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Essays on Strategic Information Transmission

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Essays on Strategic Information Transmission

A dissertation submitted in partial satisfaction of the
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by

Peicong Hu

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2021

The dissertation of Peicong Hu is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2021

DEDICATION

To my family and friends.

EPIGRAPH

The chief enemy of creativity is good sense.

— Pablo Picasso

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ABSTRACT OF THE DISSERTATION

Essays on Strategic Information Transmission

by

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Doctor of Philosophy in Economics

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Professor Joel Sobel, Co-Chair
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The dissertation studies several topics about strategic information transmission, in particular, how the outcome is influenced by cognitive capacity and communication cost; and how a decision maker should organize the procedure of requesting advice from multiple experts.

In Chapter 1, I analyze how a principal should influence an agent's incentive in processing information about multiple issues when they have conflict about relative importance. I show that because it is costly for the agent to process information, it is not necessarily beneficial for the principal to provide a higher reward for better quality of information processed (even when rewards do not involve payout from the principal) or to request the agent to process more information. I characterize when the benefit of more attention induced by a higher reward or

more information available would be dominated by the cost of attention distortion, and show that the result is not monotonic in the agent's cost of attention and the relevance between issues.

In Chapter 2, we consider a manager's problem about requesting support from multiple experts to implement one (of many) projects. The game in which the manager consults experts simultaneously typically has multiple equilibria, including one in which the manager's favorite project is supported by some expert. In the leading case, we show that only one equilibrium outcome survives iterative deletion of weakly dominated strategies which is the experts' most preferred equilibrium. We identify sequential procedures that perform equally well as this equilibrium from the manager's perspective.

In Chapter 3, we study a voluntary disclosure game in which a firm discloses a signal about the future cash flow subject to proprietary costs or uncertainty about signal endowment, and rationally inattentive investors allocate their attention to disclosures. We find that for low levels of attention, more attention facilitates communication and increases disclosure; for high levels of attention, more attention better identifies, and therefore deters, unfavorable disclosure.

In Chapter 4, we examine the impact of a sender's communication cost on information transmission by introducing cost to the cheap talk model. We show that the sender's cost, imprecision of his signal, and disagreement over actions between players could lead to better communication outcomes. A moderate cost makes the sender's message more credible to the receiver, while less signal precision or more disagreement motivates the sender to provide more information.

Chapter 1

Multidimensional Information and Rational Inattention

Abstract: This paper studies how a principal should influence an agent's incentive in processing information about multiple correlated issues. The principal agrees with the agent about appropriate actions to be taken given the states, though the players have potentially different preferences about relative importance of the issues. I show that because it is costly for the agent to process information, it is not necessarily beneficial for the principal to provide a higher reward for better quality of information processed (even when the rewards do not involve payout from the principal) or to request the agent to process more information. I characterize when the benefit of more attention induced by a higher reward or more information available would be dominated by the cost of distorting attention allocation, and show that the result is not monotonic in the agent's cost of attention and the relevance between issues.
Keywords: Information disclosure, Rational inattention, Costly communication.
JEL Codes: D82, D83, D91.

1.1 Introduction

We are in a world with abundant information. Information is important for decision making, but has to be processed by spending time and costly effort. As a result, many decision makers (henceforth, the principal) rely on experts (henceforth, the agent) to process (and communicate) information to them. If there is information about a single issue, it is in the principal's interest to motivate the agent to process more information. The principal knows the informational environment and will incorporate any useful information into decision making. If, however, there are multiple issues that are relevant to the players' payoff, the agent with limited attention needs to trade off what information to process. Then a higher motivation to process information can influence what issue appears to be more important to the agent and his attention allocation. Therefore it becomes a question whether the principal should offer high powered incentives to the agent or request the agent to process information about multiple issues.

If the agent ranks importance of issues in the same way as the principal, he will allocate his attention across issues as desired by the principal. Hence it is beneficial for the principal to induce more attention from the agent. But if they disagree about relative importance of issues, inducing more attention can have negative consequences of distorting the agent's attention allocation. The agent decides how he will allocate his attention based on what issue is more important and what information is available to him. Hence the principal might induce the agent to misallocate his attention (from the principal's perspective) when motivating the agent to process more information by influencing his payoff and availability of information. This suggests a possibility that principal may choose to limit information provided in order to influence what the agent pays attention to, which could never be beneficial for her unless agent has limited attention and different ranks over issues. This paper provides analysis about how the principal should trade off between more information processed and better attention allocation when the players disagree about relative importance of issues. I show that the optimal choice of the principal is not monotonic in the agent's cost of attention and relevance between issues.

The conflict about relative importance of issues is common in many situations and there are a variety of reasons for there to be systematic differences in preferences. From the perspective of a manager in high-technology or pharmaceutical companies, projects of basic and applied research are both important. But subordinate analysts are more interested in the applied research that promises an immediate return. From the perspective of instructors, higher-order cognitive skills, such as analysis, are as important as lower-level memorizations. But students tend to be biased toward testable lower-level skills that were the focus of exams. In these situations where the players disagree about relative importance of issues, it is important for the principal to understand how she should influence information processing of the agent.

I consider a model in which the agent processes information about a correlated[1] multi-dimensional state of the world. I provide a simple characterization about the agent's attention strategy that generalizes Kőszegi and Matějka (2020) to correlated states and distinct variances

[1] In pharmaceutical companies, the applied research (i.e., conducting clinical trials of a drug) may generate basic insights into basic research (i.e., the physiology of a disease), and vice versa.

of different dimensions. Then I specialize the model to the particular problem studied in the paper. The agent processes information about a two-dimensional state and the principal can affect the agent's payoff or determine information available to him. For example, the principal can provide promotion-based incentives to reward the agent for good quality of information processed (Cockburn et al., 1999) or control what datasets the agent can get access to.[2] To focus on information acquisition, I assume that the principal has selected an agent who agrees with her about what action to take given the information. The players, however, assign potentially different weights to the loss from suboptimal actions in each dimension.

First, I study whether it is beneficial for the principal to motivate information processing of the agent by providing a higher reward for better information. In particular, I examine how the principal should trade off the benefit of more attention induced by a higher reward against the cost of inefficient attention allocation (from the principal's point of view). I abstract away from any transfers and assume that the principal can adjust the agent's loss from suboptimal actions for free.[3] I show that providing a higher reward will lead to a higher payoff for the principal when agent's cost of processing information is either below a lower cutoff or above a higher cutoff. When the cost is sufficiently low, the agent's effort is highly sensitive to the increasing loss from suboptimal actions. He will be able to make inferences from (a lot) more information acquired about one dimension to compensate (the loss) in the other dimension from reallocated attention. When the cost is sufficiently high, the agent acquires little information (unless he loses much from suboptimal actions). As a result, little attention will be reallocated when his (relative) stake in the two dimensions changes. Further, additional information is highly valuable when the amount of attention is low, due to diminishing returns to learning. Therefore, more information processed dominates inefficient attention reallocation. Nevertheless, the latter effect could be dominant at intermediate cost and the principal could be worse off by increasing the agent's loss from suboptimal actions (relative to the highest payoff he may attain).

[2] For instance, the manager knows whether the data is about basic research or applied research (and can withhold data from the analyst), but does not know the data content that has to be processed by the analyst.

[3] Monetary contracts are absent in most of the theories about delegated expertise (see Holmstrom (1978, 1980); Armstrong (1995); Dessein (2002); Szalay (2005); Alonso and Matouschek (2008)).

The result offers one explanation for the increasingly "balanced" incentive for the agent. There has been more attempts to balance incentives across dimensions in organizations, even at the expense of lower effort level. For example, many companies have adopted a systematic balanced scorecard approach to executive remuneration and significantly shifted focus away from directly commercializable work to long-term value creation. The increasing emphasis on balanced incentives is consistent with my prediction. I show that when technological advances reduce the cost of information processing (but attention is still a scarce resource), it could be in the company's interest to weaken employees' incentive to process certain information, because the gain from more balanced attention allocation outweighs the loss from (slightly) lower attention level.

The principal can influence the agent's information processing by controlling what is available to the agent as well. Withholding information about one dimension enhances the marginal value to learn about the other dimension if the two dimensions are correlated. I compare principal's payoff when she makes information about both dimensions available to the agent and when she provides information about one dimension only. I show that it could benefit the principal to withhold the information that the agent is biased toward at intermediate cost of attention (and imperfect correlation between states of the two dimensions). When cost is low, agent will devote adequate attention to information about both dimensions if all is provided. So there is no need to restrict information provision. As cost increases, the gain from more information being provided declines, because the agent is not able to assimilate that much information. But the loss from "biased" learning grows, because the agent will "sacrifice" less important things first given scarce attention. Finally, when cost is extremely high, the agent is reluctant to process information that is less valuable to him. Hence the benefit will be minimal from "forcing" the agent to learn about this information by limiting his access to other information. Therefore it is only beneficial to withhold information at intermediate cost.

Simon (2019)'s comment has resonance here: "The problem of information provision is to design intelligent information-filtering systems". I show that the principal intervenes in order to improve quality of information that is of the highest priority from her perspective. So disagreement about relative importance of issues is the fundamental reason why some information should be filtered out given that the agent will optimally use the information acquired.

The remainder of the paper is organized as follows. Section 1.2 describes papers most closely related to my work. Section 1.3 presents the basic model. Section 1.4 examines the effect of changing the agent's payoff. Section 1.5 contains the analysis about limiting information. The appendix contains proofs that are not in the main text.

1.2 Related Literature

"Rational inattention" has been useful in explaining a variety of economic problems since the seminal work of Sims (2003) and has been applied to macroeconomics, finance, microeconomics, and political economy (Mackowiak et al., 2018). The multidimensional learning problem is recognized as one of the major problems that involve choice of attention. A profit-maximizing firm decides whether to attend to idiosyncratic shock or nominal shock when setting prices (Mackowiak and Wiederholt, 2009). An investor decides what asset payoffs to learn about when forming portfolios (Van Nieuwerburgh and Veldkamp, 2010). A voter decides what policy issue to focus on when making voting decisions (Matějka and Tabellini, 2017). A consumer decides what consumption opportunities to evaluate when allocating expenditures (Kőszegi and Matějka, 2020). Besides the information choice of an agent (who has limited attention), my paper investigates how a principal should control the agent's payoff and information availability to influence agent allocating attention to multiple issues.

To my knowledge, there are three papers that study persuasion under rational inattention (Bloedel and Segal, 2018; Lipnowski et al., 2020a,b). Lipnowski et al. (2020a) studies a model in which there are three ordered states. A benevolent principal who has the same material preferences decides how to communicate to an agent with rational inattention. The principal will limit information about the moderate state to increase the agent's loss if he slacks and hence to enhance the marginal value of learning. In the same three-state example, another paper by Lipnowski et al. (2020b) that is contemporaneous to this paper finds that the optimal information policy involves truthfully revealing the moderate state only when cost is small but revealing the extreme states only when cost grows. Bloedel and Segal (2018) find that if the principal and the agent have misaligned preferences, the principal would withhold information for two reasons: to manipulate the agent's attention and to bias the agent's decision. My paper considers a game with *two-dimensional* state of the world and analyzes how conflict in relative importance across dimensions between the principal and the agent would affect principal's strategy. Treating the decision problem about each dimension as one issue, I show that it would never benefit the principal to exclude either issue if they hold the same opinion about relative importance. So I highlight the role of preference weights about different issues in strategic communication. Moreover, the discussion about influencing agent's loss from suboptimal actions, which serves as another tool to manage agent's attention apart from withholding information, elaborates the trade-off in attention allocation between multiple *correlated* dimensions further. Lu (2019a) examines the trade-off in the financial reporting system between a noisy summary and precise details. The summary has less information content but requires less capacity to process. The tension that influences the principal's optimal strategy is not different preferences about relative importance of multiple decisions.[4] Bertomeu et al. (2020) study a voluntary disclosure game in which the investor optimally allocates fixed attention capacity. There is no moral hazard that is central to this model.

The literature about evidence exclusion provides insights about withholding information presented to the jury in trials. Lester et al. (2012) describe an example in which a piece of evidence or a jury with superior ability could reduce the marginal benefit to seek other information, which leads to a strictly worse outcome for the society. In contrast, I show that a lower cost for the agent to acquire information will always benefit the principal in this model,

[4] In Lu (2019a), the investor (agent) has a single decision that depends on the adaptation and coordination motives.

as the agent can *flexibly* choose any learning strategy.[5] Given the flexible information choice, when they have exactly the same material preferences, it will harm the principal to exclude either dimension. Bull and Watson (2019) study the merit of excluding *misleading* hard evidence in the court when the litigant and fact-finder have beliefs that are out of alignment as another mechanism for withholding evidence to improve accuracy in the deliberations of a jury.

Multidimensional analysis in strategic decision making dates from the seminal paper of Holmström and Milgrom (1991) that studies a principal-agent model (*without communication*). We can make an analogy that the agent gives out attention (input) to exchange for useful information (output). A similar trade-off in this paper is that an increase in an agent's stake in any one task (i.e., increasing weight of loss) will cause some reallocation of attention away from other tasks. But the trade-off, more attention to one dimension leading to less attention to the other, is not caused by interplay between inputs (i.e., how effort in one task influences cost of the other task) in the information context. Rather, the trade-off is driven by correlation between outputs, that is information of one dimension infers something about the other dimension and makes the information of the other dimension less valuable. This has different implications for the results. In Holmström and Milgrom (1991), if principal wants agent to do better on one thing, then a reward on the other thing will be harmful. In contrast, I find that when information processing is subject to moral hazard, if principal wants the agent to do better on one thing, it could be beneficial for her to increase reward on the other issue. I characterized when the agent will end up with a more precise belief if principal increases reward on the other issue, and when it is the other way around. In addition, the agent can flexibly design how to process information in my model.

[5]Instead of the need to read the whole piece of evidence, the agent can choose how much information he will process for each dimension in this model.

1.3 The Benchmark Model

This section describes the general model in which an agent with a linear-quadratic utility function processes information about multi-dimensional states at some cost and then takes an action on each dimension. Proposition 1 characterizes the agent's optimal attention strategy. The simple characterization generalizes the result in Kőszegi and Matějka (2020) to states that are correlated across dimensions and/or have distinct variances.[6]

1.3.1 The Preliminaries

There are two players, principal and agent. The principal is potentially informed of the realization of an N-dimensional state of the world $\boldsymbol{x} \in \mathbb{R}^N$. Assume that the prior uncertainty about $\boldsymbol{x}$ is multivariate Gaussian, i.e., $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, \Psi)$ where the mean is normalized to zero and Ψ is the variance-covariance matrix. In contrast to Kőszegi and Matějka (2020) where the matrix Ψ is equal to ψI for some constant ψ and I being the identity matrix, I did not impose any assumption about the variance-covariance matrix.

I first analyze the subgame in which the agent processes information assuming all is made available to him. In Section 1.4, I examine how the principal should influence the agent's preference parameter assuming all information is provided. In Section 1.5, I examine what information the principal should provide to the agent assuming the preference parameter of the agent is fixed. The subgame (starting with the agent's move) proceeds as follows. The agent chooses an information structure about the joint distribution of the state $\boldsymbol{x}$ and the signal he observes. Then the state is realized and the agent observed a (noisy) signal subject to his attention choice. Finally, the agent takes actions $\boldsymbol{a} \in \mathbb{R}^N$ and their payoffs are realized.

For Gaussian priors, Sims (2003) shows that the optimal signal for an agent with a linear-quadratic utility function is Gaussian as well among all possible information structures.

[6]To my knowledge, Kőszegi and Matějka (2020) is the first paper dropping restrictive assumptions about the signal structure and derives the optimal one explicitly. But they assume that the states are independent across dimensions and the variance in each dimension is the same.

Furthermore, the posterior uncertainty generated by the optimal signaling structure is Gaussian and has a constant, i.e., independent of realizations, variance-covariance matrix. Because all information is assumed to be made available to the agent, his posterior belief generated by the optimal signal is Gaussian. Hence, without loss of generality, I simplify the agent's problem by the following reduced form of the attention cost.

The optimal signal features the agent choosing a constant posterior variance-covariance matrix Σ subject to the no-forgetting constraint that $\Psi - \Sigma$ is positive semi-definite.[7] The no-forgetting constraint requires that the posterior has to be (weakly) more precise than the prior. Let the posterior distribution of the state $\boldsymbol{x}$ be $\mathcal{N}(\tilde{\boldsymbol{x}}, \Sigma)$, where $\tilde{\boldsymbol{x}}$ is mean of the posterior belief and Σ is the variance-covariance matrix. Denoting by $|\cdot|$ the determinant of a matrix, the cost of attention for the multivariate Gaussian distribution is given by

$$
\begin{aligned}
&(c/2)(\log(2\pi e)^2|\Psi| - \log(2\pi e)^2|\Sigma|) \\
=&(c/2)(\log|\Psi| - \log|\Sigma|),
\end{aligned}
$$

where $c \geq 0$ is the agent's unit attention cost for reduction in the entropy of his belief.[8]

First, I provide an analytical solution to the agent's problem of attention allocation in the general model. The agent's payoff is given by

$$
u^A(\boldsymbol{a}, \boldsymbol{x}, \Sigma) = \underbrace{-(\boldsymbol{a} - \boldsymbol{x})' R (\boldsymbol{a} - \boldsymbol{x})}_{\text{agent's loss from suboptimal actions}} - \underbrace{(c/2)(\log|\Psi| - \log|\Sigma|)}_{\text{agent's information cost}},
$$

where R is a (real-valued) symmetric positive-definite matrix in the agent's utility function.[9] The matrix R represents the interaction of losses from misperception across different dimensions or potentially different weights the agent assigns to the losses in these dimensions.

[7] The literature has well established that "choosing the signal variances is the same as choosing a variance of the posterior belief" (Veldkamp, 2011a). Further, one rule for Bayesian updating with normal variables is that the precision of the posterior belief is the sum of the precisions of prior and signals.

[8] I adopted the simplest (linear) entropy cost function to illustrate the intuitions. The comparative statics results in Sections 1.4 and 1.5 are robust with respect to a broader range of alternatives, including the convex entropy-based cost function.

[9] The specification is equivalent to the utility function in Kőszegi and Matějka (2020).

The solution concept is subgame perfect Nash equilibrium.

1.3.2 Agent's attention strategy

The solution to the agent's problem is based on the water-filling algorithm used in information theory (see Telatar (1999); Cover and Thomas (2006)). If the attention cost is extremely high, the agent will not process any information. If the cost is somewhat lower, the agent will choose to process information about a single dimension that is a linear combination ("*principal component*") of the state vector. If the cost is even lower, the agent will process information about more dimensions.

The agent maximizes his expected utility $u^A(\boldsymbol{a},\boldsymbol{x},\Sigma)$ that depends on the vector of states $\boldsymbol{x}$, the vector of actions $\boldsymbol{a}$, and his information cost. Let us solve the agent's problem backwards.

Given a posterior belief about the state realization $\boldsymbol{x} \sim \mathcal{N}(\tilde{\boldsymbol{x}},\Sigma)$, the agent will choose actions $\boldsymbol{a}$ equal to the posterior mean $\tilde{\boldsymbol{x}}$ in order to minimize the quadratic loss.

Lemma 1. *Given a posterior belief about the state $\boldsymbol{x} \sim \mathcal{N}(\tilde{\boldsymbol{x}},\Sigma)$, the agent's actions $\boldsymbol{a}$ are equal to the posterior means $\tilde{\boldsymbol{x}}$.*

Suppressing the constant term of the prior entropy, the agent's problem is

$$\max_{\Sigma} \quad \underbrace{-\mathbb{E}_{\tilde{\boldsymbol{x}}}[\mathbb{E}_{\boldsymbol{x}}((\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x})|\tilde{\boldsymbol{x}})]}_{\text{Loss from imperfect posteriors}} \quad + \quad \underbrace{(c/2)\log|\Sigma|}_{\substack{\text{Information cost}\\ \text{(excluding constant prior entropy)}}} \tag{1.1}$$
$$\text{subject to} \quad \Psi - \Sigma \text{ is positive semi-definite.}$$

The term $\tilde{\boldsymbol{x}}-\boldsymbol{x}$ is the misperception. By the Cholesky decomposition, the symmetric positive-definite matrix R can be decomposed to a product of an upper triangular matrix H with positive diagonal entries and its transpose, i.e., $R = H'H$.

Lemma 2. *Let $\Phi = H\Psi H'$ and $\Xi = H\Sigma H'$. The agent's problem can be written as*

$$\max_{\Xi}\{-Tr(\Xi)+(c/2)\log|\Xi|\} \qquad \textit{subject to } \Phi-\Xi \textit{ is positive semi-definite.} \tag{1.2}$$

Then I transform the coordinates to work on independent states across dimensions and show that they are independent in the posterior as well. The matrix Φ is symmetric by definition and thus has an orthonormal basis of eigenvectors. Let $\boldsymbol{v}^i$ be an orthonormal basis of eigenvectors of Φ with the eigenvalue corresponding to $\boldsymbol{v}^i$ denoted by Λ_i, where $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_N$. Let U be the (unitary) matrix that consists of the eigenvectors of Φ and Λ be the diagonal matrix such that the $(ii)^{th}$ entry is given by Λ_i.

Proposition 1. *The optimal attention strategy is to acquire independent signals of $\boldsymbol{v}^i \cdot \boldsymbol{q}$ such that the posterior variance of $\boldsymbol{v}^i \cdot \boldsymbol{q}$ is $\min(c/2, \Lambda_i)$, where $\boldsymbol{q} =: H\boldsymbol{x}$.*

This signal is optimal without any restriction about the agent's signal structure. For example, the eigenvectors are not restricted to be the same for prior and posterior variance matrices. But I show that $\boldsymbol{v}^i$ $(i = 1, \ldots, N)$ are actually the eigenvectors of posterior variance-covariance matrix generated by the optimal signal as well after the transformation. In other words, the agent acquires information about $\boldsymbol{v}^i \cdot \boldsymbol{q}$ separately for each dimension, subject to the no-forgetting constraint that the posterior variance cannot exceed the prior variance Λ_i. The loadings given by vectors $\boldsymbol{v}^i$ captures how much loss the agent would incur from misperception in each dimension, the interaction between actions, and the correlation of states across the dimensions.

Proposition 1 shows that the agent's attention strategy has the following pattern. If the matrix Φ is diagonal, then the agent will learn the state in each dimension separately and ignore information about the dimension with variance that is not big enough. In general, when $c/2 > \Lambda_1$, the agent will not process any information. When $c/2$ is between Λ_{k+1} and Λ_k for $1 \leq k \leq N-1$, the agent will only acquire information about the first k (transformed) dimensions that has more prior uncertainty. These dimensions will be more valuable for the agent to learn about due to diminishing returns to learning by the entropy specification. The amount of information acquired for each dimension is $\Lambda_k - c/2$. When $c/2 \leq \Lambda_N$, the targeting posterior variance $c/2$ would be smaller than the prior variances of all dimensions. The agent will hence acquire information about all dimensions.

1.3.3 Conflict interest about relative importance

I apply the characterization in Proposition 1 to the specific problem studied in this paper. Suppose that there are two dimensions. The principal and the agent both want actions to match the states, but they disagree about the relative importance of dimensions. Assume without loss of generality that the principal assigns equal weights to both dimensions, while the agent has a potential bias either toward or against dimension two.

Formally, the agent's payoff is given by

$$u^A(\boldsymbol{a},\boldsymbol{x},\Sigma) = -\left[(a_1-x_1)^2+\gamma(a_2-x_2)^2\right]-(c/2)(\log|\Psi|-\log|\Sigma|),$$

where $\gamma>0$ captures the potential bias of the agent. If $\gamma>1$, the agent thinks dimension two is more important than one. If $\gamma<1$, the agent would value the accurate decision in dimension one more. Given this payoff, the matrix R has $R_{11}=1, R_{12}=R_{21}=0, R_{22}=\gamma$; the matrix H has the same entries as R except $H_{22}=(\gamma)^{1/2}$. The principal's payoff is given by

$$u^P(\boldsymbol{a},\boldsymbol{x}) = -\left[(a_1-x_1)^2+(a_2-x_2)^2\right].$$

Recall that Λ_i and $\boldsymbol{v}^i$ are the eigenvalue and eigenvector of the matrix $\Phi = H\Psi H'$, respectively. I close this section with the principal's expected payoff when all information is available to the agent.

Lemma 3. *Let σ_1^2 and σ_2^2 be the prior variance of states x_1 and x_2. Let v_1^1 be the loading of x_1 on the transformed dimension with larger variance. The principal's expected payoff assuming all information is available to the agent is given by*

$$\begin{cases} -(c/2)[1+(1/\gamma)] & \textit{if } c/2 \le \Lambda_2 \\ -(\sigma_1^2+\sigma_2^2)+[\Lambda_1-(c/2)][(v_1^1)^2+(1/\gamma)(1-(v_1^1)^2)] & \textit{if } \Lambda_2 < c/2 \le \Lambda_1 \\ -(\sigma_1^2+\sigma_2^2) & \textit{if } c/2 > \Lambda_1 \end{cases} \cdot$$

It is helpful to understand the lemma by considering the case in which there is no correlation between dimensions in the prior, i.e., Ψ is diagonal. Then the matrix Φ is diagonal as well in the specific problem here.[10] This implies that the agent processes information about the two dimensions separately, because learning one thing is not useful to infer about the other. When cost is small enough, the agent processes information about both dimensions. When cost is somewhat larger, the agent will only process the (more valuable) information about the dimension with larger variance.

If there is correlation between dimensions, the agent will be able to infer something about one when learning about the other. Hence, we transform the coordinates to examine two "synthetic" dimensions, one with larger variance that captures the most uncertainty of the original two dimensions and the other with smaller variance that captures the rest of the uncertainty. If $c/2$ is less than or equal to the smaller variance Λ_2, the agent acquires two signals and the posterior variances of the two original dimensions x_1 and x_2 are $c/2$ and $c/(2\gamma)$, respectively. So the principal's expected payoff is given by $-[c/2+c/(2\gamma)]$. If $c/2$ is between Λ_2 and Λ_1, the agent only acquires a signal about one transformed dimension. The first term $-(\sigma_1^2+\sigma_2^2)$ is principal's expected payoff without information. The second term is the gain from information acquired about the (transformed) dimension with larger variance. If $c/2$ is greater than Λ_1, it is not worthwhile for the agent to acquire any information, because the prior variances have been lower than the target variance $c/2$. Figure 1.1 illustrates the principal's expected payoff.

In Sections 1.4 and 1.5, I discuss two strategies for the principal to "improve" the agent's learning process. The principal trades off inducing more attention devoted by the agent against the distortion in attention allocation.

[10]Note that the matrix H is diagonal when R is diagonal. It follows that Φ is diagonal, because the product of diagonal matrices is diagonal.

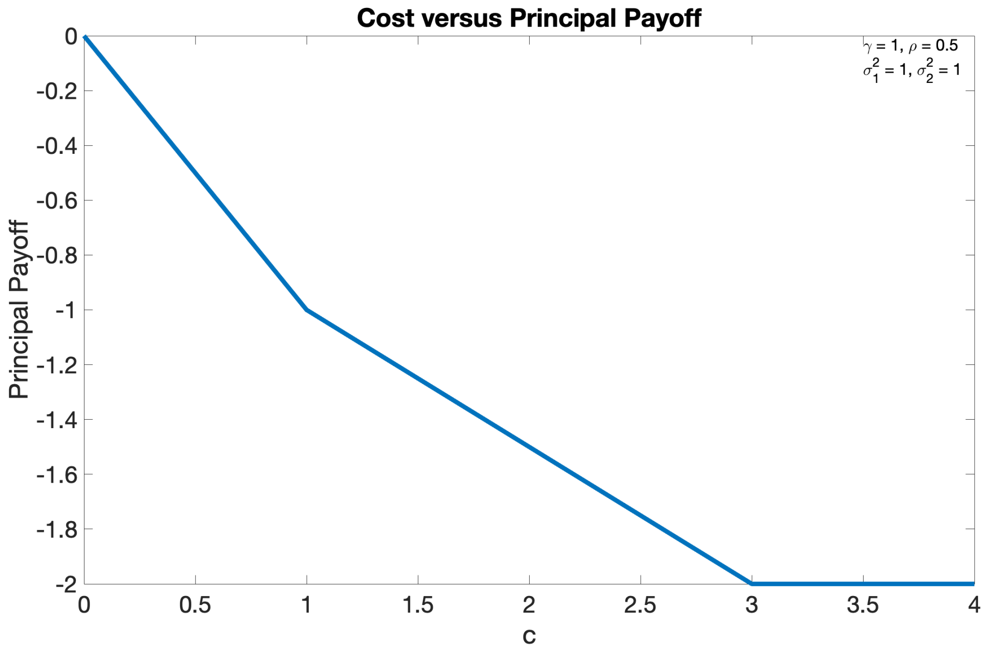

Figure 1.1: c versus u^P

1.4 Change agent's payoff

This section examines how the principal should influence the agent's weight of loss, in particular, his bias parameter γ. Instead of providing the equilibrium weight of loss which depends on the extent to which the principal has influences over the agent's payoff, I do comparative statics on the principal's expected payoff with respect to γ and find that it could be worse for the principal to increase the agent's weight.[11] Intuitively, increasing γ would induce the agent to devote more attention as he has a higher stake on one of the dimensions (and same on the other). The agent, however, would also shift some of his attention to dimension two that was allocated to one before the change. As a result, the principal benefits from more attention paid by the agent in processing information, but has to bear the (potentially negative) consequence of inducing an "unbalanced" incentive scheme that distorts the agent's attention allocation.

[11]Most results in this section are *local* in the sense that my focus will be the effect on the margin, because I assume that the principal has limited ability to influence agent's preferences. On one hand, if there is no limits, the problem will be trivial. It will be certainly in the principal's interest if she could bring agent's loss to extremely high when the two dimensions are sufficiently correlated. The agent will have an incentive to learn precisely about dimension two when γ is big enough and make inferences about the other dimension. On the other hand, the limits are realistic. The principal usually does not have an ability to arbitrarily influence agent's preferences.

The problem of “balancing employees’ incentives” arises in science-based industries such as pharmaceuticals, because firms investing in technological innovation face a difficult trade-off in allocating time and resources between longterm, “basic” or “fundamental” research activities and short-term, directly commercializable work (Cockburn et al., 1999). Investments in basic research have no immediate payoff, but contribute to the firm’s long run research capabilities. Applied research, on the other hand, promises a more immediate return from developing marketable products. It has been documented that a number of large pharmaceutical companies tend to provide more “balanced” incentives to employees in conducting basic and applied research nowadays. My theory provides one explanation for the trend toward a more balanced incentive. Further, it is common that the companies use promotion-based incentives for basic research. Hence the influence of principal modeled by affecting the agent’s weight γ applies here, where direct transfers are not available.

Suppose that the principal could marginally change agent’s weight of loss γ before state realization. Proposition 2 shows that if the two dimensions are perfectly correlated or have no correlation at all, it strictly benefits the principal to enhance agent’s incentive of learning about dimension two, regardless of the value of γ.

Proposition 2. *If* $\rho = \pm 1$*, agent’s posterior variances in both dimensions are decreasing in* γ*. If* $\rho = 0$*, agent’s posterior variance in dimension two is decreasing in* γ *but posterior variance in dimension one would be constant. In both cases, principal’s payoff is increasing in* γ*.*

When the information is perfectly correlated, the principal is essentially enhancing agent’s incentive to learn about both dimensions. When the information is completely uncorrelated, the agent learns more about dimension two but acquires the same amount of information about dimension one. As a result, the principal gains a higher payoff.

The results are not as straightforward when states of the two dimensions are *imperfectly* correlated. Proposition 3 shows that the result is definite when $\gamma \leq 1$.

Proposition 3. *Suppose that* $\gamma \leq 1$*. When* $c/2 \leq \Lambda_1$*, the principal’s payoff is strictly increasing* *in γ. When $c/2 > \Lambda_1$, the agent does not process any information until Λ_1 reaches the threshold $c/2$.*

There are two consequences on learning when γ increases. On one hand, agent will be motivated to devote more attention to process information. The decisions in both dimensions will be more informed fixing the allocation of attention. On the other hand, agent will shift some attention to the increasingly important dimension. When $\gamma < 1$, this would align their preferences across issues. The two effects add together, leading to a higher payoff for the principal when raising the agent's weight on the issue in which he has less interest.

The effect of increasing agent's weight γ, however, is not clear if γ is greater than one. Given that agent is already biased toward dimension two, reallocation of attention would act against the benefit from more attention from the principal's point of view. I show that principal will be better off by raising the agent's weight on the issue that he is biased toward, except at intermediate cost of attention and intermediate correlation between states.

Proposition 4. *Suppose that $0 < |\rho| < 1$ and $\gamma > 1$. (i) When $c/2 \leq \Lambda_2$, the agent's posterior variance in dimension one is constant and posterior variance in dimension two is decreasing in γ. (ii) When $\Lambda_2 < c/2 \leq \Lambda_1$, the agent's posterior variance in dimension one is increasing in γ at lower cost and decreasing in γ at higher cost. The posterior variance in dimension two is always decreasing in γ. (iii) When $c/2 > \Lambda_1$, the agent does not process any information until Λ_1 reaches the threshold $c/2$.*

The effect of increasing γ is ambiguous at intermediate cost (and imperfect correlation). When γ is greater than 1, so that the agent is biased toward dimension two, the two effects are countervailing from the principal's perspective when raising further the weight on this dimension: more information processed but even more unbalanced allocation of attention. The principal's payoff depends on the interaction of these two forces. Further, their relative strength depends on cost of attention and correlation between states. At intermediate cost, a higher reward modeled by an increase in γ shifts a lot of attention from the other dimension, though little additional information will be processed.

Let us take a further look at case (ii) where attention reallocation could be dominant. A higher weight on the agent's preferred issue is harmful for principal at intermediate cost, because the agent's belief about the issue in which he has less interest becomes less precise as the weight increases. On one hand, the cost is big enough, so the agent will not process a lot of expensive information and faces a real trade-off. A higher weight on the agent preferred issue will further reallocate attention from the other issue, because the agent will sacrifice less important things first given scarce resources. On the other hand, the cost is not very big, so the agent does pay quite a bit of attention ex ante. If he has a higher weight on his preferred issue, then a significant amount of attention will be shifted from the other issue. Furthermore, additional information is less valuable when the agent already devotes quite a bit of attention, because of diminishing returns to learning. Therefore, the additional attention induced by a higher weight is not enough to compensate the principal's loss from unbalanced learning.

Compared to Proposition 2, the principal can be worse off by increasing γ when issues (corresponding to the two dimensions) are *imperfectly* correlated. A higher weight on the agent's preferred issue is harmful for principal at intermediate correlation. A higher correlation means more valuable information and more attention reallocation. On one hand, the correlation is big enough. Then the agent can make a lot of inferences across issues. When a higher weight on the issue that the agent is biased toward induces more attention to this issue, lower attention will be needed for the other issue. This means that a lot of attention will be shifted to the agent preferred issue and the principal incurs high loss from unbalanced learning. On the other hand, the correlation is not very big. Then knowing one does not directly inform the other. I show that the loss from imprecision about the other issue cannot be made up by a better knowledge about the agent preferred issue. This is because the information is not valuable enough for the agent to devote a lot more attention as weight increases, because it does not perfectly inform him of both dimensions.

Proposition 4 suggests that it is more important to balance employees' incentive when information processing is no longer tremendously costly, which provides an explanation for the trend toward a more balanced incentive. The result is subject to an alternative interpretation. If the principal selects among a group of agents with various (non-pecuniary) preferences to acquire information about two projects, I show why and when it is not optimal to appoint the one with too strong interest in one project even if he does not have less interest in the other project than rest of the agents. For example, it is more harmful at recent lower information cost to appoint an empire-building manager who has unbalanced incentives and cares too much about large scale merger and acquisition projects. The appropriate choice should be made based on agent's *relative* preferences across projects (compared to the principal).

1.5 Control information availability

Besides influencing the agent's weight of loss, it is natural for the principal to consider a substitutable policy: what information should be made available to the agent? Should the principal withhold information if knowing that the agent will take an action that is desired for him? Information control serves as a substitutable tool for the principal when performance of the agent is hard to be contracted or made him responsible for. This section address what information could be ever optimally banned from the agent and characterize conditions under which it is sensible for the principal to withhold information.

I assume that the principal chooses whether to provide perfect information or no information on each dimension before any state realization.[12] Upon observing the principal's choice, the agent decides his attention strategy (that is a joint distribution between the signal he observes and the true signal). When the principal withholds information about one dimension but provides perfect information about the other, the resulting posterior beliefs would still be Gaussian *if agent optimally chooses the signal*. This will be clear from the decomposition below that transforms the

[12]A straightforward extension would be: The principal generates a signal s_i for each dimension with some normally distributed noise (noises are allowed to be correlated across dimensions). Then the agent decides how he will learn about the signal s_i. So the agent's choice of acquiring information about x_i studied in the paper is a special case of the problem where the signal s_i is perfect. The extension is feasible in the multivariate Gaussian environment. But given the leading motivation that the agent processes information on behalf of the uninformed principal, the analysis in Section 1.5 will be a natural benchmark.

two dimensional problem into one with a random loss from suboptimal actions on the disclosed dimension plus a constant loss from another independent dimension. So the theorem in Sims (2003) applies here as well, implying that it is without loss of generality to restrict our attention to an Gaussian signal (on the disclosed dimension). Hence the agent is still choosing a posterior variance-covariance matrix subject to a no-forgetting constraint.

1.5.1 Agent's strategy if one dimension is excluded

Suppose that the principal only provides the perfect signal about x_i, but withholds any information in dimension j. For the state $\boldsymbol{x} = \begin{pmatrix} x_i \\ x_j \end{pmatrix}$ such that $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, \Psi)$ where $\Psi = \begin{pmatrix} \sigma_i^2 & \rho\sigma_i\sigma_j \\ \rho\sigma_i\sigma_j & \sigma_j^2 \end{pmatrix}$, we can write in terms of two independent random variables z_i and z_j:

$$x_i = \sigma_i z_i \qquad \text{and} \qquad x_j = \sigma_j(\rho z_i + \sqrt{1-\rho^2} z_j).$$

In the matrix form,

$$\begin{pmatrix} x_i \\ x_j \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ \sigma_j\rho & \sigma_j\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} z_i \\ z_j \end{pmatrix}.$$

The prior variance-covariance matrix of $\boldsymbol{z} = \begin{pmatrix} z_i \\ z_j \end{pmatrix}$ is the identity matrix I. The agent's problem could be solved in a simple way with $\boldsymbol{z}$ being the argument, as the principal's signal does not involve anything about z_j.

Lemma 4. *If only the signal about x_1 is available, the optimal information strategy is to acquire this signal such that the posterior variance about x_1 is given by $\sigma_1^2 \min(1, \frac{c}{2(\sigma_1^2+\gamma\rho^2\sigma_2^2)})$. If only the signal about x_2 is available, the optimal information strategy is to acquire this signal such that the posterior variance about x_2 is given by $\sigma_2^2 \min(1, \frac{c}{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)})$.*

The agent's loss could be written as a quadratic function of a Gaussian random variable z_i. So the optimal signal for the agent is still Gaussian. Lemma 4 shows how agent chooses the posterior variance of x_i to trade off his loss from misperception against the information cost.

The principal's expected payoff if she provides the signal about dimension one only or about dimension two only follows from Lemma 4. I compare it with her payoff when all information is provided to the agent in the next section.

1.5.2 Principal's strategy of information provision

If states of the two dimensions are perfectly correlated, the agent could "precisely" infer the realization of one dimension if observing that the other. Essentially, agent only needs access to the information about one of the dimensions only. If states are independent across dimensions, agent processes information about the two dimensions separately. So if the principal excludes the signal of some dimension, the agent will not be able to learn anything about this dimension but still learn the same amount of information about the other dimension. Hence it is not in the principal's interest to withhold any information in either case. This section investigates the potential merit of withholding information (about some dimension) by assuming that the states are imperfectly correlated, i.e., $0 < |\rho| < 1$. In this case, the agent's marginal value of learning about one dimension is enhanced by excluding the other one because of substitutability.

If withholding information benefits the principal, it must be that the agent shifts some attention to the issue less important to him (but still important to the principal). This happens when the agent does not get access to information that is more important to him.

The next result shows that the principal should never withhold the information in which the agent has less interest. Without loss of generality, assume that the agent is biased toward dimension two, i.e., $\gamma > 1$.

Proposition 5. *For $\rho \neq \pm 1$, the principal's payoff from providing both signals is strictly higher than her payoff from providing only the signal about x_2, unless no information is acquired in both regimes. For $\rho = \pm 1$, the principal's payoff from providing both signals is exactly equal to* *her payoff from providing the signal about either dimension.*

Recall that there are two underlying forces that influence principal's payoff. On one hand, when being provided with more information, agent is able to formulate an attention strategy that is at least as well as, if not better than, having access to information about one dimension only. Because agent still values information about dimension one, the agent would attach a higher value to learning if he is able to freely construct his attention strategy with all information available. Hence agent is willing to devote more effort to learning overall when information is fully disclosed. On the other hand, the principal has a higher stake in the decision quality on dimension one relative to dimension two than the agent. Since agent incurs loss from suboptimal actions on dimension one, he will acquire some information about this dimension if being provided.[13] Although withholding information about dimension one would induce the agent to acquire more information about dimension two to compensate the loss, what agent learns instead could not substitute for the lost information about dimension one from the principal's perspective.[14] Therefore, principal should never withhold information about the dimension to which the agent attached less importance.

It, however, could benefit the principal to exclude dimension two in certain conditions, because the effect of the two forces above now countervails each other. Restricting information to dimension one only would induce agent to acquire less information overall but what is learned will be more valuable to the principal. Proposition 6 suggests the condition under which withholding the signal about x_2 could be beneficial to the principal.

Proposition 6. *The principal attains the highest benefit from withholding the signal about* x_2 *at* $c/2 = \Lambda_2$.

Figure 1.2 illustrates that withholding information is most likely to be beneficial (for the principal) at the cutoff point Λ_2.[15] The principal's payoff attains the highest possible value

[13]The marginal value of learning is high at minimal attention.

[14]The agent will learn less to "compensate the loss" on dimension one than the principal would.

[15]The horizontal axis of Figure 1.2 represents the cost parameter c, instead of $c/2$. So the two kink points of the blue line ("providing both dimensions") are $2\Lambda_2$ and $2\Lambda_1$.

when $c = 0$, i.e., there is no cost, and is strictly higher than the payoff when information about only one dimension is provided. But for $c/2$ less than Λ_2, principal's payoff is decreasing at the highest rate when information about both dimensions is provided. It is possible that the blue line ("providing both dimensions") crosses the red line ("providing dimension one only") and principal gets a higher payoff by exclusion starting from there. I show that the slope of the blue line will be flatter than the red line when $c/2$ is greater than Λ_2. Further, the two lines must cross again because the principal's payoff with full availability of information is still positive when agent stops learning with access to only one dimension.

It could be beneficial for the principal to withhold the signal about agent preferred issue at intermediate cost. On one hand, the cost is big enough. So the agent will not process a lot of expensive information and faces a real trade-off. Given scarce resources, the agent will first sacrifice less important things. This means that without restriction, the agent will sacrifice information less importance to him for a higher precision about his preferred issue. As a result, if the agent is instead forced to learn about the other issue exclusively, his belief will be much more precise. The more balanced knowledge is a gain for the principal. On the other hand, the cost is not very big. Even if the agent does not get access to the information most valuable to him, he is still willing to process other information. Further, the agent devotes quite a bit of attention at cost that is not big. If information about agent preferred issue is restricted, a significant amount of attention will then be shifted to the other issue. Therefore, the benefit from attention reallocation outweighs the loss from lower attention level.

It could be beneficial for the principal to withhold the signal about agent preferred issue at intermediate correlation. On one hand, the correlation is small enough. Then the agent will not be able to make perfect inferences about the issue less important to him from other information. But when all information is provided, the agent will focus on his preferred issue. So his belief about the issue of secondary importance to him will not be precise. Therefore, there is a potential benefit from withholding the signal about agent preferred issue because of the biased learning process. On the other hand, the correlation is not very small. Then the agent can make some inferences about his preferred issue from other information. As a result, when he does not get access to the information most valuable to him, the agent has an incentive to process more other information. So he learns a lot about the issue that he would otherwise ignore. Hence the principal's benefit from withholding information attains the highest at intermediate correlation.

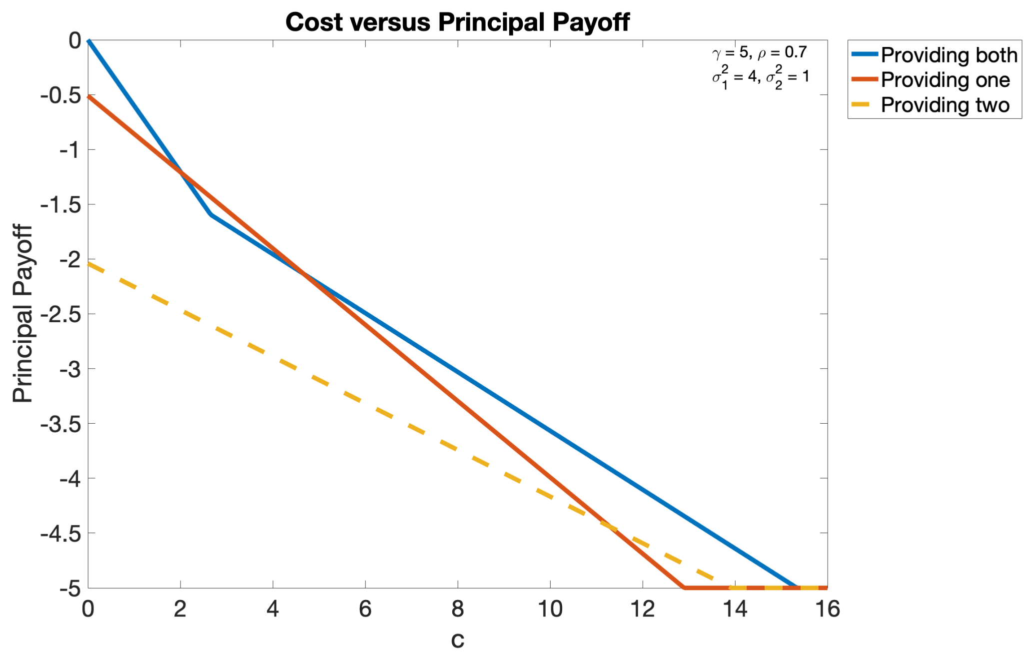

Figure 1.2: c versus u^P

Corollary 1. *If* $\gamma = 1$, *then principal never benefits from excluding either dimension.*

Corollary 1 confirms the idea that it is players' misaligned interest in relative importance of issues that creates potential benefit of withholding information. In contrast to Lester et al. (2012), the principal should never withhold any information if she attaches the same value as the agent to each dimension. Lester et al. (2012) assume that the jury ("agent") has to process the whole piece of evidence to be able to use the information. Instead, the agent is free to choose any amount of information in each dimension in this model. If the jury does not incur a *fixed* cost in processing each piece of evidence but could choose the precision of interpretation (which implies different cost), I show that it is not in the principal's interest to withhold any information (when they agree over the relative importance of each dimension).

We could think of the principal's exclusion choice from another perspective. When principal withholds information about a particular dimension, she is essentially raising agent's attention cost to infinity in this dimension. So the exclusion strategy is the flip side of those in Section 1.4, in the sense that principal is influencing the agent's cost now instead of agent's benefit from learning. Roughly, by increasing agent's cost in his preferred dimension, principal reduces the relative cost in the other dimension. This superficial analogy implies that principal might gain from agent reallocating his attention, which is traded off against the decline in total effort elicited from the agent.

The result has some implications for observations in the real world. My result suggests that if the student is biased toward some topic, such as the exam-related topic, it is possibly not optimal to cover everything in lectures even if time permits. There has been much concern lately about law schools failing to prepare students for legal practice and suggested a change in the focus of legal education (Kuehn, 2017). My result offers one recommendation. Learning law is difficult for many, if not all students. I show that for students who incur a lower cost to acquire knowledge, it could be beneficial to withhold the content they are biased toward. This is confirmed by anecdotes that in relatively lower ranked schools, a lot more time of the last year is spent going over things that students need to know for the bar exam than in top schools, though these specific rules may not be the most important things in the real-life practice. In contrast, the class of highest ranked schools is focused on broad principles underlying legal doctrine and the ways in which the doctrine evolves, while providing less preparation to students for the exam. Students are encouraged to transfer what they learned about fundamental principles to the applications that appear in the exam, instead of being told how to solve them directly.

1.6 Appendix A

Lemma 1. *Given a posterior belief about the state* $\boldsymbol{x} \sim \mathcal{N}(\tilde{\boldsymbol{x}}, \Sigma)$*, the agent's actions* $\boldsymbol{a}$ *are equal to the posterior means* $\tilde{\boldsymbol{x}}$*.*

Proof. The utility-maximizing actions are $\tilde{\boldsymbol{x}}$, because certainty equivalence applies in a quadratic setup (Kőszegi and Matějka, 2020). □

Lemma 2. *Let $\Phi = H\Psi H'$ and $\Xi = H\Sigma H'$. The agent's problem can be written as*

$$\max_{\Xi}\{-Tr(\Xi)+(c/2)\log|\Xi|\} \qquad \textit{subject to } \Phi - \Xi \textit{ is positive semi-definite.} \tag{1.2}$$

Proof. First, we show the loss term is equal to $Tr(\Xi)$. Let $Tr(\cdot)$ denote the trace of a matrix. The loss $(\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x})$ for the state $\boldsymbol{x}$ and posterior mean $\tilde{\boldsymbol{x}}$ can be written as

$$\begin{aligned} &(\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x}) \\ =&Tr((\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x})) \\ =&Tr(R(\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})'). \end{aligned} \tag{1.3}$$

Because R is a symmetric positive-definite matrix, it can be factorized into a lower triangular matrix (with positive diagonal entries) L and its transpose L', i.e., $R = LL'$, by the Cholesky decomposition. Let $H \equiv L'$. So $R = H'H$ and Eq. (1.3) becomes

$$\begin{aligned} &(\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x}) \\ =&Tr(R(\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})') \\ =&Tr(H'H(\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})') \\ =&Tr(H(\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})'H'). \end{aligned}$$

By linearity of expectation, $\mathbb{E}_{\tilde{\boldsymbol{x}}}[\mathbb{E}_{\boldsymbol{x}}((\tilde{\boldsymbol{x}}-\boldsymbol{x})'R(\tilde{\boldsymbol{x}}-\boldsymbol{x})|\tilde{\boldsymbol{x}})] = \mathbb{E}_{\tilde{\boldsymbol{x}}}[\mathbb{E}_{\boldsymbol{x}}(Tr(H(\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})'H')|\tilde{\boldsymbol{x}})] = \mathbb{E}_{\tilde{\boldsymbol{x}}}[Tr(H\mathbb{E}_{\boldsymbol{x}}((\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})'|\tilde{\boldsymbol{x}})H')] = \mathbb{E}_{\tilde{\boldsymbol{x}}}[Tr(H\Sigma H')] = Tr(H\Sigma H')$, because $\Sigma = \mathbb{E}_{\boldsymbol{x}}((\tilde{\boldsymbol{x}}-\boldsymbol{x})(\tilde{\boldsymbol{x}}-\boldsymbol{x})'|\tilde{\boldsymbol{x}})$ for all $\tilde{\boldsymbol{x}}$.

The agent's optimization problems in (1.1) and (1.2) are equivalent further because the second terms differ in a constant. By the definition of Ξ,

$$\log|\Xi| = \log|H\Sigma H'| = \log|H||\Sigma||H'| = \log|H| + \log|\Sigma| + \log|H'|,$$

where H is a constant matrix.

Lastly, the constraints are equivalent. The "no forgetting constraint" requires that $\mathbf{y}'(\Phi - \Xi)\mathbf{y} \geq 0$ for any non-zero column vector $\mathbf{y}$, which is equal to $\mathbf{y}'H(\Psi-\Sigma)H'\mathbf{y}$. If $\Psi - \Sigma$ is positive semidefinite, $\mathbf{z}'(\Psi-\Sigma)\mathbf{z} \geq 0$ for any non-zero column vector $\mathbf{z}$. Substituting $H'\mathbf{y}$ for $\mathbf{z}$,[16] we get

$$0 \leq (H'\mathbf{y})'(\Psi-\Sigma)(H'\mathbf{y}) = \mathbf{y}'H(\Psi-\Sigma)H'\mathbf{y} = \mathbf{y}'(\Phi-\Xi)\mathbf{y}.$$

Then $\Phi - \Xi$ is positive semidefinite as well. Hence the constraints in two problems are equivalent.

Therefore, the agent's problem can be equivalently expressed as in (1.2). ☐

Proposition 1. *The optimal attention strategy is to acquire independent signals of* $\boldsymbol{v}^i \cdot \boldsymbol{q}$ *such that the posterior variance of* $\boldsymbol{v}^i \cdot \boldsymbol{q}$ *is* $\min(c/2, \Lambda_i)$*, where* $\boldsymbol{q} =: H\boldsymbol{x}$*.*

Proof. By Lemma 2, I transformed the problem into optimization over the posterior variance Ξ. In terms of the signal the agent would acquire, I show that Ξ should be diagonal in the basis of eigenvectors $\boldsymbol{v}^i$. Decomposing the matrix Φ gives $\Phi = U\Lambda U'$, where U is the unitary matrix with columns being eigenvectors $\boldsymbol{v}^i$ such that $UU' = I$, and Λ is a diagonal matrix with its elements Λ_{ii} being the eigenvalues Λ_i of Φ. We can write the objective function as

$$\begin{aligned}
-Tr(\Xi) + (c/2)\log|\Xi| &= -Tr(UU'\Xi) + (c/2)\log|UU'\Xi| \\
&= -Tr(U'\Xi U) + (c/2)\log(|U||U'||\Xi|) \\
&= -Tr(U'\Xi U) + (c/2)\log(|U'||\Xi||U|) \\
&= -Tr(U'\Xi U) + (c/2)\log|U'\Xi U| \\
&= -Tr(S) + (c/2)\log|S|,
\end{aligned}$$

[16] We assume that no signal in any dimension is redundant, which implies the full-rank condition.

where $S \equiv U'\Xi U$ is the posterior variance-covariance matrix in the basis of eigenvectors of Φ. Then $\Xi = (UU')\Xi(UU') = U(U'\Xi U)U' = USU'$. The "no-forgetting" constraint takes the form of $U'(\Phi - \Xi)U = U'(U\Lambda U')U - U'\Xi U = \Lambda - S$ being positive semi-definite. We then show that the optimal matrix S is diagonal similar to Kőszegi and Matějka (2020).

Suppose that the optimal S is not diagonal. Let S^D be the diagonal matrix that has the same diagonal entries as S, i.e., $S^D_{ii} = S_{ii}$ for $i = 1, \ldots, N$ and $S^D_{ij} = 0$ for $i \neq j$.

(i) Because $\Lambda - S$ is positive semi-definite, $\mathbf{y}'(\Lambda - S)\mathbf{y} \geq 0$ for any non-zero column vector $\mathbf{y}$. Let $y_i = 1$ and $y_j = 0$ for $j \neq i$. Then we can see that $\mathbf{y}'(\Lambda - S)\mathbf{y} = y_i^2(\Lambda - S)_{ii} \geq 0$. This implies that $S_{ii} \leq \Lambda_{ii}$ for $i = 1, \ldots, N$. Hence $\Lambda - S^D$ is positive semi-definite as well.

(ii) Note that $-Tr(S^D) = -\sum_{i=1}^N S_{ii} = -Tr(S)$.

(iii) But $|S| < \Pi_{i=1}^N S_{ii} = |S^D|$ by Hadamard's inequality. So $(c/2)\log|S| < (c/2)\log|S^D|$, meaning that it would be more costly to acquire a signal having correlated dimensions given the same loss.

Hence the diagonal matrix S^D gives the agent a strictly higher payoff. The contradiction shows that the optimal posterior variance-covariance matrix has to be diagonal.

We can then find the posterior of each dimension separately, as the problem reduces to optimization over diagonal matrices:

$$\max_{S_{ii} \leq \Lambda_{ii}} -\sum_{i=1}^N S_{ii} + \sum_{i=1}^N (c/2)\log S_{ii}.$$

Because the objective function is strictly concave over the entire domain of S_{ii} ($i = 1, \ldots, N$), we use the first-order approach to find the interior solution of the optimization problem. The first order condition with respect to S_{ii} gives $-1 + c/(2S_{ii}) = 0$.[17] Hence the optimal matrix S subject to the constraint that posterior variances cannot exceed the prior variances is given by $S_{ii} = \min(c/2, \Lambda_i)$ and $S_{ij} = 0$ for $i \neq j$. □

[17] The expected return on the margin is the same when learning about any of the dimensions.

The posterior variance-covariance matrix Ξ of the vector $\boldsymbol{q}$ is given by

$$\begin{aligned}\Xi &= U \begin{pmatrix} \min(c/2,\Lambda_1) & 0 \\ 0 & \min(c/2,\Lambda_2) \end{pmatrix} U' \\ &= \begin{pmatrix} \min(c/2,\Lambda_1)(v_1^1)^2+\min(c/2,\Lambda_2)(v_1^2)^2 & \min(c/2,\Lambda_1)v_1^1v_2^1+\min(c/2,\Lambda_2)v_1^2v_2^2 \\ \min(c/2,\Lambda_1)v_1^1v_2^1+\min(c/2,\Lambda_2)v_1^2v_2^2 & \min(c/2,\Lambda_1)(v_2^1)^2+\min(c/2,\Lambda_2)(v_2^2)^2 \end{pmatrix}.\end{aligned} \tag{1.4}$$

The following claim will be useful throughout the analysis.

Claim 1. *The prior variances of the principal components are*

$$\begin{aligned}\Lambda_1 =& \frac{1}{2}\left(\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}\right); \\ \Lambda_2 =& \frac{1}{2}\left(\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}\right).\end{aligned}$$

Furthermore, the eigenvectors satisfy $v_1^2 = v_2^1$ and $v_2^2 = -v_1^1$.

Let $M = (\gamma\sigma_2^2 - \sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$. The eigenvalues can be obtained immediately. Further, simple calculations show that

$$\begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}} \\ \sqrt{1-\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}} \end{pmatrix}$$

$$\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{1+\left(\sqrt{M^2+1}-M\right)^2}} \\ -\sqrt{1-\frac{1}{1+\left(\sqrt{M^2+1}-M\right)^2}} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}} \\ -\sqrt{\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}} \end{pmatrix} = \begin{pmatrix} v_2^1 \\ -v_1^1 \end{pmatrix}.$$

Lemma 3. *Let σ_1^2 and σ_2^2 be the prior variance of states x_1 and x_2. Let v_1^1 be the loading of x_1 on the transformed dimension with larger variance. The principal's expected payoff assuming* *all information is available to the agent is given by*

$$\begin{cases} -(c/2)[1+(1/\gamma)] & \textit{if } c/2 \leq \Lambda_2 \\ -(\sigma_1^2+\sigma_2^2)+[\Lambda_1-(c/2)][(v_1^1)^2+(1/\gamma)(1-(v_1^1)^2)] & \textit{if } \Lambda_2 < c/2 \leq \Lambda_1 \\ -(\sigma_1^2+\sigma_2^2) & \textit{if } c/2 > \Lambda_1 \end{cases}.$$

Proof. Let $\hat{\sigma}_1$ and $\hat{\sigma}_2$ be the posterior variance of x_1 and x_2. This result immediately follows from Equation (1.4) and Lemma 1:

$$\begin{aligned} u^P(\hat{\sigma}_1,\hat{\sigma}_2) &= -\hat{\sigma}_1^2-\hat{\sigma}_2^2 \\ &= -\Xi_{11}-\Xi_{22}/\gamma \\ &= -[(\min(\frac{c}{2},\Lambda_1)(v_1^1)^2+\min(\frac{c}{2},\Lambda_2)(v_1^2)^2)+(1/\gamma)(\min(\frac{c}{2},\Lambda_1)(v_2^1)^2 \\ &\quad +\min(\frac{c}{2},\Lambda_2)(v_2^2)^2)] \\ &= -([\min(\frac{c}{2},\Lambda_1)(v_1^1)^2+\min(\frac{c}{2},\Lambda_2)(1-(v_1^1)^2)]+ \\ &\quad (1/\gamma)[\min(\frac{c}{2},\Lambda_1)(1-(v_1^1)^2)+\min(\frac{c}{2},\Lambda_2)(v_1^1)^2]) \\ &= \begin{cases} -(c/2)[1+(1/\gamma)] & \text{if } c/2 \leq \Lambda_2 \\ -(\sigma_1^2+\sigma_2^2)+[\Lambda_1-(c/2)][(v_1^1)^2+(1/\gamma)(1-(v_1^1)^2)] & \text{if } \Lambda_2 < c/2 \leq \Lambda_1 \\ -(\sigma_1^2+\sigma_2^2) & \text{if } c/2 > \Lambda_1 \end{cases}. \end{aligned}$$

□

The following claims will be useful in proving Proposition 3.

Claim 2. *For* $0 < |\rho| < 1$, $\partial\Lambda_1/\partial\gamma > 0$ *and* $\partial\Lambda_2/\partial\gamma > 0$.

Proof. The derivative of Λ_1 with respect to γ is

$$
\begin{aligned}
&\partial\Lambda_1/\partial\gamma \\
=&\frac{\partial}{\partial\gamma}[(1/2)\left(\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}\right)] \\
=&\frac{\sigma_2^2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}+(\gamma\sigma_2^2-\sigma_1^2+2\rho^2\sigma_1^2))}{2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2})} \\
&\begin{cases}
\geq \sigma_2^2/2 & \text{if } \gamma\sigma_2^2-\sigma_1^2+2\rho^2\sigma_1^2 \geq 0 \\
= [\sigma_2^2/2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2})][\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2} & \\
\quad -\sqrt{((\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2)-4\sigma_1^4\rho^2(1-\rho^2)}] & \text{Otherwise.}
\end{cases}
\end{aligned}
\tag{1.5}
$$

If $\gamma\sigma_2^2-\sigma_1^2+2\rho^2\sigma_1^2<0$, the second term is positive unless $|\rho|=0$ or 1. So Λ_1 is strictly increasing in γ.

The derivative of Λ_2 with respect to γ is

$$
\begin{aligned}
&\partial\Lambda_2/\partial\gamma \\
=&\frac{\partial}{\partial\gamma}[(1/2)\left(\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}\right)] \\
=&(1/2)(\sigma_2^2-\frac{2\sigma_2^2(\gamma\sigma_2^2-\sigma_1^2)+4\rho^2\sigma_1^2\sigma_2^2}{2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2})}) \\
=&\frac{\sigma_2^2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}-\gamma\sigma_2^2+\sigma_1^2-2\rho^2\sigma_1^2)}{2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2})} \\
&\begin{cases}
\geq \sigma_2^2/2 & \text{if } \gamma r_2\sigma_2^2-\sigma_1^2+2\rho^2\sigma_1^2 \leq 0 \\
= [\sigma_2^2/2(\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2})][\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2} & \\
\quad -\sqrt{((\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2)-4\sigma_1^4\rho^2(1-\rho^2)}] & \text{Otherwise.}
\end{cases}
\end{aligned}
\tag{1.6}
$$

If $\gamma\sigma_2^2-\sigma_1^2+2\rho^2\sigma_1^2>0$, the second term is positive unless $|\rho|=0$ or 1. So Λ_2 is strictly increasing in γ. □

Claim 3. $\partial(v_1^1)^2/\partial\gamma < 0$.

Proof. Recall that $M = (\gamma\sigma_2^2 - \sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$.

$$\begin{aligned}\partial(v_1^1)^2/\partial\gamma =& (\partial(v_1^1)^2/\partial M)(\partial M/\partial\gamma)\\ =& (\frac{\partial}{\partial M}\frac{1}{1+\big(\sqrt{M^2+1}+M\big)^2})(\frac{\partial}{\partial\gamma}(\frac{\sqrt{\gamma}\sigma_2}{2\rho\sigma_1} - \frac{\sigma_1}{2\sqrt{\gamma}\rho\sigma_2}))\\ =& -\frac{2\big(\sqrt{M^2+1}+M\big)(M/\sqrt{M^2+1}+1)}{[1+\big(\sqrt{M^2+1}+M\big)^2]^2}[\frac{1}{2\rho}(\frac{\sigma_2}{2\sqrt{\gamma}\sigma_1} + \frac{\sigma_1}{2\sqrt{\gamma^3}\sigma_2})] < 0.\end{aligned}$$

□

Proposition 3. *Suppose that $\gamma \leq 1$. When $c/2 \leq \Lambda_1$, the principal's payoff is strictly increasing in γ. When $c/2 > \Lambda_1$, the agent does not process any information until Λ_1 reaches the threshold $c/2$.*

Proof. (i) When $\Lambda_1 > \Lambda_2 > c/2$, the principal's payoff is equal to $-(c/2)(1+1/\gamma)$. It is clear that if we marginally increase γ, principal's payoff will increase. The uncertainty about x_1 will remain the same, while the posterior variance $c/(2\gamma)$ of x_2 declines. If we increase γ, it will always be the case that $c/2 < \Lambda_2 < \Lambda_1$.

(ii) When $\Lambda_1 > c/2 > \Lambda_2$, the marginal change of principal's payoff in γ is given by:

$$\begin{aligned}\frac{\partial u^P}{\partial\gamma} = & \underbrace{(v_1^1)^2(\frac{\partial\Lambda_1}{\partial\gamma}) + \frac{1}{\gamma}(1-(v_1^1)^2)[\frac{\partial\Lambda_1}{\partial\gamma} - \frac{1}{\gamma}(\Lambda_1 - \frac{c}{2})]}_{\text{change in principal's payoff due to more agent's effort if fixing loadings}}\\ + & \underbrace{(\Lambda_1 - \frac{c}{2})(1-\frac{1}{\gamma})\frac{\partial(v_1^1)^2}{\partial\gamma}}_{\text{decline in principal's payoff due to reallocation of attention}}.\end{aligned} \tag{1.7}$$

The prior variance Λ_1 is increasing in γ, i.e., $\partial\Lambda_1/\partial\gamma > 0$ by Claim 2, implying that more effort is expensed. Recall that the fraction of information acquired by the agent about each dimension is given by the respective loadings, v_1^1 and v_2^1. So the principal's payoff becomes higher if we *fix the loadings of each dimension*. The second term captures the effect that agent substitutes attention away from dimension one to dimension two, because $\partial(v_1^1)^2/\partial\gamma < 0$ by Claim 3. Hence when $\gamma \leq 1$, the principal gains from more balanced attention of the agent as well.

(iii) When $\Lambda_2 < \Lambda_1 \leq c/2$, the principal's payoff is equal to $-(\sigma_1^2+\sigma_2^2)$, which will be constant in γ. □

Proposition 4. *Suppose that* $0 < |\rho| < 1$ *and* $\gamma > 1$. *(i) When* $c/2 \leq \Lambda_2$, *the agent's posterior variance in dimension one is constant and posterior variance in dimension two is decreasing in* γ. *(ii) When* $\Lambda_2 < c/2 \leq \Lambda_1$, *the agent's posterior variance in dimension one is increasing in* γ *at lower cost and decreasing in* γ *at higher cost. The posterior variance in dimension two is always decreasing in* γ. *(iii) When* $c/2 > \Lambda_1$, *the agent does not process any information until* Λ_1 *reaches the threshold* $c/2$.

