UC San Diego
UC San Diego Previously Published Works

Title
Asymptotic behavior of a critical fluid model for a multiclass processor sharing queue via relative entropy

Permalink
https://escholarship.org/uc/item/46p372sw

Journal
Queueing Systems, 93(3-4)

ISSN
0257-0130

Authors
Mulvany, Justin A
Puha, Amber L
Williams, Ruth J

Publication Date
2019-12-01

DOI
10.1007/s11134-019-09629-8

Copyright Information
This work is made available under the terms of a Creative Commons Attribution-NonCommercial-NoDerivatives License, availalbe at
https://creativecommons.org/licenses/by-nc-nd/4.0/

Peer reviewed

eScholarship.org Powered by the California Digital Library
University of California

ASYMPTOTIC BEHAVIOR OF A CRITICAL FLUID MODEL FOR A MULTICLASS PROCESSOR SHARING QUEUE VIA RELATIVE ENTROPY

JUSTIN A. MULVANY*, AMBER L. PUHA†, AND RUTH J. WILLIAMS‡

ABSTRACT. This work concerns the asymptotic behavior of critical fluid model solutions for a multiclass processor sharing queue under general distributional assumptions. Such critical fluid model solutions are measure valued functions of time. We prove that critical fluid model solutions converge to the set of invariant states as time goes to infinity, uniformly for all initial conditions lying in certain relatively compact sets. This generalizes an earlier single class result of Puha and Williams to the more complex multiclass setting. In particular, several new challenges are overcome, including formulation of a suitable relative entropy functional and identifying a convenient form of the time derivative of the relative entropy applied to trajectories of critical fluid model solutions.

Keywords. queueing, multiclass processor sharing, critical fluid model, fluid model asymptotics, relative entropy

1. INTRODUCTION

In this paper, we study the asymptotic behavior of critical fluid model solutions for a multiclass processor sharing queue under general distributional assumptions. A main result in the paper is a uniform convergence result: the distance between the time t value of a critical fluid model solution and the set of invariant states converges to zero uniformly as t tends to infinity for critical fluid model solutions with initial conditions lying in relatively compact sets of a certain form. This result is stated precisely as Theorem 3.3, and related results are proved in Theorems 3.1 and 3.2. It is anticipated the result in Theorem 3.3 will play an integral role in ongoing work proving a rigorous heavy traffic diffusion approximation for a multiclass processor sharing queue via a state space collapse argument.

*MARSHALL SCHOOL OF BUSINESS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA, 90089

†DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY SAN MARCOS, SAN MARCOS, CA, 92096

‡DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA, 92093-0112

E-mail addresses: `justin.mulvany@marshall.usc.edu, apuha@csusm.edu, williams@math.ucsd.edu`.

Date: July 2019.

Research supported in part by NSF Grants DMS-1510198 (Mulvany and Puha) and DMS-1712974 (Williams).

An important point of interest here is in the approach that is used to prove Theorem 3.3, which involves specifying a notion of relative entropy suitable for a multiclass processor sharing queue. In [2, 3], Bramson studied the asymptotic behavior of fluid model solutions associated with subcritical first-in-first-out queueing networks of Kelly type and of those associated with subcritical head-of-the-line-proportional processor sharing queueing networks using notions of relative entropy suitable for these HL networks. Traditional processor sharing with general distributional assumptions, as studied here, is not HL and a different notion of relative entropy is needed to carry out our analysis. In [14], Puha and Williams formulated such a notion of relative entropy in order to study the asymptotic behavior of critical fluid model solutions for single class processor sharing queues operating under general distributional assumptions. In the earlier work [13], also on single class processor sharing queues, Puha and Williams used renewal theoretic arguments to obtain rates of convergence results under higher moment assumptions than those required in [14]. The renewal theoretic techniques in [13] seem rather specific to a single class and rates of convergence are stronger than what is typically required in state space collapse arguments. Therefore, an important motivation for the work in [14] was to devise a methodology that might not require higher moment assumptions and might more naturally extend to the network setting. The results in this current paper further develop the methodology of [14] in this direction. In particular, we specify a related notion of relative entropy suitable for the multiclass setting and use this to prove Theorem 3.3, which generalizes [14, Theorem 3.1] to the multiclass setting. In the multiclass setting, new challenges arise that concern identifying a good form for the relative entropy functional and in computing its time derivative along trajectories of fluid model solutions.

We consider a critical fluid model for a multiclass processor sharing queue with J job classes. This fluid model is obtained as a scaling limit of a heavily loaded multiclass, single server processor sharing queue under general distributional assumptions on the interarrival and service times. Indeed, functional laws of large numbers limit theorems proved in [1], for a multiclass processor sharing queue with feedback, and in [8], for a bandwidth sharing model, justify this as a special case. In [1, 8], in the original systems, natural measure valued processes are used to track system dynamics. In particular, for a multiclass processor sharing queue, for each class j, at each time t, a measure on the nonnegative real line $\mathbb{R}_+ = [0, \infty)$ is used that has an atom at the residual processing time of each class j job in the system at time t. These measure valued processes and their fluid limits take values in the product space $\mathbf{M}^J$, where $\mathbf{M}$ is the set of finite nonnegative Borel measures on $\mathbb{R}_+$ endowed with the topology of weak convergence. Indeed, critical fluid model solutions are functions $\mu : [0, \infty) \mapsto \mathbf{M}^J$ that satisfy the conditions of the critical fluid model specified in Section 2. In particular, for each job class j, μ_j satisfies the dynamic equation (4). The set $\mathbf{I}$ of invariant states was determined in [8, Theorem 6.3 and Lemma 6.4] for a large class of bandwidth sharing models that include multiclass processor sharing as a special case. When specialized to multiclass processor sharing queues this yields that $\mathbf{I}$ is precisely the set of all constant multiples of a fundamental invariant state π specified in (14) in terms of excess life distributions for the service times.

In order to prove our major result,Theorem 3.3, we introduce a function H that maps a subset of $\mathbf{M}^J$ into the extended reals $[0,\infty]$. For this, we endow $\mathcal{J} = \{1,2,\ldots,J\}$ with the discrete topology and $\mathcal{J}\times\mathbf{M}$ with the product topology. For $\zeta\in\mathbf{M}^J$ such that each component has a finite first moment and and at least one component has a positive first moment, we define an associated probability measure $\tilde{\zeta}$ on $\mathcal{J}\times\mathbf{M}$ (see (16)). This probability measure $\tilde{\zeta}$ can be regarded as a sort of generalized excess life probability measure associated with ζ (see the remark following (16)). Under the mapping $\zeta\mapsto\tilde{\zeta}$, all nonzero invariant states map to the common probability measure $\tilde{\pi}$, which is the image of the fundamental invariant state π (see (17)). For $\zeta\in\mathbf{M}^J$ such that each component has a finite first moment and at least one component has a positive first moment, $H(\zeta)$ is defined to be the relative entropy of $\tilde{\zeta}$ with respect to $\tilde{\pi}$ (see (18)). This definition generalizes the H used in [14] to the multiclass setting. This, together with developing the properties of H necessary to prove Theorem 3.3 and providing a proof of Theorem 3.3 are the major contributions of this paper.

As in [14], a significant part of the proof of Theorem 3.3 concerns proving the absolute continuity of H as a function of time along the trajectories of critical fluid model solutions with initial conditions lying in relatively compact sets of a certain form and demonstrating that the density is nonpositive and is zero at a given time if and only if the state of the critical fluid model solution is an invariant state. A precursor to the H defined here was considered in the Master of Science Thesis of the first author [9], where numerical evidence that such a result is true was provided. In the current paper, using a more streamlined definition of H, we prove the desired absolute continuity result. This is stated as Theorem 3.2 below, where we compute the density explicitly (see (29)). This generalizes [14, Theorem 7.1] to the multiclass setting and involves nonconstant, class dependent coefficients and new terms for multiple customer classes (see (25)-(29)). The astute reader will see that the proof of Theorem 3.2 benefits from the fact that the fluid analogue of the station level workload process is constant. Hence, generalization to networks of processor sharing queues remains a topic for future work.

The paper is organized as follows. We complete the introduction by introducing some basic notation in Section 1.1. In Section 2, we define critical fluid model solutions and review some of their properties including a characterization of the invariant states. The formal definition of the relative entropy functional and main results of the paper are given in Section 3. In addition to Theorems 3.2 and 3.3, this includes Theorem 3.1, which says that H as a function of time along the trajectories of critical fluid model solutions with initial conditions lying in relatively compact sets of a certain form converges to zero uniformly as time tends to infinity. Theorem 3.1 is proved as a consequence of Theorem 3.2, properties of H developed in Section 5.1, and properties of critical fluid model solutions developed in Section 6. It is the main workhorse used to prove Theorem 3.3. The remainder of the paper is devoted to proving the three main theorems. Section 3.4 provides an overview of the organization of these proofs.

1.1. **Notation.** Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of nonnegative integers, and $\mathbb{N}$ denote the set of strictly positive integers, i.e., the natural numbers. Let $\mathbb{R}$ denote the set of real numbers. For $x \in \mathbb{R}$, let $|x|$ denote the absolute value of x. The set of nonnegative real numbers will be denoted by $\mathbb{R}_+$. Given a set $B \subset \mathbb{R}_+$, $\inf B$ denotes the infimum of B, which is taken to be infinity if $B = \emptyset$. Given a Borel set $B \subset \mathbb{R}_+$, let $1_B : \mathbb{R}_+ \to \{0,1\}$ be the indicator function that satisfies $1_B(x) = 1$ if $x \in B$, and $1_B(x) = 0$ if $x \notin B$ for all $x \in \mathbb{R}_+$. Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be the identity function on $\mathbb{R}_+$ satisfying $\chi(x) = x$ for all $x \in \mathbb{R}_+$.

Let $\mathbf{C}_b(\mathbb{R}_+)$ be the set of bounded, continuous functions defined from $\mathbb{R}_+$ to $\mathbb{R}$. Let $\mathbf{C}^1_b(\mathbb{R}_+)$ be the set of continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$ that together with their first derivatives are bounded. Given a Borel measure ζ defined on $\mathbb{R}_+$, let $L_1(\zeta)$ denote the set of Borel measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are integrable with respect to ζ. For $f \in L_1(\zeta)$, denote the integral of f with respect to ζ by $\int_{\mathbb{R}_+} f(x)d\zeta(x) = \langle f, \zeta\rangle$. Also, for any nonnegative Borel measurable function $f \notin L_1(\zeta)$, let $\langle f, \zeta\rangle$ be infinity.

Let $\mathbf{M}$ be the set of nonnegative, finite Borel measures on $\mathbb{R}_+$. Let $\mathbf{M}_0 = \{\zeta \in \mathbf{M} : \langle 1, \zeta\rangle > 0\}$ be the set of nonzero measures in $\mathbf{M}$ and $\mathbf{M}_+ = \{\zeta \in \mathbf{M}_0 : \langle 1_{(0,\infty)}, \zeta\rangle > 0\}$ be the measures in $\mathbf{M}_0$ that have support intersecting the positive real numbers. Let $\mathbf{K}$ be the set of continuous measures in $\mathbf{M}$, i.e., those that do not have atoms. Let $\mathbf{A}$ be the set of measures in $\mathbf{M}$ that are absolutely continuous with respect to Lebesgue measure. Let $\mathbf{P}$ be the set of probability measures in $\mathbf{M}$. Let $\mathbf{K}_+ = \mathbf{K} \cap \mathbf{M}_+$ and $\mathbf{A}_+ = \mathbf{A} \cap \mathbf{M}_+$. For $k \in \mathbb{N}$, let $\mathbf{M}_k$ denote the set of measures in $\mathbf{M}$ that have finite kth moment. Let $\mathbf{M}_\dagger = \mathbf{M}_1 \cap \mathbf{M}_+$ and $\mathbf{M}_\ddagger = \mathbf{M}_2 \cap \mathbf{M}_+$. Analogously, let $\mathbf{K}_\dagger = \mathbf{M}_1 \cap \mathbf{K}_+$, $\mathbf{A}_\dagger = \mathbf{M}_1 \cap \mathbf{A}_+$, $\mathbf{P}_\dagger = \mathbf{M}_\dagger \cap \mathbf{P}$ and $\mathbf{P}_\ddagger = \mathbf{M}_\ddagger \cap \mathbf{P}$. If $\zeta \in \mathbf{M}_\dagger$, then there is an associated excess life distribution $\zeta^e \in \mathbf{P} \cap \mathbf{A}$ that has a probability density function p_ζ given by

$$p_\zeta(x) = \frac{\langle 1_{(x,\infty)}, \zeta\rangle}{\langle \chi, \zeta\rangle}, \quad \text{for } x \in \mathbb{R}_+. \tag{1}$$

If $\zeta \in \mathbf{M}_\ddagger$, then $\zeta^e \in \mathbf{P}_\dagger \cap \mathbf{A}$, and so $(\zeta^e)^e \in \mathbf{P} \cap \mathbf{A}$ is well defined (see [14, Page 7]).

Fix $J \in \mathbb{N}$ and let $\mathcal{J} = \{1, 2, ..., J\}$. We will let J be the number of customer classes present in our model of a multiclass processor sharing queue. Given a set $\mathbf{S}$, denote by $\mathbf{S}^J$ the Cartesian product of $\mathbf{S}$ with itself J times. Let $\mathbf{0} \in \mathbf{M}^J$ be the J-dimensional vector of zero measures. For convenience, given $\zeta \in \mathbf{M}^J$, for $j \in \mathcal{J}$, let ζ_j denote the jth component of ζ, and let $\zeta_+ \in \mathbf{M}$ be the superposition measure of ζ, i.e., $\zeta_+ = \sum_{j=1}^J \zeta_j$. Given $\zeta \in \mathbf{M}_\dagger^J$, $\zeta^e = (\zeta_1^e, \dots, \zeta_J^e)$, where $\zeta_j^e \in \mathbf{P}$ has density p_{ζ_j} defined by (1) with ζ_j in place of ζ for each $j \in \mathcal{J}$.

The set $\mathbf{M}$ is endowed with the topology of weak convergence. That is, given a sequence $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}$, we say that $\{\zeta^n\}_{n\in\mathbb{N}}$ converges weakly to $\zeta \in \mathbf{M}$ if for all $f \in \mathbf{C}_b(\mathbb{R}_+)$, $\langle f, \zeta^n\rangle \to \langle f, \zeta\rangle$ as $n \to \infty$. In this case, we write $\zeta^n \xrightarrow{w} \zeta$ as $n \to \infty$. A metric that induces the topology of weak convergence on $\mathbf{M}$ is the Prokhorov metric. Given a Borel set $B \subset \mathbb{R}_+$ and $\epsilon > 0$, let

$$B^\epsilon = \{y \in \mathbb{R}_+ : \inf_{x \in B} |x - y| < \epsilon\}.$$

Then, for $\zeta, \eta \in \mathbf{M}$, let

$$\begin{aligned}\mathbf{d}(\zeta, \eta) = \inf\{\epsilon > 0 : \zeta(B) \leq \eta(B^\epsilon) + \epsilon \text{ and } \eta(B) \leq \zeta(B^\epsilon) + \epsilon \\ \text{for all closed sets } B \subset \mathbb{R}_+\}.\end{aligned}$$

Then $\mathbf{d}$ is the Prokhorov metric. The metric space $(\mathbf{M}, \mathbf{d})$ is a Polish space.

Given a sequence $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^J$, we say that $\{\zeta^n\}_{n\in\mathbb{N}}$ *converges weakly* to a measure $\zeta \in \mathbf{M}^J$ if $\zeta_j^n \xrightarrow{w} \zeta_j$ as $n \to \infty$ for each $j \in \mathcal{J}$. In this case, we write $\zeta^n \xrightarrow{w} \zeta$ as $n \to \infty$. The topology induced by weak convergence on $\mathbf{M}^J$ is the same as that induced by the following generalization of the Prokhorov metric on $\mathbf{M}$. Define the function $\mathbf{d}_J : \mathbf{M}^J \times \mathbf{M}^J \to \mathbb{R}_+$ as follows: for $\zeta, \eta \in \mathbf{M}^J$,

$$\mathbf{d}_J(\zeta, \eta) = \max_{j=1}^{J} \mathbf{d}(\zeta_j, \eta_j),$$

where $\mathbf{d} : \mathbf{M} \times \mathbf{M} \to \mathbb{R}_+$ is the Prokhorov metric on $\mathbf{M}$. One can see immediately that $\lim_{n\to\infty} \mathbf{d}_J(\zeta^n, \eta) = 0$ if and only if $\zeta^n \xrightarrow{w} \zeta$ as $n \to \infty$. Given a subset $\mathbf{S} \subset \mathbf{M}^J$, for $\zeta \in \mathbf{M}^J$, define

$$\mathbf{d}_J(\zeta, \mathbf{S}) = \inf_{\eta \in \mathbf{S}} \mathbf{d}_J(\zeta, \eta).$$

2. Critical Fluid Model

As stated in the introduction, we aim to study a critical fluid model for a multiclass processor sharing queue with J customer classes, where $J \in \mathbb{N}$ is fixed. In this section, we define the critical fluid model and develop some of its properties.

2.1. **Critical Parameters.**

The critical fluid model has associated critical parameters (α, ν), where $\alpha \in (0, \infty)^J$, $\nu \in \mathbf{P}_\dagger^J$ is such that $\langle 1_{\{0\}}, \nu_j \rangle = 0$ for all $j \in \mathcal{J}$ and

$$\sum_{j=1}^{J} \alpha_j \langle \chi, \nu_j \rangle = 1. \tag{2}$$

For $j \in \mathcal{J}$, $\alpha_j \in (0, \infty)$ is the rate at which class j fluid arrives to the system and $\nu_j \in \mathbf{P}_\dagger$ determines the distribution of class j fluid along $\mathbb{R}_+$ as it arrives. One regards the position of fluid in $\mathbb{R}_+$ as its remaining processing time. For $j \in \mathcal{J}$, we refer to α_j as the class j *arrival rate* and to ν_j as the class j *service time distribution.* For $j \in \mathcal{J}$, define

$$\rho_j = \alpha_j \langle \chi, \nu_j \rangle, \qquad \text{for } j \in \mathcal{J}.$$

For each $j \in \mathcal{J}$, we interpret ρ_j as the instantaneous load that class j puts on the server, measured in processing time units. From (2), we have that $\sum_{j=1}^{J} \rho_j = 1$. We assume the server processes work at rate 1. Thus, (2) implies that the total instantaneous load is equal to the rate at which fluid is processed. Henceforth, we regard the critical parameters (α, ν) as being fixed.

Fix $j \in \mathcal{J}$. The cumulative distribution function and complementary cumulative distribution function for ν_j are given by

$$N_j(x) = \langle 1_{[0,x]}, \nu_j \rangle \quad \text{and} \quad \overline{N}_j(x) = 1 - N_j(x), \quad \text{for } x \in \mathbb{R}_+,$$

respectively. Moreover, we define x_j^* as follows:

$$x_j^* = \inf\{x \in \mathbb{R}_+ : \overline{N}_j(x) = 0\}. \tag{3}$$

Recall if the set on the right is empty, then $x_j^* = \infty$. In the case when x_j^* is finite, $\overline{N}_j(x) = 0$ for all $x \geq x_j^*$ since $\overline{N}_j$ is right continuous. Additionally, $x_j^* > 0$ because ν_j does not have an atom at zero.

2.2. **Fluid Model Solutions.** We now define the concept of a fluid model solution. Let $\mathcal{C} \subset \mathbf{C}_b^1(\mathbb{R}_+)$ be given by

$$\mathcal{C} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g(0) = g'(0) = 0\}.$$

A *fluid model solution* for the critical parameters (α, ν) is a function $\mu : [0, \infty) \to \mathbf{M}^J$ that satisfies the following four conditions:

(C.1) $\mu(\cdot)$ is continuous.
(C.2) $\langle 1_{\{0\}}, \mu_j(t) \rangle = 0$ for all $j \in \mathcal{J}$ and $t \geq 0$.
(C.3) For $g \in \mathcal{C}$ and $j \in \mathcal{J}$,

$$\langle g, \mu_j(t) \rangle = \langle g, \mu_j(0) \rangle - \int_0^t \frac{\langle g', \mu_j(r) \rangle}{\langle 1, \mu_+(r) \rangle} dr + \alpha_j t \langle g, \nu_j \rangle, \tag{4}$$

for all $t < t^* = \inf\{r \geq 0 : \langle 1, \mu_+(r) \rangle = 0\}$.
(C.4) For all $t \geq t^*$, $\langle 1, \mu_+(t) \rangle = 0$.

Since the critical parameters (α, ν) are fixed, we frequently write that μ is a fluid model solution as a shorthand for μ is a fluid model solution for the critical parameters (α, ν). In addition, sometimes we will make the initial condition more explicit by writing μ is a fluid model solution with initial condition $\mu(0) = \xi$, or μ is a fluid model solution for the critical parameters (α, ν) with initial condition $\mu(0) = \xi$.

We provide a a brief intuitive explanation of (C.3) and (C.4). On the right side of (4) for each $j \in \mathcal{J}$ and $t \geq 0$, the first term represents the initial state of the class j fluid and the last term corresponds to the class j fluid that has entered the system in the time interval $(0, t]$ and been spreadout over $\mathbb{R}_+$ according to ν_j upon its arrival. The middle term on the right side of (4) accounts for the servicing or processing of class j fluid for each $j \in \mathcal{J}$ over time intervals during which the total mass in the system is not zero. In particular, provided that the total mass in the system is not zero, all fluid is processed at a rate equal to the reciprocal of the total mass in the system. Thus, the effect of fluid being processed over a short time interval of length $\delta > 0$ beginning at a time $t \geq 0$ at which the total mass in the system is not zero should be that the mass in the system has shifted approximately $\delta/\langle 1, \mu_+(t) \rangle$ units toward the origin. Thus, for such a δ and t and for $g \in \mathcal{C}$ and $j \in \mathcal{J}$, $\langle g, \mu_j(t+\delta) \rangle \approx \langle g(\cdot - \delta/\langle 1, \mu_+(t) \rangle), \mu_j(t) \rangle + \alpha_j \delta \langle g, \nu_j \rangle$, and so $\langle g, \mu_j(t+\delta) \rangle - \langle g, \mu_j(t) \rangle \approx -\langle g', \mu_j(t) \rangle \delta / \langle 1, \mu_+(t) \rangle + \alpha_j \delta \langle g, \nu_j \rangle$. This gives a heuristic explanation of (C.3). Once the total mass in the system reaches zero, it should remain zero for all time since the parameters are critical, which gives a heuristic explanation for (C.4). Further discussion of (C.1)-(C.4) can be found in [14, Page 258].

