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BIMODULES ASSOCIATED TO TWISTED MODULES OF VERTEX OPERATOR ALGEBRAS

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

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by

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Abstract

Bimodules associated to twisted modules of vertex operator algebras

by

Yiyi Zhu

Let V be a vertex operator algebra, $T \in \mathbb{N}$ and (M^k, Y_{M^k}) for $k = 1, 2, 3$ be a g_k-twisted module, where g_k are commuting automorphisms of V such that $g_k^T = 1$ for $k = 1, 2, 3$ and $g_3 = g_1 g_2$. Suppose $I(\cdot, z)$ is an intertwining operator of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$. We construct an $A_{g_1 g_2}(V)$-$A_{g_2}(V)$-bimodule $A_{g_1 g_2, g_2}(M^1)$ which determines the action of M^1 from the bottom level of M^2 to the bottom level of M^3.

To my parents.

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Chapter 1

Introduction

Ever since its appearance, vertex (operator) algebra has played an important role in conformal field theory [MS], and in the study of moonshine and Monster [FLM], [B]. A vertex operator algebra is a vector space equipped with a linear map that sends each vector to a sequence of operators. These sequences of operators satisfy some axioms, among which the most important one is the so-called Jacobi identity. While they may look very different, vertex operator algebras can be viewed as analogs of Lie algebras and commutative associative algebras. In fact, the Jacobi identity is equivalent to associativity and commutativity [FHL].

In this thesis, we mainly focus on the associative aspect of vertex operator algebras. In [Z], the famous Zhu's algebra was constructed. Given a vertex operator algebra V, Zhu constructed an associative algebra $A(V)$ which is obtained from all weight-zero components of vertex operators modulo some relations hiding in Jacobi identity. Zhu in [Z] also established a one-to-one correspondence between the set of equivalence classes of irreducible $A(V)$-modules and the set of equivalence classes of irreducible admissible V-modules. On the other hand, Zhu proved for an admissible V-module $M = \oplus_{n\in\mathbb{N}} M(n)$, the bottom level $M(0)$ is an $A(V)$-module. For an $A(V)$-module U, one can construct an admissible V-module whose bottom level is exactly U. In [DLM3], Zhu's construction was generalized to the twisted case. Given a vertex operator algebra V and an automorphism g of V with finite order T, an associative algebra $A_g(V)$ was constructed, and the notion of g-twisted V-module was introduced. It was established in [DLM3] that there is a one-to-one correspondence between the set of equivalence classes of irreducible $A_g(V)$-modules and the set of equivalence classes of irreducible admissible g-twisted V-modules. For an admissible g-twisted V-module $M = \oplus_{n\in\mathbb{N}} M(\frac{n}{T})$, the bottom level $M(0)$ is an $A_g(V)$-module. For an $A_g(V)$-module U, one can construct an admissible g-twisted V-module whose bottom level is exactly U.

The Zhu's construction can also be generalized to any V-module, see [FZ]. For this purpose, the notion of intertwining operator jumps in, see [FHL]. Let M^1, M^2 and M^3 be three V-modules. An intertwining operator of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$ is a linear map I: $M^1 \mapsto \mathrm{Hom}_{\mathbb{C}}(M^2, M^3)\{z\}$ satisfying similar axioms as in the definition of V-modules, including the Jacobi identity. Each homogeneous vector in M^1 corresponds to a sequence of operators in $\mathrm{Hom}_{\mathbb{C}}(M^2, M^3)$. Since $\mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$ has an $A(V)$-

$A(V)$-bimodule structure, in [FZ], Frankel and Zhu focus on weight-zero operators and constructed an $A(V)$-$A(V)$-bimodule $A(M^1)$, which is a quotient of M^1. As an application, a bijection between $\mathrm{Hom}_{A(V)}(A(M^1) \otimes M^2(0), M^3(0))$ and $\mathcal{V}_{M^1 M^2}^{M^3}$, the space of intertwining operators of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$, was established, see [FZ], [L1], and [L2]. Theoretically, this provides us one way to compute fusion rules. There have been many generalizations of Zhu's $A(V)$ theory (see for examples, [DJ], [DLM4], [DR], [JJ], [MT].)

The goal of this thesis is to generalize the $A(V)$-$A(V)$-bimodule construction to twisted case. Let M^i be a g_i-twisted module, $i = 1, 2, 3$. There is also the notion of intertwining operator of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$, see [X], [DLM1]. Instead of Jacobi identity, generalized Jacobi identity is required in the definition. In this thesis, we shall construct an $A_{g_3}(V)$-$A_{g_2}(V)$-bimodule $A_{g_3,g_2}(M^1)$ in the case $g_3 = g_1 g_2$, which is true if there exists a nonzero intertwining operator of this type. In the future, we will build a bijection between $\mathrm{Hom}_{A_{g_1 g_2}(V)}(A_{g_1 g_2, g_2}(M^1) \otimes M^2(0), M^3(0))$ and $\mathcal{V}_{M^1 M^2}^{M^3}$.

This thesis is organized as follows: In Section 2, we recall some definitions and results required for reading this thesis. In section 3, we present the construction of $A_{g_1 g_2, g_2}(M^1)$.

Chapter 2

Preliminary

Throughout this thesis, we denote the field of complex numbers by $\mathbb{C}$, the field of rational numbers by $\mathbb{Q}$, the ring of integers by $\mathbb{Z}$, and the set of natural numbers by $\mathbb{N}$.

§ 2.1 Formal calculus

In this section, we shall present some elementary formal calculus which is basic to the theory of vertex (operator) algebra. For more details, see [FHL] and [FLM]. Let V be a vector space. Throughout the thesis, we adopt the following notations:

$$V[z] = \left\{ \sum_{n\in\mathbb{N}} v_n z^n | v_n \in V, \text{all but finitely many } v_n = 0 \right\},$$

$$V[z^{-1}, z] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n | v_n \in V, \text{all but finitely many } v_n = 0 \right\},$$

$$V[[z]] = \left\{ \sum_{n \in \mathbb{N}} v_n z^n | v_n \in V \right\},$$

$$V[[z^{-1}, z]] = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n | v_n \in V \right\},$$

$$V((z)) = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n | v_n \in V, v_n = 0 \text{ for } n \text{ sufficiently small} \right\},$$

$$V\{z\} = \left\{ \sum_{\lambda \in \mathbb{C}} v_\lambda z^\lambda | v_\lambda \in V \right\}.$$

These notations can be extended to multivariable cases in a similar way. In the theory of vertex operator algebras, we will be dealing with a lot of formal series in the space $\mathrm{End}(V)[[z^{-1}, z]]$. These formal series are called vertex operators. We shall also need to consider formal sums of infinitely many formal series, as well as products of finitely many formal series frequently. Thus we might get formal series with coefficients represented by a infinite sum of operators in $\mathrm{End}(V)$. This could lead to formal series with coefficients represented by a infinite sum of vectors in V after apply these formal series to vectors. This is not allowed in the context of vertex operator algebras. So we need the following definitions:

Definition 2.1.1. Let V be a vector space and $(f_i)_{i \in I}$ be a family of operators in $\mathrm{End}(V)$. We say $(f_i)_{i \in I}$ is summable if for any vector $v \in V$, $\sum_{i \in I} f_i(v)$ is a finite sum. In this case, we call $\sum_{i \in I} f_i$ exists.

Definition 2.1.2. Suppose $F_i(z) = \sum_{n \in \mathbb{Z}} f_i(n) z^n, 1 \leq i \leq r$ are a finite family in

$\mathrm{End}(V)[[z^{-1}, z]]$. We say the product

$$F_1(z)F_2(z)\cdots F_r(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{n_1+n_2+\cdots+n_r=n} f_1(n_1)f_2(n_2)\cdots f_r(n_r) \right) z^n$$

exists if for every $n \in \mathbb{Z}$, the family

$$\big(f_1(n_1)f_2(n_2)\cdots f_r(n_r)\big)_{n_1+n_2+\cdots+n_r=n}$$

is summable.

We shall also need the notion of "formal limit".