Proof. (i) When $\Lambda_1 > \Lambda_2 > c/2$, the posterior variance of dimension one is constant at $c/2$, while the posterior variance $c/(2\gamma)$ of dimension two is decreasing in γ.

(ii) When $\Lambda_1 > c/2 > \Lambda_2$, the posterior variance of dimension two is always decreasing in γ because $\partial(v_1^1)^2/\partial\gamma < 0$ by Claim 3. However, posterior variance of dimension one becomes negative at $c/2$ close to Λ_2, because the positive terms involving $\partial\Lambda_2/\partial\gamma$ becomes zero. The marginal change of posterior variance of dimension one in γ is increasing in c. At $c/2$ close to Λ_1, there is minimal attention substitution and the second term in Eq (1.7) becomes zero. Hence the posterior variance of dimension one will be increasing in γ at higher cost.

(iii) When $\Lambda_2 < \Lambda_1 \leq c/2$, the principal's payoff is equal to $-(\sigma_1^2+\sigma_2^2)$, which will be constant in γ. □

Lemma 4. *If only the signal about* x_1 *is available, the optimal information strategy is to acquire this signal such that the posterior variance about* x_1 *is given by* $\sigma_1^2 \min(1, \frac{c}{2(\sigma_1^2+\gamma\rho^2\sigma_2^2)})$. *If only the signal about* x_2 *is available, the optimal information strategy is to acquire this signal such that the posterior variance about* x_2 *is given by* $\sigma_2^2 \min(1, \frac{c}{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)})$.

Proof. Let $\Omega = \begin{pmatrix} \sigma_i & 0 \\ \sigma_j\rho & \sigma_j\sqrt{1-\rho^2} \end{pmatrix}$. Then $\boldsymbol{x} = \Omega\boldsymbol{z}$. Let Υ be the posterior variance-covariance matrix of $\boldsymbol{z}$. Then $\Psi = \Omega I \Omega'$ and $\Sigma = \Omega\Upsilon\Omega'$. The cost incurred in learning is equivalent in terms of $\boldsymbol{x}$ and $\boldsymbol{z}$:

$$\begin{aligned}\frac{\lambda}{2}(\log(|\Psi|)-\log(|\Sigma|)) &= \frac{\lambda}{2}(\log(|\Omega I\Omega'|)-\log(|\Omega\Upsilon\Omega'|)) \\ &= \frac{\lambda}{2}(\log(|\Omega||I||\Omega'|)-\log(|\Omega||\Upsilon||\Omega'|)) \\ &= \frac{\lambda}{2}(\log|\Omega|+\log|I|+\log|\Omega'|-(\log|\Omega|+\log|\Upsilon|+\log|\Omega'|)) \\ &= -\frac{\lambda}{2}\log|\Upsilon|.\end{aligned}$$

Let γ_i and γ_j be a player's weight in dimensions i and j, respectively. The player's loss from agent's misperception is given by

$$\begin{aligned} & -Tr(\begin{pmatrix}\gamma_i & 0 \\ 0 & \gamma_j\end{pmatrix}\Sigma) \\ =& -Tr(\begin{pmatrix}\gamma_i & 0 \\ 0 & \gamma_j\end{pmatrix}\Omega\Upsilon\Omega') \\ =& -Tr(\Omega'\begin{pmatrix}\gamma_i & 0 \\ 0 & \gamma_j\end{pmatrix}\Omega\Upsilon) \\ =& -Tr(\begin{pmatrix}\sigma_i\gamma_i & \sigma_j\rho\gamma_j \\ 0 & \sigma_j\sqrt{1-\rho^2}\gamma_j\end{pmatrix}\Omega\Upsilon) \\ =& -Tr(\begin{pmatrix}\sigma_i^2\gamma_i+\sigma_j^2\rho^2\gamma_j & \sigma_j^2\rho\sqrt{1-\rho^2}\gamma_j \\ \sigma_j^2\rho\sqrt{1-\rho^2}\gamma_j & \sigma_j^2(1-\rho^2)\gamma_j\end{pmatrix}\Upsilon).\end{aligned}$$

If principal only discloses the realization of dimension i, agent's signal cannot involve anything that is independent of this dimension. So the posterior variance of z_j is still 1. Moreover, z_i and z_j are still independent, as the signal about z_i reveals nothing about z_j. Then Υ is given by $\begin{pmatrix}\hat{\sigma}_{z_i}^2 & 0 \\ 0 & 1\end{pmatrix}$. The loss for players is

$$-(\sigma_i^2\gamma_i+\sigma_j^2\rho^2\gamma_j)\hat{\sigma}_{z_i}^2-\sigma_j^2(1-\rho^2)\gamma_j.$$

Let the posterior variance of z_i be $\hat{\sigma}^2_{z_i}$. Since agent's cost is $-c/2\log|\Upsilon| = -(c/2)\log\hat{\sigma}^2_{z_i}$, the agent's problem is equivalent to

$$\max_{\hat{\sigma}^2_{z_i}} \; -(\sigma_i^2\gamma_i^A + \sigma_j^2\rho^2\gamma_j^A)\hat{\sigma}^2_{z_i} - \sigma_j^2(1-\rho^2)\gamma_j^A + (c/2)\log\hat{\sigma}^2_{z_i}$$

$$\text{subject to} \quad \hat{\sigma}^2_{z_i} \leq 1,$$

where 1 is the prior variance of the random variable z_i.

The solution is given by the first order condition:

$$-(\sigma_i^2\gamma_i^A + \sigma_j^2\rho^2\gamma_j^A) + (c/2)(1/\hat{\sigma}^2_{z_i}) = 0.$$

So the optimal $\hat{\sigma}^2_{z_i}$ for the agent is $\min(1, c[2(\sigma_i^2\gamma_i^A + \sigma_j^2\rho^2\gamma_j^A)])$. □

The following claims are useful for proving Proposition 5.

Claim 4. *As cost increases, principal's expected payoff declines at a higher rate until c is equal to* $\sigma_1^2 + \gamma\sigma_2^2 - \sqrt{(\sigma_1^2 - \gamma\sigma_2^2)^2 + (2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$ *when both dimensions are provided than when either one of the dimensions is provided.*

Proof. If both dimensions are provided, principal's payoff is given by $-(c/2)[1+1/\gamma]$ when c is small enough such that $c/2 < \Lambda_2 < \Lambda_1$. For one unit increase in cost, principal's payoff declines by $\frac{1+1/\gamma}{2}$. If only dimension i is provided, for one unit increase in cost, principal's payoff declines by $\frac{\sigma_i^2+\rho^2\sigma_j^2}{2(\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2)}$. The claim is proved by observing that

$$
\begin{aligned}
\frac{\sigma_i^2+\rho^2\sigma_j^2}{2(\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2)} &= \frac{1}{2}\left(\frac{\sigma_i^2}{\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2}+\frac{\rho^2\sigma_j^2}{\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2}\right)\\
&= \frac{1}{2}\left(\frac{\sigma_i^2}{\gamma_i^A\sigma_i^2}\cdot\frac{\gamma_i^A\sigma_i^2}{\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2}+\frac{\rho^2\sigma_j^2}{\gamma_j^A\rho^2\sigma_j^2}\cdot\frac{\gamma_j^A\rho^2\sigma_j^2}{\gamma_i^A\sigma_i^2+\gamma_j^A\rho^2\sigma_j^2}\right)\\
&\le \frac{1}{2}\max\left(\frac{\sigma_i^2}{\gamma_i^A\sigma_i^2},\frac{\rho^2\sigma_j^2}{\gamma_j^A\rho^2\sigma_j^2}\right)\\
&= \frac{1}{2}\max\left(\frac{1}{\gamma_i^A},\frac{1}{\gamma_j^A}\right)\\
&\le \frac{1}{2}\left(\frac{1}{\gamma_i^A}+\frac{1}{\gamma_j^A}\right).
\end{aligned}
$$

□

When $\lambda = \sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$, agent stops learning the transformed dimension that has smaller variance because the target variance exceeds the prior variance Λ_2. On the one hand, principal's payoff continuously declines, because agent learns even less about the transformed dimension that has larger variance as cost rises. On the other hand, the decreasing rate is reduced, as agent's signal precision cannot be further reduced to negative by the "no-forgetting" constraint. The decreasing rate in this region is

$$
\frac{1}{2}[(v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2)],
$$

which is less than the decreasing rate at small cost $\frac{1}{2}(1+\frac{1}{\gamma})$. Moreover, $(v_1^1)^2$ and $1-(v_1^1)^2$ are the proportion of information associated with the first and the second dimension, respectively, that principal has to forsake. I show below that the decreasing rate is the highest when only dimension one is disclosed, and the decreasing rate if disclosing both dimensions is higher than the rate when only disclosing dimension two.

Claim 5. *For* $\rho^2 < 1$, $\left(\sqrt{M^2+1}+M\right)^2 < (\gamma\sigma_2^2)/(\rho^2\sigma_1^2)$, *where* $M = (\gamma\sigma_2^2-\sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$.

Proof.

$$\begin{aligned}
\left(\sqrt{M^2+1}+M\right)^2 &= \left(\sqrt{\left(\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2+1}+\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&= \left(\sqrt{\frac{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}{4\gamma\rho^2\sigma_1^2\sigma_2^2}}+\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&< \left(\sqrt{\frac{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\sigma_1^2\sigma_2^2}{4\gamma\rho^2\sigma_1^2\sigma_2^2}}+\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&= \left(\sqrt{\frac{(\gamma\sigma_2^2+\sigma_1^2)^2}{4\gamma\rho^2\sigma_1^2\sigma_2^2}}+\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&= \left(\frac{\gamma\sigma_2^2+\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}+\frac{\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&= \left(\frac{2\gamma\sigma_2^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2 \\
&= \frac{\gamma\sigma_2^2}{\rho^2\sigma_1^2},
\end{aligned}$$

where the inequality follows from $\rho^2 < 1$. □

Proposition 5. *For $\rho \neq \pm 1$, the principal's payoff from providing both signals is strictly higher than her payoff from providing only the signal about x_2, unless no information is acquired in both regimes. For $\rho = \pm 1$, the principal's payoff from providing both signals is exactly equal to her payoff from providing the signal about either dimension.*

Proof. If $\rho = \pm 1$, $\Lambda_2 = 0$ and $(v_1^1)^2 = \sigma_1^2/(\sigma_1^2+\gamma\sigma_2^2)$. It is straightforward to check that principal's payoff is identical in both regimes and equal to $-(c/2)[\sigma_1^2/(\sigma_1^2+\gamma\sigma_2^2)+\sigma_2^2/(\sigma_1^2+\gamma\sigma_2^2)]$.

If $\rho \neq \pm 1$, we claim that principal's payoff is always strictly higher when information about both dimensions are provided unless agent never acquires any information.

(i) If $c \geq \sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$, agent acquires no information in both regimes and principal's payoff is the same.

(ii) If $2(\rho^2\sigma_1^2+\gamma\sigma_2^2) \leq c < \sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$, agent only acquires information if both dimensions are disclosed. So principal is strictly better off in this regime as the effect of learning $(\Lambda_1-c/2)((v_1^1)^2+(1/\gamma)(1-(v_1^1)^2)$ adds to her payoff.

(iii) If $\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2} \leq c < 2(\rho^2\sigma_1^2+\gamma\sigma_2^2)$, agent learns in both regimes. Note that principal's payoff is integration of the increasing rate as cost declines. If only dimension two is disclosed, principal's gain beyond her payoff with no information when unit cost is c is given by

$$
\begin{aligned}
&\int_{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)}^{c} -\frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt \\
=&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}(\frac{\sigma_2^2}{\gamma\sigma_2^2}\cdot\frac{\gamma\sigma_2^2}{\gamma\sigma_2^2+\rho^2\sigma_1^2}+\frac{\rho^2\sigma_1^2}{\rho^2\sigma_1^2}\cdot\frac{\rho^2\sigma_1^2}{\gamma\sigma_2^2+\rho^2\sigma_1^2})dt \\
=&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}[\frac{1}{\gamma}\cdot(1-\frac{1}{\gamma\sigma_2^2/\rho^2\sigma_1^2+1})+\frac{1}{\gamma\sigma_2^2/\rho^2\sigma_1^2+1}]dt.
\end{aligned}
$$

If information about both dimensions are provided, principal's gain beyond her payoff with no information when unit cost is λ is given by

$$
\begin{aligned}
&\int_{\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{c} -\frac{1}{2}((v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2))dt \\
=&\int_{c}^{\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} \frac{1}{2}((v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2))dt \\
>&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}[(v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2)]dt \\
=&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}[\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}+\frac{1}{\gamma}\cdot(1-\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2})]dt,
\end{aligned}
\tag{1.8}
$$

where $M=(\gamma\sigma_2^2-\sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$. Then we unify the lower and upper limits, and the integrand differs only in the weights of 1 and $1/\gamma$. In particular, $\left(\sqrt{M^2+1}+M\right)^2<\gamma\sigma_2^2/\rho^2\sigma_1^2$ for $\rho^2<1$ by Claim 5, i.e., the weight of the larger term 1 is higher in (1.8). It follows that

$$
\begin{aligned}
&\int_{\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{c} -\frac{1}{2}((v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2))dt \\
>&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}[\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}+\frac{1}{\gamma}\cdot(1-\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2})]dt \\
\geq&\int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}[\frac{1}{\gamma}\cdot(1-\frac{1}{(\gamma\sigma_2^2/\rho^2\sigma_1^2)+1})+\frac{1}{(\gamma\sigma_2^2/\rho^2\sigma_1^2)+1}]dt \\
=&\int_{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)}^{c} -\frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt,
\end{aligned}
$$

because $1 \geq 1/\gamma$. Hence principal gains more from providing both dimensions when $\sigma_1^2 + \gamma\sigma_2^2 - \sqrt{(\sigma_1^2 - \gamma\sigma_2^2)^2 + (2\sqrt{\gamma_2^A}\rho\sigma_1\sigma_2)^2} \leq c < 2(\rho^2\sigma_1^2 + \gamma\sigma_2^2)$.

(iv) When $0 \leq c < \sigma_1^2 + \gamma\sigma_2^2 - \sqrt{(\sigma_1^2 - \gamma\sigma_2^2)^2 + (2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$, the gain from disclosing both dimensions increases at two different intensities as cost declines, which is given by

$$\begin{aligned}
&\int_{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{c} -\frac{1}{2}(1+\frac{1}{\gamma})dt \\
&+ \int_{\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} -\frac{1}{2}((v_1^1)^2 + \frac{1}{\gamma}(1-(v_1^1)^2))dt \\
=& \int_{c}^{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} \frac{1}{2}(1+\frac{1}{\gamma})dt \\
&+ \int_{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{\sigma_1^2+\gamma\sigma_2^2+\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} \frac{1}{2}((v_1^1)^2 + \frac{1}{\gamma}(1-(v_1^1)^2))dt \\
>& \int_{c}^{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} \frac{1}{2}(1+\frac{1}{\gamma})dt \\
&+ \int_{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{1}{2}((v_1^1)^2 + \frac{1}{\gamma}(1-(v_1^1)^2))dt \\
\geq& \int_{c}^{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}} \frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt \\
&+ \int_{\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt \\
=& \int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt,
\end{aligned} \tag{1.9}$$

where the second inequality follows from Claims 4 and 5. If only dimension two is provided, principal's gain beyond her payoff with no information is still given by

$$\begin{aligned}
&\int_{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)}^{c} -\frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt \\
=& \int_{c}^{2(\rho^2\sigma_1^2+\gamma\sigma_2^2)} \frac{\sigma_2^2+\rho^2\sigma_1^2}{2(\gamma\sigma_2^2+\rho^2\sigma_1^2)}dt.
\end{aligned}$$

It is then clear that principal gains strictly more from providing both dimensions. □

Claim 6. *For* $\rho^2 < 1$, $\left(\sqrt{M^2+1}+M\right)^2 > \gamma\rho^2\sigma_2^2/\sigma_1^2$, *where* $M = (\gamma\sigma_2^2 - \sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$.

Proof. Note that

$$\begin{aligned}
&\left(\sqrt{M^2+1}+M\right)^2-\frac{\gamma\rho^2\sigma_2^2}{\sigma_1^2}\\
=&\left(\frac{\sqrt{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}+\gamma\sigma_2^2-\sigma_1^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)^2-\left(\frac{\sqrt{\gamma}\rho\sigma_2}{\sigma_1}\right)^2\\
=&\left(\frac{\sqrt{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}+\gamma\sigma_2^2-\sigma_1^2+2\gamma\rho^2\sigma_2^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right)\\
&\cdot\left(\frac{\sqrt{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}+\gamma\sigma_2^2-\sigma_1^2-2\gamma\rho^2\sigma_2^2}{2\sqrt{\gamma}\rho\sigma_1\sigma_2}\right).
\end{aligned}$$

The first term of the last equation must be positive, because $\sqrt{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}>|\gamma\sigma_2^2-\sigma_1^2|$. The second term is positive as well, because for $\rho\neq\pm1$

$$\begin{aligned}
&\sqrt{(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2}+\gamma\sigma_2^2-\sigma_1^2-2\gamma\rho^2\sigma_2^2>0\\
\Leftarrow&(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2>(\gamma\sigma_2^2-\sigma_1^2-2\gamma\rho^2\sigma_2^2)^2\\
\Leftrightarrow&(\gamma\sigma_2^2-\sigma_1^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2>(\gamma\sigma_2^2-\sigma_1^2)^2-4\gamma\rho^2\sigma_2^2(\gamma\sigma_2^2-\sigma_1^2)+4(\gamma\rho^2\sigma_2^2)^2\\
\Leftrightarrow&4(\gamma\rho\sigma_2^2)^2(\rho^2-1)<0.
\end{aligned}$$

□

Lemma 5. *If* $\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}<c<2(\sigma_1^2+\sigma_2^2\rho^2\gamma)$, *the decreasing rate of principal's payoff in cost when providing both dimensions is strictly lower than the rate when providing only dimension one for* $\rho\neq\pm1$.

Proof. The decreasing rate when providing both dimensions and only dimension one are $(1/2)[(v_1^1)^2+(1/\gamma)(1-(v_1^1)^2)]$ and $(1/2)[\sigma_1^2/(\sigma_1^2+\gamma\rho^2\sigma_2^2)+\rho^2\sigma_2^2/(\sigma_1^2+\gamma\rho^2\sigma_2^2)]$, respectively. Note that

$$\frac{1}{2}[(v_1^1)^2+\frac{1}{\gamma}(1-(v_1^1)^2)]=\frac{1}{2}\left(\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}+\frac{1}{\gamma}\left(1-\frac{1}{1+\left(\sqrt{M^2+1}+M\right)^2}\right)\right)$$

and

$$\frac{1}{2}\left(\frac{\sigma_1^2}{\sigma_1^2+\gamma\rho^2\sigma_2^2}+\frac{1}{\gamma}\frac{\gamma\rho^2\sigma_2^2}{\sigma_1^2+\gamma\rho^2\sigma_2^2}\right)=\frac{1}{2}\left(\frac{1}{1+(\gamma\rho^2\sigma_2^2/\sigma_1^2)}+\frac{1}{\gamma}\left(1-\frac{1}{1+(\gamma\rho^2\sigma_2^2/\sigma_1^2)}\right)\right).$$

So we compare $\left(\sqrt{M^2+1}+M\right)^2$ and $\gamma\rho^2\sigma_2^2/\sigma_1^2$, where $M=(\gamma\sigma_2^2-\sigma_1^2)/(2\sqrt{\gamma}\rho\sigma_1\sigma_2)$. By Claim 6, $\left(\sqrt{M^2+1}+M\right)^2>\gamma\rho^2\sigma_2^2/\sigma_1^2$. Since $1\geq 1/\gamma$, the decreasing rate in the case of providing both dimensions is lower. □

Proposition 6. *The principal attains the highest benefit from withholding the signal about x_2 at $c/2=\Lambda_2$.*

The proposition follows from Lemma 5.

Claim 7. *For $0<|\rho|<1$, $\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}>\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2$.*

Proof. The result is clear by observing that

$$\begin{aligned}&\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}>\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2\\ \Leftarrow&(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2>[\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2]^2\\ \Leftrightarrow&(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2>(\sigma_1^2-\gamma\sigma_2^2)^2+4\gamma\rho^2\sigma_1^2\sigma_2^2-4\rho^2(1-\rho^2)(\gamma\sigma_2^2)^2\\ \Leftrightarrow&4\rho^2(1-\rho^2)(\gamma\sigma_2^2)^2>0.\end{aligned}$$

□

Lemma 6. *For $\rho\neq\pm1$, $\Lambda_2<(1-\rho^2)\gamma\sigma_2^2$.*

Proof. $2\Lambda_2$ is equal to $\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}$ and $2(1-\rho^2)\gamma\sigma_2^2$ is equal to $\sigma_1^2+\gamma\sigma_2^2-(\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2)$. The lemma follows by noting that

$$\begin{aligned}&\sigma_1^2+\gamma\sigma_2^2-\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}<\sigma_1^2+\gamma\sigma_2^2-(\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2)\\ \Leftrightarrow&\sqrt{(\sigma_1^2-\gamma\sigma_2^2)^2+(2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2}>\sigma_1^2+(2\rho^2-1)\gamma\sigma_2^2,\end{aligned}$$

and the last inequality follows from Claim 7. □

Corollary 1. *If $\gamma = 1$, then principal never benefits from excluding either dimension.*

Proof. When $\rho = \pm 1$, principal's payoff is the same from providing both dimensions or only one dimension. When $\rho \neq \pm 1$, we prove that the difference in payoffs is positive when $c/2 = \Lambda_2$. It then follows that principal always gains a strictly higher payoff by providing both dimensions.

By Lemma 6, $\sigma_1^2 + \gamma\sigma_2^2 - \sqrt{(\sigma_1^2 - \gamma\sigma_2^2)^2 + (2\sqrt{\gamma}\rho\sigma_1\sigma_2)^2} < \sigma_1^2 + \gamma\sigma_2^2 - [\sigma_1^2 + (2\rho^2 - 1)\gamma\sigma_2^2]$. So the difference in payoffs is greater than

$$
\begin{aligned}
& -\frac{1}{2}(\sigma_1^2 + \gamma\sigma_2^2 - [\sigma_1^2 + (2\rho^2 - 1)\gamma\sigma_2^2]) \\
& \left((1 - \frac{1}{1 + (\gamma\rho^2\sigma_2^2/\sigma_1^2)}) + \frac{1}{\gamma}\frac{1}{1 + (\gamma\rho^2\sigma_2^2/\sigma_1^2)}\right) + \sigma_2^2(1 - \rho^2) \\
= & -\frac{1}{2}(\sigma_1^2 + \gamma\sigma_2^2 - [\sigma_1^2 + (2\rho^2 - 1)\gamma\sigma_2^2])\frac{1}{\gamma} + \sigma_2^2(1 - \rho^2) \\
= & -(1 - \rho^2)\gamma\sigma_2^2(\frac{1}{\gamma}) + \sigma_2^2(1 - \rho^2) \\
= & -(1 - \rho^2)\sigma_2^2 + \sigma_2^2(1 - \rho^2) = 0.
\end{aligned}
$$

□

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Chapter 2

Getting Permission

Abstract: We study an environment in which a manager has access to several expert advisers. Experts have the skill to carry out projects that are valuable to the manager. The manager can carry out at most one project and can do so only if at least one expert provides support. The experts have (potentially) different preferences. The game in which the manager consults experts simultaneously typically has multiple equilibria. It always includes an equilibrium in which at least one expert supports the manager's favorite project. When multiple equilibria exist, the manager's favorite equilibrium fails to survive iterative deletion of weakly dominated strategies. When projects are one dimensional (the manager cannot combine support of different projects to implement a third project that is superior to the projects supported) and payoffs are generic, only one outcome survives iterative deletion of weakly dominated strategies. It is the most preferred equilibrium from the perspective of the experts. We study the outcomes that can arise when the manager can consult the experts sequentially. We identify sequential procedures that perform well from the perspective of the manager. When projects are one dimensional and payoffs are generic, the best sequential protocol leads to the outcome that survives iterated deletion of weakly dominated strategies in the simultaneous-move game. In general, sequential consultation may be superior or inferior to simultaneous consultation.
Keywords: Expert advice, sequential decision making, persuasion.
JEL Codes: C72, D23, D82

2.1 Introduction

People often lack the skill or authority to carry out their plans without help. They rely on experts to cure their diseases, remodel their homes, and plan for their retirements. They may be unable to implement a plan of action without the approval or participation of others. We study situations in which an individual cannot carry out a task without expert assistance. We focus on applications in which conflict of interest may interfere with the ability of the individual to achieve his most preferred outcome, but he can leverage competition between experts to improve his outcome.

The elements of the model are a finite set of projects, a finite set of experts, and a manager. The experts and manager have preferences over projects. The manager wishes to carry out a project, but can do so only if an expert supports it. We are interested in the relationship between how the manager requests support and the project selected. Consider two alternative organizations. In the first organization, the manager simply asks experts to report which of the projects they will support. If no expert supports any project, then the outcome is the status quo. Otherwise, the manager implements the best project consistent with the experts' approvals. In the second organization, the manager consults experts sequentially. In the first case, provided that there are at least two experts, there is always an equilibrium in which the manager receives the support needed to carry out his favorite project. If one expert supports this project, then the manager will ignore the behavior of the other experts. So it is a best reply for all of the other agents to support the manager's favorite. Sequential consultation may not work as well for the manager. In particular, if there is a project that all experts prefer to the manager's favorite, then sequential consultation will never provide the manager with permission to carry out his favorite project.

We want to know how the manager should organize consultation to maximize his payoff. Naively, the result that the simultaneous-move game includes an equilibrium that supports the manager's favorite outcome provides an answer to this question: The manager achieves his best possible outcome by consulting simultaneously. We believe that the manager-preferred equilibrium is an implausible prediction in many cases, however. This belief leads us to investigate the simultaneous-move game in more detail. The simultaneous-move game typically has multiple Nash equilibrium outcomes. When it is not possible to combine support of different projects to implement a third project that is superior to the projects supported (we call this the case of one-dimensional projects), experts have common preferences over equilibria and these preferences are completely opposed to the preferences of the manager. That is, if manager prefers equilibrium project x to x', then all experts prefer x' to x. (If an expert preferred project x to project x', then x' could not be an equilibrium outcome because the expert who preferred x could deviate and support x.) An equilibrium refinement (iterative deletion of weakly dominated strategies) selects the experts' preferred equilibrium.[1] Hence the refinement rejects the manager's preferred outcome whenever another equilibrium exists. Section 2.4 presents the results, including a generalization to multi-dimensional projects.

On the basis of the weak-dominance refinement (and intuition), we view the experts' preferred equilibrium as the most plausible outcome of the simultaneous-move game. This raises two questions. First, what is the value of having an additional expert? If we selected the manager-preferred equilibrium, the answer to the question is simple. Going from one expert to two experts is valuable (unless the manager's favorite task is also the initial expert's favorite task). Adding a third expert, however, has no value. When we focus on the expert-preferred equilibrium, adding experts may lead to a more attractive outcome for the manager. Our characterization implies that the manager gains by adding an additional expert if doing so makes the experts' preferred equilibrium more attractive to the manager. If x is the prediction of the simultaneous-move game with a fixed set of experts, adding an additional expert benefits the manager if there exists a project x' that both the manager and the new expert prefer to x.

The second question we study is: Could an alternative organization lead to outcomes that the manager prefers more than the expert-preferred equilibrium of the simultaneous-move game? We describe a **canonical sequential protocol** in Section 2.5 and demonstrate a sense in which it performs at least as well as any other sequential protocol. We compare the performance of this protocol to simultaneous consultation. The optimal sequential protocol has a simple form. The manager approaches the first expert and asks her to approve his favorite project. If she does, the process stops. If not, he gives the second expert the opportunity to approve his favorite option. Consultation stops as soon as an expert approves a project. If all experts decline to approve a project, the manager returns to the first expert and asks her to approve his next-best project. If no

[1]The result requires an assumption that holds for generic preferences.

expert approves any project, the outcome is the status quo. An essential feature of this procedure is that the manager must be able to consult experts more than once.

Assume that it is not valuable for the manager to use approval of different projects to carry out a third project. This case arises in our formal model when the set of projects is one dimensional. We show that, when projects are one dimensional and preferences satisfy a genericity condition, a sequential game cannot be better for the manager than the simultaneous game, but that the canonical sequential protocol enables him to implement the outcome that survives iterative deletion of weakly dominated strategies in the simultaneous-move game. That is, a properly designed sequential organization does at least as well as simultaneous consultation.

A special case, which we describe in Section 2.3, provides intuition for the results. Suppose that the projects are ordered so that the manager prefers the greatest project and experts have single-peaked preferences over projects. That is, Expert i is characterized by an optimal project x_i^*. Her preferences increase for $x < x_i^*$ and decrease thereafter. In this setting, projects greater than x_i^* are weakly dominated and that the salient prediction for the simultaneous-move game is that the manager will implement the maximum of the x_i^*. Even if preferences are not single peaked, the equilibrium cannot result in a project less than the maximum x_i^* because otherwise an expert would have a profitable deviation. Will the manager actually do better? The answer is plainly yes if the maximum x_i^* is not an equilibrium task. It will fail to be an equilibrium if there exists an expert j who prefers $x_j > x_i^*$ to x_i^*. In the one-dimensional case, the basic insight of the single-peaked example remains true. Our general characterization theorem formalizes this observation.

We extend the analysis beyond the single-peaked case by relaxing the assumption that preferences are single-peaked and by studying multi-dimensional environments. When experts have general (not necessarily single-peaked) preferences, but projects are one-dimensional, the equivalence between sequential and simultaneous consultation remains true. The outcome that survives iterated deletion of weakly dominated strategies is the smallest project with the property that no expert prefers a larger project. This outcome (but nothing better for the manager) can be generated by a sequential protocol. In the multi-dimensional setting where the manager can combine approvals of x and x' to carry out a hybrid job that he strictly prefers to both x and x', the canonical sequential protocol may perform strictly better than the simultaneous-move game. In the multi-dimensional case, it is also possible that there is an equilibrium of the simultaneous-move game that the manager prefers to any equilibrium outcome of a sequential procedure. However the outcome provided by the canonical sequential protocol is at least as attractive to the manager as the worst equilibrium outcome of the simultaneous-move game.

Simultaneous protocols may benefit the manager if he can use the possibility of coordination failure to induce different experts to support different projects and then can combine their support to carry out a hybrid project that he strictly prefers to either of the individual projects. In this kind of situation, the manager needs more than one expert to carry out the equilibrium project. Section 2.6 discusses how one-dimensional environments differ from higher-dimensional ones.

The canonical sequential protocol permits the manager to restrict the actions of the experts. In Section 2.7 we permit the manager to prevent experts from approving certain projects. We show that limiting the options of the experts is strictly beneficial for the manager in both sequential and simultaneous consultations. With commitment, we show that there is a sequential procedure that does as well as (but no better) than the (refined) equilibrium of the simultaneous-move game.

We organize the remainder of the paper as follows. Section 2.2 describes the basic model. Section 2.3 illustrates the results using the special case of single-peaked preferences. Section 2.4 contains the analysis of the simultaneous-move game. Section 2.5 contains the analysis of the sequential game. Section 2.6 compares simultaneous to sequential institutions. Section 2.7 describes what happens when the manager has the power to restrict the set of projects that experts can support. Section 2.8 describes different settings that fit our model. Section 2.9 describes some of the papers related to ours. Section 2.10 contains concluding remarks. The appendix contains proofs that are not in the main text.

2.2 Underlying Strategic Environment

There is a finite set of players, who we call experts. I denotes the player set.[2] We assume that there is a finite set $X \subset \mathbb{R}^N$ available to each player. We call elements of $X \subset \mathbb{R}^N$ projects. We assume that X is ordered by the usual $\geq$ relation on $\mathbb{R}^N$. We assume that $(X, \geq)$ is a complete lattice so that $\min\{z \in X : z \geq x, x'\}$ and $\max\{z \in X : z \leq x, x'\}$ are contained in X for all x and x'. We let $x \vee x' = \min\{z \in X : z \geq x, x'\}$ and $x \wedge x' = \max\{z \in X : z \leq x, x'\}$.

For $\mathbf{x} = (x_1, \ldots, x_L)$,[3] $x_i \in X$ let $M(\mathbf{x}) = x_1 \vee \cdots \vee x_L$.[4] Each expert i has a payoff function $\tilde{u}_i : X^I \to \mathbb{R}$.[5] For each i, we assume that that there exists $u_i : X \to \mathbb{R}$ such that $\tilde{u}_i : X^I \to \mathbb{R}$ is defined as $\tilde{u}_i(\mathbf{x}) \equiv u_i(M(\mathbf{x}))$. We denote the minimum element of X by $\underline{x}$ and the maximum element by $\bar{x}$.[6]

We assume each $u_i(\cdot)$ is quasi supermodular.[7] Quasi supermodularity is a complementarity assumption that implies, roughly, that increasing one dimension of a project makes increases in another dimension more attractive. It will hold if utility is separable across components, but it will fail if one dimension substitutes for another dimension (and experts strictly prefer intermediate projects).

We study strategic interactions between the experts in this basic strategic environment. The environment is abstract. There are several ways to interpret the environment. We describe one application here and others in Section 2.8.

To gain an understanding of the strategic environment, assume that $X \subset \mathbb{R}_+$, that $\succ$ is the usual order ("greater than"), and that $\underline{x} = 0$. Assume that (in addition to the experts) there

[2] In an abuse of notation, we also let I denote the cardinality of the player set.

[3] We use boldface to denote profiles that consist of actions or strategies of multiple players.

[4] We define M on lists that contain L elements; typically $L = I$, but sometimes we apply the function to the strategies of a proper subset of the players, so $L \leq I$ is possible.

[5] Formally, the payoff function should be defined on strategy profiles. For the simultaneous game we study in Section 2.4, X^I is the set of strategy profiles. When we study sequential games in Section 2.5, the strategy sets are more general, but we still denote payoffs functions by $\tilde{u}_i$.

[6] It is straightforward to handle environments in which $X_i \neq X_j$ for some i and j. If $X_i \neq X_j$, then we can replace both sets by $X_i \cup X_j$ and extend preferences by assigning a low value to $u_i(x_i)$ for $x_i \notin X_i$. This extension rules out situations in which an expert would like to support a project but is unable to do so.

[7] The function $f : X \to \mathbb{R}$ is supermodular if $f(v \vee w) + f(v \wedge w) \geq f(v) + f(w)$ and quasi-supermodular if $f(v) \geq (>) f(v \wedge w)$ implies $f(v \vee w) \geq (>) f(w)$.

is a manager with strict preferences over X. Denote the manager's preference relationship by $\succ$.[8] Assume that if $x, x' \in X$ and $x \geq x'$, then $x \succeq x'$. We interpret $\underline{x}$ as status quo project. The manager prefers any other project to $\underline{x}$. The manager cannot implement a project different from the status quo without the assistance of at least one expert. We will study strategic environments in which experts announce which project they support. When offered a variety of projects, the manager will select the maximum (his most preferred project from the set). For this reason, we assume that experts report only a single project and that preferences over profiles $\mathbf{x} \in X^I$ depend only on the maximum of $\mathbf{x}$. That is, we study a reduced form of a game in which the manager is a strategic player who selects his favorite project among those offered by experts. In this environment, there are no natural restrictions on the experts' preferences over X. For example, let $I = 2$ and $X = \{0, .1, \ldots, .9, 1\}$, where x describes a project that generates total surplus x. If the manager cares about total surplus, then he prefers x to x' if and only if $x > x'$. But different projects may distribute the share of the surplus across experts differently. This example suggests how transfers could be compatible with our framework as long as they are included in the description of elements of X. (If all divisions of the surplus were feasible, then they would need to be added to X as additional projects. Our formulation does not permit an environment in which projects are associated with the surplus that they generate, but that once supported, the manager can freely distribute the surplus a project generates. Instead, we require that actual division of surplus also requires approval.)

We close this section with a comment on the interpretation of $\succ$. We assume that the manager has strict preferences over projects and the order represents these preferences.[9] This suggests that if the manager has complete preferences, we can take X to be a completely ordered set. Specifically, we can assume $X \subset \mathbb{R}$ and (by reordering elements of X) that $\succ$ agrees with the usual order on $\mathbb{R}$ ($>$). We wish to extend the analysis to allow for the possibility that $M(\mathbf{x}) > x_i$ for all i. When this happens, the manager can combine support from different experts to carry out a project he prefers to any single project authorized by an expert. Consequently, we want to

[8] For $x, x' \in X$, we write $x \succeq x'$ if $x \succ x'$ or $x \sim x'$. Relations $\prec$, $\not\succ$, and $\not\succeq$ are defined in the standard way.

[9] Allowing the manager to be indifferent between projects adds complexity without insight.

examine situations in which X is not completely ordered by $\geq$. When we do so, we maintain the following assumptions: $X \subset \mathbb{R}^N$; and the manager's complete preference ordering is monotonic in the sense that $x > x'$ implies that the manager prefers x to x' ($x \succ x'$).

2.3 Single-Peaked Preferences

This section illustrates our results using a special case. We postpone a formal description of consultation games to the next section.

Imagine that experts have single-peaked preferences. That is, the set of projects X is a subset of the real line and, for each expert i, there is a project x_i^* such that if $x_i^* > x > x'$, then $u_i(x_i^*) > u_i(x) > u_i(x')$ and if $x_i^* < x < x'$, then $u_i(x_i^*) > u_i(x) > u_i(x')$. If Expert i is able to sponsor any project, then she can support x_i^* and hence the manager will receive permission to implement a project $x \geq x_i^*$. It follows that any consultation scheme that permits every expert to offer approval will permit the manager to implement at least $x^* = \max x_i^*$. Can the manager do better? If there are at least two experts, there will be an equilibrium of the simultaneous-move game in which $\bar{x}$ (the manager's favorite project) is approved. This outcome arises if, for example, all experts support $\bar{x}$. If $x^* < \bar{x}$, this outcome is not possible in a subgame-perfect equilibrium of finite game of perfect information in which the manager consults experts sequentially and one at a time. A standard backward-induction argument demonstrates that if Expert i is consulted when the largest project supported is x, then she will never permit more than $\max\{x, x_i^*\}$. Hence in a sequential setting, the outcome will be x^*.

Does this mean that the simultaneous procedure is better for the manager? We believe that the answer is "no" because the outcomes that approve $x > x^*$ are implausible even in the simultaneous-move game. It is weakly dominated for Expert i to support a project $x > x_i^*$. To see this, let $x > x_i^*$ and let x_{-i} denote the maximum project supported by the other experts. If $x_{-i} \geq x$, then the choice of x versus x_i^* does not influence the final outcome. If $x > x_{-i}$, then Expert i strictly prefers $\max\{x_i^*, x_{-i}\}$ to x. Hence if experts avoid dominated strategies (or select strategies conditioning on the event that they are pivotal), then no project greater than x^* would gain approval. The manager will not be able to get support for projects he prefers to x^* using any sequential protocol, but can guarantee x^* if he consults each expert at least once.

If the preferences are not single peaked, there is value to consulting more than one expert. Let x_i^* be Expert i's favorite project and let x^* be the maximum of the x_i^* as above. For concreteness, let $x_1^* = x^*$. Assume that that there is a project $x_2 > x^*$ such that Expert 2 prefers x_2 to x^*. It is apparent that there is a conflict between the experts. On one hand, in the simultaneous-move game, Expert 1 will not settle for a project less than x^*. On the other hand, Expert 2 would prefer to permit x_2 given that x^* has been approved. Hence the ability to consult two experts can strictly improve the manager's payoff. Simultaneous consultation must lead to a project that the manager prefers to x^*. The situation is tricker under sequential consultation. If the manager consults each expert only once, starting with Expert 1, Expert 2's (credible) threat to support x_2 if x^* receives support might deter Expert 1 from supporting x^*. We demonstrate that the manager will receive support for x_2 if he consults according to a well designed protocol. The manager asks an expert to support a project and then either consults another expert or stops the process. The manager can benefit by consulting experts multiple times. For example, if Expert 1 supports x^*, then Expert 2 may be willing to support $x_2 > x^*$ if she prefers x_2 to x^*. When Expert 2's preferences are not single-peaked, this possibility may arise even if Expert 2's favorite project is less than x^*. Of course, Expert 1 may prefer project $x > x_2$ to x_2, which means that giving Expert 1 additional opportunities to approve projects may influence outcomes. We show in Section 2.5 that the manager can take advantage of differences in preferences between experts to create a sequential consultation procedure that leads to a project he prefers as much as the outcome of the simultaneous-move game.

The critical property of the equilibrium project in the one-dimensional case is that all experts must weakly prefer it to any alternative project preferred by the manager. In Section 2.4 we show that when the utility functions of the experts are one-to-one, then this property characterizes the equilibrium outcome that survives iterated deletion of weakly dominated strategies in the simultaneous-move game.[10] Section 2.5 describes a sequential procedure that generates this outcome.

Even in the single-peaked case, the manager can do better than x^* if he has the power to restrict the experts' options. For example, if one expert prefers $\bar{x}$ to $\underline{x}$, then the manager gets $\bar{x}$ by limiting the experts to supporting either $\bar{x}$ or $\underline{x}$. In Section 2.7 we discuss the value of commitment in both simultaneous and sequential consultations.

The single-peaked example is not rich enough to illustrate the difference between sequential and simultaneous consultation when X is multi dimensional. We discuss this in Section 2.6.

2.4 Simultaneous Moves

In this section, we study the game in which each expert simultaneously selects an element in X. If $\mathbf{x} = (x_1, \ldots, x_I)$ is the profile of projects, then Expert i's payoff is $\tilde{u}_i(\mathbf{x}) = u_i(M(\mathbf{x}))$. We interpret the minimum element of X, $\underline{x}$, to be a status quo. So when Expert i wishes to support no project, she uses the strategy $x_i = \underline{x}$.

Section 2.4.1 points out basic properties of the Nash Equilibria of this game. Section 2.4.2 describes the equilibrium refinement. Section 2.4.3 states our main characterization result for the simultaneous-move game. Section 2.4.4 contains examples that further illustrate the result and demonstrates the tightness of our characterization theorem.

2.4.1 Basic Properties

A profile $\mathbf{x}^* = (x_1^*, \ldots, x_I^*)$ with the property that $u_i(M(\mathbf{x}^*)) \geq u_i(M(\mathbf{x}))$ for all $\mathbf{x}$ such that $M(\mathbf{x}) \geq M(\mathbf{x}^*)$ and all i is a Nash equilibrium profile. If $\mathbf{x}^*$ is a Nash Equilibrium, we refer to $M(\mathbf{x}^*)$ as an equilibrium outcome. For any equilibrium profile $\mathbf{x}^*$, a strategy profile $\mathbf{x}$ that satisfies $x_i \leq x_i^*$ and at least two $x_j = M(\mathbf{x}^*)$ is a Nash Equilibrium. The manager obtains his

[10] When X is finite, a finite set of real numbers defines an expert's utility function. The one-to-one condition is simply the restriction that no expert is indifferent between two projects. In fact, the arguments depend on a weaker condition: each local maxima of u_i is achieved by only one project.

most preferred outcome when $M(\mathbf{x}^*)$ is equal to the maximum element in X, $\bar{x}$. Task $\bar{x}$ is always an equilibrium outcome. It can be supported by a strategy profile in which at least two experts play $\bar{x}$. Typically there are other Nash Equilibria.

We claim that the pure-strategy Nash Equilibria are Pareto ranked from the perspective of the experts when $\geq$ is a complete order. More generally, if x^* and x^{**} are equilibrium outcomes and $x^{**} \geq x^*$, then all experts prefer x^* to x^{**}. To see this, observe that if any expert preferred the outcome x^{**} to x^*, then she could deviate by using the strategy x^{**} instead of the strategy she uses in the equilibrium that leads to the outcome x^*. Consequently if $\geq$ is complete, the equilibria are Pareto ranked. In this case, the manager's preferences $\succ$ coincide with $\geq$, so that the manager's preferences (restricted to equilibria) are completely opposed to the (common) preferences of the experts. The equilibria are not necessarily Pareto ranked from the perspective of the experts if projects are partially ordered because in this case it is possible that $x^{**} \succ x^*$ but not $x^{**} \geq x^*$. Even when $\succ$ does not coincide with $\geq$, if $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are both Nash Equilibria and $M(\mathbf{x}^{**}) \geq M(\mathbf{x}^*)$, then $\tilde{u}_i(\mathbf{x}^*) \geq \tilde{u}_i(\mathbf{x}^{**})$ for all i.

2.4.2 Weak Dominance

The possibility of multiple equilibria leads us to consider a more restrictive solution concept.

Definition 1. *Given subsets $X_i' \subset X$, with $X' = \prod_{i \in I} X_i'$, Expert i's strategy $x_i^* \in X_i'$ is a best response to $\mathbf{x}_{-i} \in X'_{-i}$ relative to X_i if $\tilde{u}_i(x_i^*, \mathbf{x}_{-i}) \geq \tilde{u}_i(x_i, \mathbf{x}_{-i})$ for all $x_i \in X_i$. Expert i's strategy $x_i \in X_i'$ is weakly dominated relative to X' if there exists $x_i' \in X_i'$ such that $\tilde{u}_i(x_i, \mathbf{x}_{-i}) \leq \tilde{u}_i(x_i', \mathbf{x}_{-i})$ for all $\mathbf{x}_{-i} \in X'_{-i}$, with strict inequality for at least one $\mathbf{x}_{-i} \in X'_{-i}$.*

Definition 2. *The set $S = S_1 \times \cdots \times S_I \subset X$ survives iterated deletion of weakly dominated strategies (IDWDS) if for $m = 0, 1, 2, \ldots$, there are sets $S^m = S_1^m \times \cdots \times S_I^m$, such that $S^0 = X$, $S^m \subset S^{m-1}$ for $m > 0$; S_i^m is obtained by (possibly) removing strategies in S_i^{m-1} that are weakly dominated relative to S^{m-1}; $S^m = S^{m-1}$ if and only if for each i no strategy in S_i^{m-1} is weakly* *dominated relative to* S^{m-1}*; and* $S_i = \bigcap_{m=1}^{\infty} S_i^m$ *for each i.*[11]

For finite games, it must be the case that there exists an m such that $S^r = S^m$ for all $r > m$. There are typically many different procedures that are consistent with Definition 2. These procedures may lead to different sets that survive the process. We discuss properties that are common to all sets that survive and give conditions under which all sets that survive lead to the same maximum project.

IDWDS is a powerful concept that makes strong demands on the rationality of agents. It is also delicate – the order of deletion matters and it is sometimes poorly behaved in games with continuous strategy spaces.[12] Nevertheless, this concept appears appropriate in contexts such as ours where an individual agent's decision is relevant to her own payoff in a limited number of circumstances. Just as in voting models one wants to condition behavior on the event that a voter is pivotal, in our model, one wants to focus attention on circumstances when an expert's strategy is pivotal. Weak dominance arguments capture these strategic circumstances.[13]

We analyze the implications of applying iterated deletion of weakly dominated strategies. Sobel (2019) introduces a class of games called WID-supermodular games and describes general properties of strategies that survive the process of iteratively deleting weakly dominated strategies in these games. He shows that if $X \subset \mathbb{R}$, then the simultaneous-move game is a WID-supermodular game. It is straightforward to show that when $u_i(\cdot)$ is quasi supermodular, the simultaneous-move game is a WID-supermodular game even when X is multi dimensional.

2.4.3 Characterization

This section characterizes the outcomes that survive IDWDS in the simultaneous-move game. We begin with some general properties of the equilibrium set that follow from quasi

[11]Our notation follows these rules: superscripts denote steps in an iterative process; subscripts denotes players; arguments are components.

[12]In particular, in large games there is no guarantee that there exists a Nash equilibrium in strategies that are not weakly dominated. It is for this reason that we limit attention to finite strategy spaces.

[13]It is possible that alternative solution concepts lead to the same selection as IDWDS. A natural candidate, trembling-hand perfection, is not sufficient. It is not difficult to construct generic examples with multiple trembling-hand perfect equilibria.

supermodularity. Topkis (2011, Theorem 2.72) reports Fact 1.

Fact 1. *For any sublattice $X' \subset X$, $\arg\max_{z \in X'} u_i(z)$ is a sublattice of X.*

Hence the set of best replies to any pure strategy forms a sublattice and the smallest best response exists.

Lemma 7. *If π' and π'' are equilibrium outcomes, then $\pi' \wedge \pi''$ is an equilibrium outcome.*

Lemma 7 is obvious when X is completely ordered because in that case $\pi' \wedge \pi''$ is equal to the min of π' and π''. In general, the result is a consequence of quasi supermodularity, which guarantees that if there exists a profitable deviation from $\pi' \wedge \pi''$, then there exists a profitable deviation from π' or π''.

We have observed that the maximal project is an equilibrium outcome. Lemma 7 implies that there is a minimum equilibrium outcome. We next show that the manager prefers every project that survives iterated deletion of weakly dominated strategies, whether it is an equilibrium outcome or not, to the minimum equilibrium outcome. Before we describe the result, we define two quantities π^* and $\tilde{\pi}^*$.

Definition 3. *The smallest strictly preferred equilibrium outcome is*

$$\pi^* = \min\{\pi : u_i(\pi) > u_i(x_i) \textit{ for all } x_i > \pi \textit{ and all } i\}.$$

Definition 4. *The smallest equilibrium outcome is*

$$\tilde{\pi}^* = \min\{\pi : u_i(\pi) \geq u_i(x_i) \textit{ for all } x_i > \pi \textit{ and all } i\}.$$

These outcomes are Pareto efficient (from perspective of the experts) in the set of Nash equilibrium payoffs. "Strictly preferred" in the definition of π^* refers to the preferences of the experts (and not those of the manager).

We note several consequences of these definitions. It is immediate that $\tilde{\pi}^*$ is well defined and is equal to the smallest Nash equilibrium outcome. That π^* is well defined is a straightforward consequence of the definition. Clearly, $\pi^* \geq \tilde{\pi}^*$. If $u_i(\cdot)$ is one-to-one for each player, then $\pi^* = \tilde{\pi}^*$. We emphasize that π^* and $\tilde{\pi}^*$ differ only in non-generic cases.[14] Any strategy profile $\mathbf{x}$ that satisfies $x_i \leq \pi$ and at least two $x_j = \pi$ is a Nash Equilibrium for $\pi = \pi^*$ and $\tilde{\pi}^*$. We will show that $\tilde{\pi}^*$ is a lower bound of the set of equilibrium outcomes that survive IDWDS, i.e., all outcomes that survive IDWDS are greater than or equal to $\tilde{\pi}^*$.

We need a bit more terminology and notation to state our main result.

Definition 5. *Let*

$$\bar{x}(k) = \max\{x(k) : \textit{ there exists } x(-k) \in \mathbb{R}^{N-1} \textit{ such that } x = (x(k), x(-k)) \in X\}$$

be the largest component of a (feasible) project in dimension k. Let $\bar{x}(-k)$ be the collection of largest components in dimensions other than k.

We set $\bar{x} = (\bar{x}(1), \ldots, \bar{x}(N))$.

Definition 6. *The bounding project $\bar{\pi}^*$ is*

$$\min\{\pi : u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k)) \textit{ for all } x_i(k) > \pi(k), \textit{all dimensions } k, \textit{ and all } i\}.$$

In the appendix we show that the minimum in the definition exists (Claim 1). Denoting this value by $\bar{\pi}^*$ we show that $\bar{\pi}^*$ is an equilibrium outcome and $\bar{\pi}^* \geq \pi^*$. If X is completely ordered, then $\pi^* = \bar{\pi}^*$. Hence in the generic, one-dimensional case $\tilde{\pi}^* = \pi^* = \bar{\pi}^*$.

Example 6 (in Section 2.4.4) illustrates that it is possible that $\pi^* < \bar{\pi}^*$. Example 6 also provides insight into why the bounding projects may be strictly greater than π^* when X is not one dimensional. Each expert must decide if she is providing "too much" opportunity to the manager dimension-by-dimension. The test imposed in the definition of bounding project requires that when an expert considers reducing the k^{th} component of her strategy, she assumes that the manager faces no constraints in other dimensions. It is possible to refine the definition

[14] We say that a property is generic if it holds for an open set of Lebesgue measure one and the property is non-generic otherwise. In this paper, genericity always refers to the property that utility functions are one-to-one.

of bounding project (to obtain a weakly lower bound) by replacing $\bar{x}$ by the upper bound of strategies that survive deletion of weakly dominated strategies. We omit this discussion because it adds complexity without much insight.

We can now state the main result of this section.

Proposition 7. *If* $\mathbf{x}$ *is a strategy profile that survives IDWDS in the simultaneous-move game, then* $M(\mathbf{x}) \in [\tilde{\pi}^*, \bar{\pi}^*]$. *For any equilibrium strategy profile,* $\tilde{u}_i(\mathbf{x}) \geq u_i(\bar{\pi}^*)$ *for all i.*

The proposition bounds the set of outcomes that survive iterative deletion of weakly dominated strategies. The lower bound is the lower bound of the set of Nash equilibria. The upper bound is typically lower than the maximal project. We provide examples to demonstrate that the bounds need not survive IDWDS, but we are able to describe conditions when the bounds are tight.

To prove the proposition, we first show that there is always an outcome less than or equal to π^* that survives IDWDS. This observation follows because if $x_j \leq \pi^*$ for all $j \neq i$, then Expert i must have an undominated best reply to $\mathbf{x}_{-i}$ that is less than or equal to π^*. Next we show that strategy profiles $\mathbf{x}$ with $M(\mathbf{x}) < \tilde{\pi}^*$ must eventually be eliminated. This argument uses the definition of $\tilde{\pi}^*$ and, in particular, the fact that for any $\tilde{\pi} \not\geq \tilde{\pi}^*$, there must exist an expert i and an $x_i' > \tilde{\pi}$ such that $u_i(x_i') > u_i(\tilde{\pi})$ and we can find such a strategy that weakly dominates Expert i's strategy in $\mathbf{x}$. Finally, we show how to delete strategies that are not less than or equal to $\bar{\pi}^*$. Proving this is more involved. The argument involves constructing a strategy that dominates the smallest remaining strategy that is not below $\bar{\pi}^*$. The appendix contains the details.

Sobel (2019, Proposition 3) proves this result when $X \subset \mathbb{R}$.

Corollary 2. *If* $\bar{\pi}^* = \tilde{\pi}^*$, *then for all* $\mathbf{x}$ *that survive IDWDS,* $M(\mathbf{x}) = \pi^* = \bar{\pi}^* = \tilde{\pi}^*$.

Corollary 2 follows directly from Proposition 7.

Remark 1. *When X is one dimensional, the project that survives is unique if* $u_i(\cdot)$ *is one-to-one for all i (so that* $\pi^* = \tilde{\pi}^*$*).*

As the examples in the next section demonstrate, we cannot generally say more than that equilibrium projects must be in the interval $[\tilde{\pi}^*, \bar{\pi}^*]$. That is, the bounds are not attained in every game.

2.4.4 Examples

In this subsection we present examples that demonstrate that we cannot strengthen the conclusion of Proposition 7 and that, in general, the order of deleting strategies matter. In this case Proposition 7 identifies a unique payoff that survives IDWDS when payoff functions are generic and X is one dimensional. Hence the pathologies when projects are completely ordered are all due to ties in payoff functions. We cannot guarantee that either project π^* or $\tilde{\pi}^*$ will survive IDWDS nor can we guarantee that all payoffs that survive IDWDS are greater than or equal to $u_i(\pi^*)$.

In Examples 1 – 5 assume that there are three projects, A, B, and C. The manager prefers C to B to A and the projects are completely ordered according to these preferences (so that, for example, $M(A,B) = B$). Project A is the status quo. The specification of expert preferences determine the payoff matrices.

Example 1. *Consider the following game:*

	A	B	C
A	$2,0$	$1,2$	$1,1$
B	$1,2$	$1,2$	$1,1$
C	$1,1$	$1,1$	$1,1$

Expert preferences are given by: $u_1(A) = 2, u_1(B) = u_1(C) = 1$*: and* $u_2(A) = 0, u_2(B) = 2, u_2(C) = 1$*. We have* $\pi^* = C$ *and* $\tilde{\pi}^* = B$*. If we first discard the bottom two strategies of Expert 1 (the row player),* (A,B) *is the only strategy profile that survives IDWDS; alternatively, discarding Expert 2's A and C, leaves* (x_1, B) *for* $x_1 = A,B,C$ *(so either B or C is the project implemented). You cannot delete the profile* (A,B)*. So the set of projects that survive IDWDS* *always includes $\tilde{\pi}^*$, but may or may not include π^*. Although π^* is a project that survives IDWDS for some order of deletion, this example demonstrates that we cannot guarantee that π^* survives independent of the deletion order.*

Example 2. *Consider the following game:*

	A	B	C
A	$1,1$	$-1,0$	$1,0$
B	$-1,0$	$-1,0$	$1,0$
C	$1,0$	$1,0$	$1,0$

We have $\pi^* = C$ and $\tilde{\pi}^* = A$. If we discard Expert 2's B and C, (A,A) and (C,A) survive IDWDS. Discarding Expert 1's A and B, leads to (C,x_2) for $x_2 = A,B,C$ surviving. You cannot delete Expert 1's strategy C. Here the set of projects that survive IDWDS always includes π^*, but may or may not include $\tilde{\pi}^*$.

Taken together, the examples show that you need not select π^* or $\tilde{\pi}^*$. The examples are consistent with the observation that you will select one or the other and that there is a way of deleting weakly dominated strategies that will select both. The second claim is not true, however.

Example 3. *Consider the following game:*

	A	B	C
A	$0,0$	$0,-1$	$-1,-1$
B	$0,-1$	$0,-1$	$-1,-1$
C	$-1,-1$	$-1,-1$	$-1,-1$

We have $\pi^* = C$ and $\tilde{\pi}^* = A$. (B,A) and (A,A) are equilibria that survive IDWDS, but it is weakly dominated to enable the project C. Consequently $\tilde{\pi}^*$ is an equilibrium outcome that survives IDWDS; π^* is not an equilibrium outcome that survives IDWDS; and there is an equilibrium outcome that survives IDWDS strictly between $\tilde{\pi}^*$ and π^*.

It is also possible to construct an example in which $\tilde{\pi}^*$ is not an equilibrium outcome that survives IDWDS.

Example 4. *Consider the following game:*

	A	B	C
A	$2,1$	$0,1$	$2,0$
B	$0,1$	$0,1$	$2,0$
C	$2,0$	$2,0$	$2,0$

We have $\pi^* = C$ and $\tilde{\pi}^* = A$. The only equilibrium outcome that survives IDWDS is π^*.

The next example shows IDWDS need not bound the utility of the experts. That is, the bound on utility given in Proposition 7 requires a restriction to equilibrium strategies.

Example 5. *Consider the following game:*

	A	B	C
A	$1,0$	$-1,0$	$0,0$
B	$-1,0$	$-1,0$	$0,0$
C	$0,0$	$0,0$	$0,0$

In this example, $\pi^* = C$ and $\tilde{\pi}^* = A$. Expert 1's B strategy is weakly dominated, but all other strategies survive. Consequently, profile (A,B) survives IDWDS, although it is not an equilibrium. The project B induced by (A,B) gives Expert 1 a payoff less than $u_1(\pi^*)$.

Now we turn to a situation in which the strategies are not completely ordered.

Example 6. *Consider the following game:*

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$2,2$	$1,-1$	$-1,1$	$0,0$
$(1,0)$	$1,-1$	$1,-1$	$0,0$	$0,0$
$(0,1)$	$-1,1$	$0,0$	$-1,1$	$0,0$
$(1,1)$	$0,0$	$0,0$	$0,0$	$0,0$

In this example, the set of projects has two dimensions. If one expert supports $(1,0)$ *and the other supports* $(0,1)$*, then the manager will implement* $(1,1)$*. The experts' preferences are:* $\{0,0\} \succ_1 \{1,0\} \succ_1 \{1,1\} \succ_1 \{0,1\}$ *and* $\{0,0\} \succ_2 \{0,1\} \succ_2 \{1,1\} \succ_2 \{1,0\}$*. There are* 2 *pure-strategy equilibrium projects:* $(0,0),(1,1)$*. It is dominated for Expert* 1 *to support* 1 *on the second dimension and for Expert* 2 *to support* 1 *on the first dimension. So discard* $(1,1),(0,1)$ *for Expert* 1 *and* $(1,1),(1,0)$ *for Expert* 2*. No other strategy can be deleted. Every project survives IDWDS. In this example* $\pi^* = \tilde{\pi}^* = (0,0)$*, but* $\bar{\pi}^* > \pi^*$ *and there are strategies that survive IDWDS that exceed* π^**.*

Example 6 illustrates how the manager's most preferred project may survive iterative deletion of weakly dominated strategies in the simultaneous-move game even when the largest strategy is weakly dominated. This cannot happen in the one-dimensional case. In the example, $(1,0)$ is an unattractive project from the perspective of Expert 2. If this project is a possibility, then Expert 2 will have a justification for using $(0,1)$. Similarly, Expert 1 would not delete her strategy $(1,0)$ because she prefers $(1,1)$ to $(0,1)$. Although $(0,0)$ remains an equilibrium outcome that survives IDWDS, the manager's most preferred outcome is more robust than it is in the one-dimensional case.

2.5 Sequential Protocols

The manager's preferred outcome does not always survive iterative deletion of weakly dominated strategies when experts move simultaneously. This leaves open the question of whether the manager could do better by consulting the experts in a different way. This section discusses the issue. We begin with an example that suggests that sequential procedures may perform poorly relative to the simultaneous-move game. We then introduce a family of sequential protocols and describe a simple member of the family that (generically) performs at least as well as any other sequential protocol (from the standpoint of the manager) when X is one dimensional and is undominated by other sequential protocols in general. This protocol generates the same outcome as the simultaneous-move game when X is one dimensional. It may perform better or worse than simultaneous consultation when X is multi dimensional.

Section 2.5.1 contains an example that illustrates that simple sequential procedures may not lead to good outcomes for the manager. Section 2.5.2 defines general sequential procedures. Section 2.5.3 introduces two canonical sequential consultation procedures. Section 2.5.4 provides lower bounds on the outcomes generated by these protocols. Section 2.5.5 provides general upper bounds to the performance of sequential procedures. Combined with the results in Section 2.5.4 these show why the manager favors the canonical procedures. Section 2.5.6 discusses some comparative-statics properties.

2.5.1 Example

The definition of sequential protocol permits the manager to do three things: vary the order in which he consults experts; return to experts more than once; and commit to ending the consultation process. The following example demonstrates why these three features are important and gives some insight into the general construction.

Example 7. *There are five projects,* $0,1,\ldots,4$*. The projects are completely ordered and the manager prefer higher projects to lower ones. Project* 0 *is the status quo. There are two experts. Expert 1's utility satisfies*

$$u_1(1) > u_1(3) > u_1(0) > u_1(2) > u_1(4)$$

and Expert 2's utility satisfies

$$u_2(0) > u_2(2) > u_2(1) > u_2(4) > u_2(3).$$

The unique outcome that survives IDWDS in the simultaneous game is the manager's favorite outcome.[15] *Consider the four possible consultation sequences that consult each expert at most*

[15] Notice that this is the manager's favorite outcome even though it is Expert 1's least favorite outcome and ranks

once: consulting exactly one expert, or consulting both in either order.

Sequence	*Outcome*
Expert 1	1
Expert 2	0
Expert 1, then 2	0
Expert 2, then 1	1

The first two lines in the table are straightforward to understand. When the manager consults only one expert, the expert picks her favorite project. If he instead consults Expert 1 and then Expert 2, Expert 2 will approve project 2 if Expert 1 starts with 1; Expert 2 will approve project 4 if Expert 1 starts with 3; Expert 2 will approve project 0 if Expert 1 starts with 0; if Expert 1 starts with 4, Expert 2's action will not influence the project choice; and Expert 2 will not approve a project 3 or 4 if Expert 1 starts with 2. Hence Expert 1 does best if she approves the status quo. Similarly, if the manager asks Expert 2 first, then Expert 1, the final outcome will never be 0 or 2. So Expert 2 supports project 1 and Expert 1 does not support a higher project.

It is straightforward to confirm that returning to experts will not lead to either expert supporting another project. Hence, it appears that sequential consultation need not lead to support for π^*.

We will investigate the implications of giving the manager more control over the nature of consultation. Suppose the manager begins by asking Expert 1 "Will you support Project 4?" If Expert 1 says "yes," then the manager stops and implements his favorite project. If Expert 1 says "no," then the manager repeats the question to Expert 2. If Expert 2 says "no," then the manager returns to Expert 1 and requests approval for project 3. And so on: the manager consults experts one-by-one, asking for a support for particular projects. If all experts decline to support a project, then the manager returns to the first expert and asks for approval of the next best project.[16] When the experts play this game, we can work backwards to see that an expert will approve Project 4.

next-to-last for Expert 2. Hence, although IDWDS sometimes eliminates the manager's favorite outcome, it does not do so here. The manager benefits from differences in preferences between the experts.

[16] We are grateful to Christopher Turansick for suggesting this procedure, which is the basis for the protocols we introduce in Section 2.5.3.

Suppose that both experts have rejected Projects 2, 3, and 4. Expert 1 would support Project 1, because that is her favorite. Knowing this, Expert 2 will support Project 2 because she knows that Project 0 is not available (because Expert 1 will support Project 1). But Expert 1 prefers Project 3 to Project 2, so she will support Project 3 when asked. Finally, given that Expert 1 would support Project 3 if asked, Expert 2 supports Project 4. The main result of this section is a formal definition of this protocol and a characterization of the project it generates.

2.5.2 Preliminaries

In this section we describe general sequential consultation procedures. In each period t, the procedure selects an expert to make a choice and a choice set for that expert. After a choice, the manager either consults with another expert or stops. Formally, let $H_0 = \emptyset$, $H_t = (I_+ \times X)^t$, where X is the (finite) set of possible projects and $I_+ = \{0\} \cup I$ is the union of "0" and the set of players. Let $H = \bigcup_{t=0}^{T} H_t$ be the set of histories. If $h_t = (h_t^1, \ldots, h_t^t) \in H_t$ and $h_{t'} = (h_{t'}^1, \ldots, h_{t'}^{t'}) \in H_{t'}$, then $h_t h_{t'} \in H_{t+t'}$ is the history obtained by the natural concatenation: $h_t h_{t'} = (h^1, \ldots, h^t, h^{t+1}, \ldots, h^{t+t'})$ where

$$h^m = \begin{cases} h_t^m & \text{if } 1 \le m \le t \\ h_{t'}^{m-t} & \text{if } t < m \le t+t'. \end{cases}$$

If $h' \in H_{t'}$ and $h'' \in H_{t''}$ for $t'' \ge t'$, we say $h' \subset h''$ if there exists $h \in H_{t''-t'}$ such that $h'' = (h', h)$.

