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Semi-classical bound states of Schrödinger equations

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Abstract We study the existence of semi-classical bound states to the nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \quad x \in \mathbf{R}^N,$$

where $N \geq 3$; ε is a positive parameter; $V : \mathbf{R}^N \to [0,\infty)$ satisfies some suitable assumptions. We study two cases: If f is asymptotically linear, i.e., $\lim_{|t|\to\infty} f(t)/t = constant$, we get positive solutions. If f is superlinear and $f(u) = |u|^{p-2}u + |u|^{q-2}u$, $2^* > p > q > 2$, we obtain the existence of multiple sign-changing semi-classical bound states with a composite information on the estimates of the energies, the Morse indices and the number of nodal domains. For this purpose, we establish a mountain cliff theorem without compactness condition and apply a new sign-changing critical point theorem.

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1 Introduction

Why electrons in atoms don't just spiral into the nucleus? The stability of a bound state plays an important role. One of the historical puzzles that led to the creation of quantum mechanics was also the stability of a bound state. We have known it is the bound states of nucleons, atoms, molecules and solids that allow the world and all of life to be what it is. Basically, the bound state is one of the most important topics in quantum mechanics. So much current ongoing research and historical research using quantum mechanics involve the bound states. In some ways, it is a much more complicated problem than scattering or half-scattering phenomenons. In physics, a bound state is a composite of two or more building blocks like particles or bodies that behaves as a single object. In quantum mechanics, where the number of particles is conserved, a bound state is a state in the Hilbert space that corresponds to two or more particles whose interaction energy determines whether these particles can be separated or not. Hence, the energy of a bound state is very helpful to physicists for understanding these particles. In general, a stable bound state is said to exist in a given potential of some dimension if stationary wave functions exist. In this paper we consider the existence of bound states for the following Schrödinger equation:

$$-\varepsilon^2 \Delta u + V(x)u = f(x,u), \qquad u \in H^1(\mathbf{R}^N). \tag{1.1}$$

Equation (1.1) arises from the problem of obtaining the standing wave solutions of the nonlinear Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + (V(x) + M)\psi - |\psi|^{-1} f(x, |\psi|)\psi \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

A standing wave solution to problem (1.2) is one of the form

$$\psi(x,t) = \exp(-i\varepsilon^{-1}Mt)u(x),$$

where u is a solution of (1.1). For ε small enough, the solutions to problem (1.1) can induce the standing waves of the Schrödinger equation which are referred to as semi-classical states. Some class of solutions of (1.1) concentrate and develop spike layers and peaks around certain points in $\mathbf{R}^N$ while vanishing elsewhere as $\varepsilon \to 0$. The existence of single-peak solutions was first studied in [18] where $N = 1$ and $f = u^3$ (super linear case). A single-peak solution was constructed which concentrates around any given non-degenerate critical point of the potential $V(x)$. The higher dimension cases were considered in [27, 28]. In particular, in [28], the existence of multi-peak solutions which concentrate around any finite subsets of the nondegenerate critical points of $V(x)$ was established. The arguments in [18, 27, 28] are based on a Lyapunov-Schmidt reduction and heavily rely on the uniqueness and non-degeneracy of the positive ground state solutions (least energy solutions). In [1], they studied (1.1) and considered the concentration phenomena at isolated local minima and maxima with polynomial degeneracy. In [26], the author deals with $\mathbf{C}^1$-stable critical points of $V(x)$. See also [2, 11, 12, 20] for related results about (1.1). However, as observed by

many experts, the uniqueness and non-degeneracy of the positive ground state solutions are usually quite difficult to verify. They are known so far only for some very restricted cases on nonlinearities f in (1.1). To get the existence of positive solutions without these assumptions, the variational approach which was initiated in [29] has proved to be very successful. In [29], by the mountain pass theorem, the author proves the existence of positive solutions of (1.1) for small $\varepsilon > 0$ whenever $\liminf_{|x| \to \infty} V(x)) > \inf_{x \in \mathbb{R}^N} V(x)$. These solutions concentrate around the global minimum points of $V(x)$ when $\varepsilon \to 0$ as was shown in [37]. Later in [14], by introducing a penalization approach, the authors proved a localized version of the result in [29, 37], see also [15, 16] for related results, they do not assume the uniqueness of a least-energy ground state in a related homogeneous problem. In [8, 19], the monotonicity condition of [14] is not necessary. Also, in [7], the authors develop a new variational approach to construct localized positive solutions to (1.1) which concentrate at an isolated component of positive local minimum points of $V(x)$ as $\varepsilon \to 0$ under certain conditions on f which are "almost optimal". Similarly, no uniqueness and non-degeneracy of the positive ground state solutions are required in [7]. For studying of the problem (1.1), the above papers mainly concern the positive solutions and consider the super linear case. In the present paper, we are interested in the existence of positive solutions to (1.1) with asymptotically-linear growth. On the other hand, it seems that the existence of multiple sign-changing semi-classical bound states along with topological and geometrical properties has not been established before.

1.1. Asymptotically linear case

Consider the existence of positive solutions to the semilinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbf{R}^N, \tag{1.3}$$

where $N \geq 3$ and ε is a positive parameter; $V : \mathbf{R}^N \to [0, \infty)$ and $f : [0, \infty) \to [0, \infty)$ are non-negative continuous functions. We make the following basic assumption on V.

($\mathbf{V}_1$) There exist positive constants $R_1 < r_1 < r_2 < R_2$ such that $V(x) = 0$ for $x \in \Omega := \{x \in \mathbf{R}^N : r_1 < |x| < r_2\}$; $V(x) \geq V_0 > 0$ in Λ^c, where $\Lambda := \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$, $\Lambda^c = \mathbf{R}^N \backslash \Lambda$.

We need the following hypotheses on f, which characterize (1.3) as an asymptotically linear equation:

($\mathbf{Z}_1$) $\lim_{t \to \infty} f(t)/t = \beta_0 > 0$.

($\mathbf{Z}_2$) $\lim_{t \to 0} f(t)/t = 0$; $f(t)/t$ is increasing when $0 < t \leq t_0$, where $t_0 > 0$ is a constant.

($\mathbf{Z}_3$) $f(t)t - 2F(t) > 0$ for $t > 0$, where $F(t) = \int_0^t f(s)ds$.

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$(\mathbf{Z}_4)$ $\displaystyle \liminf_{t\to\infty} \frac{f(t)t - 2F(t)}{|t|^\alpha} \geq c > 0$, where $\alpha \in (0,2)$.

$(\mathbf{Z}_5)$ $\displaystyle \liminf_{t\to 0} \frac{f(t)t}{F(t)} = \mu > 2$.

The main result for this case under consideration is the following theorem.

Theorem 1.1. *Assume that conditions* $(V_1), (Z_1)\text{-}(Z_5)$ *hold. Then there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, equation (1.3) has a positive solution* $u_\varepsilon \in H^1(\mathbf{R}^N)$ *with* $u_\varepsilon(x) \to 0$ *as* $|x| \to \infty$ *and* $\int_{\mathbf{R}^N} \left(\varepsilon^2 |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2 \right) dx \to 0$ *as* $\varepsilon \to 0$.

1.2. Super linear case

We study the existence of multiple semi-classical sign-changing bound states to the Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-2}u + |u|^{q-2}u, \quad \text{in } \mathbf{R}^N \tag{1.4}$$

for $N \geq 3, 2^* > p > q > 2$; $V \geq 0$ is a continuous function satisfying the following basic assumption:

$(\mathbf{V}_2)$ $V(0) = 0$ and there exists a constant $R_1 > 0$ such that $V(x) \geq V_0 > 0$ if $|x| > R_1$.

The solutions of (1.4) correspond to the critical points of the $\mathbf{C}^2$-functional

$$H_\varepsilon(u) = \frac{1}{2} \int_{\mathbf{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)u^2) dx - \frac{1}{p} \int_{\mathbf{R}^N} |u|^p dx - \frac{1}{q} \int_{\mathbf{R}^N} |u|^q dx$$

for $u \in E$, where $E := \{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} V(x)u^2 dx < \infty \}$.

Theorem 1.2. *Assume* (V_2)*. Then for* $m \in \mathbb{N}$*, there exists an* $\varepsilon_0 \in (0,1)$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$*, the problem (1.4) has* m *pairs of solutions* $\{\pm u^*_{k,\varepsilon}\}_{k=2}^{m+1}$ *possessing the following composite properties:*

(1) *For each* $k \geq 2$, $u^*_{k,\varepsilon}$ *is sign-changing.*

(2) $\displaystyle H_\varepsilon(u^*_{k,\varepsilon}) \leq k \frac{q-2}{q} 2^{\frac{q}{q-2}} \varepsilon^{-\frac{4q}{q-2}}$.

(3) $\|u^*_{k,\varepsilon}\|_{H^1(\mathbf{R}^N)} \geq Ck^{\frac{2}{N(p-2)}}$ *if* $q \geq 2 + 4/N$*, where* $C > 0$ *is a constant depending only on* N, p *and* q.

(4) *The number of nodal domains of* $u^*_{k,\varepsilon}$ *is* $\leq k$.

