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Abstract

It is shown that from any unitary representation of the 15-parameter conformal group, with the scalar product (f_1, f_2), another representation can be constructed in the same linear space with the indefinite metric (f_1, Rf_2). R is the operator which represents the transformation by reciprocal radii in that space. Its eigenvalues are ± 1.

In the momentum space of the Klein-Gordon equation without rest mass, R is essentially the Hankel transformation and its eigenfunctions are Laguerre's functions.

The new representations solve the problem of representing the dilatations in Hilbert space and lead to a less singular quantization in field theory. The canonical δ function in momentum space is replaced by a Bessel function of order zero.

I. INTRODUCTION

In previous papers[1,2] we have pointed out, more or less intuitively, in which way the ultraviolet divergencies of quantum field theory might be connected with the conformal group and with conformal invariance at very high energies. But in both cases the mathematical background was not clear enough to say exactly how this might be possible. We now believe we have clarified this mathematical problem and the result is rather simple: if one starts with a unitary representation of the 15-parameter conformal group in a Hilbert space with the positive definite metric (f_1, f_2) , then one can define another representation in the same linear space with the invariant metric (f_1, Rf_2), where R is the operator which represents the transformation by reciprocal radii. R serves as metric operator with eigenvalues ± 1. The metric (f_1, Rf_2) is therefore indefinite.

We illustrate this in Sections II and III by the example of the Klein-Gordon equation without rest mass. In momentum space R is essentially the Hankel transformation, and its eigenfunctions are Laguerre's functions. In Section IV we discuss the general case by means of the commutation relations of the operator R with the generators $M_{\mu\nu}$, P_μ, K_μ, and D of the orthochronous Lorentz group, the translations, the special conformal transformations, and the dilatations. The new representations are irreducible if the corresponding unitary representations are irreducible. They also solve the old problem[1] that in physical applications the eigenvalues of the dilatation $e^{iD\alpha}$ appear to be $e^{s\alpha}$, where α is the group parameter and s a real number. This paradox is not solvable in the framework of a positive definite metric, but becomes immediately clear if one has the metric (f_1, Rf_2).

In the last Section we mention briefly how the usual quantization procedure in momentum space is affected by the indefinite metric. The crucial point is that the canonical δ function $2p_o\delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2)$ is replaced by the Bessel function $(2\pi)^{-1}J_o\left[(2p_1 \cdot p_2)^{\frac{1}{2}}\right]$. This method of quantization opens a completely new approach to the theory of quantized fields. The necessity of such a new method has been particularly emphasized by Heisenberg.[3] We think we are now in a position to build up a rigorous mathematical framework in order to study the experimental and theoretical consequences. It turns out that the introduction of such a new framework seems to be connected with the basic structure of space time.

II. AN INDEFINITE METRIC FOR THE SOLUTIONS OF THE KLEIN-GORDON EQUATION WITHOUT REST MASS

In field theory the solutions $F(x)$ of the massless Klein-Gordon equation[4]

$$F(\ x) = \partial_\mu \partial^\mu F(x) = 0 \tag{1}$$

are usually considered superpositions of plane waves, the eigenfunctions of the translation operators in x space. For positive energies $p_o = (p^2)^{\frac{1}{2}} \geqslant 0$ we have

$$F(x) = f(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{p}}{2p_o} \phi(p\ )\, e^{ipx} . \tag{2}$$

Since the Klein-Gordon equation (1) is also invariant under the special conformal group

$$x^{\mu\,'} = R\, T(c)\, Rx^\mu,\ \mu = 0,1,2,3, \tag{3}$$

where

$$Rx^\mu = -\frac{x^\mu}{x^2},\ T(c)\, x^\mu = x^\mu + c^\mu ,$$

c^μ = group parameter, we can expand solutions also in terms of the eigenfunctions of the generators, which belong to the special conformal group. Because of Eq.(3) we get these eigenfunctions by applying the operator R to the plane waves. This yields the functions

$$e^{-ihx/x^2} ,$$

where the eigenvalues $h = (h_o, \underset{\sim}{h})$ form a four vector with respect to the homogeneous Lorentz group.

