UC Berkeley

UC Berkeley Previously Published Works

Title

Extracting waves from noisy geomagnetic data – A synthetic study of equatorially trapped waves in Earth's core

Permalink

<https://escholarship.org/uc/item/47c317vm>

Authors

Knezek, N

Buffett, B

Publication Date

2019

DOI

10.1016/j.pepi.2018.10.003

Peer reviewed

Extracting waves from noisy geomagnetic data – A synthetic study of equatorially trapped waves in Earth's core

Nicholas Knezek, Bruce Buffett

Dept. of Earth and Planetary Science, University of California Berkeley, United States

ARTICLE INFO

Keywords:

Geomagnetic secular variation
Magnetohydrodynamic waves
Earth's core

ABSTRACT

Geomagnetic Secular Variation (SV) and acceleration (SA) can provide a wealth of information about fluid motions in the core. However, recovering this information from noisy data is complicated by the structure of the magnetic field at the core-mantle boundary. We demonstrate the challenge using a numerical model for equatorial waves to produce synthetic records of SV and SA. We show that interpreting magnetic oscillations directly as waves in Earth's core can lead to spurious conclusions due to coupling between non-dipole components of the magnetic field and wave structures. Using the structure of our modeled waves, we derive a set of methods to detect waves using magnetic field observations. We show that these methods can recover the wave period, longitudinal wavenumber, equatorial symmetry, latitudinal extent, and amplitude for synthetic equatorial waves in Earth's core when the wave fluid velocity is ~ 0.5 km/yr or greater.

1. Introduction

Recent high-resolution observations of Earth's magnetic field by satellite have been used to determine the field at the core-mantle boundary to a much higher level of detail than in the past. Several authors have reported short-period oscillations in the equatorial region of the core field (Finlay et al., 2016; Gillet et al., 2015b), and some have interpreted these signals as evidence for wave motions in a stratified layer in Earth's core (Chulliat et al., 2015). Prior studies have proposed the existence of a stratified layer near Earth's core-mantle boundary (CMB) using seismic (Tanaka, 2007; Helffrich and Kaneshima, 2010) or geomagnetic evidence at 30–100 year timescales (Whaler, 1980; Buffett, 2014), but recent magnetic observations represent a new and potentially rich source of information about the layer properties and CMB magnetic field.

Observed SV and SA are directly connected to flows in the core because fluid motions are the main cause of changes in the radial magnetic field at the CMB. However, our ability to observe small-scale short-timescale magnetohydrodynamic waves in Earth's core is fundamentally limited by the observable spatial and temporal scales. High-resolution satellite observations of the geomagnetic field sufficient to observe these waves have only been available since 1999 when the Oersted mission launched (Neubert et al., 2001). Since this time, a number of additional satellite missions have observed Earth's field (Reigber, 2005, 2015), and numerous core-field models have been released that take advantage of this data (Olsen, 2002; Finlay et al., 2012; Finlay et al., 2016). However, we still only have a total of ~ 19 years of high-resolution observations, which places an upper limit on the period of waves that can we can detect. Lower-mantle conductivity and

external solar-magnetospheric processes also impose limitations on the minimum timescale for detectable variations in the core magnetic field. Although the structure and strength of lower mantle conductivity is currently unknown, estimates show that the minimum resolvable timescale for large-scale (less than spherical harmonic degree 13) time-dependent flow in the core is expected to be around 3 years (Gillet et al., 2015b; Jault, 2015; Shore et al., 2016). Thus, small-scale waves in Earth's core can only currently be detected if they have periods between ~ 3 and ~ 19 years.

Similarly, the spatial resolution of observations is limited by the thick, insulating mantle and strong magnetic signals produced by crustal magnetic fields, ocean-bottom currents, and solar-magnetospheric processes. These sources influence observations of the core magnetic field (B), secular variation (SV), and secular acceleration (SA) in different ways, so that B can be reliably determined to degree 13 or 14, while useful descriptions of SV and SA can perhaps be determined up to degree 16 (Finlay et al., 2016). These spatial and temporal observation limits indicate that it may be possible to detect global waves with decadal periods propagating in the core.

2. Methods

To quantitatively assess the limits of these observations to observe waves, we first derive a description of equatorial waves in a stratified layer in Earth's core using a numerical model (Knezek and Buffett, 2018) in Section 2.1. Waves are chosen to match previous suggestions for the period and spatial extent of observed oscillatory signals in equatorial SA and SV (Chulliat et al., 2015). Then, these computed waves are used to produce synthetic SA and SV (Section 2.2) and to

<https://doi.org/10.1016/j.pepi.2018.10.003>

Received 10 March 2018; Received in revised form 23 September 2018; Accepted 4 October 2018

Available online 06 October 2018

0031-9201/ © 2018 Published by Elsevier B.V.

assess the effectiveness of previous methods used to study waves in Earth's core (Section 2.3). We propose an improved technique (Section 2.4) and describe methods to detect wave signals in SV as well as SA (Section 2.5). We also construct a noise model to demonstrate that the technique is robust to data uncertainties and other sources of error in SV (Section 2.6).

2.1. Magnetic equatorial waves

We use a hybrid finite-volume and Fourier (FVF) model to compute equatorial waves in Earth's core including the influence of magnetic fields, stratification, and Coriolis forces (see Knezek and Buffett, 2018 for details). The model produces waves that propagate in the equatorial region of a stratified layer with sub-decadal periods for a variety of layer structures and magnetic field configurations. These waves are similar to Rossby waves in Earth's atmosphere and oceans in that they are global, equatorially-trapped, and propagate in a thin stratified layer. However, the radial magnetic field interacts with the conductive fluid and modifies the wave dynamics. As fluid moves horizontally in the layer, it stretches the radial magnetic field which acts as a restoring force. Off the equator, the flow is affected by the Coriolis force which acts as a waveguide, keeping the wave energy near the equator in a manner similar to equatorially-trapped waves in Earth's oceans and atmosphere (e.g. (Matsumo, 1966)).