(5) *The augmented Morse index of* $u^*_{k,\varepsilon}$ *is* $\geq k$.

(6) $u^*_{2,\varepsilon}$ *changes sign exactly once.*

2 Abstract Theory

In this section, we shall establish two abstract theories for proving Theorems 1.1-1.2. Since we want to establish universally applicable theorems, we make some general assumptions for the abstract results though these are not strictly necessary for Theorems 1.1-1.2. Let E be a Banach space and let G_λ be a functional in $\mathbf{C}^1(E, \mathbf{R})$ of the form

$$G_\lambda(u) := A(u) + \lambda B(u) + \lambda C(u) - \beta D(u), \text{ where } \lambda > 0, \beta > 0.$$

We make the following assumptions:

($\mathbf{A}_1$) $A(u) \geq 0, B(u) \geq 0, C(u) \geq 0, D(u) \geq 0, \quad \forall u \in E.$

($\mathbf{A}_2$) One of the following conditions holds:

  (1) $B(u) + C(u) \to \infty$ as $\|u\| \to \infty$,

  (2) $A(u) - \beta D(u) \to \infty$ as $\|u\| \to \infty$,

  (3) $C(u) = D(u)$ for all u and $A(u) + \lambda B(u) \to \infty$ as $\|u\| \to \infty$.

($\mathbf{A}_3$) There exist $\mathcal{N}_0 \in \Gamma$ and $\rho_0 \in \mathbf{R}$ such that $G_\lambda(0) \leq \rho_0$ and $G_\lambda(e_0) \leq \rho_0, G_\lambda(u) \geq \rho_0$ for all $u \in \partial\mathcal{N}_0$ and all $\lambda > 0$; where $e_0 \in E\backslash\{0\}$ and

  $$\Gamma := \{\mathcal{N} \subset E : \mathcal{N} \text{ is a bounded open set of } E \text{ such that } 0 \in \mathcal{N}, e_0 \notin \overline{\mathcal{N}}\}.$$

Theorem 2.1. *Assume that (A_1)-(A_3) hold. Then for almost all $\lambda > 0$, there exists a bounded sequence $\{u_k(\lambda)\} \subset E$ such that*

$$G_\lambda(u_k(\lambda)) \to b_\lambda := \sup_{\mathcal{N}\in\Gamma}\inf_{u\in\partial\mathcal{N}} G_\lambda(u), \quad G_\lambda'(u_k(\lambda)) \to 0, \quad as\ k \to \infty,$$

where $b_\lambda \in [\rho_0, D_\lambda], D_\lambda := \max_{t\in[0,1]} G_\lambda(te_0)$. In particular, if $b_\lambda = \rho_0$, then dist $(u_k(\lambda), \partial\mathcal{N}_0) \to 0$ as $k \to \infty$.

Remark 2.1. *In deriving Theorem 2.1 we have been inspired by [21] and [32]. In [21], the author established an existence theorem of bounded (PS) sequences of "Mountain Pass Type" due to [3]. We do not know whether the energy of the mountain pass point is the same as that of Theorem 2.1 in the present paper. Theorem 2.1 here gives information of the location of the (PS) sequence. For a special case, the critical point lies on the cliff $\partial\mathcal{N}_0$. The proof of Theorem 2.1 relies on the monotonicity trick due to the pioneering papers of [21, 22, 23](see also [24, 36]) and linking methods due to [32].*

Proof. Evidently, $b_\lambda \geq \rho_0$ and b_λ is nondecreasing with respect to λ. Therefore, the derivative $b_\lambda' := db_\lambda/d\lambda$ exists for almost all λ. Throughout this section, we consider those λ such that b_λ' exists. Choose $\lambda_n \in (0, \lambda)$ such that $\lambda_n \to \lambda$ and $b_\lambda' - 1 \leq \frac{b_{\lambda_n} - b_\lambda}{\lambda_n - \lambda} \leq b_\lambda' + 1, n$ large. Following [21], we claim that there exist

$\mathcal{N}_n \in \Gamma, k_0 = k_0(\lambda) > 0$ such that $\|u\| \leq k_0(\lambda)$ whenever $G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)$ for $u \in \partial \mathcal{N}_n$. By the definition of b_{λ_n}, there exists an $\mathcal{N}_n \in \Gamma$ such that

$$\inf_{u \in \partial \mathcal{N}_n} G_\lambda(u) \geq \inf_{u \in \partial \mathcal{N}_n} G_{\lambda_n}(u) \geq b_{\lambda_n} + (\lambda_n - \lambda). \tag{2.1}$$

If $u \in \partial \mathcal{N}_n$ is such that $G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)$, then $A(u) - \beta D(u) \leq b_\lambda + \lambda$ and $B(u) + C(u) = \frac{G_\lambda(u) - G_{\lambda_n}(u)}{\lambda - \lambda_n} \leq b'_\lambda + 3$. If $C(u) = D(u)$ for all u, then $A(u) \leq \beta(b'_\lambda + 3) + b_\lambda + \lambda$. Therefore, by (A_2), there exists a constant k_0, depending only on λ, such that $\|u\| \leq k_0$. By (2.1), we see that $G_\lambda(u) \geq G_{\lambda_n}(u) \geq \inf_{u \in \partial \mathcal{N}_n} G_{\lambda_n}(u) \geq b_\lambda - (b'_\lambda + 2)(\lambda - \lambda_n)$ for all $u \in \partial \mathcal{N}_n$. We define $D(\varepsilon, \lambda) := \{u \in E : \|u\| \leq k_0 + 3, |G_\lambda(u) - b_\lambda| \leq \varepsilon\}$. We first observe that $D(\varepsilon, \lambda) \neq \emptyset$. To see this, choose n large enough such that $(b'_\lambda + 2)(\lambda - \lambda_n) < \varepsilon$. Hence, $G_\lambda(u) \geq b_\lambda - (b'_\lambda + 2)(\lambda - \lambda_n) \geq b_\lambda - \varepsilon, \forall u \in \partial \mathcal{N}_n$. If there exists an $u \in \partial \mathcal{N}_n$ such that $G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n) \leq b_\lambda + \varepsilon$, then by the claim at the beginning, we see that $\|u\| \leq k_0(\lambda)$. Therefore, $u \in D(\varepsilon, \lambda)$. Otherwise, $G_\lambda(u) > b_\lambda + (\lambda - \lambda_n)$ for all $u \in \partial \mathcal{N}_n$. It implies that $b_\lambda + (\lambda - \lambda_n) \leq \inf_{u \in \partial \mathcal{N}_n} G_\lambda(u) \leq \sup_{\mathcal{N} \in \Gamma} \inf_{u \in \partial \mathcal{N}} G_\lambda(u) = b_\lambda$, a contradiction. Therefore, $D(\varepsilon, \lambda) \neq \emptyset$. We consider firstly that $b_\lambda > \rho_0$. We now prove that $\inf\{\|G'_\lambda(u)\| : u \in D(\varepsilon, \lambda)\} = 0$. By way of negation, we assume that there exists an $\varepsilon_0 > 0$ such that $\|G'_\lambda(u)\| \geq \varepsilon_0$ for $u \in D(\varepsilon_0, \lambda)$. Without loss of generality, we may assume $\varepsilon_0 < (b_\lambda - \rho_0)/3$. Choose n large enough such that $(\lambda - \lambda_n) \leq \varepsilon_0/5, (b'_\lambda + 2)(\lambda - \lambda_n) < \varepsilon_0/5$. Then by (2.1),

$$G_\lambda(u) \geq b_\lambda - (b'_\lambda + 2)(\lambda - \lambda_n) > b_\lambda - \varepsilon_0/5 > \rho_0, \quad \forall u \in \partial \mathcal{N}_n. \tag{2.2}$$

Define

$$D^*(\varepsilon_0, \lambda) := \{u \in E : \|u\| \leq k_0 + 3, b_\lambda - \varepsilon_0 \leq G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)\}. \tag{2.3}$$

Then $D^*(\varepsilon_0, \lambda) \neq \emptyset$. Define

$$M_1 := \{u \in E : \|u\| \leq k_0 + 1, b_\lambda - \varepsilon_0/2 \leq G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)/2\},$$

$$M_2 := \{u \in E : \|u\| \leq k_0 + 1, b_\lambda - \varepsilon_0/4 \leq G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)/4\}.$$

Let $\eta(u) := \dfrac{\operatorname{dist}(u, E \backslash M_1)}{\operatorname{dist}(u, M_2) + \operatorname{dist}(u, E \backslash M_1)}$ and $Y_\lambda(u)$ be a locally Lipschitz continuous pseudo-gradient vector field for G_λ (cf. [32, Lemma 2.10.1]), i.e., a mapping satisfying $(G'_\lambda(u), Y_\lambda(u)) \geq \frac{1}{2}\|G'_\lambda(u)\|, \|Y_\lambda(u)\| \leq 1, u \in \hat{E} := \{u : G'_\lambda(u) \neq 0\}$. Consider the following initial value problem and the increasing flow: $\dfrac{d\sigma(t, u)}{dt} = \eta(\sigma(t, u))Y_\lambda(\sigma(t, u)), \sigma(0, u) = u, \quad u \in E$. Note that η vanishes on an open set containing the points where $G'_\lambda = 0$. It is well known that there exists a unique solution $\sigma(t, u)(t \geq 0)$ satisfying $\|\sigma(t, u) - u\| \leq t$ and $\dfrac{dG_\lambda(\sigma(t, u))}{dt} \geq \frac{1}{2}\eta(\sigma(t, u))\|G'_\lambda(\sigma(t, u))\| \geq 0$. Therefore, $G_\lambda(\sigma(t, u)) \geq G_\lambda(u) \geq b_\lambda - \varepsilon_0/5 > \rho_0, \quad \forall u \in \partial \mathcal{N}_n$. It follows that $\sigma(t, u) \neq 0, \sigma(t, u) \neq e_0, \quad \forall u \in \partial \mathcal{N}_n, t \geq 0$.

