UNIVERSITY OF CALIFORNIA, SAN DIEGO

# On Nilpotence and Algebraicity in Algebras Over Uncountable Fields

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Alon Regev is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2008

DEDICATION

To my family.

# EPIGRAPH

I have an idea. An idea so smart, my head would explode if I even began to know what I was talking about.

—*Peter Griffin*

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ABSTRACT OF THE DISSERTATION

**On Nilpotence and Algebraicity in Algebras Over Uncountable Fields**

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Alon Regev

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Professor Lance Small, Chair

This thesis is primarily concerned with properties of nilpotence and algebraicity in algebras over fields. We study properties of certain non-commutative polynomials, which are called the order-symmetric polynomials. We give an alternative proof to Amitsur's theorem that algebraic algebras over uncountable fields have locally bounded degree. We also prove that the associated graded algebra of a filtered algebraic algebra over an uncountable field is nil.

# Chapter 1

# Introduction

We begin by introducing some basic definitions and notations. We will deal primarily with a noncommutative associative algebra, typically denoted $A$, over a field $k$. Although some of our discussion can be applied to rings which are not algebras over fields, we usually restrict our attention to algebras. We do not, in general, assume that our rings and algebras are unital.

We will use the following notation. If $S \subseteq A$ is any subset, then $(S)_A$ will denote the ideal generated by $S$, i.e. $(S)_A = ASA$. We denote by $k\langle S\rangle_A$ the $k$-subalgebra of $A$ generated by $S$. Equivalently, $k\langle S\rangle_A$ is the smallest subalgebra of $A$ that contains $S$. Without a subscript $A$, the expression $k\langle X\rangle$ will stand for the free $k$-algebra on $X$, which we describe next. We will generally "abuse" these notations by omitting the set-notation for finite sets. For example, $(a, b)_A$ will mean $(\{a, b\})_A$.

## 1.1 The Free Algebra and Polynomials

The free associative (noncommutative) unital $k$-algebra on a (finite or infinite) set $X$ of generators is denoted $k\langle X\rangle$. This algebra can be considered as the algebra of polynomials $f(x_1, \ldots, x_m)$, with coefficients in $k$, in indeterminates $x_1, \ldots, x_m \in X$, which commute with the elements of $k$ but not with each other. For any such polynomial we can write

$$f(x_1, \ldots, x_m) =$$

$$\gamma_1 x_{i_{1,1}} x_{i_{1,2}} \cdots x_{i_{1,d_1}} + \gamma_2 x_{i_{2,1}} x_{i_{2,2}} \cdots x_{i_{2,d_2}} + \ldots + \gamma_n x_{i_{n,1}} x_{i_{n,2}} \cdots x_{i_{n,d_n}}, \qquad (1.1)$$

where $i_{s,t} \in \{1, \ldots, m\}$, $\gamma_i \in k$, and

$$r \neq s \Rightarrow (i_{r,1}, \ldots, i_{r,d_r}) \neq (i_{s,1}, \ldots, i_{s,d_s}).$$

The words $\{x_1 \ldots x_m \mid m \geq 0, x_i \in X\}$ form a basis for $k\langle X\rangle$ over $k$ (when $m = 0$ or $d_r = 0$ we have the empty word which is by convention 1). In particular, the representation (1.1) is unique (up to the order of the summands). The monomials $\gamma_j x_{i_{j,1}} \cdots x_{i_{j,d_j}}$, are called the *terms* of $f$. A polynomial of the form (1.1) is said to be *homogeneous in* $x_i$ if each of its terms has the same number of occurrences of $x_i$. It is said to be *homogeneous in total degree*, if all of its terms have the same total degree, i.e. if all $d_j$ are equal.

**Remark 1.1.** Note that there is an ambiguity in using the terms "degree" and "homogeneous". We use the term "homogeneous" to describe the polynomials in $k\langle X\rangle$, as well as elements of the homogeneous components of a graded algebra. We use the term "degree" in the context of polynomials, graded algebras, and algebraic elements. To avoid confusion we use "total degree" for polynomials in $k\langle X\rangle$ as above, and specify "homogeneous element of degree" or "degree of algebraicity" to distinguish between the latter two contexts.

Similarly to $k\langle X\rangle$, we define $k^+\langle X\rangle$ to be the free associative non-unital $k$-algebra on generators $X$. This is the algebra of noncommutative polynomials $f(x_1, \ldots, x_m)$ with coefficients in $k$ and zero constant term.

By the standard universal property, any unital $k$-algebra $A$ can be presented as

$$A = \frac{k\langle X \rangle}{J}, \tag{1.2}$$

where $X$ is a set of generators, and $J = (\{f_\rho\}_{\rho \in I})_{k\langle X \rangle}$ is the ideal generated in $k\langle X \rangle$ by the polynomials $f_\rho$, which are called the the *relations* of $A$. Similarly, any non-unital $k$-algebra $A$ can be presented as

$$A = \frac{k^+\langle X \rangle}{J}. \tag{1.3}$$

Note that $k^+\langle X \rangle$ is an ideal in $k\langle X \rangle$, but not a homomorphic image of $k\langle X \rangle$.

In this way the elements of $A$ can also be thought of as polynomials, inheriting from (1.1) the representation

$$f(a_1, \ldots, a_m) = \gamma_1 a_{i_{1,1}} a_{i_{1,2}} \cdots a_{i_{1,d_1}} + \ldots + \gamma_n a_{i_{n,1}} a_{i_{n,2}} \cdots a_{i_{n,d_n}}, \tag{1.4}$$

where $a_i = x_i + J$. Thus the set $X + J = \{x + J \mid x \in X\} \subseteq A$ generates $A$ as an algebra. Note that the representation (1.4) is no longer unique. In the discussion below, we will generally denote elements of a free algebra by $x$ (or $x_i$, etc.), and elements of a general algebra by $a$ ($a_i$, etc.).

## 1.2 Filtered and Graded Algebras

We define a filtered algebra as follows.

**Definition 1.2.** Let $A$ be $k$-algebra and let $(F_n)_{n \geq 0}$ be a sequence of subspaces of $A$. We say that $(F_n)_{n \geq 0}$ is a *filtration* of $A$ (or that $A$ is *filtered* by $(F_n)_{n \geq 0}$), if

$$F_i \subseteq F_{i+1} \text{ for all } i \geq 0, \tag{1.5}$$

$$\bigcup_{n=0}^{\infty} F_n = A, \tag{1.6}$$

and

$$F_i F_j \subseteq F_{i+j} \text{ for all } i, j \geq 0. \tag{1.7}$$

For example, the free unital algebra $k\langle X\rangle$ is filtered by the filtration

$$F_n = \{\text{polynomials in } X \text{ of total degree} \leq n\}$$

$$= \operatorname{span}_k\{x_1 \cdots x_r \mid 0 \leq r \leq n; x_i \in X\}. \tag{1.8}$$

Note that since $r = 0$ gives the empty product 1, we have $F_0 = k$.

The filtration 1.2 is called the *standard filtration* with respect to the set of generators $X$. In fact, any $k$-algebra can be given the standard filtration. Let $\frac{k\langle X\rangle}{J}$ be an arbitrary unital $k$-algebra. Then $A$ inherits the standard filtration from $k\langle X\rangle$ as follows:

$$F_n = \operatorname{span}_k\{a_1 \cdots a_r \mid 0 \leq r \leq n; a_i \in X + J\}. \tag{1.9}$$

Here the empty product stands for $1 + J$, which is the unit element of $A$.

In a similar manner to (1.2), the free nonunital algebra $k^+\langle X\rangle$ is filtered by

$$F_0 = \{0\}, \text{ and}$$

$$F_n = \operatorname{span}_k\{x_1 \cdots x_r \mid 1 \leq r \leq n; x_i \in X\}, \text{ for } n \geq 1. \tag{1.10}$$

It follows similarly that any nonunital algebra also inherits this filtration. In general, if $(F_n)_{n\geq 0}$ is any filtration if $A$, then $F_0$ is a subalgebra of $A$, and $k \subseteq F_0$ if and only if $A$ is unital.

**Remark 1.3.** This type of adjustment from unital to nonunital algebras is necessary in many parts of our discussion below. However, it is usually obvious how to make these adjustments and hence we will often omit this.

In Chapter 3 we discuss some properties of filtered algebras. We will not need to assume the filtration is the standard one.

A notion related to filtered algebras is that of graded algebras.

**Definition 1.4.** Let $A$ be a $k$-algebra and let $(A_n)_{n\geq 0}$ be a sequence of subspaces of $A$. We say that $(A_n)_{n\geq 0}$ is an $\mathbb{N}$-*grading* of $A$ (or that $A$ is $\mathbb{N}$-*graded*), if

$$A = \bigoplus_{n=0}^{\infty} A_n, \tag{1.11}$$

with

$$A_i A_j \subseteq A_{i+j} \text{ for all } i, j \geq 0. \tag{1.12}$$

The $A_i$ are called the *homogeneous components* of $A$, and any $a \in A_i$ is said to be a *homogeneous element of component degree* $i$.

In our context all graded algebras are $\mathbb{N}$-graded and we will refer to such algebras simply as "graded algebras". Note that the condition

$$A_i \cap A_j = \{0\} \text{ for all } i \neq j \tag{1.13}$$

is implied from 1.11 in the definition. This allows us to "compare homogeneous components" in any equation in a graded algebra.

