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APPROXIMATE SOLUTIONS IN LINEAR VISCOELASTICITY

BY

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Preface

This report covers work carried out for the Lawrence Radiation Laboratory, Livermore, California, under the terms of Purchase Order No. 7010600. The investigation was conducted by T. Y. Chang under the general supervision of K. S. Pister and R. L. Taylor, Professor and Assistant Professor of Civil Engineering, respectively.

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I. Introduction.

In the past few years linear viscoelastic analysis has drawn much attention due to the increasing use of plastic bonded elastomers, plastics and polymers for structural components; moreover, materials such as concrete, asphalt, soil and metals may also exhibit viscoelastic behavior. In order to use materials to their maximum capacity, the normal elastic analysis appears insufficient to provide adequate design information. Therefore, there is a definite need for the development of solution methods for viscoelastic analysis.

In the state of uniform temperature, a large class of linear viscoelastic problems can be solved by utilizing the elastic-viscoelastic analogy with the help of integral transform [1]. By the application of a direct analogy an associated elastic solution can lead to a viscoelastic solution in the transformed space. In many cases it is difficult to invert the transformed solution back to the real time space due to the complicated expressions of the viscoelastic responses. Schapery [2] has suggested two approximate methods of inversion based on the use of the Laplace transform. A further investigation of the approximate inversion methods has been made by Cost [3]. Nevertheless, the transform technique is not practical for problems with irregular boundaries and problems with the inclusion of nonuniform temperature effects. For bodies with arbitrary boundary shapes, it is impossible to obtain the analytical elastic solutions, (although Schapery's direct method of inversion may be applied to find the numerical solutions, yet the method itself is restricted to particular materials). The inclusion of a variable temperature field, in particular, complicates the analysis.

The temperature dependence of viscoelastic materials may be characterized by the time-temperature equivalence hypothesis originally proposed by Leaderman [4]. Materials which possess a single time-temperature function have been termed "thermorheologically simple" by Schwarzl and Staverman [5]. Even with this simplification, very few problems have been solved for variable temperature fields [6,7]. Various approximate methods have been suggested lately. Lianis and Valanis [8] proposed a numerical method in which the time of integration (based on an integral stress-strain law) is subdived into intervals so that the reduced time can be linearized, the solution in each time interval can then be extended to the next time interval by applying the Laplace transform in connection with the convolution theorem. Hilton and Clements [9] used a similar feature to establish an approximate elastic viscoelastic analogy for nonuniform temperature field. Direct solution by numerical integration of the stress-strain relations has been obtained by Lee and Rogers [10], where the material properties can be expressed graphically and input numerically to the computer. Schapery [11] proposed a method of successive approximations based on the approximate Laplace transform inversion.

In this dissertation a numerical method is developed for the solution of linear viscoelastic media with the inclusion of deformations of mechanical and thermal origin. The work is based on a finite element method for spatial reduction and a finite difference method for time integration. The finite element method has been widely used for elastic analysis [12,13,14]. King [15] has applied the technique to study the creep and aging effects of concrete materials.

The framework for the numerical method is set up in the early part of this dissertation. The formulations are valid for both isothermal and non-isothermal cases. Two numerical examples are presented to illustrate the method.

II. General Formulations.

In preparation for the development to follow, the basic field equations will be stated briefly to define the type of boundary value problems to be considered. The principle of virtual displacements for linear viscoelasticity is formulated for the purpose of finding the approximate force-displacement relations in the finite element analysis.

1. Field Equations for Viscoelastic Solids.

In the absence of thermo-mechanical coupling effects, the fundamental system of field equations governing the quasi-static linear theory of thermo-rheologically simple viscoelastic solids consists of linearized strain-displacement relations, equations of equilibrium and viscoelastic stress-strain laws.

Let $u_i(x,t)$, $\epsilon_{ij}(x,t)$, $\sigma_{ij}(x,t)$ be the cartesian components of displacement, infinitesimal strain and stress, respectively, at a material point $x = (x_1, x_2, x_3)$ and at the time t, and let $T(x,t)$ be the temperature field.

The strain-displacement relations are

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} \right) . \tag{2.1}$$

The equations of equilibrium are

$$\sigma_{ij,j} + f_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji} , \tag{2.2}$$

where f_i denotes the body forces.

The linear isotropic stress-strain relations in the form of relaxation integral law appears as

$$S_{ij}(x,t) = \int_{-\infty}^{t} G_1(\xi-\xi')\frac{\partial}{\partial t'} e_{ij}(x,t')\,dt'$$
$$\sigma_{kk}(x,t) = \int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\epsilon_{kk}(x,t') - 3\alpha_0\Theta(x,t')\right]dt' \quad , \qquad (2.3)$$

where S_{ij}, e_{ij} are deviatoric components of stress and strain, ϵ_{kk} the dilatation and σ_{kk} the hydrostatic stress,

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk} \quad , \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$
$$\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \quad , \quad \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} \, . \qquad (2.4)$$

The pseudo-temperature function $\Theta(x,t)$ is defined by

$$\Theta(x,t) = \frac{1}{\alpha_0}\int_{T_0}^{T(x,t)} \alpha(T')\,dT' \quad , \quad \alpha_0 = \alpha(T_0) \, . \qquad (2.5)$$

If the thermal expansion coefficient α is assumed to be temperature independent, then

$$\Theta(x,t) = T(x,t) - T_0 \quad , \qquad (2.6)$$

where $T(x,t)$ is the solution of the Fourier heat conduction equation and T_0 is the reference temperature.

Finally, the argument ξ is defined as the "reduced time"

$$\xi = \xi(x,t) = \int_0^t \phi[T(x,t')]\,dt' \quad , \quad \xi' = \xi(x,t') \quad , \qquad (2.7)$$

while $\phi(T)$ represents the "shift function" which characterizes the temperature dependence of thermo-rheologically simple materials and it has the properties:

$$\phi(T_0) = 1 \quad , \qquad \phi(T) > 1 \quad , \qquad \frac{d\phi}{dT} > 0 \quad ,$$

so that ϕ is a monotonically increasing function.

In addition, appropriate boundary conditions must be included for a properly posed boundary value problem.

2. The Principle of Virtual Displacements for the Linear Theory of Viscoelasticity.

Consider a body V in static equilibrium under the action of specified body forces f_i and surface tractions $\bar{t}_i$ which is applied on the boundary B_σ. Let δu_i be a system of virtual displacements applied on V, δu_i are regarded as functions of time and space, not to be identified with the actual displacements, and also

$$\delta \epsilon_{ij} = \frac{1}{2} (\delta u_{i,j} + \delta u_{j,i}) \tag{2.8}$$

The principle of virtual displacements for nonconservative system has been formulated in general form [16, 17]. For the case of quasi-static linear viscoelasticity, it can be expressed in a convolution integral form [18]

$$\int_V \sigma_{ij} * \delta \epsilon_{ij} \, dV = \int_{B_\sigma} \bar{t}_i * \delta u_i \, dS + \int_V f_i * \delta u_i \, dV \tag{2.9}$$

where

$$f * g = \int_{-\infty}^{t} f(t-t') \frac{d}{dt'} g(t') \, dt'$$

The proof of (2.9) is identically the same as in elasticity [16].

III. Method of Solution.

As seen in the previous section, the stress-strain relations for the thermoviscoelastic problems are formulated in terms of a reduced time. The use of a reduced time simplifies the stress-strain relations; however, it does not render the problems more tractable, since there is no longer an exact elastic-viscoelastic analogy when the temperature is a function of time and space [6].

In order to attack a typical problem in the presence of viscoelastic action and thermal variations, a method will be developed by using a finite element technique for spatial reduction and a finite difference technique for time integration.

1. Spatial Reduction.

The finite element method has been used extensively and successfully for the solution of elastic bodies. The method can be readily applied to problems of arbitrary boundaries and inhomogeneous materials.

The procedure for the analysis contained herein is based on the direct stiffness method, in which a continuous body is replaced by an assemblage of discrete elements interconnected along element interfaces. Two classes of viscoelastic problems, one-dimensional and two-dimensional will be treated in this dissertation. For one-dimensional problems, annular elements will be used to fulfill cylindrical or spherical symmetry. While for two-dimensional problems, triangular elements will be used to fit arbitrary boundaries.

1.1 One-Dimensional Problems——— A thick-walled cylinder subjected to axisymmetric temperature field and boundary conditions, and a hollow sphere subjected to point symmetric temperature field and boundary conditions are considered.

The cylinder or the sphere is divided into a number of annular elements as shown in Fig. (3-1). Within each element the approximating temperature is assumed to be uniform so that the material property is spatially independent, however, the deformation due to temperature change will be included. Starting from this approximation, the expressions for element stresses and nodal forces (on the boundaries of the element) may be derived in terms of nodal displacements through the fundamental set of field equations. By the continuity condition of radial forces at each pair of elements, a system of linear integral equations for the nodal displacements is established.

1.2 Two-Dimensional Problems——— For the case of plane stress or plane strain, a continuous body is replaced by an assemblage of triangular elements, which are interconnected along element interfaces. Linear displacement variation is assumed over the element, therefore, compatibility can be maintained by matching the two displacement components at each nodal point (e.g. i, j or k in Fig. (3-2)). A continuous stress problem is thus reduced to one of a finite number of unknown nodal displacements.

Due to the assumed displacement pattern the sides of the element remain straight before and after deformation; consequently the compatibility condition for the complete system is fulfilled. Furthermore, for

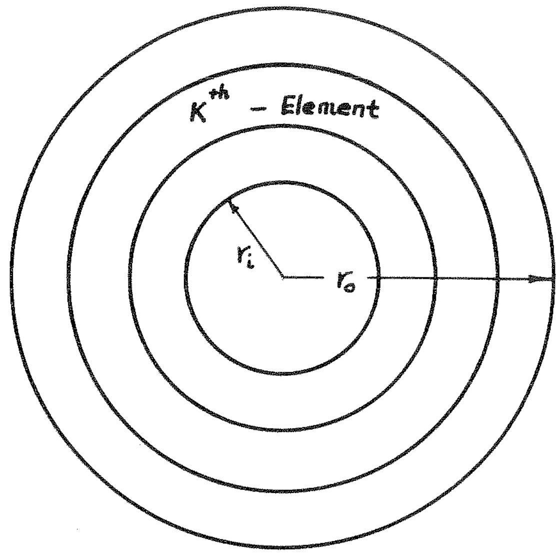

FIG.(3-1) ANNULAR ELEMENTS FOR CYLINDER

OR SPHERE

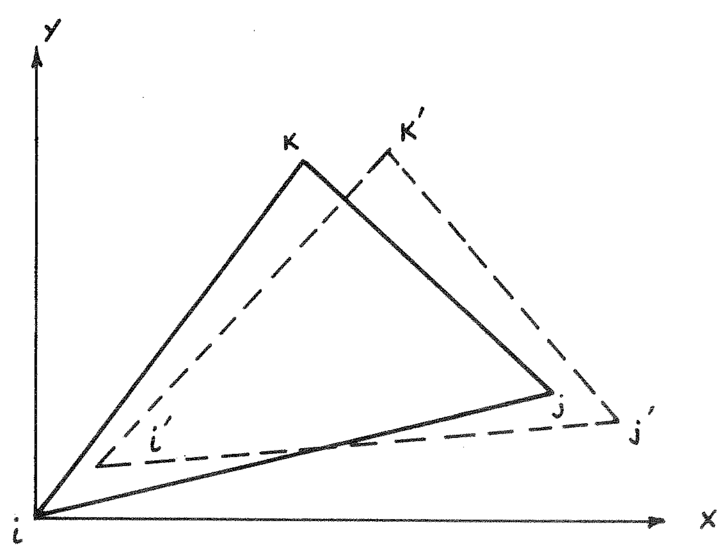

FIG.(3-2) DEFORMATION OF A TRIANGULAR

ELEMENT

bodies subjected to temperature gradients, the temperature distribution is assumed to be uniform inside each element; hence the thermally affected material properties will have no spatial dependence. Based on these approximations the relationship between the nodal displacements may be derived. From equilibrium condition of the entire body, a system of linear integral equations (for an integral constitutive law) for the nodal displacements may be established.

2. Time Integration.

In both one-dimensional analysis and two-dimensional analysis the spatial reduction of a continuous body by finite elements results in a system of linear integral equations for nodal displacements. The step-forward finite difference method will be used to reduce the integral equations to a set of linear algebraic equations, from which solutions for nodal displacements can be obtained with the aid of a computer. Once nodal displacements at different time steps are known, the stresses may be found by numerical integration of the stress-displacement relations for each element.