Fluid model solutions arise as functional law of large numbers limits of multiclass processor sharing queues under conditions of asymptotic critical loading. In particular, the single server, multiclass processor sharing queue, as studied here, coincides with a single resource, multiroute bandwidth sharing model operating under proportional fair sharing; a special case of the bandwidth sharing model considered in [8]. Thus, the fluid limit theorem in [8, Theorem 4.1], specialized to a single resource operating under proportional fair sharing, justifies regarding the critical fluid model specified here as a functional law of large numbers approximation for a heavily loaded multiclass processor sharing queue. Additionally, an extension of our fluid model for a multiclass processor sharing queue that allows feedback was introduced in [1], and [1, Theorem 3.5] also provides a justification.

Before we proceed any further, we establish some basic properties of fluid model solutions. For this, let $\alpha_+ = \sum_{j=1}^J \alpha_j$ and for each $j \in \mathcal{J}$ set $\vartheta_j = \frac{\alpha_j}{\alpha_+}\nu_j$ and

$$\vartheta = (\vartheta_1, \ldots, \vartheta_J)\,. \tag{5}$$

By (2), $\langle \chi, \vartheta_+ \rangle = 1/\alpha_+$. Thus (α_+, ϑ_+) are critical parameters for a fluid model associated with a single class processor sharing queue, as introduced in [7]. Given a fluid model solution μ for the critical parameters (α, ν) and initial condition $\mu(0) \in \mathbf{K}^J$, observe that $\mu_+ : [0, \infty) \to \mathbf{M}$ is continuous by (C.1), $\langle 1_{\{0\}}, \mu_+(t) \rangle = 0$ for all $t \in [0, \infty)$ by (C.2), and $\langle 1, \mu_+(t) \rangle = 0$ for all $t \geq t^*$ by (C.4). From (5) and summing over $j \in \mathcal{J}$ in (4), we obtain for $g \in \mathcal{C}$, for all $t < t^*$,

$$\langle g, \mu_+(t) \rangle = \langle g, \mu_+(0) \rangle - \int_0^t \frac{\langle g', \mu_+(r) \rangle}{\langle 1, \mu_+(r) \rangle} dr + \alpha_+ t \langle g, \vartheta_+ \rangle. \tag{6}$$

Thus, $\mu_+ : [0, \infty) \to \mathbf{M}$ is a fluid model solution for the fluid model associated with the single class processor sharing queue having critical parameters (α_+, ϑ_+) and initial condition $\mu_+(0) \in \mathbf{K}$. Therefore, as a consequence of the existence and uniqueness result in [7, Theorem 3.1], we obtain the following result.

Proposition 2.1. *If μ is a fluid model solution for the critical parameters (α, ν) and initial condition $\mu(0) \in \mathbf{K}^J$, then, for all $t \in [0, \infty)$,*

(1) $\mu_+(t) \in \mathbf{K}$, and so $\mu(t) \in \mathbf{K}^J$, and
(2) $\langle \chi, \mu_+(t) \rangle = \langle \chi, \mu_+(0) \rangle$.

Furthermore, if $\mu_+(0) \in \mathbf{K}_+$, then $t^* = \infty$. Otherwise, $\mu(0) = \mathbf{0}$ and $t^* = 0$.

Let μ be a fluid model solution for (α, ν) such that $\mu(0) \in \mathbf{K}^J$. Then, for $j \in \mathcal{J}$, the class j fluid queue length is given by $q_j(t) = \langle 1, \mu_j(t) \rangle$, for each $t \in [0, \infty)$. We define the fluid queue length vector for each $t \in [0, \infty)$ by $q(t) = (q_1(t), ..., q_J(t))$ and the total fluid queue length by $q_+(t) = \sum_{j=1}^J q_j(t)$ for each $t \in [0, \infty)$. Since $\mu : [0, \infty) \to \mathbf{M}^J$ is continuous, we have that q and q_+ are continuous on the time interval $[0, \infty)$.

The cumulative amount of service provided per unit of fluid by time $t \in [0,\infty)$ is given by

$$s(t) = \int_0^t \varphi(q_+(r))dr, \qquad \text{for } t \in [0,\infty), \tag{7}$$

where $\varphi(x) = 1/x$ for all $x \neq 0$, and $\varphi(0) = 0$. If $\mu_+(0) \in \mathbf{K}_+$, then q_+ is strictly positive and continuous on $[0,\infty)$, and so we have that s is continuous and differentiable on the interval $[0,\infty)$ with

$$s'(t) = \frac{1}{q_+(t)}, \qquad \text{for } t \in [0,\infty). \tag{8}$$

For each $x \in \mathbb{R}_+$, define the inverse function

$$\tau(x) = \inf\{t \in [0,\infty) : s(t) \geq x\}. \tag{9}$$

For each $j \in \mathcal{J}$, the class j fluid workload is given by $w_j(t) = \langle \chi, \mu_j(t)\rangle$, for each $t \in [0,\infty)$. From here on, we define the fluid workload by $w(t) = (w_1(t), w_2(t), ..., w_J(t))$ and total fluid workload by $w_+(t) = \sum_{j=1}^J w_j(t)$ for each $t \in [0,\infty)$. By Proposition 2.1, $w_+(t) = w_+(0)$ for all $t \in [0,\infty)$, and we shall simply denote this by w_+ henceforth.

If $\mu_+(0) \in \mathbf{K}_+$, then, for each $j \in \mathcal{J}$, $t \in [0,\infty)$ and $x \in \mathbb{R}_+$, $\mu_j(t)$ satisfies

$$\langle 1_{(x,\infty)}, \mu_j(t)\rangle \;=\; \langle 1_{(x+s(t),\infty)}, \mu_j(0)\rangle + \alpha_j \int_0^t \overline{N_j}(x + s(t) - s(r))dr. \tag{10}$$

This follows from the properties of fluid model solutions. To see this, one can observe that the argument given in [7, Lemma 4.3], which can be used to prove that (10) holds in the single class case, extends to each class $j \in \mathcal{J}$ in the multiclass setting.

Now, for $j \in \mathcal{J}$, $t \in [0,\infty)$, and $x \in \mathbb{R}_+$, let

$$M_j(t,x) = \langle 1_{[0,x]}, \mu_j(t)\rangle \quad \text{and} \quad \overline{M}_j(t,x) = \langle 1_{(x,\infty)}, \mu_j(t)\rangle.$$

We can recover the fluid performance functions from $\overline{M}_j$, for $j \in \mathcal{J}$: using (C.2), for $j \in \mathcal{J}$, $\overline{M}_j(t,0) = q_j(t)$ for all $t \in [0,\infty)$ and $w_j(t) = \int_0^\infty \overline{M}_j(t,x)dx$. Also, if $\mu_+(0) \in \mathbf{K}_+$, for $j \in \mathcal{J}$, $t \in [0,\infty)$ and $x \in \mathbb{R}_+$, by (10), we have that

$$\overline{M}_j(t,x) = \overline{M}_j(0, x + s(t)) + \alpha_j \int_0^t \overline{N}_j(x + s(t) - s(r))dr. \tag{11}$$

The following proposition is a special case of [1, Theorem 3.1]. We give an alternative proof using elements from [7].

Proposition 2.2. *Let $\xi \in \mathbf{K}^J$. A fluid model solution μ for the critical parameters (α, ν) and initial condition $\mu(0) = \xi$ exists and is unique.*

Proof. Let $\xi \in \mathbf{K}^J$. Then $\xi_+ \in \mathbf{K}$. Hence, by [7, Theorem 3.1], there exists a unique fluid model solution $\hat{\mu}^{\xi_+}$ for a fluid model of a single class processor sharing queue with critical parameters (α_+, ϑ_+) such that $\hat{\mu}^{\xi_+}(0) = \xi_+$. Let $\hat{q}^{\xi_+}(t) = \langle 1, \hat{\mu}^{\xi_+}(t)\rangle$ and $\hat{s}^{\xi_+}(t) = \int_0^t \varphi(\langle 1, \hat{\mu}^{\xi_+}(r)\rangle)dr$ for all $t \in [0,\infty)$.

To demonstrate existence for each $j \in \mathcal{J}$, define $\overline{M}_j^\xi$ via the right side of (11) using $q_+ = \hat{q}^{\xi_+}$ and $s = \hat{s}^{\xi_+}$ and, for each $t \in [0, \infty)$, let $\mu_j^\xi(t)$ be the measure in $\mathbf{M}$ such that $\langle 1_{\{0\}}, \mu_j^\xi(t)\rangle = 0$ and $\langle 1_{(x,\infty)}, \mu_j^\xi(t)\rangle = \overline{M}_j^\xi(t, x)$ for all $x \in \mathbb{R}_+$. Similarly to the proof of existence for fluid model solutions given on [7, Pages 825-826] for the single class case, it can be shown that μ^ξ is a fluid model solution satisfying (C.1)-(C.4) for the critical parameters (α, ν) with $\mu^\xi(0) = \xi$ (also see [7, Lemma 4.8]).

To demonstrate uniqueness, suppose that μ is a fluid model solution for the critical parameters (α, ν) such that $\mu(0) = \xi$. Then, since μ_+ is a fluid model solution for the critical parameters (α_+, ϑ_+) such that $\mu_+(0) = \xi_+$, it follows from [7, Theorem 3.1] that $\mu_+ = \hat{\mu}^{\xi_+}$, $q_+ = \hat{q}^{\xi_+}$ and $s = \hat{s}^{\xi_+}$. Then (11) implies that $\mu = \mu^\xi$. □

As in the proof of Proposition 2.2, given $\xi \in \mathbf{K}^J$, we sometimes denote the unique fluid model solution with initial condition ξ as μ^ξ, in which case we also append the superscript ξ to the associated functions, e.g., q^ξ, s^ξ, w^ξ, M^ξ and $\overline{M}^\xi$. Other times it will be more convenient to denote the initial condition as $\mu(0)$ and omit the superscript.

Next we establish continuity of fluid model solutions as functions of the initial condition. For this, let $\mathbf{C}([0,\infty), \mathbf{M}^J)$ denote the set of functions from $[0, \infty)$ into $\mathbf{M}^J$ that are continuous, endowed with the usual topology of uniform convergence on compact time intervals. Also, let $\mathbf{C}_{\uparrow,\infty}(\mathbb{R}_+)$ denote the set of continuous, strictly increasing functions f from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$, endowed with the topology of uniform convergence on compact sets. Define

$$\begin{aligned} \Xi(\xi) &= \mu^\xi, && \text{for } \xi \in \mathbf{K}^J, \\ \Xi_+(\xi) &= \mu_+^\xi, && \text{for } \xi \in \mathbf{K}^J. \end{aligned}$$

Proposition 2.3. $\Xi : \mathbf{K}^J \to \mathbf{C}([0,\infty), \mathbf{M}^J)$ *is continuous.*

Proof. For $J = 1$, this is [12, Theorem 3.3] (which follows from [7, Lemma 4.9] and [12, Theorem 3.4]). From this, it follows that Ξ_+ is continuous, and that Ξ is continuous at $\xi = \mathbf{0}$. Continuity of Ξ on $\mathbf{K}^J \setminus \{\mathbf{0}\}$ follows by arguing similarly to the proof of [7, Lemma 4.9]. In particular, since μ_+^ξ is a fluid model solution for the fluid model associated with the single class processor sharing queue with the critical data (α_+, ϑ_+), that proof shows that the mappings $s^\xi : \mathbf{K}^J \setminus \{\mathbf{0}\} \to \mathbf{C}_{\uparrow,\infty}(\mathbb{R}_+)$ and $\tau^\xi : \mathbf{K}^J \setminus \{\mathbf{0}\} \to \mathbf{C}_{\uparrow,\infty}(\mathbb{R}_+)$ given by (7) and (9) for μ^ξ are continuous on $\mathbf{K}^J \setminus \{\mathbf{0}\}$. From there one can verify that each coordinate of μ^ξ is continuous as a mapping from $\mathbf{K}^J \setminus \{\mathbf{0}\}$ to $\mathbf{M}$ by applying to each class $j \in \mathcal{J}$ the argument given in [7, Lemma 4.9]. This completes the proof. □

Lastly, we discuss some path properties of fluid model solutions in the next proposition.

Proposition 2.4. *Suppose that* $\xi \in \mathbf{K}^J$.

(1) *If* $\xi_+ \in \mathbf{K}_+$, *then* $\mu^\xi(t) \in \mathbf{K}_+^J$ *for all* $t \in (0, \infty)$. *In particular, if* $\xi \in \mathbf{K}_+^J$, *then* $\mu^\xi(t) \in \mathbf{K}_+^J$ *for all* $t \in [0, \infty)$.

(2) *If $\xi_+ \in \mathbf{K}_\dagger$, then $\mu^\xi(t) \in \mathbf{K}_\dagger^J$ for all $t \in (0,\infty)$. In particular, if $\xi \in \mathbf{K}_\dagger^J$, then $\mu^\xi(t) \in \mathbf{K}_\dagger^J$ for all $t \in [0,\infty)$.*

Proof. Fix $\xi \in \mathbf{K}^J$. First suppose that $\xi_+ \in \mathbf{K}_+$. Then, by Proposition 2.1, $t^* = \infty$ and $q_+^\xi(t) > 0$ for all $t \in [0,\infty)$. By applying an argument similar to the one given in the proof of [7, Proposition 4.5] to each class $j \in \mathcal{J}$, we obtain that $q_j^\xi(t) > 0$ for all $j \in \mathcal{J}$ and $t \in (0,\infty)$ (for $\xi \in \mathbf{K}_\dagger^J$, the details are provided in the proof in [9, Lemma 4.1.3]). This proves the first part of Proposition 2.4. Next suppose that $\xi_+ \in \mathbf{K}_\dagger$. Then, by the first part, $\mu^\xi(t) \in \mathbf{K}_+^J$ for all $t \in (0,\infty)$. By Proposition 2.1, $w_+^\xi = w_+^\xi(t) = w_+^\xi(0) < \infty$ for all $t \in [0,\infty)$. From all of the above and (C.2), it follows that $0 < w_j^\xi(t) < \infty$ for all $j \in \mathcal{J}$ and $t \in [0,\infty)$. □

2.3. **Invariant States.** A measure $\xi \in \mathbf{K}^J$ is an *invariant state* for critical parameters (α,ν) if the fluid model solution μ^ξ is such that $\mu^\xi(t) = \xi$ for all $t \in [0,\infty)$. The next proposition characterizes the set of invariant states. To state this, we must discuss the excess life distributions associated with the critical parameters.

Since $\nu \in \mathbf{P}_\dagger^J$, for each $j \in \mathcal{J}$, the excess life distribution ν_j^e of ν_j exists, is absolutely continuous with respect to Lebesgue measure, and has density $p_{\nu_j}(x) = \overline{N}_j(x)/\langle \chi, \nu_j\rangle$, for $x \in \mathbb{R}_+$. For convenience, for each $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$, we let

$$n_j^e(x) = p_{\nu_j}(x) = \frac{\overline{N}_j(x)}{\langle \chi, \nu_j\rangle}. \tag{12}$$

For $x \in \mathbb{R}_+$, we define the cumulative distribution function and complementary cumulative distribution function of the excess life distribution ν_j^e by

$$N_j^e(x) = \langle 1_{[0,x]}, \nu_j^e\rangle \quad \text{and} \quad \overline{N}_j^e(x) = 1 - N_j^e(x), \quad \text{for all } x \in \mathbb{R}_+, \tag{13}$$

respectively. Let

$$\pi = (\rho_1\nu_1^e, \cdots, \rho_J\nu_J^e). \tag{14}$$

Proposition 2.5. *The set of invariant states* $\mathbf{I}$ *associated with critical parameters* (α,ν) *is given by*

$$\mathbf{I} = \{\xi \in \mathbf{K}^J : \xi = c\pi,\ c \in \mathbb{R}_+\}.$$

Proposition 2.5 is a special case of [8, Theorem 6.3 and Lemma 6.4] and is also established as [9, Theorem 4.3.3]. We refer to π as the *fundamental invariant state*. The set of nonzero invariant states, denoted by $\mathbf{I}_+$, is given by

$$\mathbf{I}_+ = \{\xi \in \mathbf{K}^J : \xi = c\pi,\ c \in (0,\infty)\}.$$

Also, for convenience later on, we rewrite (11) in the following equivalent form using the notation introduced in (12). Given a fluid model solution μ such that $\mu(0) \in \mathbf{K}^J \setminus \{\mathbf{0}\}$, for each $j \in \mathcal{J}$, $t \in [0,\infty)$, and $x \in \mathbb{R}_+$,

$$\overline{M}_j(t,x) = \overline{M}_j(0, x + s(t)) + \rho_j \int_0^t n_j^e(x + s(t) - s(r))dr. \tag{15}$$

3. Main Results

Here we define the relative entropy functional and state the three main theorems of this paper. Henceforth, we make the further assumption that $\nu \in \mathbf{P}_{\ddagger}^J$, in particular, that $\langle \chi, \nu_j^e \rangle < \infty$ for all $j \in \mathcal{J}$. Then, $\langle \chi, \pi_+ \rangle < \infty$.

3.1. **The Definition of the Relative Entropy Functional.** In order to define our relative entropy functional, we consider an expansion of $\mathbf{M}$. The set $\mathcal{J}$ is endowed with the discrete topology and $\mathcal{J} \times \mathbb{R}_+$ is endowed with the product topology and associated Borel σ-algebra. We let γ denote the product of counting measure on $\mathcal{J}$ and Lebesgue measure on $\mathbb{R}_+$. We let $\widetilde{\mathbf{M}}$ denote the set of finite, nonnegative Borel measures on $\mathcal{J} \times \mathbb{R}_+$, endowed with the topology of weak convergence, and we let $\widetilde{\mathbf{P}}$ denote the subset of measures in $\widetilde{\mathbf{M}}$ that are probability measures on $\mathcal{J} \times \mathbb{R}_+$.

For $\zeta \in \mathbf{M}_\dagger$, we previously defined an associated excess life measure $\zeta^e \in \mathbf{P} \cap \mathbf{A}$ with probability density function p_ζ (see (1)). Our relative entropy function involves a generalization of the mapping $\zeta \to \zeta^e$ that maps measures in $\mathbf{M}^J$ such that the associated superposition measure is in $\mathbf{M}_\dagger$ to measures in $\widetilde{\mathbf{P}}$ that are absolutely continuous with respect to γ. For this, given $\zeta \in \mathbf{M}$, we define for $x \in \mathbb{R}_+$,

$$Z_\zeta(x) = \langle 1_{[0,x]}, \zeta \rangle \quad \text{and} \quad \overline{Z}_\zeta(x) = \langle 1_{(x,\infty)}, \zeta \rangle.$$

Then, for $\zeta \in \mathbf{M}^J$,

$$\langle \chi, \zeta_+ \rangle = \sum_{j=1}^{J} \int_{\mathbb{R}_+} \overline{Z}_{\zeta_j}(x) dx.$$

Given $\zeta \in \mathbf{M}^J$ such that $\zeta_+ \in \mathbf{M}_\dagger$, we associate with it $\tilde{\zeta} \in \widetilde{\mathbf{P}}$, which is the unique probability measure on $\mathcal{J} \times \mathbb{R}_+$ that is absolutely continuous with respect to γ and has Radon-Nikodym derivative p_ζ, where

$$p_\zeta(j,x) = \frac{\langle 1_{(x,\infty)}, \zeta_j \rangle}{\langle \chi, \zeta_+ \rangle} = \frac{\overline{Z}_{\zeta_j}(x)}{\langle \chi, \zeta_+ \rangle}, \quad \text{for } j \in \mathcal{J} \text{ and } x \in \mathbb{R}_+. \tag{16}$$

For $J = 1$ and $\zeta \in \mathbf{M}_\dagger$, $\zeta^e(A) = \tilde{\zeta}(\{1\} \times A)$ for all Borel subsets A of $\mathbb{R}_+$, and so (16) is a natural generalization of (1).

Since $\nu \in \mathbf{P}_{\ddagger}^J$, $\pi \in \mathbf{M}_{\dagger}^J$. Hence, $\widetilde{\pi} \in \widetilde{\mathbf{P}}$ is well defined. In particular, $\widetilde{\pi}$ is absolutely continuous with respect to γ and has Radon-Nikodym derivative p_π, where, for each $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$,

$$p_\pi(j,x) = \frac{\langle 1_{(x,\infty)}, \pi_j \rangle}{\langle \chi, \pi_+ \rangle} = \frac{\rho_j \langle 1_{(x,\infty)}, \nu_j^e \rangle}{\langle \chi, \pi_+ \rangle} = \frac{\rho_j \overline{N}_j^e(x)}{\langle \chi, \pi_+ \rangle}. \tag{17}$$

Observe that if $\xi \in \mathbf{I}_+$, then $\tilde{\xi} = \tilde{\pi}$.

For $x \in (0,\infty)$, let

$$h(x) = x \ln x.$$

We set $h(0) = 0$. Noting that $\lim_{x \downarrow 0} h(x) = 0$, h is a nonnegative, continuous function on $\mathbb{R}_+$. We define the multiclass relative entropy functional H as follows:

for $\zeta \in \mathbf{M}^J$ such that $\zeta_+ \in \mathbf{M}_\dagger$, define

$$H(\zeta) = \int_{\mathcal{J}\times\mathbb{R}_+} h\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) p_\pi(j,x)d\gamma(j,x), \tag{18}$$

where we adopt the convention that the integrand is $+\infty$ for $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$ such that $p_\zeta(j,x) > 0$ and $p_\pi(j,x) = 0$ and is zero for $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$ such that $p_\zeta(j,x) = p_\pi(j,x) = 0$. Then the quantity $H(\zeta)$ is the relative entropy of $\widetilde{\zeta}$ with respect to $\widetilde{\pi}$, and it takes values in $[0,\infty]$. Note that if $\langle 1_{[x_j^*,\infty)}, \zeta_j\rangle = 0$ for each $j \in \mathcal{J}$, then

$$H(\zeta) = \sum_{j=1}^{J} \int_{[0,x_j^*)} h\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) p_\pi(j,x)dx. \tag{19}$$

Otherwise, $H(\zeta) = +\infty$.