Next, we prove that $G_\lambda(\sigma(1,u)) \geq b_\lambda + (\lambda - \lambda_n), \forall u \in \partial\mathcal{N}_n$. In fact, if $u \in \partial\mathcal{N}_n$ and $G_\lambda(u) \leq b_\lambda + (\lambda - \lambda_n)/4$, then $\|u\| \leq k_0$. Furthermore, if $G_\lambda(u) \geq b_\lambda - \varepsilon_0/4$, then $u \in M_2$. Therefore, we must have

$$G_\lambda(\sigma(t,u)) \geq G_\lambda(u) > b_\lambda + (\lambda - \lambda_n)/4, \quad u \in \partial\mathcal{N}_n, u \notin M_2. \qquad (2.4)$$

If $u \in \partial\mathcal{N}_n \cap M_2$, we suppose that t_1 is the largest number such that $\sigma(t,u) \in M_2$ for $0 \leq t \leq t_1$. If $t_1 < 1$, then there exists an $s > 0$ such that $t_1 + s < 1$ and $G_\lambda(\sigma(t_1 + s, u)) \geq G_\lambda(\sigma(t_1, u)) \geq b_\lambda - \varepsilon_0/4$. Since $\sigma(t_1 + s, u) \notin M_2$, we have either $\|\sigma(t_1 + s, u)\| > k_0 + 1$ or $G_\lambda(\sigma(t_1 + s, u)) > b_\lambda + (\lambda - \lambda_n)/4$. If $G_\lambda(\sigma(t_1 + s, u)) \leq b_\lambda + (\lambda - \lambda_n)/4$, then $G_\lambda(u) \leq G_\lambda(\sigma(t_1 + s, u)) \leq b_\lambda + (\lambda - \lambda_n)/4 \leq b_\lambda + (\lambda - \lambda_n)$, which implies that $\|u\| \leq k_0$. Moreover, $G_\lambda(u) \geq b_\lambda - \varepsilon_0/5$ for $u \in \partial\mathcal{N}_n$. Then $\|\sigma(t_1 + s, u)\| \leq \|u\| + t_1 + s \leq k_0 + 1$. Combining the above arguments, we observe that $G_\lambda(\sigma(1,u)) > G_\lambda(\sigma(t_1 + s, u)) \geq b_\lambda + (\lambda - \lambda_n)/4$. If $t_1 = 1$, then $\sigma(t,u) \in M_2$ for $0 \leq t \leq 1$. Hence, $G_\lambda(\sigma(1,u)) - G_\lambda(u) \geq \frac{1}{2}\int_0^1 \eta(\sigma(s,u))\|G_\lambda'(\sigma(s,u))\|ds \geq \frac{1}{2}\varepsilon_0$. Conseqently, $G_\lambda(\sigma(1,u)) \geq \frac{1}{2}\varepsilon_0 + b_\lambda - \frac{\varepsilon_0}{5} \geq b_\lambda + \frac{(\lambda - \lambda_n)}{4}$. Combining the above arguments, we have $G_\lambda(\sigma(1,u)) \geq b_\lambda + (\lambda - \lambda_n)/4, \forall u \in \partial\mathcal{N}_n$. Let $\tilde{v}_1 := \{\sigma(1,u) : u \in \mathcal{N}_n\}$. Then it is easy to check that $\partial\tilde{\mathcal{N}}_1 := \{\sigma(1,u) : u \in \partial\mathcal{N}_n\}$. Since $0, e_0 \notin M_1$, $\eta(0) = \eta(e) = 0$. Since $\eta \equiv 0$ on $E\backslash\{M_1\}$, we see that $\sigma(1,0) = 0, \sigma(1,e_0) = e_0$. That is, $\tilde{\mathcal{N}}_1 \in \Gamma$ and $G_\lambda(u) \geq b_\lambda + (\lambda - \lambda_n)/4$ when $u \in \partial\tilde{\mathcal{N}}_1$, which contradicts the definition of b_λ. The remaining case is $b_\lambda = \rho$. Define $Q(\varepsilon, \lambda, T) := \{u \in E : \text{dist}(u, \partial\mathcal{N}_0) \leq T, |G_\lambda(u) - b_\lambda| \leq \varepsilon\}$, where $0 < T < \frac{1}{2}\min\{\text{dist}(0, \partial\mathcal{N}_0), \text{dist}(e_0, \partial\mathcal{N}_0), 1\}$. Since we cannot have $G_\lambda(u) > b_\lambda + \varepsilon$ for all $u \in \partial\mathcal{N}_0$, we see that $Q(\varepsilon, \lambda, T) \neq \emptyset$. We proceed to prove that $\inf\{\|G_\lambda'(u)\| : u \in Q(\varepsilon, \lambda, T)\} = 0$. Assume that there exists an $\varepsilon_0 > 0$ such that $\|G_\lambda'(u)\| \geq \varepsilon_0$ for $u \in Q(\varepsilon_0, \lambda, T)$. Define

$$Q^*(\varepsilon_0, \lambda, T) := \{u \in E : \text{dist}(u, \partial\mathcal{N}_0) \leq T, b_\lambda - \varepsilon_0 \leq G_\lambda(u) \leq b_\lambda + \frac{4\varepsilon_0}{5}\}.$$

Then $Q^*(\varepsilon_0, \lambda, T) \neq \emptyset$. Define

$$W_1 := \{u \in E : \text{dist}(u, \partial\mathcal{N}_0) \leq \frac{T}{2}, b_\lambda - \varepsilon_0 \leq G_\lambda(u) \leq b_\lambda + \frac{2\varepsilon_0}{5}\},$$

$$W_2 := \{u \in E : \text{dist}(u, \partial\mathcal{N}_0) \leq \frac{T}{2}, b_\lambda - \frac{\varepsilon_0}{4} \leq G_\lambda(u) \leq b_\lambda + \frac{\varepsilon_0}{5}\}$$

and $\eta^*(u) := \dfrac{\text{dist}(u, E\backslash W_1)}{\text{dist}(u, W_2) + \text{dist}(u, E\backslash W_1)}$. Let $Y_\lambda(u)$ be a locally Lipschitz continuous pseudo-gradient vector field for G_λ satisfying $(G_\lambda'(u), Y_\lambda(u)) \geq \frac{1}{2}\|G_\lambda'(u)\|$ and $\|Y_\lambda(u)\| \leq 1$. Consider the Cauchy problem $\dfrac{d\sigma(t,u)}{dt} = \eta^*(\sigma(t,u))Y_\lambda(\sigma(t,u))$ with $\sigma(0,u) = u$, $u \in E$. Then there exists a unique solution $\sigma(t,u)(t \geq 0)$ satisfying $\|\sigma(t,u) - u\| \leq t$ and $\dfrac{dG_\lambda(\sigma(t,u))}{dt} \geq \frac{1}{2}\eta^*(\sigma(t,u))\|G_\lambda'(\sigma(t,u))\| \geq 0$. By the choice of T, we see that $\sigma(t,u) \neq 0, \sigma(t,u) \neq e_0$ for $0 \leq t \leq T, u \in \partial\mathcal{N}_0$. We claim that $G_\lambda(\sigma(\frac{T}{4}, u)) \geq b_\lambda + \varepsilon_0 T/8, \forall u \in \mathcal{N}_0$. Evidently, if $u \in \partial\mathcal{N}_0$ and

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$u \notin W_2$, then $G_\lambda(\sigma(T/4, u)) \geq G_\lambda(u) > b_\lambda + \varepsilon_0/5 \geq \varepsilon_0 T/8$. If $u \in \partial \mathcal{N}_0 \cap W_2$, we assume that $t_1 \geq 0$ is the largest number such that $\sigma(t_1, u) \in W_2$. If $t_1 < T/4$, then $G_\lambda(\sigma(t_1 + s, u)) \geq G_\lambda(\sigma(t_1, u)) \geq b_\lambda - \varepsilon_0/4$. We suppose that s small enough such that $t_1 + s < T/4$. But the fact that $\sigma(t_1 + s) \notin W_2$ implies that either $\mathrm{dist}(\sigma(t_1 + s, u), \partial \mathcal{N}_0) > T/2$ or $G_\lambda(\sigma(t_1 + s, u)) > b_\lambda + \varepsilon_0/5$. However, since $\mathrm{dist}(\sigma(t_1 + s, u), \partial \mathcal{N}_0) \leq \|\sigma(t_1 + s, u) - u\| \leq t_1 + s \leq T/4$, we have $G_\lambda(\sigma(T/4, u)) \geq b_\lambda + \varepsilon_0/5$. If $t_1 = T/4$, then $\sigma(t, u) \in W_2$ for $0 \leq t \leq T/4$. Therefore, $G_\lambda(\sigma(T/4, u)) - G_\lambda(u) \geq T\varepsilon_0/8$. Thus, we see that $G_\lambda(\sigma(T/4, u)) \geq b_\lambda + T\varepsilon_0/8$. Choose $\mathcal{N}_T := \{\sigma(T/4, u) : u \in \mathcal{N}_0\}$, similar to the first case, we get a contradiction. $\qquad \square$