IV. Finite Element Method for Spatial Reduction.

In this section the element properties for a cylinder, a sphere and for two-dimensional plane strain (or plane stress) will be developed. A constant temperature field is assumed in each element to simplify the analysis.

For cylindrical and spherical elements the displacement patterns are found by satisfying the field equations and approximate stress-strain laws. For a triangular element a linear displacement variation is assumed. By use of the principle of virtual displacements, the expressions of element forces (equivalent nodal forces) in terms of displacements are derived. Finally the nodal force-displacement relationship for the complete system is constructed from the equilibrium conditions of the entire body.

1. Cylindrical Element.

Consider a cylindrical element of inner radius a and outer radius b under the influence of a temperature field of the form

$$T = T(r,t) \tag{4.1}$$

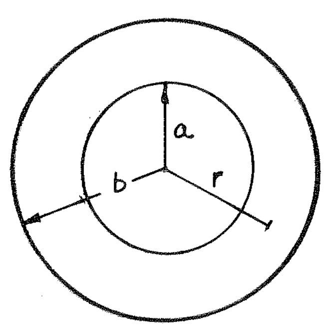

Fig. (4-1) A Typical Annular Element.

For a state of axisymmetric plane strain, the displacement components in cylindrical coordinates reduce to

$$u_r = u(r,t) \quad , \quad u_\theta = u_z = 0 \quad , \tag{4.2}$$

the strain-displacement relations yield

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u}{\partial r} \quad , \quad \epsilon_{\theta\theta} = \frac{u}{r} \\ \epsilon_{zz} &= \epsilon_{r\theta} = \epsilon_{\theta z} = \epsilon_{zr} = 0 \quad , \end{aligned} \tag{4.3}$$

and the stress tensor also simplifies so that σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are the only non-zero quantities.

The temperature distribution is assumed to be uniform in each element in order to remove the spatial difficulty in solving the equilibrium equation. The constant temperature is taken to be that at the mean radius of the element. Due to this approximation, the reduced time ξ for the element becomes

$$\xi = \xi(t) = \int_0^t \phi[T(r_m, t')]\, dt'$$

and

$$r_m = \frac{1}{2}(a+b) \quad . \tag{4.4}$$

The approximate relaxation integral law for the element in terms of stress and displacement has the form

$$\begin{aligned} \sigma_{rr}(r,t) &= \frac{1}{3}\int_{-\infty}^{t} G_1(\xi-\xi')\frac{\partial}{\partial t'}\left[2\frac{\partial u}{\partial r} - \frac{u}{r}\right]dt' + \frac{1}{3}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r} + \frac{u}{r} - 3\alpha_0\theta\right]dt' \\ \sigma_{\theta\theta}(r,t) &= \frac{1}{3}\int_{-\infty}^{t} G_1(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{2u}{r} - \frac{\partial u}{\partial r}\right]dt' + \frac{1}{3}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r} + \frac{u}{r} - 3\alpha_0\theta\right]dt' \quad . \end{aligned} \tag{4.5}$$

From the condition of axisymmetry, the only non-vanishing equation of equilibrium is

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad . \tag{4.6}$$

Rewriting the equation of equilibrium in terms of the displacement, it is found for the element

$$\int_{-\infty}^{t}\left[2G_1(\xi-\xi')+G_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right]\right\}dt' = 3\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial}{\partial r}(\alpha_0\Theta)\right]dt' \quad (4.7)$$

Assuming ξ has a unique inverse function so that one can write, $t=g(\xi)$, and also $t'=g(\xi')$, and changing the variable t' to ξ' in the above equation, yields

$$\int_{-\infty}^{\xi}\left[2G_1(\xi-\xi')+G_2(\xi-\xi')\right]\frac{\partial}{\partial \xi'}\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}(r\hat{u})\right]\right\}d\xi' = 3\int_{-\infty}^{\xi} G_2(\xi-\xi')\frac{\partial}{\partial \xi'}\left[\frac{\partial}{\partial r}(\alpha_0\hat{\Theta})\right]d\xi' , \quad (4.8)$$

where $\hat{u}=\hat{u}(x,\xi)=u(x,t)$, and likewise for $\hat{\Theta}$.

Bearing in mind that ξ is a function of t only, apply the Laplace transform on (4.8) with respect to ξ to obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}^*)\right] = S^*(p)\cdot p\cdot\frac{\partial}{\partial r}(\alpha_0\hat{\Theta}^*) , \quad (4.9)$$

where ( )* represents the transformed expression of the function,

p is the transform parameter of ξ, and

$$S^*(p) = \frac{3G_2^*(p)}{p\left[2G_1^*(p)+G_2^*(p)\right]} \quad (4.10)$$

For elasticity,

$$S^*(p) = \frac{1+\nu}{1-\nu}\,\frac{1}{p} ,$$

ν is Poisson's ratio.

Integrating (4.9) directly,

$$\hat{u}^*(r,p) = \hat{A}^*(p)\,r + \frac{\hat{B}^*(p)}{r} + S^*(p)\cdot p\cdot\frac{1}{r}\int_a^r \alpha_0\hat{\Theta}(\rho,p)\rho\,d\rho . \quad (4.11)$$

Taking the inverse transform of (4.11) and changing the variable ξ back to t for the functions $\hat{A}$, $\hat{B}$ and $\alpha_0\hat{\Theta}$, yields

$$u(r,t) = A(t)\,r + \frac{B(t)}{r} + \int_{-\infty}^{t} S(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{1}{r}\int_{a}^{r}\alpha_0\Theta(\rho,t')\rho\, d\rho\right]dt' \quad . \tag{4.12}$$

The integration constants A and B are specified in terms of nodal displacements.

Let

$$\begin{aligned} u(a,t) &= u_a(t) \quad , \quad r = a \\ u(b,t) &= u_b(t) \quad , \quad r = b \quad , \end{aligned} \tag{4.13}$$

where $u_a(t)$ and $u_b(t)$ are defined as the nodal displacements of the element.

Then, from (4.12) and (4.13), one obtains

$$\begin{aligned} A(t) &= \frac{1}{b^2-a^2}\left\{(bu_b - au_a) - \int_{-\infty}^{t} S(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\Theta(r,t')\,r\,dr\right]dt'\right\} \\ B(t) &= \frac{-ab}{b^2-a^2}(au_b - bu_a) + \frac{a^2}{b^2-a^2}\int_{-\infty}^{t} S(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\Theta(r,t')\,r\,dr\right]dt' \quad . \end{aligned} \tag{4.14}$$

By use of (4.5), (4.12) and (4.14), the element stress-nodal displacement relations may be written in matrix notation as,

$$\underset{\sim}{\sigma}(r,t) = \int_{-\infty}^{t} \underset{\approx}{K}_1(r,\xi-\xi')\frac{d}{dt'}\,\underset{\sim}{u}(t')\,dt' + \underset{\sim}{\sigma}_T(r,t) \quad , \tag{4.15}$$

where

$$\underset{\sim}{\sigma}(r,t) = \begin{bmatrix} \sigma_{rr}(r,t) \\ \sigma_{\theta\theta}(r,t) \end{bmatrix} \quad , \quad \underset{\sim}{u}(r,t) = \begin{bmatrix} u_a(t) \\ u_b(t) \end{bmatrix} \tag{4.16}$$

$$\underset{\approx}{K}_1(r,\xi-\xi') = \frac{1}{b^2-a^2}\left[\begin{array}{c|c} -aF(\xi-\xi') - \frac{ab^2}{r^2}G_1(\xi-\xi') & bF(\xi-\xi') + \frac{a^2b}{r^2}G_1(\xi-\xi') \\ \hline -aF(\xi-\xi') + \frac{ab^2}{r^2}G_1(\xi-\xi') & bF(\xi-\xi') - \frac{a^2b}{r^2}G_1(\xi-\xi') \end{array}\right] , \tag{4.17}$$

and the thermal expansion effect is

$$
\underset{\sim}{\sigma}(r,t) = \left[\begin{array}{c}
-\frac{1}{b^2-a^2}\int_{-\infty}^{t}\left[S_1(\xi-\xi')+\frac{a^2}{r^2}S_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\,\theta(r,t')\,r\,dr\right]dt' \\
-\frac{1}{r^2}\int_{-\infty}^{t} S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^r \alpha_0\,\theta(\rho,t')\,\rho\,d\rho\right]dt' \\
\hdashline
-\frac{1}{b^2-a^2}\int_{-\infty}^{t}\left[S_1(\xi-\xi')-\frac{a^2}{r^2}S_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\,\theta(r,t')\,r\,dr\right]dt' \\
+\frac{1}{r^2}\int_{-\infty}^{t} S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^r \alpha_0\,\theta(\rho,t')\,\rho\,d\rho\right]dt' - \int_{-\infty}^{t} S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\alpha_0\,\theta(r,t')\right]dt'
\end{array}\right].
\tag{4.18}
$$

In the above,

$$
\begin{aligned}
F(\xi) &= \frac{1}{3}\left[G_1(\xi) + 2G_2(\xi)\right] \\
S_1(\xi) &= \mathcal{L}^{-1}\left\{\frac{G_2^*(p)\left[G_1^*(p)+2G_2^*(p)\right]}{2G_1^*(p)+G_2^*(p)}\,,\ \xi\right\} \\
S_2(\xi) &= \mathcal{L}^{-1}\left\{\frac{3G_1^*(p)\,G_2^*(p)}{2G_1^*(p)+G_1^*(p)}\,,\ \xi\right\}
\end{aligned}
\tag{4.19}
$$

$\mathcal{L}^{-1}\{f(p),\xi\}$ represents the inverse transform.

For elasticity,

$$
\begin{aligned}
F_1(t) &= \frac{E}{(1+\nu)(1-2\nu)}\,H(t) \\
S_1(t) &= \frac{E}{(1-\nu)(1-2\nu)}\cdot H(t) \\
S_2(t) &= \frac{E}{1-\nu}\cdot H(t)\,.
\end{aligned}
\tag{4.20}
$$

The nodal forces S_a and S_b are defined per unit radian and act along the radial direction at the boundaries r=a and r=b, respectively,

$$S_a(t) = \sigma_{rr}(a,t)\cdot a$$
$$S_b(t) = \sigma_{rr}(b,t)\cdot b \quad . \tag{4.21}$$

The positive directions of nodal forces, displacements and element stresses are shown in Fig. (4-2).

Let $\delta u(r,t)$ be a continuous virtual displacement field applied to each element. For convenience, choosing δu in the form such that

$$\delta u(r,t) = \left[\frac{a}{b^2-a^2}\left(\frac{b^2}{r} - r\right)\delta u_a + \frac{b}{b^2-a^2}\left(r - \frac{a^2}{r}\right)\delta u_b\right]H(t) \quad , \tag{4.22}$$

where $\delta u_a, \delta u_b$ are the virtual displacements at r=a and r=b. H(t) is the Heaviside function. This would be the displacement field ocurring in a homogeneous, isotropic linear elastic annulus subjected to prescribed axisymmetric boundary displacement.

