Title
Topics in Quantum Topology: 3-Manifolds, BPS Series and Categorification

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Topics in Quantum Topology: 3-Manifolds, BPS Series and Categorification

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DISSERTATION

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Topics in Quantum Topology: 3-Manifolds, BPS Series and Categorification

Abstract

We analyze the two variable series invariant for knot complements originating from a categorification of the $SU(2)$ WRT invariant of closed oriented 3-manifolds. We are especially interested in examining the conjectured $\hbar$ expansion property and the q-holomonic property of the series invariant through an example of a satellite knot, namely, a cabling of the figure eight knot, which has more than twenty crossings. This cable knot result provides nontrivial evidence for the conjectures and demonstrates the robustness of integrality of the quantum invariant under the cabling operation. Furthermore, we investigate the conjectured relation between the series invariant and the ADO invariants at roots of unity. We reinforce the conjecture by presenting explicit formulas and/or an algorithm for particular ADO invariants of a class of torus knots obtained from the series invariant for complement of a knot. Additionally a one parameter deformation of ADO_3 invariants of torus knots is provided, which unifies the three original ADO_3 formulas into one formula. We also present a one variable series for closed plumbed 3-manifolds associated with a type I Lie superalgebra $osp(2|2)$ that categorifies the CGP invariant.

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To my family.

Only through hard work one can achieve one's dream.

Previously Published Work

Chapter 2 has appeared in the following paper:

"A Cable Knot and BPS Series", John Chae, arXiv:2101.11708 [math.GT].

Chapter 3 has appeared in the following paper:

"Knot Complement, ADO Invariants and their Deformations for Torus Knots", John Chae, SIGMA 16 (2020), 134, arXiv:2007.13277 [math.GT].

Chapter 4 has appeared in the following paper:

"Towards a q-series for $osp(2|2n)$", John Chae, arXiv:2106.09868 [math.GT].

Chapter 1

Introduction

1.1 Physics and topology

Quantum topology has occupied at an intersection between physics and low dimensional topology since the middle of 1980's. The interaction of these two fields was initiated in pioneering works of Witten [108, 110, 111, 112, 113, 114, 115]. These papers demonstrated that topological invariants admit physical realizations through topological quantum field theories[1] (TQFTs) [9, 71, 70, 99]. They also provide powerful machinery to compute the invariants of low dimensional manifolds ranging from link polynomials [1, 60, 61, 59], 3-manifolds, smooth 4-manifolds invariants [21, 119] and torsion invariants [91, 90, 100]. Furthermore, in some cases, a physics approach predicted the existence of new topological invariants such as the Witten-Reshetikhin-Tureav (WRT)-invariant of 3-manifolds [108], which motivated a rigorous construction of the invariant [92, 93].

From the mathematical perspective, TQFT itself was axiomatized in [4] (see [30, 75, 102] for reviews) and has been a topic for extensive mathematical research. Axiomatic TQFT synthesized topology, quantum algebra, representation theory and category theory. Furthermore, higher categories have appeared through extended TQFTs [88, 31] and have played an important role in the classification result of (extended) TQFTs, namely, the Cobordism hypothesis [6] and its proof [75].

[1] More precisely, both Schwarz and Witten (cohomological) types; the latter type often involves topological twisting.

The entire story was promoted to a homological level by an appearance of the categorification program. Its aim is to upgrade topological or algebraic systems to (higher) categories. The categorification terminology was first introduced in [20] in the context of lifting a 3-dimensional TQFT to a 4-dimensional TQFT in order to classify exotic smooth structures of 4-manifolds. In particular, categorification of link polynomials has enriched interactions between physics and low dimensional topology (see [44, 107, 79] for reviews). Since the advent of Khovanov homology [65], which categorifies the Jones polynomials of links, there has been constructions of other homological theories, for example, knot Floer homology [83, 89], Khovanov-Rozansky homology [67] and HOMFLY homology [68] categorify the well-known link polynomials: Alexander, $sl(N)$-invariants and HOMFLY polynomial, respectively. These homological theories themselves are examples of TQFTs of appropriate dimensions. Not only has categorification deepened conceptual understanding of links, it has also provided a powerful machinery to compute higher structural invariants beyond polynomial invariants. Furthermore, these advances have inspired new directions in physics, which resulted in physical realizations of the link homologies through topological string theories. The physical interpretation from knot Floer homology was found in [25]. A physical realization of Khovanov homology and Khovanov-Rozansky homology was first provided using topological string theory in [51]. Additionally, through the conifold transition, existence of the HOMFLY homology was predicted as well. In the case of Khovanov homology, a different physical system involving D-branes was constructed in [109]. A gauge theoretic approach was found in [116, 117, 118]. The physical construction of Kauffman homology exemplified the role of orientifolds [54]. Even knot homology, based on an exceptional Lie algebra, admits a physical description [28] (see Table 1 for a summary).

Polynomial	Homology	Physical Realization
Alexander	$sl(1\|1)$ Knot Floer homology	M5-M2 branes on the deformed conifold
Jones	$sl(2)$ Khovanov homology	M5-M2 branes on the deformed conifold or D3-NS5 brane system
$sl(N)$-invariants	$sl(N)$ Khovanov-Rozansky homology	M5-M2 branes on the deformed conifold
HOMFLY	HOMFLY homology	M5-M2 branes on the resolved conifold
$so(n)/sp(n)$-invariants & Kauffman	Kauffman homology	D4-brane & Orientifold system on the resolved conifold
Hyperpolynomial	e_6 homology	M5-M2 branes on the resolved conifold

Table 1.1: A summary of link invariants and their physical realizations. The choice of an orientifold type determines $so(n)$ or $sp(n)$ Lie algebra. Applications of S and T-dualities to the brane system were needed in the case of Khovanov homology (for details see [109]).

1.2 BPS series invariant for closed 3-manifolds

A topological structure of closed 3-manifolds that plays a central role in our story is $Spin^c$ structure. It is a lift of a structure group $SO(n)$ of the tangent bundle of an n-manifold M to

$$Spin^c(n) := Spin(n) \times_{\mathbb{Z}/2} S^1.$$

$Spin^c$ structures always exist for $n \leq 4$ and form an affine space over $H^2(M;\mathbb{Z})$. In other words, this structure is a principal $Spin^c(n)$-bundle over M together with a bundle map from the former to a principal $SO(n)$-bundle. The number of the structures on M is given by the order of $H_1(M)$. We focus on graph 3-manifolds (equivalently, plumbed manifolds) equipped with $Spin^c$ structures. A graph 3-manifold [105, 106] is a compact manifold whose JSJ-decomposition yields only Seifert fibered manifolds. Its 3-dimensional construction is described below (it can also appear as a boundary of graph 4-manifold).

The existence of q-series invariants called *homological blocks* $\hat{Z}_b(Y;q)$ of a closed 3-manifold Y having integrality properties was conjectured in [49] and [50]. For Y with $b_1(Y)=0$ and every $Spin^c(Y)$ structure b,

$$\Delta_b \in \mathbb{Q}, \quad c \in \mathbb{Z}_+, \quad \hat{Z}_b(Y;q) \in \frac{1}{2^c} q^{\Delta_b} \mathbb{Z}[[q]], \quad |q|<1.$$

It is a convergent q-series in the interior of a complex unit disc. The integer coefficients count BPS states of 3D $\mathcal{N}=2$ supersymmetric gauge theory with matter on Y. It is conjectured that the WRT invariant of $Y=\mathbb{Q}HS^3$ decomposes in terms of $\hat{Z}_b(Y;q)$:

Conjecture 1. ([50]) Let Y be a closed 3-manifold with $b_1(Y)=0$. Let $Spin^c(Y)$ be the set of $Spin^c$ structures on Y, with the action of $\mathbb{Z}/2$ by conjugation. Set

$$T := Spin^c(Y)/\mathbb{Z}_2.$$

The radial limit $lim_{q\to e^{i2\pi/k}} \hat{Z}_b(q)$ exists and in this limit, the WRT invariant of Y decomposes as

$$WRT[Y;k] = \frac{1}{i\sqrt{2k}} \sum_{a,b\in T} e^{i2\pi k\, lk(a,a)} \frac{1}{|W_b|} S_{ab}\, \hat{Z}_b(q)\Big|_{q\to e^{\frac{i2\pi}{k}}}$$

$$S_{ab} = \frac{e^{i2\pi k\, lk(a,b)} + e^{-i2\pi k\, lk(a,b)}}{|W_b|\sqrt{|H_1(Y;\mathbb{Z})|}} \qquad lk : Tor\, H_1(Y;\mathbb{Z}) \times Tor\, H_1(Y;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

where $W_x = Stab_{\mathbb{Z}_2}(x)$ is $\mathbb{Z}_2$ if $x=-x$ and is 1 otherwise; lk is the linking pairing that counts number of times a torsion cycle a passes through a surface Σ bounded by a multiple n of the other torsion cycle b so $lk(a,b) = \sharp(a\cap\Sigma(b))/n$. Furthermore, $\hat{Z}(Y;q)$ is supposed to admit a categorification

$$\chi[\mathcal{H}^{i,j}_{BPS}(Y;b)] = \hat{Z}_b[Y;q] = \sum_{i,j} (-1)^i\, q^j \dim \mathcal{H}^{i,j}_{BPS}(Y;b) \quad b\in Spin^c(Y)$$

This homology group $\mathcal{H}^{i,j}_{BPS}(Y;b)$ is the 3-manifold analogue of the Khovanov homology. The homology groups are identified with the BPS Hilbert space of the above mentioned 3D $\mathcal{N}=2$ supersymmetric theory.

Plumbed manifolds An explicit formula for weakly negative definite plumbed manifolds $Y(\Gamma)$ having $b_1(Y)=0$ (i.e $Y=\mathbb{Q}HS^3$) was derived in [50]. Before we state the result we describe a construction of plumbed 3-manifolds [80, 81, 82, 26, 43]. They are characterized by a weighted graph. Each vertex corresponds to a S^1-bundle over a compact g-surface and is labeled by $[g, n\in\mathbb{Z}]$, where n is the Euler number of the bundle. An edge is a gluing between two S^1-bundles in the fiber-base exchanging way. We choose all the base surfaces to be S^2's ($g=0$ suppressed from now on). For example, plumbing graphs for Lens space $L(p,q)$ and $Y=\Sigma(2,3,7)$ are

,

Figure 1.1: Graphs for $L(p,q)$ (left) and $Y=\Sigma(2,3,7)$ (right). $\{a_i\in\mathbb{Z}\}$ come from a continued fraction expansion of $p/q>1$.

An alternative perspective is the Dehn surgery presentation. A vertex is replaced by an n-framed unknot and an edge by a Hopf link. Moreover, a linking matrix corresponds to an adjacency matrix of a graph.

Invariance A plumbed manifold can be described by more than one graph, which are related by Kirby-Neumann moves. Specifically, two plumbing trees present the homeomorphic 3-manifolds if and only if they are related by a sequence of Neumann moves [26, 80, 81, 82]:

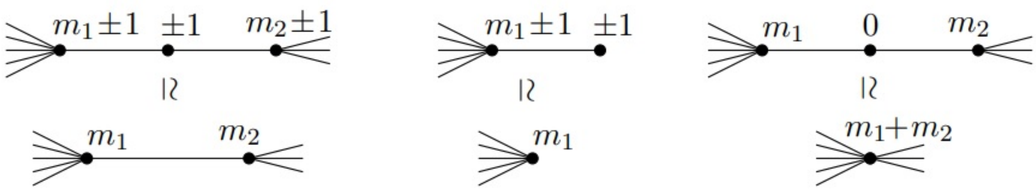


Figure 1.2: Kirby-Neumann moves that preserve a homeomorphism type of a graph manifold.

We state the formula for weakly positive definite plumbed manifolds citeGPPV:

$$\hat{Z}_b[\Gamma;q] = (-1)^{\pi} q^{\frac{3\sigma - \sum_v m_v}{4}} \prod_{v\in Vert} PV \oint_{|z_v|=1} \frac{dz_v}{i2\pi z_v} \left(z_v - \frac{1}{z_v}\right)^{2-deg(v)} \Theta_b^{-Y}(\vec{z};q),$$

where

$$\Theta_b^{-Y}(\vec{z};q) = \sum_{\vec{w}\in 2B\mathbb{Z}^L+\vec{b}} q^{-\frac{(\vec{w},B^{-1}\vec{w})}{4}} \prod_{v\in Vert} z_v^{w_v}, \quad b\in Spin^c(Y)\cong H_1(Y)$$

$$\pi = \sharp\,(\text{positive eigenvalues}), \qquad \sigma = \text{signature}(Y),$$

$$PV = \lim_{\epsilon\to 0}\frac{1}{2}\left(\oint_{|z_v|=1+\epsilon} + \oint_{|z_v|=1-\epsilon}\right)$$

Many examples have been worked out in [49, 50, 13, 17]. For plumbed manifolds having $b_1(Y) > 0$ (i.e- a graph containing a loop), Conjecture 1 and the above $\hat{Z}$ formula need a modification [16].

Physical Story The physical prediction for $\mathcal{H}^{i,j}_{BPS}(Y;b)$ originates from a brane system in M-theory given by the following setup.

Setup:

11D Spacetime:	$\mathbb{R}$	$\times$	T^*Y	$\times$	TN
N M5:	$\mathbb{R}$	$\times$	Y	$\times$	D^2
Symmetries:			"$U(1)_N$" $\times U(1)_R \times U(1)_q$		

where Y is a compact Riemannian manifold and "$U(1)_N$" exists if Y is a Seifert fibered manifold. The appearance of $T^*Y = CY_3$ is required by supersymmetry preservation for any choice of metric on Y due to McLean's theorem. Furthermore, TN is necessary to preserve supersymmetry along D^2 world volume directions and the rotational symmetries $U(1)_R \times U(1)_q$. The world-volume theory on the stack of M5 branes is 6D (2,0) theory. Dimensional reduction on Y give rises to 3D $\mathcal{N} = 2 \quad U(N)$ SCFT on $\mathbb{R} \times D^2$ denoted as $T[Y; G = U(N)]$. The symmetries $U(1)_R \times U(1)_q$ gives rise to the homological and quantum gradings on the BPS Hilbert space of $T[Y; G = U(N)]$, respectively. The boundary conditions b on $\partial D^2 = S^1$ provides the torsion grading. Therefore, we arrive at the existence of the triply-graded $\mathbb{Z} \times \mathbb{Z} \times Tor\, H_1(Y)/\mathbb{Z}_2$ homology groups:

$$\mathcal{H}_{BPS}(Y) \quad \cong \bigoplus_{\substack{b \in Tor\, H_1(Y)/\mathbb{Z}_2 \\ i \in \mathbb{Z}+\Delta_b \\ j \in \mathbb{Z}}} \mathcal{H}^{i,j}_{BPS}(Y; b)$$

The shift factor $\Delta_b(Y) \in \mathbb{Q}$ in the quantum grading is related to the d-invariant (the correction term) of the Heegaard Floer homology $HF^{\pm}(Y)$ in [48]. In the case of Y being a Seifert manifold, there is an additional grading.

Gluing two copies of the solid torus $S^1 \times D^2$ along their common boundary S^1, we can create a $S^1 \times S^2$. An important quantity that represents 3D $\mathcal{N} = 2 \quad U(N)$ theory on $S^1 \times S^2$ is the superconformal index of $T[Y]$ equivalently, its supersymmetric partition function [50]:

$$I_{sc}(q) = Tr_{\mathcal{H}^{BPS}_{S^2}}(-1)^F q^{R/2+J_3} = Z_{T[Y]}(S^1 \times_q S^2),$$

$\mathcal{H}^{BPS}_{S^2}$: the BPS sector of the Hilbert space, equivalently Q-cohomology of all physical operators

F: the fermion number R: the generator of $U(1)_R$ symmetry

J_3: the Cartan generator of the $SO(3)$ isometry of S^2

Furthermore, we let

$$\hat{Z}_b(q) := Z_{T[Y]}(S^1 \times_q D^2; b),$$

where b is a $\mathcal{N}=(0,2)$ supersymmetric boundary condition on T^2; q subscript means that, as one traverses S^1, D^2 rotates around its symmetry axis by $Arg(q)$. The splitting of $S^1\times S^2$ leads to the conjectured factorization of I_{sc}:

Conjecture 2. ([50])

$$I_{sc}[Y;q]=\sum_{b\,\in\,Tor\,H_1(Y;\mathbb{Z})/\mathbb{Z}_2}|W_b|\,\hat{Z}_b(q)\,\hat{Z}_b(1/q)\qquad\in\mathbb{Z}[[q]],$$

where $\hat{Z}_b(1/q)$ is an analytic continuation of $\hat{Z}_b(q)$ outside of a complex unit disc $|q|>1$. The conjecture has a generalization through introducing an additional parameter t, hence $I_{sc}[Y;q,t]\in\mathbb{Z}[t][[q]]$, which is called the *topologically twisted index* (the above conjecture can be recovered by setting $t=q^{\beta}$, $\beta\in\mathbb{Z}$). This generalized conjecture was verified for $Y=S^3,L(p,1)$, $O(-p)\to\Sigma_g$ in [50].

1.3 BPS series invariant for knot complements

Inspired by a categorification of the WRT invariant of a closed oriented 3-manifold, a two variable series invariant $F_K(x,q)$ for a complement of a knot M^3_K was introduced in [47]. It stems from

$$\hat{Z}_b[Y;z,n,q]\qquad b\in Spin^c(Y,\partial Y)\cong H_1(Y).$$

The label b runs over relative $Spin^c$ structures on Y whose definition is a choice of an extension of the trivial $Spin^c$ structure on a collar neighborhood of $\partial Y=\Sigma$ to a $Spin^c$ structure on Y. It is denoted by $Spin^c(Y,\partial Y)$. The space of $Spin^c(Y,\partial Y)$ forms an affine space over $H^2(Y,\partial Y)\cong H_1(Y)$. The "absolute" $Spin^c(Y)$ is isomorphic to $H^2(Y)\cong H_1(Y,\partial Y)$. There is a surjective map

$$Spin^c(Y,\partial Y)\to Spin^c(Y)$$

arising from the long exact sequence of $H^i(Y)$.

Although a rigorous definition of $F_K(x,q)$ is yet to be found, it possesses various properties such as the Dehn surgery formula and the gluing formula. This knot invariant F_K takes the form[2]

$$F_K(x,q) = \frac{1}{2}\sum_{\substack{m\geq 1\\ m \text{ odd}}}^{\infty} \big(x^{m/2} - x^{-m/2}\big) f_m(q) \in \frac{1}{2^c} q^{\Delta}\mathbb{Z}\big[x^{\pm 1/2}\big]\big[\big[q^{\pm 1}\big]\big], \tag{1.1}$$

where $f_m(q)$ are Laurent series with integer coefficients[3], $c \in \mathbb{Z}_+$ and $\Delta \in \mathbb{Q}$. Moreover, the x-variable counts the relative $\mathrm{Spin}^c\big(M_K^3, T^2\big)$-structures, which are affinely isomorphic to $H^2\big(M_K^3, T^2; \mathbb{Z}\big) \cong H_1\big(M_K^3; \mathbb{Z}\big)$; it has an infinite order, which is reflected as a series in F_K. The rational constant Δ was investigated in [48], which elucidated its intimate connection to the d-invariant (or the correction term) in certain versions of the Heegaard Floer homology ($HF^{\pm}$) for rational homology spheres. The physical interpretation of the integer coefficients in $f_m(q)$ are number of BPS states of 3d $\mathcal{N} = 2$ supersymmetric quantum field theory on M_K^3 together with boundary conditions on ∂M_K^3.

Plumbed knot complements   The series invariant F_K originates from a relative version of $\hat{Z}$ for weakly negative definite graphs. The pair consisting of a graph Γ and a distinguished vertex v_0 is called *weakly negative definite* if the corresponding matrix B is invertible and B^{-1} is negative definite on the subspace of $\mathbb{Z}^L$ spanned by the non-distinguished vertices of degree ≥ 3.
A two variable series for such graphs is

$$\hat{Z}_b[Y; z, n, q] = (-1)^{\pi} q^{\frac{3\sigma - b^T B^{-1} b}{4}} \sum_{n_v} PV \oint_{|z_v|=1} \frac{dz_v}{i2\pi z_v} \prod_{r\in Vert} (...) \prod_{(i,j)\in Edges} (...)$$

$$(r, m_r) - \text{vertex} = q^{-m_r n_r^2 - \frac{m_r}{4} - b_r n_r} z_r^{2m_r n_r + b_r} \left(z_r - \frac{1}{z_r}\right)^2$$

$$(i,j) - \text{edge} = q^{-2n_i n_j} \frac{z_i^{2n_j} z_j^{2n_i}}{(z_i - \frac{1}{z_i})(z_j - \frac{1}{z_j})},$$

where the sums and the integrals are only over $v \neq v_0$. For v_0, $n = n_{v_0}$ and $z = z_{v_0}$.

$$b \in Spin^c(Y, \partial Y) \cong (2\mathbb{Z}^L + \vec{\delta})/(2\hat{B}Z^{L-1}),$$

[2]Implicitly, there is a choice of group; originally, the group used is SU(2).
[3]They can be polynomials for monic Alexander polynomial of K (See Section 2.3)

where $\vec{\delta} = (deg(v))_{v \in Vert}$ is a degree vector and $\hat{B} = \hat{B}(\hat{\Gamma})$.

Theorem 3. ([47, Proposition 6.2]) The series $\hat{Z}_b[Y; z, n, q]$ is invariant under Neumann moves and hence an invariant of the manifold Y with torus boundary.

When $H_1(\hat{Y}; \mathbb{Z}) = 0$, $H_1(T^2) = \mathbb{Z}^2$ action on a set of $\hat{Z}_b[Y; z, n, q]$ relates different b and n values. Therefore, one of them is sufficient:

$$F_K(x, q) := \hat{Z}_0[Y; z, n = 0, q] \in \frac{1}{2^c} q^{\Delta} \mathbb{Z}[x^{1/2}, x^{-1/2}][[q^{\pm 1}]],$$

where $c \in \mathbb{Z}_+, \Delta \in \mathbb{Q}$, and $x = z^2$. In fact, using the conjugation symmetry of relative $Spin^c$ structure, $\hat{Z}_{-b}[Y; z, n, q] = -\hat{Z}_b[Y; z^{-1}, -n, q]$, F_K takes the general form (1.1).

Properties We list properties of F_K.

- TQFT characteristic: According to the definition of topological quantum field theory, to a compact oriented n-manifold M, a finite dimensional vector space over a field is associated to a boundary component of M and elements of the vector space comes from M. More precisely, they are images of M under a symmetric monoidal functor from a category of $(n+1)$-bordism to a category of vector spaces. We can view the set of invariants $\hat{Z}_b[Y; z, n, q]$ as an element in a vector space $V(T^2)$ over a Novikov field $\mathbf{k}$ associated with the torus T^2. Roughly, $V(T^2)$ is the space of functions
$$\left(\mathbb{Z} + \frac{1}{2}\right) \times \mathbb{Z} \to \mathbf{k}.$$
This implies a ($Spin^c$ decorated) TQFT for plumbed 3-manifolds.

- Gluing: For two plumbed manifolds $Y^{\pm}$ with T^2 boundary equipped with $b^{\pm} \in Spin^c(Y^{\pm}, \partial Y^{\pm})$, gluing $Y^+ \cup_{T^2} Y^- = Y$ and their $b^{\pm}$ results in
$$\hat{Z}_b[Y; q] = (-1)^{\tau} q^{\xi} \sum_n PV \oint_{|z|=1} \frac{dz}{i2\pi z} \hat{Z}_{b^-}[Y^-; z, n, q] \hat{Z}_{b^+}[Y^+; z, n, q],$$

where $\tau \in \mathbb{Z}$ and $\xi \in \mathbb{Q}$.

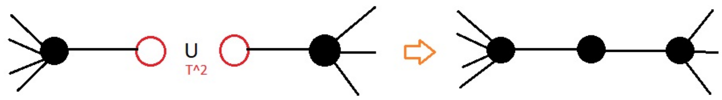


Figure 1.3: Gluing two compact oriented (plumbed) 3-manifolds with torus boundaries resulting in a closed manifold.

- Dehn Surgery: When the second graph in Figure 1.3 is a parametrized solid torus, applying the gluing formula yields the surgery formula. Specifically, for a $K \subset \hat{Y} = \mathbb{Z}HS^3$ and performing $p/r \in \mathbb{Q}^*$ surgery produces $Y_{p/r}$

$$\rightsquigarrow \hat{Z}_b[Y_{p/r}; q] = \pm q^d \mathcal{L}^{(b)}_{p/r}\left[\left(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}}\right) F_K(x,q)\right] \qquad d \in \mathbb{Q},$$

$$\mathcal{L}^{(b)}_{p/r} : x^u q^v \mapsto \begin{cases} q^{-u^2 r/p} q^v & \text{if}\, ru - b \in p\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{L}$ is the Laplace transform [5]. The above formula is a $|q| < 1$ generalization.

Several properties of F_K were conjectured as well in [47]. First of all, F_K satisfies the Melvin–Morton–Rozansky conjecture [76, 94, 95] (proven in [8]):

Conjecture 4. ([47, Conjecture 1.5]) For a knot $K \subset S^3$, the asymptotic expansion of the knot invariant $F_K\big(x, q = \mathrm{e}^{\hbar}\big)$ around $\hbar = 0$ coincides with the Melvin–Morton–Rozansky (MMR) expansion of the colored Jones polynomial in the large color limit $n \to \infty$:

$$\frac{F_K\big(x, q = \mathrm{e}^{\hbar}\big)}{x^{1/2} - x^{-1/2}} = \sum_{r=0}^{\infty} \frac{P_r(x)}{\Delta_K(x)^{2r+1}} \hbar^r, \tag{1.2}$$

where $x = e^{n\hbar}$ is fixed, n is the color of K, $P_r(x) \in \mathbb{Q}\big[x^{\pm 1}\big]$, $P_0(x) = 1$ and $\Delta_K(x)$ is the (symmetrized) Alexander polynomial of K.

Additionally motivated by the q-holonomic property of the colored Jones polynomials [38] (see Section 2.2), it was conjectured that F_K-series is q-holonomic:

Conjecture 5. ([47, Conjecture 1.6]) For any knot $K \subset S^3$, the normalized series $f_K(x,q)$ is q-holonomic. This is, it satisfies a linear recursion relation generated by the quantum A-polynomial of K $\hat{A}_K(q,\hat{x},\hat{y})$:

$$\hat{A}_K(q,\hat{x},\hat{y})f_K(x,q) = 0, \tag{1.3}$$

where $f_K := F_K(x,q)/\big(x^{1/2} - x^{-1/2}\big)$. The actions of $\hat{x}$ and $\hat{y}$ are

$$\hat{x}f_K(x,q) = xf_K(x,q) \qquad \hat{y}f_K(x,q) = f_K(xq,q).$$

Other mathematical developments of F_K [85, 86, 69] and, evidence for a relationship between F_K and the ADO link invariant [2] have been discovered in [46]. This relation is conjectured to hold for all knots and for any roots of unity:

Conjecture 6. ([46] Conjecture 3) For any knot K in S^3,

$$F_K(x,q)|_{q=\zeta_p} = \left(x^{1/2} - x^{-1/2}\right)\frac{\mathrm{ADO}_p(K;x,\zeta_p)}{\Delta_K(x^p)} \qquad \zeta_p = e^{i2\pi/p}, \quad p \in \mathbb{Z}_+.$$

This conjecture was verified for specific values of p for the right-handed trefoil and the figure eight knots [46]. Another advancement was an introduction of a refinement of $F_K(x,q)$ [29]. It was shown that $F_K(x,q)$ admits two parameter deformations through the superpolynomial [25, 32]. This led to a generalization of the above conjecture.

Conjecture 7. ([29] Conjecture 4) For any knot K in S^3, there exists a t-deformation of the symmetric ADO_p-polynomial of K for $SU(N)$,

$$\mathrm{ADO}_p^{SU(N)}[K;x,t] := \left(\Delta_K(x^p, -(-t)^p)\right)^{N-1} \lim_{q\to e^{i2\pi/p}} F_K(x,q,a=-q^N/t,t), \qquad p \in \mathbb{Z}_+$$

and $t=-1$ specialization reduces to the original $\mathrm{ADO}_p[K;x]$ (up to rescaling of x).

Examples Several prime knots up to ten crossings have been analyzed [47, 86, 69]. They include the torus knots, the figure eight knot in [47], and $\mathbf{5_1}$ in [69]. Positive braid knots ($\mathbf{10_{139}}$, $\mathbf{10_{152}}$), strongly quasipositive braids knots (m($\mathbf{10_{145}}$), $\mathbf{10_{154}}$, $\mathbf{10_{161}}$), double twist knots (m($\mathbf{5_2}$), m($\mathbf{7_3}$), m($\mathbf{7_4}$)), and a few more prime knots (m($\mathbf{7_5}$), m($\mathbf{8_{15}}$)) were examined in [86]. Among the examples, we record the result of the torus knot for application in Chapter 3.

Theorem 8. ([47, Theorem 1.2]) Let $s,t>1$ with $gcd(s,t)=1$. For the right handed torus knot $K=T(s,t)$, the series F_K is

$$F_K(x,q) = \frac{1}{2}\sum_{\substack{m\geq 1\\ m \text{ odd}}}^{\infty} \epsilon_m \left(x^{m/2} - x^{-m/2}\right) q^{\frac{m^2-(st-s-t)^2}{4st}} \tag{1.4}$$

$$\epsilon_m = \begin{cases} -1, & m \equiv st+s+t \quad \text{or} \quad st-s-t \mod 2st \\ +1, & m \equiv st+s-t \quad \text{or} \quad st-s+t \mod 2st \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 2

A cable knot and BPS series

2.1 Satellites

A (tame) knot K is a smooth embedding of S^1 into a closed 3-manifold Y.

$$i : S^1 \hookrightarrow Y \qquad K = i(S^1)$$

The ambient Y is often chosen as $\mathbb{R}^3$ or S^3. When the domain is a finite number $n \geq 1$ of S^1's , their embedding results in a link with n-components. A knot is 1-component link. According to a classification of knots embedded in 3-space, they can be categorized into three kinds:

1. Torus knot $K = T(s,t) \quad 2 \leq s \leq |t| \quad gcd(s,t) = 1$
2. Hyperbolic knot $K = 4_1,\ 5_2,\ 6_1, \cdots$
3. Satellite knot

The torus knot wraps a torus T^2 embedded in 3-space; s and t parameters are winding numbers along the standard meridian and a longitude of T^2, respectively. A hyperbolic knot has its complement/exterior as a hyperbolic 3-manifolds, whose volume is a topological invariant. A satellite knot involves the satellite operation, which creates a new knot out of an old knot. The operation consists of a pattern knot P in the interior of the solid torus $S^1 \times D^2$, a companion knot K' in the

S^3 and an canonical identification $h_{K'}$

$$h_{K'} : S^1 \times D^2 \longrightarrow \nu(K') \subset S^3, \tag{2.1}$$

where $\nu(K')$ is the tubular neighborhood of K'.

Figure 2.1: A pattern knot P (left), companion K' (center) and satellite knot $P(K')$ (right).

A well-known example of satellite knots is a cable knot $h_{K'}(P) = C_{(r,s)}(K')$ that is obtained by choosing P to be the (r, s)-torus knot pushed into the interior of the $S^1 \times D^2$. This map $h_{K'}$ has been investigated in [74, 77, 78].

2.2 Quantum torus and recursion ideal

Let $\mathcal{T}$ be a quantum torus

$$\mathcal{T} := \mathbb{C}[t^{\pm 1}] \left\langle M^{\pm 1}, L^{\pm 1} \right\rangle / (LM - t^2 ML).$$

The generators of the noncommutative ring $\mathcal{T}$ acts on a set of discrete functions, which are colored Jones polynomials $J_{K,n} \in \mathbb{Z}[t^{\pm 1}]$ in our context, as

$$MJ_{K,n} = t^{2n} J_{K,n} \qquad LJ_{K,n} = J_{K,n+1}.$$

The recursion(annihilator) ideal $\mathcal{A}_K$ of $J_{K,n}$ is the left ideal $\mathcal{A}_K$ in $\mathcal{T}$ consisting of operators that annihilates $J_{K,n}$:

$$\mathcal{A}_{J_{K,n}} := \left\{ \alpha_K \in \mathcal{T} \,|\, \alpha_K J_{K,n} = 0 \right\}.$$

As it turns out that $\mathcal{A}_K$ is not a principal ideal in general. However, by adding inverse polynomials of t and M to $\mathcal{T}$ [34], we obtain a principal ideal domain $\tilde{\mathcal{T}}$

$$\tilde{\mathcal{T}} := \left\{ \sum_{j\in\mathbb{Z}} a_j(M)L^j \middle| a_j(M) \in \mathbb{C}[t^{\pm 1}](M),\ a_j = \text{almost always} \quad 0 \right\}$$

Using $\tilde{\mathcal{T}}$ we get a principal ideal $\tilde{\mathcal{A}_K} := \tilde{\mathcal{T}}\mathcal{A}_K$ generated by a single polynomial $\hat{A}_K$

$$\hat{A}_K(t, M, L) = \sum_{j=0}^{d} a_j(t, M)L^j.$$

This is an universal property of knots (and links) that was proved in [38]:

Theorem 9. ([38]) Colored Jones polynomial of every knot K is q-holonomic. That is, there is a minimal degree q-difference operator

$$\hat{A}_K(q, M, L) = \sum_{j=0}^{d} a_j(q, M)L^j, \quad d \geq 1$$

annihilating the colored Jones polynomial $J_{K,n}, \quad n \in \mathbb{N}$,

$$\hat{A}_K(q, M, L)J_{K,n}(q) = 0.$$

In other words, $J_{K,n}$ satisfies the linear recursion relation with polynomial coefficients generated by $\hat{A}_K$. There is a multivariable generalization of the above result for links as well.

Quantum $\hat{A}_K$ polynomial [1] can be thought of as a noncommutative deformation/quantization of a classical A-polynomial of a knot [18] (see also [19]). Alternative approaches to obtain $\hat{A}_K(t, M, L)$ are by quantizing the classical A-polynomial curve using a twisted Alexander polynomial or applying the topological recursion [52]. A conjecture called AJ conjecture/quantum volume conjecture was proposed in [34, 45] via different approaches (numerically vs $SU(2)$ quantum Chern Simons theory):

[1] alternative terminology is quantum curve

Conjecture 10. ([34, 45]) For any knot $K \subset S^3$, $\hat{A}_K(t=-1, L, M)$ reduces to the (classical) A-polynomial curve $A_K(L, M)$ up to a solely M-dependent overall factor.

The conjecture was confirmed for a variety of knots [24, 34, 35, 42, 58, 73, 104, 97].

2.3 Knot polynomials

In this section we will analyze the colored Jones polynomial and the Alexander polynomial of a cable knot to show that the former satisfies the MMR expansion and the latter is monic. Furthermore, the MMR expansion enables us to read off the initial condition that is needed in Section 2.4.

For (r,2)-cabling of the figure eight knot $\mathbf{4_1}$, we set $P = T(r, 2)$ and $K' = \mathbf{4_1}$ in (2.1). The cabling formula for an unnormalized $\mathfrak{sl}_2(\mathbb{C})$ colored Jones polynomial of a $(r, 2)$-cabling of a knot K' in S^3 is [103]

$$\tilde{J}_{C_{(r,2)}(K'),n}(q) = q^{-\frac{r}{2}(n^2-1)} \sum_{w=1}^{n} (-1)^{r(n-w)} q^{\frac{r}{2}w(w-1)} \tilde{J}_{K',(2w-1)}(q), \qquad |r| > 8 \quad \text{and odd.}$$

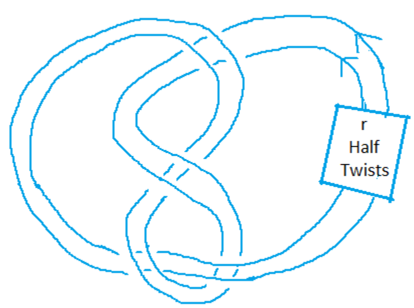


Figure 2.2: (r,2)-cable of the figure eight knot.

Its application to $K = C_{(9,2)}(\mathbf{4_1})^2$, whose diagram has 25 crossings, is

$$\tilde{J}_{K,n}(q) = q^{-\frac{9}{2}(n^2-1)} \sum_{w=1}^{n} \left[(-1)^{(n-w)} q^{\frac{9}{2}w(w-1)} [2w-1] \sum_{r=0}^{2w-2} \prod_{k=1}^{r} \left(-q^{-k} - q^{k} + q^{1-2w} + q^{2w-1} \right) \right].$$

Using the (0-framed) unknot U value together with $q = t^4$

$$J_{U,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}},$$

the first few unknot normalized polynomials $J_{K,n}(q)$ are

$$J_{K,1}(q) = 1$$

$$J_{K,2}(q) = q^2 - q + \frac{1}{q^4} + \frac{1}{q^6} - \frac{1}{q^7} + \frac{1}{q^8} - \frac{1}{q^9} + \frac{1}{q^{12}} - \frac{1}{q^{13}}$$

$$\begin{aligned} J_{K,3}(q) = {} & q^{12} - q^{11} - q^{10} + q^9 - q^8 + q^7 + q^6 - q^5 + q^2 - 1 + \frac{1}{q^8} + \frac{1}{q^{11}} - \frac{1}{q^{13}} + \frac{1}{q^{14}} - \frac{1}{q^{16}} + \frac{1}{q^{17}} \\ & - \frac{1}{q^{18}} - \frac{1}{q^{19}} + \frac{2}{q^{20}} - \frac{1}{q^{21}} + \frac{1}{q^{23}} - \frac{1}{q^{24}} + \frac{1}{q^{25}} + \frac{1}{q^{26}} - \frac{2}{q^{27}} - \frac{1}{q^{28}} + \frac{1}{q^{29}} - \frac{1}{q^{30}} + \frac{2}{q^{32}} - \frac{1}{q^{33}} \\ & - \frac{1}{q^{34}} + \frac{1}{q^{35}} \end{aligned}$$

Their $\hbar$ series are

$$\begin{aligned} J_{K,n}(e^{\hbar}) = {} & 1 + \left(6 - 6n^2\right)\hbar^2 + \left(-42 + 42n^2\right)\hbar^3 + \left(\frac{801}{2} - 462n^2 + \frac{123}{2}n^4\right)\hbar^4 \\ & + \left(-\frac{8451}{2} + 5173n^2 - \frac{1895}{2}n^4\right)\hbar^5 + \left(\frac{3111491}{60} - \frac{132779}{2}n^2 + 14986n^4 - \frac{27281}{60}n^6\right)\hbar^6 \\ & + \left(-\frac{14631401}{20} + \frac{19399417}{20}n^2 - \frac{3028829}{12}n^4 + \frac{840097}{60}n^6\right)\hbar^7 \\ & + \left(\frac{39069313501}{3360} - \frac{950122877}{60}n^2 + \frac{54585517}{12}n^4 - \frac{1725671}{5}n^6 + \frac{13273763}{3360}n^8\right)\hbar^8 + \cdots \end{aligned} \tag{2.2}$$

We see that, at each $\hbar$ order, the degree of the polynomial in n is at most the order of $\hbar$, which is an equivalent characterization of the MMR expansion of the colored Jones polynomial of a knot. Secondly, as a consequence of the cabling, odd powers of $\hbar$ appear in the expansion, which are

[2] This cabling parameters correspond to $\mathbf{9_1}$ for the pattern knot. We assume 0-framing for $\mathbf{4_1}$.

absent in the case of the figure eight knot [47]. Moreover, the coefficient polynomials for the odd $\hbar$-powers have one lower degree whereas the degree of the polynomials are the same for the even $\hbar$-powers. Hence they are unaffected by the cabling operation.

The cabling formula for the Alexander Polynomial of a knot K is [56]

$$\Delta_{C_{(p,q)}(K)}(t) = \Delta_K(t^p)\Delta_{T_{(p,q)}}(t), \quad 2 \leq p < |q| \quad \gcd(p,q) = 1,$$

where $\Delta(t)$ is the symmetrized Alexander polynomial and $T_{(p,q)}$ is the (p,q) torus knot. Note that our convention for the parameters of the torus knot are switched (i.e. $p \equiv 2, q \equiv r$). Applying the above formula to $C_{(9,2)}(\mathbf{4_1})$, we get

$$\begin{aligned}\Delta_{C_{(9,2)}(\mathbf{4_1})}(x) &= \Delta_{\mathbf{4_1}}(x^2)\Delta_{T_{(2,9)}}(x) \\ &= -x^6 - \frac{1}{x^6} + x^5 + \frac{1}{x^5} + 2x^4 + \frac{2}{x^4} - 2x^3 - \frac{2}{x^3} + x^2 + \frac{1}{x^2} - x - \frac{1}{x} + 1.\end{aligned}$$

From this Alexander polynomial its symmetric expansion about $x = 0$ (in x) and $x = \infty$ (in $1/x$) in the limit of $\hbar \to 0$ can be computed.

$$\begin{aligned}\lim_{q\to 1} 2F_K(x,q) &= 2\,\mathrm{s.e}\left(\frac{x^{1/2} - x^{-1/2}}{\Delta_K(x)}\right) \\ &= x^{11/2} - \frac{1}{x^{11/2}} + 2x^{15/2} - \frac{2}{x^{15/2}} + 5x^{19/2} - \frac{5}{x^{19/2}} + 13x^{23/2} - \frac{13}{x^{23/2}} \\ &+ 34x^{27/2} - \frac{34}{x^{27/2}} - x^{29/2} + \frac{1}{x^{29/2}} + 89x^{31/2} - \frac{89}{x^{31/2}} - 2x^{33/2} + \frac{2}{x^{33/2}} \\ &+ 233x^{35/2} - \frac{233}{x^{35/2}} - 5x^{37/2} + \frac{5}{x^{37/2}} + 610x^{39/2} - \frac{610}{x^{39/2}} + \cdots \in \mathbb{Z}\left[\left[x^{\pm 1/2}\right]\right]\end{aligned} \tag{2.3}$$

The coefficients in the expansions are integers and hence the Alexander polynomial is monic, which is a necessary condition for $f_m(q)$'s in (1.1) to be polynomials.