If $J = 1$, then (α_1, ν_1) are the critical parameters. Observe that in this case, $\pi = \nu_1^e$ and $\tilde{\pi}(\{1\}\times A) = (\nu_1^e)^e(A)$ for all Borel subsets A of $\mathbb{R}_+$. Thus, when $J = 1$, $H(\zeta)$ is the relative entropy of ζ^e with respect to $(\nu_1^e)^e$, which coincides with $H(\zeta)$ as defined in [14].

For $\zeta \in \mathbf{M}_\dagger^J$, $H(\zeta)$ can be rewritten as a certain convex combination of relative entropy functions involving measures in $\mathbf{P}$, plus a discrete relative entropy term involving probability measures on $\mathcal{J}$. This approach was taken in [9, Definition 5.2.1], which can be shown to agree with (18) for $\zeta \in \mathbf{M}_\dagger^J$. Here, in (18), we present H as a single relative entropy function involving measures in $\widetilde{\mathbf{P}}$.

We will consider H along the paths of fluid model solutions. By Proposition 2.4, given $\xi \in \mathbf{K}^J$ such that $\xi_+ \in \mathbf{K}_\dagger$, we have that $\mu_+^\xi(t) \in \mathbf{K}_\dagger$ for all $t \in [0,\infty)$. Then, given such a ξ, for each $t \in [0,\infty)$, the density $p_{\mu^\xi(t)}$ is given by (16) with $\zeta = \mu^\xi$:

$$p_{\mu^\xi(t)}(j,x) = \frac{\overline{M}_j^\xi(t,x)}{w_+^\xi} = \frac{\overline{M}_j(t,x)}{w_+}, \qquad \text{for all } (j,x) \in \mathcal{J}\times\mathbb{R}_+, \tag{20}$$

where we have suppressed the dependence on ξ in the final expression to simplify the notation. Hence, given $\xi \in \mathbf{K}^J$ such that $\xi_+ \in \mathbf{K}_\dagger$, $H(\mu^\xi(t))$ is defined for each $t \in [0,\infty)$ and we can view $H(\mu^\xi(t))$ as an extended real-valued function of $t \in [0,\infty)$. For $\xi \in \mathbf{K}^J$ such that $\xi_+ \in \mathbf{K}_\dagger$ and $t \in [0,\infty)$, let

$$\mathcal{H}^\xi(t) = H(\mu^\xi(t)). \tag{21}$$

3.2. **Uniform Convergence of the Relative Entropy Functional to Zero.** The first main result of the paper stated below as Theorem 3.1 generalizes [14, Theorem 3.2]. It concerns uniform convergence of $\mathcal{H}^\xi$ to zero for ξ lying in certain relatively compact subsets of $\mathbf{M}^J$. First we define these sets. For $u > 0$ and each $j \in \mathcal{J}$, let

$$\mathbf{K}_u(j) = \{\xi \in \mathbf{K} : \overline{Z}_\xi(x) \le u\overline{N}_j^e(x) \text{ for all } x \in \mathbb{R}_+\}. \tag{22}$$

Note that (22) is $\mathbf{K}_u$, as given in [14], adapted to each class $j \in \mathcal{J}$. So, by the justification given in [14, Page 260], the set defined in (22) is relatively compact for each $j \in \mathcal{J}$. Now, for $u > 0$, define

$$\mathbf{K}_u^{\times} = \mathbf{K}_u(1) \times \cdots \times \mathbf{K}_u(J) = \bigtimes_{j=1}^{J} \mathbf{K}_u(j). \tag{23}$$

It follows that this set is relatively compact. Notice that given $\xi \in \mathbf{K}_u^{\times}$, it is possible that $\langle 1, \xi_j \rangle = 0$ for some or all $j \in \mathcal{J}$. As in [14], we wish to exclude such elements. This leads us to define, for $u, l > 0$,

$$\mathbf{K}_{u,l}^{\times} = \mathbf{K}_u^{\times} \cap \{\xi \in \mathbf{K}^J : \langle \chi, \xi_j \rangle \geq l \text{ for all } j \in \mathcal{J}\}. \tag{24}$$

Observe that $\mathbf{K}_{u,l}^{\times} \subseteq \mathbf{K}_u^{\times}$, which implies that $\mathbf{K}_{u,l}^{\times}$ is relatively compact.

Theorem 3.1. *Let $u, l > 0$. Then, for each initial condition $\xi \in \mathbf{K}_{u,l}^{\times}$, $\mathcal{H}^{\xi}$ is nonincreasing on $[0, \infty)$. Furthermore,*

$$\lim_{t \to \infty} \sup_{\xi \in \mathbf{K}_{u,l}^{\times}} \mathcal{H}^{\xi}(t) = 0.$$

To prove Theorem 3.1, we utilize a similar strategy to that used in [14, Theorem 3.2]. Indeed, we will show that $\mathcal{H}^{\xi}$ is absolutely continuous, with respect to time, with nonpositive density, provided that the initial condition $\xi \in \mathbf{K}_{u,l}^{\times}$. In fact, as in [14], we are able to compute the density explicity, although the form is necessarily a little more involved than in [14]. For this, we define a related function K. For $a, b, x \in (0, \infty)$, define

$$k(a, b, x) = a(x - 1) - b \ln(x) + (a - b) \ln\left(\frac{a}{b}\right), \tag{25}$$

and for $a, b \in (0, \infty)$ define $k(a, b, 0) = +\infty$. Then k is a continuous, extended real-valued function on $(0, \infty)^2 \times \mathbb{R}_+$. The arguments a and b generalize the single variable version of k used in [14]. In particular, the single variable version coincides with $k(1, 1, \cdot)$. Below it is shown that k as defined here is nonnegative on $(0, \infty)^2 \times \mathbb{R}_+$ and is zero if and only if $a = b$ and $x = b/a$ (see Lemma 5.8).

Given $\zeta \in \mathbf{M}_0^J$, define

$$K(\zeta) \;=\; \int_{\mathcal{J} \times \mathbb{R}_+} k\left(\frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)}\right) n_j^e(x) d\gamma(j, x), \tag{26}$$

where for each $j \in \mathcal{J}$, n_j^e is the density of ν_j^e as given by (12) and, by convention, the integrand takes the value $+\infty$ for $j \in \mathcal{J}$ and $x \in [0, x_j^*)$ such that $\overline{Z}_{\zeta_j}(x) = 0$, or $j \in \mathcal{J}$ and $x \in [x_j^*, \infty)$ such that $\overline{Z}_{\zeta_j}(x) > 0$, and takes the value zero for $x \in [x_j^*, \infty)$ such that $\overline{Z}_{\zeta_j}(x) = 0$. Then, $K : \mathbf{M}_0^J \to [0, \infty]$ is an extended real-valued function. For $\zeta \in \mathbf{M}_0^J$ such that $\langle 1_{[x_j^*, \infty)}, \zeta_j \rangle = 0$ for all $j \in \mathcal{J}$,

$$K(\zeta) \;=\; \sum_{j=1}^{J} \int_{[0, x_j^*)} k\left(\frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)}\right) n_j^e(x) dx. \tag{27}$$

This generalizes [14, (57)]. To see this, note that if $J = 1$ and $\zeta \in \mathbf{M}_0$, then $\langle 1, \zeta_1 \rangle / \langle 1, \zeta_+ \rangle = 1$ and (2) implies that $\rho_1 = 1$, so that (27) reduces to [14, (57)]. Similarly as was done above, we consider K as a function of time along fluid model solutions. By Proposition 2.4, for $\xi \in \mathbf{K}_+^J$, $\mu^\xi(t) \in \mathbf{K}_+^J$ for all $t \in [0, \infty)$ and we set

$$\mathcal{K}^\xi(t) = K(\mu^\xi(t)), \qquad \text{for all } t \in [0, \infty). \tag{28}$$

Theorem 3.2. *Let $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}^\times$. The function $\mathcal{H}^\xi$ is finite-valued and absolutely continuous with respect to Lebesgue measure on $[0, \infty)$ with density function κ^ξ, where κ^ξ is nonpositive and is given by*

$$\kappa^\xi(t) = -\frac{1}{w_+} \mathcal{K}^\xi(t), \qquad \textit{for all } t \in [0, \infty). \tag{29}$$

In particular, $\mathcal{H}^\xi$ is nonincreasing on $[0, \infty)$. Furthermore, for $t \in [0, \infty)$, $\kappa^\xi(t) < 0$ unless

$$\mu^\xi(t) = \frac{w_+}{\langle \chi, \pi_+ \rangle} \pi. \tag{30}$$

Theorem 3.2 is a generalization of the single class result [14, Lemma 7.8] to multiclass processor sharing queues. The proof of Theorem 3.2 is given in Section 7.4. In preparation for this, a version of Theorem 3.2 for initial conditions in $\mathbf{A}_{u,l}^\times = \mathbf{A}^J \cap \mathbf{M}_{u,l}^\times$ is proved in Section 7.3 (see Lemma 7.1). It is in the proof of Lemma 7.1 that the details of the definition of $\mathcal{K}^\xi$ are leveraged. Theorem 3.2 is used in conjunction with various properties of fluid model solutions developed in Section 5 to prove Theorem 3.1 in Section 8.

3.3. **Uniform Convergence to the Set of Invariant States.** Finally, as was done in [14] for the single class processor sharing queue, we leverage Theorem 3.1, together with properties of fluid model solutions, to obtain uniform convergence of multiclass fluid model solutions to the set of invariant states provided that the corresponding initial conditions are all members of the relatively compact set $\mathbf{K}_{u,l}$, for some $u, l > 0$.

Theorem 3.3. *Let $u, l > 0$. Then,*

$$\lim_{t \to \infty} \sup_{\xi \in \mathbf{K}_{u,l}^\times} \mathbf{d}_J(\mu^\xi(t), \mathbf{I}) = 0. \tag{31}$$

Furthermore, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\xi \in \mathbf{K}_{u,l}^\times(\delta)} \sup_{t \in [0, \infty)} \mathbf{d}_J(\mu^\xi(t), \mathbf{I}) < \epsilon, \tag{32}$$

where $\mathbf{K}_{u,l}^\times(\delta) = \{\zeta \in \mathbf{K}_{u,l}^\times : \mathbf{d}_J(\zeta, \mathbf{I}) < \delta\}$.

The proof of Theorem 3.3 is similar to the proof of the analogous result stated in [14, Theorem 3.1], and is given in Section 8.

3.4. **Organization of the Remainder of the Paper.** The remaining sections of the paper are structured in the following manner. In Section 4, we introduce families of compact sets that contain fluid model solution values when the initial conditions are the relatively compact sets introduced above. In Section 5, important properties, such as continuity and finiteness, of the functions H and K are established. Such properties are needed to prove the main theorems stated above. From there, in Section 6, we proceed to establish compact containment of fluid model solutions. In Section 7, we proceed to study H and K along fluid paths by considering the time dependent functions $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$ for $\xi \in \mathbf{K}_{u,l}^\times$ for some $u, l > 0$. By utilizing the properties of H and K established in Section 5, along with the compact containment of fluid model solutions established in Section 6, we establish finiteness and continuity properties for $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$. Also in Section 7, we prove Lemma 7.1, a special case of Theorem 3.2 restricted to absolutely continuous initial conditions. We then extend Lemma 7.1 to Theorem 3.2 via an approximation argument. Finally, in Section 8, we provide proofs for Theorems 3.1 and 3.3.

4. Compact Subsets of $\mathbf{M}^J$

In this section, we introduce a family of sets analogous to the family of compact sets given in [14]. These sets will be considered throughout the paper, and are necessary for the study of the entropy functional defined in (18). In addition, we introduce some related compact sets that play integral roles in the proofs. For $u > 0$ and $j \in \mathcal{J}$, define $\mathbf{M}_u(j)$ as follows:

$$\mathbf{M}_u(j) = \{\zeta \in \mathbf{M} : \langle 1, \zeta\rangle \leq u \text{ and } \overline{Z}_\zeta(x) \leq u\overline{N}_j^e(x) \text{ for all } x \in \mathbb{R}_+\},$$

which is analogous to the set $\mathbf{M}_u$ considered in [14]. Then, for $u > 0$, define $\mathbf{M}_u^\times$ by

$$\mathbf{M}_u^\times = \bigtimes_{j=1}^{J} \mathbf{M}_u(j).$$

For $u, l > 0$ and $j \in \mathcal{J}$, define $\mathbf{M}_{u,l}(j)$ as follows:

$$\mathbf{M}_{u,l}(j) = \mathbf{M}_u(j) \cap \{\zeta \in \mathbf{M} : \langle \chi, \zeta\rangle \geq l\}, \tag{33}$$

which is analogous to the set $\mathbf{M}_{u,l}$ studied in [14]. Note that for $u, l > 0$ and $j \in \mathcal{J}$, $\mathbf{M}_{u,l}(j) \neq \emptyset$ if and only if $0 < l \leq u\langle \chi, \nu_j^e\rangle$. Now, for $u, l > 0$, let $\mathbf{M}_{u,l}^\times$ be given by

$$\mathbf{M}_{u,l}^\times = \bigtimes_{j=1}^{J} \mathbf{M}_{u,l}(j). \tag{34}$$

For $u, l > 0$, define

$$\mathbf{M}_{u,l}^+ = \{\zeta \in \mathbf{M}_u^\times : \langle \chi, \zeta_+\rangle \geq l\}, \tag{35}$$

where as always ζ_+ is the superposition measure of ζ. Note that for any $u, l > 0$, it follows that $\mathbf{M}_{u,l}^\times \subset \mathbf{M}_{u,l}^+$, since for any $\zeta \in \mathbf{M}_{u,l}^\times$, $\langle \chi, \zeta_j\rangle \geq l$ for each $j \in \mathcal{J}$ implies that $\langle \chi, \zeta_+\rangle \geq Jl \geq l$.

Recall that, for $u, l > 0$, $\mathbf{K}_u^\times$ and $\mathbf{K}_{u,l}^\times$ were defined previously in (23) and (24), respectively. Since measures in $\mathbf{K}$ have no atoms and in particular do not charge the origin, it follows that $\mathbf{K}_u^{\times} = \mathbf{K}^J \cap \mathbf{M}_u^{\times}$ and $\mathbf{K}_{u,l}^{\times} = \mathbf{K}^J \cap \mathbf{M}_{u,l}^{\times}$ for all $u, l > 0$. For $u, l > 0$, let $\mathbf{K}_{u,l}^{+} = \mathbf{K}^J \cap \mathbf{M}_{u,l}^{+}$.

Assuming that a sequence in $\mathbf{M}^J$ converges weakly to an element of $\mathbf{M}^J$, we establish conditions under which the first moments of the component measures also converge to the first moments of the components of the limiting measure. This is stated in the next lemma and follows via a generalized uniform integrability argument.

Lemma 4.1. *Suppose $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^J$ and $\zeta^n \xrightarrow{w} \zeta \in \mathbf{M}^J$ as $n \to \infty$.*

(1) If $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_u^{\times}$ for some $u > 0$, then, for each $j \in \mathcal{J}$, $\langle \chi, \zeta_j^n \rangle \to \langle \chi, \zeta_j \rangle$ as $n \to \infty$. Also, $\langle \chi, \zeta_+^n \rangle \to \langle \chi, \zeta_+ \rangle$ as $n \to \infty$.

(2) If $\{\zeta_+^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_1$, $\zeta_+ \in \mathbf{M}_1$, and $\langle \chi, \zeta_+^n \rangle \to \langle \chi, \zeta_+ \rangle$ as $n \to \infty$, then, for each $j \in \mathcal{J}$, $\langle \chi, \zeta_j^n \rangle \to \langle \chi, \zeta_j \rangle$ as $n \to \infty$.

Proof. Suppose that $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_u^{\times}$ for some $u > 0$. Then, for each $j \in \mathcal{J}$, for $x \in \mathbb{R}_+$,

$$\begin{aligned}
\langle \chi 1_{(x,\infty)}, \zeta_j^n \rangle &= x\overline{Z}_{\zeta_j^n}(x) + \int_x^\infty \overline{Z}_{\zeta_j^n}(y)dy \\
&\le u\left(x\overline{N}_j^e(x) + \int_x^\infty \overline{N}_j^e(y)dy \right) \\
&= u\langle \chi 1_{(x,\infty)}, \nu_j^e \rangle.
\end{aligned}$$

Hence, for each $j \in \mathcal{J}$,

$$\sup_{n\in\mathbb{N}} \langle \chi 1_{(x,\infty)}, \zeta_j^n \rangle \le u\langle \chi 1_{(x,\infty)}, \nu_j^e \rangle. \tag{36}$$

Since, for each $j \in \mathcal{J}$, $\zeta_j^n \xrightarrow{w} \zeta_j$ as $n \to \infty$ and $\langle \chi, \nu_j^e \rangle < \infty$, using (36) to control the tails of the integrals in $\left\langle \chi, \zeta_j^n \right\rangle$ and $\langle \chi, \zeta_j \rangle$ for each $j \in \mathcal{J}$, uniformly in n, we can show that

$$\lim_{n\to\infty} \langle \chi, \zeta_j^n \rangle = \langle \chi, \zeta_j \rangle, \qquad \text{for each } j \in \mathcal{J}. \tag{37}$$

The convergence with ζ_+^n, ζ_+ in place of ζ_j^n, ζ_j follows by summing over $j \in \mathcal{J}$. Thus the first part holds.

Suppose that $\{\zeta_+^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_1$, $\zeta_+ \in \mathbf{M}_1$, and $\langle \chi, \zeta_+^n \rangle \to \langle \chi, \zeta_+ \rangle$ as $n \to \infty$. Then, for each $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$,

$$\begin{aligned}
\lim_{x\to\infty} \sup_{n\in\mathbb{N}} \langle \chi 1_{(x,\infty)}, \zeta_j^n \rangle &\le \lim_{x\to\infty} \sup_{n\in\mathbb{N}} \langle \chi 1_{(x,\infty)}, \zeta_+^n \rangle = 0, \\
\lim_{x\to\infty} \langle \chi 1_{(x,\infty)}, \zeta_j \rangle &\le \lim_{x\to\infty} \langle \chi 1_{(x,\infty)}, \zeta_+ \rangle = 0.
\end{aligned}$$

The second part follows from this.

□

Next we utilize Lemma 4.1 to prove a compactness result.

Lemma 4.2. *Given $u, l > 0$, $\mathbf{M}_u^{\times}$, $\mathbf{M}_{u,l}^{\times}$, and $\mathbf{M}_{u,l}^{+}$ are compact.*

Proof. Let $u, l > 0$. It follows from [14, Lemma 4.6] that, for each $j \in \mathcal{J}$, $\mathbf{M}_u(j)$ and $\mathbf{M}_{u,l}(j)$ are compact. Thus, we have immediately that $\mathbf{M}_u^\times$ and $\mathbf{M}_{u,l}^\times$ are compact. It is left to show that $\mathbf{M}_{u,l}^+$ is compact.

Since $\mathbf{M}_{u,l}^+ \subset \mathbf{M}_u^\times$ and $\mathbf{M}_u^\times$ is compact, it suffices to show that $\mathbf{M}_{u,l}^+$ is closed. Suppose that $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_{u,l}^+$ and for each $j \in \mathcal{J}$, $\zeta^n \xrightarrow{w} \zeta$ as $n \to \infty$. It is enough to show that $\zeta \in \mathbf{M}_{u,l}^+$. Since $\mathbf{M}_u^\times$ is a compact set and $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_u^\times$, it follows that $\zeta \in \mathbf{M}_u^\times$. Therefore, it is enough to show that $\langle \chi, \zeta_+ \rangle \geq l$. By Lemma 4.1, (37) holds, which, along with the fact that $\zeta^n \in \mathbf{M}_{u,l}^+$ for all $n \in \mathbb{N}$, implies that

$$l \leq \lim_{n\to\infty} \langle \chi, \zeta_+^n \rangle = \langle \chi, \zeta_+ \rangle.$$

□

For $u, l > 0$, we can apply the compactness result of Lemma 4.2 to obtain a strictly positive lower bound on the total mass of elements in $\mathbf{M}_{u,l}(j)$, for $j \in \mathcal{J}$. Given $u, l > 0$, set

$$\lambda(u,l) = \min_{j=1}^{J} \left(\inf\{\langle 1, \zeta \rangle : \zeta \in \mathbf{M}_{u,l}(j)\} \right). \tag{38}$$

It follows that $\lambda(u,l) > 0$ since for each $j \in \mathcal{J}$, $\mathbf{M}_{u,l}(j)$ is compact, $\zeta \mapsto \langle 1, \zeta \rangle$ is a continuous function on $\mathbf{M}$, and $\langle 1, \zeta \rangle > 0$ for all $\zeta \in \mathbf{M}_{u,l}(j)$. Now, for $u, l, \theta > 0$, we define for each $j \in \mathcal{J}$

$$\begin{aligned} \mathbf{M}_{u,l,\theta}(j) \quad = \quad & \{\zeta \in \mathbf{M}_{u,l}(j) : \\ & \overline{Z}_\zeta(x) \geq \lambda(u,l)\rho_j \left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta) \right) \text{ for } x \in \mathbb{R}_+ \right\}, \end{aligned}$$

which is analogous to $\mathbf{M}_{u,l,\theta}$ as defined in [14]. Additionally, for $u, l, \theta > 0$, define

$$\mathbf{M}_{u,l,\theta}^\times = \bigtimes_{j=1}^{J} \mathbf{M}_{u,l,\theta}(j), \tag{39}$$

$$\mathbf{K}_{u,l,\theta}^\times = \mathbf{M}_{u,l,\theta}^\times \cap \mathbf{K}^J. \tag{40}$$

Lemma 4.3. *For $u, l, \theta > 0$, $\mathbf{M}_{u,l,\theta}^\times$ is compact.*

Proof. Lemma 4.3 follows from the fact that, for each $u, l, \theta > 0$ and $j \in \mathcal{J}$, the set $\mathbf{M}_{u,l,\theta}(j)$ is compact, which follows from [14, Lemma 5.4]. □

5. Properties of the Functions H and K

Here we develop important properties of the entropy functional H defined in (18) and the function K defined in (26). In particular, we establish that H is finite-valued and continuous on the family of compact sets defined in (34) and that K is finite-valued and continuous on the family of compact sets defined in (39). Note that, many of the results in this section are similar to results given in Section 4 and Section 7 of [14]. However, it is necessary to demonstrate that H and K, as generalized here, have the same properties as in the single class setting, as these properties are necessary for establishing Theorems 3.1, 3.2, and 3.3.

