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AN $\aleph_1$-DENSE IDEAL ON $\aleph_2$

BY

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ABSTRACT

This paper establishes the consistency of a countably complete, uniform, $\aleph_1$-dense ideal on $\aleph_2$. As a corollary, it is consistent that there exists a uniform ultrafilter D on ω_2 such that $|\omega_1^{\omega_2}/D| = \omega_1$. A general "transfer" result establishes the consistency of countably complete uniform ideal K on ω_2 such that $P(\omega_2)/K \cong P(\omega_1)/\{\text{countable sets}\}$.

0. The statement of the main theorem and its corollaries

In 1930, Ulam [U] suggested the possibility of a set X carrying a countably additive probability measure that measured each subset of X. Early results of Ulam showed, however, that such a set cannot have a small cardinality such as $\aleph_1$ or $\aleph_2$. Recent results of Gitik and Shelah [G-S] show that there is no accessible set X carrying a countably complete ideal $I \subset P(X)$ (e.g. the sets of measure 0 for some measure) such that $P(X)/I$ is separable. By their results, the strongest possible ideal property cardinals such as $\aleph_1$ or $\aleph_2$ can have is to carry a dense set of size $\aleph_1$. In this paper it is shown that it is consistent to have such an ideal on $\aleph_2$. (It was previously known from work of Woodin that it was consistent on $\aleph_1$.) The existence of such an ideal on $\aleph_2$ is shown to settle several open problems in model theory, combinatorics, and topology that date from the early 1960's.

The main result of this paper is the following theorem:

THEOREM: *Assume there is a huge cardinal. Then the following holds in a forcing extension of V:*

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There is a countably complete, weakly normal, $\aleph_1$-dense ideal K on $\aleph_2$.

This is a corollary of a general "transfer theorem" which says that assuming (roughly) the existence of a "layered" ideal on $\aleph_2$:

for any uniform ideal J on $\aleph_1$ there is a uniform ideal K on $\aleph_2$ such that:

$$P(\omega_1)/J \cong P(\omega_2)/K.$$

Further, the degree of completeness of the ideal K is equal to the degree of completeness of the ideal J.

To prove the main theorem quoted above, we first show that it is consistent to have, simultaneously, a countably complete $\aleph_1$-dense ideal on $\aleph_1$ and a sufficiently strongly layered ideal on $\aleph_2$, as well as $\Diamond$ and $\Box$. The main result then follows from the transfer theorem.

The proof of both of these results is adaptable easily to other cardinals to yield, for example, the consistency of an $\aleph_2$-complete, $\aleph_2$-dense, $\aleph_1$-closed, uniform, weakly normal ideal on ω_3.

We do not at this time know how to get the consistency of a countably complete, $\aleph_1$-dense, uniform ideal on ω_3. The obstacle is showing the consistency of a dense ideal on $\aleph_1$ together with very strongly layered ideals on $\aleph_2$ and $\aleph_3$ (simultaneously).

The main theorem contains the solution of several combinatorial problems as corollaries:

It solves "Ulam's Problem" for $\aleph_2$, by showing that it is consistent that there is a collection of $\aleph_1$ countably complete uniform measures on $\aleph_2$ such that every subset of $\aleph_2$ is measured by one of the measures.

It implies that there is an ultrafilter D on ω_2 such that the ultrapower ω^{ω_2}/D has cardinality ω_1. This solves a problem dating from the mid-1960's. (See [C-K].)

In the early 1960's, Erdős and Hajnal [E-H], investigating the chromatic properties of graphs, defined the following graph:

$$\mathfrak{G}(\kappa, \lambda) = \langle \{f | f\colon \kappa \to \lambda\}, \perp \rangle,$$

where $f \perp g$ iff $|\{\alpha\colon f(\alpha) = g(\alpha)\}| < \kappa$. This graph is of interest because of its universal properties. (See §3, "Applications", for details.) In particular, they showed that the C.H. implies that $\mathfrak{G}(\omega_2, \omega)$ has uncountable chromatic number and asked whether this graph could have chromatic number $\aleph_1$. It is an easy corollary of the ultraproduct result that in the model contructed in this paper, $\mathfrak{G}(\omega_2, \omega)$ has chromatic number $\aleph_1$.