The free unital algebra $k\langle X\rangle$ is an example of a graded algebra, with the homogeneous components

$$(k\langle X\rangle)_n$$

$$= \{\text{homogeneous polynomials in X of total degree n}\}$$

$$= \text{span}_k\{x_1 \cdots x_n \mid x_i \in X\}. \tag{1.14}$$

Consider now an arbitrary unital algebra $A = \dfrac{k\langle X\rangle}{J}$, with $J = (\{f_\rho\}_{\rho\in I})_{k\langle X\rangle}$. We can attempt to apply the same grading to $A$, namely $A_n = (k\langle X\rangle)_n + J$. The condition 1.12 is then automatically inherited from the grading of $k\langle X\rangle$. However, the condition 1.13 holds if and only if $(A_i + J) \cap (A_j + J) = J$. This is equivalent to the condition that if $a_i + a_j \in J$, with $a_i \in A_i, a_j \in A_j$, then $a_i, a_j \in J$. It follows that graded algebras are precisely those algebras that can be written in the form

$A = \dfrac{k\langle X \rangle}{J}$ with $J = (\{f_\rho\}_{\rho \in I})_{k\langle X \rangle}$, where all $f_\rho$ are homogeneous polynomials in $X$. This situation is similar for nonunital algebras.

For any graded algebra $A$, we use the notation

$$A_{\le n} = \bigoplus_{n'=0}^{n} A_{n'} \tag{1.15}$$

and

$$A_{\ge n} = \bigoplus_{n'=n}^{\infty} A_{n'}. \tag{1.16}$$

Note that if $A = \bigoplus_{n=0}^{\infty} A_n$ is a graded algebra, then we have a filtration $(F_n)_{n\ge 0}$ on $A$, where

$$F_n = A_{\le n}. \tag{1.17}$$

This is known as *the filtration induced by the grading* $(A_n)_{n\ge 0}$. In the other direction, if $A$ is filtered then it is not necessarily graded. However, for any filtered algebra we can define the following graded algebra.

**Definition 1.5.** Suppose that $A$ is a $k$-algebra filtered by $(F_n)_{n\ge 0}$. We define the *associated graded algebra* of $A$ (with respect to this filtration) to be

$$\mathrm{gr}(A) = F_0 \oplus \frac{F_1}{F_0} \oplus \frac{F_2}{F_1} \oplus \dots, \tag{1.18}$$

with multiplication defined as follows. For $p, q \ge 0$, if $a_p + F_{p-1}$ and $a_q + F_{q-1}$, with $a_p \in F_p$ and $a_q \in F_q$, are arbitrary homogeneous elements of $\mathrm{gr}(A)$ (taking $F_{-1} = \{0\}$) then we set

$$(a_p + F_{p-1})(a_q + F_{q-1}) = a_p a_q + F_{p+q-1}. \tag{1.19}$$

This multiplication is well-defined by (1.5) and (1.7) of Definition 1.2. We extend the multiplication linearly to arbitrary elements of $\mathrm{gr}(A)$.

Note that from (1.19) it follows (by induction) that if $a_{p_1}, \ldots, a_{p_n}$ satisfy $a_{p_i} \in F_{p_i}$ for all $1 \leq i \leq n$ then

$$(a_{p_1} + F_{p_1 - 1})(a_{p_2} + F_{p_2 - 1}) \cdots (a_{p_n} + F_{p_n - 1}) = a_{p_1} a_{p_2} \cdots a_{p_n} + F_{p_1 + \ldots + p_n - 1}. \quad (1.20)$$

**Remark 1.6.** In the special case where $A$ is already a graded algebra, then the associated graded algebra $\mathrm{gr}(A)$, with respect to the induced filtration 1.17, satisfies

$$\mathrm{gr}(A) \cong A \quad (1.21)$$

as graded algebras (that is, there exists an algebra isomorphism between them that maps $(\mathrm{gr}(A))_n$ to $A_n$).

In the general case, the algebra $A$ and its associated graded algebra $\mathrm{gr}(A)$ share some properties, but may be quite different in general (see eg. [12, 1.6.6-1.6.9]). In Chapter 3 we present a result that relates the properties of nilpotence and algebraicity in $A$ and $\mathrm{gr}(A)$.

A related construction that we will use is the Rees algebra.

**Definition 1.7.** Let $A$ be a filtered $k$-algebra with the filtration $(F_n)_{n \geq 0}$. Let $A[z]$ be the algebra of polynomials in a central indeterminate $z$ with coeffcients in $A$. The *Rees algebra* $R \subseteq A[z]$ (with respect to this filtration) is defined as $R = \bigoplus_{n=0}^{\infty} F_n z^n$.

Property (1.7) of Definition 1.2 ensures that this is indeed a subalgebra of $A[z]$. We note the following relation between the associated graded algebra and the Rees algebra.

**Proposition 1.8.** *Let $A$ be a filtered $k$-algebra with the filtration $(F_n)_{n \geq 0}$, let $B = \mathrm{gr}(A)$ be its associated graded algebra, and let $R \subseteq A[z]$ be the Rees algebra. Then*

$$B \cong \frac{R}{zR}. \quad (1.22)$$

*Proof.* Define a homomorphism $\psi : R \to B$ by

$$\psi(a_0 + a_1 z + \ldots + a_n z^n) = a_0 + (a_1 + F_0) + \ldots (a_n + F_{n-1}). \tag{1.23}$$

Then $\psi$ is surjective, and

$$\ker \psi = \{a_0 + a_1 z + \ldots + a_n z^n \mid a_0 + (a_1 + F_0) + \ldots (a_n + F_{n-1}) = 0_B\}$$

$$= \{a_0 + a_1 z + \ldots + a_n z^n \mid a_0 = 0, a_i \in F_{i-1} \ \forall i \geq 1\}$$

$$= \{z(a_1 + a_2 z + \ldots + a_n z^{n-1}) \mid a_i \in F_{i-1} \ \forall i \geq 1\} = zR.$$

This proves the stated isomorphism. □

## 1.3 Nilpotence

We next shift our attention to nilpotent and algebraic elements in algebras. We begin by discussing nilpotence. This notion is relevant in any ring, and therefore in this section (and only here) we refer to general rings.

**Definition 1.9.** Let $R$ be a ring. We say that $a \in R$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$. Given $a \in R$, the minimal such $n$ is called the *index of nilpotence* of $a$ (or just "the *index*" of $a$).

A natural question to ask is whether the sum of two nilpotent elements in a ring is necessarily also nilpotent. For example, this is true in commutative rings. In general, however, this is not the case, as seen from the following example.

**Example 1.10.** Let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then $a$ and $b$ are both nilpotent since $a^2 = b^2 = 0$. However,

$$a + b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not nilpotent, as $(a+b)^2$ is the identity matrix. Note that this counter-example works in rings of matrices over any ring with $0 \neq 1$.

Thus, in general, the sum of nilpotent elements in not nilpotent. However, a related property holds in any ring. We use the following standard definition.

**Definition 1.11.** We say that a subset $S \subseteq R$ of a ring $R$ is *nil* if every $s \in S$ is nilpotent.

Thus we may consider nil rings and algebras, as well as nil ideals and subspaces. Note that this notion is different from the that of a nilpotent subset of a ring, which is conventionally defined as a set in which there is a bound on the length of any nonzero word.

**Proposition 1.12.** *If $I$ and $J$ are nil two-sided ideals in a ring $R$ then $I + J$ is also nil.*

*Proof.* Let $a \in I$ and $b \in J$. Since $I$ is nil, $a$ is nilpotent, say of degree $n$. Then

$$(a + b)^n = a^n + c = c, \tag{1.24}$$

where $c$ is the sum of all the terms other than $a^n$. All of these terms include $b$ as a factor, and hence (since $J$ is a two-sided ideal containing $b$), we have $c \in J$. Thus $(a + b)^n \in J$. Since $J$ is nil, this means $(a + b)^n$ is nilpotent, and hence $(a + b)$ is nilpotent. Thus the sum of nil 2-sided ideals is a nil 2-sided ideal as well. □

Note that we have only used only the fact that $J$ is a 2-sided ideal. Therefore we actually have

**Proposition 1.13.** *If $I$ is a nil subset of $R$ and $J$ is a nil 2-sided ideal of $R$ then $I + J$ is nil.*

These propositions raise the question of whether the same conclusion would be true if both $I$ and $J$ were only assumed to be nil right ideals. This is precisely the Köthe conjecture [9], proposed in 1930. This conjecture is still open, and in fact is often considered to be one of the most important open problems in noncommutative ring theory.

An equivalent formulation of the conjecture states that if a ring has no nonzero nil two-sided ideals, then it has no nonzero nil right ideals. Another important equivalent formulation, due to Krempa [10] is that if a ring $R$ is nil, then so is any matrix ring $M_n(R)$ (the ring of $n$-by-$n$ matrices with entries in $R$).

While it is still open in general, the Köthe conjecture has been proved to be true for certain classes of algebras. One of those, as we shall see in section 1.6, is the class of algebras over uncountable fields.

## 1.4 Algebraicity

The notion of algebraicity is a generalization of nilpotence. We give the following definition.

**Definition 1.14.** Let $A$ be an algebra over a field $k$. We say that $a \in A$ is *algebraic* (over $k$) if it satisfies some nonzero polynomial with coefficients in $k$. That is, there exists a polynomial $0 \neq p(x) \in k\langle x \rangle$ such that $p(a) = 0$ in $A$. The minimal degree of such a polynomial is called the *degree of algebraicity* of $a$ (or simply the *degree* of $a$).

Note that if $a$ is algebraic of degree $d$, then there exists a minimal polynomial $m(x)$ of degree $d$ with the property that any polynomial $f(x) \in k\langle x \rangle$ satisfies

$f(a) = 0$ in $A$ if and only if $m(x)$ divides $f(x)$. If $a$ is nilpotent, this implies that the minimal polynomial is of the form $m(x) = x^d$. Thus we have

**Remark 1.15.** If $a$ is a nilpotent element of index $d$ in a $k$-algebra, then $a$ is also algebraic of degree $d$.

We note the following equivalent definition to algebraicity.