We find consistent patterns of wave motion for a variety of layer properties and magnetic fields. However, details like the period, latitudinal extent, and quality factor (a measure of wave damping) do change with the properties of the layer and field. In general, thinner layers and stronger background magnetic fields produce shorter wave periods and lower quality factors, while the latitudinal extent of the wave depends upon the distribution of magnetic field strength with latitude (see (Knezek and Buffett, 2018), Section 3.3). However, this relationship is non-linear and depends on the choice of background field structure and layer buoyancy, so a full examination of the connection between layer properties and wave structures is beyond the scope of this study. Instead, we wish to examine techniques for detecting geomagnetic signals produced by a wide variety of possible waves propagating in the stratified layer at the top of the Earth's outer core. Computing wave solutions for a wide variety of layer parameters using our FVF model is computationally expensive. Instead, we compute wave structures for a subset of layer and field structures and fit the flow fields to obtain a flexible parametric representation. By adjusting the parameters of this representation, we can adjust the flow structure and period of the waves over a wide range of conditions without having to re-compute the waves for each specific set of layer parameters.

A natural choice to fit equatorial waves are the Hermite basis functions indexed by n ,

$$\psi_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2} \quad (1)$$

because they are concentrated near $x = 0$, corresponding to the equator, and decay away from the equator. These functions are adapted from Hermite polynomials to provide an orthonormal basis over space without any weighting function.

To use them for waves on a sphere, we choose a parameter δ_θ such that $x = \theta/\delta_\theta$ where θ is latitude. Thus, δ_θ describes the width of the waves in degrees latitude.

The FVF model is formulated as an eigenvalue problem, which produces wave solution vectors that describe the real and imaginary components of flow velocity $v_{\theta,\phi}$ at each latitude θ on a grid. These are used to compute the velocity at the core-mantle boundary for a particular longitude ϕ through

$$v_{\theta,\phi}(\theta, \phi) = \Re\{[v_{\theta,\phi}^{\Re}(\theta) + iv_{\theta,\phi}^{\Im}(\theta)]e^{im\phi}\}. \quad (2)$$

We recover a flexible representation of the flow which we denote $\hat{v}$ using the Hermite basis functions

$$\hat{v}_\theta^{\Re}(\theta) = \sum_n c_{\theta,n}^{\Re} \psi_n(\theta/\delta_\theta) \quad (3)$$

$$\hat{v}_\theta^{\Im}(\theta) = \sum_n c_{\theta,n}^{\Im} \psi_n(\theta/\delta_\theta) \quad (4)$$

$$\hat{v}_\phi^{\Re}(\theta) = \sum_n c_{\phi,n}^{\Re} \psi_n(\theta/\delta_\theta) \quad (5)$$

$$\hat{v}_\phi^{\Im}(\theta) = \sum_n c_{\phi,n}^{\Im} \psi_n(\theta/\delta_\theta) \quad (6)$$

where $c_{j,n}^i$ are fit coefficients and δ_θ defines the width of the wave. These coefficients are found by minimizing

$$\sum_{i,j} (v_j^i - \hat{v}_j^i(c_{j,n}^i, \delta_\theta))^2$$

where $i = [\Re, \Im]$, $j = [\theta, \phi]$, and n is the Hermite basis function degree.

Although these waves have radial variation in their flow, fluid motions below the CMB do not produce significant observable SA and SV when the effects of magnetic diffusion are small. Therefore, we only fit flow at the core-mantle boundary with this representation.

Our FVF model produces two classes of equatorial waves with different equatorial symmetries of their flow. We denote these classes “symmetric” and “asymmetric” waves in reference to the equatorial symmetry of the geomagnetic signal they produce when advecting a pure dipole field, and examine the fundamental mode (in radial and latitudinal structure) of each class (see Fig. 1). We fit the results from the FVF model for these waves to Hermite basis functions and find that the complex coefficients $c_{j,n}^i$ are consistent across a range of realistic layer and magnetic field structures. Only the period, quality factor, and latitudinal extent (δ_θ) vary significantly as we change layer and field properties. Thus, we construct a set of symmetric and asymmetric complex coefficients $c_{j,n}^i$ that can be used to describe the complete flow structure of the wave at the CMB. These symmetric and asymmetric waves have most of their power in a single Hermite polynomial, making the flow structure of these two classes nearly orthogonal to each other. However, for accuracy, we fit each wave using the first seven Hermite basis functions to guarantee that they are well-described.

These fits can then be used to describe general symmetric and asymmetric waves by choosing a particular set of parameters including the longitudinal wavenumber m , period T , and width parameter δ_θ for the wave. Typical values of δ_θ for the waves derived using our FVF model are in the range $\delta_\theta = 10^\circ$ to 30° . See Fig. 2 for examples of wave structures with δ_θ varying across this range. Because the FVF model solves an eigenvalue problem, the amplitude of the solutions and therefore the wave velocities are unconstrained. Therefore, we estimate the wave amplitude by choosing a peak flow velocity that produces SA magnitudes consistent with the observed SA. We find that peak flow velocities of 1–2 km/yr produce consistent SA. The final parameter we must choose to fully describe these waves is their initial longitudinal location, or phase ϕ_0 . A full description of the wave solution is given by

$$v_{\theta,\phi}(\theta, \phi, t) = \Re\{[v_{\theta,\phi}^{\Re}(\theta) + iv_{\theta,\phi}^{\Im}(\theta)]e^{im(\phi+\phi_0)+i\omega t}\}. \quad (7)$$

where t is time and $\omega = 2\pi/T$ is the wave frequency. The representation in (7), based on fits to the numerical model, gives a complete and flexible description of the flow, which we use throughout the remainder of this study.