Other applications include a joint observation with Alan Dow about collectionwise normality.

A variation on the transfer theorem constructs a countably complete ideal K on ω_2 such that $P(\omega_2)/K \cong P(\omega_1)/\{\text{countable sets}\}$ and *every* ultrafilter extending K is highly nonregular. Further there is an ultrafilter D on $\aleph_2$ that is a Rudin–Keisler minimal ultrafilter on $\aleph_2$ and it has unique Rudin–Keisler predecessors on $\aleph_1$ and $\aleph_0$. This result is joint with Kanamori and Magidor, and appears in §4.

WOODIN'S THEOREM. The inspiration for the main theorem is the following theorem of Woodin:

THEOREM (Woodin [W]): *Assume $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$. If there is a normal $\aleph_1$-dense ideal on ω_1 and a normal ideal J on ω_2 such that $P(\omega_2)/J$ has a dense countably closed subset of cardinality $\aleph_2$, then there is a countably complete $\aleph_1$-dense ideal on ω_2.*

Woodin's theorem has the virtue of a relatively simple proof, however, at the time this paper is being written it is not known whether the hypothesis of Woodin's theorem are consistent. Hence, in this paper, there is a much more complicated transfer theorem, which has the advantage over Woodin's theorem that it is known that its hypothesis can be shown to be consistent using standard large cardinals.

This paper is organized as follows: In this section we give the basic definitions we will use. In §1, we show that it is consistent to have an $\aleph_1$-dense ideal on $\aleph_1$ and a very strongly layered ideal on $\aleph_2$. This result, while apparently new, uses no essentially new ideas. In §2, we prove the transfer theorem, the main new ingredient in the paper. In §3, we draw the applications as corollaries from the results of §1 and §2. In §4 we improve the results of §2 and draw some corollaries about ultrafilters under the Rudin-Keisler ordering. In §5 we list some of the many open problems in the area.

VERY STRONGLY LAYERED IDEALS ON ω_2. Let $\mathfrak{A}$ be a structure of regular cardinality κ. A **filtration** of $\mathfrak{A}$ is a continuous increasing κ- chain of elementary substructures of $\mathfrak{A}$, each of cardinality smaller than κ.

Remark: We note that any two filtrations coincide on a closed unbounded set, which also determines a filtration. Also, the requirement that each element of the chain be an elementary substructure of $\mathfrak{A}$ is, in some sense, superfluous, since this happens automatically on a closed unbounded subset of κ. Further, if we have a property that holds for a closed unbounded set of elements of the filtration, then by passing to a subsequence including only the "good" elements we can assume that the property holds for every element of the filtration.

For example, if $\mathfrak{A} \subset \mathfrak{B}$ are two structures of cardinality κ, and $\langle \mathfrak{A}_\alpha : \alpha \in \kappa \rangle$, $\langle \mathfrak{B}_\alpha : \alpha \in \kappa \rangle$ are filtrations of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, then $\langle \mathfrak{B}_\alpha \cap \mathfrak{A} : \alpha \in \kappa \rangle$ is a filtration of $\mathfrak{A}$, and hence agrees with $\langle \mathfrak{A}_\alpha : \alpha \in \kappa \rangle$ on a closed unbounded set. By passing to subsequences we can assume that $\langle \mathfrak{A}_\alpha : \alpha \in \kappa \rangle = \langle \mathfrak{B}_\alpha \cap \mathfrak{A} : \alpha \in \kappa \rangle$.

Let $\mathcal{A}, \mathcal{B}$ be partial orderings. A **projection map** from $\mathcal{A}$ to $\mathcal{B}$ is an order preserving function $\pi : \mathcal{A} \to \mathcal{B}$ such that for all $a \in \mathcal{A}$ and all $b \leq \pi(a)$ there is an $a' \leq a$ such that $\pi(a') \leq b$.