**Proposition 1.16.** *Let $A$ be a $k$-algebra. The element $a \in A$ is algebraic over $k$ if and only if* $\dim_k k\langle a\rangle_A < \infty$.

We make the following definition in analogy to Definition 1.11.

**Definition 1.17.** We say that a subset $S \subseteq A$ of a $k$-algebra $A$ is *algebraic* if every $s \in S$ is algebraic.

Note that this is not the same as the notion of an "algebraic set" as it is defined in [1] (we will refer to that as an "algebraic subvariety").

In analogy to the Köthe conjecture, we may ask whether the "algebraic Köthe conjecture" holds. That is, it may be asked whether the sum of two algebraic right-ideals also algebraic. As with the ordinary Köthe conjecture, this is true in some special cases, including algebras over uncountable fields.

We note a logical connection between the conjectures. We will use the following standard definition.

**Definition 1.18.** Let $A$ be an algebra. The *Jacobson radical* of $A$, denoted $\mathrm{Jac}(A)$, is the intersection of all maximal right ideals of $A$.

The following properties of the Jacobson radical are well known (see e.g., [7, p. 19]).

**Proposition 1.19.** *Let $A$ be an algebra and* $\mathrm{Jac}(A)$ *its Jacobson Radical. Then*

1. $\mathrm{Jac}(A)$ *is a 2-sided ideal of* $A$.

2. $\mathrm{Jac}(A)$ *contains every nil right-ideal of* $A$.

3. *If* $a \in \mathrm{Jac}(A)$ *is algebraic, then it is nilpotent.*

With these properties, we can prove the following connection between the "algebraic" and the original Köthe conjectures.

**Proposition 1.20.** *The "algebraic Köthe conjecture" implies the Köthe conjecture.*

*Proof.* Suppose that the "algebraic" conjecture holds. Let $I$ and $J$ be nil right ideals of $A$. By Proposition 1.19-2 we know that $I, J \subseteq \mathrm{Jac}(A)$. Hence by Proposition 1.19-1, we have $I + J \subseteq \mathrm{Jac}(A)$. Now let $a \in I + J$. By assumption, $a$ is algebraic, hence by Proposition 1.19-3 it is nilpotent. Thus $I + J$ is nil. □

## 1.5 The Kurosh Problem and the Golod-Shafarevich Algebra

The Kurosh problem [11], proposed in 1941, was another famous question regarding algebraicity in an algebra. The problem (in one form) asks whether every finitely generated algebraic algebra is finite dimensional. This question intrigued many mathematicians for over two decades, and several important special cases and related results were proved. One of those was the case where the degrees of algebraicity of the elements of the algebra are bounded (see e.g., [8]). However, the question was finally solved in the negative by Golod and Shafarevich in 1964.

Golod and Shafarevich [6, 5] disproved Kurosh's conjecture by constructing an infinite dimensional, finitely generated nil algebra $A$ (in fact, a family of such algebras) over any field $k$. The construction puts

$$A = \frac{k^+\langle x_1, \ldots, x_m \rangle}{(\{f_i\}_{i \geq 0})_{k^+\langle x_1, \ldots, x_m \rangle}}, \tag{1.25}$$

where $\{f_i\}_{i\geq 0}$ is an infinite set of homogeneous polynomials of $k^+\langle x_1, \ldots, x_m\rangle$. Thus $A$ is a graded algebra. The elements $f_i$ are chosen in a way that they satisfy two conditions. On the one hand, there are "enough" of them that the algebra is nil. On the other hand, there are "few enough" of them that the algebra is infinite dimensional (we discuss this further in section 2.2). More precisely, they satisfy the hypothesis of the Golod-Shafarevich theorem. The theorem states that if the total degrees $d_i$ of the $f_i$ grow rapidly enough, then the dimensions of the homogeneous components $A_n$ are exponentially increasing with $n$. In particular, this means that $A$ is infinite dimensional. Thus the Kurosh conjecture was shown to be false - over any field, there exists an algebraic (in fact, nil) finitely generated algebra which is infinite dimensional.

## 1.6 Amitsur's LBI and LBD Results

In 1955 Amitsur [1] proved several important results related to the questions discussed above, for algebras over uncountable fields. One of these states that if $A$ is a finitely generated algebra over an uncountable field, the Jacobson radical Jac($A$) is nil. From this, Amitsur obtained that these algebras satisfy the Köthe conjecture, as well as the "algebraic Köthe conjecture". As a consequence, if $A$ is a nil algebra over an uncountable field then so is any algebra of matrices with entries in $A$. Similarly, if $A$ is an algebraic algebra over an uncountable field then so is any matrix algebra
over $A$.

Another result of [1] is related to the Kurosh problem. Although Kurosh's conjecture is generally false, even for algebras over uncountable fields (as [5, 6] would later show), Amitsur found that these algebras satisfy related properties, namely the LBI and LBD properties. We define these properties and outline

Amitsur's proof.

**Definition 1.21.** If $A$ is a nil $k$-algebra, we say that $A$ is *locally of bounded index (LBI)* over $k$ if the elements of every finite dimensional $k$-subspace of $A$ have bounded index of nilpotence.

If $A$ is an algebraic $k$-algebra, we say that $A$ is *locally of bounded degree (LBD)* over $k$ if the elements of every finite dimensional $k$-subspace of $A$ have bounded degree of algebraicity.

**Theorem 1.22** ([1, Theorem 5])**.** *Let $k$ be an uncountable field and let $A$ be a $k$-algebra. If $A$ is nil then it is LBI. If $A$ is algebraic then it is LBD.*

In fact, Amitsur proved the following stronger statements:

**Theorem 1.23** ([1, Corollary 7])**.** *Let $k$ be an uncountable field and let $A$ be a $k$-algebra. Then any nil subspace of $A$ has bounded index, and any algebraic subspace of $A$ has bounded degree.*

We outline Amitsur's proof. Let $A$ be an algebra over an uncountable field $k$, and let $ka_1 + \ldots + ka_m$ be an arbitrary finite dimensional subspace of $A$. The first observation is that, for any $d \geq 0$, the sets

$\{(\beta_1, \ldots, \beta_m) \in k^m \mid \beta_1 a_1 + \ldots + \beta_m a_m \text{ is nilpotent of index } \leq d\}$ and

$\{(\beta_1, \ldots, \beta_m) \in k^m \mid \beta_1 a_1 + \ldots + \beta_m a_m \text{ is algebraic of degree } \leq d\}$

are algebraic subvarieties. That is, each of these sets can be described as the set of zeroes of a collection of polynomials in $m$ indeterminates. Next, it is shown that if $k$ is uncountable, then a countable union of proper algebraic subvarieties of $k^m$ must be a proper subset of $k^m$. It follows that every algebraic subspace must have bounded degree.

**Remark 1.24.** Note that in fact the second statement of Theorem 1.22 implies the first statement. That is, if every algebraic subspace has bounded degree of algebraicity, then every nil subspace, being algebraic, has bounded degree of algebraicity. Therefore by Remark 1.15 it has bounded index of nilpotence.

Amitsur also proved that the LBI and LBD properties are stable under field extension. That is,

**Lemma 1.25** ([1, Lemma 6]). *Let $k$ be an infinite field and let $A$ be a $k$-algebra. Let $H \supset k$ be a field extension and let $A_H = A \otimes_k H$ be the extension algebra.*

1. *If $A$ is LBI, then so is $A_H$.*

2. *If $A$ is LBD over $k$ then $A_H$ is LBD over $H$.*

As a consequence, we have the following corollary.

**Corollary 1.26.** *If $k$ is uncountable then the following hold.*

1. *If $A$ is nil then so is $A_H$.*

2. *If $A$ is algebraic over $k$ then $A_H$ is algebraic over $H$.*

Of particular interest is the case where the extension field $H$ is the field $k(x)$, that is, the field of rational functions over $k$ in $x$. Note that in this case, $A[x]$ is a subalgebra of $A_H$. This gives us the following corollaries.

**Corollary 1.27.** *Let $k$ be an infinite field and let $A$ be a $k$-algebra.*

1. *If $A$ is LBI, then so is $A[x]$.*

2. *If $A$ is LBD over $k$ then $A \otimes_k k(x)$ is LBD over $k(x)$.*

**Corollary 1.28.** *Let $k$ be an uncountable field and let $A$ be a $k$-algebra.*

1. *If $A$ is nil then so is $A[x]$.*

2. *If $A$ is algebraic over $k$ then $A \otimes_k k(x)$ is algebraic over $k(x)$.*

In [14], Smoktunowicz constructed a nil algebra $A$ over any countable field, such that $A[x]$ is not nil. Thus the assumption that $k$ is uncountable is indeed necessary.

In Section 2.3 we give a different proof of Amitsur's results 1.22 and 1.26, using the order-symmetric polynomials.

# Chapter 2

# The Order-Symmetric Polynomials

In this chapter we introduce the order-symmetric polynomials and prove some of their properties. These properties are useful in studying the questions mentioned above. We make use of the definitions from Section 1.1.

## 2.1 Definitions and Basic Properties

We begin with the definition of the order-symmetric polynomials in the free unital algebra $A = k\langle X\rangle$ (see Remark 1.3).

**Definition 2.1.** Let

$$f(x_1, \ldots, x_m) = \gamma_1 x_{i_{1,1}} x_{i_{1,2}} \cdots x_{i_{1,d}} + \ldots + \gamma_n x_{i_{n,1}} x_{i_{n,2}} \cdots x_{i_{n,d}} \tag{2.1}$$

be a polynomial which is homogeneous in total degree $d$. We say that $f$ is *order-symmetric* if it is invariant to applying any permutation to all of its terms, i.e., if

$$f(x_1, \ldots, x_m) = \gamma_1 x_{i_{1,\sigma(1)}} x_{i_{1,\sigma(2)}} \cdots x_{i_{1,\sigma(d)}} + \ldots + \gamma_n x_{i_{n,\sigma(1)}} x_{i_{n,\sigma(2)}} \cdots x_{i_{n,\sigma(d)}} \tag{2.2}$$

for any permutation $\sigma \in S_d$.