5.1. **The function H.** Before we start discussing properties of H, we define the set $\mathbf{L}$ to be an augmentation of the set of invariant states $\mathbf{I}$, given in Proposition 2.5, by allowing for the possibility of a point mass at the origin. That is, we define $\mathbf{L}$ by

$$\mathbf{L} = \{\zeta \in \mathbf{M}^J : \zeta_j = a_j\delta_0 + c\rho_j\nu_j^e \text{ for } j \in \mathcal{J}, \text{ where } a \in \mathbb{R}_+^J \text{ and } c \in \mathbb{R}_+\}.$$

For notational convenience, we define $\mathbf{L}_+$ as follows:

$$\mathbf{L}_+ = \{\zeta \in \mathbf{M}^J : \zeta_j = a_j\delta_0 + c\rho_j\nu_j^e \text{ for } j \in \mathcal{J}, \text{ where } a \in \mathbb{R}_+^J \text{ and } c > 0\}.$$

Our first result of this section states that H defined in (18) on $\{\zeta \in \mathbf{M}^J : \zeta_+ \in \mathbf{M}_\dagger\}$ is nonnegative and vanishes on $\mathbf{L}_+$. Note that this is the best result one could hope for, as (16) contains no information about the mass that the components of its argument places at the origin. For measures $\zeta \in \mathbf{M}_\dagger^J$, the result in part (ii) of Lemma 5.1 is proved in the master's thesis [9, Theorem 5.3.1] using the formulation [9, Definition 5.2.1] of H.

Lemma 5.1. *For each $\zeta \in \mathbf{M}^J$ such that $\zeta_+ \in \mathbf{M}_\dagger$,*

(i) $H(\zeta) \in [0, \infty]$;

(ii) $H(\zeta) = 0$ *if and only if* $\zeta \in \mathbf{L}_+$.

Proof. Part (i) of Lemma 5.1 follows from the fact that relative entropy is nonnegative. Let $\zeta \in \mathbf{M}^J$ be such that $\zeta_+ \in \mathbf{M}_\dagger$. If $\zeta \in \mathbf{L}_+$, then $p_\zeta = p_\pi$ and so $H(\zeta) = 0$. Conversely, suppose $H(\zeta) = 0$. Then, due to the convention for (18), $\langle 1_{[x_j^*,\infty)}, \zeta_j\rangle = 0$ for all $j \in \mathcal{J}$. By (19) and properties of h, it follows that for each $j \in \mathcal{J}$ and Lebesgue almost every $x \in [0, x_j^*)$,

$$p_\zeta(j, x) = 0 \qquad \text{or} \qquad \frac{p_\zeta(j, x)}{p_\pi(j, x)} = 1.$$

Let $c = \langle \chi, \zeta_+\rangle / \langle \chi, \pi_+\rangle$. Note that if $p_\zeta(j, y) = 0$ for some $j \in \mathcal{J}$ and $y \in \mathbb{R}_+$, then $p_\zeta(j, x) = 0$ for all $x \geq y$. For $j \in \mathcal{J}$, let $y_j^* = \inf\{x \in [0, x_j^*) : p_\zeta(j, x) = 0\}$. Then, for each $j \in \mathcal{J}$, $y_j^* \leq x_j^*$ and for each $x \in [0, y_j^*)$,

$$\frac{p_\zeta(j, x)}{p_\pi(j, x)} = 1.$$

It follows that for each $j \in \mathcal{J}$, there exists $a_j \geq 0$ such that $\zeta_j = a_j\delta_0 + c\rho_j\nu_j^e$ on $[0, y_j^*)$ and $\langle 1_{[y_j^*,\infty)}, \zeta_j\rangle = 0$. Hence, to complete the proof, it suffices to show that $y_j^* = x_j^*$ for each $j \in \mathcal{J}$. If $y_j^* < x_j^*$ for some $j \in \mathcal{J}$, then $\langle \chi, \zeta_+\rangle = c\sum_{j\in\mathcal{J}}\langle \chi 1_{[0,y_j^*)}, \nu_j^e\rangle < \langle \chi, \zeta_+\rangle$, which is a contradiction and the result follows. $\square$

The next lemma follows by applying an argument similar to the one used to prove [14, Lemma 4.2] to each class $j \in \mathcal{J}$. It provides conditions under which weak convergence of a sequence $\{\zeta^n\}_{n\in\mathbb{N}}$ in $\mathbf{M}^J$ such that $\zeta_+^n \in \mathbf{M}_\dagger$ for each $n \in \mathbb{N}$ follows from weak convergence of the corresponding sequence $\{\widetilde{\zeta}^n\}_{n\in\mathbb{N}}$ of measures in $\widetilde{\mathbf{P}}$.

Lemma 5.2. *Suppose that* $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^J$, $\{\zeta_+^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_\dagger$, $\zeta \in \mathbf{M}^J$, $\zeta_+ \in \mathbf{M}_\dagger$, *and* $\tilde{\zeta}^n \xrightarrow{w} \tilde{\zeta}$ *as* $n \to \infty$. *If, for each* $j \in \mathcal{J}$,

$$\lim_{n\to\infty} \langle 1, \zeta_j^n \rangle = \langle 1, \zeta_j \rangle \quad \textit{and} \quad \lim_{n\to\infty} \langle \chi, \zeta_+^n \rangle = \langle \chi, \zeta_+ \rangle, \tag{41}$$

then $\zeta^n \xrightarrow{w} \zeta$ *as* $n \to \infty$.

Proof. Let $f \in \mathbf{C}_b^1(\mathbb{R}_+)$. It follows from $\tilde{\zeta}^n \xrightarrow{w} \tilde{\zeta}$ and $\langle \chi, \zeta_+^n \rangle \to \langle \chi, \zeta_+ \rangle$ as $n \to \infty$ that, for each $j \in \mathcal{J}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} f'(x)\overline{Z}_{\zeta_j^n}(x)dx = \int_{\mathbb{R}_+} f'(x)\overline{Z}_{\zeta_j}(x)dx.$$

For each $j \in \mathcal{J}$ and $n \in \mathbb{N}$, an application of integration by parts yields that

$$\begin{aligned} \int_{\mathbb{R}_+} f'(x)\overline{Z}_{\zeta_j^n}(x)dx &= -f(0)\langle 1, \zeta_j^n \rangle + \langle f, \zeta_j^n \rangle, \\ \int_{\mathbb{R}_+} f'(x)\overline{Z}_{\zeta_j}(x)dx &= -f(0)\langle 1, \zeta_j \rangle + \langle f, \zeta_j \rangle. \end{aligned}$$

Since $f \in \mathbf{C}_b^1(\mathbb{R}_+)$ was arbitrary and $\mathbf{C}_b^1(\mathbb{R}_+)$ is convergence determining for the weak topology on $\mathbf{M}$, $\zeta^n \xrightarrow{w} \zeta$ as $n \to \infty$. □

Now we relate convergence to zero of H to weak convergence to the set $\mathbf{L}_+$. The following lemma is an extension of [14, Lemma 4.3].

Lemma 5.3. *Suppose that* $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^J$, $\{\zeta_+^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_\dagger$ *and* $\lim_{n\to\infty} H(\zeta^n) = 0$. *In addition, suppose there exists* $a \in \mathbb{R}_+^J$ *and* $c > 0$ *such that, for each* $j \in \mathcal{J}$,

$$\lim_{n\to\infty} \langle 1, \zeta_j^n \rangle = a_j + c\rho_j \quad \textit{and} \quad \lim_{n\to\infty} \langle \chi, \zeta_+^n \rangle = c\langle \chi, \pi_+ \rangle. \tag{42}$$

Then, as $n \to \infty$, $\zeta^n \xrightarrow{w} \zeta$, *where* $\zeta \in \mathbf{L}_+$ *is such that, for* $j \in \mathcal{J}$,

$$\zeta_j = a_j\delta_0 + c\rho_j\nu_j^e.$$

Proof. Since $\lim_{n\to\infty} H(\zeta^n) = 0$, it follows from Pinsker's inequality that $\lim_{n\to\infty} \|\tilde{\zeta}^n - \tilde{\pi}\|_{\mathrm{TV}} = 0$, where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm. Therefore, $\tilde{\zeta}^n \xrightarrow{w} \tilde{\pi}$ as $n \to \infty$. For each $j \in \mathcal{J}$, let $\zeta_j = a_j\delta_0 + c\rho_j\nu_j^e$ and set $\zeta = (\zeta_1, ..., \zeta_J)$. Note that $\zeta \in \mathbf{M}^J$ and $\zeta_+ \in \mathbf{M}_\dagger$. Moreover, for each $j \in \mathcal{J}$, $\langle 1, \zeta_j \rangle = a_j + c\rho_j$ and $\langle \chi, \zeta_+ \rangle = c\langle \chi, \pi_+ \rangle$. Given this, the result follows from Lemma 5.2. □

Now, if an element of $\mathbf{M}^J$ is near an element of $\mathbf{L}$, we wish to bound the distance to $\mathbf{I}$, the set of invariant states. The next lemma is analogous to [14, Lemma 4.4] and does just this.

Lemma 5.4. *Suppose that* $\zeta \in \mathbf{M}^J$, $\eta \in \mathbf{L}$, $\epsilon > 0$, *and* $\mathbf{d}_J(\zeta, \eta) \le \epsilon$. *Let* $c \ge 0$ *and* $a \in \mathbb{R}_+^J$ *be such that* $\eta_j = a_j\delta_0 + c\rho_j\nu_j^e$ *for* $j \in \mathcal{J}$. *Then, for each* $j \in \mathcal{J}$, $\mathbf{d}(\zeta_j, c\rho_j\nu_j^e) \le \epsilon + a_j$. *Furthermore, for each* $j \in \mathcal{J}$, $a_j \le \zeta_j([0,\epsilon)) + \epsilon$ *so that*

$$\mathbf{d}_J(\zeta, \mathbf{I}) \le \max_{j=1}^{J} \zeta_j([0,\epsilon)) + 2\epsilon.$$

Proof. Applying an identical argument as given in the proof of [14, Lemma 4.4] to each component of ζ yields $\mathbf{d}(\zeta_j, c\rho_j\nu_j^e) \leq \zeta_j([0,\epsilon)) + 2\epsilon$, for each $j \in \mathcal{J}$. The result then follows from the definition of $\mathbf{d}_J$. □

Now, we will give sufficient conditions for the lower semicontinuity of the entropy functional H. This is a necessary step in obtaining the continuity result for H later on.

Lemma 5.5. *Suppose that $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^J$, $\zeta \in \mathbf{M}^J$, $\{\zeta_+^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_\dagger$, $\zeta_+ \in \mathbf{M}_\dagger$, and $\zeta^n \xrightarrow{w} \zeta$ and $\langle \chi, \zeta_+^n\rangle \to \langle \chi, \zeta_+\rangle$ as $n \to \infty$. Then,*

$$\liminf_{n\to\infty} H(\zeta^n) \geq H(\zeta).$$

Proof. This follows similarly to the proof of [14, Lemma 4.5]. □

One reason for our interest in compact sets of the form $\mathbf{M}_{u,l}^\times$ for $u, l > 0$, is that the multiclass entropy functional is actually continuous on $\mathbf{M}_{u,l}^\times$. We state and prove this as Lemma 5.7 below. First, we prove the following ancillary result.

Lemma 5.6. *For $u, l > 0$, $\zeta \in \mathbf{M}_{u,l}^\times$, $j \in \mathcal{J}$, and $x \in [0, x_j^*)$,*

$$-e^{-1} \leq h\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) \leq h\left(\frac{u\langle\chi,\pi_+\rangle}{l\rho_j}\right), \tag{43}$$

and

$$H(\zeta) \leq h\left(\frac{u\langle\chi,\pi_+\rangle}{l\min_{j\in\mathcal{J}}\rho_j}\right). \tag{44}$$

In particular, $H(\zeta) < \infty$ for every $\zeta \in \mathbf{M}_{u,l}^\times$.

Proof. Let $u, l > 0$. Note that (43) follows similarly to the verification of (28) in [14, Lemma 4.7]. In particular, for $\zeta \in \mathbf{M}_{u,l}^\times$,

$$l \leq u\langle\chi,\nu_j^e\rangle \leq \frac{u\langle\chi,\pi_+\rangle}{\rho_j}, \tag{45}$$

for $j \in \mathcal{J}$ since $\mathbf{M}_{u,l}^\times \neq \emptyset$. So, upon using this together with (16), (17), (33), and properties of h, (43) follows. Then (44) follows from (43) and (18). □

Lemma 5.7. *Suppose $u, l > 0$. Then H is continuous on $\mathbf{M}_{u,l}^\times$.*

Proof. This follows similarly to the proof of [14, Lemma 4.8] upon using the result in Part 1 of Lemma 4.1 in place of referring to the proof of [14, Lemma 4.6] and Lemma 5.6 in place of [14, Lemma 4.7]. The details are omitted. □

5.2. **The function K.** Now, we consider properties of the function K defined in (26). As we will see in Lemma 5.12, to obtain a similar continuity result for K to that for H in Lemma 5.7, we will need to restrict the domain of K to sets of the form $\mathbf{M}^{\times}_{u,l,\theta}$, for $u, l, \theta > 0$. Furthermore, as in the single class case, we will see in Lemma 5.9 that the kernel of K is exactly $\mathbf{I}_+$. These two properties are of much utility when proving Theorem 3.1.

Recall the definition of k from (25) and set $\tilde{k}(x) = k(1, 1, x)$ for $x \in \mathbb{R}_+$. Then $\tilde{k}$ coincides with k as defined in [14]. As observed there, $\tilde{k}$ takes values in $[0, \infty]$, is strictly convex on $(0, \infty)$, $\tilde{k}(1) = 0$ and $\tilde{k}(x) > 0$ for $x \neq 1$. Also, observe that, for $a, b \in (0, \infty)$, $k(a, a, x) = a\tilde{k}(x)$ for all $x \in \mathbb{R}_+$ and $k(a, b, b/a) = a\tilde{k}(b/a)$.

The next lemma establishes some key properties of k.

Lemma 5.8. *Let $a, b \in (0, \infty)$. Then, on $[0, \infty)$, $k(a, b, \cdot)$ takes values in $[0, \infty]$ and is convex with a unique global minimum at $x = b/a$. Furthermore, $k(a, b, b/a) = 0$ if and only if $a = b$.*

Proof. It is straightforward to see that $k(a, b, \cdot)$ is convex on $[0, \infty)$ and strictly convex on $(0, \infty)$, with a unique global minimum at b/a (note, by definition, $k(a, b, 0) = +\infty$ and $\lim_{x \searrow 0} k(a, b, x) = +\infty$). The remainder follows from $k(a, b, b/a) = a\tilde{k}(b/a)$ and properties of $\tilde{k}$. □

Using the properties of k developed in the previous lemma, important properties of K are developed in the next lemma.

Lemma 5.9. *For $\zeta \in \mathbf{M}_0^J$, $K(\zeta) \in [0, \infty]$ and $K(\zeta) = 0$ if and only if $\zeta \in \mathbf{I}_+$.*

Proof. The range of K follows from the definition of K given in (26) and Lemma 5.8.

Let $\zeta \in \mathbf{M}_0^J$ be such that $K(\zeta) = 0$. Then, for each $j \in \mathcal{J}$, $\overline{Z}_{\zeta_j}(x_j^*) = 0$ and, by (27) and Lemma 5.8, for almost every $x \in [0, x_j^*)$,

$$k\left(\frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)}\right) = 0.$$

By combining this with another application of Lemma 5.8, we obtain that for each $j \in \mathcal{J}$ and for almost every $x \in [0, x_j^*)$,

$$0 = k\left(\frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)}\right) \geq k\left(\frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}, \rho_j, \frac{\rho_j \langle 1, \zeta_+ \rangle}{\langle 1, \zeta_j \rangle}\right) \geq 0,$$

and so the inequalities are all equalities. Then, using Lemma 5.8 again, for each $j \in \mathcal{J}$, $\rho_j = \frac{\langle 1, \zeta_j \rangle}{\langle 1, \zeta_+ \rangle}$ and $\overline{Z}_{\zeta_j}(x) = \langle 1, \zeta_j \rangle \overline{N}_j^e(x)$ for almost every $x \in [0, x_j^*)$. Since $\overline{Z}_{\zeta_j}$ and $\overline{N}_j^e$ are right continuous, it follows that the latter holds for all $x \in [0, x_j^*)$ and $j \in \mathcal{J}$. Thus, for each $j \in \mathcal{J}$, $\langle 1_{(0,\infty)}, \zeta_j \rangle = \overline{Z}_{\zeta_j}(0) = \langle 1, \zeta_j \rangle \overline{N}_j^e(0) = \langle 1, \zeta_j \rangle$ and so $\zeta_j(\{0\}) = 0$. It follows that $\zeta_j = c\rho_j \nu_j^e$ for each $j \in \mathcal{J}$, where $c = \langle 1, \zeta_+ \rangle$. Then $\zeta \in \mathbf{I}_+$.

Now, suppose that $\zeta \in \mathbf{I}_+$. Then, for some $c > 0$, $\zeta_j = c\rho_j \nu_j^e$ for each $j \in \mathcal{J}$. Thus, for each $j \in \mathcal{J}$, $\langle 1, \zeta_j \rangle = c\rho_j$ and $\langle 1, \zeta_j \rangle / \langle 1, \zeta_+ \rangle = \rho_j$. Furthermore, $\overline{Z}_{\zeta_j}(x) = c\rho_j \langle 1_{(x,\infty)}, \nu_j^e \rangle$, for each $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$. In this case, $K(\zeta)$ reduces to

$$K(\zeta) = \sum_{j=1}^{J} \rho_j \int_0^{x_j^*} \tilde{k}\,(1)\, n_j^e(x) dx = 0,$$

as required. $\square$

Now, we proceed to establish continuity of $K(\cdot)$ on certain compact sets. First, we give sufficient conditions for the lower semicontinuity of K.

Lemma 5.10. *Suppose that $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}_0^J$, $\zeta \in \mathbf{M}_0^J$, and $\zeta^n \xrightarrow{w} \zeta$. Then,*

$$\liminf_{n\to\infty} K(\zeta^n) \geq K(\zeta). \tag{46}$$

In particular, $K(\cdot)$ is lower semicontinuous on $\mathbf{M}_0^J$.

Proof. Since for each $j \in \mathcal{J}$, $\zeta_j^n \xrightarrow{w} \zeta_j$ and $\zeta_j \in \mathbf{M}_0^J$, we have that $\lim_{n\to\infty} \langle 1, \zeta_j^n \rangle = \langle 1, \zeta_j \rangle > 0$, for each $j \in \mathcal{J}$, and so $\lim_{n\to\infty} \langle 1, \zeta_+^n \rangle = \langle 1, \zeta_+ \rangle > 0$. Also, the assumptions imply that for each $j \in \mathcal{J}$, for almost every $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \overline{Z}_{\zeta_j^n}(x) = \overline{Z}_{\zeta_j}(x). \tag{47}$$

There are two cases to consider. Case (i): Suppose that $\overline{Z}_{\zeta_j}(x_j^*) > 0$ for some $j \in \mathcal{J}$. Fix such a $j \in \mathcal{J}$. By the monotonicity and right continuity of $\overline{Z}_{\zeta_j}$, $\overline{Z}_{\zeta_j}(\cdot) > 0$ on an open interval containing x_j^*. Hence, $K(\zeta) = +\infty$ and (47) holds for some $x \geq x_j^*$ in that interval. Then, for all n sufficient large, $\overline{Z}_{\zeta_j^n}(x) > 0$ and $\overline{Z}_{\zeta_j^n}(\cdot) > 0$ on an open interval containing x, and so $K(\zeta^n) = +\infty$. Thus (46) holds. Case (ii): Suppose that $\overline{Z}_{\zeta_j}(x_j^*) = 0$ for all $j \in \mathcal{J}$. For each $n \in \mathbb{N}$,

$$K(\zeta^n) \geq \sum_{j=1}^{J} \int_{[0,x_j^*)} k\left(\frac{\langle 1, \zeta_j^n \rangle}{\langle 1, \zeta_+^n \rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j^n}(x)}{\langle 1, \zeta_j^n \rangle \overline{N}_j^e(x)} \right) n_j^e(x) dx.$$

The result follows from the continuity of the extended real-valued function $k(\cdot, \rho_j, \cdot)$ on $(0, \infty) \times \mathbb{R}_+$, (47), Fatou's lemma and (27). $\square$

The next result was stated and proved in [14, Lemma 7.4]. We state it here as a proposition for convenience.

Proposition 5.1. *Given a probability density function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with associated cumulative distribution function $G : \mathbb{R}_+ \to [0, 1]$, let $\overline{G}(x) = 1 - G(x)$ for all $x \in \mathbb{R}_+$, and set*

$$y^* = \inf\{x \in \mathbb{R}_+ : \overline{G}(x) = 0\}.$$

Then,

$$-\int_0^{y^*} \ln\left(\overline{G}(x)\right) g(x) dx = 1.$$

The following proposition is a direct extension of [14, Lemma 7.5] to the multiclass case. So, we omit the proof and instead refer the reader to the proof of [14, Lemma 7.5], which can be applied to each class $j \in \mathcal{J}$.

Proposition 5.2. *For each $j \in \mathcal{J}$ and $\theta > 0$, let*
$$g_j(\theta, x) = -\ln\left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta)\right), \quad x \in [0, x_j^*).$$
Then, for each $j \in \mathcal{J}$ and $\theta > 0$, $g_j(\theta, \cdot) \geq 0$ and $g_j(\theta, \cdot) \in L_1(\nu_j^e)$.

Note that Proposition 5.2 will be used when establishing the continuity result for K stated in Lemma 5.12 below. First, we show that K is finite on the compact sets $\mathbf{M}_{u,l,\theta}^{\times}$, for $u, l, \theta > 0$.

Lemma 5.11. *Let $u, l, \theta > 0$ and $\lambda(u,l) > 0$ be given by (38).*

(1) For each $\zeta \in \mathbf{M}_{u,l}^{\times}$, $j \in \mathcal{J}$ and $x \in [0, x_j^*)$,
$$0 \leq \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)} \leq \frac{u}{\lambda(u,l)}. \tag{48}$$
In particular, the function $\frac{\overline{Z}_{\zeta_j}(\cdot)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(\cdot)} \in L_1(\nu_j^e)$ for each $j \in \mathcal{J}$.