Next, we are going to introduce a new theorem concerning the sign-changing critical point with a composite information. Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space with the corresponding norm $\| \cdot \|$, $\mathcal{P} \subset E$ be a closed convex (positive) cone. We call the elements outside $\pm \mathcal{P}$ sign-changing. Let Z be a subspace of E with $E = Z^\perp \oplus Z$, $\dim Z^\perp = k - 1, k \geq 2$. The nontrivial elements of Z are sign-changing. We assume that $\mathcal{P}$ is weakly closed in the sense that $\mathcal{P} \ni u_k \rightharpoonup u$ weakly in $(E, \| \cdot \|)$ implies $u \in \mathcal{P}$. Suppose that there is another norm $\| \cdot \|_*$ of E such that $\|u\|_* \leq C_* \|u\|$ for all $u \in E$, here $C_* > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_* \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \| \cdot \|)$. Let $G \in \mathbf{C}^1(E, \mathbb{R})$ be an even functional and the gradient G' be of the form $G'(u) = u - \mathcal{G}(u)$, where $\mathcal{G} : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E \backslash \mathcal{K}$, $\mathcal{K}[a, b] := \{u \in \mathcal{K} : G(u) \in [a, b]\}, G^a := \{u \in E : G(u) \leq a\}$. For $\mu_0 > 0$, define $\mathcal{D}_{\mu_0} := \{u \in E : \mathrm{dist}(u, \mathcal{P}) < \mu_0\}$. Set $\mathcal{D} := \mathcal{D}_{\mu_0} \cup (-\mathcal{D}_{\mu_0})$, $\mathcal{S} = E \backslash \mathcal{D}$. We make the following assumptions.

(B$_1$) $\mathcal{G}(\pm \mathcal{D}_{\mu_0}) \subset \pm \mathcal{D}_\mu$ for some $\mu \in (0, \mu_0)$; $\mathcal{K} \cap (E \backslash \{-\mathcal{P} \cup \mathcal{P}\}) \subset \mathcal{S}$.

(B$_2$) For any $a, b > 0$, $G^a \cap \{u \in E : \|u\|_* = b\}$ is $\| \cdot \|$-bounded.

(B$_3$) There exists a $\rho > 0$ such that $\alpha_k := \inf\limits_{\|u\|_* = \rho, \, u \in Z} G > -\infty$.

(B$_4$) $\lim\limits_{u \in Y, \, \|u\| \to \infty} G(u) = -\infty$, for any subspace $Y \subset E$ with $\dim Y < \infty$.

In applications, conditions (B_1)-(B_4) are satisfied readily. In particular, the condition similar to (B_1) has been introduced in [5, 9, 10]. The functional G is said to satisfy the $(w^*$-PS$)$ condition on $[a, b]$ if for any sequence $\{u_n\}$ such that $G(u_n) \to c \in [a, b]$ and $G'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|G'(u_n)\| \|u_n\| \to \infty$. Let $Y \subset E$ be a subspace of E with finite dimensional $\dim Y \geq k$. Define

$$\beta_k := \inf\limits_{\substack{Y \subset E, \\ k \leq \dim Y < \infty}} \sup\limits_{Y} G.$$

Theorem 2.2. *(See [39]) Assume (B_1)-(B_4) hold and $G \in \mathbf{C}^2$ is even. If there is a $\lambda_0 > 0$ such that G satisfies $(w^*$-PS$)$ condition on $[\alpha_k, \beta_k + \lambda_0]$, then G has a sign-changing critical point $u^* \in \mathcal{S}$ with $G(u^*) \in [\alpha_k, \beta_k]$ and the augmented Morse index of u^* is $\geq k$.*

3 Positive bound states for asymptotically linear Schrödinger equations

Since we are concerned with positive solutions, we may assume, from now on, that $f(t) = 0$ for $t \leq 0$. For a constant $\delta_0 > 2$ large enough, by (Z_2), there exists an $a_0 > 0$ small enough such that

$$\delta_0 := \frac{V_0}{\frac{f(a_0)}{a_0}} \leq \frac{V_0}{\frac{f(s)}{s}} \quad \text{for } s \in (0, a_0]. \tag{3.1}$$

We let $p(x, t) = \xi_\Lambda(x) f(t) + (1 - \xi_\Lambda(x)) \bar{f}(t)$, where ξ_Λ is the characteristic function on Λ; $\bar{f}(t) = f(t)$ if $t \leq a_0$; $\bar{f}(t) = V_0 t / \delta_0$ for $t > a_0$. It is easy to check that $p(x, t)$ satisfies the following condition:

$$0 \leq 2P(x, t) \leq p(x, t)t \leq \frac{V(x)t^2}{\delta_0} \quad \text{for } x \in \Lambda^c, t \in \mathbf{R}, \tag{3.2}$$

where $P(x, t) = \int_0^t p(x, s) ds$. We will be working in the following Hilbert space $E := \{u \in H^1_{rad}(\mathbf{R}^N) : \int_{\mathbf{R}^N} V u^2 dx < \infty\}$ endowed with the norm $\|u\| = \left(\int_{\mathbf{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x) u^2) dx\right)^{1/2}$ and inner product $\langle \cdot, \cdot \rangle$, where $H^1_{rad}(\mathbf{R}^N) := \{u \in H^1(\mathbf{R}^N) : u \text{ is radially symmetric}\}$. We first study the critical points of the $\mathbf{C}^1$ functional defined by

$$G_\lambda(u) = \frac{\lambda}{2} \int_{\mathbf{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x) u^2) dx - \int_{\mathbf{R}^N} P(x, u) dx, \quad u \in E, \lambda \in [1, 2].$$

Then the critical points of G_λ correspond to the solutions of the equation

$$-\varepsilon^2 \Delta u + V(x) u = \frac{1}{\lambda} p(x, u), \quad \text{in } \mathbf{R}^N. \tag{3.3}$$

Lemma 3.1. *There exist numbers $\rho_0 > 0, r_0 > 0$ such that $G_\lambda(u) \geq \rho_0$ for $u \in \mathcal{N}_0 := \{u \in E : \|u\| = r_0\}$ uniformly for $\lambda \in [1, 2]$.*

Proof. We first recall the Strauss Inequality (see [35] or [25]): $|u(x)| \leq \frac{2\pi \|u\|_{H^{1,2}}}{|x|^{1/2}}, \forall x \in \mathbf{R}^N \setminus \{0\}$. By the boundedness of Λ and Poincaré Inequality, we have that

$$\int_\Lambda u^2 dx \leq c \int_{\mathbf{R}^N} |\nabla u|^2 dx \leq$$

$$(\frac{1+c}{\varepsilon^2} + 1) \int_{\mathbf{R}^N} (|\varepsilon \nabla u|^2 + V(x) u^2) dx \leq (\frac{1+c}{\varepsilon^2} + 1) \|u\|^2. \tag{3.4}$$

Therefore,

$$\|u\|^2_{H^{1,2}} \leq \int_{\mathbf{R}^N} |\nabla u|^2 dx + \frac{1}{V_0} \int_{\Lambda^c} V(x) u^2 dx + \int_\Lambda u^2 dx \leq$$

$$\frac{c}{\varepsilon^2} \max\{1, \frac{1}{V_0}\}\|u\|^2 \leq \frac{c}{\varepsilon^2}\|u\|^2. \tag{3.5}$$

By (Z_2), for any $\varepsilon' > 0$, there is a $\delta > 0$ such that $F(t) \leq \varepsilon' t^2/2$ for all $|t| \leq \delta$. Therefore, $\int_\Lambda F(u)dx \leq \frac{\varepsilon'}{2}\int_\Lambda u^2 dx$ for $\|u\| \leq r_0$, where $r_0 > 0$ is a constant (small enough) depending on ε'. Consequently, by (3.2) and (3.4),

$$
\begin{aligned}
G_\lambda(u) &= (\int_\Lambda + \int_{\Lambda^c})\Big(\frac{\lambda}{2}(|\varepsilon\nabla u|^2 + V(x)u^2)dx - P(x,u)\Big)dx \\
&\geq \frac{\lambda}{2}\int_{\mathbf{R}^N}(\varepsilon|\nabla u|^2 + V(x)u^2)dx - \int_\Lambda F(u)dx - \frac{1}{2\delta_0}\int_{\Lambda^c}V(x)u^2 dx \\
&\geq \frac{1}{2}\int_{\mathbf{R}^N}\Big(|\varepsilon\nabla u|^2 + (1-\frac{1}{\delta_0})V(x)u^2\Big)dx - \frac{\varepsilon'}{2}\int_\Lambda u^2 dx.
\end{aligned}
$$