Definition 7. *A sequential protocol is a mapping $P = (P_I, P_X) : H \to I_+ \times 2^X$ such that for all $h, h_t \in H$,*

$$P_I(h_T) = 0 \textit{ for all } h_T \in H_T, \tag{2.1}$$

$$P_I(h_t) = 0 \Longrightarrow P_I(h_t h) = 0, \tag{2.2}$$

$$P_I(h_t) = 0 \Longrightarrow P_X(h_t) = \emptyset, \tag{2.3}$$

and

$$P_I(h_t) \neq 0 \Longrightarrow \underline{x} \in P_X(h_t). \tag{2.4}$$

The manager observes a history, h_t. He then decides whether to stop the process ($P_I(h_t) = 0$) or to consult Expert i ($P_I(h_t) = i$). Condition (2.1) expresses the fact that the protocol must stop after T periods. Condition (2.2) means that once the decision maker stops the process, he cannot restart it. $P_X(h_t)$ describes the set of choices available to the expert consulted. Condition (2.3) states that no choices are available when the process stops. Condition (2.4) states that when the manager consults an expert, the expert always has the option to support the status quo $\underline{x}$. We limit attention to deterministic protocols that end after a finite number of periods.

A sequential protocol (we shorten this to "protocol") induces a game in which the players are the experts. Player i's strategy specifies a project as a function of h_t for each h_t such that $P_I(h_t) = i$. Given a history of length t, $h_t = (h_t^1, \ldots, h_t^t)$, let $i_t(h_t) = (i_t^1, \ldots, i_t^t)$ be the list of experts consulted and $p_t(h_t) = (p_t^1, \ldots, p_t^t)$ be the list of projects supported (the projection of h_t onto X^t), and let $\mu(h_t) = p_t^1 \vee \cdots \vee p_t^t$. A strategy profile $\mathbf{s} = (s_1, \ldots, s_I)$ determines projects $\bar{p}_t(\mathbf{s}) = (\bar{p}_t^1(\mathbf{s}), \ldots, \bar{p}_t^t(\mathbf{s}))$ and histories $\bar{h}_t(\mathbf{s})$ for $t = 1, \ldots, T$ where $\bar{h}_1(\mathbf{s}) = (P_I(\emptyset), \bar{p}_1(\mathbf{s})) = (P_I(\emptyset), s_{P_I(\emptyset)}(\emptyset))$, $\bar{p}_2(\mathbf{s}) = (\bar{p}_1(\mathbf{s}), s_{P_I(\bar{h}_1(\mathbf{s}))}(\bar{h}_1(\mathbf{s})))$, $\bar{h}_2(\mathbf{s}) = (\bar{h}_1(\mathbf{s}), (P_I(\bar{h}_1(\mathbf{s})), \bar{p}_2^2(\mathbf{s})))$, and, in general, $\bar{p}_k(\mathbf{s}) = (\bar{p}_{k-1}(\mathbf{s}), s_{P_I(\bar{h}_{k-1}(\mathbf{s}))}(\bar{h}_{k-1}(\mathbf{s})))$, $\bar{h}_k(\mathbf{s}) = (\bar{h}_{k-1}(\mathbf{s}), (P_I(\bar{h}_{k-1}(\mathbf{s})), \bar{p}_k^k(\mathbf{s})))$.[17] Expert i's payoff as a function of the strategy profile is $\tilde{u}_i(\mathbf{s}) = u_i(\mu(\bar{h}_T(\mathbf{s})))$. We say that a project π is generated by a sequential protocol if the induced game has a strategy profile that survives IDWDS in which π is implemented. Formally, π is generated by a sequential protocol if there exists a strategy profile s that survives IDWDS such that $\pi = p_T^1(\mathbf{s}) \vee \cdots \vee p_T^T(\mathbf{s})$. A project π is uniquely generated by a sequential protocol if π is the only project generated by the protocol.

The specification of the game assumes that the manager implements $\mu(\bar{h}_T(\mathbf{s}))$. One can imagine games in which the manager does not do this. If the manager implements $\gamma(h)$ given the history h, then our specification of the game requires setting $\tilde{u}_i(\mathbf{s}) = u_i(\gamma(\bar{h}_T(\mathbf{s})))$ and a project π would be generated by a sequential protocol if there exists a strategy profile that survives IDWDS

[17] These formula require a specification of s_0 because it is possible that $P_I(h) = 0$. We set $s_0(h) = \mu(h)$.

such that $\pi = \gamma(\bar{h}_T(\mathbf{s}))$. In fact, we do consider an alternative in which the manager cannot combine supports from different experts and instead implements his most preferred project from $\{p_T^1(\mathbf{s}), \ldots, p_T^T(\mathbf{s})\}$.

The formal definition of sequential protocols defines P on "too many" histories. The protocol determines which expert to consult next and (possibly) restricts the sets of projects that an expert can support at any point. Assuming that the manager can commit to the consultation procedure, we can concentrate on behavior defined on a subset of allowable histories.

Definition 8. *A history h_t is allowable if $p_t^{t'}(h_t) \in P_X(h_{t'-1})$ and $i_t^{t'}(h_t) \in P_I(h_{t'-1})$ for $t' = 1, \ldots, t$.*

A history is allowable if it specifies projects that are elements in available choice sets and in which the experts are as specified by the protocol. We can restrict strategies to be defined on the set of allowable histories.

Definition 9. *A protocol is finite if there exists T such that $P_I(h_T) = 0$ for all allowable $h_T \in H_T$.*

Definition 10. *A protocol is constrained if there exists h_t such that $P_I(h_t) \neq 0$ and $P_X(h_t) \neq X$. A protocol that is not constrained is unconstrained.*

A protocol is unconstrained if whenever the manager consults an expert, the expert can support any project. The protocol that we described in Section 2.5.1 is constrained because, whenever consulted, an expert can support either the status quo or a single alternative. We will show that the ability to constrain choices is valuable to the manager and we do not want to limit attention to unconstrained protocols. We would like to point out a restriction that our procedure will satisfy.

Definition 11. *A protocol is neutral if for each history h, i, and $x \succ \mu(h)$, there exists $h' \subset h$ such that $i = P_I(h')$ and $x \in P_X(h')$.*

Neutrality prevents the manager from "skipping" an option in a protocol. Suppose that the protocol generates the project π. Neutrality requires that at some point in the consultation process leading to π every agent had an opportunity to support any project the manager prefers to π. In order to implement a non-neutral protocol, the manager must be able to commit to resist the temptation to try to get support for a project he prefers to the one he actually implements. We view neutrality as a restriction on the manager's commitment power.

In the protocol described in Section 2.5.1, an expert cannot support any project whenever she is consulted, but there is an allowable history in which she has the opportunity to support any project. Furthermore, when an expert receives the opportunity to support x, all of experts have had a chance to approve more highly ranked projects. Hence the protocol is neutral. Neutrality guarantees that the protocol does not ignore options.

The manager can gain by using a constrained protocol. In the next section we describe a protocol that is constrained and neutral. This protocol may perform better than simultaneous consultation in multi-dimensional models. We trace this improved performance to the ability to limit the set of projects that experts can support during a stage of the consultation. We will show how to improve performance of simultaneous protocols by giving the manager additional authority. We also show that if the manager must use unconstrained protocols, then simultaneous consultation does at least as well as sequential consultation.

The ability to create a protocol assumes that the manager has commitment power. Commitment enters into the analysis in at least three ways. First, we assume that the action that the manager takes is always the best available given the strategy of the experts. By making this assumption, we implicitly assume that the manager cannot commit to implementing a project that he likes less than a project he could implement. Second, the manager might find it attractive to consult experts in an order different from what the protocol recommends or continue consultations when the protocol specifies termination. Third, we assume that the manager can restrict the set of projects that an expert can support.[18] An extreme way to do this would be to exclude some elements of x from every choice set. This strong form of commitment is valuable. If the manager is able to rule out approvals of a particular project at any point in a sequential protocol, then he

[18]We always require that the protocol include the status quo as an option. That is, the manager cannot force an expert to support a project.

should be able to rule out that project in a simultaneous-move game. Under some conditions, the manager may have this power. In Section 2.7 we demonstrate that if the manager can limit consultations to a proper subset of projects (or, equivalently, can commit to not implementing some projects even if they receive support), then he can do better than the bounds established for simultaneous games or for what is possible with neutral protocols. For example, if a single expert prefers $\bar{x}$ to $\underline{x}$, then the manager can guarantee his favorite option by refusing to permit experts to support intermediate projects (or by refusing to implement these projects when supported). If the manager can never restrict the choice set of an expert, then unconstrained protocols are the appropriate consultation procedures. Neutral protocols give the manager a bit more power than general unconstrained protocols. In the next subsection we present and study the properties of an intuitive unconstrained protocol and a constrained, but neutral, protocol. These protocols are equivalent when projects are one dimensional, but the constrained protocol performs better for the manager in general.

One way to restrict commitment ability is to study sequential games in which the manager can make choices during the consultation procedure. For example, one can imagine an environment in which the manager first selects an expert, offers her a set of projects, and, based on her choice, decides whether to terminate the consultation procedure or continue the consultation procedure. If he terminates, he must select a project. If he continues, he must decide who to consult and how to constrain the expert's choices. Standard notions of sequential rationality (subgame perfection or rationalizability) would impose the restriction that the manager selects the maximum of all projects approved (we make this assumption in the definition of sequential protocols). The power to terminate consultations and restrict choices will be valuable and some projects may not arise as equilibria in a game in which the manager is an active player. We do not pursue the analysis of these extended games for two reasons. First, we believe that it is realistic to assume that managers have some commitment power. It is common to have rules governing consultation procedures.[19] Second, and related, we are interested in protocols that

[19]Robert's Rules of Order (Robert III et al., 2020), which establishes rules governing who can speak and what can be discussed in a meeting is a leading example.

perform well even when the manager does not know the preferences of the experts. If we studied a game in which managers were active players, standard solution concepts would require that preferences are common knowledge. Instead, we study the implications of different restrictions on the set of protocols available.

2.5.3 The Canonical Sequential Protocols

This subsection introduces two canonical sequential protocols. The next subsection will explain the importance and limits of these protocols.

Assume that the manager has strict preferences over projects. Specifically, suppose that the projects can be ranked $\bar{x} = \pi_K \succ \pi_{K-1} \succ \cdots \succ \pi_1 = \underline{x}$.

Definition 12. *The canonical sequential protocol (CP) has the properties*

- $T = KI$

  and if $t = mI + r$*, for* $r = 0, \ldots, I-1$ *and* $m = 0, \ldots, K-1$*, then*

- $$P_I(h_t) = \begin{cases} 0 & \text{if } \mu(h_t) \succeq \pi_{K-m} \\ r+1 & \text{otherwise} \end{cases}.$$

- $$P_X(h_t) = \begin{cases} \emptyset & \text{if } P_I(h_t) = 0 \\ \{\pi_1, \pi_{K-m}\} & \text{if } P_I(h_t) \neq 0 \end{cases}.$$

Let us describe the behavior of the canonical sequential protocol after sensible histories. The manager approaches experts in sequence. The first time he approaches, he asks experts to support a project and stops if any of the experts supports his favorite, π_K. If all of the experts decline to support the first project, then the manager goes back to Expert 1 and asks for support of his next most preferred project. The manager continues to consult until the experts, stopping after his m^{th} consultation with an expert if he has received enough support to implement a project at least as good as the m^{th} best project. The canonical protocol is neutral and constrained.

Definition 13. *The unconstrained canonical sequential protocol (UCP) has the properties*

- $T = KI$

  and if $t = mI + r$*, for* $r = 0,\ldots,I-1$ *and* $m = 0,\ldots,K-1$*, then*

- $$P_I(h_t) = \begin{cases} 0 & \text{if } \mu(h_t) \succeq \pi_{K-m} \\ r+1 & \text{otherwise} \end{cases}.$$

- $$P_X(h_t) = \begin{cases} \emptyset & \text{if } P_I(h_t) = 0 \\ X & \text{if } P_I(h_t) \neq 0 \end{cases}.$$

In the unconstrained canonical sequential protocol (UCP), the manager follows CP, but if the manager consults Expert i, then she can support any project. UCP and CP have the same stopping rule. This means, for example, when the manager first consults Expert 1 in the UCP, Expert 1 can support π_{K-1}. The protocol then specifies that the manager will continue to consult with the other experts. If one of them supports π_K (or any project π such that $\pi \vee \pi_{K-1} \succeq \pi_K$), the consultation stops. Otherwise, consultation stops as soon as all experts have been consulted once and the manager implements project π_{K-1}.

Observe that the CP and UCP depend directly on the manager's preferences (that is, the order of projects in the protocol depends on $\succ$). The preferences of the manager do play a role in the simultaneous-move game ($M(\mathbf{x})$ determines payoffs and $M(\mathbf{x})$ depends on $\succ$), but the strategies in the simultaneous-move game do not depend on the manager's preferences.

The next example demonstrates that CP may perform differently than UCP.

Example 8. *There are 6 projects, of the form* (i, j) *for* $i = 1,2,3$ *and* $j = 1,2$*, and 2 experts whose preferences are*

$$u_1(1,1) > u_1(1,2) > u_1(3,1) > u_1(2,1) > u_1(3,2) > u_1(2,2)$$

and

$$u_2(2,1) > u_2(1,1) > u_2(2,2) > u_2(3,1) > u_2(1,2) > u_2(3,2).$$

These preferences satisfy quasi supermodularity. In this case, $\tilde{\pi}^* = \pi^* = \bar{\pi}^* = (3,1)$. *Consequently, the outcome of the simultaneous-move game is* $(3,1)$. *Assume that the manager's preferences are*

$$(3,2) \succ (2,2) \succ (3,1) \succ (1,2) \succ (2,1) \succ (1,1).$$

One can verify that the outcome of the CP is $(3,2)$. *(This will be a consequence of Propositions 9.) Why does the sequential protocol yield a better result for the manager? When played simultaneously,* $(3,2)$ *is dominated for Expert 2 and* $(2,1)$ *cannot be dominated. Furthermore, Expert 2 will eventually delete* $(2,2)$ *and* $(1,2)$. *As long as Expert 2 thinks that it is possible that Expert 1 will support* $(3,1)$*, Expert 2 will avoid using strategies that support "2" in the second dimension. But once Expert 2 deletes strategies of the form* $(x(1),2)$*, Expert 1 will have no reason to support "2" in the second dimension. In the CP, however, if both experts fail to support* $(3,2)$*, then the manager will offer the experts the chance to support* $(2,2)$ *without danger of* $(3,2)$ *being selected. Expert 2 is willing to support* $(2,2)$ *because she understands that if she fails to do so, Expert 1 will support* $(3,1)$. *But then Expert 1 will be motivated to support* $(3,2)$ *in the first round because she knows that if she does not, then Expert 2 will support* $(2,2)$ *before either expert has the opportunity to support* $(1,1)$ *or* $(2,1)$.

The example identifies two differences between sequential and simultaneous procedures. On one hand, $(3,1)$ *is the outcome of the UCP. Expert 1 can support* $(3,1)$ *in the first round. Expert 2 realizes that if she supports* $(2,2)$ *in the second round, then the outcome will be* $(3,2)$. *Consequently, this protocol generates the outcome* $(3,1)$*, because once* $(3,1)$ *has been approved, neither expert would support a project that permits the manager to carry out* $(3,2)$. *It is for* *this reason we say that CP requires commitment ability. The manager would like to be able to decline to implement* $(3,1)$ *following an "impossible" history in which an expert supports* $(3,1)$ *in the first round of consultation.*

On the other hand, one can obtain $(3,2)$ *in the equilibrium of a simultaneous-move game provided that project implemented given the strategy profile* (x_1, x_2) *is* $\max_{\succ}\{x_1, x_2\}$ *(and not* $x_1 \vee x_2$*, which may be strictly preferred to both* x_1 *and* x_2*). That is, the ability to commit not to combine projects may be valuable to the manager. In this case,* $(3,2)$ *is the only outcome that survives IDWDS in the simultaneous-move game. In particular,* $(3,1)$ *cannot be the outcome because if Export 1 supports* $(3,1)$*, Expert 2 will support* $(2,2)$ *and the result will be the maximum of* $(3,1)$ *and* $(2,2)$ *with respect to the manager's preference (*$(2,2) = \max_{\succ}\{(3,1),(2,2)\}$*) and not* $(3,2) = (3,1) \vee (2,2)$*.*[20]

2.5.4 Lower Bounds for Canonical Sequential Protocols

We describe the performance of (CP) and (UCP) in this section.

Proposition 8. *If* π *is a project that survives IDWDS in the game determined by UCP, then* $\pi \geq \tilde{\pi}^*$.

Proposition 8 establishes a lower bound on the manager's outcome for sequential games. For every $\pi < \tilde{\pi}^*$, there will be an expert who strictly prefers a higher project. From this observation, it is straightforward to show that a project π such that $\pi < \tilde{\pi}^*$ cannot be generated by UCP. To prove the proposition, we must further show that if $\pi \not\geq \tilde{\pi}^*$, then π cannot be the project induced by a strategy profile that survives IDWDS. We construct a dominating strategy by showing that it is possible to increase the action of one expert at her final turn to move. When $\pi \not\geq \tilde{\pi}^*$, it is possible to find such a deviation. The altered strategy will either not change the outcome or immediately terminate the consultation process at an outcome strictly better for the deviating expert.

[20] $(3,2)$ would also be the outcome of UCP if the manager can commit to implementing $\max_{\succ}\{x_1, x_2\}$ rather than $x_1 \vee x_2$.

To describe the performance of CP we need two new terms. In the definitions, we write "$\min_{\succ}$" to denote minimization with respect to the manager's preferences $\succ$.

Definition 14. *The smallest strict CP outcome is*

$$\pi^*_{\succ} = \min_{\succ}\{\pi : u_i(\pi) > u_i(x_i) \text{ for all } x_i \succ \pi \text{ and all } i\}.$$

Definition 15. *The smallest CP outcome is*

$$\tilde{\pi}^*_{\succ} = \min_{\succ}\{\pi : u_i(\pi) \geq u_i(x_i) \text{ for all } x_i \succ \pi \text{ and all } i\}.$$

The quantities $\pi^*_{\succ}$ ($\tilde{\pi}^*_{\succ}$) and π^* ($\tilde{\pi}^*$) differ because $>$ does not coincide with $\succ$ when X is not one dimensional. The idea of the new bounds is to use the manager's preferences to provide a complete order over projects even when projects are multi dimensional. The set of x_i such that $x_i > \pi$ is identical to the set x_i such that $x_i \succ \pi$ in the one-dimensional case because the manager's preferences are monotonic. In general, $x_i > \pi$ implies $x_i \succ \pi$, but not conversely. Consequently, $\pi^*_{\succ} \geq \pi^*$ and $\tilde{\pi}^*_{\succ} \geq \tilde{\pi}^*$, with equality in the one-dimensional case. Furthermore, $\pi^*_{\succ} \geq \tilde{\pi}^*_{\succ}$. If $u_i(\cdot)$ is one-to-one for each player, then $\pi^*_{\succ} = \tilde{\pi}^*_{\succ}$.

We can provide a lower bound to the performance of CP similar (but generally higher) than the bound for UCP established in Proposition 8.

Proposition 9. *If π is a project that survives IDWDS in the game determined by CP, then $\pi \succeq \tilde{\pi}^*_{\succ}$.*

2.5.5 General Bounds on Sequential Consultation

We have focused on the canonical protocol in the previous subsection. This subsection explains what is and is not possible using general protocols.

Proposition 10. *If $x > \pi^*$, then there exists no unconstrained sequential protocol that generates the project x in a pure-strategy, subgame-perfect equilibrium. If $x \succ \pi^*_{\succ}$, then there exists no* *neutral sequential protocol that generates the project x in a pure-strategy, subgame-perfect equilibrium.*

Proposition 10 follows from backward induction. If the protocol is unconstrained, then after each history that supports no more than π^*, it is never a best response to support more than π^*. So if the first expert anticipates that the final project supported will be more than π^*, then she can do strictly better by approving π^* and no one else will add more to π^*. Consequently there will never be projects greater than π^* in equilibrium. Proving the second sentence in the proposition is more involved. The appendix contains the proof; here we indicate the idea behind the proof. Because experts can always support only the status quo, there is a terminal history in which the outcome generated is the status quo. If the protocol is neutral, then every expert has an opportunity to support $\pi^*_{\succeq}$. We can show that this means that every expert i is guaranteed a payoff of at least $u_i(\pi^*_{\succeq})$, which guarantees that $\pi^*_{\succeq} \succeq x$.

Example 10 shows that it is possible to generate projects greater than $\pi^*_{\succeq}$ (and hence π^*) using a protocol that is not neutral.

Proposition 11. *For any unconstrained protocol, there is always an outcome that survives IDWDS that is not greater than π^*.*

Proposition 11 follows from backward induction and the observation that if π^* has already been supported, no expert wishes to support a project that would lead to a strictly higher outcome. Hence if an expert supports π^* when it is her turn (and π^* is in her choice set), then she knows that there will exist undominated strategies for future experts that involve no other project being supported.

Sequential protocols generate perfect-information games. There is a close connection between subgame-perfect equilibria in perfect-information games and outcomes that survive IDWDS. So we can use Proposition 10 to obtain a characterization for our solution concept.

Proposition 12. *Fix a sequential protocol. If the experts' utility functions are one-to-one, there is a unique outcome that survives IDWDS.*

Corollary 3. *Assume the experts' utility functions are one-to-one. If the protocol is unconstrained, then no outcome that survives IDWDS is greater than* π^**. If the protocol is neutral, then the manager prefers no outcome that survives IDWDS to* $\pi^*_{\succ}$*.*

The first assertion in Corollary 3 is a consequence of Proposition 11, which shows existence of a project that satisfies IDWDS no greater than π^*, and Proposition 12, which shows that there is no other project that survives IDWDS. The second assertion follows from Proposition 10 and Proposition 12.

When X is one dimensional, our results characterize what is possible using neutral sequential protocols and provide a sense in which the canonical protocol is optimal for the manager. In the generic, one-dimensional case, the results of this subsection show that every neutral protocol generates only projects less than or equal to π^*. Combined with Propositions 8 and 9, this means that UPC and PC perform at least as well as any neutral sequential protocol from the manager's point of view. When X has more than one dimension, the results are incomplete in two ways. First, there is a difference between "less than or equal to" and "not larger than." The propositions leave open the possibility that there may be an unconstrained protocol that generates an outcome that is not comparable to π^*. Example 9 demonstrates that this possibility really arises. Second, it is possible that a protocol that is not neutral may generate an outcome strictly greater than π^*. Example 10 illustrates this possibility.

Example 9. *There are six projects. They take the form* (i, j) *for* $i = 1,2,3$ *and* $j = 1,2$*. There are three experts whose preferences are*

$$u_1(1,1) > u_1(1,2) > u_1(3,1) > u_1(3,2) > u_1(2,1) > u_1(2,2),$$

$$u_2(2,1) > u_2(1,1) > u_2(2,2) > u_2(3,1) > u_2(1,2) > u_2(3,2),$$

and

$$u_3(1,1) > u_3(1,2) > u_3(2,1) > u_3(3,1) > u_3(3,2) > u_3(2,2).$$

These preferences satisfy quasi supermodularity. In this case, $\tilde{\pi}^* = \pi^* = \bar{\pi}^* = (3,1)$*. Consider the protocol in which the manager consults all three experts in order. Expert 1 is consulted first. Expert 1's favorite project is* $(1,1)$*, but if she supports* $(1,1)$*, then Expert 2 will support* $(2,1)$*, which will be the final outcome because Expert 3 prefers* $(2,1)$ *to all higher projects. If Expert 1 supports* $(1,2)$*, then this will be the final outcome (if Expert 2 supports a higher project, then* $(3,2)$ *will be the outcome). Hence the protocol generates project* $(1,2)$*. The project* $(1,2)$ *is not comparable to* $(3,1)$*, which is the outcome of the unconstrained canonical sequential protocol. Hence if* $(1,2) \succ (3,1)$ *it is possible for the manager to do better using an alternative to the unconstrained canonical sequential protocol.*

This result is consistent with the findings of this section. The protocol generates a unique outcome and this outcome is not greater than π^**. It demonstrates that it may be possible to generate an outcome that is not comparable to* π^**. Hence the manager prefers the protocol of asking Expert 1, then 2, then 3 to the unconstrained canonical sequential protocol (that leads to the project* π^**) if he prefers* $(1,2)$ *to* $(3,1)$*.*

Example 10. *There are three projects,* $C \succ B \succ A$*, with* A *the status quo. There are two experts whose preferences are*

$$u_1(B) > u_1(C) > u_1(A) \text{ and } u_2(A) > u_2(B) > u_2(C).$$

This is a generic, one-dimensional game in which $\tilde{\pi}^* = \pi^* = \pi^*_{\succ} = B$*. Consider the protocol in which the manager consults Expert 1 first and offers her the option of supporting A or C. After consulting Expert 1, the protocol specifies that the manager consult Expert 2, who can support any project. The protocol stops after this consultation. This protocol is constrained, but not neutral. It generates project C, because Expert 1 is willing to support C to avoid the implementation of the status quo. CP, UCP, and the simultaneous-move game generate B.*

Example 10 demonstrates that if the manager can design the protocol with knowledge of the experts' preferences, then he may be able to generate outcomes he prefers to those provided by CP. The protocol has the property that Expert 1 is never given the opportunity to support π^*. This restriction is not necessary. For example, the protocol could specify that the manager consult Expert 1 a second time if some expert supported either B or C earlier. What is essential is that Expert 1 cannot support project B when Expert 2 supports A. Hence the protocol requires that the manager can commit not to ask Expert 1 about project B before "settling" for project A. Neutrality rules out this kind of commitment power. The construction of the protocol requires that the manager knows the preferences of the experts. The protocol would perform poorly if

$$u_1(B) > u_1(A) > u_1(C) \text{ and } u_2(A) > u_2(B) > u_2(C),$$

because in this case it generates outcome A while CP and UCP generate B.

In Section 2.7 we discuss optimal protocols when the manager has commitment power.

2.5.6 Comparative Statics

In this section we make a few observations about the value of adding experts.

Adding an expert cannot harm the manager in the sense that if π is a project for the original set of experts that survives IDWDS in the simultaneous-move game, a project at least as good as π for the manager will survive if additional players are added; the new player need not be consulted in a sequential protocol. Li and Norman (2018a) show that adding an expert may hurt the decision maker if the expert must be inserted in a particular place.

In situations in which our bounds are tight ($\bar{\pi}^* = \tilde{\pi}^* = \pi^*$) adding an additional expert can only be beneficial if doing so increases one of these quantities. An expert that does not increase one of these quantities is **redundant**. It is clear that if preferences are single-peaked, all experts except the one with the greatest peak is redundant. More generally, if there are a pair of experts i and j such that for all $x' \succ x$, $u_i(x') > u_i(x)$ whenever $u_j(x') \geq u_j(x)$, then Expert j is redundant.

2.6 Comparing Outcomes in Simultaneous and Sequential Games

This section reviews the connection between the simultaneous and sequential procedures.

When projects are one dimensional and the experts' preferences are one-to-one, our results are clear. CP and UCP provide the best outcome among all sequential protocols and this outcome is identical to the outcome generated by the simultaneous-move game. When projects are completely ordered and experts have non-generic preferences, the results are less clearcut.[21] We have seen (Example 3) that when $\pi^* > \tilde{\pi}^*$, it is possible that the outcome π^* may fail to survive IDWDS in the simultaneous-move game. The same phenomenon is possible in sequential protocols.[22]

When projects are multi dimensional, giving the manager the ability to commit to not implementing hybrid projects eliminates the advantage that CP has over simultaneous consultation.

Finally, we note that while the simultaneous-move game may have an outcome that the manager prefers to the outcome of the canonical sequential procedure, the sequential procedure always has an equilibrium outcome that is at least as good for the manager as some equilibrium of the simultaneous-move game.

When X is multi dimensional, CP may or may not be superior to the simultaneous protocol. One can trace the advantage of CP to the power that the manager has to limit the projects that an expert can approve. Indeed, it is straightforward to show that the bounds $\bar{\pi}^*$ and $\tilde{\pi}^*$ established in Proposition 7 can be replaced by $\pi^*_{\succ}$ and $\tilde{\pi}^*_{\succ}$ if the strategy profile $\mathbf{x}$ induces the manager's most preferred project from $\{x_1, \ldots, x_I\}$ instead of the (potentially preferred) project $x_1 \vee \cdots \vee x_I$. In this way, the advantage of CP depends on the ability to rule out hybrid projects that the manager can implement using the support of more than one expert. If the manager could

[21] We have not constructed examples in which the simultaneous-move game generates an outcome better than the outcomes of a canonical sequential protocol or vice versa.

[22] We can show that if there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is an order of deleting weakly dominated strategies such that project π^* survives in the simultaneous-move game, CP, and UCP.

commit not to implement hybrid projects in sequential consultation, then UCP would generate the same outcomes as CP for generic payoffs.[23]

We introduced the unconstrained canonical protocol to analyze sequential games in which the manager lacked the power to restrict the experts' approvals. When projects are multi dimensional, the simultaneous-move game may generate outcomes superior to CP. Example 6 illustrated this possibility. We argued that the manager's most preferred project survives IDWDS in the simultaneous-move game. Neither CP nor UCP can generate this outcome because no expert wants to be the first to support a project other than the status quo ($(0,0)$).

One reason the results are incomplete is that the characterization of the set of possible outcomes in simultaneous-move games is looser. The project $\bar{\pi}^*$ may exceed π^* and, although $\bar{\pi}^*$ is an upper bound, it may not be attained. We can characterize the outcome of CP more precisely, but cannot say much about when CP performs better than the simultaneous-move game.

Permitting multi-dimensional projects adds the possibility that the manager can exploit complementarities to combine different projects supported by different experts into a third project that he likes better than either of the two individual projects. This suggests that the manager's preferred equilibrium in the simultaneous-move game is the result of a coordination problem and that other equilibria exist. Proposition 13 confirms this intuition.

Proposition 13. *The simultaneous game always generates an outcome less than or equal to* π^*.

Proposition 13 follows from general properties of the simultaneous-move game. It is straightforward to check that the simultaneous-move game is (WID) supermodular as defined in Sobel (2019). Sobel (2019, Theorem 6) demonstrates that in these games there exists a pure-strategy Nash equilibrium that is a lower bound to the set of strategies that survive IDWDS. Claim 3, which we state and prove in the Appendix, implies that a strategy profile that generates an outcome less than or equal to π^* always survives IDWDS. Consequently the lower bound to the set of strategies that survive IDWDS must generate an outcome no greater than π^*.

[23] We are not sure if this claim holds for non-generic payoffs.

Proposition 13 identifies a way in which CP is superior to the simultaneous game. Proposition 9 guarantees that outcomes generated by CP are at least $\tilde{\pi}^*_{\succ}$. Because $\pi^*_{\succ} \geq \pi^*$ and $\pi^*_{\succ} = \tilde{\pi}^*_{\succ}$ for generic preferences for the experts, Proposition 13 implies that CP always generates a project at least as attractive to the manager as some outcome that satisfies IDWDS in the simultaneous-move game.

2.7 Commitment

Suppose that experts have single-peaked preferences. We have argued that in both simultaneous and sequential games, the equilibrium outcome permits the manager to implement project x^*, the maximum of the experts' peaks. What if the manager knows that at least one expert prefers the manager's favorite project $\bar{x}$ to the status quo? If the manager is able to limit the responses of the experts, then he can achieve his most preferred outcome. In a simultaneous-move game, he does so by limiting experts to either supporting $\bar{x}$ or not. In a sequential game, he does so by asking experts, in sequence, whether they support $\bar{x}$ and promising to maintain the status quo if no expert approves $\bar{x}$. That is, the protocol implements the status quo if no expert supports $\bar{x}$. This protocol requires that the manager have the ability to make a commitment not to ask for approval of intermediate projects if no one supports $\bar{x}$. The logic behind these assertions is familiar: If experts must decide between (only) $\bar{x}$ and the status quo, an expert who prefers $\bar{x}$ will support it.

We emphasize that the ability to obtain outcomes greater than x^* depends on commitment power. In the simultaneous-move game, the manager must be able to prevent experts from supporting intermediate projects or resist the temptation to implement an intermediate project if it is the best thing supported by the experts. In the sequential-move game, the manager must be able to commit to implementing the status quo (rather than a project in $(\underline{x}, \bar{x})$) in the event that no expert supports $\bar{x}$.

Hence the characterization of outcomes derived in Proposition 7 does not apply if the manager knows something about the preferences of experts and can make commitments. This observation does not rely on the assumption that preferences are single peaked.

We assume in this section that X is completely ordered and u_i is one-to-one for each i.

Let $X_0 = \{\underline{x}\}$ and define X_k inductively by

$$X_k = \{x \in X : \text{ there exists } i \text{ and } x_m \in X_m \text{ for } m < k \text{ such that } u_i(x) > u_i(x_m) \text{ and } x > x_m\}.$$

Projects in X_1 are those that some expert prefers to the status quo. A project π is in X_n if there is a collection of $k+1 \leq n$ projects $\underline{x} = \pi_0 < \pi_1 < \cdots < \pi_k = \pi$ such that some expert prefers π_j to π_{j-1} for $j = 1, \ldots, k$. To construct this **chain of preference**, we argue by induction. If $\pi \in X_1$, then we take $\pi_1 = \pi$. In general, assume that it is possible to construct a chain of preference for all $\pi \in X_m$ for $m < n$. We will show that it is also possible for n. Begin with $\pi \in X_n$, use the definition of X_n to find $m < n$ and $\pi_m \in X_m$ with the properties that $\pi_m < \pi$ and there is an i such that $u_i(\pi) > u_i(\pi_m)$ and then apply the induction hypothesis to find a chain of preference from $\underline{x} = \pi_0$ to π_m.

By definition, $X_{n-1} \subset X_n$. Because X is finite, there exists $n^\dagger$ such that $X_n = X_{n^\dagger}$ for all $n > n^\dagger$. Let $X^\dagger = X_{n^\dagger}$. Let $\pi^\dagger$ be the manager's preferred outcome in $X^\dagger$. We assert that the manager with knowledge of the preferences of the experts and commitment ability can implement $\pi^\dagger$ in either a simultaneous or sequential game. The reason is simple. Project $\pi^\dagger$ is equal to the equilibrium outcome if the manager restricts the projects only to those in a chain of preference.

Call a subset, $X^c \subset X$ **admissible** if $\underline{x} \in X^c$. X^c determines a simultaneous-move game in which experts simultaneously select elements of X^c; the maximum strategy determines the outcome as in the standard simultaneous-move game; and the experts' utility functions are restrictions of $\tilde{u}_i$ to $X^c \times \cdots \times X^c$. Call such a game the simultaneous-move game restricted to X^c.

Assume that the manager can require the experts to support only projects in X^c, how well can he do? If X is completely ordered and u_i is one-to-one for each i, then for each admissible X^c, the simultaneous-move game restricted to X^c has a unique outcome that survives IDWDS. Call this outcome $\tilde{\pi}(X^c)$. So the best that the manager can do by restricting outcomes is $\max\{\tilde{\pi}(X^c) : X^c \text{ is admissible}\}$. The next result states that this value is equal to $\pi^\dagger$. That is, if the manager has commitment power, then the best project he can generate is his most preferred project in $X^\dagger$.

Proposition 14. *Suppose X is completely ordered and u_i is one-to-one for each i. The best outcome that the manager can achieve by restricting strategies in a simultaneous-move game is $\pi^\dagger$. That is,*

$$\pi^\dagger = \max\{\tilde{\pi}(X^c) : X^c \textit{ is admissible}\}.$$

The proposition does not identify the restriction X^c that leads to $\pi^\dagger$. It is tempting to conjecture that the manager can restrict to strategies in $X^\dagger$, but this need not be the case. For example, if for each project $x \neq \underline{x}$ there exists an expert who prefers x to $\underline{x}$, then $X^\dagger = X$. The manager can attain his most preferred outcome $\bar{x}$ by setting $X^c = \{\underline{x}, \bar{x}\}$, but typically he would not attain $\bar{x}$ without some restrictions on X.[24] The fact that $\pi^\dagger \in X^\dagger$ guarantees that there is a minimal collection of projects, starting at $\pi_0 = \underline{x}$ and going to $\pi_K = \pi^\dagger$ such that some expert prefers π_k to π_{k-1} for each k. Restricting to this collection guarantees the outcome $\pi^\dagger$. The harder part of the proof of Proposition 14 is to show that no other restriction can do better for the manager than $\pi^\dagger$. We prove this by showing that any outcome that survives IDWDS must be in $X^\dagger$.

There is a parallel result for sequential protocols. A sequential protocol with the restricted strategy space $X^c \subset X$ is a sequential protocol in which the manager only requests experts to approve projects in X^c. The protocol operates by asking experts, in order, whether they approve the manager's most-preferred task in X^c. If they all refuse, then he asks them to approve his next favorite project. The protocol continues until either some expert approves or all projects are rejected and the status quo results.

[24] To be concrete, assume that experts have common preferences that rank $\underline{x}$ lowest, but otherwise are monotonically decreasing. That is, for $x' > x'' \neq \underline{x}$, $u_i(x'') > u_i(x') > u_i(\underline{x})$ for all i.

Proposition 15. *Suppose X is completely ordered and u_i is one-to-one for each i. There exists a sequential game in which $\pi^\dagger$ is the unique outcome that survives IDWDS. Furthermore, there is no sequential protocol with restricted strategy spaces that achieves a higher outcome.*

Proposition 15 is a consequence of two simple observations. First, because sequential protocols always give experts the freedom to endorse no project and no expert would support a project not in $X^\dagger$ if it could induce a project in $X^\dagger$, the outcome of the protocol must be an element of $X^\dagger$. Second, if one restricts to a chain of preference ending in $\pi^\dagger$, then CP generates $\pi^\dagger$.

When X is completely ordered but u_i are not necessarily one-to-one, our characterization results are not as sharp, but the basic message of the earlier results remains true. We have bounds for the outcomes that can survive IDWDS. Commitment will be beneficial to the manager if commitment leads to a game with smallest equilibrium outcome greater than $\tilde{\pi}^*$. Any restriction that helps the manager in the simultaneous-move game will also lead to a beneficial restriction for sequential consultation. Of course, the exact equivalence between simultaneous and sequential institutions breaks down when ties are possible.

When X is not completely ordered, we can treat projects as completely ordered (by the preferences of the manager). Versions of Propositions 14 and 15 apply in this setting. (If it is not feasible to combine projects – or if the manager can commit to selecting his most preferred single project of those approved rather than forming hybrid projects, then we preserve the equivalence between simultaneous and sequential procedures.).

Finally we note that in addition to commitment power, the manager needs to know the set of preferences of the experts in order to figure out his preferred restriction on reports. The analysis of the simultaneous-move game and the sequential protocol do not require this knowledge. (Preferences must be common knowledge between the players of the game. That is, the experts must know the preferences of other experts.) The constructions in this section do not require that the manager knows which expert has which preference order. If the manager had this information, it is conceivable that he could further improve the outcome by restricting different experts to different subsets of projects.[25]

2.8 Alternative Interpretations of the Model

We have described the model in which a manager requires the assistance of an expert to carry out a task. The formal model is open to other interpretations. We describe some of them in this section.

2.8.1 Location Choice

In September 2017, Amazon announced that it was planning to build new headquarters for the company to supplement the main operations center in Seattle. The company claimed that the project would bring investment and employment to the host region and requested bids from different locations. Two hundred thirty eight locations across North America responded to the announcement. Although Amazon originally claimed to be looking to find a single location for its new headquarters, in November 2018 it announced that it would make substantial investments in two areas, Northern Virginia and New York City (Stevens et al., 2018).[26]

Location choices of this kind are common. Firms decide where to place new operations. Movie producers consider several locations for filming. Sports teams periodically move to new cities. Organizations must decide where to hold conferences. These bargaining problems share features of our formal model. The firm looking for a location plays the role of the manager in our model. The experts represent the locations themselves. The firm seeks to leverage competition between locations to obtain a better deal (tax subsidies or reduced regulation). It is intuitive that expanding the number of potential locations is beneficial to the firm. It is not clear how the firm should structure the bidding process. The Amazon example suggests the relevance of

[25] We do not have an example in which the ability to restrict different experts to different strategies is strictly beneficial, but our arguments do not rule out this possibility.

[26] In February 2019, probably in response to local opposition, Amazon announced that it would not move forward with plans to build new headquarters in New York City (see Goodman (2019)).

a multi-dimensional formulation. Amazon attempted to use competition between locations to combine elements of offers from two different locations.

Location i can only agree to proposals that involve placing the new operation in Location i. For example, Virginia cannot make an agreement that involves placing Amazon's Headquarters in San Francisco. Although our formal model assumes that all locations can agree to any proposal and that it takes only one agreement to carry out a proposal, our model applies with a simple modification. We assume that all locations can propose anything, but that agreements that involve placing the operation at any other location are dominated. This modification (combined with the use of an equilibrium refinement that restricts the use of dominated strategies) permits our general formulation to apply to location-choice problems.

Quite similar to the location-choice problem is a contracting problem in which a decision maker seeks to find someone to perform a job. As in the location-choice problem, in order to describe the situation accurately within our framework we must assume that one contractor will not commit other contractors to perform the job. It is also sensible to assume that the decision maker can divide the job over several contractors, suggesting that a multi-dimensional model is relevant.

In practice, decision makers structure the negotiation process in a way that combines features of simultaneous and sequential games. The decision maker may ask for locations to submit bids simultaneously, but after receiving initial bids, the decision maker may refine agreements. Our model suggests that the second round of negotiations need not benefit the decision maker. We can think of at least two reasons not included in our model why it may be beneficial to go beyond simultaneous bargaining. First, there could be incomplete information. Information revealed in an initial stage may make coordination easier. Second, specifying a complete contract may be costly. There may be efficiency gains to institutions that identify a small set of locations. Once these locations have been identified, they can specify the details of offers more completely.

2.8.2 Project Choice

Imagine that the manager is a government official who is responsible for selecting and implementing a policy.[27] To perform this function, the official must hire an expert to carry out the chosen policy (only an expert has the expertise to do this). Hence the manager must have the support of at least one expert to carry out a specific project. The manager and experts may have different preferences over policies. The simultaneous-move game is one in which experts simultaneously describe projects that they are willing to implement. We interpret sequential protocols are those in which the manager offers employment contracts (that specify a policy choice) with the understanding that experts can decline contracts. Our results demonstrate (at least in the leading case of completely ordered projects) that the manager is indifferent between offering contracts in sequence and requesting a group of experts to bid on what projects they would be willing to implement. Our results on incompletely ordered projects identify the potential value of hiring a team of experts to carry out a project. In our model, having access to multiple experts may benefit the manager. The benefit comes from complementarities in the preferences of experts rather than in their skills (because all experts are able to implement any project).

2.8.3 Asking for Permission

Inés Moreno de Barreda suggests that one can interpret an expert's strategy as permission to undertake certain activities. (If Expert i supports x_i, then the manager – who we think of as a decision maker in this application – can pursue any activity less than or equal to x_i.) Imagine the decision maker is a teenager and the experts are parents. The teenager requires a parent to give permission for an activity (the permission could be in the form of signing a waiver that allows the teenager to go on a school trip or permission to use a family car or to stay out late). Alternatively, a manager may need to secure necessary inputs from one of many divisions. The different divisions may be semi-autonomous and have different preferences. In these settings it

[27] We thank Joel Watson for this interpretation.

is natural to assume that direct transfers are not feasible.

Our analysis identifies an equilibrium in which the decision maker receives permission to do anything he wishes, but also points out that this prediction is often implausible and identifies the equilibrium preferred by the experts as a more robust prediction.

Under the interpretation of asking for permission, it makes sense to study outcomes when the decision maker requires approval of more than one of the experts. A full analysis of this variation is beyond the scope of this paper, but our analysis extends naturally to a setting in which the decision maker requires unanimous approval rather than the approval of only one expert. To treat this case, we assume that the strategy profile $\mathbf{x}$ generates $x_1 \wedge \cdots \wedge x_I$. That is, the minimum of the strategies becomes the action available to the decision maker. When experts move simultaneously, the game again has multiple equilibria; these equilibria are Pareto ranked from the perspective of the experts (higher equilibrium outcomes are preferred to lower ones); dominance arguments select the expert-preferred outcome. In contrast to our benchmark model, the decision maker shares the experts' preference over equilibria, which provides a further argument for the selection argument that we make.

2.8.4 Bayesian Persuasion

We can interpret the model as a description of persuasion with many Senders. Assume that there is an underlying state of the world and experts provide the decision maker with "experiments" – procedures that produce for each state of the world a probability distribution over a set of signals observable by the decision maker. The decision maker then makes a decision based on the signals he observes (and knowledge of the experiments and the prior distribution on the state of the world). This interpretation is consistent with the model of competition in persuasion in Gentzkow and Kamenica (2016).

Let us describe the connection in somewhat more detail. We restrict attention to finite environments. In any Bayesian Persuasion problem, there is a given state space, Θ. We create a new state space $\Theta^* \equiv \Theta \times T$ where $(\theta, t) \in \Theta^*$, t is uniformly distributed on a finite set T, independent of θ. A partition of Θ^* is an experiment (in the sense that observing an element of the partition generates a posterior distribution on Θ). Provided that we allow only finitely many experiments, the Bayesian Persuasion model translates into our framework: We must only interpret strategy sets as partitions. The strategy set has a lattice structure: If x and x' are two partitions, then $x \vee x'$ is the common refinement ($\{Q = P \cap P'$ for $P \in x, P' \in x'\}$) and $x \wedge x'$ is the finest coarsening (a partition z such that for all $P \in x$ ($P' \in x'$) there exists $Q \in z$ ($Q' \in z$) such that $P \subset Q$ ($P' \subset Q'$) and there is no finer partition with this property). In this way, our model captures any finite Bayesian Persuasion problem. In making the transformation, we emphasize that some of our results rely on quasi supermodularity. It is straightforward to give conditions that guarantee quasi supermodularity in simple Bayesian Persuasion problems, but in general the condition is restrictive. Furthermore, the upper bound that we provide in Proposition 7 depends on the assumption that X is a cartesian product. The lattice structure derived from identifying X with (finite) partitions and $\vee$ with refinement need not have this structure. Consequently our full characterization does not apply.[28]

Gentzkow and Kamenica (2017) and Gentzkow and Kamenica (2016) study a model in which experts simultaneously choose how much to communicate to a decision maker in a Bayesian Persuasion framework. In these models, the decision maker wants to know the value of the state of the world and the strategies of experts are arbitrary signals (joint probability distributions on the state and message received by the decision maker). Gentzkow and Kamenica (2016) shows that adding an agent may decrease the amount of information revelation, but provides a condition under which increasing the number of experts increases the amount of information revealed. In our environment, additional experts are always valuable because the minimal equilibrium disclosure is increasing in the number of experts. Gentzkow and Kamenica do not focus on equilibrium selection, but they note the existence of multiple equilibria and the tendency of experts to prefer less disclosure. Li and Norman (2018b) studies a sequential version of the Gentzkow and Kamenica model. They provide an existence and partial characterization

[28]The technical problem is that the lattice induced by partitions may fail to be distributive ($x \vee (y \wedge z)$ need not equal $(x \vee y) \wedge (x \vee z)$).

result. They show that sequential persuasion results in no more informative equilibria than simultaneous persuasion. Li and Norman (2018a) also note that the order of disclosure matters, pointing out it is possible that adding an expert into a sequence may decrease the amount of information disclosure. In addition to the different interpretation of the nature of the disclosure game, our analysis contributes an equilibrium refinement of the simultaneous game and studies the optimal order of consultation in the sequential game. Alp et al. (2021) investigates a variation to the Gentzkow and Kamenica (2017) and Gentzkow and Kamenica (2016) environment. In this paper, two experts simultaneously select experiments. Each expert then observes the outcome of (only) her own experiment and sends a cheap-talk message to a decision maker. The decision maker, knowing the experiments and the messages (but not the realized outcomes of the experiment), then selects an action. The paper identifies situations (including the case in which experts have identical preferences) in which the set of equilibria to this game always includes the equilibrium of the game that Gentzkow and Kamenica study in which experts must reveal the outcome of their experiments truthfully. The analysis shares the feature of identifying environments in which the availability of multiple experts does not guarantee attractive outcomes for the decision maker.

2.8.5 Unawareness

Decision makers often take actions about things that they do not understand. When someone experiences pain, he may consult a doctor, who describes treatment options. When a car breaks down, the driver may ask a mechanic for advice. When a firm considers a product innovation, it may consult division managers before deciding a marketing strategy. In these settings the behavior of the expert creates options for the decision maker. In the case of a consultation with a doctor or a mechanic, the "project" may be a treatment option. In the case of an executive, the project may be marketing strategies or demonstrations of the valuable qualities of the new good. The actions of the expert may equip a department chair with arguments to persuade administrators to hire the job candidate. As in the basic interpretation of the model, experts are individuals capable of permitting a decision maker to do something that would not be feasible without consultation. Once a doctor describes a surgical procedure, a mechanic suggests a repair, or a marketing advisor describes an ad campaign, the approach becomes available to the decision maker. Without advice, the decision maker would not be able to implement the option. Conflict of interest between decision maker and experts creates the possibility that the expert will not help the decision maker pursue a particular project. Because of the possibility of incomplete disclosure, the decision maker might gain from consulting more than one expert.

This interpretation shares with our model the idea that the decision maker can take advantage of an option only if provided by an expert. On one hand, our sequential protocol is hard to interpret in this context because it is not clear what it means for a decision maker to request a project if he is not aware of it. On the other, it is easy to imagine the decision maker approaching an expert with a vague description that requires expertise to be implemented fully.

In a different context, Auster and Pavoni (2021) model unawareness of strategic options in a way that is similar to this interpretation of the model.

2.8.6 Disclosure

Krishna and Morgan (2001a) and Krishna and Morgan (2001c) study competition in disclosure in a cheap-talk setting. When experts report simultaneously, they construct a fully revealing equilibrium, but they show that full disclosure need not be an equilibrium when experts move sequentially. As in our model, the simultaneous-move cheap-talk game has multiple equilibria. In contrast to our model, weak dominance arguments are not sufficient to eliminate full-disclosure outcomes in interesting cases. Nevertheless, our results suggest that one cannot rely to simultaneous cheap talk to reveal strictly more information than sequential consultation.

Milgrom and Roberts (1986) study disclosure in a model of verifiable information. In this setting, full disclosure is an equilibrium with only a single expert in leading cases (see also Grossman (1981a) and Milgrom (1981a)). The logic behind the full disclosure result in games with verifiable information is different from the reasons for full disclosure in cheap-talk games (or for the existence of an equilibrium that supports the manager's favorite outcome in our model). In verifiable information model, the uninformed player can draw inferences from a player's disclosure decision. In leading special cases, these inferences imply that the decision maker will have skeptical beliefs, which in turn leads to full disclosure in equilibrium. In contrast, in both cheap-talk models and in our framework, in a simultaneous-move game, the decision maker is not strategic. The choices of experts create options for the decision maker, but the decision maker cannot use these choices to make inferences about the state of the world.[29]

Another way to see the difference is that the decision maker's favorite outcome is the unique equilibrium outcome in the verifiable information setting, but typically one of many equilibrium outcomes in our model.

Heifetz et al. (2020) shares features of hard-evidence models and our model. Their paper applies the concept of prudent rationalizability to disclosure games with hard evidence. They provide an example of a two-dimensional disclosure game in which prudent rationalizability is sufficient to guarantee the standard unraveling result if players are fully aware of the states, but is not when they are unaware of one of the dimensions. The force behind the predictions in Heifetz et al. (2020) has more in common with the Grossman (1981a) and Milgrom (1981a) than with our model. Like the literature on disclosure, Heifetz et al. (2020) exploits the ability of the decision maker to draw inferences from disclosures. Unawareness interferes with the ability to draw inferences.

We can interpret our model in which a finite set of experts have access to an identical, partially ordered set of "facts." They play a game in which facts are disclosed. Their payoffs depend on the maximum (component wise when the set of facts is multi dimensional) disclosure. The decision maker makes a decision that depends on the maximum disclosure. The experts have preferences over the decision. In our setting, the decision maker cannot take an action without explicit support from an expert. Viewed in terms of unawareness, the manager in our model cannot take advantage of a possibility unless at least one expert mentions it.

[29] Even if we interpret our model as a description of Bayesian Persuasion, the decision maker learns about the state of the world by observing outcomes of experiments, not by drawing inferences from the experiments selected.

2.9 Related Literature

We know of several papers that compare simultaneous to sequential interactions in different contexts. Dekel and Piccione (2000) compare simultaneous to sequential voting institutions. There are a finite number of voters and two options. Voters can either vote for or against the status quo. Voters do not know their valuations, but receive private signals. Dekel and Piccione compare the equilibria of games in which voters cast votes simultaneously to those in which votes are sequential. They show that a symmetric informative equilibrium of the simultaneous game is an equilibrium to any sequential game. Weaker results hold for asymmetric equilibria.[30] Although this paper reaches a conclusion that is similar to ours, we do not see a formal connection between the analyses. Dekel and Piccione's model focuses on the possibility of learning something about the state from the behavior of other voters. Our experts lack private information. Our equivalence result requires an equilibrium refinement and commitment power in the design of sequential mechanisms. Schummer and Velez (2021) identify conditions under which social choice functions that can be implemented in truthful strategies when players move simultaneously cannot be truthfully implemented when players move sequentially. The context is quite different from our paper, but it suggests environments in which sequential procedures will perform less well than simultaneous ones.

Doval and Ely (2020) characterize all equilibria that can arise from some information structure and some extensive form (for a fixed set of players and preferences over final outcomes). In their construction, they introduce a "canonical extensive form" that is sufficient to generate any equilibrium. In a canonical extensive form, each player moves at most once. Our construction requires that an individual player may move more than once. The reason for this difference is that Doval and Ely's construction requires a partial commitment assumption that requires that once a player has made an action choice, that player can have no other payoff relevant moves. This assumption does not hold in our model.

[30]Dekel and Piccione (2014) study a voting model in which the timing of votes is a strategic choice.

Glazer and Rubinstein (1996) show that given a normal game form (strategies and players) one can construct an extensive game form that is equivalent to the normal game form and for any specification of preferences, the normal-form game is dominance solvable if and only if the extensive-form game is solvable by backward induction. Glazer and Rubinstein argue that the transformation makes it easier to carry out the process of removing dominated strategies, suggesting that the extensive-form game is easier to play. Our construction associates with a dominance-solvable normal-form game an extensive-form game using a communication protocol, but it is typically not the case that the extensive-form game is equivalent to the normal-form game.

Our work shares a basic motivation with the enormous literature on principal-agent problems, but with a considerably different focus. Inefficiency arises in the principal-agent model because the principal cannot observe the agent's action, but the ability to make monetary transfers gives the principal a strong tool to motivate the agent. In our model, there is complete information and limited transfers. The manager may fail to obtain his favorite outcome because his preferences differ from the experts' preferences. He must use competition between experts rather than monetary transfers to motivate the experts to assist him.

2.10 Discussion

One of our goals in this study was to identify features that favored simultaneous versus sequential consultation procedures. Our results suggest that the choice of organization does not matter. Nevertheless, even in our setting, there are differences between the procedures.

A feature of the consultation procedures that we describe is that the manager need not know anything about the preferences of the experts in order to operate them. That is, they are robust institutions because they need not be tailored to individual preferences. Of course, our predictions require that experts know the preferences of other agents (and that they assume the other agents avoid dominated strategies). Section 2.7 demonstrates how the manager might be able to take advantage of additional information about experts' preferences if he can limit the set of projects.

We have identified a sequential protocol that performs at least as well as simultaneous consultation. Sequential procedures may be superior to the simultaneous-move game in another way if the manager has information about preferences. The simultaneous-move game requires that all experts (or, at least, a subset of experts needed to make π^* the minimum equilibrium outcome) participate actively. If the manager knows $\pi^*_{\succ}$, however, the sequential protocol that initially asks for support of $\pi^*_{\succ}$ (and, then follows the canonical sequential protocol) generates $\pi^*_{\succ}$ with only one consultation. The ability to consult more experts encourages the first expert to support $\pi^*_{\succ}$. The canonical protocol begins by asking experts to support $\bar{x}$, so unless $\bar{x} = \pi^*_{\succ}$ it must consult every expert at least once. Even if the manager does not know $\pi^*_{\succ}$, there is an equilibrium of UCP in which the first expert consulted supports the project generated by the protocol, but unless this project is $\bar{x}$, the manager will not know enough to stop consultations at this point.

There appear to be natural settings in which the manager prefers simultaneous procedures and other settings in which sequential procedures are the norm. On one hand, editors in economics typically consult several reviewers simultaneously to obtain reports on a submission.[31] Committee deliberations, anonymous voting, and obtaining multiple bids for a construction contract have features of simultaneous procedures. On the other hand, it is frequent to consult medical experts in sequence. The Amazon negotiations described in Section 2.8 mix features of simultaneous and sequential consultation. A full understanding of the relative merits of simultaneous and sequential procedures requires a richer model. Two directions seem promising. Certainly adding costs (whether direct payment or waiting times) to consultation will change the analysis, presumably in the direction of favoring sequential procedures. In our model experts cannot learn from each other. In a variation of the model in which experts have different strategy sets or where the action of one expert influences the set of actions available to subsequent agents,

[31]This institution varies across disciplines and even across editors in economics.

the manager may favor sequential over simultaneous procedures.

We have studied two ways to organize consultations. It is natural to ask whether there is an organization that is superior to either the simultaneous or sequential procedures that we examined. We lack a complete answer to this question, which differs from standard implementation questions for several reasons. First, we must study something more general than implementation in Nash equilibrium (because otherwise it is straightforward to obtain the manager's favorite outcome). Second, in the standard implementation problem, the decision maker's set of actions is fixed. We assume that what the manager can do depends on the behavior of the experts (the manager cannot take an action unless an expert supports it). Finally, standard mechanism design gives the designer commitment power, but imposes individual rationality constraints. To study commitment in our model, we wish to prevent the designer from ruling out the status quo if no expert supports an alternative, but this condition is a restriction on the experts' collective response. It does not translate into a lower-bound on utility independent of the behavior of other experts.

2.11 Appendix B

Lemma 7. *If π' and π'' are equilibrium outcomes, then $\pi' \wedge \pi''$ is an equilibrium outcome.*

Proof. If π' and π'' are equilibrium outcomes, then $u_i(\pi'') \geq u_i((\pi' \wedge \hat{\pi}) \vee \pi'')$ and $u_i(\pi') \geq u_i(\pi' \vee \hat{\pi})$ for any $\hat{\pi}$. It follows from quasi supermodularity that

$$u_i((\pi' \wedge \hat{\pi}) \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}) \tag{2.5}$$

and

$$u_i(\pi' \wedge \hat{\pi}) \geq u_i(\hat{\pi}). \tag{2.6}$$

Now assume that $\hat{\pi} \geq \pi' \wedge \pi''$. Consequently, $(\pi' \wedge \hat{\pi}) \wedge \pi'' = \pi' \wedge \pi''$ so that inequality (2.5) implies

$$u_i(\pi' \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}). \tag{2.7}$$

It follows from (2.6) and (2.7) that $u_i(\pi' \wedge \pi'') \geq u_i(\hat{\pi})$ so that $\pi' \wedge \pi''$ is an equilibrium outcome.

□

Claim 1. *Let* π *satisfy* $u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k))$ *for all* $x_i(k) > \pi(k)$*, all dimensions* k*, and all* i*. Then* π *is an equilibrium project such that* $u_i(\pi) > u_i(x_i)$ *for all* $x_i > \pi$ *and all* i*.*

Proof. Fix a project $\pi' > \pi$ and an arbitrary expert i. It follows that $\pi'(k) \geq \pi(k)$ in all dimensions and the inequality is strict on at least one dimension. We will show that $u_i(\pi) > u_i(\pi')$. Without loss of generality, suppose that $\pi'(1) > \pi(1)$, where $\pi(1)$ is first component of π and $\pi(-1)$ are the components of π in dimensions other than 1. We will show by induction on r that

$$u_i(\pi) > u_i(\pi'(1), \ldots, \pi'(r), \pi(r+1), \ldots, \pi(n)) \tag{2.8}$$

for $r = 1, \ldots, n$. The claim then follows from inequality (2.8) for $r = n$.

Because $u_i(\pi(1), \bar{x}(-1)) > u_i(\pi'(1), \bar{x}(-1))$, it must be that $u_i(\pi) > u_i(\pi'(1), \pi(-1))$ by quasi supermodularity. Hence inequality (2.8) holds for $r = 1$. Assume inequality (2.8) for $r \leq k$. It suffices to show that it holds for $r = k+1$.

Because $u_i(\pi(k+1), \bar{x}(-(k+1))) \geq u_i(\pi'(k+1), \bar{x}(-(k+1)))$,

$$u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi'(k+1), \pi(k+2), \ldots, \pi(n)) \tag{2.9}$$

by quasi supermodularity. Inequality (2.9) and quasi supermodularity imply that

$$u_i(\pi'(1), \ldots, \pi'(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\pi'(1), \ldots, \pi'(k+1), \pi(k+2), \ldots, \pi(n)). \tag{2.10}$$

It follows from inequality (2.8) for $r = k$ and inequality (2.10) that inequality (2.8) holds for $r = k+1$, which completes the proof. □

Proposition 7. *If* $\mathbf{x}$ *is a strategy profile that survives IDWDS in the simultaneous-move game, then* $M(\mathbf{x}) \in [\tilde{\pi}^*, \bar{\pi}^*]$*. For any equilibrium strategy profile,* $\tilde{u}_i(\mathbf{x}) \geq u_i(\bar{\pi}^*)$ *for all* i*.*

Denote the set of strategies that survive IDWDS by S. We prove the proposition in a series of steps, which we state and prove as claims.

Claim 2. *For all* $\mathbf{x} \in S$ *and every* i*, there exists* $x_i \in S_i$ *that is a best response to* $\mathbf{x}$ *relative to* X*.*

Proof. The result is clear if every best response to $\mathbf{x}$ relative to X has not yet been deleted. If the best response to $\mathbf{x}$ has been deleted, then it was deleted by a strategy that weakly dominates it. This strategy must be a best reply to $\mathbf{x}$. □

Claim 3. *There exists a strategy profile* $\mathbf{x} \in S$ *such that* $M(\mathbf{x}) \leq \pi^*$*.*

Proof. Suppose that after r iterations, there exists a strategy profile $\mathbf{x}$ satisfying the condition in the claim. In the next iteration, every agent must have a strategy that is a best response to $\mathbf{x}$ by Claim 2. Suppose that the best response for Expert i to $\mathbf{x}$ is some $x_i' \not\leq \pi^*$. We will argue to a contraction. Because $M(x_i', \mathbf{x}_{-i}) = x_i' \vee M(\mathbf{x}_{-i}) = M(x_i' \vee M(\mathbf{x}_{-i}), \mathbf{x}_{-i})$,

$$\tilde{u}_i(x_i', \mathbf{x}_{-i}) = u_i(M(x_i', \mathbf{x}_{-i})) = u_i(M(x_i' \vee M(\mathbf{x}_{-i}), \mathbf{x}_{-i})) = \tilde{u}_i(x_i' \vee M(\mathbf{x}_{-i}), \mathbf{x}_{-i}). \tag{2.11}$$

It follows from (2.11) that $x_i'' \equiv x_i' \vee M(\mathbf{x}_{-i})$ is also a best response to $\mathbf{x}$. Note that because $x_i'' > M(\mathbf{x}_{-i})$,

$$M(x_i'', \mathbf{x}_{-i}) = x_i''. \tag{2.12}$$

In addition, note that

$$M(x_i'' \wedge \pi^*, \mathbf{x}_{-i}) = (x_i'' \wedge \pi^*) \vee M(\mathbf{x}_{-i}) = x_i'' \wedge \pi^*, \tag{2.13}$$

where second equality holds because $M(\mathbf{x}_{-i}) < x_i''$ and $M(\mathbf{x}_{-i}) \leq \pi^*$ imply that $M(\mathbf{x}_{-i}) \leq x_i'' \wedge \pi^*$.

Because $x_i' \not\leq \pi^*$ and $x_i'' \geq x_i'$, $x_i'' \not\leq \pi^*$. Therefore, $x_i'' \vee \pi^* > \pi^*$. By definition of π^*, $u_i(\pi^*) > u_i(x_i)$ for all $x_i > \pi^*$, it follows that

$$u_i(\pi^*) > u_i(x_i'' \vee \pi^*). \tag{2.14}$$

Hence it must be that

$$\tilde{u}_i(x_i'' \wedge \pi^*, \mathbf{x}_{-i}) = u_i(M(x_i'' \wedge \pi^*, \mathbf{x}_{-i})) = u_i(x_i'' \wedge \pi^*) > u_i(x_i'') = u_i(M(x_i'', \mathbf{x}_{-i})) = \tilde{u}_i(x_i'', \mathbf{x}_{-i}) \tag{2.15}$$

where the second equation follows from (2.13), the inequality follows by (2.14) and quasi supermodularity, and the third equation follows from (2.12). Expressions (2.11) and (2.15) combine to show that x_i' is not a best response to $\mathbf{x}_{-i}$, which is a contradiction. We conclude that no strategy $x_i'' \nleq \pi^*$ can do at least as well as $x_i'' \wedge \pi^*$ against $\mathbf{x}$. Therefore, all best responses to $\mathbf{x}$ are less than or equal to π^* and a strategy less than or equal to π^* must remain for each agent. □

Next we show that all projects that survive IDWDS are greater than or equal to the smallest preferred equilibrium outcome.

Claim 4. *If* $\mathbf{x} \in S$*, then* $M(\mathbf{x}) \geq \tilde{\pi}^*$.

Proof. Let $\tilde{\pi} \equiv \wedge_{\mathbf{x} \in S} M(\mathbf{x})$ be the meet of all outcomes that survive IDWDS. We wish to show that $\tilde{\pi} \geq \tilde{\pi}^*$. In order to reach a contradiction, assume that $\tilde{\pi} \ngeq \tilde{\pi}^*$. By the definition of $\tilde{\pi}^*$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } x_i > \tilde{\pi} \text{ such that } u_i(x_i) > u_i(\tilde{\pi}), \tag{2.16}$$

because otherwise $\tilde{\pi}^*$ would not be the smallest equilibrium outcome. Let

$$x_i' = \min\{\arg\max_{x_i \geq \tilde{\pi}} u_i(x_i)\}$$

be the smallest best response of Expert i to $\tilde{\pi}$, which is well defined by Fact 1. It is apparent that $x_i' > \tilde{\pi}$ and so $x_i'(k) > \tilde{\pi}(k)$ for some k. Select $\tilde{\mathbf{x}} \in S$ such that $M(\tilde{\mathbf{x}})(k) = \tilde{\pi}(k)$. This is possible by the definition of $\tilde{\pi}$. We assert that $x_i' \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. The assertion is sufficient to prove the claim because it means that if (2.16) holds, then we can show an element in S is weakly dominated. This contradicts the definition of S.