2.2. Generation of secular variation and acceleration

Fluid motions at the core-mantle boundary advect the main magnetic field, contributing to geomagnetic secular variation (SV) and secular acceleration (SA). Secular variation (i.e. $\partial_t B_r$) is governed by the induction equation

$$\partial_t B_r = -\nabla_H \cdot (\vec{v}_H B_r) + \frac{\eta}{r} \nabla^2 B_r \quad (8)$$

Fig. 1. Horizontal wave structures for two equatorial wave modes using Hermite fits derived from our FVF numerical model solutions. Structures shown have $\delta_\theta = 30^\circ$, and a longitudinal wavenumber $m = 6$. Colors represent divergence (red) and convergence (blue), and arrows show horizontal flow structure. Waves are denoted “symmetric” (a) and “asymmetric” (b) in reference to the SV and SA they produce when advecting a pure dipole magnetic field (see Fig. 3a, b). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 2. Longitudinal flow (v_θ) structure variation for the symmetric wave with different values of δ_θ . Peak flow velocity normalized to one.

where B_r is the radial field at the CMB, $\vec{v}_H$ is the horizontal fluid velocity, and η is magnetic diffusivity (Holme, 2015). We ignore radial fluid motions v_r , as they vanish at the CMB and $v_r \ll v_H$ within the stratified layer below. In a stratified layer the diffusion term is dominated by the radial length scale L_r . Typical layers examined have $L_r \sim 50$ km to 300 km, while typical horizontal scales are $L_H \sim R/m \sim 300$ km for $m = 6$, where R is the radius of the core. We obtain a characteristic timescale for magnetic diffusion of $\tau \sim L_r^2/\eta \sim 80$ yrs for the thinnest layers and 316 yrs for the thickest layers, both much larger than the ~ 8 yr period of waves examined. This indicates the error introduced by ignoring magnetic diffusion should be small for periodic motions like these waves. Barrois et al. (2017) finds that diffusion contributes up to 20% of SV across all scales, placing an upper bound on the error introduced by this approximation. Expanding the horizontal divergence in (8),

$$\partial_t B_r = -\vec{v}_H \cdot \nabla_H B_r - B_r \nabla_H \cdot \vec{v}_H \quad (9)$$

explicitly shows the advection and divergence terms.

Secular acceleration is similarly generated by fluid motions and can be computed by taking the time derivative of secular variation in (8). When the diffusive term is dropped for the reasons outlined above we obtain

$$\partial_t^2 B_r = -\nabla_H \cdot (\vec{a}_H B_r) - \nabla_H \cdot (\vec{v}_H \partial_t B_r) \quad (10)$$

where $\vec{a}_H = \partial_t \vec{v}_H$. With terms expanded we compute secular acceleration using

$$\partial_t^2 B_r = -\vec{a}_H \cdot \nabla_H B_r - B_r \nabla_H \cdot \vec{a}_H - \vec{v}_H \cdot \nabla_H \partial_t B_r - \partial_t B_r \nabla_H \cdot \vec{v}_H \quad (11)$$

One key aspect of these equations is that the SA and SV depends upon the fluid velocities ($\vec{v}_H$) and accelerations ($\vec{a}_H$) of the wave as well as the structure of the background magnetic field B_r . When the background B_r is simple, such as with a pure dipole field, the waves described in Section 2.1 produce signals from which it is easy to determine their longitudinal wavenumber and symmetry across the equator (see Fig. 3a and b). This would allow one to assess the structure of these waves directly from the observed SV and SA. However, when the same waves are used to produce SV and SA using the observed B_r up to degree 14 from the CHAOS-6 model (Finlay et al., 2016) they produce complex signals from which neither the longitudinal wavenumber or equatorial symmetry are apparent (see Fig. 3c, d). This effect is similar to that observed by Cox et al. (2016), who explore whether global torsional waves can couple with heterogeneous background magnetic fields to produce localized signals in SV resembling geomagnetic jerks.

Coupling between flows and the background magnetic field also introduce errors into these calculations. We are only able to observe the main field up to degree ~ 14 , but flows coupling with unobserved magnetic fields with degree > 14 will also contribute to the SV observed in the core.

2.3. Period-wavenumber observations

Previous attempts to overcome the complexities in SV and SA have relied on a two-dimensional Fourier transform to convert the data from a time-longitude domain to period-wavenumber (PWN) space. Chulliat et al. (2015) used this method to examine signals near the equator by summing (differencing) time-longitude slices of observed SA at 5, 10, and 15 degrees latitude above and below the equator to compute the amplitude of symmetric (asymmetric) waves in PWN space. They then attribute peaks in these symmetric (asymmetric) SA PWN plots to symmetric (asymmetric) equatorial waves in Earth's core. However, the results of the previous section raise questions about the correspondence between peaks in PWN space and the presence of waves in the core. We

Fig. 3. Secular acceleration produced by symmetric (a, c) and asymmetric (b, d) wave modes advecting a pure dipole field (a, b) and the observed geomagnetic field (c, d) in 2010 (Finlay et al., 2016). Note that these symmetric and asymmetric waves are those depicted in Fig. 1, and that waves couple to small-scale structures in the observed geomagnetic field to produce complex signals. Waves used for these computations have a longitudinal wavenumber $m = 6$ and $\delta_\theta = 14^\circ$.

can assess the validity of this approach using known wave motions.

To compute a PWN plot, the input data must only have dimensions of time and longitude. Chulliat et al. (2015) selects SA data at 0, 5, 10, and 15 degrees latitude and uses sums or differences of the values north and south of the equator to examine symmetric and asymmetric waves. This approach discards data from all other latitudes, including those near the equator that should contain information about wave motions. In order to fully utilize this information, we generalize this approach by adopting continuous weight functions over latitude that are either symmetric or asymmetric across the equator. We use the first ($n = 0$) Hermite basis function (1)

$$\psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} \quad (12)$$

for symmetric waves and the second ($n = 1$) function

$$\psi_1(x) = \sqrt{2} \pi^{-\frac{1}{4}} x e^{-\frac{1}{2}x^2} \quad (13)$$

for asymmetric waves, where $x = \theta/\delta_\theta$. With these weight functions, we can adjust δ_θ to maximize the amount of data from wave motions included in our analysis and exclude signals far from the equator unlikely to be due to waves.

To test whether it is possible to reliably differentiate between symmetric and asymmetric waves using a PWN transform, we compute the PWN of the SA for different combinations of wave symmetry and weighting function symmetry (Fig. 4). For this analysis, all parameters other than symmetry are held constant between the two waves, including the peak wave velocity and latitudinal extent.