Hence, $G_\lambda(u) \geq \rho_0$ for $\|u\| = r_0$, where $\rho_0 > 0, r_0 > 0$ depend on ε. $\qquad\square$

Lemma 3.2. *There exists an $e_0 \in E\backslash\{0\}$ such that $G_\lambda(e_0) < 0$ uniformly for $\lambda \in [1,2]$. Moreover, $e_0(x) \geq \bar{e}_0 > 0$ on $\Omega_0 \subset \Omega$, where Ω is given in (V).*

Proof. Consider $\beta_0 > 0$ and the conditions on V, we may choose R_1, R_2 appropriately such that $\beta_0 > 3\max_\Lambda V(x)$. Now choose $\phi_0 \in C_0^\infty(\Lambda)$ so that $\phi_0 \geq c_0 > 0$ for all $x \in \Omega_0 \subset \mathrm{supp}\,\phi_0$ and $\Omega_0 \subset \Omega$. We confine ε to the open interval $\Big(0, \frac{\beta_0\int_\Lambda|\phi_0|^2 dx}{6\int_\Lambda|\nabla\phi_0|^2 dx}\Big)$. Next, we prove that $G_\lambda(t_0\phi_0) < 0$ for some t_0 large enough. By way of negation, if there exists $t_n \to \infty$ such that $G_\lambda(t_n\phi_0) \geq 0$, then

$$0 \leq 1 - \frac{\int_{\mathbf{R}^N}P(x,t_n\phi_0)dx}{t_n^2\|\phi_0\|^2} \leq 1 - \frac{\int_\Lambda\frac{1}{2}\beta_0 t_n^2|\phi_0|^2 dx + \int_\Lambda H(t_n\phi_0)dx}{t_n^2\|\phi_0\|^2},$$

where $H(t) = \int_0^t h(s)ds$, $h(s) = f(s) - \beta_0 s$. Since $h(t) = o(t)$ as $t \to \infty$, we have that $0 \leq 1 - \frac{\beta_0}{2}\int_\Lambda\frac{|\phi_0|^2}{\|\phi_0\|^2}dx$. It implies that $\frac{\beta_0}{2}\int_\Lambda|\phi_0|^2 dx \leq \int_\Lambda\Big(\varepsilon^2|\nabla\phi_0|^2 + V(x)\phi_0^2\Big)dx$, which contradicts the choice of $\varepsilon > 0$. So we may choose $e_0 = t_0\phi_0$ such that $G_\lambda(e_0) < 0$ uniformly for $\lambda \in [1,2]$, where $e_0(x) \geq t_0 c_0 := \bar{e}_0 > 0$ for all $x \in \Omega_0 \subset \Omega$. $\qquad\square$

Remark 3.1. *By the proof of Lemma 3.2, ϕ_0 is independent of ε and λ; e_0 is independent of λ and ε for $\varepsilon \in \Big(0, \frac{\beta_0\int_\Lambda|\phi_0|^2 dx}{6\int_\Lambda|\nabla\phi_0|^2 dx}\Big)$ and $\lambda \in [1,2]$.*

Lemma 3.3. *There exists an ε^* small enough such that for all $\varepsilon \in (0, \varepsilon^*)$ and for almost all $\lambda \in [1,2]$, there exists $u_{\lambda,\varepsilon}$ satisfying $G_\lambda(u_{\lambda,\varepsilon}) = b_\lambda \geq \rho_0 > 0$, $G'_\lambda(u_{\lambda,\varepsilon}) = 0$.*

Proof. By Theorem 2.1, for almost all $\lambda \in [1,2]$, there exists a bounded (PS)-sequence $\{u_n\}$ such that $\sup_n\|u_n\| < \infty, G_\lambda(u_n) \to b_\lambda$, $G'_\lambda(u_n) \to$

0, as $n \to \infty$ and $0 < \rho_0 \leq b_\lambda := \sup_{\mathcal{N} \in \Gamma} \inf_{u \in \partial \mathcal{N}} G_\lambda(u) \leq D_\lambda = \sup_{s \in [0,1]} G_\lambda(se_0) = G_\lambda(s_0 e_0)$ with $s_0 \in (0,1)$, $G_\lambda'(s_0 e_0)e_0 = 0$. We may assume that $u_n \to u$ weakly in E, $u_n \to u$ a.e. in $\mathbf{R}^N$. We claim that for any $\varepsilon' > 0$, there exists an $R > 0$ large enough such that

$$\limsup_{n \to \infty} \int_{|x| \geq R} (|\varepsilon \nabla u_n|^2 + V(x)u_n^2)dx < \varepsilon'. \tag{3.6}$$

In fact, by similar arguments as that in [14], we choose $\eta_R \in C_0^\infty(\mathbf{R}^N)$ such that $\eta_R = 0$ for $|x| < R/2$; $\eta_R = 1$ for $|x| \geq R$ and $|\nabla \eta_R| \leq c/R$. Then

$$
\begin{aligned}
o(1) &= \langle G_\lambda'(u_n), u_n \eta_R \rangle \\
&= \int_{|x| \geq R/2} (\varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2)\eta_R dx + \int_{R/2 \leq |x| < R} u_n |\nabla u_n| \nabla \eta_R dx \\
&\quad - \int_{R/2 \leq |x|} g(x, u_n)u_n \eta_R dx \\
&\geq \int_{|x| \geq R/2} (\varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2)\eta_R dx - \|u_n\|_2 \|\nabla u_n\|_2 \frac{c}{R} \\
&\quad - \frac{1}{\delta_0} \int_{|x| \geq R/2} V(x)u_n^2 \eta_R dx.
\end{aligned}
$$

Therefore,

$$(1 - \frac{1}{\delta_0}) \int_{|x| \geq R} (\varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2)\eta_R dx \leq o(1) + \|u_n\|_2 \|\nabla u_n\|_2 \frac{c}{R},$$

that is, $\int_{|x| \geq R} (\varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2)\eta_R dx < \varepsilon'$ by taking R large. Since $p : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a continuous and radially symmetric function and $u_n \to u$ weakly in H_{rad}^1, then for given positive constants a and b, it is easy to check that

$$\int_{a < |x| < b} p(x, u_n)u_n dx \to \int_{a < |x| < b} p(x, u)u dx,$$

$$\int_{a < |x| < b} P(x, u_n)dx \to \int_{a < |x| < b} P(x, u)dx,$$

as $n \to \infty$. Writing $\int_{\mathbf{R}^N} \Big(p(x, u_n)u_n - p(x, u)u\Big)dx$ as

$$\Big(\int_{|x| < R_1} + \int_{R_1 \leq |x| < R_2} + \int_{|x| \geq R_2}\Big)\Big(p(x, u_n)u_n - p(x, u)u\Big)dx.$$

It is easy to prove that each term of the above goes to zero. On the other hand, since $|u_n(x)| \leq c$ on Λ by the Strauss Inequality, we see that $\int_\Lambda \Big(p(x, u_n)v - p(x, u)v\Big) \to 0$ as $n \to \infty, \forall v \in E$. Furthermore, $\int_{\Lambda^c} \Big(p(x, u_n)v - p(x, u)v\Big)^2 dx \leq$

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$\int_{\Lambda^c} \frac{V_0(|u_n|+|u|)^2}{\delta_0} dx \leq c$. Therefore, by the Brezis-Lieb Theorem (cf. [6]), we see that $\int_{\Lambda^c} \left(p(x, u_n)v - p(x, u)v \right) dx \to 0$. Hence $\int_{\mathbf{R}^N} \left(p(x, u_n)v - p(x, u)v \right) dx \to 0$ as $n \to \infty, \forall v \in E$. Combining these, we have that $u_n \to u$ strongly in E and $G'_\lambda(u) = 0$. $\qquad \square$

Lemma 3.4. $\|u_{\lambda,\varepsilon}\| \to 0$ as $\varepsilon \to 0$ uniformly for $\lambda \in [1, 2]$.