It remains to show that $x_i' \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. For any $\mathbf{x}_{-i}$ such that $M(\mathbf{x}_{-i}) \geq x_i' \vee \tilde{x}_i$,

$$\tilde{u}_i(x_i' \vee \tilde{x}_i, \mathbf{x}_{-i}) = u_i(M(\mathbf{x}_{-i})) = \tilde{u}_i(\tilde{x}_i, \mathbf{x}_{-i}).$$

For any $\mathbf{x}_{-i}$ such that $M(\mathbf{x}_{-i}) \not\geq x_i' \vee \tilde{x}_i$, i's utility from playing $x_i' \vee \tilde{x}_i$ and $\tilde{x}_i$ are $u_i((x_i' \vee \tilde{x}_i) \vee M(\mathbf{x}_{-i})) = u_i(x_i' \vee (\tilde{x}_i \vee M(\mathbf{x}_{-i})))$ and $u_i(\tilde{x}_i \vee M(\mathbf{x}_{-i}))$, respectively. We now claim that

$$\tilde{u}_i(x_i' \vee \tilde{x}_i, \mathbf{x}_{-i}) = u_i(x_i' \vee (\tilde{x}_i \vee M(\mathbf{x}_{-i}))) \geq u_i(\tilde{x}_i \vee M(\mathbf{x}_{-i})) = \tilde{u}_i(\tilde{x}_i, \mathbf{x}_{-i}). \tag{2.17}$$

Hence $x_i' \vee \tilde{x}_i$ is weakly better than $\tilde{x}_i$. The equations in (2.17) follow from the definition of $\tilde{u}_i$. The inequality would hold by quasi supermodularity if

$$u_i(x_i') \geq u_i(x_i' \wedge (\tilde{x}_i \vee M(\mathbf{x}_{-i}))). \tag{2.18}$$

Inequality (2.18) is satisfied. Because $\tilde{x}_i \vee M(\mathbf{x}_{-i}) \geq \tilde{\pi}$ by the definition of $\tilde{\pi}$ and $x_i' > \tilde{\pi}$, $x_i' \wedge (\tilde{x}_i \vee M(\mathbf{x}_{-i})) \geq \tilde{\pi}$. Inequality (2.18) now follows from the definition of x_i'.

It follows from (2.16) and the definition of x_i' that $x_i' > \tilde{\pi}$. Also $x_i' > x_i' \wedge M(\tilde{\mathbf{x}}) \geq \tilde{\pi}$, and so the definition of x_i' also implies that $u_i(x_i') > u_i(x_i' \wedge M(\tilde{\mathbf{x}}))$. Consequently, $u_i(x_i' \vee M(\tilde{\mathbf{x}})) > u_i(M(\tilde{\mathbf{x}}))$ by quasi supermodularity. It follows that

$$\tilde{u}_i(x_i' \vee \tilde{x}_i, \tilde{\mathbf{x}}_{-i}) = u_i(x_i' \vee M(\tilde{\mathbf{x}})) > u_i(M(\tilde{\mathbf{x}})) = \tilde{u}_i(\tilde{\mathbf{x}}) = \tilde{u}_i(\tilde{x}_i, \tilde{\mathbf{x}}_{-i}). \tag{2.19}$$

Inequality (2.19) guarantees that $x_i' \vee \tilde{x}_i$ is strictly better than $\tilde{x}_i$ when $\mathbf{x}_{-i} = \tilde{\mathbf{x}}_{-i}$. Establishing this completes the proof that $x_i' \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. □

All projects that survive IDWDS are greater than or equal to $\tilde{\pi}^*$. Next, we show that all projects that survive IDWDS are less than or equal to the bounding project.

Claim 5. *If $\mathbf{x} \in S$, then $M(\mathbf{x}) \leq \bar{\pi}^*$.*

Proof. Let S_i^r be the set of strategies remaining for i after r rounds of deleting strategies. For each i let $P_i^r = \{s_i^r \in S_i^r : s_i^r \not\leq \bar{\pi}^*\}$. If there exists r such that $\bigcup_i P_i^r = \emptyset$, then the proof is complete. Otherwise, there is at least one dimension k such that $s_i^r(k) > \bar{\pi}^*(k)$ for some $s_i^r \in P_i^r$ and some i. Let $Q^r(k) = \bigcup_i \{s_i^r \in P_i^r : s_i^r(k) > \bar{\pi}^*(k)\}$. Let $z^r \in \arg\min\{s^r(k) : s^r \in Q^r(k)\}$. If $\bigcup_i P_i^r \neq \emptyset$, then z^r exists and $z^r \in P_j^r$ for some j; we write $z^r = z_j^r$ to indicate that $z_j^r \in P_j^r$. We claim that z_j^r is weakly dominated by $(\bar{\pi}^*(k), z_j^r(-k))$.[32]

For any $\mathbf{x}$ such that $M(\mathbf{x}_{-j}(k)) \geq z_j^r(k)$,

$$\tilde{u}_j((\bar{\pi}^*(k), z_j^r(-k)), \mathbf{x}_{-j}) = u_j(M(\mathbf{x}_{-j}(k)), M(z_j^r(-k), \mathbf{x}_{-j}(-k))) = \tilde{u}_j(z_j^r, \mathbf{x}_{-j}).$$

This follows because $z_j^r(k) > \bar{\pi}^*(k)$.

For any $\mathbf{x}$ such that $M(\mathbf{x}_{-j}(k)) < z_j^r(k)$, $M(\mathbf{x}_{-j}(k)) \leq \bar{\pi}^*(k)$ by the definition of z_j^r. Hence, Expert j's utility from using z_j^r is $u_j(z_j^r(k), M(z_j^r(-k), \mathbf{x}_{-j}(-k)))$, while j's utility from using $(\bar{\pi}^*(k), z_j^r(-k))$ is $u_j(\bar{\pi}^*(k), M(z_j^r(-k), \mathbf{x}_{-j}(-k)))$. We claim that

$$u_j(\bar{\pi}^*(k), M(z_j^r(-k), \mathbf{x}_{-j}(-k))) > u_j(z_j^r(k), M(z_j^r(-k), \mathbf{x}_{-j}(-k))). \tag{2.20}$$

It follows from the definition of $\bar{\pi}^*$ that $u_j(\bar{\pi}^*(k), \bar{\mathbf{x}}(-k)) > u_j(z_j^r(k), \bar{\mathbf{x}}(-k))$, therefore Inequality (2.20) follows from quasi supermodularity. That is, j does strictly better using $(\bar{\pi}^*(k), z_j^r(-k))$ than z_j^r whenever the k^{th} dimension of $M(\mathbf{x}_{-j})$ is less than $z_j^r(k)$. Because there always exists a strategy in which $M(\mathbf{x}_{-j}(k))$ is less than $z_j^r(k)$ by Claim 3 and $\pi^* \leq \bar{\pi}^*$, $(\bar{\pi}^*(k), z_j^r(-k))$ must be strictly better than z_j^r against one strategy profile that survives IDWDS. Consequently, $(\bar{\pi}^*(k), z_j^r(-k))$ weakly dominates z_j^r. Therefore, z_j^r must eventually be deleted. We conclude that there must exist an r^* such that $P_i^{r^*} = \emptyset$ for all i, which establishes the result. □

Proposition 7. *If* $\mathbf{x}$ *is a strategy profile that survives IDWDS in the simultaneous-move game, then* $M(\mathbf{x}) \in [\tilde{\pi}^*, \bar{\pi}^*]$. *For any equilibrium strategy profile,* $\tilde{u}_i(\mathbf{x}) \geq u_i(\bar{\pi}^*)$ *for all i.*

Proof. Claim 4 guarantees that all $\mathbf{x} \in S$ satisfy $M(\mathbf{x}) \geq \tilde{\pi}^*$. Claim 5 guarantees that all $\mathbf{x} \in S$ satisfy $M(\mathbf{x}) \leq \bar{\pi}^*$. This establishes the first part of the Proposition. Given any $\mathbf{x} \in S$, it follows

[32] If $X \subset \mathbb{R}$, $(\bar{\pi}^*(k), z_j^r(-k)) = \bar{\pi}^* = \pi^*$.

from Claim 2 that each player has a surviving strategy that is a best response to $\mathbf{x}_{-i}$ relative to the full strategy set. Because $x_i = \bar{\pi}^*$ leads to payoff $u_i(\bar{\pi}^*)$ for Expert i against any surviving strategy by Claim 5, the second part of the proposition follows. □

Proposition 8. *If π is a project that survives IDWDS in the game determined by UCP, then $\pi \geq \tilde{\pi}^*$.*

Proof. Let π be a project that survives IDWDS. Suppose it is generated by the strategy profile $\hat{\mathbf{s}}$. Suppose that $\pi \ngeq \tilde{\pi}^*$. We will show that there exists an expert i such that $\hat{s}_i$ is weakly dominated.

By the definition of $\tilde{\pi}^*$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } x_i > \pi \text{ such that } u_i(x_i) > u_i(\pi). \tag{2.21}$$

Find a history $\hat{h}$ consistent with $\hat{\mathbf{s}}$ such that $P_I(\hat{h}) = i$ and if i's play at h is $\hat{s}_i(\hat{h})$ then there is no undominated strategy profile that consults i again. It is possible to find such a history because $\pi \ngeq \tilde{\pi}^*$ implies that the manager must consult every expert at least once and because the protocol never consults an expert more than K times.

Consider an alternative strategy of Expert i, in which

$$s_i'(h) = \begin{cases} x_i' & \text{if } h = \hat{h} \\ \hat{s}_i(h) & \text{otherwise} \end{cases},$$

where x_i' solves $\max u_i(x_i)$ subject to $x_i > \pi$. We know that x_i' exists and satisfies and $u_i(x_i') > u_i(\pi)$ by (2.21). By the definition of $\hat{h}$, if Expert i supports x_i' after $\hat{h}$, the protocol must stop. (We know that the protocol would stop in the next round with the outcome π, so it must stop immediately when Expert i supports something strictly better for the manager than π.) It follows that s_i' weakly dominates $\hat{s}_i$. The strategy s_i' does exactly as well as $\hat{s}_i$ for any strategy profile that does not induce the history $\hat{h}$. We know that some strategy profile does induce $\hat{h}$ and, by construction, Expert i does strictly better in any such case. Consequently, any outcome $\pi \ngeq \tilde{\pi}^*$ must be generated by a strategy profile in which one player uses a weakly dominated strategy. This establishes the proposition. $\square$

Proposition 9. *If* π *is a project that survives IDWDS in the game determined by CP, then* $\pi \succeq \tilde{\pi}^*_{\succ}$.

Proof. Let π be a project that survives IDWDS. Suppose it is generated by the strategy profile $\hat{\mathbf{s}}$. Suppose that $\pi \prec \tilde{\pi}^*_{\succ}$. We take π to be a minimal such project (that is, no $\pi' \prec \pi$ survives IDWDS). We will show that there exists an expert i such that $\hat{s}_i$ is weakly dominated.

By the definition of $\tilde{\pi}^*_{\succ}$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } x_i \succ \pi \text{ such that } u_i(x_i) > u_i(\pi). \tag{2.22}$$

Using (2.22), select i and $x_i \in \arg\max_{\{\pi':\pi' \succ \pi\}} u_i(\pi')$ so that $u_i(x_i) > u_i(\pi)$. Find a possible history $\hat{h}$ consistent with $\hat{\mathbf{s}}$ such that $P_I(\hat{h}) = i$ and $x_i \in P_X(\hat{h})$. Because $x_i \succ \pi$, such a history exists and $\hat{s}_i(h)$ does not support x_i.

Consider an alternative strategy of Expert i, in which

$$s'_i(h) = \begin{cases} x_i & \text{if } h = \hat{h} \\ \hat{s}_i(h) & \text{otherwise} \end{cases},$$

By the definition of $\hat{h}$ and CP, if Expert i supports x_i after $\hat{h}$, the protocol must stop. It follows that s'_i weakly dominates $\hat{s}_i$. The strategy s'_i does exactly as well as $\hat{s}_i$ for any strategy profile that does not induce the history $\hat{h}$. We know that some strategy profile does induce $\hat{h}$. We claim that Expert i does at least as well in such a case. To see this, note that by construction, Expert i prefers project x_i to any other project $\pi' \succ \pi$. Furthermore, no project $\pi' \prec \pi$ survives IDWDS by the definition of π. Finally, because $\hat{\mathbf{s}}$ generates π and $u_i(x_i) > u_i(\pi)$, s'_i performs better than $\hat{s}_i$ in once instance and at least as well in all instances.

Consequently, any outcome $\pi \not\succeq \tilde{\pi}^*_{\succ}$ must be generated by a strategy profile in which one player uses a weakly dominated strategy. This establishes the proposition. $\square$

Proposition 10. *If $x > \pi^*$, then there exists no unconstrained sequential protocol that generates the project x in a pure-strategy, subgame-perfect equilibrium. If $x \succ \pi^*_\succ$, then there exists no neutral sequential protocol that generates the project x in a pure-strategy, subgame-perfect equilibrium.*

Proof. Proposition 10 follows from backward induction. If the protocol is unconstrained, then after each history that supports no more than π^*, it is never a best response to disclose more than π^*. So if the first expert anticipates that the final project supported will be more than π^*, then she can do strictly better by approving π^* and no one else will add more to π^*. Consequently there will never be projects greater than π^* in equilibrium.

Assume that the protocol is neutral. If $\pi^* = \underline{x}$, then all experts prefer $\underline{x}$ to any other project and hence $\underline{x}$ will be the outcome of any protocol. Assume that $\pi^* > \underline{x}$. Protocols must always permit an expert to support the status quo when consulted. Consequently, there exists a terminal history $\underline{h}$ such that $\mu(\underline{h}) = \underline{x}$. Neutrality implies that for each i there is a history $h \subset \underline{h}$ such that $\pi^*_\succ \in P_X(h)$ and $i = P_I(h)$. Furthermore, Expert i can guarantee a payoff of at least $u_i(\pi^*_\succ)$ in the subgame that follows h by supporting $\pi^*_\succ$ (after which no other expert would support a larger project). Hence, by the definition of $\pi^*_\succ$, the outcome $\pi(h)$ of this subgame must satisfy $\pi^*_\succ \succeq \pi(h)$. Let h^* be a minimal subhistory of $\underline{h}$ with the property that the outcome $\pi(h^*)$ of the subgame determined by h^* satisfies $\pi^*_\succ \succeq \pi(h^*)$. That is, h^* satisfies $h^* \subset \underline{h}$ and has the property that if $h' \subset h^*$ and the outcome of the subgame determined by h' satisfies $\pi^*_\succ \succeq \pi(h')$, then $h' = h^*$.

Either $h^* = \emptyset$, or $\pi(h^*) = \pi^*_\succ$, or $\pi^*_\succ \succ \pi(h^*)$. We will show that $h^* = \emptyset$. If $\pi(h^*) = \pi^*_\succ$, then $h^* = \emptyset$ because otherwise there is an expert i^* who moves immediately before the subgame and Expert i^* can guarantee a payoff of at least $u_{i^*}(\pi^*_\succ)$ by inducing h^*. Hence her equilibrium action must generate a project that the manager prefers less than $\pi^*_\succ$, which contradicts the minimality of h^*. If $\pi^*_\succ \succ \pi(h^*)$, then it follows from neutrality that either $h^* = \emptyset$ or there is a history $h' \subset h^*$, $h' \neq h^*$ in which $\pi^*_\succ \in P_X(h')$. But the existence of h' would contradict the minimality of h^* because Expert $P_I(h')$ could guarantee the project $\pi^*_\succ$ in the subgame starting from h'. Hence Expert $P_I(h')$ must obtain utility at least $u_{P_I(h')}(\pi^*_{\succ})$ and therefore $\pi^*_{\succ} \succeq \pi(h')$. □

Proposition 11. *For any unconstrained protocol, there is always an outcome that survives IDWDS that is not greater than* π^**.*

Proof. Fix a protocol. We say that a history generates project π if π is the maximum of projects supported in the history. Because anything greater than π^* is strictly worse than π^* for every expert, supporting π^* is the unique best reply at any history that generates π^* provided that the manager will consult no further experts independent of the choice. Consequently, at least one of her strategies that survive IDWDS must involve supporting π^* at histories that generate π^*. By backward induction, there must exist a strategy profile that survives IDWDS such that experts will choose π^* in any history that generates project π^*, because each expert's unique best response to these histories is to support no further projects when the expert expects future experts to support no new projects. It follows that one of the final projects supported will be π^* if the first expert supports π^*. So if the strategy π^* survives IDWDS for the first expert, then the proof is complete. If the first expert's strategy π^* has been deleted, it is weakly dominated by a strategy that leads to a weakly higher payoff for the first expert against all remaining strategies of the other experts. Consequently at least one project that survives IDWDS does at least as well as π^* for the first expert. By definition of π^*, the first expert strictly prefers π^* to anything greater than π^*. Hence there exists a project that survives IDWDS that is no greater than π^*. □

Proposition 12. *Fix a sequential protocol. If the experts' utility functions are one-to-one, there is a unique outcome that survives IDWDS.*

Proof. If an expert is indifferent between two strategies given any strategy profile of other players, the two strategies must lead to the same project because utility functions are one-to-one. So the other experts are indifferent between the two profiles (differing only in that expert's action) as well. Hence the **transference of decisionmaker indifference** (TDI) condition in Marx and Swinkels (1997) holds.

Furthermore, it is clear that there is a unique outcome from backward induction in this game (with no ties). Marx and Swinkels (1997) show that if an extensive-form game with perfect information that satisfies TDI has a unique backward-induction payoff, then any full reduction by weak dominance also contains only that payoff. Therefore there is a unique project that survives IDWDS, which is the unique backward-induction solution. □

Proposition 14. *Suppose X is completely ordered and u_i is one-to-one for each i. The best outcome that the manager can achieve by restricting strategies in a simultaneous-move game is $\pi^\dagger$. That is,*

$$\pi^\dagger = \max\{\tilde{\pi}(X^c) : X^c \textit{ is admissible}\}.$$

Proof. Proposition 7 applies to the simultaneous-move game restricted to X^c. Hence $\tilde{\pi}(X^c)$ is well defined and equal to

$$\min\{\pi : u_i(\pi) > u_i(x_i) \text{ for all } x_i \in X^c \text{ such that } x_i > \pi \text{ and all } i\}$$

(because X is linearly ordered, one can replace $x_i > \pi$ by $x_i \succ \pi$ in this expression).

Because $\pi^\dagger \in X^\dagger$, there exists a finite set of projects $\{\pi_0, \pi_1, \ldots, \pi_K\}$ such that $\underline{x} = \pi_0 < \pi_1 < \cdots < \pi_K = \pi^\dagger$; $\pi_i \in X^\dagger$ for all $i = 0, \ldots, K$ and

$$\text{for each } k = 1, \ldots, K \text{ there exists } i \text{ such that } u_i(\pi_k) > u_i(\pi_{k-1}). \tag{2.23}$$

Let $\tilde{X}$ denote the set $\{\pi_0, \pi_1, \ldots, \pi_K\}$. $\pi^\dagger$ must be the manager's most preferred element in $\tilde{X}$ because $\tilde{X} \subset X^\dagger$ and $\pi^\dagger$ is the manager's most preferred element of $X^\dagger$. By (2.23), $\tilde{\pi}(\tilde{X}) = \pi^\dagger$. It remains to show that for all admissible X^c, the manager prefers $\pi^\dagger$ to $\tilde{\pi}(X^c)$.

Recall that $X_0 = \{\underline{x}\}$ and X_k is defined inductively by

$$X_k = \{x \in X : \text{ there exists } i \text{ and } x_m \in X_m \text{ for } m < k \text{ such that } x > x_m \text{ and } u_i(x) > u_i(x_m)\}$$

and that $\tilde{\pi}(X^c)$ is defined to be the project that survives IDWDS when experts are restricted to projects in X^c. We will show that for any admissible X^c for which the manager prefers $\tilde{\pi}(X^c)$ to $\pi^\dagger$, there exists k such that $\tilde{\pi}(X^c) \in X_k$. Hence $\pi^c \equiv \tilde{\pi}(X^c) \in X^\dagger$ and, by definition of $\pi^\dagger$, the manager prefers $\pi^\dagger$ to π^c. This establishes the proposition.

We prove the claim by constructing a sequence $\{\pi_0^c, \ldots, \pi_k^c\}$ such that $\pi_0^c = \underline{x}$, $\pi_{k'}^c > \pi_{k'-1}^c$ for $k' = 1, \ldots, k$, $\pi_{k'}^c \in X_{k'} \cap X^c$, $\pi_k^c = \pi^c$.

Set $\pi_0^c = \underline{x}$. If $\underline{x} = \pi^c$, then we set $k = 0$ and we are done. Otherwise, assume that we have constructed $\{\pi_0^c, \ldots, \pi_m^c\}$ such that $\pi_0^c = \underline{x}$, $\pi_{m'}^c > \pi_{m'-1}^c$ for $m' = 1, \ldots, m$, $\pi_{m'}^c \in X_{m'} \cap X^c$. If $\pi_m^c = \pi^c$, then we set $k = m$ and we are done. Otherwise, observe that $\pi_m^c = \tilde{\pi}(\{\pi_0^c, \ldots, \pi_m^c\})$ because π_m^c is the only element of $\{\pi_0^c, \ldots, \pi_m^c\}$ for which $u_i(\pi_{m'}^c) > u_i(\pi_{m''}^c)$ for all $m'' > m'$. It must be that $\tilde{\pi}(X^c) \geq \tilde{\pi}(\{\pi_0^c, \ldots, \pi_m^c\})$ because $\tilde{\pi}(X^c) \geq \pi^\dagger$ by hypothesis, $\pi^\dagger$ is the largest element of $X^\dagger$, and $\{\pi_0^c, \ldots, \pi_m^c\} \subset X^\dagger$. Therefore, $\pi_m^c \neq \pi^c$ implies $\tilde{\pi}(X^c) > \pi_m^c$. Consequently, there must be $\pi_{m+1}^c > \pi_m^c$ such that $\pi_{m+1}^c \in X_{m+1} \cap X^c$. Otherwise, the outcome of the game restricted to X^c could be no greater than π_m^c. Hence we can continue to add new elements to the sequence $\{\pi_m^c\}$ until we have included π^c. Because X is finite, the process must eventually include π^c.

□

Proposition 15. *Suppose X is completely ordered and u_i is one-to-one for each i. There exists a sequential game in which $\pi^\dagger$ is the unique outcome that survives IDWDS. Furthermore, there is no sequential protocol with restricted strategy spaces that achieves a higher outcome.*

Proof. The manager can obtain $\pi^\dagger$ by restricting the strategy space to a chain of preference ending in $\pi^\dagger$ and using CP. It remains to show that the manager cannot do better than $\pi^\dagger$. Consider the game generated by a protocol. Because preferences are generic, there will be a unique subgame perfect equilibrium project. Call a history h nice if the equilibrium outcome induced in the subgame determined by h is in $X^\dagger$. There exist nice histories. For example, a history in which no project has been supported and all actions at this history are terminal is nice. It suffices to show that $\emptyset$ is a nice history. Let $h^* \neq \emptyset$ be a nice history. We claim that the history h that immediately precedes h^* is nice. Because nice histories exist, the claim proves that $\emptyset$ is a nice history and proves the proposition. To establish the claim, note thatbecause h^* is nice, the outcome induced in the subgame determined by h^* and equilibrium strategies is in $X^\dagger$ and that the expert, call her Expert i, consulted initially in the subgame generated by h has an action that generates history h^*. If the equilibrium specifies that Expert i take another action, it must be that Expert i prefers the outcome induced to the outcome induced following history h^*. Consequently, the equilibrium outcome induced in the subgame determined by h is in $X^\dagger$ and therefore h a nice history.

$\square$

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Chapter 3

Disclosure and Investor Inattention

Abstract: Investors have a finite capacity to organize all information they receive from financial disclosures. Under rational inattention, we show that investor attention capacity affects the probability of disclosure. In the model, an informed firm makes a strategic voluntary disclosure subject to proprietary costs (Verrecchia 1983a) or uncertainty about information endowment (Dye 1985) and investors optimally allocate their attention as a function of their conjectures about the disclosure strategy. Our main result is that the probability of disclosure is inverse U-shaped in investor attention: for low levels of attention, more attention facilitates communication and increases disclosure; for high levels of attention, more attention better identifies, and therefore deters, unfavorable voluntary disclosure. We provide preliminary empirical evidence that the relationship between investor attention and management forecast is concave, using institutional ownership as a proxy for investor attention.
Keywords: disclosure, inattention, constraints, communication, voluntary, theory.
JEL Codes: D83; G14; M4.

3.1 Introduction

In a standard model with rational investors using all public information, economic agents use all available sources of information to make optimal decisions. Challenges to the theory have been widely documented and call for renewed interest in theories where investors cannot fully process the rich and diverse information released to the market (Blankespoor et al. 2020). Prior research focuses on behavioral models where some agents use a miscalibrated model when updating their beliefs. For example, inattentive investors may be unaware of certain sources of information (Hirshleifer and Teoh 2003) leading to persistent misvaluations of accounting numbers. This approach can explain a variety of observable features in the financial market (Daniel et al. 2002; Barberis and Thaler 2003; Hirshleifer et al. 2004; Banerjee and Kremer 2010).

In this study, we explore a close cousin of behavioral models, known as "rational" inattention, and examine its implication in the context of disclosure theory. As in behavioral models, investors subject to rational inattention cannot correctly process all public information; however, in this approach, investors are cognizant of the limitation and treat attention as a capacity constraint that can be allocated efficiently, see, e.g., Sims (2003), Veldkamp (2011b) and Maćkowiak et al. (2018). The main purpose of this approach is to discipline the model so that the allocation of attention endogenously responds to the qualities of the information. This is of particular interest in voluntary disclosure theory because (a) disclosures are strategic and, therefore, choices over what information to disclose respond to how investors allocate their attention, (b) in comparative statics that affect the disclosure process, investors will presumably re-adjust their attention toward signals that are more informative about fundamentals.

In the model, a firm makes a disclosure subject to disclosure costs (Verrecchia 1983a) or uncertainty about information endowment (Dye 1985), with an objective to increase market prices. We deviate from the standard model by assuming that investors cannot price the firm using all the information contained in the disclosure but have a finite capacity to mentally represent information. Specifically, we use a model of rational inattention that maintains the (partitional) structure of disclosure games and such that investors can only recall a finite number of messages or memory, see Gray and Neuhoff (1998) or chapter 4 in Rubinstein (1998). Investors program how to classify disclosures or non-disclosures in this finite memory as a function of their expectations of the disclosure process. Inattention affects the non-disclosure price and the price for the marginal discloser which, in turn, affects the disclosure threshold away from the fully rational model. Our main contribution is to jointly solve for the allocation of attention and the frequency and nature of disclosures in this framework.

Inattention has two countervailing effects on incentives to disclose. First, inattentive investors respond less to public information and, therefore, weaken the link between price and disclosure which, all other things equal, will reduce voluntary disclosure. Second, inattention will increase price reaction to the lowest disclosed information (or marginal type) because inattentive investors may inaccurately classify this disclosed signal with more favorable states. This increases incentives to disclose at the marginal discloser. Combining both forces, we determine that the link between attention capacity and voluntary disclosure is inverse U-shaped. Disclosure first increases for very low levels of attention in which inattention is an impairment to communication, and then decreases as more attention reduces price reaction to unfavorable news. In particular, for sufficiently high levels of attention, firms always disclose less when subject to more attention.

We develop supplementary theoretical results that offer novel testable implications linking proxies of attention capacity and disclosure frictions. We show that disclosure frictions affect whether disclosure increases or decreases in inattention. The model explicitly captures how attention is differentially allocated for changes in disclosure frictions. In environments where frictions are higher and most unfavorable events are unreported, attention is reallocated so that investors price firms more accurately conditional on disclosure. This implication differs from standard disclosure theory in which disclosures, when they occur, reflect the private information of the firm. In extensions, we find that inattention may reduce incentives to acquire private information and, in the multi-period model of Einhorn and Ziv (2008), attention is reallocated as a function of past disclosures. Further analyses with the normal distribution also reveal, as intuitive, that attention is more concentrated toward more likely disclosures near the mode of the distribution.

We develop a simple empirical application, which examines the relation between likelihood of management forecast and investor attention proxied by institutional ownership.[1] This application does not intend to be a complete test of the theory but offers preliminary evidence on the main prediction of our study. In univariate analyses, we sort firms into both deciles and quintiles based on institutional ownership measured immediately before management forecasts. We find that the likelihood of managers' making a forecast is increasing in the first 4 (8) quantiles

[1] Our results are also robust to using an alternative measure of institutional ownership that adjusts for long-term strategic institutional investors who may have lower incentives to acquire and process management forecasts (Ali et al., 2008; Miao et al., 2016).

(deciles) of institutional ownership, and drops in the 5^{th} (9^{th} and 10^{th}) quintile (deciles). We also estimate polynomial ordinary least squares (OLS) and logistic (Logit) models that include both linear and squared term of institutional ownership as well as industry and year fixed effects and firm-level controls. We find that the linear term of institutional ownership is significantly positive while the squared term significantly negative, which lends preliminary support to our theoretical prediction of a hump shape relation between disclosure and investor attention.

Understanding inattention is a critical, and yet not fully understood, topic in accounting research with much to be discovered as to how inattention shapes financial communication. Our results speak to defining tests over one of the three categories of attention in the framework of Blankespoor et al. (2020). They decompose attention in three mental processes: awareness ("knowledge of the existence of a disclosure"), acquisition ("extraction of the signal from the disclosure") and integration ("mapping of the signal into firm value"). Uncertainty about information endowment (Dye 1985) is mathematically equivalent to awareness in models such as ours, because whether the firm cannot disclose because it is uninformed or discloses but its message is not received implies the same belief structure. As intuitive, lower awareness unambiguously increases strategic non-disclosure. Our main result is about acquisition, given that investors extract and simplify information from reports, possibly confounding multiple reports as a coarse message. Acquisition, we show, implies a non-monotonic link between acquisition capacity constraints and disclosure. Left for further research, our model does not capture integration because investors in our model always correctly form an expectation about value from an extracted (coarse) signal.

Financial communication has increased over time, facilitated by the free and instant access to corporate filings on the EDGAR system (Liu 2020), the dissemination by the financial press and, more recently, the implementation of machine-readable eXtensible Business Reporting Language (XBRL) in financial statements and footnotes (Blankespoor 2019). With the growth in online communication, financial communication now takes the form of an extensive documentation of conference calls (webcasts and transcripts), a wide net of unstructured disclosures

(Blankespoor et al. 2014), or Google searches (Da et al. 2011). This information overload is unlikely to be met with increased investor time, creating a need to understand how inattention may pose limits on how public information is reflected into price. At a conceptual level, full development of the theory will explain that more accounting information or footnotes, on their own, does not increase market efficiency if it is not organized with the proper means of delivery and with better financial education.

In practice, we also observe that many companies which garner high levels of investor attention do not necessarily choose forthcoming levels of disclosure, even though their market leadership and quality of information systems make a proprietary cost or information endowment explanation somewhat less persuasive. Companies such as Alphabet, Facebook, Tesla, or Groupon are frequently noted in the financial press to be less than forthcoming and unpredictable in their financial communications and sometimes openly note an unwillingness to report. For example, the CEO of Tesla Elon Musk comments in a 2018 email to employees that "Being public also subjects us to the quarterly earnings cycle that puts enormous pressure on Tesla to make decisions that may be right for a given quarter, but not necessarily right for the long term." Our model provides one channel that may explain this pattern, noting that firms with a very high level of attention may disclose less.

The model also has implications about the role of regulators in facilitating access to information, for example, via the better organization of financial communications and accounting numbers (e.g., the XBRL mandate or structuring of accounting numbers in the income statement). It is generally assumed that increasing investor attention would benefit communication. We show here that a small amount of inattention starting from a fully rational market will always increase disclosure. Hence, we argue, more broadly, that increasing attention may come with a trade-off and reduce incentives by firms to disclose information voluntarily. This echoes long-standing concerns by firms to have greater control over their reporting process.

Our theoretical analysis contributes to a growing literature in accounting, discussing the role of attention in understanding financial communications (Blankespoor et al. 2020). While linking to this literature in its entirety is far beyond our scope, we note below a few studies that closely relate to our results.

Extending the model of misreporting of Chen et al. (2007), Chen et al. (2017) develop a model of bilateral "it takes two to tango" model of costly attention, in which firms make a disclosure clarity choice and investors make an attention choice revealing the existence of manipulated numbers. As in our paper, the choice of attention by investors is a function of the communication strategy made by the firm. However, their model and focus are quite different from ours. In their model, the choice of clarity is part of a signaling game which jointly affects investors' attention and which projects are financed. They show how additional mandatory disclosure can change the outcome of the game from a separating equilibrium in which investment decisions are efficient, to a pooling equilibrium in which firms choose low clarity. In other words, our primary focus in this paper is whether more investor attention can reduce communication; their focus, by contrast, is whether more mandatory disclosure may discourage joint efforts to communicate.

While there is an extensive literature in economics and finance considering rational inattention (Sims, 2010; Veldkamp, 2011b; Gabaix, 2019), this type of approach is relatively novel in accounting. Two recent studies model attention in terms of an entropy constraint, bounding the amount of information that can be transferred from public signals. Jiang and Yang (2017) consider a game in which a privately-informed but impatient firm seeks to maximize proceeds from issuing equity. In this type of model, absent an accounting system, the firm must reduce its equity to signal its type. When the information released by the accounting system is subject to entropy constraint, they show that different accounting reports must always prescribe different lower bounds akin to a conservative reporting system which identifies the lowest possible outcomes. This result emerges in their study because the signaling inefficiency increases in the distance from the lower bound.[2]

[2]While inattention is a special case of behavioral cognition constraint, there are other studies in the literature that focus on other types of behavioral effects which impact the response of a sender to information, e.g., ambiguity aversion (Caskey 2009; Budanova et al. 2020), disagreement (Bloomfield and Fischer 2011; Banerjee 2011), non-monetary investment preferences (Friedman and Heinle 2016) or the self-fulfilling anticipation of a price bubble

To our knowledge, the only other study specifically focusing on inattention and disclosure is by Lu (2019b). His primary focus is on the effect of investor inattention on aggregation in financial statements. In his model, the firm may use an aggregated signal or supplement the signal with disaggregated details, in an economy subject to strategic complementarities. He shows how additional details in this environment can lead investors subject to inattention to over-emphasize certain details that are privately, but not socially, desirable; on the other hand, removing details can aggravate coordination failures by coordinating all investors on the same simplified (but correlated) signals. A key difference between this approach and ours is that we model attention to the realization of a signal, while his model focuses on attention to particular subcomponents of the information.

While our model features truthful communication by the firm and is not a cheap talk game, our approach using a partitional (imprecise) model of investor attention draws heavily from the methods in the cheap talk literature (Crawford and Sobel 1982; Farrell and Rabin 1996; Stocken 2013). Within this literature, Fischer and Stocken (2001a) show that more informed senders may decrease the receivers' information through its effect on the sender's partition. Likewise, in our model, more investor attention, which (presumably) should increase communication, may change the disclosure strategy of the sender and reduce effective communication. Other studies such as Morgan and Stocken (2003), Kumar et al. (2012), Bertomeu and Marinovic (2016) or Liang et al. (2018) provide applications of cheap talk in models of financial communication.

Lastly, our model aims to show that factors that intuitively increase communication may, in the context of a strategic game between sender and receiver, imply a (testable) non-monotonic relation between communication and disclosure and, as such, rationalize mixed empirical results. We briefly note several recent studies below that suggest an hump shape relationship between characteristics of disclosure and various frictions. Fang et al. (2017) show theoretically and empirically that the response of earnings to restatements is concave in the prevalence of restatements in an industry, if both the noise in the reporting process and the

(Fischer et al. 2016). Similar to rational inattention, these models can be jointly interpreted as deviations from the prediction of a traditional rational model and a behavioral assumption about how players *optimally* solve the game.

cost of manipulation are driven by a common characteristic. Samuels et al. (2020) consider the effect of public scrutiny, admittedly a reduction to obstacles to communication, on misreporting. Noting that scrutiny increases market response to disclosure, hence, payoffs to misreporting, they show and test that misreporting is inverse U-shaped in public scrutiny. In the context of voluntary disclosures, Kim et al. (2020) show that characteristics of the business increasing both the probability of receiving private information and the cost of publicly revealing this information can explain the non-linear relationships between disclosure and characteristics found empirically. Aghamolla et al. (2019) document evidence that the relationship between disclosure and earnings is, contrary to standard models, inverse U-shaped. They show that, in equilibrium, high-ability managers counter-signal by withholding guidance.

3.2 The Model

3.2.1 Assumptions

The model features an owner-manager ("the firm") and boundedly rational investors: investors in our model have a finite capacity to save and recall messages. The firm generates an expected cash flow $\tilde{v}$ with realizations v drawn from a probability distribution with mean μ and full support on an interval normalized to $[0,1]$, and probability density function $f(.)$. As in Dye (1985), Jung and Kwon (1988) and Beyer and Dye (2020), there is a probability $p \in (0,1]$ that the firm observes v. Then, the firm can disclose $d \in \{\text{“}ND\text{”}, \text{“}s\text{”}\}$ where "ND" stands for non-disclosure and "s" stands for truthful disclosure. When the firm does not observe the signal, it has no means to credibly convey it is uninformed and must disclose $d =$ "ND." As in Verrecchia (1983a), disclosure involves a cost which reduces the surplus of the owner by $c \geq 0$. The objective of the firm is to maximize the market price $P(d)$ minus disclosure costs. For all results stated in the formal analysis, we require the existence of a friction, i.e., if $c = 0$, the probability of information endowment p must be strictly less than one.

In traditional voluntary disclosure models, investors form expectations using all information contained in the disclosure $P(d) = \mathbb{E}(\tilde{v}|d)$. That is, all the informational content of d can be processed by investors to predict v. We develop here an extension of this model in which d is observable subject to capacity constraints to classify, recall, and use information. Specifically, investors can only remember $I > 1$ different messages, where I is their capacity to process information. This representation follows what Gray and Neuhoff (1998) refer as a quantization of the information into a finite number of bits (see example below) and, for our purpose, offers a model of inattention that meshes well with discrete features of voluntary disclosure equilibria. We define investors' information as a partition $\{A_i\}_{i=1}^{I}$ of the message space $[0,1] \cup$ "ND", i.e., such that $\cup_{i=1}^{I} A_i =$ "ND" $\cup [0,1]$ and $A_i \cap A_j = \emptyset$ for any $i \neq j$. The partition corresponds to information sets in decision theory and means that investors cannot distinguish between disclosures located in the same information set A_i. Importantly, while I is an exogenous measure of investors' attention capacity, the choice of the partition will be made endogenous. As I becomes large, the ability of investors to distinguish messages converges to the traditional model with fully-rational prices.[3]

Example: Consider the following machine representation of investors' information processing. The disclosure must be classified using a finite memory capacity that must be encoded into memory bits (a number equal to 0 or 1). If investors have only one bit of capacity, they can only distinguish between two information signals, or $I = 2$. With two bits, investors can classify information as 00, 01, 10 or 11, corresponding to $I = 4$. More generally, with b bits of memory, the corresponding number of elements in the partition is $I = 2^b$; vice-versa, a value of I corresponds to a memory of $[\ln I / \ln 2]$ bits (ignoring integer constraints).

For information sets $\{A_i\}_{i=1}^{I}$, the market price forms as the best estimate of v conditional on this coarse understanding of the disclosure. Then, the market price forms based on this

[3]We represent the set of investors as a single investor subject to bounded rationality, in the sense of the firm making a take-it-or-leave-it offer to a boundedly rational investor. In practice, however, the market may feature multiple investors and, in these settings, we could think about the optimal partition for a set of investors as the intersection of individual partitions using $I = nI'$ as the set of message separated by the market as a whole if n investors can each distinguish between I' messages.

partition, i.e.,

$$P(i|D(\tilde{v}) \in A_i) \equiv \mathbb{E}[\tilde{v}|D(\tilde{v}) \in A_i], \tag{3.1}$$

where $D(\cdot)$ is the anticipated disclosure strategy as a function of v and has c.d.f. $G(.)$.

We further restrict the analysis to (intuitive) partitions in which investors' information sets preserve the ordinal ranking of cash flows.[4] Given that no-disclosure must lead to the worst prior in this type of model, we assign the no-disclosure event to the first element of the partition A_1 and denote the associated price, in short-hand, by $P(i) \equiv P(i|D(\tilde{v}) \in A_i)$. A formal definition is given below.

Definition 16. *A partition $\{A_i\}_{i=1}^{I}$ is monotonic if there exists an increasing sequence $\{a_i\}_{i=1}^{I-1}$ given a_0 and a_I such that: (a) $A_1 = \{ND\} \cup [a_0, a_1)$, (b) for each $i \in [2, I]$, $A_i = [a_{i-1}, a_i)$.*

Since we focus exclusively on monotonic partitions, the information set will now be represented as a sequence $\{a_i\}_{i=1}^{I-1}$. Investors are aware of the capacity constraint and choose $\{a_i\}_{i=1}^{I-1}$ in the best possible manner to make their inference correct. To capture (in reduced-form) a penalty for incorrect inferences, we assume that investors face an ex-ante quadratic loss function

$$L(D) = \underbrace{p\int_0^1 (v - P(i|D(v) \in A_i))^2 f(v)dv}_{\text{Loss when firm receives a signal}} + \underbrace{(1-p)\int_0^1 (v - P(1))^2 f(v)dv}_{\text{Loss when firm does not receive a signal}}, \tag{3.2}$$

where $f(v)$ is the probability density of the cash flow v. This preference can also be interpreted as the receiver matching the state, e.g., Chakraborty and Harbaugh (2010). We might also interpret the partition as an analyst or financial expert (receiver) obtaining the signal and mapping it into a recommendation understood by investor as a coarse message, under the assumption that the expert is evaluated more favorably when the message is more precise. We provide a micro foundation for the loss in the Appendix.

[4] In the case of uniform distributions, we show in Section 3.3 that the optimal information set is in the form of a monotonic partition. However, this may not be the case in general.

Example (cont.): Although the memory of the machine is limited to b bits, it can be programmed in advance to process information in a certain manner, leading to the encoding of the various disclosed messages into different sequences of zeros and ones. For example, when observing no disclosure, the machine may encode it as a sequence of zeros (A_1 in the model). Note also that the machine is perfectly able to recognize the initial message it needs to encode, but its information storage capacity is bounded.

The number of different elements of the partition $I > 1$ is an exogenous parameter capturing investors' capacity constraints. Although investors are limited in their ability to process disclosures, they are entirely rational in terms of (a) understanding the limitation, (b) anticipating the equilibrium disclosure strategy, (c) making rational choices about which events they should classify more precisely.

3.2.2 Equilibrium

The timeline of the model is as follows: simultaneously, investors choose their information sets $\{a_i\}_{i=1}^{I-1}$ and the firm chooses the disclosure policy with t denoting the minimum disclosed cash flow when informed (aka, disclosure cutoff). Then the message is sent and payoffs realize.

Definition 17. *An equilibrium Γ is given by a disclosure cutoff $t \in [0,1]$, where $D(v) =$ "ND" if the firm gets no signal or $v < t$ and $D(v) = v$ if $v \geq t$, and an investor partition $\{a_i\}_{i=1}^{I-1}$ such that:*

1. *For any v, the firm discloses optimally given the anticipated investor attention:*

$$P(i|D(v) \in A_i) - c \cdot 1_{D(v) \neq ND} = \max\{P(1), P(i|v \in A_i) - c\}.$$

2. *Conditional on the anticipated disclosure policy $D(\cdot)$, investors set their attention optimally:*

$$\{a_i\}_{i=1}^{I-1} \in \arg\min\{p \cdot \int_0^1 (v - P(D(v) \in A_i))^2 f(v)dv + (1-p) \cdot \int_0^1 (v - P(1))^2 f(v)dv\}.$$

The notion of partitional information structure represents a natural restriction about how investors process information (Krishna and Morgan 2001b; Ivanov 2010a,b; Dworczak and Martini 2019; Kolotilin and Zapechelnyuk 2019; Kolotilin and Li 2019). We focus on the most-informative equilibrium to model the maximum feasible level of communication. For simplicity, we state a definition below in terms of the equilibrium that maximizes the probability of disclosure, hereafter maximal equilibrium. It can be shown that a maximal equilibrium minimizes pricing error.[5]

Definition 18. *An equilibrium is maximal if there is no other monotonic equilibrium with a strictly lower disclosure cutoff t.*

As is common in communication games, there can be equilibria with the same beliefs and payoffs (hence, equivalent) but using different messages. In Definition 19 below, we say that two equilibria are equivalent under these circumstances and, in the rest of our analysis, do not distinguish between equilibria in the same equivalence class.

Definition 19. *Two equilibria Γ and Γ' are equivalent if*

$$\int_{d\in A_i} \mathbb{E}(\tilde{v}|D(\tilde{v})=d)\mathbf{d}G(d) = \int_{d\in A_i} \mathbb{E}(\tilde{v}|D'(\tilde{v})=d)\mathbf{d}G(d)$$

and, if $c>0$, $\{v : D(v) = \text{“}ND\text{”}\} = \{v : D'(v) = \text{“}ND\text{”}\}$.

The next Lemma provides an intuitive application of this definition. For any equilibrium with $t \neq a_1$, no disclosure is ever made below the disclosure threshold t and prices are constant for any disclosure below a_1. Hence, for any equilibrium with $t \neq a_1$, there exists an equivalent equilibrium with $t' = a_1' = \max(t, a_1)$, such that the upper bound of the first information set coincide. Equipped with this observation, we set the upper bound of the first element A_1 of the partition equal to the disclosure threshold, i.e., $t = a_1$, in later analyses.

Lemma 8. *For any equilibrium Γ, there exists an equivalent equilibrium Γ' such that $a_1' = t'$.*

[5] While our model does not involve cheap talk (i.e., disclosures are truthful), this property is common in many communication equilibria with partitional signals; see Fischer and Stocken (2001a) for another example. Other studies such as Hart et al. (2017) and Rappoport (2020) focus on receiver-preferred equilibria.

3.2.3 No strategic withholding benchmark

We solve a benchmark in which the manager is non-strategic and always discloses when receiving information. Investors locate each element of the partition $A_i = [a_{i-1}, a_i)$ to minimize the pricing error:

$$
\begin{aligned}
(K_0) : \{a_i\}_{i=1}^{I-1} \in \arg \min_{\{\hat{a}_i\}_{i=0}^{I}} \{ & p \sum_{i=2}^{I} \int_{\hat{a}_{i-1}}^{\hat{a}_i} (v - \mathbb{E}[\tilde{v}|\hat{a}_{i-1} \leq \tilde{v} < \hat{a}_i])^2 f(v)dv \\
& + (1-p) \int_{\hat{a}_1}^{1} (v - P(1))^2 f(v)dv + \int_{\hat{a}_0}^{\hat{a}_1} (v - P(1))^2 f(v)dv \}, \\
& \text{s.t. } P(1) = \frac{pF(\hat{a}_1)\mathbb{E}(\tilde{v}|\hat{a}_0 \leq \tilde{v} < \hat{a}_1) + (1-p)\mathbb{E}(\tilde{v})}{pF(\hat{a}_1) + (1-p)}, \hat{a}_0 = 0, \hat{a}_I = 1.
\end{aligned}
\tag{3.3}
$$

Lemma 9. *A solution $\{a_i^\dagger\}$ to program (K_0) satisfies*

$$
a_i^\dagger = \frac{\mathbb{E}[\tilde{v}|a_i^\dagger \leq \tilde{v} < a_{i+1}^\dagger] + \mathbb{E}[\tilde{v}|a_{i-1}^\dagger \leq \tilde{v} < a_i^\dagger]}{2} \tag{3.4}
$$

for $i = 2, \ldots, I-1$.

The cutoffs chosen for $a_i^\dagger$ $(i = 2, \ldots, I-1)$ can be reinterpreted as equalizing pricing errors in any two contiguous elements of the partition at each side of a_i, that is:[6]

$$
-(\mathbb{E}[\tilde{v}|a_{i-1}^\dagger \leq \tilde{v} < a_i^\dagger] - a_i^\dagger)^2 = -(\mathbb{E}[\tilde{v}|a_i^\dagger \leq \tilde{v} < a_{i+1}^\dagger] - a_i^\dagger)^2.
$$

For the first cutoff $a_1^\dagger$, the conditional expectation is slightly different because the message $A_1 = [a_0^\dagger, a_1^\dagger]$ may also be the result of not receiving information. Adapting equation (3.4), the

[6]This characterization draws an interesting analogy to Equation (1) in Morgan and Stocken (2003) in which a sender cares about a weighted average of price and accuracy. The accuracy component in their model implies, expectedly, a very similar condition which equates the errors across information sets. The price incentive implies that the pricing error must be increasing in price while, by contrast, the pricing error is constant in our model and the number of elements in the partition is exogenously specified in terms of the degree of bounded rationality.

first cutoff $a_1^\dagger$ is given by[7]

$$\frac{\partial \Delta_1}{\partial a_1}|_{a_1=a_1^\dagger} = 0, \tag{3.5}$$

where

$$\begin{aligned}\Delta_1 =& p\int_{a_1}^{a_2^\dagger}(v-\mathbb{E}[\tilde{v}|a_1 \le \tilde{v} < a_2^\dagger])^2 f(v)dv \\ &+(1-p)\int_{a_1}^{1}(v-P(1))^2 f(v)dv + \int_0^{a_1}(v-P(1))^2 f(v)dv\end{aligned}$$

includes the terms in (K_0) that depend on a_1. Naturally, when the firm always receives information $p=1$, equation (3.5) simplifies to equation (3.4) evaluated at $I=1$, that is,

$$a_i^\dagger = \frac{\mathbb{E}[\tilde{v}|a_1^\dagger \le \tilde{v} < a_2^\dagger] + \mathbb{E}[\tilde{v}|0 \le \tilde{v} < a_1^\dagger]}{2}.$$

The following technical assumption guarantees that this solution to program (K_0) is unique, which is similar to the "Monotonicity" condition in Crawford and Sobel (1982).

Assumption 1 (Monotonicity I)**.** *For $I \ge 1$, if two sequences $a \equiv \{a_i\}_{i=0}^I$ and $a' \equiv \{a_i'\}_{i=0}^I$ satisfy Equations (3.4) and (3.5) with $a_{I-1} < a'_{I-1} < a_I = a'_I$, then $a_i < a'_i$ for all $0 \le i \le I-1$.*[8]

3.2.4 Discussion

The model of information classification is designed to reflect investors' inability to process all relevant information. While this model has an intuitive interpretation in terms of reducing the message space, it is also technically convenient in the special context of disclosure theory: disclosure equilibria partition the state space into a disclosure and a non-disclosure region. Hence, bounded rationality works to coarsen the state space further but otherwise maintains the partitional structure of the communication game. Below, we discuss some of the key assumptions

[7] We maintain in the benchmark the assumption that uninformed firms must be classified in A_1 because the main role of this preliminary is to help state the solution to the problem with strategic withholding. Naturally, investors could do even better by classifying non-disclosures with disclosures near the unconditional mean; however, this type of solution would not be feasible with strategic withholding because the non-disclosure message must always generate the lowest posterior.

[8] Consider the solution to (3.3) when $\tilde{v}$ is Uniform and rearrange terms $a_{i+1}^\dagger = 2a_i^\dagger - a_{i-1}^\dagger$ implying that Monotonicity I is satisfied for the case of the Uniform distribution.

in this setting.

(i) A different approach is to model attention capacity in terms of a maximal reduction in entropy (Sims, 2003). This criterion may alter the nature of the game because, with a bound in (differential) entropy, investors will never be able to rule out any state with certainty regardless of a disclosure or non-disclosure - thus, implying a setting perceived by investors as noisy disclosure and no longer has a partitional nature, see, e.g., Jiang and Yang (2017) or Lu (2019b). To our knowledge, the properties of voluntary disclosure games when investors have entropy constraints have not yet been worked out but present interesting research opportunities in this area. Sims (2003) also discusses finite codes as a foundation for entropy (p. 668-669), noting that entropy can be recovered as the information recovered from a finite code observed over a continuous time. This formulation suggests that a finite code may represent a single disclosure event, while entropy may reflect the information collected over a given time horizon composed of many disclosure events.

(ii) In the baseline model, we use I as a measure of the collective ability of investors to distinguish messages: for example, in the form of the intersection of all partitions chosen by each individual investor as it would be efficient for investors to choose non-overlapping information sets. Other interpretations may feature institutional aspects of information providers in which the message is discrete. For example, financial auditors issue an unqualified, qualified or adverse opinion; rating agencies rate debt issues on a scale; stock analysts issue a stock recommendation within a scale. This is also true for quality certifications outside of financial reporting (Dranove and Jin 2010): restaurants, hospitals and movies may receive qualitative grades. This type of coarse partition may be desirable if small investors or consumers have limited ability to process more complex (or continuous) message spaces.

(iii) We present the analysis in terms of investor-driven capacity constraints but a different model may involve manager-driven capacity constraints if, say, the manager can only use I separate messages when disclosing to the market. If the firm has some means to pre-commit itself to a level of disclosure (Heinle and Verrecchia 2016; Suijs and Wielhouwer 2019; Aghamolla et al. 2019), the firm will be better off committing to complete inattention to reduce disclosure costs. However, if the firm cannot credibly commit to attention, it can be readily verified that the maximal equilibrium in the manager attention model will coincide with the baseline investor attention model. Hence, I can also be thought of as the maximum feasible attention by investors and the firm.

3.3 Uniform Payoffs

We lay out the intuitions in the context of $\tilde{v}$ being uniformly distributed and the only friction is a non-zero cost $c > 0$ of disclosure. As we will show next, this specification captures the main trade-offs of the model in closed-form. Another interesting property of the uniform model is that it can be formally shown that the information sets formed by investors must be a monotone partition (thus demonstrating that monotonic partitions do not seem pathological in simple settings), as we claim next.

Proposition 16. *When $\tilde{v}$ is uniformly distributed, all equilibrium information structures induce monotone partitions on the state space.*

The intuition for Proposition 16 is that the investors can always reduce the pricing error by modifying a non-monotone partition. Two prior studies, by Bergemann et al. (2012) and Kos (2012), prove this property using the single-crossing properties of cheap talk with an upper bound on the number of possible messages. Information sets have this form in our model but for different reasons: there is no single-crossing property and disclosures are verifiable; instead, the interval structure are selected because they minimize the pricing error of an uninformed receiver.

Next, we derive equilibrium in this game. Absent strategic withholding, investors optimally separate the state space in intervals of equal length, so that

$$a_i^\dagger = i/I. \tag{3.6}$$

We need to verify if this (ideal) information structure is feasible when the firm can strategically withhold. Specifically, for this to be sustainable in an equilibrium, the firm must report $v \geq a_1^\dagger$, that is,

$$\underbrace{\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1^\dagger)}_{=a_1^\dagger/2} \leq \underbrace{\mathbb{E}(\tilde{v}|a_1^\dagger \leq \tilde{v} \leq a_2^\dagger)}_{=(a_1^\dagger+a_2^\dagger)/2} - c.$$

Reinjecting the values of $a_i^\dagger$ from (3.6), this partition is feasible as long as incentives to strategically withhold are not too high, that is, if $c \leq 1/I$. Intuitively, when the friction is small, the pooling over low strategic types in the non-disclosure region A_1 required by the voluntary disclosure game is less than the pooling directly caused by investor inattention.

Suppose next that $c > 1/I$. Then, the disclosure threshold $t = a_1$ must be set strictly higher than $a_1^\dagger$. The optimal information structure for investors having $I-1$ messages to learn about the remaining state space $[t,1]$ is to set, likewise to (3.6), $I-1$ intervals of equal size on $[t,1]$, i.e., for any $i \geq 2$,

$$a_i = t + (1-t)\frac{i-1}{I-1}. \tag{3.7}$$

The maximal equilibrium will prescribe setting t as low as possible, which should involve making the firm exactly indifferent between withholding and disclosing when $v = t$, that is

$$\underbrace{\mathbb{E}(\tilde{v}|\tilde{v} \leq t)}_{=t/2} = \underbrace{\mathbb{E}(\tilde{v}|t \leq \tilde{v} \leq a_2)}_{=(t+a_2)/2} - c,$$

which simplifies to $a_2 = 2c$. Note that a threshold $2c$ would be the disclosure threshold in a fully rational model, but since the threshold here is $t = a_1 < a_2 = 2c$, we know that, in this case, there is more disclosure with rational inattention: put differently, inattentive investors induce the firm to disclose information that would have been withheld if investors had infinite attention capacity.

Using Equation (3.7) to recover a_2 and solving for t readily yields the following equilibrium.

Proposition 17. *The maximal equilibrium is given as follows:*

(i) If $c \leq 1/I$, $t = 1/I$ and $a_i = i/I$.

(ii) If $c > 1/I$, $a_1 = t = \frac{2c(I-1)-1}{I-2}$ and, for $i > 1$, $a_i = \frac{2c(I-i)+i-2}{I-2}$.

This equilibrium has two core properties illustrating how inattention affects the characteristics of the voluntary disclosure equilibrium.

First, in classic disclosure models, investors are fully attentive to all disclosures and, therefore, the uncertainty that may remain after a disclosure event is not affected by strategic behavior. In the inattention model, by contrast, a higher disclosure cost implies a higher threshold t. This, in turn, implies that disclosures above t receive more attention and lead to more accurate prices. Put differently, as attention is optimally allocated, investors trade off more inaccuracy due to non-disclosure with more *accurate* pricing conditional on disclosure. As in standard disclosure models, the withholding region is (weakly) the least precise but the inattention model predicts an inverse relationship between the frequency of disclosure and the degree of attention to each disclosure.

Second, the equilibrium has a central comparative static that ties how the degree of inattention affects the probability of disclosure. We illustrate the trade-off in Figure 3.1 by varying the degree of inattention. At the maximal level of inattention ($I = 2$), $t = 1/2$ means that only above-average outcomes are disclosed. As the degree of attention increases (up to $I = 1/c$), the cutoff t decreases: intuitively, the partition of the message space becomes more precise as the market becomes more attentive. We refer to this first part of the trade-off as the "informativeness effect" of attention. As the degree of attention increases further (from $I = 1/c$ onward), the cutoff point t *increases*. The intuition for this region is better obtained by considering a decrease in inattention: when investors are inattentive, they classify incorrectly the marginal discloser

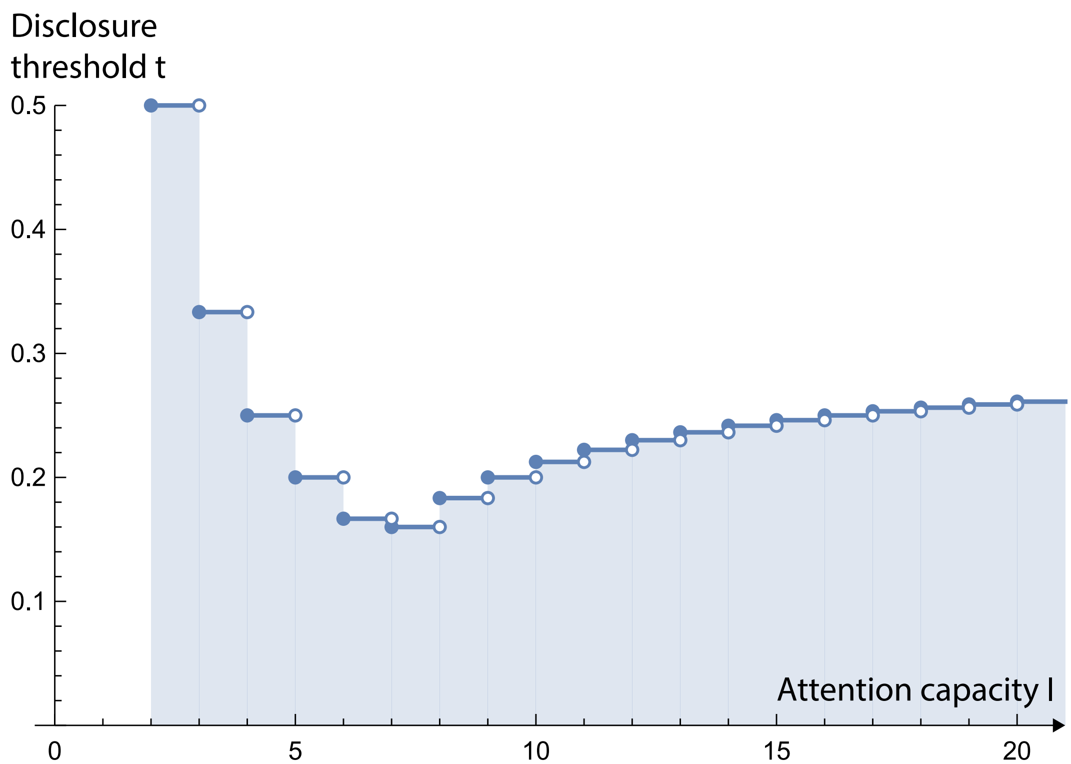


Figure 3.1: Disclosure threshold and attention capacity ($c = .15$)

as a better firm $[a_1, a_2)$, leading to more incentives to disclose; we refer to this second part of the trade-off as the "marginal discloser effect" of attention. The disclosure threshold, i.e., the probability of non-disclosure, is plotted as a function of the degree of attention in Figure 3.1.

To explain the non-monotonicity further, Figure 3.2 illustrates the change in cutoffs when $I = 2, 3, 4$ for $c = 1/3$. Up to $I = 3$, investors are implementing their ideal message space with three signals (i.e., subdividing the message space in three equal intervals). The voluntary disclosure problem does not affect the determination of the cutoff t and, as a result, the informativeness effect must dominate as the precision of all intervals increases. Starting at $I = 4$, however, the voluntary disclosure equilibrium prescribes $t > 1/I$ and there is a loss in precision due to strategic withholding. Then, the marginal discloser effect dominates as incentives to disclose decrease with more attention. In summary, the relationship between attention and the probability of disclosure $1 - t$ is inverse U-shaped, with the maximal probability of disclosure achieved at $I = [1/c]$ or $I = [1/c] + 1$.

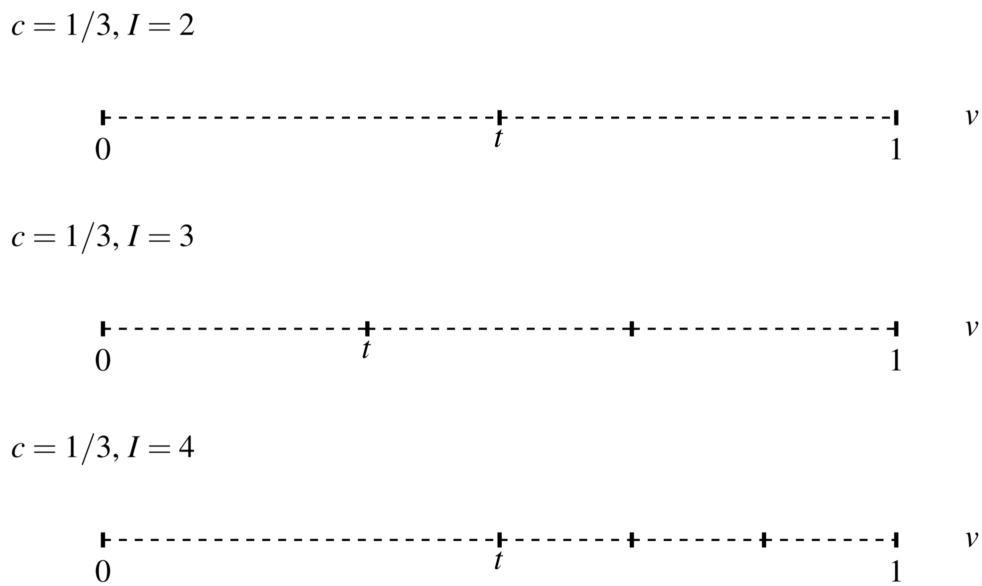


Figure 3.2: Disclosure cutoffs

3.4 General Analysis

3.4.1 Preliminaries

We prove next these results in the general model, lifting the assumption that $\tilde{v}$ is uniform and allowing for a non-zero probability $1 - p > 0$ of not receiving information. The next statement formally demonstrates that the maximal equilibrium minimizes the pricing error.

Proposition 18. *The equilibrium with the lowest disclosure cutoff gives investors the highest expected payoff over all equilibria that induce interval partitions.*

Given that investors make an equilibrium conjecture about the threshold t, we can rewrite the objective function of investors as a program $K(t)$ which consists of choosing all but the first element of the partition to minimize pricing error over *disclosed* values:

$$K(t) : \{a_i\}_{i=2}^{I-1} \in \arg \min_{\{\hat{a}_i\}_{i=2}^{I-1}} \sum_{i=2}^{I} \int_{\hat{a}_{i-1}}^{\hat{a}_i} (v - \hat{P}(i))^2 f(v) dv, \tag{3.8}$$

$$\text{s.t. } \hat{a}_1 = t,\ \hat{a}_I = 1, \text{ and } \hat{P}(i) = \mathbb{E}[\tilde{v} | \hat{a}_{i-1} \leq \tilde{v} < \hat{a}_i].$$

Lemma 10 characterizes an optimal choice a_i in the above program. The proof is identical to Lemma 9, hence omitted, except that the information sets are optimized starting at the disclosure threshold t and over $I-1$ intervals.

Lemma 10. *For any $i \geq 2$,*

$$a_i = \frac{\mathbb{E}[\tilde{v}|a_i \leq \tilde{v} < a_{i+1}] + \mathbb{E}[\tilde{v}|a_{i-1} \leq \tilde{v} < a_i]}{2}. \tag{3.9}$$

3.4.2 Main Analysis

We are now equipped to characterize a solution of the model. We proceed in two simple steps that closely follow the argument in the uniform model.

First, in what follows next, we show that the maximal equilibrium coincides with the benchmark partition $\{a_i^\dagger\}$ if $A_1^\dagger = [a_0^\dagger, a_1^\dagger) \cup \{ND\}$ can be sustained as the withholding region. To verify this, it must be verified that values in the next information set $v \in A_2^\dagger = [a_1^\dagger, a_2^\dagger)$ would not be strategically withheld. That is,

$$\mathbb{E}(\tilde{v}|\tilde{v} \in (a_1^\dagger, a_2^\dagger)) - c \geq \frac{pF(a_1^\dagger)\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1^\dagger) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1^\dagger) + (1-p)}, \tag{3.10}$$

where the right-hand side is the non-disclosure price in Jung and Kwon (1988).

When condition (3.10) is satisfied, which means that the disclosure threshold with fully rational investors is lesser or equal than $a_1^\dagger$, investors will respond to any $t < a_1^\dagger$ by increasing the cutoff of the first information set to $a_1^\dagger$. The firm will, of course, respond by increasing the disclosure threshold to $t = a_1^\dagger$, implying a maximal equilibrium that coincides with the model without disclosure frictions and similar to Proposition 17 (i).

Second, suppose that Equation (3.10) is not satisfied, in which case the partition preferred by the investor is too fine and would encourage the manager to withhold some $v > a_1^\dagger$. Recall then that the maximal equilibrium is the equilibrium with the smallest disclosure cutoff $t = a_1$, which involves a choice of $a_1 = t > a_1^\dagger$ binding the withholding constraint:

$$\mathbb{E}(\tilde{v}|\tilde{v} \in [a_1, a_2)) - c = \frac{pF(a_1)\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1) + (1-p)}, \tag{3.11}$$

corresponding to Proposition 17 (ii) where the withholding region A_1 is driven by the binding incentive constraint on the cutoff. The next theorem summarizes these observations and is the main result of our study.

Theorem 1. *Let $\overline{p}$ be the probability cutoff such that (3.10) is met at equality.*

(i) If Equation (3.10) is satisfied (i.e., $p \geq \overline{p}$), $\{a_i^\dagger\}$ is the maximal equilibrium and the manager discloses when informed with $v \geq a_1^\dagger$;

(ii) Otherwise, the equilibrium disclosure cutoff t is strictly greater than $a_1^\dagger$ and the maximal equilibrium $\{a_i\}$ satisfies equations (3.9) and (3.11).

