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Robust and Optimum Fractional Factorial Designs

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Applied Statistics

by

Fu ze Huang

August 2010

Dissertation Committee:
Dr. Subir Ghosh, Chairperson
Dr. Bajis Dodin
Dr. Jun Li

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The Dissertation of Fu ze Huang is approved:

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ABSTRACT OF THE DISSERTATION

Robust and Optimum Fractional Factorial Designs

by

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Doctor of Philosophy, Graduate Program in Applied Statistics
University of California, Riverside, August 2010
Dr. Subir Ghosh, Chairperson

This thesis is devoted to the study of robust and optimum fractional factorial designs. We consider models that contain the general mean, main effects, and k two-factor interactions for 2^m fractional factorial experiments. We define S_i to be the set of all $(1 \times m)$ vectors, with elements 1 and -1 of weight i, where the weight of a vector is the number of nonzero elements in it. We present the robustness property of two classes of designs $D = \{S_0, S_1, S_{m-1}, S_m\}$ and $D_1 = \{S_0, S_1, S_2, S_m\}$ with respect to any t runs as well as a specific set of t runs in the sense that the full estimation capacity of the designs remain when we delete any t runs as well as specific t runs from the designs D and D_1. The number of runs are $(2 + m)$ and $\left(2 + m + \binom{m}{2}\right)$ in designs D and D_1 respectively.

We introduce a general structure $\boldsymbol{M}$ for the information matrices of a class of models possibly describing the data from a fractional factorial experiment with m factors each at two levels and n runs. We characterize all the eigenvalues and eigenvectors for such matrices $\boldsymbol{M}$. For $m = 4$ we establish the robustness property of the design $D_7 = \{S_0, S_1, S_3\}$. The runs of D_7 are contained in design D when $m = 4$. We show all the information matrices from design D_7 and designs obtained from D_7 by deleting some runs are special cases of $\boldsymbol{M}$.

Let $\mathcal{D}_{\mathfrak{T}}$ be the class of designs with n runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. We prove that if $T^* \in \mathcal{D}_{\mathfrak{T}}$

is E-optimum, then $\binom{T^*}{-T^*}$ is optimum design with respect to $AMCR$ and $GMCR$ in $\in \mathcal{F}_{\mathfrak{T}}$. Furthermore, if T^* is D- and A- optimum with a special structure for $\boldsymbol{X}'_{\boldsymbol{1T^*}}\boldsymbol{X}_{\boldsymbol{1T^*}}$ we prove $\binom{T^*}{-T^*}$ is GD, AD, GT, and AT optimal in $\mathcal{F}_{\mathfrak{T}}$.

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Chapter 1

Introduction

1.1 Factorial Experiments

Experiments are often performed by investigators across a variety of fields to determine the dependence response on one or more treatment factors. Experimental design is important for collecting meaningful data in drawing efficient inferences from the data. One important situation where experimental designs are widely used is a factorial experiment. A factorial experiment consists of a number of factors with each factor having several levels. In a full factorial experiment all the combinations of levels of the factors are investigated. Many leading statisticians have contributed to the construction and analysis of factorial experiments. Sir R.A. Fisher(1926) argued that factorial designs were more efficient than studying one factor at a time. In the Rothamsted experimental station R.A. Fisher and F. Yates used factorial designs in agricultural and biological experiments and contributed significantly to the analysis of experiments, including factorial experiments. Subsequently factorial experiments were used and studied by R.C. Bose, G.E.P. Box, J.P. Burman, O. Kempthorne, R.L. Plackett, C.R. Rao, G. Taguchi, W.J. Youden, and many others.

Factorial experiments allow us to study the main effects of each factor as well as the interactions of factors on the response. The main effect of a factorial experiment is defined to be the change in the response produced by the change in the levels of the factors. Interactions between factors are present when the change in response between the levels of one factor is not the same at all levels of the other factors. Factorial designs have a major advantage over the one-factor-at-a-time strategy that was used extensively in practice. In an one-factor-at-a-time strategy experimenters fail to consider any possible interactions between factors. Ignoring an interaction which is large, introduces “bias” in the inference on the main effects. When interactions between factors are present we must examine the effects of a factor at different levels of other factors.

Example 1.1

In practice we often encounter factorial experiments that involve several factors with each factor at two levels. This is referred to as a 2^m factorial experiment, where m is the number of factors. The levels are usually coded as 1 and -1 in the design and they represent the high and low levels of each factor respectively. To illustrate, we consider a 2^4 factorial experiment(Montgomery 2001) where an engineer wants to study the effect of four factors on the surface roughness of a machined part. The factors (and their levels) are A = tool angle $(12, 15^o)$, B = cutting fluid viscosity (300, 400), C = feed rate 910, 15 in/min), and D = cutting fluid cooler used (no, yes). The data from this experiment (with the factors coded to the usual 1, -1 levels) are shown in Table 1.1.

Table 1.1: 2^4 Factorial experiment

	Factors				Treatment	Surface
Run	A	B	C	D	Combinations	Roughness
1	-1	-1	-1	-1	(1)	0.00340
2	1	-1	-1	-1	a	0.00362
3	-1	1	-1	-1	b	0.00301
4	1	1	-1	-1	ab	0.00182
5	-1	-1	1	-1	c	0.00280
6	1	-1	1	-1	ac	0.00290
7	-1	1	1	-1	bc	0.00252
8	1	1	1	-1	abc	0.00160
9	-1	-1	-1	1	d	0.00336
10	1	-1	-1	1	ad	0.00344
11	-1	1	-1	1	bd	0.00308
12	1	1	-1	1	abd	0.00184
13	-1	-1	1	1	cd	0.00269
14	1	-1	1	1	acd	0.00284
15	-1	1	1	1	bcd	0.00253
16	1	1	1	1	abcd	0.00163

The goal of this factorial experiment is to study the effect of each factor as well as the interaction effects on surface roughness. If all the interaction effects are zero and suppose main effect B cutting fluid viscosity is not zero, then changing the level from low to high of factor B has an effect on the surface roughness. Now if the interaction AB is not zero, then it would be erroneous to just examine the main effects A and B. We must examine the effects of factor A at different levels of factors B and vise versa.

1.2 Fractional Factorial Experiment

In a full factorial experiment, all main effects as well as all the interactions can be estimated. However, the number of runs needed for a full factorial experiment increase dramatically with the number of factors and levels. For example, when we consider a full 2^8 factorial experiment the number of runs required is 256. In practice it is often not possible to run such a large experiment. In many experimental settings, experimenters can reasonably assume that higher order interactions are negligible and thus are able to estimate the main effects and lower order interactions with only a fraction of the total number of runs needed in a full factorial experiment.

In a fractional factorial experiment, a fraction of all possible runs is chosen in such a way that the desired inferences can be drawn effectively. We can have fractional factorial design with one-fourth fraction of all the runs for 2^8 experiment. The number of runs becomes 64 which is a fraction of the 256 total runs. Assuming three or higher order interactions to be zero, a fraction of runs may permit us to estimate the general mean, main effects and two order factor interactions. In general the treatments to be included in the experiment are determined by the assumption of which factorial effects are zero and which are nonzero. We would like to unbiasedly estimate the nonzero factorial effects and thus we need to choose a set of runs that can achieve this goal.

1.3 Resolution of a Fractional Factorial Experiment

Design resolution for a fractional factorial experiment was introduced and investigated in Box and Hunter(1961). The Resolution III, IV, and V plans are widely used in practice. Resolution III plans permit us to unbiasedly estimate the general mean and main effect under the assumption that two-factor and higher order interactions are zero. Resolution IV plans allow us to estimate the general mean and main effects in the presence of two-factor interactions under the assumption that three-factor and higher order interactions are zero. Resolution V plans permit us to unbiasedly estimate the general mean, main effect and two-factor interactions under the assumption that three-factor and higher order interactions are zero.

1.4 Thesis Outline

Model identification and discrimination is fundamental in statistics. At the planning stage, the experimenter does not know which model is the best model. The experimenter may have some feelings that a set of models may be appropriate. But from this set the experimenter does not know which model is the best model at the planning stage. Since the experimenter does not know which is the best model then it is important that we determine the class of designs that can unbiasedly estimate all the parameters in the set of models. Furthermore we would like to determine the best design within the class that can unbiasedly estimate all the parameters in the set of models.

In Chapter 2, we discuss linear models for full factorial experiment as well as fractional factorial experiments. We also discuss D-, A- and E- optimality under the ordinary linear models. In Chapter 3, we discuss search design and search linear model. We discuss the optimality criterion functions for search linear models. In Chapter 4, we review an important theorem on robustness of designs against incomplete data. For nested fractional factorial designs we present the structure of a matrix whose rows are orthogonal to the columns of the design matrix $\boldsymbol{X}$. In Chapter 5, we determine the robustness property of design $D = \{S_0, S_1, S_{m-1}, S_m\}$ with respect to any t runs as well as a specific set of t runs, where S_i consists of all runs with (i) -1 and $(m - i)$ +1. In Chapter 6, we present the robustness property of design $D_1 = \{S_0, S_1, S_2, S_m\}$ with respect to any t runs as well as a specific set of t runs. In Chapter 7, we present a general structure $\boldsymbol{M}$ for the information matrices of a class of models possibly describing the data from a fractional factorial experiment with m factors each at two levels and n runs, where $n < 2^m$. We characterize all the eigenvalues for such matrices $\boldsymbol{M}$. In Chapter 8, for a class of matrices we determine the relation between minimum eigenvalue of a matrix in the class and the minimum eigenvalues of its principal sub-matrices. We study the optimality of foldover designs obtained from a main effects plan that are optimum w.r.t. one or more of A-, D-, and E- optimality criterion functions. In Chapter 9, we obtain optimum designs for two sub-classes of foldover designs. In Chapter 10, we present optimum fractional factorial designs for $m = 4$. The numbers of runs we consider are $n = 6$, 7, 8, 9, 10, 11 and 12. The values of k the number of two-factor interactions are 1, 2, 3, 4, 5 and 6. We discuss hierarchical designs. For $n = 10$, we obtain non-isomorphic classes for designs with full estimation capacity for $k = 1, 2, 3, 4$ and 5.

Chapter 2

Ordinary Linear Model

2.0 Summary

In this chapter we discuss the ordinary linear model and least squares estimators for the parameters in this linear model. We present the linear model for the full 2^m factorial experiment as well as models for fractional factorial experiments. We also discuss the optimality under these modes with three widely used D-, A- and E- optimality criteria.

2.1 Ordinary Linear Model

In many experimental settings, investigators would like to know the dependence of the response variable on a set of controllable factors. Consider example 1.1, the engineer may want to find the relationship between response variable, surface roughness and the four factors: tool angle, cutting fluid viscosity, feed rate, and cutting fluid cooler used. We can study the relationship between the response variable and the factors by considering an appropriate model. There are many types of models that we may consider. In this chapter we discuss the ordinary linear model, which is linear in the model parameters.

We consider the linear model

$$\begin{aligned} E(y_i) &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} \ldots + \beta_p x_{ip}, \quad i = 1, \ldots, n, \\ Cov(y_i, y_j) &= \sigma^2 \;\; for\ i = j, \\ &= 0 \quad for\ i \neq j, \end{aligned} \tag{2.1}$$

where y_i is the i^{th} observation, $\{\beta_0, \beta_1, \ldots, \beta_p\}$ are the unknown β parameters and $\boldsymbol{x_i} = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ is the vector of known values depending on the levels of factors. We can express (2.1) in matrix form

$$\begin{aligned} E(\boldsymbol{y}) &= \boldsymbol{X\beta}, \\ Var(\boldsymbol{y}) &= \sigma^2 \boldsymbol{I}, \end{aligned} \tag{2.2}$$

where $\boldsymbol{y'} = (y_1, \ldots, y_n)$ is response vector, $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)$ is the vector of unknown β parameters, $\boldsymbol{X}_{(n\times(p+1))} = (\boldsymbol{j}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_n})'$, $\boldsymbol{j} = (1, \ldots, 1)'$ and $\boldsymbol{I}$ is the $(n \times n)$ identity matrix.

2.2 Least Squares Estimator

In this section we consider the least square estimator for the unknown vector of parameters $\boldsymbol{\beta}$ in (2.2). The least squares estimator is the best linear unbiased estimator in the class of linear unbiased estimators. We denote $\hat{\boldsymbol{\beta}}$ the least squares estimator of $\boldsymbol{\beta}$. In the least squares setting we minimize

$$S(\boldsymbol{\beta}) = \boldsymbol{(y - X\beta)'(y - X\beta)} \tag{2.3}$$

with respect to $\boldsymbol{\beta}$, thus

$$\hat{\boldsymbol{\beta}} = \underset{\beta}{\text{argmin}}\ S(\boldsymbol{\beta}) = (\boldsymbol{y} - \boldsymbol{X\beta})'(\boldsymbol{y} - \boldsymbol{X\beta}). \tag{2.4}$$

If the determinant of the $\boldsymbol{X'X}$ matrix is not zero then

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y}. \tag{2.5}$$

Since $\hat{\boldsymbol{\beta}}$ is a linear combination of the elements of the response vector $\boldsymbol{y}$ then we obtain its variance-covariance matrix. The variance-covariance matrix for $\hat{\boldsymbol{\beta}}$ is

$$Var(\hat{\boldsymbol{\beta}}) = Var\big((\boldsymbol{X'X})^{-1}\boldsymbol{X'y}\big) = \sigma^2(\boldsymbol{X'X})^{-1}. \tag{2.6}$$

Since σ^2 in (2.6) is unknown we present two estimators for σ^2. The unbiased estimator for σ^2 is

$$s^2 = \frac{1}{n-(p+1)}\big[\boldsymbol{y}'\big(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X'X})^{-1}\boldsymbol{X}'\big)\boldsymbol{y}\big]. \tag{2.7}$$

Under the normality assumption for the response, the maximum likelihood estimator for σ^2 is

$$\hat{\sigma}^2 = \frac{n-(p+1)}{n}s^2. \tag{2.8}$$

We routinely examine the residual plots as well as the adjusted R^2 to check for model adequacy. If more than one model seems appropriate then we can selected the best model using model selection criteria such as Akaike's information criterion(AIC) or Bayesian information criterion(BIC). In order to use (AIC) or (BIC) we must assume a distribution for the response.

2.3 Linear Model for 2^m Factorial Experiment

We now consider linear models for a complete factorial experiment with m factors $\{x_1, x_2, \ldots, x_m\}$. The model we consider contains the general mean, main effects, and all factor interactions

$$\begin{aligned} E(y_{ij}) =& \beta_0 + \beta_1 x_{i1} + \beta_1 x_{i1} + \ldots + \beta_m x_{im} \\ & + \beta_{12} x_{i1} x_{i2} + \ldots + \beta_{12\ldots m} x_{i1} x_{i2} \ldots x_{im}, \quad i = 1, \ldots, T; j = 1, ..., r_i, \\ & Cov(y_{ij}, y_{i'j'}) = \sigma^2 \;\; for\ (ij) = (i'j'), \\ & \qquad = 0 \;\; for\ (ij) \neq (i'j'), \end{aligned} \tag{2.9}$$

where $(x_{i1}, x_{i2}, \ldots, x_{im})$ is the i^{th} treatment, y_{ij} is the j^{th} observation for the i^{th} treatment and the β's are the unknown parameters. The total number of observations $n = \sum_{i=1}^{T} r_i$. The β_0 corresponds to the general mean, $(\beta_1, \ldots, \beta_m)$ corresponds to the main effects, $(\beta_{12}, \ldots \beta_{(m-1)(m)})$ corresponds to the two-factor interactions and so on. We can write (2.9) in matrix form given in (2.2). For a 2^m factorial experiment it is well-known that the regression coefficient is one-half the corresponding factor effect estimates.

Example 2.1

We consider the full factorial experiment in example 1.1 with the model given in (2.9). The $\boldsymbol{X}$ matrix is (16×16). We obtain $\hat{\boldsymbol{\beta}}$ for the parameters as well as the information matrix

$$(10^4) \times \hat{\boldsymbol{\beta}} = \begin{pmatrix} 26.9313 \\ \hline\hline -2.31875 \\ -4.38125 \\ -2.54375 \\ -0.15625 \\ \hline\hline -3.00625 \\ 0.35625 \\ -0.08125 \\ 0.69375 \\ 0.33125 \\ -0.00625 \\ \hline\hline 0.41875 \\ 0.03125 \\ 0.16875 \\ -0.06875 \\ \hline\hline -0.06875 \end{pmatrix},$$

and

$$\boldsymbol{X'X} = 16\boldsymbol{I}_{(16\times16)}.$$

From $\hat{\boldsymbol{\beta}}$ all estimates correspond to three and higher order interactions are small compared to the main effects. This suggests that these interactions may be negligible. The estimates for the two-factor interactions $\hat{\beta}_{13}$, $\hat{\beta}_{14}$, $\hat{\beta}_{23}$, $\hat{\beta}_{24}$ and $\hat{\beta}_{34}$ are small relative to the main effects. They may be negligible. The estimate $\hat{\beta}_{12}$ is comparable to the main effect and thus it may not be negligible. Furthermore the estimate $\hat{\beta}_4$ is small relative to estimates of the other main effects. It may also be negligible. Since the information matrix is a diagonal matrix the $\hat{\beta}$'s are uncorrelated. Moreover the variances of the $\hat{\beta}$'s are constant. Since our model is saturated we need more runs to be able to estimate σ^2.

2.4 Linear Models for 2^m Fractional Factorial Experiment

In a fractional factorial experiment, we take a fraction of the treatments from the full factorial design. We cannot unbiasedly estimate all the parameters in the model given in (2.9) with a fractional factorial experiment. We assume some of the higher order interactions are zero. We consider linear models for fractional factorial experiment with m factors $\{x_1, x_2, \ldots, x_m\}$. There are many possible models we may consider.

We discuss a widely used model that contains the general mean, main effects and two-factor interactions

$$
\begin{aligned}
E(y_i) =& \beta_0 + \beta_1 x_{i1} + \beta_1 x_{i1} + \ldots + \beta_m x_{im} \\
&+ \beta_{12} x_{i1} x_{i2} + \ldots + \beta_{(m-1)m} x_{i(m-1)} x_{im}, \quad i = 1, \ldots, n, \\
Cov(y_i, y_j) &= \sigma^2 \quad for\ i = j, \\
&= 0 \quad for\ i \neq j,
\end{aligned}
\tag{2.10}
$$

where $\{x_{i1}, x_{i2}, \ldots, x_{im}\}$ is the i^{th} treatment, y_i is the observation corresponds to the i^{th} treatment and the β's are the unknown parameters. The total number of observations is n. The β_0 corresponds to the general mean, $\{\beta_1, \ldots, \beta_m\}$ correspond to the main effects and $\{\beta_{12}, \ldots \beta_{(m-1)(m)}\}$ correspond to two-factor interactions.

Example 2.2

We consider the following fractional factorial design

$$D_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The design D_2 is a 3/4 fraction of 2^4 full factorial design. Furthermore it is a resolution V plan thus we can unbiasedly estimate all the parameter in the model given in (2.10). The observation vector is given in table 1.1. The least squares estimates for the parameters in the model given in (2.10) as well as the information matrix are presented:

$$(10^4) \times \hat{\boldsymbol{\beta}} = \left(\begin{array}{r} 26.750 \\ \hline -2.425 \\ -4.275 \\ -2.550 \\ 0.200 \\ \hline -2.825 \\ 0.650 \\ -0.150 \\ 0.775 \\ 0.025 \\ -0.175 \end{array}\right),$$

$$\boldsymbol{X'X} = \left(\begin{array}{r|rrrr|rrrrrr} 12 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 12 & 0 & 0 & 0 & 2 & 2 & 2 & -2 & -2 & -2 \\ 2 & 0 & 12 & 0 & 0 & 2 & -2 & -2 & 2 & 2 & -2 \\ 2 & 0 & 0 & 12 & 0 & -2 & 2 & -2 & 2 & -2 & 2 \\ 2 & 0 & 0 & 0 & 12 & -2 & -2 & 2 & -2 & 2 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 & 12 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & -2 & 2 & -2 & 0 & 12 & 0 & 0 & 4 & 0 \\ 0 & 2 & -2 & -2 & 2 & 0 & 0 & 12 & 4 & 0 & 0 \\ 0 & -2 & 2 & 2 & -2 & 0 & 0 & 4 & 12 & 0 & 0 \\ 0 & -2 & 2 & -2 & 2 & 0 & 4 & 0 & 0 & 12 & 0 \\ 0 & -2 & -2 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 12 \end{array}\right).$$

The estimates $\hat{\beta_{13}}$, $\hat{\beta_{14}}$, $\hat{\beta_{23}}$, $\hat{\beta_{24}}$ and $\hat{\beta_{13}}$ are small relative to the main effects. These two-factor interaction effects may be negligible. The two-factor interaction effect AB is comparable to the main effect and is unlikely to be negligible. The estimate for the main effect D is small compared to the other main effects thus it may be negligible. From the $\boldsymbol{X'X}$ matrix we see the correlations between $\hat{\beta}_0$ and the $\hat{\beta}$'s for the two-factor interactions are zero. The correlations between $\hat{\beta}$'s for the main effects are zero. The $\hat{\beta}$'s for the main effects and the $\hat{\beta}$'s for the two-factor interactions are correlated. The $\hat{\beta}$'s for the two-factor interactions are uncorrelated except when the two two-factor interactions are disjoint. We determine the value of $\hat{\sigma}^2$ to be 3.2×10^{-9}.

2.5 Optimality Under Ordinary Linear Model

The theory of optimal designs under ordinary linear model was introduced formally in Kiefer(1959). An optimum design means that is best with respect to one or more criterion functions. To illustrate, let us consider designs with n runs from a 2^m factorial experiment. The number of designs with n runs is $\binom{2^m}{n}$. Let $\mathcal{D}$ be the class of designs with n runs that can unbiasedly estimate the parameters in the model given in (2.10). Thus we need $n \geq 1 + m + \binom{m}{2}$. The goal is to obtain a design in $\mathcal{D}$ that is optimal with respect to one or more criterion functions.

We discuss three widely use criterion functions: D-optimal criterion, A-optimal criterion, and E-optimal criterion. The D-optimal design minimizes the determinant of the matrix $\boldsymbol{(X'X)^{-1}}$. It has shown that D-optimal design minimizes the volume of the joint confidence region of the estimators for the β parameters. An A-optimal design minimizes the trace of the matrix $\boldsymbol{(X'X)^{-1}}$. An A-optimal design minimizes the sum of the variances of the estimators for the β parameters. An E-optimal design minimizes the maximum characteristic root of the matrix $\boldsymbol{(X'X)^{-1}}$. This means the E- optimum design minimizes the maximum variance of the estimate of the normalized linear function of the β parameters. In general a design is optimal with respect to one criterion does not imply that it is optimal with resect to other criteria. In some special cases, a design can be D-, A-, and E-optimal.

Example 2.3

To illustrate we consider the following two fractional factorial designs $D_{2.1}$ and $D_{2.2}$ with 8 runs. The model contains the general mean, main effects and the (12) two-factor interaction.

$$D_{2.1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \qquad D_{2.2} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

The determinant, trace and maximum characteristic root of the matrix $\boldsymbol{X'X}$ are given in Table 2.1.

