text{VI.12})$$

Further, in a "light-cone" gauge, say,

$$A^+ = 0, \quad M(-L) = 1.$$

(VI.13)

Q_T breaks up into two separately conserved pieces: $M(-L) = 1$

implies $\dot{G}(-L) = 0$. Thus with $\dot{Q} \equiv g \int_{-L}^{+L} dx^- J^+$,

$$\bar{Q} = -G(-L) = Q - G(L),$$

(VI.14)

is conserved. Note also that, in this gauge, Eqs. (VI.8) imply

$A^+(-L) = 0$. We will say more about all this below.

By steps similar to that of Section IV, we have also isolated a "useful" Hamiltonian ("light-cone" gauge):

$$H = \int dx^- \text{Tr} \left\{ \left[G(L) - g \int_x^L dy^- J^+(y^-) \right]^2 \right\} + H_{\text{OF}},$$

(VI.15)

which can be put in the "usual" form, except that there is present a term $\text{Tr} [G(L)D]$, as in the abelian case.

DISCUSSION

Note that, in general, the quark-plus-gluon part of the color is simply not conserved. We can, however, work in a sector where it is conserved by choosing in-states which satisfy (say in "light-cone" gauge)

$$Q = G(L) = \bar{Q} = -G(-L) = 0.$$

(VI.16)

If $G(\pm L, -\infty) = 0$, they remain zero forever (by Eq. (VI.8)). Hence, on these states, $\dot{Q} = 0$. We will refer to this sector as the $\mathcal{H} = 0$ sector.

It is not hard to compare the naive and surface approaches. In our gauge $[A^+ = 0, A^+(-L) = 0]$, we find the precise connection,

$$G(L) = -b + \frac{1}{2} Q \quad \text{or} \quad Q + 2b = 2\bar{Q}.$$

(VI.17)

We now understand that (the naive) b is essentially a surface variable, and why $Q + 2b$ was conserved (when $A^+(-L) = 0$). We also see that choosing in-states for the surface model which satisfy Eq. (VI.16) is equivalent to setting $b = 0$. The $\mathcal{H} = 0$ states are then the "usual" states of the naive approach.

On the other hand, in the surface model, all states are singlet, and deserve investigation. In the naive approach (say with the correct stress-tensor energy), this corresponds to keeping $b \neq 0$ and not worrying about (naive) charge conservation (the "real" charge is conserved and zero.) The extra states (beyond the "usual" states), are characterized in general by $\mathcal{H}(\pm L) \neq 0$, and all involve surface excitation. We are not clear at present whether any of these states can meaningfully survive the $L \rightarrow \infty$ limit. Indeed, such is not likely because it apparently involves infinite color separation. In the box, however, the spectrum should certainly be considered as a function of $\mathcal{H}(\pm L)$. Of course, $\mathcal{H}(\pm L) \neq 0$ is not a simple "angle": The new variables are removed in the $\mathcal{H} = 0$

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In four dimensions, it may be possible to use these ideas to study sectors where the fields hold up. Of course, new covariant physics, if any, is not expected to be so simple as a static "angle".

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FIGURE CAPTIONS

Fig. 1. The Space Grid.

Fig. 1

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