Using the principle of virtual displacement (2.9), yields

$$\int_{-\infty}^{t} 2\pi\left[S_a(t-t')\frac{\partial}{\partial t'}(\delta u_a H(t')) + S_b(t-t')\frac{\partial}{\partial t'}(\delta u_b H(t'))\right] dt'$$
$$= \int_{-\infty}^{t}\int_{a}^{b}\left\{\sigma_{rr}(t-t')\frac{\partial}{\partial t'}[\delta\epsilon_{rr}(r,t')] + \sigma_{\theta\theta}(t-t')\frac{\partial}{\partial t'}[\delta\epsilon_{\theta\theta}(r,t')]\right\}2\pi r\,dr\,dt' . \tag{4.23}$$

Substituting (4.15) and (4.21) into (4.23), and performing the integration, one may find the relation between nodal forces and nodal displacements,

$$\underset{\sim}{S}(t) = \int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi')\frac{d}{dt'}\underset{\sim}{u}(t')\,dt' + \underset{\sim}{S}_T(t) \quad , \tag{4.24}$$

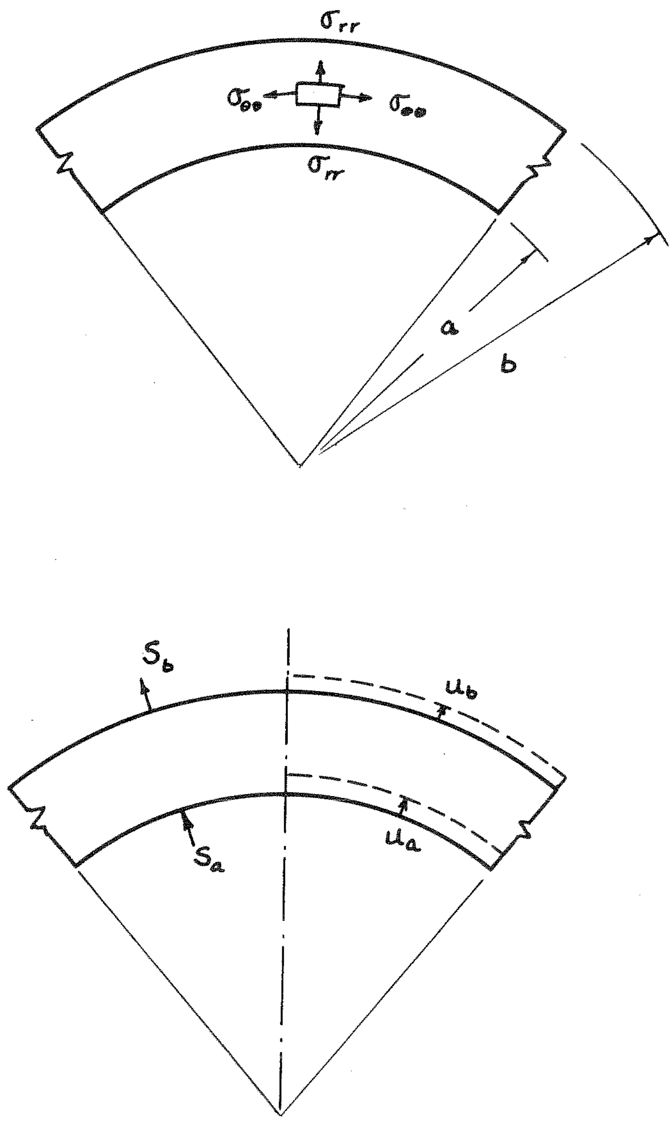

FIG.(4-2) DESCRIPTION OF A TYPICAL ELEMENT

where

$$\underset{\sim}{S}(t) = \begin{bmatrix} S_a(t) \\ S_b(t) \end{bmatrix}$$

$\int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi')\frac{d}{dt'}(\quad)\, dt'$ may be defined as the stiffness operator for the element, and $\underset{\approx}{K}$ the stiffness matrix, with

$$\underset{\approx}{K}(\xi-\xi') = \frac{1}{b^2-a^2}\begin{bmatrix} a^2F(\xi-\xi')+b^2G_1(\xi-\xi') & -ab[F(\xi-\xi')+G_1(\xi-\xi')] \\ -ab[F(\xi-\xi')+G_1(\xi-\xi')] & b^2F(\xi-\xi')+a^2G_1(\xi-\xi') \end{bmatrix} \tag{4.25}$$

Note that $\quad K_{12} = K_{21}$

Finally,
$$\underset{\sim}{S}_T(t) = \frac{2}{b^2-a^2}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\,\Theta(r,t')\,r\,dr\right]dt' \cdot \begin{bmatrix} a \\ -b \end{bmatrix}. \tag{4.26}$$

2. Spherical Element.

A typical spherical element of inner radius a and outer radius b is subjected to the effect of temperature T=T(r,t). Because of the point symmetry of the problem, the displacement field simplifies to

$$u_r = u(r,t) \quad , \quad u_\theta = u_\phi = 0 . \tag{4.27}$$

In this section, the field equations for the spherical element are developed. The derivation of these equations is similar to that given in Section 1.

The approximate stress-displacement relations are

$$\begin{aligned}\sigma_{rr}(r,t) &= \frac{2}{3}\int_{-\infty}^{t} G_1(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r}-\frac{u}{r}\right]dt' + \frac{1}{3}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r}+\frac{2u}{r}-3\alpha_0\Theta\right]dt' \\ \sigma_{\theta\theta}(r,t) &= -\frac{1}{3}\int_{-\infty}^{t} G_1(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r}-\frac{u}{r}\right]dt' + \frac{1}{3}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial u}{\partial r}+\frac{2u}{r}-3\alpha_0\Theta\right]dt' \\ \sigma_{\phi\phi}(r,t) &= \sigma_{\theta\theta}(r,t) \quad ,\end{aligned} \tag{4.28}$$

where the reduced time "ξ" has the same definition as (4.4).

The equation of equilibrium in terms of displacement can be written

$$\begin{aligned}&\frac{1}{3}\int_{-\infty}^{t}\left[2G_1(\xi-\xi')+G_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left\{\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right]\right\}dt' \\ &= \int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\frac{\partial}{\partial r}(\alpha_0\Theta)\right]dt' .\end{aligned} \tag{4.29}$$

Let $u_a(t)$ and $u_b(t)$ be the nodal displacements at r=a and r=b for the spherical element, the relationship between element stresses and nodal displacements is found to be

$$\underset{\sim}{\sigma}(r,t) = \int_{-\infty}^{t} \underset{\approx}{K}_1(r,\xi-\xi')\frac{d}{dt'}\underset{\sim}{u}(t')\,dt' + \underset{\sim}{\sigma}_T(r,t) , \tag{4.30}$$

where

$$\underset{\sim}{\sigma}(r,t) = \begin{bmatrix}\sigma_{rr}(r,t)\\ \sigma_{\theta\theta}(r,t)\end{bmatrix} , \quad \sigma_{\phi\phi} = \sigma_{\theta\theta} , \quad \underset{\sim}{u}(t) = \begin{bmatrix}u_a(t)\\ u_b(t)\end{bmatrix} , \tag{4.31}$$

$$\underset{\approx}{K}_1(r,\xi-\xi') = \frac{1}{b^3-a^3}\begin{bmatrix} -\frac{2a^2b^3}{r^3}G_1(\xi-\xi')-a^2G_2(\xi-\xi') & \frac{2a^3b^2}{r^3}G_1(\xi-\xi')+b^2G_2(\xi-\xi') \\ \frac{a^2b^3}{r^3}G_1(\xi-\xi')-a^2G_2(\xi-\xi') & -\frac{a^3b^2}{r^3}G_1(\xi-\xi')+b^2G_2(\xi-\xi')\end{bmatrix} , \tag{4.32}$$

and the thermal expansion effect is

$$\underset{\sim}{\sigma}_T(r,t)=\begin{bmatrix} -\frac{1}{b^3-a^3}\int_{-\infty}^{t}\left[S_1(\xi-\xi')+\frac{2a^3}{r^3}S_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\Theta(r,t')r^2dr\right]dt' \\ -\frac{2}{r^3}\int_{-\infty}^{t}S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^r \alpha_0\Theta(\rho,t')\rho^2d\rho\right]dt' \\ \hline -\frac{1}{b^3-a^3}\int_{-\infty}^{t}\left[S_1(\xi-\xi')-\frac{a^3}{r^3}S_2(\xi-\xi')\right]\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\Theta(r,t')r^2dr\right]dt' \\ +\frac{1}{r^3}\int_{-\infty}^{t}S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^r \alpha_0\Theta(\rho,t')\rho^2d\rho\right]dt' - \int_{-\infty}^{t}S_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\alpha_0\Theta(r,t')\right]dt' \end{bmatrix} . \tag{4.33}$$

The definitions of $S_1(\xi)$ and $S_2(\xi)$ are the same as defined in (4.20).

Let $S_a(t)$ and $S_b(t)$ be the nodal forces perunit solid angle acting along the radial direction at r=a and r=b, respectively,

$$\begin{aligned} S_a(t) &= a^2\sigma_{rr}(a,t) \\ S_b(t) &= b^2\sigma_{rr}(b,t) \quad . \end{aligned} \tag{4.34}$$

Similarly, the nodal force-nodal displacement relation can be derived in integral form,

$$\underset{\sim}{S}(t) = \int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi')\frac{d}{dt'}\underset{\sim}{u}(t')\,dt' + \underset{\sim}{S}_T(t) \quad , \tag{4.35}$$

where

$$\underset{\sim}{S}(t) = \begin{bmatrix} S_a(t) \\ S_b(t) \end{bmatrix} ,$$

$\int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi')\frac{d}{dt'}(\quad)\,dt'$ is defined as the stiffness operator for the spherical element, and $\underset{\approx}{K}$ the stiffness matrix, and

$$\underset{\approx}{K}(\xi-\xi') = \frac{1}{b^3-a^3}\begin{bmatrix} 2ab^3G_1(\xi-\xi')+a^4G_2(\xi-\xi') & -a^2b^2[2G_1(\xi-\xi')+G_2(\xi-\xi')] \\ -a^2b^2[2G_1(\xi-\xi')+G_2(\xi-\xi')] & 2a^3bG_1(\xi-\xi')+b^4G_2(\xi-\xi') \end{bmatrix}. \tag{4.36}$$

The thermal effect is

$$\underset{\sim}{S}_T(t) = \frac{3}{b^3-a^3}\int_{-\infty}^{t} G_2(\xi-\xi')\frac{\partial}{\partial t'}\left[\int_a^b \alpha_0\,\Theta(r,t')\,r^2dr\right]dt' \begin{bmatrix} a^2 \\ -b^2 \end{bmatrix}. \tag{4.37}$$

3. <u>Triangular Element.</u>

The formulations of the triangular element method with 3-nodal points in elastic media are established in detail by Wilson [13]. For linear viscoelasticity the development of the finite element method will be similar to that of elasticity except that an additional variable, time t, is involved due to the viscoelastic behavior. A typical triangular element is shown in Fig. (4-3). Let x, y be the local material coordinates for the element, a_j, a_k, b_j, b_k be the dimensions of the element.

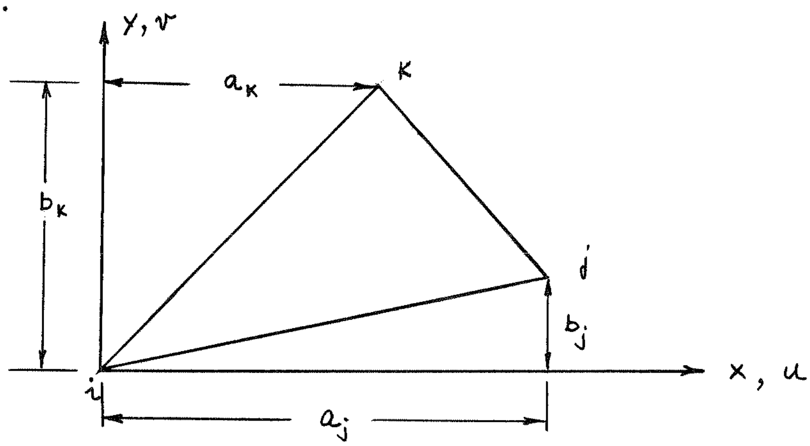

Fig. (4-3) Element Dimensions

The displacement field in the element is assumed to be a linear function of space with time varying coefficients, therefore

$$\begin{aligned} u(x, y, t) &= u_i + C_1(t)\, x + C_2(t)\, y \\ v(x, y, t) &= v_i + C_3(t)\, x + C_4(t)\, y \end{aligned} \qquad (4.38)$$

where u_i, v_i are the displacement components at point i along the positive x and y directions respectively. Furthermore let $\underset{\sim}{u}$ be the nodal displacement vector and $\underset{\sim}{s}$ be the nodal force vector, defined by

$$\underset{\sim}{u} = \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix} , \qquad \underset{\sim}{s} = \begin{bmatrix} s_x^i \\ s_y^i \\ s_x^j \\ s_y^j \\ s_x^k \\ s_y^k \end{bmatrix} , \tag{4.39}$$

and let $\underset{\sim}{\epsilon}$ be the element strain vector and $\underset{\sim}{\sigma}$ be the stress vector,

$$\underset{\sim}{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ 2\epsilon_{xy} \end{bmatrix} , \qquad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} . \tag{4.40}$$

From the definition of infinitesimal strain (2.1) and the assumed displacement pattern (4.38), the element strain and nodal displacement relation may be established through a transformation matrix $\underset{\approx}{A}$:

$$\underset{\sim}{\epsilon} = \underset{\approx}{A}\,\underset{\sim}{u} , \tag{4.41}$$

where

$$\underset{\approx}{A} = \frac{1}{a_j b_k - a_k b_j} \begin{bmatrix} b_j - b_k & 0 & b_k & 0 & -b_j & 0 \\ 0 & a_k - a_j & 0 & -a_k & 0 & a_j \\ a_k - a_j & b_j - b_k & -a_k & b_k & a_j & -b_j \end{bmatrix} . \tag{4.42}$$