If $\mathcal{A}$ and $\mathcal{B}$ are Boolean Algebras and $\mathcal{B} \subset \mathcal{A}$ is a regular subalgebra (i.e. maximal antichains in $\mathcal{B}$ are maximal antichains in $\mathcal{A}$), and $\pi : \mathcal{A} \setminus \{0\} \to \mathcal{B} \setminus \{0\}$ is a projection map that is the identity on $\mathcal{B}$, then it is easy to check that for all $a \in \mathcal{A}, \pi(a) \geq a$. Projection maps rarely are homomorphisms (they don't preserve meets), but they do preserve descending meets: If $\mathcal{D}$ is linearly ordered by $\geq$ and $\wedge D$ exists in $\mathcal{A}$, then $\wedge \pi``D$ exists in $\mathcal{B}$ and $\pi(\wedge \mathcal{D}) = \wedge(\pi``\mathcal{D})$.

We will attempt to use standard notation throughout this paper. We will write $\mathrm{Col}(\kappa, \lambda)$ for the Levy partial ordering collapsing λ to have cardinality κ with conditions of size $< \kappa$. We will write $\mathrm{Col}(\kappa, < \lambda)$ for the Levy partial ordering collapsing every ordinal $< \lambda$ to have cardinality κ. We will use the notation $\mathrm{Add}(\kappa)$ for the partial ordering adding a single Cohen subset to κ with conditions of size $< \kappa$. We will write $\mathrm{S}(\kappa, \lambda)$ for the "Silver Collapse" of λ to be κ^+. Conditions here are partial functions $p : \kappa \times \lambda \to \lambda$ such that for all α, β, $p(\alpha, \beta) < \beta$, $|\mathrm{dom}(p)| \leq \kappa$ and there is a $\delta < \kappa$, such that $\mathrm{dom}(p) \subset \delta \times \lambda$. The ordering on the Silver Collapse is inclusion. (See [Ku] for a good explication of this partial ordering.)

We now make one of our main definitions:

Definition: A normal $\aleph_2$-complete ideal I on ω_2 is **very strongly layered** iff

$$P(\omega_2)/I = \bigcup \{B_\alpha : \alpha < \omega_3\}$$

where:

(1) The sequence $\langle B_\alpha : \alpha < \omega_3 \rangle$ is increasing and continuous, and for all α, $|B_\alpha| = \omega_2$. (In other words $\langle B_\alpha : \alpha < \omega_3 \rangle$ is a filtration.)

(2) There is a dense set $D \subset P(\omega_2)/I$ that is closed under descending ω-sequences and finite non-zero meets (i.e. if $\{d_1, \ldots, d_n\}$ are in D and $\wedge d_i \neq 0$ then $\wedge d_i \in D$).

(3) For $\alpha \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})$, B_α is a regular subalgebra of $P(\omega_2)/I$. Further there is a family of projection maps $\{\pi_\alpha \colon \alpha \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})\}$ such that $\pi_\alpha \colon D \to (D \cap B_\alpha)$, $\pi_\alpha \restriction (D \cap B_\alpha)$ is the identity, and for $\alpha < \beta$ we have $\pi_\alpha \circ \pi_\beta = \pi_\alpha$ (i.e. the projection maps commute).

We will denote $D \cap B_\alpha$ by D_α. We note that if we have the continuum hypothesis, then, by passing to a subsequence if necessary, we may assume that for $\alpha \in \mathrm{cof}(\omega_2)$, D_α is closed under finite non-zero meets and descending ω-sequences and that D_α is dense in B_α.