Consider the following polynomials in $k\langle x_1, \dots, x_m \rangle$.

**Definition 2.2.** For any $i_1, \dots, i_m \in \mathbb{N}$, we denote

$$p_{i_1,\dots,i_m}(x_1, \dots, x_m) = \sum_{(t_1,\dots,t_s)\in S} x_{t_1} \cdots x_{t_s}, \tag{2.3}$$

where $s = i_1 + \dots + i_m$ and

$$S = \left\{ (t_1, \dots, t_s) \in \mathbb{N}^s \;\middle|\; |\{r \mid t_r = j\}| = i_j \;\forall j \in \{1, \dots, m\} \right\}. \tag{2.4}$$

That is, $p_{i_1,\dots,i_m}(x_1, \dots, x_m)$ is the sum of all the $\dfrac{(i_1 + \dots + i_m)!}{i_1! \cdots i_m!}$ different words consisting of exactly $i_j$ occurrences of $x_j$ for each $j$. We take $p_{0,\dots,0}(x_1, \dots, x_m) = 1$.

It is readily seen that each $p_{i_1,\dots,i_m}$ is order-symmetric. In fact, we now show that any order-symmetric polynomial can be written using these polynomials.

**Proposition 2.3.** *Let $f(x_1, \dots, x_m)$ be a polynomial which is homogeneous of degree $i_j$ in each of the $x_j$. If $f$ is order-symmetric then*

$$f(x_1, \dots, x_m) = \gamma p_{i_1,\dots,i_m}(x_1, \dots, x_m) \tag{2.5}$$

*for some $\gamma \in k$. In particular, every order-symmetric polynomial is a linear combination of polynomials of the form $p_{i_1,\dots,i_m}(x_1, \dots, x_m)$.*

*Proof.* The uniqueness of the representation (2.1) implies that in order for $f(x_1, \dots, x_m)$ to be order-symmetric, the $\gamma_i$ must all be equal. That is,

$$f(x_1, \dots, x_m) = \gamma(w_1(x_1, \dots, x_m) + \dots + w_n(x_1, \dots, x_m)), \tag{2.6}$$

where the $w_i(x_1, \dots, x_m)$ are different words of the same length $d$, each consisting of exactly $i_j$ occurrences of $x_j$ for all $j \in \{1, \dots, m\}$.

It remains to show that all possible such $w_i(x_1, \dots, x_m)$ must be included in (2.6). Suppose this is not the case, and suppose that the word $w_0(x_1, \dots, x_m)$ has

$i_j$ occurrences of $x_j$ for all $j$, but $w_0$ does not appear in (2.6). Let $\sigma \in S_d$ be any permutation that takes the monomial $w_0$ to $w_1$. Applying this permutation to (2.6), we arrive at

$$w_1(x_1, \ldots, x_m) + \ldots + w_n(x_1, \ldots, x_m) = v_1(x_1, \ldots, x_m) + \ldots + v_n(x_1, \ldots, x_m), \quad (2.7)$$

where $v_i$ is the monomial obtained from applying the permutation $\sigma$ to $w_i$. Now, since by assumption the left-hand side of (2.7) does not include $w_0$, the right-hand side will not include $w_1$. But the left-hand side does include $w_1$, so we have reached a contradiction. Thus $f(x_1, \ldots, x_m)$ can be written as a constant times $p_{i_1,\ldots,i_m}(x_1, \ldots, x_m)$. The last statement folllows from the fact that every polynomial which is homogeneous in total degree can be written as a sum of polynomials which are homogeneous in all of the $x_i$. □

**Example 2.4.** Consider the case $m = 2$. Let $A = k\langle x, y\rangle$. Then some order-symmetric polynomials are

$$p_{1,1}(x, y) = xy + yx, \quad (2.8)$$

$$p_{2,1}(x, y) = x^2y + xyx + yx^2, \quad (2.9)$$

and for any $n \geq 0$,

$$p_{n,0}(x, y) = x^n,\ p_{0,n}(x, y) = y^n. \quad (2.10)$$

We consider some basic properties of the order-symmetric polynomials. Note that if $i_j = 0$ for some $j$, then we can omit the variable $x_j$. For example,

$$p_{i_1,\ldots,i_{m-1},0}(x_1, \ldots, x_m) = p_{i_1,\ldots,i_{m-1}}(x_1, \ldots, x_{m-1}). \quad (2.11)$$

Thus we can express any $p_{i_1,\ldots,i_m}$ as an order-symmetric polynomial of the same form, in which all $i_j \geq 1$. We prove the following recurrence relation.

**Proposition 2.5.** *Suppose $i_j \geq 1$ for all $j$. Then for any $r \in \{0, \ldots, m\}$,*

$$p_{i_1,\ldots,i_m}(x_1, \ldots, x_m) = \sum_{r=1}^{m} x_r p_{i_1,\ldots,i_{r-1},i_r-1,i_{r+1},\ldots,i_m}(x_1, \ldots, x_m). \tag{2.12}$$

*Proof.* Let $w$ be a word containing exactly $i_j$ occurrences of $x_j$ for all $j$. Then we can write $w = x_r w'$ for some $r \in \{1, \ldots, m\}$, where $w'$ contains exactly $i_r - 1$ occurrences of $x_r$ and $i_j$ occurrences of $x_j$ for $j \neq r$. Conversely, every such $x_r w'$ can be written as such $w$, and the relation follows. □

The span of all order-symmetric polynomials $p_{i_1,\ldots,i_m}(x_1, \ldots, x_m)$ for which $i_1 + \ldots + i_m = n$, will play an important role below, making the following definitions useful.

**Definition 2.6.** For any $n \geq 0$, any field $k$ and any $x_1, \ldots, x_m$ we denote

$$P_{n;k}(x_1, \ldots, x_m) = \mathrm{span}_k\{p_{i_1,\ldots,i_m}(x_1, \ldots, x_m) \mid i_1 + \ldots + i_m = n\}. \tag{2.13}$$

When there is no danger of confusion, we omit the subscript $k$.

We also define

$$\begin{aligned} P_{\leq n}(x_1, \ldots, x_m) &= \mathrm{span}_k\{p_{i_1,\ldots,i_m}(x_1, \ldots, x_m) \mid i_1 + \ldots + i_m \leq n\} \\ &= \bigoplus_{n'=0}^{n} P_{n'}(x_1, \ldots, x_m) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} P_{\geq n}(x_1, \ldots, x_m) &= \mathrm{span}_k\{p_{i_1,\ldots,i_m}(x_1, \ldots, x_m) \mid i_1 + \ldots + i_m \geq n\} \\ &= \bigoplus_{n'=n}^{\infty} P_{n'}(x_1, \ldots, x_m). \end{aligned} \tag{2.15}$$

We note that

$$\dim_k P_n(x_1, \ldots, x_m) = \Big|\{p_{i_1,\ldots,i_m}(x_1, \ldots, x_m) \mid i_1 + \ldots + i_m = n\}\Big| = \binom{m+n-1}{m-1} \tag{2.16}$$

and

$$\dim_k P_{\leq n}(x_1, \ldots, x_m) = \sum_{j=0}^{n} \dim_k P_j = \sum_{j=0}^{n} \tbinom{j+m-1}{m-1} = \tbinom{n+m}{m}. \tag{2.17}$$

At this point we extend our discussion from the free algebra $k\langle X\rangle$ to an arbitrary unital algebra $A = \dfrac{k\langle X\rangle}{J}$ with $J = (\{f_\rho\}_{\rho\in I})_{k\langle X\rangle}$. The algebra $A$ "inherits" the order-symmetric polynomials from $k\langle X\rangle$ in the following way. If $a_1, \ldots, a_m \in A$ are the images in $A$ of some $x_1, \ldots, x_m \in k\langle X\rangle$, then $p_{i_1,\ldots,i_m}(a_1, \ldots, a_m)$ is the image of $p_{i_1,\ldots,i_m}(x_1, \ldots, x_m)$, and we refer to such elements as order-symmetric polynomials in $A$. Of course, since the uniqueness property of $k\langle X\rangle$ is not inherited, the image of an order-symmetric polynomial may also be an image of a polynomial which is not order-symmetric. Also, we may have $p_{i_1,\ldots,i_m}(a_1, \ldots, a_m) = 0$.

The spaces $P_n(a_1, \ldots, a_m) \subseteq A$ are similarly obtained. The definitions (2.14) give us

$$\begin{aligned} P_{\leq n}(a_1, \ldots, a_m) &= \operatorname{span}_k\{p_{i_1,\ldots,i_m}(a_1, \ldots, a_m) \mid i_1 + \ldots + i_m \leq n\} \\ &= \sum_{n'=0}^{n} P_{n'}(a_1, \ldots, a_m) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} P_{\geq n}(a_1, \ldots, a_m) &= \operatorname{span}_k\{p_{i_1,\ldots,i_m}(a_1, \ldots, a_m) \mid i_1 + \ldots + i_m \geq n\} \\ &= \sum_{n'=n}^{\infty} P_{n'}(a_1, \ldots, a_m). \end{aligned} \tag{2.19}$$

In place of equations 2.16 and 2.17 we have the inequalities

$$\dim_k P_n(a_1, \ldots, a_m) \leq \binom{m+n-1}{m-1} \tag{2.20}$$

and

$$\dim_k P_{\leq n}(a_1, \ldots, a_m) \leq \binom{n+m}{m}. \tag{2.21}$$

Using this notation, we conclude from Proposition 2.5 that

$$P_n(x_1, \ldots, x_m) \subseteq (P_{n-1}(x_1, \ldots, x_m))_{k\langle X\rangle}. \tag{2.22}$$

In $A$, this implies the following.