For non-homogeneous temperature problems, the temperature distribution inside the element is assumed to be uniform, consequently the reduced time ξ for each element is independent of position in the element. In the state of plane strain, the stress-strain relationship is reduced from (2.3) to

$$\underset{\sim}{\sigma}(t) = \int_{-\infty}^{t} \underset{\approx}{K}_1(\xi-\xi') \frac{d}{dt'} \underset{\sim}{\epsilon}(t')\,dt' + \underset{\sim}{\sigma}_T(t) , \tag{4.43}$$

where

$$\underset{\approx}{K}_1(\xi-\xi') = \begin{bmatrix} \frac{1}{3}\left[2G_1(\xi-\xi')+G_2(\xi-\xi')\right] & \frac{1}{3}\left[G_2(\xi-\xi')-G_1(\xi-\xi')\right] & 0 \\ \frac{1}{3}\left[G_2(\xi-\xi')-G_1(\xi-\xi')\right] & \frac{1}{3}\left[2G_1(\xi-\xi')+G_2(\xi-\xi')\right] & 0 \\ 0 & 0 & \frac{1}{2}G_1(\xi-\xi') \end{bmatrix} , \tag{4.44}$$

the thermal expansion effect is

$$\underset{\sim}{\sigma}_T(t) = \int_{-\infty}^{t} G_2(\xi-\xi') \frac{\partial}{\partial t'}(\alpha_0\Theta)\,dt' \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} . \tag{4.45}$$

For plane stress,

$$\underset{\approx}{K}_1(\xi-\xi') = \begin{bmatrix} H_1(\xi-\xi') & H_2(\xi-\xi') & 0 \\ H_2(\xi-\xi') & H_1(\xi-\xi') & 0 \\ 0 & 0 & \frac{1}{2}G_1(\xi-\xi') \end{bmatrix} \tag{4.46}$$

$$\underset{\sim}{\sigma}_T(t) = \int_{-\infty}^{t} S_2(\xi-\xi') \frac{\partial}{\partial t'}(\alpha_0\Theta)\,dt' \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} , \tag{4.47}$$

where

$$\begin{aligned} H_1(\xi) &= \mathcal{L}^{-1}\left\{ \frac{G_1^*(p)\left[G_1^*(p)+2G_2^*(p)\right]}{2G_1^*(p)+G_2^*(p)} , \xi \right\} \\ H_2(\xi) &= \mathcal{L}^{-1}\left\{ \frac{G_1^*(p)\left[G_2^*(p)-G_1^*(p)\right]}{2G_1^*(p)+G_2^*(p)} , \xi \right\} \\ S_2(\xi) &= \mathcal{L}^{-1}\left\{ \frac{3G_1^*(p)G_2^*(p)}{2G_1^*(p)+G_2^*(p)} , \xi \right\} . \end{aligned} \tag{4.48}$$

The expressions of element stresses in terms of nodal displacements are derived from (4.41) and (4.43)

$$\underset{\sim}{\sigma}(t) = \int_{-\infty}^{t} \underset{\approx}{K}_1(\xi-\xi') \cdot \underset{\approx}{A} \cdot \frac{d}{dt'} \underset{\sim}{u}(t') + \underset{\sim}{\sigma}_T(t) \quad (4.49)$$

Use the principle of virtual work to find the relation between the nodal forces $\underset{\sim}{s}$ and the nodal displacements $\underset{\sim}{u}$. Suppose that the element is subjected to a system of virtual nodal displacements δu , such that

$$\delta \underset{\sim}{\epsilon} = \underset{\approx}{A}\, \delta \underset{\sim}{u} \quad (4.50)$$

From (2.9) one may formulate the area integral for the element,

$$\int_{Area} \underset{\sim}{\sigma}^T * \delta \underset{\sim}{\epsilon} \; da = \underset{\sim}{s}^T * \delta \underset{\sim}{u} , \quad (4.51)$$

where $(\;)^T$ is the matrix transpose

Let

$$\delta \underset{\sim}{u} = \underset{\sim}{u}^v \cdot H(t) , \quad (4.52)$$

where $\underset{\sim}{u}^v$ is a virtual nodal displacement vector which is independent of time. Substitution of $\underset{\sim}{\sigma}$, $\delta\underset{\sim}{\epsilon}$ and $\delta\underset{\sim}{u}$ from (4.49), (4.50) and (4.52) into equation (4.51) yields

$$\frac{1}{2}(a_K b_j - a_j b_K)\left[-\int_{-\infty}^{t} \underset{\approx}{K}_1(\xi-\xi') \underset{\approx}{A} \cdot \frac{d}{dt'} \underset{\sim}{u}(t')\, dt' + \underset{\sim}{\sigma}_T(t)\right]^T \cdot \underset{\approx}{A} \cdot \underset{\sim}{u}^v = \underset{\sim}{s}^T \cdot \underset{\sim}{u}^v . \quad (4.53)$$

Since $\underset{\sim}{u}^v$ can be arbitrary, (4.53) yields the nodal force-displacement expressions ,

$$\underset{\sim}{s}(t) = \int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi') \frac{d}{dt'} \underset{\sim}{u}(t')\, dt' + \underset{\sim}{s}_T(t) , \quad (4.54)$$

where

$$\underset{\approx}{K}(\xi-\xi') = \frac{1}{2}(a_K b_j - a_j b_K) \cdot \underset{\approx}{A}^T \cdot \underset{\approx}{K}_1(\xi-\xi') \cdot \underset{\approx}{A} \quad , \qquad (4.55)$$

and

$$\underset{\sim}{S}_T(t) = \frac{1}{2}(a_K b_j - a_j b_K) \cdot \underset{\approx}{A}^T \cdot \underset{\sim}{\sigma}_T(t) \qquad (4.56)$$

$\int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi') \frac{d}{dt'}(\quad) \, dt'$ is defined as the stiffness operator of the triangular element and $\underset{\approx}{K}$ the stiffness matrix.

4. Integral Equations for the Complete Continuum

The element nodal force-displacement relations in integral form have been formulated in the previous section (equations (4.24), (4.35) and (4.54)). Once the element properties are specified, the entire structure is considered to be composed of the elements as the structural components. Therefore, from the equilibrium conditions of the complete system, the relationship between applied nodal forces and resulting nodal displacements can be obtained by superposition of element stiffness matrices around a nodal point,

$$\int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi') \frac{d}{dt'} \underset{\sim}{u}(t')\, dt' = \underset{\sim}{R}(t') \quad , \qquad (4.57)$$

where $\int_{-\infty}^{t} \underset{\approx}{K}(\xi-\xi') \frac{d}{dt'}(\quad)\, dt'$ is the stiffness operator for the entire structure, the elements of the matrix $\underset{\approx}{K}(\xi-\xi')$ are functions of time and temperature, and serve as the Kernel function for the integral operator. $\underset{\sim}{R}(t)$ is an applied nodal force vector and/or thermal load vector.

For linear materials, the stiffness matrix $\underset{\approx}{K}$ is symmetrical and by appropriate nodal sequencing can be restricted to a banded width about the principal diagonal.

In a computer analysis of the problem, the symmetric and narrow band properties simplify the computations and programming procedures.

V. Finite Difference Approximation.

Reduction of a continuous body into a system of finite elements results in a set of linear integral equations (4.57) with nodal displacements as unknowns. In general it is difficult to obtain a closed solution to a large system of simultaneous integral equations, especially when they are not in simple convolutional form (as in the presence of temperature gradients). However, a numerical solution can be effected conveniently by application of finite difference methods. The use of step-forward numerical integration process leads to a system of linear algebraic equations which may be solved for the unknowns at different time steps. The numerical calculations may be accomplished with the aid of a high speed electronic computer.

Many standard difference formulae, (for instance, trapezoidal rule, Simpson's rule, etc.), are available to approximate an integral expression. To obtain the solution of the integral equations (4.57), the standard difference formulae, however, can not be readily applied to the integral expressions where the integrands involve the derivatives of the unknown functions. In this section, a modified difference formula is suggested and some simplifications are made under certain restrictions. The section concludes with a discussion of solution accuracy.

1. Numerical Integration.

The finite difference technique was used by Hopkins and Hamming [19] to solve a convolution integral equation interrelating the creep and relaxation functions, and also by Rogers and Lee [10] for viscoelastic analysis involving a single integral equation. A similar method will be used for the present analysis.

Departing from the matrix equations for convenience, a single integral equation will be studied,

$$\int_{-\infty}^{t} K(\xi-\xi')\frac{d}{dt'}u(t')\,dt' = R(t) \quad . \tag{5.1}$$

Assuming $u(t)\equiv 0, \forall t<0$ and removing the contribution at t=0, yields

$$\int_{0^+}^{t} K(\xi-\xi')\frac{d}{dt'}u(t')\,dt' = R(t) - K(\xi)\,u(0^+) \quad . \tag{5.2}$$

The present time t is divided into n intervals by the integration time t_i, i=1,2,......, n+1, with $t_1=0$ and $t_{n+1}=t$. Accordingly the reduced time ξ is also divided into n intervals ξ_i, i=1,2,......,n+1. The integral expression in (5.2) may then be written as

$$\int_{0^+}^{t} K(\xi-\xi')\frac{d}{dt'}u(t')\,dt' = \sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} K(\xi-\xi')\frac{d}{dt'}u(t')\,dt' \, . \tag{5.3}$$

If the function K is continuous for $t' \in [t_i, t_{i+1}]$ and the time interval is chosen to be small enough such that the derivative of u(t) does not change sign inside $[t_i, t_{i+1}]$, then, from the second mean value theorem, the integral in (5.3) may be approximated by

$$\int_{t_i}^{t_{i+1}} K(\xi-\xi')\frac{d}{dt'}u(t') \simeq \frac{1}{6}\left[K(\xi_{n+1}-\xi_i)+4K(\xi_{n+1}-\xi_{i+\frac{1}{2}})+K(\xi_{n+1}-\xi_{i+1})\right]\cdot\int_{t_i}^{t_{i+1}} \frac{du(t')}{dt'}dt'$$

$$= \frac{1}{6}\left[K(\xi_{n+1}-\xi_i)+4K(\xi_{n+1}-\xi_{i+\frac{1}{2}})+K(\xi_{n+1}-\xi_{i+1})\right]\cdot\left[u(t_{i+1})-u(t_i)\right] . \tag{5.4}$$

Equation (5.4) will be called the 3-point approximation formula for integration.

By virtue of (5.2) - (5.4), it follows that,

$$u(0^+) = [K(0)]^{-1}\cdot R(0) \qquad \text{for} \quad t = t_1 = 0 \tag{5.5}$$

$$u(t_{n+1}) \simeq A^{-1}\Big\{R(t_{n+1}) - K(\xi_{n+1})u(0^+) - \frac{1}{6}\sum_{i=1}^{n-1}\left[K(\xi_{n+1}-\xi_i)+4K(\xi_{n+1}-\xi_{i+\frac{1}{2}}) + K(\xi_{n+1}-\xi_{i+1})\right]\left[u(t_{i+1})-u(t_i)\right]\Big\} + u(t_n) \qquad \text{for} \quad t>0 \tag{5.6}$$

where

$$A = \frac{1}{6}\left[K(\xi_{n+1}-\xi_n)+4K(\xi_{n+1}-\xi_{n+\frac{1}{2}})+K(0)\right] . \tag{5.7}$$

As one may see from (5.5), the solution at the time of interest t_{n+1} is obtained by summing over the entire history. To this end, in a computer solution it would be necessary to store all the previous values of the dependent variables. Thus, it is possible to exceed the computer storage or overtax the computer in solving problems with a large number of unknowns. Simplications can be made through certain restrictions.

1.1. Solutions Based on Measured Relaxation (or Creep) Data of Viscoelastic Materials with Limited Memory —— Practically most viscoelastic materials exhibit finite memory behavior. If the material function, taking the relaxation function as shown in Fig. (5-1) for instance, effectively reaches its equilibrium state in short time t_ϵ so that the number of summations required is within the capacity of computer storage, then it is only necessary to store a certain number of previous solutions for solutions at large times.