Table 2.1: Comparison of fractional factorial designs $D_{2.1}$ and $D_{2.2}$

Comparison of Fractional Factorial Design		
	$D_{2.2}$	$D_{2.1}$
$\text{Det}\big((\boldsymbol{X'X})^{-1}\big)$	3.815×10^{-6}	2.035×10^{-5}
$\text{Trace}\big((\boldsymbol{X'X})^{-1}\big)$	0.750	1.583
$\text{MaxCR}\big((\boldsymbol{X'X})^{-1}\big)$	0.125	0.933

From Table 2.1, the determinant, trace and maximum characteristic root of the matrix $(\boldsymbol{X'X})^{-1}$ are smaller for $D_{2.2}$ than $D_{2.1}$. Thus $D_{2.2}$ is better than $D_{2.1}$ with respect to A-, D-, and E- optimality.

Chapter 3

Search Design and Search Linear Model

3.0 Summary

In this chapter we present search design and the search linear model. We discuss how to obtain search designs. We also discuss optimality under search linear model along with an example.

3.1 Search Design and Search Linear Model

Search design and the search linear model was introduced in Srivastava(1975) and subsequently in Srivastava(1976), Srivastava and Ghosh(1976, 1977) Gupta(1979), Ohnishi and Shirakura(1985), Shirakura, Takahashi, and Srivastava(1996) and so on. In many experiments, it is reasonable to assume that most of the higher order interactions are negligible and while a few of the lower order interactions are nonnegligible. The goal is to construct designs that can identify and estimate the non-negligible interaction effects.

In an full factorial experiment, the factorial effects can be partitioned into to three classes:

C1. The effects we want to estimate such as the main effects.

C2. The effects we are interested in but there are only few that are non-negligible such as two and three-factor interactions.

C3. The remaining effects such as high order interactions, which we consider negligible.

Since we do not know which of the lower order interactions are non-negligible we assume k of the lower order interactions such as the two-factor or three-factor interactions are non-negligible.

We consider the following model

$$\boldsymbol{y} = \boldsymbol{X_1}\boldsymbol{\beta_1} + \boldsymbol{X_2}\boldsymbol{\beta_2} + \boldsymbol{\epsilon} \tag{3.1}$$

where $\boldsymbol{y}$ is the response vector, $\boldsymbol{X_1}$, $\boldsymbol{X_2}$ are known matrices of order $(n \times \upsilon_1)$ and $(n \times \upsilon_2)$ respectively, $\boldsymbol{\beta_1}$ is a $(\upsilon_1 \times 1)$ vector of unknown parameters consisting of the general mean μ and the effects in C1, $\boldsymbol{\beta_2}$ is a $(\upsilon_2 \times 1)$ vector of unknown parameters consisting of the interaction effects in C2 and $\boldsymbol{\epsilon}$ is the $(n \times 1)$ vector of errors with

$$E(\boldsymbol{\epsilon}) = \boldsymbol{0} \qquad Var(\boldsymbol{\epsilon}) = \boldsymbol{\sigma^2 I} \tag{3.2}$$

where $\sigma^2\boldsymbol{I}$ represents the variance-convariance matrix for $\boldsymbol{\epsilon}$ under the error structure of the completely randomized design.

Let D be the design which has given rise to the observations $\boldsymbol{y}$ and k is small relative to υ_2 in C2. If we can estimate $\boldsymbol{\beta_1}$ and the k non-negligible effects of $\boldsymbol{\beta_2}$, then D is said to be a search design of resolving power $(\boldsymbol{\beta_1}; \boldsymbol{\beta_2}; k)$ and the model in (3.1) is said to be a search linear model(Srivastava 1975). Srivastava(1975) provided the necessary condition for a search design in the stochastic case and the necessary and sufficient in the noiseless case.

Theorem 1 *Given model (3.1), a necessary condition for a design D to be a search design of resolving power ($\boldsymbol{\beta_1}$; $\boldsymbol{\beta_2}$; k) is*

$$rank(X_1, X_{20}) = \upsilon_1 + 2k \tag{3.3}$$

for every $(n \times 2k)$ submatrix $\boldsymbol{X_{20}}$ of $\boldsymbol{X_2}$.

Theorem 2 *In the noiseless class where $\boldsymbol{\epsilon} \equiv \mathbf{0}$ in* (3.1) *and* (3.2) *the condition in* (3.3) *is necessary and sufficient.*

In the noiseless case we can identify and estimate the non-negligible effects in $\boldsymbol{\beta_2}$ with probability one. But in the stochastic case we can usually identify the non-negligible effects in $\boldsymbol{\beta_2}$ with probability less than one [Ghosh and Teschmacher(2002)]. In Srivastava and Ghosh(1977) they obtained a theorem that can be used to check the conditions of a search design.

Theorem 3 *Consider the models in* (3.1) *and suppose that rank*($\boldsymbol{X_1}$) $= \upsilon_1$. *Then condition in* (3.3) *holds if and only for every* $(n \times 2k)$ *submatrix* $\boldsymbol{X_{20}}$ *and* $\boldsymbol{X_2}$ *we have*

$$rank[\boldsymbol{X'_{20}X_{20} - X'_{20}X_1(X'_1X_1)^{-1}X'_1X_{20}}] = 2k. \tag{3.4}$$

3.2 How to Obtain Search Designs and Examples

There are several methods of obtaining search designs.

1. Start with the smallest design D_1 that can estimate ($\boldsymbol{\beta_1}$) in (3.1) with ($\boldsymbol{\beta_2}$) absent.

Then add treatments so that the larger design D satisfies the conditions of search design.

2. Start with a design D_1 that can estimate $(\boldsymbol{\beta_1})$ in (3.1), then show that design D_1 satisfies the conditions of search design.

Example 3.1

The following table is presented in Srivastava and Ghosh(1977). The designs presented are resolution V.1 plans for $4 \leq m \leq 8$ factors. These are resolution V.1 plans because we can estimate all main effects and all two-factor interactions as well as $k = 1$ three-factor interaction.

Table 3.1: Resolution V.1 search design designs $4 \leq m \leq 8$

Srivastava and Ghosh (1977)				
m	υ_1	υ_2	n	$\boldsymbol{\lambda'}$
4	11	5	15	11110
5	16	16	21	111010
6	22	42	28	1110010
7	29	99	36	11100010
8	37	219	45	111000010

In table 3.1, $\boldsymbol{\lambda'}$ is describe as follows. Let S_i be the set of all $(1 \times m)$ vectors, with elements 0 and 1 of weight i $(i = 0, 1, \ldots, m)$. The weight of a vector is the number of nonzero elements in it. The designs are denoted by $\boldsymbol{\lambda'} = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ where λ_i is the number of times the set S_i occurs in the design(Ghosh, 1977).

3.3 Optimality Under Search Linear Model

In the search model described in section (3.1) we assume that k lower order interactions are non-negligible but we do not know which k lower order interactions are non-negligible. For many experiments it is reasonable to assume three or higher order interactions are negligible and thus we take lower order interactions to be two-factor interactions.

We consider the following models

$$E(\boldsymbol{y}) = \beta_0\boldsymbol{j}+\boldsymbol{X_1\beta_1} + \boldsymbol{X_2^{(i)}\beta_2^{(i)}}, \quad i = 1, \ldots, \binom{\binom{m}{2}}{k}, \tag{3.5}$$

$$Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I},$$

where $\boldsymbol{y}$ is the observation vector, β_0 is the general mean, $\boldsymbol{\beta_1}$ is the vector of main effects, $\boldsymbol{\beta_2^{(i)}}$ is the vector of k two-factor interactions, $\boldsymbol{j}$ is the vector of ones, $\boldsymbol{X_1}$ $(n \times m)$ depends on the design and $\boldsymbol{X_2}$ $(n \times k)$ depends on the design and the k two-factor interactions in $\boldsymbol{\beta_2^{(i)}}$. We denote the matrix $\boldsymbol{X^{(i)}}$ of dimension $\big(n \times p\big)$ where $p = (1 + m + k)$ as

$$\boldsymbol{X^{(i)}} = \big[\boldsymbol{j, X_1, X_2^{(i)}}\big], \quad i = 1, \ldots, \binom{\binom{m}{2}}{k}. \tag{3.6}$$

A design D with n runs has full estimation capacity(FEC) if it can unbiasedly estimate the parameters in all the models in (3.5). Let $\mathcal{D}$ be the class of n run fractional factorial designs with FEC. When k is even and (≥ 2), the model (3.5) is the search linear model(Srivastava, 1975) and design D is the search design in searching for and identifying $(k/2)$ non-negligible two-factor interactions. When k is odd and (≥ 3), the model in (3.5) is search linear model and design D is the search design in searching for and identifying $((k-1)/2)$ nonnegligible two-factor interactions. Let $\boldsymbol{\beta^{(i)}} = (\mu, \boldsymbol{\beta_1, \beta_2^{(i)}})'$, the least square estimator of $\boldsymbol{\beta^{(i)}}$ as $\hat{\boldsymbol{\beta}}^{(i)}$, and $\boldsymbol{V^{(i)}} = \sigma^{-2}Var(\hat{\boldsymbol{\beta}}^{(i)}) = \boldsymbol{(X^{(i)\prime}X^{(i)})^{-1}}$. We use the following criterion functions (Srivastava, 1977) to compare designs in $\mathcal{D}$;

$AD =$ Arithmetic mean of the determinants of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

$AT =$ Arithmetic mean of the traces of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

$AMCR =$ Arithmetic mean of max. eigenvalues of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

$GD =$ Geometric mean of the determinants of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

$GT =$ Geometric mean of the traces of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

$GMCR =$ Geometric mean of the max. eigenvalues of $\boldsymbol{V^{(i)}}$, $i = 1, \ldots, \upsilon$,

where $\upsilon = \binom{\binom{m}{2}}{k}$. A fractional factorial plan D is $\mathcal{D}$ is optimum with respect to a criterion function if the value of the criterion function is minimum for the designs in $\mathcal{D}$.

Example 3.2

To illustrate we us consider the follow search designs $D_{3.1}$ and $D_{3.2}$

$$D_{3.1} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \qquad D_{3.2} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

Designs $D_{3.1}$ and $D_{3.2}$ have FEC for $k = 2$ under the model in (3.5). Thus $D_{3.1}$ and $D_{3.2}$ are search designs for the models in (3.5) when $k = 1$. In Table 3.2 for $k = 1$, we present the determinant , trace, maximum characteristic roots of the $(X'X)^{-1}$ for each model as well as the six criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$.

Table 3.2: Comparison of designs $D_{3.1}$ and $D_{3.2}$ w.r.t. six criterion functions

Comparison of Fractional Factorial Design									
		M_1	M_2	M_3	M_4	M_5	M_6	AM	GM
	$Det^*(\boldsymbol{X'X}^{-1})$	5.549	5.549	5.549	7.629	7.629	7.629	6.589	6.506
$D_{3.1}$	$Tra(\boldsymbol{X'X}^{-1})$	0.864	0.864	0.864	1.000	1.000	1.000	0.932	0.929
	$MCR(\boldsymbol{X'X}^{-1})$	0.283	0.283	0.283	0.427	0.427	0.427	0.348	0.355
	$Det^*(\boldsymbol{X'X}^{-1})$	8.719	8.719	8.719	7.629	7.629	7.629	8.174	8.156
$D_{3.2}$	$Tra(\boldsymbol{X'X}^{-1})$	1.071	1.071	1.071	1.000	1.000	1.000	1.036	1.035
	$MCR(\boldsymbol{X'X}^{-1})$	0.500	0.500	0.500	0.427	0.427	0.427	0.463	0.462

We denote AM and GM to be the arithmetic and geometric mean respectively and we denote $Det^*(\boldsymbol{X'X}^{-1})$ to be $10^6 \times Det(\boldsymbol{X'X}^{-1})$. For design $D_{3.1}$ the values for the six criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$ are smaller than $D_{3.2}$. Thus design $D_{3.1}$ is better then $D_{3.2}$ with respect to AD, AT, $AMCR$, GD, GT and $GMCR$. A designs is optimal with respect to a criterion function for some values k but it may or may not be optimal for other values of k as will see later in chapter 8.

Chapter 4

Robustness of Factorial Designs

4.0 Summary

In this chapter we review an important theorem on robustness of designs against incomplete data. We use this theorem in chapters 5 and 6 when we consider estimation capacity and robustness of two optimum classes of fractional factorial designs. We present an example for a 2^m fractional factorial design that is robust against missing any two observations for a model. We also present an example of a fractional factorial design that is robust against missing any one observation for all the models when $k = 1$. We obtain a matrix whose rows are orthogonal to the columns of the design matrix. We present for nested fractional factorial designs the structure the matrix thus obtained.

4.1 On Robustness of Design Against Incomplete Data

In this section we review a theorem [Ghosh(1979)] on the robustness of designs against incomplete data. The theorem characterizes the robustness property against missing of any t observations. We consider the linear model

$$\begin{aligned} E(\boldsymbol{y}) &= \boldsymbol{X\beta} \\ Var(\boldsymbol{y}) &= \sigma^2 \boldsymbol{I_n} \\ Rank(\boldsymbol{X}) &= p \end{aligned} \tag{4.1}$$

where $\boldsymbol{y}(n \times 1)$ is the observation vector, $\boldsymbol{X}(n \times p)$ is the design matrix, $\boldsymbol{\beta}(p \times 1)$ is the vector of parameters in the model and σ^2 is a constant.

Definition:[Ghosh(1979)] A design under the model in (4.1) is said to be robust against missing any of t observation if the $\big((n-t) \times p\big)$ matrix obtained from $\boldsymbol{X}$ by deleting any t rows has rank p.

Definition: A matrix $\boldsymbol{B}$ is to have the property P_t if no t columns of $\boldsymbol{B}$ are linearly dependent.

For $n = p + k$, where $k \geq t$, there is a matrix $\boldsymbol{Z}(k \times n)$ with rank k satisfying

$$\boldsymbol{ZX} = \boldsymbol{0}. \tag{4.2}$$

The characterization of the robustness property is given in the following theorem.

Theorem 1 *[Ghosh(1979)] Let T be the design under* (4.1) *with $n = p+k$ observations, where $k \geq t$. Then T is robust against missing of any t observations if and only if the matrix $\boldsymbol{Z}$ in* (4.2) *has the property P_t.*

Proof:

Suppose $\boldsymbol{Z}$ has P_t and let

$$\boldsymbol{X} = \left(\frac{\boldsymbol{X_1}}{\boldsymbol{X_2}} \right), \quad \boldsymbol{Z} = (\boldsymbol{Z_1} | \boldsymbol{Z_2}), \tag{4.3}$$

where $\boldsymbol{X_1}(t \times p)$, $\boldsymbol{X_2}\big((n-t) \times p\big)$, $\boldsymbol{Z_1}(k \times t)$, and $\boldsymbol{Z_2}\big(k \times (n-t)\big)$.
Thus from (4.2) and (4.3) we have

$$\boldsymbol{Z_1 X_1} + \boldsymbol{Z_2 X_2} = \boldsymbol{0}. \tag{4.4}$$

Now suppose

$$\boldsymbol{Z_1} = \left(\frac{\boldsymbol{Z_{11}}}{\boldsymbol{Z_{12}}} \right), \quad \boldsymbol{Z_2} = \left(\frac{\boldsymbol{Z_{21}}}{\boldsymbol{Z_{22}}} \right), \tag{4.5}$$

where $\boldsymbol{Z_{11}}(t \times t)$, $\boldsymbol{Z_{12}}\big((k-t) \times t\big)$, $\boldsymbol{Z_{21}}\big(k \times (n-t)\big)$, and $\boldsymbol{Z_{22}}\big((k-t) \times (n-t)\big)$.
Furthermore suppose $Rank(\boldsymbol{Z_{11}}) = t$. Thus we obtain

$$\boldsymbol{Z_{11} X_1} + \boldsymbol{Z_{21} X_2} = \boldsymbol{0}. \tag{4.6}$$

Hence

$$\boldsymbol{X_1} = -\boldsymbol{Z_{11}^{-1} Z_{21} X_2}. \tag{4.7}$$

Thus the rows of $\boldsymbol{X_1}$ are linear combinations of $\boldsymbol{X_2}$. Consequently $\boldsymbol{X_2}$ has rank p. Since $\boldsymbol{Z}$ has P_t, similarly this is true for any t rows of $\boldsymbol{X}$. Hence design T is robust.

Suppose the design T is robust against missing any t observations. Then there is a $\big(t \times (n-t)\big)$ matrix $\boldsymbol{D}$ satisfying

$$\boldsymbol{X_1} = \boldsymbol{D X_2}. \tag{4.8}$$

It follows from (4.2), (4.3) and (4.6) that there is a matrix $\boldsymbol{U}(t \times k)$ such that

$$\boldsymbol{U C_1} = \boldsymbol{I_t}, \quad \boldsymbol{U C_2} = -\boldsymbol{D}. \tag{4.9}$$

Thus the $Rank(\boldsymbol{C_1}) = t$. Hence $\boldsymbol{C}$ has P_t. ■

The result is easy to use for $t = 1$ and $t = 2$. The two cases for $t = 1$ and $t = 2$ are given in the following corollaries.

Corollary 1 *Suppose $t = 1$. The design T is robust against missing any one observations iff $\boldsymbol{C}(k \times n)$ has the property P_1; equivalently no column vector in $\boldsymbol{C}$ is the null vector.*

Corollary 2 *Suppose $t = 2$. The design T is robust against missing any two observations iff $\boldsymbol{C}(k \times n)$ has the property P_2; equivalently no column vector in $\boldsymbol{C}$ is the null vector and no columns of $\boldsymbol{C}$ are proportional to each other.*

Example 4.1

We consider the resolution V plan $\{S_0, S_1, S_2, S_3\}$ for $m = 4$. The number of runs in the design is $n = 15$ and the number of parameters in the model is $p = 11$. Thus our $\boldsymbol{Z}$ matrix is (4×15). The $\boldsymbol{Z}$ is presented below:

$$\boldsymbol{Z} = \begin{pmatrix} 1 & -3 & -3 & -3 & -3 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & -2 & 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & -2 & 2 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & -1 & -1 \end{pmatrix}. \quad (4.10)$$

It can be checked that matrix $\boldsymbol{Z}$ has P_2 but not P_3. The columns 1, 6 and 11 of the $\boldsymbol{Z}$ are linearly dependent. Thus designs obtained by deleting two any two runs are resolution V plans.

Example 4.2

We consider the fractional factorial plan $\{S_0, S_1, S_4, S_5\}$ for $m = 5$ and $k = 1$. The model contains the main effects, as well as the (12) two-factor interaction. The number of runs in the design is $n = 12$ and the number of parameters in the model is $p = 7$ thus our $\boldsymbol{Z}$ matrix is (5×12). The $\boldsymbol{Z}$ is presented below:

$$\boldsymbol{Z} = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{4.11}$$

We observe the matrix $\boldsymbol{Z}$ has P_1 but not P_2. The columns 2 and 8 of the $\boldsymbol{Z}$ matrix are linearly dependent. Thus we can delete any one run from the design and still be able to unbiasedly estimate all the parameters in the model. Since the $\boldsymbol{Z}$ matrices of all other models are isomorphic with respect to column permutations thus the designs obtained by deleting one run have full estimation capacity for $k = 1$. Furthermore we can delete some two runs and the resulting designs can estimate all the parameters in this model. We can delete S_0 and any one run from S_1, S_4 or S_5 and still be able to unbiasedly estimate all the parameters in the model.

4.2 Z Matrices of Nested Designs

In this section we consider two designs D_2 and D_3 such that $D_2 \supset D_3$. Let $\boldsymbol{X_2}(n_2 \times p)$ and $\boldsymbol{X_3}(n_3 \times p)$ be the design matrices associated with designs D_2 and D_3. Suppose the matrices $\boldsymbol{X_2}$ and $\boldsymbol{X_3}$ have rank p. Since $D_2 \supset D_3$ then $n_2 > n_3$. First we determine $\boldsymbol{Z_3}\big((n_3 - p) \times n\big)$ satisfies $\boldsymbol{Z_3 X_3 = 0}$. Let

$$\boldsymbol{X_3} = \begin{pmatrix} \boldsymbol{X_{31}} \\ \hline \boldsymbol{X_{32}} \end{pmatrix}, \quad \boldsymbol{Z_3 = (Z_{31} | Z_{32})} \tag{4.12}$$

where $\boldsymbol{Z_{31}}\big((n-p)\times p\big)$, $\boldsymbol{Z_{32}}\big((n-p)\times(n-p)\big)$, $\boldsymbol{X_{31}}(p\times p)$, and $\boldsymbol{X_{32}}\big((n-p)\times p\big)$. Suppose the rank($\boldsymbol{X_{31}}$) $= p$ and we can let $\boldsymbol{X_{31}} = \boldsymbol{I}$ then we have

$$(\boldsymbol{Z_{31}}|\boldsymbol{I})\left(\begin{array}{c}\boldsymbol{X_{31}}\\ \hline \boldsymbol{X_{32}}\end{array}\right) = \boldsymbol{0}. \tag{4.13}$$

Thus

$$\boldsymbol{Z_{31}} = -\boldsymbol{X_{31}^{-1}}\boldsymbol{X_{32}}. \tag{4.14}$$

Hence

$$\boldsymbol{Z_3} = (-\boldsymbol{X_{31}^{-1}}\boldsymbol{X_{32}}|\boldsymbol{I_{(n-p)}}). \tag{4.15}$$

Since rank($\boldsymbol{X_3}$) $= p$, then we can always permute the rows of $\boldsymbol{X_3}$ so that rank($\boldsymbol{X_{31}}$) $= p$.