A solution of the Klein-Gordon equation (1) in terms of these eigenfunctions is

$$g(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{x^2} \int \frac{d^3\underset{\sim}{h}}{2h_o} \psi(h)\, e^{-ihx/x^2}, \; h_o = (\underset{\sim}{h}^2)^{\frac{1}{2}} \geqslant 0. \tag{4}$$

We define the scalar product between the solutions (2) and (4) in the usual way:

$$\begin{aligned} \langle f|g\rangle &= i\int d^3\underset{\sim}{x}(f^*\partial_o g - \partial_o f^* g)_{x^o=0} \\ &= \frac{1}{(2\pi)^3}\int d^3\underset{\sim}{x} \frac{d^3\underset{\sim}{p}}{2p_o} \frac{d^3\underset{\sim}{h}}{2h_o} \phi^*(p)\left(\frac{p_o}{\underset{\sim}{x}^2} + \frac{h_o}{(\underset{\sim}{x}^2)^2}\right) e^{ip\underset{\sim}{x}} e^{i\underset{\sim}{h}\underset{\sim}{x}/\underset{\sim}{x}^2} \psi(h) . \end{aligned} \tag{5}$$

Making use of the substitution $\underset{\sim}{x} \to \underset{\sim}{x}/\underset{\sim}{x}^2$ under the integral, one can see by direct calculations that the quantity

$$R(p,h) = \frac{1}{(2\pi)^3}\int d^3x \left(\frac{p_o}{x^2} + \frac{h_o}{(x^2)^2}\right) e^{ipx} e^{-ihx/x^2} \qquad (6)$$

has the properties

$$R(p,h) = R(h,p) = R(p,h)^*, \qquad (7)$$

and

$$\int \frac{d^3h}{2h_o} R(p_1,h)\, R(p_2,h) = 2p_o\, \delta(p_1 - p_2) \quad . \qquad (8)$$

The relation (8) means that $R(p,h)$ is a unitary operator in a Hilbert space H of the functions $\phi(p)$, with the scalar product

$$(\phi_1,\phi_2) = \int \frac{d^3p}{2p_o}\, \phi_1^*(p)\, \phi_2(p) \quad .$$

Within this scalar product the relation (8) reads

$$(R\phi_1, R\phi_2) = (\phi_1,\phi_2) \quad .$$

Because of Eq.(7) $R(p,h)$ is also Hermitian. But since it is unitary and therefore bounded, it is even self-adjoint;

$$(\phi_1,R\phi_2) = (R\phi_1,\phi_2) \quad ,$$

for all elements $\phi_1, \phi_2 \in H$.

The possible eigenvalues of a unitary and self-adjoint operator are ± 1, and therefore the metric

$$\langle f|g\rangle = (\phi, R\psi)$$

is indefinite. If we define the operation R in connection with the solutions F(x) of Eq. (1) by

$$RF(x) = \frac{1}{x^2} F(Rx),$$

then R(p,h) is nothing other than this transformation in momentum space; and since $R^2 = 1$, this shows in a similiar way how the above eigenvalues occur.

III. DIAGONALIZATION OF THE METRIC R(p,h)

Since the operator $R(p,h)$ is unitary and self-adjoint, its eigenfunctions $e_{\pm}$, where

$$Re_{\pm} = \pm e_{\pm} ,$$

are elements of the Hilbert space. In order to determine these eigenfunctions and to get more insight into the structure of the metric, it is very convenient to expand all quantities in terms of spherical harmonics Y_{ℓ}^{m}. The reason is that the operator R commutes with the rotations and that both transformations therefore can be diagonalized simultaneously. With

$$\psi(\underset{\sim}{h}) = \sum_{\ell, m} \psi_{\ell m}(h_o)\, Y_{\ell}^{m}(\theta, \phi) ,$$

$$\int d\Omega\, Y_{\ell_1}^{*m_1}\, Y_{\ell_2}^{m_2} = \delta_{\ell_1 \ell_2}\, \delta_{m_1 m_2} ,$$

$$e^{i\underset{\sim}{p}\underset{\sim}{x}} = e^{ip_o r \cos \gamma_1} ,$$

$$= \sum_{\ell=0} (2\ell+1)\, i^{\ell} \left(\frac{\pi}{2p_o r}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(p_o r)\, P_{\ell}(\cos \gamma_1) , \quad \text{and}$$

$$e^{-i\underset{\sim}{h}\cdot\underset{\sim}{x}/\underset{\sim}{x}^2} = e^{-ih_o \cos \gamma_2 / r}$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1)\, (-i)^{\ell} \left(\frac{\pi r}{2h_o}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}\left(\frac{h_o}{r}\right) P_{\ell}(\cos \gamma_2) ,$$

where the J_ν are Bessel functions and the P_l Lengendre polynomials, and applying the addition theorem for the Legendre polynomials and the formula[5]

$$\int_0^\infty dr\, J_{\ell+\frac{1}{2}}\left(\frac{a}{r}\right) J_{\ell+\frac{1}{2}}(br) = b^{-1} J_{2\ell+1}\left[2(ab)^{\frac{1}{2}}\right] \quad ,$$

we get

$$R(p,h) = \frac{1}{2\pi} (p_o h_o)^{-\frac{1}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, J_{2\ell+1}\left[2(p_o h_o)^{\frac{1}{2}}\right] P_\ell(z) \;, \qquad (9)$$

where $z = \cos(\underset{\sim}{p},\underset{\sim}{h})$.