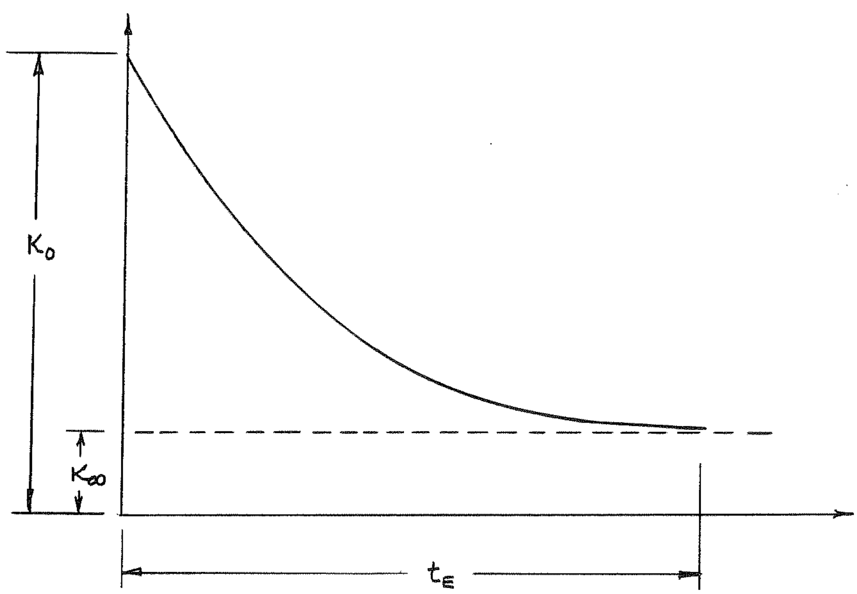

Fig. (5-1) Relaxation Modulus

Due to the above assumption, the expression of displacement (5.6) for large time may be reduced to

$$u(t_{n+1}) = A^{-1}\Big\{R(t_{n+1}) - K(\xi_{n+1})u(0^+) - \frac{1}{6}\sum_{i=m}^{n-1}\big[K(\xi_{n+1}-\xi_i) + 4K(\xi_{n+1}-\xi_{i+\frac{1}{2}}) + K(\xi_{n+1}-\xi_{i+1})\big]\big[u(t_{i+1}) - u(t_i)\big] - K_\infty\big[u(t_m) - u(0^+)\big]\Big\} + u(t_n) \tag{5.8}$$

where K_∞ : Equilibrium modulus, a constant.

K_0 : Initial modulus.

n-m-1 = Number of summations required for $0 \leq t' \leq t_E$.

The advantages in the formulation of (5.8) are two fold:

1) It is only necessary to store a certain number of solutions in order to perform the numerical time integrations.

2) Material functions can be taken directly from test data or graphs, therefore; there is no need to approximate the experimental results of material properties by closed form expressions.

1.2 Solutions Based on the Material Functions Which Are Expressed by Dirichlet or Prony Series——Exponential series expressions have often been used to approximate the rheological data for viscoelastic materials. As an example, the relaxation modulus may be approximated by

$$E(t) = \sum_{i=0}^{m} E_i \exp(-t/\tau_i) \tag{5.9}$$

Schapery [2] has taken advantage of this expression to propose an effective collocation scheme of approximate Laplace inversion for viscoelastic analysis. A further advantage of this expression is as follows.

Replacing $K(\xi-\xi')$ by $\sum K_i \exp(-\frac{\xi-\xi'}{\tau_i})$ (due to(5.9)) for the integral expression in (5.2), yields

$$\int_{0^+}^{t} K(\xi-\xi')\frac{d}{dt'}u(t')dt' = \int_{0^+}^{t}\Big[\sum_{\ell=0}^{m} K_\ell \exp(-\frac{\xi-\xi'}{\tau_\ell})\Big]\frac{d}{dt'}u(t')dt'$$
$$= \sum_{\ell=0}^{m} K_\ell \exp(-\frac{\xi}{\tau_\ell})\Big[\int_{0^+}^{t_n} \exp(-\frac{\xi'}{\tau_\ell})\frac{d}{dt'}u(t')dt'$$
$$+ \int_{t_n}^{t_{n+1}} \exp(-\frac{\xi'}{\tau_\ell})\frac{d}{dt'}u(t')dt'\Big] . \qquad (5.10)$$

Again the integrals in the above equation may be approximated by the difference formulas (5.3) and (5.4). One may see from the expression inside the bracket of (5.10) that the sum at time t_{n+1} can be obtained by adding the contribution between t_n and t_{n+1} to the sum at time t_n; therefore, for the solutions at the time of interest, there is no need to sum all the way back to the starting time.

2. Accuracy of Solution.

The use of the finite element method and the finite difference method contains certain approximations. Consequently, the computed solution must differ to some extent from the true solution. It is the objective of this section to discuss the sources of error. The main sources of error appear as:

i). Error due to the application of the finite element method, (especially when a constant temperature is assumed in each element).

ii). Rounding error in the computing process.

iii). Truncation error due to replacing the integral by an approximate difference formula.

iv). The propagation of errors from early time to later time.

It was pointed out [13] that the solution obtained from the finite element method converges to the true solution as the dimensions of the elements are successively reduced and provided that the displacement compatibility between elements is retained. Suppose the error from the use of finite element technique and the rounding error being kept small, the error introduced by the finite difference approximation will then be examined.

In order to find the truncation error due to the use of difference formula, rewrite the integral expression (5.4) by Simpson's approximation,

$$I = \int_{t_i}^{t_{i+1}} K(\xi_{n+1}-\xi')\frac{d}{dt'}u(t')\,dt' = \int_{t_i}^{t_{i+1}} K(t_{n+1},t')\frac{d}{dt'}u(t')\,dt'$$

$$= \frac{h}{6}\left[K(t_{n+1},t_i)u'(t_i) + 4K(t_{n+1},t_{i+\frac{1}{2}})u'(t_{i+\frac{1}{2}}) + K(t_{n+1},t_{i+1})u'(t_{i+1})\right] + e_1 \, , \qquad (5.11)$$

where

$$u'(t_i) = \frac{d}{dt'} u(t') \Big|_{t'=t_i} \quad , \quad t_{i+\frac{1}{2}} = \frac{t_i + t_{i+1}}{2}$$

$$h = t_{i+1} - t_i \tag{5.12}$$

$$e_1 = -\frac{1}{90}\left(\frac{h}{2}\right)^5 \frac{d^4}{dt'^4}\left[K(t_{n+1},t')\,u'(t')\right] ,$$ truncation error

due to numerical integration formula.

Using forward difference to approximate $u'(t_i)$, central difference to approximate $u'(t_{i+\frac{1}{2}})$ and backward difference to approximate $u'(t_{i+1})$ in (5.11), yields

$$I = \frac{h}{6}\Big\{K(t_{n+1},t_i)\Big[\frac{u(t_{i+1})-u(t_i)}{h} - \frac{h}{2}u''(\theta_1)\Big] + 4K(t_{n+1},t_{i+\frac{1}{2}})\Big[\frac{u(t_{i+1})-u(t_i)}{h}$$

$$-\frac{h^2}{12}u''(\theta_2)\Big] + K(t_{n+1},t_{i+1})\cdot\Big[\frac{u(t_{i+1})-u(t_i)}{h} + \frac{h}{2}u''(\theta_3)\Big]\Big\} + e_1$$

$$= \frac{h}{6}\left[K(t_{n+1},t_i) + 4K(t_{n+1},t_{i+\frac{1}{2}}) + K(t_{n+1},t_{i+1})\right]\cdot\left[u(t_{i+1}) - u(t_i)\right] + E_i ,$$

or

$$I = \frac{h}{6}\left[K(\xi_{n+1}-\xi_i) + 4K(\xi_{n+1}-\xi_{i+\frac{1}{2}}) + K(\xi_{n+1}-\xi_{i+1})\right]\cdot\left[u(t_{i+1}) - u(t_i)\right] + E_i \tag{5.13}$$

where $E_i = e_1 + e_2$

e_2 : truncation error due to the derivative approximation,

$$e_2 = \frac{h^3}{12}\left[K'(t_{n+1},\theta)\,u''(\theta)\right] - \frac{h^3}{18}\left[K(t_{n+1},t_{i+\frac{1}{2}})\,u'''(\theta_3)\right] \tag{5.14}$$

$$t_i \leq \theta_1 , \theta_2 , \theta_3 , \theta \leq t_{i+1} .$$

Assuming that K has bounded derivatives up to the fourth and u has bounded derivatives up to the fifth, it follows that the truncation error E_i is of $O(h^3)$, therefore, the numerical integration method is of $O(h^2)$. On the other hand if the dependent variable u is a slowly varying function in time so that the difference formula for $u'(t)$ is a good approximation, then the truncation error due to integration formula dominates. Hence the truncation error E_i is of $O(h^5)$ and the method may become $O(h^4)$.

The truncation error from the finite difference formula is a local matter. A knowledge of local error is not sufficient to determine the accuracy of the extended process. For, during the step-by-step numerical integration, the error at each time step will affect the solutions at later time steps. In particular if such errors tend to accumulate rapidly, it will cause instability in the solution. To study the propagation of errors, the coefficients of early time solutions in (5.6) will be examined,

$$u(t_{n+1}) = \gamma_1 u(t_n) + \gamma_2 u(t_{n-1}) + \cdots + \gamma_i u(t_{n-i}) + \cdots \tag{5.15}$$

where

$$\gamma_1 = \frac{K_0 - K_2 + 4(K_{1/2} - K_{3/2})}{A}$$

$$\gamma_2 = \frac{K_1 - K_3 + 4(K_{3/2} - K_{5/2})}{A} \tag{5.16}$$

$$\cdots\cdots\cdots$$

$$\gamma_i = \frac{K_{i-1} - K_{i+1} + 4(K_{i-\frac{1}{2}} - K_{i+\frac{1}{2}})}{A} ,$$

and

$$A = K_0 + 4K_{1/2} + K_1$$
$$K_0 = K(0) \quad , \quad K_{1/2} = K(\xi_{n+1} - \xi_{n+\frac{1}{2}}) \ , \quad K_1 = K(\xi_{n+1} - \xi_n) \ , \quad K_{3/2} = K(\xi_{n+1} - \xi_{n-\frac{1}{2}}) \tag{5.17}$$
$$------ \ .$$

Since the function K is either a monotonically decreasing function concave upward, or a monotonically increasing function concave downward, then

$$|\gamma_1| < 1 \quad , |\gamma_2| < 1 \quad , ------ , \ |\gamma_i| < 1 \ , \ ---- \quad . \tag{5.18}$$

Furthermore,

$$\left| \frac{\gamma_{i+1}}{\gamma_i} \right| = \left| \frac{K_i - K_{i+2} + 4(K_{i+\frac{1}{2}} - K_{i+3/2})}{K_{i-1} - K_{i+1} + 4(K_{i-\frac{1}{2}} - K_{i+1/2})} \right| < 1 \quad . \tag{5.19}$$

If the time interval $\Delta t/\tau$ tends to zero, $\gamma_1, \gamma_2, \cdots \gamma_i,$ will also tend to vanish. Thus the effect of error from the early steps to the later steps will die out rapidly.

To illustrate the convergence of the finite difference formula, a convolutional integral equation will be studied,

$$\int_{-\infty}^{t} G(t-t') \frac{d}{dt'} J(t') dt' = 1 \ . \tag{5.20}$$

Assuming the relaxation function is a five element model with the following expression

$$G(t) = 0.9 + 3 e^{-t/\tau} + 3 e^{-5t/\tau} \ . \tag{5.21}$$

The plot of G(t) is shown in Fig. (5-2), where it is noted that G(t) has a steep initial slope. The creep function may be found by the Laplace transform, and is given by the expression,

$$J(t) = 1.1111 - 0.9057\, e^{-0.2046\, t/\tau} - 0.0605\, e^{-3.1866\, t/\tau} \qquad (5.22)$$

Equation (5.20) is also solved by applying the difference formula (5.4) and the 2-point approximation formula,

$$\int_{t_i}^{t_{i+1}} G(t_{n+1}-t')\,\frac{d}{dt'}\,J(t')\,dt' \simeq \frac{1}{2}\left[G(t_{n+1}-t_{i+1}) + G(t_{n+1}-t_i)\right]\left[u(t_{i+1}) - u(t_i)\right]. \qquad (5.23)$$

The solutions for different time meshes are plotted in Fig. (5-3). As the time mesh becomes smaller, the solution converges to the analytical solution monotonically and the solution for $\Delta t/\tau = 0.5$ is indistinguishable from the analytical solution. Values obtained from 3-point approximation with $\Delta t/\tau = 1.0$ are even better than those from 2-point approximation at finer time mesh, $\Delta t/\tau = 0.5$.