Next we determine $\boldsymbol{Z_2}\big((n_2-p)\times n\big)$ satisfies $\boldsymbol{Z_2}\boldsymbol{X_2} = \boldsymbol{0}$. Since $\boldsymbol{X_2} \supset \boldsymbol{X_3}$ then we have

$$\boldsymbol{X_2} = \left(\begin{array}{c}\boldsymbol{X_3}\\ \hline \boldsymbol{X_{21}}\end{array}\right). \tag{4.16}$$

where $\boldsymbol{X_{21}}((n_2-n_3)\times p)$. We take $\boldsymbol{Z_2}$ to have the form

$$\boldsymbol{Z_2} = \left(\begin{array}{c|c}\boldsymbol{Z_3} & \boldsymbol{0}\\ \hline \boldsymbol{Z_{21}} & \boldsymbol{Z_{22}}\end{array}\right). \tag{4.17}$$

We choose $\boldsymbol{Z_{21}}$ and $\boldsymbol{Z_{22}}$ so that the rank($\boldsymbol{Z_2}$) $= (n_2-p)$. Since the rank($\boldsymbol{Z_3}$) $= (n_3-p)$ and rank($\boldsymbol{Z_2}$)=rank($\boldsymbol{Z_3}$)+rank($\boldsymbol{Z_{22}}$) hence rank($\boldsymbol{Z_2}$) $= (n_2-p)$ if and only if rank$(\boldsymbol{Z_{22}}) = (n_2 - n_3)$. We choose $\boldsymbol{Z_{22}} = \boldsymbol{I}$ then we must find $\boldsymbol{Z_{21}}$ satisfies

$$\boldsymbol{Z_{21}} \boldsymbol{X_3} + \boldsymbol{X_{21}} = \mathbf{0}. \tag{4.18}$$

We let

$$\boldsymbol{Z_{21}} = (\boldsymbol{Z_{21}^{(1)}} | \boldsymbol{Z_{21}^{(2)}}), \quad \boldsymbol{X_3} = \left(\begin{array}{c} \boldsymbol{X_{31}} \\ \hline \boldsymbol{X_{32}} \end{array} \right). \tag{4.19}$$

then

$$\boldsymbol{Z_{21}} \boldsymbol{X_3} = \boldsymbol{Z_{21}^{(1)}} \boldsymbol{X_{31}} + \boldsymbol{Z_{21}^{(2)}} \boldsymbol{X_{32}}. \tag{4.20}$$

We choose $\boldsymbol{Z_{21}^{(2)}} = \mathbf{0}$ then

$$\boldsymbol{Z_{21}} \boldsymbol{X_3} + \boldsymbol{X_{21}} = \boldsymbol{Z_{21}^{(1)}} \boldsymbol{X_{31}} + \boldsymbol{X_{21}} = \mathbf{0}. \tag{4.21}$$

Thus

$$\boldsymbol{Z_{21}^{(1)}} = -\boldsymbol{X_{21}} \boldsymbol{X_{31}^{-1}}. \tag{4.22}$$

Hence

$$\boldsymbol{Z_2} \boldsymbol{X_2} = \left(\begin{array}{c|c|c} -\boldsymbol{X_{31}^{-1}} \boldsymbol{X_{32}} & \boldsymbol{I} & \mathbf{0} \\ \hline -\boldsymbol{X_{21}} \boldsymbol{X_{31}^{-1}} & \mathbf{0} & \boldsymbol{I} \end{array} \right) \left(\begin{array}{c} \boldsymbol{X_{31}} \\ \hline \boldsymbol{X_{32}} \\ \hline \boldsymbol{X_{21}} \end{array} \right) = \mathbf{0}. \tag{4.23}$$

Example 4.3

We consider the following $\boldsymbol{X_2}$ matrix for $m = 4$ and $k = 1$

$$\boldsymbol{X_2} = \left(\begin{array}{c} \boldsymbol{X_{31}} \\ \hline \boldsymbol{X_{32}} \\ \hline \boldsymbol{X_{21}} \end{array}\right) = \left(\begin{array}{rrrrrr} 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \end{array}\right), \qquad (4.24)$$

$$\boldsymbol{Z_2} = \left(\begin{array}{rrrrrr|cc|cc} 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & -1/2 & -1/2 & -1/2 & 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & -1 & -1/2 & 0 & 0 & 0 & 1 \end{array}\right). \qquad (4.25)$$

Example 4.4

We consider the following $\boldsymbol{X_2}$ matrix for $m = 5$ and $k = 2$

$$\boldsymbol{X_2} = \left(\begin{array}{c} \boldsymbol{X_{31}} \\ \hline \boldsymbol{X_{32}} \\ \hline \boldsymbol{X_{21}} \end{array} \right) = \left(\begin{array}{rrrrrrrr} 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ \hline 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \end{array} \right), \tag{4.26}$$

$$\boldsymbol{Z_2} = \left(\begin{array}{rrrrrrrr|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ \hline -1/3 & 0 & -1/3 & 0 & -2/3 & -1/3 & 1/3 & 1/3 & 0 & 0 & 1 & 0 \\ 1/3 & 0 & 1/3 & 0 & -1/3 & -2/3 & -1/3 & -1/3 & 0 & 0 & 0 & 1 \end{array} \right). \tag{4.27}$$

Chapter 5

Estimation Capacity and Robustness of Design D

5.0 Summary

In this chapter, we present the robustness property of design $D = \{S_0, S_1, S_{m-1}, S_m\}$ with respect to any t runs as well as a specific set of t runs. We consider the estimation capacity of design D. We determine design D has maximum full estimation capacity for $k = 3$. The designs obtained from D by deleting one run also have full estimation capacity for $k = 3$. The designs obtained from D by deleting two runs may or may not have full estimation capacity for $k = 3$. The designs obtained from D by deleting three runs do not have full estimation capacity for $k = 3$. The designs obtained from D by deleting two or three runs may or may not have full estimation capacity for $k = 2$. The designs obtained from D by deleting a run from S_1 and a run from S_{m-1} where the two runs are not foldover do not have full estimation capacity for $k = 1$. All other designs obtained by deleting two runs from D have full estimation for $k = 1$. The designs obtained by deleting three runs from D may or may not have full estimation capacity for $k = 1$.

5.1 Determination of Estimation Capability for D

We consider the foldover design $D = \{S_0, S_1, S_{m-1}, S_m\}$ and the models

$$E(\boldsymbol{y}) = \mu\boldsymbol{j} + \boldsymbol{X_1}\boldsymbol{\beta_1} + \boldsymbol{X_2^{(i)}}\boldsymbol{\beta_2^{(i)}}, \quad i = 1, \ldots, \binom{\binom{m}{2}}{k}, \tag{5.1}$$

$$Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I},$$

where $\boldsymbol{y}$ is the observation vector, μ is the general mean, $\boldsymbol{\beta_1}$ is the vector of main effects, $\boldsymbol{\beta_2^{(i)}}$ is the vector of k two-factor interactions, $\boldsymbol{j}$ is the vector of ones, $\boldsymbol{X_1}$ $(n\times m)$ depends on the design and $\boldsymbol{X_2}$ $(n\times k)$ depends on the design D and the k two-factor interactions in $\boldsymbol{\beta_2^{(i)}}$. We denote FEC = Full Estimation Capability, k^* = the maximum value of k for a design to have FEC under the model (5.1). We determine the k^* for the design D as well as designs obtained from D by deleting runs. We denote the matrix $\boldsymbol{X^{(i)}}$ of dimension $\big(n \times p\big)$ where $p = (1 + m + k)$ as

$$\boldsymbol{X^{(i)}} = \big[\boldsymbol{j}, \boldsymbol{X_1}, \boldsymbol{X_2^{(i)}}\big], \quad i = 1, \ldots, \binom{\binom{m}{2}}{k}. \tag{5.2}$$

The matrix $\boldsymbol{X^{(i)}}$ has the rank p if and only if there is a matrix $\boldsymbol{Z}$ of dimension $\big((n - p) \times n\big)$ with rank $(n - p)$ and satisfies $\boldsymbol{Z}\boldsymbol{X^{(i)}} = 0$.

First we consider the class of models in (5.1) with the elements of $\boldsymbol{\beta_2^{(i)}}$ as k two-factor interactions from

$$C = \big(12, 13, ..., 1m_1, (m_1 + 1)(m_1 + 2), (m_1 + 3)(m_1 + 4), \ldots, (m_2 - 1)(m_2)\big). \tag{5.3}$$

where $k = \big[(m_1 - 1) + 1/2(m_2 - m_1)\big]$, $2 \leq m_1 \leq m_2 \leq m$. The following theorem determine the estimation capacity of design D for this class of models.

Theorem 1 *For the model in* (5.1) *with the elements of* $\boldsymbol{\beta_2^{(i)}}$ *as* k *two-factor inter-*

actions from C in (5.3), the design D has the ability to unbiasedly estimate all the β parameters.

Proof:

We consider the $\boldsymbol{Z}$ matrix presented below:

$$\boldsymbol{Z} = \left(\begin{array}{c} \boldsymbol{Z_1}, \boldsymbol{Z_2} \end{array} \right), \tag{5.4}$$

where

$$\boldsymbol{Z_1} = \left(\begin{array}{c|cccc|cccccccc|cccc}
a_{11} & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & \dots & 1 \\
a_{12} & 0 & -1 & \dots & -1 & -1 & 0 & -1 & 0 & \dots & -1 & 0 & -1 & -1 & \dots & -1 \\
\hline
0 & 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
 & & & \vdots & & & & & & \vdots & & & & & \vdots & \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 & 0 & 0 & \dots & 0 \\
\hline
-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
 & & & \vdots & & & & & & \vdots & & & & & \vdots & \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1
\end{array} \right),$$

$$
\boldsymbol{Z_2} = \left(\begin{array}{cccc|cccccccc|ccccc|c}
0 & -1 & \dots & -1 & 0 & -1 & 0 & -1 & \dots & 0 & -1 & 0 & 0 & 0 & \dots & 0 & a_{21} \\
1 & 0 & \dots & 0 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 & a_{22} \\
\hline
0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & -1 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & -1 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 & -1 \\
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1
\end{array}\right),
$$

$a_{11} = -[m - 1/2(m_2 + m_1)]$, $a_{12} = 1/2(m_2 + m_1 - 4)$, $a_{21} = [m - 1/2(m_2 - m_1 + 4)]$ and $a_{22} = 1/2(m_1 - m_2)$. Let $\boldsymbol{X}$ be the design matrix, we show $\boldsymbol{ZX} = 0$. First we show row 1 of $\boldsymbol{Z}$ is orthogonal to the columns of $\boldsymbol{X}$. We define $\boldsymbol{Z_{11}}$ and $\boldsymbol{Z_{12}}$ to be the first row of $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$ respectively. The elements in $\boldsymbol{Z_{11}}$ and $\boldsymbol{Z_{12}}$ are corresponding elements if their corresponding runs in S_1 and S_{m-1} are foldover runs. We see that if the element is 1 in $\boldsymbol{Z_{11}}$ then its corresponding element must be 0 in $\boldsymbol{Z_{12}}$. Similarly if the element is 0 in $\boldsymbol{Z_{11}}$ then its corresponding element is -1 in $\boldsymbol{Z_{12}}$. The product of row 1 of $\boldsymbol{Z}$ and $\boldsymbol{j}$ is

$$a_{11} + a_{12} + (m - m_2 - m_1) + 2 = 0. \tag{5.5}$$

In any columns of S_1 there is exactly one -1 and in any columns of S_{m-1} there is exactly one 1. When we take the product of row 1 of $\boldsymbol{Z}$ and columns of $\boldsymbol{X_1}$, the -1 in the columns of S_1 is multiply to 1 or 0 in $\boldsymbol{Z_{11}}$ and this depends on the columns. Their corresponding element in $\boldsymbol{Z_{12}}$ is multiply to 1 in S_{m-1}. In both cases the product of row 1 of $\boldsymbol{Z}$ and columns of $\boldsymbol{X_1}$ is

$$a_{11} - a_{12} + m - 2 = 0. \tag{5.6}$$

We can see that the product of row 1 of $\boldsymbol{Z}$ and a column of $\boldsymbol{X_2}$ is

$$a_{11} + a_{12} + (m - m_2 - m_1) + 2 = 0. \tag{5.7}$$

Similarly the product of row 2 of $\boldsymbol{Z}$ and $\boldsymbol{j}$ is

$$a_{21} + a_{22} - (m - (m_2 + m_1)) + 2 = 0. \tag{5.8}$$

The product of row 2 of $\boldsymbol{Z}$ and any columns of $\boldsymbol{X_1}$ is

$$a_{21} - a_{22} - m + 2 = 0. \tag{5.9}$$

The product of row 2 of $\boldsymbol{Z}$ and any columns of $\boldsymbol{X_2}$ is

$$a_{21} + a_{22} - (m - (m_2 + m_1)) + 2 = 0. \tag{5.10}$$

The first two rows of $\boldsymbol{Z}$ are orthogonal to $\boldsymbol{X}$. From the structure of the remaining rows of matrix $\boldsymbol{Z}$ and the symmetry of the $\boldsymbol{X}$ matrix it is obvious the remaining rows of matrix $\boldsymbol{Z}$ are orthogonal to the $\boldsymbol{X}$ matrix. Furthermore the matrix $\boldsymbol{Z}$ has rank $\left[m + 2 - 1/2(m_1 + m_2)\right]$ thus design D allows us to estimate unbiasedly all the β parameters in (5.1) with the k elements of $\boldsymbol{\beta_2^{(i)}}$ from C in (5.3) . ■

We now consider a general class C of the type in (5.3) as

$$C = \left(i_{12}, i_{13}, ..., i_{m_1}, i_{(m_1+1)(m_1+2)}, i_{(m_1+3)(m_1+4)}, \ldots, i_{(m_2-1)(m_2)}\right). \tag{5.11}$$

The $\boldsymbol{Z}$ matrices for models with k elements of $\boldsymbol{\beta}_2^{(i)}$ from C in (5.11) are isomorphic to the $\boldsymbol{Z}$ matrix in (5.4).

Corollary 1 *Theorem 1 is also true for the k elements of $\boldsymbol{\beta}_2^{(i)}$ from C in (5.11).*

Next we determine k^*, the maximum value of k for design D to have FEC for the models in (5.1). The result is given in the following theorem.

Theorem 2 *The design D has the properties below under the models in (5.1).*

(a) FEC for $k^* = 1$, $m = 2$

(b) FEC for $k^* = 3$, $m \geq 3$.

Proof:

First we consider the model in (5.1) for $k = 4$ and the two factor interactions 12, 13, 24, and 34. The dimension of the $\boldsymbol{X}$ matrix is $\big((2m+2)\times(m+5)\big)$. We obtain a $\big((m-2)\times(2m+2)\big)$ matrix $\boldsymbol{Z}$ satisfying $\boldsymbol{ZX} = 0$ as

$$\boldsymbol{Z} = \left(\begin{array}{c|cccc|cccc|cccc|cccc|c}
-a & 1 & 0 & 0 & 1 & 1 & 1 & \dots & 1 & 0 & -1 & -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ \hline
a & 0 & -1 & -1 & 0 & -1 & -1 & \dots & -1 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & -1 \\ \hline
-1 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & -1 \\
 & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1
\end{array}\right) \tag{5.12}$$

where $a = (m-3)$. The rank of $\boldsymbol{Z}$ is $(m-2)$. Thus $\boldsymbol{X}$ matrix has rank less than $(m+5)$. Hence the design D does not have FEC for $k = 4$. Now we consider models for $k = 3$. The models can be partitioned into maximum of five classes depending on the number of factors m. The classes are determined by the three two-factor interactions in the models. The five classes are presented below:

$$
\begin{aligned}
C1 &= (i_1i_2, i_1i_3, i_1i_4),\\
C2 &= (i_1i_2, i_1i_3, i_2i_3),\\
C3 &= (i_1i_2, i_1i_3, i_2i_4),\\
C4 &= (i_1i_2, i_1i_3, i_4i_5),\\
C5 &= (i_1i_2, i_3i_4, i_5i_6).
\end{aligned}
\tag{5.13}
$$

For the design D, all the models within a class in (5.13) are isomorphic to each other by renaming of factors. We can consider one model in each class by taking $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, $i_4 = 4$, $i_5 = 5$, and $i_6 = 6$. We denote the column of the $\boldsymbol{Z}$ matrix corresponding to S_0 as r_0. The column of the $\boldsymbol{Z}$ matrix corresponding to the i^{th} run of S_1 as r_i. The column of the $\boldsymbol{Z}$ matrix corresponding to the i^{th} run of S_{m-1} as r_i^c. The column of the $\boldsymbol{Z}$ matrix corresponding to S_m as r_0^c. The matrix $\boldsymbol{Z}_{\mathbf{(12,13,14)}}$ is obtained from (5.4) for $m_1 = 4$ and $m_2 = 4$. The matrix $\boldsymbol{Z}_{\mathbf{(12,13,45)}}$ is obtained from (5.4) for $m_1 = 3$ and $m_2 = 5$. The matrix $\boldsymbol{Z}_{\mathbf{(12,34,56)}}$ is obtained from (5.4) for $m_1 = 2$ and $m_2 = 6$. The matrix $\boldsymbol{Z}_{\mathbf{(12,13,23)}}$ is obtained from $\boldsymbol{Z}_{\mathbf{(12,13)}}$ and $\boldsymbol{Z}_{\mathbf{(12,13)}}$ is obtained from (5.4) for $m_1 = 3$ and $m_2 = 3$. The first row of $\boldsymbol{Z}_{\mathbf{(12,13,23)}}$ is the difference of row 1 and row 2 of $\boldsymbol{Z}_{\mathbf{(12,13)}}$. The remaining $(m-3)$ rows of $\boldsymbol{Z}_{\mathbf{(12,13,23)}}$ are identical to the last $(m-3)$ rows of $\boldsymbol{Z}_{\mathbf{(12,13)}}$. The matrix $\boldsymbol{Z}_{\mathbf{(12,13,24)}}$ is obtained from $\boldsymbol{Z}_{\mathbf{(12,13)}}$. The first row of $\boldsymbol{Z}_{\mathbf{(12,13,24)}}$ is identical to the first row of $\boldsymbol{Z}_{\mathbf{(12,13)}}$. The second row of $\boldsymbol{Z}_{\mathbf{(12,13,24)}}$ is the sum of row 2 and row 3 of $\boldsymbol{Z}_{\mathbf{(12,13)}}$. The remaining $(m-4)$ rows of $\boldsymbol{Z}_{\mathbf{(12,13,24)}}$ are identical to the last $(m-4)$ rows of $\boldsymbol{Z}_{\mathbf{(12,13)}}$.

Within each class of models in (5.13) the $\boldsymbol{Z}$ matrices are isomorphic to column permutations. Consider the two models with the interactions $\{12, 13, 14\}$ and $\{12, 23, 24\}$ from class $C1$. To obtain $\boldsymbol{Z}_{\mathbf{(12,23,24)}}$ from $\boldsymbol{Z}_{\mathbf{(12,13,14)}}$ we simply switch the columns r_1

with r_2 and r_1^c with r_2^c in $\boldsymbol{Z}_{(\mathbf{12,13,14})}$. We determine all the $\boldsymbol{Z}$ matrices for $k = 3$ have rank $(m-2)$ thus all the $\boldsymbol{X}^{(i)}$ matrices in (5.2) have rank $(m+4)$. Hence D has FEC for $k^* = 3$ for all $m \geq 3$. The case for $m = 2$ with $k = 1$ is obvious. ■

5.2 Designs Obtained from D with FEC for $k = 3$

In this section we determine which designs obtained from D by deleting a certain number of runs have FEC for $k^* = 3$. We define $S_1^{(-i)}$ to be i runs deleted from S_1 for $i = 1, ..., m$. Similarly we denote $S_{m-1}^{(-i)}$ to be i runs deleted from S_{m-1} for $i = 1, ..., m$. The results are given in the following three theorems.

Theorem 3 *The designs obtain by deleting any one run from D have FEC when* $k^* = 3$, $m \geq 3$ *under the model in (5.1) .*

Proof:

All $\boldsymbol{Z}$ matrices for $k = 3$ have P_1 for $m \geq 3$. From Theorem 1 in chapter 4 the designs have FEC for $k^* = 3$, $m \geq 3$. ■

Theorem 4 *Let* $D(1)$ *be the design obtain from* D *by deleting* $(S_0, S_1^{(-1)})$ *or* $(S_{m-1}^{(-1)}, S_m)$. *Let* $D(2)$ *be the design obtain from* D *by deleting* $(S_0, S_{m-1}^{(-1)})$ *or* $(S_1^{(-1)}, S_m)$. *Let* $D(3)$ *be the design obtain from* D *by deleting* (S_0, S_m). *The designs* $D(1)$, $D(2)$, *and* $D(3)$ *are the only designs obtained by deleting two runs from* D *with FEC for* $k^* = 3$, *when* $m = 5$ *and* $m \geq 7$ *under the model in* (5.1) .

Proof:

We consider in all the $\boldsymbol{Z}$ matrices the column r_0 and another column. We see that these two columns are linearly independent except for $m = 4$ and $m = 6$. The designs within the class $D(1)$ and $D(2)$ are isomorphic by renaming factors or levels. From Theorem 1 in chapter 4 $D(1)$, $D(2)$ and $D(3)$ have FEC for $k^* = 3$ when $m = 5$ and $m \geq 7$. From Theorem 1 in chapter 4 for $\boldsymbol{Z}_{\mathbf{(12,13,23)}}$ we cannot delete two runs from S_1 or two runs from S_{m-1}. Similarly we cannot delete one run from S_1 and one run from S_{m-1}. ■

Theorem 5 *The designs obtain by deleting any three run from D do not have FEC for $k^* = 3$ under the model in* (5.1).

Proof:

From theorem 3, it is suffice to show we cannot deleting S_0, S_m and a run from S_1. For $m = 5$ and $\boldsymbol{Z}_{\mathbf{(12,13,24)}}$, the three columns r_0, r_0^c and r_5 are linearly dependent. For $m \geq 7$ and $Z_{(12,34,56)}$ the three columns r_0, r_0^c and r_m are linearly dependent. Thus from Theorem 1 in chapter 4 we cannot delete any three runs from D and have FEC for $k^* = 3$. ■

5.3 Designs Obtained from D with FEC for $k = 2$

Now we consider models for $k = 2$. The models can be partitioned into two isomorphic classes. The classes are given below:

$$C1 = (i_1 i_2, i_1 i_3),$$
$$C2 = (i_1 i_2, i_3 i_4). \qquad (5.14)$$

We can consider one models from each class by let $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, and $i_4 = 4$. The $\boldsymbol{Z}$ matrices of these two models are special cases of (5.4). From the $\boldsymbol{Z}$ matrices we obtain results given in the following table:

Table 5.1: Designs obtained from D by deleting runs with FEC for $k = 2$

Designs With Full Estimation Capacity ($m \geq 4$)		
k	Number of runs deleted	Run(s) Deleted
2	1	Any one run
2	2	S_0, $S_1^{(-1)}$
2	2	S_0, $S_{m-1}^{(-1)}$
2	2	S_0, S_m, $m \geq 5$
2	2	$S_1^{(-1)}$, S_m
2	2	$S_{m-1}^{(-1)}$, S_m
2	2	$S_1^{(-1)}$, its foldover in S_{m-1}
2	3	S_0, $S_1^{(-1)}$, its foldover in S_{m-1}, $m \geq 5$
2	3	S_0, $S_1^{(-1)}$, S_m
2	3	S_0, $S_{m-1}^{(-1)}$, S_m
2	3	$S_1^{(-1)}$, its foldover in S_{m-1}, S_m, $m \geq 5$

For $m = 3$ it is easy to check that we can deleted any one run from D and any pair of foldover runs from D and still have FEC for $k = 2$.

5.4 Designs Obtained from D with FEC for $k = 1$

Now we consider models for $k = 1$. There is only one isomorphic class of models. The class is given below:

$$C1 = (i_1 i_2). \tag{5.15}$$

We can consider one model from this class by taking $i_1 = 1$ and $i_2 = 2$. The $\boldsymbol{Z}$ matrix for this class is a special case of (5.4). From the $\boldsymbol{Z}$ matrix we obtain the results given in the following table:

Table 5.2: Designs obtained from D by deleting runs with FEC for $k = 1$

Designs With Full Estimation Capacity ($m \geq 4$)		
k	Number of runs deleted	Run(s) Deleted
1	1	Any one run
1	2	All two runs except $\left(S_1^{(-1)}\right.$, nonfoldovers in $\left.S_{m-1}\right)$
1	3	S_0, $S_1^{(-1)}$, S_m
1	3	S_0, $S_{m-1}^{(-1)}$, S_m
1	3	S_0, $S_1^{(-1)}$, its foldover in S_{m-1}
1	3	$S_1^{(-1)}$, its foldover in S_{m-1}, S_m
1	3	S_0, $S_1^{(-2)}$, $m \geq 5$
1	3	$S_{m-1}^{(-2)}$, S_m, $m \geq 5$
1	3	S_0, $S_{m-1}^{(-2)}$
1	3	$S_1^{(-2)}$, S_m
1	3	$S_1^{(-3)}$
1	3	$S_{m-1}^{(-3)}$

For $m = 3$ it is easy to check that we can deleted any one run from D and still have FEC for $k = 1$. We can delete two runs from S_1 or S_{m-1}. Furthermore we can delete a run from S_1 and a run from S_{m-1}, where the two runs are not foldovers.