**Proposition 2.7.** *If* $P_n(a_1, \ldots, a_m) = 0$ *then* $P_{\geq n}(a_1, \ldots, a_m) = 0$.

## 2.2 Relation to Nilpotence and Algebraicity

The connection between the order-symmetric polynomials and the properties of nilpotence and algebraicity can be seen from the following equation, which is a noncommutative generalization of the binomial formula.

$$(\xi_1 x_1 + \ldots + \xi_m x_m)^n = \sum_{i_1 + \ldots\ldots + i_m = n} \xi_1^{i_1} \cdots \xi_m^{i_m} p_{i_1, \ldots, i_m}(x_1, \ldots, x_m). \qquad (2.23)$$

A key observation which relates the order-symmetric polynomials with properties of nilpotence is the following.

**Proposition 2.8.** *Let* $A$ *be an algebra over an infinite field* $k$*, and let* $a_1, \ldots, a_m \in A$*. Then the subspace* $ka_1 + \ldots + ka_m \subseteq A$ *is nil of bounded index* $\leq d$ *if and only if* $P_d(a_1, \ldots, a_m) = \{0\}$.

Kaplansky [8] used this fact in studying nil algebras of bounded degree. We follow his method (see [8, Lemma 1]) of using a Vandermonde matrix below.

Golod and Shafarevich [5, 6] also used this in constructing their counterexample. They used the forward implication of the proposition to ensure that their algebra is indeed nil, (by including in the relations order-symmetric polynomials of progressively larger sets $(a_1, \ldots, a_m)$). They also counted these polynomials (as in (2.16)) to show that few enough of these polynomials are needed, and therefore the Golod-Shafarevich Theorem applies.

The order-symmetric polynomials are also related to algebraicity in an algebra, through the following observation, which is a generalization of Proposition 1.16.

**Proposition 2.9** ([3, Lemma 16(i)]). *Let $A$ be an algebra over an infinite field $k$, and let $a_1, \ldots, a_m \in A$. Then $ka_1 + \ldots + ka_m$ is algebraic of bounded degree if and only if* $\dim_k P_{\geq 0}(a_1, \ldots, a_m) < \infty$.

These results will follow from Corollary 2.18 below. In preparation for proving this lemma, we begin with the following standard result. Note that the settings of Lemma 2.10 and the subsequent results are vector spaces in general, and not necessarily algebras.

**Lemma 2.10.** *Let $V$ be a vector space over a field $k$, let $v_0, \ldots, v_d \in V$. Assume $|k| \geq d+1$ and let $\xi_0, \ldots, \xi_d \in k$ be $d+1$ distinct field elements. If*

$$\sum_{i=0}^{d} \xi_j^i v_i = 0 \tag{2.24}$$

*for all $j \in \{0, \ldots, d\}$, then $v_i = 0$ for all $i \in \{0, \ldots, n\}$.*

*Proof.* We write the equations 2.24 in matrix form as follows:

$$\begin{pmatrix} 1 & \xi_0 & \cdots & \xi_0^d \\ 1 & \xi_1 & \cdots & \xi_1^d \\ 1 & \vdots & \ddots & \vdots \\ 1 & \xi_d & \cdots & \xi_d^d \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The multiplying matrix is a Vandermonde matrix, known to be invertible when the $\xi_i$ are distinct. Multiplying by the inverse matrix, we obtain $v_i = 0$ for all $i \in \{0, \ldots, d\}$. □

We note the following generalization of Lemma 2.10.

**Lemma 2.11.** *Let $V$ be a vector space over a field $k$, let $v_0, \ldots, v_d \in V$ and let $\xi_0, \ldots, \xi_d \in k$ be $d+1$ distinct field elements. Let $W$ be a subspace of $V$. If*

$$\sum_{i=0}^{d} \xi_j^i v_i \in W \tag{2.25}$$

*for all $j \in \{0, \ldots, d\}$, then $v_i \in W$ for all $i \in \{0, \ldots, d\}$.*

*Proof.* This follows by applying Lemma 2.10 to $v_0 + W, \ldots, v_d + W \in V/W$. $\square$

The next generalization is to replace the $d+1$ distinct field elements $\xi_j$ with $(d+1) + (d+1)^2 + \ldots + (d+1)^m$ field elements $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$, which are not necessarily all distinct but satisfy the following condition.

**Definition 2.12.** Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$. We say that $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are *stepwise-distinct* if all $\xi_{i_1}$ are distinct and (for $m \geq 2$)

$$\xi_{i_1,\ldots,i_{r-1},i} = \xi_{i_1,\ldots,i_{r-1},j} \Rightarrow i = j \tag{2.26}$$

for all $i, j \in \{1, \ldots, m\}$, $r \in \{2, \ldots, m\}$ and $i_1, \ldots, i_{r-1} \in \{0, \ldots, d\}$.

We use this Definition 2.12 for the purpose of stating Lemma 2.14 in the most generality. The following specific choice will actually often be sufficient.

**Remark 2.13.** If $\xi_0, \ldots, \xi_d \in k$ are distinct and $\xi_{i_1,\ldots,i_r} = \xi_{i_r}$ for each $r \in \{1, \ldots, m\}$ and $i_1, \ldots, i_r \in \{0, \ldots, d\}$ then $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. In particular, it is always possible to choose stepwise-distinct $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$, with $i_1, \ldots, i_m \in \{0, \ldots, d\}$, as long as $k$ has at least $d+1$ elements.

**Lemma 2.14.** *Let $V$ be a vector space over a field $k$, with $|k| \geq d+1$. Let $v_{n_1,\ldots,n_m} \in V$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. Let $W$ be a subspace of $V$. If*

$$\sum_{n_1=0}^{d} \cdots \sum_{n_m=0}^{d} \xi_{i_1}^{n_1} \xi_{i_1,i_2}^{n_2} \cdots \xi_{i_1,\ldots,i_m}^{n_m} v_{n_1,\ldots,n_m} \in W \tag{2.27}$$

*for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, then $v_{n_1,\ldots,n_m} \in W$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$.*

*Proof.* The proof is by induction on $m$. The case $m = 1$ is just Lemma 2.11. Now suppose $m \geq 2$ and the assertion holds for $m - 1$.

Let $i_1, \ldots, i_{m-1} \in \{0, \ldots, d\}$. Then for all $i \in \{0, \ldots, d\}$,

$$\sum_{n_m=0}^{d} \xi_{i_1,\ldots,i_{m-1},i}^{n_m} \Big( \sum_{n_1=0}^{d} \cdots \sum_{n_{m-1}=0}^{d} \xi_{i_1}^{n_1} \xi_{i_1,i_2}^{n_2} \cdots \xi_{i_1,\ldots,i_{m-1}}^{n_{m-1}} v_{n_1,\ldots,n_m} \Big) \in W. \tag{2.28}$$

By the stepwise-distinctness assumption, the $\xi_{i_1,\ldots,i_{m-1},i}$ are distinct for $i \in \{0, \ldots, d\}$. Thus by Lemma 2.11,

$$\sum_{n_1=0}^{d} \cdots \sum_{n_{m-1}=0}^{d} \xi_{i_1}^{n_1} \xi_{i_1,i_2}^{n_2} \cdots \xi_{i_1,\ldots,i_{m-1}}^{n_{m-1}} v_{n_1,\ldots,n_m} \in W \tag{2.29}$$

for all $n_m$ and for all $i_1, \ldots, i_{m-1} \in \{0, \ldots, d\}$. The result now follows by induction.

□

Note the following immediate corollary.

**Corollary 2.15.** *Let $V$ be a vector space over a field $k$ with $|k| \geq d+1$. Let $v_{n_1,\ldots,n_m} \in V$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$. Let $W$ be a subspace of $V$. If*

$$\sum_{n_1=0}^{d} \cdots \sum_{n_m=0}^{d} \xi_1^{n_1} \xi_2^{n_2} \cdots \xi_m^{n_m} v_{n_1,\ldots,n_m} \in W \tag{2.30}$$

*for all $\xi_1, \ldots, \xi_m \in k$, then $v_{n_1,\ldots,n_m} \in W$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$.*

*Proof.* In view of Remark 2.13, this statement is an immediate consequence of Lemma 2.14. □

The following corollary will be useful in applying the results of Lemma 2.14 to the order-symmetric polynomials.

**Corollary 2.16.** *Let $V$ be a vector space over a field $k$, with $|k| \geq d+1$. Let $u_{n_1,\ldots,n_m} \in V$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$ with $n_1 + \ldots + n_m = d$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. Let $W$ be a subspace of $V$. If*

$$\sum_{n_1+\ldots+n_m=d} \xi_{i_1}^{n_1} \xi_{i_1,i_2}^{n_2} \cdots \xi_{i_1,\ldots,i_m}^{n_m} u_{n_1,\ldots,n_m} \in W \tag{2.31}$$

*for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, then $u_{n_1,\ldots,n_m} \in W$ for all $n_1, \ldots, n_m \in \{0, \ldots, d\}$ with $n_1 + \ldots + n_m = d$.*

*Proof.* Set $v_{n_1,\ldots,n_m} = u_{n_1,\ldots,n_m}$ when $n_1 + \ldots + n_m = d$ and $v_{n_1,\ldots,n_m} = 0$ otherwise, and apply Lemma 2.14. □

We can now apply this corollary in the setting of an algebra, and draw the desired conclusion on the order-symmetric polynomials.