A symmetric wave examined using a symmetric weight function (12) produces a single strong signal at the correct period and wavenumber (Fig. 4a). Likewise, an asymmetric wave examined using an asymmetric weight function (13) recovers the correct wave parameters (Fig. 4d). However, an asymmetric wave also produces significant power in the PWN plot when using a symmetric weight function, as seen in (Fig. 4c). There are multiple peaks of similar magnitude at various wavenumbers, with very little power at the true wavenumber used. One small piece of good news is that these spurious peaks are all a factor of five smaller than the true peak observed from the symmetric wave using a symmetric weight function. However, when a symmetric wave is examined using an asymmetric weight function, it produces multiple spurious peaks in the PWN spectrum (Fig. 4b), many of which have a similar magnitude to the single peak obtained from an

asymmetric wave.

This result illustrates that the PWN transformation cannot reliably distinguish signals from symmetric and asymmetric equatorial waves, even in the simplest possible case of only a single wave mode. This result is robust when using a wide variety of weight-functions, including constant weight with latitude, square functions, and averaging observations at particular latitudes north and south of the equator as used in (Chulliat et al., 2015).

The situation is worse when multiple waves are present (as is likely in Earth's core), as it becomes impossible to determine if a signal arises from a single asymmetric wave (Fig. 3d) or multiple symmetric waves (Fig. 3b). Also, the strength of the FFT is directly related to the amplitudes of the waves, so even the weak signals seen in Fig. 3c could result from a single asymmetric wave (as in this case) or multiple low-amplitude symmetric waves.

2.4. Wave correlations with observations

Instead of simply transforming observed SA to look for wave signals, we can utilize our knowledge of the structure of our equatorial waves to look for correlations between signals produced by our waves and observed SA. First, we select the wave by specifying a representative value for δ_θ and choose the wavenumber (m), period (T), and phase (ϕ_0). Then we compute the SA_{wave} for our choice of wave and evaluate its correlation with the observed SA_{obs} .

We calculate the correlation between SA_{obs} and the SA_{wave} by first applying a weight function w_θ across latitude to each to obtain $X = w_\theta \cdot SA_{\text{obs}}$ and $Y = w_\theta \cdot SA_{\text{wave}}$. Then, we compute the correlation between the weighted datasets using

$$\rho_{X,Y} = \sum_{\theta, \phi, t} \frac{(X - \mu_X)(Y - \mu_Y)}{\sigma_X \sigma_Y} \quad (14)$$

where μ and σ are the mean and standard deviation of each weighted dataset. We evaluate this cross-correlation across a range of periods T , wavenumbers m , and the full range of phases ϕ_0 from 0 to 2π . This allows us to find the peak correlation across these three parameters in the ranges of interest. With this method, we can focus on either symmetric or asymmetric waves by choosing only one type of wave to generate the SA_{wave} .

To demonstrate this method, we create 15 years of synthetic

Fig. 4. Period-wavenumber (PWN) transformations of SA produced by waves advecting the observed geomagnetic field. Waves were computed with $m = 6$, period = 7.5 yrs, $\delta_\theta = 14^\circ$; and are either symmetric (a, b) or asymmetric (c, d). PWN transformations were computed using weighting functions that are symmetric (a, c) or asymmetric (b, d). Note that the symmetric flow transformed with a symmetric weight function gives a single strong signal recovering the wave properties, but that the same wave also produces a weaker signal in many wavenumbers when examined with an asymmetric weighting function. Likewise, the asymmetric wave produces power in many wavenumbers when examined with a symmetric weight function, and only produces a weak signal of similar magnitude to the symmetric wave using asymmetric weights.

Fig. 5. Correlation of symmetric (a) and asymmetric (b) waves across different wavenumbers and periods with a single SA signal produced by advecting the observed field using four different waves (see text). Stars on the plot designate the period and wavenumber for each wave. Note that both plots are produced using the same SA signal, yet this method clearly distinguishes between symmetric and asymmetric waves with little spurious power bleeding between symmetries.

observations of SA using four waves and use them to advect the observed geomagnetic field simultaneously. Two waves are symmetric and have periods and wavenumbers of $T = 7.5$ yrs, $m = 6$ and $T = 8.5$ yrs, $m = -4$, respectively. The other two waves are asymmetric with $T = 10.5$ yrs, $m = 6$ and $T = 9.5$ yrs, $m = -3$. All waves have $v_0 = 1$ km/yr, while their δ_θ are randomly chosen between from 8° to 16° and phases distributed uniformly between 0° and 360° . The computed correlations shown in Fig. 5 show clear recovery of all four wave modes. Other peaks in T and m have much less power.

2.5. Steady flow and residual SV

The correlation analysis can also be applied to SV. There are several reasons for examining SV in addition to SA. First, we can obtain a check against random noise producing spurious wave detections as SV is more well-resolved than SA at smaller length scales and suffers fewer artifacts near the start and end of the time series in models that regularize their solutions (Finlay et al., 2016). Second, simultaneous detection of waves with identical properties in both SA and SV is much less likely to occur

due to chance. To see why, note that some processes in the core, such as steady fluid motions, produce power dominantly in SV. Likewise, other processes such as small-amplitude transient fluid motions produce little SV but significant SA. Each of these processes might individually produce signals in either SA or SV that could be interpreted as wave motions. However, simultaneous wave correlations in both SA and SV must result from a process that has a precise relationship between fluid velocity and acceleration (e.g., $\partial_t v = i\omega v$).

To apply the correlation analysis to SV, we first need to account for the contribution from large-scale steady fluid motions in the core. While these fluid motions do not significantly contribute to observed SA, a large-scale steady flow explains a significant fraction of SV (see Section 2.2). These flows have been estimated to have peak velocities on the order of ~ 10 – 20 km/yr (Holme, 2015). For comparison, equatorial waves with periods of ~ 10 years and $m = 6$ require peak flow velocities of only ~ 1 km/yr to produce a representative $|SA| \sim 2 \mu\text{T/yr}$ in the equatorial region of Earth's core. As this velocity is an order of magnitude less than estimates for large-scale steady flow in the core, we might expect this flow to obscure signals from wave motions in observed SV.