To summarize Theorem 1, investors will try to set their ideal information sets $\{a_i^\dagger\}$. But this is only feasible if there are limited incentives to withhold $A_2^\dagger$ - which, in turn, requires the market to be sufficiently skeptical after a non-disclosure to decrease their beliefs when observing $A_1^\dagger$. As is well-known in this type of model, this can only occur if firms are expected to be informed, i.e., likely to be strategically withholding, when reporting in $A_1^\dagger$. When firms are likely to be uninformed, this equilibrium is no longer sustainable because firms with $v \in A_2^\dagger$ will be better-off withholding (and pretend to be uninformed). Then, the disclosure cutoff must increase to $t = a_1 > a_1^\dagger$ to satisfy the indifference condition of the marginal discloser (3.11). The remaining cutoffs $\{a_i\}$ for $i \geq 2$ are then set according to (3.9) to minimize pricing errors over the disclosure region $[a_1, 1]$.

For reasons similar to Crawford and Sobel (1982) and the assumed monotonicity condition in (1), it is possible for the necessary conditions in (3.9) and (3.11) to have multiple solutions because while these are second-order sequences subject to two boundary points, $a_0 = 0$ and $a_I = 1$, the equilibrium equations are non-linear. While these seem to describe pathological cases, we formally show below that the Monotonicity condition can be adapted to the current setting so that these conditions are necessary and sufficient.

Assumption 2 (Monotonicity II)**.** *Given $c > 0$ and $0 < p < 1$, for $I \geq 1$, if two sequences $a \equiv \{a_i\}_{i=0}^{I}$ and $a' \equiv \{a'_i\}_{i=0}^{I}$ satisfy Equations* (3.9) *and* (3.11) *with equality such that $a_{I-1} < a'_{I-1} < a_I = a'_I$, then $a_i < a'_i$ for all $0 \leq i \leq I-1$.*

The monotonicity assumption guarantees that, as for the case with low disclosure frictions, a solution exists and is unique.

Corollary 4. *Suppose that Assumption 2 holds. Then, if Equation (3.10) is not satisfied (i.e., $p < \overline{p}$), the maximal equilibrium is uniquely given by the solution to (3.9) and (3.11).*

The main results will continue to hold with an arbitrary, e.g., unbounded, support as long as we adjust the boundary conditions a_0 and a_I when solving for (3.9)-(3.11) in Theorem 1. Below in Figure 3.3, we plot a numerical example with the normal distribution, comparing the optimal partition under complete information (dashed) versus under strategic withholding (solid). The equilibrium features a large strategic withholding region followed by compressed information sets in the disclosure region. Note also that, in the example of the normal distribution, investors set more precise information sets near the median distribution over events that have greater likelihood.

3.4.3 Properties of the equilibrium

We discuss next the key properties of the model, generalizing the observations made in the case of the uniform distribution. To begin with, we demonstrate a few results that establish several standard insights of classic voluntary disclosure models - shown to be preserved with minor adjustments for any degree of inattention.

Proposition 19. *The voluntary disclosure cutoff t increases in the disclosure cost c and decreases in the probability of being informed p.*

When strategic withholding constrains investor learning, the probability of strategic withholding is affected by the friction in a manner similar to the traditional models - even though

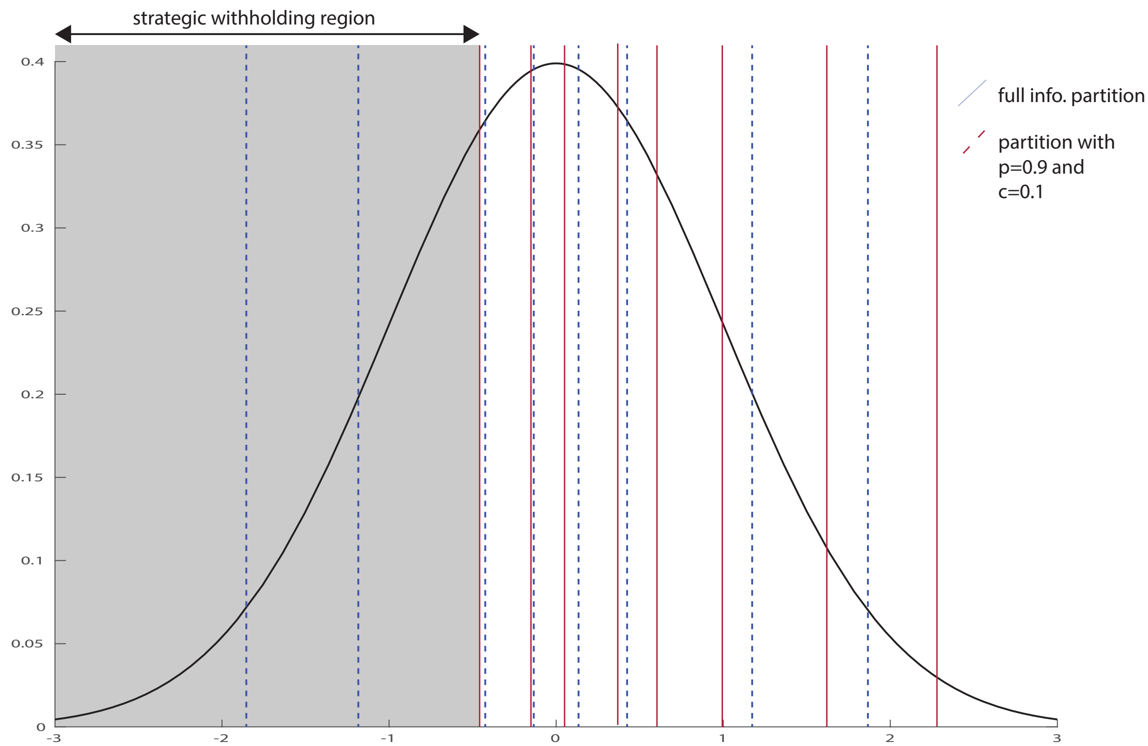


Figure 3.3: Rational Inattention: Normal Distribution

the disclosure cutoff need not be set at the same location. Interestingly, note that the standard disclosure model would always predict that non-disclosure implies the least precise beliefs. Neither statements need to hold with rational inattention and it may be the case that uncertainty is higher conditional on disclosure than conditional on non-disclosure. The choice of cutoffs $\{a_i\}$ in (3.9) equates the pricing error for the *marginal type* located at $v = a_i$, this needs not hold for the average type in an information set and, therefore, when comparing $Var(\tilde{v}|A_1)$ to $Var(\tilde{v}|A_i)$ with $i \geq 2$.

We turn next to new insights unique to the inattention setting.

Proposition 20. *The expected pricing error is increasing in the disclosure cost c and decreasing in the probability of being informed p. As an example, in the special case of uniform cash flows $\tilde{v}$,*

(i) The pricing error conditional on disclosure is decreasing in the disclosure cost c and

increasing in the probability of being informed p;

(ii) For sufficiently large cost, the expected pricing error is first strictly decreasing and then strictly increasing in attention capacity I. The pricing error conditional on disclosure is strictly decreasing in attention capacity I for I sufficiently large, i.e., when Inequality (3.10) *does not hold.*

In Proposition 20, we show how inattention reallocates investors' information sets between disclosure and non-disclosure regions. Apart from the results that are true for any general distributions, there are some interesting comparative statics that hold under uniform distributions. The pricing error conditional on disclosure decreases when a disclosure friction increases, which illustrates the trade-off between less precise non-disclosure and more precise disclosures. Investors who cannot observe well strategically withheld low events pay more attention to fewer disclosed news: in other words, inattention creates an inherent trade-off between frequency of disclosure and (perceived) quality of disclosure.

The next proposition summarizes the key main result from our analysis and demonstrates how disclosure varies as a function of inattention.

Proposition 21. *The disclosure cutoff t is first strictly decreasing and then strictly increasing in the partition size I.*

Proposition 21 generalizes the observations made in the context of the uniform distributions (where all elements of the investor partition *conditional on disclosure* are of equal size) to the case of general distributions. The effect of attention on the probability of disclosure is a simple inverse U-shaped relationship with, first, the probability of disclosure being increasing for low levels of attention and, then, decreasing for high levels of attention. The probability of disclosure is maximal at an interior level of investor attention binding Equation (3.10) and is the point at which the non-disclosure region corresponds exactly to how unconstrained investors would have chosen the lowest element of the partition.

Below, we state an additional result in the context of $c = 0$, i.e., with only uncertainty about information endowment. In this type of model with fully rational investors, Acharya et al. (2011) and Guttman et al. (2014) demonstrate that the equilibrium satisfies the "minimum" principle, that is, minimizes the price conditional on any possible disclosure cutoff. We show below that the minimum principle may be upset in the presence of rational inattention.

Proposition 22. *Suppose $c = 0$. There exists at most a single level of attention I_m such that the minimum principle and, subject to $I \in \mathbb{N}$ being an integer, is not generic, i.e., the set of parameters $p \in (0,1)$ such that the minimum principle holds has zero mass. If the cutoff t when $I = 2$ is lower than the cutoff in the rational model (Jung and Kwon 1988), the non-disclosure belief is always strictly higher under rational inattention for any $I \geq 2$.*

To explain this result, note that the minimum principle is a generalization of the unravelling principle (**?**) in the presence of a disclosure friction. The principle relies on the ability of an informed firm to separate (by disclosing) which, in turn, causes skepticism in beliefs following non-disclosure. Reformulated, the minimum principle, just like the unravelling principle, states that any equilibrium features the maximal rationalizable skepticism. Rational inattention works as a constraint on the ability of informed firms to separate, thus reducing the equilibrium skepticism. Counter-intuitively, the higher non-disclosure belief under this constraint implies that strategically withholding firms achieve a higher surplus than under the rational model. In particular, Proposition 22 implies that investor inattention benefits (on average) strategic firms at the expense of firms that were truly uninformed.

3.5 Discussions

3.5.1 Information Acquisition

In a seminal study, Shavell (1994) shows that voluntary disclosure induces excessive information acquisition because informed firms have discretion to strategically disclose. Rational inattention can interact with this effect: as firms cannot "freely" strategically disclose because investors are not allocating enough attention, incentives for excess information acquisition may be muted. We discuss this idea formally below.

As in Shavell (1994), assume that, ex-ante, information has social value (otherwise, any reduction in information acquisition is socially beneficial). Let v be a productivity signal, with density $f(.)$, and let x be an investment. The firm's market value is then given by $vx - \psi(x)$, where $\psi(0) = \psi'(0) = 0$ and $\psi''(\cdot) > 0$. Let $x^*(v)$ be the optimal investment at v. As benchmark, we restate Proposition 5 of Shavell (1994) below.

Lemma 11. *The value of information to firms V exceeds the social value of information V^*.*

We show next that this problem can be mitigated if investors have limited attention.

Proposition 23. *Suppose that there is only acquisition cost and no disclosure cost, i.e., $c = 0$. For **any** finite information capacity, the value of information to firms is less than the full-information case.*

The proof is provided in the appendix that utilizes the minimum principle (Acharya et al., 2011) and the properties of the equilibrium. Proposition 23 demonstrates that inattention reduces incentives to acquire information. This does not mean, however, that inattention necessarily increases social welfare. When information is useful to determine the optimal level of investment, it is socially desirable for investors to understand more information *given that it has been obtained by the firm*. Hence, there is a trade-off between information precision and acquisition cost. If investors are able to allocate more attention to firms' disclosure, the quality of information potentially increases and more informed decisions could be made, which is socially beneficial.

3.5.2 Dynamics

Rational inattention also has multi-period implications. Consider a two-period simplified version of the model by Einhorn and Ziv (2008). The cash flows realized at the end of each period are independent across periods and publicly observed. At the beginning of each period, the firm potentially receives a (noisy) signal s that is equal to the cash flow v in the current period with probability $q(v) > 0$ and a pure error with probability $1-q(v) > 0$.[9] Assume that the firm cannot distinguish between signals about the cash flow and errors. Let G be the probability distribution of the error that is independent of the cash flow v. The probability that the firm receives a signal in period t is $p_t \in (0,1)$ $(t = 1,2)$, where $p_1 = \lambda$, $p_2 = \lambda_0$ if the firm does not have a signal in period 1; $p_2 = \lambda_1 > \lambda_0$ if the firm has a signal in period 1. The investors' attention capacities are I_1 and I_2, respectively, in periods 1 and 2.

Proposition 24. *Let A_1^1 be the first element of the investor's information set in period 1. Let a_1^2 be the first cutoff that the investor selects in period two. Let t_1 be the disclosure threshold in period one. The cutoff a_1^2 will be lower if the investor does not observe A_1^1 in period one or if the realized cash flow in period one falls below the disclosure threshold t_1 (when the observation about the signal is A_1^1).*

At the end of period 1, investors will update their beliefs about the second-period signal endowment of the firm to

$$p_2 = \begin{cases} \lambda_1 & \text{if investors does not observe } A_1^1 \text{ in period 1} \\ \phi(v_1) = \frac{1-\lambda}{M}\lambda_0 + (1-\frac{1-\lambda}{M})\lambda_1 & \text{otherwise,} \end{cases}$$

where $M = (1-\lambda) + \lambda \Pr(s_1 < t_1 | v_1)$. For any information set different from A_1^1, it is certain that the firm received a signal in this period conditional on the disclosure. In turn, this implies that the firm is more likely to be informed again in period 2, and investors will pay more attention to lower disclosures and choose a lower cutoff a_1^2 in period 2 (than if the first-period observation is A_1^1) by Proposition 19.

Similarly, if the realized cash flow v_1 in period 1 is higher than t_1, the probability $\Pr(s_1 < t_1 | v_1)$ that s_1 is lower than the equilibrium threshold t_1 (and the firm then withdraws the low signal) will be lower, because a signal lower than t_1 can only be generated by error. So

[9] The probability of signal being informative could potentially depend on the actual state. Our result hold for any function $q(\cdot)$ as long as $q(v) > 0$ for all v.

when forming the belief p_2, there is less weight assigned to the case where the firm conceals a low signal in period 1 (given the observation A_1^1). Then $\phi(v_1)$ will be smaller because $\lambda_1 > \lambda_0$. Hence the investors will pay less attention to lower disclosures and choose a higher a_1^2 in period 2. In summary, if information endowments are correlated, attention will be serially correlated as well and vary over time as a function of disclosures and realized signals.

3.6 Empirical Application

3.6.1 Sample Selection

Our main theoretical prediction is that firm disclosure has an inverse-U shaped relation with investor attention. We develop preliminary evidence about this prediction using management forecast as a proxy for firm disclosure. Management forecasts are voluntary disclosures and managers face substantial uncertainty in making forecasts about future earnings. Moreover, management forecasts are released as part of earnings conference calls which are highly publicized and and discussed by the financial press. Management forecasts generally garner more significant price reactions than most other types of firm disclosures (Beyer et al. 2010).

We present the definitions and sources of our main variables in Table 1 and sample selection procedures in Table 2. We start with all annual earnings announcements made by the U.S. firms for fiscal years ending between January 1^{st}, 2004 and December 31^{st}, 2016 obtained from the Institutional Brokers' Estimate System (I/B/E/S) earnings announcement database.[10] We construct a sample of earnings per share (EPS).[11] Our sample starts with 67,239 firm-year

[10] Our sample starts from 2004 due to two significant regulatory changes in the U.S. in 2000 and 2002, which have fundamentally changed both managers' incentives to disclose information and process of collecting management forecasts. Since August 2000, Regulation Fair Disclosure (Reg FD) has shut down most private communications between managers and financial analysts. Consequently, Reg FD have increased the frequency of public managerial forecasts. In addition, since July 2002, the Sarbanes-Oxley Act (SOX) has dramatically increased internal controls and management responsibilities. From a data-collection perspective, SOX also requires conference calls to be recorded in transcript form, which allows for much more convenient identification of management forecasts.

[11] Earnings in I/B/E/S are reported as pro-forma earnings calculated under the same accounting principles for both analysts' and managers' forecasts (Bertomeu et al. 2019). We choose to use raw EPS since it is the actual nominal variable being forecasted by managers and analysts and has been kept within a similar range across firms (Cheong and Thomas, 2011). EPS, that have been adjusted for stock splits, are more problematic since its magnitude tends to

observations and 10,945 unique firms. We only keep observations with non-missing current and prior year earnings announcement dates, which are used to construct a time window for management forecasts.

We merge earnings announcements with management forecasts which are acquired from the I/B/E/S management forecast guidance (CIG) database using I/B/E/S unique tickers and forecast period end dates. We can match 70,198 management forecasts to the I/B/E/S earnings announcement sample. As in Bertomeu et al. (2019), we further require all forecasts to be made after the prior year's earnings announcement date but at least six months before the current period end date, which shrinks the number of annual management forecasts to 28,787. The majority of management forecasts is bundled with earnings announcements and takes place between 10 to 11 months before the current fiscal year end. Since our theory is silent on how frequently managers forecast within a period, we only retain the earliest management forecast for periods with more than one forecast.[12]

We obtain information on stock prices from CRSP, accounting fundamentals from Compustat, and institutional ownership from Thomson Reuters. After the merge, our sample shrinks to 50,703 firm-year observations, 7,864 unique firms, and 11,451 management forecasts. Lastly, we drop firms that either always or never make a forecast since these firms probably have committed to a fixed forecast policy for reasons out of the scope of our model. Our final sample has a total of 16,508 firm-year observation, 2,583 unique firms, and 7,392 management forecasts. As shown in Table 3.3, 45% of all firm-years have management forecasts in our sample. A median firm-year in our sample has institutional ownership of 75%, leverage ratio of 53%, market capitalization of 1.01 billion U.S. dollars, and book-to-market ratio of 53%. 15% of all firm-years report negative earnings, and 68% have an increase in EPS.

decline as firms split their shares.

[12]Note that management forecasts in I/B/E/S have already been adjusted for the number of shares. We construct the raw earnings forecasts by multiplying forecasts in I/B/E/S with I/B/E/S adjustment factor, which is recovered using the ratio of raw earnings to adjusted earnings in the I/B/E/S earnings database.

3.6.2 Proxies for Investor Attention

A critical empirical challenge in testing our theory is to construct a plausible proxy for investors' attention. The proxy should capture investors' aggregate capacity constraints to extract managers' forecasts. Institutional investors hold more than 70% of the common shares of NYSE/NASDAQ/AMEX stocks as of 2012 (Kempf et al., 2017).[13] Moreover, institutional investors generally have both better skills and stronger incentives than retail investors to proactively acquire and process management forecasts. Hence, our firm-level proxy for investors' capacity should be increasing with the amount of influence institutional investors have on managers. Secondly, since the number of messages (I) investors can remember is set before managers' disclosure in our model, our empirical proxy for investors' capacity should be measured *prior to* managers' forecasts.

With these considerations, we use the percentage of institutional ownership measured immediately before management forecasts as a firm-level proxy for investor attention. Higher institutional ownership correspond to higher capacity by investors to acquire and process managers' voluntary disclosure. We will refer to this measure as $Capacity\ (percent)$ for the rest of the paper.

Institutions that hold significant stakes ($> 5\%$) usually have strategic considerations and are less likely to acquire and trade on management forecasts. Following Ali et al. (2008) and Miao et al. (2016), our second measure $Capacity\ (ratio)$ refines the first measure by adjusting for long-term institutional ownership:

$$Capacity\ (ratio) = \frac{Ins - Ins(LT)}{1 - Ins(LT)},$$

where Ins is the percentage of institutional ownership and $Ins(LT)$ is the percentage shareholdings by institutions that own more than 5% of shares.

Admittedly, our empirical proxies might be related to how likely managers make a fore-

[13] Institutional investors interact and communicate with firm managers their demands of disclosure. In contrast, retail investors' demands are much more opaque to managers (Basu et al., 2020).

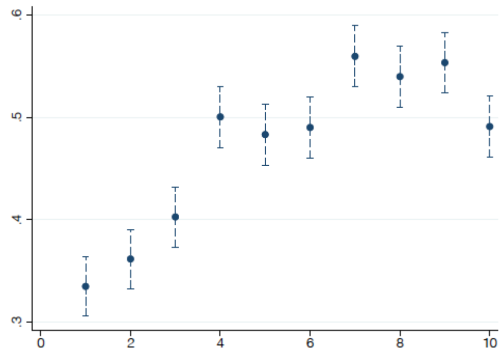
cast for reasons other than what we conjecture in the model. In other words, $Capacity\ (percent)$ and $Capacity\ (ratio)$ might affect management forecasts through channels other than investor attention. For example, institutional investors may demand more voluntary disclosure to balance their portfolio or combine public disclosures with their own private information (Cheynel and Levine 2020). If we fail to find an inverse U-shaped relation between $Capacity$ and management forecast in the data, it could either be: 1) the channel through investor attention predicted by our theory does not exist; 2) the other channels add sufficiently substantial noise into $Capacity$ such that our tests do not have enough power to detect our theoretical channel. Our empirical proxies may capture other determinants of management forecasts, which could be biased against us finding an inverse U-shaped relation between $Capacity$ and management forecasts in the data. However, other determinants of management forecasts have no reason to produce the inverse U-shaped relation on their own.

3.6.3 Empirical Analysis

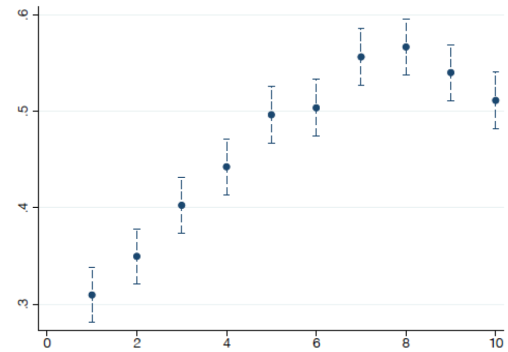
Investor Attention and Management Forecast

We begin by graphically presenting the relation between investor attention and managers' likelihood of making a forecast. In Figure 3.4, we sort firms into either ten deciles or five quintiles based on $Capacity\ (percent)$ and $Capacity\ (ratio)$ in year $t-1$. Within each decile or quintile, we calculate and report the average probability of managers making a forecast in year t. The 95% confidence intervals are plotted around the mean values for each decile and quintile. Consistent with an inverse U-shape relation predicted by the theory, we find that the likelihood of making a forecast increases in the first 4 (8) quintiles (deciles), and then declines in the 5^{th} (9^{th} and 10^{th}) quintile (deciles).

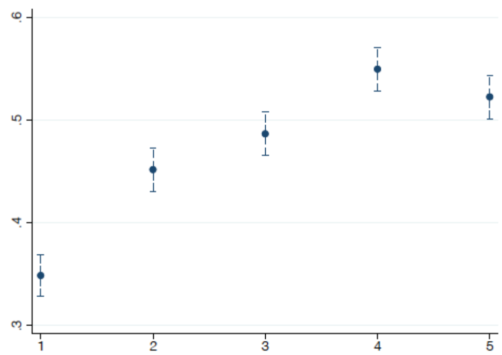
We conduct next additional tests to lend further support to the theoretical prediction. First, we estimate a polynomial regression model which includes both proxies for investors' bounded capacity ($Capacity\ (percent)$ and $Capacity\ (ratio)$) and their respective squared terms to test if investor bounded capacity has a hump-shaped relation with management forecast. The



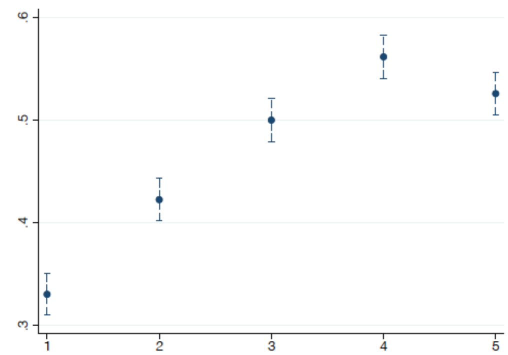
(a) Institutional Ownership ($Capacity$ ($percent$))



(b) Institutional Ownership ($Capacity$ ($Ratio$))



(c) Institutional Ownership ($Capacity$ ($percent$))



(d) Institutional Ownership ($Capacity$ ($ratio$))

Note: Figure 4 plots percentage of firms with management forecasts in year t across deciles (sub-figures a and b) and quintiles (sub-figures c and d) of investor attention which is proxied by either $Capacity$ ($percent$) (sub-figures a and c) or $Capacity$ ($ratio$) (sub-figures b and d) measured in year $t-1$. All sub-figures plot the 95% confidence interval around the mean values for each decile or quintile.

Figure 3.4: Likelihood of Management Forecast Across Deciles and Quintiles of Institutional Ownership

hump-shaped relation will be supported if: 1) the estimated coefficients of $Capacity$ ($percent$) and $Capacity$ ($ratio$) are significantly positive; 2) the estimated coefficients of the squared terms of $Capacity$ ($percent$) and $Capacity$ ($ratio$) are significantly negative. Our polynomial regression model is specified as follow:

$$MF_{i,t} = \alpha_t + \alpha_j + \beta\, Capacity_{i,t-1} + \gamma\, Capacity^2_{i,t-1} + Controls_{i,t-1} + \varepsilon_{i,t}, \qquad (3.12)$$

where α_t is year fixed effect and α_j 4-digit SIC industry fixed effect. The dependent variable $MF_{i,t}$ equals to one if a firm i makes a forecast on future earnings in year t. The variables of interest are $Capacity_{i,t}$ and $Capacity^2_{i,t-1}$. All independent variables are lagged one period relative to management forecasts. Standard errors are clustered by firm to account for potential transitory shocks that are correlated across time for a specific firm. In addition, to capture firm-level variables that can influence manager's decision to forecast, we control for firm size with *Size*, growth opportunities with *Book to Market*, leverage with *Leverage Ratio*, whether a firm reports negative earnings with *Loss*, whether a firm has an increase in earnings per share with *EPS increase*, the absolute value of the change in earnings per share with *Abs. EPS Change*.[14]

The polynomial regression results are presented in Table 3.4. Panel A reports results from estimating our polynomial model with OLS and Panel B results from a Logit regression. For both panels, we show results from the same set of six different specifications. Columns 1 and 2 on both panels estimate a univariate regression. Columns 3 and 4 include both year and industry fixed effects, which control for common time trends and persistent differences across industries, respectively. Lastly, columns 5 and 6 further control for firm-level characteristics. The estimated coefficients of control variables are generally consistent with prior literature on management forecasts.[15] The positive estimated coefficients of *EPS Increase*, *Leverage Ratio*, and *Size* suggest that well-performing, highly-levered, and large firms are more likely to issue management forecasts. Besides, the negative estimated coefficients of *Loss*, *Abs EPS Change*, and *Book to Market* imply that firms with poor and volatile financial performances and with fewer growth opportunities are less likely to issue management forecasts.

Consistent with our predicted inverse U-shaped relation, we find that the estimated coefficients of $Capacity_{i,t-1}$ are significantly positive and the coefficients of the squared term - $Capacity^2_{i,t-1}$ are significantly negative across all of our six different specifications. These formal statistical tests, along with patterns in the raw data shown in Figure 3.4, add to the credibility of our primary theoretical prediction.

[14] Please see Table 3.1 for more details on variable construction.

[15] For example: Cheng et al. 2013, Goodman et al. 2014, Li and Yang 2016, Tsang et al. 2019, Guan et al. 2020, Basu et al. (2020), and Abramova et al. (2020), etc.

Investor Attention and Management Forecast using an alternative research design

In addition to the polynomial regressions above, we adopt an alternative research design to lend further support to our theoretical prediction. More precisely, we estimate a spline regression that treats the relation between the likelihood of management forecast and investor attention as piecewise linear. In other words, we estimate a separate slope for each side of a threshold τ of investor attention as follows:

$$MF_{i,t} = \alpha_t + \alpha_j + \beta_1(Capacity_{i,t-1} - \tau < 0) + \beta_2(Capacity_{i,t-1} - \tau \geq 0) + Controls_{i,t-1} + \varepsilon_{i,t}.$$

If our theoretical prediction holds, we expect to see that the slope between likelihood of management forecast and institutional ownership to be significantly positive (negative) if institutional ownership is below (above) the threshold τ (i.e., $\beta_1 > 0$ and $\beta_2 < 0$).

However, the major challenge of estimating a spline regression is that we need first to specify the threshold τ, which our theory is silent on. We approach this challenge in two ways. Firstly, by eyeballing Figure 3.4, we conjecture that the threshold is around the 80^{th} percentile of both $Capacity(percent)$ and $Capacity(ratio)$ because the probability of management forecasts declines in the 5^{th} (9^{th} and 10^{th}) quintile (deciles). For robustness, we set $\tau = 70^{th}, 75^{th}, 80^{th}, 85^{th}$ percentile of both $Capacity(percent)$ and $Capacity(ratio)$.

Our results estimated from the spline regression are consistent with our inverse U-shaped relation prediction. Table 3.5 Panel A reports results using $Capacity(ratio)$ across four pre-specified values of τ and Panel B reports results using $Capacity(percent)$. Across four different thresholds τ and two proxies for investor attention ($Capacity(ratio)$ and $Capacity(percent)$, we find a statistically significant positive slope between management forecast and investor attention for values of investor attention that are below the thresholds τ (i.e.,$Capacity - \tau < 0$). In addition, the slope between management forecast and investor attention for values of investor attention that are above the thresholds τ is significantly negative in all specifications (i.e.,$Capacity - \tau \geq 0$).

Our second approach employs the multivariate adaptive regression spline (MARS) method, which simultaneously determines the optimal threshold τ^* and the sign of the slope on either side of τ^*. This statistical method is developed by Friedman (1991) and has been recently applied by Samuels et al. (2020) to test an inverse-U relation predicted by their model. The primary advantage of MARS is that it does not require a pre-specified threshold τ by researchers. Instead, MARS searches for the optimal threshold τ^*, which minimizes the mean-squared errors of our spline regression model.

Again, our empirical results estimated from the MARS method are consistent with our theoretical prediction and are reported in Table 3.5 Panel C. We report results using $Capacity(ratio)$ as a proxy in column 1 and $Capacity(percent)$ in column 2. First, the optimal threshold τ^* that minimizes the mean squared errors of our spline regression model corresponds to 79^{th} percentile of $Capacity(ratio)$ and 81^{th} percentile of $Capacity(percent)$. The optimal threshold τ^* matches and confirms our conjectured τ at around 80^{th} percentile from patterns in the raw data. Second, similar to our results using pre-specified thresholds τ, the slope for values of investor attention that are below (above) the estimated optimal threshold τ^* is significantly positive (negative). In other words, for values of investor attention that are below either the 79^{th} percentile of $Capacity(ratio)$ and 81^{th} percentile of $Capacity(percent)$, an increase in investor attention is associated with a higher likelihood of management forecast. In contrast, for values of investor attention that are above the estimated optimal threshold τ^*, investor attention is negatively associated with managers' likelihood of making a forecast.

Three Types of Institutional Investor Attention and Management Forecast

Our empirical tests above provide robust evidence that the likelihood of management forecast has an inverse U-shaped relation with institutional investor attention. To paint a more granular picture of the roles played by different types of institutional investors, we follow Bushee and Noe (2000) to classify institutional investors into one the three categories: quasi-indexers, transient investors, and dedicated investors.[16]

[16] Bushee (1998) and Bushee and Noe (2000) use principal component analysis to construct factors that capture institutional investors' average size of stake in their portfolio firms and degree of portfolio turnover. Similar

Similar to our graphical analysis above, we start by plotting the probability of management forecasts across ten deciles of each of the three types of institutional investor ownership. Sub-figure a of figure 3.5 sorts firms by quasi-indexers' ownership, sub-figure b by transient investors, and sub-figure c by dedicated investors. The probability of management forecast is positively associated with all three types of institutional ownership for low levels of institutional ownership. In particular, the likelihood of management forecasts is monotonically increasing in the bottom eight deciles of quasi-indexers' ownership. Our result on quasi-indexers is consistent with the finding in the literature that quasi-indexers have a strong preference for management forecasts and firms cater to quasi-indexers' demands.[17] In addition, we document three novel associations. First, the probability of management forecast declines in the 9^{th} and 10^{th} deciles of ownership by quasi-indexers, suggesting that sufficiently high levels of quasi-indexers' ownership reduces managers' incentives to forecast. Second, the probability of management forecast does not respond to changes in transient investors' ownership in the top eight deciles. Third, the probability of management forecast increases in the bottom five deciles of dedicated investors' ownership and declines thereafter, which is a clear inverse U-shaped relation consistent with our theoretical prediction.

Lastly, we provide formal statistical tests on the relation between management forecast and different types of institutional investor ownership. We re-estimate equation (3.12) by replacing $Capacity_{i,t-1}$ with each of the three types of institutional investor ownership at $t-1$. Table 3.6 presents the results from our regressions with the full-set of firm controls as well as year and industry fixed effects. The main takeaway is that: while all three types of institutional

institutional investors are grouped together into one of the three clusters: dedicated, quasi-indexers, and transient investors. Dedicated investors generally have significant stakes in a small number of firms and hold their stakes for a long period of time. Quasi-indexers consist of passive index funds and active funds that have a diverse portfolio of companies, trade infrequently, and closely benchmark against indexes. Lastly, transient investors trade frequently on a select of firms, and they use short-run strategies (Basu et al., 2020).

[17] Relevant papers in the literature include: Boone and White (2015), Bird and Karolyi (2016), Schoenfeld (2017), Basu et al. (2020), and Abramova et al. (2020), etc). Quasi-indexers generally hold a well-diversified portfolio and hence, face enormous costs in collecting private information on their portfolio firms. In addition, quasi-indexers' tracking strategies limit their ability to trade on private information. Consequently, quasi-indexers demand higher firm transparency with more public disclosures, which reduces the information asymmetry between them and their portfolio firms and lowers the costs of monitoring portfolio firms.

investor ownership are positively associated with management forecasts, the inverse U-shaped relation is primarily driven by dedicated investor ownership (i.e., the estimated coefficient of $Dedicated^2$ is statistically significant at 1% level and with the highest magnitude).[18]

3.7 Conclusion

Inattention is a complex behavioral constraint that can, in its application to capital markets, restrict how much information is incorporated into price. In this study, we examine how investor inattention affects strategic withholding in a standard model of voluntary disclosure. Inattention is jointly determined with disclosure choices. On the one hand, inattention alters how prices respond to disclosure and can either increase or decrease incentives to withhold. On the other hand, investors allocate their attention as a function of their expectations in the disclosure game. Our primary result is that disclosure first increases and then decreases in investors' attention capacity. We also show how the informativeness of disclosures as perceived by market participants changes as a function of attention capacity and market frictions.

Our model presents only first steps into the role of inattention, when reading through the lens of an otherwise generic disclosure theory. This presents advantages and disadvantages. The advantage is that the general properties of these models are well-understood with perfect attention. Hence, we can easily observe the incremental effect of inattention in a manner that extends existing insights. A disadvantage is that our model only intends to develop one applied setting of inattention, but disclosure models, on their own, do not aim to represent all forms of communication, in particular regulated and audited financial reports.

Having noted these, many questions are left open for future research in manners that would, likely, not require a model of voluntary disclosure. As an example, further research may consider the role of enforcement and its effect on investor attention. In particular, whether

[18] Consistent with the apparent inverse U-shaped relation in the raw data, the estimated coefficient of the linear term *Dedicated* is not statistically significant. This insignificant result is in line with Abramova et al. (2020) that attention by non-passive investors (i.e., investors other than quasi-indexers) does not have a significant impact on whether firms make a forecast.

enforcement may allocate attention away from the manipulative activities (Schantl and Wagenhofer 2020). We also do not know the interactions between mandatory and voluntary disclosure (Einhorn 2005) in the context of finite attention capacity. Finally, while our primary purpose has been to present the theory and offer some tentative empirical facts, more empirical tests are required to validate the theory. Inattention, even "rational" inattention, violates semi-strong market efficiency in that not all public information is reflected into price (Fama 1970). There is still disagreement between proponents of the efficient market hypothesis and behavioral finance as to whether such violations is significant enough, especially given that new technologies have increased how to organize massive datasets, while simultaneously allowing for broader use of statistics and machine learning to summarize information.

3.8 Appendix C

3.8.1 Proofs

Lemma 8. *For any equilibrium Γ, there exists an equivalent equilibrium Γ' such that $a_1' = t'$.*

Lemma 8, which is proved in steps below, demonstrates that we can restrict attention to equilibria in which $a_1 = t$ where the lowest element of the partition exactly coincides with no-disclosure.

Lemma 12. *The following statements hold:*

(i) Let $\tilde{x}$ be a continuous random variable on an open interval Y of $[0,1]$. Let $h(\cdot)$ be the density of $\tilde{x}$. Then for any $b \in Y$,

$$\begin{aligned}&\int_Y (x - \mathbb{E}(\tilde{x}|\tilde{x} \in Y))^2 h(x)dx \\ &> \int_{Y\cap[0,b]} (x - \mathbb{E}(\tilde{x}|\tilde{x} \in Y \cap [0,b]))^2 h(x)dx + \int_{Y\cap(b,1]} (x - \mathbb{E}(\tilde{x}|\tilde{x} \in Y \cap (b,1]))^2 h(x)dx.\end{aligned} \tag{3.13}$$

(ii) In any maximal equilibrium Γ, (ii.a) $t \in [a_1, a_2)$ and (ii.b) there exists an equivalent equilibrium Γ' such that $a'_1 = t'$ and $a'_i = a_i$ for any $i \geq 2$.

Part (i) of Lemma 12 shows that it is always strictly better for the investors to be able to choose a partition with more elements (no matter where the cutoff is).

Proof of Part (i). Let $Y_1 = Y \cap [0,b]$ and $Y_2 = Y \cap (b,1]$. Then $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. Let $E_1 = \mathbb{E}(x|x \in Y_1)$ and $E_2 = \mathbb{E}(x|x \in Y_2)$. Let $m = \int_{Y_1} h(x)dx = (\int_{Y_1} h(x)dx)/(\int_Y h(x)dx) \in (0,1)$ and $1-m = \int_{Y_2} h(x)dx = (\int_{Y_2} h(x)dx)/(\int_Y h(x)dx) \in (0,1)$. Then $\mathbb{E}(x|x \in Y) = E_1 m + E_2(1-m)$. So the LHS of Eq (3.13) can be written as

$$
\begin{aligned}
&\int_Y (x - \mathbb{E}(x|x \in Y))^2 h(x)dx \\
=&\int_Y (x - E_1 m - E_2(1-m))^2 h(x)dx \\
=&\int_Y x^2 h(x)dx - 2(E_1 m + E_2(1-m)) \int_Y xh(x)dx + (E_1 m + E_2(1-m))^2 \\
=&\int_Y x^2 h(x)dx - (E_1 m + E_2(1-m))^2.
\end{aligned}
$$

The RHS of Eq (3.13) can be written as

$$
\begin{aligned}
&\int_{Y\cap[0,b]} (x - \mathbb{E}(x|x \in Y \cap [0,b]))^2 h(x)dx + \int_{Y\cap(b,1]} (x - \mathbb{E}(x|x \in Y \cap (b,1]))^2 h(x)dx \\
=&\int_{Y_1} (x - E_1)^2 h(x)dx + \int_{Y_2} (x - E_2)^2 h(x)dx \\
=&[\int_{Y_1} x^2 h(x)dx + \int_{Y_2} x^2 h(x)dx] - 2[E_1 \int_{Y_1} xh(x)dx + E_2 \int_{Y_2} xh(x)dx] + [E_1^2 m + E_2^2(1-m)] \\
=&\int_Y x^2 h(x)dx - 2[E_1^2 m + E_2^2(1-m)] + [E_1^2 m + E_2^2(1-m)] \\
=&\int_Y x^2 h(x)dx - (E_1^2 m + E_2^2(1-m)).
\end{aligned}
$$

Because the quadratic function is strictly convex, $(mE_1 + (1-m)E_2)^2 < mE_1^2 + (1-m)E_2^2$. Hence the LHS of Eq (3.13) is strictly greater than the RHS of Eq (3.13), which completes the proof. □

We then prove (ii.a) by the following two results. First, we show that the first equilibrium cutoff a_1 is less than or equal to t.

Claim 6. *In any equilibrium, $a_1 \leq t$.*

Proof. Given an equilibrium information set $\{a_i\}_{i=1}^{I-1}$, suppose that $t < a_1$. Then the investors receive message *ND* if the firm gets no signal or a signal less than t; message v for v greater than or equal to t. Let

$$q \equiv (1-p) + p\int_{a_0}^{t} f(v)dv$$

and

$$r \equiv (1-p) + p\int_{a_0}^{a_1} f(v)dv.$$

By Eq. (3.1),

$$\begin{aligned}
P(1) &= [q/r][\mu\frac{1-p}{q} + \int_{a_0}^{t} v\frac{f(v)}{(q-(1-p))/p}dv\frac{q-(1-p)}{q}] + [(r-q)/r][\int_{t}^{a_1} v\frac{f(v)}{(r-q)/p}dv] \\
&= \mu\frac{1-p}{r} + \frac{p}{r}\int_{a_0}^{t} vf(v)dv + \frac{p}{r}\int_{t}^{a_1} vf(v)dv \\
&= \mu\frac{1-p}{r} + \frac{p}{r}\int_{a_0}^{a_1} vf(v)dv.
\end{aligned}$$

If the firm changes the disclosure policy to $t' = a_1$, then the firm saves cost $c[p\int_t^{a_1} f(v)dv]$ > 0 for $p > 0$. But the first element $P(1)'$ is

$$\begin{aligned}
P(1)' &= \mu\frac{1-p}{r} + \int_{a_0}^{a_1} v\frac{f(v)}{(r-(1-p))/p}dv\frac{r-(1-p)}{r} \\
&= \mu\frac{1-p}{r} + \frac{p}{r}\int_{a_0}^{a_1} vf(v)dv \\
&= P(1).
\end{aligned}$$

It is clear that the market price does not change in other partition elements under these two disclosure policies, because the firm always reveals the signal if it exists. So by Eq. (3.2), the policy t' gives the firm an expected off higher than the policy t by the amount $c[p\int_t^{a_1} f(v)dv]$. Hence it must be that $t \geq a_1$, i.e., there is no value in disclosing below a_1. □

Next, we show that the second cutoff a_2 is greater than t.

Claim 7. *In any equilibrium, $a_2 > t$.*

Proof. Fix an equilibrium information set $\{a_i\}_{i=1}^{I-1}$. Suppose that $a_2 \leq t$. Consider another information set $\{a'_i\}_{i=1}^{I-1}$ such that $a'_i = a_{i+1}$ for $i = 1, \ldots, I-2$ and $a'_{I-1} > a'_{I-2} = a_{I-1}$ for some $a_{I-1} < a'_{I-1} < \infty$. We claim that the investors will get a strictly higher expected payoff from $\{a'_i\}_{i=1}^{I-1}$ than $\{a_i\}_{i=1}^{I-1}$ given the firm's disclosure cutoff t. Let $a_j \leq t < a_{j+1}$ for $j \geq 2$.

It is clear that the price in the elements $k = j+1, \ldots, I-2$ of $\{a'_i\}_{i=1}^{I-1}$ or $k+1 = j+2, \ldots, I-1$ of $\{a_i\}_{i=1}^{I-1}$ satisfy $\hat{P}(k)' = (\int_{a'_{k-1}}^{a'_k} vf(v)dv)/(\int_{a'_{k-1}}^{a'_k} f(v)dv) = (\int_{a_k}^{a_{k+1}} vf(v)dv)/(\int_{a_k}^{a_{k+1}} f(v)dv) = \hat{P}(k+1)$. So the market will respond with the same price for values between $a'_j = a_{j+1} > t$ and $a'_{I-2} = a_{I-1}$ when the firm gets a signal. Then by Eq (3.2), the investors' expected payoff will be the same in this case.

Furthermore, when the firm gets a signal, the value will be revealed only if $v \geq t$. Since $a_j \leq t$, the investors are able to distinguish the firm's signal (if revealed) from *ND* under $\{a_i\}_{i=1}^{I-1}$. So the price $\hat{P}(1)$ for the disclosure *ND* (under $\{a_i\}_{i=1}^{I-1}$) is determined by

$$\begin{aligned}
\hat{P}(1) &= \mu\frac{1-p}{(1-p)+p\int_0^t f(v)dv} + \int_0^t v\frac{f(v)}{\int_0^t f(v)dv}dv\frac{p\int_0^t f(v)dv}{(1-p)+p\int_0^t f(v)dv} \\
&= \frac{1-p}{(1-p)+p\int_0^t f(v)dv}\mu + \frac{p}{(1-p)+p\int_0^t f(v)dv}\int_0^t vf(v)dv.
\end{aligned}$$

Since $j \geq 2$, $a'_1 \leq a'_{j-1} = a_j \leq t$. Then the investors are able to distinguish the firm's signal from *ND* under $\{a'_i\}_{i=1}^{I-1}$ as well and the price $\hat{P}(1)'$ for the disclosure *ND* is exactly the same as $\hat{P}(1)$ because of the same expression. Hence the market will respond with price $\hat{P}(1)' = \hat{P}(1)$ if the firm's signal value is below t or if there is no signal. Then the investors' expected payoff will be the same as well.

Let us now consider the cases where $t \leq v < a'_j = a_{j+1}$ and the firm gets a signal. Since

$a'_{j-1} = a_j \leq t < a_{j+1} = a'_j$, the price $\hat{P}(j+1)$ and $\hat{P}(j)'$ are determined by

$$\hat{P}(j+1) = \int_t^{a_{j+1}} v \frac{f(v)}{\int_t^{a_{j+1}} f(v)dv} dv = \int_t^{a'_j} v \frac{f(v)}{\int_t^{a'_j} f(v)dv} dv = \hat{P}(j)'.$$

So the market will respond with price $\hat{P}(j)' = \hat{P}(j+1)$ when the firm's signal is between t and $a'_j = a_{j+1}$. Then the investors' expected payoff is still the same in this case.

Finally, investors will get a strictly higher expected payoff for signals greater than $a'_{I-2} = a_{I-1}$ by Lemma 12 (i). Because the distribution f has a positive measure in this region, the investors can do strictly better from $\{a'_i\}_{i=1}^{I-1}$ than $\{a_i\}_{i=1}^{I-1}$, which contradicts the equilibrium assumption. Therefore we conclude that $a_2 > t$. □

Part (ii.b) of Lemma 12 implies that every equilibrium outcome can be supported by a strategy profile involving $t = a_1$. Hence it is without loss of generality to restrict our attention to monotonic equilibria in which $t = a_1$.

Proof of Part (ii.b). By Claims 6 and 7, the equilibrium disclosure policy satisfies $a_1 \leq t < a_2$. We show that if there is an equilibrium in which $a_1 < t < a_2$, there is another equilibrium in which $a'_1 = t$ with everything else the same. Consider the proposed strategy of investors $\{a'_i\}_{i=1}^{I-1}$ such that $a'_1 = t$ and $a'_j = a_j$ for $j = 2, \ldots, I-1$.

First, we show that the firm has no incentive to deviate from t given $\{a'_i\}_{i=1}^{I-1}$. Because $a'_1 = t$, the firm does not want to disclose more, i.e. choosing a lower cutoff, by the similar argument as the proof of Lemma 6. If the firm gains by choosing a larger cutoff, it is then greater than a'_1 and also a_1. Note that the firm gets the same expected payoff from t given the two information sets, because $a_1 < t < a_2$ and $a'_1 = t < a'_2 = a_2$. Moreover, the firm still gets the same expected payoff from any cutoff t' greater than t, because $a_1 < a'_1 < t'$ (*ND* is sent if there is no signal or the value is less than t' and the two information sets only differ in the first cutoff). So if there is a profitable deviation to a larger disclosure cutoff under $\{a'_i\}_{i=1}^{I-1}$, there must be a profitable deviation to a larger disclosure cutoff under $\{a_i\}_{i=1}^{I-1}$, which contradicts the equilibrium assumption. Hence the firm has no incentive to deviate from t.

Next, we show that the investors have no incentive to deviate from $\{a_i'\}_{i=1}^{I-1}$ given t. It is clear that the investors get the same expected payoff from $\{a_i'\}_{i=1}^{I-1}$ and from $\{a_i\}_{i=1}^{I-1}$ given t. So if investors have a profitable deviation from $\{a_i'\}_{i=1}^{I-1}$, they must also have a profitable deviation from $\{a_i\}_{i=1}^{I-1}$, which contradicts the equilibrium assumption. Hence the investors have no incentive to deviate from $\{a_i'\}_{i=1}^{I-1}$. Therefore, we have shown that $(t, \{a_i'\}_{i=1}^{I-1})$ is an equilibrium strategy profile. □

Lemma 9. *A solution* $\{a_i^\dagger\}$ *to program* (K_0) *satisfies*

$$a_i^\dagger = \frac{\mathbb{E}[\tilde{v}|a_i^\dagger \leq \tilde{v} < a_{i+1}^\dagger] + \mathbb{E}[\tilde{v}|a_{i-1}^\dagger \leq \tilde{v} < a_i^\dagger]}{2} \tag{3.4}$$

for $i = 2, \ldots, I-1$.

Proof. Assume that the firm will fully disclose the signal when being perfectly informed. The investors minimize the ex-ante loss function given by Eq (3.3). The objective function is rewritten below:

$$\begin{aligned} &p\sum_{i=2}^{I}\int_{\hat{a}_{i-1}}^{\hat{a}_i}(v-\mathbb{E}[\tilde{v}|\hat{a}_{i-1}\leq\tilde{v}<\hat{a}_i])^2 f(v)dv \\ &+(1-p)\int_{\hat{a}_1}^{1}(v-P(1))^2 f(v)dv+\int_{\hat{a}_0}^{\hat{a}_1}(v-P(1))^2 f(v)dv, \end{aligned}$$

where $P(1) = \frac{pF(\hat{a}_1)\mathbb{E}(\tilde{v}|\hat{a}_0\leq\tilde{v}<\hat{a}_1)+(1-p)\mathbb{E}(\tilde{v})}{pF(\hat{a}_1)+(1-p)}$, $\hat{a}_0 = 0$, and $\hat{a}_I = 1$. The conditional expectation is given by $\mathbb{E}[\tilde{v}|a_{i-1} \leq \tilde{v} < a_i] = \int_{a_{i-1}}^{a_i} vf(v)dv / \int_{a_{i-1}}^{a_i} f(v)dv$. Then the first term of Equation (3.3) can be written as

$$\begin{aligned} &p\sum_{i=2}^{I}\int_{\hat{a}_{i-1}}^{\hat{a}_i}(v-\mathbb{E}[\tilde{v}|\hat{a}_{i-1}\leq\tilde{v}<\hat{a}_i])^2 f(v)dv \\ =&p\sum_{i=2}^{I}[\int_{\hat{a}_{i-1}}^{\hat{a}_i}v^2 f(v)dv-2\,\mathbb{E}[\tilde{v}|\hat{a}_{i-1}\leq\tilde{v}<\hat{a}_i]\int_{\hat{a}_{i-1}}^{\hat{a}_i}vf(v)dv+\mathbb{E}[\tilde{v}|\hat{a}_{i-1}\leq\tilde{v}<\hat{a}_i]^2\int_{\hat{a}_{i-1}}^{\hat{a}_i}f(v)dv] \\ =&p\sum_{i=2}^{I}[\int_{\hat{a}_{i-1}}^{\hat{a}_i}v^2 f(v)dv-2(\int_{\hat{a}_{i-1}}^{\hat{a}_i}vf(v)dv)^2/(\int_{\hat{a}_{i-1}}^{\hat{a}_i}f(v)dv)+(\int_{\hat{a}_{i-1}}^{\hat{a}_i}vf(v)dv)^2/(\int_{\hat{a}_{i-1}}^{\hat{a}_i}f(v)dv)] \\ =&p\sum_{i=2}^{I}[\int_{\hat{a}_{i-1}}^{\hat{a}_i}v^2 f(v)dv-(\int_{\hat{a}_{i-1}}^{\hat{a}_i}vf(v)dv)^2/(\int_{\hat{a}_{i-1}}^{\hat{a}_i}f(v)dv)] \end{aligned}$$

The cutoff $\hat{a}_i$ for $i = 2, \ldots, I-1$ only appears in the first term of Equation (3.3). Given the other cutoffs, each $\hat{a}_i$ ($i = 2, \ldots, I-1$) minimizes

$$\int_{\hat{a}_{i-1}}^{\hat{a}_i} v^2 f(v)dv - (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)^2/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)$$
$$+\int_{\hat{a}_i}^{\hat{a}_{i+1}} v^2 f(v)dv - (\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)^2/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv).$$

Assume that all terms are differentiable and the conditions for Dominated Convergence Theorem are satisfied. So we can take first order condition with respect to $\hat{a}_i$ ($i = 2, \ldots, I-1$) and interchange derivatives and integrals. By Leibniz integral rule,

$$\begin{aligned}
&\hat{a}_i^2 f(\hat{a}_i) - [2(\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)(\hat{a}_i f(\hat{a}_i))/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv) - (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)^2(f(\hat{a}_i))/\\
&(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)^2] + (-\hat{a}_i^2 f(\hat{a}_i)) - [-2(\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)(\hat{a}_i f(\hat{a}_i))/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv)\\
&+(\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)^2(f(\hat{a}_i))/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv)^2]\\
=&2\hat{a}_i f(\hat{a}_i)[(\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv) - (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)]\\
&+f(\hat{a}_i)[((\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv))^2 - ((\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv))^2]\\
=&f(\hat{a}_i)[(\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv) - (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)]\\
&[2\hat{a}_i - (\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv) - (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)] = 0.
\end{aligned}$$

Because $f(a_i) > 0$ for all $a_i \in [0,1]$ and $(\int_{a_i}^{a_{i+1}} vf(v)dv)/(\int_{a_i}^{a_{i+1}} f(v)dv) = \mathbb{E}[\tilde{v}|a_i \le \tilde{v} < a_{i+1}] > \mathbb{E}[\tilde{v}|a_{i-1} \le \tilde{v} < a_i] = (\int_{a_{i-1}}^{a_i} vf(v)dv)/(\int_{a_{i-1}}^{a_i} f(v)dv)$, the optimal $\hat{a}_i$ ($i = 2, \ldots, I-1$) satisfies

$$\begin{aligned}
\hat{a}_i &= [(\int_{\hat{a}_i}^{\hat{a}_{i+1}} vf(v)dv)/(\int_{\hat{a}_i}^{\hat{a}_{i+1}} f(v)dv) + (\int_{\hat{a}_{i-1}}^{\hat{a}_i} vf(v)dv)/(\int_{\hat{a}_{i-1}}^{\hat{a}_i} f(v)dv)]/2\\
&= (\mathbb{E}[\tilde{v}|\hat{a}_i \le \tilde{v} < \hat{a}_{i+1}] + \mathbb{E}[\tilde{v}|\hat{a}_{i-1} \le \tilde{v} < \hat{a}_i])/2.
\end{aligned}$$

for $i = 2, \ldots, I-1$. Next, we derive the solution to $\hat{a}_1$ that are involved in both the endpoints of integration and $P(1)$. The terms that involve $\hat{a}_1$ are

$$
\begin{aligned}
& p\int_{\hat{a}_1}^{\hat{a}_2}(v-\mathbb{E}[\tilde{v}|\hat{a}_1 \leq \tilde{v} < \hat{a}_2])^2 f(v)dv + (1-p)\int_{\hat{a}_1}^{1}(v-P(1))^2 f(v)dv \\
& + \int_{\hat{a}_0}^{\hat{a}_1}(v-P(1))^2 f(v)dv \\
= & p[\int_{\hat{a}_1}^{\hat{a}_2} v^2 f(v)dv - (\int_{\hat{a}_1}^{\hat{a}_2} vf(v)dv)^2/(\int_{\hat{a}_1}^{\hat{a}_2} f(v)dv)] + (1-p)\int_{\hat{a}_1}^{1}(v-P(1))^2 f(v)dv \\
& + \int_{\hat{a}_0}^{\hat{a}_1}(v-P(1))^2 f(v)dv.
\end{aligned}
$$

We take first order condition with respect to $\hat{a}_1$:

$$
\begin{aligned}
& p(-\hat{a}_1^2 f(\hat{a}_1)) - p[-2(\int_{\hat{a}_1}^{\hat{a}_2} vf(v)dv)(\hat{a}_1 f(\hat{a}_1))/(\int_{\hat{a}_1}^{\hat{a}_2} f(v)dv) + (\int_{\hat{a}_1}^{\hat{a}_2} vf(v)dv)^2(f(\hat{a}_1))/ \\
& (\int_{\hat{a}_1}^{\hat{a}_2} f(v)dv)^2] + (1-p)[-(\hat{a}_1 - P(1))^2 f(\hat{a}_1) + \int_{\hat{a}_1}^{1} 2f(v)(v-P(1))(-\frac{\partial}{\partial \hat{a}_1}P(1))dv] + \\
& (\hat{a}_1 - P(1))^2 f(\hat{a}_1) + \int_{\hat{a}_0}^{\hat{a}_1} 2f(v)(v-P(1))(-\frac{\partial}{\partial \hat{a}_1}P(1))dv \\
= & -p\hat{a}_1^2 f(\hat{a}_1) + p(\int_{\hat{a}_1}^{\hat{a}_2} vf(v)dv f(\hat{a}_1)/\int_{\hat{a}_1}^{\hat{a}_2} f(v)dv)[2\hat{a}_1 - \int_{\hat{a}_1}^{\hat{a}_2} vf(v)dv/\int_{\hat{a}_1}^{\hat{a}_2} f(v)dv] \\
& + p(\hat{a}_1 - P(1))^2 f(\hat{a}_1) - \frac{\partial}{\partial \hat{a}_1}P(1)[(1-p)\int_{\hat{a}_1}^{1} 2(v-P(1))f(v)dv \\
& + \int_{0}^{\hat{a}_1} 2(v-P(1))f(v)dv] = 0,
\end{aligned}
$$

where $P(1) = \frac{pF(\hat{a}_1)\mathbb{E}(\tilde{v}|0 \leq \tilde{v} < \hat{a}_1) + (1-p)\mathbb{E}(\tilde{v})}{pF(\hat{a}_1) + (1-p)}$ and

$$
\frac{\partial}{\partial \hat{a}_1}P(1) = \frac{p^2 f(\hat{a}_1)F(\hat{a}_1)(\hat{a}_1 - \mathbb{E}(\tilde{v}|0 \leq \tilde{v} < \hat{a}_1)) + (1-p)pf(\hat{a}_1)(\hat{a}_1 - \mathbb{E}(\tilde{v}))}{(pF(\hat{a}_1) + (1-p))^2}.
$$

Hence the interior solution is characterized by Equation (3.5). By continuity of the loss function, the minimum either attains at the interior solution where the first order condition holds or at 0, because $\hat{a}_1 = 1$ is clearly dominated by $\hat{a}_1 = 0$ given Equation (3.4). Therefore, for the ideal information set of investors, Equation (3.4) is satisfied, and either Equation (3.5) or the corner solution $\hat{a}_1 = 0$ holds. □

Proposition 16. *When $\tilde{v}$ is uniformly distributed, all equilibrium information structures induce monotone partitions on the state space.*

Proof. We show that in the optimal information set, the set that induces any price must be connected with Lebesgue measure one. Suppose that price p is induced by disclosures in intervals $(k-n,k)$ and $(k+x,k+x+m)$, where $m,n>0$, and $x \geq 0$. The price formed by rational expectation is $p=(k-n+k+k+x+k+x+m)/4 = k+(m-n)/4+x/2$. We show that x must be zero in the optimal information set. The expected pricing error from these intervals is given by

$$\begin{aligned}
&\int_{k-n}^{k}(v-p)^2dv+\int_{k+x}^{k+x+m}(v-p)^2dv\\
=&\frac{1}{3}[v-(k+(m-n)/4+x/2)]^3|_{k-n}^{k}+\frac{1}{3}[v-(k+(m-n)/4+x/2)]^3|_{k+x}^{k+x+m}\\
=&\frac{1}{3}[-((m-n)/4+x/2)^3+(n+(m-n)/4+x/2)^3\\
&+(x+m-((m-n)/4+x/2))^3)-(x-((m-n)/4+x/2))^3)]\\
=&\frac{1}{3}[-((m-n)/4+x/2)^3+((m-n)/4+x/2)^3+3((m-n)/4+x/2)^2n\\
&+3((m-n)/4+x/2)n^2+n^3+(x-((m-n)/4+x/2))^3\\
&+3(x-((m-n)/4+x/2))^2m+3(x-((m-n)/4+x/2))m^2+m^3\\
&-(x-((m-n)/4+x/2))^3]\\
=&\frac{1}{3}[3((m-n)/4+x/2)n((m-n)/4+x/2+n)\\
&+3(x-((m-n)/4+x/2))m(x-((m-n)/4+x/2)+m)+m^3+n^3]\\
=&(\frac{x}{2}+\frac{m-n}{4})(\frac{x}{2}+\frac{m+3n}{4})n+(\frac{x}{2}-\frac{m-n}{4})(\frac{x}{2}+\frac{3m+n}{4})m+\frac{1}{3}(m^3+n^3)\\
=&\frac{m+n}{4}x^2+\frac{(m+n)^2}{4}x-\frac{3(m-n)^2(m+n)}{16}+\frac{1}{3}(m^3+n^3).
\end{aligned}$$

It is then clear that x should be minimized at zero. Furthermore, the pricing error from these intervals that induce the same price is strictly increasing in x. Hence if there is a set that is not connected in the state space, we can always permute the intervals so that each set is connected, which reduces the expected pricing error. □

Lemma 13 demonstrates that types withholding their signals in a monotonic equilibrium are the lowest ones, which is useful in proving Proposition 18.

Lemma 13. *If a type v induces the nondisclosure price in a monotonic equilibrium, then the nondisclosure price is induced by all types below v as well.*

Proof. Suppose that a type $v' < v$ induces a price p' different than the nondisclosure price in a monotonic equilibrium. The nondisclosure price cannot be lower than p' in a monotonic equilibrium, because the nondisclosure price is induced by higher informed types and all uninformed types. If the nondisclosure price is higher than p', then the type v' will have a strict incentive to deviate to nondisclosure, which contradicts the equilibrium definition. So the nondisclosure price is induced by all types below v in equilibrium. □

Proposition 18. *The equilibrium with the lowest disclosure cutoff gives investors the highest expected payoff over all equilibria that induce interval partitions.*

Proof. We focus on equilibria with interval structures in which the induced price is the same in each interval. Let t be the highest type in the maximal equilibrium that induces the nondisclosure price.[19] Suppose that investors gain a strictly higher payoff in another equilibrium with a higher cutoff type $t' > t$. Because any type above t will disclose the signal if being informed in the maximal equilibrium by Lemma 13 and $t' > t$, the investor can raise a_1 and induce the same information set as the "better" equilibrium, which would generate a strictly higher payoff than the maximal equilibrium by the hypothesis. The profitable deviation implies that the maximal "equilibrium" strategy profile is actually not an equilibrium. Hence the maximal equilibrium (equilibrium with the lowest cutoff) is optimal among all equilibria from the perspective of investors. □

[19] If there is no disclosure cost (but some probability that the firm is not informed), some types below t might disclose *but would still induce the nondisclosure price*. They are indifferent between disclosure or not because the same price is induced. If there is a positive disclosure cost, no type below t will disclose her signal.

Lemma 14. *If Inequality (3.10) does not hold, the equilibrium disclosure cutoff t is greater than $a_1^\dagger$.*

Proof. By Lemma 8, we focus on the equilibrium in which $t = a_1$ without loss of generality. Suppose by contradiction that Inequality (3.10) does not hold in equilibrium, i.e.,

$$\mathbb{E}(\tilde{v}|\tilde{v} \in [a_1^\dagger, a_2^\dagger)) - c < \frac{pF(a_1^\dagger)\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1^\dagger) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1^\dagger) + (1-p))},$$

and the disclosure cutoff t is less than or equal to $a_1^\dagger$. By Lemma 9, the investors must best respond to a cutoff no greater than $a_1^\dagger$ by choosing the information set $\{a_i^\dagger\}_{i=1}^{I-1}$. The firm will then be strictly worse off if revealing the signal in $[a_1^\dagger, a_2^\dagger)$ than concealing it by the hypothesis. The profitable deviation for the firm shows that the disclosure cutoff t must be greater than $a_1^\dagger$. □

Lemma 15. *In the maximal equilibrium, the firm is either indifferent between disclosing the signal in A_2 or withholding it, or $t = a_1^\dagger$.*

Proof. Because we restrict our attention to equilibrium with $t = a_1$, it must be that

$$\mathbb{E}(\tilde{v}|\tilde{v} \in [a_1, a_2)) - c \geq \frac{pF(a_1)\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1) + (1-p))},$$

because firm with signal in A_2 is willing to disclose. If Inequality (3.10) holds, then $t = a_1^\dagger$ by the definition of $a^\dagger$. If Inequality (3.10) does not hold, we claim that the weak inequality above must hold with equality in the *maximal* equilibrium. Suppose not. We show that there is another equilibrium with a strictly lower cutoff.

Consider an equilibrium with cutoff t and information set $\{a_i\}_{i=1}^{I-1}$ that is given by Equation (3.8). In other words, $a_2, \ldots, a_{I-1}$ are given by Equation (3.9). Because it is assumed that $\mathbb{E}(\tilde{v}|\tilde{v} \in [a_1, a_2)) - c > \frac{pF(a_1)\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1) + (1-p))}$, by continuity there are $t' < t$ arbitrarily close to t and an information set $\{a_i'\}_{i=1}^{I-1}$ satisfying $a_1' = t'$ and Equation (3.9) such that $\mathbb{E}(\tilde{v}|\tilde{v} \in [a_1', a_2')) - c \geq \frac{pF(a_1')\mathbb{E}(\tilde{v}|\tilde{v} \leq a_1') + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1') + (1-p))}$. It is clear that the firm is best responding to $\{a_i'\}_{i=1}^{I-1}$.

Further, we claim that the investors are best responding as well. By Lemma 14, the disclosure cutoff t is greater than $a_1^\dagger$ when Inequality (3.10) does not hold. Recall that $a_1^\dagger$ minimizes the investors' expected pricing error (given that $a_2,\ldots,a_{I-1}$ are determined by Equation (3.9)). By convexity, the expected pricing error is increasing in a_1 for a_1 greater than $a_1^\dagger$. Hence the investors minimize pricing error by choosing a_1' equal to t' and, together with Lemma 10, are best responding to the firm's disclosure strategy given by t'. Therefore, we construct an equilibrium with a strictly lower cutoff, implying that the original one is not the maximal equilibrium, which completes the proof. □

Lemma 16. *Suppose that Assumption 2 holds. For two sequences $a \equiv \{a_i\}_{i=0}^I$ and $a' \equiv \{a_i'\}_{i=0}^I$ ($I \geq 2$) that solve Equations* (3.9) *and* (3.11)*, $a_0 = a_0' < a_1' < a_1$ implies that $a_i < a_i'$ for all $2 \leq i \leq I$.*

Proof. First, we show that $a_0 = a_0' < a_1' < a_1$ implies $a_2 < a_2'$. Choose $\{a_i''\}_{i=0}^2$ with $a_1' = a_1'' < a_2'' = a_2$ so that Equation (3.11) holds. Then $a_1' = a_1'' < a_1$. By Assumption 2, $a_0'' < a_0$. So $a_0 = a_0' > a_0''$. It follows that

$$\begin{aligned}&\frac{1-p}{(1-p)+pF(a_1')}\mu+\frac{pF(a_1')}{(1-p)+pF(a_1')}\mathbb{E}[\tilde{v}|a_0'\leq\tilde{v}<a_1']\\>&\frac{1-p}{(1-p)+pF(a_1'')}\mu+\frac{pF(a_1'')}{(1-p)+pF(a_1'')}\mathbb{E}[\tilde{v}|a_0''\leq\tilde{v}<a_1''].\end{aligned}$$

Then $\mathbb{E}[\tilde{v}|a_1' \leq \tilde{v} < a_2'] - c > \mathbb{E}[\tilde{v}|a_1'' \leq \tilde{v} < a_2''] - c$ by Equation (3.11). Hence $a_2' > a_2'' = a_2$.