Definition 2.1.3. Let $f(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} f(m,n) z_1^m z_2^n \in \big(\mathrm{End}(V)\big)[[z_1^{-1}, z_2^{-1}, z_1, z_2]]$. We say that $\lim\limits_{z_1\to z_2} f(z_1, z_2)$ exists if for every $n \in \mathbb{Z}$, the family $\big(f(m, n-m)\big)_{m\in\mathbb{Z}}$ is summable.

Definition 2.1.4. (binomial expansion convention) Throughout the thesis, for any real number α, we define $(z_1 + z_2)^\alpha$ to be the formal series

$$(z_1 + z_2)^\alpha = \sum_{k\in\mathbb{N}} \binom{\alpha}{k} z_1^{\alpha-k} z_2^k,$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Note that

$$(z_1 + z_2)^\alpha \neq (z_2 + z_1)^\alpha$$

for $\alpha \notin \mathbb{N}$.

A distinguished formal series in $\mathbb{C}[[z, z^{-1}]]$ which plays an important role in formal calculus is the formal delta function,

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

Below are several properties of $\delta(z)$, see chapter 2 and chapter 8 in [FLM] or chapter 2 in [LL].

(1). $z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) = z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)$.

(2). $z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)$.

(3). Let $f(z_1, z_2) \in \big(\mathrm{End}(V)\big)[[z_1^{-1}, z_2^{-1}, z_1, z_2]]$ be such that $\lim_{z_1\to z_2} f(z_1, z_2)$ exists. Then

$$f(z_1, z_2)\delta\left(\frac{z_1}{z_2}\right) = f(z_1, z_1)\delta\left(\frac{z_1}{z_2}\right) = f(z_2, z_2)\delta\left(\frac{z_1}{z_2}\right).$$

(4). Let $f(z_1, z_2) \in \big(\mathrm{End}(V)\big)[[z_1^{-1}, z_2^{-1}, z_1, z_2]]$ be such that $\lim_{z_1\to z_2} f(z_1, z_2)$ exists. Then

$$f(z_1, z_2)\delta\left(\frac{z_1+z_0}{z_2}\right) = f(z_1, z_1+z_0)\delta\left(\frac{z_1+z_0}{z_2}\right).$$

We will use these properties of delta function a lot in calculation in Chapter 3.

§ 2.2 Vertex operator algebras and their modules

Below are some definitions related to vertex (operator) algebras, see [B], [DLM2], [FHL], [FLM], and [LL].

Definition 2.2.1. A vertex algebra is a triple $(V, Y, \mathbf{1})$, where V is a vector space, Y: $V \mapsto (\mathrm{End}V)[[z, z^{-1}]]$, $v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$, is a linear map and vacuum vector $\mathbf{1} \in V$, satisfying the following conditions:

(V1). Truncation condition: for $u, v \in V$, $Y(u, z)v = \sum_{n \in \mathbb{Z}} u_n v z^{-n-1}$ and $u_n v = 0$ for sufficiently large n;

(V2). Vacuum property: $Y(\mathbf{1}, z) = id_V$;

(V3). Creation property: $Y(u, z)\mathbf{1} \in V[[z]]$ and $\lim_{z \to 0} Y(u, z)\mathbf{1} = u$ for $u \in V$;

(V4). Jacobi identity: for $u, v \in V$,

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta(\frac{-z_2 + z_1}{z_0})Y(v, z_2)Y(u, z_1)$$
$$= z_1^{-1}\delta(\frac{z_2 + z_0}{z_1})Y(Y(u, z_0)v, z_2).$$

As a consequence of Jacobi identity, we have the following associativity and commutativity.

Associativity: for any $a, c \in V$, there is a nonnegative integer r such that for all $b \in V$

$$(z_0 + z_2)^r Y(a, z_0 + z_2)Y(b, z_2)c = (z_2 + z_0)^r Y(Y(a, z_0)b, z_2)c.$$

Commutativity: for any $a, b \in V$, there is a nonnegative integer k such that for any $c \in V$

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2)c = (z_1 - z_2)^k Y(b, z_2)Y(a, z_1)c.$$

In fact, the Jacobi identity is equivalent to the commutativity and the associativity, see [FHL] and [LL].

Example 2.2.2. Let A be a unital commutative associative algebra with a derivation D. Define

$$Y(a,z)b = (e^{zD})b \quad \text{for any } a, b \in A.$$

Then $(A, Y, 1)$ is a vertex algebra, see [B].

Definition 2.2.3. A vertex operator algebra is a quadruple $(V, Y, \mathbf{1}, \omega)$, where $(V, Y, \mathbf{1})$ is a vertex algebra, and $\omega \in V$ satisfying the following conditions:

(V5). $V = \coprod_{n\in\mathbb{Z}} V_n$, $\dim V_n < \infty$ and $V_n = 0$ if $n \ll 0$;

(V6). Virasoro relations:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V$$

where $Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, $c_V \in \mathbb{C}$;

(V7). $L(0)$-eigenspace decomposition: $L(0)u = nu$ for $u \in V_n$. That is, the weight of u, denoted by wt$u$, is the corresponding eigenvalue of $L(0)$;

(V8). $L(-1)$-derivative property: $Y(L(-1)u, z) = \frac{d}{dz}Y(u, z)$;

Example 2.2.4. For affine vertex operator algebras, see [FZ] and [LL]. For Virasoro vertex operator algebras, see [LL] and for lattice vertex operator algebras, see [FLM] and [LL].

Definition 2.2.5. An automorphism g of a vertex operator algebra V is a linear automorphism of V such that:

(1). $gY(u,z)g^{-1} = Y(gu,z)$ for all $u \in V$;

(2). $g\omega = \omega$.

Definition 2.2.6. A weak module of vertex operator algebra V is a vector space M, equipped with a linear map $Y_M : V \mapsto (\mathrm{End}M)[[z^{-1}, z]]$ such that:

(M1). $Y_M(\mathbf{1}, z) = id_M$;

(M2). For $u \in V$ and $w \in M$, $Y_M(u,z)w = \sum_{n\in\mathbb{Z}} u_n w z^{-n-1}$ and $u_n w = 0$ if $n \gg 0$;

(M3). For $u, v \in V$,

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta(\frac{-z_2 + z_1}{z_0})Y_M(v,z_2)Y_M(u,z_1)$$
$$= z_1^{-1}\delta(\frac{z_2 + z_0}{z_1})Y_M(Y(u,z_0)v, z_2).$$

Definition 2.2.7. A weak V-module M is called admissible if the following hold:

(1). It is equipped with an $\mathbb{N}$-grading, $M = \oplus_{n\in\mathbb{N}} M_n$;

(2). For homogeneous $u \in V$, $u_n M_m \subseteq M_{m+\mathrm{wt}u-n-1}$.

That is, M is an $\mathbb{N}$-graded space and for homogeneous $u \in V$, u_n is a homogeneous map of degree $\mathrm{wt}u - n - 1$. In particular, $u_{\mathrm{wt}u-1}$ preserves all homogeneous subspaces of M, which we denote by $o_M(u)$, following [Z].

Definition 2.2.8. A weak V-module M is called ordinary if the following hold:

(1). $M = \oplus_{\lambda \in \mathbb{C}} M_\lambda$ with $\dim M_\lambda < \infty$, $M_{\lambda+n} = 0$ for all sufficiently negative integer n;

(2). $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$.

It's straightforward to show that ordinary modules are admissible. By V-modules we always mean ordinary V-modules.

§ 2.3 Twisted modules and associative algebras

Let V be a vertex operator algebra, g an automorphism of V with order $T < \infty$. Then

$$V = \oplus_{r=0}^{T-1} V^r,$$

where

$$V^r = \{v \in V \mid gv = e^{\frac{2\pi i r}{T}} v\}.$$

The following definitions and results can be found in [DLM3], [Z].