Chapter 6

Estimation Capacity and Robustness of Design D_1

6.0 Summary

In this chapter, we present the robustness property of design $D_1 = \{S_0, S_1, S_2, S_m\}$ with respect to any t runs as well as a specific set of t runs. We consider the estimation capacity of design D_1. We determine design D_1 is a resolution V plan. The designs obtained from D_1 by deleting one run is also a resolution V plan. The designs obtained from D_1 by deleting two runs may or may not have full estimation capacity for $k = \left(\binom{m}{2} - 1\right)$. The designs obtained from D_1 by deleting two runs may or may not have full estimation capacity for $k = \left(\binom{m}{2} - 2\right)$. The designs obtained from D_1 by deleting three runs do not have full estimation capacity for $k = \left(\binom{m}{2} - 2\right)$.

6.1 Determination of Estimation Capability for D_1

We first show that the design D_1 is a resolution V plan. Assuming three or higher factor interactions are negligible, our model in (5.1) with $k = \binom{m}{2}$ has the elements of $\boldsymbol{\beta_2^{(i)}}$ as all two-factor interactions. The dimension of the $\boldsymbol{X}$ is $(n \times p)$, where $n = \left(2 + m + \binom{m}{2}\right)$ and $p = \left(1 + m + \binom{m}{2}\right)$ Consequently design D_1 is a resolution V plan if the $\boldsymbol{X}$ matrix has rank p. To prove matrix $\boldsymbol{X}$ has rank p we show that the $\boldsymbol{Z}$ matrix is $(1 \times n)$ with rank 1. The following theorem establishes the estimation capacity and the robustness of D_1 with one run deleted.

Theorem 1 *The design D_1 and designs obtain from D_1 by deleting one run are all resolution V plans.*

Proof:

The vector $\boldsymbol{Z}$

$$\boldsymbol{Z} = \left(\begin{array}{c|ccc|ccc|c} -\frac{(m-1)(m-2)}{2} & (m-2) & \dots & (m-2) & -1 & \dots & -1 & 1 \end{array} \right), \tag{6.1}$$

has rank 1 and satisfies $\boldsymbol{ZX} = 0$. Thus the $\boldsymbol{X}$ matrix has rank p. Hence D_1 is a resolution V plan. For $m \geq 3$, the $\boldsymbol{Z}$ vector has the property P_1 in Theorem 1 from chapter 4 in the sense that designs obtained from D_1 by deleting a run are resolution V plans. ■

6.2 Structure of $\boldsymbol{Z}$ matrices

We consider the model in (5.1) for $k = \left(\binom{m}{2} - 1\right)$ with the interaction (12) excluded. The $\boldsymbol{Z}$ matrix for this model is presented below:

$$\boldsymbol{Z}_{\overline{\mathbf{12}}} = \left(\begin{array}{c|cccc|ccc|c} -\frac{m(m-3)}{2} & (m-3) & (m-3) & (m-2) & \dots & 0 & -1 & \dots & 1 \\ \hline 1 & -1 & -1 & 0 & \dots & 1 & 0 & \dots & 0 \end{array} \right). \tag{6.2}$$

We can label the runs of S_1 as $1, 2, ..., m$ and the runs of S_2 as $12, 13, ..., (m-1)(m)$. The numbers indicate the positions of the -1 in the runs. For row 2 of the $Z_{\overline{12}}$ matrix, element corresponding to S_0 is 1, the elements corresponding to 1 and 2 of S_1 is -1 and the elements corresponding to 12 of S_2 is 1. All other elements are zero. To obtain the first row of $\boldsymbol{Z}_{\overline{\mathbf{12}}}$ we add row two of $\boldsymbol{Z}_{\overline{\mathbf{12}}}$ to the $\boldsymbol{Z}$ vector in (6.1). Similarly we can generate the $\boldsymbol{Z}$ matrix of any model. Next we consider two models in (5.1) for $k = \left(\binom{m}{2} - 2\right)$ with the interaction (12,13) and (12,34) excluded. The $\boldsymbol{Z}$ matrices are given respectively below:

$$\boldsymbol{Z}_{\overline{\mathbf{12,13}}} = \left(\begin{array}{c|cccccc|cccccc|c} a & b & c & c & d & \dots & d & 0 & 0 & -1 & -1 & \dots & -1 & 1 \\ \hline 1 & -1 & -1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 1 & -1 & 0 & -1 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{array}\right), \tag{6.3}$$

$$\boldsymbol{Z}_{\overline{\mathbf{12,34}}} = \left(\begin{array}{c|cccccc|ccccccc|c} a & c & c & c & d & d & \dots & 0 & -1 & \dots & -1 & 0 & -1 & \dots & 1 \\ \hline 1 & -1 & -1 & 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline 1 & 0 & 0 & -1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \end{array}\right), \tag{6.4}$$

where $a = -\frac{m^2-3m-2}{2}$, $b = (m-4)$, $c = (m-3)$, and $d = (m-2)$. To obtain the row 1 of $\boldsymbol{Z}_{\overline{\mathbf{12,13}}}$, we sum the last two rows and $\boldsymbol{Z}_{\overline{\mathbf{12,13}}}$ and add to the $\boldsymbol{Z}$ vector in (6.1). This is also true for $\boldsymbol{Z}_{\overline{\mathbf{12,34}}}$. Similarly we can obtain $\boldsymbol{Z}$ matrix of any model for D_1.

6.3 Designs with FEC for $k = \left(\binom{m}{2} - 1\right)$

We define $S_2^{(-i)}$, $i = 1, \dots, \binom{m}{2}$ to be i runs deleted from S_2. For $k = \left(\binom{m}{2} - 1\right)$ we can delete at most two runs from D_1 and still have FEC. We see $\boldsymbol{Z}_{\overline{\mathbf{12}}}$ in (6.2) has P_1 but not P_2. The designs with FEC after one or two runs are deleted is given in the following table:

Table 6.1: Designs obtained from D_1 by deleting runs with FEC for $k = \left(\binom{m}{2} - 1\right)$

Designs With Full Estimation Capacity ($m \geq 4$)		
k	Number of runs deleted	Run(s) Deleted
$\left(\binom{m}{2} - 1\right)$	1	Any run
$\left(\binom{m}{2} - 1\right)$	2	S_0, $S_1^{(-1)}$
$\left(\binom{m}{2} - 1\right)$	2	S_0, $S_2^{(-1)}$
$\left(\binom{m}{2} - 1\right)$	2	S_0 and S_m, $m \geq 3$

6.4 Designs with FEC for $k = \left(\binom{m}{2} - 2\right)$

For $k = \left(\binom{m}{2} - 2\right)$ we cannot delete more than three runs from D_1 and still have FEC. We see $\boldsymbol{Z}_{\overline{\mathbf{12,13}}}$ in (6.3) and $\boldsymbol{Z}_{\overline{\mathbf{12,13}}}$ in (6.4) has P_1 but not P_2. We cannot delete three runs and have FEC. The designs with FEC after one or two are deleted is given in the following table:

Table 6.2: Designs obtained from D_1 by deleting runs with FEC for $k = \left(\binom{m}{2} - 2\right)$

Designs With Full Estimation Capacity ($m \geq 4$)		
k	Number of runs deleted	Run(s) Deleted
$\left(\binom{m}{2} - 2\right)$	1	Any run
$\left(\binom{m}{2} - 2\right)$	2	S_0, $S_1^{(-1)}$
$\left(\binom{m}{2} - 2\right)$	2	S_0, $S_2^{(-1)}$
$\left(\binom{m}{2} - 2\right)$	2	S_0 and S_m, $m \geq 3$

Chapter 7

General Structure *M* for Information Matrices

7.0 Summary

In this chapter, we present a general structure M for the information matrices of a class of models possibly describing the data from a fractional factorial experiment with m factors each at two levels and n runs, where $n < 2^m$. When the information matrices M have the full rank for all the models in the class, we have the full estimation capacity for the fractional factorial designs [Srivastava(1977)]. We are characterizing all the eigenvalues for such matrices $\boldsymbol{M}$. We use these eigenvalues for establishing the optimality properties of the fractional factorial designs [Ghosh and Tian(2006)]. We establish the robustness property of the design $D_7 = \{S_0, S_1, S_3\}$ [Ghosh(1977)] in the sense that the full estimation capacity of the designs remain when we delete any t runs as well as specific t runs from the design D_7.

7.1 Matrix $\boldsymbol{M}$ and Spectrum

We consider design D_7 and models in (5.1). Design D_7 has maximum estimation capacity for $k = 3$ in the sense that we can find the least squares estimate of the factorial effect parameters in all $\binom{6}{3}$ models. The information matrices for the $\binom{6}{3}$ models can be represented by matrix $\boldsymbol{M}$.

We consider the following general structure $\boldsymbol{M}$ whose elements are integers and p_1, p_2, $\dots p_r$ are nonnegative integers:

$$\boldsymbol{M} = \left(\begin{array}{c|c|c|c} (a-a_1)I_{p_1} + a_1 J_{p_1} & b_2^{(1)} J_{(p_1 \times p_2)} & \dots & b_r^{(1)} J_{(p_1 \times p_r)} \\ \hline b_2^{(1)} J_{(p_2 \times p_1)} & (a-a_2)I_{p_2} + a_2 J_{p_2} & \dots & b_r^{(2)} J_{(p_2 \times p_r)} \\ \hline \ddots & \ddots & \ddots & \ddots \\ \hline b_r^{(1)} J_{(p_r \times p_1)} & b_r^{(2)} J_{(p_r \times p_2)} & \dots & (a-a_r)I_{p_r} + a_r J_{p_r} \end{array}\right). \quad (7.1)$$

The dimensions of matrix M are $\big((p_1+...+p_r)\times(p_1+...+p_r)\big)$. We obtain from (7.1) r distinct eigenvalues. The eigenvalues and their multiplicities are given in the following table:

Table 7.1: The r eigenvalues of $\boldsymbol{M}$ and Multiplicity

Eigenvalues	Multiplicity
$\lambda_1 = (a - a_1)$	$(p_1 - 1)$
$\lambda_2 = (a - a_2)$	$(p_2 - 1)$
$\vdots$	$\vdots$
$\lambda_r = (a - a_r)$	$(p_r - 1)$

To find the remaining r eigenvalues we consider the following equation:

$$\boldsymbol{MU} = \lambda \boldsymbol{U}, \tag{7.2}$$

where $\boldsymbol{U}' = [\,\overbrace{1,\ldots,1}^{p_1}, \overbrace{u_2,\ldots,u_2}^{p_2}, \ldots, \overbrace{u_r,\ldots,u_r}^{p_r}\,]$.

Equivalently we have the following r equations:

$$\begin{aligned}
a + (p_1-1)a_1 + p_2 b_2^{(1)} u_2 + p_3 b_3^{(1)} u_3 + \ldots + p_r b_r^{(1)} u_r &= \lambda, \\
p_1 b_2^{(1)} + (a + (p_2-1)a_2) u_2 + p_3 b_3^{(2)} u_3 + \ldots + p_r b_r^{(2)} u_r &= \lambda u_2, \\
p_1 b_3^{(1)} + p_2 b_3^{(2)} u_2 + (a + (p_3-1)a_3) u_3 + \ldots + p_r b_r^{(3)} u_r &= \lambda u_3, \\
\vdots \qquad\qquad & \qquad \vdots \\
p_1 b_r^{(1)} + p_2 b_r^{(2)} u_2 + p_3 b_r^{(3)} u_3 + \ldots + (a + (p_r-1)a_r) u_r &= \lambda u_r.
\end{aligned} \tag{7.3}$$

Consequently we have

$$\begin{aligned}
\frac{p_1 b_2^{(1)} + (a + (p_2-1)a_2) u_2 + p_3 b_3^{(2)} u_3 + \ldots + p_r b_r^{(2)} u_r}{a + (p_1-1)a_1 + p_2 b_2^{(1)} u_2 + p_3 b_3^{(1)} u_3 + \ldots + p_r b_r^{(1)} u_r} &= u_2, \\
\frac{p_1 b_3^{(1)} + p_2 b_3^{(2)} u_2 + (a + (p_3-1)a_3) u_3 + \ldots + p_r b_r^{(3)} u_r}{a + (p_1-1)a_1 + p_2 b_2^{(1)} u_2 + p_3 b_3^{(1)} u_3 + \ldots + p_r b_r^{(1)} u_r} &= u_3, \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\
\frac{p_1 b_r^{(1)} + p_2 b_r^{(2)} u_2 + p_3 b_r^{(3)} u_3 + \ldots + (a + (p_r-1)a_r) u_r}{a + (p_1-1)a_1 + p_2 b_2^{(1)} u_2 + p_3 b_3^{(1)} u_3 + \ldots + p_r b_r^{(1)} u_r} &= u_r.
\end{aligned} \tag{7.4}$$

We can solve equations (7.4) for $[u_2, u_3, \ldots, u_r]$.

Example 7.1

We now consider the special case where $p_3 = \ldots = p_r = 0$. We have

$$M = \left(\begin{array}{ccccc|ccccc} a & b & b & \ldots & b & c & c & c & \ldots & c \\ b & a & b & \ldots & b & c & c & c & \ldots & c \\ & \ddots & \vdots & & & & \ddots & \vdots & & \\ b & b & \ldots & b & a & c & c & c & \ldots & c \\ \hline c & c & c & \ldots & c & a & d & d & \ldots & d \\ c & c & c & \ldots & c & d & a & d & \ldots & d \\ & \ddots & \vdots & & & & \ddots & \vdots & & \\ c & c & c & \ldots & c & d & d & d & \ldots & a \end{array}\right), \tag{7.5}$$

where $b = a_1$, $c = b_2^{(1)}$, and $d = a_2$.

From (7.5) we obtain the following eigenvalues:

$\lambda_1 = (a - b)$ multiplicity $(p_1 - 1)$,

$\lambda_2 = (a - d)$ multiplicity $(p_2 - 1)$.

From (7.2) we have

$$a + (p_1 - 1)b + p_2 c u_2 = \lambda, \tag{7.6}$$

and

$$p_1 c + \big(a + (p_2 - 1)d\big)u_2 = \lambda u_2. \tag{7.7}$$

Thus

$$\frac{p_1 c + \big(a + (p_2 - 1)d\big)u_2}{a + (p_1 - 1)b + p_2 c u_2} = u_2, \tag{7.8}$$

and

$$p_2cu_2^2 + \big((p_1-1)b-(p_2-1)d\big)u_2 - p_1c = 0. \tag{7.9}$$

We have

$$u_2 = \frac{-(p_1-1)b+(p_2-1)d \pm \sqrt{\big((p_1-1)b-(p_2-1)d\big)^2 + 4p_1p_2c}}{2p_2c}. \tag{7.10}$$

Hence

$$u_2^{(1)} = \frac{\alpha}{p_2}, \tag{7.11}$$

$$u_2^{(2)} = -\frac{p_1}{\alpha}, \tag{7.12}$$

and

$$\lambda_3 = a + (p_1-1)b + p_2cu_2^{(1)} = (a-b) + p_1b + \alpha c, \tag{7.13}$$

$$\lambda_4 = a + (p_1-1)b + p_2cu_2^{(2)} = (a-b) + p_1b - \frac{p_1p_2c}{\alpha}, \tag{7.14}$$

where $\alpha = \frac{-(p_1-1)b+(p_2-1)d\pm\sqrt{\big((p_1-1)b-(p_2-1)d\big)^2+4p_1p_2c}}{2c}$.

The eigenvectors correspond to λ_3 and λ_4 are given respectively below:

$$\boldsymbol{U_3'} = \Big[\,\overbrace{1,\ldots,1}^{p_1},\overbrace{\frac{\alpha}{p_2},\ldots,\frac{\alpha}{p_2}}^{p_2}\,\Big], \tag{7.15}$$

$$U'_{\mathbf{4}} = \Big[\, \overbrace{1, \ldots, 1}^{p_1}, \overbrace{-\frac{p_1}{\alpha}, \ldots, -\frac{p_1}{\alpha}}^{p_2} \,\Big]. \tag{7.16}$$

Example 7.2

We consider the following design $D_{7.1}$ derived from D_7 by deleting the following two runs $(-1, 1, 1, 1)$ and $(1, -1, 1, 1)$. We consider the models in (5.1) with any one of the following two-factor interactions $\{AC, AD, BC, BD\}$.

$$D_{7.1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} \tag{7.17}$$

The $\boldsymbol{X'X}$ matrix can be represented by the $\boldsymbol{M}$ matrix in (7.1) after some row and column permutations. We have $p_1 = 1$, $p_2 = 2$, $p_3 = 3$ and the following equation:

$$\left(\begin{array}{c|cc|ccc} 7 & 3 & 3 & 1 & 1 & 1 \\ \hline 3 & 7 & -1 & 1 & 1 & 1 \\ 3 & -1 & 7 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 7 & -1 & -1 \\ 1 & 1 & 1 & -1 & 7 & -1 \\ 1 & 1 & 1 & -1 & -1 & 7 \end{array}\right) \left(\begin{array}{c} 1 \\ \hline u_2 \\ u_2 \\ \hline u_3 \\ u_3 \\ u_3 \end{array}\right) = \lambda \left(\begin{array}{c} 1 \\ \hline u_2 \\ u_2 \\ \hline u_3 \\ u_3 \\ u_3 \end{array}\right). \tag{7.18}$$

Thus we have

$$u_2 = \frac{3 + 6u_2 + 3u_3}{7 + 6u_2 + 3u_3} \tag{7.19}$$

and

$$u_3 = \frac{1 + 2u_2 + 5u_3}{7 + 6u_2 + 3u_3}. \tag{7.20}$$

From (7.19) and (7.20) we obtain

$$6u_2^2 + u_2 - 3 = 3u_3(1 - u_2) \tag{7.21}$$

and

$$3u_3^2 + 2u_3 - 1 = -(1 + u_3)(1 - 3u_3) = 2u_2(1 - 3u_3). \tag{7.22}$$

From (7.21) and (7.22) we have

If $1 - 3u_3 = 0$ and $1 + u_2 \neq 0$ then $u_2 = 2/3$ and $u_3 = 1/3$,

If $1 - 3u_3 = 0$ and $1 + u_2 = 0$ then $u_2 = -1$ and $u_3 = 1/3$,

If $1 - 3u_3 \neq 0$ then $u_2 = 0$ and $u_3 = -1$.

The eigenvalues and eigenvectors of matrix $\boldsymbol{M}$ are given below:

$$\begin{aligned}
\lambda_1 = 8, &\quad \boldsymbol{U_1'} = \left[0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0, 0, 0\right], \\
\lambda_2 = 8, &\quad \boldsymbol{U_2'} = \left[0, 0, 0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right], \\
\lambda_3 = 8, &\quad \boldsymbol{U_3'} = \left[0, 0, 0, 0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right], \\
\lambda_4 = 12, &\quad \boldsymbol{U_4'} = \left[1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right], \\
\lambda_5 = 2, &\quad \boldsymbol{U_5'} = \left[1, -1, -1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right], \\
\lambda_6 = 4, &\quad \boldsymbol{U_6'} = \left[1, 0, 0, -1, -1, -1\right].
\end{aligned}$$

Example 7.3

We consider the following design $D_{7.2}$ that is not derived from D_7. We consider the model in (5.1) with the two-factor interactions $\{CD\}$.

$$D_{7.2} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \tag{7.23}$$

The $\boldsymbol{X'X}$ matrix can be represented by the $\boldsymbol{M}$ matrix in (7.1) after some row and column permutations. We have $p_1 = 3$, $p_2 = 2$, $p_3 = 1$ and the following equation:

$$\left(\begin{array}{ccc|cc|c} 9 & 1 & 1 & -1 & -1 & -3 \\ 1 & 9 & 1 & -1 & -1 & -3 \\ 1 & 1 & 9 & -1 & -1 & -3 \\ \hline -1 & -1 & -1 & 9 & -3 & -1 \\ -1 & -1 & -1 & -3 & 9 & -1 \\ \hline -3 & -3 & -3 & -1 & -1 & 9 \end{array}\right)\left(\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline u_2 \\ u_2 \\ \hline u_3 \end{array}\right) = \lambda \left(\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline u_2 \\ u_2 \\ \hline u_3 \end{array}\right). \tag{7.24}$$

Thus we have

$$u_2 = \frac{-3 + 6u_2 - u_3}{11 - 2u_2 - 3u_3} \tag{7.25}$$

and

$$u_3 = \frac{-9 - 2u_2 + 9u_3}{11 - 2u_2 - 3u_3}. \tag{7.26}$$

From (7.25) and (7.26) we obtain

$$\frac{-8(u_2 + 3)(4u_2^2 - 5u_2 - 1)}{(-1 + 3u_2)^2} = 0. \tag{7.27}$$

From (7.27) we have

$$u_2 = -3 \text{ and } u_2 = \tfrac{5}{8} \pm \tfrac{1}{8}\sqrt{41}.$$

The eigenvalues and eigenvectors of matrix $\boldsymbol{M}$ are given below:

$$\begin{aligned}
\lambda_1 &= 8, & \boldsymbol{U_1'} &= \left[\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0, 0, 0, 0\right],\\
\lambda_2 &= 8, & \boldsymbol{U_2'} &= \left[\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}, 0, 0, 0\right],\\
\lambda_3 &= 12, & \boldsymbol{U_3'} &= \left[0, 0, 0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right],\\
\lambda_4 &= 8, & \boldsymbol{U_4'} &= \left[1, 1, 1, -3, -3, 3\right],\\
\lambda_5 &= 9-\sqrt{41}, & \boldsymbol{U_5'} &= \left[1, 1, 1, \left(\tfrac{5}{8}+\tfrac{1}{8}\sqrt{41}\right), \left(\tfrac{5}{8}+\tfrac{1}{8}\sqrt{41}\right), \left(\tfrac{1}{4}+\tfrac{1}{4}\sqrt{41}\right)\right],\\
\lambda_6 &= 9+\sqrt{41}, & \boldsymbol{U_6'} &= \left[1, 1, 1, \left(\tfrac{5}{8}-\tfrac{1}{8}\sqrt{41}\right), \left(\tfrac{5}{8}-\tfrac{1}{8}\sqrt{41}\right), \left(\tfrac{1}{4}-\tfrac{1}{4}\sqrt{41}\right)\right].
\end{aligned}$$

7.2 Estimation Capacity and Robustness of D_7

Design D_7 has maximum estimation capacity for $k = 3$. Furthermore we can delete runs from D_7 and some resultant designs have full estimation capacity for $k = 1$ and $k = 2$. The results are given in the following table.