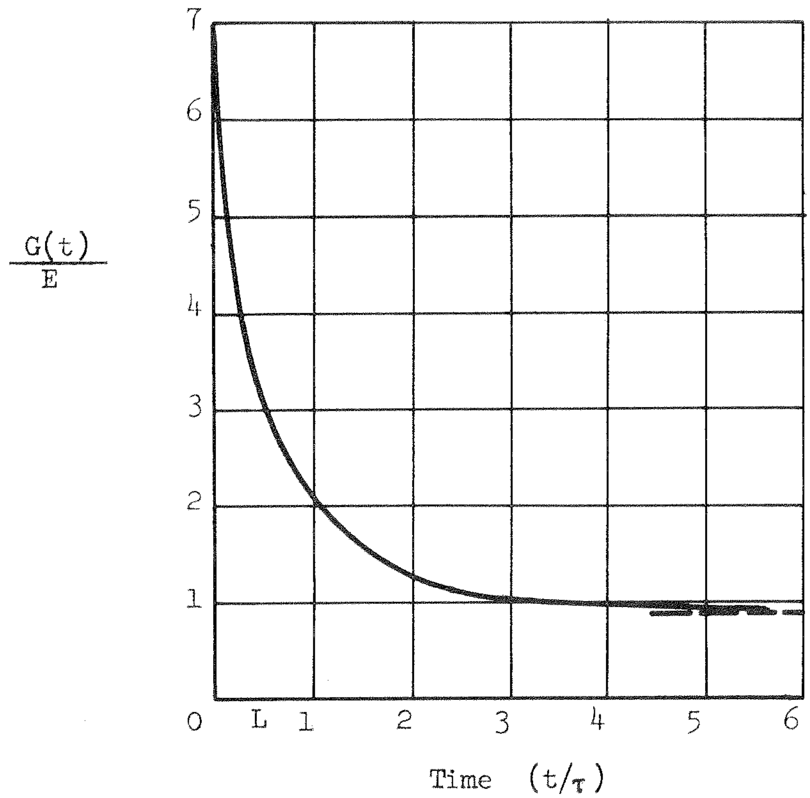

FIG.(5-2) RELAXATION MODULUS

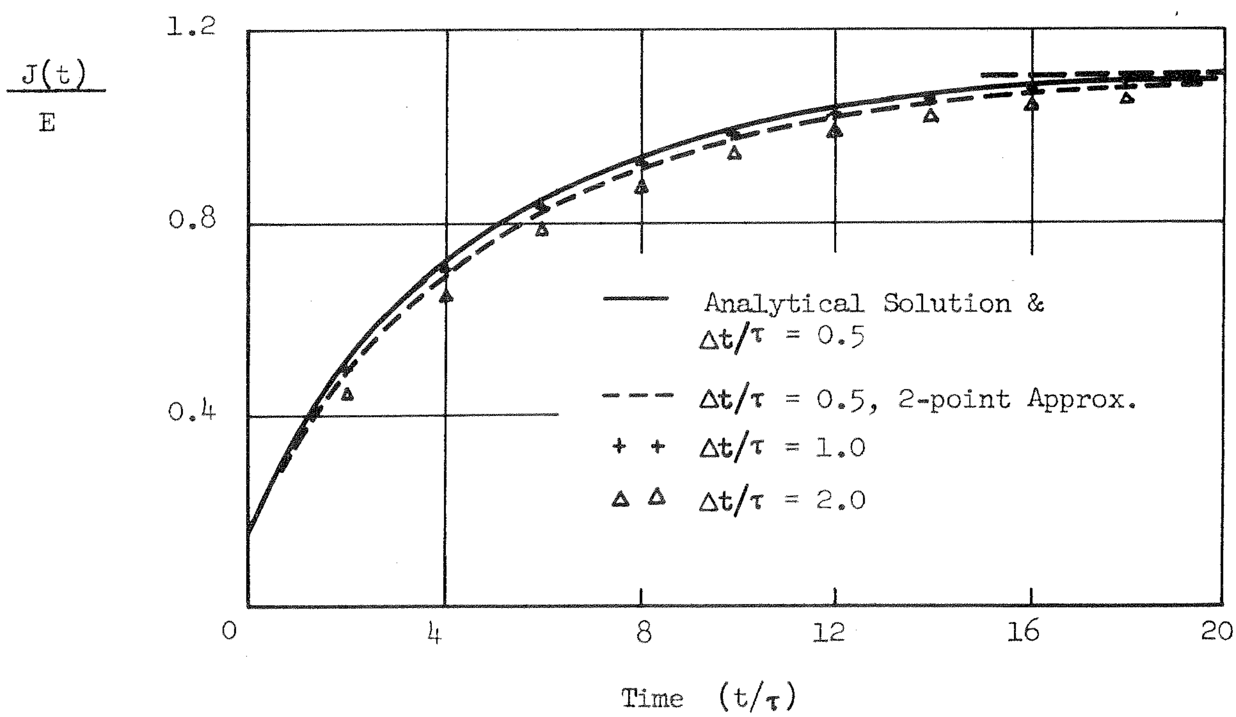

FIG.(5-3) CREEP FUNCTION

VI. Numerical Examples.

1. Infinite Cylinder Bonded to a Thin Elastic Case.

A viscoelastic hollow cylinder bonded to a thin elastic case (as shown in Fig. (6-1)) will be studied. The cylinder is internally pressurized and stress free at the outer boundary. Three different temperature conditions are considered:

i). A uniform temperature field.

ii). A steady state temperature field where the outer boundary temperature is elevated 40° F, hence

$$\Theta(r,t) = 36.0828 \ln r \cdot H(t) \tag{6.1}$$

iii). A steady state temperature field where the inner boundary temperature is elevated 40° F, hence

$$\Theta(r,t) = 36.0828 \cdot \ln(3.03/r) \cdot H(t) \tag{6.2}$$

For the present purpose it is assumed that the steady state temperature has been reached before load application. As no particular application is intended, the following hypothetical mechanical properties are used,

$$\begin{aligned} G_1(\xi)/E &= 0.9 + 3e^{-\xi/\tau} + 3e^{-5\xi/\tau} \\ G_2(\xi)/E &= 100 \cdot H(\xi) \quad , \end{aligned} \tag{6.3}$$

where E is a material constant, τ is a characteristic time.

The shift function is taken as

$$\varphi(T) = 10^{0.005(T-T_0)} \tag{6.4}$$

The viscoelastic portion of the cylinder is divided into 10 equal elements. All the calculations are carried out through a computer program. Different time steps are tried; it is found that the real time interval of 0.5τ is sufficiently accurate for isothermal case and that of 0.25τ for steady state temperature. The solution is continued until $t=6.5\tau$, at this point all the stresses are close to the asymptotic values and the responses are changing quite slowly.

As a direct check on the accuracy of the numerical method, the radial stress at the interface of the viscoelastic core and the elastic case under uniform temperature is found analytically. The analytical solution and the approximate solution are plotted in Fig.(6-2) for comparison. As one may see, there is no distinct difference between them. The distribution of radial stresses and tangential stresses in the cylinder are also shown in Fig. (6-3) and Fig. (6-4), respectively. The stresses under the temperature effect reach the asymptotic values much faster than those in the isothermal state, especially when the temperature on the pressurized boundary is elevated. It should be noted here that the tangential stresses under the effect of non-uniform temperature are not continuous between the elements because of the discrepancy in temperature assumption. The tangential stresses at the interfaces of the elements are further approximated by averaging those from the related elements.