Definition 2.3.1. A weak g-twisted V-module M is a vector space equipped with a linear map $Y_M \colon V \mapsto (\mathrm{End} M)\{z\}$ such that:

(M1). For $u \in V^r, w \in M$, $Y_M(u,z)w = \sum_{n \in \frac{r}{T}+\mathbb{Z}} u_n w z^{-n-1}$ and $u_n w = 0$ if $n \gg 0$;

(M2). $Y_M(\mathbf{1}, z) = id_M$;

(M3). Twisted Jacobi identity: for $u \in V^r, v \in V$,

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta(\frac{-z_2 + z_1}{z_0})Y_M(v, z_2)Y_M(u, z_1)$$

$$= z_1^{-1}\delta(\frac{z_2 + z_0}{z_1})(\frac{z_2 + z_0}{z_1})^{\frac{r}{T}}Y(Y(u, z_0)v, z_2).$$

Definition 2.3.2. A weak g-twisted V-module M is called admissible if the following hold:

(1). $M = \oplus_{n\in\frac{1}{T}\mathbb{N}}M_n$;

(2). For homogeneous $u \in V$, $u_nM_m \subseteq M_{m+\mathrm{wt}u-n-1}$.

For u_n in $Y_M(u, z)$ where $n \in \frac{1}{T}\mathbb{Z}$, again, like in untwisted case, we set

$$\mathrm{wt}u_n = \mathrm{wt}u - n - 1$$

and set

$$o_M(u) = u_{\mathrm{wt}u-1}.$$

As we can see, if M is admissible, then

$$u_nM_n \subseteq M_{n+\mathrm{wt}u_n}$$

and $o_M(u)$ stabilizes each homogeneous subspace M_n.

Definition 2.3.3. A g-twisted V-module M is a weak g-twisted V-module carrying a $\mathbb{C}$-grading such that the following hold:

(1). $M = \oplus_{\lambda \in \mathbb{C}} M_\lambda$ with $\dim M_\lambda < \infty$, $M_{\lambda + \frac{n}{T}} = 0$ for all sufficiently negative integers n;

(2). $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$.

Like in untwisted case, a g-twisted V-module is admissible.

Following [DLM3], for $r \in \mathbb{N}$, define $\delta(r) = 1$ if $r \equiv 0 \pmod{T}$ and $\delta(r) = 0$ otherwise. For $u \in V^r, v \in V$, define

$$u \circ_g v = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}u - 1 + \delta(r) + \frac{r}{T}}}{z^{1+\delta(r)}} Y(u,z)v.$$

Let $O_g(V)$ be the linear span of all $u \circ_g v$ and $(L(-1) + L(0))u$, define $A_g(V)$ to be the quotient space $V/O_g(V)$. Also define

$$u *_g v = \begin{cases} \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}u}}{z} Y_M(u,z)v, & \text{if } u \in V^r \text{ with } r = 0 \\ 0, & \text{otherwise} \end{cases}$$

The following results were obtained in [DLM3].

Proposition 2.3.4. *$V^r \subseteq O_g(V)$ for $0 < r < T$.*

Proposition 2.3.4 tells us that $A_g(V)$ is a quotient of V^0.

Theorem 2.3.5. *Let V be a vertex operator algebra, g an automorphism of V with finite order T. Then $A_g(V)$ is an associative algebra with respect to the operation $*_g$. Furthermore, $\mathbf{1} + O_g(V)$ acts as the identity and $\omega + O_g(V)$ lies in the center of $A_g(V)$.*

Theorem 2.3.6. *Let V be a vertex operator algebra, g an automorphism of V with finite order T. Let $M = \oplus_{n\in\mathbb{N}} M_{\frac{n}{T}}$ be an admissible g-twisted V-module. Then*

$$o_M(u)o_M(v) = o_M(u *_g v)$$

and

$$o(u') = 0$$

hold in End(M_0) for every $u, v \in V$ and $u' \in O_g(V)$. Thus, the bottom level M_0 is a left $A_g(V)$-module with $u + O_g(V)$ acting as $o_M(u)$.

Remark 2.3.7. If $g = id$, then g-twisted V-modules are just V-modules and $A_g(V)$ coincides with Zhu's algebra $A(V)$ which was constructed in [Z]. In this special case, the untwisted version of theorem 2.3.5 and 2.3.6 were established in [Z].

§ 2.4 Intertwining operators

Let g_k ($k = 1, 2, 3$) be three commuting automorphisms of vertex operator algebra V and $T \in \mathbb{N}$, a finite number such that $g_k^T = 1$ for $k = 1, 2, 3$. Let (M^k, Y_{M^k}) be a g_k-twisted ($k = 1, 2, 3$) V-module. Since $g_1 g_2 = g_2 g_1$, we have the following common eigenspace decomposition:

$$V = \oplus_{0\le j_1, j_2 < T} V^{(j_1, j_2)},$$

where

$$V^{(j_1,j_2)} = \{v \in V \mid g_k v = e^{\frac{2\pi i j_k}{T}} v, k = 1, 2\}.$$

The following definition can be found in [X], [DLM1].

Definition 2.4.1. An intertwining operator of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$ is a linear map $I(\cdot, z)$: $M^1 \mapsto (\mathrm{Hom}(M^2, M^3))\{z\}$ such that:

(1) For $w^1 \in M^1, w^2 \in M^2$, $I(w^1, z)w^2 = \sum_{n\in\mathbb{C}} w^1_n w^2 z^{-n-1}$ and $w^1_{c+n}w^2 = 0$ for a fixed $c \in \mathbb{C}$, $n \gg 0$, and $n \in \mathbb{Q}$.

(2) Generalized Jacobi identity: for $u \in V^{(j_1,j_2)}$, $w_1 \in M^1$, $w_2 \in M^2$, and $0 \leq j_1, j_2 \leq T-1$,

$$\begin{aligned} &z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})(\frac{z_1 - z_2}{z_0})^{\frac{j_1}{T}} Y_{M^3}(u, z_1)I(w_1, z_2)w_2 \\ -&z_0^{-1}\delta(\frac{-z_2 + z_1}{z_0})(\frac{z_2 - z_1}{z_0})^{\frac{j_1}{T}} I(w_1, z_2)Y_{M^2}(u, z_1)w_2 \\ =&z_1^{-1}\delta(\frac{z_2 + z_0}{z_1})(\frac{z_2 + z_0}{z_1})^{\frac{j_2}{T}} I(Y_{M^1}(u, z_0)w_1, z_2)w_2 \end{aligned}$$

(3) $L(-1)$-derivative property: for $w_1 \in M^1$,

$$I(L(-1)w_1, z) = \frac{d}{dz}I(w_1, z).$$

Remark 2.4.2. Note that $Y(\cdot, z)$ acting on V is an example of an intertwining operator of type $\begin{pmatrix} V \\ VV \end{pmatrix}$ and $Y_M(\cdot, z)$ acting on a g-twisted module M is an example of an intertwining operator of type $\begin{pmatrix} M \\ VM \end{pmatrix}$.

Denote the space of intertwining operators of type $\left(\begin{array}{c} M^3 \\ \\ M^1 M^2 \end{array}\right)$ by $\mathcal{V}_{M^1 M^2}^{M^3}$ and set

$$N_{M^1 M^2}^{M^3} = \dim \mathcal{V}_{M^1 M^2}^{M^3}.$$

These numbers are called fusion rules associated to these data. If

$$N_{M^1 M^2}^{M^3} > 0,$$

then

$$g_3 = g_1 g_2$$

(see [X]).

For the rest of the thesis, let (M^k, Y_{M^k}) $(k = 1, 2, 3)$ be a g_k-twisted V-module such that

$$M^k = \oplus_{n \in \mathbb{N}} M^k_{h_k + \frac{n}{T}},$$

where

$$L(0) \mid_{M^k_{h_k + \frac{n}{T}}} = (h_k + \frac{n}{T}) \operatorname{Id},$$

and

$$g_3 = g_1 g_2.$$

For convenience, we denote $M^k_{h_k + \frac{n}{T}}$ by $M^k(\frac{n}{T})$ and for $w \in M^k(\frac{n}{T})$, we set

$$\deg w = \frac{n}{T}.$$

Let $I(\cdot, z)$ be an intertwining operator in $\mathcal{V}_{M^1 M^2}^{M^3}$, we have the following associativity.