Table 7.2: Designs obtained from D_7 by deleting runs with FEC for $k = 1$ and 2

Designs With Full Estimation Capacity		
k	Number of runs deleted	Run(s) Deleted
1	1	Any one run
1	2	$S_1^{(-2)}$
1	2	$S_1^{(-1)}$ and its foldover in S_3
1	2	S_0, $S_1^{(-1)}$
1	2	S_0, $S_3^{(-1)}$
1	3	$S_1^{(-3)}$
2	1	$S_1^{(-1)}$
2	1	$S_3^{(-1)}$

7.3 Information Matrices and Spectrum

All the information matrices associated with design D_7 and designs given in Table 7.2 can be obtained from matrix $\boldsymbol{M}$ in (7.1). The values of $p_i's$, the elements of the $X'X$ matrices and the eigenvalues are given in Table 7.3, Table 7.4 and Table 7.5. We denote models(Ms) in Table 7.3. When there is no factor in common between the two-factor interactions in the models we denote these models as disjoint pair(DP) in the tables. We denote others(O) to be other than disjoint pairs in Table 7.3. In Table 7.4 and Table 7.5 we denote runs deleted to be RD. We denote runs $S_0 = (1,1,1,1)$, $S_1(1) = (-1,1,1,1)$, $S_1(2) = (1,-1,1,1)$, $S_1(3) = (1,1,-1,1)$, $S_3(1) = (-1,-1,-1,1)$ and $S_3(4) = (1,-1,-1,-1)$. We denote the elements of $\boldsymbol{M}$ as follow: $b = a_{(1)}$, $c = b_2^{(1)}$, $e = b_3^{(1)}$, $h = b_4^{(1)}$, $d = a_2$, $f = b_3^{(2)}$, $i = b_4^{(2)}$, $g = a_3$ and $j = b_4^{(3)}$ in Table 7.4.

Table 7.3: Information matrices and eigenvalues from D_7

D_7															
k	Ms	p_1	p_2	a	b	c	d	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
1	all	6	0	9	1	NA	NA	8	8	8	8	8	14	NA	NA
2	DP	5	2	9	1	1	-7	8	8	8	8	16	1.16	13.84	NA
2	O	7	0	9	1	NA	NA	8	8	8	8	8	8	15	NA
3	DP	6	2	9	1	1	-7	8	8	8	8	8	16	1.07	14.93
3	O	8	0	9	1	NA	NA	8	8	8	8	8	8	8	16

Table 7.4: Information matrices from design obtained from D_7

Designs Obtained from D_7												
RD	k	models	p_1	p_2	p_3	p_4	a	b	c	d	e	f	g
S_0	1	all	6	0	0	0	8	0	NA	NA	NA	NA	NA
$S_1(1)$	1	(AB),(AC),	2	4	0	0	8	0	2	0	NA	NA	NA
$S_1(1)$	1	(BC),(BD),(CD)	1	5	0	0	8	NA	2	0	NA	NA	NA
$S_3(1)$	1	(AB),(AC),(BC)	3	3	0	0	8	0	2	0	NA	NA	NA
$S_3(1)$	1	(AD),(BD),(CD)	2	4	0	0	8	0	2	0	NA	NA	NA
$S_1(1)$ $S_1(2)$	1	(CD)	2	4	0	0	7	3	1	-1	NA	NA	NA
$S_1(1)$ $S_1(2)$	1	(AB)	2	3	1	0	7	3	1	-1	1	3	NA
$S_1(1)$ $S_1(2)$	1	(AC),(AD) (BC),(BD)	1	2	3	0	7	NA	3	-1	1	1	-1
$S_1(1)$ $S_3(4)$	1	all	1	3	2	0	7	NA	3	-1	1	1	3
S_0 $S_1(1)$	1	all	1	5	0	0	7	NA	1	-1	NA	NA	NA
S_0 $S_3(1)$	1	all	3	3	0	0	7	-1	1	-1	NA	NA	NA

Designs Obtained from D_7 (Table 7.4 con't)													
RD	k	models	p_1	p_2	p_3	p_4	a	b	c	d	e	f	g
$S_1(1)$ $S_1(2)$ $S_1(3)$	1	(AD) (BD) (CD)	2	2	2	0	6	-2	2	2	0	0	-2
$S_1(1)^*$ $S_1(2)$ $S_1(3)$	1	(AB) (AC) (BC)	2	2	1	1	6	2	0	-2	2	0	NA
$S_1(1)$	2	(AB,AC) (AB,AD) (AC,AD)	4	3	0	0	8	0	2	0	NA	NA	NA
$S_1(1)$	2	(AB,BC) (AB,BD) (AC,BD) (AC,CD) (AD,BD) (AD,CD)	2	5	0	0	8	0	2	0	NA	NA	NA
$S_1(1)$	2	(BC,BD) (BC,CD) (BD,CD)	1	6	0	0	8	NA	2	0	NA	NA	NA
$S_1(1)^{**}$	2	DP	4	1	1	1	8	0	2	NA	2	0	NA

For $(*)$, $h = 0$, $i = 2$ and $j = 4$.

For $(**)$, $h = 0$, $i = 2$ and $j = -6$.

Designs Obtained from D_7 (Table 7.4 con't)													
RD	k	models	p_1	p_2	p_3	p_4	a	b	c	d	e	f	g
$S_3(1)$	2	(AD,BD) (AD,CD) (BD,CD)	5	2	0	0	8	0	2	0	NA	NA	NA
$S_3(1)$	2	all others	3	4	0	0	8	0	2	0	NA	NA	NA

Table 7.5: Eigenvalues for information matrices in Table 7.4

Eigenvalues for Information Matrices in Table 7.4												
RD	k	models	p_1	p_2	p_3	p_4	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
S_0	1	all	6	0	0	0	8	8	8	8	8	8
$S_1(1)$	1	(AB),(AC),(AD)	2	4	0	0	8	8	8	8	2.34	13.66
$S_1(1)$	1	(BC),(BD),(CD)	1	5	0	0	8	8	8	8	3.53	12.47
$S_3(1)$	1	(AB),(AC),(BC)	3	3	0	0	8	8	8	8	2	14
$S_3(1)$	1	(AD),(BD),(CD)	2	4	0	0	8	8	8	8	2.34	13.66
$S_1(1)$ $S_1(2)$	1	(CD)	2	4	0	0	8	8	8	4	2.88	11.12
$S_1(1)$ $S_1(2)$	1	(AB)	2	3	1	0	8	8	8	4	0.60	13.40
$S_1(1)$ $S_1(2)$	1	(AC),(AD) (BC),(BD)	1	2	3	0	8	8	8	4	2	12
$S_1(1)$ $S_3(4)$	1	all	1	3	2	0	8	8	8	4	0.60	13.40
S_0 $S_1(1)$	1	all	1	5	0	0	8	8	8	8	2	8
S_0 $S_3(1)$	1	all	3	3	0	0	8	8	8	8	2	8

Eigenvalues for Information Matrices (Table 7.5 con't)													
RD	k	models	p_1	p_2	p_3	p_4	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$S_1(1)$ $S_1(2)$ $S_1(3)$	1	(AD) (BD) (CD)	2	2	2	0	8	8	4	4	1.53	10.47	NA
$S_1(1)$ $S_1(2)$ $S_1(3)$	1	(AB) (AC) (BC)	2	2	1	1	8	8	4	4	0.34	11.66	NA
$S_1(1)$	2	(AB, AC) (AB,AD) (AC,AD)	4	3	0	0	8	8	8	8	8	1.07	14.93
$S_1(1)$	2	(AB,BC) (AB,BD) (AC,BD) (AC,CD) (AD,BD) (AD,CD)	2	5	0	0	8	8	8	8	8	1.68	14.33
$S_1(1)$	2	(BC,BD) (BC,CD) (BD,CD)	1	6	0	0	8	8	8	8	8	3.10	12.90
$S_1(1)$	2	DP	4	1	1	1	8	8	8	0.76	3.58	12.42	15.25

Eigenvalues for Information Matrices (Table 7.5 con't)														
RD	k	models	p_1	p_2	p_3	p_4	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$S_3(1)$	2	(AD,BD) (AD,CD) (BD,CD)	5	2	0	0	8	8	8	8	8	1.68	14.33
$S_3(1)$	2	all others	3	4	0	0	8	8	8	8	8	1.07	14.93

Chapter 8

A Class of Optimum Designs for $k = 1$

8.0 Summary

In this chapter we consider the models in (5.1) for $k = 1$. We denote $\boldsymbol{X_{2i}} = \left[\boldsymbol{j}, \boldsymbol{X_2^{(i)}}\right]$ for $i = 1, \ldots, \binom{m}{2}$. We write the design matrix $\boldsymbol{X^{(i)}} = \left[\boldsymbol{X_1}, \boldsymbol{X_{2i}}\right]$. We show for all designs with general m, n and $k = 1$ the matrix $\boldsymbol{X'_{2i}X_{2i}}$ is a principal sub-matrix of $\boldsymbol{X'_1X_1}$. When the $\boldsymbol{X'_{2i}X_{2i}}$ matrix is a principle sub-matricx of $\boldsymbol{X'_1X_1}$ we proved the minimum eigenvalue of $\boldsymbol{X'_1X_1}$ is less than or equal to minimum eigenvalue of $\boldsymbol{X'_{2i}X_{2i}}$. For foldover designs with $k = 1$ we proved the minimum eigenvalue of $\boldsymbol{X^{(i)\prime}X^{(i)}}$ is equal to the minimum eigenvalue of $\boldsymbol{X'_1X_1}$. We consider the class of designs $\mathcal{D}_{\mathfrak{T}}$ with n runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. We prove that if $T^* \in \mathcal{D}_{\mathfrak{T}}$ is E-optimum, then $\binom{T^*}{-T^*}$ is optimum design with respect to $AMCR$ and $GMCR$ in $\mathcal{F}_{\mathfrak{T}}$. Furthermore, if T^* is D- and A- optimum with a special structure for $\boldsymbol{X'_{1T^*}X_{1T^*}}$ we prove $\binom{T^*}{-T^*}$ is optimum with respect to GD, AD, GT, and AT in $\mathcal{F}_{\mathfrak{T}}$.

8.1 Sub-Matrices of $X_1'X_1$ and Eigenvalues

In this section we investigate the relationship between $\boldsymbol{X_1'X_1}$ and $\boldsymbol{X_{2i}'X_{2i}}$ and their minimum eigenvalues. For all designs, whether foldover or not, we consider general m, n and $k = 1$. For $i = 1, ..., \binom{m}{2}$, we can partition $\boldsymbol{X^{(i)\prime}X^{(i)}}$ in this manner

$$\boldsymbol{X^{(i)\prime}X^{(i)}} = \left(\begin{array}{c|c} \boldsymbol{X_1'X_1} & \boldsymbol{X_1'X_{2i}} \\ \hline \boldsymbol{X_{2i}'X_1} & \boldsymbol{X_{2i}'X_{2i}} \end{array}\right),$$

where the rows and columns of $\boldsymbol{X_1}$ represent the main effects, the rows and columns of $\boldsymbol{X_{2i}}$ represent the general mean and the two factor interaction of the i^{th} model. We consider $k = 1$ with the interaction of the i^{th} and j^{th} factor in the model, then the matrices $\boldsymbol{X_1'X_1}$ and $\boldsymbol{X_{2i}'X_{2i}}$ are

$$\boldsymbol{X_1'X_1} = \begin{pmatrix} n & \boldsymbol{x_1'x_2} & \dots & \boldsymbol{x_1'x_m} \\ \boldsymbol{x_2'x_1} & n & \dots & \boldsymbol{x_2'x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{x_m'x_1} & \dots & \boldsymbol{x_m'x_{m-1}} & n \end{pmatrix},$$

and

$$\boldsymbol{X_{2i}'X_{2i}} = \begin{pmatrix} n & \boldsymbol{x_i'x_j} \\ \boldsymbol{x_j'x_i} & n \end{pmatrix}.$$

We can obtain $\boldsymbol{X_{2i}'X_{2i}}$ from $\boldsymbol{X_1'X_1}$ by extracting the i^{th} and j^{th} entries of the i^{th} row and j^{th} and i^{th} entries of the j^{th} row. Thus the matrix $\boldsymbol{X_{2i}'X_{2i}}$ is a principal sub-matrix of $\boldsymbol{X_1'X_1}$. We proved the following two lemmas in view of the above observations.

Lemma 1 *The minimum eigenvalue of* $\boldsymbol{X_1'X_1}$ *is always less than or equal to the minimum eigenvalue of* $\boldsymbol{X_{2i}'X_{2i}}$*, for* $i = 1, ..., \binom{m}{2}$.

Proof:

For all $i = 1, ..., \binom{m}{2}$, $\boldsymbol{X'_{2i}X_{2i}}$ is a principal sub-matrix of $\boldsymbol{X'_1X_1}$. Let $\underline{\boldsymbol{b}}' = (\underline{\boldsymbol{b}}'_1, \underline{\boldsymbol{b}}'_2)$ and $\underline{\boldsymbol{b}}^{*'} = (\boldsymbol{0}, \underline{\boldsymbol{b}}'_2)$, since $\boldsymbol{X'_{2i}X_{2i}}$ is a sub-matrix of $\boldsymbol{X'_1X_1}$, then

$$\begin{aligned}\lambda_{min}(\boldsymbol{X'_1X_1}) = \min_{\underline{\boldsymbol{b}}} \frac{\underline{\boldsymbol{b}}'(\boldsymbol{X'_1X_1})\underline{\boldsymbol{b}}}{\underline{\boldsymbol{b}}'\underline{\boldsymbol{b}}} \leq \min_{\underline{\boldsymbol{b}}^*} \frac{\underline{\boldsymbol{b}}^{*'}(\boldsymbol{X'_1X_1})\underline{\boldsymbol{b}}^*}{\underline{\boldsymbol{b}}^{*'}\underline{\boldsymbol{b}}^*} &= \min_{\underline{\boldsymbol{b}}_2} \frac{\underline{\boldsymbol{b}}'_2(\boldsymbol{X'_2X_2})\underline{\boldsymbol{b}}_2}{\underline{\boldsymbol{b}}'_2\underline{\boldsymbol{b}}_2} \\ &= \lambda_{min}(\boldsymbol{X'_{2i}X_{2i}}), \qquad (8.1)\end{aligned}$$

where λ_{min} denote the minimum eigenvalue of a matrix. ■

Lemma 2 *For all foldover designs with general m, n (even), and k = 1, with i = 1, ..., $\binom{m}{2}$ the minimum eigenvalue of $\boldsymbol{X^{(i)'}X^{(i)}}$ is equal to the minimum eigenvalue of $\boldsymbol{X'_1X_1}$.*

Proof:

For $i = 1, ..., \binom{m}{2}$ and for any foldover design the structure of the matrices $\boldsymbol{X^{(i)'}X^{(i)}}$ are block diagonal with 2 blocks, the structure is presented below

$$\boldsymbol{X^{(i)'}X^{(i)}} = \left(\begin{array}{c|c} \boldsymbol{X'_1X_1} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{X'_{2i}X_{2i}} \end{array}\right).$$

Since $\boldsymbol{X^{(i)'}X^{(i)}}$ are block diagonal matrices then the spectrum of $\boldsymbol{X^{(i)'}X^{(i)}}$ is the comprise of the spectra of $\boldsymbol{X'_1X_1}$ and $\boldsymbol{X'_{2i}X_{2i}}$. Since we have $\lambda_{min}(\boldsymbol{X'_1X_1}) \leq \lambda_{min}(\boldsymbol{X'_{2i}X_{2i}})$ for all i, from lemma 1 this implies $\lambda_{min}(\boldsymbol{X^{(i)'}X^{(i)}}) = \lambda_{min}(\boldsymbol{X'_1X_1})$. ■

The minimum eigenvalue of $\boldsymbol{X^{(i)'}X^{(i)}}$ is independent of i. Consequently, for all foldover designs, we do not have to consider $\boldsymbol{X_{2i}}$ to find the minimum eigenvalue of $\boldsymbol{X^{(i)'}X^{(i)}}$. We can find it from $\boldsymbol{X_1}$. This is a major advantage because the $\boldsymbol{X^{(i)}}$ matrices for all $\binom{m}{2}$ models have $\boldsymbol{X_1}$ sub-matrix in common and $\boldsymbol{X_{2i}}$ sub-matrices will not be the same for most of the cases.

Example 8.1

Let us consider the following design $D_{8.1}$ with $k = 1$ two-factor interaction,

$$D_{8.1} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

Design $D_{8.1}$ is not a foldover design. There are two classes of models. Within each class the matrices $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$ are identical. There are three models in each class. The matrix $\boldsymbol{X}_1'\boldsymbol{X}_1$ is

$$\boldsymbol{X}_1'\boldsymbol{X}_1 = \begin{pmatrix} 6 & 2 & 0 & -2 \\ 2 & 6 & 0 & -2 \\ 0 & 0 & 6 & 0 \\ -2 & -2 & 0 & 6 \end{pmatrix},$$

and the matrices $\boldsymbol{X}_{2i}'\boldsymbol{X}_{2i}$ are

$$\boldsymbol{W}_1 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix},$$

$$\boldsymbol{W}_2 = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{X}_1'\boldsymbol{X}_1$ are $\{4, 4, 6, 10\}$, the eigenvalues of $\boldsymbol{W}_1$ are $\{4, 8\}$ and the eigenvalues of $\boldsymbol{W}_2$ are $\{6, 6\}$. The minimum eigenvalue of $\boldsymbol{X}_1'\boldsymbol{X}_1$ is less than or equal to the minimum eigenvalues of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$.

Example 8.2

Let us consider the following design $D_{8.2}$ with $k = 1$ two-factor interaction,

$$D_{8.2} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

Design $D_{8.2}$ is a foldover design and there are two classes of models. Within each class the matrices $\boldsymbol{X^{(i)\prime}X^{(i)}}$ are identical. The $\boldsymbol{X_1'X_1}$ matrix is

$$\boldsymbol{X_1'X_1} = \begin{pmatrix} 10 & -2 & -2 & 2 \\ -2 & 10 & 2 & -2 \\ -2 & 2 & 10 & -2 \\ 2 & -2 & -2 & 10 \end{pmatrix},$$

and the $\boldsymbol{X_{2i}'X_{2i}}$ matrices are

$$\boldsymbol{W_1} = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix},$$

$$\boldsymbol{W_2} = \begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{X}_1'\boldsymbol{X}_1$ are $\{8, 8, 8, 16\}$, the eigenvalues of $\boldsymbol{W}_1$ are $\{8, 12\}$ and the eigenvalues of $\boldsymbol{W}_2$ are $\{8, 12\}$. The minimum eigenvalue of $\boldsymbol{X}_1'\boldsymbol{X}_1$ is less than or equal to the minimum eigenvalue of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. Moreover the minimum eigenvalue of $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$ is equal to the minimum eigenvalue of $\boldsymbol{X}_1'\boldsymbol{X}_1$ as asserted by lemma 2.

8.2 E-, $AMCR$ and $GMCR$ Optimum Designs

In this section we consider a design T^* that is E-optimum and investigate the optimality of the foldover design $\binom{T^*}{-T^*}$ with respect to $AMCR$ and $GMCR$. The result is given in the following theorem.

Theorem 1 *Let $\mathcal{D}_{\mathfrak{T}}$ be the class of designs with n runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. If T^* is E-optimum in $\mathcal{D}_{\mathfrak{T}}$, then $f_{T^*} = \binom{T^*}{-T^*}$ is optimum design with respect to $AMCR$ and $GMCR$ in $\mathcal{F}_{\mathfrak{T}}$.*

Proof:

Suppose T^* is E-optimum design in $\mathcal{D}_{\mathfrak{T}}$ and $f_T \in \mathcal{F}_{\mathfrak{T}}$ then for $i = 1, ..., \binom{m}{2}$

$$\boldsymbol{X}_{f_T}^{(i)\prime}\boldsymbol{X}_{f_T}^{(i)} = \left(\begin{array}{c|c} \boldsymbol{X}_{1f_T}'\boldsymbol{X}_{1f_T} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{X}_{2if_T}'\boldsymbol{X}_{2if_T} \end{array}\right), \tag{8.2}$$

and

$$(\boldsymbol{X}_{f_T}^{(i)\prime}\boldsymbol{X}_{f_T}^{(i)})^{-1} = \left(\begin{array}{c|c} (\boldsymbol{X}_{1f_T}'\boldsymbol{X}_{1f_T})^{-1} & \boldsymbol{0} \\ \hline \boldsymbol{0} & (\boldsymbol{X}_{2if_T}'\boldsymbol{X}_{2if_T})^{-1} \end{array}\right). \tag{8.3}$$

Since

$$\boldsymbol{X_{1f_T}} = \left(\frac{\boldsymbol{X_{1T}}}{\boldsymbol{-X_{1T}}}\right), \tag{8.4}$$

then

$$\boldsymbol{X'_{1f_T} X_{1f_T} = 2X'_{1T} X_{1T}}, \tag{8.5}$$

and

$$\boldsymbol{X_{2if_T}} = \left(\frac{\boldsymbol{X_{2iT}}}{\boldsymbol{X_{2iT}}}\right), \tag{8.6}$$

then

$$\boldsymbol{X'_{2if_T} X_{2if_T} = 2X'_{2iT} X_{2iT}}. \tag{8.7}$$

Since

$$\lambda_{min}(\boldsymbol{X'_{1f_T} X_{1f_T}}) \leq \lambda_{min}(\boldsymbol{X'_{2if_T} X_{2if_T}}), \tag{8.8}$$

$$\tag{8.9}$$

then

$$\lambda_{max}(\boldsymbol{X'_{1f_T} X_{1f_T}})^{-1} \geq \lambda_{max}(\boldsymbol{X'_{2if_T} X_{2if_T}})^{-1}. \tag{8.10}$$

We must choose T in $\mathcal{D}_{\mathfrak{T}}$ so that the maximum eigenvalue (MEV) of $\left(\boldsymbol{X}'_{\boldsymbol{1f_T}}\boldsymbol{X}_{\boldsymbol{1f_T}}\right)^{-1}$ is minimum. Since T^* is E- optimum in $\mathcal{D}_{\mathfrak{T}}$, then $f_{T^*} = \binom{T^*}{-T^*}$ gives minimum maximum eigenvalue of $\left(\boldsymbol{X}'_{\boldsymbol{1f_T}}\boldsymbol{X}_{\boldsymbol{1f_T}}\right)^{-1}$ in $\mathcal{F}_{\mathfrak{T}}$. This result is in fact model independent, meaning that it is true for all $\binom{m}{2}$ models. Hence $f_{T^*} = \binom{T^*}{-T^*}$ is optimum with respect to $AMCR$ and $GMCR$ in $\mathcal{F}_{\mathfrak{T}}$. ■

If we choose a main effects plan T^* that is E- optimum in $\mathcal{D}_{\mathfrak{T}}$, then $f_{T^*} = \binom{T^*}{-T^*}$ is optimum design with respect to $AMCR$ and $GMCR$ in $\mathcal{F}_{\mathfrak{T}}$. We illustrate with the following example.

Example 8.3

Let us consider the following designs

$$D_{8.3} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix},$$

and

$$D_{8.4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

The design $D_{8.3}$ is an $D-$, $A-$ and $E-$ optimal main effect plan while design $D_{8.4}$ is not optimal. Let $D_{8.3f} = \begin{pmatrix} D_{8.3} \\ -D_{8.3} \end{pmatrix}$ and $D_{8.4f} = \begin{pmatrix} D_{8.4} \\ -D_{8.4} \end{pmatrix}$ we determined $D_{8.3f}$ and $D_{8.4f}$ have FEC for $k = 1$. The values of the six criterion functions for the designs $D_{8.3f}$ and $D_{8.4f}$ is given in Table 8.1.