(2) For $\zeta \in \mathbf{M}_{u,l,\theta}^{\times}$, $j \in \mathcal{J}$ and $x \in [0, x_j^*)$, $\overline{Z}_{\zeta_j}(x) > 0$ and
$$0 \leq -\ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle}\right) \leq -\ln\left(\frac{\lambda(u,l)}{u}\rho_j\left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta)\right)\right). \tag{49}$$
In particular, the function $\ln\left(\frac{\overline{Z}_{\zeta_j}(\cdot)}{\langle 1, \zeta_j \rangle}\right) \in L_1(\nu_j^e)$ for each $j \in \mathcal{J}$.

Therefore, for $\zeta \in \mathbf{M}_{u,l,\theta}^{\times}$, $K(\zeta) < \infty$.

Proof. Fix $u, l, \theta > 0$. For $\zeta \in \mathbf{M}_{u,l}^{\times}$, $j \in \mathcal{J}$ and $x \in [0, x_j^*)$, by definition of $\mathbf{M}_{u,l}^{\times}$ and $\lambda(u,l) > 0$, we have $\langle 1, \zeta_j \rangle \geq \lambda(u,l)$ and $\overline{Z}_{\zeta_j}(x) \leq u\overline{N}_j^e(x)$. Thus, (48) and part 1 hold.

Henceforth, fix $\zeta \in \mathbf{M}_{u,l,\theta}^{\times}$. For $j \in \mathcal{J}$ and $x \in [0, x_j^*)$, by definition of $\mathbf{M}_{u,l,\theta}^{\times}$ we have that
$$0 < \lambda(u,l)\rho_j\left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta)\right) \leq \overline{Z}_{\zeta_j}(x). \tag{50}$$
Then, (49) follows since $\langle 1, \zeta_j \rangle \leq u$ for each $j \in \mathcal{J}$. The remainder of part 2 follows from (49) and Proposition 5.2.

From (48), we obtain that for each $j \in \mathcal{J}$,
$$-1 \leq \int_0^{x_j^*} \left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)} - 1\right) n_j^e(x)dx \leq \left(\frac{u}{\lambda(u,l)} - 1\right). \tag{51}$$
Furthermore, by Proposition 5.1, it follows that for each $j \in \mathcal{J}$,
$$-\int_0^{x_j^*} \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle \overline{N}_j^e(x)}\right) n_j^e(x)dx = -\int_0^{x_j^*} \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j \rangle}\right) n_j^e(x)dx - 1. \tag{52}$$

Combining (51), (52) and part 2 with (27) yields that $K(\zeta) < \infty$.

□

We are now ready to state and prove the continuity result for K.

Lemma 5.12. *For each $u, l, \theta > 0$, K is continuous on $\mathbf{M}^{\times}_{u,l,\theta}$.*

The proof of Lemma 5.12 given next is a modest generalization of the proof of [14, Lemma 7.6].

Proof. Fix $u, l, \theta > 0$. Recall, $\mathbf{M}^{\times}_{u,l,\theta}$ is compact. Suppose $\{\zeta^n\}_{n\in\mathbb{N}} \subset \mathbf{M}^{\times}_{u,l,\theta}$ is such that $\zeta^n \xrightarrow{w} \zeta \in \mathbf{M}^{\times}_{u,l,\theta}$ as $n \to \infty$. Then, we have that $\lim_{n\to\infty}\langle 1, \zeta^n_j\rangle = \langle 1, \zeta_j\rangle \geq \lambda(u, l) > 0$, for each $j \in \mathcal{J}$, and so $\lim_{n\to\infty}\langle 1, \zeta^n_+\rangle = \langle 1, \zeta_+\rangle > 0$. Thus, for each $j \in \mathcal{J}$,

$$\lim_{n\to\infty}\left(\frac{\langle 1, \zeta^n_j\rangle}{\langle 1, \zeta^n_+\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta^n_j\rangle}{\rho_j\langle 1, \zeta^n_+\rangle}\right) = \left(\frac{\langle 1, \zeta_j\rangle}{\langle 1, \zeta_+\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta_j\rangle}{\rho_j\langle 1, \zeta_+\rangle}\right).$$

Hence, there exists N such that for all $j \in \mathcal{J}$ and $n \geq N$,

$$\left(\frac{\langle 1, \zeta^n_j\rangle}{\langle 1, \zeta^n_+\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta^n_j\rangle}{\rho_j\langle 1, \zeta^n_+\rangle}\right) \leq 2\left(\frac{\langle 1, \zeta_j\rangle}{\langle 1, \zeta_+\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta_j\rangle}{\rho_j\langle 1, \zeta_+\rangle}\right). \tag{53}$$

Also, the assumptions imply that for each $j \in \mathcal{J}$, for almost all $x \in \mathbb{R}_+$, $\lim_{n\to\infty}\overline{Z}_{\zeta^n_j}(x) = \overline{Z}_{\zeta_j}(x)$. Since for each $j \in \mathcal{J}$, $k(\cdot, \rho_j, \cdot)$ is continuous on $(0, \infty) \times \mathbb{R}_+$, it follows that for almost every $x \in [0, x^*_j)$,

$$\lim_{n\to\infty} k\left(\frac{\langle 1, \zeta^n_j\rangle}{\langle 1, \zeta^n_+\rangle}, \rho_j, \frac{\overline{Z}_{\zeta^n_j}(x)}{\langle 1, \zeta^n_j\rangle\overline{N}^e_j(x)}\right) = k\left(\frac{\langle 1, \zeta_j\rangle}{\langle 1, \zeta_+\rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j\rangle\overline{N}^e_j(x)}\right). \tag{54}$$

By definition of $\mathbf{M}^{\times}_{u,l,\theta}$ for each $j \in \mathcal{J}$, for all $x \in [0, x^*_j)$ and $n \in \mathbb{N}$,

$$\frac{\lambda(u, l)\rho_j\left(\overline{N}^e_j(x) - \overline{N}^e_j(x + \theta)\right)}{u\overline{N}^e_j(x)} \leq \frac{\overline{Z}_{\zeta^n_j}(x)}{\langle 1, \zeta^n_j\rangle\overline{N}^e_j(x)} \leq \frac{u}{\lambda(u, l)}. \tag{55}$$

It follows from the definitions of $\lambda(u, l)$ and u that $\lambda(u, l) \leq u$. So, the upper bound in (55) is at least one. Additionally, notice that the lower bound in (55) is at most one. Thus, by Lemma 5.8, (55), the definition of k, and (53), for each $j \in \mathcal{J}$, for all

$x \in [0, x_j^*)$ and $n \geq N$, we have

$$
\begin{aligned}
0 &\leq k\left(\frac{\langle 1, \zeta_j^n\rangle}{\langle 1, \zeta_+^n\rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j^n}(x)}{\langle 1, \zeta_j^n\rangle \overline{N}_j^e(x)}\right) \\
&\leq \frac{\langle 1, \zeta_j^n\rangle}{\langle 1, \zeta_+^n\rangle}\left(\frac{u}{\lambda(u,l)} - 1\right) - \rho_j \ln\left(\frac{\lambda(u,l)\rho_j\left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta)\right)}{u\overline{N}_j^e(x)}\right) \\
&\quad + \left(\frac{\langle 1, \zeta_j^n\rangle}{\langle 1, \zeta_+^n\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta_j^n\rangle}{\rho_j\langle 1, \zeta_+^n\rangle}\right) \\
&\leq \left(\frac{u}{\lambda(u,l)} - 1\right) - \rho_j \ln\left(\frac{\lambda(u,l)\rho_j\left(\overline{N}_j^e(x) - \overline{N}_j^e(x+\theta)\right)}{u\overline{N}_j^e(x)}\right) \qquad (56) \\
&\quad + 2\left(\frac{\langle 1, \zeta_j\rangle}{\langle 1, \zeta_+\rangle} - \rho_j\right)\ln\left(\frac{\langle 1, \zeta_j\rangle}{\rho_j\langle 1, \zeta_+\rangle}\right).
\end{aligned}
$$

For each $j \in \mathcal{J}$ and $x \in [0, x_j^*)$, let $y_j(x)$ equal the right hand side of the inequality in (56). Then, for each $j \in \mathcal{J}$, $y_j(x) \geq 0$ for all $x \in [0, x_j^*)$ and by Propositions 5.1 and 5.2, $y_j \in L_1(\nu_j^e)$. Therefore, since $j \in \mathcal{J}$ was arbitrary, (54), Lemma 5.11, (27) and the dominated convergence theorem together imply that $\lim_{n\to\infty} K(\zeta^n) = K(\zeta)$. □

5.3. **An Alternative Representation for** K**.** In this section, we develop an alternative representation for $K(\zeta)$ for $\zeta \in \mathbf{A}_{u,l,\theta}^{\times} = \mathbf{A}^J \cap \mathbf{M}_{u,l,\theta}^{\times}$ for $u, l, \theta > 0$ (see Lemma 5.15 below). For this, we will leverage the following elementary fact.

Lemma 5.13. *For $i = 1, 2$, let $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ be a probability density function on $\mathbb{R}_+$, let $G_i : \mathbb{R}_+ \to [0,1]$ denote the associated cumulative distribution function, define $\overline{G}_i(x) = 1 - G_i(x)$ for all $x \in \mathbb{R}_+$ and let*

$$y_i^* = \inf\{x \in \mathbb{R}_+ : \overline{G}_i(x) = 0\}.$$

Assume that
$\lim_{x \nearrow y_1^*} \ln\left(\overline{G}_1(x)\right)\overline{G}_2(x) = 0$. *Then*

$$-\int_0^{y_1^*} \ln\left(\overline{G}_1(x)\right) g_2(x)dx = \int_0^{y_1^*} \frac{\overline{G}_2(x)}{\overline{G}_1(x)} g_1(x)dx.$$

Proof. The result follows as an application of integration by parts. □

Next, we introduce some notation. Given $\zeta \in \mathbf{A}$, let $z_\zeta(\cdot)$ denote the density of ζ. For $u, l, \theta > 0$ and $\zeta \in \mathbf{A}_{u,l,\theta}^{\times}$, define

$$
\ell_{\zeta,1}(j,x) = \begin{cases} \ln\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right)\rho_j n_j^e(x), & \text{if } j \in \mathcal{J} \text{ and } x \in [0, x_j^*), \\ 0, & \text{if } j \in \mathcal{J} \text{ and } x \in [x_j^*, \infty). \end{cases}
$$

$$
\ell_{\zeta,2}(j,x) = \begin{cases} \ln\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right)\frac{z_{\zeta_j}(x)}{\langle 1, \zeta_+\rangle}, & \text{if } j \in \mathcal{J} \text{ and } x \in [0, x_j^*), \\ 0, & \text{if } j \in \mathcal{J} \text{ and } x \in [x_j^*, \infty). \end{cases}
$$

Observe that since $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$, the support of p_ζ coincides with the support of p_π. Indeed, by part 2 of Lemma 5.11, $\overline{Z}_{\zeta_j}(x) > 0$ for $x \in [0, x_j^*)$ and $j \in \mathcal{J}$ and since $\zeta \in \mathbf{A}^{\times}_{u,l,\theta} \subset \mathbf{A}^{\times}_{u}$, $\overline{Z}_{\zeta_j}(x) = 0$ for $x \in [x_j^*, \infty)$ and $j \in \mathcal{J}$. Hence, the argument of the natural logarithm above is never zero. Thus, for $u, l, \theta > 0$ and $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$, $\ell_{\zeta,1}$ and $\ell_{\zeta,2}$ are well defined on $\mathcal{J} \times \mathbb{R}_+$.

Lemma 5.14. *Let $u, l, \theta > 0$. For $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$, we have $\ell_{\zeta,1}, \ell_{\zeta,2} \in L_1(\gamma)$ and*

$$\int_{\mathcal{J}\times\mathbb{R}_+} \ell_{\zeta,1}(j,x) d\gamma(j,x) = \sum_{j=1}^{J} \rho_j \int_{[0,x_j^*)} \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j\rangle \overline{N}^e_j(x)}\right) n^e_j(x) dx$$
$$+ \sum_{j=1}^{J} \rho_j \ln\left(\frac{\langle 1,\zeta_j\rangle}{\rho_j \langle 1, \zeta_+\rangle}\right) + \ln\left(\frac{\langle \chi, \pi_+\rangle \langle 1, \zeta_+\rangle}{\langle \chi, \zeta_+\rangle}\right), \tag{57}$$

$$\int_{\mathcal{J}\times\mathbb{R}_+} \ell_{\zeta,2}(j,x) d\gamma(j,x) = \sum_{j=1}^{J} \int_{[0,x_j^*)} \frac{\langle 1,\zeta_j\rangle}{\langle 1,\zeta_+\rangle} \left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j\rangle \overline{N}^e_j(x)} - 1\right) n^e_j(x) dx$$
$$+ \sum_{j=1}^{J} \frac{\langle 1,\zeta_j\rangle}{\langle 1,\zeta_+\rangle} \ln\left(\frac{\langle 1,\zeta_j\rangle}{\rho_j \langle 1, \zeta_+\rangle}\right) + \ln\left(\frac{\langle \chi, \pi_+\rangle \langle 1, \zeta_+\rangle}{\langle \chi, \zeta_+\rangle}\right). \tag{58}$$

Moreover, there exist positive, finite constants $C_1(u,l,\theta)$ and $C_2(u,l,\theta)$ such that for $i = 1, 2$ and any $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$

$$\int_{\mathcal{J}\times\mathbb{R}_+} |\ell_{\zeta,i}(j,x)|\, d\gamma(j,x) \le C_i(u,l,\theta).$$

Proof. Fix $u, l, \theta > 0$ and $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$. Let $\lambda(u,l)$ be given by (38). Then $\lambda(u,l) \le \langle 1, \zeta_+\rangle \le u$. Since $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$, for $i = 1, 2$,

$$\int_{\mathcal{J}\times\mathbb{R}_+} \ell_{\zeta,i}(j,x) d\gamma(j,x) = \sum_{j=1}^{J} \int_{[0,x_j^*)} \ell_{\zeta,i}(j,x) dx.$$

Also, for $j \in \mathcal{J}$ and $x \in [0, x_j^*)$,

$$\ln\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) = \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1,\zeta_j\rangle}\right) - \ln\left(\overline{N}^e_j(x)\right)$$
$$+ \ln\left(\frac{\langle 1,\zeta_j\rangle}{\rho_j \langle 1, \zeta_+\rangle}\right) + \ln\left(\frac{\langle \chi, \pi_+\rangle \langle 1, \zeta_+\rangle}{\langle \chi, \zeta_+\rangle}\right). \tag{59}$$

Since $0 < \langle \chi, \pi_+\rangle < \infty$, $0 < \rho_j < 1$ and $\lambda(u,l) \le \langle 1, \zeta_j\rangle \le u$ for each $j \in \mathcal{J}$, and $Jl \le \langle \chi, \zeta_+\rangle \le u\langle \chi, \nu^e_+\rangle$, there exists a finite, positive constant $C_3(u,l,\theta) > 0$ such that

$$\left|\ln\left(\frac{\langle 1,\zeta_j\rangle}{\rho_j \langle 1, \zeta_+\rangle}\right)\right| + \left|\ln\left(\frac{\langle \chi, \pi_+\rangle \langle 1, \zeta_+\rangle}{\langle \chi, \zeta_+\rangle}\right)\right| \le C_3(u,l,\theta). \tag{60}$$

The result for $i = 1$ is an immediate consequence of (59), (49) in part 2 of Lemma 5.11, Propositions 5.2 and 5.1, and (60). Next we verify the result for $i = 2$. By Proposition 5.1,

$$\begin{aligned}\sum_{j=1}^{J} \int_{[0,x_j^*)} \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1,\zeta_j\rangle}\right) \frac{z_{\zeta_j}(x)}{\langle 1,\zeta_+\rangle} dx \\ &= \sum_{j=1}^{J} \frac{\langle 1,\zeta_j\rangle}{\langle 1,\zeta_+\rangle} \int_{[0,x_j^*)} \ln\left(\frac{\overline{Z}_{\zeta_j}(x)}{\langle 1,\zeta_j\rangle}\right) \frac{z_{\zeta_j}(x)}{\langle 1,\zeta_j\rangle} dx \\ &= -1 = \sum_{j=1}^{J} \frac{\langle 1,\zeta_j\rangle}{\langle 1,\zeta_+\rangle} \int_{[0,x_j^*)} n_j^e(x) dx. \qquad (61)\end{aligned}$$

Since $0 \le \lim_{x\nearrow x_j^*} -\ln\left(\overline{N}_j^e(x)\right) \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1,\zeta_+\rangle} \le \lim_{x\nearrow x_j^*} -u \ln\left(\overline{N}_j^e(x)\right) \frac{\overline{N}_j^e(x)}{\langle 1,\zeta_+\rangle} = 0$, Lemma 5.13 and part 1 of Lemma 5.11 imply that

$$\begin{aligned}\sum_{j=1}^{J} \int_{[0,x_j^*)} -\ln\left(\overline{N}_j^e(x)\right) \frac{z_{\zeta_j}(x)}{\langle 1,\zeta_+\rangle} dx \\ &= \sum_{j=1}^{J} \frac{\langle 1,\zeta_j\rangle}{\langle 1,\zeta_+\rangle} \int_{[0,x_j^*)} \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1,\zeta_j\rangle \overline{N}_j^e(x)} n_j^e(x) dx \\ &\le \frac{u}{\lambda(u,l)}. \qquad (62)\end{aligned}$$

The existence of $C_2(u, l, \theta)$ follows from (59), (61), (62) and (60), and (58) follows from (59), (61) and (62). □

Given $u, l, \theta > 0$ and $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$, for $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$, define

$$u_\zeta(j,x) = \frac{z_{\zeta_j}(x)}{\langle 1,\zeta_+\rangle} - \rho_j n_j^e(x). \qquad (63)$$

Also, recall that $h(x) = x\ln(x)$ for $x \in \mathbb{R}_+$, where $h(0) = 0$. Then, $h'(x) = \ln(x) + 1$ for $x \in (0, \infty)$.

Lemma 5.15. *Let $u, l, \theta > 0$. There exists a finite, positive constant $C(u, l, \theta)$ such that for all $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$,*

$$\int_{\mathcal{J}\times\mathbb{R}_+} \left| h'\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) u_\zeta(j,x) \right| d\gamma(j,x) \le C(u,l,\theta),$$

where, by convention, the integrand takes the value zero for all $x \in [x_j^*, \infty)$ and $j \in \mathcal{J}$. Moreover, for each $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$,

$$K(\zeta) = \int_{\mathcal{J}\times\mathbb{R}_+} h'\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) u_\zeta(j,x) d\gamma(j,x).$$

Proof. Fix $u, l, \theta > 0$. The first statement follows from Lemma 5.14 and the fact that for all $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$,

$$\int_{\mathcal{J}\times\mathbb{R}_+} |u_\zeta(j,x)|\, d\gamma(j,x) \leq \int_{\mathcal{J}\times\mathbb{R}_+} \left(\frac{z_{\zeta_j}(x)}{\langle 1, \zeta_+\rangle} + \rho_j n^e_j(x) \right) d\gamma(j,x) = 2.$$

For the second statement, fix $\zeta \in \mathbf{A}^{\times}_{u,l,\theta}$. Note that

$$\int_{\mathcal{J}\times\mathbb{R}_+} u_\zeta(j,x) d\gamma(j,x) = 1 - 1 = 0.$$

This together with the definitions of $\ell_{\zeta,1}$ and $\ell_{\zeta,2}$ and the definition of k given in (25) implies that

$$\int_{\mathcal{J}\times\mathbb{R}_+} h'\left(\frac{p_\zeta(j,x)}{p_\pi(j,x)}\right) u_\zeta(j,x) d\gamma(j,x) = \int_{\mathcal{J}\times\mathbb{R}_+} \left(\ell_{\zeta,2}(j,x) - \ell_{\zeta,1}(j,x)\right) d\gamma(j,x)$$
$$= \int_{\mathcal{J}\times\mathbb{R}_+} k\left(\frac{\langle 1, \zeta_j\rangle}{\langle 1, \zeta_+\rangle}, \rho_j, \frac{\overline{Z}_{\zeta_j}(x)}{\langle 1, \zeta_j\rangle \overline{N}^e_j(x)} \right) n^e_j(x) d\gamma(j,x),$$

and the result follows by (26). □

6. Compact Containment of Fluid Model Solutions

As in the single class case, we wish to prove existence of constants $u^*, l^* > 0$ such that fluid model solutions with initial conditions in $\mathbf{K}^{\times}_{u,l}$ for some $u, l > 0$, remain in $\mathbf{K}^{\times}_{u^*,l^*}$ for all time. For $J = 1$, this is given by [14, Corollary 5.1]. For $J > 1$, one must take care to address the fact that the class level fluid workload function $w_j(\cdot)$ is not necessarily constant for each $j \in \mathcal{J}$. We give the details for this case in Lemma 6.4. In Lemma 6.5, we extend [14, Corollary 5.4] to $J > 1$ and obtain a uniform time by which a fluid model solution must enter a set of the form $\mathbf{K}^{\times}_{u^*,l^*,\theta^*}$, a fact that will be utilized in the proof of Theorem 3.1 in Section 8. Finally, in Corollary 6.5, we extend [14, Corollary 5.2] to $J > 1$ and provide a useful upper bound on the mass near the origin for each class.

Recall that, if μ is a fluid model solution with $\mu(0) \in \mathbf{K}^J$ and $\mu_+(0) \in \mathbf{K}_\dagger$, $q_j(t) = \langle 1, \mu_j(t)\rangle$ for all $j \in \mathcal{J}$ and $0 < w_+(t) = \langle \chi, \mu_+(t)\rangle = w_+(0) < \infty$ for all $t \geq 0$. The following lemma establishes an upper bound on fluid queue length.

Lemma 6.1. *Let μ be a fluid model solution with $\mu(0) \in \mathbf{K}^J$ and $\mu_+(0) \in \mathbf{K}_\dagger$. Set*

$$u = \frac{3\max(q_+(0), 6\alpha_+ w_+(0))}{2}.$$

Then, for all $j \in \mathcal{J}$ and $t \geq 0$, $q_j(t) \leq q_+(t) \leq u$.

Proof. Recall that μ_+ is a fluid model solution for a single class processor sharing queue with critical parameters (α_+, ϑ_+) such that $\mu_+(0) \in \mathbf{K}_\dagger$. Thus, we can apply [14, Lemma 5.1] to obtain the result. □

The next result is an intermediate compact containment result that will be used below to verify Lemma 6.4.