Next, we show that for $I \geq 3$, $a_i < a_i'$ for all $2 \leq i \leq I$. Suppose by way of contradiction that $a_j \geq a_j'$ for some $3 \leq j \leq I$; suppose further that j is the smallest index greater than 2 such that this inequality is satisfied, so that $a_i < a_i'$ for all i such that $2 \leq i < j$. Because $a_2' > a_2$, there must be at least one index $2 \leq i < j$ such that $a_i < a_i'$. Choose $\{\hat{a}_i''\}_{i=0}^j$ with $a_{j-1}' = \hat{a}_{j-1}'' < \hat{a}_j'' = a_j$ so that Equation (3.9) and (3.11) hold. Then $a_{j-1} < a_{j-1}' = \hat{a}_{j-1}''$ by the definition of j. By Assumption 2, $a_i < \hat{a}_i''$ for all $0 \leq i \leq j-1$. Furthermore, $\hat{a}_j'' = a_j \geq a_j'$ and

$a'_{j-1} = \hat{a}''_{j-1}$ by the assumption and definition. Then by Equation (3.9),

$$\begin{aligned}\mathbb{E}[v|a'_{j-2} \leq v < a'_{j-1}] =& 2a'_{j-1} - \mathbb{E}[v|a'_{j-1} \leq v < a'_j] \\ \geq& 2\hat{a}''_{j-1} - \mathbb{E}[v|\hat{a}''_{j-1} \leq v < \hat{a}''_j] \\ =& \mathbb{E}[v|\hat{a}''_{j-2} \leq v < \hat{a}''_{j-1}],\end{aligned}$$

which implies that $a'_{j-2} \geq \hat{a}''_{j-2}$. So $a'_i \geq \hat{a}''_i$ for all $0 \leq i \leq j-2$ by Assumption 2.[20] Hence $a'_i \geq \hat{a}''_i > a_i$ for all $0 \leq i \leq j-2$, particularly $a'_1 > a_1$, which leads to a contradiction. □

Let $\{\bar{a}_i\}_{i=0}^{I}$ be the sequence that satisfies Equation (3.9) and (3.11). When the cost c is small or probability of being informed p is large, the investors choose the information structure $\{a_i^\dagger\}_{i=1}^{I-1}$; When c is large or p is small, the equilibrium partition is given by $\{\bar{a}_i\}_{i=1}^{I-1}$. The following lemmas are useful to prove Proposition 19. Lemma 17 shows that $\bar{a}_i$ increases in c for $i = 1, \ldots, I-1$. We consider two sequences with cost $c > c'$ and show that $a_i^c > a_i^{c'}$ for all $i = 1, \ldots, I-1$ by contradiction.

Lemma 17. *Given I and p, $\bar{a}_1$ of the sequence $\{\bar{a}_i\}_{i=0}^{I}$ is strictly increasing in c.*

Proof. Suppose not. Then there exists $c > c'$ such that $\bar{a}_{I-1}(c) < \bar{a}_{I-1}(c')$. This is because otherwise $\bar{a}_1(c) > \bar{a}_1(c')$ by Assumption 2. Let $\{\bar{a}'_i(c')\}_{i=0}^{I}$ be another sequence that satisfies $\mathbb{E}[\tilde{v}|\bar{a}'_1(c') \leq \tilde{v} < \bar{a}'_2(c')] - c' = \frac{1-p}{(1-p)+pF(\bar{a}'_1(c'))}\mu + \frac{pF(\bar{a}'_1(c'))}{(1-p)+pF(\bar{a}'_1(c'))}\mathbb{E}[\tilde{v}|\bar{a}'_0(c') \leq \tilde{v} < \bar{a}'_1(c')]$ and $\bar{a}'_i(c') = \bar{a}_i(c)$ for $i = 1, \ldots, I$. Then $\bar{a}'_i(c') < \bar{a}_i(c')$ for all $0 \leq i \leq I-1$ by Assumption 2, particularly $\bar{a}'_0(c') < \bar{a}_0(c') = \bar{a}_0(c)$. But observe that

[20] This follows immediately by Assumption 2 if $a'_{j-2} > \hat{a}''_{j-2}$. If $a'_{j-2} = \hat{a}''_{j-2}$, $a'_i = \hat{a}''_i$ for all $i \leq j-1$ by a straightforward induction argument on Equation (3.9) and the continuity assumption about prior density.

$$
\begin{aligned}
&\mathbb{E}[\tilde{v}|\bar{a}_0'(c') \leq \tilde{v} < \bar{a}_1(c)] \\
=&\mathbb{E}[\tilde{v}|\bar{a}_0'(c') \leq \tilde{v} < \bar{a}_1'(c')] \\
=&(\mathbb{E}[\tilde{v}|\bar{a}_1'(c') \leq \tilde{v} < \bar{a}_2'(c')] - c' - \frac{1-p}{(1-p)+pF(\bar{a}_1'(c'))}\mu)/\frac{pF(\bar{a}_1'(c'))}{(1-p)+pF(\bar{a}_1'(c'))} \\
>&(\mathbb{E}[\tilde{v}|\bar{a}_1'(c') \leq \tilde{v} < \bar{a}_2'(c')] - c - \frac{1-p}{(1-p)+pF(\bar{a}_1'(c'))}\mu)/\frac{pF(\bar{a}_1'(c'))}{(1-p)+pF(\bar{a}_1'(c'))} \\
=&(\mathbb{E}[\tilde{v}|\bar{a}_1(c) \leq \tilde{v} < \bar{a}_2(c)] - c - \frac{1-p}{(1-p)+pF(\bar{a}_1(c))}\mu)/\frac{pF(\bar{a}_1(c))}{(1-p)+pF(\bar{a}_1(c))} \\
=&\mathbb{E}[\tilde{v}|\bar{a}_0(c) \leq \tilde{v} < \bar{a}_1(c)],
\end{aligned}
$$

where the first and the third equalities follow from the construction $\bar{a}_i'(c') = \bar{a}_i(c)$ for $i = 1, \ldots, I-1$, the second and the fourth equalities follow from the definition of the sequences, and the inequality follows from $c > c'$. Then $\bar{a}_0'(c') > \bar{a}_0(c)$, which implies a contradiction. Hence $\bar{a}_i(c) > \bar{a}_i(c')$ for $1 \leq i \leq I-1$ by Assumption 2. □

The next two lemmas show that $\bar{a}_i$ decreases in p for $i = 1, \ldots, I-1$. When the probability of getting a signal is small, the firm withholds the bad signal as if no signal was received, similar to the intuition in Dye (1985) and Jung and Kwon (1988). When the probability of getting a signal is large, the firm knows that no disclosure will be interpreted as an extremely bad signal and hence would like to disclose more. But anticipating that investors will not choose the first cutoff to be lower than $a_1^\dagger$, the firm will set the disclosure cutoff t exactly to be $a_1^\dagger$ to save the cost.

Lemma 18. *For any sequence $\{\bar{a}_i\}_{i=0}^{I}$ with $a_0 = 0$ and $a_I = 1$ such that Equations* (3.9) *and* (3.11) *hold,* $\mathbb{E}[\tilde{v}|\bar{a}_1 \leq \tilde{v} < \bar{a}_2] - c < \mu$.

Proof. This lemma follows directly from Equation (3.11). Observe that

$$
\begin{aligned}
\mu &= \mathbb{E}[\tilde{v}|\bar{a}_0 \leq \tilde{v} < \bar{a}_1]F(\bar{a}_1) + \mathbb{E}[\tilde{v}|\bar{a}_1 \leq \tilde{v} \leq \bar{a}_I](1 - F(\bar{a}_1)) \\
&> \mathbb{E}[\tilde{v}|\bar{a}_0 \leq \tilde{v} < \bar{a}_1]F(\bar{a}_1) + \mathbb{E}[\tilde{v}|\bar{a}_0 \leq \tilde{v} < \bar{a}_1](1 - F(\bar{a}_1)) = \mathbb{E}[\tilde{v}|\bar{a}_0 \leq \tilde{v} < \bar{a}_1]
\end{aligned}
$$

under the continuous distribution F with strictly positive density everywhere. Hence we must have

$$\begin{aligned}\mathbb{E}[\tilde{v}|\bar{a}_1 \leq \tilde{v} < \bar{a}_2] - c &= \frac{1-p}{(1-p)+pF(\bar{a}_1)}\mu + \frac{pF(\bar{a}_1)}{(1-p)+pF(\bar{a}_1)}\mathbb{E}[\tilde{v}|\bar{a}_0 \leq \tilde{v} < \bar{a}_1] \\ &< \frac{1-p}{(1-p)+pF(\bar{a}_1)}\mu + \frac{pF(\bar{a}_1)}{(1-p)+pF(\bar{a}_1)}\mu = \mu.\end{aligned}$$

□

Lemma 19. *Given I and c, $\bar{a}_1$ of the sequence $\{\bar{a}_i\}_{i=0}^{I}$ is strictly decreasing in p.*

Proof. Suppose not. Then there exists $p > p'$ such that $\bar{a}_{I-1}(p) > \bar{a}_{I-1}(p')$. This is because otherwise $\bar{a}_1(p) < \bar{a}_1(p')$ by Assumption 2. Let $\{\bar{a}'_i(p')\}_{i=0}^{I}$ be another sequence that satisfies $\mathbb{E}[\tilde{v}|\bar{a}'_1(p') \leq \tilde{v} < \bar{a}'_2(p')] - c = \frac{1-p'}{(1-p')+p'F(\bar{a}'_1(p'))}\mu + \frac{p'F(\bar{a}'_1(p'))}{(1-p')+p'F(\bar{a}'_1(p'))}\mathbb{E}[\tilde{v}|\bar{a}'_0(p') \leq \tilde{v} < \bar{a}'_1(p')]$ and $\bar{a}'_i(p') = \bar{a}_i(p)$ for $i = 1,\ldots,I$. Then $\bar{a}'_i(p') > \bar{a}_i(p')$ for all $0 \leq i \leq I-1$ by Assumption 2, particularly $\bar{a}'_0(p') > \bar{a}_0(p') = \bar{a}_0(p)$. But observe that

$$\begin{aligned}
&\mathbb{E}[\tilde{v}|\bar{a}'_0(p') \leq \tilde{v} < \bar{a}_1(p)] \\
=&\mathbb{E}[\tilde{v}|\bar{a}'_0(p') \leq \tilde{v} < \bar{a}'_1(p')] \\
=&(\mathbb{E}[\tilde{v}|\bar{a}'_1(p') \leq \tilde{v} < \bar{a}'_2(p')] - c - \frac{1-p'}{(1-p')+p'F(\bar{a}'_1(p'))}\mu)/\frac{p'F(\bar{a}'_1(p'))}{(1-p')+p'F(\bar{a}'_1(p'))} \\
=&(\mathbb{E}[\tilde{v}|\bar{a}'_1(p') \leq \tilde{v} < \bar{a}'_2(p')] - c - [1 - \frac{p'F(\bar{a}'_1(p'))}{(1-p')+p'F(\bar{a}'_1(p'))}]\mu)/\frac{p'F(\bar{a}'_1(p'))}{(1-p')+p'F(\bar{a}'_1(p'))} \\
=&(\mathbb{E}[\tilde{v}|\bar{a}'_1(p') \leq \tilde{v} < \bar{a}'_2(p')] - c - \mu)/(\frac{p'F(\bar{a}'_1(p'))}{(1-p')+p'F(\bar{a}'_1(p'))}) + \mu \\
<&(\mathbb{E}[\tilde{v}|\bar{a}'_1(p') \leq \tilde{v} < \bar{a}'_2(p')] - c - \mu)/(\frac{pF(\bar{a}'_1(p'))}{(1-p)+pF(\bar{a}'_1(p'))}) + \mu \\
=&(\mathbb{E}[\tilde{v}|\bar{a}_1(p) \leq \tilde{v} < \bar{a}_2(p)] - c - \mu)/(\frac{pF(\bar{a}_1(p))}{(1-p)+pF(\bar{a}_1(p))}) + \mu \\
=&(\mathbb{E}[\tilde{v}|\bar{a}_1(p) \leq \tilde{v} < \bar{a}_2(p)] - c - \frac{1-p}{(1-p)+pF(\bar{a}_1(p))}\mu)/\frac{pF(\bar{a}_1(p))}{(1-p)+pF(\bar{a}_1(p))} \\
=&\mathbb{E}[\tilde{v}|\bar{a}_0(p) \leq \tilde{v} < \bar{a}_1(p)],
\end{aligned}$$

where the first and the fifth equalities follow from the construction $\bar{a}'_i(c') = \bar{a}_i(c)$ for $i = 1, \ldots, I-1$, the second and the seventh equalities follow from the definition of the sequences, and the inequality follows from $p > p'$ and Lemma 18. Then $\bar{a}'_0(p') < \bar{a}_0(p)$, which implies a contradiction. Hence $\bar{a}_i(p) < \bar{a}_i(p')$ for $1 \leq i \leq I-1$ by Assumption 2. □

Proposition 19. *The voluntary disclosure cutoff t increases in the disclosure cost c and decreases in the probability of being informed p.*

Proof. By Lemma 8, we can restrict attention to equilibria in which $t = a_1$ without loss of generality. If the cost is small enough so that Inequality (3.10) holds, $t = a_1^\dagger$ which is constant in c. If the cost is large enough such that Inequality (3.10) no longer holds, the equilibrium cutoffs are given by $\{\bar{a}_i\}_{i=0}^I$. By Lemma 17, $\bar{a}_1$ is strictly increasing in the cost c. Hence the disclosure cutoff t is (weakly) increasing in c overall and strictly increasing when c is large.

Similarly, if the probability is large enough so that Inequality (3.10) holds, then $t = a_1^\dagger$ which is decreasing in the probability p. If the probability is small enough such that Inequality (3.10) no longer holds, the equilibrium cutoffs are given by $\{\bar{a}_i\}_{i=0}^I$. By Lemma 19, $\bar{a}_1$ is strictly decreasing in p as well. Hence the disclosure cutoff t is decreasing in p. □

Proposition 20. *The expected pricing error is increasing in the disclosure cost c and decreasing in the probability of being informed p. As an example, in the special case of uniform cash flows $\tilde{v}$,*

(i) The pricing error conditional on disclosure is decreasing in the disclosure cost c and increasing in the probability of being informed p;

(ii) For sufficiently large cost, the expected pricing error is first strictly decreasing and then strictly increasing in attention capacity I. The pricing error conditional on disclosure is strictly decreasing in attention capacity I for I sufficiently large, i.e., when Inequality (3.10) *does not hold.*

The pricing error is U-shaped if we assume uniform (though the optimal attention is not necessarily at the level where the firm discloses most). In general, it is ambiguous (and clearly not monotonic). When investors have more attention, they would not incorrectly classify the marginal discloser into better firms. The firm then has less incentive to disclose because of less price in response, which is detrimental to the quality of investors' information and can become a dominant force that affects pricing error under some distribution.

Proof of general distributions. When Inequality (3.10) is satisfied, the pricing error is determined by $\{a_i^\dagger\}_{i=0}^I$ and not affected by the cost c or probability p.

When Inequality (3.10) is not satisfied, the cutoffs are given by Equations (3.9) and (3.11). In the maximal equilibrium, all types below t induce the nondisclosure price and types above t disclose their signals. By Proposition 19, the cutoff t increases in the cost c and decreases in the probability p of being informed. So when cost increases or probability decreases, the disclosure threshold becomes higher and the investors cannot do better, because fewer types are providing information. We show further that the expected pricing error is *strictly* increasing in the extent of frictions.

The expected pricing error when Inequality (3.10) does not hold is given by

$$\begin{aligned}\mathbb{E}L \equiv & p\sum_{j=2}^{I}\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1}\leq\tilde{v}<a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}+(1-p)\int_{a_1}^{1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v}\\ &+\int_{a_0}^{a_1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v},\end{aligned}$$

where $P(1)=\frac{pF(a_1)\mathbb{E}(\tilde{v}|a_0\leq\tilde{v}<a_1)+(1-p)\mathbb{E}(\tilde{v})}{pF(a_1)+(1-p)}$. Note that $a_1,\ldots,a_{I-1}$ are all functions of c and p.

The derivative of $\mathbb{E}L$ with respect to c is given by the chain rule,

$$
\begin{aligned}
&\frac{d\mathbb{E}L}{dc} \\
=&p\sum_{j=2}^{I-1}\frac{da_j}{dc}\frac{d}{da_j}\Big(\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1}\le\tilde{v}<a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v} \\
&+\int_{a_j}^{a_{j+1}}(\mathbb{E}[\tilde{v}|a_j\le\tilde{v}<a_{j+1}]-\tilde{v})^2 f(\tilde{v})d\tilde{v}\Big) \\
&+\frac{da_1}{dc}\frac{d}{da_1}\Big(p\int_{a_1}^{a_2}(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-\tilde{v})^2 f(\tilde{v})d\tilde{v}+(1-p)\int_{a_1}^{1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v} \\
&+\int_{a_0}^{a_1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v}\Big)
\end{aligned}
\tag{3.14}
$$

Since $\mathbb{E}[\tilde{v}|a^t_{j-1}\le\tilde{v}<a^t_j]$ and $P(1)$ are the investor's rational pricing to a signal that would minimize the pricing error, and since $a_0\equiv 0, a_I\equiv 1$, it follows by the Envelope Theorem that for $j=2,\ldots,I-1$,

$$
\begin{aligned}
&\frac{d}{da_j}\Big(\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1}\le\tilde{v}<a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}+\int_{a_j}^{a_{j+1}}(\mathbb{E}[\tilde{v}|a_j\le\tilde{v}<a_{j+1}]-\tilde{v})^2 f(\tilde{v})d\tilde{v}\Big) \\
=&f(a_j)[(\mathbb{E}[\tilde{v}|a_{j-1}\le\tilde{v}<a_j]-a_j)^2-(\mathbb{E}[\tilde{v}|a_j\le\tilde{v}<a_{j+1}]-a_j)^2];
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{da_1}\Big(p\int_{a_1}^{a_2}(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-\tilde{v})^2 f(\tilde{v})d\tilde{v}+(1-p)\int_{a_1}^{1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v} \\
&+\int_{a_0}^{a_1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v}\Big) \\
=&f(a_1)[(a_1-P(1))^2-p(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-a_1)^2-(1-p)(a_1-P(1))^2].
\end{aligned}
$$

Because $(\mathbb{E}[\tilde{v}|a_{j-1}\le\tilde{v}<a_j]-a_j)^2=(\mathbb{E}[\tilde{v}|a_j\le\tilde{v}<a_{j+1}]-a_j)^2$ for all $j=2,\ldots,I-1$ by (3.9), Equation (3.14) is simplified to

$$
\frac{d\mathbb{E}L}{dc}=\frac{da_1}{dc}f(a_1)p[(P(1)-a_1)^2-(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-a_1)^2]. \tag{3.15}
$$

By Proposition 19, $da_1/dc>0$. Furthermore, $(P(1)-a_1^\dagger)^2=(\mathbb{E}[\tilde{v}|a_1^\dagger\le\tilde{v}<a_2^\dagger]-a_1^\dagger)^2$ in the unconstrained problem. As c increases, a_1 will increase but a_2 will decrease by Proposition 19 and

Lemma 16. It follows from the continuity of $(P(1)-a_1)^2-(\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2$ with respect to a_1 that $(P(1)-a_1)^2 > (\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2$ for $a_1 > a_1^{\dagger}$.[21] So $(P(1)-a_1)^2-(\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2 > 0$ and $d\mathbb{E}L/dc > 0$ by Equation (3.15).

The comparative static analysis with respect to p uses the similar argument except that $da_1/dp < 0$. The extra term that is the partial derivative of $\mathbb{E}L$ with respect to p is

$$\sum_{j=2}^{I}\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1} \leq \tilde{v} < a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}-\int_{a_1}^{1}(\tilde{v}-P(1))^2 f(\tilde{v})d\tilde{v} < 0.$$

It is hence clear that $d\mathbb{E}L/dp < 0$. □

Proof of Part (i) of the uniform case. The perceived quality of disclosure, however, increases in c and decreases in p. Let $\mathbb{E}(L|v \geq t)$ denote the pricing error conditional on disclosure which is given by

$$\mathbb{E}(L|v \geq t) \equiv \frac{1}{1-F(a_1)}\sum_{j=2}^{I}\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1} \leq \tilde{v} < a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}.$$

Similarly because $(\mathbb{E}[\tilde{v}|a_{j-1} \leq \tilde{v} < a_j]-a_j)^2 = (\mathbb{E}[\tilde{v}|a_j \leq \tilde{v} < a_{j+1}]-a_j)^2$ for $j = 2,\ldots,I-1$, the derivative of $\mathbb{E}(L|v \geq t)$ with respect to c can be simplified to

$$\begin{aligned}
&\frac{d\mathbb{E}(L|v \geq t)}{dc}\\
=&\frac{da_1}{dc}[-\frac{1}{1-F(a_1)}f(a_1)(\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2\\
&+\frac{f(a_1)}{(1-F(a_1))^2}\sum_{j=2}^{I}\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1} \leq \tilde{v} < a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}]\\
=&\frac{da_1}{dc}\frac{f(a_1)}{1-F(a_1)}[\frac{1}{1-F(a_1)}\sum_{j=2}^{I}\int_{a_{j-1}}^{a_j}(\mathbb{E}[\tilde{v}|a_{j-1} \leq \tilde{v} < a_j]-\tilde{v})^2 f(\tilde{v})d\tilde{v}\\
&-(\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2]\\
=&\frac{da_1}{dc}\frac{f(a_1)}{1-F(a_1)}[\mathbb{E}(L|v \geq t)-(\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2].
\end{aligned}$$

It is clear that the sign of the derivative depends on the average pricing error in the disclosure

[21] Note that given a_1 and $a_0 = 0$, a_2 is determined by Equation (3.11). As $a_2,\ldots,a_{I-1}$ declines, there is no a_1 such that $(P(1)-a_1)^2 = (\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2]-a_1)^2$ with $a_0 = 0$ by Assumption 1.

region and the pricing error at the point a_1. Hence it is in general ambiguous, which depends on the probability density function f. But if the distribution is uniform, $\mathbb{E}(L|v \geq t) < (\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - a_1)^2$. The result follows because a_1 is a boundary point and the pricing error in the interior of the partition elements is smaller than the pricing error at the boundary. So the average pricing error is smaller as well. This implies that $d\,\mathbb{E}(L|v \geq t)/dc < 0$ from $da_1/dc > 0$ by Proposition 19. Hence it is clear in the case of uniform distribution that the pricing error conditional on disclosure declines in the disclosure cost c, which supports our intuition.

Likewise, we can perform exactly the same analysis for the comparative statics with respect to p. The result is ambiguous in general as well, but the pricing error conditional on disclosure strictly increases in p in the uniform case when Equation (3.10) is not satisfied. □

The pricing error *conditional* on disclosure is not part of the *expected* pricing error from the disclosure region. In the latter case, the pricing error from disclosure strictly decreases (increases) in cost (probability) for any general distribution (that satisfies Assumption 2).

Proof of Part (ii) of the uniform case: (a) Expected pricing error. Finally, let us consider the comparative statics with respect to attention capacity I. We show next that for given cost and probability of being informed, it is not necessary that the investors would strictly prefer equilibrium partitions with more steps (larger I's).[22] The comparative statics of expected pricing error with respect to I are ambiguous, but we find that it is U-shaped if we assume uniform distribution. In general, it is ambiguous (and clearly not monotonic). When investors have more attention, they would not incorrectly classify the marginal discloser as better firms. The firm then has less incentive to disclose because of less price in response, which is detrimental to the quality of investors' information and can become a dominant force for sufficiently large capacity.

It is clear that the expected pricing error is decreasing in the partition size if the information set is given by $\{a_i^\dagger\}_{i=1}^{I-1}$, i.e., if Inequality (3.10) is satisfied, because the partition is the *unique* optimal information set with more attention capacity and more disclosure. We will

[22] Still, the investors base their pricing choice on rational expectations and the prior distribution is fixed. Nevertheless, the equilibria with more steps are not, ceteris paribus, more informative.

examine below the change in expected pricing error as I increases when Inequality (3.10) does not hold.

Fix p and c, and let $\bar{a}(I)$ be the maximal equilibrium of size I. We shall argue that $\bar{a}(I)$ can be continuously deformed to the (maximal) equilibrium of size $I+1$ and express how the expected pricing error changes throughout the deformation.

Let $a^t \equiv (a_0^t, a_1^t, \ldots, a_{I+1}^t)$ be the partition that satisfies

$$\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - c = \frac{pF(a_1)\,\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v})}{(1-p) + pF(a_1)} \tag{3.16}$$

for $i = 1$ and

$$a_i = \frac{\mathbb{E}[\tilde{v}|a_i \leq \tilde{v} < a_{i+1}] + \mathbb{E}[\tilde{v}|a_{i-1} \leq \tilde{v} < a_i]}{2}. \tag{3.17}$$

for $i = 2, \ldots, I-1$ with $a_0^t = 0$, $a_1^t = t$, and $a_{I+1}^t = 1$. If $t = \bar{a}_1(I)$ then $a_I^t = 1$, and if $t = \bar{a}_1(I+1)$ then $a^t = \bar{a}(I+1)$ and (3.9) is satisfied for all $i = 2, \ldots, I$. We will next write down the partial derivative of the expected pricing error $\mathbb{E}L(t)$ with respect to t when $t \in [\bar{a}_1(I), \bar{a}_1(I+1)]$, which is a non-degenerate interval by Lemma 21.

By definition, $\mathbb{E}L(t)$ is given by

$$\begin{aligned}\mathbb{E}L(t) \equiv & p \sum_{j=2}^{I+1} \int_{a_{j-1}^t}^{a_j^t} (\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - \tilde{v})^2 f(\tilde{v}) d\tilde{v} + (1-p) \int_{a_1^t}^{1} (\tilde{v} - P(1))^2 f(\tilde{v}) d\tilde{v} \\ & + \int_{a_0}^{a_1^t} (\tilde{v} - P(1))^2 f(\tilde{v}) d\tilde{v},\end{aligned}$$

where $P(1) = \frac{pF(a_1^t)\mathbb{E}(\tilde{v}|a_0 \leq \tilde{v} < a_1^t) + (1-p)\mathbb{E}(\tilde{v})}{pF(a_1^t) + (1-p)}$. The Envelope Theorem yields

$$\begin{aligned}\frac{d\,\mathbb{E}L(t)}{dt} = & p \sum_{j=2}^{I} f(a_j^t) \frac{da_j^t}{dt} [(\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - a_j^t)^2 - (\mathbb{E}[\tilde{v}|a_j^t \leq \tilde{v} < a_{j+1}^t] - a_j^t)^2] \\ & + f(a_1^t) \frac{da_1^t}{dt} [(a_1^t - P(1))^2 - p(\mathbb{E}[\tilde{v}|a_1^t \leq \tilde{v} < a_2^t] - a_1^t)^2 - (1-p)(a_1^t - P(1))^2].\end{aligned}$$

Note that $(\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - a_j^t)^2 = (\mathbb{E}[\tilde{v}|a_j^t \leq \tilde{v} < a_{j+1}^t] - a_j^t)^2$ for $j = 2, \ldots, I-1$ by

(3.9). Then we can simplify the expression to

$$\begin{aligned}\frac{d\,\mathbb{E}L(t)}{dt} =& pf(a_1^t)\frac{da_1^t}{dt}[(a_1^t-P(1))^2-(\mathbb{E}[\tilde{v}|a_1^t\le\tilde{v}<a_2^t]-a_1^t)^2]\\ &+pf(a_I^t)\frac{da_I^t}{dt}[(\mathbb{E}[\tilde{v}|a_{I-1}^t\le\tilde{v}<a_I^t]-a_I^t)^2-(\mathbb{E}[\tilde{v}|a_I^t\le\tilde{v}<a_{I+1}^t]-a_I^t)^2].\end{aligned}$$

So the change in the expected pricing error when I increases to $I+1$ is given by

$$\begin{aligned}\Delta(\mathbb{E}L) =& \int_{\bar{a}_1(I)}^{\bar{a}_1(I+1)}\frac{d\,\mathbb{E}L(t)}{dt}dt\\ =& p\int_{\bar{a}_1(I)}^{\bar{a}_1(I+1)} f(a_1^t)\frac{da_1^t}{dt}[(a_1^t-P(1))^2-(\mathbb{E}[\tilde{v}|a_1^t\le\tilde{v}<a_2^t]-a_1^t)^2]dt\\ &+p\int_{\bar{a}_1(I)}^{\bar{a}_1(I+1)} f(a_I^t)\frac{da_I^t}{dt}[(\mathbb{E}[\tilde{v}|a_{I-1}^t\le\tilde{v}<a_I^t]-a_I^t)^2-(\mathbb{E}[\tilde{v}|a_I^t\le\tilde{v}<a_{I+1}^t]-a_I^t)^2]dt\\ =& p\int_{\bar{a}_1(I)}^{\bar{a}_1(I+1)}[(a_1-P(1))^2-(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-a_1)^2]f(a_1)da_1\\ &-p\int_{\bar{a}_I(I+1)}^{1}[(\mathbb{E}[\tilde{v}|a_{I-1}\le\tilde{v}<a_I]-a_I)^2-(\mathbb{E}[\tilde{v}|a_I\le\tilde{v}<a_{I+1}]-a_I)^2]f(a_I)da_I.\end{aligned} \tag{3.18}$$

Let us take a closer look at the two terms in (3.18). First,

$$(a_1-P(1))^2-(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-a_1)^2>0$$

for all $a_1\in(\bar{a}_1(I),\bar{a}_1(I+1)]$ when c is sufficiently large by Equation (3.10).[23] Further,

$$(\mathbb{E}[\tilde{v}|a_{I-1}\le\tilde{v}<a_I]-a_I)^2-(\mathbb{E}[\tilde{v}|a_I\le\tilde{v}<a_{I+1}]-a_I)^2>0$$

for all $a_I\in[\bar{a}_I(I+1),\bar{a}_I(I))$.

When I is sufficiently large, $|\bar{a}_1(I+1)-\bar{a}_1(I)|>|\bar{a}_I(I)-\bar{a}_I(I+1)|$ and $(\bar{a}_1(I)-P(1))^2-(\mathbb{E}[\tilde{v}|\bar{a}_1(I)\le\tilde{v}<\bar{a}_2(I)]-\bar{a}_1(I))^2=\min_{a_1\in(\bar{a}_1(I),\bar{a}_1(I+1)]}(a_1-P(1))^2-(\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]-a_1)^2>(\mathbb{E}[\tilde{v}|\bar{a}_{I-1}(I)\le\tilde{v}<\bar{a}_I(I)]-\bar{a}_I(I))^2-(\mathbb{E}[\tilde{v}|\bar{a}_I(I)\le\tilde{v}<1]-\bar{a}_I(I))^2=\max_{a_I\in(\bar{a}_I(I+1),\bar{a}_I(I))]}(\mathbb{E}[\tilde{v}|a_{I-1}\le\tilde{v}<a_I]-a_I)^2-(\mathbb{E}[\tilde{v}|a_I\le\tilde{v}<a_{I+1}]-a_I)^2$. Because $f(a_1)=$

[23] When $c=0$, $P(1)=\mathbb{E}[\tilde{v}|a_1\le\tilde{v}<a_2]$. Then the expected pricing error is strictly decreasing in I for all I.

$f(a_I)$, $\Delta\mathbb{E}L$ is positive. $\square$

Proof of Part (ii) of the uniform case: (b) Pricing error conditional on disclosure. The pricing error conditional on disclosure is given by

$$\mathbb{E}(L|v \geq t) \equiv \frac{1}{1-F(a_1^t)} \sum_{j=2}^{I+1} \int_{a_{j-1}^t}^{a_j^t} (\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - \tilde{v})^2 f(\tilde{v}) d\tilde{v}.$$

The Envelope Theorem yields

$$\begin{aligned}
&\frac{d\,\mathbb{E}(L|v \geq t)}{dt} \\
=&\frac{da_1^t}{dt}[-\frac{1}{1-F(a_1^t)} f(a_1^t)(\mathbb{E}[\tilde{v}|a_1^t \leq \tilde{v} < a_2^t] - a_1^t)^2 \\
&+ \frac{f(a_1^t)}{(1-F(a_1^t))^2} \sum_{j=2}^{I+1} \int_{a_{j-1}^t}^{a_j^t} (\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - \tilde{v})^2 f(\tilde{v}) d\tilde{v}] \\
&+ \frac{da_I^t}{dt} \frac{f(a_I^t)}{1-F(a_1^t)} [(\mathbb{E}[\tilde{v}|a_{I-1}^t \leq \tilde{v} < a_I^t] - a_I^t)^2 - (\mathbb{E}[\tilde{v}|a_I^t \leq \tilde{v} < a_{I+1}^t] - a_I^t)^2] \\
=&\frac{da_1^t}{dt} \frac{f(a_1^t)}{1-F(a_1^t)} [\frac{1}{1-F(a_1^t)} \sum_{j=2}^{I+1} \int_{a_{j-1}^t}^{a_j^t} (\mathbb{E}[\tilde{v}|a_{j-1}^t \leq \tilde{v} < a_j^t] - \tilde{v})^2 f(\tilde{v}) d\tilde{v} \\
&- (\mathbb{E}[\tilde{v}|a_1^t \leq \tilde{v} < a_2^t] - a_1^t)^2] + \frac{da_I^t}{dt} \frac{f(a_I^t)}{1-F(a_1^t)} [(\mathbb{E}[\tilde{v}|a_{I-1}^t \leq \tilde{v} < a_I^t] - a_I^t)^2 \\
&- (\mathbb{E}[\tilde{v}|a_I^t \leq \tilde{v} < a_{I+1}^t] - a_I^t)^2] \\
=&\frac{da_1^t}{dt} \frac{f(a_1^t)}{1-F(a_1^t)} [\mathbb{E}(L|v \geq t) - (\mathbb{E}[\tilde{v}|a_1^t \leq \tilde{v} < a_2^t] - a_1^t)^2] \\
&+ \frac{da_I^t}{dt} \frac{f(a_I^t)}{1-F(a_1^t)} [(\mathbb{E}[\tilde{v}|a_{I-1}^t \leq \tilde{v} < a_I^t] - a_I^t)^2 - (\mathbb{E}[\tilde{v}|a_I^t \leq \tilde{v} < a_{I+1}^t] - a_I^t)^2].
\end{aligned}$$

Note that $a_1^t = t$. For the uniform distribution, $\mathbb{E}(L|v \geq t) < (\mathbb{E}[\tilde{v}|a_1^t \leq \tilde{v} < a_2^t] - a_1^t)^2$ and $(\mathbb{E}[\tilde{v}|a_{I-1}^t \leq \tilde{v} < a_I^t] - a_I^t)^2 > (\mathbb{E}[\tilde{v}|a_I^t \leq \tilde{v} < a_{I+1}^t] - a_I^t)^2$. So the first term is negative. When Inequality (3.10) holds, both t and a_I^t decreases in I by Lemma 20. So $da_I^t/dt > 0$ and the second term is positive. Hence the sign of $d\,\mathbb{E}(L|v \geq t)/dt$ is ambiguous and the effect of change in attention capacity on pricing error conditional on disclosure is indeterminate overall. When

Inequality (3.10) does not hold, $da_I^t/dt < 0$. In this case, both terms are negative. So $d\mathbb{E}(L|v \geq t)/dt < 0$ and the pricing error conditional on disclosure is decreasing in I by Proposition 21. $\square$

The next two lemmas will be used to show Proposition 21.

Lemma 20. *Let $a_0 = 0$ and $a_I = 1$. For unconstrained information sets $a^\dagger(I)$ with size I and $a^\dagger(I+1)$ with size $I+1$, $a_{i-1}^\dagger(I) < a_i^\dagger(I+1) < a_i^\dagger(I)$ for all $i = 1, \ldots, I$.*

Proof. That $a_{i-1}^\dagger(I) < a_i^\dagger(I+1)$ follows from Assumption 1. If $a_{i-1}^\dagger(I) \geq a_i^\dagger(I+1)$ for some $i = 1, \ldots, I$, then $a_{I-1}^\dagger(I) \geq a_I^\dagger(I+1)$ by Assumption 1. This leads to a contradiction of $a_0^\dagger(I) = a_0^\dagger(I+1) = 0$.

That $a_i^\dagger(I+1) < a_i^\dagger(I)$ for $i = 1, \ldots, I$ is by induction on I. For $I = 1$, the lemma is vacuously true. Suppose that $I > 1$ and that the conclusion of the Lemma is true for all $i = 1, \ldots, I-1$. Let $a^\dagger(I+1)$ and $a^\dagger(I)$ be as in the statement of the Lemma. Suppose by way of contradiction that $a_j^\dagger(I+1) \geq a_j^\dagger(I)$ for some j such that $0 < j < I$; suppose further that j is the smallest index greater than 0 such that this inequality is satisfied, so that $a_i^\dagger(I+1) < a_i^\dagger(I)$ for all i such that $0 < i < j$. Let ${}^x a \equiv ({}^x a_j, {}^x a_{j+1}, \ldots, {}^x a_I)$ be the partial partition that satisfies (3.4) for $i = j+1, \ldots, I-1$ with ${}^x a_I = a_I^\dagger(I) = 1$ and ${}^x a_{I-1} = x$. It follows from Assumption 1 and continuity of ${}^x a$ in x that there is an $\tilde{x} < a_I^\dagger(I+1)$ such that $a_j^\dagger(I+1) = {}^{\tilde{x}} a_j$. Let ${}^{\tilde{x}} a \equiv \tilde{a}$. We assumed that $a_j^\dagger(I+1) \geq a_j^\dagger(I)$. So $\tilde{a}_i \geq a_i^\dagger(I)$ for $j \leq i \leq I-1$ by Assumption 1. This implies that there is a unique $\tilde{a}_{j-1} \in [0, \tilde{a}_j)$ such that $\mathbb{E}[\tilde{v}|\tilde{a}_j \leq \tilde{v} < \tilde{a}_{j+1}] = 2\tilde{a}_j - \mathbb{E}[\tilde{v}|\tilde{a}_{j-1} \leq \tilde{v} < \tilde{a}_j]$ and $\mathbb{E}[\tilde{v}|\tilde{a}_j \leq \tilde{v} < \tilde{a}_{j+1}] \leq 2\tilde{a}_j - \mathbb{E}[\tilde{v}|a \leq \tilde{v} < \tilde{a}_j]$ for $a \leq \tilde{a}_{j-1}$. Then $\mathbb{E}[\tilde{v}|\tilde{a}_j \leq \tilde{v} < \tilde{a}_{j+1}] \leq 2\tilde{a}_j - \mathbb{E}[\tilde{v}|a_{j-1}^\dagger(I) \leq \tilde{v} < \tilde{a}_j]$ by Assumption 1. Further because $a_j^\dagger(I+1) = \tilde{a}_j$,

$$\begin{aligned}
\mathbb{E}[\tilde{v}|\tilde{a}_j \leq \tilde{v} < a_{j+1}^\dagger(I+1)] =& \mathbb{E}[\tilde{v}|a_j^\dagger(I+1) \leq \tilde{v} < a_{j+1}^\dagger(I+1)] \\
=& 2a_j^\dagger(I+1) - \mathbb{E}[\tilde{v}|a_{j-1}^\dagger(I+1) \leq \tilde{v} < a_j^\dagger(I+1)] \\
=& 2\tilde{a}_j - \mathbb{E}[\tilde{v}|a_{j-1}^\dagger(I+1) \leq \tilde{v} < \tilde{a}_j] \\
>& 2\tilde{a}_j - \mathbb{E}[\tilde{v}|a_{j-1}^\dagger(I) \leq \tilde{v} < \tilde{a}_j] \\
\geq& \mathbb{E}[\tilde{v}|\tilde{a}_j \leq \tilde{v} < \tilde{a}_{j+1}],
\end{aligned}$$

where the second equality follows from the definition of the sequence $\{a_i^\dagger(I+1)\}_{i=0}^{I+1}$ and the first inequality follows from $a_{j-1}^\dagger(I+1) < a_{j-1}^\dagger(I)$. So $a_{j+1}^\dagger(I+1) > \tilde{a}_{j+1}$. But $a_{j+1}^\dagger(I+1) < \tilde{a}_{j+1}$ by the induction hypothesis, because $a_j^\dagger(I+1) = \tilde{a}_j$ and $a_{I+1}^\dagger(I+1) = \tilde{a}_I$. Hence the contradiction establishes the desired conclusion. □

In particular, we show that $a_1^\dagger(I+1) < a_1^\dagger(I)$, which is equal to the disclosure cutoffs under these two attention levels.

Lemma 21. *Let* $a_0 = 0$ *and* $a_I = 1$. *For partitions* $\bar{a}(I)$ *with size* I *and* $\bar{a}(I+1)$ *with size* $I+1$ *that satisfy Equations* (3.9) *and* (3.11), $\bar{a}_1(I) < \bar{a}_1(I+1)$.

Proof. This lemma follows directly from Lemma 16. Consider two partitions $\{\bar{a}_i(I)\}_{i=0}^{I}$ with size I and $\{\bar{a}_i(I+1)\}_{i=0}^{I+1}$ with size $I+1$ such that Equations (3.9) and (3.11) are satisfied. Suppose by way of contradiction that $\bar{a}_1(I) \geq \bar{a}_1(I+1)$. Then $\bar{a}_i(I) \leq \bar{a}_i(I+1)$ for all $2 \leq i \leq I$ by Lemma 16, which contradicts $\bar{a}_I(I) = \bar{a}_{I+1}(I+1) = 1$. □

The only way to have more partition elements is to increase the first cutoff $\bar{a}_1$ (and all subsequent cutoffs will decline). Intuitively, if the partition is finer, the firm's gain from inducing a slightly higher price $\mathbb{E}[\tilde{v}|\bar{a}_1 \leq \tilde{v} < \bar{a}_2]$ becomes smaller, which is outweighed by the cost of disclosure.

Proposition 21. *The disclosure cutoff* t *is first strictly decreasing and then strictly increasing in the partition size* I.

Proof. By Lemma 8, we restrict attention to equilibria in which $t = a_1$ without loss of generality. By Lemma 20, the first disclosure cutoff $a_1^\dagger(I)$ of the optimal information set is strictly decreasing in I. By Lemma 21, the first cutoff $\bar{a}_1(I)$ of the sequence $\{\bar{a}_i(I)\}_{i=0}^{I}$ is strictly increasing in I. So if the investors have very limited attention such that Inequality (3.10) holds, then $t = a_1^\dagger$ is strictly decreasing in I. If the investors are able to pay a lot of attentions to the signal such that Inequality (3.10) does not hold, the equilibrium cutoffs are given by $\{\bar{a}_i\}_{i=0}^{I}$ and $\bar{a}_1$ is strictly increasing in I. Hence the disclosure cutoff t is first strictly decreasing and then strictly increasing in the partition size I. □

Our comparative statics results with respect to the disclosure cost c and probability of being informed p would still hold for cash flow v following a general distribution on an unbounded support. In this case, we focus on the equilibrium with the lowest disclosure threshold. Without imposing the Monotonicity Conditions, it is possible that there are multiple solutions to the optimal information set. Hence multiple equilibria could arise.

Proposition 25. *The voluntary disclosure threshold t increases in the disclosure cost c and decreases in the probability of being informed p.*

Proof. When the constraints are slack, the disclosure threshold t does not depend on the cost c or the probability p. When the constraints bind, the equilibrium information set is given by Equations (3.9) and (3.11). So the equilibrium threshold is a function of c and p. We define L as follows:

$$L = \mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - c - [\frac{pF(a_1)\,\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v})}{(1-p) + pF(a_1)}]. \tag{3.19}$$

When a_1 and a_2 are part of the (constrained) equilibrium, $L = 0$ by Eq (3.11). We will first sign the derivative $\partial L/\partial a_1$, $\partial L/\partial p$, and $\partial L/\partial c$ in order to determine the sign of $\partial a_1/\partial p$ and $\partial a_1/\partial c$ by the Implicit Function Theorem.

First, let us find the sign of $\partial L/\partial a_1$. If a_1 is close to zero, $\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - c$ is no greater than $\mathbb{E}[\tilde{v}|0 \leq \tilde{v} < a_2]$, while $\frac{pF(a_1)\,\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v})}{(1-p)+pF(a_1)} \geq \mathbb{E}(\tilde{v}) - \varepsilon$ for $\varepsilon > 0$ arbitrarily small by continuity. When $I \geq 3$, a_2 is less than 1, which implies that $\mathbb{E}[\tilde{v}|0 \leq \tilde{v} < a_2] < \mathbb{E}(\tilde{v}) - \varepsilon$ for $\varepsilon > 0$ small enough. Hence, if a_1 is close to zero, L must be less than zero. If a_1 is close to the upper bound 1, a_2 must be close to the upper bound as well and $\mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - c \geq 1 - \varepsilon' - c$ for $\varepsilon' > 0$ arbitrarily small. Moreover, $\frac{pF(a_1)\,\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v})}{(1-p)+pF(a_1)}$ is no greater than $\mathbb{E}(\tilde{v})$. Whenever there is some type v' who would like to disclose the type, we can find ε' small enough such that $1 - \varepsilon' > v'$ and all types above $1 - \varepsilon'$ (including $1 - \varepsilon'$) would all (strictly) prefer to disclose. Hence $1 - \varepsilon' - c > \mathbb{E}(\tilde{v})$. It follows that $L = \mathbb{E}[\tilde{v}|a_1 \leq \tilde{v} < a_2] - c - [\frac{pF(a_1)\,\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v})}{(1-p)+pF(a_1)}] > 0$ if a_1 is close to 1. This shows that there is at least one value of a_1 such that $L = 0$ holds.

Furthermore, because $L < 0$ when a_1 is close to zero, the function L must cross zero from below at the first solution of a_1 to $L = 0$. So $\partial L/\partial a_1 > 0$.

The derivative of L with respect to p is given by

$$
\begin{aligned}
&\partial L/\partial p \\
=&-[(F(a_1)\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] - \mathbb{E}(\tilde{v}))((1-p) + pF(a_1)) - (F(a_1)-1) \\
&(pF(a_1)\mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1] + (1-p)\mathbb{E}(\tilde{v}))]/[((1-p)+pF(a_1))^2] \\
=&\frac{F(a_1)(\mathbb{E}(\tilde{v}) - \mathbb{E}[\tilde{v}|a_0 \leq \tilde{v} < a_1])}{((1-p)+pF(a_1))^2} > 0.
\end{aligned}
$$

It is then clear that $\partial a_1/\partial p = -(\partial L/\partial p)/(\partial L/\partial a_1) < 0$ and the disclosure threshold is strictly decreasing in p in the constrained case.

The derivative of L with respect to c is given by

$$\partial L/\partial c = -1 < 0.$$

Hence $\partial a_1/\partial c = -(\partial L/\partial c)/(\partial L/\partial a_1) > 0$ and the disclosure threshold is strictly increasing in c in the constrained case. □

Lemma 11. *The value of information to firms V exceeds the social value of information V^*.*

Proof. Let $\mu =: \mathbb{E}(\tilde{v})$ be the mean value of future cash flows. The social value of information V^* is given by

$$
\begin{aligned}
V^* &= \int_0^1 [(vx^*(v) - \psi(x^*(v))) - (vx^*(\mu) - \psi(x^*(\mu)))]f(v)dv \\
&= \int_0^1 (vx^*(v) - \psi(x^*(v)))f(v)dv - [x^*(\mu)\int_0^1 vf(v)dv - \psi(x^*(\mu))] \\
&= \int_0^1 (vx^*(v) - \psi(x^*(v)))f(v)dv - (\mu x^*(\mu) - \psi(x^*(\mu))).
\end{aligned}
$$

The firm's private value of information is greater than the social value, i.e., $V > V^*$, as shown below:

$$
\begin{aligned}
V &= \int_t^1 [(vx^*(v) - \psi(x^*(v))) - (\mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))]f(v)dv \\
&> \int_0^1 [(vx^*(v) - \psi(x^*(v))) - (\mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))]f(v)dv \\
&= \int_0^1 (vx^*(v) - \psi(x^*(v)))f(v)dv - (\mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND)))) \\
&> \int_0^1 (vx^*(v) - \psi(x^*(v)))f(v)dv - (\mu x^*(\mu) - \psi(x^*(\mu))) = V^*,
\end{aligned}
$$

where the first inequality follows because $vx^*(v) - \psi(x^*(v)) > (<)\, \mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND)))$ for $v > (<)\, t$, and the second inequality follows because $\mu > \mathbb{E}(\tilde{v}|ND)$ and $vx^*(v) - \psi(x^*(v))$ is strictly increasing in v by the Envelope Theorem. □

Proposition 23. *Suppose that there is only acquisition cost and no disclosure cost, i.e., $c = 0$. For* ***any*** *finite information capacity, the value of information to firms is less than the full-information case.*

Proof. Consider the firm's private value of information V_f when investors only have finite information capacity I. Let t be the equilibrium disclosure threshold in the rational (full attention) model. Let t^f be the equilibrium disclosure threshold when the attention capacity is I. Because the firm will disclose any signal above $t^f = a_1$ (and conceal otherwise), the value at capacity I is given by

$$
\begin{aligned}
V_f &= \int_{t^f}^{1} [(P(i|D(v) \in A_i)x^*(P(i|D(v) \in A_i)) - \psi(x^*(P(i|D(v) \in A_i)))) - (P(1)x^*(P(1)) \\
&\quad - \psi(x^*(P(1))))] f(v)dv \\
&= \int_{t^f}^{1} (P(i|D(v) \in A_i)x^*(P(i|D(v) \in A_i)) - \psi(x^*(P(i|D(v) \in A_i)))) f(v)dv \\
&\quad - \int_{t^f}^{1} (P(1)x^*(P(1)) - \psi(x^*(P(1)))) f(v)dv \\
&= \sum_{i=2}^{I} (\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])) - \psi(x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])))) \\
&\quad (F(a_i) - F(a_{i-1})) - \int_{t^f}^{1} (P(1)x^*(P(1)) - \psi(x^*(P(1)))) f(v)dv,
\end{aligned}
$$

where the third equality follows by the formation of market price. Because $vx^*(v) - \psi(x^*(v))$ is strictly convex in v by the convexity of $\psi(\cdot)$,

$$
\begin{aligned}
&\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])) - \psi(x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i]))) \\
&< \int_{a_{i-1}}^{a_i} (vx^*(v) - \psi(x^*(v))) \frac{f(v)}{F(a_i) - F(a_{i-1})} dv
\end{aligned}
$$

for $i = 2, \ldots, I$ by Jensen's Inequality. It is follows that

$$
\begin{aligned}
&\sum_{i=2}^{I} (\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i])) \\
&\quad - \psi(x^*(\mathbb{E}(\tilde{v}|\tilde{v} \in (a_{i-1}, a_i]))))(F(a_i) - F(a_{i-1})) \\
&< \int_{a_1}^{1} (vx^*(v) - \psi(x^*(v))) f(v)dv \\
&= \int_{t^f}^{1} (vx^*(v) - \psi(x^*(v))) f(v)dv,
\end{aligned}
$$

where the equality is implied from the equilibrium condition. Hence the firm's value with finite capacity I satisfies

$$\begin{aligned}
V_f &< \int_{t^f}^{1} [(vx^*(v) - \psi(x^*(v))) - (P(1)x^*(P(1)) - \psi(x^*(P(1))))] f(v)dv \\
&\leq \int_{t^f}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv \\
&\leq \int_{t}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv = V,
\end{aligned}$$

where V is the firm's private value of information in the full attention case. The second inequality above holds because $P(1) \geq \mathbb{E}(\tilde{v}|ND)$ (the non-disclosure price in the case of full attention) by the minimum principle and the function $vx^*(v) - \psi(x^*(v))$ is strictly increasing in v. The third inequality holds because

(1) If $t^f \geq t$,

$$\begin{aligned}
&\int_{t^f}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv \\
=&\int_{t}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv - \\
&\int_{t}^{t^f} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv.
\end{aligned} \tag{3.20}$$

Since $vx^*(v) - \psi(x^*(v))$ is strictly increasing in v, $vx^*(v) - \psi(x^*(v)) > \mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND)))$ for all $v > \mathbb{E}(\tilde{v}|ND) = t$, where $\mathbb{E}(\tilde{v}|ND) = t$ follows from the equilibrium condition in the case of full attention. Hence the second term of Equation (3.20) is nonnegative and the result follows.

(2) If $t^f < t$,

$$\begin{aligned}
&\int_{t^f}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv \\
=&\int_{t}^{1} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv + \\
&\int_{t^f}^{t} [(vx^*(v) - \psi(x^*(v))) - (E(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) - \psi(x^*(\mathbb{E}(\tilde{v}|ND))))] f(v)dv.
\end{aligned} \tag{3.21}$$

Since $vx^*(v) - \psi(x^*(v))$ is strictly increasing in v, $vx^*(v) - \psi(x^*(v)) < \mathbb{E}(\tilde{v}|ND)x^*(\mathbb{E}(\tilde{v}|ND)) -$

$\psi(x^*(\mathbb{E}(\tilde{v}|ND)))$ for all $v < \mathbb{E}(\tilde{v}|ND) = t$. Hence the second term of Equation (3.21) is negative and the result follows.

Therefore, the firm gains less from acquiring information, which implies the role of inattention in reducing excessive information acquisition. □

Proposition 24. *Let A_1^1 be the first element of the investor's information set in period 1. Let a_1^2 be the first cutoff that the investor selects in period two. Let t_1 be the disclosure threshold in period one. The cutoff a_1^2 will be lower if the investor does not observe A_1^1 in period one or if the realized cash flow in period one falls below the disclosure threshold t_1 (when the observation about the signal is A_1^1).*

Proof. We prove the first part of the proposition. Suppose that the investor observes something other than A_1^1 in the first period. Then they know that the firm does obtain a signal and disclose. So the probability p_2 that the firm receives a signal in period 2 is λ_1. Because $\lambda_1 > \lambda_0$ and the probability $\phi(v_1)$ if A_1^1 is instead observed is a weighted average of λ_0 and λ_1, the probability p_2 attains the highest possible value when the observation is not A_1^1. By Proposition 19, with a higher belief about the signal endowment, the investor will choose a lower cutoff a_1^2 in period 2 in equilibrium.

We next show that if v_1 in period one is lower than t_1, the cutoff a_1^2 in period 2 will be lower than the case in which $v_1 \geq t_1$. First, the realization of v_1 (given the *observed disclosure*) does not affect the investor's choice of information set and firm's disclosure in period two if the investor observes something other than A_1^1 in period one, because the cash flows between the two periods are independent and p_1 is known to be λ_1. If the investor observes A_1^1 in period one, the firm either receives no signal or withdraws a low signal. The probability p_2 that the firm gets a signal in period two is given by $\frac{1-\lambda}{M}\lambda_0 + (1 - \frac{1-\lambda}{M})\lambda_1$, where $M = (1-\lambda) + \lambda \Pr(s_1 < t_1 | v_1)$. The conditional probability is equal to

$$\Pr(s_1 < t_1 | v_1) = \begin{cases} q(v_1) + (1 - q(v_1)) \int_0^{t_1} dG & \text{if } v_1 < t_1 \\ (1 - q(v_1)) \int_0^{t_1} dG & \text{if } v_1 \geq t_1. \end{cases}$$

If the cash flow is less than t_1, $\Pr(s_1 < t_1|v_1)$ will be higher and M will be larger, because $q(v_1) + (1 - q(v_1)) \int_0^{t_1} dG \geq \int_0^{t_1} dG > (1 - q(v_1')) \int_0^{t_1} dG$ for any $v_1' \geq t_1$ by $q > 0$ and $\int_0^{t_1} dG \leq 1$. It follows that the weight $\frac{1-\lambda}{M}$ will be smaller and the complement $1 - \frac{1-\lambda}{M}$ will be larger. Further because $\lambda_1 > \lambda_0$, $\phi(v_1)$ must be larger. Hence the investor believes that the firm has a higher chance to get a signal in period 2, i.e., p_2 is larger. By Proposition 19, with a higher belief about the signal endowment, the investor will choose a lower cutoff a_1^2 in period 2 in equilibrium, which completes the proof. $\square$

3.8.2 Tables and Figures in Section 3.6

Table 3.1: Variable Definitions

Variables	Definitions	Sources
Dependent Variables		
$Management\ Forecast\ (MF_{i,t})$	Indicator equals to 1 if a firm i makes a forecast in year t, and zero otherwise.	I/B/E/S
Variables of Interest		
$Capacity\ (percent)_{i,t-1}$	Percentage of institutional ownership in year $t-1$	Thomson Reuters
$Capacity\ (percent)^2_{i,t-1}$	Squared term of $Capacity\ (percent)$ in year $t-1$	Thomson Reuters
$Capacity\ (ratio)_{i,t-1}$	$(Ins - Ins(LT))/(1 - Ins(LT))$ where Ins is institutional ownership and $Ins(LT)$ ownership by by institutions with > 5 % of shares	Thomson Reuters
$Capacity\ (ratio)^2_{i,t-1}$	Squared term of $Capacity\ (ratio)$ in year $t-1$	Thomson Reuters
Control Variables		
$EPS\ Increase_{i,t-1}$	Indicator equals to one if firm i reports an increase in earnings per share from year $t-2$ to $t-1$	Compustat
$Abs.\ EPS\ Change_{i,t-1}$	Absolute value of the change in earnings per share from year $t-2$ to $t-1$	Compustat
$Book\ to\ Market_{i,t-1}$	Book value of equity / market value of equity	Compustat/CRSP
$Size_{i,t-1}$	Natural log of market capitalization	CRSP
$Loss_{i,t-1}$	Indicator equals to 1 if earnings < 0 in year $t-1$	Compustat
$Leverage\ Ratio_{i,t-1}$	Total liabilities divided by total assets measured for firm i in year $t-1$	Compustat

Table 3.2: Sample Selection

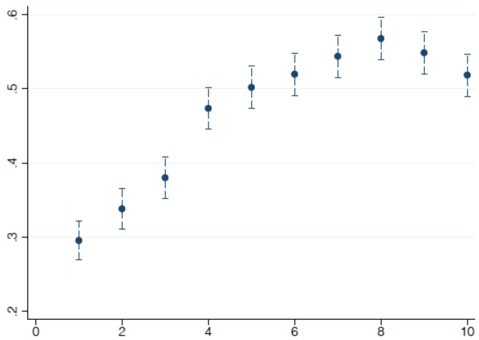
	Details	# Firm-Year	# Firms	# MF
Step 1	I/B/E/S EA sample (US firms) 1/1/2004 - 12/31/2016	67,239	10,945	
	a) : Non-missing current or prior EA date	62,359	10,035	
Step 2	I/B/E/S CIG sample 1/1/2004 - 12/31/2016			
	a) : Matched to I/B/E/S EA	62,359	10,035	70,198
	b) : Keep management forecasts (MF) after prior EA date and at least 6 months before current period end	62,359	10,035	28,787
	c) : Keep only earliest MF	62,359	10,035	12,769
Step 3	Keep obs. with data from CRSP, Compustat, Thompson Reuters	50,703	7,864	11,451
Step 4	Other Sample Selections:			
	a) : Drop firms that always forecast	46,748	7,339	7,392
	b) : Drop firms that never forecast	16,508	2,583	7,392
	Total	**16,508**	**2,583**	**7,392**

Note: this table summarizes our sample selection procedures. Annual earnings announcements (EA) and management forecasts (MF) are obtained from I/B/E/S. Firms in our sample must also have information on prices from CRSP, fundamentals from Compustat, and institutional ownership from Thompson Reuters.

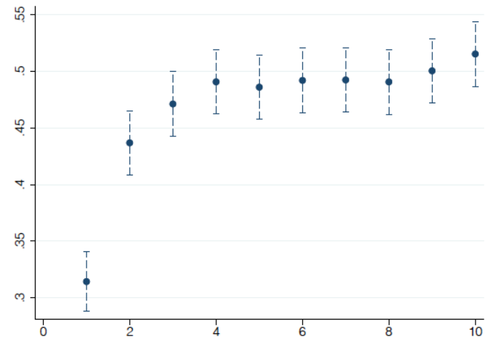
Table 3.3: Summary Statistics

	N	mean	sd	p10	p25	p50	p75	p90
Management Forecast	16,508	0.45	0.50	0.00	0.00	0.00	1.00	1.00
Ins Holding Ratio	15,901	0.64	0.30	0.17	0.42	0.68	0.86	0.98
Ins Holding Percent	15,901	0.69	0.28	0.26	0.53	0.75	0.90	0.99
Quasi-indexer	16,508	0.45	0.24	0.03	0.29	0.49	0.63	0.74
Transient	16,508	0.14	0.10	0.00	0.06	0.13	0.21	0.28
Dedicated	16,508	0.04	0.06	0.00	0.00	0.00	0.06	0.12
Earnings Per Share	16,508	1.50	3.04	-0.31	0.32	1.15	2.27	3.72
Total Asset	16,508	8,867.31	27,606.47	110.86	326.13	1,178.02	4,585.02	16,931.30
Leverage Ratio	16,459	0.53	0.23	0.22	0.36	0.53	0.70	0.86
Book to Market	15,323	0.67	5.71	0.17	0.28	0.47	0.73	1.08
Market Cap	15,372	6,145.79	20,694.78	106.27	316.50	1,013.20	3,449.15	12,566.46
Return on Assets	16,508	0.01	0.16	-0.09	0.01	0.04	0.08	0.12
Loss	16,508	0.15	0.36	0.00	0.00	0.00	0.00	1.00
EPS Increase	16,508	0.68	0.47	0.00	0.00	1.00	1.00	1.00

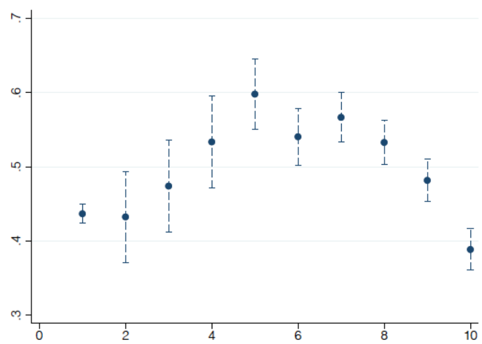
Note: this table reports summary statistics on our sample. Management Forecast is an indicator variable which equals to one when a firm makes a forecast. 45% of all firm-years have management forecasts. Information on institutional ownership is obtained from Thomson Reuters. Firm fundamentals including earnings per share, total asset, leverage ratio, return on assets are from Compustat. Market capitalization is calculated as the product of number of shares times closing price obtained from CRSP.



(a) *Quasi-indexer* $(t-1)$



(b) *Transient* $(t-1)$



(c) *Dedicated* $(t-1)$

Note: Figure 5 plots percentage of firms with management forecasts in year t across deciles of ownership by *Quasi-indexers*(sub-figure a), *Transient* investors (sub-figure b), and *Dedicated* investors in year $t-1$. All sub-figures plot the 95% confidence interval around the mean values for each decile.

Figure 3.5: Likelihood of Management Forecast Across Deciles of Three Types of Institutional Ownership

Table 3.4: Investor Attention and Management Forecast

This table presents results from estimating the relation between investor attention and the likelihood of management forecast using the following specification:

$$MF_{i,t} = \alpha_t + \alpha_j + \beta Capacity_{i,t-1} + \gamma Capacity^2_{i,t-1} + Controls_{i,t-1} + \epsilon_{i,t}$$

where α_t is year fixed effect and α_j industry fixed effect. The dependent variable $MF_{i,t}$ equals to one if a firm i makes a forecast on future earnings in year t. The variable of interest is $Capacity^2_{i,t-1}$, which is the squared term of lagged either *Capacity* (*percent*) and *Capacity* (*ratio*) at firm level. All independent variables are lagged one period relative to management forecasts. t statistics are in parentheses and all standard errors are clustered at firm level. * indicates statistical significance at the 10% level, ** at the 5% level, and *** at the 1% level.
Our theory predicts an inverse-U relation between investor attention and management forecast. This prediction would be empirically supported with a significantly negative estimated coefficient γ. Panel A reports results from ordinary least squares regressions (OLS) and Panel B logistic (Logit) regressions.

Panel A: Ordinary Least Squares Regressions

	(1)	(2)	(3)	(4)	(5)	(6)
Dep. Var.	Management Forecast ($MF_{i,t}$)					
$Capacity\ (ratio)_{i,t-1}$	0.610***		0.603***		0.109***	
	(11.33)		(11.66)		(3.81)	
$Capacity\ (ratio)^2_{i,t-1}$	−0.271***		−0.256***		−0.012***	
	(-5.91)		(-6.04)		(-4.22)	
$Capacity\ (percent)_{i,t-1}$		0.536***		0.470***		0.126***
		(6.67)		(6.11)		(3.96)
$Capacity\ (percent)^2_{i,t-1}$		−0.217***		−0.149**		−0.00046***
		(-3.03)		(-2.20)		(-3.89)
$Loss_{i,t-1}$					−0.180***	−0.179***
					(-9.22)	(-9.25)
$EPS\ Increase_{i,t-1}$					0.0198**	0.0191**
					(2.16)	(2.08)
$Abs.\ EPS\ Change_{i,t-1}$					−0.022***	−0.0225***
					(-3.17)	(-3.21)
$Leverage\ Ratio_{i,t-1}$					0.0620	0.0599
					(1.43)	(1.38)
$Size_{i,t-1}$					0.0342***	0.0338***
					(5.21)	(5.12)
$Book\ to\ Market_{i,t-1}$					−0.0440**	−0.0475**
					(-2.34)	(-2.51)
$Constant$	0.203***	0.208***	-0.0126	-0.0180	-0.0291	-0.0380
	(13.92)	(10.84)	(-0.11)	(-0.15)	(-0.19)	(-0.25)
Year FE			✓	✓	✓	✓
Industry FE			✓	✓	✓	✓
Controls					✓	✓
Observations	16,508	16,508	16,508	16,508	16,508	16,508
Adjusted R^2	0.035	0.026	0.146	0.138	0.193	0.192

Table 3.4: Investor Attention and Management Forecast, continued.

Panel B Logit Regressions

	(1)	(2)	(3)	(4)	(5)	(6)
Dep. Var.	Management Forecast ($MF_{i,t}$)					
$Capacity\ (ratio)_{i,t-1}$	2.865***		3.241***		0.787***	
	(10.50)		(10.80)		(4.42)	
$Capacity\ (ratio)^2_{i,t-1}$	−1.368***		−1.502***		−0.139***	
	(−6.23)		(−6.44)		(−2.82)	
$Capacity\ (percent)_{i,t-1}$		2.448***		2.466***		1.384***
		(6.66)		(6.22)		(12.29)
$Capacity\ (percent)^2_{i,t-1}$		−1.066***		−0.894***		−0.00499***
		(−3.41)		(−2.68)		(−12.04)
$Constant$	−1.334***	−1.279***	−2.641***	−2.640***	−2.343***	−2.443***
	(−16.87)	(−13.10)	(−3.23)	(−3.21)	(−2.64)	(−3.10)
Year FE			✓	✓	✓	✓
Industry FE			✓	✓	✓	✓
Controls					✓	✓
Observations	16,508	16,508	16,508	16,508	16,508	16,508
Adjusted R^2	0.035	0.026	0.146	0.138	0.193	0.192

Table 3.5: Results from Spline Regressions

This table presents our results from estimating a spline regression that treats the relation between the likelihood of management forecast and investor attention as piecewise linear. We estimate a separate slope for each side of a threshold τ of investor attention as follows:

$$MF_{i,t} = \alpha_t + \alpha_j + \beta_1(Capacity_{i,t-1} - \tau < 0) + \beta_2(Capacity_{i,t-1} - \tau \geqslant 0) + Controls_{i,t-1} + \epsilon_{i,t}.$$

If our theoretical prediction holds, we expect to see that $\beta_1 > 0$ and $\beta_2 < 0$. By eyeballing Figure 4, we conjecture that the threshold is around the 80^{th} percentile of both *Capacity(percent)* and *Capacity(ratio)*. For robustness, we set $\tau = 70^{th}, 75^{th}, 80^{th}, 85^{th}$ percentile of both *Capacity(ratio)* (Panel A) and *Capacity(percent)* (Panel B). Panel C presents our results estimated from the Multivariate Adaptive Regression Spline (MARS) method.