**Corollary 2.17.** *Let $A$ be a $k$-algebra with $|k| \geq d+1$ and let $a_1, \ldots, a_m \in A$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. Let $W$ be a subspace of $A$. If*

$$(\xi_{i_1} a_1 + \xi_{i_1,i_2} a_2 + \ldots + \xi_{i_1,\ldots,i_m} a_m)^d \in W \tag{2.32}$$

*for all $i_r \in \{0, \ldots, d\}$ then $P_d(a_1, \ldots, a_m) \subseteq W$.*

*Proof.* This follows from formula 2.23 and from applying Corollary 2.16 with $u_{n_1,\ldots,n_m} = p_{n_1,\ldots,n_m}(a_1, \ldots, a_m)$. □

**Corollary 2.18.** *Let $A$ be a $k$-algebra with $|k| \geq d+1$ and let $a_1, \ldots, a_m \in A$. Let $W \subseteq A$ be a subspace of $A$. If*

$$(\xi_1 a_1 + \ldots + \xi_m a_m)^d \in W \tag{2.33}$$

*for all $\xi_1, \ldots, \xi_m \in k$ then $P_d(a_1, \ldots, a_m) \subseteq W$.*

*Proof.* Choose stepwise-distinct $\xi_{i_1}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots i_m \in \{0, \ldots, d\}$ as in Remark 2.13, and apply Corollary 2.17. □

We are now in position to readily prove Proposition 2.8 which was stated above.

*Proof.* Note that $ka_1 + \ldots + ka_m$ is nil of bounded index if and only if there exists $d \geq 1$ such that $(\xi_1 a_1 + \ldots + \xi_m a_m)^d = 0$ for all $\xi_1, \ldots, \xi_m \in k$. Thus the "if" part of the proposition follows from formula 2.23, and the reverse assertion is a direct consequence of Corollary 2.18, with $W = \{0\}$. □

Proposition 2.9 will also follow from Corollary 2.18. However we will also need the following stronger statement.

**Lemma 2.19.** *Let $A$ be a $k$-algebra with $|k| \geq d+1$ and let $a_1, \ldots, a_m \in A$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. If $\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m$ is algebraic of degree at most $d$ for all $i_1, \ldots, i_m \in \{0, \ldots, d\}$ then $P_d(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m)$.*

*Proof.* Suppose that $\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,i_2,\ldots,i_m}a_m$ is algebraic of degree $d' \leq d$. Then

$$(\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m)^{d'} \in$$
$$\operatorname{span}_k\{(\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m)^c \mid 0 \leq c \leq d'-1\}. \tag{2.34}$$

Multiplying both sides by $(\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m)^{d-d'}$ we obtain that

$$(\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m)^{d} \in$$
$$\operatorname{span}_k\{(\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m)^c \mid 0 \leq c \leq d-1\}$$
$$\subseteq P_{\leq d-1}, \tag{2.35}$$

where in the last containment we have used (2.23). The assertion now follows from Corollary 2.17, with $W = P_{\leq d-1}$. □

We can generalize Lemma 2.19 as follows.

**Lemma 2.20.** *Let $D \geq d$. Let $A$ be a $k$-algebra with $|k| \geq D+1$ and let $a_1, \ldots, a_m \in A$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, D\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct. If $\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m$ is algebraic of degree at most $d$ for all $i_1, \ldots, i_m \in \{0, \ldots, D\}$ then $P_D(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m)$.*

*Proof.* This follows from repeated application of Lemma 2.19 as follows. We first apply Lemma 2.19 verbatim, since the hypothesis here is stronger. This gives us

$$P_d(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m) \tag{2.36}$$

Next, we apply the lemma with $d + 1$ replacing $d$. We can do this since the hypothesis remains true assuming that $D \geq d + 1$. This gives us

$$P_{d+1}(a_1, \ldots, a_m) \subseteq P_{\leq d}(a_1, \ldots, a_m) \tag{2.37}$$

Combining (2.36) and (2.37) gives

$$P_{d+1}(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m) \tag{2.38}$$

The result follows from continuing this process, i.e. by applying Lemma 2.19 while replacing $d$ by $d + 2$, by $d + 3$ and so on up to $D$. □

This gives us the following corollary.

**Corollary 2.21.** *Let $A$ be an algebra over an infinite field $k$, and let $a_1, \ldots, a_m \in A$. If $ka_1 + \ldots + ka_m$ is algebraic of degree at most $d$ then $P_{\geq 0}(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m)$ and in particular, $\dim_k P_{\geq 0} < \infty$.*

*Proof.* By Remark 2.13 we can apply Lemma 2.20 for each for each $D \geq d$, and the result follows. □

Corollary 2.21 gives us one direction required for Proposition 2.9. The other direction will follow from the next lemma. We first introduce the notation

$$M_{d,m} = \binom{d + m - 1}{m}, \tag{2.39}$$

and recall from (2.17) that

$$\dim_k P_{\leq d-1}(a_1, \ldots, a_m) \leq M_{d,m}. \tag{2.40}$$

**Lemma 2.22.** *If $P_{\leq M_{d,m}}(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m)$ for some $d \geq 1$ then the subspace $ka_1 + \ldots + ka_m$ is algebraic of degree at most $M_{d,m}$.*

*Proof.* Let $a = \xi_1 a_1 + \ldots + \xi_m a_m \in ka_1 + \ldots + ka_m$. Then by assumption, for any $0 \leq n \leq M_{d,m}$ we have (using (2.23))

$$a^n \in P_{\leq M_{d,m}} \subseteq P_{\leq d-1}(a_1, \ldots, a_m). \tag{2.41}$$

Thus

$$\dim_k \operatorname{span}_k\{a^n \mid 0 \leq n \leq M_{d,m}\} \leq \dim_k P_{\leq d-1}(a_1, \ldots, a_m) \leq M_{d,m}. \tag{2.42}$$

Therefore the set $\{a^n \mid 0 \leq n \leq M_{d,m}\}$ is linearly dependent over $k$, and hence $a$ is algebraic of degree at most $M_{d,m}$. □

The following corollaries are immediate.

**Corollary 2.23.** *If $P_{\geq 0}(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m)$ for some $d \geq 1$ then $ka_1 + \ldots + ka_m$ is algebraic of degree at most $M_{d,m}$.*

**Corollary 2.24.** *If $\dim_k P_{\geq 0}(a_1, \ldots, a_m) < \infty$ then $ka_1 + \ldots + ka_m$ is algebraic of bounded degree.*

We are now able to prove Proposition 2.9, which was stated above. Indeed, this follows immediately from Corollary 2.21 and Corollary 2.24.

Returning to Corollary 2.23, it may be asked whether the degree bound of $M_{d,m}$ can be improved, and in particular, whether the converse of Corollary 2.21 holds. The answer to the latter question is negative, as seen from the following example.

**Example 2.25** ([4])**.** Let $k$ be any field, and let $A$ be a field extension of degree 3 (e.g., $k = \mathbb{Q}$, and $A = \mathbb{Q}(\sqrt[3]{2})$). Let $a \in A$ be an element of degree 3 (e.g., $a = \sqrt[3]{2}$)), and let $b = a^2$. Then $P_{\geq 0}(a, b) \subseteq A = \operatorname{span}_k\{1, a, b\} = P_{\leq 1}(a, b)$ (in fact equality holds here). However, it is not true that every element of $ka + kb$ is algebraic of degree 2, i.e., the converse of Corollary 2.21 (with $m = d = 2$) fails.

## 2.3 Alternative proof of Amitsur's results

We are now ready to use the properties of the order-symmetric polynomials from Section 2.2 to give another proof of Amitsur's results which were discussed in Section 1.6.

Let $A$ be a $k$-algebra, and let $a_1, \ldots, a_m \in A$. We first derive a relation which is a generalization of of [1, Corollary 6].

**Lemma 2.26.** *Let $A$ be a $k$-algebra with $|k| \geq M_{d,m} + 1$ and let $a_1, \ldots, a_m \in A$. Let $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, M_{d,m}\}$, and assume $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m}$ are stepwise-distinct.*

*If $\xi_{i_1}a_1 + \xi_{i_1,i_2}a_2 + \ldots + \xi_{i_1,\ldots,i_m}a_m$ is algebraic of degree at most $d$ for any $i_1, \ldots, i_m \in \{0, \ldots, M_{d,m}\}$ then the subspace $ka_1 + \ldots + ka_m$ is algebraic of degree at most $M_{d,m}$.*

*Proof.* By Lemma 2.20 (with $D = M_{d,m}$), we have

$$P_{\leq M_{d,m}}(a_1, \ldots, a_m) \subseteq P_{\leq d-1}(a_1, \ldots, a_m). \tag{2.43}$$

Therefore the conclusion follows from Lemma 2.22. □

We now prove Amitsur's Theorem 1.23 that if $k$ is uncountable, then every algebraic finite dimensional subspace of a $k$-algebra has bounded degree of algebraicity.

*Proof.* Let $ka_1 + \ldots + ka_m$ be a finite dimensional subspace of $A$. We use induction on the dimension $m$. The statement clearly holds when $m = 1$. Suppose then that $m \geq 2$.

For each integer $d \geq 0$ let

$$W_d = \left\{\omega \in k \;\middle|\; a_1 + ka_2 + \ldots + ka_{m-1} + \omega a_m \text{ is algebraic of degree} \leq d\right\}. \tag{2.44}$$

If $|W_d| \leq M_{d,m}$ for all $d \geq 0$, then $\bigcup_{d\geq 0} W_d$ is countable, and so $\bigcup_{d\geq 0} W_d \neq k$. Therefore there exists $\omega_0 \in k$ such that $a_1 + ka_2 \ldots + ka_{m-1} + \omega_0 a_m$ is algebraic of unbounded degree. But then $k(\omega_0 a_m + a_1) + ka_2 \ldots + ka_{m-1}$ is algebraic of unbounded degree, which is impossible by induction.