Proposition 2.4.3. *(Associativity) For homogeneous $u \in V^{(j_1,j_2)}$, $w_1 \in M^1$ and $w_2 \in M^2(0)$, we have*

$$(z_0+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}Y_{M^3}(u,z_0+z_2)I(w_1,z_2)w_2$$
$$=(z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}I(Y_{M^1}(u,z_0)w_1,z_2)w_2.$$

Proof. For homogeneous $u \in V^{(j_1,j_2)}$, $w_1 \in M^1$ and $w_2 \in M^2(0)$, applying $\mathrm{Res}_{z_1}z_1^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}$ to the generalized Jacobi identity gives

$$\mathrm{Res}_{z_1}z_1^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}z_0^{-1}\delta(\frac{z_1-z_2}{z_0})(\frac{z_1-z_2}{z_0})^{\frac{j_1}{T}}Y_{M^3}(u,z_1)I(w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_1}z_1^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}z_1^{-1}\delta(\frac{z_2+z_0}{z_1})(\frac{z_2+z_0}{z_1})^{\frac{j_2}{T}}I(Y_{M^1}(u,z_0)w_1,z_2)w_2.$$

Since (see [FLM])

$$z_0^{-1}\delta(\frac{z_1-z_2}{z_0})(\frac{z_1-z_2}{z_0})^{\frac{j_1}{T}}=z_1^{-1}\delta(\frac{z_0+z_2}{z_1})(\frac{z_0+z_2}{z_1})^{-\frac{j_1}{T}}$$

we have

$$(z_0+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}Y_{M^3}(u,z_0+z_2)I(w_1,z_2)w_2$$
$$=(z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}I(Y_{M^1}(u,z_0)w_1,z_2)w_2.$$

This completes the proof. □

The following proposition is an easy corollary of the generalized Jacobi identity and $L(-1)$-derivative property. (See [FHL] and [FZ].)

Proposition 2.4.4. *Let*

$$I^\circ(\cdot,z)=z^{h_1+h_2-h_3}I(\cdot,z).$$

Then for $w_1 \in M^1$,

$$I^{\circ}(w_1, z) \in (Hom(M^2, M^3))[[z^{\frac{1}{T}}, z^{-\frac{1}{T}}]].$$

Set

$$I^{\circ}(w_1, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} w_1(n) z^{-n-1},$$

then for every homogeneous $w_1 \in M^1$, $n \in \frac{1}{T}\mathbb{Z}$ *and* $m \in \frac{1}{T}\mathbb{N}$,

$$w_1(n)M^2(m) \subseteq M^3(m + \deg w_1 - n - 1).$$

Denote $w_1(\deg w_1 - 1)$ by $o_I(w_1)$. It's obvious that $I^{\circ}(\cdot, z)$ also satisfies the generalized Jacobi identity and associativity.

Chapter 3

Bimodules associated to twisted modules

With the same setting as in chapter 2, in this chapter, we will construct an $A_{g_1g_2}(V)$-$A_{g_2}(V)$-bimodule $A_{g_1g_2,g_2}(M^1)$ and explain why we define it in such way. For $r \in N$, denote the remainder of r divided by T by $[r]$.

For homogeneous $u \in V$ and $w_1 \in M^1$, we define

$$u \circ_{g_1g_2,g_2} w_1 = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z^{1+\delta(j_1,j_2)-\frac{j_1}{T}}} Y_{M^1}(u,z)w_1,$$

where $u \in V^{(j_1,j_2)}$ and

$$\delta(j_1,j_2)=\begin{cases}1, & j_2=0\\ 1, & j_2\neq 0,\ j_1+j_2\geq T\\ 0, & j_2\neq 0,\ j_1+j_2<T\end{cases}$$

Note that $\delta(0,j_2)=\delta(j_2)$.

Define

$$u *_{g_1g_2,g_2} w_1=\begin{cases}\mathrm{Res}_z Y_{M^1}(u,z)w_1\frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z^{1-\frac{j_1}{T}}}, & j_1+j_2\equiv 0 \pmod T\\ 0, & \text{otherwise}\end{cases}$$

Define

$$w_1 *_{g_2,g_1g_2} u=\begin{cases}\mathrm{Res}_z Y_{M^1}(u,z)w_1\frac{(1+z)^{\mathrm{wt}u-1}}{z^{1-\frac{j_1}{T}}}, & j_2=0\\ 0, & \text{otherwise}\end{cases}$$

Let $O'_{g_1g_2,g_2}(M^1)$ be the subspace of M^1 spanned by all $u\circ_{g_1g_2,g_2} w_1$.

Remark 3.0.1. Let $g_1=g_2=1$, then $\circ_{g_1g_2,g_2}$, $*_{g_1g_2,g_2}$ and $*_{g_2,g_1g_2}$ give the same products as in [FZ], where the authors constructed an $A(V)$-$A(V)$-bimodule $A(M)$; Let $M^1=V$, then these three products give the same construction as in [DLM3], where the authors constructed an associative algebra $A_g(V)$.

Lemma 3.0.2. *For homogeneous $u\in V$, $w_1\in M^1$, $m,n\in\mathbb{Z}$ and $m\geq n\geq 0$, we have*

$$\mathrm{Res}_z\frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}+n}}{z^{1+\delta(j_1,j_2)-\frac{j_1}{T}+m}}Y_{M^1}(u,z)w_1\in O'_{g_1g_2,g_2}(M^1).$$

The proof of lemma 3.0.2 is fairly standard (cf. [DLM3] and [Z]).

Lemma 3.0.3. *For $u \in V$, $u *_{g_1g_2,g_2} O'_{g_1g_2,g_2}(M^1) \subseteq O'_{g_1g_2,g_2}(M^1)$.*

Proof. It suffices to show it holds for homogeneous $u \in V^{(j_1,j_2)}$, where $j_1 + j_2 \equiv 0 \pmod T$. Let $u \in V^{(j_1,j_2)}$, $v \in V^{(j_3,j_4)}$, $w_1 \in M^1$ and $j_1 + j_2 \equiv 0 \pmod T$, then

$$
\begin{aligned}
& u *_{g_1g_2,g_2} (v \circ_{g_1g_2,g_2} w_1) \\
=& \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(u,z_1)Y_{M^1}(v,z_2)w_1 \\
=& \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(v,z_2)Y_{M^1}(u,z_1)w_1 \\
&+ \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} z_1^{-1}\delta(\frac{z_2+z_0}{z_1}) \\
& \left(\frac{z_2+z_0}{z_1}\right)^{\frac{j_1}{T}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
\equiv& \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} \frac{(1+z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{(z_2+z_0)^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} \\
& Y_{M^1}(Y(u,z_0)v,z_2)w_1 \left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right) \\
\equiv& \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} \sum_{i\geq 0} \binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} z_0^i (1+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}-i} \sum_{j\geq 0} \\
& \binom{-1+\frac{j_1}{T}}{j} z_0^j z_2^{-1+\frac{j_1}{T}-j} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
& \left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right) \\
\equiv& \mathrm{Res}_{z_2} \sum_{i\geq 0}\sum_{j\geq 0} \binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} \binom{-1+\frac{j_1}{T}}{j} Y_{M^1}(u_{i+j}v,z_2)w_1
\end{aligned}
$$

$$\frac{(1+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}-i+\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1-\frac{j_1}{T}+j+1+\delta(j_3,j_4)-\frac{j_3}{T}}}\left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right)$$

$$\equiv \mathrm{Res}_{z_2}\sum_{i\geq 0}\sum_{j\geq 0}\binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i}\binom{-1+\frac{j_1}{T}}{j}Y_{M^1}(u_{i+j}v,z_2)w_1$$