Panel A: *Capacity(ratio)* with pre-specified τ

	(1) $\tau = 70^{th} pctile$	(2) $\tau = 75^{th} pctile$	(3) $\tau = 80^{th} pctile$	(4) $\tau = 85^{th} pctile$
$Capacity(ratio) - \tau < 0$	0.413***	0.415***	0.412***	0.395***
	(19.91)	(14.93)	(15.51)	(15.58)
$Capacity(ratio) - \tau \geqslant 0$	-0.212***	-0.226***	-0.245***	-0.274***
	(-3.80)	(-2.90)	(-2.70)	(-2.59)
Year FE	✓	✓	✓	✓
4-digit SIC FE	✓	✓	✓	✓
Controls	✓	✓	✓	✓
Observations	16,508	16,508	16,508	16,508
Adjusted R^2	0.149	0.154	0.155	0.152

Panel B: *Capacity(percent)* with pre-specified τ

	(1) $\tau = 70^{th} pctile$	(2) $\tau = 75^{th} pctile$	(3) $\tau = 80^{th} pctile$	(4) $\tau = 85^{th} pctile$
$Capacity(ratio) - \tau < 0$	0.371***	0.370***	0.371***	0.368***
	(12.56)	(13.08)	(13.69)	(14.09)
$Capacity(ratio) - \tau \geqslant 0$	-0.285**	-0.334**	-0.439***	-0.409**
	(-2.42)	(-2.51)	(-2.74)	(-2.11)
Year FE	✓	✓	✓	✓
4-digit SIC FE	✓	✓	✓	✓
Controls	✓	✓	✓	✓
Observations	16,508	16,508	16,508	16,508
Adjusted R^2	0.145	0.146	0.148	0.147

Panel C: Multivariate Adaptive Regression Spline (MARS) with optimal threshold τ^*

	(1) $\tau^* = 79^{th}$ *pctile of Capacity(ratio)*	(2) $\tau^* = 81^{th}$ *pctile of Capacity(percent)*
$Capacity\ (ratio) - \tau < 0$	0.397***	
	(15.24)	
$Capacity\ (ratio) - \tau \geqslant 0$	-0.208**	
	(-2.49)	
$Capacity\ (percent) - \tau < 0$		0.366***
		(13.87)
$Capacity\ (percent) - \tau \geqslant 0$		-0.303**
		(-2.01)
Year FE	✓	✓
4-digit SIC FE	✓	✓
Controls	✓	✓
Observations	16,508	16,508
Adjusted R^2	0.153	0.159

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Table 3.6: Three Types of Institutional Investor Ownership and Management Forecast

This table presents results from estimating the relation between each of the three types of institutional investor ownership and the likelihood of management forecast using the following specification:

$$MF_{i,t} = \alpha_t + \alpha_j + \beta Capacity_{i,t-1} + \gamma Capacity^2_{i,t-1} + Controls_{i,t-1} + \epsilon_{i,t}$$

where α_t is year fixed effect and α_j industry fixed effect. The dependent variable $MF_{i,t}$ equals to one if a firm i makes a forecast in year t. We replace $Capacity_{i,t-1}$ and $Capacity^2_{i,t-1}$ with either ownership by *Quasi-indexer*, *Transient*, or *Dedicated* investors at $t-1$ and their squared terms, respectively. Standard errors are clustered at firm-level and t statistics are in parentheses. * indicates statistical significance at the 10% level, ** at the 5% level, and *** at the 1% level.

	Management Forecast		
	(1)	(2)	(3)
Quasi-indexer	0.485***		
	(4.30)		
Quasi-indexer^2	-0.102		
	(-0.85)		
Transient		0.879***	
		(4.78)	
Transient^2		-1.387***	
		(-3.11)	
Dedicated			0.385
			(1.28)
Dedicated^2			-3.792***
			(-2.80)
Constant	0.0140	0.148	0.194
	(0.07)	(0.79)	(1.06)
Year FE	✓	✓	✓
4-digit SIC FE	✓	✓	✓
Controls	✓	✓	✓
Observations	16,508	16,508	16,508
Adjusted R^2	0.189	0.172	0.170

3.8.3 Micro foundation of the loss function

Consider an economy with a firm and one representative investor. The firm is operated by a manager (he) who maximizes the firm's value. The representative investor (she) is risk-neutral and has unlimited access to capital. She decides how much capital to invest in the firm after release of the accounting report, but before the realization of cash flows.[24]

The firm has an investment opportunity with diminishing-returns-to-scale in capital. The terminal net cash flow $\tilde{\omega}$ is determined by capital k invested and a productivity shock $\tilde{\theta} \in [0,1]$. Given the realized productivity θ, the net cash flow is given by

$$\omega = 2\theta k - k^2.$$

The realized productivity is potentially observable to the firm's manager only and can be verifiably disclosed to the investor who chooses capital investment conditional on information about θ.

Preferences To maximize the expected net cash flow, the investor chooses attention allocated to the firm's signal and the capital investment. To maximize the expected stock price (net of any proprietary cost), the firm chooses whether to disclose the signal.

Timeline Timeline of the game is as follows. There are four dates $t \in \{0,1,2,3\}$. At $t = 0$, the firm's manager decides whether to disclose a verifiable signal about productivity θ and the investor decides his information set simultaneously. The manager chooses the signal d from $\{\theta, ND\}$ subject to proprietary costs (Verrecchia 1983) or uncertainty about information endowment (Dye 1985). The representative investor chooses the attention allocation that is given by the information set $\{A_i\}_{i=1}^{I}$. At $t = 1$, the manager discloses the signal if deciding to do so and the investor observes the partition element A_i such that the disclosure d is in A_i. At $t = 2$,

[24] Our setup does not require any private information, and a representative investor is a good approximation to a capital market that allocates attention and capital efficiently given the publicly available information. The representative investor can be interpreted as the aggregation of all market participants, including investors, financial analysts, and the news media who are subject to similar attention constraints in their learning processes.

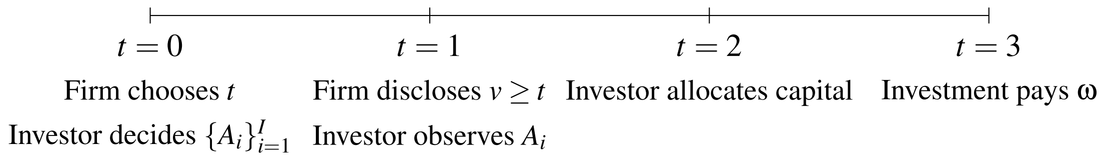

Figure 3.6: Time line

the investor invests capital in the firm. At the last date $t=3$, the cash flow from the investment is realized and distributed to the investor.

The solution concept is subgame perfect equilibrium.

Attention Allocation We solve the model by backward induction. Given the observed partition element A_i *(not d)*, the expected payoff of the investor is

$$\begin{aligned} \mathbb{E}(\tilde{\omega}|d \in A_i) =& \mathbb{E}(2\tilde{\theta}k - k^2|d \in A_i) \\ =& 2\,\mathbb{E}(\tilde{\theta}|d \in A_i)k - k^2. \end{aligned} \tag{3.22}$$

It is clear that the investor would choose the optimal amount of capital $k_i^*(A_i) = \mathbb{E}(\tilde{\theta}|d \in A_i)$ as a function of the partition element he observes. So the investor's payoff wth productivity θ is

$$\begin{aligned} & 2\theta\,\mathbb{E}(\tilde{\theta}|d \in A_i) - (\mathbb{E}(\tilde{\theta}|d \in A_i))^2 \\ =& -(\theta - \mathbb{E}(\tilde{\theta}|d \in A_i))^2 + \theta^2. \end{aligned}$$

The expectation $\mathbb{E}(\tilde{\theta}|d \in A_i)$ is determined in equilibrium, i.e., the investor correctly anticipates the disclosure decision of the manager. Assuming the monotonic partition, we have

$$\mathbb{E}(\tilde{\theta}|d \in A_i) = \int_{a_{i-1}}^{a_i} \theta f(\theta) d\theta / \int_{a_{i-1}}^{a_i} f(\theta) d\theta$$

for $i = 2, \ldots I$; and

$$\mathbb{E}(\tilde{\theta}|d \in A_1) = (p \int_0^{a_1} \theta f(\theta) d\theta + (1-p) \int_0^1 \theta f(\theta) d\theta) / (p \int_0^{a_1} f(\theta) d\theta + (1-p)).$$

Lastly, the investor's attention choice can be written as

$$\mathbb{E}(\omega^*) = -[p\int_0^1 (\theta - \mathbb{E}(\tilde{\theta}|D(\theta) \in A_i))^2 f(\theta)d\theta$$
$$+ (1-p)\int_0^1 (\theta - \mathbb{E}(\tilde{\theta}|d \in A_1))^2 f(\theta)d\theta] + \int_0^1 \theta^2 f(\theta)d\theta.$$

An equivalent objective is to maximize

$$-[p\int_0^1 (\theta - \mathbb{E}(\tilde{\theta}|D(\theta) \in A_i))^2 f(\theta)d\theta + (1-p)\int_0^1 (\theta - \mathbb{E}(\tilde{\theta}|d \in A_1))^2 f(\theta)d\theta].$$

Hence we provide a foundation for our quadratic loss function.

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Chapter 4

Cheaper talk

Abstract: Technological advancement has been lowering the cost of information provision. Thanks to new information technologies, everyone with access to the Internet can provide information at minimal cost. In this paper, we introduce a fixed cost of "talking" into the canonical cheap talk model and allow the sender to choose whether to "talk" or not. We explore the following questions: (1) Is a sender with more accurate information or a smaller bias necessarily more valuable to the receiver? (2) Does a lower "talking cost" always benefit the receiver? (3) Will the sender choose ex ante the communication technology that is optimal for communication? The main results are: (1) A sender with less accurate information or a larger bias can be more valuable to the receiver by being more motivated to provide information. (2) Too high a talking cost discourages information provision, whereas too low a cost can reduce the effectiveness of communication. (3) A sender may prefer a "cheaper" but less effective communication technology ex ante.
Keywords: Cheap talk; costly information provision; communication; noisy signal; bias
JEL codes: D80, D81, D83

4.1 Introduction

It is now extremely easy for people to make statements available to a wide audience. Anyone with access to the Internet can post their ideas on social media. It was not always like this. Historically, information used to be recorded by handwriting and distributed manually. The invention of mechanical means of reproducing writing, for example the press printing, substantially lowered the cost of distributing information. Nevertheless, it was not until the emergence of the Internet that the cost of information provision dramatically declines. This paper explores the implications of the information provision cost in a cheap talk model: Two players care about an unknown state of the world and an action to be taken. The informed party (the "sender") potentially provides information to the decision maker (the "receiver") who takes the action. Two scenarios are considered: the sender is unbiased (i.e., he agrees with the receiver on what action is preferred conditional on the state) but can be imperfectly informed, or the sender is perfectly informed but can be biased. We deviate from the canonical cheap talk model by assuming that the sender incurs a fixed cost if he sends a message ("talk") and by allowing the sender to decide whether to talk or not. While players dislike the loss from a poorly informed action, the sender bears the cost of communication. We ask the following three questions: (1) Is a sender with more accurate information or a smaller bias necessarily more valuable to the receiver? (2) Does a lower talking cost always benefit the receiver? (3) Will the sender choose ex ante the communication technology (indexed by the talking cost) that is optimal for communication?

To examine the implications of inaccurate information, we assume that the sender can be imperfectly informed and that, in any state of the world, the players agree on which action is preferred (Section 4.3). There however remains a conflict between the sender and the receiver because only the sender pays the communication cost. Communication between the players can be perfect (i.e. fully-revealing about the sender's signal) if the sender sends a message. We show that a less-informed sender can benefit the receiver by being more willing to supply information than a better-informed sender. Informally, due to a more disperse posterior distribution, a less-informed sender views extreme states as more likely. If losses from large mistakes are extremely high, given the same posterior expectation, a less-informed sender would expect bigger loss from not communicating with the receiver than a better-informed sender. Hence, a sender with less accurate information may provide information more often than a better informed sender, and is possible to be more valuable to the receiver ex ante.

To examine the implications of the conflict of interests, we study a model where the sender is perfectly informed but disagrees with the receiver on favorite actions (Section 4.4). We examine how the receiver's expected payoff changes with the size of the bias. We show that a more severe disagreement can make the sender more motivated to talk. It is therefore possible that a more biased sender can deliver a higher payoff to the receiver in expectation.

Although too high a cost discourages information provision, a moderate talking cost can lead to more effective communication in the presence of disagreement. Intuitively, the ineffectiveness of strategic communication comes from the following tension: the sender tends to exaggerate should the receiver believe him. The receiver will hence not take the sender's statement at face value, resulting in limited communication. Nevertheless, if it is costly for the sender to talk, the sender would only be motivated to talk when the action induced by "silence" is a big mistake to take given the state. As a result, communication credibly indicates that the state is large, which better aligns the sender's and the receiver's preferences.[1]

If a moderate talking cost can facilitate communication, will the sender prefer a communication technology featuring a moderate talking cost ex ante? To answer this question, we assume the sender can choose a talking cost before observing the state and compare the sender's favorite talking cost with that of the receiver. We show that the receiver and the sender have different rankings of preferred talking costs. Since a higher talking cost in general implies a lower ex ante probability of talking, it is ambiguous how the cost affects the sender's ex ante expenditure of talking. The comparison between the sender's and receiver's favorite communication technology is also in general ambiguous.

Our main findings are: (1) Absent difference in preferences over the actions conditional on states ("no disagreement"), a sender with less accurate information - modeled as more "disperse" posterior belief - can be more willing to talk. We provide a necessary condition under which the receiver is better off communicating with a less-informed sender than a better-informed one. In the presence of disagreement, a sender with a larger bias can also be more motivated to talk than a less-biased sender. Therefore, the receiver does not necessarily prefer a sender with more accurate information or a smaller bias. (2) If the players disagree on favorite actions, a moderate talking cost can mitigate disagreement and improve communication relative to no cost. Too high a cost discourages information provision, whereas too low a cost can reduce the effectiveness of communication; (3) Ex ante, a sender may prefer a "cheaper" communication

[1] There are multiple equilibrium state-action distributions. We focus on the type of equilibria where sending no message induces the lowest action. We defer a more complete characterization and discussion to the appendix.

technology that is suboptimal in terms of the receiver's expected payoff. The comparison between the players' favorite talking cost is however ambiguous in general.

Although the model does not fully capture all the main forces in play when it comes to communication over the media, it allows us to highlight some important trade-offs and serves as a benchmark. By identifying several situations in which seemingly "intuitive" comparative statics results fail to hold, insights gained from the model could shed light on the implications of the emergence of new media. Thanks to blogs and social media, anyone with access to the Internet can easily provide information. Such information, nevertheless, is oftentimes inaccurate, and the communication is affected by personal preferences. The problems of inaccuracy and bias are well recognized by the audience.[2] Survey results show that social media receives the lowest level of trust among different types of media.[3] Among the very people who consume news on social media, 57% expect news there to be largely inaccurate.[4] The following question naturally arises: why is social media news valuable to the audience despite the inaccuracy and bias? We show that lower accuracy or larger bias does not necessarily translate into lower value of which a sender is to a receiver even in this simplest communication setting. The counter-intuitive phenomenon mentioned above is in fact readily rationalizable. Our result hinges on the existence of a strictly positive talking cost. Given the talking cost, we highlight the novel trade-off between the sender's "quality" and his willingness-to-talk.

Behind the abundance of low quality information providers is the substantial drop in the cost of information provision. As our second finding and its two-sender extension reveal, compared to traditional media that features a moderate talking cost, the trivial cost implied

[2] "When asked for reasons why they do not trust news organizations, Americans top categories of answers largely focus on inaccuracy and bias." Source: Knight Foundation and Gallup. https://www.knightfoundation.org/reports/indicators-of-news-media-trust

[3] In 2016, 82% and 76% U.S. adults have at least some trust in local and national news organizations respectively, while only 34% do so in social media. Source: Pew Research Center. https://www.pewresearch.org/fact-tank/2018/07/30/newsroom-employment-dropped-nearly-a-quarter-in-less-than-10-years-with-greatest-\decline-at-newspapers/

[4] Source: Pew Research Center. https://www.journalism.org/2018/09/10/news-use-across-social-media-platforms-2018/pj_2018-09-10_social-media-news_0-01/

by social media can consolidate bias and worsen communication. This is consistent with the documented declining trust in news media.[5] Our last finding states that a sender in general prefers a different talking cost from what is the best for communication effectiveness. We view this result as consistent with the fact that different types of media coexist.

Related Literature

In this paper, we explore under what conditions a receiver would prefer a sender with a less accurate signal. This question puts our paper in line with the literature that discusses how a decision-maker's payoff changes with information quality (e.g. Blackwell (1953), Lehmann (1988), Persico (2000), Athey and Levin (2018)). In particular, we use the concept of "accuracy" developed by Lehmann (1988) and used in Persico (2000) to order different information structures. Furthermore, our paper is also related to the literature studying how the distribution of optimal actions changes with information quality (e.g. Ganuza and Penalva (2006), Ganuza and Penalva (2010), Johnson and Myatt (2006)). The aforementioned literature mainly focuses on a decision-making environment where there is no strategic interaction and information quality affects the decision-maker's payoff only through the relevant distribution.

Another line of related literature studies the implication of information quality when the receiver accesses information through a strategic sender. In this literature, information quality also affects the decision-maker's payoff directly through the sender's strategy. Even the signal can be ordered by accuracy, the strategic tension implies that the information received by the receiver is no longer monotonic in the signal accuracy. Fischer and Stocken (2001b) demonstrate this non-monotonicity in a canonical cheap talk model where the players disagree on favorite actions. In this paper, we show that an arguably weaker strategic tension - the existence of a talking cost - can break the monotonicity in the absence of disagreement. Blume et al. (2007) also shows that adding noise to the sender's message can improve communication effectiveness

[5] According to a survey by Knight Foundation and Gallup, 69% of U.S. adults say their trust in the news media has decreased in the past decade. Source:https://www.knightfoundation.org/reports/indicators-of-news-media-trust

by moving players' preferences closer, but the underlying reason for this effect is different from that in our model: In our model, when "silence" induces the lowest action, an upwardly biased sender finds it less profitable to induce a higher action when talking is costly; In Blume et al. (2007), a noisy message distorts the receiver's posterior mean towards prior mean. A noisy message sent in low states therefore brings the receiver's posterior mean closer to the sender's favorite action.

Szalay (2005) studies a delegation problem with costly information acquisition. He shows that the receiver can be better off if the sender is not allowed to take actions close to the prior optimal action. Although different in setting and fundamental driving forces, Szalay (2005) is related to our paper in the following sense: In Szalay (2005), the sender is more motivated to pay the cost to acquire information because the loss of making a mistake is more severe if the prior optimal action is removed. In our paper, a sender with a less accurate signal can be more motivated to supply information because his expected loss is more severe if the receiver takes the default action.

The second part of the paper studies the situation in which the players disagree on favorite actions. We build on the canonical cheap talk model by Crawford and Sobel (1982). The main difference is that we endogenize the "talking" decision by introducing a fixed cost incurred by the sender when a message is sent. There are various ways to introduce a cost into a communication model in the literature, such as state-dependent cost (e.g. "lying cost" as in Kartik (2009)), partition-dependent cost (e.g. "diagnosis cost" as in Cremer et al. (2007)), and message-dependent cost (e.g. Austen-Smith and Banks (2002), Kartik (2007), Hertel and Smith (2013)) which is the case in our model. In Austen-Smith and Banks (2002) and Kartik (2007), the sender can freely choose the cost associated with each message. With the power to choose any cost for any message, the sender can achieve almost full revelation. In a closely related paper, Hertel and Smith (2013) study a cheap-talk model where there are discrete messages differing in the cost to send and characterize the equilibria. Our model can be viewed as a special version of theirs in the sense that we have one free message and all other messages cost the same. The equilibria therefore share similar patterns: the cutoff types are indifferent between two adjacent equilibrium messages taking into account the corresponding costs. Our special cost structure however allows us to further explore comparative statics questions with respect to the cost. We show that the talking cost brings about two opposites forces: on the one hand, consistent with Austen-Smith and Banks (2002) and Hertel and Smith (2013), costly talking mitigates the disagreement on favorite actions. On the other hand, it disincentivizes the sender to supply information. As a consequence, we find that the relationship between the receiver's payoff and the sender's talking cost is in general not monotonic.

Our model is related to the costly verifiable disclosure literature studied by Jovanovic (1982), Verrecchia (1983b), Hedlund (2015), and Kartik et al. (2017), but differs in implications. In this literature, messages are verifiable and the sender's favorite action is typically independent of the state. The equilibrium threshold of disclosure is strictly increasing in the proprietary cost. If the cost rises, the receiver will obtain less disclosure and be worse off. In contrast, we suggest a distinct comparative statics result with respect to the cost when messages are not verifiable. Because the communication outcome conditional on disclosure can be enhanced by a moderate cost that mitigates the conflict between the sender and receiver, the monotonic result of information transmission in the disclosure model breaks down. The receiver can benefit from a moderate cost relative to no cost. Hedlund (2015) shows that the unraveling result about verifiable information derived by Milgrom (1981b) and Grossman (1981b) is robust to costly reporting. Kartik et al. (2017) develop a theory of multi-sender voluntary disclosure and analyze how the cost of information disclosure influences the competition between senders.

After setting up the formal model in Section 4.2, we analyze the model to study the implications of signal accuracy (Section 4.3) and disagreement between the players (Section 4.4). All proofs are in the Appendix.

4.2 The Model

We consider an environment where there is one receiver R (*she*) who demands information and one sender S (*he*) who potentially supplies information. Both players are interested in learning about an uncertain state θ. Assume the state space is an interval on $\mathbb{R}$ and denote it by Θ. The players' prior follows some distribution $H(\theta)$ with mean μ_h and support Θ.

The sender observes a potentially noisy signal $\psi \in \Theta \subset \mathbb{R}$ that is informative about the true state. Let λ be a parameter that reflects the signal accuracy. Higher λ implies higher signal accuracy. Let $F^{\lambda}(\theta, \psi)$ be the joint distribution of θ and ψ when the signal accuracy is λ and $f^{\lambda}(\theta, \psi)$ be the density function which is assumed to be continuous.

Upon observing the signal, the sender can decide whether to send a message m and what message to send. The special message $m^0 \in M = \Theta$ denotes the sender's choice of not sending a message. The receiver then takes an action $a \in A = \Theta$ upon receiving the message (or receiving no message). The strategy of a sender with signal accuracy λ is $m^{\lambda} : \Psi \mapsto M$ and the strategy of the receiver is $a(m) : M \mapsto A$.[6]

The players want to minimize loss from making a mistake. Their specific payoffs are given by:

$$\text{Receiver:} \qquad u^R(a, \theta) = -L(|\theta - a|);$$

$$\text{Sender:} \qquad u^S(a, \theta) = -L(|\theta + b - a|) - c \cdot \mathbf{1}_{m \neq m^0},$$

where L is the loss function with $L(0) = L'(0) = 0$, $L'(x) > 0$ and $L''(\cdot) > 0$, $\forall x > 0$. Both players prefer higher actions to be taken when the state is higher. The curvature of L reflects the size of loss associated with a wrong action. The loss becomes larger as the mistake becomes bigger, i.e., as the gap between the action and the state grows wider. The sender in particular may prefer a higher action than what the receiver prefers, represented by an upward bias $b \geq 0$. In addition, the sender incurs a cost $c > 0$ if $m \neq m^0$, i.e., costly communication.

[6]Although it reduces the set of equilibria, restricting attention to pure strategies does not affect the set of state-action distributions that can be induced by an equilibrium profile.

Let $\sigma^* = (a^*(m), m^{\lambda *}(\psi))$ denote an equilibrium. The equilibrium concept we are using is Bayesian Nash equilibrium.

4.2.1 Preliminaries

We assume all relevant expected values exist. We introduce some useful notation. Let

$$a^S(\psi;\lambda) \in \arg\max_a \mathbb{E}_\theta[-L(|a-\theta-b|)|\psi;\lambda]$$

be a favorite action of the sender when he observes a signal ψ with accuracy λ. The existence of $a^S(\psi,\lambda)$ comes from the assumptions of the loss function $L(\cdot)$.

Also define

$$a^R(\underline{\theta},\bar{\theta};\lambda) \in \begin{cases} \arg\max_a \mathbb{E}(-L(|a-\theta|)|\mathbb{E}(\theta|\psi;\lambda) \in [\underline{\theta},\bar{\theta}]), & \text{if } \underline{\theta} \neq \bar{\theta} \\ \arg\max_a \mathbb{E}(-L(|a-\theta|)|\mathbb{E}(\theta|\psi;\lambda) = \underline{\theta}), & \text{if } \underline{\theta} = \bar{\theta} \end{cases} \tag{4.1}$$

as a favorite action of the receiver when she thinks the sender's posterior expected state is within the interval $[\underline{\theta},\bar{\theta}]$.

Finally, define $U^S(a,\psi;\lambda) = \mathbb{E}_\theta[u^S(a,\theta)|\psi;\lambda]$. This is the sender's expected payoff from an action a if he observes signal ψ.

We sometimes suppress λ when doing so would cause no confusion.

4.3 Signal Accuracy

Can it ever be the case that a sender with a less accurate signal is more valuable to the receiver? To explore this question, we shut down the influence of the bias by assuming $b = 0$ and focus on how the sender's signal accuracy (indexed by λ) affects the receiver's ex ante payoff.[7] As we show in this section, a sender with a less accurate signal can benefit the receiver by being more willing to talk. After characterizing a particular class of equilibria, we provide a necessary and sufficient condition under which a less-informed sender is more willing to talk, which can possibly make such a sender more valuable to the receiver despite the less accurate signal.

We maintain the following assumption in this section.

Assumption 3. $b = 0$.

To order information structures, we adopt the "accuracy" notion in Lehmann (1988). Lehmann (1988) considers the following class of distributions $F^{\lambda}(\theta, \psi)$.

Assumption 4. *(Milgrom and Weber, 1982)* θ *and* ψ *are affiliated. That is,* $\theta' > \theta, \psi' > \psi \Rightarrow f^{\lambda}(\theta', \psi') f(\theta, \psi) \geq f^{\lambda}(\theta, \psi') f^{\lambda}(\theta', \psi)$.

Roughly speaking, affiliation between θ and ψ implies that a high realization of ψ is associated with higher values of θ.

The following definition orders information structures and formalizes what we mean by "accuracy".

Definition 20. *(Lehmann, 1988) Given two signal structures indexed by* λ'', λ'*, we say the signal structure* λ'' *is more* accurate *than the signal structure* λ' *if* $T_{\lambda'',\lambda',\theta}(\psi) \equiv F^{\lambda''(-1)}$ $(F^{\lambda'}(\psi|\theta)|\theta)$ *is nondecreasing in* θ *for every* ψ.

In other words, $T_{\lambda'',\lambda',\theta}(\psi|\theta;\lambda')$ is distributed as $\psi|\theta;\lambda''$. For example, a normally distributed signal is more accurate than another normally distributed signal if the former has lower variance than the latter.

Definition 20 concerns the conditional signal distribution. The payoff relevant distributions - the posterior distributions and the distribution of posteriors induced by a signal structure - nevertheless hinge on both the conditional signal distribution and the prior distribution. To impose more structure on the payoff relevant distributions, we first formalize the notion of "dispersion".

[7] In the presence of disagreement, i.e., $b > 0$, the receiver's payoff is in general not monotonic in the sender's signal accuracy. Fischer and Stocken (2001b) show this within a special class of information structure.

Definition 21. *(Ganuza and Penalva, 2006; Johnson and Myatt, 2006) A distribution* $F^{\lambda''}$ *is more single-crossing disperse (denoted by* $F^{\lambda''} \succ_{SC} F^{\lambda'}$*) than a distribution* $F^{\lambda'}$ *if there exists* x_0 *such that* $F^{\lambda''}(x_0) > F^{\lambda'}(x_0) \Rightarrow F^{\lambda''}(x) \geq F^{\lambda'}(x), \forall x \geq x_0$.

Let $\mu(\psi;\lambda) \equiv \mathbb{E}(\theta|\psi;\lambda)$ be the mean of the sender's posterior after observing the signal ψ with accuracy λ. Given ψ, the posterior distribution can be written as $G^{\lambda}(\theta|\mu)$ with the density function $g^{\lambda}(\theta|\mu)$, where $\mu = \mu(\psi;\lambda)$. Let $\Phi^{\lambda}(\mu)$ be the distribution of the posterior mean μ when the signal accuracy is λ, and $\phi^{\lambda}(\mu)$ be the density function.

To gain tractability, we impose Assumption 5 in this section. Discussion about this assumption is postponed to the end of this section.

Assumption 5.

1. $\Theta = \mathbb{R}$.

2. $\forall \mu, g^{\lambda}(\theta|\mu)$ *is symmetric with respect to* μ*;* $g^{\lambda}(\theta|\mu) = g^{\lambda}(\theta + x|\mu + x), \forall x$.

3. $\phi^{\lambda}(\mu)$ *is unimodal and symmetric with respect to the prior mean* μ_h.

Note that the density functions of posterior belief have the same shape but only differ in their locations. The distribution of posterior distributions induced by an information structure indexed by λ can therefore be reduced to the distribution of posterior means $\Phi^{\lambda}(\mu)$.

The following assumption is key to Proposition 27 and Corollary 5. It states that any posterior distribution of a less-informed sender is more disperse in the sense of Definition 21 than that of a better-informed sender, and that the distribution of posterior mean of a better-informed sender is more disperse than that of a less-informed sender.

Assumption 6. $\forall \lambda'' > \lambda'$*, for any given* μ*,* $G^{\lambda'}(\theta|\mu) \succ_{SC} G^{\lambda''}(\theta|\mu)$*. Furthermore,* $\Phi^{\lambda''}(\mu) \succ_{SC} \Phi^{\lambda'}(\mu)$.

As a leading example, conditions in Assumption 5 and Assumption 6 are satisfied if both the prior distribution and signals' conditional distributions are normal. That is, the prior distribution is given by $\theta \sim N(\mu_h, \sigma_h)$ and the conditional signal distribution is given by $\psi|\theta \sim N(\theta, \frac{1}{\lambda})$.

Lemma 22. *The sender's favorite action is the mean of the posterior plus the bias, i.e.,* $a^S(\psi;\lambda) = \mu(\psi;\lambda)$ *for* $b = 0$.

Given that the players have the same preferences for favorite actions, the only conflict between them is that the receiver always wants more information but the sender does not always "talk" due to the cost. More specifically, given a receiver's strategy $a(m)$, the sender's best response is such that he will not talk if the favorite action $a^S(\psi;\lambda)$ is sufficiently close to the action induced by no message $a(m^0)$. The following lemma states that the posterior means for which the sender does not talk constitute an interval.

Lemma 23. *Given a strategy* $a(m)$ *of receiver, let* $\tilde{m}^\lambda(\psi)$ *be a best response of the sender. There exists an interval* $[\theta_1(\lambda), \theta_2(\lambda)]$ *such that* $\tilde{m}^\lambda(\psi) = m^0$ *if and only if* $\mu(\psi;\lambda) \in [\theta_1(\lambda), \theta_2(\lambda)]$.

Provided that a message is sent, communication could be perfect (i.e., fully-revealing), since the players agree on favorite actions. It is therefore not difficult to find an equilibrium of this game from Lemma 23.[8]

Proposition 26. *The following strategy profile is an equilibrium profile.*

$$m^{\lambda *}(\psi) = \begin{cases} m^0 & \text{if } \mu(\psi;\lambda) \in [\theta_1(\lambda), \theta_2(\lambda)] \\ \mu(\psi;\lambda) & \text{otherwise} \end{cases}$$

$$a^*(m) = \begin{cases} a_0 \equiv a^R(\theta_1, \theta_2) & \text{if } m = m^0 \\ m & \text{otherwise .} \end{cases}$$

For the rest of this paper we will refer to $[\theta_1(\lambda), \theta_2(\lambda)]$ - the set of favorite actions for which the sender with signal accuracy λ does not send a message - as the **silence interval**. We

[8] As in Crawford and Sobel (1982), there exists many other equilibria outcomes where the communication (when the sender talks) is less effective than it can possibly be. Nevertheless, if we restrict attention to the class of equilibria where communication is perfect if the sender talks, the equilibrium outcome implied by Proposition 26 is unique. Further discussion about the multiplicity can be found in Appendix B.

also refer to a_0 - the action taken by the receiver when she receives no message - as the **default action**.

The following lemma pins down the position of the default action a_0 and suggests that a_0 is independent of the sender's signal accuracy.

Lemma 24. $a_0 = \mu_h$.

Lemma 24 hinges on the symmetry of $\phi^{\lambda}(\mu)$. Because the two cutoff sender types $\theta_1(\lambda)$ and $\theta_2(\lambda)$ are indifferent between talking and not talking, they must be symmetric about the default action. Therefore, the default action - the receiver's posterior mean if she receives no message - must be the midpoint of the silence interval.

Proposition 26 and Lemma 24 together describe what happens in the equilibrium: if the sender talks, the receiver will be able to take the favorite action as if she observes the signal herself. If the sender does not talk, the receiver takes some default action as if she is making a decision under her prior belief. The sender talks if and only if the favorite action turns out to be sufficiently "surprising", that is, sufficiently far away from what the players expect under their prior belief. If the favorite action is close to the default action, the sender is unwilling to talk, which impedes the receiver taking the optimal action, because the loss from taking a slightly wrong action is low and does not justify paying the talking cost.

Now we can start to examine how a sender with a less accurate signal could benefit the receiver. The following lemma serves as a benchmark and suggests that in the absence of a talking cost, the receiver will always prefer a sender with a more accurate signal. The lemma holds by definition. It directly follows from Theorem 5.1 in Lehmann (1988).

Lemma 25. *If $c = 0$, the receiver's ex ante payoff is increasing in λ.*

As an implication of Lemma 25, the only possibility for a less-informed sender to be more valuable to the receiver is that when $c > 0$, such a sender might be more willing to talk ex post, reflected by a smaller "silence interval" in the equilibrium. In Eq. (4.2) below, $I(\mu, \lambda)$ is the difference between the sender's payoff if the receiver acts as if she can observe the sender's signal and that if the receiver takes the default action a_0. $I(\mu;\lambda)$ can hence be viewed as the **ex post benefit of talking**. The sender is willing to talk if and only if the ex post benefit of talking, i.e., the reduction in loss given by Equation (4.2), exceeds the cost of talking.

$$I(\mu,\lambda) \equiv \mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi;\lambda)=\mu] - \mathbb{E}_\theta[L(|\mu-\theta|)|\mu(\psi;\lambda)=\mu] \tag{4.2}$$

At an equilibrium of the form described in Proposition 26, a sender talks if and only if the ex post benefit of talking $I(\mu;\lambda)$ is no lower than the cost of talking. That is, $\mu \notin [\theta_1(\lambda),\theta_2(\lambda)]$ if and only if $I(\mu;\lambda) \geq c$. In particular, the two boundary sender types of the silence interval, namely $\theta_1(\lambda)$ and $\theta_2(\lambda)$, are indifferent between talking at a cost and not talking at no cost. That is, $I(\theta_1(\lambda);\lambda) = I(\theta_2(\lambda);\lambda) = c$.

For any given c, the magnitude of $I(\mu;\lambda)$ determines the size of the silence interval. Specifically, for some λ',λ'', if $I(\mu;\lambda') > I(\mu;\lambda''), \forall\mu$, then the silence interval associated with λ' is smaller than that associated with λ'', i.e., $|\theta_2(\lambda')-\theta_1(\lambda')| < |\theta_2(\lambda'')-\theta_1(\lambda'')|$.

We next examine the sender's value to the receiver. Remember that the two players have the same preferences with respect to the action when $b=0$. The receiver's payoff can be decomposed into a basic part guaranteed by always taking the default action and an incremental part thanks to the sender's information transmission.

Lemma 26. *The receiver's payoff can be decomposed as follows:*

$$U^R(\lambda) = -\mathbb{E}_\theta[L(|a_0-\theta|)] + I^+(\lambda) \tag{4.3}$$

where

$$I^+(\lambda) \equiv \int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} I(\mu;\lambda)\phi^\lambda(\mu)d\mu = \int_{I(\mu;\lambda)\geq c} I(\mu;\lambda)\phi^\lambda(\mu)d\mu$$

$I^+(\lambda)$ can be viewed as the sender's (ex ante) value to the receiver. We can also express as below the ex ante probability that a sender indexed by λ sends a message:

[8] Note that inducing the ex post favorite action gives the sender a fixed expected payoff regardless of the signal. As the signal moves away from the default action, the ex post benefit of talking increases not because the favorite action becomes better, but because the default action becomes worse.

$$P(\lambda) \equiv \int \mathbf{1}_{I(\mu;\lambda)\geq c} \phi^{\lambda}(\mu) d\mu \tag{4.4}$$

Note that λ affects $I^{+}(\lambda)$ and $P(\lambda)$ through two channels: (1) It affects the quality of the sender's messages by affecting the distribution of the sender's posterior distributions represented by $\phi^{\lambda}(\mu)$. $\phi^{\lambda}(\mu)$ is more disperse for a sender with a more accurate signal. This is because such a sender puts more weight on the signal, less weight on the prior and hence updates belief more aggressively; (2) It affects the sender's willingness to talk in terms of the set of μ such that $I(\mu;\lambda) \geq c$.

We are interested in the sign of the following terms:

$$\begin{aligned} \frac{\partial I^{+}(\lambda)}{\partial \lambda} &= \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c} I(\mu;\lambda) \phi^{\lambda}(\mu)}{\partial \lambda} d\mu \\ &= \underbrace{\int \frac{\partial \phi^{\lambda}(\mu) I(\mu;\lambda)}{\partial \lambda} \mathbf{1}_{I(\mu;\lambda)\geq c} d\mu}_{\text{The effect of } \lambda \text{ on message quality}} + \underbrace{\int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial \lambda} \phi^{\lambda}(\mu) I(\mu;\lambda) d\mu}_{\text{The effect of } \lambda \text{ on the sender's willingness to talk}} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \frac{\partial P(\lambda)}{\partial \lambda} &= \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c} \phi^{\lambda}(\mu)}{\partial \lambda} d\mu \\ &= \int \frac{\partial \phi^{\lambda}(\mu)}{\partial \lambda} \mathbf{1}_{I(\mu;\lambda)\geq c} d\mu + \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial \lambda} \phi^{\lambda}(\mu) d\mu. \end{aligned} \tag{4.6}$$

In Equation (4.5), the first term can be interpreted as how the sender's message quality changes with his signal accuracy fixing the favorite actions for which the sender does talk. The second term represents the relationship between the sender's willingness to talk and his signal accuracy. Remember that the two boundary types of the silence interval $[\theta_1(\lambda), \theta_2(\lambda)]$ feature $I(\theta_i(\lambda);\lambda) = c, i = 1,2$. If $\frac{\partial I(\mu;\lambda)}{\partial \lambda} < 0$, a sender with a less accurate signal will have a higher benefit from talking and hence a smaller equilibrium silence interval, i.e., $\theta_2(\lambda) - \theta_1(\lambda)$ is smaller.

The following proposition describes how the sender's signal accuracy affects the size of the silence interval.

Proposition 27.

1. *If L'' is strictly increasing, $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} < 0$ and the size of the silence interval $\theta_2(\lambda) - \theta_1(\lambda)$ is increasing in λ.*

2. *If L'' is strictly decreasing, $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} > 0$ and the size of the silence interval $\theta_2(\lambda) - \theta_1(\lambda)$ is decreasing in λ.*

3. *If L'' is constant, $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} = 0$ and the size of the silence interval $\theta_2(\lambda) - \theta_1(\lambda)$ is constant over λ.*

To provide some intuition, suppose without loss of generality $\mu > a_0$. Recall that the sender's payoff excluding the talking cost is given by $-L(|\theta - a|)$. Note that compared to the scenario where the state is close to μ, the default action a_0 is far from being optimal if the state is extremely high (i.e., $\theta \gg \mu > a_0$). Compared to a better-informed sender, a less-informed sender has a more disperse posterior and hence deems extremely high states as more likely. Whether the less-informed sender is more motivated to talk then depends on how fast the loss increases as the realized state θ becomes further away from a_0 (thus a_0 becomes further away from being optimal).Therefore the convexity of $L(\cdot)$ matters.

The reason why the third order derivative is relevant has to do with the fact that, compared to a better-informed sender, a less-informed sender also views extremely low states (i.e., $\theta \ll a_0 < \mu$) as more likely. If the realized θ is extremely low, the ideal action μ would be inferior to a_0. In the proof, we pair up states of equal distance to μ. Any pair of states in the form of $\mu + x$ and $\mu - x$ are viewed as equally likely for the sender due to the symmetric $g^{\lambda}(\theta|\mu)$. Conditional on the state being either $\mu + x$ or $\mu - x$, to find the expected benefit of talking, i.e., the expected benefit of inducing the action μ instead of a_0, one compares the change in payoff as the action changes from a_0 to μ when the state is $\mu + x$ with the change when the state is $\mu - x$.

Given x, this expected benefit depends on $L''(\cdot)$, because it determines how the change in $L(\cdot)$ depends on the value of the argument as the argument changes by $\mu - a_0$. Note that compared to a better-informed sender, a less-informed sender deems larger x as more likely and smaller x less likely. For example, a perfectly informed sender views $x = 0$ with probability 1 and an imperfectly informed sender views $x > 0$ with strictly positive probability. The comparison in the expected benefit of talking between a less-informed sender and a better-informed sender can be translated into a comparison about the expected benefit of talking conditional on $\theta \in \{\mu - x, \mu + x\}$ for different values of x. To be more specific, consider a pair of states $\mu + x$ and $\mu - x$ for some $x > 0$ such that $\mu - x < a_0$,[9] If the state is $\mu + x$, by changing the action from a_0 to μ, the sender's payoff changes by $-L(\mu + x - \mu) - [-L(\mu + x - a_0)] = L(x + \mu - a_0) - L(x)$. If the state is $\mu - x$, by changing the action from a_0 to μ, the sender's payoff changes by $-L(|\mu - x - \mu|) - [-L(|\mu - x - a_0|)] = L(x + a_0 - \mu) - L(x)$. Conditional on the state being either $\mu + x$ or $\mu - x$, the expected payoff change is then

$$
\begin{aligned}
&0.5[L(x+\mu-a_0)-L(x)+L(x+a_0-\mu)-L(x)]\\
=&0.5[\int_0^{\mu-a_0} L'(x+y)dy+\int_0^{-(\mu-a_0)} L'(x+y)dy],
\end{aligned}
$$

which is strictly positive due to $L''(\cdot) > 0$. The size of this expected payoff change depends on $L''(\cdot)$ locally at the interval $[x - (\mu - a_0), x + \mu - a_0]$. Note that if and only if $L'''(\cdot) = 0$, $L''(\cdot)$ is the same everywhere, and therefore $0.5[L(x + \mu - a_0) - L(x) + L(x + a_0 - \mu) - L(x)]$ is independent of x. Consequently, the overall expected benefit of changing the action from a_0 to μ across all values of x is independent of the distribution of x. Since the difference in the sender's signal accuracy essentially implies different distributions of x, if and only if $L'''(\cdot) = 0$, the overall expected benefit of talking and changing the action from a_0 to μ is independent of the sender's signal accuracy.

The following corollary states that the receiver prefers a sender with a more accurate signal if $L''' \leq 0$.

[9] A similar argument holds for the case where $\mu - x > a_0$.

Corollary 5. *If $L''' \leq 0$, the receiver prefers a sender with a more accurate signal. A sender with a more accurate signal is also more likely to talk ex ante.*

If $L''' \leq 0$, compared to a less-informed sender, a sender with a more accurate signal is more likely to talk ex ante for two reasons: First, as suggested by Proposition 27, the set of favorite actions for which he is willing to talk is larger. Second, fixing the set of favorite actions for which a sender is willing to talk, a sender with a more accurate signal is still more likely to talk ex ante. This is because the distribution of his favorite actions, or posterior means, is more disperse and hence more likely to fall out of the silence interval.[10]

If $L''' > 0$, while a sender with a more accurate signal features higher information quality, a sender with a less accurate signal could benefit the receiver by being more willing to talk (i.e., the second terms of Equation (4.5) and Equation (4.6) are negative). But in general it remains ambiguous whether a sender with a more accurate signals is more likely to talk *ex ante* and whether a receiver would prefer such a sender ex ante. That is, $P(\lambda)$ and $I^+(\lambda)$ may not be monotonic in λ.

To show that it is possible for a sender with a less accurate signal to be more likely to talk ex ante and to generate a higher expected payoff to the receiver, we provide the following numerical example generated by simulation.

Example 11. *Suppose that the prior distribution follows $\theta \sim \mathcal{N}(0, 1.5)$ and the signal ψ conditional on state θ is normally distributed as $\psi|\theta \sim \mathcal{N}(\theta, \frac{1}{\lambda})$. Further, the loss is given by $L(|a-\theta|) = (a-\theta)^4$ and the cost is $c = 10$. Figure 4.1 depicts $P(\lambda)$ and $I^+(\lambda)$.*[11]

Discussion: Assumption 5

The assumption that $\phi^\lambda(\mu)$ is unimodal and symmetric is key to Proposition 26 and Lemma 24. If μ is uniformly distributed, there exists multiple equilibria differing in the position

[10] As discussed in Ganuza and Penalva (2010), if $F^{\lambda''}(\theta, \psi)$ is more accurate than $F^{\lambda'}(\theta, \psi)$, $\Phi^{\lambda''}$ is a mean-preserving spread of $\Phi^{\lambda'}$, but $\Phi^{\lambda''}(\mu) \succ_{SC} \Phi^{\lambda'}(\mu)$ is not necessarily true. The condition that $\Phi^{\lambda''}$ is a mean-preserving spread of $\Phi^{\lambda'}$ is not sufficient for Corollary 5, because the function $I(\mu;\lambda)1_{I(\mu;\lambda)\geq c}$ is not convex in μ. Therefore a stronger assumption - Assumption 6 - is needed.

[11] Since the integration in $P(\lambda)$ and $I^+(\lambda)$ can only be done numerically, we cannot draw continuous curves.

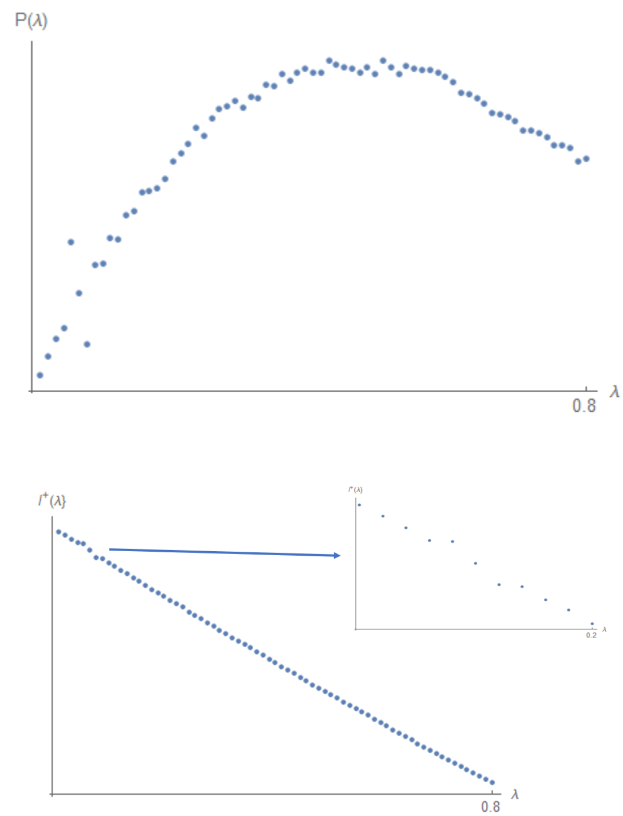

Figure 4.1: $P(\lambda), I^+(\lambda)$ in Example 11

of the silence interval and the default action. The same can happen if $\phi^\lambda(\mu)$ has multiple peaks. If $\phi^\lambda(\mu)$ is unimodal but not symmetric, the general existence of an equilibrium where communication is perfect when the sender talks is not guaranteed.

The symmetry of $g^\lambda(\theta|\mu)$ plays an important role in Proposition 27. Proposition 27 can fail if $g^\lambda(\theta|\mu)$ is sufficiently skewed. To see this, consider the example where $\mu > a_0$. If the very low states (i.e., $\theta \ll a_0 < \mu$), in which changing the default action to the favorite action would incur a huge loss, are weighted heavier probability-wise than the very high states (i.e., $\theta \gg \mu > a_0$) in which taking the favorite action leads to much gain, then a less-informed sender would be less willing to talk when $L''' > 0$.

The symmetry of $g^\lambda(\theta|\mu)$ is also related to the assumption of the unbounded state space. If we instead assume that the state space is bounded and the relevant distributions are truncated accordingly, the posterior distributions will typically be asymmetric. Consequently, whether a less-informed sender is more willing to talk will depend on the specific favorite action. If $a_0 < \mu = \sup\Theta$, the less informed sender is less willing to talk if $L''' > 0$. This is because there does not exist any state $\theta \gg \mu$ such that the gain by taking action μ instead of a_0 is large enough to balance the loss in the case where $\theta \ll a_0 < \mu$. An unbounded state space guarantees that for any μ, there always exists extreme states in the support of $g^{\lambda}(\theta|\mu)$ where changing the default action to the favorite action leads to a huge gain.

4.4 Disagreement on Favorite Actions

We next explore the implications of the disagreement on favorite actions for the receiver's expected payoff. The main results in this section highlight the following possibilities: (1) A sender with a larger bias can be more motivated to talk and hence be more valuable to the receiver; (2) In the presence of disagreement, the existence of a moderate talking cost can mitigate disagreement and hence improve the communication; (3) Ex ante, the sender does not necessarily prefer a communication technology that is preferred by the receiver if the expected talking expenditure is too high.

To focus on the implication of disagreement, we shut down the influence of inaccuracy of signals by maintaining the following assumption in this section.

Assumption 7. *The sender is biased ($b > 0$) but perfectly informed, i.e.,* $\psi = \theta$.

We then drop the symbol λ for the rest of analysis.

We next assume that the space of the players' favorite actions is bounded as in the canonical cheap talk model. As we will show in this section, equilibria of this game feature a partition structure similar to that in the canonical cheap talk model. An unbounded state space would imply that there might be infinitely many partitions in an equilibrium, which makes it difficult to compare informativeness across equilibrium outcomes.

Assumption 8. $\Theta = [-T, T]$.

We know from the canonical cheap talk model that any equilibrium takes the form of a partition on the state space in which the cutoff sender type is indifferent between the two actions closest to its type. The partition elements $\theta_0 < \ldots < \theta_N$ satisfy a difference equation which is essentially an arbitrage condition. The introduction of one free and many costly messages changes the standard equilibrium characterization because there will be a silence interval associated with a costless message. At its boundaries, the sender is indifferent between paying the talking cost to induce one action and paying no cost to induce another (default) action, which implies that the sender strictly prefers the costly action and is indifferent only because of the talking cost. The following proposition describes a class of equilibria of this game.

Proposition 28. *There exists a positive integer $L(b,c)$ such that, given any l such that $1 \leq l \leq L(b,c)$, for every N with $l \leq N \leq N(b,c,l)$ for some integer $N(b,c,l)$, there exists at least one equilibrium $(a^*(m), m^*(\theta))$, where $m^*(\theta)$ is uniform, supported on $[\theta_k, \theta_{k+1}]$,[12] if $\theta \in (\theta_k, \theta_{k+1})$, for $1 \leq k \leq N-1$ such that $k \notin \{l-1, l\}$,*

$$-\mathbb{E}_\theta[L(|a^R(\theta_k,\theta_{k+1}) - \theta + b|)|\theta = \theta_k] = -\mathbb{E}_\theta[L(|a^R(\theta_{k-1},\theta_k) - \theta + b|)|\theta = \theta_k], \tag{4.7}$$

for $k = l-1$ if $2 \leq l \leq N$,

$$-\mathbb{E}_\theta[L(|a^R(\theta_{l-1},\theta_l) - \theta - b|)|\theta = \theta_{l-1}] = -\mathbb{E}_\theta[L(|a^R(\theta_{l-2},\theta_{l-1}) - \theta - b|)|\theta = \theta_{l-1}] - c, \tag{4.8}$$

for $k = l$ if $1 \leq l \leq N-1$,

$$-\mathbb{E}_\theta[L(|a^R(\theta_l,\theta_{l+1}) - \theta - b|)|\theta = \theta_l] - c = -\mathbb{E}_\theta[L(|a^R(\theta_{l-1},\theta_l) - \theta - b|)|\theta = \theta_l], \tag{4.9}$$

and

$$a^*(m) = a^R(\theta_k,\theta_{k+1}) \qquad \textit{for all } m \in (\theta_k,\theta_{k+1}); \tag{4.10}$$

[12]We allow the interval to be degenerate, i.e., we allow for the possibility that $\theta_k = \theta_{k+1}$.

$$\theta_0 = -T \qquad and \tag{4.11}$$

$$\theta_N = T. \tag{4.12}$$

Further,

$$a^R(\theta_{k-1}, \theta_k) < a^S(\theta_k) = a^S(\theta_k) < a^R(\theta_k, \theta_{k+1}). \tag{4.13}$$

We let (θ_{l-1}, θ_l) denote the silence interval. The proof is parallel to the proof of Theorem 1 in **?**. The canonical cheap talk model is known for having multiple equilibria, in particular, multiple equilibrium state-action distributions associated with different sizes of the equilibrium partition. The presence of a talking cost introduces a new type of multiplicity associated with the location of the silence interval.

For the rest of this section, we focus on the set of equilibria that satisfy the description of Proposition 28 and feature the silence interval being the leftmost one (i.e., $l = 1$). In equilibria with $l = 1$, an upwardly biased sender always has to pay a cost to induce his preferred action, as the default action is the lowest one. We mainly focus on this type of equilibria and conduct comparative static analysis because it provides interesting insights with respect to the role of the talking cost. In particular, our results suggest that intuitive monotonicity relationships recognized in the literature break down in the presence of a talking cost, which are however consistent with observations about communication on media described in the introduction. In Appendix C, we provide a complete characterization of equilibria of this game, and discuss how our main results in this section depend on the specific equilibrium we analyzed.

Lastly, as in Crawford and Sobel (1982), we impose the following assumption throughout the remainder of this section.

Assumption 9. *(Crawford and Sobel (1982) Assumption M, M') For a given value of b, c, let θ' and θ'' be two sequences of length N that satisfies Equations* (4.7), (4.8)*, and* (4.9)*. If $\theta'_0 = \theta''_0$ and $\theta'_1 > \theta''_1$, then $\theta'_k > \theta''_k$ for all $k \geq 2$; equivalently, if $\theta'_N = \theta''_N$ and $\theta'_{N-1} < \theta''_{N-1}$, then $\theta'_k < \theta''_k$ for all $k \leq N - 2$.*

4.4.1 The Motivating Effect of Disagreement

This section discusses the comparative statics with respect to the size of the bias. We show that in the presence of a talking cost, the receiver can be better off if the sender has a larger bias. More specifically, a sender with a larger bias may find the default action more undesirable and is hence more motivated to supply (costly) information in order to avoid the default action being taken. As an implication, a more biased sender can be more valuable to the receiver by being more likely to provide information.

We first introduce some useful notation. To simplify things, we treat an equilibrium as a two-step equilibrium if exactly one sender type induces a certain action and the rest types induce another. That is, if one equilibrium partition element is a singleton.

Definition 22. *Let $C(b)$ be the set of talking costs in which a non-babbling equilibrium (i.e., equilibrium with at least two steps) exists.*

There exists a threshold of talking cost above which the sender never talks. As an important part of the aforementioned non-monotonicity, we show that the threshold of communication increases with the size of the bias. That is, the more severe the bias is, the higher cost the sender is willing to bear to communicate with the receiver.

Lemma 27 suggests that $C(b)$ is an interval.

$$\bar{c}(b) \equiv \max_{x\in[-T,T]} \{\mathbb{E}_\theta[L(|\theta+b-a^R(-T,x)||\mu(\psi)=x)]-\mathbb{E}_\theta[L(|a^R(x,T)-\theta-b||\mu(\psi)=x)]\}.^{13} \tag{4.14}$$

Lemma 27. *If $c, c' \in C(b)$ such that $c < c'$, then $c'' \in C(b)$ for any $c < c'' < c'$.*

Let $\bar{c}(b)$ be the least upper bound of $C(b)$. The following proposition implies that a sender with a larger bias can be more motivated to talk.

Proposition 29. *$\bar{c}(b)$ is increasing in b.*

In other words, if a non-babbling equilibrium exists when the bias is b_1 and the cost is c, then it also exists when the bias is some $b_2 > b_1$ at the cost c. Intuitively, compared to a sender with no bias, the default action in a babbling equilibrium is more undesirable for a sender with an upward bias when the state is high. The latter is hence more willing to bear the cost to induce a higher action instead of letting receiver take the undesirable default action. The proposition above suggests that the receiver may prefer to communicate with a more biased sender than a less biased sender, which is further illustrated by Example 1 at the end of this section.

4.4.2 The Relationship between the Receiver's Payoff and the Talking Cost

In this subsection, we examine the comparative statics with respect to the talking cost. Specifically, we study how the talking cost affects the receiver's maximal ex ante expected payoff and highlight the possibility that a higher cost can improve the communication effectiveness. If it is costly for a biased sender to recommend actions he is biased towards while costless to recommend actions he is biased against, the sender would be more credible to the receiver because the players' preferences with respect to preferred actions are more aligned. As an implication, a moderate talking cost can mitigate bias, enhance credibility, and facilitate communication. Too low a talking cost can reduce the effectiveness of communication.

If the players agree on favorite actions conditional on states (i.e., $b = 0$), it is straightforward to see that the receiver always prefers a lower talking cost because a lower talking cost implies a smaller silence interval. In contrast, in the presence of disagreement, the following two propositions suggest that the receiver may benefit from a moderate talking cost. More specifically, Proposition 30 suggests that, if meaningful communication is possible when the talking cost is zero, the receiver can obtain a higher payoff when there is a moderate talking cost compared to the zero cost benchmark.

Proposition 30. *If a non-babbling equilibrium exists when $c = 0$, there exists $\hat{c} > 0$ associated* *with which there is an equilibrium outcome preferred by the receiver over all equilibrium outcomes associated with zero cost.*

In a cheap talk game where the players disagree on favorite actions, if the receiver completely believes the biased sender, the sender would take advantage of receiver's trust and exaggerate. A rational receiver would hence not take the sender's statement at its face value, rendering the communication between them limited. Nevertheless, if it is costly for the sender to talk, the net benefit of exaggerating becomes lower. When the true state is not so high and therefore close to the default action, the sender would not be motivated to pay the cost to exaggerate. When the true state is sufficiently high (the sender's favorite action is even higher), the default action would be very undesirable for the sender and the sender would be motivated to talk. The receiver would find the sender more credible because he only talks if the state is sufficiently high. Communication between players can therefore be improved.

As a complement to Proposition 30, Proposition 31 suggests that, if there is no meaningful communication between the players when the talking cost is zero, there always exists some strictly positive c to bring about meaningful communication between the players.[14]

Proposition 31. *$\forall b$, there exists c such that a non-babbling equilibrium exists, i.e., $C(b) \neq \emptyset$.*

When talking is cheap and bias is large, effective communication between players becomes impossible. Introducing a cost turns the game into one that is similar to a signaling game, and communication becomes possible. Proposition 30 and Proposition 31 above reflect the bias-offsetting effect of the talking cost.

Nevertheless, this positive effect of the talking cost on improving the receiver's payoff is limited. In other words, the receiver's maximal ex ante expected payoff is not monotonically increasing in the talking cost. As the cost becomes overly large, the sender is disincentivized to talk. In particular, non-babbling equilibrium does not exist once the cost is higher than the

[14]Unlike Austen-Smith et al. (2000), the existence of the talking cost cannot arbitrarily increase the maximal equilibrium partition size. This is because there only exists two possible cost levels (0 and c) and hence one special message (the "silence").

threshold $\bar{c}(b)$ defined in Section 4.4.1. Example 1 at the end of this section further illustrates this non-monotonicity.

4.4.3 The Relationship between the Sender's Payoff and the Talking Cost

Section 4.4.2 has demonstrated that a moderate talking cost can facilitate communication while too low a cost can reduce communication effectiveness. Analysis in this subsection shows that the comparison between the sender's favorite cost and the cost that maximizes the communication effectiveness is ambiguous in general.

Assume the sender has to choose a technology associated with the talking cost c ex ante.[15] A higher talking cost has two effects on the sender's total payoff: First, as we have shown in the previous section, it influences the communication effectiveness, which may have a positive effect on the sender's communication payoff – the part of the sender's payoff excluding the talking expenditure, because it offsets bias; Second, it influences the ex ante total talking expenditure. Specifically, let $p(c)$ be the ex ante probability that the sender sends a costly message in the equilibrium associated with cost c. The talking expenditure can be expressed as $c \cdot p(c)$. In general, the talking expenditure is not necessarily monotonic in c. That is, increasing talking cost can lower the talking expenditure by lowering the ex ante probability of talking (at some cost).

Interestingly, a moderate bias (i.e., a higher b) can incentivize the sender to choose a more costly communication technology that facilitates communication. Let c^S_{CM} and c^S be the costs that maximize the sender's communication payoff and net payoff, respectively. Intuitively, a severe disagreement requires a high talking cost to offset. That is, c^S_{CM} is relatively high. An even higher cost decreases the probability of talking so that the talking expenditure does not increase by much or may even decrease, which is beneficial to the sender. This intuition can be best illustrated by the following example featuring uniform prior and quadratic loss function. In this example, $\max_c \bar{N}(b,c) = 2$ implies $c^S_{CM} = c^R$, where c^R is the receiver's favorite cost. The

[15] Austen-Smith and Banks (2002), Kartik (2007), and Hertel and Smith (2013) discuss situations where the sender chooses the talking cost after he learns the signal.

comparison between c^S and c^R hence completely depends on the relationship between the ex ante talking expenditure $c \cdot p(c)$ and the talking cost c.

Example 12. *Under uniform prior distribution and quadratic loss function,*[16] *if* $\max_c \bar{N}(b,c) = 2$, *then the sender's favorite cost is lower than that of the receiver's if* $b < \frac{T}{2}$, *equal to that of the receiver's if* $b = \frac{T}{2}$, *and higher than that of the receiver's if* $b > \frac{T}{2}$.

What if b is low and there then exists an equilibrium with more than two steps? The answer is ambiguous even under the assumption of the uniform prior distribution. In addition to the ambiguous effect of a higher talking cost on the sender's talking expenditure, $c^S_{CM} = c^R$ is no longer true and the comparison between them is in general ambiguous.

4.4.4 An Example

This example illustrates the main results in this section.

Example 13. *In this example,* $H(\theta)$ *is uniform on* $[-\frac{1}{2}, \frac{1}{2}]$, $U^S(a,\theta,b) \equiv -[a-(\theta+b)]^2 - c \cdot \mathbf{1}_{m \neq m^0}$, *and* $U^R(\theta) \equiv -(a-\theta)^2$, *and* $b > 0$.

The following figures illustrate how players' maximal ex ante expected payoffs change with the talking cost c for $b = \frac{1}{5}$. For these parameters, an equilibrium that satisfies Proposition 28 has at most two steps.[17]

We highlight the following observations.

1. There exists a threshold $\bar{c}(b) = \frac{1}{4} + b$ above which there is no communication, i.e., the sender does not talk. This threshold increases with b. Furthermore, for any given c, the receiver's highest possible payoff increases with b if $b < c$.

2. The relationship between the receiver's maximal ex ante expected payoff and the talking cost is not monotonic. Zero cost is not optimal in terms of the communication effectiveness

[16] The quadratic loss function assumption can be easily relaxed. We use it here for more clear illustration because the problem can be solved in closed form.

[17] In fact, the two-step equilibrium is also players' favorite equilibrium across all equilibria.

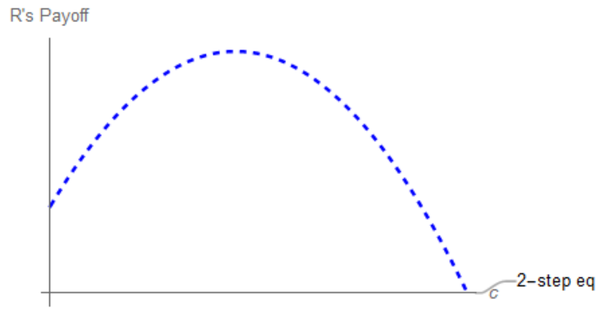

Figure 4.2: The relationship between the receiver's payoff and c for $b = \frac{1}{5}$

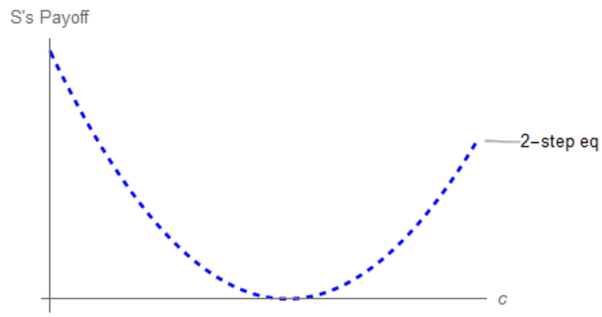

Figure 4.3: The relationship between the sender's payoff and c for $b = \frac{1}{5}$

measured by the receiver's payoff. The receiver's expected payoff is the highest when $c = b$. That is, as the bias b increases, the receiver's favorite talking cost also increases to offset the bias.

3. For a range of b, the sender's favorite talking cost is zero at which his highest ex ante expected payoff is attained, lower than the talking cost that allows for maximal communication effectiveness measured by receiver's maximal ex ante expected payoff.