Thus $|W_d| > M_{d,m}$ for some $d \geq 0$. Let $\omega_0, \ldots, \omega_{M_{d,m}}$ be distinct elements of $W_d$. Then by definition of $W_d$, for each $i \in \{0, \ldots, M_{d,m}\}$ the set

$$S = a_1 + ka_2 + \ldots + ka_{m-1}a_{m-1} + \omega_i a_m \tag{2.45}$$

is algebraic of degree at most $d$. We claim that this implies that the entire subspace $ka_1 + \ldots + ka_m$ is algebraic of degree at most $d$. Indeed, if we multiply each element of $S$ by any $\mu \in k$, we obtain that the set

$\mu a_1 + ka_2 + \ldots + ka_{m-1}a_{m-1} + \mu\omega_i a_m$ is algebraic of bounded degree at most $d$. Now choose any distinct nonzero $\mu_0, \ldots, \mu_{M_{d,m}} \in k$. Define $\xi_{i_1}, \xi_{i_1,i_2}, \ldots, \xi_{i_1,\ldots,i_m} \in k$ for all $i_1, \ldots, i_m \in \{0, \ldots, M_{d,m}\}$ by $\xi_{i_1} = \mu_{i_1}$ and $\xi_{i_1,\ldots,i_r} = \mu_{i_1}\omega_{i_r}$ for $r \geq 2$. Then $\xi_{i_1}, \ldots, \xi_{i_1,\ldots,i_m}$ satisfy the hypothesis of Lemma 2.26, and thus $ka_1 + \ldots + ka_m$ is algebraic of degree at most $M_{d,m}$. This completes the proof of Theorem 1.23. □

We can also use the results above to prove Lemma 1.25. Let $k \subseteq H$ be infinite fields and suppose that $A$ is a $k$-algebra which is LBI. Let $S$ be a finite dimensional $H$-subspace of $A_H$. We may assume without loss of generality that

$S = H(a_1 \otimes 1) + \ldots + H(a_m \otimes 1)$, since every finite-dimensional subspace of $A_H$ is contained in such a subspace. By Proposition 2.8, there exists $d$ such that $P_{d;k}(a_1, \ldots, a_m) = \{0\}$.

It follows that

$$P_{d;H}(a_1 \otimes 1, \ldots, a_m \otimes 1) = \{0\}. \tag{2.46}$$

By Proposition 2.8 this implies that $S$ is nil of bounded index. Thus $A_H$ is LBI over $H$.

Similarly, suppose $A$ is a $k$-algebra which is LBD over $k$. By Proposition 2.9,

$$\dim_k P_{\geq 0;k}(a_1, \ldots, a_m) < \infty. \tag{2.47}$$

It follows that

$$\dim_H P_{\geq 0;H}(a_1 \otimes 1, \ldots, a_m \otimes 1) < \infty, \tag{2.48}$$

and so by Proposition 2.9, $A_H$ is LBD over $H$.

# Chapter 3

# Application to Filtered and Graded Algebras

In this chapter we apply the results discussed in the previous chapters to filtered and graded algebras. We begin by describing the behavior of these algebras in relation to the nilpotence and algebraicity properties defined in Chapter 1. We then give an application to filtered algebraic algebras over uncountable fields.

## 3.1 Nilpotence and Algebraicity in Graded Algebras

We study the properties of nilpotence and algberaicity, introduced in Chapter 1, in filtered and graded algebras (also introduced in Chapter 1).

Let $A$ be a graded $k$-algebra. Recall the notation $A_{\geq 1} = A_1 \oplus A_2 \oplus \ldots$, the ideal of elements of $A$ with no scalar component. We first note the following property of these elements.

**Proposition 3.1.** *In a graded algebra $A$, an element of $A_{\geq 1}$ is algebraic if and* *only if it is nilpotent.*

*Proof.* Let $a = a_p + a_{p+1} + \ldots + a_q \in A_{\geq 1}$, with $1 \leq p \leq q$ and $a_i \in A_i$ for all $i$, and suppose $a$ is algebraic of degree $d$. Then $a^D \in \text{span}_k\{a^{d'} \mid 0 \leq d' \leq d-1\}$ for all $D \geq 0$. Let $D = \lceil \frac{q(d-1)+1}{p} \rceil$, and consider the equation

$$(a_p+\ldots+a_q)^D = \gamma_{d-1}(a_p+\ldots+a_q)^{d-1}+\ldots+\gamma_2(a_p+\ldots+a_q)^2+\gamma_1(a_p+\ldots+a_q)+\gamma_0. \tag{3.1}$$

Since $Dp > q(d-1)$, we have (using the notation from (1.15) and (1.16))

$$a^D = (a_p + \ldots + a_q)^D \in A_{\geq Dp} \bigcap A_{\leq q(d-1)} = \{0\}. \tag{3.2}$$

Thus $a$ is nilpotent (note that in fact by Remark 1.15 we have $a^d = 0$). □

For graded algebras we can define the following property, which is weaker than being nil.

**Definition 3.2.** let $A$ be a graded $k$-algebra. We say that $A$ is *graded-nil* if $A_n$ is nil for all $n \geq 1$.

Clearly, if $A_{\geq 1}$ is nil then $A$ is graded-nil. It may be asked whether the converse is true:

**Question 3.3.** *Let $A$ be a $k$-algebra. If $A$ is graded-nil, is $A_{\geq 1}$ nil?*

In 2006 Bartholdi [2] constructed an algebra over a countable field which is graded-nil but not nil. Since algebras over uncountable fields are known to satisfy the Köthe conjecture, one may intuitively expect that for algebras over uncountable fields, every graded-nil algebra is nil (since we know that in such algebras, "many" sums of nilpotent elements are nilpotent). However, Smoktunowicz [16] has recently constructed a graded-nil-but-not-nil algebra over any field.

We can nevertheless give an affirmative answer to Question 3.3 in the special case we describe next. Here we make use of the associated graded algebra, which was defined in Section 1.2. We first note the following property of filtered algebraic algebras.

**Proposition 3.4.** *Let $A$ be a filtered algebra, with a filtration $(F_n)_{n\geq 0}$. Let $B = \mathrm{gr}(A)$ be its associated graded algebra. If $A$ is algebraic then $B$ is graded-nil.*

*Proof.* Let $b_p = a_p + A_{p-1}$, with $p \geq 1$ and $a_p \in A_p$, be an arbitrary homogeneous element of $B_{\geq 1}$. By assumption, $a_p$ is algebraic, say of degree $d$. Therefore

$$a_p^d = \beta_{d-1}a_p^{d-1} + \ldots + \beta_1 a_p + \beta_0 \in F_{p(d-1)} \subseteq F_{pd-1}. \tag{3.3}$$

The definition of multiplication in $B$ now gives

$$(a_p + F_{p-1})^d = a_p^d + F_{pd-1} = 0_B. \tag{3.4}$$

□

Thus the associated graded algebra $\mathrm{gr}(A)$ of an algebraic $k$-algebra $A$ is graded-nil. We next prove that if the underlying field $k$ is uncountable, then $\mathrm{gr}(A)$ is actually nil, showing there are no graded-nil-but-not-nil algebras of this type. However, we first give the following weaker result, since there is some interest in the proof.

**Proposition 3.5.** *Let $A$ be a filtered $k$-algebra, with a filtration $(F_n)_{n\geq 0}$. Let $B = \mathrm{gr}(A)$ be its associated graded algebra. If $A$ is LBI then so is $B$. In particular, the associated graded algebra of a nil filtered algebra over an uncountable field is nil.*

We give two proofs of this proposition. The first uses the Rees algebra (see Section 1.7), while the second uses the order-symmetric polynomials.

*Proof.* (1) By Proposition 1.28, the polynomial algebra $A[z]$ is LBI. Therefore the Rees algebra $R \subset A[z]$ is LBI as well. By Proposition 1.8, $B \cong \dfrac{R}{zR}$. Thus $B$ is also LBI. The last statement now follows from Theorem 1.22. □

In preparation for the second proof, we first note the following properties of the order-symmetric polynomials in graded and filtered algebras. If the $k$-algebra $A$ is filtered by $(F_n)_{n\geq 0}$ and $a_p, \ldots, a_q \in A$ satisfy $a_i \in F_i$ for all $p \leq i \leq q$, then

$$p_{i_1,\ldots,i_m}(a_p, \ldots, a_q) \in F_{i_1 p+\ldots+i_m q}. \tag{3.5}$$

Similarly, suppose $B$ is a graded $k$-algebra. If $b_p, \ldots, b_q$ are $m$ homogeneous elements with $b_i \in B_i$ then $p_{i_1,\ldots,i_m}(b_p, \ldots, b_q) \in B_{i_1 p+\ldots+i_m q}$.

Now let $A$ be a $k$-algebra filtered by $(F_n)_{n\geq 0}$ and let $B = \mathrm{gr}(A)$ be its associated graded algebra. Let $a_p, \ldots, a_q \in A$ satisfy $a_i \in F_i$ for each $p \leq i \leq q$, and let $b_i = a_i + F_{i-1} \in B$. We first note that by (1.20), if $p \leq p_j \leq q$ for all $1 \leq j \leq n$ then

$$b_{p_1} \cdots b_{p_n} = a_{p_1} \cdots a_{p_n} + F_{p_1+\ldots+p_n-1}. \tag{3.6}$$

Therefore if this product contains exactly $i_j$ occurrences of $b_j$ for each $p \leq j \leq q$, then

$$b_{p_1} \cdots b_{p_n} = a_{p_1} \cdots a_{p_n} + F_{i_p p+\ldots+i_q q-1}. \tag{3.7}$$

Since each $p_{i_p,\ldots,i_q}(b_p, \ldots, b_q)$ is a sum of such products, we have

$$p_{i_p,\ldots,i_q}(b_p, \ldots, b_q) = p_{i_p,\ldots,i_q}(a_p, \ldots, a_q) + F_{pi_p+\ldots+qi_q-1}. \tag{3.8}$$

We now give a second proof of Proposition 3.5.