To examine whether this is true, we first fit a steady flow to the CHAOS-6 field model (Finlay et al., 2016) using the method described by Holme (2015). We choose a damping parameter to produce a reasonably smooth flow structure and minimize the residual at the CMB as this is the region we are interested in studying. The resulting steady flow velocity is shown in Fig. 6. Then, we use this steady flow to advect the observed geomagnetic field to produce SV and SA. Finally, we compare this result to SV and SA produced by representative equatorial waves. We show in Fig. 7 that steady flow produces strong SV and little SA, while the opposite is true for wave motions. The SA produced by equatorial waves is an order of magnitude larger than that produced by steady flow, even though the wave flow velocity is much smaller. The opposite is true for SV. This is easily explained by examination of Eqs. (9) and (11). Steady motions have large magnitude $\vec{v}$ compared to waves, so they would produce greater SV. On the other hand, steady flow by definition has $\vec{a} = 0$, so they produce very little SA.

Because only a small portion of total SV originates from wave motions, we seek to remove the contribution of large-scale steady flow from the total SV in order to examine wave motions. To do this, we compute the SV produced by our steady flow fit, then subtract this from the total SV to obtain

$$SV_{\text{residual}} = SV_{\text{observed}} - SV_{\text{steady flow}}.$$

This could include SV produced by the wave motions as well as transient motions unrelated to waves and effects of magnetic diffusion within the core (Barrois et al., 2017). Also, SV_{residual} will have errors introduced due to the small-scale flows and the imperfect steady-flow fit resulting from the limited temporal and spatial resolution of SV observations (Baerenzung et al., 2016; Pais and Jault, 2008; Gillet et al.,

Fig. 6. Steady flow structure fit to explain changes in the observed geomagnetic field model CHAOS-6 from 2001 to 2015.

2015a). We group all of these effects into a noise term SV_{noise} which gives

$$SV_{\text{residual}} = SV_{\text{wave}} + SV_{\text{noise}}.$$

This noise can potentially obscure any wave signals that exist in the data or cause spurious correlations that could be interpreted as waves. Indeed, applying our wave correlation method to the SV_{residual} results in strong correlations for a large number of waves across a variety of wave properties. In order to confidently identify waves, we need to quantify the level of correlation that arises from SV_{noise} .

2.6. Detection confidence in secular variation

We construct a stochastic noise model SV_{noise}^* to represent SV_{noise} arising from non-wave processes in SV_{residual} . We aggregate many realizations of this model to determine the baseline level of wave correlation expected when no waves are present. Then, we can attribute correlations above this baseline to waves with greater confidence.

Stochastic methods have been used in the past to study uncertainties in observations of Earth's magnetic field (e.g., COV-OBS.x1, Gillet et al., 2015a) and large-scale flow in the core (Gillet et al., 2015b). These techniques have been used with great success, but they are not directly applicable to the question of characterizing SV_{noise} . Instead, these models are intended to represent time variations in the total field, including contributions from a steady flow, waves, and other non-steady influences. However, we wish to evaluate signals arising in SV_{residual} with contributions from steady flow removed, which has not been previously considered in the literature.

We construct our SV_{noise}^* model by generating random gauss coefficients with an appropriate power spectrum in space that varies with time. The spatial power spectrum of total observed SV at the CMB has been well-studied (see, e.g. Finlay et al., 2016 Fig. 8). However, much of this power can be explained by large-scale steady flow in the core, which we remove in our analysis and therefore do not wish to simulate in our noise model. Examining just the portion of SV unexplained by steady flow (SV_{residual}), we find that the spatial power spectrum increases with degree l and are fairly evenly distributed across order m (see Fig. 8).

We must then find an appropriate way to treat the temporal variations of SV_{noise} . Previous authors have used an AR(1) stochastic process to find that the frequency spectrum of the total SV follows a power law distribution with exponent -2 (Lesur et al., 2018). However, as the authors of that study point out, AR(1) processes produce significant high-frequency noise which would need to be filtered to represent realistic observed SV. Also, the frequency spectrum of the total observed SV does not necessarily reflect the frequency spectrum of the SV_{noise} we wish to model.

We instead adopt an approach based on a Fourier expansion of the gauss coefficients of SV_{residual} to examine its temporal variation. This results in complex Gauss-Fourier coefficients $c_r(l, m, t)$ which are related to SV_{residual} through

$$SV_{\text{residual}}(\theta, \phi, t) = \frac{1}{T} \sum_{l=0}^{N_l} \sum_{m=-l}^l \sum_{k=0}^{N_k} c_r(l, m, k) Y_l^m(\theta, \phi) e^{i2\pi k t/T} \quad (15)$$

where l and m are spherical harmonic degree and order, k is Fourier frequency, $Y_l^m(\theta, \phi)$ is the complex spherical harmonic function, θ is colatitude, ϕ is longitude, t is time, and T is the duration of the time series. This transformation allows us to examine the spatial and temporal spectrum of SV_{residual} simultaneously.

We find that the magnitude of the coefficients increases linearly with degree l for $k = 0$, while magnitudes for $k \neq 0$ increase with l to $l \sim 12$ before beginning to decrease. Magnitudes are fairly evenly distributed across order m , and decrease with temporal frequency k (see Fig. 8). On the other hand, the phase of coefficients (i.e. the relative magnitude and sign of $g_n(l, m, k)$ and $h_n(l, m, k)$) are uniformly

Fig. 7. SV and SA produced by steady flow and wave motions (wave parameters: symmetric, $m = 6$, period = 8.5 yrs, peak velocity = 2 km/yr, $\delta_\theta = 10^\circ$). Note that steady flow produces strong SV and little SA, while wave motions produce strong SA and little SV.

distributed across all possible values with no apparent trends with l , m , or k .