The following graph depicts how the two-step equilibrium partition changes with c. The black dot is the cutoff sender type who is indifferent between the two on-equilibrium-path messages. When $c = 0$, the equilibrium partition is uneven: the first partition element is smaller than the second. This reflects the ineffectiveness in communication about higher states due to the sender's upward bias. Let the left interval be the silence interval. Increasing c moves the cutoff sender type to the right. The equilibrium partition becomes more even until $c = b$. When $c = b$, the two partition elements are of equal size. During this process, the receiver's expected payoff is first increasing until $c = b$, driven by the bias-offsetting effect. Increasing c further beyond

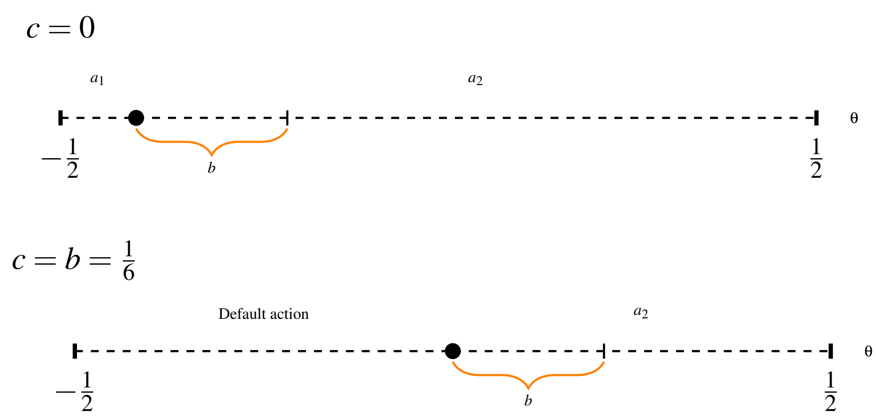

Figure 4.4: Bias-offsetting by a higher cost

$c = b$ makes the partition uneven again. This hurts the receiver because the sender is discouraged from talking, reflected by an overly large silence interval.

4.5 Appendix D

4.5.1 Proofs for Section 4.3

We first establish the following lemma which will be useful for the subsequent proofs.

Lemma 28. $\mathbb{E}[L(|a_0-\theta|)|\mu=\mu(\psi;\lambda),\lambda]-\mathbb{E}[L(|\mu-\theta|)|\mu(\psi;\lambda),\lambda]$ *is increasing in* μ *if* $\mu>a_0$*, and decreasing in* μ *if* $\mu<a_0$.

Proof. Let $I(\mu;\lambda)=\mathbb{E}[L(|a_0-\theta|)|\mu=\mu(\psi;\lambda),\lambda]-\mathbb{E}[L(|\mu-\theta|)|\mu(\psi;\lambda),\lambda]$. We show that $I(\mu;\lambda)$ increases in μ if $\mu>a_0$. A parallel argument can show that $I(\mu;\lambda)$ decreases in μ if $\mu<a_0$.

Suppose $\mu>a_0$. For arbitrary $d>0$,

$$\begin{aligned}
&I(\mu+d;\lambda)-I(\mu;\lambda)\\
=&\int_\theta\{-L(|\mu+d-\theta|)+L(|a_0-\theta|)\}g^\lambda(\theta|\mu+d)d\theta\\
&-\int_\theta\{-L(|\mu-\theta|)+L(|a_0-\theta|)\}g^\lambda(\theta|\mu)d\theta\\
=&\int_\theta\{-L(|\mu+d-(\theta+d)|)+L(|a_0-(\theta+d)|)\}g^\lambda(\theta|\mu)d\theta\\
&-\int_\theta\{-L(|\mu-\theta|)+L(|a_0-\theta|)\}g^\lambda(\theta|\mu)d\theta\\
=&\int_\theta\{-L(|a_0-\theta|)+L(|a_0-\theta-d|)\}g^\lambda(\theta|\mu)d\theta\\
=&2\int_{x\geq 0}\{L(\mu+x+d-a_0)-L(\mu+x-a_0)\}g^\lambda(\mu+x|\mu)dx>0.
\end{aligned}$$

□

We then present the omitted proofs as below.

Lemma 22. *The sender's favorite action is the mean of the posterior plus the bias, i.e.,* $a^S(\psi;\lambda)=\mu(\psi;\lambda)$ *for* $b=0$.

Proof. Given ψ, the expected payoff of the sender will be

$$\int -L(|a-\theta-b|)f^{\lambda}(\theta|\psi)d\theta.$$

By assumption, the posterior distribution $f^{\lambda}(\theta|\psi)$ is symmetric given the signal ψ. Let $p(x) = f^{\lambda}(\mu(\psi;\lambda)+x|\psi) = f^{\lambda}(\mu(\psi;\lambda)-x|\psi)$ be probability of the state that has a distance of $x > 0$ from the mean $\mu(\psi;\lambda)$. Then the expected payoff of the sender will be

$$\int_{\mathbb{R}_+} -[L(|a-\mu(\psi;\lambda)-b-x|)\mathbf{1}_{x>0} + L(|a-\mu(\psi;\lambda)-b+x|)\mathbf{1}_{x>0}$$
$$+L(|a-\mu(\psi;\lambda)-b|)\mathbf{1}_{x=0}]p(x)dx,$$

For $x=0$, it is clear that $L(|a-\mu(\psi;\lambda)-b|) = L(|\mu(\psi;\lambda)+b-\mu(\psi;\lambda)-b|) = L(0) = 0$ when $a = \mu(\psi;\lambda)+b$. We claim that $L(|a-\mu(\psi;\lambda)-b-x|)+L(|a-\mu(\psi;\lambda)-b+x|)$ is minimized for each $x > 0$ as well by setting $a = \mu(\psi;\lambda)+b$.

Let $\hat{L}(x) = L(|a-\mu(\psi;\lambda)-b-x|)+L(|a-\mu(\psi;\lambda)-b+x|) = L(|x-a+\mu(\psi;\lambda)+b|)+L(|x+a-\mu(\psi;\lambda)-b|)$ for $x > 0$. If $a = \mu(\psi;\lambda)+b$, then $\hat{L}(x) = 2L(|x|)$. In general, since $L(\cdot)$ is a convex function, we have the following condition by Jensen's inequality:

$$\begin{aligned}
\hat{L}(x) &= L(|x-a+\mu(\psi;\lambda)+b|)+L(|x+a-\mu(\psi;\lambda)-b|) \\
&= 2[(1/2)L(|x-a+\mu(\psi;\lambda)+b|)+(1/2)L(|x+a-\mu(\psi;\lambda)-b|)] \\
&\geq 2L((1/2)|x-a+\mu(\psi;\lambda)+b|+(1/2)|x+a-\mu(\psi;\lambda)-b|) \\
&\geq 2L((1/2)|x-a+\mu(\psi;\lambda)+b+x+a-\mu(\psi;\lambda)-b|) \\
&= 2L(|x|),
\end{aligned} \tag{4.15}$$

where the first inequality follows from the Jensen's inequality and the second inequality follows from the triangle inequality and $L' > 0$. Because the loss function is minimized by $a = \mu(\psi;\lambda)+b$ point wise for any $x \in \mathbb{R}_+$ and $a = \mu(\psi;\lambda)+b$ uniquely minimizes the loss when $x = 0$, $a^S = \mu(\psi;\lambda)+b$ is an favorite action of the sender.

Furthermore, the first inequality of Eq 4.15 is strict for $x > 0$ and $a \neq \mu(\psi;\lambda) + b$ when L is a strictly convex function, i.e. $L'' > 0$. In this case, $a^S = \mu(\psi;\lambda) + b$ is the unique favorite action of the sender. □

Lemma 23. *Given a strategy $a(m)$ of receiver, let $\tilde{m}^\lambda(\psi)$ be a best response of the sender. There exists an interval $[\theta_1(\lambda), \theta_2(\lambda)]$ such that $\tilde{m}^\lambda(\psi) = m^0$ if and only if $\mu(\psi;\lambda) \in [\theta_1(\lambda), \theta_2(\lambda)]$.*

Proof. By Lemma 28, if $-\mathbb{E}[L(|\mu - \theta|)|\psi,\lambda] - c \geq -\mathbb{E}[L(|a_0 - \theta|)|\psi,\lambda]$ for some $\mu(\psi;\lambda) = \mu \in \mathbb{R}$, then $-\mathbb{E}[L(|\mu' - \theta|)|\psi',\lambda] - c > -\mathbb{E}[L(|a_0 - \theta|)|\psi',\lambda]$ for all $\mu(\psi';\lambda) = \mu'$ such that $|\mu' - a_0| > |\mu - a_0|$. The lemma hence follows. □

Lemma 24. $a_0 = \mu_h$.

Proof. As the first step, we show that for any λ, the equilibrium default action $a_0 = a^*(m^0)$ is the midpoint of the interval $\Theta^0 = [\theta_1, \theta_2]$, i.e. $a_0 = \frac{\theta_1+\theta_2}{2}$.

Let signal ψ_1 and ψ_2 satisfy $\mu(\psi_1;\lambda) = \theta_1$ and $\mu(\psi_2;\lambda) = \theta_2$. By the equilibrium condition, $-\mathbb{E}_\theta\{L(|\theta_1 - \theta|)|\psi_1;\lambda\} - c = -\mathbb{E}_\theta\{L(|a_0 - \theta|)|\psi_1;\lambda\}$ and $-\mathbb{E}_\theta\{L(|\theta_2 - \theta|)|\psi_2;\lambda\} - c = -\mathbb{E}_\theta\{L(|a_0 - \theta|)|\psi_2;\lambda\}$. Note that $-\mathbb{E}_\theta\{L(|\theta_1 - \theta|)|\psi_1;\lambda\} - c = -\mathbb{E}_\theta\{L(|\theta_2 - \theta|)|\psi_2;\lambda\} - c$. Therefore

$$
\begin{aligned}
&\int_\theta L(|a_0 - \theta|) f^\lambda(\theta|\psi_1) d\theta \\
=& \int_\theta L(|a_0 - \theta|) f^\lambda(\theta|\psi_2) d\theta \\
=& \int_\theta L(|a_0 - \theta|) f^\lambda(\theta + \theta_1 - \theta_2|\psi_1) d\theta \\
=& \int_x L(|a_0 - (x + \theta_2 - \theta_1)|) f^\lambda(x|\psi_1) dx \\
=& \int_\theta L(|a_0 - (\theta + \theta_2 - \theta_1)|) f^\lambda(\theta|\psi_1) d\theta
\end{aligned}
$$

Because $L' > 0$ and $\theta_1 \neq \theta_2$, the last line above implies $a_0 + a_0 - \theta_2 + \theta_1 = 2\theta_1$, that is, $a_0 = \frac{\theta_2+\theta_1}{2}$.

We are now ready to prove the lemma. Let the equilibrium associated with λ' that has a_0 as the equilibrium default action be $\delta^{*'} = (a^{*'}, m^{*'})$. Also let $\Theta^{0'} = [\theta_1', \theta_2']$ be the "silence region" in the equilibrium outcome. The posterior of the receiver after receiving no message is as follows:

$$\phi^\lambda(\mu|\mu \in [\theta_1', \theta_2']) = \begin{cases} \frac{\phi^\lambda(\mu)}{\Phi^\lambda(\theta_2') - \Phi^\lambda(\theta_1')} & \text{if } \theta \in \Theta^{0'} \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 22,

$$\begin{aligned} a_0 &= \frac{\theta_1' + \theta_2'}{2} \\ &= E_\theta(\theta|\theta \in [\theta_1', \theta_2']) \\ &= \frac{1}{\Phi^\lambda(\theta_2') - \Phi^\lambda(\theta_1')} \int_{\theta_1'}^{\theta_2'} \theta\phi^\lambda(\theta)d\theta \\ &= \frac{1}{\Phi^\lambda(\theta_2') - \Phi^\lambda(\theta_1')} \int_0^y [(a_0 + x)\phi^\lambda(a_0 + x) + (a_0 - x)\phi^\lambda(a_0 - x)]dx \\ &= \frac{1}{\Phi^\lambda(\theta_2') - \Phi^\lambda(\theta_1')} \{a_0 \int_0^y [\phi^\lambda(a_0 + x) + \phi^\lambda(a_0 - x)]dx \\ &+ \int_0^y x[\phi^\lambda(a_0 + x) - \phi^\lambda(a_0 - x)]dx\} \\ &= a_0 + \frac{1}{\Phi^\lambda(\theta_2') - \Phi^\lambda(\theta_1')} \int_0^y x[\phi^\lambda(a_0 + x) - \phi^\lambda(a_0 - x)]dx \end{aligned} \tag{4.16}$$

where $y = a_0 - \theta_1' = \theta_2' - a_0$.

Hence $\int_0^y x[\phi^\lambda(a_0 + x) - \phi^\lambda(a_0 - x)]dx = 0$. It follows that $\phi^\lambda(\theta)$ can not be monotonic over $[\theta_1', \theta_2']$, i.e. $\mu_h \in [\theta_1', \theta_2']$. By the symmetry of ϕ^λ, $\int_0^y x[\phi^\lambda(a_0 + x) - \phi^\lambda(a_0 - x)]dx = \int_0^y x[\phi^\lambda(\mu_h + a_0 - \mu_h + x) - \phi^\lambda(\mu_h + a_0 - \mu_h - x)]dx = \int_0^y x[\phi^\lambda(\mu_h + a_0 - \mu_h + x) - \phi^\lambda(\mu_h - a_0 + \mu_h + x)]dx$. That is, $\int_0^y x\phi^\lambda(\mu_h + x + a_0 - \mu_h)dx = \int_0^y x\phi^\lambda(\mu_h + x - (a_0 - \mu_h))dx$. Since ϕ^λ is monotonic over $[\mu_h, \mu_h + y]$, $a_0 = \mu_h$.

□

Lemma 26. *The receiver's payoff can be decomposed as follows:*

$$U^R(\lambda) = -\mathbb{E}_\theta[L(|a_0-\theta|)] + I^+(\lambda) \tag{4.3}$$

where

$$I^+(\lambda) \equiv \int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} I(\mu;\lambda)\phi^\lambda(\mu)d\mu = \int_{I(\mu;\lambda)\geq c} I(\mu;\lambda)\phi^\lambda(\mu)d\mu$$

Proof.

$$\begin{aligned}
U^R(\lambda) \equiv &-\int_{\mu\in[\theta_1(\lambda),\theta_2(\lambda)]} \mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
&-\int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} E_\theta[L(|\mu-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
=&-\int_{\mu\in[\theta_1(\lambda),\theta_2(\lambda)]} \mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
&-\int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} \mathbb{E}_\theta[L(|\mu-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
&\quad+\int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} \mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
&-\int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]} \mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\phi^\lambda(\mu)d\mu \\
=&-\mathbb{E}_\mu\{\mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\} - \int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]}\{\mathbb{E}_\theta[L(|\mu-\theta|)|\mu(\psi)=\mu] \\
&-E_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda]\}\phi^\lambda(\mu)d\mu \\
=&-\mathbb{E}_\theta[L(|a_0-\theta|)] + \int_{\mu\notin[\theta_1(\lambda),\theta_2(\lambda)]}\{\mathbb{E}_\theta[L(|a_0-\theta|)|\mu(\psi)=\mu;\lambda] \\
&-\mathbb{E}_\theta[L(|\mu-\theta|)|\mu(\psi)=\mu;\lambda]\}\phi^\lambda(\mu)d\mu
\end{aligned}$$

□

Proposition 27.

1. *If L'' is strictly increasing, $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} < 0$ and the size of the silence interval $\theta_2(\lambda)-\theta_1(\lambda)$ is increasing in λ.*

2. *If L'' is strictly decreasing, $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} > 0$ and the size of the silence interval $\theta_2(\lambda)-\theta_1(\lambda)$ is*

decreasing in λ.

3. *If* L'' *is constant,* $\frac{\lambda I(\mu;\lambda)}{\partial\lambda} = 0$ *and the size of the silence interval* $\theta_2(\lambda) - \theta_1(\lambda)$ *is constant over* λ.

Proof. We prove the first statement. The other two statements follow from a similar argument.

To show that $\theta_2(\lambda) - \theta_1(\lambda)$ is increasing in λ, it suffices to show that for given μ, $I(\mu;\lambda)$ is decreasing in λ, so that $[\theta_1(\lambda), \theta_2(\lambda)] = \{\mu | I(\mu;\lambda) \leq c\}$ is larger if λ becomes larger.

Define $d = \mu - a_0$. Let $\lambda'' > \lambda'$. We are interested in the sign of the following object:

$$
\begin{aligned}
& I(\mu;\lambda'') - I(\mu;\lambda') \\
= & -\int_{\Theta} [L(|\mu - \theta|) - L(|\mu - d - \theta|)] dG^{\lambda''}(\theta|\mu) + \int_{\Theta} [L(|\mu - \theta|) - L(|\mu - d - \theta|)] dG^{\lambda'}(\theta|\mu) \\
= & -\int_{\mu+x\geq\mu} [L(|\mu - (\mu+x)|) - L(|\mu - d - (\mu+x)|) + L(|\mu - (\mu-x)|) \\
& - L(|\mu - d - (\mu-x)|)] dG^{\lambda''}(\mu+x|\mu) \\
& + \int_{\mu+x\geq\mu} [L(|\mu - (\mu+x)|) - L(|\mu - d - (\mu+x)|) + L(|\mu - (\mu-x)|) \\
& - L(|\mu - d - (\mu-x)|)] dG^{\lambda'}(\mu+x|\mu) \\
\equiv & -\int_{\mu+x\geq\mu} \mathcal{L}(x) dG^{\lambda''}(\mu+x|\mu) + \int_{\mu+x\geq\mu} \mathcal{L}(x) dG^{\lambda'}(\mu+x|\mu)
\end{aligned}
$$

where

$$\mathcal{L}(x) \equiv 2L(x) - L(|x-d|) - L(|x+d|)$$

We first claim that $G^{\lambda'}(\mu+x|\mu)$ first order stochastically dominates $G^{\lambda''}(\mu+x|\mu)$ for all $x \geq 0$. That is, $\forall x > 0$, $G^{\lambda'}(\mu+x|\mu) \leq G^{\lambda''}(\mu+x|\mu)$. Suppose otherwise, that is, $\exists x > 0$, $G^{\lambda'}(\mu+x|\mu) > G^{\lambda''}(\mu+x|\mu)$. By symmetry, $\exists x' < 0$, $G^{\lambda'}(\mu+x'|\mu) < G^{\lambda''}(\mu+x'|\mu)$. Then by the assumption that $G^{\lambda'}(\theta|\mu) \succ_{SC} G^{\lambda''}(\theta|\mu)$ for any given μ, $G^{\lambda'}(\mu+x|\mu) < G^{\lambda''}(\mu+x|\mu)$ for any x, leading to a contradiction.

It then suffices to show that $\mathcal{L}(x)$ is increasing in x. That is, $I(\mu;\lambda'') - I(\mu;\lambda') < 0$ if $\frac{d\mathcal{L}(x)}{dx} > 0$.

If $x > d$, the sign of $\frac{d\mathcal{L}(x)}{dx}$ is the same as:

$$
\begin{aligned}
&-2L'(x)+L'(|x-d|)+L'(|x+d|)\\
=&-2L'(x)+L'(x-d)+L'(x+d)\\
=&2[\frac{1}{2}L'(x+d)+\frac{1}{2}L'(x-d)]-2L'(x)\\
>&2L'[\frac{1}{2}(x+d)+\frac{1}{2}(x-d)]-2L'(x)\\
=&0
\end{aligned}
$$

where the inequality comes from $L''' > 0$.

If $0 < x < d$, the sign of $\frac{d\mathcal{L}(x)}{dx}$ is the same as:

$$
\begin{aligned}
&-2L'(x)+L'(|x-d|)+L'(|x+d|)\\
=&-2L'(x)-L'(d-x)+L'(x+d)\\
>&-2[\frac{1}{2}L'(2x)+\frac{1}{2}L'(0)]+\int_0^{2x}L''(d-x+y)dy\\
=&-L'(2x)+\int_0^{2x}L''(d-x+y)dy\\
=&-\int_0^{2x}L''(y)dy+\int_0^{2x}L''(y+d-x)dy\\
=&\int_0^{2x}\int_0^{d-x}L'''(y+z)dydz\geq 0
\end{aligned}
$$

which follows from $L'(0)=0$ and $L''' > 0$.

Therefore, if $L''' > 0$, $\frac{d\mathcal{L}(x)}{dx} > 0$ and hence $\frac{dI(\mu;\lambda)}{d\lambda} < 0$. □

Corollary 5. *If $L''' \leq 0$, the receiver prefers a sender with a more accurate signal. A sender with a more accurate signal is also more likely to talk ex ante.*

Proof.

$$\begin{aligned}
\frac{\partial I^{+}(\lambda)}{\partial\lambda} &= \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c} I(\mu;\lambda)\phi^{\lambda}(\mu)}{\partial\lambda} d\mu \\
&= \int \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} I(\mu;\lambda) d\mu + \int \frac{\partial I(\mu;\lambda)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c}\phi^{\lambda}(\mu) d\mu \\
&+ \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial\lambda}\phi^{\lambda}(\mu) I(\mu;\lambda) d\mu \\
&= 2\int_{\mu>\mu_h} \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} I(\mu;\lambda) d\mu + \int \frac{\partial I(\mu;\lambda)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c}\phi^{\lambda}(\mu) d\mu \\
&+ \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial\lambda}\phi^{\lambda}(\mu) I(\mu;\lambda) d\mu
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\frac{\partial P(\lambda)}{\partial\lambda} &= \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}\phi^{\lambda}(\mu)}{\partial\lambda} d\mu \\
&= \int \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} d\mu + \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial\lambda}\phi^{\lambda}(\mu) d\mu \\
&= 2\int_{\mu>\mu_h} \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} d\mu + \int \frac{\partial \mathbf{1}_{I(\mu;\lambda)\geq c}}{\partial\lambda}\phi^{\lambda}(\mu) d\mu
\end{aligned} \tag{4.18}$$

If $L''' < 0$, Proposition 27 suggests that the last two terms in Equation (4.17) and the last term in Equation (4.18) are positive. It suffices to show that

$$\int_{\mu>\mu_h} \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} I(\mu;\lambda) d\mu > 0$$

and

$$\int_{\mu>\mu_h} \frac{\partial\phi^{\lambda}(\mu)}{\partial\lambda}\mathbf{1}_{I(\mu;\lambda)\geq c} d\mu > 0$$

Let $\lambda'' > \lambda'$, we first claim that $\Phi^{\lambda''}(\mu)$ first order stochastically dominates $\Phi^{\lambda'}(\mu)$ for $\mu > \mu_h$. That is, $\forall\mu > \mu_h$, $\Phi^{\lambda''}(\mu) \leq \Phi^{\lambda'}(\mu)$. Suppose otherwise, that is, $\exists\mu > \mu_h$, $\Phi^{\lambda''}(\mu) > \Phi^{\lambda'}(\mu)$. By symmetry, $\exists\mu' < \mu_h$, $\Phi^{\lambda''}(\mu') < \Phi^{\lambda'}(\mu')$. Then by the assumption that $\Phi^{\lambda''}(\mu) \succ_{SC} \Phi^{\lambda'}(\mu)$, $\Phi^{\lambda''}(\mu) < \Phi^{\lambda'}(\mu)$ for any μ, leading to a contradiction.

By Lemma 28, $I(\mu;\lambda)$ is increasing in μ for $\mu > \mu_h$, and hence $I(\mu;\lambda)\mathbf{1}_{I(\mu;\lambda)\geq c}$ and

$\mathbf{1}_{I(\mu;\lambda)\geq c}$ are non-decreasing in μ for $\mu > \mu_h$. The statement then follows. □

4.5.2 Proofs for Section 4.4

The following lemma is an extension of Lemma 1 of Crawford and Sobel (1982).

Lemma 29. *If $b > 0$, then there exists an $\varepsilon > 0$ such that if ν' and ν'' are actions induced in equilibrium (i.e. on-equilibrium path actions), then $|\nu' - \nu''| \geq \varepsilon$. Further, the set of actions induced in equilibrium is finite.*

Proof. Let ν', ν'', with $\nu'' > \nu'$, be two actions induced by $m^*(\theta'), m^*(\theta'') \neq m^0$ for some θ', θ'' in equilibrium. By continuity there exists an $\bar{\theta}$ such that $U^S(\nu', \bar{\theta}) = U^S(\nu'', \bar{\theta})$. Therefore (1) $\nu' < a^S[\bar{\theta}] < \nu''$. (2) ν' is not induced by any sender type with posterior mean $\theta > \bar{\theta}$. (3) ν'' is not induced by any sender type with posterior mean $\theta < \bar{\theta}$. (2) and (3) imply that $\nu' < a^R[\bar{\theta}] < \nu''$. Let $\varepsilon = a^S[\bar{\theta}] - a^R[\bar{\theta}] = b$. Then $|\nu'' - \nu'| > \varepsilon$.

Now let $a^*(m^0)$ be the action induced by m^0 and ν' be an action induced by some message $m^*(\theta) \neq m^0$ when the signal is θ. Assume $\nu' > a^*(m^0)$ (a parallel argument holds for $\nu' < a^*(m^0)$). By continuity there exists an $\bar{\theta}'$ such that $U^S(\nu', \bar{\theta}') - c = U^S(a^*(m^0), \bar{\theta}')$. In this case, it is possible that $a^*(m^0) < a^S[\bar{\theta}] < \nu'$ or $a^*(m^0) < \nu' < a^S[\bar{\theta}]$. In the former case, similar analysis implies that $a^*(m^0) < a^R[\bar{\theta}] < \nu'$. Then there exists $\varepsilon > 0$ such that $|a^*(m^0) - \nu'| \geq \varepsilon$. In the latter case, $a^*(m^0)$ and ν' are both on the left of $a^S[\bar{\theta}]$. Since $U^S(\cdot, \bar{\theta}')$ is a continuous function and $c > 0$, there is a neighborhood $[a^*(m_0) - \delta, a^*(m_0) + \delta]$ around $a^*(m_0)$ such that for every $a \in (a^*(m_0) - \delta, a^*(m_0) + \delta)$, $|U^S(a, \bar{\theta}') - U^S(a^*(m_0), \bar{\theta}')| < c/2$. Hence $|a^*(m^0) - \nu'| \geq \delta > 0$.

The finiteness of the induced actions follow from the assumption that the action space is bounded on $[-T, T]$. □

Proposition 28. *There exists a positive integer $L(b,c)$ such that, given any l such that $1 \leq l \leq L(b,c)$, for every N with $l \leq N \leq N(b,c,l)$ for some integer $N(b,c,l)$, there exists at least one equilibrium $(a^*(m), m^*(\theta))$, where $m^*(\theta)$ is uniform, supported on $[\theta_k, \theta_{k+1}]$,[18] if $\theta \in (\theta_k, \theta_{k+1})$,* *for* $1 \leq k \leq N-1$ *such that* $k \notin \{l-1, l\}$,

$$-\mathbb{E}_\theta[L(|a^R(\theta_k, \theta_{k+1}) - \theta + b|)|\theta = \theta_k] = -\mathbb{E}_\theta[L(|a^R(\theta_{k-1}, \theta_k) - \theta + b|)|\theta = \theta_k], \tag{4.7}$$

for $k = l-1$ *if* $2 \leq l \leq N$,

$$-\mathbb{E}_\theta[L(|a^R(\theta_{l-1}, \theta_l) - \theta - b|)|\theta = \theta_{l-1}] = -\mathbb{E}_\theta[L(|a^R(\theta_{l-2}, \theta_{l-1}) - \theta - b|)|\theta = \theta_{l-1}] - c, \tag{4.8}$$

for $k = l$ *if* $1 \leq l \leq N-1$,

$$-\mathbb{E}_\theta[L(|a^R(\theta_l, \theta_{l+1}) - \theta - b|)|\theta = \theta_l] - c = -\mathbb{E}_\theta[L(|a^R(\theta_{l-1}, \theta_l) - \theta - b|)|\theta = \theta_l], \tag{4.9}$$

and

$$a^*(m) = a^R(\theta_k, \theta_{k+1}) \qquad \textit{for all } m \in (\theta_k, \theta_{k+1}); \tag{4.10}$$

$$\theta_0 = -T \qquad \textit{and} \tag{4.11}$$

$$\theta_N = T. \tag{4.12}$$

Further,

$$a^R(\theta_{k-1}, \theta_k) < a^S(\theta_k) = a^S(\theta_k) < a^R(\theta_k, \theta_{k+1}). \tag{4.13}$$

Proof. (This proof is parallel to the proof of Theorem 1 in Crawford and Sobel (1982)) We show that Equations (4.7), (4.8), (4.9), (4.11), and (4.12) form a difference equation, that it has a solution for any l such that $1 \leq l \leq L(b,c)$ and any N such that $l \leq N \leq N(b,c,l)$, and that any solution $\theta_0, \ldots, \theta_N$, together with Sender's strategy profile given in the Proposition, is a best response for S to $a^*(m)$ defined in Equation (4.10).

First, note that by (4.1) and the loss functional form, $a^R(\theta_k, \theta_{k+1})$ must be strictly

[18] We allow the interval to be degenerate, i.e., we allow for the possibility that $\theta_k = \theta_{k+1}$.

increasing in both of its arguments. Let θ^k denote the partial partition $\theta_1, \theta_2, \ldots, \theta_k$ which is strictly increasing and satisfies Equations (4.7), (4.8), and (4.9). If $k \neq l-1$ and $k \neq l$, there can be at most one value of $\theta_{k+1} > \theta_k$ that satisfies (4.7) because U^R is concave in a and $a^R(\cdot)$ is monotonic. Thus any sequence $\theta_0, \ldots, \theta_k$ determines at most one relevant $\theta_{k+1} > \theta_k$. If $k = l-1$ or l, there might exist two such θ_{k+1}, but only one satisfies the assumption of the Type I equilibria, namely (4.13).

Fix a positive integer l. Let $K(x|l) \equiv \max\{i :$ there exists x such that $-T < x < \theta_2 < \ldots < \theta_i \leq T$ and (4.7), (4.8) and (4.9) are satisfied$\}$. Note that it is possible for $K(x|l) < l$ for some l, i.e., the silence interval is not in $[-T, T]$, as we did not restrict the value of l yet. By Lemma 29, $a^R(\theta_k, \theta_{k+1}) - a^R(\theta_{k-1}, \theta_k) \geq \varepsilon$ for some $\varepsilon > 0$, hence $\theta_{k+2} - \theta_k$ is bounded above zero for any solution to (4.7), (4.8), and (4.9). Thus $K(x|l)$ is finite, well defined, and uniformly bounded, so for each l, $\sup_{-T<x\leq T} K(x|l)$ is achieved for some $\bar{x}_l \in (-T, T]$. Let $L(b,c) \equiv \max\{l : K(\bar{x}_l) \geq l\}$ be the largest partition element in which the silence interval is in $[-T, T]$. Let $N(b,c,l) \equiv K(\bar{x}_l) < \infty$ for $1 \leq l \leq L(b,c)$. It is clear that for any l, if $\sup_{-T<x\leq T} K(x|l) < l$, then $\sup_{-T<x\leq T} K(x|l+1) < l+1$. So the induction on l shows that $K(\bar{x}_l) \geq l$ for all $1 \leq l \leq L(b,c)$, because $K(\bar{x}_{L(b,c)}) \geq L(b,c)$ by definition of $L(b,c)$.

It remains to show that for any l such that $1 \leq l \leq L(b,c)$ and each N such that $l \leq N \leq N(b,c,l)$, there is a partition $\theta_0, \ldots, \theta_N$ satisfying (4.7), (4.8) and (4.9). Fix l such that $1 \leq l \leq L(b,c)$. Let $\theta^{K(x|l)}$ be the partial partition of length $K(x|l)$ that satisfies (4.7), (4.8) and (4.9) and $\theta_1^{K(x|l)} = x$. Since solution to (4.7), (4.8) and (4.9) vary continuously with respect to initial conditions, if $\theta_{K(x|l)}^{K(x|l)}$ (the last term in the partial partition $\theta^{K(x|l)}$) is less than unity, $K(\cdot|l)$ is continuous (and therefore locally constant) at x. Moreover, $K(x|l)$ can change by at most one at a discontinuity. Because $K(1|l) = 1$ for all l, $K(x|l)$ takes on all integer values between one and $N(b,c,l)$. When $l \leq K(x|l) \leq N(b,c,l)$, there is exactly one partition element in which the default action is induced. If $K(x_1|l) = N$ and $K(x|l)$ is discontinuous at $x = x_1$, then the sequence $(-T, x, \theta_2, \ldots, \theta_N)$ satisfying (4.7), (4.8), (4.9) also satisfies (4.11) and (4.12).

Now we shall argue that S 's strategy $m^*(\theta)$ prescribed in the Proposition is a best response for an S of type $\theta \in (\theta_k, \theta_{k+1})$ to R's strategy $a^*(m)$ given by (4.10). More precisely, (4.7), (4.8) and (4.9) imply that

$$U^S(a^R(\theta_k, \theta_{k+1}), \theta) = \max_j U^S(a^R(\theta_j, \theta_{j+1}), \theta), \forall \theta \in [\theta_k, \theta_{k+1}], \tag{4.19}$$

where the maximum in (4.19) is taken over $j = 0, \ldots, N-1$. To see this, note that because $a^R(\theta_k, \theta_{k+1}) > a^R(\theta_{k-1}, \theta_k)$ and U^S is concave in a, (4.7), (4.8) and (4.9) imply (4.19) for $\theta = \theta_k$. Since $U^S_{12}(\cdot) > 0$ and $\theta \in [\theta_k, \theta_{k+1}]$,

$$U^S(a^R(\theta_k, \theta_{k+1}), \theta) - U^S(a^R(\theta_i, \theta_{i+1}), \theta) \geq U^S(a^R(\theta_k, \theta_{k+1}), \theta_k) - U^S(a^R(\theta_i, \theta_{i+1}), \theta_k) \geq 0 \tag{4.20}$$

and

$$\begin{aligned} &U^S(a^R(\theta_k, \theta_{k+1}), \theta) - U^S(a^R(\theta_j, \theta_{j+1}), \theta) \\ \geq &U^S(a^R(\theta_k, \theta_{k+1}), \theta_{k+1}) - U^S(a^R(\theta_j, \theta_{j+1}), \theta_{k+1}) \geq 0 \end{aligned} \tag{4.21}$$

where (4.20) and (4.21) hold for any $0 \leq i \leq k \leq j \leq N$ and θ such that $\theta \in [\theta_k, \theta_{k+1}]$. Conversely, it is clear from this argument that, except for S-types who fall on the boundaries between steps, only signals of this kind are best responses for S.

Now consider R. Provided that S uses a strategy given in the Proposition, when R hears a message m in the step (θ_k, θ_{k+1}), R's posterior, denoted by $g(\theta|m)$ is:

$$g(\theta|m) = \frac{m^*(m|\theta)h(\theta)}{\int_{\theta_k}^{\theta_{k+1}} m^*(t|\theta)h(t)dt} = \frac{h(\theta)}{\int_{\theta_k}^{\theta_{k+1}} h(t)dt} \tag{4.22}$$

Thus her conditional expected utility when playing a is

$$\int_{\theta_k}^{\theta^{k+1}} -L(|\theta - a|)g(\theta|m)d\theta = -\frac{\int_{\theta_k}^{\theta_{k+1}} L(|\theta - a|)h(\theta)d\theta}{\int_{\theta_k}^{\theta_{k+1}} h(t)dt}. \tag{4.23}$$

Therefore, $a^R(\theta_k, \theta_{k+1})$ as defined in (4.1) is a best response for R to S's message $m \in (\theta_k, \theta_{k+1})$.

Conversely, Lemma 29 shows that any equilibrium is a partition equilibrium, and the above arguments show that any equilibrium partition, θ^k, must satisfy (4.7), (4.8), (4.9), (4.11), and (4.12) for some value of l between unity and $L(b,c)$ and some value of N between l and $N(b,c,l)$. Let a_k be the action induced when the state $\theta \in (\theta_k, \theta_{k+1})$ and let $M_k \equiv \{m : a(m) = a_k\}$; if R hears a message $m \in M_k$ in such an equilibrium, her conditional expected utility is proportional to $-\int_{\theta_k}^{\theta_{k+1}} L(|\theta - a(m)|)m^*(m|\theta)h(\theta)d\theta$. Since a_k is a best response to any message $m \in M_k$, it must also maximize

$$-\int_{\theta_k}^{\theta_{k+1}} \int_{m\in M_k} L(|\theta - a(m)|)m^*(m|\theta)h(\theta)dmd\theta = -\int_{\theta_k}^{\theta_{k+1}} L(|\theta - a(m)|)h(\theta)d\theta, \quad (4.24)$$

where the identity follows because $a(m)$ is constant over the range of integration and conditional densities integrate to unity. It follows that all equilibria are essentially equivalent to the class of equilibria in the theorem with uniform messaging rules. □

We first provide a definition and then establish a series of claims which we will use in later proofs.

Definition 23. *For fixed b and c, we shall call a sequence* $\theta(N,l|b,c) \equiv \{\theta_0(N,l|b,c),\ldots,\theta_N(N,l|b,c)\}$ *a partial partition of steps N where the silence interval is the* l^{th} *partition element if Equations* (4.7), (4.8), *and* (4.9) *are satisfied.*

Claim 8. *The function* $L(|y-\theta|)$ *is strictly increasing in* $y \geq \theta$ *and strictly decreasing in* $y < \theta$.

Proof. Because $L'(\cdot) > 0$,

$$L(|y-\theta|) - L(|y'-\theta|) = L(y-\theta) - L(y'-\theta) = \int_{y'-\theta}^{y-\theta} L'(t)dt > 0$$

for $y > y' \geq \theta$. Similarly, $L(|y-\theta|) - L(|y'-\theta|) < 0$ for $y' < y < \theta$. □

Claim 9 says that for two equilibrium partitions where the silence interval is the first element, the partition associated with higher cost features higher partition points (except for the endpoints). This is because the size of the first step increases as cost increases.

Claim 9. *Let* $\theta(N,1|b,c)$ *and* $\theta(N,1|b,c')$ *be two equilibrium partitions and* $c > c'$. *Then* $\theta_i(N,1|b,c') < \theta_i(N,1|b,c)$ *for all* $i = 1,...,N-1$.

Proof. For $N = 1$, the Claim is vacuously true. Suppose that $N > 1$. For notational simplicity, we write $\theta_i(N,1|b,c)$ in the proof as $\theta_i(c)$, because we fix b and N in the discussion below. First, we claim that $\theta_i(c') \neq \theta_i(c)$ for all i such that $0 < i < N$.

If $\theta_i(c') = \theta_i(c)$ for all $0 < i < N$, then $\theta_0(c') > \theta_0(c)$, because $-L(|a^R(\theta_0(c'),\theta_1(c')) - (\theta_1(c')+b)|) = -L(|a^R(\theta_1(c'),\theta_2(c')) - (\theta_1(c')+b)|) - c' = -L(|a^R(\theta_1(c),\theta_2(c)) - (\theta_1(c)+b)|) - c' > -L(|a^R(\theta_1(c),\theta_2(c)) - (\theta_1(c)+b)|) - c = -L(|a^R(\theta_0(c),\theta_1(c)) - (\theta_1(c)+b)|)$ by Claim 8. So there is some cutoff $i \in (0,N)$ such that $\theta_i(c') \neq \theta_i(c)$. The following argument shows that $\theta_i(c') \neq \theta_i(c)$ for all i such that $0 < i < N$. Let $j \in (1,N]$ be the index such that $\theta_i(c') = \theta_i(c)$ for all $j \leq i \leq N$ and $\theta_{j-1}(c') \neq \theta_{j-1}(c)$. We show as follows that $j < N$ leads to a contradiction. Suppose $j < N$, . then $\theta_{j-1}(c') = \theta_{j-1}(c)$ by Claim 8, because $-L(|a^R(\theta_{j-1}(c'),\theta_j(c')) - (\theta_j(c')+b)|) = -L(|a^R(\theta_j(c'),\theta_{j+1}(c')) - (\theta_j(c')+b)|) = -L(|a^R(\theta_j(c),\theta_{j+1}(c)) - (\theta_j(c)+b)|) = -L(|a^R(\theta_{j-1}(c),\theta_j(c)) - (\theta_j(c)+b)|)$ by Eq. (4.7), which leads to a contradiction. Therefore $j = N$ which is equivalent to that $\theta_i(c') \neq \theta_i(c)$ for all i such that $0 < i < N$.

Next, we show that all cutoffs in between satisfy $\theta_i(c') < \theta_i(c)$. If $\theta_{N-1}(c') < \theta_{N-1}(c)$, Assumption 9 guarantees that $\theta_i(c') < \theta_i(c)$ for all $i = 1,\ldots,N-1$ and the result holds. If $\theta_{N-1}(c') > \theta_{N-1}(c)$, then $\theta_i(c') > \theta_i(c)$ for all $i = 1,\ldots,N-1$ by Assumption 9. Let $\theta_0'(c')$ be the cutoff such that the following equation holds: $-L(|a^R(\theta_0'(c'),\theta_1(c)) - (\theta_1(c)+b)|) = -L(|a^R(\theta_1(c),\theta_2(c)) - (\theta_1(c)+b)|) - c'$. Because $-L(|a^R(\theta_0'(c'),\theta_1(c)) - (\theta_1(c)+b)|) = -L(|a^R(\theta_1(c),\theta_2(c)) - (\theta_1(c)+b)|) - c' > -L(|a^R(\theta_1(c),\theta_2(c)) - (\theta_1(c)+b)|) - c = -L(|a^R(\theta_0(c),\theta_1(c)) - (\theta_1(c)+b)|)$, it must be that $\theta_1(c) > \theta_0'(c') > \theta_0(c)$. Furthermore, note that $-L(|a^R(\theta_0(c'),\theta_1(c')) - (\theta_1(c')+b)|) = -L(|a^R(\theta_1(c'),\theta_2(c')) - (\theta_1(c')+b)|) - c'$ by Eq. (4.9) and $\theta_i(c') > \theta_i(c)$ for all $i = 1,\ldots,N-1$. Then $\theta_0(c') > \theta_0'(c')$ by Assumption 9 and the definition of $\theta_0'(c')$. So $\theta_0(c') > \theta_0'(c') > \theta_0(c)$. But $\theta_0(c') = \theta_0(c)$ by assumption, which implies a contradiction. Therefore, we proved that $\theta_{N-1}(c') < \theta_{N-1}(c)$ and that overall

$\theta_i(c') < \theta_i(c)$ for all $i = 1, \ldots, N-1$. □

Claim 10. *Let $\theta(N,1|b,c)$ and $\theta(N,1|b,c')$ be two partial partitions. If $\theta_1(N,1|b,c) = \theta_1(N,1|b,c') \equiv \theta_1$ and $c' < c < L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|)$, then $\theta_i(N,1|b,c) < \theta_i(N,1|b,c')$ for all $i = 2, \ldots, N$.*

Claim 10 is an immediate consequence of Claim 9 and Assumption 9.[19]

Claim 11. *Then the maximum type-I equilibrium partition size $N(b,c,1)$ is at least $\bar{N}(b,0)$ for any c sufficiently close to zero.*

Proof. Let $\theta(\bar{N}(b,0),1|b,0) \equiv (\theta_0,\theta_1,\ldots,\theta_{\bar{N}(b,0)})$ be an equilibrium partition equilibrium of size $\bar{N}(b,0)$ when the cost is 0. We suppress b and $l = 1$ for notational simplicity.

Let $c^\dagger(b) \equiv L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|)$. We will show that the maximum equilibrium partition size when cost is any c such that $0 < c < c^\dagger(b)$ is greater than or equal to $\bar{N}(b,0)$. In other words, there is an equilibrium of size $\bar{N}(b,0)$ when the cost is $0 < c < c^\dagger$.

Let $\bar{\theta}^P(\bar{N}(b,0),c) \equiv (\bar{\theta}^P_0,\ldots,\bar{\theta}^P_{\bar{N}(b,0)})$ be a partial partition with $\bar{\theta}^P_1 = \theta_1$. The partition is well defined, because $-L(|a^R(\theta_1,\bar{\theta}^P_2 - (\theta_1+b)|) = -L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|) + c < -L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|) + c^\dagger(b) = 0$ for $0 < c < c^\dagger(b)$. By Claim 10, $\bar{\theta}^P_i < \theta_i$ for all $i = 2,\ldots,\bar{N}(b,0)$. It follows by continuity that there exists an equilibrium with size $\bar{N}(b,0)$ when the cost is c. □

The following claim is useful for proving Corollary 7.

Claim 12. *Let $\theta(N,N|b,c)$ and $\theta(N,N|b,c')$ be two partial partitions and $c > c'$, then $\theta_N(N,N|b,c) = \theta_N(N,N|b,c')$ implies that $\theta_i(N,N|b,c') > \theta_i(N,N|b,c)$ for all $i = 1,...,N-1$.*

Claim 12 is parallel to Claim 9. The proof is similar and therefore omitted.

[19] Claim 10 parallels to Lemma 5 in Crawford and Sobel (1982). We impose the additional condition that c and c' are less than $L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|)$ to ensure that $\theta_2 > \theta_1$ is well defined by the difference equation of sender's preferences, i.e., $-L(|a^R(\theta_1,\theta_2(c)) - (\theta_1+b)|) = -L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|) + c < 0$ and $-L(|a^R(\theta_1,\theta_2(c')) - (\theta_1+b)|) = -L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|) + c' < -L(|a^R(\theta_0,\theta_1) - (\theta_1+b)|) + c < 0$.

Lemma 27. *If $c, c' \in C(b)$ such that $c < c'$, then $c'' \in C(b)$ for any $c < c'' < c'$.*

Proof. We show that given b, if there exists a two-step equilibrium when $c = \bar{c}(b)$, then there exists a two-step equilibrium for any $c \in (\inf_c C(b), \bar{c}(b)]$.

If $c < \bar{c}$, construct a sequence $\theta' = (\theta_0, \theta_1, \theta_2)$ that satisfies Equation (4.9) with $l = 1$ and $\theta_1 = \theta_2 = T$ when the cost is $c < \bar{c}$, it follows that $\theta_0 > -T$ by Claim 9.

By continuity we can find a sequence $\tilde{\theta} = (\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2)$ with $\tilde{\theta}_0 = -T$, $\tilde{\theta}_2 = T$ that satisfies Equation (4.9) with $l = 1$ which constitutes an equilibrium partition when the talking cost is $c < \bar{c}$. □

Proposition 29. *$\bar{c}(b)$ is increasing in b.*

Proof. By Claim 14 and Claim 15, when the cost is $\bar{c}(b)$, there exists a two-step equilibrium with $l = 1$. Let $(\theta_0, \theta_1, \theta_2)$ be its equilibrium partition elements where $\theta_0 = -T, \theta_2 = T$. By definition $L(|\theta_1 + b - a^R(-T, \theta_1|)) - L(|\theta_1 + b - a^R(\theta_1, T)|) = c$.

By Claim 13, $\frac{d}{db}\{L(|\theta_1 + b - a^R(-T, \theta_1|)) - L(|\theta_1 + b - a^R(\theta_1, T)|)\} > 0$. Therefore, as b increases to $b' > b$, $\bar{c}(b) < \{\mathbb{E}_\theta[L(|\theta + b' - a^R(-T, \theta_1)||\theta = \theta_1)] - \mathbb{E}_\theta[L(|a^R(\theta_1, T) - \theta - b'||\theta = \theta_1)]\}$. There hence exists $\bar{c}(b') > \bar{c}(b)$ such that $\bar{c}(b') = \{\mathbb{E}_\theta[L(|\theta + b' - a^R(-T, \theta_1)||\theta = \theta_1)] - \mathbb{E}_\theta[L(|a^R(\theta_1, T) - \theta - b'||\theta = \theta_1)]\}$, and $(\theta_0, \theta_1, \theta_2)$ constitutes a two-step equilibrium partition associated with $\bar{c}(b')$, which proves the statement. □

Proposition 30. *If a non-babbling equilibrium exists when $c = 0$, there exists $\hat{c} > 0$ associated with which there is an equilibrium outcome preferred by the receiver over all equilibrium outcomes associated with zero cost.*

Proof. The proof is similar to Theorem 3 of Crawford and Sobel (1982). Fix b and $\bar{N}(b,0)$, and let $\theta^c \equiv (\theta^c_0, \theta^c_1, \ldots, \theta^c_{\bar{N}(b,0)})$ be the partition equilibrium of size $\bar{N}(b,0)$ for $c < c^\dagger(b)$, where $c^\dagger(b) \equiv L(|a^R(\theta_0(0), \theta_1(0)) - (\theta_1(0) + b)|)$. The equilibrium exists by Claim 11. Furthermore, θ^c_i $(i = 1, \ldots, \bar{N}(b,0) - 1)$ is a strictly increasing function of c by Claim 9. We shall argue that receiver's ex ante expected payoff in the partition equilibrium of size $\bar{N}(b,0)$ is locally increasing in the cost c at 0. Because receiver obtains her highest payoff when cost is zero at the equilibrium of size $\bar{N}(b,0)$, we prove that receiver is strictly better off at some positive cost if her payoff is strictly increasing in a neighborhood of 0.

Because sender's payoff at cost zero satisfies $-L(|a^R(\theta_0^0,\theta_1^0)-(\theta_1^0+b)|) = -L(|a^R(\theta_1^0,\theta_2^0)-(\theta_1^0+b)|)$, receiver's payoff in this equilibrium satisfies

$$\begin{aligned} & -L(|a^R(\theta_0^0,\theta_1^0)-\theta_1^0|) \\ > & -L(|a^R(\theta_0^0,\theta_1^0)-(\theta_1^0+b)|) \\ = & -L(|a^R(\theta_1^0,\theta_2^0)-(\theta_1^0+b)|) \\ > & -L(|a^R(\theta_1^0,\theta_2^0)-\theta_1^0|). \end{aligned} \tag{4.25}$$

Because $[-L(|a^R(\theta_0^c,\theta_1^c)-\theta_1^c|)]-[-L(|a^R(\theta_1^c,\theta_2^c)-\theta_1^c|)]$ is a continuous function of c, there is $\hat{c}>0$ such that $[-L(|a^R(\theta_0^c,\theta_1^c)-\theta_1^c|)]-[-L(|a^R(\theta_1^c,\theta_2^c)-\theta_1^c|)]>0$ for all $0\leq c\leq\hat{c}$. We claim that receiver's payoff is strictly increasing in c when $c\in[0,\hat{c}]$, which is a non-degenerate interval.

The receiver's expected payoff is given by

$$\mathbb{E}[u^R(c)] \equiv \sum_{j=1}^{\bar{N}(b,0)} \int_{\theta_{j-1}^c}^{\theta_j^c} [-L(|a^R(\theta_{j-1}^c,\theta_j^c)-\theta|)]h(\theta)d\theta.$$

Since $a^R(\theta_{j-1}^c,\theta_j^c)$ as receiver's best response to a message in the step $[\theta_{j-1}^c,\theta_j^c]$, maximizes the jth term in the sum and since $\theta_{\bar{N}(b,0)}^c\equiv T$, the Envelope Theorem yields

$$\frac{d\,\mathbb{E}[u^R(c)]}{dc} \equiv \sum_{j=1}^{\bar{N}(b,0)-1} h(\theta_j^c)\frac{d\theta_j^c}{dc}([-L(|a^R(\theta_{j-1}^c,\theta_j^c)-\theta_j^c|)]-[-L(|a^R(\theta_j^c,\theta_{j+1}^c)-\theta_j^c|)]). \tag{4.26}$$

Claim 9 implies that $d\theta_j^c/dc>0$ for all $j=1,\ldots,\bar{N}(b,0)-1$, and

$$\begin{aligned} & [-L(|a^R(\theta_{j-1}^c,\theta_j^c)-\theta_j^c|)]-[-L(|a^R(\theta_j^c,\theta_{j+1}^c)-\theta_j^c|)] \\ \geq & [-L(|a^R(\theta_{j-1}^c,\theta_j^c)-(\theta_j^c+b)|)]-[-L(|a^R(\theta_j^c,\theta_{j+1}^c)-(\theta_j^c+b)|)] \\ = & 0 \end{aligned} \tag{4.27}$$

for $j = 2, \dots, \bar{N}(b,0)-1$. The inequality in (4.27) holds by the formulation of loss function because $a^R(\theta^c_{j-1}, \theta^c_j) < \theta^c_j < \theta^c_j + b < a^R(\theta^c_j, \theta^c_{j+1})$. The equality follows from (4.7) and the definition of θ^c. For $j = 1$, $[-L(|a^R(\theta^c_0, \theta^c_1) - \theta^c_1|)] - [-L(|a^R(\theta^c_1, \theta^c_2) - \theta^c_1|)] > 0$ for all $0 \le c \le \hat{c}$. So $\mathbb{E}[u^R(c)]$ is strictly increasing in c when $c \in [0, \hat{c}]$. □

Proposition 31. *$\forall b$, there exists c such that a non-babbling equilibrium exists, i.e., $C(b) \neq \emptyset$.*

Proof. If $\bar{N}(b,0) > 1$, then $c = 0$ is an element in C. If $\bar{N}(b,0) = 1$. Let $c = \mathbb{E}_\theta(L(|-T-\theta-b|)) - \mathbb{E}_\theta\{L(|a^R(-T,T)-\theta-b|)\}$. By Chen et al. (2008), $c > 0$. Given c, there then exists a non-babbling equilibrium with $N(b,c,1) = 2$ where the lowest type $\theta = -T$ alone separates himself and the other types pool together . Hence $c \in C(b) \neq \emptyset$. □

4.5.3 Multiple Equilibria in the $b = 0$ Situation

The equilibrium we discuss in Section 4.3 features perfect communication when there is communication (i.e. outside the silence interval). By Crawford and Sobel (1982), one can always construct equilibria where the communication is imperfect outside the silence interval. The imperfect communication takes form of partitions and can be constructed by a difference equation system similar to that in Proposition 28. Among the equilibria where communication can be imperfect, the silence interval can be anywhere in the space of states/actions.

Does the equilibrium we construct in Proposition 26 deliver the highest payoff to the receiver among all equilibria? The answer is: not necessarily. However, the equilibria where the receiver enjoys higher payoffs, if exist, are problematic. To illustrate the point, we start by considering a bounded state space with $\Theta = [-T, T]$ and normally distributed prior and signals (conditional on the states). The following strategy profile is an equilibrium profile:

$$m^*(\psi) = \begin{cases} m^0 & \text{if } \mu(\psi) \notin [-T, \theta_2] \\ \mu(\psi) & \text{otherwise} \end{cases}$$

$$a^*(m) = \begin{cases} m & \text{if } m = \mu(\psi) \in [-T, \theta_2] \\ a_0 = a^R(-T, \theta_2) & \text{otherwise} \end{cases} \quad \text{such that } -\mathbb{E}\{L(|a_0 - \theta|)|\mu(\psi;\lambda) = \theta_2\}$$

$= -\mathbb{E}\{L(|\theta_2 - \theta|)|\mu(\psi;\lambda) = \theta_2\} - c$. Such θ_2 exists by continuity.

This equilibrium is nevertheless a problematic one for the following reason: note the default action is closer to θ_2 than it is to $-T$ since the silence interval lies on the "left side" of the normal prior distribution. If the sender is indifferent between inducing a_0 costless and inducing θ_2 costly when the favorite action is θ_2, the sender would strictly prefer to induce $-T$ over a_0 when the favorite action is $-T$. For this strategy profile to be an equilibrium profile, it is critical that the receiver responds to any off-equilibrium message by the default action a_0. This equilibrium profile is therefore problematic in the sense that the sender whose favorite action is $-T$ strictly prefers costly revealing his type over the equilibrium action he receives, and it is of the receiver's interest to respond to a message that reveals this sender type instead of taking the default action a_0. Fundamentally, this strategy profile can be supported as an equilibrium because of the well-known feature of cheap talk games, namely there exists equilibria where communication is less effective than it can possibly be. The usual cheap talk game equilibrium refinement technique such as NITS Chen et al. (2008) would eliminate such equilibria.

Nevertheless, such problematic equilibria might deliver higher payoff to the players compared to the one we characterize in Proposition 26. For one thing, the length of the silence interval at the equilibrium constructed as above can be shorter than that at the equilibrium constructed in Proposition 26, depending on the shape of the prior distribution around the silence interval. For another, the players might prefer the equilibrium constructed above because the ex ante probability that the sender does not talk is low compared to the equilibrium in Proposition 26. This can happen because the silence interval of the equilibrium constructed above lies on the tail of the prior distribution.

When Θ is unbounded, one can imagine equilibria similar to the one constructed above: there is perfect communication to the right of the silence interval, and imperfect communication to the left of the silence interval. If the prior distribution has thin tails, one can put the silence interval and the adjacent imperfect communication region on the tail. As a result, the probability that there is no communication or no perfect communication can be arbitrarily low ex ante and the receiver's payoff can be arbitrarily close to what can be achieved in the full information case.

This kind of equilibria have some intuitive interpretation: the receiver "threatens" the sender that if the latter does not talk, a very low action would be taken. Since the prior probability that the default action is indeed taken is extremely small, the receiver can almost achieve the payoff associated with full information. As previously argued, such equilibria rely heavily on the off-equilibrium path beliefs. Nevertheless, since the state space is unbounded and there is no "lowest" sender type, the NITS selection does not apply to this case.

4.5.4 Equilibrium Characterization in the $b > 0$ Situation

For some range of the talking cost c, there may exist another type of equilibria featuring $a^R(\theta_{l-1},\theta_l) < a^R(\theta_l,\theta_{l+1}) \leq a^S(\theta_l)$. We refer to this type of equilibria as "Type II Equilibria", and refer to the type of equilibria described in Proposition 28 as "Type I Equilibria". More specifically, a Type II equilibrium $(a^*(m), m^*(\theta))$ can be characterized as follows:[20] $m^*(\theta)$ is uniform, supported on $[\theta_k,\theta_{k+1}]$ if $\theta \in (\theta_k,\theta_{k+1})$. For $1 \leq k \leq N-1$ such that $k \notin \{l-1,l\}$, Equation (4.7) holds; For $k = l-1$ if $2 \leq l \leq N$, Equation (4.8) holds; For $k = l$ if $1 \leq l \leq N-1$, Equation (4.9) holds; And Equations (4.10), (4.11), and (4.12) hold. Further, Equation (4.13) holds when $k \neq l$; when $k = l$,[21]

$$a^R(\theta_{l-1},\theta_l) < a^R(\theta_l,\theta_{l+1}) \leq a^S(\theta_l) \tag{4.28}$$

For this type of equilibria, the upper special cutoff type of the sender is still indifferent between not sending a message to induce the default action and sending a message to induce a slightly higher action. Different from the case of a Type I equilibrium, both actions are lower

[14] For a fixed pair of b and c, Type II equilibria, if exist at all, may exist with step N but not with step $N-1$.

[21] There is another cutoff type whose posterior mean is θ_{l-1} that is indifferent between sending and not sending a message. Nevertheless, fixing the θ_{l-1}, there is only a unique θ_l with $\theta_{l-1} < \theta_l < \theta_{l+1}$ that satisfies $-\mathbb{E}_\theta[L(|a^R(\theta_{l-1},\theta_l) - \theta - b|)|\theta = \theta_l] = -\mathbb{E}_\theta[L(|a^R(\theta_l,\theta_{l+1}) - \theta - b|)|\theta = \theta_l] - c$. This solution features $a^R(\theta_{l-1},\theta_l) < a^S(\theta_l) < a^R(\theta_l,\theta_{l+1})$.

than the cutoff type's favorite action, though the latter is higher and hence more desirable.

The corollary below shows that there is no other type of equilibrium.

Corollary 6. *Any equilibrium is equivalent in terms of state-action distribution to one in either the class of Type I equilibria or the class of Type II equilibria.*

Proof. Lemma 29 shows that any equilibrium is a partition equilibrium, and the argument in Proposition 4.13 shows that any equilibrium partition $\theta_0,\dots,\theta_N$ must satisfy Equation (4.7). (4.8), (4.9) along with Equation (4.11) and Equation (4.12) for some value N with $l+1 \leq N \leq N(b,c,l)$. Equation (4.7) and (4.8) has unique solutions for $k = 0,\dots,l,l+2,\dots,N-1$ and Equation (4.9) potentially has two solutions for $k = l+1$. $\theta_0,\dots,\theta_N$ either falls in Type I equilibrium partition or Type II equilibrium partition.

Let a_k be the action induced by an S-type θ with $\theta \in [\theta_k,\theta_{k+1}]$. For any arbitrary equilibrium, since a_k best responds to the equilibrium message sent by a sender with signal θ such that $\theta \in [\theta_k,\theta_{k+1}]$, $a_k = a^R(\theta_k,\theta_{k+1})$. It follows that all equilibria are is equivalent in terms of state-action distribution to the one with the sender's strategy given in the Proposition. □

When we consider all possible equilibria, equilibria with more partition elements do not necessarily imply higher expected payoff for the receiver when there is a strictly positive talking cost. Technically, this is because although a Type II equilibrium can have more steps than all Type I equilibria, the "extra" partition element is usually very small in size and the largest partition element associated with a Type II equilibrium can be larger than that associated with a Type I equilibrium.

We then discuss how our main results in Section 4.4 change when we consider all equilibria.

Results in Section 4.4.1

Proposition 27 holds if we assume uniform prior distribution. To prove it, we first establish the following claims.

Claim 13. *Let $W(\theta,b) \equiv L(\frac{T+\theta}{2}+b) - L(|\frac{T-\theta}{2}-b|)$ where $\theta \in [-T,T]$. $W(\theta,b)$ is increasing in θ and b. $W(\theta,b)$ is continuous in θ.*

Proof. If $\theta \leq T-2b$, $\frac{T-\theta}{2} \geq b$ and $W(\theta,b) = L(\frac{T+\theta}{2}+b) - L(\frac{T-\theta}{2}-b)$. $\frac{\partial W(\theta)}{\partial \theta} = \frac{1}{2}L'(\frac{T+\theta}{2}+b) + \frac{1}{2}L'(\frac{T-\theta}{2}-b) > 0$. Similarly, $\frac{\partial W(\theta,b)}{\partial b} > 0$.

If $\theta > T-2b$, $\frac{T-\theta}{2} < b$ and $W(\theta,b) = L(\frac{T+\theta}{2}+b) - L(b-\frac{T-\theta}{2})$. $\frac{\partial W(\theta)}{\partial \theta} = \frac{1}{2}L'(\frac{T+\theta}{2}+b) - \frac{1}{2}L'(b-\frac{T-\theta}{2}) > 0$ because $L'' > 0$ and $\frac{T+\theta}{2}+b > b-\frac{T-\theta}{2}$. Similarly, $\frac{\partial W(\theta,b)}{\partial b} > 0$.

$\lim_{\theta \to (T-2b)^-} W(\theta,b) = \lim_{\theta \to (T-2b)^+} W(\theta,b)$, so $W(\theta,b)$ is continuous in θ. □

Claim 14. *Assume uniform prior distribution. For given b,c, if there exists a two-step Type I equilibrium with $l=2$, then there exists a two-step equilibrium with $l=1$.*

Proof. Let $[-T,\theta_1,T]$ be the equilibrium partition elements such that $\theta_1 \in [-T,T]$ and $-L(\theta_1 + b - a^R(-T,\theta_1)) - c = -L(a^R(\theta_1,T) - \theta_1 - b)$. Under the uniform prior assumption, this is equivalent to: $-L(\theta_1 + b - \frac{\theta_1 - T}{2}) - c = -L(\frac{T+\theta_1}{2}+b) - c = -L(\frac{\theta_1+T}{2} - \theta_1 - b) = -L(\frac{T-\theta_1}{2} - b)$.

By Claim 13, $\forall x \in [L(b)-L(T+b), L(T-b)-L(b)]$, there exists $\theta'(x)$ such that $L(|\frac{T-\theta'}{2}-b|) - L(\frac{T+\theta'}{2}+b) = x$. Since $L(b)-L(T+b) \leq L(|\frac{T-\theta_1}{2}-b|) - L(\frac{T+\theta_1}{2}+b) \leq L(T-b)-L(b)$, c must satisfy $c \leq L(T-b)-L(b)$. Note that $c \leq L(T-b)-L(b) \leq |L(b)-L(T+b)|$, or equivalently $L(b)-L(T+b) \leq -c$, because $(T+b)-b = T > (T-b)-b = T-2b$ and $L' > 0$. There exists $\theta'(-c) \in [-T,T]$ such that $L(|\frac{T-\theta'(-c)}{2}-b|) - L(\frac{T+\theta'(-c)}{2}+b) = -c$. $[-T,\theta'(-c),T]$ constitutes a two-step equilibrium partition elements with $l=1$. □

Claim 15. *When $c = \bar{c}(b)$, there exists a two-step equilibrium.*

Proof. By definition, there exists a N-step equilibrium when $c = \bar{c}(b)$ with $N \geq 2$. By definition of the equilibrium, there exists an interval $\Theta' \subseteq \Theta = [-T,T]$ over which there exists a two-step equilibrium with a silence interval if the state space is Θ'. We prove the claim by proving that $\forall \Theta'$, if there exists a two-step equilibrium with a silence interval over Θ', then there exists a two-step equilibrium δ^* with a silence interval over Θ with $\Theta' \subseteq \Theta$. This statement is trivially true if $\Theta' = \Theta$, we therefore only assume $\Theta' \subsetneq \Theta$. If δ^* is a Type I equilibrium, the statement holds because of continuity. If δ^* is a Type II equilibrium, □

We now are ready to prove Proposition 27 assuming uniform prior distribution.

Proof. The "only if" part follows directly from the definition.

We then show the "if" part: Justified by Claim 14 and Claim 15, we restrict attention to two-step equilibria featuring $l = 1$. We show that given b, if there exists a two-step equilibrium with $l = 1$ when $c = \bar{c}(b)$, then there exists a two-step equilibrium for any $c \in [\inf_c C(b), \bar{c}(b)]$.

Under uniform prior distribution, $a^R(-T, \theta_1) = \frac{\theta_1 - T}{2}$, $a^R(\theta_1, T) = \frac{T+\theta_1}{2}$. Therefore, if there exists $\theta_1 \in [-T, T]$ such that $-L(\theta_1 + b - \frac{\theta_1 - T}{2}) = -L(\frac{T+\theta_1}{2} + b) = -L(|\frac{\theta_1 + T}{2} - \theta_1 - b|) - c = -L(|\frac{T-\theta_1}{2} - b|) - c$, $[-T, \theta_1, T]$ constitutes the partition elements of a two-step equilibrium with $l = 1$.

By Claim 13, $W(\theta_1, b) = L(\frac{T+\theta_1}{2} + b) - L(|\frac{T-\theta_1}{2} - b|)$ is continuous and increasing in θ_1. Hence fixing b, the value of $W(\theta_1, b)$ for $\theta_1 \in [-T, T]$ constitutes an interval $[W(-T, b), W(T, b)]$. Let $C(b) = [W(-T, b), W(T, b)]$ and the statement is proved. □

Results in Section 4.4.2

The "bias offsetting effect" in Section 4.4.2 depends on the position of the silence interval. For the bias-offsetting effect to work, the cost has to be imposed on the direction in which the sender tends to exaggerate. If it is instead costly for the sender to induce the lower action, i.e., if the silence interval is the rightmost one, the cost will be always detrimental to the information transmission. Intuitively, sender's upward bias implies that communication about high states is less effective than communication about low states even in the absence of the talking cost. When cost increases from zero, the sender types who originally slightly prefer the second highest action to the highest action would be attracted to induce the highest action by remaining silent, which makes communication about high states even less effective. This would hurt the receiver given her convex loss function because the biggest mistake she could possibly make becomes even bigger as talking becomes more costly.

Corollary 7. *Let the silence region be the last partition element (i.e., $l = N$). For a given number of steps (i.e., N) and given preferences (i.e., b), receiver always strictly prefers the equilibrium that is associated with less talking cost (i.e., a smaller value of c).*

The argument for Corollary 7 is similar to that of Proposition 30 and is hence omitted.

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