2.4 The recursion relation

The quantum (or noncommutative) A-polynomial of a class of cable knot $C_{(r,2)}(\mathbf{4_1})$ in S^3 having minimal L-degree is given by [96]

$$\hat{A}_K(t,M,L) = (L-1)B(t,M)^{-1}Q(t,M,L)\left(M^rL + t^{-2r}M^{-r}\right) \in \tilde{\mathcal{A}}_K \tag{2.4}$$

where

$$Q(t,M,L) = Q_2(t,M)L^2 + Q_1(t,M)L + Q_0(t,M), \quad B(t,M) := \sum_{j=0}^{2} c_j b(t, t^{2j+2}M^2)$$

$$b(t,M) = \frac{M(1+t^4M^2)(-1+t^4M^4)(-t^2+t^{14}M^4)}{t^2-t^{-2}}$$

$$c_0 = \hat{P}_0(t,t^4M^2)\hat{P}_1(t,t^6M^2), \quad c_1 = -\hat{P}_1(t,t^2M^2)\hat{P}_1(t,t^6M^2), \quad c_2 = \hat{P}_1(t,t^2M^2)\hat{P}_2(t,t^4M^2).$$

$$\begin{aligned}
Q_2(t,M) =&\ \hat{P}_2(t,t^4M^2)\,\hat{P}_1(t,t^2M^2)\,\hat{P}_0(t,t^6M^2)\\
Q_1(t,M) =&\ \hat{P}_0(t,t^4M^2)\,\hat{P}_1(t,t^6M^2)\,\hat{P}_2(t,t^2M^2) - \hat{P}_1(t,t^6M^2)\,\hat{P}_1(t,t^2M^2)\,\hat{P}_1(t,t^4M^2)\\
&+ \hat{P}_2(t,t^4M^2)\,\hat{P}_1(t,t^2M^2)\,\hat{P}_0(t,t^6M^2)\\
Q_0(t,M) =&\ \hat{P}_0(t,t^4M^2)\,\hat{P}_1(t,t^6M^2)\,\hat{P}_0(t,t^2M^2),
\end{aligned}$$

$$\begin{aligned}
\hat{P}_0(t,M) :=&\ t^6M^4(-1+t^{12}M^4)\\
\hat{P}_1(t,M) :=&\ -(-1+t^4M^2)(1+t^4M^2)\left(1-t^4M^2-t^4M^4-t^{12}M^4-t^{12}M^4-t^{12}M^6+t^{16}M^8\right)\\
\hat{P}_2(t,M) :=&\ t^{10}M^4(-1+t^4M^4).
\end{aligned}$$

For $K = C_{(9,2)}(\mathbf{4_1})$, applying (2.4) to $f_K(x,q)$ together with $x = q^n$ yields

$$\alpha(x,q)F_K(x,q) + \beta(x,q)F_K(xq,q) + \gamma(x,q)F_K(xq^2,q) + \delta(x,q)F_K(xq^3,q) + F_K(xq^4,q) = 0, \tag{2.5}$$

where $\alpha, \beta, \gamma, \delta$ functions and their $\hbar$ series are documented in Appendix A. From (2.5) we find the recursion relation for f_m.

$$
\begin{aligned}
f_{m+98}(q) = \frac{-1}{q^{\frac{109+m}{2}}\left(1-q^{\frac{87+m}{2}}\right)} \Big[& t_2\, f_{m+94} + t_4\, f_{m+90} + t_6\, f_{m+86} + t_8\, f_{m+82} + t_9\, f_{m+80} + t_{10}\, f_{m+78} \\
& + t_{11}\, f_{m+76} + t_{12}\, f_{m+74} + t_{13}\, f_{m+72} + t_{14}\, f_{m+70} + t_{15}\, f_{m+68} + t_{16}\, f_{m+66} + t_{17}\, f_{m+64} \\
& + t_{18}\, f_{m+62} + t_{19}\, f_{m+60} + t_{20}\, f_{m+58} + t_{21}\, f_{m+56} + t_{22}\, f_{m+54} + t_{23}\, f_{m+52} + t_{24}\, f_{m+50} \\
& + t_{25}\, f_{m+48} + t_{26}\, f_{m+46} + t_{27}\, f_{m+44} + t_{28}\, f_{m+42} + t_{29}\, f_{m+40} + t_{30}\, f_{m+38} + t_{31}\, f_{m+36} \\
& + t_{32}\, f_{m+34} + t_{33}\, f_{m+32} + t_{34}\, f_{m+30} + t_{35}\, f_{m+28} + t_{36}\, f_{m+26} + t_{37}\, f_{m+24} + t_{38}\, f_{m+22} \\
& + t_{39}\, f_{m+20} + t_{40}\, f_{m+18} + t_{41}\, f_{m+16} + t_{43}\, f_{m+12} + t_{45}\, f_{m+8} + t_{47}\, f_{m+4} + t_{49}\, f_m \Big] \in \mathbb{Z}[q^{\pm 1}]
\end{aligned}
\tag{2.6}
$$

where $t_v = t_v(q, q^m)$'s are listed in Appendix B. Using this recursion and the initial data documented in Appendix C, $F_K(x, q)$ can be obtained to any desired order in x.

2.5 An expansion of a knot complement

We next compute a series expansion of the F_K of complement of the cable knot K. Specifically, a straightforward computation from (2.5) yields an ordinary differential equation(ODE) for $P_m(x)$ at each $\hbar$ order. Using the initial conditions for the ODEs obtained from (2.2)

$$
P_1(1) = 0, \quad P_2(1) = 6, \quad P_3(1) = -42, \quad P_4(1) = \frac{801}{2}, \quad P_5(1) = -\frac{8451}{2}, \quad \cdots
$$

we find that, for example,

$$
\begin{aligned}
P_1(x) = \ & 5x^{12} + \frac{5}{x^{12}} - 10x^{11} - \frac{10}{x^{11}} - 13x^{10} - \frac{13}{x^{10}} + 36x^9 + \frac{36}{x^9} - 10x^8 - \frac{10}{x^8} - 16x^7 - \frac{16}{x^7} + 15x^6 \\
& + \frac{15}{x^6} - 14x^5 - \frac{14}{x^5} + 16x^4 + \frac{16}{x^4} - 18x^3 - \frac{18}{x^3} + 19x^2 + \frac{19}{x^2} - 20x - \frac{20}{x} + 20 \\
P_2(x) = \ & \frac{25x^{24}}{2} + \frac{25}{2x^{24}} - 50x^{23} - \frac{50}{x^{23}} - 14x^{22} - \frac{14}{x^{22}} + 306x^{21} + \frac{306}{x^{21}} - \frac{641x^{20}}{2} - \frac{641}{2x^{20}} - 448x^{19} \\
& - \frac{448}{x^{19}} + \frac{2011x^{18}}{2} + \frac{2011}{2x^{18}} - 358x^{17} - \frac{358}{x^{17}} - 522x^{16} - \frac{522}{x^{16}} + 612x^{15} + \frac{612}{x^{15}} - \frac{589x^{14}}{2} \\
& - \frac{589}{2x^{14}} + 508x^{13} + \frac{508}{x^{13}} - \frac{3325x^{12}}{2} - \frac{3325}{2x^{12}} + 1648x^{11} + \frac{1648}{x^{11}} + 1538x^{10} + \frac{1538}{x^{10}} - 3932x^9 \\
& - \frac{3932}{x^9} + 1574x^8 + \frac{1574}{x^8} + 1670x^7 + \frac{1670}{x^7} - 1798x^6 - \frac{1798}{x^6} + 396x^5 + \frac{396}{x^5} - \frac{1521x^4}{2} \\
& - \frac{1521}{2x^4} + 4082x^3 + \frac{4082}{x^3} - \frac{6541x^2}{2} - \frac{6541}{2x^2} - 8334x - \frac{8334}{x} + 16831.
\end{aligned}
$$

Substituting them into (1.2) results in

$$
\begin{aligned}
2F(x, e^\hbar) = & \Big(x^{11/2} - \frac{1}{x^{11/2}} + 2x^{15/2} - \frac{2}{x^{15/2}} + 5x^{19/2} - \frac{5}{x^{19/2}} + 13x^{23/2} - \frac{13}{x^{23/2}} + 34x^{27/2} - \frac{34}{x^{27/2}} \\
& - x^{29/2} + \frac{1}{x^{29/2}} + 89x^{31/2} - \frac{89}{x^{31/2}} - 2x^{33/2} + \frac{2}{x^{33/2}} + 233x^{35/2} - \frac{233}{x^{35/2}} - 5x^{37/2} \\
& + \frac{5}{x^{37/2}} + \cdots \Big) \\
& + \hbar \Big(5x^{11/2} - \frac{5}{x^{11/2}} + 12x^{15/2} - \frac{12}{x^{15/2}} + 35x^{19/2} - \frac{35}{x^{19/2}} + 104x^{23/2} - \frac{104}{x^{23/2}} + 306x^{27/2} \\
& - \frac{306}{x^{27/2}} - 15x^{29/2} + \frac{15}{x^{29/2}} + 890x^{31/2} - \frac{890}{x^{31/2}} - 36x^{33/2} + \frac{36}{x^{33/2}} + 2563x^{35/2} - \frac{2563}{x^{35/2}} \\
& - 105x^{37/2} + \frac{105}{x^{37/2}} + \cdots \Big) \\
& + \hbar^2 \Big(\frac{25}{2}x^{11/2} - \frac{25}{2}\frac{1}{x^{11/2}} + 36x^{15/2} - \frac{36}{x^{15/2}} + \frac{247}{2}x^{19/2} - \frac{247}{2}\frac{1}{x^{19/2}} + 426x^{23/2} - \frac{426}{x^{23/2}} \\
& + 1441x^{27/2} - \frac{1441}{x^{27/2}} - \frac{225}{2}x^{29/2} + \frac{225}{2}\frac{1}{x^{29/2}} + 4781x^{31/2} - \frac{4781}{x^{31/2}} - 324x^{33/2} + \frac{324}{x^{33/2}} \\
& + 2563x^{35/2} - \frac{2563}{x^{35/2}} - \frac{2207}{2}x^{37/2} + \frac{2207}{2}\frac{1}{x^{37/2}} + \cdots \Big).
\end{aligned}
$$

Comparing to the series of the figure eight knot [47], we notice that every order of $\hbar$ appears in the above series whereas the series corresponding to the figure eight knot consists of only even powers of $\hbar$ (i.e. $P_i(x) = 0$ for i odd). This difference is an effect of the torus knot whose expansion involve all powers of $\hbar$ [47]. Furthermore, the x-terms begin from $m = 11$ instead of $m = 1$ and there are gaps in their powers. Specifically, $x^{\pm 13/2}, x^{\pm 17/2}, x^{\pm 21/2}$ and $x^{\pm 25/2}$ are absent. This is a consequence of the structure of (2.3). A distinctive feature of the cable knot is that from $x^{\pm 29/2}$ the coefficients are negative. Moreover, the positive and the negative coefficients alternate from that x-power for all $\hbar$ powers. These differences persist in the higher $\hbar$-orders. We will see these differences in a manifest way in the next section.

2.6 Effects of the cabling

Since the initial data plays a core role in the recursion relation method, we discuss their features for the cable knot and then propose conjectures about them, which can be a useful guide for finding initial data for a family of the cable knots.

In the initial data (see Appendix C) for the recursion relation (2.6), we notice several differences from that of the figure eight knot [47]. Before discussing them, let us begin with the properties of the F_K that are preserved by the cabling. The initial data consists of an odd number of terms and power of q increases by one between every consecutive terms in a fixed f_m for all m's, which are also true for f_{99} and f_{101}. Additionally the reflection symmetry of coefficients is retained up to f_{43} for positive coefficients and up to f_{61} for the negative ones but of course, those f_m's do not have the complete amphichiral structure. These invariant properties are a remnant feature of the amphichiral property of the figure eight knot.

A difference is that the nonzero initial data begins from f_{11} and the gaps between the powers of x is four up to $x^{27/2}$, which is in the accordance with f_m's. These features are direct consequences of the symmetric expansion of the Alexander polynomial of the cable knot (2.3). In the case of the figure eight its coefficient functions start from f_1 and there are no such gaps. Another distinctive difference is that f_m's containing negative coefficients appear from $m = 29$. Moreover, the positive and negative coefficient f_m's alternate from f_{27} (i.e. positive coefficients for f_{27}, f_{31},

$\ldots$ and negative coefficients for f_{29}, f_{33}, $\ldots$). Furthermore, from f_{47} the reflection symmetry of the positive coefficients in the appropriate f_m's is broken. This phenomenon also occurs for the negative coefficient f_m's from $m = 65$. Breaking of the symmetry is expected since the cable knot of the figure eight is not amphichiral. The next difference is that the largest power of q in the positive coefficient f_m's for $m \geq 15$, the powers increase by $2, 2, 3, 3, 4, 4, \ldots, 11, 11$. For the negative coefficient case, the changes are $4, 4, 5, 5, 6, 6, \ldots, 11, 11$ from $m = 33$. The smallest powers of f_m's having positive coefficients exhibit their changes as $0, 0, -1, -1, -2, -2, -3, -3, \ldots$ and for those with negative coefficients the pattern is $2, 2, 1, 1, 0, 0, -1, -1, -2, -2, \ldots$.

An universal feature of the negative coefficient f_m's in the initial data is that their coefficient modulo sign is determined by the positive coefficient f_m's. For example, the absolute value of the coefficients of f_{29} is same as that of f_{11}; f_{33}'s coefficients come from that of f_{15} up to sign and so forth. Hence coefficients of f_m having negative coefficients are determined by f_{m-18}. In fact, this peculiar coefficient correlation also exists in the non-initial data f_{101} whose coefficients are correlated with that of f_{83}.

Conjecture 10. For a class of a cable knot of the figure eight $K_r = C_{(r,2)}(\mathbf{4_1}) \subset S^3$, $r > 8$ and odd having monic Alexander polynomial, the coefficient functions $\left\{f_m(q) \in \mathbb{Z}[q^{\pm 1}]\right\}$ in $F_{K_r}(x, q)$ can be classified into two (disjoint) subsets: one of them consists of elements having all positive coefficients $\left\{f_t^+(q)\right\}_{t \in I^+}$ and the other subset contains elements whose coefficients are all negative $\{f_w^-(q)\}_{w \in I^-}$. Furthermore, for every element in $\{f_w^-(q)\}$, its coefficients coincide with that of an element in $\left\{f_t^+(q)\right\}$ up to sign.

Conjecture 11. For the family of knots in Conjecture 10, every nonzero element in $\{f_m(q)\}$ consists of an odd number of terms and power of q increases(decreases) by one between every consecutive terms of the element. Moreover, coefficients of an element $f_v^-(q) \in \{f_w^-(q)\}$ agree with coefficients of $f_{v-2r}^+(q) \in \left\{f_t^+(q)\right\}$ up to sign for all $v \in I^-$.

Chapter 3

Relation to the ADO invariants

3.1 The ADO polynomials

Colored generalization of the Alexander polynomial for framed colored and oriented knot (link) was introduced in [2]. This knot invariant(ADO invariant) is based on $(1,1)$-colored tangle diagram obtained by cutting the knot (or a component of a link). From this colored and oriented tangle diagram, the ADO invariant is constructed from a non-semisimple category of module over the unrolled quantum group $\mathcal{U}^H_{\zeta_{2r}}(sl_2(\mathbb{C}))$ together with the *modified* quantum dimension ($r \in \mathbb{Z}_{\geq 2}$). We will employ the quantum algebra construction of the ADO invariants for verification of our results; the computational ingredients are summarized in Section 3.6. We give a concise review of the conceptual features of the construction [2, 101, 7].

The first ingredient is the unrolled quantum group $\mathcal{U}^H_{\zeta_{2r}}(sl_2(\mathbb{C}))$, which is a $\mathbb{C}$-algebra specialized at $q = \zeta_{2r}$; its generators and relations are

Generators:

$$E, F, K, K^{-1}, H$$

Relations:

$$KK^{-1} = K^{-1}K = 1 \quad KE = \zeta_{2r}^2 EK \quad KF = \zeta_{2r}^{-2} FK \quad [E,F] = \frac{K - K^{-1}}{\zeta_{2r} - \zeta_{2r}^{-1}}$$

$$KH = HK \qquad [H,E] = 2E \qquad [H,F] = -2F \qquad E^r = F^r = 0.$$

This algebra possess a Hopf algebra structure:

$$\Delta(E) = 1 \otimes E + E \otimes K \qquad \epsilon(E) = 0 \qquad S(E) = -EK^{-1}$$

$$\Delta(F) = K^{-1} \otimes F + F \otimes 1 \qquad \epsilon(F) = 0 \qquad S(F) = -KF$$

$$\Delta(H) = 1 \otimes H + H \otimes 1 \qquad \epsilon(H) = 0 \qquad S(H) = -H$$

$$\Delta(K) = K \otimes K \qquad \epsilon(K) = 1 \qquad S(K) = K^{-1}$$

$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1} \qquad \epsilon(K^{-1}) = 1 \qquad S(K^{-1}) = K$$

The second element of the construction of the ADO invariant is a functor RT between a category of colored oriented tangle diagrams COD and a category Rep of representations of $\mathcal{U}^H_{\zeta_{2r}}(sl_2(\mathbb{C}))$.

$$RT : COD \longrightarrow Rep.$$

The objects of COD are framed colored oriented $(1,1)$-tangle diagrams and morphisms are equivalence classes of the tangle diagrams whose equivalence relations are generated by the tangle moves(see [2] Section 2). For the target category, its objects are vector spaces V and morphisms are linear maps between them. The image of the RT functor is $RT(T) =< T > Id_V \in End_{\mathbb{C}}(V)$, which enables to define

$$\mathrm{ADO}(K)_r := d(V_\alpha; r) < T >,$$

where V_α is a vector space assigned to K (or to an open component of a link[1]) and $d(V_\alpha; r)$ is the modified quantum dimension,

$$d(V_\alpha; r) = -\zeta_{2r}^{\frac{1}{2}r(1-r)} \frac{\zeta_{2r}^{\alpha+1} - \zeta_{2r}^{-\alpha-1}}{\zeta_{2r}^{r\alpha} - \zeta_{2r}^{-r\alpha}}, \quad \alpha \in (\mathbb{C} \backslash \mathbb{Z}) \cup (r\mathbb{Z} - 1).$$

[1]ADO invariant is independent of choice of a component of a link that is cut (for details see Section 5 in [2]).

This modified dimension replaces the usual quantum trace, which vanishes in this context. Moreover, it makes $\mathrm{ADO}(K)$ an isotopy invariant.

3.2 The ADO invariants of torus knots

Recently, evidence for a relation between F_K at specific values of roots of unity and the ADO invariants were discovered for the (right-handed) trefoil, the figure eight and 5_2 knots [46]. Furthermore, this relation is conjectured to hold for any roots of unity and for all knots. Using (1.1) and Conjecture 6, close examination of torus knots $T(2, 2s+1)$ at various values of s yields an explicit formula or an algorithm for ADO_3 and ADO_4 invariants of $T(2, 2s+1)$, $s \in \mathbb{Z}_+$.

3.3 The ADO_3 invariants of $T(2, 2s+1)$

The ADO_3 invariants of $T(2, 2s+1)$ are divided in three types depending on their coefficient pattern.

1. For $K = T(2, 2s+1) = T(2,3), T(2,9), T(2,15), T(2,21), \cdots$

$$\begin{aligned}\mathrm{ADO}_3(x) = \zeta_3 x^{2s} + \zeta_3 x^{2s-1} + (\zeta_3 - \zeta_3^{-1})x^{2s-2} - \zeta_3^{-1}x^{2s-3} - \zeta_3^{-1}x^{2s-4}\\ + \zeta_3 x^{2s-6} + \zeta_3 x^{2s-7} + (\zeta_3 - \zeta_3^{-1})x^{2s-8} - \zeta_3^{-1}x^{2s-9} - \zeta_3^{-1}x^{2s-10}\\ + \cdots + (\zeta_3 - \zeta_3^{-1}) + (x \to 1/x).\end{aligned}$$

2. For $K = T(2, 2s+1) = T(2,5), T(2,11), T(2,17), T(2,23), \cdots$

$$\begin{aligned}\mathrm{ADO}_3(x) = \zeta_3^{-1} x^{2s} + \zeta_3^{-1} x^{2s-1} + (\zeta_3^{-1} - 1)x^{2s-2} - x^{2s-3} - x^{2s-4}\\ + \zeta_3^{-1} x^{2s-6} + \zeta_3^{-1} x^{2s-7} + (\zeta_3^{-1} - 1)x^{2s-8} - x^{2s-9} - x^{2s-10}\\ + \cdots - 1 + (x \to 1/x).\end{aligned}$$

3. For $K = T(2,2s+1) = T(2,7), T(2,13), T(2,19), T(2,25), \cdots$

$$\begin{aligned}\mathrm{ADO}_3(x) = x^{2s} + x^{2s-1} + (1-\zeta_3)x^{2s-2} - \zeta_3 x^{2s-3} - \zeta_3 x^{2s-4} \\ + x^{2s-6} + x^{2s-7} + (1-\zeta_3)x^{2s-8} - \zeta_3 x^{2s-9} - \zeta_3 x^{2s-10} \\ + \cdots + 1 + (x \to 1/x).\end{aligned}$$

All the explicit x terms are polynomials and power of x decreases by two after one cycle of a coefficient combination. We next move onto ADO_4 invariants, whose explicit formula can be obtained algorithmically.

3.4 The algorithm for ADO_4 invariants of $T(2, 2s+1)$

Explicit formulas for ADO_4 invariants of $T(2,2s+1)$ for $s \in \mathbb{Z}_{\geq 7}$ are constructed inductively. This subclass of torus knots are divided into four sets and each set has its own seed $\mathrm{ADO}_4[T(2,2s+1)]$ together with a pattern of coefficients that generates the invariant for higher values of $2s+1$. We present an algorithm for obtaining explicit expressions.

The algorithm

1. Beginning with x^{3s}, write a polynomial with coefficients c_i following one of the four patterns (shown below) that $T(2,2s+1)$ belong to:

$$c_{3s}x^{3s} + c_{3s-1}x^{3s-1} + c_{3s-2}x^{3s-2} + c_{3s-3}x^{3s-3} + c_{3s-4}x^{3s-4} + c_{3s-5}x^{3s-5}, \quad c_n \in \mathbb{C}.$$

2. Add a polynomial starting with x^{3s-8} with exponent pattern and coefficients given by $\mathrm{ADO}_4[T(2,2s-7)]$.

3. Remaining polynomials are determined by a mirror reflection of coefficients across the last term in the previous step beginning from the second last term. Furthermore, adjust of the

exponents of the variable x following the pattern of the Step 2 until a constant term is reached.

4. Use the Weyl symmetry to obtain $1/x$ terms.

As a consequence of the normalization factor $(x^{1/2}-x^{-1/2})$ in Conjecture 6, we obtain the symmetric version of ADO invariants. Their coefficients c_n are divided into four types:

1. $-i, -i, -i-1, -i-1, -1, -1;$ for $\{T(2,7), T(2,15), T(2,23), \cdots\}$
2. $1, 1, 1-i, 1-i, -i, -i;$ for $\{T(2,9), T(2,17), T(2,25), \cdots\}$
3. $i, i, i+1, i+1, 1, 1;$ for $\{T(2,11), T(2,19), \cdots\}$
4. $-1, -1, -1+i, -1+i, i, i;$ for $\{T(2,13), T(2,21), \cdots\}$,

where the semicolon means that the next term has a power of x lowered by three. The coefficients of the first and the third sets differ by signs as well as the second and the fourth sets. ADO_4 polynomial of the first knot in each set is a seed for the next knot in the set. This pattern continues for all the subsequent knots in each set. The fundamental seed invariants can be easily computed using (1.4).

$$\mathrm{ADO}_4[T(2,7)] = -ix^9 - ix^8 + (-1-i)x^7 + (-1-i)x^6 - x^5 - x^4 - ix^2 - i2x + 1 - i2 + (x \to 1/x).$$

$$\mathrm{ADO}_4[T(2,9)] = x^{12} + x^{11} + (1-i)x^{10} + (1-i)x^9 - ix^8 - ix^7 + x^4 - ix^2 - i2x - 1 - i2 + (x \to 1/x).$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,11)] = {} & ix^{15} + ix^{14} + (1+i)x^{13} + (1+i)x^{12} + x^{11} + x^{10} + ix^7 + ix^6 + (1+i)x^5 + (1+i2)x^4 \\ & + (1+i)x^3 + ix^2 + (i-1)x - 1 + (x \to 1/x).\end{aligned}$$

$$\mathrm{ADO}_4[T(2,13)] = -x^{18} - x^{17} + (-1+i)x^{16} + (-1+i)x^{15} + ix^{14} + ix^{13} - x^{10} - x^9 + (-1+i)x^8$$

$$+(-1+i)x^7+ix^6+(1+i)x^5+x^4+(1+i)x^3+ix^2+(i-1)x-1+i2+(x\to 1/x).$$

For completeness, we display the ADO_4 polynomials of $T(2,3)$ [46] and $T(2,5)$.

$$\mathrm{ADO}_4[T(2,3)] = ix^3+ix^2+(1+i)x+1+i2+(x\to 1/x).$$

$$\mathrm{ADO}_4[T(2,5)] = -x^6-x^5+(-1+i)x^4+(-1+i)x^3+ix^2+(1+i)x+1+(x\to 1/x).$$

3.5 Examples

Let us demonstrate the algorithm through examples. For $T(2,15)$ in the first set, the first step of the algorithm yields

$$\text{Step 1} = -ix^{21}-ix^{20}+(-1-i)x^{19}+(-1-i)x^{18}-x^{17}-x^{16}.$$

Next step is to use the coefficients from the seed $\mathrm{ADO}_4[T(2,7)]$ but its powers of x are adjusted appropriately.

$$\begin{aligned}\text{Step 2} = & -ix^{21}-ix^{20}+(-1-i)x^{19}+(-1-i)x^{18}-x^{17}-x^{16}\\ & -ix^{13}-ix^{12}+(-1-i)x^{11}+(-1-i)x^{10}-x^9-x^8-ix^6-i2x^5+(1-i2)x^4.\end{aligned}$$

Since the above expression ends in $(1-i2)x^4$, we need to reflect the coefficients about this term until a constant term is reached. This results in

$$\begin{aligned}\text{Step 3} = & -ix^{21}-ix^{20}+(-1-i)x^{19}+(-1-i)x^{18}-x^{17}-x^{16}-ix^{13}-ix^{12}\\ & +(-1-i)x^{11}+(-1-i)x^{10}-x^9-x^8-ix^6-i2x^5+(1-i2)x^4-i2x^3-ix^2-1.\end{aligned}$$

The application of the last step leads to

$$\mathrm{ADO}_4[T(2,15)] = -ix^{21}-ix^{20}+(-1-i)x^{19}+(-1-i)x^{18}-x^{17}-x^{16}-ix^{13}-ix^{12}$$

$$+(-1-i)x^{11}+(-1-i)x^{10}-x^{9}-x^{8}-ix^{6}-i2x^{5}+(1-i2)x^{4}-i2x^{3}-ix^{2}-1$$
$$+(x\to 1/x).$$

As a consistency check, $F_{T(2,15)}(x,q=\zeta_4)$ obtained from the $\mathrm{ADO}_4[T(2,15)]$ using Conjecture 6 agrees with the direct computation of $F_{T(2,15)}(x,q=\zeta_4)$ from Theorem 8.

For $T(2,17)$ in the second set, the seed invariant is $\mathrm{ADO}_4[T(2,9)]$ and application of the first and second steps produce

$$\text{Step 1} = x^{24}+x^{23}+(1-i)x^{22}+(1-i)x^{21}-ix^{20}-ix^{19}.$$

$$\text{Step 2} = x^{24}+x^{23}+(1-i)x^{22}+(1-i)x^{21}-ix^{20}-ix^{19}$$
$$+x^{16}+x^{15}+(1-i)x^{14}+(1-i)x^{13}-ix^{12}-ix^{11}+x^{8}-ix^{6}-i2x^{5}+(-1-i2)x^{4}.$$

After the reflection about x^4-term

$$\text{Step 3} = x^{24}+x^{23}+(1-i)x^{22}+(1-i)x^{21}-ix^{20}-ix^{19}+x^{16}+x^{15}+(1-i)x^{14}$$
$$+(1-i)x^{13}-ix^{12}-ix^{11}+x^{8}-ix^{6}-i2x^{5}+(-1-i2)x^{4}-i2x^{3}-ix^{2}+1.$$

The last step results in

$$\mathrm{ADO}_4[T(2,17)] = x^{24}+x^{23}+(1-i)x^{22}+(1-i)x^{21}-ix^{20}-ix^{19}+x^{16}+x^{15}+(1-i)x^{14}$$
$$+(1-i)x^{13}-ix^{12}-ix^{11}+x^{8}-ix^{6}-i2x^{5}+(-1-i2)x^{4}-i2x^{3}-ix^{2}+1+(x\to 1/x).$$

One can verify that $F_{T(2,17)}(x,q=\zeta_4)$ obtained from $\mathrm{ADO}_4[T(2,17)]$ matches with the result (at $q=\zeta_4$) of the direct method from Theorem 8.

In the third set, the seed for $T(2,19)$ is $\mathrm{ADO}_4[T(2,11)]$. Applying the first two steps yields

$$ix^{27}+ix^{26}+(1+i)x^{25}+(1+i)x^{24}+x^{23}+x^{22}+ix^{19}+ix^{18}+(1+i)x^{17}+(1+i)x^{16}+x^{15}+x^{14}$$
$$ix^{11}+ix^{10}+(1+i)x^{9}+(1+i2)x^{8}+(1+i)x^{7}+ix^{6}+(-1+i)x^{5}-x^{4}.$$

The last two steps produce

$$\mathrm{ADO}_4[T(2,19)]=ix^{27}+ix^{26}+(1+i)x^{25}+(1+i)x^{24}+x^{23}+x^{22}+ix^{19}+ix^{18}+(1+i)x^{17}+(1+i)x^{16}$$
$$+x^{15}+x^{14}+ix^{11}+ix^{10}+(1+i)x^{9}+(1+i2)x^{8}+(1+i)x^{7}+ix^{6}+(-1+i)x^{5}-x^{4}+(-1+i)x^{3}+ix^{2}$$
$$+(1+i)x+1+i2+(x\to 1/x).$$

Similarly, $\mathrm{ADO}_4[T(2,21)]$ can be obtained using $\mathrm{ADO}_4[T(2,13)]$.

$$\mathrm{ADO}_4[T(2,21)]=-x^{30}-x^{29}+(-1+i)x^{28}+(-1+i)x^{27}+ix^{26}+ix^{25}-x^{22}-x^{21}+(-1+i)x^{20}$$
$$+(-1+i)x^{19}+ix^{18}+ix^{17}-x^{14}-x^{13}+(-1+i)x^{12}+(-1+i)x^{11}+ix^{10}+(1+i)x^{9}+x^{8}$$
$$+(1+i)x^{7}+ix^{6}+(-1+i)x^{5}+(-1+i2)x^{4}+(-1+i)x^{3}+ix^{2}+(1+i)x+1+(x\to 1/x).$$

We record ADO_4 polynomials of other torus knots obtained from the algorithm together with the above results.

$$\mathrm{ADO}_4[T(2,23)]=-ix^{33}-ix^{32}-(1+i)x^{31}-(1+i)x^{30}-x^{29}-x^{28}-ix^{25}-ix^{24}-(1+i)x^{23}-(1+i)x^{22}$$
$$-x^{21}-x^{20}-ix^{17}-ix^{16}-(1+i)x^{15}-(1+i)x^{14}-x^{13}-x^{12}-ix^{10}-2ix^{9}+(1-2i)x^{8}-2ix^{7}$$
$$-ix^{6}-x^{4}-ix^{2}-2ix+(1-2i)+(x\to 1/x).$$

$$\mathrm{ADO}_4[T(2,25)]=x^{36}+x^{35}+(1-i)x^{34}+(1-i)x^{33}-ix^{32}-ix^{31}+x^{28}+x^{27}+(1-i)x^{26}$$
$$+(1-i)x^{25}-ix^{24}-ix^{23}+x^{20}+x^{19}+(1-i)x^{18}+(1-i)x^{17}-ix^{16}-ix^{15}+x^{12}-ix^{10}$$
$$-2ix^{9}-(1+2i)x^{8}-2ix^{7}-ix^{6}+x^{4}-ix^{2}-2ix-(1+2i)+(x\to 1/x).$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,27)] &= ix^{39}+ix^{38}+(1+i)x^{37}+(1+i)x^{36}+x^{35}+x^{34}+ix^{31}+ix^{30}+(1+i)x^{29}\\ &+(1+i)x^{28}+x^{27}+x^{26}+ix^{23}+ix^{22}+(1+i)x^{21}+(1+i)x^{20}+x^{19}+x^{18}+ix^{15}+ix^{14}\\ &+(1+i)x^{13}+(1+i2)x^{12}+(1+i)x^{11}+ix^{10}+(-1+i)x^{9}-x^{8}+(-1+i)x^{7}+ix^{6}+(1+i)x^{5}\\ &+(1+i2)x^{4}+(1+i)x^{3}+ix^{2}+(-1+i)x-1+(x\to 1/x).\end{aligned}$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,29)] &= -x^{42}-x^{41}+(-1+i)x^{40}+(-1+i)x^{39}+ix^{38}+ix^{37}-x^{34}-x^{33}+(-1+i)x^{32}\\ &+(-1+i)x^{31}+ix^{30}+ix^{29}-x^{26}-x^{25}+(-1+i)x^{24}+(-1+i)x^{23}+ix^{22}+ix^{21}-x^{18}-x^{17}\\ &+(-1+i)x^{16}+(-1+i)x^{15}+ix^{14}+(1+i)x^{13}+x^{12}+(1+i)x^{11}+ix^{10}+(-1+i)x^{9}+(-1+i2)x^{8}\\ &+(-1+i)x^{7}+ix^{6}+(1+i)x^{5}+x^{4}+(1+i)x^{3}+ix^{2}+(-1+i)x-1+i2+(x\to 1/x).\end{aligned}$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,31)] &= -ix^{45}-ix^{44}-(1+i)x^{43}-(1+i)x^{42}-x^{41}-x^{40}-ix^{37}-ix^{36}-(1+i)x^{35}-(1+i)x^{34}\\ &-x^{33}-x^{32}-ix^{29}-ix^{28}-(1+i)x^{27}-(1+i)x^{26}-x^{25}-x^{24}-ix^{21}-ix^{20}-(1+i)x^{19}-(1+i)x^{18}\\ &-x^{17}-x^{16}-ix^{14}-2ix^{13}+(1-2i)x^{12}-2ix^{11}-ix^{10}-x^{8}-ix^{6}-2ix^{5}+(1-2i)x^{4}-2ix^{3}-ix^{2}-1+(x\to 1/x).\end{aligned}$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,33)] &= x^{48}+x^{47}+(1-i)x^{46}+(1-i)x^{45}-ix^{44}-ix^{43}+x^{40}+x^{39}+(1-i)x^{38}+(1-i)x^{37}\\ &-ix^{36}-ix^{35}+x^{32}+x^{31}+(1-i)x^{30}+(1-i)x^{29}-ix^{28}-ix^{27}+x^{24}+x^{23}+(1-i)x^{22}\\ &+(1-i)x^{21}-ix^{20}-ix^{19}+x^{16}-ix^{14}-i2x^{13}+(-1-i2)x^{12}-i2x^{11}-ix^{10}+x^{8}-ix^{6}\\ &-i2x^{5}+(-1-i2)x^{4}-i2x^{3}-ix^{2}+1+(x\to 1/x).\end{aligned}$$

$$\begin{aligned}\mathrm{ADO}_4[T(2,35)] &= ix^{51}+ix^{50}+(1+i)x^{49}+(1+i)x^{48}+x^{47}+x^{46}+ix^{43}+ix^{42}+(1+i)x^{41}+(1+i)x^{40}+x^{39}\\ &+x^{38}+ix^{35}+ix^{34}+(1+i)x^{33}+(1+i)x^{32}+x^{31}+x^{30}+ix^{27}+ix^{26}+(1+i)x^{25}+(1+i)x^{24}+x^{23}+x^{22}\\ &+ix^{19}+ix^{18}+(1+i)x^{17}+(1+2i)x^{16}+(1+i)x^{15}+ix^{14}-(1-i)x^{13}-x^{12}-(1-i)x^{11}+ix^{10}+(1+i)x^{9}\\ &+(1+2i)x^{8}+(1+i)x^{7}+ix^{6}-(1-i)x^{5}-x^{4}-(1-i)x^{3}+ix^{2}+(1+i)x+(1+2i)+(x\to 1/x).\end{aligned}$$

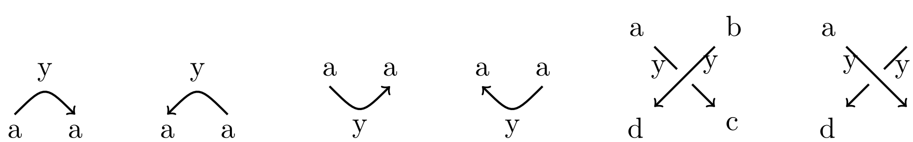
$$
\begin{aligned}
\mathrm{ADO}_4[T(2,37)] = {}& -x^{54}-x^{53}-(1-i)x^{52}-(1-i)x^{51}+ix^{50}+ix^{49}-x^{46}-x^{45}-(1-i)x^{44}-(1-i)x^{43}+ix^{42} \\
& +ix^{41}-x^{38}-x^{37}-(1-i)x^{36}-(1-i)x^{35}+ix^{34}+ix^{33}-x^{30}-x^{29}-(1-i)x^{28}-(1-i)x^{27}+ix^{26} \\
& +ix^{25}-x^{22}-x^{21}-(1-i)x^{20}-(1-i)x^{19}+ix^{18}+(1+i)x^{17}+x^{16}+(1+i)x^{15}+ix^{14}-(1-i)x^{13} \\
& -(1-2i)x^{12}-(1-i)x^{11}+ix^{10}+(1+i)x^{9}+x^{8}+(1+i)x^{7}+ix^{6}-(1-i)x^{5}-(1-2i)x^{4}-(1-i)x^{3}+ix^{2} \\
& +(1+i)x+1+(x\to 1/x).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{ADO}_4[T(2,39)] = {}& -ix^{57}-ix^{56}-(1+i)x^{55}-(1+i)x^{54}-x^{53}-x^{52}-ix^{49}-ix^{48}-(1+i)x^{47}-(1+i)x^{46} \\
& -x^{45}-x^{44}-ix^{41}-ix^{40}-(1+i)x^{39}-(1+i)x^{38}-x^{37}-x^{36}-ix^{33}-ix^{32}-(1+i)x^{31}-(1+i)x^{30} \\
& -x^{29}-x^{28}-ix^{25}-ix^{24}-(1+i)x^{23}-(1+i)x^{22}-x^{21}-x^{20}-ix^{18}-2ix^{17}+(1-2i)x^{16}-2ix^{15} \\
& -ix^{14}-x^{12}-ix^{10}-2ix^{9}+(1-2i)x^{8}-2ix^{7}-ix^{6}-x^{4}-ix^{2}-2ix+(1-2i)+(x\to 1/x).
\end{aligned}
$$

We next verify the subset of the above results through the original R-matrix of the quantum group summarized in Section 3.1.