$$\frac{(1+z_2)^{\mathrm{wt}(u_{i+j}v)-1+\delta(j_2+j_4)+\frac{[j_2+j_4]}{T}+\delta(j_4)+\delta(j_2)+j-\delta(j_2+j_4)+\frac{j_2+j_4}{T}-\frac{[j_2+j_4]}{T}}}{z_2^{1+\delta([j_1+j_3],[j_2+j_4])-\frac{[j_1+j_3]}{T}+j+1+\delta(j_3,j_4)-\frac{j_1}{T}-\frac{j_3}{T}-\delta([j_1+j_3],[j_2+j_4])+\frac{[j_1+j_3]}{T}}}$$

$$\left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right)$$

$$\equiv 0\left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right).$$

Here we explain the last congruence. We compare

$$n=\delta(j_4)+\delta(j_2)+j-\delta(j_2+j_4)+\frac{j_2+j_4}{T}-\frac{[j_2+j_4]}{T}$$

and

$$m=j+1+\delta(j_3,j_4)-\delta([j_1+j_3],[j_2+j_4])+\frac{[j_1+j_3]}{T}-\frac{j_1+j_3}{T}$$

case by case. Note that for $i,j\in\mathbb{Z}$ and $0\leq i,j<T$, $[i+j]=i+j$ when $i+j<T$ and $[i+j]=i+j-T$ when $i+j\geq T$. Since $0\leq j_k<T$ for $k=1,2,3,4$ and $j_1+j_2\equiv 0$ (mod T), we divide it into two cases:

(1) If $j_1+j_2=0$, i.e. $j_1=j_2=0$, then $n=j+1$ and $m=j+1$, $m=n$;

(2) If $j_1+j_2=T$, then $j_1,j_2\neq 0$;

When $j_2+j_4<T$: if $j_4=0$, then $n=j+1$ and $m=j+1$ whether $j_1+j_3<T$ or not. So $m=n$; If $j_4\neq 0$, then $n=j$ and $m=j+\delta(j_3,j_4)$ when $j_1+j_3<T$ and $m=j$ when $j_1+j_3\geq T$, either way we will have $m\geq n$.

When $j_2 + j_4 = T$: then $j_4 \neq 0$ and $j_3 + j_4 = j_1 + j_3$. Hence, $\delta(j_3, j_4) - \delta([j_1 + j_3], [j_2 + j_4]) + \frac{[j_1+j_3]}{T} - \frac{j_1+j_3}{T} = -1$. Therefore, $n = j$, $m = j$, $m = n$.

When $j_2 + j_4 > T$: then $j_4 \neq 0$. Then $n = j + 1$, $m = j + 1$ whether $j_1 + j_3 < T$ or not. Either case, we have $m = n$.

Now by lemma 3.0.2, the last congruence holds. □

Lemma 3.0.4. *For $u \in V$, $O'_{g_1g_2,g_2}(M^1) *_{g_2,g_1g_2} u \subseteq O'_{g_1g_2,g_2}(M^1)$.*

Proof. It suffices to show it holds for homogeneous $u \in V^{(j_1,j_2)}$, where $j_2 = 0$. Let $u \in V^{(j_1,0)}$, $v \in V^{(j_3,j_4)}$ and $w_1 \in M^1$, then

$$
\begin{aligned}
&(v \circ_{g_1g_2,g_2} w_1) *_{g_2,g_1g_2} u \\
=&\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(u,z_1)Y_{M^1}(v,z_2)w_1 \\
=&\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(v,z_2)Y_{M^1}(u,z_1)w_1 \\
&+\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) \\
&\left(\frac{z_2+z_0}{z_1}\right)^{\frac{j_1}{T}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
\equiv&\mathrm{Res}_{z_0}\mathrm{Res}_{z_2} \frac{(1+z_2+z_0)^{\mathrm{wt}u-1}}{(z_2+z_0)^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
&\left(\mathrm{mod}\ O'_{g_1g_2,g_2}(M^1)\right) \\
\equiv&\mathrm{Res}_{z_2} \sum_{i\geq 0}\sum_{j\geq 0} \binom{\mathrm{wt}u-1}{i}\binom{-1+\frac{j_1}{T}}{j} \frac{(1+z_2)^{\mathrm{wt}v-1+\delta(j_4)+\frac{j_4}{T}+\mathrm{wt}u-1-i}}{z_2^{1+\delta(j_3,j_4)-\frac{j_3}{T}+1-\frac{j_1}{T}+j}}
\end{aligned}
$$

$$
\begin{aligned}
&Y_{M^1}(u_{i+j}v, z_2)w_1 \left(\bmod O'_{g_1g_2,g_2}(M^1)\right)\\
\equiv&\mathrm{Res}_{z_2}\sum_{i\geq 0}\sum_{j\geq 0}\binom{\mathrm{wt}u-1}{i}\binom{-1+\frac{j_1}{T}}{j}Y_{M^1}(u_{i+j}v, z_2)w_1\\
&\frac{(1+z_2)^{\mathrm{wt}(u_{i+j}v)-1+\delta(j_4)+\frac{j_4}{T}+j}}{z_2^{1+\delta([j_1+j_3],j_4)-\frac{[j_1+j_3]}{T}+\delta(j_3,j_4)-\frac{j_3}{T}+1-\frac{j_1}{T}-\delta([j_1+j_3],j_4)+\frac{[j_1+j_3]}{T}+j}}\\
&\left(\bmod O'_{g_1g_2,g_2}(M^1)\right)\\
\equiv&0 \left(\bmod O'_{g_1g_2,g_2}(M^1)\right)
\end{aligned}
$$

Here we explain the last congruence. We compare

$$n = j$$

and

$$m = j + \delta(j_3, j_4) + 1 - \delta([j_1 + j_3], j_4) + \frac{[j_1 + j_3]}{T} - \frac{j_1 + j_3}{T}$$

case by case:

(1) If $j_4 = 0$, then $m = j + 1$ when $j_1 + j_3 < T$ and $m = j$ when $j_1 + j_3 \geq T$. Either way, we have $m \geq n$;

(2) If $j_4 \neq 0$, then $m = j + \delta(j_3, j_4) + 1 - \delta(j_1 + j_3, j_4) \geq j$ when $j_1 + j_3 < T$ and $m = j + \delta(j_3, j_4) - \delta(j_1 + j_3 - T, j_4) \geq j$ when $j_1 + j_3 \geq T$. Either way, we have $m \geq n$. Now by lemma 3.0.2, the last congruence holds. □

Lemma 3.0.5. *For $u, v \in V$ and $w_1 \in M^1$, $(u *_{g_1g_2,g_2} w_1) *_{g_2,g_1g_2} v - u *_{g_1g_2,g_2} (w_1 *_{g_2,g_1g_2} v) \subseteq O'_{g_1g_2,g_2}(M^1)$.*

Proof. It suffices to show it holds for homogeneous $u \in V^{(j_1,j_2)}, v \in V^{(j_3,j_4)}$, where $j_1 + j_2 \equiv 0 \pmod T$ and $j_4 = 0$. Let $u \in V^{(j_1,j_2)}$, $v \in V^{(j_3,0)}$ where $j_1 + j_2 \equiv 0$ (mod T), then

$$
\begin{aligned}
&(u *_{g_1g_2,g_2} w_1) *_{g_2,g_1g_2} v - u *_{g_1g_2,g_2} \left(w_1 *_{g_2,g_1g_2} v\right) \\
=&\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} Y_{M^1}(v,z_2)Y_{M^1}(u,z_1)w_1 \\
&- \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} Y_{M^1}(u,z_1)Y_{M^1}(v,z_2)w_1 \\
=&-\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_1)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z_1^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} z_1^{-1}\delta(\frac{z_2+z_0}{z_1}) \\
&\left(\frac{z_2+z_0}{z_1}\right)^{\frac{j_1}{T}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
=&-\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} \frac{(1+z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{(z_2+z_0)^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} z_1^{-1}\delta(\frac{z_2+z_0}{z_1}) \\
&Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
=&-\mathrm{Res}_{z_0}\mathrm{Res}_{z_2} \frac{(1+z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{(z_2+z_0)^{1-\frac{j_1}{T}}} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
=&-\mathrm{Res}_{z_0}\mathrm{Res}_{z_2} \sum_{i\geq 0} \binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} z_0^i (1+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}-i} \\
&\sum_{j\geq 0}\binom{-1+\frac{j_1}{T}}{j} z_0^j z_2^{-1+\frac{j_1}{T}-j} \frac{(1+z_2)^{\mathrm{wt}v-1}}{z_2^{1-\frac{j_3}{T}}} Y_{M^1}(Y(u,z_0)v,z_2)w_1 \\
=&-\mathrm{Res}_{z_2} \sum_{i\geq 0}\sum_{j\geq 0} \binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i}\binom{-1+\frac{j_1}{T}}{j} \frac{(1+z_2)^{\mathrm{wt}v-1+\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}-i}}{z_2^{2-\frac{j_1}{T}-\frac{j_3}{T}+j}} \\
&Y_{M^1}(u_{i+j}v,z_2)w_1
\end{aligned}
$$