Lemma 6.2. *Let $u, l > 0$. Set*

$$u^* = \frac{3u \max\left(J, 6\alpha_+ \langle \chi, \nu_+^e \rangle\right)}{2}. \tag{64}$$

Then, for all fluid model solutions μ with $\mu(0) \in \mathbf{K}_u^\times$, $\mu(t) \in \mathbf{K}_{u^*}^\times$ for all $t \in [0, \infty)$. Moreover, for all fluid model solutions μ with $\mu(0) \in \mathbf{K}_{u,l}^+$, $\mu(t) \in \mathbf{K}_{u^*,l}^+$ for all $t \in [0, \infty)$.

Proof. Fix $u, l > 0$ and let u^* be given by (64). First let μ be a fluid model solution with $\mu(0) \in \mathbf{K}_u^\times$. Then $q_+(0) \leq Ju$ and $w_+(0) \leq u\langle \chi, \nu_+^e \rangle$. Hence, by Lemma 6.1, $q_+(t) \leq u^*$ for all $t \in [0, \infty)$. Given this, applying the argument given in the proof of [14, Corollary 5.1] to each class j yields the first result. Next consider a fluid model solution μ with $\mu(0) \in \mathbf{K}_{u,l}^+ \subset \mathbf{K}_u^\times$. Then, by the first result $\mu(t) \in \mathbf{K}_{u^*}^J$ for all $t \in [0, \infty)$. In addition, by Proposition 2.1, $l \leq w_+ = w_+(t)$ for all $t \in [0, \infty)$, and the second result follows. $\square$

For $u > 0$, by the proof of Lemma 6.2, we have that u^*, as defined in (64), gives an upper bound on total fluid queue length, provided that the fluid model solution has initial condition in $\mathbf{K}_u^\times$. This is stated formally in the following corollary.

Corollary 6.1. *Let $u > 0$ and let u^* be given as in (64). Then, for all fluid model solutions μ with $\mu(0) \in \mathbf{K}_u^\times$, $q_+(t) \leq u^*$ for all $t \in [0, \infty)$.*

A second corollary of Lemma 6.2, stated below in Corollary 6.2, provides a uniform, positive lower bound on fluid queue length and a time dependent, but monotone increasing lower bound on the fluid workload in each class. For this, let $u, l > 0$ and set

$$\lambda^+(u, l) = \inf\{\langle 1, \zeta_+ \rangle : \zeta \in \mathbf{M}_{u,l}^+\}. \tag{65}$$

Lemma 6.3. *Let $u, l > 0$. Then $\lambda^+(u, l) > 0$ and $\lambda(u, l) \leq \lambda^+(u, Jl)$.*

Proof. Fix $u, l > 0$. Recall that $\langle \chi, \zeta_+ \rangle \geq l$ for all $\zeta \in \mathbf{M}_{u,l}^+$. Thus, $\langle 1, \zeta_+ \rangle > 0$ for all $\zeta \in \mathbf{M}_{u,l}^+$. Hence, $\mathbf{0} \notin \mathbf{M}_{u^*,l}^+$. By observing that $\mathbf{M}_{u,l}^+$ is compact by Lemma 4.2 and that $\zeta \mapsto \langle 1, \zeta_+ \rangle$ is a continuous function on $\mathbf{M}^J$, we obtain $\lambda^+(u, l) > 0$. Again, by compactness of $\mathbf{M}_{u,Jl}^+$, there exists $\zeta \in \mathbf{M}_{u,Jl}^+$ such that $\langle 1, \zeta_+ \rangle = \lambda^+(u, Jl)$. Fix such a $\zeta \in \mathbf{M}_{u,Jl}^+$. Then, there exists $j \in \mathcal{J}$ such that $\langle \chi, \zeta_j \rangle \geq l$. Let $\eta \in \mathbf{M}_{u,l}^\times$ be given by

$$\eta_j = \begin{cases} u\nu_i^e, & \text{if } i \neq j, \\ \zeta_j, & \text{if } i = j. \end{cases}$$

Then $\lambda(u, l) \leq \langle 1, \zeta_j \rangle \leq \langle 1, \zeta_+ \rangle = \lambda^+(u, Jl)$. $\square$

Corollary 6.2. *Let $u, l > 0$. Let u^* be given by (64) and let $\lambda^+(u^*, l)$ be given by (65) with u^* in place of u. For all fluid model solutions $\mu(\cdot)$ with $\mu(0) \in \mathbf{K}_{u,l}^+$ and* *for all* $t \in [0,\infty)$, $q_+(t) \geq \lambda^+(u^*,l) > 0$,

$$\overline{M}_j(t,x) \;\geq\; \lambda^+(u^*,l)\rho_j\left(\overline{N}^e_j(x) - \overline{N}^e_j(x+s(t))\right), \tag{66}$$

$$w_j(t) \;\geq\; \lambda^+(u^*,l)\rho_j\int_0^{s(t)}\overline{N}^e_j(x)dx. \tag{67}$$

In particular, $\overline{M}_j(t,x) > 0$ *for all* $x \in [0,x_j^*)$, $t \in (0,\infty)$, *and* $j \in \mathcal{J}$.

Proof. Fix $u,l > 0$. Let μ be a fluid model solution with $\mu(0) \in \mathbf{K}^+_{u,l}$. By Lemma 6.2, we have that $\mu(t) \in \mathbf{K}^+_{u^*,l} \subset \mathbf{M}^+_{u^*,l}$ for all $t \in [0,\infty)$. Therefore, by (65) and Lemma 6.3, at every time $t \in [0,\infty)$, $q_+(t) = \langle 1, \mu_+(t)\rangle \geq \lambda^+(u^*,l) > 0$. This together with (15) gives that, for every $j \in \mathcal{J}$, $t \in [0,\infty)$ and $x \in \mathbb{R}_+$,

$$\begin{aligned}
\overline{M}_j(t,x) &\geq \rho_j\int_0^t n^e_j(x+s(t)-s(r))dr \\
&\geq \rho_j\lambda^+(u^*,l)\int_0^t n^e_j(x+s(t)-s(r))\frac{1}{q_+(r)}dr \\
&= \rho_j\lambda^+(u^*,l)(\overline{N}^e_j(x) - \overline{N}^e_j(x+s(t))).
\end{aligned}$$

So (66) holds. Recall, $w_j(t) = \int_0^\infty \overline{M}_j(t,x)dx$ for $t \in [0,\infty)$ and $j \in \mathcal{J}$. So, (67) follows from (66) upon integrating, which completes the proof. □

Now we are ready to state and prove one of the main compact containment results of this section, Lemma 6.4. In the proof, we will leverage the intermediate result in Lemma 6.2 and its consequence Corollary 6.2.

Lemma 6.4. *Let* $u,l,T > 0$. *Let* u^* *be given by* (64) *and let* $\lambda^+(u^*,l)$ *be given by* (65) *with* u^* *in place of* u. *Define*

$$l^+ \;=\; l^+(u^*,l,T) = \lambda^+(u^*,l)\min\left\{\rho_j\int_0^{T/u^*}\overline{N}^e_j(x)dx : j\in\mathcal{J}\right\}, \tag{68}$$

$$l^* \;=\; l^*(u^*,l) = \min\{l^+(u^*,l,\hat{t}), l/2\}, \tag{69}$$

where $\hat{t} = \hat{t}(u^*,l) = l\lambda^+(u^*,l)/(2u^*)$.

(1) *If* μ *is a fluid model solution with* $\mu(0) \in \mathbf{K}^+_{u,l}$, *then* $\mu(t) \in \mathbf{K}^\times_{u^*,l^+}$ *for all* $t \geq T$.

(2) *If* μ *is a fluid model solution with* $\mu(0) \in \mathbf{K}^\times_{u,l}$, *then,* $\mu(t) \in \mathbf{K}^\times_{u^*,l^*}$ *for all* $t \in [0,\infty)$.

Proof. Fix $u,l,T > 0$. Let μ be a fluid model solution with $\mu(0) \in \mathbf{K}^+_{u,l}$. Recall, by Lemma 6.2, $\mu(t) \in \mathbf{K}^+_{u^*,l}$ for all $t \geq 0$. Thus, for all $t \in [0,\infty)$, we have that $0 < \lambda^+(u^*,l) \leq q_+(t) \leq u^*$ by Corollary 6.1. Then, for $t \geq T$, $s(t) \geq T/u^*$. This together with (67) implies that $w_j(t) \geq l^+(u^*,l,T)$ for all $t \geq T$ and $j \in \mathcal{J}$. Thus, $\mu(t) \in \mathbf{K}^\times_{u^*,l^+}$.

Now, to prove the second claim, assume that μ is such that $\mu(0) \in \mathbf{K}^{\times}_{u,l}$. By the proof of part 1, $w_j(t) \geq l^+(u^*, l, \hat{t}) \geq l^*$ for $t \geq \hat{t}$. It remains to show that $w_j(t) \geq l^*$ for all $0 \leq t < \hat{t}$ and $j \in \mathcal{J}$. By (11), for $j \in \mathcal{J}$, $t \in [0, \infty)$, and $x \in \mathbb{R}_+$, we see that

$$\overline{M}_j(t, x) \geq \overline{M}_j(0, x + s(t)).$$

Integrating both sides with respect to x over $\mathbb{R}_+$, for $j \in \mathcal{J}$ and $t \in [0, \infty)$, we obtain

$$w_j(t) \geq \int_0^\infty \overline{M}_j(0, x + s(t))dx = \int_{s(t)}^\infty \overline{M}_j(0, y)dy,$$

where the last equality following from the change of variables $y = x + s(t)$. Then, it follows that, for $j \in \mathcal{J}$ and $t \in [0, \infty)$,

$$w_j(t) \geq \int_0^\infty \overline{M}_j(0, y)dy - \int_0^{s(t)} \overline{M}_j(0, y)dy = w_j(0) - \int_0^{s(t)} \overline{M}_j(0, y)dy.$$

Now, for $j \in \mathcal{J}$ and $y \in \mathbb{R}_+$, by definition and the fact that $q_j(0) \leq u \leq u^*$ (since $\mu(0) \in \mathbf{K}^{\times}_{u,l} \subset \mathbf{M}^{\times}_{u,l}$),

$$\overline{M}_j(0, y) = \langle 1_{(y,\infty)}, \mu_j(0) \rangle \leq \langle 1_{(0,\infty)}, \mu_j(0) \rangle = q_j(0) \leq u^*.$$

So, by using the fact that $w_j(0) \geq l$ (since $\mu(0) \in \mathbf{K}^{\times}_{u,l} \subset \mathbf{M}^{\times}_{u,l}$), we see that for $j \in \mathcal{J}$ and $t \in [0, \infty)$,

$$w_j(t) \geq w_j(0) - \int_0^{s(t)} q_j(0)dy = w_j(0) - q_j(0)s(t) \geq l - u^* s(t). \tag{70}$$

For $t \in [0, \hat{t}]$, $q_+(t) \geq \lambda^+(u^*, l)$ and $s(t) \leq s(\hat{t})$ imply that

$$s(t) \leq \int_0^{\hat{t}} \frac{1}{q_+(r)} dr \leq \frac{\hat{t}}{\lambda^+(u^*, l)} = \frac{l}{2u^*}. \tag{71}$$

Therefore, for $j \in \mathcal{J}$ and $t \in [0, \hat{t}]$, by combining (70), (71), and $l/2 \geq l^*$, we have that $w_j(t) \geq l^*$. Hence, $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*}$ for all $t \in [0, \infty)$. □

Having established Lemma 6.4, we wish to obtain a uniform, positive lower bound on each component of the fluid queue length for fluid model solutions with initial conditions in $\mathbf{K}^{\times}_{u,l}$, for $u, l > 0$. This motivates the following definition. For notational convenience, given $u, l > 0$, for u^* and l^* given by (64) and (69), we define

$$\lambda^*(u, l) = \lambda(u^*, l^*), \tag{72}$$

where $\lambda(\cdot, \cdot)$ is defined in (38). The following result is an immediate consequence of Lemma 6.4.

Corollary 6.3. *Let $u, l > 0$. Then, $\lambda^*(u, l) > 0$. Moreover, for all fluid model solutions μ with $\mu(0) \in \mathbf{K}^{\times}_{u,l}$, for each $j \in \mathcal{J}$, $q_j(t) \geq \lambda^*(u, l)$ for all $t \in [0, \infty)$.*

Proof. Fix $u, l > 0$ and u^* and l^* be given by (64) and (69), respectively. The fact that $\lambda^*(u, l) > 0$ follows since $\lambda^*(u, l) = \lambda(u^*, l^*) > 0$ by the remarks immediately below (38). Given a fluid model solution μ with $\mu(0) \in \mathbf{K}^{\times}_{u,l}$, then $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*} \subset$

$\mathbf{M}^{\times}_{u^*,l^*}$ for all $t \in [0,\infty)$ by Lemma 6.4. Therefore, for each $j \in \mathcal{J}$, at every time $t \in [0,\infty)$, $q_j(t) = \langle 1, \mu_j(t)\rangle \geq \lambda^*(u,l)$. □

Next, we extend [14, Corollary 5.2] to $J > 1$ by specifying a uniform time by which a fluid model solution enters a set of the form given in (40).

Lemma 6.5. *Let $u, l, \theta > 0$. Let u^* and l^* be given as in* (64) *and* (69)*, respectively. Set $\theta^* = \theta$ and $T^* = \theta^* u^*$. If μ is a fluid model solution such that $\mu(0) \in \mathbf{K}^{\times}_{u,l}$, then $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$ for all $t \geq T^*$.*

Proof. Fix $u, l, \theta > 0$. Let μ be a fluid model solution with $\mu(0) \in \mathbf{K}^{\times}_{u,l}$. Then, by Lemma 6.4, $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*}$ for all $t \in [0,\infty)$. Also, since $\mathbf{K}^{\times}_{u,l} \subset \mathbf{K}^{+}_{u,Jl}$, it follows from (66) in Corollary 6.2 that, for all $j \in \mathcal{J}$, $t \in [0,\infty)$, and $x \in [0, x_j^*)$,

$$\overline{M}_j(t,x) \geq \lambda^+(u^*, Jl)\rho_j \left(\overline{N}^e_j(x) - \overline{N}^e_j(x + s(t))\right).$$

Then, since $l^* \leq l$, it follows from (38) and Lemma 6.3 that $\lambda(u^*, l^*) \leq \lambda(u^*, l) \leq \lambda^+(u^*, Jl)$. Hence, for all $j \in \mathcal{J}$, $t \in [0,\infty)$, and $x \in [0, x_j^*)$,

$$\overline{M}_j(t,x) \geq \lambda(u^*, l^*)\rho_j \left(\overline{N}^e_j(x) - \overline{N}^e_j(x + s(t))\right).$$

Therefore, $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*,s(t)}$ for all $t \in [0,\infty)$. But, for all $t \geq T^*$, $s(t) \geq s(T^*) \geq \theta^*$ by Corollary 6.1. So, $\mathbf{K}^{\times}_{u^*,l^*,s(t)} \subset \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$ for all $t \geq T^*$. □

Corollary 6.4. *Let $u, l > 0$ and u^* and l^* be given as in* (64) *and* (69)*, respectively. Let μ be a fluid model solution such that $\mu(0) \in \mathbf{K}^{\times}_{u,l}$. Then, for every $T^* > 0$, there exists $\theta^* > 0$ such that $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$, for all $t \geq T^*$.*

Proof. Given $T^* > 0$, let $\theta = T^*/u^*$. Let $\theta^* = \theta$. Then, $T^* = \theta^* u^*$, and so by Lemma 6.5, $\mu(t) \in \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$, for all $t \geq T^*$, as desired. □

Lemma 6.6 and Corollary 6.5, stated below, are extensions of [14, Lemma 5.2] and [14, Corollary 5.2], respectively. In particular, Corollary 6.5 provides a useful upper bound on the fluid mass near the origin. We will utilize these results in the proof of Theorem 3.3 in Section 8.

Lemma 6.6. *Let $u > 0$ and let u^* be given as in* (64)*. Let μ be a fluid model solution with $\mu(0) \in \mathbf{K}^{\times}_{u} \cap \mathbf{M}^J_0$. Then, for each $j \in \mathcal{J}$, $t \geq 0$, and $x \in \mathbb{R}_+$,*

$$M_j(t,x) \leq \overline{M}_j(0, s(t)) - \overline{M}_j(0, s(t) + x) + \alpha_j u^* x.$$

Proof. Note, by Lemma 6.2, that $\mu(t) \in \mathbf{K}^{\times}_{u^*}$ for all $t \geq 0$, where $u^* > 0$ is given by (64). Thus, by Corollary 6.1, $q_+(t) \leq u^*$ for all $t \geq 0$. From here, one can apply an analogous argument as given in the proof of [14, Lemma 5.2] to each class $j \in \mathcal{J}$ to conclude the argument. □

Corollary 6.5. *Let $u > 0$ and let u^* be given as in* (64)*. Given a fluid model solution $\mu(\cdot)$ with $\mu(0) \in \mathbf{K}^{\times}_{u}$, then, for each $j \in \mathcal{J}$, $t \geq 0$ and $x \in \mathbb{R}_+$,*

$$M_j(t,x) \leq u^* \left(\overline{N}^e_j(t/u^*) + \alpha_j x\right). \tag{73}$$

Furthermore, given $\epsilon > 0$, there exists $\delta, x^* > 0$ such that if $\mu(\cdot)$ is a fluid model solution satisfying $\mu(0) \in \mathbf{K}_u^{\times}$ and $\mathbf{d}_J(\mu(0), \mathbf{I}) < \delta$, then, for all $j \in \mathcal{J}$

$$\sup_{0 \le x \le x^*} \sup_{t \in [0,\infty)} M_j(t,x) < \epsilon. \tag{74}$$

Proof. Let $u > 0$ and let u^* be given as in (64). Consider a fluid model solution $\mu(\cdot)$ with $\mu(0) \in \mathbf{K}_u^{\times}$. Note that (73) and (74) hold trivially if $\mu(0) = \mathbf{0}$ since $\mu(t) = \mathbf{0}$ for all $t \in [0,\infty)$ in that case. Henceforth, we assume $\mu(0) \neq \mathbf{0}$. Fix $t \in [0,\infty)$, $x \in \mathbb{R}_+$ and $j \in \mathcal{J}$. By Proposition 2.1, Lemma 6.2, and Corollary 6.1, $\mu(t) \in \mathbf{K}_{u^*}^{\times} \setminus \{\mathbf{0}\}$ and $0 < q_+(t) \le u^*$ for all $t \in [0,\infty)$. Therefore, for all $t \in [0,\infty)$,

$$s(t) = \int_0^t \frac{1}{q_+(r)} dr \ge \frac{t}{u^*}.$$

So, $\overline{M}_j(0, s(t)) \le \overline{M}_j(0, t/u^*) \le u^* \overline{N}_j^e(t/u^*)$. By Lemma 6.6,

$$\begin{aligned} M_j(t,x) &\le \overline{M}_j(0, s(t)) - \overline{M}_j(0, s(t)+x) + \alpha_j u^* x \\ &\le \overline{M}_j(0, s(t)) + \alpha_j u^* x \\ &\le u^* \overline{N}_j^e(t/u^*) + \alpha_j u^* x. \end{aligned}$$

Since $j \in \mathcal{J}$, $t \in [0,\infty)$, and $x \in \mathbb{R}_+$ were arbitrary, (73) follows. Since $\mathbf{d}_J(\mu(0), \mathbf{I}) < \delta$ implies that there exists $c \ge 0$ such that $\mathbf{d}(\mu_j(0), c\rho_j \nu_j^e) < \delta$ for each $j \in \mathcal{J}$, one can apply Lemma 6.6 together with the same argument used to verify [14, Corollary 5.2] to each class $j \in \mathcal{J}$ to prove (74). $\square$

7. Relative Entropy Along Fluid Paths

We now consider properties of H and K along fluid paths. That is, we consider $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$, $\xi \in \mathbf{K}_{u,l}^{\times}$ for some $u, l > 0$, the corresponding time-dependent functions of H and K at the value of a fluid model solution. As we will see in Section 7.1, both $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$ turn out to be continuous functions of time. Section 7.2 will extend the partial differential equation considered in [14] to a system of partial differential equations, one for each class, provided that the initial condition satisfies certain continuity conditions.

One of the primary results of this section is Lemma 7.1, which is a special case of Theorem 3.2 restricted to initial conditions that lie in sets of the form

$$\mathbf{A}_{u,l}^{\times} = \mathbf{A}^J \cap \mathbf{M}_{u,l}^{\times}, \tag{75}$$

for $u, l > 0$. We state the lemma here.

Lemma 7.1. *Let $u, l > 0$ and $\xi \in \mathbf{A}_{u,l}^{\times}$. The function $\mathcal{H}^\xi$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ with density function κ^ξ, where κ^ξ is nonpositive and given by*

$$\kappa^\xi(t) = -\frac{1}{w_+} \mathcal{K}^\xi(t).$$

In particular, $\mathcal{H}^\xi$ is nonincreasing on $[0,\infty)$.

We prove Lemma 7.1 in Section 7.3. In Section 7.4, we use an approximation argument to extend Lemma 7.1 to continuous initial states, which proves Theorem 3.2.

7.1. **Finiteness and continuity of $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$.** Given a fluid model solution μ^ξ with $\xi \in \mathbf{K}_{u,l}^{\times}$, we are interested in the finiteness and continuity of the corresponding time-dependent functions $\mathcal{H}^\xi$ and $\mathcal{K}^\xi$. Specifically, we need to guarantee that $\mathcal{K}^\xi$ is locally integrable to prove Lemma 7.1.

Lemma 7.2. *Let $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}^{\times}$. The function $\mathcal{H}^\xi$ is finite and continuous on $[0,\infty)$ and the function $\mathcal{K}^\xi$ is finite and continuous on $(0,\infty)$.*

Proof. Fix $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}^{\times}$. Let $u^*, l^* > 0$ be the constants given by (64) and (69) respectively. By Lemma 6.4, $\mu^\xi(t) \in \mathbf{K}_{u^*,l^*}^{\times}$ for all $t \in [0,\infty)$. For $t \in [0,\infty)$, we have that $\mathcal{H}^\xi(t) = H(\mu^\xi(t))$. By Lemmas 5.6 and 5.7, H is finite and continuous on $\mathbf{M}_{u^*,l^*}^{\times}$. Thus, $\mathcal{H}^\xi$ is finite. Further, by property (C.1) of fluid model solutions, $t \mapsto \mu^\xi(t)$ is continuous. Then $\mathcal{H}^\xi$ is continuous since it is a composition of continuous functions.