3.6 Comparison with the R-matrix approach

We perform an independent computation of the ADO polynomial using its R-matrix formulation [2, 7] to strengthen the Conjecture 6. We summarize the ingredients for the computation [7]. A (1,1)-tangle diagram of $T(2,2s+1)$ consists of three kinds of building blocks: oriented caps, cups, and crossings, respectively.



$$
\epsilon_a[y] = 1 \qquad \epsilon^*_a[y] = q^{2a(r-1)}y^{1-r} \qquad \eta_a[y] = 1 \quad \eta^*_a[y] = q^{2a(1-r)}y^{r-1}
$$

$$
R^{a,b}_{c,d}[y] = \delta_{a-c,d-b}\,\theta_{a\geq c}\,\theta_{d\geq b}\,(-y)^{a-c}\,q^{(c-a)(a+b+1)+2cd}\,z\,y^{-d-c}\,\frac{(q^{2(a-1)}/y^2;q^{-2})_{a-c}(q^{2(b+1)};q^2)_{a-c}}{(q^{-2};q^{-2})_{a-c}}
$$

$$\left(R^{-1}\right)^{a,b}_{c,d}[y] = \delta_{a-c,d-b}\,\theta_{a\geq c}\,\theta_{d\geq b}\,(-y)^{a-c}q^{(c-a)(a+b+1)-2ab}\,z\,y^{b+a}\,\frac{(q^{2(a-1)}/y^2;q^{-2})_{a-c}(q^{2(b+1)};q^2)_{a-c}}{(q^2;q^2)_{a-c}}$$

$$\delta_{a,b} = \begin{cases} 1, & a=b \\ 0, & \text{otherwise} \end{cases} \qquad \theta_{a\geq b} = \begin{cases} 1, & a\geq b \\ 0, & \text{otherwise,} \end{cases}$$

where a,b,c,d are subset of variables $a_1,\cdots,a_m$ in the tangle diagram and $(w,q)_t$ is the q-Pochhammer symbol (see Section 3.7). The above formulas are in the same ordering as the diagrams. From these ingredients, a function that gives rise to the ADO polynomial can be defined as

$$G^{\times}_D(q,y,z,r;a_1,\cdots,a_m) := d[y]\,\delta_{a_1,0}\,\delta_{a_m,0}\prod_{crossings}R\prod_{crossings}R^{-1}\prod_{caps}\epsilon\prod_{caps}\epsilon^*\prod_{cups}\eta\prod_{cups}\eta^*,$$

$$d[y] = \prod_{j=2}^{r}\frac{1}{q^jy-q^{-j}y^{-1}} = (-y)^{r-1}q^{\frac{1}{2}r(r+1)-1}\frac{1}{(q^4y^2;q^2)_{r-1}}.$$

At $q=\zeta_{2r}$, $y=\zeta_{2r}^{\alpha}$, $z=\zeta_{2r}^{\alpha^2}$, an (unnormalized) ADO polynomial $N^r_K(\alpha)$ is

$$N^r_K(\alpha) = G^{\times}_D(\zeta_{2r},\zeta_{2r}^{\alpha},\zeta_{2r}^{\alpha^2},r;a_1,\cdots,a_m).$$

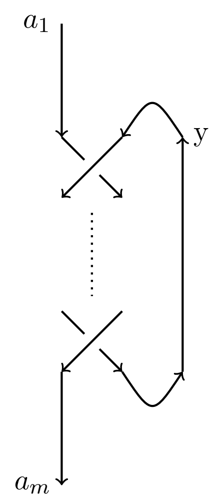
The quantity computed in [7] is a normalized version

$$\hat{N}^r_K(\alpha) := i^{1-r}(y^r-y^{-r})N^r_K(\alpha-1).$$

The change of normalization between $\hat{N}^r_K(\alpha)$ and our ADO_p for zero framed knots is

$$\mathrm{ADO}_p(x) \cong \left.\frac{\hat{N}^{r=p}_K(\alpha;y)}{y-y^{-1}}\right|_{y\to x^{1/2},\,x\to cx} \cong \mathrm{num}[N^r_K(\alpha-1;y)]\Big|_{y\to x^{1/2},\,x\to cx} \quad c\in\mathbb{C}^*.$$

where $\cong$ denotes equivalence up to an overall monomial and an overall constant. The RHS is due to the structure of $G^{\times}_D(r)$ such that $N^r_K(\alpha-1)$ always contains $(y-y^{-1})/(y^r-y^{-r})$ for any knot (for details see Section 2.4 in [7]). We denote the numerator of $N^r_K(\alpha-1;y)$ as $\mathrm{num}[N^r_K(\alpha-1;y)]$. A $(1,1)$-tangle diagram of $T(2,2s+1)$, which consists of $(2s+1)$-crossings is



The vertical dots represent the same type of crossings. Applying the formula to the diagram, we have schematically

$$G_D^{\times}(q,y,z,r;a_1,\cdots,a_m)=d[y]\,\delta_{a_1,0}\,\delta_{a_m,0}\left(\prod_{i=1}^{2s+1}R_i\right)\eta\,\epsilon^*,$$

where $m=m(s)$. The ADO polynomials for $T(2,3)$ are listed in Appendix B of [7]. Using the above relation ($c=1$), we find an agreement that

$$\hat{N}^3_{T(2,3)}(\alpha;y)=q^2(y^5-y^{-5})+q(y-y^{-1})\Rightarrow\left.\frac{\hat{N}^3_K(\alpha;y)}{y-y^{-1}}\right|_{y\to x^{1/2}}\cong\mathrm{ADO}_3[T(2,3)](x).$$

$$\hat{N}^4_{T(2,3)}(\alpha;y)=q^2(y^7-y^{-7})+(y^3-y^{-3})+q^2(y-y^{-1})\Rightarrow\left.\frac{\hat{N}^4_K(\alpha;y)}{y-y^{-1}}\right|_{y\to x^{1/2}}\cong\mathrm{ADO}_4[T(2,3)](x).$$

We next check $T(2,5)$ case. The computation of $G_D^{\times}$ yields

$$\begin{aligned}\mathrm{num}[N^3_{T(2,5)}(\alpha-1)]=&-\sqrt[3]{-1}y^8-\sqrt[3]{-1}y^6-\frac{1}{2}\sqrt[3]{-1}\left(3-i\sqrt{3}\right)y^4-\frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)y^2\\&\left.-\frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)+\left(y\to\frac{1}{y}\right)\right|_{y\to x^{1/2}}\cong\mathrm{ADO}_3[T(2,5)](x).\end{aligned}$$

We now list several more verifications of the ADO_3 formula in Section 3.5.

$$\mathrm{num}[N^3_{T(2,7)}(\alpha-1)] = -\sqrt[3]{-1}y^{12} - \sqrt[3]{-1}y^{10} - \frac{1}{2}\sqrt[3]{-1}\left(3-i\sqrt{3}\right)y^8 - \frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)y^6$$
$$-\frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)y^4 - \sqrt[3]{-1} + \left(y \to \frac{1}{y}\right)\Bigg|_{y\to x^{1/2}} \cong \mathrm{ADO}_3[T(2,7)](x).$$

$$\mathrm{num}[N^3_{T(2,9)}(\alpha-1)] = -\frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}+i\right)y^{16} - \frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}+i\right)y^{14} - \sqrt[6]{-1}\sqrt{3}y^{12}$$
$$-\frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}-i\right)y^{10} - \frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}-i\right)y^8 - \frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}+i\right)y^4 - \frac{1}{2}\sqrt[6]{-1}\left(\sqrt{3}+i\right)y^2$$
$$-\sqrt[6]{-1}\sqrt{3} + \left(y \to \frac{1}{y}\right)\Bigg|_{y\to x^{1/2}} \cong \mathrm{ADO}_3[T(2,9)](x).$$

$$\mathrm{num}[N^3_{T(2,11)}(\alpha-1)] = -(-1)^{2/3}y^{20} - (-1)^{2/3}y^{18} - \frac{1}{2}(-1)^{2/3}\left(3-i\sqrt{3}\right)y^{16} - \frac{1}{2}(-1)^{2/3}\left(1-i\sqrt{3}\right)y^{14}$$
$$-\frac{1}{2}(-1)^{2/3}\left(1-i\sqrt{3}\right)y^{12} - (-1)^{2/3}y^8 - (-1)^{2/3}y^6 - \frac{1}{2}(-1)^{2/3}\left(3-i\sqrt{3}\right)y^4 - \frac{1}{2}(-1)^{2/3}\left(1-i\sqrt{3}\right)y^2$$
$$-\frac{1}{2}(-1)^{2/3}\left(1-i\sqrt{3}\right) + \left(y \to \frac{1}{y}\right)\Bigg|_{y\to x^{1/2}} \cong \mathrm{ADO}_3[T(2,11)](x).$$

$$\mathrm{num}[N^3_{T(2,13)}(\alpha-1)] = \frac{1}{2}\left(-1-i\sqrt{3}\right)y^{24} + \frac{1}{2}\left(-1-i\sqrt{3}\right)y^{22} + \frac{1}{2}\left(-3-i\sqrt{3}\right)y^{20} - y^{18} - y^{16}$$
$$+\frac{1}{2}\left(-1-i\sqrt{3}\right)y^{12} + \frac{1}{2}\left(-1-i\sqrt{3}\right)y^{10} + \frac{1}{2}\left(-3-i\sqrt{3}\right)y^8 - y^6 - y^4 + \frac{1}{2}\left(-1-i\sqrt{3}\right) + \left(y \to \frac{1}{y}\right)\Bigg|_{y\to x^{1/2}}$$
$$\cong \mathrm{ADO}_3[T(2,13)](x).$$

$$\mathrm{num}[N^3_{T(2,15)}(\alpha-1)] = \frac{1}{2}\left(1-i\sqrt{3}\right)y^{28} + \frac{1}{2}\left(1-i\sqrt{3}\right)y^{26} - i\sqrt{3}y^{24} + \frac{1}{2}\left(-1-i\sqrt{3}\right)y^{22}$$
$$+\frac{1}{2}\left(-1-i\sqrt{3}\right)y^{20} + \frac{1}{2}\left(1-i\sqrt{3}\right)y^{16} + \frac{1}{2}\left(1-i\sqrt{3}\right)y^{14} - i\sqrt{3}y^{12} + \frac{1}{2}\left(-1-i\sqrt{3}\right)y^{10}$$
$$+\frac{1}{2}\left(-1-i\sqrt{3}\right)y^8 + \frac{1}{2}\left(1-i\sqrt{3}\right)y^4 + \frac{1}{2}\left(1-i\sqrt{3}\right)y^2 - i\sqrt{3} + \left(y \to \frac{1}{y}\right)\Bigg|_{y\to x^{1/2}} \cong \mathrm{ADO}_3[T(2,15)](x).$$

$$\mathrm{num}[N^3_{T(2,17)}(\alpha-1)] = \frac{1}{2}i\left(\sqrt{3}+i\right)y^{32} + \frac{1}{2}i\left(\sqrt{3}+i\right)y^{30} + i\sqrt{3}y^{28} + \frac{1}{2}\left(1+i\sqrt{3}\right)y^{26}$$
$$+\frac{1}{2}\left(1+i\sqrt{3}\right)y^{24} + \frac{1}{2}i\left(\sqrt{3}+i\right)y^{20} + \frac{1}{2}i\left(\sqrt{3}+i\right)y^{18} + i\sqrt{3}y^{16} + \frac{1}{2}\left(1+i\sqrt{3}\right)y^{14}$$

$$+\frac{1}{2}\left(1+i\sqrt{3}\right)y^{12}+\frac{1}{2}i\left(\sqrt{3}+i\right)y^{8}+\frac{1}{2}i\left(\sqrt{3}+i\right)y^{6}+i\sqrt{3}y^{4}+\frac{i\sqrt{3}}{y^{4}}+\frac{1}{2}\left(1+i\sqrt{3}\right)y^{2}$$
$$+\frac{1}{2}\left(1+i\sqrt{3}\right)+\left(y\to\frac{1}{y}\right)\bigg|_{y\to x^{1/2}}\cong \mathrm{ADO}_3[T(2,17)](x).$$

We next verify ADO_4 polynomials.

$$\mathrm{num}[N^4_{T(2,7)}(\alpha-1)]=-\sqrt[3]{-1}y^{12}-\sqrt[3]{-1}y^{10}-\frac{1}{2}\sqrt[3]{-1}\left(3-i\sqrt{3}\right)y^{8}-\frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)y^{6}$$
$$-\frac{1}{2}\sqrt[3]{-1}\left(1-i\sqrt{3}\right)y^{4}-\sqrt[3]{-1}+\left(y\to\frac{1}{y}\right)\bigg|_{y\to x^{1/2}}\cong \mathrm{ADO}_4[T(2,7)](x).$$

$$\mathrm{num}[N^4_{T(2,9)}(\alpha-1)]=-\sqrt[4]{-1}y^{24}-\sqrt[4]{-1}y^{22}-(1-i)\sqrt[4]{-1}y^{20}-(1-i)\sqrt[4]{-1}y^{18}+(-1)^{3/4}y^{16}+(-1)^{3/4}y^{14}$$
$$-\sqrt[4]{-1}y^{8}+(-1)^{3/4}y^{4}+2(-1)^{3/4}y^{2}+(1+2i)\sqrt[4]{-1}+\left(y\to\frac{1}{y}\right)\bigg|_{y\to x^{1/2}}\cong \mathrm{ADO}_4[T(2,9)](x).$$

$$\mathrm{num}[N^4_{T(2,11)}(\alpha-1)]=iy^{30}+iy^{28}+(1+i)y^{26}+(1+i)y^{24}+y^{22}+y^{20}+iy^{14}+iy^{12}+(1+i)y^{10}+(1+2i)y^{8}$$
$$+(1+i)y^{6}+iy^{4}-(1-i)y^{2}-1+\left(y\to\frac{1}{y}\right)\bigg|_{y\to x^{1/2}}\cong \mathrm{ADO}_4[T(2,11)](x).$$

$$\mathrm{num}[N^4_{T(2,13)}(\alpha-1)]=(-1+i)y^{36}-(1-i)y^{34}+2iy^{32}+2iy^{30}+(1+i)y^{28}+(1+i)y^{26}-(1-i)y^{20}-(1-i)y^{18}$$
$$+2iy^{16}+2iy^{14}+(1+i)y^{12}+2y^{10}+(1-i)y^{8}+2y^{6}+(1+i)y^{4}+2iy^{2}+(1+3i)$$
$$+\left(y\to\frac{1}{y}\right)\bigg|_{y\to x^{1/2}}\cong \mathrm{ADO}_4[T(2,13)](x).$$

We move onto the deformation of the ADO polynomial.

3.7 Deformed ADO_3 invariants of $T(2, 2s+1)$

A link between superpolynomial defined in [25] and F_K was discovered in [29]. Specifically, two parameter refinement $F_K(x, q, a, t)$ was introduced, which motivated to define t-deformed ADO polynomial. This deformation introduces one more variable to the original ADO polynomial ADO(x,t); as a consequence, it is a colored version of the t-deformed Alexander polynomial $\Delta(x, t)$ that can distinguish chirality of torus knots. In this section we present t-deformed version of ADO_3 polynomials for $T(2, 2s+1)$ knots.

Reduced superpolynomial for negative torus knots carrying symmetric representation S^r of $SU(N)$ is stated in [32]:

$$\mathcal{P}_{S^r}[T(2,-(2s+1));q,a,t] = \left(\frac{a}{q}\right)^{pr} \sum_{k_1=0}^{r} \sum_{k_2=0}^{k_1} \cdots \sum_{k_s=0}^{k_{s-1}} q^{(2r+1)(k_1+\cdots+k_s)-\sum_{i=1}^{s} k_{i-1}k_i}\, t^{2(k_1+\cdots+k_s)} \times \frac{(q^r;q^{-1})_{k_1}(-at/q;q)_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k_{s-1} \\ k_s \end{bmatrix}_q,$$

$$(w;q)_m := \prod_{i=1}^{m}(1-wq^{i-1}) \qquad \begin{bmatrix} w \\ n \end{bmatrix}_q := \frac{(q;q)_w}{(q;q)_n (q;q)_{w-n}},$$

where $s \in \mathbb{Z}_+$, r is the dimension of S^r and $k_0 \equiv r$. Note that the convention for negative torus knot in [29] is $T(2, 2s+1)$ for $s \in \mathbb{Z}_+$, which is opposite of the convention used in this article. In [29], it was shown that $\mathcal{P}_{S^r}$ can be converted into a two parameter deformation of F_K by replacing q^r by x and dropping the overall factor $(a/q)^{pr}$:

$$F_{T(2,-(2s+1))}(x,q,a,t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_s=0}^{k_{s-1}} x^{2(k_1+\cdots+k_s)-k_1}\, q^{(k_1+\cdots+k_s)-\sum_{i=2}^{s} k_{i-1}k_i}\, t^{2(k_1+\cdots+k_s)} \times \frac{(x;q^{-1})_{k_1}(-at/q;q)_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k_{s-1} \\ k_s \end{bmatrix}_q.$$

Fixing $a = q^N$ and $t = -1$, $F_K(x, q, a, t)$ becomes the original $F_K(x, q)$ for torus knots.[2] Different specialization of a, namely, $a = -t^{-1}$ yields a refined Alexander polynomial [29],

$$F_K(x, q, -t^{-1}, t) = \Delta_K(x, t).$$

Using Conjecture 7, a refined ADO$_3$ polynomial for $T(2, 2s+1)$, $s \in \mathbb{Z}_+$ is

$$\mathrm{ADO}_3[T(2,2s+1); x, t] = (tx)^{2s} + \frac{\zeta_3^{-1}}{t}(tx)^{2s-1} + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)(tx)^{2s-2} - \frac{\zeta_3}{t}(tx)^{2s-3} - \frac{1}{t^2}(tx)^{2s-4}$$

$$+(tx)^{2s-6} + \frac{\zeta_3^{-1}}{t}(tx)^{2s-7} + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)(tx)^{2s-8} - \frac{\zeta_3}{t}(tx)^{2s-9} - \frac{1}{t^2}(tx)^{2s-10} + \cdots + O\left(\frac{1}{tx}\right),$$

where $O(1/tx)$-terms are determined by the t-deformed Weyl symmetry of the ADO$_p$ invariant,

$$\mathrm{ADO}_p^{SU(2)}(1/x, t) = \mathrm{ADO}_p^{SU(2)}(\zeta_p^{-2} t^{-2} x, t).$$

The suppressed polynomial terms follow the same power and coefficient patterns of the previous terms. The three formulas for the original ADO$_3[T(2, 2s+1); x]$ coalesce into one formula through the t-deformation. We next present a few examples.

- $K = T(2, 5)$

We start from $F_K(x, q, a, t)$ for $T(2, -5)$,

$$F_{T(2,-5)}(x, q, a, t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} x^{2(k_1+k_2)-k_1}\, q^{k_1+k_2-k_1k_2}\, t^{2(k_1+k_2)} \frac{(x; q^{-1})_{k_1} \left(-\frac{at}{q}; q\right)_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q .$$

We next apply the mirror map to reverse the orientation of K,

$$x \mapsto 1/x \quad q \mapsto 1/q \quad a \mapsto 1/a \quad t \mapsto 1/t.$$

Setting $a = -1/t$, we get a refined Alexander polynomial of K (upon multiplication by an overall

[2]Specifically, additional manipulations are needed to arrive at $F_K(x, q)$ for torus knots ([29] Subsection 5.2).

monomial),

$$\Delta_K(x,t) = t^2x^2 + \frac{1}{t^2x^2} - \frac{1}{t^2x} - x + 1.$$

Further fixing $t = -1$, it reduces to the Alexander polynomial of K. Moreover, this refined polynomial possess the t-deformed Weyl symmetry for the refined Alexander polynomial,

$$\Delta_K(1/x,t) = \Delta_K(x/t^2,t).$$

A refined ADO_3 polynomial of K is computed via Conjecture 7 as

$$\mathrm{ADO}_3[T(2,5);x,t] = (tx)^4 + \frac{\zeta_3^{-1}}{t}(tx)^3 + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)(tx)^2 - \frac{\zeta_3}{t}(tx) - \frac{1}{t^2} - \frac{\zeta_3^{-1}}{t}\frac{1}{(tx)}$$
$$+ \left(\frac{1}{t^2} - \zeta_3\right)\frac{1}{(tx)^2} + \frac{\zeta_3^{-1}}{t}\frac{1}{(tx)^3} + \zeta_3\frac{1}{(tx)^4}.$$

This formula carries the t-deformed Weyl symmetry of the ADO_3 invariant. Moreover, fixing $t = -1$ and rescaling $x \mapsto \zeta_3^2 x$, the refined polynomial becomes the original ADO_3 polynomial,

$$\zeta_3^{-1}x^4 + \zeta_3^{-1}x^3 + (\zeta_3^{-1} - 1)x^2 - x - 1 + (x \to 1/x).$$

- $K = T(2,7)$

Two parameter deformation of F_K for $T(2,-7)$ is

$$F_{T(2,-7)}(x,q,a,t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2} x^{2(k_1+k_2+k_3)-k_1} q^{k_1+k_2+k_3-k_1k_2-k_2k_3}\, t^{2(k_1+k_2+k_3)}$$
$$\times \frac{(x;q^{-1})_{k_1}\left(-\frac{at}{q};q\right)_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q \begin{bmatrix} k_2 \\ k_3 \end{bmatrix}_q .$$

A refined Alexander polynomial of K having the refined Weyl symmetry is

$$\Delta_{T(2,7)}(x,t) = -t^3x^3 - \frac{1}{t^3x^3} + \frac{1}{t^3x^2} + tx^2 - tx - \frac{1}{tx} + \frac{1}{t}.$$

A refined ADO_3 polynomial of K is

$$\mathrm{ADO}_3[T(2,7);x,t] = (tx)^6 + \frac{\zeta_3^{-1}}{t}(tx)^5 + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)(tx)^4 - \frac{\zeta_3}{t}(tx)^3 - \frac{1}{t^2}(tx)^2 + 1$$

$$-\frac{\zeta^{-1}}{t^2}\frac{1}{(tx)^2} - \frac{\zeta}{t}\frac{1}{(tx)^3} + \left(\frac{\zeta^{-1}}{t^2} - 1\right)\frac{1}{(tx)^4} + \frac{\zeta}{t}\frac{1}{(tx)^5} + \frac{1}{(tx)^6}.$$

This polynomial possess the t-deformed Weyl symmetry of the ADO_3 invariant and after specializing $t = -1$ and rescaling $x \mapsto \zeta_3^2 x$, it becomes

$$x^6 + x^5 + (1-\zeta_3)x^4 - \zeta_3x^3 - \zeta_3x^2 + 1 + (x \to 1/x),$$

which is the original ADO_3 polynomial for K.

- $K = T(2,9)$

A refined Alexander polynomial of K carrying the refined Weyl symmetry is

$$\Delta_{T(2,9)}(x,t) = t^4x^4 + \frac{1}{t^4x^4} - \frac{1}{t^4x^3} - t^2x^3 + t^2x^2 + \frac{1}{t^2x^2} - \frac{1}{t^2x} - x + 1.$$

A refined ADO_3 polynomial of K is

$$\mathrm{ADO}_3[T(2,9);x,t] = (tx)^8 + \frac{\zeta_3^{-1}}{t}(tx)^7 + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)(tx)^6 - \frac{\zeta_3}{t}(tx)^5 - \frac{1}{t^2}(tx)^4$$

$$+(tx)^2 + \frac{\zeta_3^{-1}}{t}(tx) + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right) + \frac{1}{t}\frac{1}{tx} + \zeta_3^2\frac{1}{(tx)^2} - \frac{\zeta_3}{t^2}\frac{1}{(tx)^4} - \frac{1}{t}\frac{1}{(tx)^5} + \left(\frac{\zeta_3}{t^2} - \zeta_3^{-1}\right)\frac{1}{(tx)^6}$$

$$+\frac{1}{t}\frac{1}{(tx)^7} + \zeta_3^2\frac{1}{(tx)^8}.$$

This polynomial is invariant under the refined Weyl symmetry of the ADO_3 invariant and becomes the original ADO_3 polynomial after setting $t=-1$ and rescaling $x \mapsto \zeta_3^2 x$,

$$\zeta_3 x^8 + \zeta_3 x^7 + (\zeta_3 - \zeta_3^{-1})x^6 - \zeta_3^{-1}x^5 - \zeta_3^{-1}x^4 + \zeta_3 x^2 + \zeta_3 x + \left(\zeta_3 - \zeta_3^{-1}\right) + (x \to 1/x).$$

Chapter 4

An extension to Lie superalgebra

A topological invariant q-series for graph 3-manifolds associated with a type I Lie superalgebra $sl(m|n)$ was introduced recently in [33]. This q-series denoted as $\hat{Z}_{ab}$ is labeled by a pair of $Spin^c$ structures of the manifolds as opposed to one label for a q-series ($\hat{Z}_b$) corresponding to a classical Lie algebra [85]. Another core difference is that the q-series invariant $\hat{Z}_{ab}$ decomposes an extension of the WRT invariant, namely, the CGP invariant based on a Lie superalgebra [14]. In the most general topological setting, the CGP invariant is a topological invariant of a triple consisting of a closed oriented 3-manifold, a link and a certain cohomology class [1]. The construction of the CGP invariant associated to a Lie (super)algebra involves a new facet: the *modified* quantum dimension, which was first introduced in [40] for the quantum groups of Lie superalgebra of type I and then for the quantum groups of $sl(2)$ at roots of unity [41]. In general, for a complex Lie superalgebra, the standard quantum dimension vanishes, which makes the WRT invariant of links and 3-manifolds trivial. The modified quantum dimension overcomes the obstacle. In the case of a complex simple Lie algebra, the modified quantum dimension enables to extend the WRT invariant to the above mentioned triplet. Furthermore, for a fixed Lie (super)algebra, the modified dimension can be defined for semisimple and nonsemisimple ribbon categories. Specific examples of the former type were given in [41] and [36]. The latter type, which is relevant to this paper, was first dealt with in [37]. And then it was expanded into a Lie superalgebra in [3] and [55] in which finite dimensional

[1] It is denoted $\omega \in H^1(Y;\mathbb{Z}/2\mathbb{Z})$, which induces coloring on a surgery link; this link is not part of the triplet [14].

irreducible representations of the (unrolled) quantum groups of $sl(m|n)\,(m \neq n)$ at roots of unity was analyzed (the latter paper focuses on $sl(2|1)$). The CGP invariant contains a variety of information such as the multivariable Alexander polynomial and the ADO polynomial of a link. The relation between the CGP invariant and the WRT invariant for $sl(2;\mathbb{C})$ was conjectured in [15] and was proved for certain classes of 3-manifolds.

4.1 Background

The twist θ, the S-matrix and the modified quantum dimension d for the quantum group $\mathcal{U}_h(g)$ of $g =$ type I $= sl(m|n), osp(2|2n)$ are given in [40], where h is a formal variable related to $q = e^{h/2}$:

$$\theta_V(\lambda) = q^{\langle \lambda, \lambda+2\rho \rangle} 1_V,$$

where λ is the highest weight of a highest weight $\mathcal{U}_h(g)$-module V coloring the a link component and $\rho = \rho_0 - \rho_1$ and $\langle -, - \rangle$ is the bilinear form (see Appendix D for the details of representation theoretic concepts).

$$S(V, V') = \varphi_{\mu+\rho}(sch(V(\lambda))),$$

where λ and μ are weights of (irreducible) $\mathcal{U}_h(g)$-modules V and V', respectively, coloring each link component. Moreover, $sch(V(\lambda))$ is a supercharacter of V and φ_β is a map from a group ring $\mathbb{Z}[\Lambda]$ to $\mathbb{C}[[h]]$[2]. The modified quantum dimension d is

$$d(\mu)S(\lambda, \mu) = d(\lambda)S(\mu, \lambda).$$

In the case of $sl(m|n)$, for the unrolled quantum group at roots of unity $\mathcal{U}_l^H(sl(m|n))\,(m \neq n)$, the above formulas modify [3]:

$$\theta_V(\lambda; l) = \xi^{\langle \lambda, \lambda+\pi \rangle} 1_V \qquad \xi = e^{\frac{i2\pi}{l}} \quad l \geq m+n-1$$

[2] $e^\alpha \mapsto q^{2\langle \alpha, \beta \rangle}$ for any weight β

$$S(V,V';l) = \varphi_{\lambda+\frac{\pi}{2}}(sch(V(\lambda))), \qquad \pi = 2\rho - 2l\rho_{\bar{0}} \in \Lambda_R$$
$$= \xi^{2\langle\lambda+\frac{\pi}{2},\mu+\frac{\pi}{2}\rangle} \prod_{\alpha\in\Delta_{\bar{0}}^+} \frac{\{l\langle\lambda+\frac{\pi}{2},\alpha\rangle\}_\xi}{\{\langle\lambda+\frac{\pi}{2},\alpha\rangle\}_\xi} \prod_{\alpha\in\Delta_{\bar{1}}^+} \left\{\left\langle\lambda+\frac{\pi}{2},\alpha\right\rangle\right\}_\xi$$
$$d(\mu;l)S(\lambda,\mu;l) = d(\lambda;l)S(\mu,\lambda;l).$$

The second equality in the S-matrix element assumes V is a simple $\mathcal{U}_l^H(sl(m|n))$-module that is typical having dimension D. The changes of the formula are due to the different pivotal structure of $\mathcal{U}_l^H(sl(m|n))$. Since the modifications are on the representation theory level and don't seem to involve unique features of $sl(m|n)$, we *assume* that the same modifications occur for $osp(2|2n)$. We next summarize the concepts involved in the homological blocks $\hat{Z}_{ab}$ associated with a Lie superalgebra [33].

Generic graph In type I Lie superalgebra case, plumbing graph used in practice needs to satisfy certain conditions due to the functional form of the edge factor in the integrand of $\hat{Z}$; a plumbing graph is called generic, if

1. at least one vertex of a graph has degree > 2
2. $V|_{deg>2} \neq U \sqcup W$ such that $B^{-1}_{IJ} = 0$ for some I $\in V$, J $\in W$,

where V is a set of vertices of a graph.

Good Chamber For $\hat{Z}$ associated with type I Lie superalgebra, there is a notion of a good chamber introduced in [33]. Existence of a good chamber guarantees that $\hat{Z}$ is a power series with integer coefficients. Specifically, an adjacency matrix B of a generic plumbing graph needs to admit a good chamber. It turns out that $osp(2|2)$ requires positive definite (generic) plumbing graphs, its criteria for good chamber existence shown below are opposite of $sl(2|1)$ criteria in [33]. Good chamber exists, if there exists a vector α whose components are

$$\alpha_I = \pm 1, \quad I \in V|_{deg\neq 2}$$

such that

$$B^{-1}_{IJ}\alpha_I\alpha_J \geq 0 \quad \forall I \in V|_{deg=1}, \quad J \in V|_{deg\neq 2}$$

$$B^{-1}_{IJ}\alpha_I\alpha_J > 0 \quad \forall I, J \in V|_{deg=1}, \quad I \neq J$$

$$X_{IJ} \quad \text{is copositive} \qquad X_{IJ} = B^{-1}_{IJ}\alpha_I\alpha_J \quad I, J \in V|_{deg>2}$$

are satisfied. Furthermore, a good chamber corresponding to such α is

$$deg(I) = 1 : \begin{cases} |y_I|^{\alpha_I} < 1 \\ |z_I|^{\alpha_I} > 1 \end{cases}$$

$$deg(I) > 2 : \left|\frac{y_I}{z_I}\right|^{\alpha_I} < 1.$$

<u>Chamber Expansion</u> The series expansions for the good chambers for $osp(2|2)$ turn out to be the same as that of $sl(2|1)$ in [33]. We record here the expansions.

- degree(I) $= 2 + K > 2$

$$\left(\tfrac{(1-z_I)(1-y_I)}{y_I - z_I}\right)^K = \begin{cases} (z_I - 1)^K (1 - y_I^{-1})^K \sum_{r_I=0}^{\infty} \frac{(r_I+1)(r_I+2)\cdots(r_I+K-1)}{(K-1)!} (z_I/y_I)^{r_I}, & |z_I| < |y_I| \\ (1 - y_I)^K (1 - z_I^{-1})^K \sum_{r_I=0}^{\infty} \frac{(r_I+1)(r_I+2)\cdots(r_I+K-1)}{(K-1)!} (y_I/z_I)^{r_I}, & |z_I| > |y_I| \end{cases}$$

- degree(I) $= 1$

$$\tfrac{y_I - z_I}{(1-z_I)(1-y_I)} = \begin{cases} \sum_{r_I=1}^{\infty} y_I^{r_I} + \sum_{r_I=0}^{\infty} z_I^{-r_I}, & |y_I| < 1, |z_I| > 1 \\ -\sum_{r_I=0}^{\infty} y_I^{-r_I} - \sum_{r_I=1}^{\infty} z_I^{r_I}, & |y_I| > 1, |z_I| < 1 \end{cases}$$

4.2 Result

For closed oriented plumbed 3-manifolds $Y(\Gamma)$ having $b_1(Y) = 0$ and $a, b \in Spin^c(Y) \cong H_1(Y)$, $\hat{Z}_{ab}[Y;q]$ associated with $osp(2|2)$ is [12]

$$\hat{Z}^{osp(2|2)}_{ab}[Y;q] = (-1)^{\pi} \int_{\Omega} \prod_{I\in V} \frac{dz_I}{i2\pi z_I}\frac{dy_I}{i2\pi y_I}\left(\frac{y_I - z_I}{(1-z_I)(1-y_I)}\right)^{2-deg(I)} \sum_{\substack{n\in BZ^L+a \\ m\in BZ^L+b}} q^{-2nB^{-1}m} \prod_{J\in V} z_J^{m_J} y_J^{n_J}$$

$$\in \mathbb{Q} + q^{\Delta_{ab}}\mathbb{Z}[[q]], \qquad \Delta_{ab} \in \mathbb{Q} \qquad H_1(Y) = \mathbb{Z}^L/B\mathbb{Z}^L \qquad |q| < 1$$

where $B = B(\Gamma)$ is an adjacency matrix of a generic plumbing graph Γ. Moreover, the convergent q-series in a complex unit disc requires Γ to be positive definite plumbing graphs inside a complex unit disc. This is opposite of the $sl(m|n)$ case (and for $su(n)$). The edge factor in the integrand turns out to be the same as that of $sl(2|1)$ [3]. The integration prescription states that one chooses a chamber for an expansion of the integrand using the chamber expansion in the previous section and then picks constant terms in y_I and z_I.

4.3 Examples

We apply the above formula to $\mathbb{Z}HS^3$ and $\mathbb{Q}HS^3$. Each $\hat{Z}$ is either even or odd power series and the regularized constants are given by the zeta function $\zeta(s)$ or the Hurwitz zeta function $\zeta(s,x)$ (see [33] for details).

- $Y = S^3$ The adjacency matrix of a generic plumbing graph of S^3 (Figure 4.1 in Section 4.4) is

$$B(\Gamma) = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad DetB = 1 \qquad |H_1(Y)| = |DetB|$$

[3] This seems to be no longer true in higher rank cases

The two chambers are

$$\alpha = \pm(1, -1, -1, -1).$$

$$\hat{Z}_{00}^{osp(2|2)}[Y;q] = 1+2\zeta(-1)+2\zeta(0)-2q^2-4q^4-4q^6-6q^8-4q^{10}-8q^{12}-4q^{14}-8q^{16}-6q^{18}+\cdots$$

- $Y = \Sigma(2,3,7)$ The adjacency matrix of a (generic) plumbing graph (Figure 4.2 in Section 4.4) is

$$B(\Gamma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 7 \end{pmatrix} \qquad DetB = 1$$

The two chambers are

$$\alpha = \pm(1, -1, -1, -1).$$

$$\begin{aligned} \hat{Z}_{00}^{osp(2|2)} &= 1 + 2\zeta(-1) + 2\zeta(0) - 2q^4 - 2q^6 - 4q^8 - 2q^{10} - 6q^{12} - 4q^{14} - 6q^{16} - 6q^{18} - 8q^{20} \\ &\quad - 4q^{22} - 10q^{24} - 6q^{26} - 8q^{28} + \cdots \end{aligned}$$

- $Y = M(-1|\frac{1}{2}, \frac{1}{3}, \frac{1}{8})$ The adjacency matrix of a (generic) plumbing graph (Figure 4.3 in Section 4.4) is

$$B(\Gamma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 8 \end{pmatrix} \qquad DetB = 2$$

The two chambers are

$$\alpha = \pm(1, -1, -1, -1).$$

The independent $\hat{Z}_{ab}$ are

$$\hat{Z}_{00} = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^4 - 4q^8 - 2q^{10} - 4q^{12} - 2q^{14} - 6q^{16} - 2q^{18} - 6q^{20} - 2q^{22} - 8q^{24}$$

$$- 2q^{26} - 4q^{28} + \cdots$$

$$\hat{Z}_{11} = -2q^3 - 2q^5 - 2q^7 - 4q^9 - 4q^{11} - 2q^{13} - 6q^{15} - 4q^{17} - 4q^{19} - 6q^{21} - 4q^{23} - 6q^{27} + \cdots$$

$$\hat{Z}_{01} = 2\zeta(-1, 1/2) + \zeta(0, 1/2) - q^2 - q^4 - 3q^6 - 2q^8 - 3q^{10} - 4q^{12} - 4q^{14} - 3q^{16} - 5q^{18} - 5q^{20}$$
$$- 4q^{22} - 5q^{24} - 4q^{26} - 5q^{28} + \cdots$$

where the labels denote the last components of elements of $H_1(Y)$.

- $Y = M(-1|\frac{1}{2}, \frac{1}{3}, \frac{1}{9})$ The adjacency matrix of a (generic) plumbing graph (Figure 4.4 in Section 4.4) is

$$B(\Gamma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 9 \end{pmatrix} \qquad Det B = 3$$

The independent $\hat{Z}^{osp(2|2)}_{ab}[Y]$ are

$$\hat{Z}_{00} = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^6 - 2q^8 - 4q^{12} - 2q^{14} - 2q^{16} - 4q^{18} - 2q^{20} - 2q^{22} - 6q^{24} - 2q^{26}$$
$$- 2q^{28} + \cdots$$

$$\hat{Z}_{11} = q^{\frac{1}{3}} \left(-q - q^3 - 3q^5 - 2q^7 - 3q^9 - 2q^{11} - 5q^{13} - 2q^{15} - 4q^{17} - 2q^{19} - 5q^{21} - 4q^{23} - 2q^{25}\right.$$
$$\left.-q^{27} - 3q^{29} + \cdots\right)$$

$$\hat{Z}_{22} = q^{\frac{1}{3}} \left(-q - q^3 - 3q^5 - 2q^7 - 3q^9 - 2q^{11} - 5q^{13} - 2q^{15} - 4q^{17} - 2q^{19} - 5q^{21} - 4q^{23} - 2q^{25}\right.$$
$$\left.-q^{27} - 3q^{29} + \cdots\right)$$

$$\hat{Z}_{01} = 3\zeta(-1, 1/2) + \zeta(0, 1/2) - 2q^4 - q^6 - 2q^8 - 3q^{10} - 3q^{12} - 2q^{14} - 4q^{16} - 2q^{18} - 5q^{20} - 3q^{22}$$
$$- 2q^{24} - 2q^{26} - 3q^{28} + \cdots$$

$$\hat{Z}_{02} = 3\zeta(-1, 1/2) + \zeta(0, 1/2) - q^2 - q^4 - q^6 - 3q^8 - 2q^{10} - 2q^{12} - 3q^{14} - 4q^{16} - 2q^{18} - 4q^{20}$$
$$- 2q^{22} - 3q^{24} - 2q^{26} - 4q^{28} + \cdots$$

$$\hat{Z}_{12} = q^{-\frac{1}{3}} \left(-2q^3 - 2q^5 - 2q^7 - 2q^9 - 4q^{11} - 2q^{13} - 4q^{15} - 2q^{17} - 6q^{19} - 2q^{21} - 4q^{23} - 2q^{25}\right.$$
$$\left.-4q^{27} + \cdots\right)$$

We next check the topological invariance of the above results.

4.4 Invariance checks

The invariance of $\hat{Z}_{ab}$ under the Kirby-Neumman moves (Figure 1.2) for the exhibited graphs and the graphs mentioned in the captions of Figure 3 and 4 were verified.

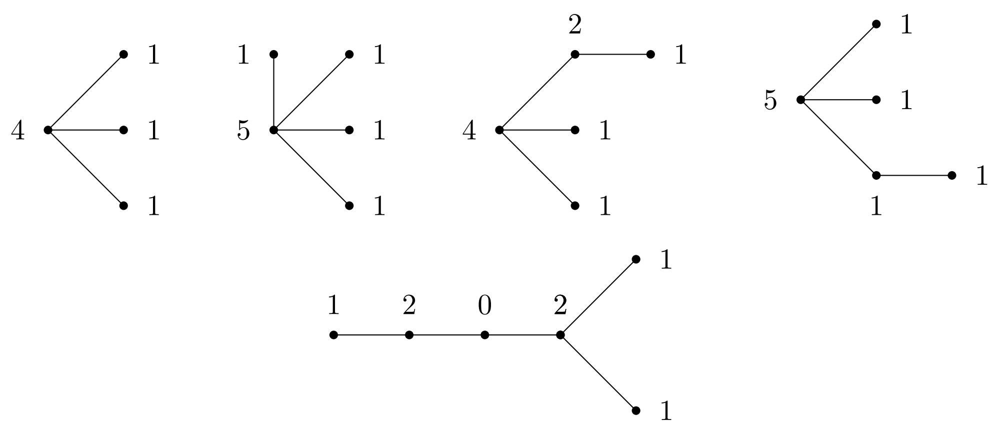


Figure 4.1: A generic plumbing graph of S^3 (left most) and its equivalent graphs

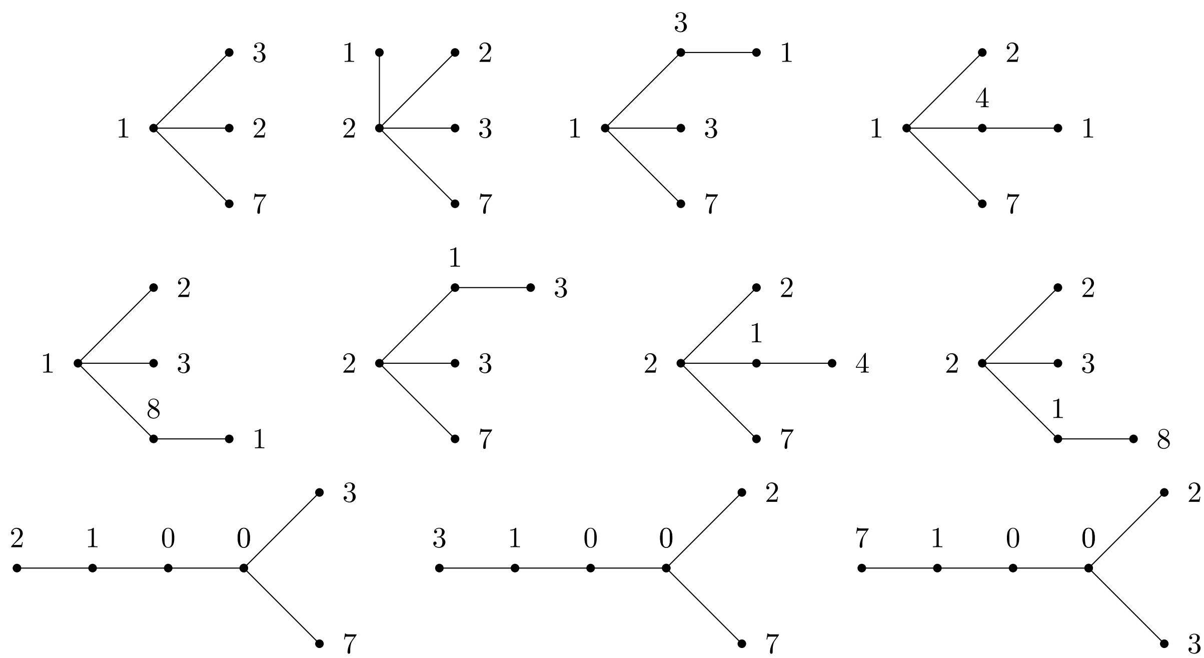


Figure 4.2: A generic plumbing graph of $\Sigma(2, 3, 7)$ (the first graph) and its equivalent graphs

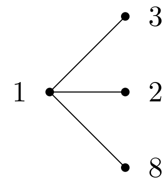


Figure 4.3: A generic plumbing graph of $M(1|\frac{1}{2},\frac{1}{3},\frac{1}{8})$ and its equivalent graphs can be obtained from Figure 4.2 by replacing the weight 7 by 8 or 8 by 9.