Layered ideals were first defined in [F-M-S]. A κ-complete ideal I on a regular cardinal κ is **layered** iff there is a filtration of the quotient $P(\kappa)/I = \bigcup\{B_\alpha \colon \alpha < \kappa^+\}$ such that for a stationary set of $\alpha \in \mathrm{cof}(\kappa)$, B_α is a regular subalgebra of B. If I is a layered ideal on κ then I is κ^+ saturated: If $A \subset P(\kappa)/I$ is a maximal antichain, then for some $\alpha \in \mathrm{cof}(\kappa)$, $A \cap B_\alpha$ is a maximal antichain and B_α is a regular subalgebra of $P(\kappa)/I$. Hence, $A = A \cap B_\alpha$ and thus has cardinality $\leq \kappa$.

We can easily define the notion of a **very strongly layered ideal** on κ^+ by replacing "ω_2" by "κ^+", and "σ-closed" by "$< \kappa^+$-closed".

For technical reasons that occur later we need the following easy fact:

Fact: Assume the C.H. If $\mathbb{P}$ is an ω-closed partial ordering that collapses $\aleph_2$ and has cardinality $\aleph_2$ then $\mathbb{P}$ has a dense set isomorphic to $\mathrm{Col}(\omega_1, \omega_2)$.

If we have a very strongly layered ideal we may assume that B_0 is a regular subalgebra of B that collapses ω_2, and thus we can assume that $D_0 \supset D_0'$ where D_0' is dense in D_0 and $D_0' \cong \mathrm{Col}(\omega_1, \omega_2)$.

Finally, we remark that the ideal J on ω_2 hypothesized in Woodin's theorem, namely a normal ideal J on ω_2 such that $P(\omega_2)/J$ has a dense countably closed set of size $\aleph_2$, is easily seen to be very strongly layered.

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1. The consistency result

In the late 1970's Woodin showed that it is consistent to have an $\aleph_1$-dense ideal on $\aleph_1$, assuming the consistency of of the theory "$ZF + AD_{\mathbb{R}} + \theta$ is regular". Later Woodin [W] improved this result showing, assuming the consistency of an almost huge cardinal, that the following is consistent: "for all $\aleph_2$- saturated partial orderings $\mathbb{P}$ that collapse $\aleph_1$, there is a countably complete ideal I on ω_1 such that $P(\omega_1)/I$ has a dense set isomorphic to $\mathbb{P}$".

At cardinals above $\aleph_1$ there are serious obstructions. Kunen [Ku] proved that there can be no uniform countably complete $\aleph_2$-saturated ideal on any cardinal κ with $\aleph_\omega \leq \kappa \leq \aleph_{\omega_1}$. In [F-M], it is shown that this theorem is sharp at the upper end of the interval, namely that assuming the consistency of a supercompact cardinal, one can construct a model where there is a countably complete $\aleph_2$-saturated ideal on $\aleph_{\omega_1+1}$. The sharpness of the other inequality is still an open problem. However, as stated in the first section we make progress on this problem by showing that it is consistent to have an $\aleph_1$-dense ideal on $\aleph_2$.

To prove our main result we need to extend Woodin's theorem to get an $\aleph_1$-dense ideal on ω_1 consistent with a very-strongly layered ideal on $\aleph_2$.

In this section we prove the following theorem:

THEOREM 1.1: *Let j_0 and j_1 be almost huge embeddings with critical points κ_0 and κ_1, respectively. Suppose that $j_0(\kappa_0) = \kappa_1$ and that $\kappa_2 = j_1(\kappa_1)$ is Mahlo. Then there is a partial ordering $\mathbb{P}$ such that there is a definable subclass W of $V^{\mathbb{P}}$ satisfying:*

(1) $ZFC + G.C.H. + \Diamond_{\omega_1} + \Diamond_{\omega_2}(\mathrm{cof}(\omega_1)) + \Box_{\omega_2}$.

(2) *There is an $\aleph_1$-dense ideal J on $\aleph_1$.*

(3) *There is a very-strongly-layered ideal I on $\aleph_2$.*

Remarks: The existence of two embeddings satisfying the hypothesis of the theorem is an easy consequence of the existence of a huge cardinal. It is also the case that we can get a model with a very strongly layered ideal on $\aleph_2$ satisfying the Woodin conclusion that: "for all $\aleph_2$-saturated partial orderings $\mathbb{P}$ of cardinality $\aleph_2$ that collapse $\aleph_1$, there is a countably complete ideal I on ω_1 such that $P(\omega_1)/I$ has a dense set isomorphic to $\mathbb{P}$". This involves some additional preliminary forcing. (See [W].)