$$= -\operatorname{Res}_{z_2} \sum_{i\geq 0}\sum_{j\geq 0} \binom{\mathrm{wt} u - 1 + \delta(j_2) + \frac{j_2}{T}}{i}\binom{-1+\frac{j_1}{T}}{j} \frac{(1+z_2)^{\mathrm{wt}(u_{i+j}v)-1+\delta(j_2)+\frac{j_2}{T}+j}}{z_2^{2-\frac{j_1+j_3}{T}+j}}$$

$$Y_{M^1}(u_{i+j}v, z_2)w_1$$

$$\in O'_{g_1g_2,g_2}(M^1).$$

Again, we used lemma 3.0.2 in the last step. □

To construct the desired bimodule, we need to modulo out a bigger subspace than $O'_{g_1g_2,g_2}(M^1)$ from M^1. Let $O''_{g_1g_2,g_2}(M^1)$ be the linear span of all $(u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1 - u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)$, $w_1 *_{g_2,g_1g_2} (v *_{g_2} u) - (w_1 *_{g_2,g_1g_2} v) *_{g_2,g_1g_2} u$, $u' *_{g_1g_2,g_2} w_1$ and $w_1 *_{g_2,g_1g_2} v'$, where $u, v \in V$, $u' \in O_{g_1g_2}(V)$, $v' \in O_{g_2}(V)$ and $w_1 \in M^1$. Let

$$O_{g_1g_2,g_2}(M^1) = O'_{g_1g_2,g_2}(M^1) + O''_{g_1g_2,g_2}(M^1),$$

and

$$A_{g_1g_2,g_2}(M^1) = M^1/O_{g_1g_2,g_2}(M^1).$$

Lemma 3.0.6. *For $a \in V$, $a*_{g_1g_2,g_2}O_{g_1g_2,g_2}(M^1) \subseteq O_{g_1g_2,g_2}(M^1)$ and $O_{g_1g_2,g_2}(M^1)*_{g_2,g_1g_2} a \subseteq O_{g_1g_2,g_2}(M^1)$.*

Proof. It suffices to show $a*_{g_1g_2,g_2}O''_{g_1g_2,g_2}(M^1) \subseteq O_{g_1g_2,g_2}(M^1)$ and $O''_{g_1g_2,g_2}(M^1)*_{g_2,g_1g_2} a \subseteq O_{g_1g_2,g_2}(M^1)$ due to lemmas 3.0.3 and 3.0.4. We verify it for all 4 types of spanning vectors in $O''_{g_1g_2,g_2}(M^1)$.

For $u, v \in V$ and $w_1 \in M^1$,

$$a *_{g_1g_2,g_2} \left((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1 - u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\right)$$

$$=a *_{g_1g_2,g_2} \left((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1\right) - a *_{g_1g_2,g_2} \left(u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\right)$$

by the definition of $O''_{g_1g_2,g_2}(M^1)$,

$$\in\left(a *_{g_1g_2} (u *_{g_1g_2} v)\right) *_{g_1g_2,g_2} w_1 - \left(a *_{g_1g_2} u\right) *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1) + O''_{g_1g_2,g_2}(M^1)$$

$$=\left((a *_{g_1g_2} u) *_{g_1g_2} v\right) *_{g_1g_2,g_2} w_1 - \left(a *_{g_1g_2} u\right) *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1) + O''_{g_1g_2,g_2}(M^1)$$

$$=\left(a *_{g_1g_2} u\right) *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1) - \left(a *_{g_1g_2} u\right) *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1) + O''_{g_1g_2,g_2}(M^1)$$

$$=O''_{g_1g_2,g_2}(M^1) \subseteq O_{g_1g_2,g_2}(M^1).$$

The second equality holds because $A_{g_1g_2}(V)$ is associative, see theorem 2.3.5, thus

$$a *_{g_1g_2} (u *_{g_1g_2} v) \in (a *_{g_1g_2} u) *_{g_1g_2} v + O_{g_1g_2}(V),$$

and

$$O_{g_1g_2}(V) *_{g_1g_2,g_2} w_1 \in O''_{g_1g_2,g_2}(M^1)$$

by the definition of $O''_{g_1g_2,g_2}(M^1)$.

For $u, v \in V$ and $w_1 \in M^1$,

$$\left((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1 - u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\right) *_{g_2,g_1g_2} a$$

$$=\left((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1\right) *_{g_2,g_1g_2} a - \left(u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\right) *_{g_2,g_1g_2} a$$

applying lemma 3.0.5 to both terms,

$$\in(u *_{g_1g_2} v) *_{g_1g_2,g_2} \left(w_1 *_{g_2,g_1g_2} a\right) - u *_{g_1g_2,g_2} \left((v *_{g_1g_2,g_2} w_1) *_{g_2,g_1g_2} a\right) + O'_{g_1g_2,g_2}(M^1)$$

by the definition of $O''_{g_1g_2,g_2}(M^1)$ and applying lemma 3.0.5 to the second term,

$$\in u *_{g_1g_2,g_2} \big(v *_{g_1g_2,g_2} (w_1 *_{g_2,g_1g_2} a)\big) - u *_{g_1g_2,g_2} \big(v *_{g_1g_2,g_2} (w_1 *_{g_2,g_1g_2} a)\big)$$
$$+ u *_{g_1g_2,g_2} O'_{g_1g_2,g_2}(M^1) + O'_{g_1g_2,g_2}(M^1) + O''_{g_1g_2,g_2}(M^1)$$

by lemma 3.0.3,

$$\subseteq O_{g_1g_2,g_2}(M^1).$$

Similarly, one can prove

$$a *_{g_1g_2,g_2} \big(w_1 *_{g_2,g_1g_2} (v *_{g_2} u) - (w_1 *_{g_2,g_1g_2} v) *_{g_2,g_1g_2} u\big) \in O_{g_1g_2,g_2}(M^1)$$

and

$$\big(w_1 *_{g_2,g_1g_2} (v *_{g_2} u) - (w_1 *_{g_2,g_1g_2} v) *_{g_2,g_1g_2} u\big) *_{g_2,g_1g_2} a \in O_{g_1g_2,g_2}(M^1).$$

For $u' \in O_{g_1g_2}(V)$ and $w_1 \in M^1$,

$$a *_{g_1g_2,g_2} (u' *_{g_1g_2,g_2} w_1)$$
$$\in (a *_{g_1g_2} u') *_{g_1g_2,g_2} w_1 + O''_{g_1g_2,g_2}(M^1)$$

since $O_{g_1g_2}(V)$ is an ideal of V with respect to $*_{g_1g_2}$,

$$\subseteq O_{g_1g_2}(V) *_{g_1g_2,g_2} w_1 + O''_{g_1g_2,g_2}(M^1)$$
$$\subseteq O''_{g_1g_2,g_2}(M^1) \subseteq O_{g_1g_2,g_2}(M^1).$$