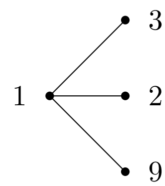


Figure 4.4: A generic plumbing graph of $M(1|\frac{1}{2},\frac{1}{3},\frac{1}{9})$ and its equivalent graphs can be obtained from Figure 4.2 by replacing the weight 7 by 9 or 8 by 10.

Appendix

A The details of the recursion I

We write the coefficient functions $\alpha, \beta, \gamma, \delta$ of the recursion relation for the F_K in (2.5).

$$\alpha(x,q) = \frac{N(\alpha)}{D(\alpha)},$$

$$\begin{aligned}
D(\alpha) = \ & q^{66}x^{33} - q^{64}x^{31} + q^{63}x^9\left(x^{22}-x^{20}\right) + q^{62}x^9\left(-2x^{22}-x^{20}\right) - q^{61}x^{29} \\
& + q^{60}x^9\left(2x^{20}-x^{18}\right) + q^{59}x^9\left(-2x^{20}+2x^{18}+x^{16}\right) + q^{58}x^9\left(x^{20}+x^{18}\right) \\
& + q^{57}x^9\left(-x^{20}+2x^{18}+x^{16}\right) + q^{56}x^9\left(-2x^{18}+2x^{16}+x^{14}\right) + q^{55}x^9\left(3x^{18}-x^{16}-x^{14}\right) \\
& + q^{54}x^9\left(-2x^{18}+2x^{16}+x^{14}-x^{12}\right) + q^{53}x^9\left(x^{18}-x^{16}-x^{14}+x^{12}\right) \\
& + q^{52}x^9\left(3x^{16}-2x^{14}-2x^{12}\right) + q^{51}x^9\left(-2x^{16}+2x^{14}-x^{10}\right) \\
& + q^{50}x^9\left(2x^{16}-2x^{14}-3x^{12}+2x^{10}\right) + q^{49}x^9\left(x^{14}-2x^{10}\right) + q^{48}x^9\left(x^{10}-2x^{14}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ q^{47}x^9\left(2x^{14}-3x^{12}-2x^{10}+2x^8\right)+q^{46}x^9\left(-x^{14}+2x^{10}-2x^8\right)\\
&+ q^{45}x^9\left(-2x^{12}-2x^{10}+3x^8\right)+q^{44}x^9\left(x^{12}-x^{10}-x^8+x^6\right)\\
&+ q^{43}x^9\left(-x^{12}+x^{10}+2x^8-2x^6\right)+q^{42}x^9\left(-x^{10}-x^8+3x^6\right)\\
&+ q^{41}x^9\left(x^{10}+2x^8-2x^6\right)+q^{40}x^9\left(x^8+2x^6-x^4\right)+q^{39}x^9\left(x^6+x^4\right)\\
&+ q^{38}x^9\left(x^8+2x^6-2x^4\right)+q^{37}x^9\left(2x^4-x^6\right)-q^{36}x^{13}+q^{35}x^9\left(-x^4-2x^2\right)\\
&+ q^{34}x^9\left(x^2-x^4\right)-q^{33}x^{11}+q^{31}x^9
\end{aligned}
$$

$$
\begin{aligned}
N(\alpha) =\ & q^{61}\left(-x^{24}\right)+2q^{57}x^{22}-q^{56}x^{22}+q^{55}x^{22}+q^{54}x^{20}-q^{53}x^{20}+2q^{52}x^{20}-2q^{51}x^{20}\\
&- q^{50}\left(x^{18}-x^{20}\right)-q^{49}\left(-x^{20}-2x^{18}\right)-q^{48}\left(3x^{18}-x^{20}\right)+2q^{47}x^{18}-2q^{46}x^{18}\\
&- q^{45}\left(x^{18}+2x^{16}\right)-q^{44}\left(2x^{18}-2x^{16}\right)-q^{43}\left(3x^{16}-x^{18}\right)+q^{42}x^{16}\\
&- q^{41}\left(2x^{16}-x^{14}\right)-q^{40}\left(2x^{14}-x^{16}\right)-q^{39}\left(2x^{16}-2x^{14}\right)-q^{38}\left(x^{16}+x^{14}\right)\\
&+ 2q^{37}x^{14}-q^{36}\left(x^{16}+2x^{14}-x^{12}\right)-q^{35}\left(x^{12}-2x^{14}\right)-q^{34}\left(-x^{14}-2x^{12}\right)\\
&- q^{33}x^{14}-q^{32}\left(-x^{14}-3x^{12}\right)-q^{31}x^{14}-q^{30}x^{10}-q^{29}\left(-3x^{12}-x^{10}\right)-q^{28}x^{10}\\
&- q^{27}\left(-2x^{12}-x^{10}\right)-q^{26}\left(x^{12}-2x^{10}\right)-q^{25}\left(-x^{12}+2x^{10}+x^8\right)+2q^{24}x^{10}\\
&- q^{23}\left(x^{10}+x^8\right)-q^{22}\left(2x^8-2x^{10}\right)-q^{21}\left(2x^{10}-x^8\right)-q^{20}\left(2x^8-x^{10}\right)\\
&+ q^{19}x^8-q^{18}\left(3x^8-x^6\right)-q^{17}\left(2x^6-2x^8\right)-q^{16}\left(2x^8+x^6\right)-2q^{15}x^6+2q^{14}x^6\\
&- q^{13}\left(3x^6-x^4\right)-q^{12}\left(-2x^6-x^4\right)-q^{11}\left(x^6-x^4\right)-2q^{10}x^4+2q^9x^4-q^8x^4+q^7x^4\\
&+ q^6x^2-q^5x^2+2q^4x^2-1.
\end{aligned}
$$

$$\beta(x,q)=\frac{N(\beta)}{D(\beta)},$$

$$
\begin{aligned}
D(\beta) =\ & q^{159/2}x^{41}-q^{155/2}x^{39}-q^{149/2}x^{37}-q^{99/2}x^{21}-q^{93/2}x^{19}+q^{89/2}x^{17}+q^{153/2}x^{17}\left(x^{22}-x^{20}\right)\\
&+ q^{151/2}x^{17}\left(-2x^{22}-x^{20}\right)+q^{147/2}x^{17}\left(2x^{20}-x^{18}\right)+q^{143/2}x^{17}\left(x^{20}+x^{18}\right)\\
&+ q^{125/2}x^{17}\left(x^{14}-2x^{10}\right)+q^{123/2}x^{17}\left(x^{10}-2x^{14}\right)+q^{105/2}x^{17}\left(x^6+x^4\right)\\
&+ q^{101/2}x^{17}\left(2x^4-x^6\right)+q^{97/2}x^{17}\left(-x^4-2x^2\right)+q^{95/2}x^{17}\left(x^2-x^4\right)
\end{aligned}
$$

$$+ q^{145/2}x^{17}\left(-2x^{20}+2x^{18}+x^{16}\right)+q^{141/2}x^{17}\left(-x^{20}+2x^{18}+x^{16}\right)$$
$$+ q^{139/2}x^{17}\left(-2x^{18}+2x^{16}+x^{14}\right)+q^{137/2}x^{17}\left(3x^{18}-x^{16}-x^{14}\right)$$
$$+ q^{131/2}x^{17}\left(3x^{16}-2x^{14}-2x^{12}\right)+q^{129/2}x^{17}\left(-2x^{16}+2x^{14}-x^{10}\right)$$
$$+ q^{119/2}x^{17}\left(-x^{14}+2x^{10}-2x^{8}\right)+q^{117/2}x^{17}\left(-2x^{12}-2x^{10}+3x^{8}\right)$$
$$+ q^{111/2}x^{17}\left(-x^{10}-x^{8}+3x^{6}\right)+q^{109/2}x^{17}\left(x^{10}+2x^{8}-2x^{6}\right)+q^{107/2}x^{17}\left(x^{8}+2x^{6}-x^{4}\right)$$
$$+ q^{103/2}x^{17}\left(x^{8}+2x^{6}-2x^{4}\right)+q^{135/2}x^{17}\left(-2x^{18}+2x^{16}+x^{14}-x^{12}\right)$$
$$+ q^{133/2}x^{17}\left(x^{18}-x^{16}-x^{14}+x^{12}\right)+q^{127/2}x^{17}\left(2x^{16}-2x^{14}-3x^{12}+2x^{10}\right)$$
$$+ q^{121/2}x^{17}\left(2x^{14}-3x^{12}-2x^{10}+2x^{8}\right)+q^{115/2}x^{17}\left(x^{12}-x^{10}-x^{8}+x^{6}\right)$$
$$+ q^{113/2}x^{17}\left(-x^{12}+x^{10}+2x^{8}-2x^{6}\right)$$

$$N(\beta) = -\left(x^{41}-x^{40}\right)q^{79}-x^{38}q^{77}-\left(x^{38}+x^{36}\right)q^{76}-\left(-2x^{39}+2x^{38}+x^{36}\right)q^{75}-\left(x^{39}-x^{38}\right)q^{74}$$
$$-\left(-x^{39}+x^{38}-x^{36}-x^{34}\right)q^{73}-\left(-x^{37}-3x^{34}-x^{32}\right)q^{72}-\left(x^{37}-x^{36}-2x^{34}-x^{32}\right)q^{71}$$
$$-\left(-2x^{37}+x^{36}-x^{34}-2x^{32}\right)q^{70}-\left(2x^{37}-2x^{36}-2x^{34}\right)q^{69}$$
$$-\left(-x^{37}+x^{36}+x^{35}-x^{34}+2x^{30}\right)q^{68}-\left(-x^{37}+x^{36}-2x^{35}+x^{34}+x^{30}+x^{28}\right)q^{67}$$
$$-\left(-x^{37}+x^{36}+3x^{35}-2x^{34}+4x^{30}+x^{28}\right)q^{66}-\left(-2x^{35}+x^{32}+x^{30}+2x^{28}\right)q^{65}$$
$$-\left(2x^{35}-4x^{34}-2x^{32}+3x^{30}+x^{28}\right)q^{64}$$
$$-\left(x^{35}-2x^{34}+2x^{33}-3x^{32}+x^{30}+3x^{28}-x^{26}\right)q^{63}$$
$$-\left(2x^{35}-2x^{34}-2x^{33}+3x^{30}-x^{28}\right)q^{62}$$
$$-\left(-x^{35}+x^{34}+3x^{33}-x^{32}+2x^{30}+4x^{28}-2x^{26}-x^{24}\right)q^{61}$$
$$-\left(-x^{33}+3x^{32}+7x^{30}+x^{28}-x^{24}\right)q^{60}$$
$$-\left(2x^{33}-x^{32}-x^{31}+7x^{30}+6x^{28}-3x^{26}-x^{24}\right)q^{59}$$
$$-\left(-x^{33}+2x^{31}+3x^{30}+3x^{28}+x^{26}-2x^{24}\right)q^{58}$$
$$-\left(2x^{33}-2x^{32}-2x^{31}+3x^{30}+4x^{28}-5x^{26}-x^{24}+x^{22}\right)q^{57}$$
$$-\left(x^{33}-x^{32}+x^{31}+x^{30}-x^{28}-2x^{26}-4x^{24}-x^{22}\right)q^{56}$$

$$- \left(-2x^{31} + 2x^{30} + x^{28} - 9x^{26} - 2x^{24}\right) q^{55}$$
$$- \left(x^{33} - x^{32} + 2x^{31} + x^{30} - x^{29} + 2x^{28} - 4x^{26} - 8x^{24} + x^{22} + x^{20}\right) q^{54}$$
$$- \left(-2x^{31} + 3x^{30} + x^{29} + 3x^{28} - 8x^{26} - 3x^{24} - x^{22}\right) q^{53}$$
$$- \left(-x^{31} + 2x^{30} - 2x^{29} + 4x^{28} - x^{26} - 8x^{24} + x^{22} + 2x^{20}\right) q^{52}$$
$$- \left(x^{31} + 3x^{28} - 3x^{26} - 2x^{24}\right) q^{51}$$
$$- \left(-x^{31} + x^{30} - 3x^{29} + 3x^{28} - 4x^{26} - 7x^{24} + 5x^{22} + 2x^{20} - x^{18}\right) q^{50}$$
$$- \left(x^{31} - x^{30} - 7x^{26} - 4x^{24} - x^{22} + 4x^{20} + 2x^{18}\right) q^{49}$$
$$- \left(x^{28} + x^{27} - 5x^{26} - 11x^{24} + 4x^{22} + 3x^{20} - x^{18}\right) q^{48}$$
$$- \left(-3x^{29} + 3x^{28} - x^{27} - 4x^{26} - 9x^{24} - 2x^{22} + 5x^{20} + x^{18}\right) q^{47}$$
$$- \left(x^{28} + x^{27} - 3x^{26} - 7x^{24} + 3x^{22} + 3x^{20} - x^{18}\right) q^{46}$$
$$- \left(-2x^{29} + 2x^{28} - x^{27} - 4x^{24} + x^{22} + 8x^{20} - 2x^{16}\right) q^{45}$$
$$- \left(x^{29} - x^{28} - 2x^{27} - x^{26} - 5x^{24} + 7x^{22} + 6x^{20} + x^{18} + x^{16}\right) q^{44}$$
$$- \left(-x^{29} + x^{28} + 2x^{27} - 3x^{26} + x^{25} - 6x^{24} + 6x^{22} + 11x^{20} + x^{18} - 2x^{16}\right) q^{43}$$
$$- \left(-2x^{27} + x^{26} - 5x^{24} + 2x^{22} + 9x^{20} + 2x^{18}\right) q^{42}$$
$$- \left(x^{27} - x^{26} + x^{25} - 5x^{24} + 11x^{20} - 3x^{16} + x^{14}\right) q^{41}$$
$$- \left(-2x^{27} + x^{26} + 2x^{25} - 3x^{24} + 6x^{20} + 2x^{18} - 3x^{16} - 2x^{14}\right) q^{40}$$
$$- \left(2x^{27} - 2x^{26} - x^{25} - 3x^{24} + 2x^{22} + 6x^{20} - 3x^{16} + x^{14}\right) q^{39}$$
$$- \left(-x^{27} + x^{26} + 2x^{25} - 3x^{24} + 11x^{20} - 5x^{16} - x^{14}\right) q^{38}$$
$$- \left(-x^{25} + 2x^{22} + 9x^{20} + 2x^{18} - 5x^{16} + x^{14}\right) q^{37}$$
$$- \left(3x^{25} - 2x^{24} - x^{23} + x^{22} + 11x^{20} + 6x^{18} - 6x^{16} - 3x^{14} + x^{12}\right) q^{36}$$
$$- \left(-2x^{25} + x^{24} + 2x^{23} + x^{22} + 6x^{20} + 7x^{18} - 5x^{16} - x^{14} - x^{12}\right) q^{35}$$
$$- \left(2x^{25} - 2x^{24} + x^{23} + 8x^{20} + x^{18} - 4x^{16} + 2x^{12}\right) q^{34}$$
$$- \left(2x^{23} - x^{22} + 3x^{20} + 3x^{18} - 7x^{16} - 3x^{14} + x^{12}\right) q^{33}$$
$$- \left(-2x^{23} + x^{22} + 5x^{20} - 2x^{18} - 9x^{16} - 4x^{14} + 3x^{12}\right) q^{32}$$
$$- \left(3x^{23} - x^{22} - x^{21} + 3x^{20} + 4x^{18} - 11x^{16} - 5x^{14} + x^{12}\right) q^{31}$$

$$
\begin{aligned}
&-\left(-2x^{23}+2x^{22}-x^{21}+4x^{20}-x^{18}-4x^{16}-7x^{14}-x^{10}\right)q^{30}\\
&-\left(x^{23}-x^{22}-x^{21}+2x^{20}+5x^{18}-7x^{16}-4x^{14}+3x^{12}+x^{10}\right)q^{29}\\
&-\left(2x^{21}-2x^{16}-3x^{14}+3x^{12}\right)q^{28}-\left(-2x^{21}+2x^{20}+x^{18}-8x^{16}-x^{14}+4x^{12}+2x^{10}\right)q^{27}\\
&-\left(x^{21}-x^{18}-3x^{16}-8x^{14}+3x^{12}+3x^{10}\right)q^{26}\\
&-\left(-x^{21}+x^{20}+x^{18}-8x^{16}-4x^{14}+2x^{12}+x^{10}-x^{8}\right)q^{25}\\
&-\left(-x^{19}-2x^{16}-9x^{14}+x^{12}+2x^{10}\right)q^{24}-\left(x^{19}-x^{18}-4x^{16}-2x^{14}-x^{12}+x^{10}-x^{8}\right)q^{23}\\
&-\left(-2x^{19}+x^{18}-x^{16}-5x^{14}+4x^{12}+3x^{10}-2x^{8}\right)q^{22}-\left(-2x^{16}+x^{14}+3x^{12}+3x^{10}\right)q^{21}\\
&-\left(-x^{16}-3x^{14}+6x^{12}+7x^{10}-x^{8}\right)q^{20}-\left(-x^{16}+x^{12}+7x^{10}+3x^{8}\right)q^{19}\\
&-\left(x^{17}-x^{16}-2x^{14}+4x^{12}+2x^{10}-x^{8}+x^{6}\right)q^{18}-\left(-x^{12}+3x^{10}-2x^{6}\right)q^{17}\\
&-\left(-x^{14}+3x^{12}+x^{10}-3x^{8}-2x^{6}\right)q^{16}-\left(x^{12}+3x^{10}-2x^{8}-4x^{6}\right)q^{15}\\
&-\left(2x^{12}+x^{10}+x^{8}\right)q^{14}-\left(x^{12}+4x^{10}-2x^{6}+x^{4}\right)q^{13}-\left(x^{12}+x^{10}+x^{6}+x^{4}\right)q^{12}\\
&-\left(2x^{10}-x^{6}+x^{4}\right)q^{11}-\left(-2x^{6}-2x^{4}\right)q^{10}-\left(-2x^{8}-x^{6}+x^{4}\right)q^{9}\\
&-\left(-x^{8}-2x^{6}-x^{4}\right)q^{8}-\left(-x^{8}-3x^{6}\right)q^{7}-\left(-x^{6}-x^{4}+x^{2}\right)q^{6}+x^{2}q^{5}-\left(x^{4}+2x^{2}\right)q^{4}\\
&-\left(x^{4}+x^{2}\right)q^{3}-x^{2}q^{2}+1.
\end{aligned}
$$

$$\gamma(x,q)=\frac{N(\gamma)}{D(\gamma)},$$

$$
\begin{aligned}
D(\gamma)= {}& q^{85}x^{41}-q^{83}x^{39}+q^{82}x^{17}\left(x^{22}-x^{20}\right)+q^{81}x^{17}\left(-2x^{22}-x^{20}\right)-q^{80}x^{37}\\
&+q^{79}x^{17}\left(2x^{20}-x^{18}\right)+q^{78}x^{17}\left(-2x^{20}+2x^{18}+x^{16}\right)+q^{77}x^{17}\left(x^{20}+x^{18}\right)\\
&+q^{76}x^{17}\left(-x^{20}+2x^{18}+x^{16}\right)+q^{75}x^{17}\left(-2x^{18}+2x^{16}+x^{14}\right)\\
&+q^{74}x^{17}\left(3x^{18}-x^{16}-x^{14}\right)+q^{73}x^{17}\left(-2x^{18}+2x^{16}+x^{14}-x^{12}\right)\\
&+q^{72}x^{17}\left(x^{18}-x^{16}-x^{14}+x^{12}\right)+q^{71}x^{17}\left(3x^{16}-2x^{14}-2x^{12}\right)\\
&+q^{70}x^{17}\left(-2x^{16}+2x^{14}-x^{10}\right)+q^{69}x^{17}\left(2x^{16}-2x^{14}-3x^{12}+2x^{10}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{68}x^{17}\left(x^{14}-2x^{10}\right)+q^{67}x^{17}\left(x^{10}-2x^{14}\right)\\
&+ q^{66}x^{17}\left(2x^{14}-3x^{12}-2x^{10}+2x^{8}\right)+q^{65}x^{17}\left(-x^{14}+2x^{10}-2x^{8}\right)\\
&+ q^{64}x^{17}\left(-2x^{12}-2x^{10}+3x^{8}\right)+q^{63}x^{17}\left(x^{12}-x^{10}-x^{8}+x^{6}\right)\\
&+ q^{62}x^{17}\left(-x^{12}+x^{10}+2x^{8}-2x^{6}\right)+q^{61}x^{17}\left(-x^{10}-x^{8}+3x^{6}\right)\\
&+ q^{60}x^{17}\left(x^{10}+2x^{8}-2x^{6}\right)+q^{59}x^{17}\left(x^{8}+2x^{6}-x^{4}\right)+q^{58}x^{17}\left(x^{6}+x^{4}\right)\\
&+ q^{57}x^{17}\left(x^{8}+2x^{6}-2x^{4}\right)+q^{56}x^{17}\left(2x^{4}-x^{6}\right)-q^{55}x^{21}\\
&+ q^{54}x^{17}\left(-x^{4}-2x^{2}\right)+q^{53}x^{17}\left(x^{2}-x^{4}\right)-q^{52}x^{19}+q^{50}x^{17}
\end{aligned}
$$

.

$$
\begin{aligned}
N(\gamma) = {}& x^{49}q^{98}-x^{47}q^{96}+\left(-x^{47}-x^{45}\right)q^{95}+\left(-2x^{47}-x^{45}\right)q^{94}+x^{47}q^{93}\\
&+\left(-x^{47}+x^{45}+x^{43}\right)q^{92}+\left(3x^{43}+x^{41}\right)q^{91}+\left(x^{45}+2x^{43}+x^{41}\right)q^{90}\\
&+\left(-x^{45}+x^{43}+2x^{41}\right)q^{89}+\left(2x^{45}+2x^{43}\right)q^{88}+\left(-x^{45}+x^{43}-2x^{39}\right)q^{87}\\
&+\left(-x^{45}-x^{43}-x^{39}-x^{37}\right)q^{86}+\left(-x^{45}+2x^{43}-4x^{39}-x^{37}\right)q^{85}\\
&+\left(-x^{41}-x^{39}-2x^{37}\right)q^{84}+\left(4x^{43}+2x^{41}-3x^{39}-x^{37}\right)q^{83}\\
&+\left(2x^{43}+3x^{41}-x^{39}-3x^{37}+x^{35}\right)q^{82}+\left(2x^{43}-x^{40}-3x^{39}+x^{37}\right)q^{81}\\
&+\left(-x^{43}+x^{41}-2x^{39}-4x^{37}+2x^{35}+x^{33}\right)q^{80}+\left(-3x^{41}-7x^{39}+x^{38}-x^{37}+x^{33}\right)q^{79}\\
&+\left(x^{41}-7x^{39}-x^{38}-6x^{37}+x^{36}+3x^{35}+x^{33}\right)q^{78}\\
&+\left(-3x^{39}+2x^{38}-3x^{37}+x^{36}-x^{35}+2x^{33}\right)q^{77}\\
&+\left(2x^{41}-3x^{39}+x^{38}-4x^{37}+x^{36}+5x^{35}+x^{33}-x^{31}\right)q^{76}\\
&+\left(x^{41}-x^{39}+x^{38}+x^{37}-2x^{36}+2x^{35}+x^{34}+4x^{33}+x^{31}\right)q^{75}\\
&+\left(-2x^{39}-x^{37}+x^{36}+9x^{35}-2x^{34}+2x^{33}-x^{32}\right)q^{74}\\
&+\left(x^{41}-x^{39}-2x^{37}-x^{36}+4x^{35}-2x^{34}+8x^{33}-x^{31}-x^{29}\right)q^{73}\\
&+\left(-3x^{39}-3x^{37}+8x^{35}-4x^{34}+3x^{33}-x^{32}+x^{31}\right)q^{72}\\
&+\left(-2x^{39}-4x^{37}-x^{36}+x^{35}+8x^{33}-2x^{32}-x^{31}-x^{30}-2x^{29}\right)q^{71}
\end{aligned}
$$

$$+ \left(-3x^{37} + 3x^{35} - 2x^{34} + 2x^{33} + x^{30}\right) q^{70}$$
$$+ \left(-x^{39} - 3x^{37} + x^{36} + 4x^{35} + x^{34} + 7x^{33} - x^{32} - 5x^{31} - 2x^{29} + x^{28} + x^{27}\right) q^{69}$$
$$+ \left(x^{39} + x^{36} + 7x^{35} - x^{34} + 4x^{33} + 3x^{32} + x^{31} + 2x^{30} - 4x^{29} - x^{28} - 2x^{27}\right) q^{68}$$
$$+ \left(-x^{37} + 5x^{35} - 2x^{34} + 11x^{33} - x^{32} - 4x^{31} + 3x^{30} - 3x^{29} + 2x^{28} + x^{27}\right) q^{67}$$
$$+ \left(-3x^{37} + 4x^{35} - x^{34} + 9x^{33} + 2x^{31} + x^{30} - 5x^{29} + x^{28} - x^{27} + x^{26}\right) q^{66}$$
$$+ \left(-x^{37} + 3x^{35} - 2x^{34} + 7x^{33} - 3x^{32} - 3x^{31} + 2x^{30} - 3x^{29} + 3x^{28} + x^{27} - 2x^{26}\right) q^{65}$$
$$+ \left(-2x^{37} - 3x^{34} + 4x^{33} - 2x^{32} - x^{31} + x^{30} - 8x^{29} + x^{28} + x^{26} + 2x^{25}\right) q^{64}$$
$$+ \left(x^{37} + x^{35} - x^{34} + 5x^{33} + x^{32} - 7x^{31} + 3x^{30} - 6x^{29} - x^{27} - x^{26} - x^{25}\right) q^{63}$$
$$+ \left(-x^{37} + 3x^{35} + 6x^{33} - 6x^{31} + 3x^{30} - 11x^{29} + 3x^{28} - x^{27} + x^{26} + 2x^{25} - 2x^{24}\right) q^{62}$$
$$+ \left(-x^{35} + 5x^{33} - 2x^{31} + 7x^{30} - 9x^{29} + 3x^{28} - 2x^{27} - 3x^{26} + x^{24}\right) q^{61}$$
$$+ \left(x^{35} + 5x^{33} + 7x^{30} - 11x^{29} + 4x^{28} - x^{26} + 3x^{25} - 2x^{24} - x^{23}\right) q^{60}$$
$$+ \left(-x^{35} + 3x^{33} + 2x^{30} - 6x^{29} + 3x^{28} - 2x^{27} + 3x^{25} + 2x^{23} - x^{22}\right) q^{59}$$
$$+ \left(2x^{35} + 3x^{33} - 2x^{32} - 2x^{31} + 3x^{30} - 6x^{29} + 2x^{28} - 3x^{26} + 3x^{25} - 3x^{24} - x^{23} + 2x^{22}\right) q^{58}$$
$$+ \left(-x^{35} + 3x^{33} - x^{32} + x^{30} - 11x^{29} + x^{28} - 4x^{26} + 5x^{25} - 3x^{24} + x^{23} - x^{22}\right) q^{57}$$
$$+ \left(-x^{32} - 2x^{31} + 3x^{30} - 9x^{29} - x^{28} - 2x^{27} - 5x^{26} + 5x^{25} - 3x^{24} - x^{23} + x^{22}\right) q^{56}$$
$$+ \left(2x^{33} - x^{31} + x^{30} - 11x^{29} + 4x^{28} - 6x^{27} - 7x^{26} + 6x^{25} - 5x^{24} + 3x^{23} - x^{22} - x^{21} + x^{20}\right) q^{55}$$
$$+ \left(-x^{33} - x^{31} + 4x^{30} - 6x^{29} + 3x^{28} - 7x^{27} - 4x^{26} + 5x^{25} - 5x^{24} + x^{23} + x^{21}\right) q^{54}$$
$$+ \left(2x^{33} + x^{30} - 8x^{29} + 6x^{28} - x^{27} - 3x^{26} + 4x^{25} - 6x^{24} + x^{22} - 2x^{21} + 2x^{20}\right) q^{53}$$
$$+ \left(x^{31} + 2x^{30} - 3x^{29} + x^{28} - 3x^{27} - x^{26} + 7x^{25} - 5x^{24} + 3x^{23} + x^{22} - x^{21}\right) q^{52}$$
$$+ \left(-x^{31} - 5x^{29} + 4x^{28} + 2x^{27} - 8x^{26} + 9x^{25} - 4x^{24} + 4x^{23} + 2x^{22} - 3x^{21} + 2x^{20}\right) q^{51}$$
$$+ \left(x^{31} - 3x^{29} - x^{28} - 4x^{27} - 4x^{26} + 11x^{25} - 7x^{24} + 5x^{23} - x^{21} + 4x^{20} + x^{18}\right) q^{50}$$
$$+ \left(-2x^{31} - 4x^{29} + 3x^{28} + x^{27} - 9x^{26} + 4x^{25} - 9x^{24} + 7x^{23} + 2x^{22} + 3x^{20} + x^{19} - x^{18}\right) q^{49}$$
$$+ \left(x^{31} - 2x^{29} + x^{28} - 5x^{27} - 2x^{26} + 7x^{25} - 11x^{24} + 4x^{23} - 3x^{21} + 5x^{20} - x^{19}\right) q^{48}$$
$$+ \left(2x^{28} - 5x^{26} + 2x^{25} - 4x^{24} + 3x^{23} - 3x^{21} + 3x^{20} + x^{18}\right) q^{47}$$
$$+ \left(-2x^{29} + x^{28} - x^{27} + x^{26} + 8x^{25} - 7x^{24} + x^{23} + 2x^{22} - 4x^{21} + 8x^{20} - 2x^{19} - x^{18} - x^{16}\right) q^{46}$$

$$+ \left(x^{28} + x^{27} - 3x^{26} + 3x^{25} - 2x^{24} + 8x^{23} + 6x^{22} - 3x^{21} + 6x^{20} - 3x^{19} + x^{18} - x^{16}\right) q^{45}$$
$$+ \left(-x^{29} - x^{27} + 8x^{25} - 8x^{24} + 4x^{23} + 7x^{22} - 2x^{21} + 11x^{20} - x^{19} + x^{17} - x^{16}\right) q^{44}$$
$$+ \left(-2x^{26} + 2x^{25} - 3x^{24} + 9x^{23} + x^{22} - x^{21} + 9x^{20} - 2x^{19} + 5x^{18} - 2x^{16}\right) q^{43}$$
$$+ \left(x^{27} + 4x^{25} - 8x^{24} + 2x^{23} + 3x^{22} + x^{21} + 11x^{20} - x^{19} - x^{18} + x^{17} - x^{16}\right) q^{42}$$
$$+ \left(-x^{27} - x^{26} + x^{25} - 2x^{24} + 5x^{23} - 2x^{22} - 4x^{21} + 6x^{20} - 3x^{19} + 4x^{18} + 2x^{17} - 4x^{16}\right) q^{41}$$
$$+ \left(2x^{25} - 4x^{24} - x^{23} + 4x^{22} - 3x^{21} + 6x^{20} - 3x^{19} - 2x^{18} - 2x^{16} - x^{14}\right) q^{40}$$
$$+ \left(x^{25} - x^{24} + 3x^{23} - x^{22} - 6x^{21} + 11x^{20} - 7x^{19} + 3x^{18} + x^{17} - 8x^{16}\right) q^{39}$$
$$+ \left(x^{25} - 2x^{24} + 5x^{22} - x^{21} + 9x^{20} - 7x^{19} + x^{18} - 3x^{17} - 3x^{16} - 2x^{14}\right) q^{38}$$
$$+ \left(x^{25} - x^{24} + 2x^{23} - 4x^{21} + 11x^{20} - 2x^{19} + 7x^{18} + x^{17} - 8x^{16} - x^{15}\right) q^{37}$$
$$+ \left(-x^{24} + x^{22} + x^{21} + 6x^{20} - 3x^{19} + 6x^{18} - 2x^{16} + 2x^{15} - 3x^{14} + x^{12}\right) q^{36}$$
$$+ \left(-x^{24} + x^{23} - x^{22} - 3x^{21} + 8x^{20} - x^{19} + 2x^{18} + 3x^{17} - 7x^{16} + 2x^{15} + x^{14} + x^{12}\right) q^{35}$$
$$+ \left(x^{22} - x^{21} + 3x^{20} - 3x^{19} + 2x^{17} - 4x^{16} + 4x^{15} - 5x^{14} + 2x^{12}\right) q^{34}$$
$$+ \left(-2x^{21} + 5x^{20} - x^{19} - x^{17} - 11x^{16} - 2x^{14} + x^{12}\right) q^{33}$$
$$+ \left(-x^{22} - x^{21} + 3x^{20} - 4x^{19} + 2x^{18} - 9x^{16} + 2x^{15} - 9x^{14} - x^{13} + 3x^{12}\right) q^{32}$$
$$+ \left(x^{22} - x^{21} + 4x^{20} - x^{19} - 7x^{16} - x^{15} - 4x^{14} - x^{13} - x^{12}\right) q^{31}$$
$$+ \left(2x^{20} - 2x^{19} + 2x^{18} - 4x^{16} + x^{15} - 8x^{14} - x^{13} + 4x^{12}\right) q^{30}$$
$$+ \left(x^{18} - 5x^{16} + 2x^{15} - x^{14} + 2x^{13} + x^{12} + 2x^{10}\right) q^{29}$$
$$+ \left(2x^{20} + x^{18} + 2x^{17} - 6x^{16} + x^{15} - 3x^{14} - x^{13} + 6x^{12} + x^{10}\right) q^{28}$$
$$+ \left(x^{17} - 5x^{16} + 2x^{15} - 4x^{14} + x^{13} + 3x^{12} + 4x^{10}\right) q^{27}$$
$$+ \left(x^{20} - x^{18} + x^{17} - 5x^{16} + 3x^{15} - 7x^{14} + 4x^{12} + x^{10}\right) q^{26}$$
$$+ \left(x^{18} - 3x^{16} + x^{15} - 5x^{14} + x^{13} - x^{12} - x^{11} + 3x^{10} - x^{8}\right) q^{25}$$
$$+ \left(-x^{18} - 3x^{16} - 4x^{14} + x^{12} + x^{11} + x^{10} - x^{8}\right) q^{24}$$
$$+ \left(2x^{18} - 3x^{16} - 3x^{14} - x^{13} + 2x^{12} - 2x^{11} + 3x^{10} - 2x^{8}\right) q^{23}$$
$$+ \left(-x^{18} - x^{13} + 3x^{12} - x^{11} + 2x^{10}\right) q^{22} + \left(-2x^{16} - x^{14} + 4x^{12} - x^{11} + 7x^{10}\right) q^{21}$$
$$+ \left(x^{16} - 3x^{14} + 3x^{12} + 7x^{10}\right) q^{20} + \left(-2x^{16} + x^{14} + 3x^{12} + 3x^{10} + x^{9}\right) q^{19}$$

$$+\left(-x^{14}+3x^{10}+x^{8}-x^{6}\right)q^{18}+\left(x^{14}+x^{12}+x^{10}-2x^{8}-3x^{6}\right)q^{17}$$
$$+\left(-2x^{14}+3x^{12}+2x^{10}-3x^{8}-2x^{6}\right)q^{16}+\left(x^{14}+x^{12}+x^{10}-x^{6}\right)q^{15}$$
$$+\left(2x^{12}+3x^{10}-x^{8}-2x^{6}\right)q^{14}+\left(-x^{12}+2x^{10}+3x^{8}-x^{6}+x^{4}\right)q^{13}$$
$$+\left(x^{12}-x^{8}+x^{6}+x^{4}\right)q^{12}+\left(x^{10}-2x^{6}\right)q^{11}+\left(-x^{10}-2x^{8}-x^{4}\right)q^{10}$$
$$+\left(-x^{8}-4x^{6}\right)q^{9}+\left(-2x^{6}-x^{4}\right)q^{8}+\left(-x^{8}-2x^{6}+x^{4}\right)q^{7}+\left(x^{6}-2x^{4}+x^{2}\right)q^{6}$$
$$+\left(x^{4}+x^{2}\right)q^{5}+\left(x^{4}+2x^{2}\right)q^{4}+\left(x^{4}-x^{2}\right)q^{3}+x^{2}q^{2}-1.$$

$$\delta(x,q)=\frac{N(\delta)}{D(\delta)},$$

$$\begin{aligned}
D(\delta)=\ & q^{129/2}x^{31}-2q^{125/2}x^{29}-2q^{67/2}x^{11}+q^{63/2}x^{9}+q^{123/2}x^{9}\left(x^{20}-x^{18}\right)\\
&+q^{121/2}x^{9}\left(x^{18}-2x^{20}\right)+q^{119/2}x^{9}\left(x^{16}-2x^{18}\right)\\
&+q^{117/2}x^{9}\left(4x^{18}-2x^{16}\right)+q^{115/2}x^{9}\left(4x^{16}-2x^{18}\right)\\
&+q^{77/2}x^{9}\left(4x^{6}-2x^{4}\right)+q^{75/2}x^{9}\left(4x^{4}-2x^{6}\right)+q^{73/2}x^{9}\left(x^{6}-2x^{4}\right)\\
&+q^{71/2}x^{9}\left(x^{4}-2x^{2}\right)+q^{69/2}x^{9}\left(x^{2}-x^{4}\right)+q^{113/2}x^{9}\left(x^{18}-3x^{16}+2x^{14}\right)\\
&+q^{111/2}x^{9}\left(-x^{18}+4x^{16}-3x^{14}\right)+q^{109/2}x^{9}\left(-3x^{16}+5x^{14}-x^{12}\right)\\
&+q^{107/2}x^{9}\left(4x^{16}-5x^{14}+2x^{12}\right)+q^{105/2}x^{9}\left(-2x^{16}+5x^{14}-4x^{12}\right)\\
&+q^{101/2}x^{9}\left(5x^{14}-7x^{12}+2x^{10}\right)+q^{99/2}x^{9}\left(-3x^{14}+7x^{12}-4x^{10}\right)\\
&+q^{97/2}x^{9}\left(2x^{14}-7x^{12}+4x^{10}\right)+q^{95/2}x^{9}\left(4x^{12}-7x^{10}+2x^{8}\right)\\
&+q^{93/2}x^{9}\left(-4x^{12}+7x^{10}-3x^{8}\right)+q^{91/2}x^{9}\left(2x^{12}-7x^{10}+5x^{8}\right)\\
&+q^{87/2}x^{9}\left(-4x^{10}+5x^{8}-2x^{6}\right)+q^{85/2}x^{9}\left(2x^{10}-5x^{8}+4x^{6}\right)\\
&+q^{83/2}x^{9}\left(-x^{10}+5x^{8}-3x^{6}\right)+q^{81/2}x^{9}\left(-3x^{8}+4x^{6}-x^{4}\right)\\
&+q^{79/2}x^{9}\left(2x^{8}-3x^{6}+x^{4}\right)+q^{103/2}x^{9}\left(x^{16}-5x^{14}+4x^{12}-x^{10}\right)\\
&+q^{89/2}x^{9}\left(-x^{12}+4x^{10}-5x^{8}+x^{6}\right)
\end{aligned}$$