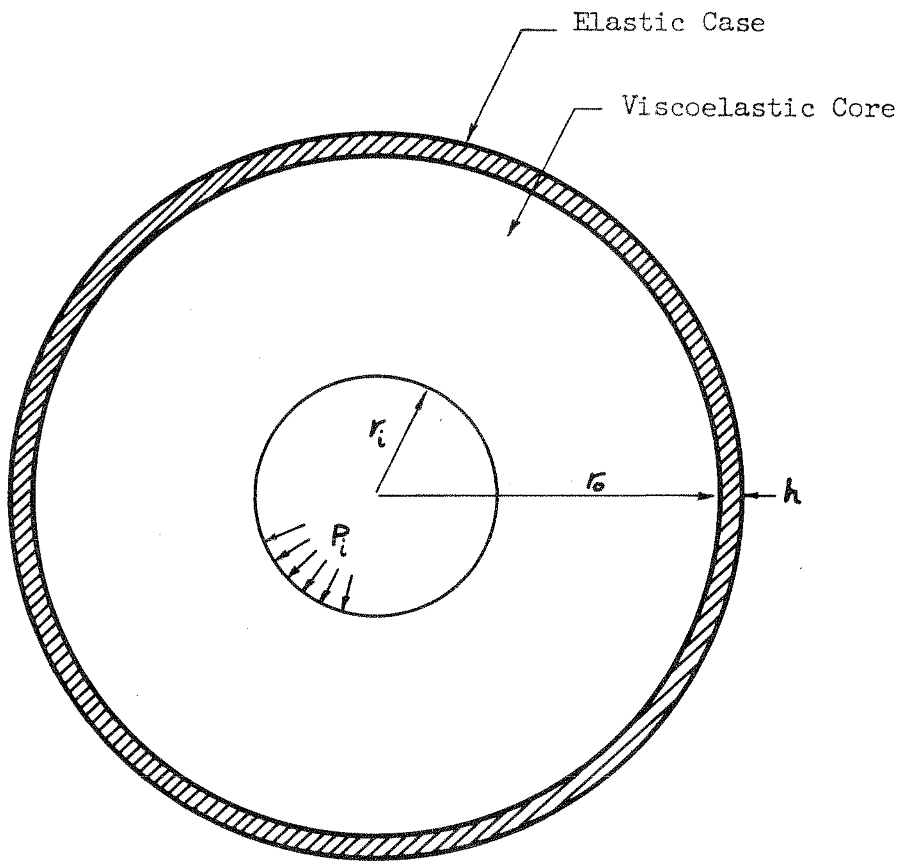

FIG.(6-1) CROSS-SECTION OF THE HOLLOW CYLINDER

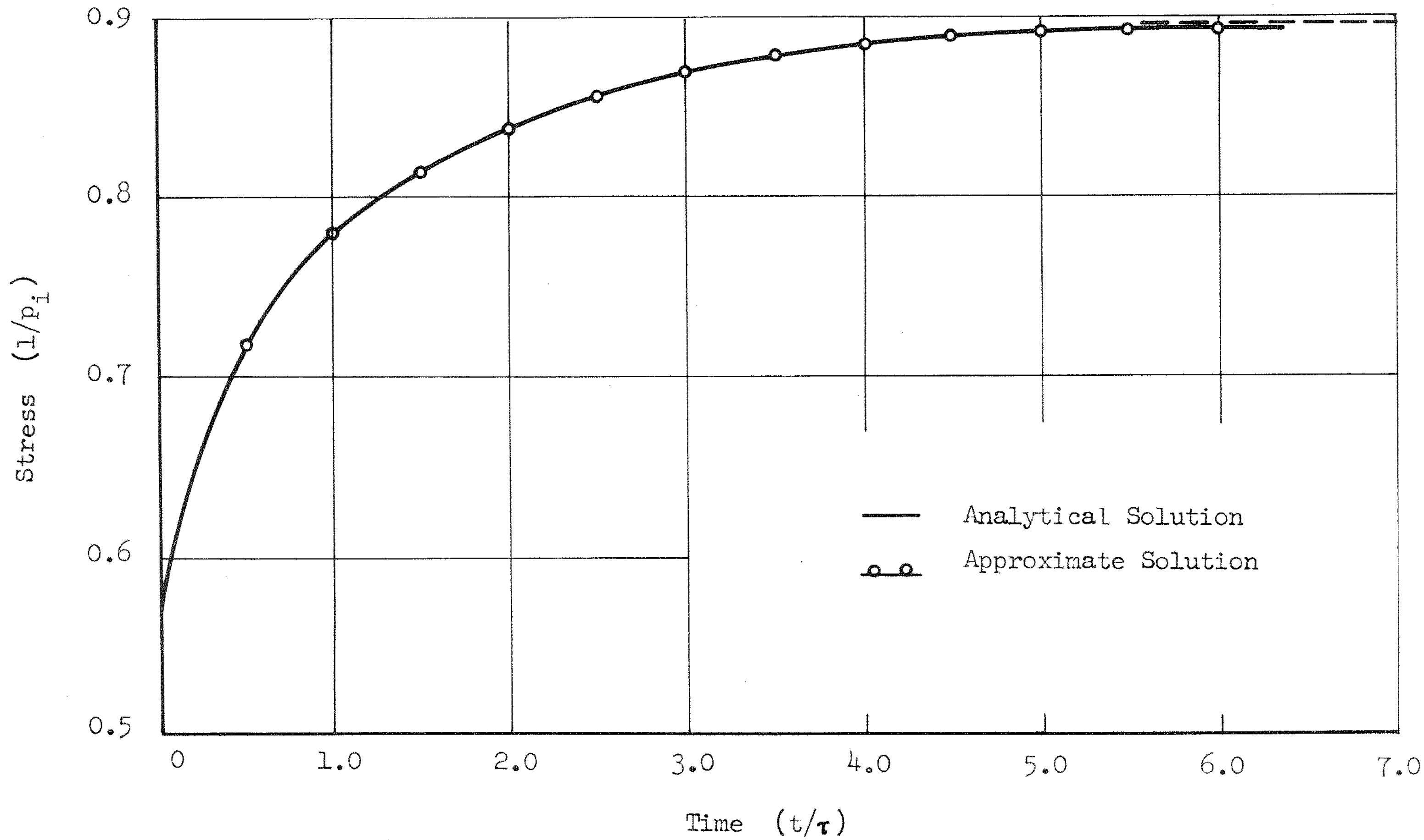

FIG.(6-2) RADIAL STRESS AT THE CASE-CORE INTERFACE

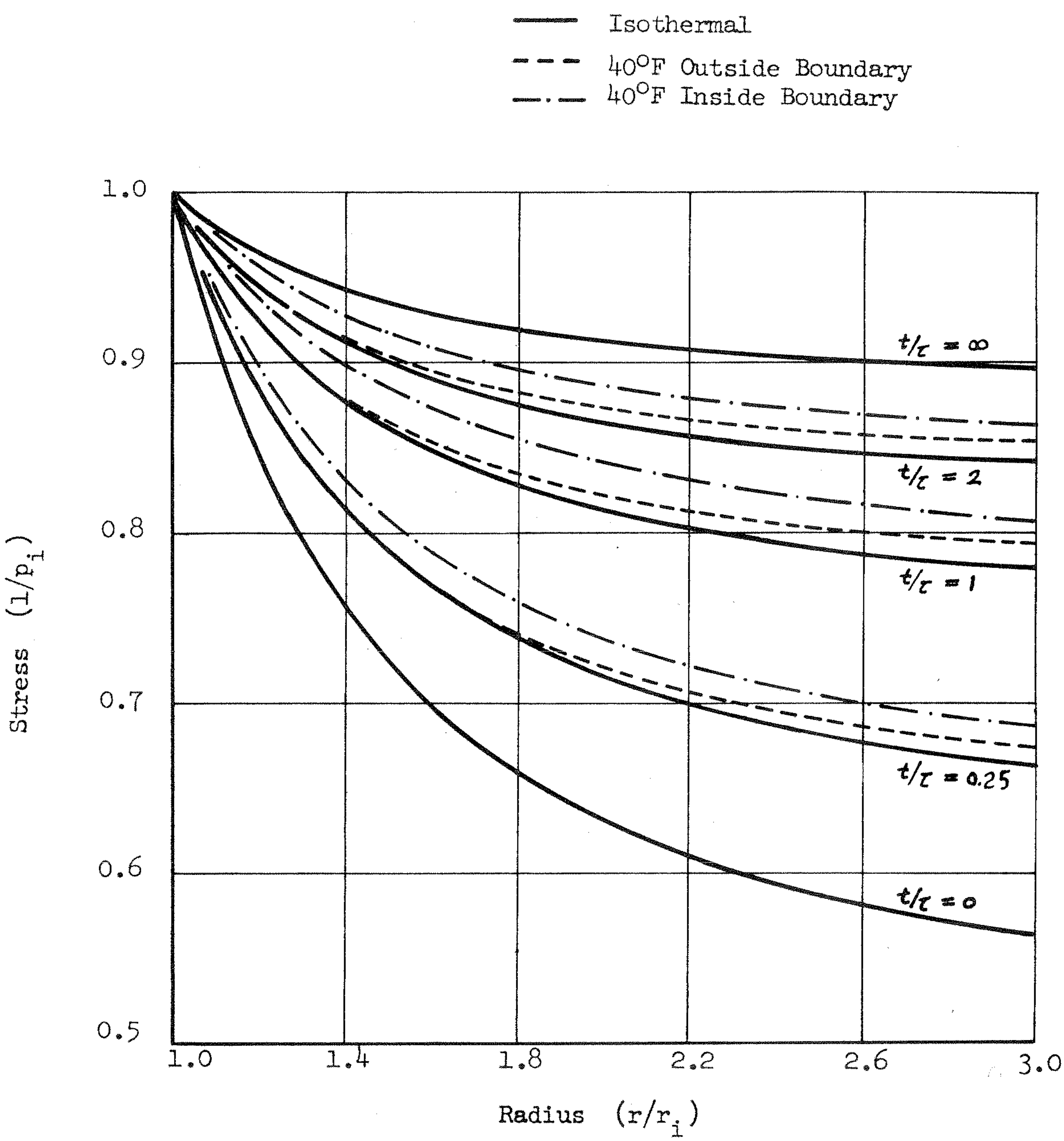

FIG.(6-3) RADIAL STRESS IN CYLINDER

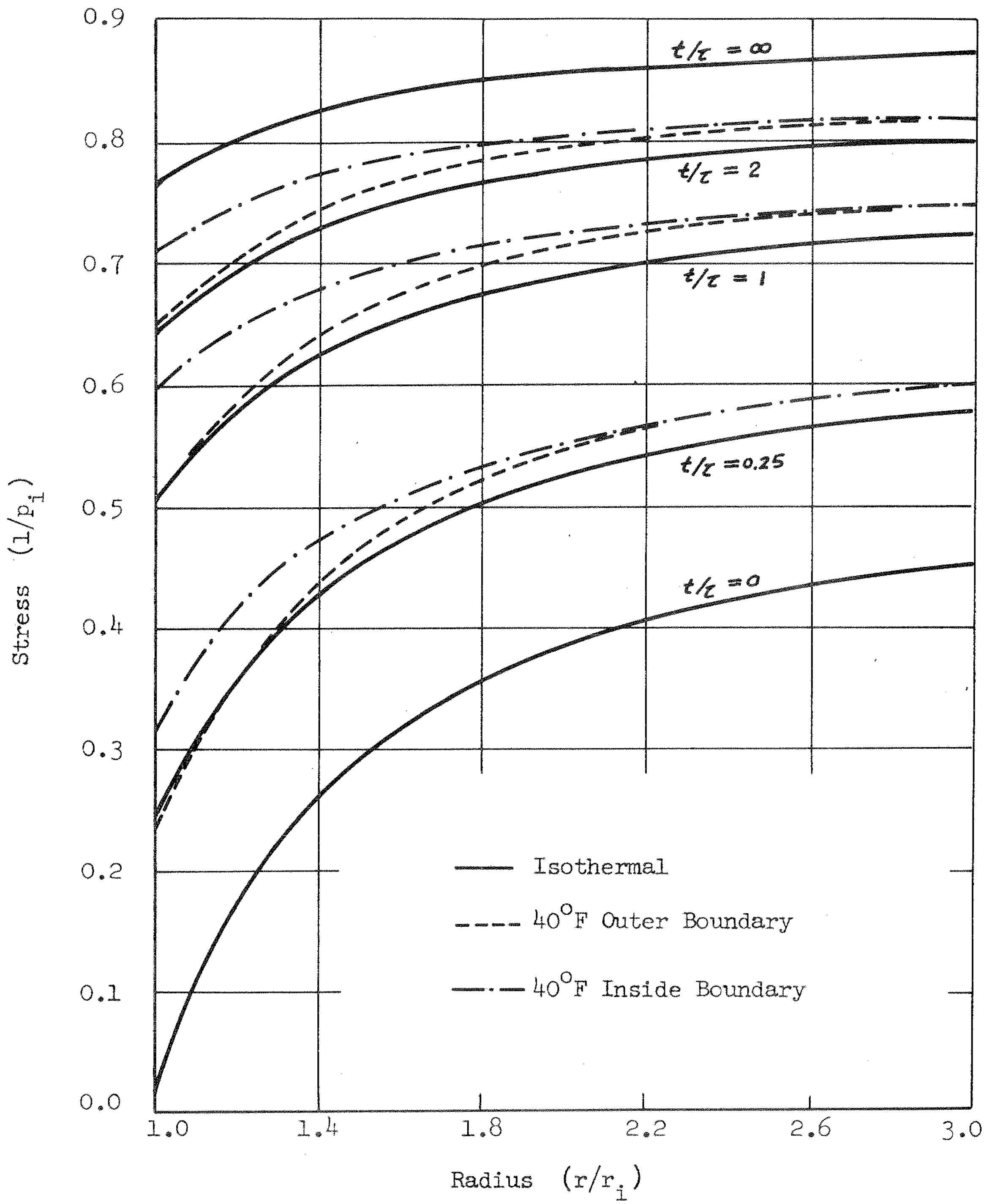

FIG.(6-4) TANGENTIAL STRESS IN CYLINDER

2. A Star Grain Bonded to A Rigid Case.

A grain section (as shown in Fig. (6-5)) bonded to a rigid case will be studied. In the present analysis, a constant internal pressure and uniform temperature field is considered. For simplicity the problem is treated as plane strain; accordingly, only a segment of the grain will be analyzed.

The segment is idealized as a system of 220 triangular elements and 132 nodal points. The internal pressure is approximated by concentrated forces acting at nodal points along the inner boundary. Points on the radial boundaries of the segment are fixed along the tangential direction and points on the outer boundary are prevented from moving along any direction.

Again, as no particular application is intended, the material properties to be used herein are the same as defined by equations (6.3) where the corresponding Poisson's ratio varies with time from 0.45 to 0.493.

Of particular interest are the variations of the hoop strain across the section A-B, the hoop stress and strain at the star point A and the contact stresses (radial stress and hoop stress) between the grain and the rigid case, (as shown in Figs. (6-6), (6-7) and (6-8) respectively.) It is of interest to note the change of stress state from tension to compression (Fig. (6-6)) and the high strain concentration (Fig. (6-7)) at the star point A. The stress distribution in the grain at large time becomes a near hydrostatic state (except the region near the star point) since the material is near incompressible.