For $v' \in O_{g_2}(V)$ and $w_1 \in M^1$,

$$(u' *_{g_1 g_2, g_2} w_1) *_{g_2, g_1 g_2} a$$

by lemma 3.0.5,

$$\begin{aligned}
&\in u' *_{g_1 g_2, g_2} (w_1 *_{g_2, g_1 g_2} a) + O'_{g_1 g_2, g_2}(M^1) \\
&\subseteq O''_{g_1 g_2, g_2}(M^1) + O'_{g_1 g_2, g_2}(M^1) \\
&= O_{g_1 g_2, g_2}(M^1)
\end{aligned}$$

Similarly, we can prove

$$a *_{g_1 g_2, g_2} (w_1 *_{g_2, g_1 g_2} v') \in O_{g_1 g_2, g_2}(M^1)$$

and

$$(w_1 *_{g_2, g_1 g_2} v') *_{g_2, g_1 g_2} a \in O_{g_1 g_2, g_2}(M^1).$$

The proof is completed. □

Theorem 3.0.7. *$A_{g_1 g_2, g_2}(M^1)$ is an $A_{g_1 g_2}(V)$-$A_{g_2}(V)$-bimodule with left action $*_{g_1 g_2, g_2}$ and right action $*_{g_2, g_1 g_2}$.*

Proof. Combining lemmas 3.0.3, 3.0.4, 3.0.5 and 3.0.6, we see it immediately. □

Remark 3.0.8. Consider two special cases. (1): $g_1 = g_2 = 1$. This special case was dealt with in [FZ]. The $A(V)$-$A(V)$-bimodule $A(M^1)$ constructed in [FZ] is just $A_{1,1}(M^1)$; (2): $M^1 = V$. In this case, $g_1 = 1$ and any intertwining operator $I(\cdot, z) \in \mathcal{V}_{M^1 M^2}^{M^3}$ is just a g_2-twisted V-module map. The associative algebra $A_{g_2}(V)$ constructed in [DLM3] is just $A_{g_2,g_2}(V)$. It's worth to point out that in these two special cases, they were both able to prove that $O_{g_1g_2,g_2}(M^1) = O'_{g_1g_2,g_2}(M^1)$. It's reasonable to conjecture that $O_{g_1g_2,g_2}(M^1) = O'_{g_1g_2,g_2}(M^1)$ holds in general, but we are not able to prove it in this thesis.

Next, we are going to explain our bimodule construction by connecting it to representation theory. Suppose $I(\cdot, z) \in \mathcal{V}_{M^1M^2}^{M^3}$. Recall that $I^\circ(w_1, z) = \sum_{n\in\frac{1}{T}\mathbb{Z}} w_1(n)z^{-n-1}$ and $o_I(w_1) = w_1(\deg w_1 - 1)$.

Lemma 3.0.9. *For homogeneous $u \in V^{(j_1,j_2)}$, $w_1 \in M^1$ and $w_2 \in M^2(0)$, $o_I(u *_{g_1g_2,g_2} w_1)w_2 = o_{M^3}(u)o_I(w_1)w_2$, $o_I(w_1 *_{g_2,g_1g_2} u)w_2 = o_I(w_1)o_{M^2}(u)w_2$.*

Proof. By the definition of twisted module, action $*_{g_1g_2,g_2}$, and action $*_{g_2,g_1g_2}$:

if

$$j_1 + j_2 \not\equiv 0 \pmod{T},$$

then

$$o_I(u *_{g_1g_2,g_2} w_1)w_2 = 0 = o_{M^3}(u)o_I(w_1)w_2;$$

If

$$j_2 \neq 0,$$

then

$$o_I(w_1 *_{g_2,g_1g_2} u)w_2 = 0 = o_I(w_1)o_{M^2}(u)w_2.$$

So it suffices to prove the two identities for $j_1 + j_2 \equiv 0 \pmod T$ and $j_2 = 0$ respectively.

With the help of associativity, for homogeneous $u \in V^{(j_1,j_2)}$, $j_1 + j_2 \equiv 0 \pmod T$, homogeneous $w_1 \in M^1$ and $w_2 \in M^2(0)$, we have

$$
\begin{aligned}
& o_{M^3}(u) o_I(w_1) w_2 \\
=& \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_1^{\mathrm{wt}u-1} z_2^{\deg w_1 - 1} Y_{M^3}(u, z_1) I^\circ(w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) z_1^{\mathrm{wt}u-1} z_2^{\deg w_1 - 1} Y_{M^3}(u, z_1) I^\circ(w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_1^{-1} \delta\left(\frac{z_0 + z_2}{z_1}\right) z_1^{\mathrm{wt}u-1} z_2^{\deg w_1 - 1} Y_{M^3}(u, z_1) I^\circ(w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} (z_0 + z_2)^{\mathrm{wt}u-1} z_2^{\deg w_1 - 1} Y_{M^3}(u, z_0 + z_2) I^\circ(w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} z_2^{\deg w_1 - 1} (z_0 + z_2)^{-\delta(j_2) - \frac{j_2}{T}} (z_0 + z_2)^{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}} Y_{M^3}(u, z_0 + z_2) I^\circ(w_1, z_2) w_2
\end{aligned}
$$

Expanding $(z_0 + z_2)^{-\delta(j_2) - \frac{j_2}{T}}$, we can see that only the first term $z_0^{-\delta(j_2) - \frac{j_2}{T}}$ remains after applying Res_{z_2}.

$$
\begin{aligned}
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} z_2^{\deg w_1 - 1} z_0^{-\delta(j_2) - \frac{j_2}{T}} (z_0 + z_2)^{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}} Y_{M^3}(u, z_0 + z_2) I^\circ(w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} z_2^{\deg w_1 - 1} z_0^{-\delta(j_2) - \frac{j_2}{T}} (z_2 + z_0)^{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}} I^\circ(Y_{M^1}(u, z_0) w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} \sum_{i \geq 0} \binom{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}}{i} z_2^{\deg w_1 - 1 + \mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T} - i} z_0^{-\delta(j_2) - \frac{j_2}{T} + i} \\
& I^\circ(Y_{M^1}(u, z_0) w_1, z_2) w_2 \\
=& \mathrm{Res}_{z_2} \sum_{i \geq 0} \binom{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}}{i} z_2^{\deg w_1 - 1 + \mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T} - i} I^\circ(u_{i - \delta(j_2) - \frac{j_2}{T}} w_1, z_2) w_2 \\
=& o_I \left(\sum_{i \geq 0} \binom{\mathrm{wt}u - 1 + \delta(j_2) + \frac{j_2}{T}}{i} u_{i - \delta(j_2) - \frac{j_2}{T}} w_1 \right) w_2
\end{aligned}
$$

$$=o_I\left(\mathrm{Res}_z Y_{M^1}(u,z)w_1\frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z^{\delta(j_2)+\frac{j_2}{T}}}\right)w_2$$
$$=o_I\left(\mathrm{Res}_z Y_{M^1}(u,z)w_1\frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z^{1-\frac{j_1}{T}}}\right)w_2$$

The last equality holds because when $j_1+j_2\equiv 0 \pmod{T}$, either $j_1=j_2=0$ or $j_1+j_2=T$, so $\delta(j_2)+\frac{j_2}{T}=1-\frac{j_1}{T}$.