$$
\begin{aligned}
N(\delta) = {} & -x^{39}q^{83} + 2x^{37}q^{81} + \left(x^{35} - x^{37}\right) q^{80} + \left(2x^{37} - x^{35}\right) q^{79} + \left(x^{37} + 2x^{35} - x^{33}\right) q^{78} \\
& + \left(x^{37} - 4x^{35} + 2x^{33}\right) q^{77} - 4x^{33}q^{76} + \left(-2x^{35} + 2x^{33} - 2x^{31}\right) q^{75} \\
& + \left(3x^{31} - 4x^{33}\right) q^{74} + \left(-x^{35} + 2x^{33} - 4x^{31} + x^{29}\right) q^{73} + \left(-2x^{33} + 4x^{31} - 2x^{29}\right) q^{72} \\
& + \left(x^{35} + 2x^{33} - 3x^{31} + 4x^{29}\right) q^{71} + \left(x^{35} - x^{33} + 5x^{31} - 2x^{29} + x^{27}\right) q^{70} \\
& + \left(-3x^{33} - 3x^{31} + 6x^{29} - 2x^{27}\right) q^{69} + \left(-2x^{33} + x^{31} - 4x^{29} + 3x^{27}\right) q^{68} \\
& + \left(-2x^{33} + 5x^{29} - 3x^{27}\right) q^{67} + \left(-3x^{33} + 5x^{27} - 2x^{25}\right) q^{66} \\
& + \left(-x^{33} + 3x^{31} + 3x^{29} - 5x^{27} + 2x^{25}\right) q^{65} \\
& + \left(3x^{31} + 3x^{29} + 3x^{27} - 4x^{25}\right) q^{64} + \left(x^{31} + 4x^{29} + 2x^{25} - x^{23}\right) q^{63} \\
& + \left(4x^{29} + x^{27} - 4x^{25} + 2x^{23}\right) q^{62} + \left(x^{29} - x^{27} - 2x^{23}\right) q^{61} \\
& + \left(-2x^{31} + 3x^{29} - 2x^{27} - 4x^{25} + x^{23}\right) q^{60} \\
& + \left(-x^{31} + x^{29} - 3x^{25} - 3x^{23} + x^{21}\right) q^{59} + \left(-x^{31} + 5x^{29} - 4x^{27} - 3x^{25} + x^{23}\right) q^{58} \\
& + \left(2x^{29} + 3x^{27} - 7x^{25} - 2x^{23} + 2x^{21}\right) q^{57} + \left(5x^{29} - 2x^{27} - 2x^{23} - x^{21}\right) q^{56} \\
& + \left(x^{29} + 4x^{27} - 6x^{25} + x^{23} + 3x^{21}\right) q^{55} + \left(2x^{29} - 4x^{27} + x^{25} - 5x^{23} + 3x^{21} + x^{19}\right) q^{54} \\
& + \left(3x^{27} - 12x^{25} + 2x^{23} + x^{21} - x^{19}\right) q^{53} + \left(-3x^{27} - 8x^{23} + 5x^{21} + x^{19}\right) q^{52} \\
& + \left(3x^{27} - 12x^{25} + 3x^{23} + 2x^{19}\right) q^{51} + \left(x^{27} + x^{25} - 11x^{23} + 8x^{21} - x^{17}\right) q^{50} \\
& + \left(2x^{27} - 8x^{25} + 8x^{23} - 3x^{21} + 3x^{19} - x^{17}\right) q^{49} + \left(x^{27} - 8x^{23} + 13x^{21} - x^{17}\right) q^{48} \\
& + \left(x^{27} - 5x^{25} + 6x^{23} - 2x^{21} + 9x^{19} - 2x^{17}\right) q^{47} \\
& + \left(-x^{25} - 8x^{23} + 15x^{21} - 2x^{19}\right) q^{46} + \left(-3x^{25} + 2x^{23} - 5x^{21} + 11x^{19} - 4x^{17}\right) q^{45} \\
& + \left(-7x^{23} + 11x^{21} - 4x^{19} + x^{17} - x^{15}\right) q^{44} + \left(-x^{25} + x^{23} - 4x^{21} + 11x^{19} - 7x^{17}\right) q^{43} \\
& + \left(-4x^{23} + 11x^{21} - 5x^{19} + 2x^{17} - 3x^{15}\right) q^{42} + \left(-2x^{21} + 15x^{19} - 8x^{17} - x^{15}\right) q^{41} \\
& + \left(-2x^{23} + 9x^{21} - 2x^{19} + 6x^{17} - 5x^{15} + x^{13}\right) q^{40} + \left(-x^{23} + 13x^{19} - 8x^{17} + x^{13}\right) q^{39} \\
& + \left(-x^{23} + 3x^{21} - 3x^{19} + 8x^{17} - 8x^{15} + 2x^{13}\right) q^{38} + \left(-x^{23} + 8x^{19} - 11x^{17} + x^{15} + x^{13}\right) q^{37} \\
& + \left(2x^{21} + 3x^{17} - 12x^{15} + 3x^{13}\right) q^{36} + \left(x^{21} + 5x^{19} - 8x^{17} - 3x^{13}\right) q^{35} \\
& + \left(-x^{21} + x^{19} + 2x^{17} - 12x^{15} + 3x^{13}\right) q^{34} \\
& + \left(x^{22} + x^{21} + 3x^{19} - 5x^{17} + x^{15} - 4x^{13} + 2x^{11}\right) q^{33}
\end{aligned}
$$

$$+\left(3x^{19}+x^{17}-6x^{15}+4x^{13}+x^{11}\right)q^{32}+\left(-2x^{20}-x^{19}-2x^{17}-2x^{13}+5x^{11}\right)q^{31}$$
$$+\left(x^{20}+2x^{19}-x^{18}-2x^{17}-7x^{15}+3x^{13}+2x^{11}\right)q^{30}$$
$$+\left(-2x^{20}+x^{18}+x^{17}-3x^{15}-4x^{13}+5x^{11}-x^{9}\right)q^{29}$$
$$+\left(x^{19}-2x^{18}-3x^{17}+x^{16}-3x^{15}+x^{11}-x^{9}\right)q^{28}$$
$$+\left(4x^{18}+x^{17}-2x^{16}-4x^{15}-2x^{13}+3x^{11}-2x^{9}\right)q^{27}$$
$$+\left(-2x^{18}-2x^{17}+4x^{16}-x^{13}+x^{11}\right)q^{26}$$
$$+\left(x^{18}+2x^{17}-3x^{16}-4x^{15}+2x^{14}+x^{13}+4x^{11}\right)q^{25}$$
$$+\left(-x^{18}-x^{17}+4x^{16}+2x^{15}-3x^{14}+4x^{11}+x^{9}\right)q^{24}$$
$$+\left(-3x^{16}-4x^{15}+5x^{14}+3x^{13}-x^{12}+3x^{11}+3x^{9}\right)q^{23}$$
$$+\left(4x^{16}+2x^{15}-5x^{14}-5x^{13}+2x^{12}+3x^{11}+3x^{9}-x^{7}\right)q^{22}$$
$$+\left(-2x^{16}-2x^{15}+5x^{14}+5x^{13}-4x^{12}-3x^{7}\right)q^{21}$$
$$+\left(x^{16}-5x^{14}-3x^{13}+4x^{12}+5x^{11}-x^{10}-2x^{7}\right)q^{20}$$
$$+\left(5x^{14}+3x^{13}-7x^{12}-4x^{11}+2x^{10}+x^{9}-2x^{7}\right)q^{19}$$
$$+\left(-3x^{14}-2x^{13}+7x^{12}+6x^{11}-4x^{10}-3x^{9}-3x^{7}\right)q^{18}$$
$$+\left(2x^{14}+x^{13}-7x^{12}-2x^{11}+4x^{10}+5x^{9}-x^{7}+x^{5}\right)q^{17}$$
$$+\left(4x^{12}+4x^{11}-7x^{10}-3x^{9}+2x^{8}+2x^{7}+x^{5}\right)q^{16}$$
$$+\left(-4x^{12}-2x^{11}+7x^{10}+4x^{9}-3x^{8}-2x^{7}\right)q^{15}$$
$$+\left(2x^{12}+x^{11}-7x^{10}-4x^{9}+5x^{8}+2x^{7}-x^{5}\right)q^{14}$$
$$+\left(-x^{12}+4x^{10}+3x^{9}-5x^{8}-4x^{7}+x^{6}\right)q^{13}+\left(-4x^{10}-2x^{9}+5x^{8}+2x^{7}-2x^{6}-2x^{5}\right)q^{12}$$
$$+\left(2x^{10}-5x^{8}-4x^{7}+4x^{6}\right)q^{11}+\left(-x^{10}+5x^{8}+2x^{7}-3x^{6}-4x^{5}+x^{3}\right)q^{10}$$
$$+\left(-3x^{8}-x^{7}+4x^{6}+2x^{5}-x^{4}+x^{3}\right)q^{9}+\left(2x^{8}-3x^{6}-x^{5}+x^{4}+2x^{3}\right)q^{8}$$
$$+\left(4x^{6}+x^{5}-2x^{4}-x^{3}\right)q^{7}+\left(-2x^{6}+4x^{4}+2x^{3}\right)q^{6}+\left(x^{6}-2x^{4}\right)q^{5}$$
$$+\left(x^{4}-2x^{2}-x\right)q^{4}+\left(x^{2}-x^{4}\right)q^{3}-2x^{2}q^{2}+1.$$

The $\hbar$ expansions of the above coefficients functions are the following.

$$\alpha(x,e^{\hbar}) = -\frac{1}{x^9} + \frac{\left(5x^{16}-2x^{14}-37x^{12}-40x^{10}+28x^8+104x^6+71x^4-2x^2-31\right)}{x^9\left(x^2-1\right)\left(x^2+1\right)\left(x^{12}-2x^8-4x^6-2x^4+1\right)}\hbar$$
$$+\frac{1}{2x^9\left(x^2-1\right)\left(x^2+1\right)^2\left(x^{12}-2x^8-4x^6-2x^4+1\right)^2}\Big(-25x^{30}+11x^{28}+729x^{26}$$
$$+1509x^{24}-1068x^{22}-8092x^{20}-9514x^{18}+5430x^{16}+27546x^{14}+31834x^{12}+12988x^{10}$$
$$-7284x^8-10941x^6-3681x^4+1069x^2+961\Big)\hbar^2+O\left(\hbar^3\right)$$

$$\beta(x,e^{\hbar}) = \frac{1}{x^{17}}\left(-x^{17}+x^{16}-2x^{14}-3x^{12}+2x^{10}+7x^8+2x^6-3x^4-2x^2+1\right)$$
$$-\frac{1}{2x^{17}\left(x^{16}-3x^{12}-4x^{10}+4x^6+3x^4-1\right)}\Big(-x^{33}+x^{32}+4x^{31}-16x^{30}+47x^{29}$$
$$-68x^{28}+44x^{27}+116x^{26}-56x^{25}+538x^{24}-172x^{23}+156x^{22}-115x^{21}-1439x^{20}$$
$$+4x^{19}-1936x^{18}+53x^{17}+460x^{16}+2924x^{14}+1891x^{12}-924x^{10}-1622x^8-244x^6$$
$$+472x^4+164x^2-89\Big)\hbar$$
$$+\frac{1}{8x^{17}\left(x^{16}-3x^{12}-4x^{10}+4x^6+3x^4-1\right)^2}\Big(-x^{49}+x^{48}+72x^{47}-146x^{46}+1486x^{45}$$
$$-1561x^{44}+1824x^{43}+3126x^{42}-5745x^{41}+21810x^{40}-16576x^{39}+2724x^{38}-1926x^{37}$$
$$-123674x^{36}+40840x^{35}-171040x^{34}+58980x^{33}+217052x^{32}+8872x^{31}+754786x^{30}$$
$$-54198x^{29}+388419x^{28}-56320x^{27}-1005850x^{26}-8985x^{25}-1674366x^{24}+21264x^{23}$$
$$-305650x^{22}+13582x^{21}+1557339x^{20}-360x^{19}+1533466x^{18}-2809x^{17}-91108x^{16}$$
$$-1042960x^{14}-532634x^{12}+214404x^{10}+281730x^8+22926x^6-61681x^4-13466x^2$$
$$+7921\Big)\hbar^2+O\left(\hbar^3\right)$$

$$\gamma(x,e^{\hbar}) = -\frac{1}{x^{17}}+\frac{2}{x^{15}}+\frac{3}{x^{13}}-\frac{2}{x^{11}}-\frac{7}{x^9}+x^8+\frac{1}{x^8}-\frac{2}{x^7}-2x^6-\frac{2}{x^6}+\frac{3}{x^5}-3x^4-\frac{3}{x^4}+\frac{2}{x^3}$$
$$+2x^2+\frac{2}{x^2}-\frac{1}{x}+7$$

$$
\begin{aligned}
&+\frac{1}{x^{17}\left(x^{16}-3x^{12}-4x^{10}+4x^{6}+3x^{4}-1\right)}\Big(13x^{41}-19x^{39}-47x^{37}-4x^{35}+55x^{33}+4x^{32}\\
&+84x^{31}-19x^{30}+220x^{29}-59x^{28}+239x^{27}+86x^{26}-230x^{25}+401x^{24}-733x^{23}\\
&+134x^{22}-446x^{21}-913x^{20}+300x^{19}-1251x^{18}+487x^{17}+198x^{16}+68x^{15}+1665x^{14}\\
&-155x^{13}+1085x^{12}-55x^{11}-514x^{10}+31x^{9}-895x^{8}-130x^{6}+265x^{4}+89x^{2}-50\Big)\hbar\\
&-\frac{1}{2x^{17}(x+1)^2\left(x^2+1\right)^2\left(x^2+x+1\right)^2\left(x^{10}-x^9+x^7-3x^6+2x^5-3x^4+x^3-x+1\right)^2}\\
&\times\Big(-169x^{55}-338x^{54}-322x^{53}-306x^{52}+782x^{51}+1870x^{50}+3243x^{49}+4616x^{48}\\
&+2305x^{47}+10x^{46}-6935x^{45}-14061x^{44}-14879x^{43}-16534x^{42}-1889x^{41}+14715x^{40}\\
&+25683x^{39}+47565x^{38}+23918x^{37}+1153x^{36}-45216x^{35}-145535x^{34}-170229x^{33}\\
&-268495x^{32}-246695x^{31}-146963x^{30}-41626x^{29}+348444x^{28}+598018x^{27}\\
&+1011439x^{26}+1301463x^{25}+1268658x^{24}+1261033x^{23}+686816x^{22}+216091x^{21}\\
&-382307x^{20}-933501x^{19}-1003328x^{18}-1098973x^{17}-710409x^{16}-351521x^{15}\\
&-16653x^{14}+316520x^{13}+329665x^{12}+349894x^{11}+209685x^{10}+70993x^{9}+1007x^{8}\\
&-69940x^{7}-54805x^{6}-39670x^{5}-18796x^{4}+2078x^{3}+3539x^{2}+5000x+2500\Big)\hbar^2\\
&+O\left(\hbar^3\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta(x,e^{\hbar}) =\ &\frac{1}{x^9}-x^8-\frac{1}{x^8}+2x^6+\frac{2}{x^6}+3x^4+\frac{3}{x^4}-2x^2-\frac{2}{x^2}-7\\
&-\frac{1}{2x^{9}\left(x^{16}-3x^{12}-4x^{10}+4x^{6}+3x^{4}-1\right)}\Big(37x^{33}-52x^{31}-152x^{29}+8x^{27}+278x^{25}\\
&+272x^{23}+161x^{21}+72x^{19}-396x^{17}+63x^{16}-900x^{15}-505x^{13}-189x^{12}+488x^{11}\\
&-252x^{10}+710x^{9}+80x^{7}+252x^{6}-260x^{5}+189x^{4}-88x^{3}+55x-63\Big)\hbar\\
&+\frac{1}{8x^{9}\left(x^{16}-3x^{12}-4x^{10}+4x^{6}+3x^{4}-1\right)^2}\Big(-1369x^{49}+1354x^{47}+8793x^{45}+2874x^{43}\\
&-27186x^{41}-37188x^{39}+24730x^{37}+92688x^{35}+28292x^{33}+3969x^{32}-127602x^{31}\\
&-110123x^{29}-23814x^{28}+143906x^{27}-31752x^{26}+276198x^{25}+35721x^{24}+13802x^{23}
\end{aligned}
$$

$$
\begin{aligned}
&+127008x^{22}-321803x^{21}+87318x^{20}-260586x^{19}-95256x^{18}+96260x^{17}\\
&-206388x^{16}+256992x^{15}-95256x^{14}+95722x^{13}+87318x^{12}-82404x^{11}+127008x^{10}\\
&-77298x^{9}+35721x^{8}+354x^{7}-31752x^{6}+21177x^{5}-23814x^{4}+3874x^{3}-3025x\\
&\left.+3969\right)\hbar^2+O\left(h^3\right).
\end{aligned}
$$

B The details of the recursion II

This is a list of the coefficient functions $t_v(q,q^m)$ of the f_m-recursion relation (2.6).

$$
\begin{aligned}
t_2(q,q^m)=\ & q^{\frac{m+109}{2}}\Big(q^{\frac{m+83}{2}}-q^{\frac{m+85}{2}}+2q^{\frac{m+87}{2}}+q^{\frac{m+89}{2}}+q^{\frac{m+91}{2}}-q^4+q^3-2q^2-q-1\Big)\\
t_4(q,q^m)=\ & q^{\frac{m+107}{2}}\Big(q^{\frac{m+79}{2}}+q^{\frac{m+81}{2}}+q^{\frac{m+83}{2}}-2q^{\frac{m+85}{2}}+q^{\frac{m+87}{2}}-q^{\frac{m+89}{2}}-q^{\frac{m+93}{2}}+q^{\frac{m+97}{2}}+q^{\frac{m+99}{2}}\\
&-q^{10}-q^9-q^8+2q^7-q^6+q^5+q^3-q-1\Big)\\
t_6(q,q^m)=\ & -q^{\frac{m+109}{2}}\Big(-q^{\frac{m+75}{2}}+2q^{\frac{m+77}{2}}+2q^{\frac{m+79}{2}}+4q^{\frac{m+81}{2}}+2q^{\frac{m+85}{2}}-q^{\frac{m+87}{2}}+q^{\frac{m+89}{2}}+2q^{\frac{m+91}{2}}\\
&+q^{\frac{m+93}{2}}+2q^{\frac{m+95}{2}}+3q^{\frac{m+97}{2}}+q^{\frac{m+99}{2}}+q^{12}-2q^{11}-2q^{10}-4q^9-2q^7+q^6-q^5-2q^4\\
&-q^3-2q^2-3q-1\Big)\\
t_8(q,q^m)=\ & -q^{m+89}-q^{m+91}-2q^{m+92}-q^{m+94}+3q^{m+95}-q^{m+96}-3q^{m+98}-2q^{m+99}+q^{m+100}\\
&-2q^{m+105}-q^{m+106}-q^{m+107}+q^{\frac{3m}{2}+\frac{283}{2}}+q^{\frac{m+107}{2}}+q^{\frac{m+109}{2}}+2q^{\frac{m+111}{2}}-q^{\frac{m+121}{2}}\\
&+2q^{\frac{m+123}{2}}+3q^{\frac{m+125}{2}}+q^{\frac{m+129}{2}}-3q^{\frac{m+131}{2}}+q^{\frac{m+133}{2}}+2q^{\frac{m+137}{2}}+q^{\frac{m+139}{2}}+q^{\frac{m+143}{2}}-q^{19}\\
t_9(q,q^m)=\ & q^{m+99}-q^{\frac{3(m+95)}{2}}\\
t_{10}(q,q^m)=\ & -q^{23}\Big(q^{m+65}-q^{m+66}-2q^{m+68}-3q^{m+69}-q^{m+70}-2q^{m+71}-q^{m+72}-3q^{m+73}\\
&-3q^{m+74}-7q^{m+75}-7q^{m+76}-2q^{m+77}-3q^{m+78}-q^{m+79}-3q^{m+80}-q^{m+81}-4q^{m+82}\\
&-q^{m+83}-2q^{m+84}+q^{\frac{3m}{2}+\frac{229}{2}}+2q^{\frac{3m}{2}+\frac{233}{2}}+2q^{\frac{m+65}{2}}+q^{\frac{m+67}{2}}+4q^{\frac{m+69}{2}}+q^{\frac{m+71}{2}}\\
&+3q^{\frac{m+73}{2}}+q^{\frac{m+75}{2}}+3q^{\frac{m+77}{2}}-q^{\frac{3(m+77)}{2}}+2q^{\frac{m+79}{2}}+7q^{\frac{m+81}{2}}+7q^{\frac{m+83}{2}}+3q^{\frac{m+85}{2}}+3q^{\frac{m+87}{2}}\\
&+q^{\frac{m+89}{2}}+2q^{\frac{m+91}{2}}+q^{\frac{m+93}{2}}+3q^{\frac{m+95}{2}}+2q^{\frac{m+97}{2}}+q^{\frac{m+101}{2}}-q^{\frac{m+103}{2}}-q^2+q-2\Big)\\
t_{11}(q,q^m)=\ & q^{m+89}\Big(q^{\frac{m+99}{2}}-q^{\frac{m+101}{2}}+2q^{\frac{m+103}{2}}+q^{\frac{m+105}{2}}+q^{\frac{m+107}{2}}-q^{12}+q^{11}-2q^{10}-q^9-q^8\Big)\\
t_{12}(q,q^m)=\ & -q^{26}\Big(-q^{m+60}+q^{m+61}-2q^{m+62}-q^{m+63}-3q^{m+64}-q^{m+65}-3q^{m+67}-3q^{m+68}
\end{aligned}
$$

$$
\begin{aligned}
&- 4q^{m+69} - 3q^{m+70} - 2q^{m+71} - q^{m+72} + q^{m+73} - 4q^{m+74} - 3q^{m+75} - 6q^{m+76} - q^{m+77} \\
&- 4q^{m+78} + q^{m+79} - 3q^{m+80} - q^{m+81} - 2q^{m+82} - q^{m+83} - q^{m+84} + q^{\frac{3m}{2}+\frac{215}{2}} \\
&+ q^{\frac{3m}{2}+\frac{217}{2}} + 2q^{\frac{3m}{2}+\frac{221}{2}} - q^{\frac{3m}{2}+\frac{223}{2}} + q^{\frac{m+57}{2}} + q^{\frac{m+59}{2}} + 2q^{\frac{m+61}{2}} + q^{\frac{m+63}{2}} + 3q^{\frac{m+65}{2}} \\
&- q^{\frac{m+67}{2}} + 4q^{\frac{m+69}{2}} + q^{\frac{m+71}{2}} + q^{\frac{3(m+71)}{2}} + 6q^{\frac{m+73}{2}} - 2q^{\frac{3(m+73)}{2}} + 3q^{\frac{m+75}{2}} + q^{\frac{3(m+75)}{2}} \\
&+ 4q^{\frac{m+77}{2}} - q^{\frac{m+79}{2}} + q^{\frac{m+81}{2}} + 2q^{\frac{m+83}{2}} + 3q^{\frac{m+85}{2}} + 4q^{\frac{m+87}{2}} + 3q^{\frac{m+89}{2}} + 3q^{\frac{m+91}{2}} + q^{\frac{m+95}{2}} \\
&+ 3q^{\frac{m+97}{2}} + q^{\frac{m+99}{2}} + 2q^{\frac{m+101}{2}} - q^{\frac{m+103}{2}} + q^{\frac{m+105}{2}} - q^6 - q^5 - q^4 + 2q^3 - 2q^2 + q - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{13}(q,q^m) =\ & q^{m+84}\Big(q^{\frac{m+99}{2}} + q^{\frac{m+101}{2}} + q^{\frac{m+103}{2}} - 2q^{\frac{m+105}{2}} + q^{\frac{m+107}{2}} - q^{\frac{m+109}{2}} - q^{\frac{m+113}{2}} + q^{\frac{m+117}{2}} \\
&+ q^{\frac{m+119}{2}} - q^{20} - q^{19} - q^{18} + 2q^{17} - q^{16} + q^{15} + q^{13} - q^{11} - q^{10}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{14}(q,q^m) =\ & q^{30}\Big(q^{m+55} - 2q^{m+56} + q^{m+57} - q^{m+58} + q^{m+59} - 3q^{m+60} - q^{m+61} - 3q^{m+63} - 4q^{m+64} \\
&- 5q^{m+65} - 7q^{m+66} - 4q^{m+67} - 3q^{m+68} - q^{m+69} - 8q^{m+70} - 4q^{m+71} - 9q^{m+72} \\
&- 2q^{m+73} - 5q^{m+74} + q^{m+75} - 3q^{m+76} - 2q^{m+78} - q^{m+80} - q^{\frac{3m}{2}+\frac{199}{2}} + q^{\frac{3m}{2}+\frac{203}{2}} \\
&+ 2q^{\frac{3m}{2}+\frac{205}{2}} + 3q^{\frac{3m}{2}+\frac{209}{2}} - 2q^{\frac{3m}{2}+\frac{211}{2}} + q^{\frac{m+53}{2}} + 2q^{\frac{m+57}{2}} + 3q^{\frac{m+61}{2}} - q^{\frac{m+63}{2}} + 5q^{\frac{m+65}{2}} \\
&+ 2q^{\frac{m+67}{2}} + 2q^{\frac{3(m+67)}{2}} + 9q^{\frac{m+69}{2}} - 2q^{\frac{3(m+69)}{2}} + 4q^{\frac{m+71}{2}} + q^{\frac{3(m+71)}{2}} + 8q^{\frac{m+73}{2}} + q^{\frac{m+75}{2}} \\
&+ 3q^{\frac{m+77}{2}} + 4q^{\frac{m+79}{2}} + 7q^{\frac{m+81}{2}} + 5q^{\frac{m+83}{2}} + 4q^{\frac{m+85}{2}} + 3q^{\frac{m+87}{2}} + q^{\frac{m+91}{2}} + 3q^{\frac{m+93}{2}} - q^{\frac{m+95}{2}} \\
&+ q^{\frac{m+97}{2}} - q^{\frac{m+99}{2}} + 2q^{\frac{m+101}{2}} - q^{\frac{m+103}{2}} + q^7 - 2q^6 - q^5 - 2q^4 + 2q^3 - 3q^2 + 2q - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{15}(q,q^m) =\ & q^{m+81}\Big(q^{\frac{m+99}{2}} - 2q^{\frac{m+101}{2}} - 2q^{\frac{m+103}{2}} - 4q^{\frac{m+105}{2}} - 2q^{\frac{m+109}{2}} + q^{\frac{m+111}{2}} - q^{\frac{m+113}{2}} - 2q^{\frac{m+115}{2}} \\
&- q^{\frac{m+117}{2}} - 2q^{\frac{m+119}{2}} - 3q^{\frac{m+121}{2}} - q^{\frac{m+123}{2}} - q^{24} + 2q^{23} + 2q^{22} + 4q^{21} + 2q^{19} - q^{18} + q^{17} \\
&+ 2q^{16} + q^{15} + 2q^{14} + 3q^{13} + q^{12}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{16}(q,q^m) =& -2q^{m+85} + q^{m+86} - 2q^{m+87} - 3q^{m+89} - 3q^{m+90} - 3q^{m+91} - 5q^{m+92} - 5q^{m+93} \\
&- 6q^{m+94} - 5q^{m+95} - 4q^{m+96} - 7q^{m+97} - 9q^{m+98} - 11q^{m+99} - 4q^{m+100} - 7q^{m+101} \\
&- 2q^{m+102} - 8q^{m+103} - 3q^{m+104} - 8q^{m+105} - 2q^{m+106} - 4q^{m+107} - q^{m+108} - 2q^{m+109} \\
&- q^{m+110} - q^{m+111} - q^{m+112} + q^{\frac{3m}{2}+\frac{253}{2}} - q^{\frac{3m}{2}+\frac{257}{2}} + 2q^{\frac{3m}{2}+\frac{259}{2}} + 3q^{\frac{3m}{2}+\frac{263}{2}} - 2q^{\frac{3m}{2}+\frac{265}{2}} \\
&+ q^{\frac{3(m+83)}{2}} + 2q^{\frac{3(m+85)}{2}} - q^{\frac{3(m+87)}{2}} + 2q^{\frac{3(m+89)}{2}} + q^{\frac{m+113}{2}} + q^{\frac{m+115}{2}} + q^{\frac{m+117}{2}} + 2q^{\frac{m+119}{2}} \\
&+ q^{\frac{m+121}{2}} + 4q^{\frac{m+123}{2}} + 2q^{\frac{m+125}{2}} + 8q^{\frac{m+127}{2}} + 3q^{\frac{m+129}{2}} + 8q^{\frac{m+131}{2}} + 2q^{\frac{m+133}{2}} + 7q^{\frac{m+135}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+4q^{\frac{m+137}{2}}+11q^{\frac{m+139}{2}}+9q^{\frac{m+141}{2}}+7q^{\frac{m+143}{2}}+4q^{\frac{m+145}{2}}+5q^{\frac{m+147}{2}}+6q^{\frac{m+149}{2}}+5q^{\frac{m+151}{2}}\\
&+5q^{\frac{m+153}{2}}+3q^{\frac{m+155}{2}}+3q^{\frac{m+157}{2}}+3q^{\frac{m+159}{2}}+2q^{\frac{m+163}{2}}-q^{\frac{m+165}{2}}+2q^{\frac{m+167}{2}}-q^{44}-q^{42}\\
&-2q^{41}+q^{40}-2q^{39}+q^{38}-3q^{37}+2q^{36}-2q^{35}\\
t_{17}(q,q^m)=\ &q^{55}\Big(q^{m+35}+q^{m+36}+2q^{m+37}-q^{m+42}+2q^{m+43}+3q^{m+44}+q^{m+46}-3q^{m+47}+q^{m+48}\\
&+2q^{m+50}+q^{m+51}+q^{m+53}-q^{\frac{3m}{2}+\frac{145}{2}}-q^{\frac{3m}{2}+\frac{151}{2}}-q^{\frac{3m}{2}+\frac{155}{2}}-2q^{\frac{3m}{2}+\frac{161}{2}}+q^{\frac{3m}{2}+\frac{163}{2}}\\
&-2q^{\frac{3m}{2}+\frac{173}{2}}-q^{\frac{3m}{2}+\frac{175}{2}}+q^{2m+123}-q^{\frac{m+1}{2}}-q^{\frac{3(m+47)}{2}}-2q^{\frac{3(m+49)}{2}}+3q^{\frac{3(m+51)}{2}}-3q^{\frac{3(m+53)}{2}}\\
&-q^{\frac{3(m+59)}{2}}\Big)\\
t_{18}(q,q^m)=\ &-q^{m+84}+2q^{m+85}-q^{m+86}+q^{m+87}-q^{m+88}+q^{m+90}+q^{m+91}+2q^{m+92}+2q^{m+94}\\
&+2q^{m+97}+6q^{m+98}+7q^{m+99}+q^{m+100}+3q^{m+101}-2q^{m+102}+4q^{m+103}-q^{m+104}\\
&+5q^{m+105}+q^{m+107}-q^{m+108}+q^{m+109}-q^{m+111}+q^{m+112}-q^{\frac{3m}{2}+\frac{245}{2}}+q^{\frac{3m}{2}+\frac{247}{2}}\\
&-2q^{\frac{3m}{2}+\frac{251}{2}}+2q^{\frac{3m}{2}+\frac{253}{2}}+q^{\frac{3m}{2}+\frac{257}{2}}-2q^{\frac{3m}{2}+\frac{259}{2}}-q^{\frac{3m}{2}+\frac{263}{2}}+q^{\frac{3(m+81)}{2}}-q^{\frac{3(m+83)}{2}}\\
&-2q^{\frac{3(m+85)}{2}}+2q^{\frac{3(m+87)}{2}}-q^{\frac{m+117}{2}}+q^{\frac{m+119}{2}}-q^{\frac{m+123}{2}}+q^{\frac{m+125}{2}}-q^{\frac{m+127}{2}}-5q^{\frac{m+131}{2}}\\
&+q^{\frac{m+133}{2}}-4q^{\frac{m+135}{2}}+2q^{\frac{m+137}{2}}-3q^{\frac{m+139}{2}}-q^{\frac{m+141}{2}}-7q^{\frac{m+143}{2}}-6q^{\frac{m+145}{2}}-2q^{\frac{m+147}{2}}\\
&-2q^{\frac{m+153}{2}}-2q^{\frac{m+157}{2}}-q^{\frac{m+159}{2}}-q^{\frac{m+161}{2}}+q^{\frac{m+165}{2}}-q^{\frac{m+167}{2}}+q^{\frac{m+169}{2}}-2q^{\frac{m+171}{2}}\\
&+q^{\frac{m+173}{2}}-q^{49}+q^{48}-q^{47}+q^{46}+2q^{45}-2q^{44}+2q^{43}-q^{42}+2q^{41}-2q^{40}+q^{39}\\
t_{19}(q,q^m)=\ &q^{\frac{m}{2}+24}\Big(-2q^{\frac{m}{2}+66}-q^{\frac{m}{2}+67}-4q^{\frac{m}{2}+68}-q^{\frac{m}{2}+69}-3q^{\frac{m}{2}+70}-q^{\frac{m}{2}+71}-3q^{\frac{m}{2}+72}-2q^{\frac{m}{2}+73}\\
&-7q^{\frac{m}{2}+74}-7q^{\frac{m}{2}+75}-3q^{\frac{m}{2}+76}-3q^{\frac{m}{2}+77}-q^{\frac{m}{2}+78}-2q^{\frac{m}{2}+79}-q^{\frac{m}{2}+80}-3q^{\frac{m}{2}+81}\\
&-2q^{\frac{m}{2}+82}-q^{\frac{m}{2}+84}+q^{\frac{m}{2}+85}-q^{m+\frac{197}{2}}+q^{m+\frac{199}{2}}+2q^{m+\frac{203}{2}}+3q^{m+\frac{205}{2}}+q^{m+\frac{207}{2}}\\
&+2q^{m+\frac{209}{2}}+q^{m+\frac{211}{2}}+3q^{m+\frac{213}{2}}+3q^{m+\frac{215}{2}}+7q^{m+\frac{217}{2}}+7q^{m+\frac{219}{2}}+2q^{m+\frac{221}{2}}+3q^{m+\frac{223}{2}}\\
&+q^{m+\frac{225}{2}}+3q^{m+\frac{227}{2}}+q^{m+\frac{229}{2}}+4q^{m+\frac{231}{2}}+q^{m+\frac{233}{2}}+2q^{m+\frac{235}{2}}-q^{\frac{3m}{2}+148}+q^{\frac{3m}{2}+149}\\
&-2q^{\frac{3m}{2}+150}+q^{71/2}-q^{69/2}+2q^{67/2}\Big)\\
t_{20}(q,q^m)=\ &q^{m+84}+2q^{m+86}+2q^{m+88}+4q^{m+89}+3q^{m+90}+5q^{m+91}+3q^{m+92}+8q^{m+93}+6q^{m+94}\\
&+11q^{m+95}+9q^{m+96}+11q^{m+97}+6q^{m+98}+6q^{m+99}+11q^{m+100}+9q^{m+101}+11q^{m+102}\\
&+6q^{m+103}+8q^{m+104}+3q^{m+105}+5q^{m+106}+3q^{m+107}+4q^{m+108}+2q^{m+109}
\end{aligned}
$$

$$
\begin{aligned}
&+2q^{m+111}+q^{m+113}-q^{\frac{3m}{2}+\frac{235}{2}}-2q^{\frac{3m}{2}+\frac{239}{2}}-2q^{\frac{3m}{2}+\frac{253}{2}}-q^{\frac{3m}{2}+\frac{257}{2}}+q^{\frac{3(m+79)}{2}}\\
&-3q^{\frac{3(m+81)}{2}}-3q^{\frac{3(m+83)}{2}}+q^{\frac{3(m+85)}{2}}-q^{\frac{m+119}{2}}-2q^{\frac{m+123}{2}}-2q^{\frac{m+127}{2}}-4q^{\frac{m+129}{2}}-3q^{\frac{m+131}{2}}\\
&-5q^{\frac{m+133}{2}}-3q^{\frac{m+135}{2}}-8q^{\frac{m+137}{2}}-6q^{\frac{m+139}{2}}-11q^{\frac{m+141}{2}}-9q^{\frac{m+143}{2}}-11q^{\frac{m+145}{2}}\\
&-6q^{\frac{m+147}{2}}-6q^{\frac{m+149}{2}}-11q^{\frac{m+151}{2}}-9q^{\frac{m+153}{2}}-11q^{\frac{m+155}{2}}-6q^{\frac{m+157}{2}}-8q^{\frac{m+159}{2}}-3q^{\frac{m+161}{2}}\\
&-5q^{\frac{m+163}{2}}-3q^{\frac{m+165}{2}}-4q^{\frac{m+167}{2}}-2q^{\frac{m+169}{2}}-2q^{\frac{m+173}{2}}-q^{\frac{m+177}{2}}+q^{55}-q^{54}+2q^{53}+3q^{51}\\
&+3q^{48}+2q^{46}-q^{45}+q^{44}
\end{aligned}
$$

$$
\begin{aligned}
t_{21}(q,q^m)=\ &q^{\frac{m}{2}+18}\Big(-q^{\frac{m}{2}+69}-q^{\frac{m}{2}+70}-2q^{\frac{m}{2}+71}-q^{\frac{m}{2}+72}-3q^{\frac{m}{2}+73}+q^{\frac{m}{2}+74}-4q^{\frac{m}{2}+75}-q^{\frac{m}{2}+76}\\
&-6q^{\frac{m}{2}+77}-3q^{\frac{m}{2}+78}-4q^{\frac{m}{2}+79}+q^{\frac{m}{2}+80}-q^{\frac{m}{2}+81}-2q^{\frac{m}{2}+82}-3q^{\frac{m}{2}+83}-4q^{\frac{m}{2}+84}\\
&-3q^{\frac{m}{2}+85}-3q^{\frac{m}{2}+86}-q^{\frac{m}{2}+88}-3q^{\frac{m}{2}+89}-q^{\frac{m}{2}+90}-2q^{\frac{m}{2}+91}+q^{\frac{m}{2}+92}-q^{\frac{m}{2}+93}+q^{m+\frac{201}{2}}\\
&-q^{m+\frac{203}{2}}+2q^{m+\frac{205}{2}}+q^{m+\frac{207}{2}}+3q^{m+\frac{209}{2}}+q^{m+\frac{211}{2}}+3q^{m+\frac{215}{2}}+3q^{m+\frac{217}{2}}+4q^{m+\frac{219}{2}}\\
&+3q^{m+\frac{221}{2}}+2q^{m+\frac{223}{2}}+q^{m+\frac{225}{2}}-q^{m+\frac{227}{2}}+4q^{m+\frac{229}{2}}+3q^{m+\frac{231}{2}}+6q^{m+\frac{233}{2}}+q^{m+\frac{235}{2}}\\
&+4q^{m+\frac{237}{2}}-q^{m+\frac{239}{2}}+3q^{m+\frac{241}{2}}+q^{m+\frac{243}{2}}+2q^{m+\frac{245}{2}}+q^{m+\frac{247}{2}}+q^{m+\frac{249}{2}}-q^{\frac{3m}{2}+148}\\
&-q^{\frac{3m}{2}+149}+2q^{\frac{3m}{2}+150}-2q^{\frac{3m}{2}+151}+q^{\frac{3m}{2}+152}-q^{\frac{3m}{2}+153}+q^{93/2}+q^{91/2}+q^{89/2}\\
&-2q^{87/2}+2q^{85/2}-q^{83/2}+q^{81/2}-q^{\frac{3(m+98)}{2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{22}(q,q^m)=\ &q^{m+85}-q^{m+86}+q^{m+88}-q^{m+89}+q^{m+90}+5q^{m+92}-q^{m+93}+4q^{m+94}-2q^{m+95}\\
&+3q^{m+96}+q^{m+97}+7q^{m+98}+6q^{m+99}+2q^{m+100}+2q^{m+103}+2q^{m+105}+q^{m+106}\\
&+q^{m+107}-q^{m+109}+q^{m+110}-q^{m+111}+2q^{m+112}-q^{m+113}-q^{\frac{3m}{2}+\frac{229}{2}}-2q^{\frac{3m}{2}+\frac{233}{2}}\\
&+q^{\frac{3m}{2}+\frac{235}{2}}+2q^{\frac{3m}{2}+\frac{239}{2}}-2q^{\frac{3m}{2}+\frac{241}{2}}+q^{\frac{3m}{2}+\frac{245}{2}}-q^{\frac{3m}{2}+\frac{247}{2}}+2q^{\frac{3(m+77)}{2}}-2q^{\frac{3(m+79)}{2}}\\
&-q^{\frac{3(m+81)}{2}}+q^{\frac{3(m+83)}{2}}+q^{\frac{m+123}{2}}-2q^{\frac{m+125}{2}}+q^{\frac{m+127}{2}}-q^{\frac{m+129}{2}}+q^{\frac{m+131}{2}}-q^{\frac{m+135}{2}}\\
&-q^{\frac{m+137}{2}}-2q^{\frac{m+139}{2}}-2q^{\frac{m+143}{2}}-2q^{\frac{m+149}{2}}-6q^{\frac{m+151}{2}}-7q^{\frac{m+153}{2}}-q^{\frac{m+155}{2}}-3q^{\frac{m+157}{2}}\\
&+2q^{\frac{m+159}{2}}-4q^{\frac{m+161}{2}}+q^{\frac{m+163}{2}}-5q^{\frac{m+165}{2}}-q^{\frac{m+169}{2}}+q^{\frac{m+171}{2}}-q^{\frac{m+173}{2}}+q^{\frac{m+177}{2}}\\
&-q^{\frac{m+179}{2}}+q^{60}-2q^{59}+2q^{58}-q^{57}+2q^{56}-2q^{55}+2q^{54}+q^{53}-q^{52}+q^{51}-q^{50}
\end{aligned}
$$