We use the Gauss-Fourier coefficients of SV_{residual} as the basis for our SV_{noise}^* model. This approach could include contributions from any

waves that exist in the core, but this has the effect of raising the threshold for detection. We assume that the Fourier components of the complex Gauss coefficient are drawn from a normal distribution where the mean and variance (denoted μ_{fit} and σ_{fit}^2) are inferred from the

Fig. 8. Spectrum of the SV_{residual} . Each dot is the magnitude of an individual $c_r(l, m, k)$ (see (15)), plotted with respect to degree l , with colors corresponding to order m . Subplots show increasing Fourier frequencies k . Trends are seen in l and k , but no trend is seen with m (colors). Solid lines represent the trend of the mean magnitude of coefficients with l , fitted with linear ($k = 0$) or quadratic ($k > 0$) functions, while dotted lines show bounds of one standard deviation (see text for details).

complex coefficients of SV_{residual} for the CHAOS-6 model (Specific details are given below). Thus the amplitude of the coefficients of the noise model (denoted $c_n(l, m, k)$) are drawn from a normal distribution

$$|c_n(l, m, k)| \sim \mathcal{N}(\mu_{fit}(l, k), \sigma_{fit}^2(l, k)). \quad (16)$$

This allows us to construct our model without knowledge of the temporal power spectrum of SV_{residual} a priori, and does not introduce spurious high-frequency noise as an AR(1) process would.

There is considerable scatter in the means and variances computed from the coefficients at fixed l and k and variable m . More importantly, there are relatively few m values at low degree. Consequently, we adopt simple functional forms for μ_{fit} and σ_{fit} that depend on degree l and frequency k . We compute the mean $\mu(l, k)$ of the magnitude of each observed coefficient across m for each l and k . Then, we fit these means with a linear fit vs l for the constant Fourier term ($k = 0$) and a quadratic fit vs l for the time-varying ($k \neq 0$) terms, weighted by the number of m coefficients at each degree l .

$$\mu_{fit}(l, k) = \begin{cases} \alpha_0 + \alpha_1 l & \text{for } k = 0 \\ \alpha_0 + \alpha_1 l + \alpha_2 l^2 & \text{for } k \neq 0 \end{cases}$$

We perform a similar analysis for the standard deviation σ . We compute the sample standard deviation $\sigma(l, k)$ around $\mu(l, k)$ to provide an estimate of the coefficient variance across order m . We assume the standard deviation varies linearly across l to find $\sigma_{fit}(l, k)$, again weighted by the number of m coefficients at each l .

Using $\mu_{fit}(l, k)$ and $\sigma_{fit}(l, k)$ in (16), we generate a random magnitude for each complex coefficient of the noise model $|c_n(l, m, k)|$. Then, we generate a random phase for each coefficient $\gamma(l, m, k)$ by drawing from a uniform distribution on the interval 0 to 2π

$$\gamma(l, m, k) \sim \text{Unif}[0, 2\pi). \quad (17)$$

We combine the magnitude and phase to obtain a set of coefficients

$$c_n(l, m, k) = |c_n(l, m, k)|e^{i\gamma(l, m, k)}$$

which are then transformed to physical space and time using an equation similar to (15). Finally, an additional constraint is imposed that the root-mean-square amplitude of the generated SV_{noise}^* must match that of the observed SV_{residual} in the equatorial region by scaling the coefficients of each realization by a constant factor.

To determine whether wave signals are visible in SV_{residual} over this background noise, we produce 20 realizations of our SV_{noise}^* model with no synthetic waves added and compute the wave correlation across m and period. Then, for each point in (m, period) space we fit a normal distribution

Fig. 9. Period-wavenumber plot showing the threshold at which a correlation with SV_{residual} is likely to be a wave. Values shown are the 95% quintile of correlation with the synthetic noise model representing SV_{residual} . See text for details.

Fig. 10. Period-wavenumber plot showing the ratio of the correlation signal from a wave to the background correlation threshold. The signal uses a wave with a period of 7.5 years, $m = 6$, peak velocity of 1 km/yr, and includes one realization of the residual SV noise model. The background correlation threshold is a smoothed version of that shown in Fig. 9. Note that the wave correlation is almost twice the background correlation for the wave, with only small locations elsewhere where the ratio rises above one near the bottom of the plot.

to the histogram of these computed correlations to estimate the 95% quintile of correlation when no waves are present, which we plot in Fig. 9. This represents a floor for detection of wave motions using the wave correlation analysis for SV_{residual} . Any correlation below this threshold is likely due to spurious correlations with SV_{noise} , not the SV_{wave} we wish to find. The detection threshold varies from around 0.01 for wave periods of 5 years to 0.1 for wave periods of 12 years.

We use this detection threshold to examine the minimum wave velocity required for detection in SV . Using a peak flow velocity of 1 km/yr, we find a signal-to-noise ratio of almost 2 recovering the synthetic wave properties. In this analysis, there are only a few places elsewhere on the plot where the signal to noise ratio barely rises above one due to signals arising from the SV_{noise}^* model (see Fig. 10). Repeating this calculation across a range of velocities, we find that a peak wave flow velocity of approximately 0.3 km/yr is required for a wave with these parameters to appear above the background 95% noise level. This compares favorably with the $|v| \sim 1$ km/yr needed to explain the magnitude of observed equatorial SA. We find that the wave velocity needed for detection does not vary with the longitudinal wavenumber m , but does change with the wave period. We find that the wave velocity needed to rise above our detection threshold remains around 0.3 km/yr for wave periods between 3 and 7 years, then increases approximately linearly with the wave period to ~ 0.7 km/yr for waves with a 12-year period.

3. Results

The methods described above are shown to be able to recover wave period (T), longitudinal wavenumber (m), and equatorial symmetry from both SA (Fig. 5) and SV (Fig. 10). We show below that this technique can also accurately recover latitudinal extent (δ_θ) (Section 3.1), phase (ϕ_0) (Section 3.2), and peak flow velocity (v_0) (Section 3.3) for each wave in a set of waves.