Next we turn our attention to $\mathcal{K}^\xi$. It suffices to verify finiteness and continuity on (t,∞) for each $t > 0$. For this, fix $t > 0$. Set $\theta = t/u^*$. Let θ^* and T^* be as in the statement of Lemma 6.5. Then $T^* = t$. Hence, by Lemma 6.5, $\mu^\xi(r) \in \mathbf{K}_{u^*,l^*,\theta^*}^{\times} \subset \mathbf{M}_{u^*,l^*,\theta^*}^{\times}$ for all $r \geq t$. For each $r \in (t,\infty)$, $\mathcal{K}^\xi(r) = \mathcal{K}(\mu^\xi(r))$. The result follows from property (C.1) of fluid model solutions and Lemmas 5.11 and 5.12.

□

7.2. **Partial derivatives for fluid model solutions.** In the single class setting, a key step toward obtaining a proof of the result analogous to Lemma 7.1 ([14, Lemma 7.8]), was the study of a partial differential equation involving the value of a fluid model solution at time $t \in [0,\infty)$. In the multiclass case, we will study a system of partial differential equations, one associated with each class $j \in \mathcal{J}$. As done in [14], we restrict our attention to absolutely continuous initial states, i.e., we will limit ourselves to fluid model solutions μ with $\mu(0) \in \mathbf{A}^J$. The analysis of the partial differential equation, and other relevant results, carried out in [14] extends to the multiclass setting in a relatively straightforward manner, as described here.

First, given a critical fluid model solution μ with $\mu(0) \in \mathbf{A}^J$, we establish that $\mu(t) \in \mathbf{A}^J$ for each $t \in [0,\infty)$, and thus that there exists of a density for $\mu_j(t)$ for each $j \in \mathcal{J}$ and $t \in [0,\infty)$.

Proposition 7.1. *Let μ be a critical fluid model solution with $\mu(0) \in \mathbf{A}^J$. For each $t \in [0,\infty)$, $\mu(t) \in \mathbf{A}^J$. In particular, for each $j \in \mathcal{J}$ and $t \in [0,\infty)$, $\mu_j(t)$ has density $m_j(t,\cdot)$ (with respect to Lebesgue measure), where for all $x \in \mathbb{R}_+$, $m_j(t,x) = 0$ if $\mu(0) = \mathbf{0}$ and*

$$m_j(t,x) = m_j(0, x + s(t)) + \alpha_j \langle 1_{(x,x+s(t)]}(\cdot) q_+(\tau(x + s(t) - \cdot)), \nu_j \rangle, \tag{76}$$

otherwise. Thus, for each $t \in [0, \infty)$,

$$\frac{\partial \overline{M}_j(t,x)}{\partial x} = -m_j(t,x), \quad \textit{for almost all } x \in \mathbb{R}_+. \tag{77}$$

Furthermore, for each $j \in \mathcal{J}$, *if* $\nu_j \in \mathbf{K}$ *and* $m_j(0,\cdot)$ *is continuous on* $\mathbb{R}_+$, *then* $m_j(t,x)$ *is continuous as a function of* $x \in \mathbb{R}_+$ *for each* $t \in [0,\infty)$ *and continuous as a function of* $t \in [0,\infty)$ *for each* $x \in \mathbb{R}_+$, *in which case* (77) *holds for all* $t \in [0,\infty)$ *and* $x \in \mathbb{R}_+$, *where partial derivatives at* $x = 0$ *are from the right.*

Proof. If $\mu(0) = \mathbf{0}$, then by (C.4) in the definition of a fluid model solution, $\langle 1, \mu_j(t)\rangle = 0$ for all $t \in [0,\infty)$ and $j \in \mathcal{J}$, and the result follows. Otherwise, $\mu_+(0) \in \mathbf{A}_+$ and $t^* = \infty$ by Proposition 2.1. In this case, by using (15) and applying to each class $j \in \mathcal{J}$ an argument similar to that given in the proof of [14, Lemma 7.9], one can verify that (76) and the remainder of the statements hold. □

In Proposition 7.1, $t \in [0,\infty)$ was fixed and a density was obtained for each such t. The next lemma considers fixed $x \in \mathbb{R}_+$ and a density with respect to time is obtained for each such x. The proof of Proposition 7.2 is similar to the proof of [14, Lemma 7.10] applied to each class $j \in \mathcal{J}$, and so we omit the details.

Proposition 7.2. *Let* μ *be a fluid model solution with* $\mu(0) \in \mathbf{A}^J \setminus \{\mathbf{0}\}$. *For each* $j \in \mathcal{J}$, *for each fixed* $x \in \mathbb{R}_+$, *the function* $\overline{M}_j(\cdot, x)$ *is absolutely continuous (with respect to Lebesgue measure) and has density function* $-u_{\mu(\cdot)}(j,x)$, *where, for each* $t \in [0,\infty)$, $u_{\mu(t)}(j,x)$ *is defined as in* (63) *with* $\mu(t)$ *in place of* ζ. *In particular, for each* $j \in \mathcal{J}$ *and* $x \in \mathbb{R}_+$,

$$-u_{\mu(t)}(j,x) = \rho_j n_j^e(x) - \frac{m_j(t,x)}{q_+(t)}, \quad \textit{for all } t \in [0,\infty).$$

Consequently, for each $j \in \mathcal{J}$ *and* $x \in \mathbb{R}_+$, *for almost every* $t \in [0,\infty)$, $\dfrac{\partial \overline{M}_j(t,x)}{\partial t}$ *exists and satisfies*

$$\frac{\partial \overline{M}_j(t,x)}{\partial t} = -u_{\mu(t)}(j,x). \tag{78}$$

Furthermore, for each $j \in \mathcal{J}$, *if* $\nu_j \in \mathbf{K}$ *and* $\mu_j(0)$ *has a continuous density* $m_j(0,\cdot)$, *then* (78) *holds for all* $x \in \mathbb{R}_+$ *and* $t \in [0,\infty)$.

Combining Propositions 7.1 and 7.2, we immediately obtain the following corollary.

Corollary 7.1. *Suppose that* $\nu \in \mathbf{K}^J$ *and* $\mu(\cdot)$ *is a fluid model solution such that* $\mu(0) \in \mathbf{A}^J \setminus \{\mathbf{0}\}$ *and* $m_j(0,\cdot)$ *is continuous for each* $j \in \mathcal{J}$. *Then, for every* $t \in [0,\infty)$ *and* $x \in \mathbb{R}_+$,

$$\frac{\partial \overline{M}_j(t,x)}{\partial t} = \rho_j n_j^e(x) + \frac{1}{q_+(t)} \frac{\partial \overline{M}_j(t,x)}{\partial x}. \tag{79}$$

While we do not use the result in Corollary 7.1 in the proof of Lemma 7.1, we state it here because it provides a rigorous connection between the system of PDEs

(79) and the fluid model solutions studied here. An analogue of the system of PDEs (79) was used in [10] to study stability properties of subcritical bandwidth sharing models. In that work, the authors of [10] assume that their fluid model solutions are absolutely continuous with respect to Lebesgue measure for all time and that their densities are sufficiently smooth in (t,x) for their PDE to be satisfied. They also assume that no component of a fluid model solution reaches zero before all components reach zero. The bandwidth sharing model with one link and multiple routes operating under proportional fair sharing corresponds to a multiclass processor sharing queue. For this case, with critical loading, our Corollary 7.1 provides conditions under which a version of the system of PDEs assumed to hold in [10] does in fact hold.

Since for $j \in \mathcal{J}$, class j fluid work fluctuates in time, it is advantageous to know when its derivative exists. In fact, the derivative $\dot{w}_j$ exists almost everywhere when Proposition 7.2 holds. We state and prove this result in the following corollary.

Corollary 7.2. *Let μ be a fluid model solution with $\mu(0) \in \mathbf{A}^J \setminus \{\mathbf{0}\}$. Then, for each $j \in \mathcal{J}$, $w_j(\cdot)$ is continuously differentiable with derivative*

$$\rho_j - \frac{q_j(\cdot)}{q_+(\cdot)}. \tag{80}$$

Moreover, $\sum_{j=1}^J \dot{w}_j(\cdot) = 0$.

Proof. Let μ be a fluid model solution with $\mu(0) \in \mathbf{A}^J \setminus \{\mathbf{0}\}$. For any $t \in [0,\infty)$, by Proposition 7.2 and an application of Fubini's theorem, we obtain for each $j \in \mathcal{J}$,

$$\begin{aligned}
w_j(t) - w_j(0) &= \int_0^{x_j^*} \left(\overline{M}_j(t,x) - \overline{M}_j(0,x)\right) dx \\
&= \int_0^{x_j^*} \int_0^t \left(\rho_j n_j^e(x) - \frac{m_j(v,x)}{q_+(v)}\right) dv dx \\
&= \int_0^t \int_0^{x_j^*} \left(\rho_j n_j^e(x) - \frac{m_j(v,x)}{q_+(v)}\right) dx dv \\
&= \int_0^t \left(\rho_j - \frac{q_j(v)}{q_+(v)}\right) dv.
\end{aligned}$$

Since $q_j(\cdot)$ is continuous for each $j \in \mathcal{J}$, it follows from the fundamental theorem of calculus that $w_j(\cdot)$ is differentiable with derivative given by the continuous function in (80) for each $j \in \mathcal{J}$. Summing (80) over $j \in \mathcal{J}$, it follows that $\sum_{j=1}^J \dot{w}_j(t) = 1 - 1 = 0$ for all $t \in [0,\infty)$ (which reflects the fact that since $w_+(t) = w_+$ for all $t \in [0,\infty)$, $\dot{w}_+(t) = 0$ for all $t \in [0,\infty)$). □

7.3. **Proof of Lemma 7.1.**

Proof of Lemma 7.1. Fix $u, l > 0$. Let $\xi \in \mathbf{A}^{\times}_{u,l}$. Let $u^*, l^*, \lambda^* = \lambda^*(u,l) > 0$ be given by (64), (69) and (72), respectively. Fix $0 < r \le t < \infty$. By Corollary 6.4 and

Proposition 7.1, there exists θ^* such that $\mu^\xi(v) \in \mathbf{A}^{\times}_{u^*,l^*,\theta^*}$ for all $v \geq r$. By Lemma 5.15, Fubini's theorem, and (78) in Proposition 7.2, we have

$$\int_r^t \mathcal{K}^\xi(v)dv = \int_r^t \int_{\mathcal{J}\times\mathbb{R}_+} h'\left(\frac{p_{\mu^\xi(v)}(j,x)}{p_\pi(j,x)}\right) u_{\mu^\xi(v)}(j,x)d\gamma(j,x)dv$$
$$= -\int_{\mathcal{J}\times\mathbb{R}_+} \int_r^t h'\left(\frac{p_{\mu^\xi v)}(j,x)}{p_\pi(j,x)}\right) \frac{\partial}{\partial v}\overline{M}^\xi_j(v,x)dvd\gamma(j,x).$$

Upon writing $\overline{M}^\xi_j(v,x) = w_+ p_{\mu^\xi(v)}(j,x)$ for $r \leq v \leq t$, $j \in \mathcal{J}$ and $x \in \mathbb{R}_+$, we see that

$$\int_r^t \mathcal{K}^\xi(v)dv$$
$$= -w_+ \int_{\mathcal{J}\times\mathbb{R}_+} \int_r^t h'\left(\frac{p_{\mu^\xi(v)}(j,x)}{p_\pi(j,x)}\right)\left(\frac{\frac{\partial}{\partial v} p_{\mu^\xi(v)}(j,x)}{p_\pi(j,x)}\right) p_\pi(j,x)dvd\gamma(j,x)$$
$$= -w_+ \int_{\mathcal{J}\times\mathbb{R}_+} \left[h\left(\frac{p_{\mu^\xi(v)}(j,x)}{p_\pi(j,x)}\right) p_\pi(j,x)\right]_{v=r}^{v=t} d\gamma(j,x)$$
$$= -w_+(\mathcal{H}^\xi(t) - \mathcal{H}^\xi(r)).$$

On letting $r \searrow 0$, using the continuity of $\mathcal{H}^\xi(\cdot)$, the nonnegativity of $\mathcal{K}^\xi(\cdot)$, along with monotone convergence, we see that the result holds for $r = 0$. $\square$

7.4. **Proof of Theorem 3.2.** This follows similarly to how [14, Theorem 7.1] follows from [14, Lemma 7.8], using an approximation argument that exploits the continuity of fluid model solutions in the initial condition established in Proposition 2.3.

As in [14, Section 7.4], given $n \in \mathbb{N}$, let $\phi_n \in \mathbf{C}_b(\mathbb{R}_+)$ be such that $\phi_n \geq 0$, $\phi_n(x) = 0$ for all $x \in (-\infty, -1/n] \cup [1/n, \infty)$ and $\int_\mathbb{R} \phi_n(x)dx = 1$. Given $\xi \in \mathbf{K}^J$, $n \in \mathbb{N}$, and $j \in \mathcal{J}$, set $\xi^n_j = \phi_n * \xi_j$ restricted to $\mathbb{R}_+$. Here $*$ denotes the usual convolution operator. In particular, for each $n \in \mathbb{N}$, $\xi^n = (\xi^n_1, \dots, \xi^n_J)$, where for each $j \in \mathcal{J}$, ξ^n_j is the Borel measure on $\mathbb{R}_+$ that is absolutely continuous with respect to Lebesgue measure with density $d^n_j(x) = \int_{\mathbb{R}_+} \phi_n(y-x)d\xi_j(y)$ for $x \in \mathbb{R}_+$. Then, given $n \in \mathbb{N}$, $j \in \mathcal{J}$, and $f \in \mathbf{C}_b(\mathbb{R}_+)$, by Fubini's theorem,

$$\langle f, \xi^n_j\rangle = \langle f * \phi_n, \xi_j\rangle,$$

where $(f * \phi_n)(y) = \int_{\mathbb{R}_+} f(x)\phi_n(y-x)dx$ for $y \in \mathbb{R}_+$, the same as the usual convolution on $\mathbb{R}$ with $f \equiv 0$ on $(-\infty, 0)$. Given $\xi \in \mathbf{K}^J$, we refer to $\{\xi^n\}_{n\in\mathbb{N}}$ as the approximating sequence. It follows by the proof of [14, Lemma 7.12], that $\xi^n_j \xrightarrow{w} \xi_j$ as $n \to \infty$ for each $j \in \mathcal{J}$. Hence, $\xi^n \xrightarrow{w} \xi$ as $n \to \infty$, and so, by Proposition 2.3, as $n \to \infty$,

$$\mu^{\xi^n}(\cdot) = \Xi(\xi^n) \to \Xi(\xi) = \mu^\xi(\cdot). \tag{81}$$

To ease the notation in this subsection, given $\xi \in \mathbf{K}^J$ with approximating sequence $\{\xi^n\}_{n\in\mathbb{N}}$, we let $\mu^n = \mu^{\xi^n}$ for each $n \in \mathbb{N}$ and $\mu = \mu^\xi$.

Next, we establish Lemma 7.3, which is a technical lemma concerning path properties of μ^n, $n \in \mathbb{N}$. For this, given $u, l, \theta > 0$, let

$$\mathbf{A}_u^{\times} = \mathbf{A}^J \cap \mathbf{M}_u^{\times} \qquad \text{and} \qquad \mathbf{A}_{u,l,\theta}^{\times} = \mathbf{A}^J \cap \mathbf{M}_{u,l,\theta}^{\times}.$$

To motivate the statement of Lemma 7.3, fix $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}^{\times}$. While $\xi \in \mathbf{K}_{u,l}^{\times}$, it is possible that the convolution operation has shifted the mass of ξ such that $\xi^n \notin \mathbf{A}_{u'}^{\times}$ for any $u' > 0$ and $n \in \mathbb{N}$. To overcome this, we will show in Lemma 7.3 that given $T > 0$, there is an N and a $\tilde{u}^*$ (depending on T) such that $\mu^n(t) \in \mathbf{A}_{\tilde{u}^*}^{\times}$ for some $\tilde{u}^* > 0$ for all $t \geq T$ and $n \geq N$. Next, in order to execute the approximation argument, given $T > 0$, we wish to consider $\mathcal{H}^n = \mathcal{H}^{\xi^n}$ for n sufficiently large and to express its increments $\mathcal{H}^n(t) - \mathcal{H}^n(r)$, for $t \geq r \geq T$ in terms of the density $-\mathcal{K}^n(\cdot)/w_+$, where $\mathcal{K}^n = \mathcal{K}^{\xi^n}$. By Lemma 7.1, it suffices to establish that given $T > 0$, $\mu^n(t) \in \mathbf{A}_{\tilde{u}^*,\tilde{l}^*}^{\times}$ for some $\tilde{u}^*, \tilde{l}^* > 0$ for all $t \geq T$ and $n \geq N$ (depending on T, but not t). Finally, we wish to let n tend to infinity and leverage continuity properties of H and K to prove Theorem 3.2. Therefore, it suffices to establish that for each $T > 0$, $\mu^n(t)$ is contained in a suitable relatively compact set on which both H and K are finite and continuous for all $t \geq T$ and n sufficiently large, i.e., a set of the form $\mathbf{A}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}^{\times} \subset \mathbf{M}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}^{\times}$ for some $\tilde{u}^*, \tilde{l}^*, \tilde{\theta}^* > 0$. This is done in the next lemma, which generalizes [14, Lemma 7.14].

Lemma 7.3. *Let $u, l > 0$. Given $T > 0$, there exist positive constants $\tilde{u}^*$, $\tilde{l}^*$, and $\tilde{\theta}^*$, and $N \in \mathbb{N}$ (depending on T) such that $\mu_t^n \in \mathbf{A}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}^{\times}$ for each $\xi \in \mathbf{K}_{u,l}^{\times}$, $n \geq N$ and $t \geq T$.*

Proof. Fix $u, l, T > 0$ and $\xi \in \mathbf{K}_{u,l}^{\times}$. By applying [14, Lemma 7.12] to each class, for all each $j \in \mathcal{J}$ and $n \in \mathbb{N}$,

$$\left\langle 1, \xi_j^n \right\rangle \leq \langle 1, \xi_j \rangle \leq u, \tag{82}$$

$$\left\langle \chi, \xi_j^n \right\rangle \geq \langle \chi, \xi_j \rangle - \frac{\langle 1, \xi_j \rangle}{n}, \tag{83}$$

and

$$\left\langle \chi, \xi_j^n \right\rangle \leq \langle \chi, \xi_j \rangle + \frac{\langle 1, \xi_j \rangle}{n} \leq u \left\langle \chi, \nu_j^e \right\rangle + \frac{u}{n} \leq u \left(\left\langle \chi, \nu_+^e \right\rangle + 1 \right). \tag{84}$$

Let

$$u_0 = \frac{3 \max\left(u, 6\alpha_+ u(\left\langle \chi, \nu_+^e \right\rangle + 1)\right)}{2}.$$

By (82), (84) and Lemma 6.1, $q_j^n(t) \leq q_+^n(t) \leq u_0$ for all $t \in [0, \infty)$ and $n \in \mathbb{N}$. Therefore, $s^n(t) \geq t/u_0$ for all $t \in [0, \infty)$ and $n \in \mathbb{N}$. Fix $N' \in \mathbb{N}$ such that $(T/3u_0) - (1/N') > 0$. Then, for $n \geq N'$,

$$s^n(T/3) - \frac{1}{n} \geq \frac{T}{3u_0} - \frac{1}{n} \geq \frac{T}{3u_0} - \frac{1}{N'} > 0.$$

Moreover, for each $n \geq N'$, $j \in \mathcal{J}$, and $x \in \mathbb{R}_+$,

$$\overline{M}_j^n(0,x) = \left\langle 1_{(x,\infty)}, \xi_j^n \right\rangle \leq \left\langle 1_{(x-1/n)^+}, \xi_j \right\rangle = \overline{M}_j(0,(x-1/n)^+),$$

(see proof of [14, Lemma 7.12]). Therefore, for each $n \geq N'$, $j \in \mathcal{J}$, and $x \in \mathbb{R}_+$,

$$\begin{aligned}\overline{M}_j^n(0, x + s^n(T/3)) &\leq \overline{M}_j(0, x + s^n(T/3) - 1/n)\\ &\leq \overline{M}_j(0,x) \leq u\overline{N}_j^e(x) \leq u_0\overline{N}_j^e(x).\end{aligned}$$

Hence, for each $n \geq N'$, $j \in \mathcal{J}$, and $x \in \mathbb{R}_+$, by (15), $0 < q_+(t) \leq u_0$ for all $t \in [0,\infty)$ and $n \in \mathbb{N}$, and $0 < \rho_j \leq 1$ for each $j \in \mathcal{J}$,

$$\begin{aligned}\overline{M}_j^n(T/3, x) &= \overline{M}_j^n(0, x + s^n(T/3)) + \rho_j \int_0^{T/3} n_j^e(x + s^n(T/3) - s^n(v))dv\\ &\leq \overline{M}_j^n(0, x + s^n(T/3)) + \rho_j \int_0^{T/3} n_j^e(x + s^n(T/3) - s^n(v))\frac{u_0}{q_+(v)}dv\\ &\leq u_0\overline{N}_j^e(x) + u_0\left(\overline{N}_j^e(x) - \overline{N}_j^e(x + s^n(T/3))\right) \leq 2u_0\overline{N}_j^e(x).\end{aligned}$$

By Proposition 7.1, $\mu^n(T/3) \in \mathbf{A}^J$ for each $n \in \mathbb{N}$. Combining this with the above, we obtain that $\mu^n(T/3) \in \mathbf{A}_{2u_0}^{\times}$ for all $n \geq N'$.