$$
\begin{aligned}
t_{23}(q,q^m)=\ &q^{\frac{m}{2}+13}\Big(q^{\frac{m}{2}+74}+2q^{\frac{m}{2}+76}+3q^{\frac{m}{2}+78}-q^{\frac{m}{2}+79}+5q^{\frac{m}{2}+80}+2q^{\frac{m}{2}+81}+9q^{\frac{m}{2}+82}+4q^{\frac{m}{2}+83}\\
&+8q^{\frac{m}{2}+84}+q^{\frac{m}{2}+85}+3q^{\frac{m}{2}+86}+4q^{\frac{m}{2}+87}+7q^{\frac{m}{2}+88}+5q^{\frac{m}{2}+89}+4q^{\frac{m}{2}+90}+3q^{\frac{m}{2}+91}
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{\frac{m}{2}+93} + 3q^{\frac{m}{2}+94} - q^{\frac{m}{2}+95} + q^{\frac{m}{2}+96} - q^{\frac{m}{2}+97} + 2q^{\frac{m}{2}+98} - q^{\frac{m}{2}+99} + q^{m+\frac{205}{2}} - 2q^{m+\frac{207}{2}} \\
&+ q^{m+\frac{209}{2}} - q^{m+\frac{211}{2}} + q^{m+\frac{213}{2}} - 3q^{m+\frac{215}{2}} - q^{m+\frac{217}{2}} - 3q^{m+\frac{221}{2}} - 4q^{m+\frac{223}{2}} - 5q^{m+\frac{225}{2}} \\
&- 7q^{m+\frac{227}{2}} - 4q^{m+\frac{229}{2}} - 3q^{m+\frac{231}{2}} - q^{m+\frac{233}{2}} - 8q^{m+\frac{235}{2}} - 4q^{m+\frac{237}{2}} - 9q^{m+\frac{239}{2}} - 2q^{m+\frac{241}{2}} \\
&- 5q^{m+\frac{243}{2}} + q^{m+\frac{245}{2}} - 3q^{m+\frac{247}{2}} - 2q^{m+\frac{251}{2}} - q^{m+\frac{255}{2}} + 2q^{\frac{3m}{2}+148} + q^{\frac{3m}{2}+149} + 2q^{\frac{3m}{2}+150} \\
&- 2q^{\frac{3m}{2}+151} + 3q^{\frac{3m}{2}+152} - 2q^{\frac{3m}{2}+153} + q^{\frac{3m}{2}+154} + q^{109/2} - 2q^{107/2} - q^{105/2} - 2q^{103/2} \\
&+ 2q^{101/2} - 3q^{99/2} + 2q^{97/2} - q^{95/2} - q^{\frac{3(m+98)}{2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{24}(q,q^m) = &-q^{m+85} - q^{m+86} - q^{m+87} - 2q^{m+88} - q^{m+89} - 4q^{m+90} - 2q^{m+91} - 8q^{m+92} - 3q^{m+93} \\
&- 8q^{m+94} - 2q^{m+95} - 7q^{m+96} - 4q^{m+97} - 11q^{m+98} - 9q^{m+99} - 7q^{m+100} - 4q^{m+101} \\
&- 5q^{m+102} - 6q^{m+103} - 5q^{m+104} - 5q^{m+105} - 3q^{m+106} - 3q^{m+107} - 3q^{m+108} - 2q^{m+110} \\
&+ q^{m+111} - 2q^{m+112} - 2q^{\frac{3m}{2}+\frac{227}{2}} + 3q^{\frac{3m}{2}+\frac{229}{2}} + 2q^{\frac{3m}{2}+\frac{233}{2}} - q^{\frac{3m}{2}+\frac{235}{2}} + q^{\frac{3m}{2}+\frac{239}{2}} \\
&+ 2q^{\frac{3(m+75)}{2}} - q^{\frac{3(m+77)}{2}} + 2q^{\frac{3(m+79)}{2}} + q^{\frac{3(m+81)}{2}} + 2q^{\frac{m+129}{2}} - q^{\frac{m+131}{2}} + 2q^{\frac{m+133}{2}} + 3q^{\frac{m+137}{2}} \\
&+ 3q^{\frac{m+139}{2}} + 3q^{\frac{m+141}{2}} + 5q^{\frac{m+143}{2}} + 5q^{\frac{m+145}{2}} + 6q^{\frac{m+147}{2}} + 5q^{\frac{m+149}{2}} + 4q^{\frac{m+151}{2}} + 7q^{\frac{m+153}{2}} \\
&+ 9q^{\frac{m+155}{2}} + 11q^{\frac{m+157}{2}} + 4q^{\frac{m+159}{2}} + 7q^{\frac{m+161}{2}} + 2q^{\frac{m+163}{2}} + 8q^{\frac{m+165}{2}} + 3q^{\frac{m+167}{2}} + 8q^{\frac{m+169}{2}} \\
&+ 2q^{\frac{m+171}{2}} + 4q^{\frac{m+173}{2}} + q^{\frac{m+175}{2}} + 2q^{\frac{m+177}{2}} + q^{\frac{m+179}{2}} + q^{\frac{m+181}{2}} + q^{\frac{m+183}{2}} - 2q^{64} + 2q^{63} \\
&- 3q^{62} + q^{61} - 2q^{60} + q^{59} - 2q^{58} - q^{57} - q^{55}
\end{aligned}
$$

$$
\begin{aligned}
t_{25}(q,q^m) = \ & q^{\frac{m}{2}+6}\Big(q^{\frac{m}{2}+79} + q^{\frac{m}{2}+80} + q^{\frac{m}{2}+81} + 2q^{\frac{m}{2}+82} + q^{\frac{m}{2}+83} + 4q^{\frac{m}{2}+84} + 2q^{\frac{m}{2}+85} + 8q^{\frac{m}{2}+86} \\
&+ 3q^{\frac{m}{2}+87} + 8q^{\frac{m}{2}+88} + 2q^{\frac{m}{2}+89} + 7q^{\frac{m}{2}+90} + 4q^{\frac{m}{2}+91} + 11q^{\frac{m}{2}+92} + 9q^{\frac{m}{2}+93} + 7q^{\frac{m}{2}+94} \\
&+ 4q^{\frac{m}{2}+95} + 5q^{\frac{m}{2}+96} + 6q^{\frac{m}{2}+97} + 5q^{\frac{m}{2}+98} + 5q^{\frac{m}{2}+99} + 3q^{\frac{m}{2}+100} + 3q^{\frac{m}{2}+101} + 3q^{\frac{m}{2}+102} \\
&+ 2q^{\frac{m}{2}+104} - q^{\frac{m}{2}+105} + 2q^{\frac{m}{2}+106} - 2q^{m+\frac{215}{2}} + q^{m+\frac{217}{2}} - 2q^{m+\frac{219}{2}} - 3q^{m+\frac{223}{2}} - 3q^{m+\frac{225}{2}} \\
&- 3q^{m+\frac{227}{2}} - 5q^{m+\frac{229}{2}} - 5q^{m+\frac{231}{2}} - 6q^{m+\frac{233}{2}} - 5q^{m+\frac{235}{2}} - 4q^{m+\frac{237}{2}} - 7q^{m+\frac{239}{2}} - 9q^{m+\frac{241}{2}} \\
&- 11q^{m+\frac{243}{2}} - 4q^{m+\frac{245}{2}} - 7q^{m+\frac{247}{2}} - 2q^{m+\frac{249}{2}} - 8q^{m+\frac{251}{2}} - 3q^{m+\frac{253}{2}} - 8q^{m+\frac{255}{2}} \\
&- 2q^{m+\frac{257}{2}} - 4q^{m+\frac{259}{2}} - q^{m+\frac{261}{2}} - 2q^{m+\frac{263}{2}} - q^{m+\frac{265}{2}} - q^{m+\frac{267}{2}} - q^{m+\frac{269}{2}} + q^{\frac{3m}{2}+149} \\
&+ 2q^{\frac{3m}{2}+150} - q^{\frac{3m}{2}+151} + 2q^{\frac{3m}{2}+152} - q^{\frac{3m}{2}+153} + 3q^{\frac{3m}{2}+154} - 2q^{\frac{3m}{2}+155} + 2q^{\frac{3m}{2}+156} \\
&- q^{133/2} - q^{129/2} - 2q^{127/2} + q^{125/2} - 2q^{123/2} + q^{121/2} - 3q^{119/2} + 2q^{117/2} - 2q^{115/2} \\
&+ q^{\frac{3(m+98)}{2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{26}(q,q^m) =\ & q^{123/2}\Big(-q^{\frac{m}{2}+5} + 2q^{\frac{m}{2}+6} - q^{\frac{m}{2}+7} + q^{\frac{m}{2}+8} - q^{\frac{m}{2}+9} + 3q^{\frac{m}{2}+10} + q^{\frac{m}{2}+11} + 3q^{\frac{m}{2}+13} \\
& + 4q^{\frac{m}{2}+14} + 5q^{\frac{m}{2}+15} + 7q^{\frac{m}{2}+16} + 4q^{\frac{m}{2}+17} + 3q^{\frac{m}{2}+18} + q^{\frac{m}{2}+19} + 8q^{\frac{m}{2}+20} + 4q^{\frac{m}{2}+21} \\
& + 9q^{\frac{m}{2}+22} + 2q^{\frac{m}{2}+23} + 5q^{\frac{m}{2}+24} - q^{\frac{m}{2}+25} + 3q^{\frac{m}{2}+26} + 2q^{\frac{m}{2}+28} + q^{\frac{m}{2}+30} - q^{m+\frac{51}{2}} \\
& - 2q^{m+\frac{55}{2}} - 3q^{m+\frac{59}{2}} + q^{m+\frac{61}{2}} - 5q^{m+\frac{63}{2}} - 2q^{m+\frac{65}{2}} - 9q^{m+\frac{67}{2}} - 4q^{m+\frac{69}{2}} - 8q^{m+\frac{71}{2}} \\
& - q^{m+\frac{73}{2}} - 3q^{m+\frac{75}{2}} - 4q^{m+\frac{77}{2}} - 7q^{m+\frac{79}{2}} - 5q^{m+\frac{81}{2}} - 4q^{m+\frac{83}{2}} - 3q^{m+\frac{85}{2}} - q^{m+\frac{89}{2}} \\
& - 3q^{m+\frac{91}{2}} + q^{m+\frac{93}{2}} - q^{m+\frac{95}{2}} + q^{m+\frac{97}{2}} - 2q^{m+\frac{99}{2}} + q^{m+\frac{101}{2}} + q^{\frac{3m}{2}+48} - 2q^{\frac{3m}{2}+49} \\
& + 3q^{\frac{3m}{2}+50} - 2q^{\frac{3m}{2}+51} + 2q^{\frac{3m}{2}+52} + q^{\frac{3m}{2}+53} + 2q^{\frac{3m}{2}+54} - q^{\frac{3m}{2}+55} - q^{15/2} + 2q^{13/2} \\
& - 3q^{11/2} + 2q^{9/2} - 2q^{7/2} - q^{5/2} - 2q^{3/2} + \sqrt{q}\Big) \\
t_{27}(q,q^m) =\ & q^{\frac{m}{2}+1}\Big(-q^{\frac{m}{2}+84} + q^{\frac{m}{2}+85} - q^{\frac{m}{2}+87} + q^{\frac{m}{2}+88} - q^{\frac{m}{2}+89} - 5q^{\frac{m}{2}+91} + q^{\frac{m}{2}+92} - 4q^{\frac{m}{2}+93} \\
& + 2q^{\frac{m}{2}+94} - 3q^{\frac{m}{2}+95} - q^{\frac{m}{2}+96} - 7q^{\frac{m}{2}+97} - 6q^{\frac{m}{2}+98} - 2q^{\frac{m}{2}+99} - 2q^{\frac{m}{2}+102} - 2q^{\frac{m}{2}+104} \\
& - q^{\frac{m}{2}+105} - q^{\frac{m}{2}+106} + q^{\frac{m}{2}+108} - q^{\frac{m}{2}+109} + q^{\frac{m}{2}+110} - 2q^{\frac{m}{2}+111} + q^{\frac{m}{2}+112} - q^{m+\frac{219}{2}} \\
& + 2q^{m+\frac{221}{2}} - q^{m+\frac{223}{2}} + q^{m+\frac{225}{2}} - q^{m+\frac{227}{2}} + q^{m+\frac{231}{2}} + q^{m+\frac{233}{2}} + 2q^{m+\frac{235}{2}} + 2q^{m+\frac{239}{2}} \\
& + 2q^{m+\frac{245}{2}} + 6q^{m+\frac{247}{2}} + 7q^{m+\frac{249}{2}} + q^{m+\frac{251}{2}} + 3q^{m+\frac{253}{2}} - 2q^{m+\frac{255}{2}} + 4q^{m+\frac{257}{2}} - q^{m+\frac{259}{2}} \\
& + 5q^{m+\frac{261}{2}} + q^{m+\frac{265}{2}} - q^{m+\frac{267}{2}} + q^{m+\frac{269}{2}} - q^{m+\frac{273}{2}} + q^{m+\frac{275}{2}} - q^{\frac{3m}{2}+148} + q^{\frac{3m}{2}+149} \\
& - q^{\frac{3m}{2}+150} - 2q^{\frac{3m}{2}+151} + 2q^{\frac{3m}{2}+152} - 2q^{\frac{3m}{2}+153} + q^{\frac{3m}{2}+154} - 2q^{\frac{3m}{2}+155} + 2q^{\frac{3m}{2}+156} \\
& - q^{\frac{3m}{2}+157} - q^{149/2} + q^{147/2} - q^{145/2} + q^{143/2} + 2q^{141/2} - 2q^{139/2} + 2q^{137/2} - q^{135/2} \\
& + 2q^{133/2} - 2q^{131/2} + q^{129/2} + q^{\frac{3(m+98)}{2}}\Big) \\
t_{28}(q,q^m) =\ & q^{133/2}\Big(-q^{\frac{m}{2}+3} + q^{\frac{m}{2}+4} - 2q^{\frac{m}{2}+5} - q^{\frac{m}{2}+6} - 3q^{\frac{m}{2}+7} - q^{\frac{m}{2}+8} - 3q^{\frac{m}{2}+10} - 3q^{\frac{m}{2}+11} \\
& - 4q^{\frac{m}{2}+12} - 3q^{\frac{m}{2}+13} - 2q^{\frac{m}{2}+14} - q^{\frac{m}{2}+15} + q^{\frac{m}{2}+16} - 4q^{\frac{m}{2}+17} - 3q^{\frac{m}{2}+18} - 6q^{\frac{m}{2}+19} \\
& - q^{\frac{m}{2}+20} - 4q^{\frac{m}{2}+21} + q^{\frac{m}{2}+22} - 3q^{\frac{m}{2}+23} - q^{\frac{m}{2}+24} - 2q^{\frac{m}{2}+25} - q^{\frac{m}{2}+26} - q^{\frac{m}{2}+27} \\
& + q^{m+\frac{41}{2}} + q^{m+\frac{43}{2}} + 2q^{m+\frac{45}{2}} + q^{m+\frac{47}{2}} + 3q^{m+\frac{49}{2}} - q^{m+\frac{51}{2}} + 4q^{m+\frac{53}{2}} + q^{m+\frac{55}{2}} \\
& + 6q^{m+\frac{57}{2}} + 3q^{m+\frac{59}{2}} + 4q^{m+\frac{61}{2}} - q^{m+\frac{63}{2}} + q^{m+\frac{65}{2}} + 2q^{m+\frac{67}{2}} + 3q^{m+\frac{69}{2}} + 4q^{m+\frac{71}{2}} \\
& + 3q^{m+\frac{73}{2}} + 3q^{m+\frac{75}{2}} + q^{m+\frac{79}{2}} + 3q^{m+\frac{81}{2}} + q^{m+\frac{83}{2}} + 2q^{m+\frac{85}{2}} - q^{m+\frac{87}{2}} + q^{m+\frac{89}{2}} - q^{\frac{3m}{2}+41} \\
& + q^{\frac{3m}{2}+42} - 2q^{\frac{3m}{2}+43} + 2q^{\frac{3m}{2}+44} - q^{\frac{3m}{2}+45} - q^{\frac{3m}{2}+46} - q^{\frac{3m}{2}+47} + q^{13/2} - q^{11/2}
\end{aligned}
$$

$$+ 2q^{9/2} - 2q^{7/2} + q^{5/2} + q^{3/2} + \sqrt{q}\Big)$$

$$t_{29}(q,q^m) = -q^{\frac{m+137}{2}}\Big(- q^{m+40} - 2q^{m+42} - 2q^{m+44} - 4q^{m+45} - 3q^{m+46} - 5q^{m+47} - 3q^{m+48}$$
$$- 8q^{m+49} - 6q^{m+50} - 11q^{m+51} - 9q^{m+52} - 11q^{m+53} - 6q^{m+54} - 6q^{m+55} - 11q^{m+56}$$
$$- 9q^{m+57} - 11q^{m+58} - 6q^{m+59} - 8q^{m+60} - 3q^{m+61} - 5q^{m+62} - 3q^{m+63} - 4q^{m+64}$$
$$- 2q^{m+65} - 2q^{m+67} - q^{m+69} - q^{\frac{3m}{2}+\frac{149}{2}} + 2q^{\frac{3m}{2}+\frac{151}{2}} + 3q^{\frac{3m}{2}+\frac{155}{2}} + 3q^{\frac{3m}{2}+\frac{161}{2}}$$
$$- q^{\frac{3m}{2}+\frac{167}{2}} + q^{\frac{3m}{2}+\frac{169}{2}} + q^{\frac{m+31}{2}} + 2q^{\frac{m+35}{2}} + 2q^{\frac{m+39}{2}} + 4q^{\frac{m+41}{2}} + 3q^{\frac{m+43}{2}} + 5q^{\frac{m+45}{2}}$$
$$+ 3q^{\frac{m+47}{2}} + 8q^{\frac{m+49}{2}} + q^{\frac{3(m+49)}{2}} + 6q^{\frac{m+51}{2}} + 11q^{\frac{m+53}{2}} + 9q^{\frac{m+55}{2}} + 2q^{\frac{3(m+55)}{2}} + 11q^{\frac{m+57}{2}}$$
$$+ 6q^{\frac{m+59}{2}} + 6q^{\frac{m+61}{2}} + 11q^{\frac{m+63}{2}} + 9q^{\frac{m+65}{2}} + 11q^{\frac{m+67}{2}} + 6q^{\frac{m+69}{2}} + 8q^{\frac{m+71}{2}} + 3q^{\frac{m+73}{2}}$$
$$+ 5q^{\frac{m+75}{2}} + 3q^{\frac{m+77}{2}} + 4q^{\frac{m+79}{2}} + 2q^{\frac{m+81}{2}} + 2q^{\frac{m+85}{2}} + q^{\frac{m+89}{2}} - q^{11} + q^{10} - 2q^{9} - 3q^{7}$$
$$- 3q^{4} - 2q^{2} + q - 1\Big)$$

$$t_{30}(q,q^m) = q^{147/2}\Big(- q^{\frac{m}{2}+1} - 2q^{\frac{m}{2}+3} - 3q^{\frac{m}{2}+4} - q^{\frac{m}{2}+5} - 2q^{\frac{m}{2}+6} - q^{\frac{m}{2}+7} - 3q^{\frac{m}{2}+8} - 3q^{\frac{m}{2}+9}$$
$$- 7q^{\frac{m}{2}+10} - 7q^{\frac{m}{2}+11} - 2q^{\frac{m}{2}+12} - 3q^{\frac{m}{2}+13} - q^{\frac{m}{2}+14} - 3q^{\frac{m}{2}+15} - q^{\frac{m}{2}+16} - 4q^{\frac{m}{2}+17}$$
$$- q^{\frac{m}{2}+18} - 2q^{\frac{m}{2}+19} + q^{m/2} + 2q^{m+\frac{33}{2}} + q^{m+\frac{35}{2}} + 4q^{m+\frac{37}{2}} + q^{m+\frac{39}{2}} + 3q^{m+\frac{41}{2}} + q^{m+\frac{43}{2}}$$
$$+ 3q^{m+\frac{45}{2}} + 2q^{m+\frac{47}{2}} + 7q^{m+\frac{49}{2}} + 7q^{m+\frac{51}{2}} + 3q^{m+\frac{53}{2}} + 3q^{m+\frac{55}{2}} + q^{m+\frac{57}{2}} + 2q^{m+\frac{59}{2}}$$
$$+ q^{m+\frac{61}{2}} + 3q^{m+\frac{63}{2}} + 2q^{m+\frac{65}{2}} + q^{m+\frac{69}{2}} - q^{m+\frac{71}{2}} - 2q^{\frac{3m}{2}+33} + q^{\frac{3m}{2}+34} - q^{\frac{3m}{2}+35} + 2q^{5/2}$$
$$- q^{3/2} + \sqrt{q}\Big)$$

$$t_{31}(q,q^m) = q^{\frac{m+145}{2}}\Big(q^{m+35} - q^{m+36} + q^{m+38} - q^{m+39} + q^{m+40} + 5q^{m+42} - q^{m+43} + 4q^{m+44} - 2q^{m+45}$$
$$+ 3q^{m+46} + q^{m+47} + 7q^{m+48} + 6q^{m+49} + 2q^{m+50} + 2q^{m+53} + 2q^{m+55} + q^{m+56} + q^{m+57}$$
$$- q^{m+59} + q^{m+60} - q^{m+61} + 2q^{m+62} - q^{m+63} + 2q^{\frac{3m}{2}+\frac{131}{2}} - 2q^{\frac{3m}{2}+\frac{133}{2}} - 2q^{\frac{3m}{2}+\frac{137}{2}}$$
$$+ 2q^{\frac{3m}{2}+\frac{139}{2}} - q^{\frac{3m}{2}+\frac{143}{2}} + q^{\frac{3m}{2}+\frac{145}{2}} + q^{\frac{3m}{2}+\frac{149}{2}} + q^{\frac{m+23}{2}} - 2q^{\frac{m+25}{2}} + q^{\frac{m+27}{2}} - q^{\frac{m+29}{2}}$$
$$+ q^{\frac{m+31}{2}} - q^{\frac{m+35}{2}} - q^{\frac{m+37}{2}} - 2q^{\frac{m+39}{2}} - 2q^{\frac{m+43}{2}} - q^{\frac{3(m+43)}{2}} + q^{\frac{3(m+45)}{2}} - 2q^{\frac{3(m+47)}{2}}$$
$$- 2q^{\frac{m+49}{2}} - q^{\frac{3(m+49)}{2}} - 6q^{\frac{m+51}{2}} - 7q^{\frac{m+53}{2}} - q^{\frac{m+55}{2}} - 3q^{\frac{m+57}{2}} + 2q^{\frac{m+59}{2}} - 4q^{\frac{m+61}{2}} + q^{\frac{m+63}{2}}$$

$$
\begin{aligned}
&- 5q^{\frac{m+65}{2}} - q^{\frac{m+69}{2}} + q^{\frac{m+71}{2}} - q^{\frac{m+73}{2}} + q^{\frac{m+77}{2}} - q^{\frac{m+79}{2}} + q^{10} - 2q^9 + 2q^8 - q^7 + 2q^6 \\
&- 2q^5 + 2q^4 + q^3 - q^2 + q - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{32}(q,q^m) = &-q^{m+90} - q^{m+91} - 2q^{m+92} + q^{m+97} - 2q^{m+98} - 3q^{m+99} - q^{m+101} + 3q^{m+102} - q^{m+103} \\
&- 2q^{m+105} - q^{m+106} - q^{m+108} + q^{\frac{3m}{2}+\frac{209}{2}} + q^{\frac{m+153}{2}} + q^{\frac{m+157}{2}} + 2q^{\frac{m+159}{2}} + q^{\frac{m+163}{2}} \\
&- 3q^{\frac{m+165}{2}} + q^{\frac{m+167}{2}} + 3q^{\frac{m+171}{2}} + 2q^{\frac{m+173}{2}} - q^{\frac{m+175}{2}} + 2q^{\frac{m+185}{2}} + q^{\frac{m+187}{2}} + q^{\frac{m+189}{2}} - q^{80}
\end{aligned}
$$

$$
\begin{aligned}
t_{33}(q,q^m) = &\ q^{\frac{m+151}{2}}\Big(- q^{m+30} - q^{m+31} - q^{m+32} - 2q^{m+33} - q^{m+34} - 4q^{m+35} - 2q^{m+36} - 8q^{m+37} \\
&- 3q^{m+38} - 8q^{m+39} - 2q^{m+40} - 7q^{m+41} - 4q^{m+42} - 11q^{m+43} - 9q^{m+44} - 7q^{m+45} \\
&- 4q^{m+46} - 5q^{m+47} - 6q^{m+48} - 5q^{m+49} - 5q^{m+50} - 3q^{m+51} - 3q^{m+52} - 3q^{m+53} \\
&- 2q^{m+55} + q^{m+56} - 2q^{m+57} + 2q^{\frac{3m}{2}+\frac{115}{2}} + 3q^{\frac{3m}{2}+\frac{119}{2}} - q^{\frac{3m}{2}+\frac{121}{2}} - q^{\frac{3m}{2}+\frac{125}{2}} + 2q^{\frac{3m}{2}+\frac{127}{2}} \\
&+ q^{\frac{3m}{2}+\frac{133}{2}} + 2q^{\frac{m+19}{2}} - q^{\frac{m+21}{2}} + 2q^{\frac{m+23}{2}} + 3q^{\frac{m+27}{2}} + 3q^{\frac{m+29}{2}} + 3q^{\frac{m+31}{2}} + 5q^{\frac{m+33}{2}} \\
&+ 5q^{\frac{m+35}{2}} + 6q^{\frac{m+37}{2}} + 5q^{\frac{m+39}{2}} - 2q^{\frac{3(m+39)}{2}} + 4q^{\frac{m+41}{2}} + 2q^{\frac{3(m+41)}{2}} + 7q^{\frac{m+43}{2}} + q^{\frac{3(m+43)}{2}} \\
&+ 9q^{\frac{m+45}{2}} + 11q^{\frac{m+47}{2}} + 4q^{\frac{m+49}{2}} + 7q^{\frac{m+51}{2}} + 2q^{\frac{m+53}{2}} + 8q^{\frac{m+55}{2}} + 3q^{\frac{m+57}{2}} + 8q^{\frac{m+59}{2}} \\
&+ 2q^{\frac{m+61}{2}} + 4q^{\frac{m+63}{2}} + q^{\frac{m+65}{2}} + 2q^{\frac{m+67}{2}} + q^{\frac{m+69}{2}} + q^{\frac{m+71}{2}} + q^{\frac{m+73}{2}} - 2q^9 + 2q^8 - 3q^7 \\
&+ q^6 - 2q^5 + q^4 - 2q^3 - q^2 - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{34}(q,q^m) = &\ q^{\frac{m+163}{2}}\Big(- q^{\frac{m+23}{2}} - 3q^{\frac{m+25}{2}} - 2q^{\frac{m+27}{2}} - q^{\frac{m+29}{2}} - 2q^{\frac{m+31}{2}} - q^{\frac{m+33}{2}} + q^{\frac{m+35}{2}} - 2q^{\frac{m+37}{2}} \\
&- 4q^{\frac{m+41}{2}} - 2q^{\frac{m+43}{2}} - 2q^{\frac{m+45}{2}} + q^{\frac{m+47}{2}} + q^{12} + 3q^{11} + 2q^{10} + q^9 + 2q^8 + q^7 - q^6 + 2q^5 \\
&+ 4q^3 + 2q^2 + 2q - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{35}(q,q^m) = &-q^{\frac{m+161}{2}}\Big(q^{m+25} + 2q^{m+27} + 3q^{m+29} - q^{m+30} + 5q^{m+31} + 2q^{m+32} + 9q^{m+33} + 4q^{m+34} \\
&+ 8q^{m+35} + q^{m+36} + 3q^{m+37} + 4q^{m+38} + 7q^{m+39} + 5q^{m+40} + 4q^{m+41} + 3q^{m+42} \\
&+ q^{m+44} + 3q^{m+45} - q^{m+46} + q^{m+47} - q^{m+48} + 2q^{m+49} - q^{m+50} - q^{\frac{3m}{2}+\frac{95}{2}} + 2q^{\frac{3m}{2}+\frac{97}{2}} \\
&+ 2q^{\frac{3m}{2}+\frac{101}{2}} - 2q^{\frac{3m}{2}+\frac{103}{2}} - 2q^{\frac{3m}{2}+\frac{107}{2}} + q^{\frac{3m}{2}+\frac{109}{2}} + q^{\frac{m+9}{2}} - 2q^{\frac{m+11}{2}} + q^{\frac{m+13}{2}} - q^{\frac{m+15}{2}} \\
&+ q^{\frac{m+17}{2}} - 3q^{\frac{m+19}{2}} - q^{\frac{m+21}{2}} - 3q^{\frac{m+25}{2}} - 4q^{\frac{m+27}{2}} - 5q^{\frac{m+29}{2}} - 7q^{\frac{m+31}{2}} - 4q^{\frac{m+33}{2}} - 3q^{\frac{3(m+33)}{2}} \\
&- 3q^{\frac{m+35}{2}} - q^{\frac{3(m+35)}{2}} - q^{\frac{m+37}{2}} - 8q^{\frac{m+39}{2}} - 4q^{\frac{m+41}{2}} - 9q^{\frac{m+43}{2}} - 2q^{\frac{m+45}{2}} - 5q^{\frac{m+47}{2}} \\
&+ q^{\frac{m+49}{2}} - 3q^{\frac{m+51}{2}} - 2q^{\frac{m+55}{2}} - q^{\frac{m+59}{2}} + q^7 - 2q^6 + 3q^5 - 2q^4 + 2q^3 + q^2 + 2q - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{36}(q,q^m) =&\ q^{\frac{m+169}{2}}\Big(q^{\frac{m+19}{2}}+q^{\frac{m+21}{2}}-q^{\frac{m+25}{2}}-q^{\frac{m+29}{2}}+q^{\frac{m+31}{2}}-2q^{\frac{m+33}{2}}+q^{\frac{m+35}{2}}+q^{\frac{m+37}{2}}+q^{\frac{m+39}{2}}\\
&-q^{10}-q^9+q^7+q^5-q^4+2q^3-q^2-q-1\Big)\\
t_{37}(q,q^m) =& -q^{\frac{m+167}{2}}\Big(-q^{m+20}-q^{m+21}-2q^{m+22}-q^{m+23}-3q^{m+24}+q^{m+25}-4q^{m+26}-q^{m+27}\\
&-6q^{m+28}-3q^{m+29}-4q^{m+30}+q^{m+31}-q^{m+32}-2q^{m+33}-3q^{m+34}-4q^{m+35}\\
&-3q^{m+36}-3q^{m+37}-q^{m+39}-3q^{m+40}-q^{m+41}-2q^{m+42}+q^{m+43}-q^{m+44}-q^{\frac{3m}{2}+\frac{83}{2}}\\
&+2q^{\frac{3m}{2}+\frac{85}{2}}+q^{\frac{3m}{2}+\frac{89}{2}}+q^{\frac{3m}{2}+\frac{91}{2}}+q^{\frac{m+5}{2}}-q^{\frac{m+7}{2}}+2q^{\frac{m+9}{2}}+q^{\frac{m+11}{2}}+3q^{\frac{m+13}{2}}+q^{\frac{m+15}{2}}\\
&+3q^{\frac{m+19}{2}}+3q^{\frac{m+21}{2}}+4q^{\frac{m+23}{2}}+3q^{\frac{m+25}{2}}+2q^{\frac{m+27}{2}}+q^{\frac{3(m+27)}{2}}+q^{\frac{m+29}{2}}-2q^{\frac{3(m+29)}{2}}\\
&-q^{\frac{m+31}{2}}+q^{\frac{3(m+31)}{2}}+4q^{\frac{m+33}{2}}+3q^{\frac{m+35}{2}}+6q^{\frac{m+37}{2}}+q^{\frac{m+39}{2}}+4q^{\frac{m+41}{2}}-q^{\frac{m+43}{2}}+3q^{\frac{m+45}{2}}\\
&+q^{\frac{m+47}{2}}+2q^{\frac{m+49}{2}}+q^{\frac{m+51}{2}}+q^{\frac{m+53}{2}}-q^6+q^5-2q^4+2q^3-q^2-q-1\Big)\\
t_{38}(q,q^m) =&\ q^{\frac{m+179}{2}}\Big(q^{\frac{m+15}{2}}+q^{\frac{m+17}{2}}+2q^{\frac{m+19}{2}}-q^{\frac{m+21}{2}}+q^{\frac{m+23}{2}}-q^4-q^3-2q^2+q-1\Big)\\
t_{39}(q,q^m) =&\ q^{\frac{m}{2}+88}\Big(-q^{\frac{m}{2}+1}-2q^{\frac{m}{2}+3}-3q^{\frac{m}{2}+4}-q^{\frac{m}{2}+5}-2q^{\frac{m}{2}+6}-q^{\frac{m}{2}+7}-3q^{\frac{m}{2}+8}-3q^{\frac{m}{2}+9}\\
&-7q^{\frac{m}{2}+10}-7q^{\frac{m}{2}+11}-2q^{\frac{m}{2}+12}-3q^{\frac{m}{2}+13}-q^{\frac{m}{2}+14}-3q^{\frac{m}{2}+15}-q^{\frac{m}{2}+16}-4q^{\frac{m}{2}+17}\\
&-q^{\frac{m}{2}+18}-2q^{\frac{m}{2}+19}+q^{m/2}+2q^{m+\frac{33}{2}}+q^{m+\frac{35}{2}}+4q^{m+\frac{37}{2}}+q^{m+\frac{39}{2}}+3q^{m+\frac{41}{2}}+q^{m+\frac{43}{2}}\\
&+3q^{m+\frac{45}{2}}+2q^{m+\frac{47}{2}}+7q^{m+\frac{49}{2}}+7q^{m+\frac{51}{2}}+3q^{m+\frac{53}{2}}+3q^{m+\frac{55}{2}}+q^{m+\frac{57}{2}}+2q^{m+\frac{59}{2}}\\
&+q^{m+\frac{61}{2}}+3q^{m+\frac{63}{2}}+2q^{m+\frac{65}{2}}+q^{m+\frac{69}{2}}-q^{m+\frac{71}{2}}-2q^{\frac{3m}{2}+33}+q^{\frac{3m}{2}+34}-q^{\frac{3m}{2}+35}+2q^{5/2}\\
&-q^{3/2}+\sqrt{q}\Big)\\
t_{40}(q,q^m) =&\ q\Big(q^{\frac{m+185}{2}}-q^{m+98}\Big)\\
t_{41}(q,q^m) =&\ q^{177/2}\Big(-q^{\frac{m}{2}+4}+q^{m+\frac{1}{2}}+q^{m+\frac{5}{2}}+2q^{m+\frac{7}{2}}+q^{m+\frac{11}{2}}-3q^{m+\frac{13}{2}}+q^{m+\frac{15}{2}}+3q^{m+\frac{19}{2}}\\
&+2q^{m+\frac{21}{2}}-q^{m+\frac{23}{2}}+2q^{m+\frac{33}{2}}+q^{m+\frac{35}{2}}+q^{m+\frac{37}{2}}-q^{\frac{3m}{2}+14}-q^{\frac{3m}{2}+15}-2q^{\frac{3m}{2}+16}\\
&+q^{\frac{3m}{2}+21}-2q^{\frac{3m}{2}+22}-3q^{\frac{3m}{2}+23}-q^{\frac{3m}{2}+25}+3q^{\frac{3m}{2}+26}-q^{\frac{3m}{2}+27}-2q^{\frac{3m}{2}+29}-q^{\frac{3m}{2}+30}\\
&-q^{\frac{3m}{2}+32}+q^{2m+\frac{57}{2}}\Big)\\
t_{43}(q,q^m) =&\ q^{m+54}\Big(-q^{\frac{m+99}{2}}-3q^{\frac{m+101}{2}}-2q^{\frac{m+103}{2}}-q^{\frac{m+105}{2}}-2q^{\frac{m+107}{2}}-q^{\frac{m+109}{2}}+q^{\frac{m+111}{2}}\\
&-2q^{\frac{m+113}{2}}-4q^{\frac{m+117}{2}}-2q^{\frac{m+119}{2}}-2q^{\frac{m+121}{2}}+q^{\frac{m+123}{2}}+q^{50}+3q^{49}+2q^{48}+q^{47}+2q^{46}\\
&+q^{45}-q^{44}+2q^{43}+4q^{41}+2q^{40}+2q^{39}-q^{38}\Big)
\end{aligned}
$$

$$
\begin{aligned}
t_{45}(q,q^m) =&\ q^{m+53}\Big(q^{\frac{m+99}{2}}+q^{\frac{m+101}{2}}-q^{\frac{m+105}{2}}-q^{\frac{m+109}{2}}+q^{\frac{m+111}{2}}-2q^{\frac{m+113}{2}}+q^{\frac{m+115}{2}}+q^{\frac{m+117}{2}}\\
&+q^{\frac{m+119}{2}}-q^{50}-q^{49}+q^{47}+q^{45}-q^{44}+2q^{43}-q^{42}-q^{41}-q^{40}\Big)\\
t_{47}(q,q^m) =&\ q^{m+96}\Big(q^{\frac{m+15}{2}}+q^{\frac{m+17}{2}}+2q^{\frac{m+19}{2}}-q^{\frac{m+21}{2}}+q^{\frac{m+23}{2}}-q^4-q^3-2q^2+q-1\Big)\\
t_{49}(q,q^m) =&\ q^{m+98}-q^{\frac{3(m+69)}{2}}.
\end{aligned}
$$

C The Initial data

We record the initial data $\{f_m(q)\in\mathbb{Z}[q^{\pm 1}]\,|\,m=1,\cdots,97\}$ for the f_m-recursion relation (2.6).