Table 8.1: Values of six criterion functions for $D_{8.3f}$ and $D_{8.4f}$

Values of Six Criterion Functions for $D_{8.3f}$ and $D_{8.4f}$						
Design	AD	AT	$AMCR$	GD	GT	$GMCR$
$D_{8.3f}$	1.272×10^{-6}	0.646	0.125	1.272×10^{-6}	0.646	0.125
$D_{8.4f}$	2.756×10^{-6}	0.913	0.427	2.721×10^{-6}	0.912	0.427

From Table 8.1, the values of $AMCR$ and $GMCR$ for design $D_{8.3f}$ is smaller that than of $D_{8.4f}$. Moreover the values of AD, AT , GD and GT for design $D_{8.3f}$ is smaller that than of $D_{8.4f}$. We will consider AD, AT , GD and GT in the sections 8.3 and 8.4.

8.3 D-, AD and GD Optimum Designs

In this section we consider T^* that is a D- optimum plan in $\mathcal{D}_{\mathfrak{T}}$. We investigate the optimality of $f_{T^*} = \binom{T^*}{-T^*}$ with respect to GD and AD for a special structures of $\boldsymbol{X'_{1T^*}X_{1T^*}}$ matrix. The result is given in the following theorem.

Theorem 2 *Let $\mathcal{D}_{\mathfrak{T}}$ be the class of designs with n(odd) runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. If T^* is D-optimum in $\mathcal{D}_{\mathfrak{T}}$ and the non-diagonal elements of $\boldsymbol{X'_{1T^*}X_{1T^*}}$ are all -1 or 1 then $f_{T^*} = \binom{T^*}{-T^*}$ is optimum design with respect to GD and AD in $\mathcal{F}_{\mathfrak{T}}$.*

Proof:

We prove the case for GD, the proof for AD is analogous. Suppose T^* is D-optimum in $\mathcal{D}_{\mathfrak{T}}$ and the non-diagonal elements of $\boldsymbol{X'_{1T^*}X_{1T^*}}$ are all -1 or 1. For any $T \in \mathcal{D}_{\mathfrak{T}}$, let the columns of $\boldsymbol{X_{1T}}$ represent the main effects, the columns of $\boldsymbol{X_{2iT}}$ represent the general mean and the two factor interactions of the i^{th} model, $i = 1, ..., \binom{m}{2}$. The matrix

$\boldsymbol{X}^{(i)}_{\boldsymbol{f_T}}$ for $f_T = \binom{T}{-T} \in \mathcal{F}_{\mathfrak{T}}$ is given as

$$\boldsymbol{X}^{(i)}_{\boldsymbol{f_T}} = \left(\begin{array}{c|c} \boldsymbol{X_{1T}} & \boldsymbol{X_{2iT}} \\ \hline -\boldsymbol{X_{1T}} & \boldsymbol{X_{2iT}} \end{array}\right). \tag{8.11}$$

Since f_T is a foldover design then $\boldsymbol{X}^{(i)\prime}_{\boldsymbol{f_T}}\boldsymbol{X}^{(i)}_{\boldsymbol{f_T}}$ are block diagonal matrices

$$\boldsymbol{X}^{(i)\prime}_{\boldsymbol{f_T}}\boldsymbol{X}^{(i)}_{\boldsymbol{f_T}} = \left(\begin{array}{c|c} \boldsymbol{2X^{\prime}_{1T}X_{1T}} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{2X^{\prime}_{2iT}X_{2iT}} \end{array}\right). \tag{8.12}$$

Since $\boldsymbol{X}^{(i)\prime}_{\boldsymbol{f_T}}\boldsymbol{X}^{(i)}_{\boldsymbol{f_T}}$ are block diagonal matrices then we can express GD as

$$GD = \left[[det(2\boldsymbol{X^{\prime}_{1T}X_{1T}})]^{\binom{m}{2}} \prod_{i=1}^{\binom{m}{2}} det(2\boldsymbol{X^{\prime}_{2iT}X_{2iT}})\right]^{-\frac{1}{\binom{m}{2}}}. \tag{8.13}$$

To minimizes GD it is suffice to choose $T \in \mathcal{D}_{\mathfrak{T}}$ so that we maximize

$$det(\boldsymbol{X^{\prime}_{1T}X_{1T}}) \prod_{i=1}^{\binom{m}{2}} det(\boldsymbol{X^{\prime}_{2iT}X_{2iT}}). \tag{8.14}$$

Since $T^* \in \mathcal{D}_{\mathfrak{T}}$ is D- optimum then $\forall T \in \mathcal{D}_{\mathfrak{T}}$

$$det(\boldsymbol{X^{\prime}_{1T^*}X_{1T^*}}) \geq det(\boldsymbol{X^{\prime}_{1T}X_{1T}}), \tag{8.15}$$

and given that $\boldsymbol{X^{\prime}_{2iT}X_{2iT}}$ are principal sub-matrices of $\boldsymbol{X^{\prime}_{1T}X_{1T}}$ with non-diagonal entries of $\boldsymbol{X^{\prime}_{1T^*}X_{1T^*}}$ are all 1 and -1 then

$$\prod_{i=1}^{\binom{m}{2}} det(\boldsymbol{X^{\prime}_{2iT^*}X_{2iT^*}}) = [n^2-1]^{\binom{m}{2}} \geq \prod_{i=1}^{\binom{m}{2}} det(\boldsymbol{X^{\prime}_{2iT}X_{2iT}}). \tag{8.16}$$

Thus we have

$$det(\boldsymbol{X'_{1T^*}X_{1T^*}})\prod_{i=1}^{\binom{m}{2}} det(\boldsymbol{X'_{2iT^*}X_{2iT^*}}) \geq det(\boldsymbol{X'_{1T}X_{1T}})\prod_{i=1}^{\binom{m}{2}} det(\boldsymbol{X'_{2iT}X_{2iT}}). \quad (8.17)$$

Consequently $T^* \in \mathcal{D}_{\mathfrak{T}}$ that maximize (8.14), is equivalent to $f_{T^*} = \binom{T^*}{-T^*} \in \mathcal{F}_{\mathfrak{T}}$ that minimize GD. Hence f_{T^*} is optimum with respect to GD in $\mathcal{F}_{\mathfrak{T}}$. ■

In example 8.1, the $\boldsymbol{X'_{1T}X_{1T}}$ matrix for design $D_{8.3}$ is

$$\boldsymbol{X'_{1T}X_{1T}} = \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 \end{pmatrix}.$$

Thus the values of AD and GD for design $D_{8.3f}$ is smaller that than of $D_{8.4f}$ as seen in Table 8.1.

8.4 A-, GT and AT Optimum designs

In this section we consider T^* that is an A- optimum plan in $\mathcal{D}_{\mathfrak{T}}$. We investigate the optimality of $f_{T^*} = \binom{T^*}{-T^*}$ with respect to GT and AT for a special structures of $\boldsymbol{X'_{1T^*}X_{1T^*}}$ matrix. The result is given in the following theorem.

Theorem 3 *Let $\mathcal{D}_{\mathfrak{T}}$ be the class of designs with n(odd) runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. If T^* is A-optimum in $\mathcal{D}_{\mathfrak{T}}$ and the non-diagonal elements of $\boldsymbol{X'_{1T^*}X_{1T^*}}$ are all -1 or 1 then $f_{T^*} = \binom{T^*}{-T^*}$ is optimum design with respect to GT and AT in $\mathcal{F}_{\mathfrak{T}}$.*

Proof:

We will prove the case of GT the proof for AT is similar. Suppose T^* is A- optimum in $\mathcal{D}_{\mathfrak{T}}$ and the non-diagonal elements of $\boldsymbol{X}'_{\boldsymbol{1T^*}}\boldsymbol{X}_{\boldsymbol{1T^*}}$ are all -1 or 1. For any $T \in \mathcal{D}_{\mathfrak{T}}$, let $\boldsymbol{X}_{\boldsymbol{1T}}$ represent the main effects, the columns of $\boldsymbol{X}_{\boldsymbol{2iT}}$ represent the general mean and the two factor interactions of the i^{th} model. Since $\boldsymbol{X}^{\boldsymbol{(i)}}_{\boldsymbol{f_T}}$ $i = 1, ..., \binom{m}{2}$ are block diagonal matrix, then for any $f_T = \binom{T}{-T} \in \mathcal{F}_{\mathfrak{T}}$ we can express GT as

$$GT = \left[\prod_{i=1}^{\binom{m}{2}} \left[trace(\boldsymbol{2X}'_{\boldsymbol{1T}}\boldsymbol{X}_{\boldsymbol{1T}})^{-1} + trace(2\boldsymbol{X}'_{\boldsymbol{2iT}}\boldsymbol{X}_{\boldsymbol{2iT}})^{-1}\right]\right]^{\frac{1}{\binom{m}{2}}}. \tag{8.18}$$

Since T^* is A- optimal in $\mathcal{D}_{\mathfrak{T}}$ then $\forall T \in \mathcal{D}_{\mathfrak{T}}$

$$trace(\boldsymbol{X}'_{\boldsymbol{1T^*}}\boldsymbol{X}_{\boldsymbol{1T^*}})^{-1} \leq trace(\boldsymbol{X}'_{\boldsymbol{1T}}\boldsymbol{X}_{\boldsymbol{1T}})^{-1}, \tag{8.19}$$

and since off-diagonal elements of the principal sub-matrices comprised of 1 or -1, then

$$trace(\boldsymbol{X}'_{\boldsymbol{2iT^*}}\boldsymbol{X}_{\boldsymbol{2iT^*}})^{-1} \leq trace(\boldsymbol{X}'_{\boldsymbol{2iT}}\boldsymbol{X}_{\boldsymbol{2iT}})^{-1}, \tag{8.20}$$

thus

$$GT_{T^*} \leq GT_T, \tag{8.21}$$

hence the foldover design $f_{T^*} = \binom{T^*}{-T^*}$ is optimum design with respect to GT in $\mathcal{F}_{\mathfrak{T}}$. ■

Chapter 9

Optimum Designs in Two Sub-classes for $k = 1$

9.0 Summary

We consider two classes of foldover designs. For $N = 2n$, $m = 2d$ and $k = 1$, we consider the class of foldover designs $\mathcal{D}_{\mathbf{1}} \subset \mathcal{F}_{\mathfrak{T}}$ giving $\boldsymbol{X^{(i)\prime}X^{(i)}}$ to be block diagonal matrix with two block matrices, $\boldsymbol{X_1^{\prime}X_1}$ whose rows and columns represent the main effects and $\boldsymbol{X_{2i}^{\prime}X_{2i}}$ whose rows and columns represent the general mean and the i^{th} two-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. In this class the matrix $\boldsymbol{X_1^{\prime}X_1}$ is a block diagonal matrix with d (2×2) block matrices of the same structure. In each block of the (2×2) matrix, the diagonal elements is n and the off-diagonal elements is a. For N, m even and $k = 1$, we consider the class of foldover designs $\mathcal{D}_{\mathbf{2}} \subset \mathcal{F}_{\mathfrak{T}}$ give rise to $\boldsymbol{X^{(i)\prime}X^{(i)}}$ to be block diagonal matrix with two block matrices, $\boldsymbol{X_1^{\prime}X_1}$ whose rows and columns represent the main effects and $\boldsymbol{X_{2i}^{\prime}X_{2i}}$ whose rows and columns represent the general mean and the i^{th} two-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. In this class the off-diagonal elements is a and the diagonal elements is n in the $\boldsymbol{X_1^{\prime}X_1}$ matrix. Given $a^2 \geq a_0$ and a_0 is a nonnegative number. Let a_1 be the smallest even integer that is greater than

or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. For $\mathcal{D}\mathbf{1}$ and $\mathcal{D}\mathbf{2}$ if $a_0 = 0$ then $a = 0$ minimizes the determinant, trace and maximum eigenvalue of $(\boldsymbol{X_1'X_1})^{-1}$ as well as the optimality criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$. For $\mathcal{D}\mathbf{1}$, if $a_0 > 0$ then a_1 and a_2 minimize the determinant, trace and maximum eigenvalue of $(\boldsymbol{X_1'X_1})^{-1}$ as well as the optimality criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$. For $\mathcal{D}\mathbf{2}$, if $a_0 > 0$ then a_1 minimizes the determinant and maximum eigenvalue of $(\boldsymbol{X_1'X_1})^{-1}$ as well as the optimality criterion functions GD, AD, AT, GT, $GMCR$, and $AMCR$.

9.1 A-, D- and E- Optimum Designs in $\mathcal{D}\mathbf{1}$

For N even, $m = 2d$ and $k = 1$, we consider the subclass of foldover designs $\mathcal{D}\mathbf{1} \subset \mathcal{F}_{\mathfrak{T}}$ give rise to $\boldsymbol{X^{(i)'}X^{(i)}}$ to be block diagonal matrix with two block matrices. The rows and columns of the matrix $\boldsymbol{X_1'X_1}$ represent the main effects and the rows and columns of $\boldsymbol{X_{2i}'X_{2i}}$ represent the general mean and the i^{th} two-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. The matrix $\boldsymbol{X_1'X_1}$ is a block diagonal matrix with d block matrices of the same structure

$$\boldsymbol{X_1'X_1} = \begin{pmatrix} N & a & 0 & 0 & \dots & 0 & 0 \\ a & N & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & N & a & 0 & \dots & 0 \\ 0 & 0 & a & N & 0 & \dots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & N & a \\ 0 & 0 & 0 & \dots & 0 & a & N \end{pmatrix}, \tag{9.1}$$

where $a^2 \geq a_0$ and a_0 is a nonnegative number. Since we are considering a class of foldover designs then a must be an even number or a negative even number. Since

$k = 1$, then the matrices $\boldsymbol{X'_{2i}X_{2i}}$, $i = 1, 2, ..., \binom{m}{2}$ must be principal sub-matrices of $\boldsymbol{X'_1X_1}$. From structure of $\boldsymbol{X'_1X_1}$ matrix, there are two types of $\boldsymbol{X'_{2i}X_{2i}}$ matrices. There are d of

$$\boldsymbol{W_1} = \begin{pmatrix} N & a \\ a & N \end{pmatrix}, \tag{9.2}$$

and there are $(2d^2 - 2d)$ of

$$\boldsymbol{W_2} = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}. \tag{9.3}$$

From $\boldsymbol{X'_1X_1}$ matrix we determine

$$f(a) = det(\boldsymbol{X'_1X_1})^{-1} = [(N-a)(N+a)]^{-d}, \tag{9.4}$$

$$h(a) = trace(\boldsymbol{X'_1X_1})^{-1} = \frac{2dN}{N^2 - a^2}, \tag{9.5}$$

$$g(a) = \lambda_{min}(\boldsymbol{X'_1X_1})^{-1} = \begin{cases} \frac{1}{N-a} & \text{if } a \geq 0 \\ \frac{1}{N+a} & \text{if } a \leq 0 \end{cases}. \tag{9.6}$$

Since the matrix $(\boldsymbol{X'_1X_1})^{-1}$ is positive definite then $(2 - N) \leq a \leq (N - 2)$ for the functions f, g and h. We obtain the following theorem for minimizing the determinant, trace and maximum eigenvalue of $(\boldsymbol{X'_1X_1})^{-1}$.

Theorem 1 *Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. If $a_0 = 0$, then $a = 0$ minimizes the determinant, trace and maximum eigenvalue of $(\boldsymbol{X'_1X_1})^{-1}$. If $a_0 \geq 0$, then a_1 and a_2 minimize the* *determinant, trace and maximum eigenvalue of* $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}$.

Proof:

Suppose $a^2 \geq a_0 = 0$, the functions f, g and h are minimize when $a = 0$. Now suppose $a^2 \geq a_0 > 0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. Since the functions f, g and h are monotonically decreasing on the interval $[(2-N), 0]$ and monotonically increasing on the interval $[0, (N-2)]$ thus a_1 and a_2 minimize the determinant, trace and maximum eigenvalue of $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}$. ■

9.2 Optimum Designs for Six Criterion Functions in $\mathcal{D}_1$

In this section we consider the optimality criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$ for designs in $\mathcal{D}_1$ with the matrices given in (9.1), (9.2) and (9.3). The two classes of $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$ are

$$(\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)})_1 = \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}_1 \end{pmatrix}, \tag{9.7}$$

and

$$(\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)})_2 = \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}_2 \end{pmatrix}. \tag{9.8}$$

The numbers of $(\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)})_1$ and $(\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)})_2$ matrices are d and $2d^2-2d$ respectively. We can express the six criterion functions as follows:

$$GD(a) = \left\{ \left[(N^2-a^2)^{-(d+1)} \right]^d \left[(N^2-a^2)^{-d}(N)^{-2} \right]^{2d^2-2d} \right\}^{\frac{1}{2d^2-d}}, \tag{9.9}$$

$$AD(a) = \frac{d(N^2 - a^2)^{-(d+1)} + (2d^2 - 2d)[(N^2 - a^2)^{-d} N^{-2}]}{2d^2 - d}, \tag{9.10}$$

$$GT(a) = \left\{ \left[\frac{2(d+1)N}{N^2 - a^2} \right]^d \left[\frac{2dn}{N^2 - a^2} + \frac{2}{N} \right]^{2d^2 - 2d} \right\}^{\frac{1}{2d^2 - d}}, \tag{9.11}$$

$$AT(a) = \frac{d\left[\frac{2(d+1)N}{N^2 - a^2}\right] + (2d^2 - 2d)\left[\frac{2dN}{N^2 - a^2} + \frac{2}{N}\right]}{2d^2 - d}, \tag{9.12}$$

$$GMCR(a) = AMCR(a) = \begin{cases} \frac{1}{N-a} & \text{if } a \geq 0, \\ \frac{1}{N+a} & \text{if } a \leq 0. \end{cases} \tag{9.13}$$

We obtain the following theorem for minimizing the six criterion functions.

Theorem 2 *Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. If $a_0 = 0$, then $a = 0$ minimizes the six criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$. If $a_0 \geq 0$, then a_1 and a_2 minimize the six criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$.*

Proof:

Suppose $a^2 \geq a_0 = 0$, then the functions GD, AD, GT, AT, $GMCR$, and $AMCR$ are minimize when $a = 0$. Suppose $a^2 \geq a_0 > 0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. Since the functions f, g and h are monotonically decreasing on the interval $[(2 - N), 0]$ and monotonically increasing on the interval $[0, (N - 2)]$ thus a_1 and a_2 minimize the functions GD, AD, GT, AT, $GMCR$, and $AMCR$. ■

For $a_0 = 0$, then designs in $\mathcal{D}_1$ given rise to (9.1) with $a = 0$ is optimum with respect to AD, AT, $AMCR$, GD, GT and $GMCR$. Similarly for $a_0 > 0$, then designs in $\mathcal{D}_1$

given rise to (9.1) with $a = a_1$ is optimum with respect to AD, AT, $AMCR$, GD, GT and $GMCR$.

Example 9.1

For $N = 12$, $m = 4$, and $k = 1$ let us consider the following foldover design $D_{9.1}$,

$$D_{9.1} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

We have the following structures for $\boldsymbol{X^{(i)\prime} X^{(i)}}$

$$\boldsymbol{(X^{(i)\prime} X^{(i)})_1} = \left(\begin{array}{cccc|cc} 12 & -4 & 0 & 0 & 0 & 0 \\ -4 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & -4 & 0 & 0 \\ 0 & 0 & -4 & 12 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 12 & -4 \\ 0 & 0 & 0 & 0 & -4 & 12 \end{array}\right),$$

and

$$
(\boldsymbol{X^{(i)\prime}X^{(i)}})_{\mathbf{2}} = \left(\begin{array}{cccc|cc}
12 & -4 & 0 & 0 & 0 & 0 \\
-4 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & -4 & 0 & 0 \\
0 & 0 & -4 & 12 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 12
\end{array}\right).
$$

There are $d = 2$ and $(2d^2 - 2d) = 4$ of $(\boldsymbol{X^{(i)\prime}X^{(i)}})_{\mathbf{1}}$ and $(\boldsymbol{X^{(i)\prime}X^{(i)}})_{\mathbf{2}}$ respectively. Given $a^2 \geq a_0 = 10$, if we consider all foldover designs with the structure of $\boldsymbol{X_1^{\prime}X_1}$ given in (9.1) then $D_{9.1}$ is optimum design with respect to all six criterion functions in this class of designs.

Example 9.2

For $N = 8$, $m = 4$, and $k = 1$ let us consider the following foldover design $D_{9.2}$,

$$
D_{9.2} = \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
-1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 \\
-1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1
\end{pmatrix}.
$$

We have the following structure for $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$

$$\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)} = \left(\begin{array}{cccc|cc} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right).$$

Given $a^2 \geq a_0 = 0$, if we consider all foldover designs with the structure of $\boldsymbol{X_1'X_1}$ given in (9.1) then $D_{9.2}$ is optimum design with respect to all six criterion functions in this class of designs.

9.3 E-, $AMCR$ and $GMCR$ Optimum Designs in $\mathcal{D}_2$

For N, m even and $k = 1$, we consider the subclass of foldover designs $\mathcal{D}_2 \subset \mathcal{F}_{\mathfrak{T}}$ give rise to $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ to be block diagonal matrix with two block matrices, $\boldsymbol{X_1'X_1}$ whose rows and columns represent the main effects and $\boldsymbol{X_{2i}'X_{2i}}$ whose rows and columns represent the general mean and the i^{th} *two*-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. The matrix $\boldsymbol{X_1'X_1}$ is

$$\boldsymbol{X_1'X_1} = \begin{pmatrix} N & a & a & a \dots & a & a \\ a & N & a & a \dots & a & a \\ a & a & N & a & a \dots & a \\ a & a & a & N & a \dots & a \\ \vdots & & \vdots & \ddots & & \vdots \\ a & a & a \dots & a & N & a \\ a & a & a \dots & a & a & N \end{pmatrix} \qquad (9.14)$$

where $a^2 \geq a_0$ and a_0 is a nonnegative number. Since we are considering foldover designs then a must be an even number or a negative even number. Since $k = 1$, then the matrices $\boldsymbol{X_{2i}'X_{2i}}$, $i = 1, 2, ..., \binom{m}{2}$ must be principal sub-matrices of $\boldsymbol{X_1'X_1}$. From structure of $\boldsymbol{X_1'X_1}$ there is one class of matrices $\boldsymbol{X_{2i}'X_{2i}}$

$$\boldsymbol{X_{2i}'X_{2i}} = \begin{pmatrix} N & a \\ a & N \end{pmatrix}. \qquad (9.15)$$

We obtain the following theorem for minimizing $AMCR$, $GMCR$ and the maximum eigenvalue of $\boldsymbol{(X_1'X_1)^{-1}}$.