While we believe this theorem to be new, it requires no essential new ideas to prove. Accordingly, we only sketch the proof. We refer the reader to [W] for more details on the construction of $\aleph_1$-dense ideals, and to [F-M-S] for more details on the construction of layered ideals.

Proof: We will use the following standard forcing fact:

Fact: Let κ be an inaccessible cardinal and $\mathbb{P}$ be a partial ordering that is κ-c.c., collapses κ to be ω_1 and is such that if τ is a $\mathbb{P}$-term for a subset of ω, then the least complete subalgebra of $\mathbb{P}$ deciding τ has cardinality less then κ. Then for all V-generic objects $G \subset \mathbb{P}$, there is a forcing extension V' of $V[G]$ such that in V' there is a V-generic object $H \subset \mathrm{Col}(\omega, < \kappa)$ such that $P(\omega)^{V[G]} = P(\omega)^{V[H]}$.

We construct our model W in several stages. We first collapse κ_0 to be $\aleph_1$ using the Levy collapse. We then move to the choiceless inner model $V(\mathbb{R})$. In $V(\mathbb{R})$, we build the "universal" Kunen collapse $\mathbb{Q}$ of κ_1 to be ω_2. After forcing with this collapse we get a model V_2 which has a normal, $\aleph_1$ dense ideal on $\aleph_1$. We force over V_2 with the Silver collapse making κ_2 into $\aleph_3$ to get a model V_3 with a layered ideal on $\aleph_2$ that has a countably closed dense subset. Since the Silver Collapse doesn't add new subsets of $\aleph_1$, the ideal on $\aleph_1$ remains $\aleph_1$-dense. Shooting a closed unbounded set through the stationary set witnessing the layering yields a model with a very strongly layered ideal. To finish, we add $\square_{\aleph_2}$, using the canonical conditions of size $\aleph_2$. Since these conditions don't add new subsets of $\aleph_2$, we preserve the property of being very strongly layered.

The main points of the proof are:

(1) to see that Woodin's arguments for the consistency of an $\aleph_1$-dense ideal can be carried out to show that V_2 has such an ideal, and

(2) that W has a very-strongly layered ideal on $\aleph_2$.

CLAIM 1.2: *Let j be an almost-huge embedding with critical point κ_0 and $j(\kappa_0) = \kappa_1$. Let $C_0 \subset \mathrm{Col}(\omega, < \kappa_0)$ be generic over V. Let $V_1 = V(\mathbb{R}) \subset V[C_0]$. Let $\mathbb{Q} = \mathrm{Add}(\omega_1) * \mathbb{Q}'$ be a κ_1-c.c. partial ordering in V_1 that is countably closed, has cardinality κ_1, and collapses κ_1 to be $\aleph_2$. Then for all $V(\mathbb{R})$-generic $G \subset \mathbb{Q}$, $V(\mathbb{R})[G] \models$ Z.F.C. + $\diamondsuit_{\omega_1}$ + "there is a normal $\aleph_1$-dense ideal J on $\aleph_1$".*

Proof: Let $\mathbb{R}_0 = P(\omega) \cap V[C_0]$. Since $\mathrm{Add}(\omega_1)$ adds a wellordering of $\mathbb{R}$ in a canonical way, we see that any generic $H \subset \mathbb{Q}$ can be decomposed into $G_0 * G_1$, and $V_1[G_0] \models$ *Z.F.C.*