$$f_1(q)=f_3(q)=f_5(q)=f_7(q)=f_9(q)=0$$

$$f_{11}(q)=\ q^5 \qquad f_{13}(q)=0 \qquad f_{15}(q)=\ 2q^6 \qquad f_{17}(q)=0$$

$$f_{19}(q)=\ q^6+3q^7+q^8 \qquad f_{21}(q)=0$$

$$f_{23}(q)=\ 2q^6+2q^7+5q^8+2q^9+2q^{10} \qquad f_{25}(q)=0$$

$$
\begin{aligned}
f_{27}(q) =&\ q^5+3q^6+4q^7+5q^8+8q^9+5q^{10}+4q^{11}+3q^{12}+q^{13}\\
f_{29}(q) =& -q^{15}\\
f_{31}(q) =&\ 2q^{16}+2q^{15}+6q^{14}+7q^{13}+10q^{12}+10q^{11}+15q^{10}+10q^9+10q^8+7q^7+6q^6+2q^5+2q^4\\
f_{33}(q) =& -2q^{18}\\
f_{35}(q) =&\ q^{20}+3q^{19}+4q^{18}+7q^{17}+11q^{16}+15q^{15}+18q^{14}+21q^{13}+23q^{12}+27q^{11}+23q^{10}+21q^9\\
&+18q^8+15q^7+11q^6+7q^5+4q^4+3q^3+q^2\\
f_{37}(q) =& -q^{22}-3q^{21}-q^{20}\\
f_{39}(q) =&\ 2q^{24}+2q^{23}+6q^{22}+8q^{21}+13q^{20}+16q^{19}+26q^{18}+29q^{17}+38q^{16}+41q^{15}+48q^{14}\\
&+48q^{13}+56q^{12}+48q^{11}+48q^{10}+41q^9+38q^8+29q^7+26q^6+16q^5+13q^4+8q^3+6q^2\\
&+2q+2\\
f_{41}(q) =& -2q^{26}-2q^{25}-5q^{24}-2q^{23}-2q^{22}
\end{aligned}
$$

$$
\begin{aligned}
f_{43}(q) = {} & q^{29} + 3q^{28} + 4q^{27} + 7q^{26} + 13q^{25} + 17q^{24} + 25q^{23} + 33q^{22} + 43q^{21} + 54q^{20} + 67q^{19} + 77q^{18} \\
& + 88q^{17} + 97q^{16} + 104q^{15} + 108q^{14} + 115q^{13} + 108q^{12} + 104q^{11} + 97q^{10} + 88q^{9} + 77q^{8} \\
& + 67q^{7} + 54q^{6} + 43q^{5} + 33q^{4} + 25q^{3} + 17q^{2} + 13q + 7 + \frac{4}{q} + \frac{3}{q^2} + \frac{1}{q^3} \\
f_{45}(q) = {} & -q^{31} - 3q^{30} - 4q^{29} - 5q^{28} - 8q^{27} - 5q^{26} - 4q^{25} - 3q^{24} - q^{23} \\
f_{47}(q) = {} & 3q^{34} + 2q^{33} + 6q^{32} + 8q^{31} + 14q^{30} + 19q^{29} + 30q^{28} + 38q^{27} + 55q^{26} + 66q^{25} + 87q^{24} \\
& + 102q^{23} + 129q^{22} + 145q^{21} + 172q^{20} + 186q^{19} + 210q^{18} + 219q^{17} + 237q^{16} + 238q^{15} \\
& + 251q^{14} + 238q^{13} + 237q^{12} + 219q^{11} + 210q^{10} + 186q^{9} + 172q^{8} + 145q^{7} + 129q^{6} + 102q^{5} \\
& + 87q^{4} + 66q^{3} + 55q^{2} + 38q + 30 + \frac{19}{q} + \frac{14}{q^2} + \frac{8}{q^3} + \frac{6}{q^4} + \frac{2}{q^5} + \frac{2}{q^6} \\
f_{49}(q) = {} & -2q^{36} - 2q^{35} - 6q^{34} - 7q^{33} - 10q^{32} - 10q^{31} - 15q^{30} - 10q^{29} - 10q^{28} - 7q^{27} - 6q^{26} - 2q^{25} \\
& - 2q^{24} \\
f_{51}(q) = {} & q^{40} + 5q^{39} + 4q^{38} + 7q^{37} + 13q^{36} + 19q^{35} + 27q^{34} + 39q^{33} + 52q^{32} + 72q^{31} + 93q^{30} + 118q^{29} \\
& + 146q^{28} + 182q^{27} + 214q^{26} + 254q^{25} + 296q^{24} + 338q^{23} + 377q^{22} + 419q^{21} + 452q^{20} \\
& + 486q^{19} + 511q^{18} + 531q^{17} + 542q^{16} + 554q^{15} + 542q^{14} + 531q^{13} + 511q^{12} + 486q^{11} \\
& + 452q^{10} + 419q^{9} + 377q^{8} + 338q^{7} + 296q^{6} + 254q^{5} + 214q^{4} + 182q^{3} + 146q^{2} + 118q \\
& + 93 + \frac{72}{q} + \frac{52}{q^2} + \frac{39}{q^3} + \frac{27}{q^4} + \frac{19}{q^5} + \frac{13}{q^6} + \frac{7}{q^7} + \frac{4}{q^8} + \frac{3}{q^9} + \frac{1}{q^{10}} \\
f_{53}(q) = {} & -q^{42} - 3q^{41} - 4q^{40} - 7q^{39} - 11q^{38} - 15q^{37} - 18q^{36} - 21q^{35} - 23q^{34} - 27q^{33} - 23q^{32} \\
& - 21q^{31} - 18q^{30} - 15q^{29} - 11q^{28} - 7q^{27} - 4q^{26} - 3q^{25} - q^{24} \\
f_{55}(q) = {} & 2q^{46} + 3q^{45} + 9q^{44} + 9q^{43} + 14q^{42} + 20q^{41} + 33q^{40} + 42q^{39} + 62q^{38} + 81q^{37} + 110q^{36} \\
& + 137q^{35} + 182q^{34} + 219q^{33} + 277q^{32} + 329q^{31} + 399q^{30} + 460q^{29} + 545q^{28} + 612q^{27} \\
& + 705q^{26} + 778q^{25} + 869q^{24} + 935q^{23} + 1022q^{22} + 1073q^{21} + 1143q^{20} + 1178q^{19} + 1224q^{18} \\
& + 1232q^{17} + 1259q^{16} + 1232q^{15} + 1224q^{14} + 1178q^{13} + 1143q^{12} + 1073q^{11} + 1022q^{10} \\
& + 935q^{9} + 869q^{8} + 778q^{7} + 705q^{6} + 612q^{5} + 545q^{4} + 460q^{3} + 399q^{2} + 329q + 277 + \frac{219}{q} \\
& + \frac{182}{q^2} + \frac{137}{q^3} + \frac{110}{q^4} + \frac{81}{q^5} + \frac{62}{q^6} + \frac{42}{q^7} + \frac{33}{q^8} + \frac{20}{q^9} + \frac{14}{q^{10}} + \frac{8}{q^{11}} + \frac{6}{q^{12}} + \frac{2}{q^{13}} + \frac{2}{q^{14}} \\
f_{57}(q) = {} & -2q^{48} - 2q^{47} - 6q^{46} - 8q^{45} - 13q^{44} - 16q^{43} - 26q^{42} - 29q^{41} - 38q^{40} - 41q^{39} - 48q^{38}
\end{aligned}
$$

$$
\begin{aligned}
& - 48q^{37} - 56q^{36} - 48q^{35} - 48q^{34} - 41q^{33} - 38q^{32} - 29q^{31} - 26q^{30} - 16q^{29} - 13q^{28} - 8q^{27} \\
& - 6q^{26} - 2q^{25} - 2q^{24}
\end{aligned}
$$

$$
\begin{aligned}
f_{59}(q) = {} & q^{53} + 3q^{52} + 6q^{51} + 9q^{50} + 18q^{49} + 21q^{48} + 31q^{47} + 41q^{46} + 58q^{45} + 80q^{44} + 108q^{43} \\
& + 142q^{42} + 186q^{41} + 235q^{40} + 296q^{39} + 367q^{38} + 452q^{37} + 543q^{36} + 652q^{35} + 767q^{34} \\
& + 899q^{33} + 1036q^{32} + 1187q^{31} + 1339q^{30} + 1506q^{29} + 1669q^{28} + 1839q^{27} + 2000q^{26} \\
& + 2162q^{25} + 2307q^{24} + 2449q^{23} + 2567q^{22} + 2676q^{21} + 2756q^{20} + 2822q^{19} + 2856q^{18} \\
& + 2879q^{17} + 2856q^{16} + 2822q^{15} + 2756q^{14} + 2676q^{13} + 2567q^{12} + 2449q^{11} + 2307q^{10} \\
& + 2162q^{9} + 2000q^{8} + 1839q^{7} + 1669q^{6} + 1506q^{5} + 1339q^{4} + 1187q^{3} + 1036q^{2} + 899q \\
& + 767 + \frac{652}{q} + \frac{543}{q^2} + \frac{452}{q^3} + \frac{367}{q^4} + \frac{296}{q^5} + \frac{235}{q^6} + \frac{186}{q^7} + \frac{142}{q^8} + \frac{108}{q^9} + \frac{80}{q^{10}} + \frac{58}{q^{11}} + \frac{41}{q^{12}} \\
& + \frac{29}{q^{13}} + \frac{19}{q^{14}} + \frac{13}{q^{15}} + \frac{7}{q^{16}} + \frac{4}{q^{17}} + \frac{3}{q^{18}} + \frac{1}{q^{19}}
\end{aligned}
$$

$$
\begin{aligned}
f_{61}(q) = {} & - q^{55} - 3q^{54} - 4q^{53} - 7q^{52} - 13q^{51} - 17q^{50} - 25q^{49} - 33q^{48} - 43q^{47} - 54q^{46} - 67q^{45} \\
& - 77q^{44} - 88q^{43} - 97q^{42} - 104q^{41} - 108q^{40} - 115q^{39} - 108q^{38} - 104q^{37} - 97q^{36} - 88q^{35} \\
& - 77q^{34} - 67q^{33} - 54q^{32} - 43q^{31} - 33q^{30} - 25q^{29} - 17q^{28} - 13q^{27} - 7q^{26} - 4q^{25} - 3q^{24} \\
& - q^{23}
\end{aligned}
$$

$$
\begin{aligned}
f_{63}(q) = {} & 2q^{60} + 2q^{59} + 7q^{58} + 11q^{57} + 18q^{56} + 25q^{55} + 42q^{54} + 50q^{53} + 70q^{52} + 91q^{51} + 124q^{50} \\
& + 158q^{49} + 215q^{48} + 268q^{47} + 349q^{46} + 431q^{45} + 541q^{44} + 655q^{43} + 807q^{42} + 954q^{41} \\
& + 1148q^{40} + 1341q^{39} + 1579q^{38} + 1810q^{37} + 2096q^{36} + 2363q^{35} + 2686q^{34} + 2989q^{33} \\
& + 3336q^{32} + 3655q^{31} + 4026q^{30} + 4343q^{29} + 4706q^{28} + 5011q^{27} + 5345q^{26} + 5608q^{25} \\
& + 5901q^{24} + 6103q^{23} + 6327q^{22} + 6459q^{21} + 6599q^{20} + 6641q^{19} + 6702q^{18} + 6641q^{17} \\
& + 6599q^{16} + 6459q^{15} + 6327q^{14} + 6103q^{13} + 5901q^{12} + 5608q^{11} + 5345q^{10} + 5011q^{9} \\
& + 4706q^{8} + 4343q^{7} + 4026q^{6} + 3655q^{5} + 3336q^{4} + 2989q^{3} + 2686q^{2} + 2363q + 2096 \\
& + \frac{1810}{q} + \frac{1579}{q^2} + \frac{1341}{q^3} + \frac{1148}{q^4} + \frac{954}{q^5} + \frac{807}{q^6} + \frac{655}{q^7} + \frac{541}{q^8} + \frac{431}{q^9} + \frac{349}{q^{10}} + \frac{268}{q^{11}} + \frac{215}{q^{12}} \\
& + \frac{158}{q^{13}} + \frac{123}{q^{14}} + \frac{88}{q^{15}} + \frac{66}{q^{16}} + \frac{45}{q^{17}} + \frac{34}{q^{18}} + \frac{20}{q^{19}} + \frac{14}{q^{20}} + \frac{8}{q^{21}} + \frac{6}{q^{22}} + \frac{2}{q^{23}} + \frac{2}{q^{24}}
\end{aligned}
$$

$$
f_{65}(q) = - 3q^{62} - 2q^{61} - 6q^{60} - 8q^{59} - 14q^{58} - 19q^{57} - 30q^{56} - 38q^{55} - 55q^{54} - 66q^{53} - 87q^{52}
$$

$$
\begin{aligned}
&- 102q^{51} - 129q^{50} - 145q^{49} - 172q^{48} - 186q^{47} - 210q^{46} - 219q^{45} - 237q^{44} - 238q^{43} \\
&- 251q^{42} - 238q^{41} - 237q^{40} - 219q^{39} - 210q^{38} - 186q^{37} - 172q^{36} - 145q^{35} - 129q^{34} \\
&- 102q^{33} - 87q^{32} - 66q^{31} - 55q^{30} - 38q^{29} - 30q^{28} - 19q^{27} - 14q^{26} - 8q^{25} - 6q^{24} - 2q^{23} \\
&- 2q^{22}
\end{aligned}
$$

$$
\begin{aligned}
f_{67}(q) =\, &q^{68} + 3q^{67} + 4q^{66} + 9q^{65} + 15q^{64} + 25q^{63} + 36q^{62} + 53q^{61} + 70q^{60} + 101q^{59} + 126q^{58} \\
&+ 166q^{57} + 214q^{56} + 279q^{55} + 350q^{54} + 446q^{53} + 560q^{52} + 698q^{51} + 858q^{50} + 1052q^{49} \\
&+ 1268q^{48} + 1525q^{47} + 1812q^{46} + 2137q^{45} + 2500q^{44} + 2913q^{43} + 3351q^{42} + 3843q^{41} \\
&+ 4369q^{40} + 4940q^{39} + 5536q^{38} + 6182q^{37} + 6838q^{36} + 7536q^{35} + 8239q^{34} + 8962q^{33} \\
&+ 9682q^{32} + 10416q^{31} + 11111q^{30} + 11806q^{29} + 12458q^{28} + 13081q^{27} + 13639q^{26} \\
&+ 14163q^{25} + 14598q^{24} + 14985q^{23} + 15278q^{22} + 15501q^{21} + 15624q^{20} + 15689q^{19} \\
&+ 15624q^{18} + 15501q^{17} + 15278q^{16} + 14985q^{15} + 14598q^{14} + 14163q^{13} + 13639q^{12} \\
&+ 13081q^{11} + 12458q^{10} + 11806q^{9} + 11111q^{8} + 10416q^{7} + 9682q^{6} + 8962q^{5} + 8239q^{4} \\
&+ 7536q^{3} + 6838q^{2} + 6182q + 5536 + \frac{4940}{q} + \frac{4369}{q^{2}} + \frac{3843}{q^{3}} + \frac{3351}{q^{4}} + \frac{2913}{q^{5}} + \frac{2500}{q^{6}} \\
&+ \frac{2137}{q^{7}} + \frac{1812}{q^{8}} + \frac{1525}{q^{9}} + \frac{1268}{q^{10}} + \frac{1052}{q^{11}} + \frac{858}{q^{12}} + \frac{698}{q^{13}} + \frac{560}{q^{14}} + \frac{444}{q^{15}} + \frac{348}{q^{16}} + \frac{273}{q^{17}} \\
&+ \frac{207}{q^{18}} + \frac{156}{q^{19}} + \frac{116}{q^{20}} + \frac{86}{q^{21}} + \frac{60}{q^{22}} + \frac{43}{q^{23}} + \frac{29}{q^{24}} + \frac{19}{q^{25}} + \frac{13}{q^{26}} + \frac{7}{q^{27}} + \frac{4}{q^{28}} + \frac{3}{q^{29}} + \frac{1}{q^{30}}
\end{aligned}
$$

$$
\begin{aligned}
f_{69}(q) =\, &- q^{70} - 5q^{69} - 4q^{68} - 7q^{67} - 13q^{66} - 19q^{65} - 27q^{64} - 39q^{63} - 52q^{62} - 72q^{61} - 93q^{60} \\
&- 118q^{59} - 146q^{58} - 182q^{57} - 214q^{56} - 254q^{55} - 296q^{54} - 338q^{53} - 377q^{52} - 419q^{51} \\
&- 452q^{50} - 486q^{49} - 511q^{48} - 531q^{47} - 542q^{46} - 554q^{45} - 542q^{44} - 531q^{43} - 511q^{42} \\
&- 486q^{41} - 452q^{40} - 419q^{39} - 377q^{38} - 338q^{37} - 296q^{36} - 254q^{35} - 214q^{34} - 182q^{33} \\
&- 146q^{32} - 118q^{31} - 93q^{30} - 72q^{29} - 52q^{28} - 39q^{27} - 27q^{26} - 19q^{25} - 13q^{24} - 7q^{23} \\
&- 4q^{22} - 3q^{21} - q^{20}
\end{aligned}
$$

$$
\begin{aligned}
f_{71}(q) =\, &2q^{76} + 2q^{75} + 6q^{74} + 9q^{73} + 17q^{72} + 24q^{71} + 41q^{70} + 57q^{69} + 84q^{68} + 110q^{67} + 151q^{66} \\
&+ 194q^{65} + 261q^{64} + 322q^{63} + 417q^{62} + 517q^{61} + 656q^{60} + 804q^{59} + 1004q^{58} + 1219q^{57} \\
&+ 1503q^{56} + 1802q^{55} + 2180q^{54} + 2585q^{53} + 3082q^{52} + 3605q^{51} + 4238q^{50} + 4897q^{49}
\end{aligned}
$$

$$
\begin{aligned}
&+5676q^{48}+6485q^{47}+7425q^{46}+8387q^{45}+9491q^{44}+10605q^{43}+11861q^{42}+13119q^{41}\\
&+14514q^{40}+15887q^{39}+17393q^{38}+18850q^{37}+20424q^{36}+21923q^{35}+23519q^{34}\\
&+25002q^{33}+26562q^{32}+27969q^{31}+29425q^{30}+30697q^{29}+31988q^{28}+33059q^{27}\\
&+34130q^{26}+34947q^{25}+35742q^{24}+36264q^{23}+36746q^{22}+36941q^{21}+37095q^{20}\\
&+36941q^{19}+36746q^{18}+36264q^{17}+35742q^{16}+34947q^{15}+34130q^{14}+33059q^{13}\\
&+31988q^{12}+30697q^{11}+29425q^{10}+27969q^{9}+26562q^{8}+25002q^{7}+23519q^{6}+21923q^{5}\\
&+20424q^{4}+18850q^{3}+17393q^{2}+15887q+14514+\frac{13119}{q}+\frac{11861}{q^{2}}+\frac{10605}{q^{3}}+\frac{9491}{q^{4}}\\
&+\frac{8387}{q^{5}}+\frac{7425}{q^{6}}+\frac{6485}{q^{7}}+\frac{5676}{q^{8}}+\frac{4897}{q^{9}}+\frac{4238}{q^{10}}+\frac{3605}{q^{11}}+\frac{3082}{q^{12}}+\frac{2585}{q^{13}}+\frac{2180}{q^{14}}+\frac{1801}{q^{15}}\\
&+\frac{1500}{q^{16}}+\frac{1215}{q^{17}}+\frac{997}{q^{18}}+\frac{793}{q^{19}}+\frac{641}{q^{20}}+\frac{499}{q^{21}}+\frac{396}{q^{22}}+\frac{299}{q^{23}}+\frac{234}{q^{24}}+\frac{171}{q^{25}}+\frac{130}{q^{26}}+\frac{92}{q^{27}}\\
&+\frac{69}{q^{28}}+\frac{46}{q^{29}}+\frac{34}{q^{30}}+\frac{20}{q^{31}}+\frac{14}{q^{32}}+\frac{8}{q^{33}}+\frac{6}{q^{34}}+\frac{2}{q^{35}}+\frac{2}{q^{36}}\\
f_{73}(q)=&-2q^{78}-3q^{77}-9q^{76}-9q^{75}-14q^{74}-20q^{73}-33q^{72}-42q^{71}-62q^{70}-81q^{69}-110q^{68}\\
&-137q^{67}-182q^{66}-219q^{65}-277q^{64}-329q^{63}-399q^{62}-460q^{61}-545q^{60}-612q^{59}\\
&-705q^{58}-778q^{57}-869q^{56}-935q^{55}-1022q^{54}-1073q^{53}-1143q^{52}-1178q^{51}-1224q^{50}\\
&-1232q^{49}-1259q^{48}-1232q^{47}-1224q^{46}-1178q^{45}-1143q^{44}-1073q^{43}-1022q^{42}\\
&-935q^{41}-869q^{40}-778q^{39}-705q^{38}-612q^{37}-545q^{36}-460q^{35}-399q^{34}-329q^{33}\\
&-277q^{32}-219q^{31}-182q^{30}-137q^{29}-110q^{28}-81q^{27}-62q^{26}-42q^{25}-33q^{24}-20q^{23}\\
&-14q^{22}-8q^{21}-6q^{20}-2q^{19}-2q^{18}\\
f_{75}(q)=&\,q^{85}+3q^{84}+4q^{83}+7q^{82}+15q^{81}+21q^{80}+35q^{79}+51q^{78}+75q^{77}+104q^{76}+148q^{75}\\
&+193q^{74}+259q^{73}+334q^{72}+431q^{71}+542q^{70}+692q^{69}+849q^{68}+1056q^{67}+1296q^{66}\\
&+1588q^{65}+1925q^{64}+2337q^{63}+2800q^{62}+3358q^{61}+3992q^{60}+4725q^{59}+5553q^{58}\\
&+6509q^{57}+7563q^{56}+8759q^{55}+10081q^{54}+11549q^{53}+13152q^{52}+14925q^{51}+16826q^{50}\\
&+18908q^{49}+21128q^{48}+23514q^{47}+26035q^{46}+28729q^{45}+31522q^{44}+34474q^{43}\\
&+37517q^{42}+40677q^{41}+43891q^{40}+47204q^{39}+50512q^{38}+53878q^{37}+57207q^{36}\\
&+60521q^{35}+63744q^{34}+66919q^{33}+69921q^{32}+72821q^{31}+75512q^{30}+78028q^{29}
\end{aligned}
$$

$$+ 80280q^{28} + 82330q^{27} + 84052q^{26} + 85533q^{25} + 86669q^{24} + 87516q^{23} + 87998q^{22}$$
$$+ 88202q^{21} + 87998q^{20} + 87516q^{19} + 86669q^{18} + 85533q^{17} + 84052q^{16} + 82330q^{15}$$
$$+ 80280q^{14} + 78028q^{13} + 75512q^{12} + 72821q^{11} + 69921q^{10} + 66919q^{9} + 63744q^{8}$$
$$+ 60521q^{7} + 57207q^{6} + 53878q^{5} + 50512q^{4} + 47204q^{3} + 43891q^{2} + 40677q + 37517$$
$$+ \frac{34474}{q} + \frac{31522}{q^2} + \frac{28729}{q^3} + \frac{26035}{q^4} + \frac{23514}{q^5} + \frac{21128}{q^6} + \frac{18908}{q^7} + \frac{16826}{q^8} + \frac{14925}{q^9}$$
$$+ \frac{13152}{q^{10}} + \frac{11549}{q^{11}} + \frac{10081}{q^{12}} + \frac{8759}{q^{13}} + \frac{7563}{q^{14}} + \frac{6507}{q^{15}} + \frac{5551}{q^{16}} + \frac{4719}{q^{17}} + \frac{3984}{q^{18}} + \frac{3345}{q^{19}}$$
$$+ \frac{2784}{q^{20}} + \frac{2311}{q^{21}} + \frac{1896}{q^{22}} + \frac{1550}{q^{23}} + \frac{1255}{q^{24}} + \frac{1008}{q^{25}} + \frac{801}{q^{26}} + \frac{636}{q^{27}} + \frac{494}{q^{28}} + \frac{383}{q^{29}} + \frac{293}{q^{30}}$$
$$+ \frac{221}{q^{31}} + \frac{164}{q^{32}} + \frac{122}{q^{33}} + \frac{88}{q^{34}} + \frac{62}{q^{35}} + \frac{43}{q^{36}} + \frac{29}{q^{37}} + \frac{19}{q^{38}} + \frac{13}{q^{39}} + \frac{7}{q^{40}} + \frac{4}{q^{41}} + \frac{3}{q^{42}} + \frac{1}{q^{43}}$$
$$f_{77}(q) = - q^{87} - 3q^{86} - 6q^{85} - 9q^{84} - 18q^{83} - 21q^{82} - 31q^{81} - 41q^{80} - 58q^{79} - 80q^{78} - 108q^{77}$$
$$- 142q^{76} - 186q^{75} - 235q^{74} - 296q^{73} - 367q^{72} - 452q^{71} - 543q^{70} - 652q^{69} - 767q^{68}$$
$$- 899q^{67} - 1036q^{66} - 1187q^{65} - 1339q^{64} - 1506q^{63} - 1669q^{62} - 1839q^{61} - 2000q^{60}$$
$$- 2162q^{59} - 2307q^{58} - 2449q^{57} - 2567q^{56} - 2676q^{55} - 2756q^{54} - 2822q^{53} - 2856q^{52}$$
$$- 2879q^{51} - 2856q^{50} - 2822q^{49} - 2756q^{48} - 2676q^{47} - 2567q^{46} - 2449q^{45} - 2307q^{44}$$
$$- 2162q^{43} - 2000q^{42} - 1839q^{41} - 1669q^{40} - 1506q^{39} - 1339q^{38} - 1187q^{37} - 1036q^{36}$$
$$- 899q^{35} - 767q^{34} - 652q^{33} - 543q^{32} - 452q^{31} - 367q^{30} - 296q^{29} - 235q^{28} - 186q^{27}$$
$$- 142q^{26} - 108q^{25} - 80q^{24} - 58q^{23} - 41q^{22} - 29q^{21} - 19q^{20} - 13q^{19} - 7q^{18} - 4q^{17} - 3q^{16}$$
$$- q^{15}$$
$$f_{79}(q) = 2q^{94} + 2q^{93} + 6q^{92} + 8q^{91} + 15q^{90} + 23q^{89} + 38q^{88} + 53q^{87} + 83q^{86} + 112q^{85} + 159q^{84}$$
$$+ 211q^{83} + 290q^{82} + 372q^{81} + 492q^{80} + 621q^{79} + 795q^{78} + 986q^{77} + 1237q^{76} + 1509q^{75}$$
$$+ 1870q^{74} + 2256q^{73} + 2747q^{72} + 3291q^{71} + 3968q^{70} + 4708q^{69} + 5618q^{68} + 6618q^{67}$$
$$+ 7815q^{66} + 9126q^{65} + 10683q^{64} + 12365q^{63} + 14337q^{62} + 16460q^{61} + 18898q^{60}$$
$$+ 21518q^{59} + 24498q^{58} + 27655q^{57} + 31208q^{56} + 34961q^{55} + 39122q^{54} + 43481q^{53}$$
$$+ 48283q^{52} + 53251q^{51} + 58671q^{50} + 64247q^{49} + 70248q^{48} + 76369q^{47} + 82909q^{46}$$
$$+ 89489q^{45} + 96453q^{44} + 103408q^{43} + 110671q^{42} + 117835q^{41} + 125260q^{40}$$

$$
\begin{aligned}
&+132470q^{39}+139865q^{38}+146953q^{37}+154109q^{36}+160849q^{35}+167589q^{34}\\
&+173771q^{33}+179860q^{32}+185309q^{31}+190550q^{30}+195053q^{29}+199295q^{28}\\
&+202693q^{27}+205766q^{26}+207955q^{25}+209750q^{24}+210625q^{23}+211113q^{22}\\
&+210625q^{21}+209750q^{20}+207955q^{19}+205766q^{18}+202693q^{17}+199295q^{16}\\
&+195053q^{15}+190550q^{14}+185309q^{13}+179860q^{12}+173771q^{11}+167589q^{10}\\
&+160849q^{9}+154109q^{8}+146953q^{7}+139865q^{6}+132470q^{5}+125260q^{4}+117835q^{3}\\
&+110671q^{2}+103408q+96453+\frac{89489}{q}+\frac{82909}{q^{2}}+\frac{76369}{q^{3}}+\frac{70248}{q^{4}}+\frac{64247}{q^{5}}+\frac{58671}{q^{6}}\\
&+\frac{53251}{q^{7}}+\frac{48283}{q^{8}}+\frac{43481}{q^{9}}+\frac{39122}{q^{10}}+\frac{34961}{q^{11}}+\frac{31208}{q^{12}}+\frac{27655}{q^{13}}+\frac{24497}{q^{14}}+\frac{21515}{q^{15}}\\
&+\frac{18894}{q^{16}}+\frac{16453}{q^{17}}+\frac{14324}{q^{18}}+\frac{12348}{q^{19}}+\frac{10658}{q^{20}}+\frac{9093}{q^{21}}+\frac{7772}{q^{22}}+\frac{6564}{q^{23}}+\frac{5551}{q^{24}}+\frac{4631}{q^{25}}\\
&+\frac{3880}{q^{26}}+\frac{3194}{q^{27}}+\frac{2643}{q^{28}}+\frac{2148}{q^{29}}+\frac{1755}{q^{30}}+\frac{1401}{q^{31}}+\frac{1133}{q^{32}}+\frac{889}{q^{33}}+\frac{707}{q^{34}}+\frac{544}{q^{35}}+\frac{425}{q^{36}}+\frac{318}{q^{37}}\\
&+\frac{247}{q^{38}}+\frac{178}{q^{39}}+\frac{134}{q^{40}}+\frac{95}{q^{41}}+\frac{70}{q^{42}}+\frac{46}{q^{43}}+\frac{34}{q^{44}}+\frac{20}{q^{45}}+\frac{14}{q^{46}}+\frac{8}{q^{47}}+\frac{6}{q^{48}}+\frac{2}{q^{49}}+\frac{2}{q^{50}}\\
f_{81}(q)=&-2q^{96}-2q^{95}-7q^{94}-11q^{93}-18q^{92}-25q^{91}-42q^{90}-50q^{89}-70q^{88}-91q^{87}-124q^{86}\\
&-158q^{85}-215q^{84}-268q^{83}-349q^{82}-431q^{81}-541q^{80}-655q^{79}-807q^{78}-954q^{77}\\
&-1148q^{76}-1341q^{75}-1579q^{74}-1810q^{73}-2096q^{72}-2363q^{71}-2686q^{70}-2989q^{69}\\
&-3336q^{68}-3655q^{67}-4026q^{66}-4343q^{65}-4706q^{64}-5011q^{63}-5345q^{62}-5608q^{61}\\
&-5901q^{60}-6103q^{59}-6327q^{58}-6459q^{57}-6599q^{56}-6641q^{55}-6702q^{54}-6641q^{53}\\
&-6599q^{52}-6459q^{51}-6327q^{50}-6103q^{49}-5901q^{48}-5608q^{47}-5345q^{46}-5011q^{45}\\
&-4706q^{44}-4343q^{43}-4026q^{42}-3655q^{41}-3336q^{40}-2989q^{39}-2686q^{38}-2363q^{37}\\
&-2096q^{36}-1810q^{35}-1579q^{34}-1341q^{33}-1148q^{32}-954q^{31}-807q^{30}-655q^{29}-541q^{28}\\
&-431q^{27}-349q^{26}-268q^{25}-215q^{24}-158q^{23}-123q^{22}-88q^{21}-66q^{20}-45q^{19}-34q^{18}\\
&-20q^{17}-14q^{16}-8q^{15}-6q^{14}-2q^{13}-2q^{12}\\
f_{83}(q)=&\,q^{104}+3q^{103}+4q^{102}+7q^{101}+13q^{100}+22q^{99}+31q^{98}+49q^{97}+70q^{96}+104q^{95}\\
&+143q^{94}+200q^{93}+267q^{92}+362q^{91}+469q^{90}+615q^{89}+785q^{88}+1004q^{87}+1255q^{86}\\
&+1572q^{85}+1934q^{84}+2385q^{83}+2896q^{82}+3518q^{81}+4232q^{80}+5092q^{79}+6058q^{78}
\end{aligned}
$$

$$
\begin{aligned}
&+7209q^{77}+8518q^{76}+10050q^{75}+11783q^{74}+13792q^{73}+16049q^{72}+18640q^{71}\\
&+21530q^{70}+24809q^{69}+28458q^{68}+32550q^{67}+37056q^{66}+42074q^{65}+47562q^{64}\\
&+53611q^{63}+60179q^{62}+67353q^{61}+75086q^{60}+83471q^{59}+92432q^{58}+102059q^{57}\\
&+112288q^{56}+123190q^{55}+134666q^{54}+146812q^{53}+159497q^{52}+172808q^{51}\\
&+186601q^{50}+200949q^{49}+215696q^{48}+230912q^{47}+246398q^{46}+262233q^{45}\\
&+278215q^{44}+294397q^{43}+310547q^{42}+326745q^{41}+342737q^{40}+358588q^{39}\\
&+374042q^{38}+389156q^{37}+403679q^{36}+417684q^{35}+430885q^{34}+443382q^{33}\\
&+454907q^{32}+465550q^{31}+475049q^{30}+483527q^{29}+490725q^{28}+496781q^{27}\\
&+501455q^{26}+504899q^{25}+506903q^{24}+507651q^{23}+506903q^{22}+504899q^{21}\\
&+501455q^{20}+496781q^{19}+490725q^{18}+483527q^{17}+475049q^{16}+465550q^{15}\\
&+454907q^{14}+443382q^{13}+430885q^{12}+417684q^{11}+403679q^{10}+389156q^{9}\\
&+374042q^{8}+358588q^{7}+342737q^{6}+326745q^{5}+310547q^{4}+294397q^{3}+278215q^{2}\\
&+262233q+246398+\frac{230912}{q}+\frac{215696}{q^{2}}+\frac{200949}{q^{3}}+\frac{186601}{q^{4}}+\frac{172808}{q^{5}}+\frac{159497}{q^{6}}\\
&+\frac{146812}{q^{7}}+\frac{134666}{q^{8}}+\frac{123190}{q^{9}}+\frac{112288}{q^{10}}+\frac{102059}{q^{11}}+\frac{92432}{q^{12}}+\frac{83469}{q^{13}}+\frac{75084}{q^{14}}\\
&+\frac{67347}{q^{15}}+\frac{60171}{q^{16}}+\frac{53597}{q^{17}}+\frac{47543}{q^{18}}+\frac{42044}{q^{19}}+\frac{37018}{q^{20}}+\frac{32495}{q^{21}}+\frac{28392}{q^{22}}\\
&+\frac{24722}{q^{23}}+\frac{21428}{q^{24}}+\frac{18511}{q^{25}}+\frac{15904}{q^{26}}+\frac{13620}{q^{27}}+\frac{11597}{q^{28}}+\frac{9840}{q^{29}}+\frac{8299}{q^{30}}+\frac{6972}{q^{31}}\\
&+\frac{5820}{q^{32}}+\frac{4841}{q^{33}}+\frac{3994}{q^{34}}+\frac{3281}{q^{35}}+\frac{2677}{q^{36}}+\frac{2175}{q^{37}}+\frac{1748}{q^{38}}+\frac{1400}{q^{39}}+\frac{1110}{q^{40}}+\frac{875}{q^{41}}\\
&+\frac{683}{q^{42}}+\frac{528}{q^{43}}+\frac{403}{q^{44}}+\frac{307}{q^{45}}+\frac{229}{q^{46}}+\frac{170}{q^{47}}+\frac{124}{q^{48}}+\frac{90}{q^{49}}+\frac{62}{q^{50}}\frac{43}{q^{51}}+\frac{29}{q^{52}}+\frac{19}{q^{53}}+\frac{13}{q^{54}}\\
&+\frac{7}{q^{55}}+\frac{4}{q^{56}}+\frac{3}{q^{57}}+\frac{1}{q^{58}}
\end{aligned}
$$

$$
\begin{aligned}
f_{85}(q)=&-q^{106}-3q^{105}-4q^{104}-9q^{103}-15q^{102}-25q^{101}-36q^{100}-53q^{99}-70q^{98}-101q^{97}\\
&-126q^{96}-166q^{95}-214q^{94}-279q^{93}-350q^{92}-446q^{91}-560q^{90}-698q^{89}-858q^{88}\\
&-1052q^{87}-1268q^{86}-1525q^{85}-1812q^{84}-2137q^{83}-2500q^{82}-2913q^{81}-3351q^{80}\\
&-3843q^{79}-4369q^{78}-4940q^{77}-5536q^{76}-6182q^{75}-6838q^{74}-7536q^{73}-8239q^{72}\\
&-8962q^{71}-9682q^{70}-10416q^{69}-11111q^{68}-11806q^{67}-12458q^{66}-13081q^{65}
\end{aligned}
$$

$$- 13639q^{64} - 14163q^{63} - 14598q^{62} - 14985q^{61} - 15278q^{60} - 15501q^{59} - 15624q^{58}$$
$$- 15689q^{57} - 15624q^{56} - 15501q^{55} - 15278q^{54} - 14985q^{53} - 14598q^{52} - 14163q^{51}$$
$$- 13639q^{50} - 13081q^{49} - 12458q^{48} - 11806q^{47} - 11111q^{46} - 10416q^{45} - 9682q^{44}$$
$$- 8962q^{43} - 8239q^{42} - 7536q^{41} - 6838q^{40} - 6182q^{39} - 5536q^{38} - 4940q^{37} - 4369q^{36}$$
$$- 3843q^{35} - 3351q^{34} - 2913q^{33} - 2500q^{32} - 2137q^{31} - 1812q^{30} - 1525q^{29} - 1268q^{28}$$
$$- 1052q^{27} - 858q^{26} - 698q^{25} - 560q^{24} - 444q^{23} - 348q^{22} - 273q^{21} - 207q^{20} - 156q^{19}$$
$$- 116q^{18} - 86q^{17} - 60q^{16} - 43q^{15} - 29q^{14} - 19q^{13} - 13q^{12} - 7q^{11} - 4q^{10} - 3q^{9} - q^{8}$$

$$f_{87}(q) = 2q^{114} + 2q^{113} + 6q^{112} + 8q^{111} + 14q^{110} + 21q^{109} + 39q^{108} + 50q^{107} + 77q^{106} + 109q^{105}$$
$$+ 156q^{104} + 209q^{103} + 293q^{102} + 383q^{101} + 516q^{100} + 666q^{99} + 868q^{98} + 1099q^{97} + 1409q^{96}$$
$$+ 1747q^{95} + 2193q^{94} + 2695q^{93} + 3322q^{92} + 4028q^{91} + 4907q^{90} + 5882q^{89} + 7077q^{88}$$
$$+ 8415q^{87} + 10018q^{86} + 11812q^{85} + 13959q^{84} + 16323q^{83} + 19124q^{82} + 22236q^{81}$$
$$+ 25858q^{80} + 29865q^{79} + 34504q^{78} + 39597q^{77} + 45439q^{76} + 51840q^{75} + 59094q^{74}$$
$$+ 67006q^{73} + 75920q^{72} + 85556q^{71} + 96322q^{70} + 107928q^{69} + 120758q^{68}$$
$$+ 134506q^{67} + 149618q^{66} + 165686q^{65} + 183217q^{64} + 201771q^{63} + 221834q^{62}$$
$$+ 242944q^{61} + 265638q^{60} + 289332q^{59} + 314626q^{58} + 340899q^{57} + 368717q^{56}$$
$$+ 397416q^{55} + 427625q^{54} + 458540q^{53} + 490853q^{52} + 523715q^{51} + 557774q^{50}$$
$$+ 592154q^{49} + 627553q^{48} + 662951q^{47} + 699119q^{46} + 735003q^{45} + 771326q^{44}$$
$$+ 807011q^{43} + 842855q^{42} + 877642q^{41} + 912241q^{40} + 945436q^{39} + 978047q^{38}$$
$$+ 1008868q^{37} + 1038799q^{36} + 1066518q^{35} + 1093018q^{34} + 1117004q^{33} + 1139426q^{32}$$
$$+ 1159030q^{31} + 1176852q^{30} + 1191564q^{29} + 1204290q^{28} + 1213761q^{27} + 1221060q^{26}$$
$$+ 1224988q^{25} + 1226724q^{24} + 1224988q^{23} + 1221060q^{22} + 1213761q^{21} + 1204290q^{20}$$
$$+ 1191564q^{19} + 1176852q^{18} + 1159030q^{17} + 1139426q^{16} + 1117004q^{15} + 1093018q^{14}$$
$$+ 1066518q^{13} + 1038799q^{12} + 1008868q^{11} + 978047q^{10} + 945436q^{9} + 912241q^{8} + 877642q^{7}$$
$$+ 842855q^{6} + 807011q^{5} + 771326q^{4} + 735003q^{3} + 699119q^{2} + 662951q + 627553 + \frac{592154}{q}$$

$$
\begin{aligned}
&+\frac{557774}{q^{2}}+\frac{523715}{q^{3}}+\frac{490853}{q^{4}}+\frac{458540}{q^{5}}+\frac{427625}{q^{6}}+\frac{397416}{q^{7}}+\frac{368717}{q^{8}}+\frac{340899}{q^{9}}\\
&+\frac{314626}{q^{10}}+\frac{289331}{q^{11}}+\frac{265635}{q^{12}}+\frac{42940}{q^{13}}+\frac{221827}{q^{14}}+\frac{201758}{q^{15}}+\frac{183198}{q^{16}}+\frac{165659}{q^{17}}\\
&+\frac{149579}{q^{18}}+\frac{134454}{q^{19}}+\frac{120686}{q^{20}}+\frac{107835}{q^{21}}+\frac{96204}{q^{22}}+\frac{85410}{q^{23}}+\frac{75738}{q^{24}}+\frac{66792}{q^{25}}+\frac{58840}{q^{26}}\\
&+\frac{51544}{q^{27}}+\frac{45101}{q^{28}}+\frac{39220}{q^{29}}+\frac{34085}{q^{30}}+\frac{29413}{q^{31}}+\frac{25372}{q^{32}}+\frac{21725}{q^{33}}+\frac{18593}{q^{34}}+\frac{15781}{q^{35}}\\
&+\frac{13405}{q^{36}}+\frac{11270}{q^{37}}+\frac{9487}{q^{38}}+\frac{7904}{q^{39}}+\frac{6591}{q^{40}}+\frac{5430}{q^{41}}+\frac{4488}{q^{42}}+\frac{3651}{q^{43}}+\frac{2984}{q^{44}}+\frac{2399}{q^{45}}\\
&+\frac{1939}{q^{46}}+\frac{1533}{q^{47}}+\frac{1227}{q^{48}}+\frac{953}{q^{49}}+\frac{750}{q^{50}}+\frac{573}{q^{51}}+\frac{444}{q^{52}}+\frac{331}{q^{53}}+\frac{254}{q^{54}}+\frac{182}{q^{55}}+\frac{137}{q^{56}}+\frac{96}{q^{57}}\\
&+\frac{70}{q^{58}}+\frac{46}{q^{59}}+\frac{34}{q^{60}}+\frac{20}{q^{61}}+\frac{14}{q^{62}}+\frac{8}{q^{63}}+\frac{6}{q^{64}}+\frac{2}{q^{65}}+\frac{2}{q^{66}}
\end{aligned}
$$

$$
\begin{aligned}
f_{89}(q) =& -2q^{116}-2q^{115}-6q^{114}-9q^{113}-17q^{112}-24q^{111}-41q^{110}-57q^{109}-84q^{108}\\
&-110q^{107}-151q^{106}-194q^{105}-261q^{104}-322q^{103}-417q^{102}-517q^{101}-656q^{100}\\
&-804q^{99}-1004q^{98}-1219q^{97}-1503q^{96}-1802q^{95}-2180q^{94}-2585q^{93}-3082q^{92}\\
&-3605q^{91}-4238q^{90}-4897q^{89}-5676q^{88}-6485q^{87}-7425q^{86}-8387q^{85}-9491q^{84}\\
&-10605q^{83}-11861q^{82}-13119q^{81}-14514q^{80}-15887q^{79}-17393q^{78}-18850q^{77}\\
&-20424q^{76}-21923q^{75}-23519q^{74}-25002q^{73}-26562q^{72}-27969q^{71}-29425q^{70}\\
&-30697q^{69}-31988q^{68}-33059q^{67}-34130q^{66}-34947q^{65}-35742q^{64}-36264q^{63}\\
&-36746q^{62}-36941q^{61}-37095q^{60}-36941q^{59}-36746q^{58}-36264q^{57}-35742q^{56}\\
&-34947q^{55}-34130q^{54}-33059q^{53}-31988q^{52}-30697q^{51}-29425q^{50}-27969q^{49}\\
&-26562q^{48}-25002q^{47}-23519q^{46}-21923q^{45}-20424q^{44}-18850q^{43}-17393q^{42}\\
&-15887q^{41}-14514q^{40}-13119q^{39}-11861q^{38}-10605q^{37}-9491q^{36}-8387q^{35}\\
&-7425q^{34}-6485q^{33}-5676q^{32}-4897q^{31}-4238q^{30}-3605q^{29}-3082q^{28}-2585q^{27}\\
&-2180q^{26}-1801q^{25}-1500q^{24}-1215q^{23}-997q^{22}-793q^{21}-641q^{20}-499q^{19}-396q^{18}\\
&-299q^{17}-234q^{16}-171q^{15}-130q^{14}-92q^{13}-69q^{12}-46q^{11}-34q^{10}-20q^{9}-14q^{8}-8q^{7}\\
&-6q^{6}-2q^{5}-2q^{4}
\end{aligned}
$$

$$
\begin{aligned}
f_{91}(q) =& \, q^{125}+3q^{124}+4q^{123}+7q^{122}+13q^{121}+19q^{120}+31q^{119}+46q^{118}+71q^{117}+99q^{116}\\
&+140q^{115}+192q^{114}+268q^{113}+357q^{112}+479q^{111}+629q^{110}+827q^{109}+1058q^{108}
\end{aligned}
$$