For homogeneous $u\in V^{(j_1,0)}$, homogeneous $w_1\in M^1$ and $w_2\in M^2(0)$, we have

$$o_I(w_1 *_{g_2,g_1g_2} u)w_2$$
$$=o_I(\sum_{i\geq 0}\binom{\mathrm{wt}u-1}{i}u_{i-1+\frac{j_1}{T}}w_1)w_2$$
$$=\sum_{i\geq 0}\binom{\mathrm{wt}u-1}{i}(u_{i-1+\frac{j_1}{T}}w_1)(\deg w_1+\mathrm{wt}u-i-\frac{j_1}{T}-1)w_2$$
$$=\mathrm{Res}_{z_2}\sum_{i\geq 0}\binom{\mathrm{wt}u-1}{i}z_2^{\deg w_1+\mathrm{wt}u-i-\frac{j_1}{T}-1}I^\circ(u_{i-1+\frac{j_1}{T}}w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}\sum_{i\geq 0}\binom{\mathrm{wt}u-1}{i}z_2^{\deg w_1+\mathrm{wt}u-i-\frac{j_1}{T}-1}I^\circ(z_0^{i-1+\frac{j_1}{T}}Y_{M^1}(u,z_0)w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_2+z_0)^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}z_0^{-1+\frac{j_1}{T}}I^\circ(Y_{M^1}(u,z_0)w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(z_2+z_0)^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}z_0^{-1+\frac{j_1}{T}}z_1^{-1}\delta(\frac{z_2+z_0}{z_1})I^\circ(Y_{M^1}(u,z_0)w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}z_0^{-1+\frac{j_1}{T}}z_1^{-1}\delta(\frac{z_2+z_0}{z_1})I^\circ(Y_{M^1}(u,z_0)w_1,z_2)w_2$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}z_0^{-1+\frac{j_1}{T}}z_0^{-1}\delta(\frac{z_1-z_2}{z_0})(\frac{z_1-z_2}{z_0})^{\frac{j_1}{T}}Y_{M^3}(u,z_1)I^\circ(w_1,z_2)w_2$$
$$-\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}z_0^{-1+\frac{j_1}{T}}z_0^{-1}\delta(\frac{-z_2+z_1}{z_0})(\frac{z_2-z_1}{z_0})^{\frac{j_1}{T}}I^\circ(w_1,z_2)Y_{M^2}(u,z_1)w_2$$
$$=\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{\mathrm{wt}u-1}z_2^{\deg w_1-\frac{j_1}{T}}(z_2-z_1)^{-1+\frac{j_1}{T}}I^\circ(w_1,z_2)Y_{M^2}(u,z_1)w_2$$
$$=o_I(w_1)o_{M^2}(u)w_2.$$

The last equality holds because $z_1^{\mathrm{wt}u}Y_{M^2}(u,z_1)w_2 \in M^2[[z_1^{\frac{1}{T}}]]$. $\square$

Proposition 3.0.10. *For all* $w \in O_{g_1g_2,g_2}(M^1)$, $o_I(w)|_{M^2(0)} = 0$.

Proof. First we prove it for $w \in O'_{g_1g_2,g_2}(M^1)$. Let $u \in V^{(j_1,j_2)}$ be homogeneous, $w_1 \in M^1$ and $w_2 \in M^2(0)$.

$$
\begin{aligned}
& o_I(u \circ_{g_1g_2,g_2} w_1)w_2 \\
=& o_I\big(\mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}}{z^{1+\delta(j_1,j_2)-\frac{j_1}{T}}} Y_{M^1}(u,z)w_1\big)w_2 \\
=& \sum_{i\geq 0}\binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} \\
& (u_{i-1-\delta(j_1,j_2)+\frac{j_1}{T}}w_1)(\deg w_1+\mathrm{wt}u-i+\delta(j_1,j_2)-\frac{j_1}{T}-1)w_2 \\
=& \mathrm{Res}_{z_2}\sum_{i\geq 0}\binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} z_2^{\deg w_1+\mathrm{wt}u-i+\delta(j_1,j_2)-\frac{j_1}{T}-1} \\
& I^{\circ}(u_{i-1-\delta(j_1,j_2)+\frac{j_1}{T}}w_1,z_2)w_2 \\
=& \mathrm{Res}_{z_0}\mathrm{Res}_{z_2}\sum_{i\geq 0}\binom{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}}{i} z_0^{i-1-\delta(j_1,j_2)+\frac{j_1}{T}} z_2^{\deg w_1+\mathrm{wt}u-i+\delta(j_1,j_2)-\frac{j_1}{T}-1} \\
& I^{\circ}(Y_{M^1}(u,z_0)w_1,z_2)w_2 \\
=& \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} z_0^{-1-\delta(j_1,j_2)+\frac{j_1}{T}} z_2^{\deg w_1+\delta(j_1,j_2)-\delta(j_2)-\frac{j_1+j_2}{T}}(z_2+z_0)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}} \\
& I^{\circ}(Y_{M^1}(u,z_0)w_1,z_2)w_2 \\
=& \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} z_0^{-1-\delta(j_1,j_2)+\frac{j_1}{T}} z_2^{\deg w_1+\delta(j_1,j_2)-\delta(j_2)-\frac{j_1+j_2}{T}}(z_0+z_2)^{\mathrm{wt}u-1+\delta(j_2)+\frac{j_2}{T}} \\
& Y_{M^3}(u,z_0+z_2)I^{\circ}(w_1,z_2)w_2
\end{aligned}
$$

Since the power of z_2 in $I^\circ(w_1, z_2)w_2$ is $\geq -\text{deg}w_1$ and $\delta(j_1, j_2) - \delta(j_2) - \frac{j_1+j_2}{T} > -1$, we get 0 after evaluating Res_{z_2}.

For those vectors in $O''_{g_1g_2,g_2}(M^1)$, we prove it for one case:

$$o_I\big((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1 - u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\big) \mid_{M^2(0)} = 0,$$

the proof for other cases are similar. Use lemma 3.0.9 and remark 3.0.8,

$$\begin{aligned}
& o_I\big((u *_{g_1g_2} v) *_{g_1g_2,g_2} w_1 - u *_{g_1g_2,g_2} (v *_{g_1g_2,g_2} w_1)\big) \mid_{M^2(0)} \\
=& o_{M^3}(u *_{g_1g_2} v) o_I(w_1) \mid_{M^2(0)} - o_{M^3}(u) o_I(v *_{g_1g_2,g_2} w_1) \mid_{M^2(0)} \\
=& o_{M^3}(u) o_{M^3}(v) o_I(w_1) \mid_{M^2(0)} - o_{M^3}(u) o_{M^3}(v) o_I(w_1) \mid_{M^2(0)} \\
=& 0
\end{aligned}$$

This completes the proof. □

By theorem 2.3.6, $M^2(0)$ is a left $A_{g_2}(V)$ module and $M^3(0)$ is a left $A_{g_1g_2}(V)$ module, hence $\text{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$ would be an $A_{g_1g_2}(V) - A_{g_2}(V)$ bimodule with the following left and right actions:

$$\big(u + O_{g_1g_2}(V)\big) \cdot f = o_{M^3}(u) \circ f,$$

$$f \cdot \big(v + O_{g_2}(V)\big) = f \circ o_{M^2}(v)$$

where $f \in \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$, $u + O_{g_1g_2}(V) \in A_{g_1g_2}(V)$ and $v + O_{g_2}(V) \in A_{g_2}(V)$. Consider the set

$$S_I := \{o_I(w_1)|_{M^2(0)} : w_1 \in M^1\}.$$

It is a subspace of $\mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$. Lemma 3.0.9 tells us that S_I is actually a subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$. Regarding o_I as a linear map from M^1 to S_I, we obtain that $M^1/ker\ o_I \cong S_I$ also has an $A_{g_1g_2}(V)$-$A_{g_2}(V)$-bimodule structure. From lemma 3.0.10, we see that

Proposition 3.0.11. *For every intertwining operator $I(\cdot, z) \in \mathcal{V}_{M^1M^2}^{M^3}$, there exists an $A_{g_1g_2}(V)$-$A_{g_2}(V)$-bimodule epimorphism from $A_{g_1g_2,g_2}(M^1)$ to S_I.*

Proof. By proposition 3.0.10, $O_{g_1g_2,g_2}(M^1) \subseteq ker\ o_I$. The statement follows immediately. □

Remark 3.0.12. Though not a perfect explanation, proposition 3.0.10 and 3.0.11 do give us a clue why we should modulo $O_{g_1g_2,g_2}(M^1)$ out. We have a series of $A_{g_1g_2}(V) - A_{g_2}(V)-$bimodules, i.e. S_I, where I ranges through all intertwining operators of type $\mathcal{V}_{M^1M^2}^{M^3}$. But these S_I's are not good enough, because they rely on the choice of I. We want something that is universal or at least independent of the choice of I. Proposition 3.0.11 makes $A_{g_1g_2,g_2}(M^1)$ a good candidate.

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