The series on the right-hand side is the expansion of

$$\frac{1}{2\pi} J_o\left[(2p\cdot h)^{\frac{1}{2}}\right], \; p\cdot h = p_o h_o - \underset{\sim}{p}\,\underset{\sim}{h} \;,$$

in terms of Lengendre polynomials. This can be seen from the orthogonality of these polynomials and the relation[6]

$$\int_0^1 du\, u\, P_\ell(1-2u^2)\, J_o(uv) = v^{-1} J_{2\ell+1}(v) \quad .$$

We therefore have

$$R(p,h) = \frac{1}{2\pi} J_o\left[(2p\cdot h)^{\frac{1}{2}}\right] \quad . \qquad (10)$$

Since $J_o(u)$ is a function of u^2 it is not necessary to specify the sign of $(2ph)^{\frac{1}{2}}$.

The Eq.(9) shows that the transformation $R(p,h)$ is essentially the Hankel transformation.[7] From Eq.(9) it follows that the integral equation for the radial part $e_l(p_o)$ of the eigenfunctions $e(\underset{\sim}{p})$ of $R(p,h)$ has the form

$$(p_o)^{\frac{1}{2}}\, e_\ell(p_o) = \epsilon \int_0^\infty dh_o\, J_{2\ell+1}\left[2(p_o h_o)^{\frac{1}{2}}\right] h_o^{\frac{1}{2}} e_\ell(h_o) \;, \qquad \epsilon = \pm 1 \;. \tag{11}$$

The complete orthogonal system of eigensolutions for this equation is known.[8] They are

$$e_{n\ell}(p_o) = e^{-p_o}(2p_o)^\ell\, L_n^{(2\ell+1)}(2p_o)\, N^{-\frac{1}{2}} \;, \tag{12}$$

$$\epsilon = (-1)^n,\; n = 0,1,\cdots \;,$$

with

$$8N = \Gamma(2\ell+2)\begin{pmatrix} 2\ell+n+1 \\ n \end{pmatrix} \;.$$

The functions $L_n^{(\alpha)}(p_o)$ are Laguerre's polynomials:[9]

$$L_n^{(\alpha)}(p_o) \sum_{k=0}^{n} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{(-p_o)^k}{k!} \;.$$

For fixed l and m we therefore have an infinite set of eigenfunctions $e_{nl}(p_o)$ with the eigenvalues $(-1)^n$. The $e_{nl}(p_o)$ are so normalized that

$$\int_0^\infty \left(\frac{dp_o}{2}\right) p_o \, e_{n_1\ell}(p_o) \, e_{n_2\ell}(p_o) = \delta_{n_1 n_2} \quad .$$

But with the scalar product (5) we have

$$\langle e_{n_1\ell_1 m_1} | e_{n_2\ell_2 m_2} \rangle = (-1)^{n_1} \delta_{n_1 n_2} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \quad , \qquad (13)$$

where

$$e_{n\ell m}(\underset{\sim}{p}) = e_{n\ell}(p_o) \, Y_\ell^m(\theta, \phi) \quad .$$

IV. TRANSFORMATION PROPERTIES OF THE INDEFINITE METRIC

In order to study the transformation properties of the scalar product (5) under the 15-parameter conformal group, we consider the generators $M_{\mu\nu}$, P_μ, K_μ, and D of the homogeneous Lorentz group, the translations, the special conformal transformations (3), and the dilatations

$$x^{\mu'} = \rho x^\mu \,, \quad \rho > 0, \quad \mu = 0,1,2,3. \tag{14}$$

It follows from the definition of R in Eq. (3) that the following relations hold:

$$R\, M_{\mu\nu} R = M_{\mu\nu} \,, \tag{15a}$$

$$R\, P_\mu R = K_\mu \,, \tag{15b}$$

$$R\, K_\mu R = P_\mu \,, \tag{15c}$$

$$R\, D\, R = -D \,. \tag{15d}$$

If we now have any unitary representation[10-12] of the conformal group with the scalar product (f_1,f_2), in which Rf_2 is defined, then

$$\langle f_1 | f_2 \rangle \equiv (f_1, Rf_2) \tag{16}$$

constitutes an invariant indefinite metric in the same linear space. This can be seen as follows: since the $M_{\mu\nu}$ commute with R, we have

$$(M_{\mu\nu} f_1, Rf_2) = (f_1, M_{\mu\nu} Rf_2)$$

$$= (f_1, R M_{\mu\nu} f_2) .$$

This means that the $M_{\mu\nu}$ are Hermitian with respect to (f_1, f_2) and with respect to $\langle f_1 | f_2 \rangle$.