Proof. First, we note that

$$b_\lambda \leq G_\lambda(s_0 e_0) \leq \int_\Omega s_0^2 \varepsilon^2 |\nabla e_0|^2 dx - \int_\Omega F(s_0 e_0) dx.$$

Furthermore, note $G'_\lambda(s_0 e_0)e_0 = 0$, we have that

$$\lambda \varepsilon^2 \int_\Omega |\nabla e_0|^2 = \int_\Omega \frac{f(s_0 e_0)e_0}{s_0} dx \geq \bar{e}_0^2 \int_{\Omega_0} \frac{f(s_0 e_0)}{s_0 e_0} dx.$$

Since $s_0 \in (0, 1)$ and $\lambda \in [1, 2]$, by Remark 3.1, we conclude that $s_0 \to 0$ as $\varepsilon \to 0$ uniformly for $\lambda \in [1, 2]$. Therefore,

$$0 < \rho_0 \leq b_\lambda \leq D_\lambda \leq \int_\Omega s_0^2 \varepsilon^2 |\nabla e_0|^2 dx \to 0, \quad \frac{b_\lambda}{\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0 \qquad (3.7)$$

uniformly for $\lambda \in [1, 2]$. Since, by (Z_3)-(Z_5), there exists an $R_0 > 0$ such that

$$
\begin{aligned}
b_\lambda &= G_\lambda(u_{\lambda,\varepsilon}) - \frac{1}{2}G'_\lambda(u_{\lambda,\varepsilon})u_{\lambda,\varepsilon} \\
&\geq \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \leq R_0\}} \left(\frac{1}{2}f(u_{\lambda,,\varepsilon})u_{\lambda,\varepsilon} - F_\lambda(u_{\lambda,\varepsilon}) \right) dx \\
&\quad + \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} \left(\frac{1}{2}f(u_{\lambda,\varepsilon})u_{\lambda,\varepsilon} - F(u_{\lambda,\varepsilon}) \right) dx \\
&\geq c \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \leq R_0\}} |u_{\lambda,\varepsilon}|^\mu dx + c \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} |u_{\lambda,\varepsilon}|^\alpha dx. \ (3.8)
\end{aligned}
$$

On the other hand, if we choose $q = \frac{2^*(1-\alpha/2)}{2^*-\alpha}$, then $q \in (0, 1)$ and

$$
\begin{aligned}
&\int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} |u_{\lambda,\varepsilon}|^2 dx \\
&\leq \left(\int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} |u_{\lambda,\varepsilon}|^\alpha dx \right)^{2(1-q)/\alpha} \left(\int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} |u_{\lambda,\varepsilon}|^{2^*} dx \right)^{2q/2^*} \\
&\leq c \left(b_\lambda \right)^{2(1-q)/\alpha} \|u_{\lambda,\varepsilon}\|^{2q}. \qquad (3.9)
\end{aligned}
$$

Condition (Z_5) implies that $f(t)t \leq c|t|^m$ for t small enough, where $m > \mu$. Therefore, by combining (3.8)-(3.9) and conditions (Z_1)-(Z_5), we have

$$
\begin{aligned}
\int_\Lambda f(u_{\lambda,\varepsilon})u_{\lambda,\varepsilon} dx &\leq c \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \geq R_0\}} |u_{\lambda,\varepsilon}|^2 dx + c \int_{\Lambda \cap \{|u_{\lambda,\varepsilon}| \leq R_0\}} |u_{\lambda,\varepsilon}|^m dx \\
&\leq c \left(b_\lambda \right)^{2(1-q)/\alpha} \|u_{\lambda,\varepsilon}\|^{2q} + c b_\lambda. \qquad (3.10)
\end{aligned}
$$

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Noting that $G'_\lambda(u_{\lambda,\varepsilon}) = 0$, we see that

$$\lambda \int_{\mathbf{R}^N} (\varepsilon^2 |\nabla u_{\lambda,\varepsilon}|^2 + V(x)u_{\lambda,\varepsilon}^2)dx \leq \int_\Lambda p(x, u_{\lambda,\varepsilon})u_{\lambda,\varepsilon}dx + \int_{\Lambda^c} \frac{V(x)u_{\lambda,\varepsilon}^2}{\delta_0}dx.$$

It follows that

$$\int_{\mathbf{R}^N} (\varepsilon^2 |\nabla u_{\lambda,\varepsilon}|^2 + (1 - \frac{1}{\delta_0})V(x)u_{\lambda,\varepsilon}^2)dx \leq \int_\Lambda p(x, u_{\lambda,\varepsilon})u_{\lambda,\varepsilon}dx = \int_\Lambda f(u_{\lambda,\varepsilon})u_{\lambda,\varepsilon}dx.$$
$$(3.11)$$

By (3.10)-(3.11), $\|u_{\lambda,\varepsilon}\|^2 \leq c\left(b_\lambda\right)^{2(1-q)/\alpha} \|u_{\lambda,\varepsilon}\|^{2q} + cb_\lambda \to 0$ as $\varepsilon \to 0$.
Hence,
$$\frac{\|u_{\lambda,\varepsilon}\|^2}{\varepsilon^2} \leq \left(\frac{b_\lambda}{\varepsilon^2}\right)^{2(1-q)/\alpha} \frac{\|u_{\lambda,\varepsilon}\|^{2q}}{\varepsilon^{2-4(1-q)/\alpha}} + \frac{cb_\lambda}{\varepsilon^2}.$$

By (3.5)-(3.7) and the fact that $2 > 2q > 2 - 4(1-q)/\alpha$, we have that $\|u_{\lambda,\varepsilon}\| \to 0$ as $\varepsilon \to 0$ uniformly for $\lambda \in [1, 2]$. $\qquad\square$

Lemma 3.6. *There exists an* $u_\varepsilon \neq 0$ *such that* $G'_1(u_\varepsilon) = 0, G_1(u_\varepsilon) \geq \rho_0$. *Particularly,* $\|u_\varepsilon\| \to 0$ *as* $\varepsilon \to 0$.

Proof. Since $\|u_{\lambda,\varepsilon}\| \to 0$ as $\varepsilon \to 0$ uniformly for λ, we let $\lambda \to 1$. Using the same arguments as that in Lemma 3.4, we may prove that $u_{\lambda,\varepsilon} \to u_\varepsilon$ strongly in E as $\lambda \to 1$ and $G'_1(u_\varepsilon) = 0, G_1(u_\varepsilon) \geq \rho_0$. By the same reasoning as that in Lemma 3.5, $\|u_\varepsilon\| \to 0$ as $\varepsilon \to 0$. $\qquad\square$

Proof of Theorem 1.1. We prove that u_ε is the solution of (1.3) when ε is small enough. For each $\varepsilon > 0$, by the Strauss Inequality, we have that $m_\varepsilon = \max_{\bar\Lambda} u_\varepsilon(x) \to 0$ as $\varepsilon \to 0$. Therefore, there is an $\varepsilon_0 > 0$ such that $m_\varepsilon < a_0$ for all $\varepsilon \in (0, \varepsilon_0)$, where a_0 is given in (3.1). Now

$$\int_{\mathbf{R}^N \setminus \bar\Lambda} \left(\varepsilon^2 |\nabla (u_\varepsilon - a_0)^+|^2 + V(x)u_\varepsilon(u_\varepsilon - a_0)^+\right)dx = \int_{\mathbf{R}^N \setminus \bar\Lambda} p(x, u_\varepsilon)(u_\varepsilon - a_0)^+dx,$$

where $w^+ = \max\{0, w\}$. Since by (3.2), $V(x)u_\varepsilon(u_\varepsilon - a_0)^+ - p(x, u_\varepsilon)(u_\varepsilon - a_0)^+ \geq 0, \quad \forall x \in \Lambda^c$, we have $\int_{\mathbf{R}^N \setminus \bar\Lambda} \varepsilon^2 |\nabla (u_\varepsilon - a_0)^+|^2 dx = 0$, it follows that $u_\varepsilon \leq a_0$ for all $x \in \mathbf{R}^N \setminus \bar\Lambda$, and hence $p(x, u_\varepsilon) = f(u_\varepsilon)$ for all $x \in \mathbf{R}^N \setminus \bar\Lambda$. Thus for all $\varepsilon \in (0, \varepsilon_0), u_\varepsilon$ is a solution of (1.3). $\qquad\square$

4 Sign-changing bound states for superlinear Schrödinger equations

In this section, we study the sign-changing bound states for the Schrödinger equation (1.4) and prove Theorem 1.2. Let $\beta = 1/\varepsilon^2$. Then (1.4) becomes as

$$-\Delta u + \beta V(x)u = \beta|u|^{p-2}u + \beta|u|^{q-2}u, \quad x \in \mathbf{R}^N. \tag{4.1}$$

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Let $H^1(\mathbf{R}^N)$ be the Sobolev space with the norm $\|u\|_{H^1(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + u^2) dx \right)^{1/2}$. The solutions of (4.1) correspond to the critical points of the $\mathbf{C}^2$-functional

$$G_\beta(u) = \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla u|^2 + \beta V(x) u^2) dx - \frac{\beta}{p} \int_{\mathbf{R}^N} |u|^p dx - \frac{\beta}{q} \int_{\mathbf{R}^N} |u|^q dx,$$

for $u \in E$, where $E := \{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} V(x) u^2 dx < \infty\}$, which is a Hilbert space equipped with the inner product $\langle u, v \rangle_E := \int_{\mathbf{R}^N} (\nabla u \nabla v + V(x) uv) dx$ and the associated norm $\|u\|_E := \langle u, v \rangle_E^{1/2}$. Let $\langle u, v \rangle_\beta := \int_{\mathbf{R}^N} (\nabla u \nabla v + \beta V(x) uv) dx$ and the associated norm $\|u\|_\beta := \langle u, v \rangle_\beta^{1/2}$. Then $\|\cdot\|_E$ and $\|\cdot\|_\beta$ are equivalent. Let $\Omega_n := \{x \in \mathbf{R}^N : |x| < n\}$ and $E_n := H_0^1(\Omega_n)$ with the inner product $\langle u, v \rangle = \int_{\Omega_n} (\nabla u \cdot \nabla v + V(x) uv) dx$ and the corresponding norm $\|u\| = \langle u, u \rangle^{1/2}$, which is equivalent to the standard norm of $H_0^1(\Omega_n)$. Consider the following problem approximating (4.1):

$$\begin{cases} -\Delta u + \beta V(x) u & = & \beta |u|^{p-2} u + \beta |u|^{q-2} u & \text{in } \Omega_n, \\ u & = & 0 & \text{on } \partial \Omega_n. \end{cases} \qquad (4.2)$$