For $n \geq N'$ and $t \in [0,\infty)$, set

$$\tilde{\mu}^n(t) = \mu^n(T/3 + t).$$

Then, for all $n \geq N'$, $\tilde{\mu}^n$ is a fluid model solution such that $\tilde{\mu}_0^n \in \mathbf{A}_{2u_0}^{\times}$. Let

$$\tilde{u} = 3u_0 \max(J, 6\alpha_+ \left\langle \chi, \nu_+^e \right\rangle).$$

Then by Lemma 6.2, for all $n \geq N'$, $\tilde{\mu}^n(t) \in \mathbf{A}_{\tilde{u}}^{\times}$ for all $t \in [0,\infty)$. Since $\xi \in \mathbf{K}_{u,l}^{\times}$, it follows by part 2 of Proposition 2.1, (82), and (83) that

$$\begin{aligned}\left\langle \chi, \tilde{\mu}_+^n(0) \right\rangle &= \left\langle \chi, \mu_+^n(T/3) \right\rangle = \left\langle \chi, \xi_+^n \right\rangle\\ &\geq \left\langle \chi, \xi_+ \right\rangle - \frac{\left\langle 1, \xi_+ \right\rangle}{n} \geq J\left(l - \frac{u}{n}\right).\end{aligned} \tag{85}$$

Let N be such that $N \geq N'$ and $l - u/n > 0$ for all $n \geq N$. Observe that $\tilde{\mu}^n(0) \in \mathbf{A}_{\tilde{u},J(l-u/N)}^{+} = \mathbf{A}^J \cap \mathbf{M}_{\tilde{u},J(l-u/N)}^{+}$. Set $\tilde{l} = l^+(\tilde{u}, J(l-u/N), T/3)$, where l^+ is defined as in (68). Then, by part 1 of Lemma 6.4 and the fact that $\tilde{\mu}^n(t) \in \mathbf{A}_{\tilde{u}}^{\times}$ for all $t \in [0,\infty)$, $\tilde{\mu}^n(t) \in \mathbf{A}_{\tilde{u},\tilde{l}}^{\times}$ for all $t \geq T/3$ and $n \geq N$.

Now, for $n \geq N$ and $t \in [0,\infty)$, define

$$\hat{\mu}^n(t) = \tilde{\mu}^n(T/3 + t) = \mu^n(2T/3 + t).$$

Then, $\hat{\mu}^n(0) \in \mathbf{A}_{\tilde{u},\tilde{l}}^{\times}$. Let

$$\tilde{u}^* = \frac{3\tilde{u}\max(J, 6\alpha_+ \left\langle \chi, \nu_+^e \right\rangle)}{2}$$

and let

$$\tilde{l}^* = \min\{l^+(\tilde{u}^*, \tilde{l}, \tilde{t}), \tilde{l}/2\},$$

where $l^+(\tilde{u}^*, \tilde{l}, \tilde{t})$ is defined as in (68) with $\tilde{u}^*$, $\tilde{l}$ and $\tilde{t}$ in place of u^*, l and T respectively, for $\tilde{t} = \tilde{t}(\tilde{u}^*, \tilde{l}) = \tilde{l}\lambda^+(\tilde{u}^*, \tilde{l})/(2\tilde{u}^*)$. By part 2 of Lemma 6.4, it follows that $\hat{\mu}^n(t) \in \mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*}$ for all $t \in [0, \infty)$ and $n \geq N$. By Corollary 6.4, there exists $\tilde{\theta}^*$ such that $\hat{\mu}^n(t) \in \mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}$ for all $t \geq T/3$ and $n \geq N$. But, for $n \geq N$ and $t \geq T$, we have that

$$\mu^n(t) = \tilde{\mu}^n(t - T/3) = \hat{\mu}^n(t - 2T/3) \in \mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*},$$

since $t \geq T$ implies $t - 2T/3 \geq T/3$. The result follows. □

Proof of Theorem 3.2. Let $u, l > 0$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. The first part of Theorem 3.2 can be proved similarly to the proof of [14, Theorem 7.1]. In particular, given $0 < r < t < \infty$, one can use Lemmas 7.1 and 7.3 to obtain that for some $\tilde{u}^*, \tilde{l}^*, \tilde{\theta}^* > 0$ and $N \in \mathbb{N}$, $\mu^n(v) \in \mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}$ for all $v \geq r$ and $n \geq N$ and $\mathcal{H}^n(t) - \mathcal{H}^n(r) = \frac{-1}{w_+}\int_r^t \mathcal{K}^n(v)dv$ for all $n \geq N$. Then using (81), continuity of $H(\cdot)$ on $\mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*} \subset \mathbf{M}^{\times}_{\tilde{u}^*,\tilde{l}^*}$ (see Lemma 5.7), continuity of $K(\cdot)$ on $\mathbf{A}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*} \subset \mathbf{M}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}$ (see Lemma 5.12), compactness of $\mathbf{M}^{\times}_{\tilde{u}^*,\tilde{l}^*,\tilde{\theta}^*}$ (see Lemma 4.3), and bounded convergence, one can pass to the limit to obtain $\mathcal{H}^{\xi}(t) - \mathcal{H}^{\xi}(r) = \frac{-1}{w_+}\int_r^t \mathcal{K}^{\xi}(v)dv$. Since $0 < r < t < \infty$ were arbitrary, the first part of Theorem 3.2 follows. From this and Lemma 5.9, it follows that $\mathcal{H}^{\xi}$ is nonincreasing. Let $t \in [0, \infty)$. Also, by Lemma 5.9, $\mathcal{K}^{\xi}(t) = 0$ if and only if $\mu^{\xi}(t) \in \mathbf{I}_+$. Thus, $\mu^{\xi}(t) = c\pi$ for some $c > 0$, and then by Proposition 2.1, $w_+ = c\langle \chi, \pi \rangle$, which proves (30). □

8. Proofs of Theorems 3.1 and 3.3

Having established Theorem 3.2, various properties of H and K in Section 5, and of fluid model solutions in Section 6, the proofs of Theorems 3.1 and 3.3 follow similarly to the proofs of [14, Theorem 3.2] and [14, Theorem 3.1], respectively. However, since these two results are principal findings of this work, we provide the details below.

Proof of Theorem 3.1. Let $u, l > 0$ be such that $\mathbf{K}^{\times}_{u,l} \neq \emptyset$ and $\varepsilon > 0$. We need to show that there exists a uniform time $T > 0$ such that $\mathcal{H}^{\xi}(t) < \varepsilon$ for all $t \geq T$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. For this, let u^*, l^* and $\lambda^* = \lambda^*(u, l)$ be given by (64), (69), and (72), respectively. For $\theta > 0$, set $\theta^* = \theta$ and $T^* = \theta^* u^*$. Then, by Lemma 6.5, $\mu^{\xi}(t) \in \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$ for all $t \geq T^*$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. Note that $\mathbf{K}^{\times}_{u^*,l^*,\theta^*} \neq \emptyset$. Define

$$\mathbf{H}^{\times}_{\varepsilon} = \{\zeta \in \mathbf{M}^{\times}_{u^*,l^*,\theta^*} : H(\zeta) \geq \varepsilon\}.$$

Observe that if $\xi \in \mathbf{K}^{\times}_{u,l}$ and $t \geq T^*$ is such that $\mu^{\xi}(t) \notin \mathbf{H}^{\times}_{\varepsilon}$, then, since $\mu^{\xi}(t) \in \mathbf{K}^{\times}_{u^*,l^*,\theta^*}$, it must follow that $H(\mu^{\xi}(t)) < \varepsilon$. Hence, it suffices to show that there exists $T \geq 0$ such that $\mu^{\xi}(t) \notin \mathbf{H}^{\times}_{\varepsilon}$ for all $t \geq T$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. By Theorem 3.2, $\mathcal{H}^{\xi}$ is nonincreasing for all initial conditions $\xi \in \mathbf{K}^{\times}_{u,l}$. So, we only need to find a single time $T \geq T^*$, at which $\mu^\xi(T) \notin \mathbf{H}_\varepsilon^\times$ for all $\xi \in \mathbf{K}_{u,l}^\times$ in order to conclude that $\mu^\xi(t) \notin \mathbf{H}_\varepsilon^\times$ for all $t \geq T$ and $\xi \in \mathbf{K}_{u,l}^\times$.

Observe that $\mathbf{H}_\varepsilon^\times \subset \mathbf{M}_{u^*,l^*,\theta^*}^\times$, a compact set by Lemma 4.3. Therefore, the closure $\overline{\mathbf{H}}_\varepsilon^\times$ of $\mathbf{H}_\varepsilon^\times$ is a compact subset of $\mathbf{M}_{u^*,l^*,\theta^*}^\times \subset \mathbf{M}_{u^*,l^*}^\times$. We have by Lemma 5.7 that H is a continuous function on $\mathbf{M}_{u^*,l^*}^\times$. Hence, for all $\zeta \in \overline{\mathbf{H}}_\varepsilon^\times$, we must have that $H(\zeta) \geq \varepsilon$. Thus, for all $\zeta \in \overline{\mathbf{H}}_\varepsilon^\times$, $\zeta \notin \mathbf{L}$ by Lemma 5.1 and the fact that $\zeta \neq \mathbf{0}$. Thus, for all $\zeta \in \overline{\mathbf{H}}_\varepsilon^\times$, it follows that $K(\zeta) > 0$ by Lemma 5.9. However, K is lower semicontinuous by Lemma 5.10 and $\overline{\mathbf{H}}_\varepsilon^\times$ is compact, and so K achieves its minimum on $\overline{\mathbf{H}}_\varepsilon^\times$. Therefore, $K(\zeta) \geq \Delta$ for some $\Delta \in (0,1)$ for all $\zeta \in \overline{\mathbf{H}}_\varepsilon^\times$. Set

$$T = T^* + \frac{u^* \langle \chi, \nu_+^e \rangle}{\Delta} h\left(\frac{u^* \langle \chi, \pi_+ \rangle}{l^* \min_{j \in \mathcal{J}} \rho_j}\right).$$

Note that by (45), $u \leq u^*$ (see (64)), and $l^* \leq l$ (see (69)), so the argument of h in the previous line is at least one. In order to complete the proof, it is enough to show that for each initial condition $\xi \in \mathbf{K}_{u,l}^\times$, there is a corresponding $t \in [0,T]$ that satisfies $\mu(t) \notin \mathbf{H}_\varepsilon^\times$.

Suppose that $\xi \in \mathbf{K}_{u,l}^\times$ and $t \geq T^*$ are such that $\mu^\xi(r) \in \mathbf{H}_\varepsilon^\times$ for each $r \in [T^*, t]$. In order to complete the proof, it is enough to show that $t < T$. Since $\mu^\xi(r) \in \mathbf{H}_\varepsilon^\times$ for each $r \in [T^*, t]$, it follows that $K(\mu^\xi(r)) \geq \Delta$ for each $r \in [T^*, t]$. So, for each $r \in [T^*, t]$,

$$\kappa^\xi(r) = -\frac{1}{w_+}\mathcal{K}^\xi(r) = -\frac{1}{w_+}K(\mu^\xi(t)) \leq -\frac{\Delta}{u^* \langle \chi, \nu_+^e \rangle}.$$

Therefore, by Theorem 3.2, the fact that $\mu^\xi(T^*) \in \mathbf{H}_\varepsilon^\times \subset \mathbf{M}_{u^*,l^*,\theta^*}^\times \subset \mathbf{M}_{u^*,l^*}^\times$ and Lemma 5.6,

$$\begin{aligned}
\varepsilon \leq \mathcal{H}^\xi(t) &\leq -\frac{\Delta}{u^* \langle \chi, \nu_+^e \rangle}(t - T^*) + \mathcal{H}^\xi(T^*) \\
&\leq -\frac{\Delta}{u^* \langle \chi, \nu_+^e \rangle}(t - T^*) + h\left(\frac{u^* \langle \chi, \pi_+ \rangle}{l^* \min_{j \in \mathcal{J}} \rho_j}\right) \\
&= \frac{\Delta}{u^* \langle \chi, \nu_+^e \rangle}(T - t).
\end{aligned}$$

This calculation implies that $T - t > 0$, or rather that $t < T$, as needed. □

Now, after proving Theorem 3.1, we are ready to prove Theorem 3.3 in a similar manner to that in the proof of [14, Theorem 3.1]. First, for $u, l, \delta > 0$, recall that

$$\mathbf{K}_{u,l}^\times(\delta) = \{\zeta \in \mathbf{K}_{u,l}^\times : \mathbf{d}_J(\zeta, \mathbf{I}) < \delta\}.$$

The following lemma will be of use in the proof of Theorem 3.3.

Lemma 8.1. *Let* $u, l > 0$. *Then,*

$$\lim_{t\to\infty} \sup_{\xi \in \mathbf{K}^{\times}_{u,l}} \mathbf{d}_J(\mu^{\xi}(t), \mathbf{L}) = 0.$$

Furthermore, given $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sup_{\xi \in \mathbf{K}^{\times}_{u,l}(\delta)} \sup_{t \in [0,\infty)} \mathbf{d}_J(\mu^{\xi}(t), \mathbf{L}) < \varepsilon.$$

Proof. Fix $u, l > 0$. Let $u^*, l^* > 0$ be given as in (64) and (69), respectively. Then, by Lemma 6.4, $\mu^{\xi}(t) \in \mathbf{K}^{\times}_{u^*,l^*}$ for all $\xi \in \mathbf{K}^{\times}_{u,l}$ and $t \in [0,\infty)$. Given $x \in \mathbb{R}_+$, let

$$\mathbf{D}^{\times}_{x} = \{\zeta \in \mathbf{M}^{\times}_{u^*,l^*} : \mathbf{d}_J(\zeta, \mathbf{L}) \geq x\} \quad \text{and} \quad \mathbf{H}^{\times}_{x} = \{\zeta \in \mathbf{M}^{\times}_{u^*,l^*} : H(\zeta) \geq x\}.$$

For any given $x \in \mathbb{R}_+$, $\mathbf{D}^{\times}_{x}$ is a closed subset of a compact set and is therefore compact itself. Moreover, by its definition, for $x > 0$, $\mathbf{D}^{\times}_{x} \cap \mathbf{L} = \emptyset$. But, we know by Lemma 5.1 that, for $\zeta \in \mathbf{M}^J$ such that $\zeta_+ \in \mathbf{M}_{\dagger}$, $H(\zeta) = 0$ if and only if $\zeta \in \mathbf{L}_+$. Thus, if $x > 0$, $H(\zeta) > 0$ for all $\zeta \in \mathbf{D}^{\times}_{x}$. By Lemma 5.7, H is continuous on $\mathbf{M}^{\times}_{u^*,l^*}$, and so it must be continuous on $\mathbf{D}^{\times}_{x} \subset \mathbf{M}^{\times}_{u^*,l^*}$. Then, H is a continuous function on a compact set, and so it must achieve its minimum. Therefore, for each $x > 0$ there exists a $y = y(x) > 0$ such that $H(\zeta) \geq y$ for every $\zeta \in \mathbf{D}^{\times}_{x}$. Fix $\varepsilon > 0$ and let $\Delta = \Delta(\varepsilon) > 0$ be such that

$$\mathbf{D}^{\times}_{\varepsilon} \subset \mathbf{H}^{\times}_{\Delta}.$$

By Theorem 3.1, there exists $T > 0$ such that $\mathcal{H}^{\xi}(t) < \Delta$ for all $t \geq T$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. Therefore, $\mu^{\xi}(t) \notin \mathbf{H}^{\times}_{\Delta}$ for all $t \geq T$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. This forces $\mathbf{d}_J(\mu^{\xi}(t), \mathbf{L}) < \varepsilon$ for all $t \geq T$ and $\xi \in \mathbf{K}^{\times}_{u,l}$. This proves the first statement.

Since $\mathbf{M}^{\times}_{u,l}$ is compact and H is continuous on $\mathbf{M}^{\times}_{u,l}$, we see that H is uniformly continuous on $\mathbf{M}^{\times}_{u,l}$. Moreover, for $\zeta \in \mathbf{M}^{\times}_{u,l}$, $H(\zeta) = 0$ if and only if $\zeta \in \mathbf{L}_+ \cap \mathbf{M}^{\times}_{u,l}$. Thus, there exists $\delta > 0$ such that $H(\zeta) < \Delta$ for all $\zeta \in \mathbf{M}^{\times}_{u,l}$ satisfying $\mathbf{d}_J(\zeta, \mathbf{L}_+) < \delta$, where $\Delta = \Delta(\varepsilon)$ is given in the paragraph above. Observe that if $\xi \in \mathbf{K}^{\times}_{u,l}(\delta) \subset \mathbf{M}^{\times}_{u,l}$, then $\mathbf{d}_J(\xi, \mathbf{I}) < \delta$ implies $\mathbf{d}_J(\xi, \mathbf{I}_+) < \delta$. Since $\mathbf{I}_+ \subset \mathbf{L}_+$, it follows that $\mathbf{d}_J(\xi, \mathbf{L}_+) < \delta$ for all $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$. Therefore, $\mathcal{H}^{\xi}(0) = H(\xi) < \Delta$ when $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$. By Theorem 3.2, $\mathcal{H}^{\xi}$ is nonincreasing, and so

$$H(\mu^{\xi}(t)) = \mathcal{H}^{\xi}(t) \leq \mathcal{H}^{\xi}(0) = H(\xi) < \Delta,$$

for every $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0,\infty)$. Therefore, $\mu^{\xi}(t) \notin \mathbf{H}^{\times}_{\Delta}$ for all $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0,\infty)$. It follows that $\mu^{\xi}(t) \notin \mathbf{D}^{\times}_{\varepsilon}$ for all $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0,\infty)$. But, $\mu^{\xi}(t) \in \mathbf{K}^{\times}_{u^*,l^*} \subset \mathbf{M}^{\times}_{u^*,l^*}$ for all $\xi \in \mathbf{K}^{\times}_{u,l}(\delta) \subset \mathbf{K}^{\times}_{u,l}$ and $t \in [0,\infty)$. Hence, $\mathbf{d}_J(\mu^{\xi}(t), \mathbf{L}) < \varepsilon$ for all $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0,\infty)$. □

Proof of Theorem 3.3. Fix $u, l, \varepsilon > 0$ and let u^* and l^* be given by (64) and (69) respectively. We must show that there exists $T, \delta > 0$ such that if either $\xi \in \mathbf{K}^{\times}_{u,l}$

and $t \geq T$ or $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0, \infty)$, then $\mathbf{d}_J(\mu^\xi(t), \mathbf{I}) < \varepsilon$. Let $\theta > 0$ be such that $u^* \overline{N}^e_j(\theta) < \varepsilon/6$ for each $j \in \mathcal{J}$. Set $T' = u^*\theta$. Notice that $u^* \overline{N}^e_j(t/u^*) < \varepsilon/6$ for all $j \in \mathcal{J}$ and $t \geq T'$. Let $x' > 0$ be such that $u^* \alpha_j x' \leq \varepsilon/6$ for all $j \in \mathcal{J}$. By (74) in Corollary 6.5, there exists $\delta' > 0$ and $0 < x_0 < \min(x', \varepsilon/3)$ such that for any $\xi \in \mathbf{K}^{\times}_{u,l}(\delta')$, $j \in \mathcal{J}$, and $t \in [0, \infty)$,

$$M^\xi_j(t, x_0) \leq \frac{\varepsilon}{3}.$$

From (73) in Corollary 6.5, it follows that, if $\xi \in \mathbf{K}^{\times}_{u,l}$, $j \in \mathcal{J}$, and $t \geq T'$, then

$$M^\xi_j(t, x_0) \leq u^* \overline{N}^e_j(t/u^*) + u^* \alpha_j x_0 < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

By Lemma 8.1, there exists $T \geq T'$ and $0 < \delta \leq \delta'$ such that if either $\xi \in \mathbf{K}^{\times}_{u,l}$ and $t \geq T$ or $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0, \infty)$, then

$$\mathbf{d}_J(\mu^\xi(t), \mathbf{L}) < x_0.$$

Therefore, by Lemma 5.4, we see that if $\xi \in \mathbf{K}^{\times}_{u,l}$ and $t \geq T$ or $\xi \in \mathbf{K}^{\times}_{u,l}(\delta)$ and $t \in [0, \infty)$, then

$$\mathbf{d}_J(\mu^\xi(t), \mathbf{I}) < \max_{j=1}^{J} M^\xi_j(t, x_0) + 2x_0 < \varepsilon,$$

as desired. $\square$

References

[1] Ben Tahar, A. and Jean-Marie, A.: The Fluid Limit of the Multiclass Processor Sharing Queue. *Queueing Syst. Theory and Appl.* **71**, 347-404 (2012).

[2] Bramson, M.: Convergence to Equilibria for Fluid Models of FIFO Queueing Networks. *Queueing Syst. Theory and Appl.* **22**, 5-45 (1996).

[3] Bramson, M.: Convergence to Equilibria for Fluid Models of Head-of-the-Line Proportional Processor Sharing Queueing Networks. *Queueing Syst. Theory and Appl.* **22**, 5-45 (1996).

[4] Bramson, M.: State Space Collapse with Applications to Heavy Traffic Limits for Multiclass Queueing Networks. *Queueing Syst. Theory and Appl.* **30**, 89-148 (1998).

[5] Durrett, R.T.: Probability Theory and Examples. Cambridge University Press, New York (2010).

[6] Gromoll, H.C.: Diffusion Approximation for a Processor Sharing Queue in Heavy Traffic. *Ann. Appl. Probab.* **14**, 555-611 (2004).

[7] Gromoll, H.C., Puha, A. L. and Williams, R.J.: The Fluid Limit of a Heavily Loaded Processor Sharing Queue. *Ann. Appl. Probab.* **12**, 797-859 (2002).

[8] Gromoll, H.C. and Williams, R.J.: Fluid Limits for Networks with Bandwidth Sharing and General Document Size Distributions. *Ann. Appl. Probab.* **19**, 243-280 (2009).

[9] Mulvany, J.A.: A Relative Entropy Approach to Study Critical Fluid Model Solutions for Multiclass Processor Sharing Queues. *Master's Thesis. California State University San Marcos* (2018).

[10] Paganini, F. and Tang, A. and Ferragut, A. and Andrew, L. L. H.: Stability of Networks under General File Size Distribution with Alpha Fair Rate Allocation. *Proceedings Allerton Conference*, UIUC, IL (2009).

[11] Paganini, F. and Tang, A. and Ferragut, A. and Andrew, L. L. H.: Network Stability Under Alpha Fair Bandwidth Allocation with General File Size Distribution. *IEEE Transactions on Automatic Control*, **57**, 579–591 (2012).

[12] Puha, A.L. and Stoylar, A.L. and Williams, R.J.: The Fluid Limit of an Overloaded Processor Sharing Queue. *Math. Oper. Res.* **31**, 316-350, (2006).

[13] Puha, A.L. and Williams, R.J.: Invariant States and Rates of Convergence for a Critical Fluid Model of a Processor Sharing Queue. *Ann. Appl. Probab.* **14**, 517-554 (2004).

[14] Puha, A.L. and Williams, R.J.: Asymptotic Behavior of a Critical Fluid Model for a Processor Sharing Queue via Relative Entropy. *Stochastic Syst.* **6**, 251-300 (2016).