It follows further from Eqs. (15b) and (15c) that the operators

$$A_\mu = \frac{1}{\sqrt{2}} (P_\mu + K_\mu) \quad \text{and} \quad B_\mu = \frac{1}{i\sqrt{2}} (P_\mu - K_\mu) , \quad \mu = 0,1,2,3,$$

are Hermitian with respect to the metric $\langle f_1 | f_2 \rangle$, if the P_μ and K_μ are Hermitian with respect to (f_1, f_2) . This means that

$$(A_\mu f_1, Rf_2) = (f_1, R A_\mu f_2) \quad \text{and} \quad (B_\mu f_1, Rf_2) = (f_1, R B_\mu f_2) ,$$

if

$$(P_\mu f_1, f_2) = (f_1, P_\mu f_2) \quad \text{and} \quad (K_\mu f_1, f_2) = (f_1, K_\mu f_2) .$$

The P_μ and K_μ themselves are not Hermitian with respect to $\langle f_1 | f_2 \rangle$.

Because of the relation (15d) we have

$$(Df_1, Rf_2) = (f_1, RDf_2)$$

if D is skew Hermitian with respect to (f_1, f_2).

This last result is particularly important, since it solves immediately an old puzzle:[1] In the physical applications the eigenvalues of the operator $e^{iD\alpha}$, where α is the group parameter, always appear to be real. They have the form $e^{s\alpha}$, where s is a real number. This expresses the physical situation that a dilatation means a multiplication by a real factor, not a multiplication by a phase factor which we would have in the case of a unitary representation! But if D is skew Hermitian with respect to (f_1, f_2), then its eigenvalues are purely imaginary, and that solves the problem mentioned above. This feature is one of the main physical reasons why one has to consider representations of the conformal group in a Hilbert space with an indefinite metric.

It is obvious that the representations of the conformal group, discussed here, are irreducible, if the corresponding unitary representations are irreducible.

As to the discontinuous transformations as space reflection P and time reversal T, it follows immediately from the definition of R that RP = PR and RT = TR. They therefore cause no new problems in this context.

Although our starting point was the Klein-Gordon equation without rest mass, the considerations of this section obviously do not depend on the rest mass zero. They are valid for any unitary representation of the conformal group. It is a different question, whether a certain field equation in x space is conformal invariant or not.

V. QUANTIZATION

Besides solving the puzzle how the dilatations have to be represented in the Hilbert space, the most fundamental consequences of the indefinite metric seem to be those which affect the canonical quantization of field theories. The indefinite metric opens the door to a completely new approach to this crucial problem, and one hopes it will solve the current fundamental difficulties.

Let us consider the creation and annihilation operators $a^+(h)$ and $a(p)$ with the commutation relations

$$\left[a^+(h_1),\ a^+(h_2)\right] = 0\ ,\ \left[a(p_1),\ a(p_2)\right] = 0\ ,$$
$$\left[a(p),\ a^+(h)\right] = R(p,h) = \frac{1}{2\pi} J_o\left[(2p\cdot h)^{\frac{1}{2}}\right]\ . \tag{17}$$

Here $a^+(h)$ is the Hermitian adjoint to $a(p)$ with respect to the metric $(\phi_1, R\phi_2)$. This means

$$a^+ = R\, a^* R,$$

where a^* is the Hermitian adjoint of a with respect to (ϕ_1, ϕ_2).

By means of the functions $e_{nlm}(\underset{\sim}{p})$ of Eq. (13) we can define the operators

$$a^+_{n\ell m} = \int \frac{d^3\underset{\sim}{h}}{2h_o} e_{n\ell m}(\underset{\sim}{h})\, a^+(h),\quad a_{n\ell m} = \int \frac{d^3\underset{\sim}{p}}{2p_o} e^*_{n\ell m}(\underset{\sim}{p})\, a(p)\ .$$

Because of Eqs. (17) they have the following commutation relations:

$$\left[a^{+}_{n_1\ell_1 m_1},\ a^{+}_{n_2\ell_2 m_2}\right] = 0, \quad \left[a_{n_1\ell_1 m_1},\ a_{n_2\ell_2 m_2}\right] = 0 \ ,$$

$$\left[a_{n_1\ell_1 m_1},\ a^{+}_{n_2\ell_2 m_2}\right] = (-1)^{n_1}\,\delta_{n_1 n_2}\,\delta_{\ell_1\ell_2}\,\delta_{m_1 m_2} \ .$$

By means of these relations we can construct the Hilbert space in the usual way if a vacuum $|0\rangle$, with $\langle 0|0\rangle = 1$, is given. But this Hilbert space has an indefinite metric.

The essential difference between our quantization procedure and the canonical one is given by the right-hand side of the last commutator in Eqs. (17). The canonical singular δ - function $2p_0\delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2)$ is replaced by the regular Bessel function of order zero. A field theory, quantized by this method, is therefore also less singular than in the canonical case. The crucial question is, of course, whether this new quantization procedure leads to reasonable physical results.

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FOOTNOTES AND REFERENCES

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