The solutions of (4.2) correspond to the critical points of the $\mathbf{C}^2$-functional

$$G_{n,\beta}(u) = \frac{1}{2} \int_{\Omega_n} (|\nabla u|^2 + \beta V(x) u^2) dx - \frac{\beta}{p} \int_{\Omega_n} |u|^p dx - \frac{\beta}{q} \int_{\Omega_n} |u|^q dx,$$

for $u \in H_0^1(\Omega_n)$. Let $0 < \lambda_1(\Omega_n) < \lambda_2(\Omega_n) \leq \cdots \leq \lambda_k(\Omega_n) \leq \cdots$ be the eigenvalues counted with their multiplicities of the following Dirichlet zero-boundary value problem on Ω_n: $-\Delta u + \beta V(x) u = \lambda u$, and let $\{e_{n,j}\}_{j=1}^\infty \in H_0^1(\Omega_n)$ be the corresponding eigenfunctions with $\|e_{n,j}\|_2 = 1$. Let $\mathcal{P} := \{u \in E_n : u(x) \geq 0 \text{ for a.e. } x \in \Omega_n\}$. Then $\mathcal{P}$ is a positive cone of E_n. We define $\mathcal{D}_{\mu_0} := \{u \in E_n : \text{dist}(u, \mathcal{P}) < \mu_0\}$. Similar to [5, 33, 38], (B_1) is satisfied for G_n. We choose $\|\cdot\|_* = \|\cdot\|_p$, then it is easy to check that (B_2)-(B_4) are satisfied for $G_{n,\beta}$. Let $Z^\perp := \text{span}\{e_{n,1}, \cdots, e_{n,k-1}\}, k \geq 2$, then $E_n := Z^\perp \oplus Z$, where $Z := (Z^\perp)^\perp$. By Theorem 2.2, we find a sequence $u_{n,k,\beta} \in E_n$ such that

(1) for each $k \geq 2$, $\pm u_{n,k,\beta}$ are sign-changing solutions of (4.2);

(2) the augmented Morse index of $u_{n,k,\beta}$ is $\geq k$;

(3) $0 < G_{n,\beta}(u_{n,k,\beta}) \leq \inf_{\substack{Y \subset E_n, \\ k \leq \dim Y < \infty}} \sup_Y G_{n,\beta}$.

Note that $\inf\{\int_{\mathbf{R}^N} |\nabla u|^2 dx : u \in \mathbf{C}_0^\infty(\mathbf{R}^N), \|u\|_q = 1\} = 0$. For any $k \geq 2$, we choose $w_j \in \mathbf{C}_0^\infty(\mathbf{R}^N)$ with $\|w_j\|_q = 1$, $\int_{\mathbf{R}^N} |\nabla w_j|^2 dx < 1$ for $j = 1, ..., k$. We may want $\text{supp} \, w_j \cap \text{supp} w_i = \emptyset$ for $i \neq j$ and $\text{supp} \, w_j \subset \Omega_{n_k}$ for all $j = 1, ..., k$. For $0 < s < \frac{2(p-q)}{(p-2)(2N+2q-Nq)}$, we set $\theta_j(x) = w_j(\beta^s x)$ and $Y_k := \text{span}\{\theta_1, ..., \theta_k\}$. Then there exists a constant n_k depending on k such that

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$Y_k \subset E_n$ for $n \geq n_k$ (we assume that $\beta \geq 1$). For any $u = \sum_{i=1}^{k} c_i \theta_i \in Y_k$, we have that $G_{n,\beta}(u) = \sum_{i=1}^{k} G_{n,\beta}(c_i \theta_i)$ if $n \geq n_k$. Recall that $V(0) = 0$, there is a β_k (depending on k) such that if we consider $\beta \geq \beta_k$, we have that $V(\beta^{-s} x) \leq \|w_i\|_2^{-2}$ for all $x \in \Omega_{n_k}, i = 1, ..., k$. Therefore,

$$
\begin{aligned}
&G_{n,\beta}(t\theta_i) \\
&= \frac{t^2}{2} \left(\beta^{2s-sN} \int_{\Omega_n} |\nabla w_i|^2 dx + \beta^{1-sN} \int_{\Omega_n} V(\beta^{-s} x)|w_i|^2 dx \right) \\
&\quad - \frac{t^p \beta^{1-sN}}{p} \int_{\Omega_n} |w_i|^p dx - \frac{t^q \beta^{1-sN}}{q} \int_{\Omega_n} |w_i|^q dx \\
&\leq \frac{q-2}{2q} \left(\beta^{2s-sN} + \beta^{1-sN} \right)^{q/(q-2)} \beta^{2(sN-1)/(q-2)}.
\end{aligned}
$$

Therefore, by item (3) above, we have

Lemma 4.1. *We have*

$$
0 < G_{n,\beta}(u_{n,k,\beta}) \leq k \frac{q-2}{2q} \left(\beta^{2s-sN} + \beta^{1-sN} \right)^{q/(q-2)} \beta^{2(sN-1)/(q-2)}.
$$

Therefore, we may assume that $G_{n,\beta}(u_{n,k,\beta}) \to c_{k,\beta} \geq 0$ as $n \to \infty$. On the other hand, the number of nodal domains of $u_{n,k,\beta}$ is $\leq k$.

Proof. We just estimate the number of nodal domains of $u_{n,k,\beta}$. If the number of nodal domains of $u_{n,k,\beta}$ is $> k$, we denote such domains by $\Omega_{n,1}, \cdots, \Omega_{n,k+1}$. Let $\theta_i(x) = u_{n,k,\beta}(x)$ if $x \in \Omega_{n,i}$ and $\theta_i(x) = 0$ otherwise, then $\theta_i \in E_n$. Let $v_{n,k,\beta} := u_{n,k,\beta} - \sum_{i=1}^{k} \theta_i$, we have that $0 < G_{n,\beta}(u_{n,k,\beta}) = G_{n,\beta}(v_{n,k,\beta}) + \sum_{i=1}^{k} G_{n,\beta}(\theta_i)$ and $G_{n,\beta}(v_{n,k,\beta}) > 0$ (up to an appropriate choice of $\Omega_{n,j}$). Note that $\langle G'_{n,\beta}(\theta_i), \theta_i \rangle = \langle G'_{n,\beta}(u_{n,k,\beta}), \theta_i \rangle = 0$ and that $G_{n,\beta}(\theta_i) = \sup_{t \in \mathbb{R}} G_{n,\beta}(t\theta_i)$. Let $X := span\{\theta_1, \cdots, \theta_k\}$. We obtain $G_{n,\beta}(u_{n,k,\beta}) \leq \inf_{\substack{Y \subset E_n, \\ k \leq \dim Y < \infty}} \sup_Y G_{n,\beta} \leq$

$\sup_X G_{n,\beta} = \sum_{i=1}^{k} G_{n,\beta}(\theta_i)$. But the last term is $G_{n,\beta}(u_{n,k,\beta}) - G_{n,\beta}(v_{n,k,\beta})$, it follows a contradiction. $\qquad \square$

Proof of Theorem 1.2. Due to (V_2), $V(x)$ has a positive lower-bound outside a big ball, together with Lemma 4.1, it is a routine to show that $\{u_{n,k,\beta}\}_{n=1}^{\infty}$ is bounded in E. We assume $u_{n,k,\beta} \to u_{k,\beta}^*$ weakly in E, in $L^p(\mathbf{R}^N)$ and in $L^q(\mathbf{R}^N)$; strongly in $L_{loc}^p(\mathbf{R}^N)$ and $L_{loc}^q(\mathbf{R}^N)$ for $2 < q < p < 2^*$; $u_{n,k,\beta}(x) \to u_{k,\beta}^*(x)$ for a.e. $x \in \mathbf{R}^N$. Let $\xi(t) : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\xi(t) = 1$ if $t \leq 1$ and $\xi(t) = 0$ if $t \geq 2$. Let $\hat{u}_{n,k,\beta}(x) = \xi(3|x|/n)u_{k,\beta}^*(x)$.