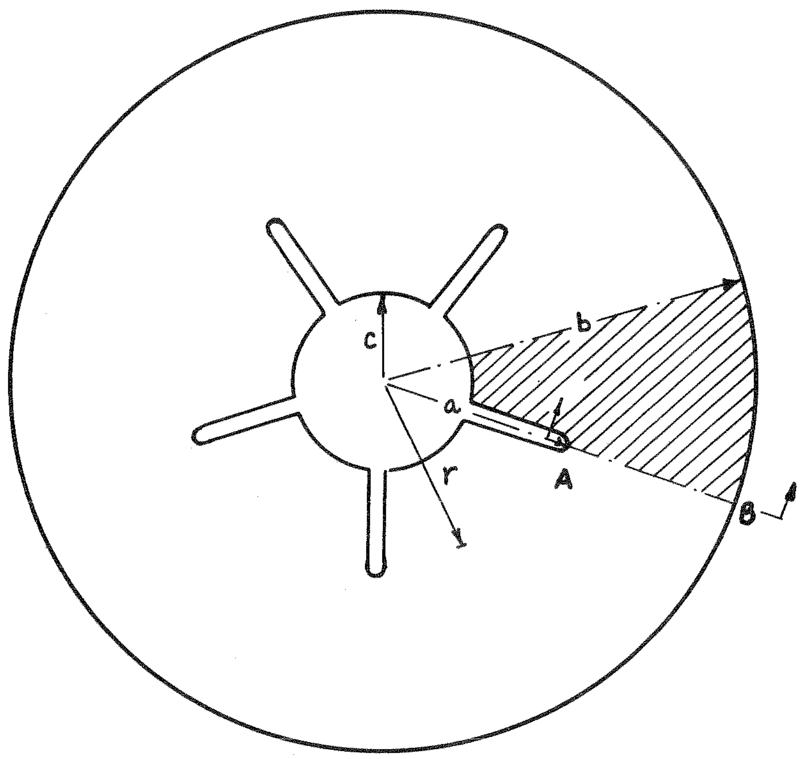

FIG.(6-5) STAR GRAIN SECTION

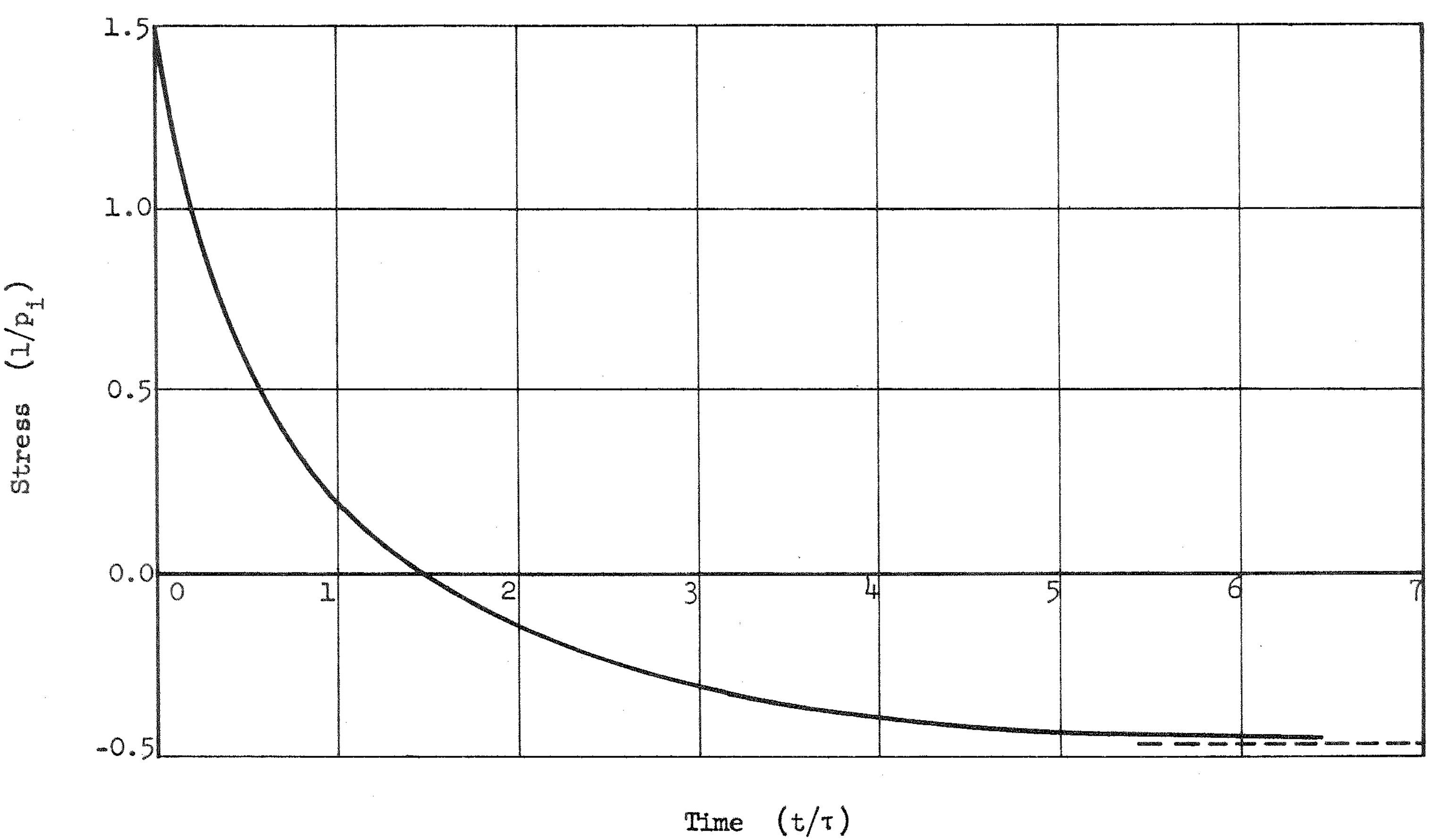

FIG.(6-6) HOOP STRESS AT THE STAR POINT "A"

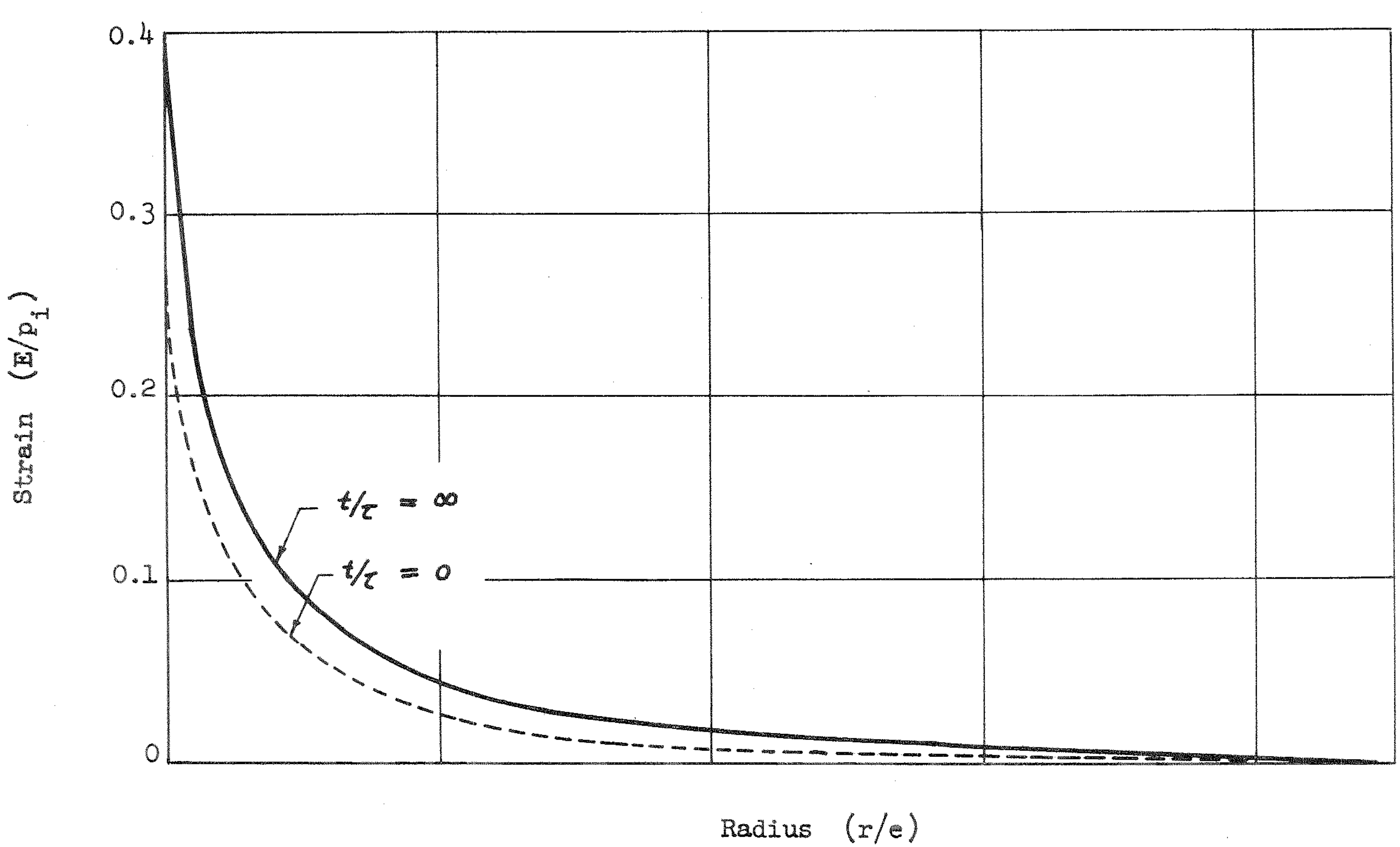

FIG.(6-7) HOOP STRAIN FOR SECTION A-B

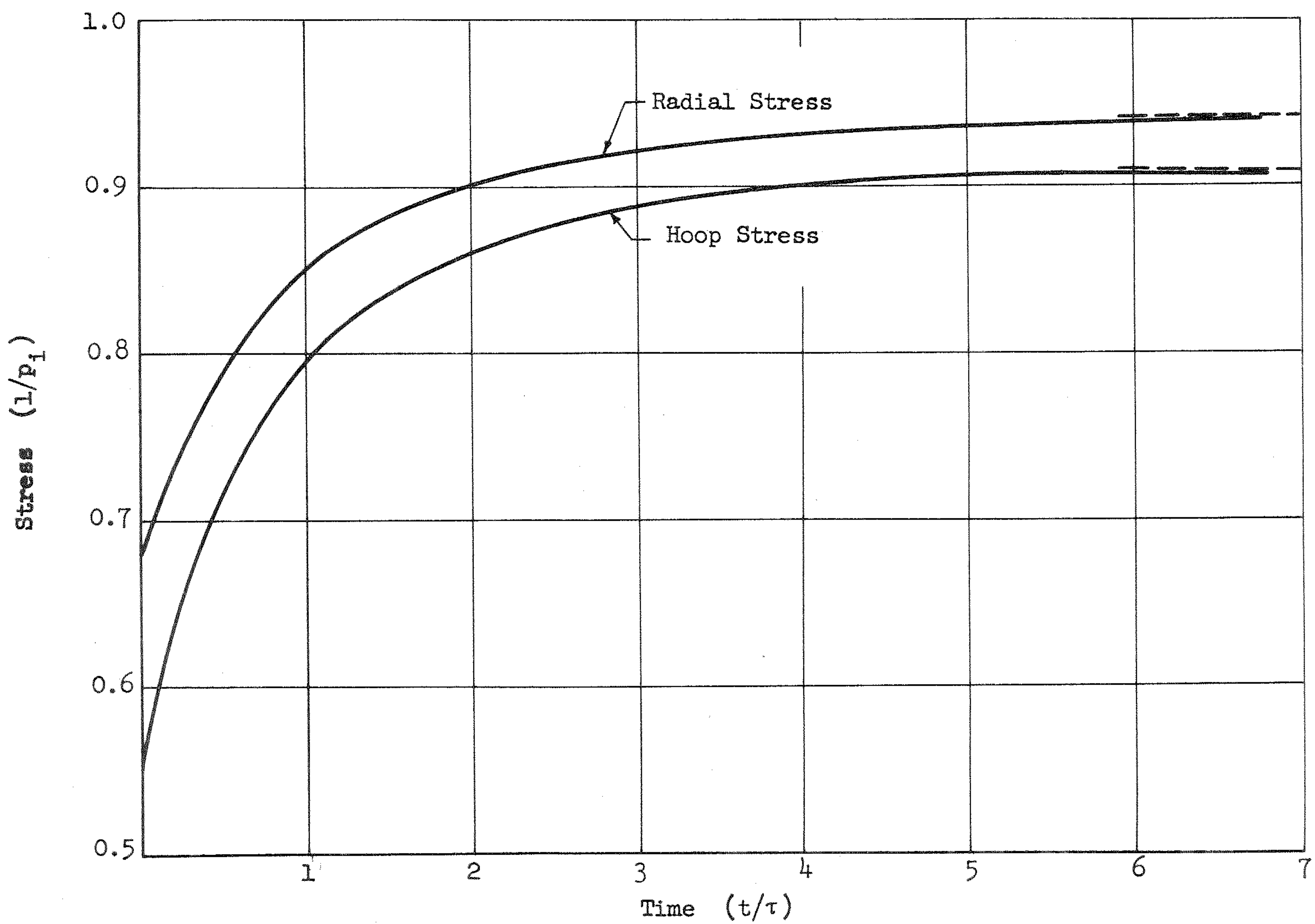

FIG.(6-8) CONTACT STRESSES AT B

VII. Conclusions

A numerical method of solution for stress analysis in viscoelastic media has been formulated and some representative results are obtained. As pointed out before, the method is readily applicable to problems of inhomogeneous, anisotropic materials. As far as numerical integration is concerned, the method can be extended to the application of visco-elastic media with more general behavior, e.g. the material may be described by a function of the form $f(t,T)$, where the variable T may represent the effect of temperature in thermoviscoelastic problems or the aging effects of the material. Furthermore, problems with moving boundary conditions present no difficulty in the analysis.

In addition, a heat conduction problem may also be solved by use of the finite element method [20]. Once the temperature variations in the solid are found numerically, they can be used as the input data in a computer program of the stress problem.

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