$$+ 1363q^{107} + 1719q^{106} + 2169q^{105} + 2697q^{104} + 3350q^{103} + 4104q^{102} + 5031q^{101}$$
$$+ 6098q^{100} + 7380q^{99} + 8855q^{98} + 10609q^{97} + 12604q^{96} + 14959q^{95} + 17634q^{94}$$
$$+ 20749q^{93} + 24285q^{92} + 28371q^{91} + 32975q^{90} + 38270q^{89} + 44207q^{88} + 50966q^{87}$$
$$+ 58539q^{86} + 67106q^{85} + 76634q^{84} + 87349q^{83} + 99224q^{82} + 112469q^{81}$$
$$+ 127078q^{80} + 143276q^{79} + 161026q^{78} + 180595q^{77} + 201941q^{76} + 225292q^{75}$$
$$+ 250636q^{74} + 278211q^{73} + 307938q^{72} + 340096q^{71} + 374601q^{70} + 411670q^{69}$$
$$+ 451230q^{68} + 493505q^{67} + 538336q^{66} + 585975q^{65} + 636236q^{64} + 689293q^{63}$$
$$+ 744968q^{62} + 803437q^{61} + 864387q^{60} + 928022q^{59} + 994012q^{58} + 1062457q^{57}$$
$$+ 1133007q^{56} + 1205771q^{55} + 1280263q^{54} + 1356612q^{53} + 1434305q^{52} + 1513364q^{51}$$
$$+ 1593272q^{50} + 1674065q^{49} + 1755076q^{48} + 1836383q^{47} + 1917323q^{46} + 1997860q^{45}$$
$$+ 2077343q^{44} + 2155778q^{43} + 2232395q^{42} + 2307259q^{41} + 2379622q^{40} + 2449467q^{39}$$
$$+ 2516105q^{38} + 2579591q^{37} + 2639132q^{36} + 2694879q^{35} + 2746118q^{34} + 2792934q^{33}$$
$$+ 2834697q^{32} + 2871604q^{31} + 2902963q^{30} + 2929084q^{29} + 2949378q^{28} + 2964116q^{27}$$
$$+ 2972831q^{26} + 2975915q^{25} + 2972831q^{24} + 2964116q^{23} + 2949378q^{22} + 2929084q^{21}$$
$$+ 2902963q^{20} + 2871604q^{19} + 2834697q^{18} + 2792934q^{17} + 2746118q^{16} + 2694879q^{15}$$
$$+ 2639132q^{14} + 2579591q^{13} + 2516105q^{12} + 2449467q^{11} + 2379622q^{10} + 2307259q^{9}$$
$$+ 2232395q^{8} + 2155778q^{7} + 2077343q^{6} + 1997860q^{5} + 1917323q^{4} + 1836383q^{3}$$
$$+ 1755076q^{2} + 1674065q + 1593272 + \frac{1513364}{q} + \frac{1434305}{q^{2}} + \frac{1356612}{q^{3}} + \frac{1280263}{q^{4}}$$
$$+ \frac{1205771}{q^{5}} + \frac{1133007}{q^{6}} + \frac{1062457}{q^{7}} + \frac{994012}{q^{8}} + \frac{928020}{q^{9}} + \frac{864385}{q^{10}} + \frac{803431}{q^{11}} + \frac{744960}{q^{12}}$$
$$+ \frac{689279}{q^{13}} + \frac{636216}{q^{14}} + \frac{585942}{q^{15}} + \frac{538294}{q^{16}} + \frac{493443}{q^{17}} + \frac{451149}{q^{18}} + \frac{411560}{q^{19}} + \frac{374464}{q^{20}}$$
$$+ \frac{339914}{q^{21}} + \frac{307719}{q^{22}} + \frac{277934}{q^{23}} + \frac{250307}{q^{24}} + \frac{224893}{q^{25}} + \frac{201481}{q^{26}} + \frac{180050}{q^{27}} + \frac{160414}{q^{28}}$$
$$+ \frac{142571}{q^{29}} + \frac{126300}{q^{30}} + \frac{111600}{q^{31}} + \frac{98289}{q^{32}} + \frac{86327}{q^{33}} + \frac{75561}{q^{34}} + \frac{65963}{q^{35}} + \frac{57361}{q^{36}} + \frac{49742}{q^{37}}$$
$$+ \frac{42975}{q^{38}} + \frac{37011}{q^{39}} + \frac{31743}{q^{40}} + \frac{27147}{q^{41}} + \frac{23107}{q^{42}} + \frac{19606}{q^{43}} + \frac{16561}{q^{44}} + \frac{13937}{q^{45}} + \frac{11669}{q^{46}}$$
$$+ \frac{9740}{q^{47}} + \frac{8077}{q^{48}} + \frac{6675}{q^{49}} + \frac{5486}{q^{50}} + \frac{4486}{q^{51}} + \frac{3644}{q^{52}} + \frac{2951}{q^{53}} + \frac{2368}{q^{54}} + \frac{1892}{q^{55}} + \frac{1500}{q^{56}} + \frac{1181}{q^{57}}$$

$$+\frac{921}{q^{58}}+\frac{717}{q^{59}}+\frac{548}{q^{60}}+\frac{417}{q^{61}}+\frac{315}{q^{62}}+\frac{235}{q^{63}}+\frac{172}{q^{64}}+\frac{126}{q^{65}}+\frac{90}{q^{66}}+\frac{62}{q^{67}}+\frac{43}{q^{68}}+\frac{29}{q^{69}}+\frac{19}{q^{70}}$$
$$+\frac{13}{q^{71}}+\frac{7}{q^{72}}+\frac{4}{q^{73}}+\frac{3}{q^{74}}+\frac{1}{q^{75}}$$

$$\begin{aligned}
f_{93}(q) = &-q^{127}-3q^{126}-4q^{125}-7q^{124}-15q^{123}-21q^{122}-35q^{121}-51q^{120}-75q^{119}-104q^{118}\\
&-148q^{117}-193q^{116}-259q^{115}-334q^{114}-431q^{113}-542q^{112}-692q^{111}-849q^{110}\\
&-1056q^{109}-1296q^{108}-1588q^{107}-1925q^{106}-2337q^{105}-2800q^{104}-3358q^{103}-3992q^{102}\\
&-4725q^{101}-5553q^{100}-6509q^{99}-7563q^{98}-8759q^{97}-10081q^{96}-11549q^{95}-13152q^{94}\\
&-14925q^{93}-16826q^{92}-18908q^{91}-21128q^{90}-23514q^{89}-26035q^{88}-28729q^{87}\\
&-31522q^{86}-34474q^{85}-37517q^{84}-40677q^{83}-43891q^{82}-47204q^{81}-50512q^{80}\\
&-53878q^{79}-57207q^{78}-60521q^{77}-63744q^{76}-66919q^{75}-69921q^{74}-72821q^{73}\\
&-75512q^{72}-78028q^{71}-80280q^{70}-82330q^{69}-84052q^{68}-85533q^{67}-86669q^{66}\\
&-87516q^{65}-87998q^{64}-88202q^{63}-87998q^{62}-87516q^{61}-86669q^{60}-85533q^{59}\\
&-84052q^{58}-82330q^{57}-80280q^{56}-78028q^{55}-75512q^{54}-72821q^{53}-69921q^{52}\\
&-66919q^{51}-63744q^{50}-60521q^{49}-57207q^{48}-53878q^{47}-50512q^{46}-47204q^{45}\\
&-43891q^{44}-40677q^{43}-37517q^{42}-34474q^{41}-31522q^{40}-28729q^{39}-26035q^{38}\\
&-23514q^{37}-21128q^{36}-18908q^{35}-16826q^{34}-14925q^{33}-13152q^{32}-11549q^{31}\\
&-10081q^{30}-8759q^{29}-7563q^{28}-6507q^{27}-5551q^{26}-4719q^{25}-3984q^{24}-3345q^{23}\\
&-2784q^{22}-2311q^{21}-1896q^{20}-1550q^{19}-1255q^{18}-1008q^{17}-801q^{16}-636q^{15}-494q^{14}\\
&-383q^{13}-293q^{12}-221q^{11}-164q^{10}-122q^{9}-88q^{8}-62q^{7}-43q^{6}-29q^{5}-19q^{4}\\
&-13q^{3}-7q^{2}-4q-3-\frac{1}{q}
\end{aligned}$$

$$\begin{aligned}
f_{95}(q) = &\,2q^{136}+2q^{135}+6q^{134}+8q^{133}+14q^{132}+20q^{131}+35q^{130}+49q^{129}+76q^{128}+105q^{127}\\
&+156q^{126}+206q^{125}+289q^{124}+379q^{123}+515q^{122}+672q^{121}+887q^{120}+1138q^{119}\\
&+1475q^{118}+1860q^{117}+2365q^{116}+2946q^{115}+3685q^{114}+4531q^{113}+5591q^{112}\\
&+6799q^{111}+8287q^{110}+9977q^{109}+12021q^{108}+14342q^{107}+17123q^{106}+20257q^{105}\\
&+23972q^{104}+28151q^{103}+33050q^{102}+38548q^{101}+44941q^{100}+52085q^{99}+60337q^{98}
\end{aligned}$$

$$+ 69525q^{97} + 80055q^{96} + 91750q^{95} + 105073q^{94} + 119793q^{93} + 136462q^{92} + 154833q^{91}$$
$$+ 175489q^{90} + 198171q^{89} + 223526q^{88} + 251247q^{87} + 282067q^{86} + 315613q^{85} + 352682q^{84}$$
$$+ 392876q^{83} + 437042q^{82} + 484702q^{81} + 536796q^{80} + 592760q^{79} + 653584q^{78} + 718641q^{77}$$
$$+ 788970q^{76} + 863838q^{75} + 944364q^{74} + 1029680q^{73} + 1120951q^{72} + 1217232q^{71}$$
$$+ 1319695q^{70} + 1427250q^{69} + 1541148q^{68} + 1660134q^{67} + 1785484q^{66} + 1915810q^{65}$$
$$+ 2052400q^{64} + 2193702q^{63} + 2341053q^{62} + 2492696q^{61} + 2649991q^{60} + 2811050q^{59}$$
$$+ 2977212q^{58} + 3146405q^{57} + 3320028q^{56} + 3495803q^{55} + 3675148q^{54} + 3855646q^{53}$$
$$+ 4038708q^{52} + 4221770q^{51} + 4406293q^{50} + 4589527q^{49} + 4772986q^{48} + 4953828q^{47}$$
$$+ 5133575q^{46} + 5309264q^{45} + 5482517q^{44} + 5650256q^{43} + 5814175q^{42} + 5971158q^{41}$$
$$+ 6122947q^{40} + 6266410q^{39} + 6403409q^{38} + 6530749q^{37} + 6650426q^{36} + 6759292q^{35}$$
$$+ 6859416q^{34} + 6947700q^{33} + 7026370q^{32} + 7092358q^{31} + 7148031q^{30} + 7190426q^{29}$$
$$+ 7222011q^{28} + 7239947q^{27} + 7246857q^{26} + 7239947q^{25} + 7222011q^{24} + 7190426q^{23}$$
$$+ 7148031q^{22} + 7092358q^{21} + 7026370q^{20} + 6947700q^{19} + 6859416q^{18} + 6759292q^{17}$$
$$+ 6650426q^{16} + 6530749q^{15} + 6403409q^{14} + 6266410q^{13} + 6122947q^{12} + 5971158q^{11}$$
$$+ 5814175q^{10} + 5650256q^{9} + 5482517q^{8} + 5309264q^{7} + 5133575q^{6} + 4953828q^{5}$$
$$+ 4772986q^{4} + 4589527q^{3} + 4406293q^{2} + 4221770q + 4038708 + \frac{3855646}{q} + \frac{3675148}{q^2}$$
$$+ \frac{3495803}{q^3} + \frac{3320028}{q^4} + \frac{3146405}{q^5} + \frac{2977211}{q^6} + \frac{2811047}{q^7} + \frac{2649987}{q^8} + \frac{2492689}{q^9}$$
$$+ \frac{2341040}{q^{10}} + \frac{2193683}{q^{11}} + \frac{2052371}{q^{12}} + \frac{1915769}{q^{13}} + \frac{1785426}{q^{14}} + \frac{1660054}{q^{15}} + \frac{1541040}{q^{16}}$$
$$+ \frac{1427108}{q^{17}} + \frac{1319509}{q^{18}} + \frac{1216997}{q^{19}} + \frac{1120655}{q^{20}} + \frac{1029313}{q^{21}} + \frac{943912}{q^{22}} + \frac{863295}{q^{23}} + \frac{788318}{q^{24}}$$
$$+ \frac{717874}{q^{25}} + \frac{652685}{q^{26}} + \frac{591724}{q^{27}} + \frac{535609}{q^{28}} + \frac{483363}{q^{29}} + \frac{435536}{q^{30}} + \frac{391207}{q^{31}} + \frac{350843}{q^{32}}$$
$$+ \frac{313613}{q^{33}} + \frac{279905}{q^{34}} + \frac{248940}{q^{35}} + \frac{221077}{q^{36}} + \frac{195604}{q^{37}} + \frac{172813}{q^{38}} + \frac{152077}{q^{39}} + \frac{133640}{q^{40}}$$
$$+ \frac{116937}{q^{41}} + \frac{102194}{q^{42}} + \frac{88894}{q^{43}} + \frac{77233}{q^{44}} + \frac{66769}{q^{45}} + \frac{57661}{q^{46}} + \frac{49518}{q^{47}} + \frac{42492}{q^{48}} + \frac{36241}{q^{49}}$$
$$+ \frac{30888}{q^{50}} + \frac{26151}{q^{51}} + \frac{22133}{q^{52}} + \frac{18588}{q^{53}} + \frac{15617}{q^{54}} + \frac{13003}{q^{55}} + \frac{10834}{q^{56}} + \frac{8941}{q^{57}} + \frac{7388}{q^{58}} + \frac{6032}{q^{59}}$$

$$+\frac{4939}{q^{60}}+\frac{3988}{q^{61}}+\frac{3233}{q^{62}}+\frac{2579}{q^{63}}+\frac{2069}{q^{64}}+\frac{1625}{q^{65}}+\frac{1289}{q^{66}}+\frac{996}{q^{67}}+\frac{779}{q^{68}}+\frac{592}{q^{69}}+\frac{457}{q^{70}}+\frac{338}{q^{71}}$$
$$+\frac{258}{q^{72}}+\frac{185}{q^{73}}+\frac{138}{q^{74}}+\frac{96}{q^{75}}+\frac{70}{q^{76}}+\frac{46}{q^{77}}+\frac{34}{q^{78}}+\frac{20}{q^{79}}+\frac{14}{q^{80}}+\frac{8}{q^{81}}+\frac{6}{q^{82}}+\frac{2}{q^{83}}+\frac{2}{q^{84}}$$

$$\begin{aligned}
f_{97}(q) = & -2q^{138}-2q^{137}-6q^{136}-8q^{135}-15q^{134}-23q^{133}-38q^{132}-53q^{131}-83q^{130}-112q^{129}\\
& -159q^{128}-211q^{127}-290q^{126}-372q^{125}-492q^{124}-621q^{123}-795q^{122}-986q^{121}\\
& -1237q^{120}-1509q^{119}-1870q^{118}-2256q^{117}-2747q^{116}-3291q^{115}-3968q^{114}-4708q^{113}\\
& -5618q^{112}-6618q^{111}-7815q^{110}-9126q^{109}-10683q^{108}-12365q^{107}-14337q^{106}\\
& -16460q^{105}-18898q^{104}-21518q^{103}-24498q^{102}-27655q^{101}-31208q^{100}-34961q^{99}\\
& -39122q^{98}-43481q^{97}-48283q^{96}-53251q^{95}-58671q^{94}-64247q^{93}-70248q^{92}\\
& -76369q^{91}-82909q^{90}-89489q^{89}-96453q^{88}-103408q^{87}-110671q^{86}-117835q^{85}\\
& -125260q^{84}-132470q^{83}-139865q^{82}-146953q^{81}-154109q^{80}-160849q^{79}-167589q^{78}\\
& -173771q^{77}-179860q^{76}-185309q^{75}-190550q^{74}-195053q^{73}-199295q^{72}-202693q^{71}\\
& -205766q^{70}-207955q^{69}-209750q^{68}-210625q^{67}-211113q^{66}-210625q^{65}-209750q^{64}\\
& -207955q^{63}-205766q^{62}-202693q^{61}-199295q^{60}-195053q^{59}-190550q^{58}-185309q^{57}\\
& -179860q^{56}-173771q^{55}-167589q^{54}-160849q^{53}-154109q^{52}-146953q^{51}-139865q^{50}\\
& -132470q^{49}-125260q^{48}-117835q^{47}-110671q^{46}-103408q^{45}-96453q^{44}-89489q^{43}\\
& -82909q^{42}-76369q^{41}-70248q^{40}-64247q^{39}-58671q^{38}-53251q^{37}-48283q^{36}\\
& -43481q^{35}-39122q^{34}-34961q^{33}-31208q^{32}-27655q^{31}-24497q^{30}-21515q^{29}\\
& -18894q^{28}-16453q^{27}-14324q^{26}-12348q^{25}-10658q^{24}-9093q^{23}-7772q^{22}-6564q^{21}\\
& -5551q^{20}-4631q^{19}-3880q^{18}-3194q^{17}-2643q^{16}-2148q^{15}-1755q^{14}-1401q^{13}\\
& -1133q^{12}-889q^{11}-707q^{10}-544q^{9}-425q^{8}-318q^{7}-247q^{6}-178q^{5}-134q^{4}-95q^{3}\\
& -70q^{2}-46q-34-\frac{20}{q}-\frac{14}{q^{2}}-\frac{8}{q^{3}}-\frac{6}{q^{4}}-\frac{2}{q^{5}}-\frac{2}{q^{6}}.
\end{aligned}$$

Using the recursion relation (2.5) and the above initial data, we obtain, for example, f_{99} and f_{101}.

$$
\begin{aligned}
f_{99}(q) = {} & q^{148} + 3q^{147} + 4q^{146} + 7q^{145} + 13q^{144} + 19q^{143} + 29q^{142} + 45q^{141} + 64q^{140} + 97q^{139} \\
& + 137q^{138} + 192q^{137} + 262q^{136} + 363q^{135} + 475q^{134} + 632q^{133} + 828q^{132} + 1079q^{131} \\
& + 1385q^{130} + 1785q^{129} + 2257q^{128} + 2859q^{127} + 3574q^{126} + 4454q^{125} + 5499q^{124} \\
& + 6782q^{123} + 8269q^{122} + 10076q^{121} + 12182q^{120} + 14689q^{119} + 17588q^{118} + 21024q^{117} \\
& + 24957q^{116} + 29578q^{115} + 34862q^{114} + 40991q^{113} + 47977q^{112} + 56047q^{111} + 65162q^{110} \\
& + 75627q^{109} + 87436q^{108} + 100875q^{107} + 115973q^{106} + 133090q^{105} + 152223q^{104} \\
& + 173796q^{103} + 197846q^{102} + 224787q^{101} + 254715q^{100} + 288123q^{99} + 325026q^{98} \\
& + 366027q^{97} + 411206q^{96} + 461113q^{95} + 515887q^{94} + 576158q^{93} + 642003q^{92} + 714125q^{91} \\
& + 792633q^{90} + 878190q^{89} + 970956q^{88} + 1071647q^{87} + 1180302q^{86} + 1297720q^{85} \\
& + 1423977q^{84} + 1559750q^{83} + 1705123q^{82} + 1860827q^{81} + 2026797q^{80} + 2203794q^{79} \\
& + 2391721q^{78} + 2591201q^{77} + 2802128q^{76} + 3025118q^{75} + 3259834q^{74} + 3506904q^{73} \\
& + 3765949q^{72} + 4037374q^{71} + 4320724q^{70} + 4616400q^{69} + 4923670q^{68} + 5242901q^{67} \\
& + 5573242q^{66} + 5914833q^{65} + 6266753q^{64} + 6629069q^{63} + 7000524q^{62} + 7381167q^{61} \\
& + 7769650q^{60} + 8165746q^{59} + 8567991q^{58} + 8976142q^{57} + 9388421q^{56} + 9804559q^{55} \\
& + 10222689q^{54} + 10642306q^{53} + 11061511q^{52} + 11479791q^{51} + 11894949q^{50} + 12306545q^{49} \\
& + 12712390q^{48} + 13111827q^{47} + 13502683q^{46} + 13884422q^{45} + 14254688q^{44} + 14613062q^{43} \\
& + 14957266q^{42} + 15286787q^{41} + 15599512q^{40} + 15895110q^{39} + 16171340q^{38} + 16428123q^{37} \\
& + 16663441q^{36} + 16877174q^{35} + 17067534q^{34} + 17234699q^{33} + 17376881q^{32} + 17494540q^{31} \\
& + 17586169q^{30} + 17652279q^{29} + 17691688q^{28} + 17705222q^{27} + 17691688q^{26} + 17652279q^{25} \\
& + 17586169q^{24} + 17494540q^{23} + 17376881q^{22} + 17234699q^{21} + 17067534q^{20} + 16877174q^{19} \\
& + 16663441q^{18} + 16428123q^{17} + 16171340q^{16} + 15895110q^{15} + 15599512q^{14} + 15286787q^{13} \\
& + 14957266q^{12} + 14613062q^{11} + 14254688q^{10} + 13884422q^{9} + 13502683q^{8} + 13111827q^{7}
\end{aligned}
$$

$$+12712390q^6+12306545q^5+11894949q^4+11479791q^3+11061511q^2+10642306q$$
$$+10222689+\frac{9804559}{q}+\frac{9388421}{q^2}+\frac{8976140}{q^3}+\frac{8567989}{q^4}+\frac{8165740}{q^5}+\frac{7769642}{q^6}$$
$$+\frac{7381153}{q^7}+\frac{7000504}{q^8}+\frac{6629035}{q^9}+\frac{6266708}{q^{10}}+\frac{5914767}{q^{11}}+\frac{5573154}{q^{12}}+\frac{5242778}{q^{13}}$$
$$+\frac{4923512}{q^{14}}+\frac{4616185}{q^{15}}+\frac{4320456}{q^{16}}+\frac{4037025}{q^{17}}+\frac{3765518}{q^{18}}+\frac{3506363}{q^{19}}+\frac{3259179}{q^{20}}$$
$$+\frac{3024311}{q^{21}}+\frac{2801174}{q^{22}}+\frac{2590053}{q^{23}}+\frac{2390380}{q^{24}}+\frac{2202215}{q^{25}}+\frac{2024987}{q^{26}}+\frac{1858731}{q^{27}}$$
$$+\frac{1702760}{q^{28}}+\frac{1557064}{q^{29}}+\frac{1420988}{q^{30}}+\frac{1294384}{q^{31}}+\frac{1176647}{q^{32}}+\frac{1067621}{q^{33}}+\frac{966613}{q^{34}}$$
$$+\frac{873484}{q^{35}}+\frac{787622}{q^{36}}+\frac{708780}{q^{37}}+\frac{636395}{q^{38}}+\frac{570257}{q^{39}}+\frac{509784}{q^{40}}+\frac{454786}{q^{41}}+\frac{404747}{q^{42}}$$
$$+\frac{359428}{q^{43}}+\frac{318385}{q^{44}}+\frac{281421}{q^{45}}+\frac{248074}{q^{46}}+\frac{218188}{q^{47}}+\frac{191387}{q^{48}}+\frac{167469}{q^{49}}+\frac{146120}{q^{50}}$$
$$+\frac{127189}{q^{51}}+\frac{110365}{q^{52}}+\frac{95530}{q^{53}}+\frac{82425}{q^{54}}+\frac{70921}{q^{55}}+\frac{60819}{q^{56}}+\frac{52021}{q^{57}}+\frac{44322}{q^{58}}+\frac{37655}{q^{59}}$$
$$+\frac{31873}{q^{60}}+\frac{26892}{q^{61}}+\frac{22594}{q^{62}}+\frac{18928}{q^{63}}+\frac{15778}{q^{64}}+\frac{13110}{q^{65}}+\frac{10841}{q^{66}}+\frac{8928}{q^{67}}+\frac{7315}{q^{68}}+\frac{5975}{q^{69}}$$
$$+\frac{4844}{q^{70}}+\frac{3913}{q^{71}}+\frac{3143}{q^{72}}+\frac{2510}{q^{73}}+\frac{1989}{q^{74}}+\frac{1570}{q^{75}}+\frac{1227}{q^{76}}+\frac{955}{q^{77}}+\frac{737}{q^{78}}+\frac{562}{q^{79}}+\frac{425}{q^{80}}+\frac{321}{q^{81}}$$
$$+\frac{237}{q^{82}}+\frac{174}{q^{83}}+\frac{126}{q^{84}}+\frac{90}{q85}+\frac{62}{q^{86}}+\frac{43}{q^{87}}+\frac{29}{q^{88}}+\frac{19}{q^{89}}+\frac{13}{q^{90}}+\frac{7}{q^{91}}+\frac{4}{q^{92}}+\frac{3}{q^{93}}+\frac{1}{q^{94}}$$
$$f_{101}(q)=-q^{150}-3q^{149}-4q^{148}-7q^{147}-13q^{146}-22q^{145}-31q^{144}-49q^{143}-70q^{142}-104q^{141}$$
$$-143q^{140}-200q^{139}-267q^{138}-362q^{137}-469q^{136}-615q^{135}-785q^{134}-1004q^{133}$$
$$-1255q^{132}-1572q^{131}-1934q^{130}-2385q^{129}-2896q^{128}-3518q^{127}-4232q^{126}$$
$$-5092q^{125}-6058q^{124}-7209q^{123}-8518q^{122}-10050q^{121}-11783q^{120}-13792q^{119}$$
$$-16049q^{118}-18640q^{117}-21530q^{116}-24809q^{115}-28458q^{114}-32550q^{113}-37056q^{112}$$
$$-42074q^{111}-47562q^{110}-53611q^{109}-60179q^{108}-67353q^{107}-75086q^{106}-83471q^{105}$$
$$-92432q^{104}-102059q^{103}-112288q^{102}-123190q^{101}-134666q^{100}-146812q^{99}$$
$$-159497q^{98}-172808q^{97}-186601q^{96}-200949q^{95}-215696q^{94}-230912q^{93}$$
$$-246398q^{92}-262233q^{91}-278215q^{90}-294397q^{89}-310547q^{88}-326745q^{87}$$
$$-342737q^{86}-358588q^{85}-374042q^{84}-389156q^{83}-403679q^{82}-417684q^{81}$$
$$-430885q^{80}-443382q^{79}-454907q^{78}-465550q^{77}-475049q^{76}-483527q^{75}$$

$$
\begin{aligned}
&- 490725q^{74} - 496781q^{73} - 501455q^{72} - 504899q^{71} - 506903q^{70} - 507651q^{69} \\
&- 506903q^{68} - 504899q^{67} - 501455q^{66} - 496781q^{65} - 490725q^{64} - 483527q^{63} \\
&- 475049q^{62} - 465550q^{61} - 454907q^{60} - 443382q^{59} - 430885q^{58} - 417684q^{57} \\
&- 403679q^{56} - 389156q^{55} - 374042q^{54} - 358588q^{53} - 342737q^{52} - 326745q^{51} \\
&- 310547q^{50} - 294397q^{49} - 278215q^{48} - 262233q^{47} - 246398q^{46} - 230912q^{45} \\
&- 215696q^{44} - 200949q^{43} - 186601q^{42} - 172808q^{41} - 159497q^{40} - 146812q^{39} \\
&- 134666q^{38} - 123190q^{37} - 112288q^{36} - 102059q^{35} - 92432q^{34} - 83469q^{33} - 75084q^{32} \\
&- 67347q^{31} - 60171q^{30} - 53597q^{29} - 47543q^{28} - 42044q^{27} - 37018q^{26} - 32495q^{25} \\
&- 28392q^{24} - 24722q^{23} - 21428q^{22} - 18511q^{21} - 15904q^{20} - 13620q^{19} - 11597q^{18} \\
&- 9840q^{17} - 8299q^{16} - 6972q^{15} - 5820q^{14} - 4841q^{13} - 3994q^{12} - 3281q^{11} - 2677q^{10} \\
&- 2175q^{9} - 1748q^{8} - 1400q^{7} - 1110q^{6} - 875q^{5} - 683q^{4} - 528q^{3} - 403q^{2} - 307q - 229 \\
&- \frac{170}{q} - \frac{124}{q^2} - \frac{90}{q^3} - \frac{62}{q^4} - \frac{43}{q^5} - \frac{29}{q^6} - \frac{19}{q^7} - \frac{13}{q^8} - \frac{7}{q^9} - \frac{4}{q^{10}} - \frac{3}{q^{11}} - \frac{1}{q^{12}}.
\end{aligned}
$$

D Type I Lie superalgebra and its quantization

We give a summary of the representation theory of $osp(2|2n)$ and the quantum group $U_h(\text{type I})$ in [40][4]. For $osp(2|2n)$, the set of positive roots is $\Delta^+ = \Delta^+_{\bar 0} \cup \Delta^+_{\bar 1}$ with

$$
\Delta^+_{\bar 0} = \{\delta_i \pm \delta_j | 1 \le i < j \le n\} \cup \{2\delta_i\} \quad \text{and} \quad \Delta^+_{\bar 1} = \{\epsilon \pm \delta_i\}, \qquad n \in \mathbb{Z}_+,
$$

where $\{\epsilon, \delta_1, \cdots, \delta_n\}$ form the dual basis of the Cartan subalgebra. Their inner products are

$$
\langle \epsilon, \epsilon \rangle = 1 \qquad \langle \delta_i, \delta_j \rangle = -\delta_{ij} \qquad \langle \epsilon, \delta_j \rangle = 0
$$

The half sums of positive roots are

$$
\rho_0 = \sum_i (n+1-i)\delta_i, \quad \rho_1 = n\epsilon \quad \text{and} \quad \rho = \rho_0 - \rho_1.
$$

[4]For reviews on Lie superalgebras, see [62, 63, 64]

The fundamental weights are

$$w_1 = \epsilon \qquad w_{k+1} = \epsilon + \sum_{i=1}^{k} \delta_i \quad \text{and} \quad k = 1, \cdots, n.$$

The weight λ_a^c decomposes as

$$\lambda_a^c = aw_1 + c_1 w_2 + \cdots + c_n w_{n+1},$$

where $c \in \mathbb{N}^{r-1}, a \in \mathbb{C}$

Let g be a Lie superalgebra of type I, $sl(m|n)$ or $osp(2|2n)\,(m \neq n)$. $U_h(g)$ is the $\mathbb{C}[[h]]$-Hopf superalgebra generated by

$$E_i, F_i, h_i, \quad i = 1, \cdots, r = m + n - 1 \quad \text{or} \quad n + 1$$

satisfying the relations

$$[h_i, h_j] = 0, \quad [h_i, E_j] = A_{ij} E_j, \quad [h_i, F_j] = A_{ij} F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad E_s^2 = F_s^2 = 0$$

and the quantum Serre-type relations; they are quadratic, cubic and quartic relations among E_i or F_i (see [120] Definition 4.2.1). A_{ij} is $r \times r$ Cartan matrix and $\{s\} = \tau \subset \{1, \cdots, r\}$ determining the parity of the generators. E_s, F_s are the only odd generators. Moreover, the (anti)commutator is $[x, y] := xy - (-1)^{\bar{x}\bar{y}} yx$. The Hopf algebra structure is

$$\Delta(E_i) = \Delta(E_i) \otimes 1 + q^{-h_i} \otimes \Delta(E_i), \quad \epsilon(\Delta(E_i)) = 0, \quad S(\Delta(E_i)) = -q^{h_i} \Delta(E_i)$$

$$\Delta(F_i) = \Delta(F_i) \otimes 1 + q^{-h_i} \otimes \Delta(F_i), \quad \epsilon(\Delta(F_i)) = 0, \quad S(\Delta(E_i)) = -\Delta(F_i) q^{h_i}$$

$$\Delta(h_i) = \Delta(F_i) \otimes 1 + 1 \otimes \Delta(h_i), \quad \epsilon(\Delta(h_i)) = 0, \quad S(\Delta(h_i)) = -h_i$$

E Derivations

We setup for the derivations of the ingredients in the CGP invariant formula using Appendix D and outline the derivation of $\hat{Z}^{osp(2|2)}$ in Section 4.2.

The root lattice Λ_R of $osp(2|2)$ are generated by 2δ and $\epsilon - \delta$, hence, two dimensional. The pivotal element π

$$\begin{aligned} \pi &= 2(\rho_0 - \rho_1) - 2l\rho_0 \in \Lambda_R \\ &= -2(\epsilon - \delta) - l(2\delta) \end{aligned}$$

This implies that

$$K_\pi = K_1^{-l} K_2^{-2} \in U_\xi^H(osp(2|2))$$

The weight λ of V is

$$\begin{aligned} \lambda &= \mu_1 w_1 + \mu_2 w_2 \\ &= (\mu_1 + \mu_2)\epsilon + \mu_2 \delta, \end{aligned}$$

where $w_1 = \epsilon$ and $w_2 = \epsilon + \delta$. Under the assumption mentioned in Section 4.1, we arrive at

$$\theta_V(\vec{\mu}; l) = \xi^{2l\mu_2} \xi^{\mu_1^2 + 2\mu_1\mu_2 - 4\mu_2 - 2\mu_1}\, 1_V \qquad l = \text{odd and} \geq 3.$$

$$\begin{aligned} S(\vec{\mu}, \vec{\mu}'; l) &= \xi^{2l(\mu_2 + \mu_2')} \xi^{2\left(\mu_1\mu_1' + \mu_1\mu_2' + \mu_1'\mu_2\right) - 2\left(\mu_1 + \mu_1' + 2\mu_2 + 2\mu_2'\right)} \\ &\quad \times \frac{\{2l(\mu_2' + 1 - l)\}}{\{2(\mu_2' + 1 - l)\}} \{\mu_1' - 2 + l\} \{\mu_1' + 2\mu_2' - l\} \\ d(\vec{\mu}; l) &= \frac{\{2(\mu_2 + 1 - l)\}}{\{2l(\mu_2 + 1 - l)\}} \frac{1}{\{\mu_1 - 2 + l\}\{\mu_1 + 2\mu_2 - l\}}, \\ \{x\}_\xi &:= \xi^x - \xi^{-x} \quad \xi = q^{1/2} = e^{i2\pi/l}, \end{aligned}$$

where (μ_1, μ_2) and (μ_1', μ_2') are the coefficients in the weight decompositions of λ and μ for V and V', respectively (see Appendix D). For the S-matrix, notations for V and V' are switched.

In order to apply the derivation of the superalgebra $\hat{Z}$ given in [33], we need to adapt the above three expressions into a plumbing graph setup. We first let

$$\vec{\alpha} = (\alpha_1 = \mu_1 - 2 + l,\, \alpha_2 = 2\mu_2 + 2 - 2l) \quad \in \mathbb{C}^2$$

Then

$$\theta_V = \xi^{\alpha_1^2 + \alpha_1\alpha_2}$$

$$S' = \xi^{2\alpha_1\alpha_1' + \alpha_1\alpha_2' + \alpha_2\alpha_1'} \frac{\{l\alpha_2'\}_\xi \{\alpha_1'\}_\xi \{\alpha_1' + \alpha_2'\}_\xi}{\{\alpha_2'\}_\xi}$$

We next shift α_1 and α_2 by s and t, respectively. And then we associate θ_V to each vertex. This leads to

$$\theta_{V_{\alpha^I_{s^I t^I}}} = \xi^{(\mu_1 - 2 + s)^I (\mu_1 + 2\mu_2 + s + t)^I}.$$

Similarly, the S-matrix elements between to vertices I and J of graphs contain

$$\prod_{(I,J) \in Edges} S'(\alpha^J_{s^J t^J}, \alpha^I_{s^I t^I}) \supset \xi^{\sum_{IJ} B_{IJ} (\mu_1 - 2 + s)^I (\mu_1 + 2\mu_2 + s + t)^J},$$

which is the relevant part for the derivation. For the modified quantum dimension $d(\vec{\alpha})$

$$d(\vec{\alpha}) = \frac{\{\alpha_2\}_\xi}{\{l\alpha_2\}_\xi \{\alpha_1\}_\xi \{\alpha_1 + \alpha_2\}_\xi},$$

after shifting by s and t as above and some manipulations, the roots of unity dependent factor becomes

$$d(\vec{\alpha}) \supset \frac{\xi^{2(\mu_1 + 2\mu_2 + s + t)} - \xi^{2(\mu_1 + s - 2)}}{\left(1 - \xi^{2(\mu_1 + 2\mu_2 + s + t)}\right)\left(1 - \xi^{2(\mu_1 + s - 2)}\right)}$$

We define coordinates of the 2D Cartan subalgebra of $osp(2|2)$ to be

$$y = \xi^{2(\mu_1 + 2\mu_2 + s + t)} \qquad z = \xi^{2(\mu_1 + s - 2)}$$

Then the modified quantum dimension for a graph contains

$$d(y,z) \supset \frac{y_I - z_I}{(1-y_I)(1-z_I)}$$

We next sketch the derivation of $\hat{Z}^{osp(2|2)}$ in Section 4.2 by applying the procedure in Appendix D of [33]. For a closed oriented graph 3-manifold Y equipped with $\omega \in H^1(Y;\mathbb{Q}/\mathbb{Z}\times\mathbb{Q}/\mathbb{Z})$, the CGP invariant for a pair (Y,ω) in which Y is presented by Dehn surgery is

$$N_l(Y,\omega) = \sum_{s^I,t^I=0}^{l-1} d(\alpha^{I_0}_{s^{I_0}t^{I_0}}) \prod_{I\in Vert} d(\alpha^I_{s^I t^I}) \left\langle \theta_{V_{\alpha^I_{s^I t^I}}} \right\rangle^{B_{II}} \prod_{(I,J)\in Edges} S'(\alpha^J_{s^J t^J}, \alpha^I_{s^I t^I})$$

$$\omega \in H^1(Y;\mathbb{Q}/\mathbb{Z}\times\mathbb{Q}/\mathbb{Z}) \cong B^{-1}\mathbb{Z}^L/\mathbb{Z}^L \times B^{-1}\mathbb{Z}^L/\mathbb{Z}^L$$

$$\omega([m_I]) = \mu^I = (\mu_1^I, \mu_2^I) \in \mathbb{Q}/\mathbb{Z}\times\mathbb{Q}/\mathbb{Z}, \qquad \sum_J B_{IJ}\mu^J = 0 \mod \mathbb{Z}\times\mathbb{Z},$$

where m_I is a meridian of I-th component L_I of a surgery link L and $[m_I]$ is its homology class and B is a linking matrix of L. A color of L_I is set by $\omega([m_I])$. The origin of ω can be found in the footnote in Section 1.3 of [14]. After substituting the ingredients, the right hand side becomes

$$\begin{aligned} N_l(Y,\omega) = &\frac{1}{l^{L+1}} \prod_{I\in V} \left(e^{i2\pi\mu_1^I} - e^{-i2\pi\mu_1^I}\right)^{deg(I)-2} \\ &\times \sum_{a^I,b^I\in\mathbb{Z}^L/l\mathbb{Z}^L} F\left(\left\{\xi^{2(\mu_1+2\mu_2+a)}, \xi^{2(\mu_1+b)}\right\}_{I\in V}\right) \xi^{\sum_{IJ} B_{IJ}(\mu_1+2\mu_2+a)^I(\mu_1+b)^J}, \end{aligned}$$

where $a^I = s^I + t^I$, $b^I = s^I - 2$ and

$$F(y,z) = \prod_{I\in V} \left(\frac{y_I - z_I}{(1-y_I)(1-z_I)}\right)^{2-deg(I)}.$$

We next expand $F(y,z)$, which modifies the summand as

$$\sum_{a^I,b^I\in\mathbb{Z}^L/l\mathbb{Z}^L} \xi^{\sum_{IJ} B_{IJ}(\mu_1+2\mu_2+a)^I(\mu_1+b)^J + 2\sum_I n_I(\mu_1+2\mu_2+a)^I + m_I(\mu_1+b)^I}$$

And then we recast it in terms of matrices by forming

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \qquad r = \begin{pmatrix} b \\ a \end{pmatrix} \qquad p = \begin{pmatrix} B(\mu_1 + 2\mu_2) + 2m \\ B\mu_1 + 2n \end{pmatrix},$$

This enables us to apply the appropriate Gauss reciprocity formula

$$\sum_{r \in \mathbb{Z}^L/l\mathbb{Z}^L} exp\left(\frac{i2\pi}{l} r^T \mathcal{M} r + \frac{i2\pi}{l} p^T r\right) =$$
$$\frac{e^{i\pi\sigma(\mathcal{M})/4}(l/2)^{N/2}}{|Det\mathcal{M}|^{1/2}} \sum_{\delta \in \mathbb{Z}^L/2\mathcal{M}\mathbb{Z}^L} exp\left(-\frac{i\pi l}{2}\left(\delta + \frac{p}{l}\right)^T \mathcal{M}^{-1}\left(\delta + \frac{p}{l}\right)\right),$$

where $\mathcal{M}$ is a non-degenerate symmetric $2L \times 2L$ matrix over $\mathbb{Z}$ and $\sigma(\mathcal{M})$ is its signature. From the p-quadratic term in the exponential we read off

$$RHS \supset \xi^{-4m^T B^{-1} n},$$

which becomes the q-term in the summand in Section 4.2 after we analytically continue from a complex unit circle into an interior of an unit disc coordinatized by q.

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