3.1. Latitudinal extent of waves

The latitudinal extent δ_θ of the equatorial waves can vary greatly depending upon the layer and magnetic field properties. This means that the reliable recovery of the latitudinal extent of the waves can provide important insights. Also, an accurate estimate is required to

Fig. 11. Recovery of latitudinal extent (δ_θ) of a symmetric and asymmetric wave from correlations with synthetic SA. The synthetic SA includes two symmetric and two asymmetric waves with periods around 8 years, amplitudes 1 km/yr, δ_θ between 8° and 20° , and various longitudinal wavenumbers and phases. The symmetric wave examined has a true $\delta_\theta = 8^\circ$, which is recovered precisely by the peak correlation. For the asymmetric wave, peak correlation occurs for $\delta_\theta = 15^\circ$, very close to the actual value of $\delta_\theta = 14^\circ$.

obtain useful results from the wave correlation analysis described in Section 2.4. To estimate the latitudinal extent of these waves, we find the peak correlation across an array of values for δ_θ at the period, wavenumber, and phase corresponding to a particular wave. We show this technique for two wavenumbers in Fig. 11. These results were obtained using the synthetic SA produced by four simultaneous waves. All waves have periods around eight years, amplitudes of 1 km/yr, δ_θ between 8° and 20° , longitudinal wavenumbers of $-6, 3, 2$, and 6 , and uniformly random phases. The symmetric wave shown has a true $\delta_\theta = 8^\circ$, which is recovered precisely by the peak correlation. Peak correlation occurs for the asymmetric wave at $\delta_\theta = 15^\circ$, very close to the actual value of $\delta_\theta = 14^\circ$. By repeating this analysis for a large sample of waves with random parameters, we find that this technique recovers the true δ_θ within 2° in approximately 90% of the cases.

Notably, the correlation declines away from the peak much more quickly for the asymmetric wave compared to the symmetric wave. We believe this is because asymmetric waves have more complex flow structures than symmetric waves – small changes in δ_θ create larger discrepancies in the direction and magnitude of flow at any particular point, leading to a faster decline in the correlation with changes in δ_θ .

3.2. Phase determination with secular variation

We recover the phase of a particular wave by the wave correlation analysis with either SA or SV described in Section 2.4. While the correlation in the SV signal is generally lower than with SA due to the additional sources of noise described in Section 2.5, the shape of the peak correlation between SA and SV are remarkably similar. We show the correlation analysis for SA and SV produced by a wave with $m = 6$, a period of 7.5 years, and a phase of 120° in Fig. 12, demonstrating clear recovery of the correct wave properties, even with the inclusion of SV noise.

Wave phase can be a robust indication of waves when a consistent value is observed in both SA and SV. The observations have limited resolution in space ($l < \sim 14$) and time (periods ~ 3 to 19 years) (see Section 1), so resolution improves at larger time and spatial scales. On the other hand, detection of oscillations becomes more confident with more observed wave periods, so short-period waves are easier to detect. Thus, we might expect to preferentially detect waves with intermediate

periods $T \sim 10$ yrs and low spatial wavenumbers (e.g., $m < \sim 6$). This is precisely where waves are reported. Transient processes in the core produce both SA and SV, leading to the possibility of non-waves producing signals that correlate with similar wavenumbers and periods in both SA and SV. Also, equatorial processes unrelated to waves might give similar estimates for the latitudinal extent. However, observational constraints do not bias any particular value of phase, allowing for all phases to appear uniformly when computing wave correlations with random SA and SV data. Thus, it is unlikely that the computed phase would match between SA and SV unless a wave generated the observations.

To see why, note that general fluid motions in the core create SA and SV through (9) and (11), which couple all the Fourier coefficients describing the flow with all the coefficients of the background magnetic field magnitude and gradients. Because of this coupling, components across all m and l values could potentially contribute to a correlation analysis performed at a single wavenumber m , causing peak correlations at random phases for any particular set of wave parameters examined. Thus, if the same phase produces the strongest correlations for a particular set of wave parameters in both SA and SV, it provides strong evidence of wave motion.

3.3. Peak flow velocity determination

The last parameter needed to describe a set of observed waves are the amplitudes or peak flow velocity of each wave mode. Under the assumptions made in Section 2.2, the total SA produced by a set of waves is simply a linear combination of the SA_n produced by each wave mode n . Thus, in order to find the best-fit amplitudes C_n for a set of N waves, we find the best-fit coefficients to minimize the residual

$$Err = \left\langle SA_{\text{obs.}}(\theta, \phi, t) - \sum_n C_n SA_n(\theta, \phi, t) \right\rangle_{rms} \quad (18)$$

where $SA_{\text{obs.}}$ is the observed secular acceleration, and SA_n is the secular acceleration produced for each wave, with wavenumber, period, phase, and latitudinal extent determined using the methods described in previous sections. Then, it is straightforward to convert from the amplitude C_n to the peak wave velocity of each mode (recall that this is unspecified by the eigenvalue problem). Because the computed wave modes each have most ($\sim 85\%$) of their power in a single Hermite function, they are nearly orthogonal. Therefore, there is very little interaction between different wave mode's contribution to observations, since their structures do not significantly overlap. This means that the global minimum of the error is convex and has an easily identified minimum. We demonstrate this in Fig. 13, showing the smoothly concave residual for the amplitudes of two waves, with the minimum recovering the amplitude of each synthetic wave precisely. This technique is easily extended to as many wave modes as required. Additionally, this method is robust to over-fitting the number of waves, again because of the orthogonality of the wave modes. If we fit five waves to the same synthetic SA produced above using four waves, the analysis recovers the four nonzero wave amplitudes precisely and produces zero amplitude for the extraneous wave.

4. Conclusions

We examine techniques for extracting information about equatorial waves in Earth's core from observations of geomagnetic secular acceleration and variation. We describe a plausible suite of equatorial waves using a numerical magnetohydrodynamic model and use them to produce synthetic SA and SV. We demonstrate that coupling between the waves motions and the complex background field render straightforward methods of analysis ineffective, and describe a set of techniques to

Fig. 12. Correlation in SA and SV with noise as a function of phase and period. The synthetic SA and SV datasets are produced using a single symmetric wave with $T = 7.5$ yrs, $\phi_0 = 120^\circ$ (marked on each plot with a star), and $v_0 = 1$ km/yr. The correlation is computed using a symmetric wave. Note that the peak of the correlation occurs at the same period and phase in both SA and SV, despite the fact that the peak correlation in SV is only ~ 0.25 due to the inclusion of noise.