Theorem 3 *Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. If $a_0 = 0$ then $a = 0$ minimizes AMCR, GMCR and the maximum eigenvalues of $\boldsymbol{(X_1'X_1)^{-1}}$. If $a_0 > 0$ then a_1 minimizes AMCR, GMCR and the maximum eigenvalues of $\boldsymbol{(X_1'X_1)^{-1}}$.*

Proof:

Suppose $a^2 \geq a_0$, let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. Since $\boldsymbol{X^{(i)\prime}X^{(i)}}$, $i = 1, 2, ..., \binom{m}{2}$ are identical, then $AMCR = GMCR$. Furthermore minimizing $AMCR$ and $GMCR$ is equivalent to maximizing the minimum eigenvalue of $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$. The minimum eigenvalue of $(\boldsymbol{X}_1'\boldsymbol{X}_1)$ is

$$g(a) = \lambda_{min}(\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}) = \begin{cases} N - a & \text{if } 0 \leq a \leq N-2, \\ N + (m-1)a & \text{if } (-N/(m-1) < a \leq 0. \end{cases} \tag{9.16}$$

For $a_0 = 0$, we see $a = 0$ maximizes the minimum eigenvalue $\boldsymbol{X}_1'\boldsymbol{X}_1$ and $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$. Hence $a = 0$ minimizes $AMCR$, $GMCR$ and the maximum eigenvalues of $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}$. Since

$$|g'(-a)| > |g'(a)|, \qquad a \in (0, N/(m-1)), \tag{9.17}$$

then

$$g(a) > g(-a), \qquad a \in (0, N/(m-1)). \tag{9.18}$$

Thus for $a_0 > 0$, a_1 maximizes the minimum eigenvalue of $\boldsymbol{X}_1'\boldsymbol{X}_1$ and $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$. Hence a_1 minimizes $AMCR$ and $GMCR$. ■

For $a_0 = 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = 0$ are E-, $AMCR$ and $GMCR$ optimum. Similarly for $a_0 > 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = a_1$ are E-, $AMCR$ and $GMCR$ optimum.

9.4 D-, AD and GD Optimum Designs in $\mathcal{D}_2$

In this section we obtain D-, AD and GD optimum designs in $\mathcal{D}_2$. We obtain the following theorem for minimizing AD, GD and the determinant of $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}$.

Theorem 4 *Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. If $a_0 = 0$ then $a = 0$ minimizes GD, AD and the determinant of*

$(\boldsymbol{X_1'X_1})^{-1}$. *If* $a_0 > 0$, *then* a_1 *minimizes* GD, AD *and the determinant of* $(\boldsymbol{X_1'X_1})^{-1}$.

Proof:

Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. Since the matrices $\boldsymbol{X^{(i)'}X^{(i)}}$, $i = 1, 2, ..., \binom{m}{2}$ are identical, then $AD = GD$. Hence minimizing AD and GD is equivalent to maximizing the determinant of $\boldsymbol{X^{(i)'}X^{(i)}}$. We consider the the cases for $m \geq 3$, the case for $m = 2$ is obvious. First we consider the determinant of $\boldsymbol{X_1'X_1}$ given below:

$$f(a) = det(\boldsymbol{X_1'X_1}) = (N-a)^{m-1}(N+(m-1)a), \quad a \in ((-N/(m-1), N-2] \tag{9.19}$$

The derivative of f is

$$f'(a) = -m(m-1)a(N-a)^{m-2}. \tag{9.20}$$

Thus we have

$$\begin{aligned} f^{'}(a) > 0 \quad &\text{if} \quad (-N/m-1) < a < 0, \\ f^{'}(a) = 0 \quad &\text{if} \quad a = 0, \\ f^{'}(a) < 0 \quad &\text{if} \quad 0 < a \leq (N-2). \end{aligned} \tag{9.21}$$

For $a_0 = 0$, we see $a = 0$ maximizes the determinant of $\boldsymbol{X_1'X_1}$. Equivalently $a = 0$ minimizes the determinant of $(\boldsymbol{X_1'X_1})^{-1}$.

For $a \in (0, N/(m-1))$

$$|f'(-a)| = (m-1)ma(N+a)^{m-2} > (m-1)ma(N-a)^{m-2} = |f'(a)|, \tag{9.22}$$

then

$$f(a) > f(-a), \quad a \in (0, N/(m-1)) \tag{9.23}$$

Hence for $a_0 > 0$, a_1 maximizes the determinant of $\boldsymbol{X_1'X_1}$. Equivalently a_1 minimizes the determinant of $\boldsymbol{(X_1'X_1)^{-1}}$.

Now we consider the determinant of $\boldsymbol{X^{(i)\prime}X^{(i)}}$

$$h(a) = det(\boldsymbol{X^{(i)\prime}X^{(i)}}) = f(a)f_1(a) = f(a)(N^2 - a^2). \tag{9.24}$$

Since f_1 is monotonically increasing on $(-N/(m-1), 0)$, reach its maximum at $a = 0$ and monotonically decreasing on $(0, N-2)$, then for $a_0 = 0$, $a = 0$ maximizes the determinant of $\boldsymbol{X^{(i)\prime}X^{(i)}}$. Equivalently $a = 0$ minimizes the AD and GD. Furthermore for $a_0 > 0$, a_1 maximizes the determinant of $\boldsymbol{X^{(i)\prime}X^{(i)}}$, hence minimizes AD and GD.

■

For $a_0 = 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = 0$ are D-, AD and GD optimum. Similarly for $a_0 > 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = a_1$ are A-, AD and GD optimum.

9.5 A- AT and GT Optimum Designs in $\mathcal{D}_2$

In this section we obtain A-, AT and GT optimum designs in $\mathcal{D}_2$. We obtain the following theorem for minimizing AT, GT and the trace of $\boldsymbol{(X_1'X_1)^{-1}}$.

Theorem 5 *Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. If $a_0 = 0$ then $a = 0$ minimizes AT, GT and the trace of $\boldsymbol{(X_1'X_1)^{-1}}$. If $a_0 > 0$, then a_1 minimizes AT, GT and the trace of $\boldsymbol{(X_1'X_1)^{-1}}$.*

Proof:

Suppose $a^2 \geq a_0$ and let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$. Since the matrices $\boldsymbol{X^{(i)\prime}X^{(i)}}$, $i = 1, 2, ..., \binom{m}{2}$ are identical, then $AT = GT$. Hence minimizing AT and GT is equivalent to minimizing the trace of $\boldsymbol{(X^{(i)\prime}X^{(i)})^{-1}}$. We consider the the cases for $m \geq 3$.

First we consider the trace of $\boldsymbol{(X_1^{\prime}X_1)^{-1}}$ given below:

$$f(a) = \frac{m[N + (m-2)a]}{[N + (m-1)a][N - a]}, \qquad a \in ((-N/(m-1), N-2]. \tag{9.25}$$

The derivative of f is

$$f'(a) = \frac{am(m-1)(am - 2a + 2N)}{(N + am - a)^2(a - N)^2}. \tag{9.26}$$

Thus we have

$$\begin{aligned} f^{'}(a) &> 0 \quad \text{if} \quad (-N/m-1) < a < 0, \\ f^{'}(a) &= 0 \quad \text{if} \quad a = 0, \\ f^{'}(a) &< 0 \quad \text{if} \quad 0 < a \leq (N-2). \end{aligned} \tag{9.27}$$

For $a_0 = 0$, we see $a = 0$ minimizes the trace of $\boldsymbol{(X_1^{\prime}X_1)^{-1}}$.

For $a \in (0, N/(m-1))$

$$f(-a) - f(a) = \frac{2a^3(m^2 - 3m + 2)}{(a^2 - N^2)((a(m-1))^2 - N^2)}. \tag{9.28}$$

Since

$$a^2 - N^2 < 0, \tag{9.29}$$

$$a((m-1))^2 - N^2 < 0, \tag{9.30}$$

and

$$2a^3(m^2 - 3m + 2) > 0 \tag{9.31}$$

then

$$f(-a) > f(a), \quad a \in (0, N/(m-1)). \tag{9.32}$$

Thus for $a_0 > 0$, $a = a_1$ minimizes the trace of $(\boldsymbol{X_1'X_1})^{-1}$.

Next we consider the trace of $(\boldsymbol{X^{(i)\prime}X^{(i)}})^{-1}$

$$h(a) = trace(\boldsymbol{X^{(i)\prime}X^{(i)}})^{-1} = f(a)f_1(a) = f(a)\frac{2n}{N^2 - a^2}. \tag{9.33}$$

Since f_1 is monotonically decreasing on $(-N/(m-1), 0)$, reach its minimum at $a = 0$ and monotonically increasing on $(0, N-2)$, then for $a_0 = 0$, $a = 0$ minimizes the trace of $(\boldsymbol{X^{(i)\prime}X^{(i)}})^{-1}$. Equivalently $a = 0$ minimizes the AT and GT. Furthermore for $a_0 > 0$, a_1 minimizes the trace of $(\boldsymbol{X^{(i)\prime}X^{(i)}})^{-1}$ hence minimizes AD and GD. ■

For $a_0 = 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = 0$ are A-, AT and GT optimum. Similarly for $a_0 > 0$, then designs in $\mathcal{D}_2$ given rise to (9.14) with $a = a_1$ are A-, AD and GD optimum.

Example 9.3 Consider the following foldover design

$$D_{9.3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The matrix $\boldsymbol{X_1'X_1}$ is

$$\boldsymbol{X_1'X_1} = \begin{pmatrix} 10 & 2 & 2 & 2 \\ 2 & 10 & 2 & 2 \\ 2 & 2 & 10 & 2 \\ 2 & 2 & 2 & 10 \end{pmatrix}.$$

For $a^2 > 0$, the design $D_{9.3}$ is optimum with respect to AD, AT, $AMCR$, GD, GT and $GMCR$ in $\mathcal{D}_2$.

Chapter 10

Optimum Fractional Factorial Designs for $m = 4$

10.0 Summary

In this chapter we present optimum fractional factorial designs for $m = 4$. The numbers of runs we consider are $n = 6$, 7, 8, 9, 10, 11 and 12. The values of k, the number of two-factor interactions are 1, 2, 3, 4, 5 and 6. In [Ghosh and Tian(2006)] they present optimum designs for maximum values of k. We discuss hierarchical designs. For designs with n runs we can partition the designs with full estimation capacity into non-isomorphic classes with respect to the criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$. Within each class the values of all the six criterion functions are identical. As an example we consider designs with ten runs. We partitioned these designs into isomorphic classes with respect to all six criteria. The models contain the general mean, main effects and $k = 1, 2, 3, 4$ and 5 two-factor interactions. We present a table for each k. For the models containing general mean, main effects, and $k = 1, 2, 3, 4$ and 5 two-factor interactions, designs within isomorphic classes can be obtained from one another by renaming factors or levels of the factors. We cannot transform a design from one class by renaming factors or levels of the factors to design of another class.

10.1 Optimum Designs for $m = 4$ when $k \leq 6$

In this section we present fractional factorial optimum designs for $m = 4$ with the number of runs $n = 6, 7, 8, 9, 10, 11$ and 12. The number of two-factor interactions $k = 1, 2, 3, 4, 5$ and 6 for the models in (5.1). For a fractional factorial design with n runs the total number of designs is $\binom{2^m}{n}$. Let $\mathcal{D}$ denote the class of designs with n runs that have FEC. We would like to find the designs that given the smallest values for one or more of the six criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$. For a 2^4 factorial experiment, we consider the positions with 1 in a run or equivalently the factors with high level in a run. We then denote a run by the positions with 1. For example, in the run $(1, -1, 1, -1)$, 1 is appearing in position 1 and 3 and hence we denote this run by 13. We denote the run $(-1, -1, -1, -1)$ by 0[Ghosh, and Tian(2006)]. In Table 10.1 we present optimum designs, the number of runs in the designs, the number of designs, the values of k and which criterion functions the designs are the best.

Table 10.1: Optimum designs for $m = 4$

Optimum Designs for $m = 4$				
Design	n	ND	k_{opt}	Optimal w.r.t.
$D_{(6)} = (123, 124, 134, 234, 1, 0)$	6	64	1	All
$D_{(7)} = (123, 124, 134, 234, 1, 2, 3)$	7	16	1	All
$D_{(8)} = (123, 124, 134, 234, 1, 2, 3, 4)$	8	2	1	All
$D_{(8.1)} = (1234, 123, 124, 134, 1, 2, 3, 4)$	8	64	2	AD,AT, GD,GT
$D_{(8.2)} = (123, 124, 12, 13, 24, 34, 3, 4)$	8	24	2	AMCR, GMCR
$D_{(9)} = (1234, 123, 124, 134, 234, 1, 2, 3, 4)$	9	16	1,2,3	All
$D_{(10)} = (1234, 123, 124, 134, 234, 12, 1, 2, 3, 4)$	10	48	1	All
$D_{(10.1)} = (123, 124, 134, 234, 14, 23, 1, 2, 3, 4)$	10	8	1	AMCR, GMCR
			2,3	All
$D_{(10.2)} = (123, 124, 134, 234, 12, 13, 14, 2, 3, 4)$	10	64	4,5	All
$D_{(11)} = (1234, 123, 124, 134, 234, 12, 13, 1, 2, 3, 4)$	11	64	1	All
$D_{(11.1)} = (123, 124, 134, 234, 12, 13, 1, 2, 3, 4, 0)$	11	64	1	AMCR, GMCR
$D_{(11.2)} = (123, 124, 134, 234, 14, 23, 34, 1, 2, 3, 4)$	11	48	2,3	All
$D_{(11.3)} = (1234, 124, 134, 234, 12, 13, 23, 1, 2, 3, 4)$	11	16	4,5,6	All

Optimum Designs for $m = 4$ (Table 10.1 con't)				
Design	n	ND	k_{opt}	Optimal w.r.t.
$D_{(12)} = (1234, 123, 124, 134, 234, 12, 13, 14, 1, 2, 3, 4)$	12	32	1,4,5,6	All
$D_{(12.1)} = (123, 124, 134, 234, 13, 14, 23, 24, 1, 2, 3, 4)$	12	96	1	AMCR, GMCR
$D_{(12.2)} = (123, 124, 134, 234, 13, 14, 23, 24, 1, 2, 3, 4)$	12	12	1	AMCR, GMCR
			2,3	All
$D_{(12.3)} = (1234, 123, 124, 134, 23, 24, 34, 1, 2, 3, 4, 0)$	12	64	4,5	AMCR, GMCR
			6	All
$D_{(12.4)} = (1234, 124, 134, 234, 12, 13, 14, 23, 2, 3, 4, 0)$	12	24	4,5	AMCR, GMCR
			6	All

We denote k_{opt} to be the values of k for which the design is optimum. From Table 10.1, design $D_{(8)}$ is the optimum design for $n = 8$ and $k = 1$ with respect to all six criterion functions but it is not optimal for $k = 2$. We note from chapter 5, Table 5.1 design $D_{(8)}$ does not have FEC for $k = 2$. For $n = 10$, design $D_{(10.1)}$ is the optimum design for $k = 2$ and 3 with respect to all six criterion functions but it is not optimal for $k = 5$ and 6. We see from the Table 10.1 both designs $D_{(10)}$ and $D_{(10.1)}$ are optimum with respect to $AMCR$ and $GMCR$ for $k = 1$. Meaning the values of $AMCR$ and $GMCR$ are identical for both designs $D_{(10)}$ and $D_{(10.1)}$. Furthermore $D_{(10)}$ is optimum design for all six criterion functions when $k = 1$ but design $D_{(10.1)}$ is not. For $n = 12$, designs $D_{(12)}$, $D_{(12.3)}$ and $D_{(12.4)}$ are optimum designs with respect to all six criterion functions for $k = 4$, 5 and 6. But designs $D_{(12)}$, $D_{(12.3)}$ and $D_{(12.4)}$ are not isomorphic in the sense that we cannot obtain one design from the other by renaming factors or levels.

10.2 Hierarchical Designs

Many of the designs in Table 10.1 are nested. For example let us consider optimum designs for $k = 1$ w.r.t to all six criterion functions. We see $D_{(7)} \subseteq D_{(8)} \subseteq D_{(9)} \subseteq D_{(10)} \subseteq D_{(11)} \subseteq D_{(12)}$ in the sense that runs of $D_{(7)}$ are in $D_{(8)}$, the runs of $D_{(8)}$ are in $D_{(9)}$, the runs of $D_{(9)}$ are in $D_{(10)}$, the runs of $D_{(10)}$ are in $D_{(11)}$ and the runs of $D_{(11)}$ are in $D_{(12)}$. From Table 10.1, $D_{(10)} \subseteq D_{(11.2)}$ in the sense the runs of $D_{(10.1)}$ are in $D_{(11.2)}$. Further more $D_{(8.1)} \subseteq D_{(9)}$ and $D_{(11.3)} \subseteq D_{(12.4)}$. In Table 10.2, we present the values of the six criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$ for the designs in Table 10.1.

10.3 Values of the Criterion Functions in Table 10.1

In this section we present the values of the six criterion functions AD, AT, $AMCR$, GD, GT and $GMCR$ for the designs in Table 10.1.

Table 10.2: Values of criterion functions for designs in Table 10.1

Optimum Designs and Values of Criterion Functions							
k	Design	AD	AT	$AMCR$	GD	GT	$GMCR$
1	$D_{(6)}$	1.526×10^{-4}	2.625	1.784	1.221×10^{-4}	2.372	1.381
1	$D_{(7)}$	1.526×10^{-5}	1.125	0.500	1.526×10^{-6}	1.125	0.500
1	$D_{(8)}$	3.815×10^{-6}	0.750	0.125	3.815×10^{-6}	0.750	0.125
1	$D_{(9)}$	2.180×10^{-6}	0.696	0.125	2.180×10^{-6}	0.696	0.125
1	$D_{(10)}$	1.254×10^{-6}	0.644	0.125	1.254×10^{-6}	0.644	0.125
1	$D_{(10.1)}$	1.272×10^{-6}	0.646	0.125	1.272×10^{-6}	0.646	0.125
1	$D_{(11)}$	7.266×10^{-7}	0.592	0.125	7.266×10^{-7}	0.592	0.125
1	$D_{(11.1)}$	7.387×10^{-7}	0.595	0.125	7.385×10^{-7}	0.595	0.125
1	$D_{(12)}$	4.239×10^{-7}	0.542	0.125	4.239×10^{-7}	0.542	0.125
1	$D_{(12.1)}$	4.319×10^{-7}	0.545	0.125	4.319×10^{-7}	0.545	0.125
1	$D_{(12.2)}$	4.415×10^{-7}	0.549	0.125	4.408×10^{-7}	0.549	0.125
2	$D_{(8.1)}$	1.806×10^{-6}	1.472	0.755	1.551×10^{-6}	1.426	0.677
2	$D_{(8.2)}$	3.688×10^{-6}	1.608	0.427	3.642×10^{-6}	1.607	0.427
2	$D_{(9)}$	3.942×10^{-7}	0.953	0.273	3.313×10^{-7}	0.922	0.184
2	$D_{(10.1)}$	1.567×10^{-7}	0.795	0.170	1.524×10^{-7}	0.792	0.153
2	$D_{(11.2)}$	7.935×10^{-8}	0.721	0.152	7.847×10^{-8}	0.720	0.144
2	$D_{(12.2)}$	4.106×10^{-8}	0.653	0.133	4.083×10^{-8}	0.652	0.131

Optimum Designs and Values of Criterion Functions (Table 10.2 con't)							
k	Design	AD	AT	$AMCR$	GD	GT	$GMCR$
3	$D_{(9)}$	8.345×10^{-8}	1.388	0.610	6.847×10^{-8}	1.334	0.418
3	$D_{(10.1)}$	2.384×10^{-8}	1.025	0.306	2.259×10^{-8}	1.017	0.261
3	$D_{(11.2)}$	9.735×10^{-9}	0.889	0.229	9.524×10^{-9}	0.887	0.210
3	$D_{(12.2)}$	7.202×10^{-9}	1.639	0.736	6.335×10^{-9}	1.583	0.625
4	$D_{(10.1)}$	7.202×10^{-9}	1.640	0.736	6.335×10^{-9}	1.583	0.625
4	$D_{(11.3)}$	1.389×10^{-9}	1.097	0.250	1.388×10^{-9}	1.097	0.250
4	$D_{(12)}$	4.967×10^{-10}	0.954	0.250	4.932×10^{-10}	0.954	0.250
4	$D_{(12.3)}$	5.189×10^{-10}	0.966	0.250	5.182×10^{-10}	0.966	0.250
4	$D_{(12.4)}$	5.381×10^{-10}	0.976	0.250	5.326×10^{-10}	0.976	0.250
5	$D_{(10.1)}$	2.328×10^{-9}	2.875	1.784	1.863×10^{-9}	2.646	1.381
5	$D_{(11.3)}$	1.863×10^{-10}	1.288	0.250	1.863×10^{-10}	1.288	0.250
5	$D_{(12)}$	5.821×10^{-11}	1.125	0.250	5.821×10^{-11}	1.125	0.250
5	$D_{(12.3)}$	6.014×10^{-11}	1.133	0.250	6.012×10^{-11}	1.133	0.250
5	$D_{(12.4)}$	6.144×10^{-11}	1.139	0.250	6.107×10^{-11}	1.139	0.250
6	$D_{(11.3)}$	2.587×10^{-11}	1.486	0.250	2.587×10^{-11}	1.486	0.250
6	$D_{(12)}$	7.276×10^{-12}	1.313	0.250	7.276×10^{-12}	1.313	0.250
6	$D_{(12.3)}$	7.276×10^{-12}	1.313	0.250	7.276×10^{-12}	1.313	0.250
6	$D_{(12.4)}$	7.276×10^{-12}	1.313	0.250	7.276×10^{-12}	1.313	0.250

10.4 Non-isomorphic Classes for $k = 5$

In this section we consider models with general mean, main effects and five two-factor interactions. We determined there are 272 designs from the 8008 designs that have the ability to estimate all the parameters in the six models. These designs with full estimation capability can be partitioned into four non-isomorphic classes with respect to all six criteria. These four classes of designs are sorted from the best to the worst in terms of AT. Within each class the values of all six criterion functions are identical. We denote S1, S2, S3, S4 to be all runs in S_1, S_2, S_3 and S_4 respectively. We denote S1{(1),(2)} to be two runs from S_1 . The numbers {(1),(2)} indicates the position of the -1 in the runs from S_1. The (1) refers to (-1,1,1,1) and (2) refers to (1 -1, 1 1). Similarly S2{(1,2),(1,3)} denotes two runs from S_2. The numbers {(1,2),(1,3)} indicates the positions of the -1 in the runs from S_2. The (1,2) refers to (-1,-1,1,1) and (1,3) refers to (-1,1,-1,1). For S3{(3),(4)} denotes two runs from S_3. The numbers {(3),(4)} indicate the position of the +1 in the runs from S_3. The (3) refers to (-1 -1 1 -1) and (4) refers to (-1 -1 -1 1). The number of designs in each non-isomorphic class is denoted by ND. These four classes of designs are sorted from the best to the worst in terms of AT.

Table 10.3: Non-isomorphic designs ($n = 10, m = 4, k = 5$)

Non-isomorphic Designs (n = 10, m = 4, k = 5)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} S3{(3),(2),(1)} S4	64	2.3283	2.8750	1.7844	1.8626	2.6458	1.3811
S1 S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(1)} S4	96	3.2596	3.0833	1.8499	2.9568	2.9953	1.6857
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2),(1)} S4	16	3.7253	3.2500	2.0727	3.7253	3.2500	2.0727
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2)} S4	96	3.2596	3.5000	2.3973	2.9568	3.3666	2.1466

We denote AD^* to be $10^9 \times AD$ and GD^* to be $10^9 \times GD$.

10.5 Non-isomorphic Classes for $k = 4$

In this section we consider models with general mean, main effects and four two-factor interactions. We determined there are $1,232$ designs from the $8,008$ designs that have the ability to estimate all the parameters in the fifteen models. These designs with full estimation capability can be partitioned into eight non-isomorphic classes with respect to all six criteria. These eight classes of designs are sorted from the best to the worst in terms of AT.