*Proof.* (2) Let $S$ be a finite dimensional subspace of $B$. Without loss of generality we may assume $S = B_p \oplus \ldots \oplus B_q$, with $0 \leq p \leq q$, since every finite dimensional subspace of $B$ is contained in a subspace of this form. Let $b = b_p + \ldots + b_q$ be any element of $S$, where $b_i \in \dfrac{F_i}{F_{i-1}}$ (we take $F_{-1} = \{0\}$). That is, for each $i \in \{p, \ldots, q\}$, we have $b_i = a_i + F_{i-1}$, with $a_i \in F_i$. By assumption, the subspace $F_p + \ldots + F_q \subseteq A$ is nil of bounded index, say $d$. Therefore by Proposition 2.8, we have $P_d(a_p, \ldots, a_q) = \{0\}$, i.e. $p_{i_p,\ldots,i_q}(a_p, \ldots, a_q) = 0$ for $i_p + \ldots + i_q = d$.

By (3.8) this gives us that $p_{i_p,\dots,i_q}(b_p,\dots,b_q) = 0 + F_{pi_p+\dots+qi_q-1} = 0_B$ whenever $i_p+\dots+i_q = d$, i.e. $P_d(b_p,\dots,b_q) = \{0_B\}$. Therefore by Proposition 2.8, we have $(b_p+\dots+b_q)^d = 0$. This shows that $S$ is nil of bounded index $d$. Thus $B$ is LBI. □

We now prove the following more general theorem.

**Theorem 3.6.** *Let $A$ be a filtered $k$-algebra, with a filtration $(F_n)_{n\geq 0}$. Let $B = \mathrm{gr}(A)$ be its associated graded algebra. If $A$ is LBD, then $B_{\geq 1}$ is LBI. In particular, if $k$ is uncountable and $A$ is a filtered algebraic $k$-algebra then* $\mathrm{gr}(A)_{\geq 1}$ *is nil (and LBI).*

*Proof.* Let $S$ be a finite dimensional subspace of $B$. We may assume $S = B_p \oplus \dots \oplus B_q$, with $1 \leq p \leq q$, since every finite dimensional subspace of $B_{\geq 1}$ is contained in a subspace of this form. Let $b = b_p+\dots+b_q$ be any element of $S$, where $b_i \in \dfrac{F_i}{F_{i-1}}$ (we take $F_{-1} = \{0\}$). That is, for each $i \in \{p,\dots,q\}$, we have $b_i = a_i + F_{i-1}$, with $a_i \in F_i$. By assumption, the subspace $F_p+\dots+F_q$ is algebraic of bounded degree, say $d$. Therefore by Corollary 2.21, we have

$$P_{\geq 0}(a_p,\dots,a_q) \subseteq P_{\leq d-1}(a_p,\dots,a_q). \tag{3.9}$$

By (3.5) this implies

$$P_{\geq 0}(a_p,\dots,a_q) \subseteq F_{(d-1)q}. \tag{3.10}$$

Now let

$$N = \left\lceil \frac{(d-1)q}{p} + 1 \right\rceil \tag{3.11}$$

and suppose $i_p+\dots+i_q \geq N$. Then

$$pi_p+\dots+qi_q - 1 \geq pN - 1 \geq (d-1)q. \tag{3.12}$$

Therefore

$$p_{i_p,\dots,i_q}(a_p,\dots,a_q) \in F_{(d-1)q} \subseteq F_{pi_p+\dots+qi_q-1}, \tag{3.13}$$

and so

$$p_{i_p,\ldots,i_q}(b_p,\ldots,b_q) = p_{i_p,\ldots,i_q}(a_p,\ldots,a_q) + F_{pi_p+\ldots+qi_q-1} = 0_B. \tag{3.14}$$

Thus $P_N(b_1,\ldots,b_n) = \{0_B\}$, and it follows from 2.8 that $(b_p+\ldots+b_q)^N = 0$. This shows that $S$ is nil of bounded index $N$. Thus $B$ is LBI. $\square$

**Remark 3.7.** We can use the bound $N$ of (3.11) to draw the following conclusion from the proof of Theorem 3.6. If we are given that each $F_i$ is algebraic of bounded degree at most $d_i$, then each $\frac{F_1}{F_0} \oplus \cdots \oplus \frac{F_r}{F_{r-1}}$ is nil of bounded index at most $(d_r-1)r+1 \leq rd_r$.

## 3.2 Some concluding remarks

In the discussion above we used the properties of the subspaces $P_n$, to draw conclusions about nilpotence and algebraicity in algebras.

We generalize Definition 2.6 as follows. We use the notation

$$M_j(x_1,\ldots,x_m) = \{y_1\cdots y_j \mid y_i \in \{x_1,\ldots,x_m\}\} \tag{3.15}$$

and

$$M_{\leq j}(x_1,\ldots,x_m) = \bigcup_{1\leq i\leq j} M_i(x_1,\ldots,x_m) \tag{3.16}$$

**Definition 3.8.** For any $n,j \geq 0$, any field $k$ and any $x_1,\ldots,x_m$ we denote

$$P^{(j)}_{n;k}(x_1,\ldots,x_m) = P_{n;k}(M_j(x_1,\ldots,x_m)) \tag{3.17}$$

and

$$P^{(\leq j)}_{n;k}(x_1,\ldots,x_m) = P_{n;k}(M_{\leq j}(x_1,\ldots,x_m)) \tag{3.18}$$

Note that $P^{(j)}_{n;k}(x_1,\ldots,x_m)$ is well-defined since $P_{n;k}$ does not depend on the order in which the its arguments (here, the elements of $M_j(x_1.\ldots,x_n)$) are listed.

For example,

$$P_2^{(2)}(x, y) = P_2(x^2, xy, yx, y^2) \tag{3.19}$$

and

$$P_{\geq 0}^{(\leq 2)}(x, y) = P_{\geq 0}(x, y, x^2, xy, yx, y^2). \tag{3.20}$$

With these notations, the following characterizations follow from Propositions 2.8 and 2.9.

**Proposition 3.9.** *Let $k$ be an infinite field and let $A$ be a $k$-algebra generated by $a_1, \ldots, a_m$.*

1. *Suppose $A$ is graded, and $a_i$ are homogeneous elements, all having the same degree. Then*

   *$A$ is graded-nil, each homgeneous component having bounded index*

   $\Leftrightarrow$ *for each $r \geq 1$ there exists $n_r \geq 1$ such that $P_{n_r}^{(r)}(a_1, \ldots, a_m) = \{0\}$.*

2. *$A$ is LBI*

   $\Leftrightarrow$ *for each $r \geq 1$ there exists $n_r \geq 1$ such that $P_{n_r}^{(\leq r)}(a_1, \ldots, a_m) = \{0\}$.*

3. *$A$ is LBD $\Leftrightarrow \dim_k P_{\geq 0}^{(\leq r)}(a_1, \ldots, a_m) < \infty$ for all $r \geq 0$.*

It is conceivable that these characterizations may be used to answer questions regarding nilpotence and algebraicity in algebras. For example, the similarity between parts 1 and 2 of Proposition 3.9 may be helpful in constructing graded-nil-but-not-nil algebras or in finding conditions that such algebras must satisfy.

# Bibliography


[1] S.A. Amitsur. Algebras over infinite fields. *Proceedings of the American Mathematical Society*, 7(1):35–48, Feb. 1956.

[2] L. Bartholdi. Branch rings, thinned rings, tree enveloping rings. *Israel J. Math.*, 154:93–139, 2006.

[3] G. Bergman. Radicals, tensor products and algebraicity. *Israel Mathematical Conference Proceedings*, 1:150–192, 1989.

[4] G. Bergman. private communication, 2007.

[5] E. S. Golod. On nil algebras and finitely approximable p-groups. *Izv. Akad. Nauk SSSR*, (28):273–276, 1964. English Translation: AMS Translations, Ser. 2, 48 (1965) 103-106.

[6] E. S. Golod and I.R. Šafarevič. On class field towers. *Izv. Akad. Nauk SSSR*, (28):261–272, 1964. English Translation: AMS Translations, Ser. 2, 48 (1965) 91-102.

[7] N. Jacobson. *Structure of Rings*, volume 37 of *AMS Colloquium Publications*. AMS, 1964.

[8] I. Kaplansky. On a problem of Kurosch and Jacobson. *Bull. Amer. Math. Soc.*, 52(12):1033–1035, 1946.

[9] G. Köthe. Die struktur der ringe, deren restklassenring nach dem radikal vollstndig reduzibel ist. *Math. Zeitschrift*, 32:161–186, 1930.

[10] J. Krempa. Logical connections between some open problems concerning nil rings. *Fundamental Mathematicae*, 76:121–130, 1972.

[11] A. G. Kurosh. Problems in ring theory connected with the Burnside problem on periodic groups. *Izv. Akad. Nauk SSSR*, 5:233–240, 1941.

[12] J. C. Robson and J. C. McConnell. *Noncommutative Noetherian Rings*. John Wiley and Sons, 1987.

[13] L. H. Rowen. Koethe's conjecture. *Israel Mathematical Conference Proceedings*, 1:193–202, 1989.

[14] A. Smoktunowicz. Polynomial rings over nil rings need not be nil. *J. Algebra*, 233:427–436, 2000.

[15] A. Smoktunowicz. Some results in noncommutative ring theory. *Proc. of the ICM*, Vol. 2: invited talks:259–269, August 2006.

[16] A. Smoktunowicz. private communication, 2008.