Fig. 13. Wave velocity fit using SA produced by two waves. The true amplitudes of the synthetic waves marked by a white star denoting 1.5 and 2.0 km/yr peak flow velocities. The goodness of fit is calculated as the minimum error divided by the error at a particular point so that a value of 1.0 denotes the best-fit, and smaller values have an increasing error. Note that because the system is linear and the waves are mostly orthogonal, the residual is convex in amplitude space with only one global best fit. Two waves are shown for visualization purposes, but this fit method can be easily extended to multiple dimensions to fit several waves.

recover a robust description of these waves from observations of SA and SV.

We use these methods to explore the current observational limits for waves of various periods and wavenumbers. Using a stochastic noise model based on observations, we show that waves can be detected in SV with peak flow velocities as small as 0.3 km/yr, which compares favorably with the ~ 1 km/yr velocities required to explain observed SA. We point out that the phase of a wave provides strong evidence of wave motion when independently derived using both SA and SV. Finally, we demonstrate a technique to simultaneously determine the amplitudes of a set of multiple waves from a single dataset. Together, these methods demonstrate that the detailed study of equatorial waves requires careful analysis, but is possible using recent observations of SA and SV.

Acknowledgments

We thank Richard Holme for graciously sharing code and methods used to obtain steady flow inversions from geomagnetic observations. We thank two anonymous reviewers for their helpful comments to improve this manuscript. This research was supported by grant EAR-1430526 from the National Science Foundation.

References

- Baerenzung, J., Holschneider, M., Lesur, V., 2016. The flow at the Earth's core-mantle boundary under weak prior constraints. *J. Geophys. Res.: Solid Earth* 121.
- Barrois, O., Gillet, N., Aubert, J., Oct. 2017. Contributions to the geomagnetic secular variation from a reanalysis of core surface dynamics. *Geophys. J. Int.* 211 (1), 50–68.
- Buffett, B., Mar. 2014. Geomagnetic fluctuations reveal stable stratification at the top of the Earth's core. *Nature* 507 (7493), 484–487.
- Chulliat, A., Alken, P., Maus, S., 2015. Fast equatorial waves propagating at the top of the Earth's core. *Geophys. Res. Lett.* 42 (9), 3321–3329.
- Cox, G.A., Livermore, P.W., Mound, J.E., Jun. 2016. The observational signature of modelled torsional waves and comparison to geomagnetic jerks. *Phys. Earth Planet. Inter.* 255, 50–65.
- Finlay, C.C., Jackson, A., Gillet, N., Olsen, N., 2012. Core surface magnetic field evolution 2000–2010. *Geophys. J. Int.* 189 (2), 761–781.
- Finlay, C.C., Olsen, N., Kotsiaros, S., Gillet, N., Tffner-Clausen, L., 2016. Recent geomagnetic secular variation from Swarm and ground observatories as estimated in the CHAOS-6 geomagnetic field model. *Earth Planets Space* 68 (1) 112–112.
- Gillet, N., Barrois, O., Finlay, C.C., 2015a. Stochastic forecasting of the geomagnetic field from the COV-OBS.x1 geomagnetic field model, and candidate models for IGRF-12. *Earth Planets Space* 67 (1) 71–71.
- Gillet, N., Jault, D., Finlay, C.C., Jun. 2015b. Planetary gyre, time-dependent eddies, torsional waves, and equatorial jets at the Earth's core surface. *J. Geophys. Res.: Solid Earth* 120 (6), 3991–4013.
- Helfrich, G., Kaneshima, S., 2010. Outer-core compositional stratification from observed core wave speed profiles. *Nature* 468 (7325), 807–812.
- Holme, R., 2015. Large-scale flow in the core. In: *Treatise on Geophysics*, vol. 8. Elsevier, pp. 91–113.
- Jault, D., Jul. 2015. Illuminating the electrical conductivity of the lowermost mantle from below. *Geophys. J. Int.* 202 (1), 482–496.
- Knežek, N., Buffett, B., Apr. 2018. Influence of magnetic field configuration on magnetohydrodynamic waves in Earth's core. *Phys. Earth Planet. Inter.* 277, 1–9.
- Lesur, V., Wardinski, I., Baerenzung, J., Holschneider, M., Mar. 2018. On the frequency spectra of the core magnetic field Gauss coefficients. *Phys. Earth Planet. Inter.* 276, 145–158.
- Matsuno, T., 1966. Quasi-geostrophic motions in the equatorial area. *J. Meteorol. Soc. Japan. Ser. II* 44 (1), 25–43.
- Neubert, T., Manda, M., Hulot, G., Von Frese, R., Frimdhall, E., Jrgensen, J.L., Friis-Christensen, E., Stauning, P., Olsen, N., Risbo, T., 2001. Ørsted satellite captures high-precision geomagnetic field data. *Eos* 82 (7), 81–88.
- Olsen, N., 2002. A model of the geomagnetic field and its secular variation for epoch 2000 estimated from Ørsted data. *Geophys. J. Int.* 149 (2), 454–462.
- Olsen, N., Hulot, G., Lesur, V., Finlay, C.C., Beggan, C., Chulliat, A., Sabaka, T.J., Floberghagen, R., Friis-Christensen, E., Haagmans, R., Kotsiaros, S., Lühr, H., Tffner-

- Clausen, L., Vigneron, P., 2015. The Swarm Initial Field Model for the 2014 geo-magnetic field. *Geophys. Res. Lett.* 42 (4), 1092–1098.
- Pais, M.A., Jault, D., May 2008. Quasi-geostrophic flows responsible for the secular variation of the Earth's magnetic field. *Geophys. J. Int.* 173 (2), 421–443.
- Reigber, C., 2005. *Earth Observation with CHAMP-Results from Three Years in Orbit*.
- Shore, R.M., Whaler, K.A., Macmillan, S., Beggan, C., Velínský, J., Olsen, N., 2016. Decadal period external magnetic field variations determined via eigenanalysis. *J. Geophys. Res.: Space Phys.* 121 (6), 5172–5184.
- Tanaka, S., Jul. 2007. Possibility of a low P-wave velocity layer in the outermost core from global SmKS waveforms. *Earth Planet. Sci. Lett.* 259 (3), 486–499.
- Whaler, K.A., 1980. Does the whole of the Earth's core convect? *Nature* 287 (5782), 528–530.