Table 10.4: Non-isomorphic designs $(n = 10, m = 4, k = 4)$

Non-isomorphic Designs (n = 10, m = 4, k = 4)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} S3{(3),(2),(1)} S4	64	7.2022	1.6396	0.73550	6.3351	1.5832	0.6249
S1 S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(1)} S4	96	10.100	1.8313	0.85863	9.1686	1.7920	0.7826
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2)} S4	96	11.093	1.9937	1.0704	10.056	1.9392	0.9618
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 3),(2, 4)} S3{(2),(1)} S4	192	14.239	2.0097	0.92492	14.117	2.0019	0.8987

Non-isomorphic Designs (n = 10, m = 4, k = 4)(Table 10.4 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 4),(3, 4)} S3{(3),(1)} S4	192	13.245	2.0653	1.0466	12.871	2.0453	0.9931
S1{(1),(2),(3)} S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(2),(1)} S4	384	13.080	2.0681	1.0460	12.528	2.0407	0.9984
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2),(1)} S4	16	13.908	2.0917	1.1125	13.740	2.0820	1.0901
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(3, 4)} S3{(4),(1)} S4	192	13.742	2.1389	1.1178	13.479	2.1108	1.0359

We denote AD^* to be $10^9 \times AD$ and GD^* to be $10^9 \times GD$.

10.6 Non-isomorphic Classes for $k = 3$

In this section we consider models with general mean, main effects and three two-factor interactions. We determined there are $2,248$ designs from the $8,008$ designs that have the ability to estimate all the parameters in the twenty models. These designs with full estimation capability can be partitioned into fifteen non-isomorphic classes with respect to all six criteria. These fifteen classes of designs are sorted from the best to the worst in terms of AT.

Table 10.5: Non-isomorphic designs ($n = 10, m = 4, k = 3$)

Non-isomorphic Designs (n = 10, m = 4, k = 3)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(3, 4)} S3	8	2.3842	1.0250	0.3061	2.2586	1.0174	0.2612
S1 S2{(1, 2),(1, 3)} S3	48	2.5545	1.0536	0.3354	2.3825	1.0428	0.2782
S1 S2{(1, 2),(1, 3)} S3{(3),(2),(1)} S4	64	3.4068	1.1644	0.4325	3.2287	1.1502	0.4034
S1 S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(1)} S4	96	4.5461	1.2812	0.5062	4.2866	1.2691	0.4810
S1 S2{(1, 2),(3, 4)} S3{(3),(2),(1)} S4	192	4.3780	1.3134	0.5766	4.0672	1.2861	0.5125
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2)} S4	96	5.1870	1.3829	0.6267	4.8054	1.3604	0.5741
S1 S2{(1, 2),(2, 3)} S3{(4),(2),(1)} S4	192	4.9754	1.4064	0.6682	4.4964	1.3642	0.5710

Non-isomorphic Designs (n = 10, m = 4, k = 3) (Table 10.5 con't)

Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} {(3, 4)} S3{(4),(1)} S4	384	5.6970	1.4167	0.6173	5.4621	1.4038	0.5852
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(1)} S4	192	5.7680	1.4228	0.6233	5.6113	1.4147	0.6011
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(4),(3)}	192	6.2784	1.4517	0.6012	6.1958	1.4458	0.5750
S1 S2{(1, 2),(1, 3)} {(2, 3),(3, 4)} S3{(1)} S4	192	6.0884	1.4559	0.6401	5.9412	1.4489	0.6148
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(3),(1)} S4	192	6.6360	1.4642	0.5909	6.5627	1.4608	0.5718
S1 S2{(1, 2),(1, 3)} {(2, 4)} S3{(4),(3),(2)}	192	5.6192	1.4846	0.7390	5.3736	1.4619	0.6852

Non-isomorphic Designs (n = 10, m = 4, k = 3) (Table 10.5 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} {(1, 4),(3, 4)} S3{(4),(3)}	192	6.7949	1.5167	0.6925	6.7108	1.5094	0.6683
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2),(1)} S4	16	6.9936	1.5583	0.7659	6.9514	1.5536	0.7542

We denote AD^* to be $10^8 \times AD$ and GD^* to be $10^8 \times GD$.

10.7 Non-isomorphic Classes for $k = 2$

In this section we consider models with general mean, main effects and two two-factor interactions. We determined there are $3,768$ designs from the $8,008$ designs that have the ability to estimate all the parameters in the fifteen models. These designs with full estimation capability can be partitioned into twenty five non-isomorphic classes with respect to all six criteria. These twenty five classes of designs are sorted from the best to the worst in terms of AT.

Table 10.6: Non-isomorphic designs ($n = 10, m = 4, k = 2$)

Non-isomorphic Designs (n = 10, m = 4, k = 2)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(3, 4)} S3	8	1.5668	0.7946	0.1695	1.5237	0.7917	0.1534
S1 S2{(1, 2),(1, 3)} S3	48	1.6651	0.8139	0.1902	1.5811	0.8072	0.1615
S1 S2{(1, 2),(1, 3)} S3{(3),(2),(1)} S4	64	2.0618	0.8832	0.2875	2.0235	0.8806	0.2798
S1 S2{(1, 2),(3, 4)} S3{(3),(2),(1)} S4	192	2.4194	0.9437	0.3443	2.3509	0.9383	0.3284
S1 S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(1)} S4	96	2.5894	0.9598	0.3452	2.5165	0.9557	0.3353
S1 S2{(1, 2),(2, 3)} S3{(4),(2),(1)} S4	192	2.5892	0.9767	0.3816	2.4624	0.9655	0.3512
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2)} S4	96	2.8837	1.0183	0.4146	2.7216	1.0054	0.3827
S1 S2{(1, 2),(1, 3)} {(3, 4)} S3{(4),(1)} S4	384	3.1213	1.0407	0.4202	3.0451	1.0358	0.4089

Non-isomorphic Designs (n = 10, m = 4, k = 2) (Table 10.6 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} {(2, 4)} S3{(4),(3),(2)}	192	2.9843	1.0428	0.4484	2.9130	1.0368	0.4316
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(1)} S4	192	3.1687	1.0490	0.4240	3.1089	1.0447	0.4106
S1 S2{(1, 2),(1, 3)} {(2, 3),(3, 4)} S3{(1)} S4	192	3.2728	1.0633	0.4366	3.1951	1.0589	0.4265
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(4),(3)}	192	3.4145	1.0637	0.4150	3.3919	1.0627	0.4118
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(4),(3),(2)}	192	3.2152	1.0817	0.4868	3.0504	1.0669	0.4526
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 3),(2, 4)} S3{(3),(1)} S4	192	3.6205	1.0834	0.4215	3.5434	1.0813	0.4210
S1{(1),(2),(3) } S2{(1, 3),(1, 4)} {(2, 3),(2, 4)} S3{(3),(2),(1)}	48	3.6785	1.0923	0.4268	3.5379	1.0886	0.4268
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 3),(3, 4)} S3{(4),(3),(2)}	192	3.5551	1.0939	0.4448	3.5117	1.0891	0.4242

Non-isomorphic Designs (n = 10, m = 4, k = 2) (Table 10.6 con't)							
Designs	ND	AD^{*}	AT	$AMCR$	GD^{*}	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} {(1, 4),(3, 4)} S3{(4),(3)}	192	3.6711	1.1156	0.4758	3.6353	1.1114	0.4606
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(3),(1)} S4	192	3.7721	1.1162	0.4538	3.7440	1.1154	0.4509
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 4)} S3{(3),(2),(1)}	192	3.7111	1.1171	0.4815	3.6228	1.1123	0.4720
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} {(2, 4)} S3{(3),(1)}	192	4.1408	1.1286	0.4205	4.1024	1.1274	0.4197
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4)} S3{(4),(3),(1)} S4	192	3.7794	1.1389	0.5007	3.7457	1.1332	0.4835
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 4)} S3{(3),(2),(1)} S4	192	3.8241	1.1391	0.5016	3.7483	1.1347	0.4918

Non-isomorphic Designs (n = 10, m = 4, k = 2) (Table 10.6 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4)} S3{(3),(2),(1)} S4	64	3.8821	1.1652	0.5388	3.8473	1.1592	0.5205
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} S3 S4	64	3.8821	1.1894	0.5818	3.8473	1.1860	0.5702
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2),(1)} S4	16	4.2386	1.2229	0.6027	4.2386	1.2229	0.6020

We denote AD^* to be $10^7 \times AD$ and GD^* to be $10^7 \times GD$.

10.8 Non-isomorphic Classes for $k = 1$

In this section we consider models with general mean, main effects and two two-factor interactions. We determined there are $6,520$ designs from the $8,008$ designs that have the ability to estimate all the parameters in the six models. These designs with full estimation capability can be partitioned into forty non-isomorphic classes with respect to all six criteria. These forty classes of designs are sorted from the best to the worst in terms of AT.

Table 10.7: Non-isomorphic designs ($n = 10, m = 4, k = 1$)

Non-isomorphic Designs (n = 10, m = 4, k = 1)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} S3	48	1.2543	0.6439	0.1250	1.2542	0.6439	0.1250
S1 S2{(1, 2),(3, 4)} S3	8	1.2716	0.6458	0.1250	1.2716	0.6458	0.1250
S1 S2{(1, 2),(1, 3)} S3{(3),(2),(1)} S4	64	1.4760	0.6887	0.2056	1.4712	0.6884	0.2041
S1 S2{(1, 2),(2, 3)} S3{(4),(2),(1)} S4	192	1.6835	0.7268	0.2443	1.6586	0.7244	0.2366
S1 S2{(1, 2),(3, 4)} S3{(3),(2),(1)} S4	192	1.7108	0.7300	0.2478	1.6985	0.7290	0.2450

Non-isomorphic Designs (n = 10, m = 4, k = 1) (Table 10.7 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 4)} {(3, 4)} S3{(4),(1)} S4	96	1.7361	0.7339	0.2487	1.7139	0.7323	0.2449
S1 S2{(1, 2),(1, 3)} {(1, 4)} S3{(3),(1)} S4	96	1.7776	0.7450	0.2495	1.6965	0.7378	0.2157
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2)} S4	96	1.7922	0.7516	0.2727	1.7267	0.7444	0.2545
S1 S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(1)} S4	96	1.8381	0.7574	0.2627	1.7406	0.7486	0.2232
S1 S2{(1, 4),(2, 3)} {(3, 4)} S3{(4),(1)} S4	384	2.0119	0.7812	0.2960	1.9596	0.7772	0.2867
S1 S2{(1, 2),(1, 3)} {(2, 3),(3, 4)} S3{(1)} S4	192	1.9781	0.7814	0.2995	1.9464	0.7785	0.2908
S1 S2{(1, 2),(1, 3)} {(2, 3)} S3{(4),(3),(2)}	192	1.9777	0.7835	0.3070	1.9481	0.7807	2.988

Non-isomorphic Designs (n = 10, m = 4, k = 1) (Table 10.7 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1 S2{(1, 2),(1, 3)} {(3, 4)} S3{(4),(1)} S4	384	2.0212	0.7865	0.2941	1.9994	0.7846	0.2874
S1 S2{(1, 2),(1, 3)} {(2, 4)} S3{(4),(3),(2)}	192	2.0175	0.7886	0.3101	1.9986	0.7867	0.3043
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(1)} S4	192	2.0452	0.7913	0.2919	2.0215	0.7888	0.2816
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 3),(3, 4)} S3{(4),(3),(2)}	192	2.1798	0.8021	0.2910	2.1750	0.8015	0.2881
S1{(1),(2),(3)} S2{(1, 2),(1, 4)} {(2, 4),(3, 4)} S3{(4),(3)} S4	32	2.1617	0.8104	0.3125	1.9699	0.7935	0.2500
S1{(1),(2),(3)} S2{(1, 2),(1, 4)} {(2, 4)} S3{(4),(3),(1)} S4	192	2.2273	0.8153	0.3262	2.1623	0.8107	0.3169
S1 S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(4),(3)}	192	2.2953	0.8166	0.2881	2.2850	0.8160	0.2842

Non-isomorphic Designs (n = 10, m = 4, k = 1) (Table 10.7 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4)} S3{(4),(3),(1)} S4	192	2.2106	0.8196	0.3174	2.2032	0.8185	0.3088
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(3, 4)} S3{(4),(3)} S4	192	2.2567	0.8218	0.3337	2.2013	0.8179	0.3259
S1{(1),(2),(3)} S2{(1, 2),(2, 4)} {(3, 4)} S3{(4),(3),(1)} S4	192	2.3172	0.8263	0.3263	2.2586	0.8226	0.3197
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 4)} S3{(3),(1)} S4	384	2.3182	0.8295	0.3313	2.2797	0.8268	0.3250
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 3),(2, 4)} S3{(4),(2),(1)}	48	2.4069	0.8393	0.3445	2.3403	0.8360	0.3405
S1 S2{(1, 2),(1, 3)} {(1, 4),(3, 4)} S3{(4),(3)}	192	2.3431	0.8403	0.3384	2.3222	0.8370	0.3270
S1{(1),(2),(3)} S2{(1, 2),(1, 4)} {(2, 4)} S3{(3),(2),(1)} S4	48	2.3610	0.8472	0.3448	2.3193	0.8407	0.3220

Non-isomorphic Designs (n = 10, m = 4, k = 1) (Table 10.7 con't)							
Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1{(1),(2),(3)} S2{(1, 3),(1, 4)} {(2, 3),(2, 4)} S3{(3),(1)} S4	192	2.4947	0.8603	0.3599	2.4779	0.8598	0.3594
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(4),(1)} S4	192	2.5310	0.8606	0.3531	2.5152	0.8600	0.3520
S1{(1),(2),(3)} S2{(1, 2),(1, 4)} {(2, 3),(2, 4)} S3{(3),(1)} S4	384	2.6009	0.8617	0.3317	2.5771	0.8601	0.3261
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4)} S3{(3),(2),(1)} S4	64	2.4341	0.8646	0.3794	2.4317	0.8640	0.3764
S1{(1),(2),(3) } S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(3),(1)} S4	192	2.4763	0.8648	0.3614	2.4645	0.8642	0.3591
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(4),(3),(1)}	384	2.6009	0.8656	0.3420	2.5771	0.8639	0.3349
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} S3{(3),(1)} S4	192	2.4947	0.8668	0.3778	2.4779	0.8658	0.3762

Non-isomorphic Designs (n = 10, m = 4, k = 1) (Table 10.7 con't)

Designs	ND	AD^*	AT	$AMCR$	GD^*	GT	$GMCR$
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} S3 S4	64	2.4341	0.8753	0.4059	2.4317	0.8751	0.4054
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(1, 4),(2, 3)} {(2, 4)} S3{(3),(1)}	192	2.7849	0.8847	0.3272	2.7721	0.8840	0.3265
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 4),(3, 4)} S3{(3),(2)} S4	96	2.6340	0.8884	0.3872	2.5847	0.8864	0.3852
S1{(1),(2),(3)} S2{(1, 2),(1, 3)} {(2, 4)} S3{(3),(2),(1)} S4	192	2.6701	0.9084	0.4273	2.6548	0.9071	0.4246
S1{(1),(2)} S2{(1, 2),(1, 3)} {(1, 4),(3, 4),} S3{(4),(3),(2)} S4	32	2.9973	0.9107	0.3872	2.8836	0.9063	0.3852
S1{(1),(2),(3) } S2{(1, 3),(1, 4)} {(2, 3),(2, 4)} S3{(3),(2),(1)}	48	2.7551	0.9132	0.4268	2.7209	0.9124	0.4268
S1{(1),(2),(3) } S2{(1, 2),(1, 3)} {(2, 3)} S3{(3),(2),(1)} S4	16	3.0141	1.0000	0.5343	3.0141	1.0000	0.5343

We denote AD^* to be $10^6 \times AD$ and GD^* to be $10^6 \times GD$.

Chapter 11

Conclusion

We presented the robustness property of design $D = \{S_0, S_1, S_{m-1}, S_m\}$ with respect to any t runs as well as a specific set of t runs. We determined design D has maximum full estimation capacity for $k = 3$. The designs obtained from D by deleting one run also have full estimation capacity for $k = 3$. The designs obtained from D by deleting two runs may or may not have full estimation capacity for $k = 3$. The designs obtained from D by deleting three runs do not have full estimation capacity for $k = 3$. The designs obtained from D by deleting two or three runs may or may not have full estimation capacity for $k = 2$. The designs obtained from D by deleting a run from S_1 and a run from S_{m-1} where the two runs are not foldover do not have full estimation capacity for $k = 1$. All other designs obtained by deleting two runs from D have full estimation for $k = 1$. The designs obtained by deleting three runs from D may or may not have full estimation capacity for $k = 1$.

We determined the robustness property of design $D_1 = \{S_0, S_1, S_2, S_m\}$ with respect to any t runs as well as a specific set of t runs. We considered the estimation capacity of design D_1. We determined design D_1 is a resolution V plan. The designs obtained from D_1 by deleting one run is also a resolution V plan. The designs obtained from D_1 by deleting two runs may or may not have full estimation capacity for $k = \left(\binom{m}{2} - 1\right)$.

The designs obtained from D_1 by deleting two runs may or may not have full estimation capacity for $k = \left(\binom{m}{2} - 2\right)$. The designs obtained from D_1 by deleting three runs do not have full estimation capacity for $k = \left(\binom{m}{2} - 2\right)$.

We presented a general structure M for the information matrices of a class of models possibly describing the data from a fractional factorial experiment with m factors each at two levels and n runs, where $n < 2^m$. When the information matrices M have the full rank for all the models in the class, we have the full estimation capacity for the fractional factorial designs. We characterized all the eigenvalues for such matrices $\boldsymbol{M}$. We used these eigenvalues for establishing the optimality properties of the fractional factorial designs. We established the robustness property of the design $D_7 = \{S_0, S_1, S_3\}$ in the sense that the full estimation capacity of the designs remain when we delete any t runs as well as specific t runs from the design D_7. All the information matrices associated with design D_7 and designs obtained from D_7 by deleting some runs are special cases of $\boldsymbol{M}$.

For the models in (5.1) with $k = 1$. We denoted $\boldsymbol{X_{2i}} = \left[\boldsymbol{j}, \boldsymbol{X_2^{(i)}}\right]$ for $i = 1, \ldots, \binom{m}{2}$. We wrote the design matrix $\boldsymbol{X^{(i)}} = \left[\boldsymbol{X_1}, \boldsymbol{X_{2i}}\right]$. We showed for all designs with general m, n and $k = 1$ the matrix $\boldsymbol{X'_{2i}X_{2i}}$ is a principal sub-matrix of $\boldsymbol{X'_1X_1}$. When $\boldsymbol{X'_{2i}X_{2i}}$ matrix is a principle sub-matricx of $\boldsymbol{X'_1X_1}$ we proved the minimum eigenvalue of $\boldsymbol{X'_1X_1}$ is less than or equal to minimum eigenvalue of $\boldsymbol{X'_{2i}X_{2i}}$. For foldover designs with $k = 1$ we proved the minimum eigenvalue of $\boldsymbol{X^{(i)\prime}X^{(i)}}$ is equal to the minimum eigenvalue of $\boldsymbol{X'_1X_1}$. We considered the class of designs $\mathcal{D}_{\mathfrak{T}}$ with n runs for estimating the main effects only and let $\mathcal{F}_{\mathfrak{T}}$ be the class of foldover designs with $2n$ runs, n runs from $T \in \mathcal{D}_{\mathfrak{T}}$ and another n runs from $-T$, having full estimation capacity for $k = 1$. We proved that if $T^* \in \mathcal{D}_{\mathfrak{T}}$ is E-optimum, then $\binom{T^*}{-T^*}$ is optimum design with respect to $AMCR$ and $GMCR$ in $\in \mathcal{F}_{\mathfrak{T}}$. Furthermore, if T^* is D- and A- optimum with a special structure for $\boldsymbol{X'_{1T^*}X_{1T^*}}$ we proved $\binom{T^*}{-T^*}$ is optimum with respect to GD, AD, GT, and AT in $\mathcal{F}_{\mathfrak{T}}$.

For two classes of foldover designs. For $N = 2n$ even, $m = 2d$ and $k = 1$, we considered the class of foldover designs $\mathcal{D}\mathbf{1} \subset \mathcal{F}_{\mathfrak{T}}$ giving $\boldsymbol{X^{(i)\prime}X^{(i)}}$ to be block diagonal matrix with two block matrices, $\boldsymbol{X_1^{\prime}X_1}$ whose rows and columns represent the main effects and $\boldsymbol{X_2^{(i)\prime}X_2^{(i)}}$ whose rows and columns represent the general mean and the i^{th} two-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. In this class the matrix $\boldsymbol{X_1^{\prime}X_1}$ is a block diagonal matrix with d (2×2) block matrices of the same structure. In each block of the (2×2) matrix, the diagonal elements is n and the off-diagonal elements is a. For $N = 2n$, m even and $k = 1$, we considered the class of foldover designs $\mathcal{D}\mathbf{2} \subset \mathcal{F}_{\mathfrak{T}}$ give rise to $\boldsymbol{X^{(i)\prime}X^{(i)}}$ to be block diagonal matrix with two block matrices, $\boldsymbol{X_1^{\prime}X_1}$ whose rows and columns represent the main effects and $\boldsymbol{X_2^{(i)\prime}X_2^{(i)}}$ whose rows and columns represent the general mean and the i^{th} two-factor interaction, $i = 1, 2, ..., \binom{m}{2}$. In this class the off-diagonal elements is a and the diagonal elements is n in the $\boldsymbol{X_1^{\prime}X_1}$ matrix. Given $a^2 \geq a_0$ and a_0 is a nonnegative number. Let a_1 be the smallest even integer that is greater than or equal to $\sqrt{a_0}$ and $a_2 = -a_1$. For $\mathcal{D}\mathbf{1}$ and $\mathcal{D}\mathbf{2}$ if $a_0 = 0$ then $a = 0$ minimizes the determinant, trace and maximum eigenvalue of $\boldsymbol{(X_1^{\prime}X_1)^{-1}}$ as well as the optimality criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$. For $\mathcal{D}\mathbf{1}$, if $a_0 > 0$ then a_1 and a_2 minimize the determinant, trace and maximum eigenvalue of $\boldsymbol{(X_1^{\prime}X_1)^{-1}}$ as well as the optimality criterion functions GD, AD, GT, AT, $GMCR$, and $AMCR$. For $\mathcal{D}\mathbf{2}$, if $a_0 > 0$ then a_1 minimizes the determinant and maximum eigenvalue of $\boldsymbol{(X_1^{\prime}X_1)^{-1}}$ as well as the optimality criterion functions GD, AD, $GMCR$, and $AMCR$.

We obtained optimum fractional factorial designs for $m = 4$. The numbers of runs we considered are n = 6, 7, 8, 9, 10, 11 and 12. The values of k the number of two-factor interactions are 1, 2, 3, 4, 5 and 6. We presented hierarchical designs. In [Ghosh and Tian(2006)], they presented optimum designs for maximum k, we presented optimum designs for all values of k. For the designs with four factors and ten runs.

We partitioned these designs into isomorphic classes with respect to all six criteria. The models contained the general mean, main effects and $k = 1, 2, 3, 4$ and 5 two-factor interactions. We presented a table for each k. We have found that for models containing general mean, main effects, and $k = 1, 2, 3, 4$ and 5 two-factor interactions, designs within isomorphic classes can be obtained from one another by renaming factors or levels of the factors. We cannot transform a design from one class by renaming factors or levels of the factors to design of another class.

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