Combining Brézis-Lieb Lemma (see [6]):

$$
\begin{aligned}
&G_{n,\beta}\left(u_{n,k,\beta} - \hat{u}_{n,k,\beta}\right) \\
&= G_{n,\beta}\left(u_{n,k,\beta}\right) - G_{n,\beta}(\hat{u}_{n,k,\beta}) \\
&\quad - \int_{\mathbf{R}^N} \left(\nabla(u_{n,k,\beta} - \hat{u}_{n,k,\beta})\nabla\hat{u}_{n,k,\beta}dx + \beta V(x)(u_{n,k,\beta} - \hat{u}_{n,k,\beta})\hat{u}_{n,k,\beta}\right)dx \\
&\quad + \frac{\beta}{p}\int_{\mathbf{R}^N}\left(|u_{n,k,\beta}|^p - |u_{n,k,\beta} - \hat{u}_{n,k,\beta}|^p - |\hat{u}_{n,k,\beta}|^p\right)dx \\
&\quad + \frac{\beta}{q}\int_{\mathbf{R}^N}\left(|u_{n,k,\beta}|^q - |u_{n,k,\beta} - \hat{u}_{n,k,\beta}|^q - |\hat{u}_{n,k,\beta}|^q\right)dx \\
&\to c_{k,\beta} - G_\beta(u_{n,\beta}^*)
\end{aligned}
$$

as $n \to \infty$. Further, it is easy to check that $\langle G'_{n,\beta}(u_{n,k,\beta} - \hat{u}_{n,k,\beta}), (u_{n,k,\beta} - \hat{u}_{n,k,\beta})\rangle \to 0$ as $n \to \infty$. Note that

$$
\begin{aligned}
&G_{n,\beta}\left(u_{n,k,\beta} - \hat{u}_{n,k,\beta}\right) - \frac{1}{2}\langle G'_{n,\beta}(u_{n,k,\beta} - \hat{u}_{n,k,\beta}), (u_{n,k,\beta} - \hat{u}_{n,k,\beta})\rangle \\
&\geq (\frac{1}{2} - \frac{1}{p})\beta\int_{\Omega_n} |u_{n,k,\beta} - \hat{u}_{n,k,\beta}|^p dx.
\end{aligned}
$$

It follows that

$$
\|u_{n,k,\beta} - \hat{u}_{n,k,\beta}\|_p \leq \left(\frac{2p(c_{k,\beta} - G_\beta(u_{k,\beta}^*))}{\beta(p-2)} + o(1)\right)^{1/p}. \tag{4.3}
$$

We claim that there is a $\beta_k^* > \beta_k$ such that for all $\beta > \beta_k^*$ we have that

$$
u_{n,k,\beta} \to u_{k,\beta}^* \quad \text{strongly in } E, \ n \to \infty. \tag{4.4}
$$

Otherwise, $u_{n,k,\beta} \nrightarrow u_{k,\beta}^*$ in E for some $\beta \to \infty$. Thus, $\liminf_{n\to\infty}\|u_{n,k,\beta} - \hat{u}_{n,k,\beta}\| > 0$. Recall that $u_{n,k,\beta} - \hat{u}_{n,k,\beta} \to 0$ in L^2_{loc}. Since there is a constant $C(p)$ (depending on p only) such that

$$
\|u_{n,k,\beta} - \hat{u}_{n,k,\beta}\|_p^2 \leq C(p)\int_{\mathbf{R}^N}\left(|\nabla(u_{n,k,\beta} - \hat{u}_{n,k,\beta})|^2 + (u_{n,k,\beta} - \hat{u}_{n,k,\beta})^2\right)dx.
$$

Combining (V_2), we deduce that

$$
\|u_{n,k,\beta} - \hat{u}_{n,k,\beta}\|_p^2 \leq C(p,V)\beta\int_{\mathbf{R}^N} |u_{n,k,\beta} - \hat{u}_{n,k,\beta}|^p dx + o(1), \tag{4.5}
$$

where and in the sequel, we denote by $C(p,V)$ the constant depending on p and $V(x)$ only. Thus, $o(1) + \|u_{n,k,\beta} - \hat{u}_{n,k,\beta}\|^p \geq (\beta C(p,V))^{-1/(p-2)}$ as $n \to \infty$. By (4.3), we have

$$
(\beta C(p,V))^{-1/(p-2)} \leq \left(\frac{2p(c_{k,\beta} - G_\beta(u_{k,\beta}^*))}{\beta(p-2)}\right)^{1/p} + o(1) \tag{4.6}
$$

as $n \to \infty$. Further, we observe that $G_\beta(u^*_{k,\beta}) \geq 0$, by Lemma 4.1,

$$
\begin{aligned}
& C(p,V)\beta^{1-p/(p-2)} \\
& \leq c_{k,\beta} - G_\beta(u^*_{k,\beta}) \\
& \leq k\frac{q-2}{2q}\left(\beta^{2s-sN} + \beta^{1-sN}\right)^{q/(q-2)}\beta^{2(sN-1)/(q-2)}.
\end{aligned}
$$

This is impossible if β large enough. So, our claim in (4.4) is true, that is, there is a $\beta^*_k > \beta_k$ large enough, such that for $\beta > \beta^*_k$, $u_{n,k,\beta} \to u^*_{k,\beta}$ strongly in E. Then $\pm u^*_{k,\beta}$ are solutions of (4.1) such that

$$
G_\beta(u^*_{k,\beta}) \leq k\frac{q-2}{2q}\left(\beta^{2s-sN} + \beta^{1-sN}\right)^{q/(q-2)}\beta^{2(sN-1)/(q-2)}. \tag{4.7}
$$

The number of nodal domains of $u^*_{k,\beta}$ is $\leq k$; the augmented Morse index of $u^*_{k,\beta}$ is $\geq k$. We claim that $u^*_{k,\beta}$ is still sign-changing. Indeed, since $\pm u_{n,k,\beta}$ are sign-changing solutions of (4.2), we have

$$
\int_{\Omega_n}(|\nabla u^\pm_{n,k,\beta}|^2 + \beta V(x)(u^\pm_{n,k,\beta})^2)dx = \beta\int_{\Omega_n}|u^\pm_{n,k,\beta}|^p dx + \beta\int_{\Omega_n}|u^\pm_{n,k,\beta}|^q dx
$$

$$
\leq C_1(\|u^\pm_{n,k,\beta}\|^p_{H^1(\mathbb{R}^N)} + \|u^\pm_{n,k,\beta}\|^q_{H^1(\mathbb{R}^N)}), \tag{4.8}
$$

where $w^\pm = \max\{\pm w, 0\}$ and $C_1 > 0$ is a constant independent of n, k. When n large enough, the first integral in (4.8) is $\geq C_2\|u^\pm_{n,k,\beta}\|^2_{H^1(\mathbb{R}^N)}$ for a constant $C_2 > 0$ independent of n and k. Hence, we may conclude that $u^*_{k,\beta}$ must be sign-changing. Finally, since the augmented Morse index of $u_{n,k,\beta}$ is $\geq k$, we let $\{\mu_{n,j}\}_{j=1}^\infty$ be the sequence of the eigenvalues (repeated with their multiplicities) of the operator $w \mapsto (-\Delta w + V(x)w - (p-1)|u_{n,k,\beta}|^{p-2}w - (q-1)|u_{n,k,\beta}|^{q-2}w)$ with Dirichlet zero-boundary condition on $\partial\Omega_n$, then the cardinality $\sharp\{j \in \mathbb{N} : \mu_{n,j} \leq 0\} \geq k$. On the other hand, let $W(x) = a(x) - (p-1)|u_{n,k,\beta}|^{p-2} - (q-1)|u_{n,k,\beta}|^{q-2}$ in Ω_n and $W(x) = 0$ outside Ω_n, by [31, Theorem 3], there is a constant C_N depending on N only such that $\sharp\{j \in \mathbb{N} : \mu_{n,j} \leq 0\} \leq C_N\int_{\mathbb{R}^N}\left(W^-(x)\right)^{N/2}dx$, where $W^-(x) = (p-1)|u_{n,k,\beta}|^{p-2} + (q-1)|u_{n,k,\beta}|^{q-2} - V(x)$ if $(p-1)|u_{n,k,\beta}|^{p-2} + (q-1)|u_{n,k,\beta}|^{q-2} - V(x) \geq 0$ and $W^-(x) = 0$ otherwise. Hence,

$$
k \leq C_N\left((p-1)^{N/2}\int_{\Omega_n}|u_{n,k,\beta}|^{N(p-2)/2}dx + (q-1)^{N/2}\int_{\Omega_n}|u_{n,k,\beta}|^{N(q-2)/2}dx\right)
$$

for all $n \geq 1$. If $q \geq 2 + 4/N$, by the Sobolev embedding theorem, we see that $\|u^*_{k,\beta}\|^{N(p-2)/2}_{H^1(\mathbb{R}^N)} + \|u^*_{k,\beta}\|^{N(q-2)/2}_{H^1(\mathbb{R}^N)} \geq C(p,q,N)k$, where $C(p,q,N) > 0$ is a constant depending on p, q and N only. It is easy to check that $\|u^*_{k,\beta}\|_{H^1(\mathbb{R}^N)} \geq C(p,q,N)k^{2/(N(p-2))}$. Note that $u^*_{k,\beta} := u^*_{k,\varepsilon}$ is also a solution of (1.4). Further, since $G_\beta(u) = \beta H_\varepsilon(u)$, by combining (4.7) and noting that s may be chosen arbitrarily small, we see that $H_\varepsilon(u^*_{k,\varepsilon}) \leq k\frac{q-2}{q}2^{\frac{q}{q-2}}\varepsilon^{-\frac{4q}{q-2}}$. This finishes the proof of Theorem 1.2. $\qquad\square$

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