Let $G \subset \mathbb{Q}$ be V_1-generic. Let $C \subset \mathrm{Col}(\omega, \omega_1)$ be $V_1[G]$-generic and $\mathbb{R}_1 = P(\omega) \cap V_1[G * C]$. Then by the "Fact", $\mathbb{R}_1 = P(\omega) \cap V[C_1]$ for some V-generic $C_1 \subset \mathrm{Col}(\omega, < \kappa_1)$. Standard ideas show that j can be extended to a

$$\hat{j}\colon V(\mathbb{R}_0) \to M(\mathbb{R}_1)$$

by setting $\hat{j}(\tau^{V(\mathbb{R}_0)}) = \tau^{M(\mathbb{R}_1)}$. (We do not need the generic object H to define the elementary embedding, the "Fact" is used to prove that this definition, given in $V_1[G * C]$, yields an elementary embedding.)

We work in $V_1[G*C]$. For $\kappa_0 < \alpha < \kappa_1$, let $m_\alpha = \hat{j}``(G \cap V_\alpha)$. Then each m_α is in $M(\mathbb{R}_1)$. Let $\langle x_\alpha \colon \alpha < \kappa_1\rangle$ be an enumeration of the V_1-terms for elements of $P(\kappa_0) \cap V_1[G]$ such that for a closed unbounded set of α and all $\beta < \alpha \in \kappa_1$, $x_\beta \in V_1[G \cap V_\alpha]$. Note that for $\beta < \kappa_1$, $\langle x_\alpha \colon \alpha < \beta\rangle$ is in $M(\mathbb{R}_1)$. Let $\blacktriangle$ be a well-ordering of $\hat{j}(\mathbb{Q}')$ in $M(\mathbb{R}_1)^{j(\mathrm{Add}(\omega_1))}$. Define a descending sequence $\langle p_\alpha \colon \alpha < \kappa_1\rangle \subset \hat{j}(\mathbb{Q})$ such that:

(1) for each α,γ, if $x_\alpha \in V_1[G \cap V_\gamma]$, then $p_{\alpha+1} \cap M_{\hat{j}(\gamma)}$ decides $\| \kappa_0 \in \hat{j}(x_\alpha) \|$.
(2) If $p_\alpha \in V_{\hat{j}(\gamma)}$, then p_α is compatible with m_γ.
(3) If $p_{\alpha+1} = q_0^\alpha * q_1^\alpha \in \mathrm{Add}(\omega_1) * \mathbb{Q}'$ then $q_0^\alpha \Vdash q_1^\alpha$ is the $\blacktriangle$-least element of $\mathbb{Q}'$ so that $p_{\alpha+1} < p_\alpha$ and $p_{\alpha+1}$ has q_0^α as its first coordinate and satisfies (1) and (2).

Using (3), and the fact that $\langle q_0^\alpha \colon \alpha < \beta\rangle \in M(\mathbb{R}_1)$, one can check that for all $\beta < \kappa_1$ the sequence $\langle p_\alpha \colon \alpha < \beta\rangle \in M(\mathbb{R}_1)$.

The sequence $\langle p_\alpha \colon \alpha < \kappa_1\rangle$ induces an ultrafilter U on $P(\kappa_0) \cap V_1[G]$ that is κ_0-complete for sequences that lie in $V_1[G]$. Define an ideal J in $V_1[G]$ by putting $x \in J$ iff $\| x \in U \| = 0$, where the boolean value is taken $B(\mathrm{Col}(\omega, \omega_1))$. Equivalently, $x \in J^\smile$ iff $\|$ there is an $\alpha, p_\alpha \Vdash \kappa_0 \in \hat{j}(x) \| = 1$.

To see that J is a normal ideal, let $\langle x_\beta \colon \beta < \kappa_0\rangle \in V_1[G]$ be a sequence of elements of $J^\smile$. Then for all β, $\|$for some $\alpha, p_\alpha \Vdash \kappa_0 \in \hat{j}(x_\beta) \| = 1$. Since, in $V_1[G][C]$, $\mathrm{cof}(\kappa_1) > \kappa_0$ and $\langle p_\alpha \colon \alpha \in \kappa_1\rangle$ is a descending sequence, and the forcing yielding $V_1[G][C]$ is κ_1-c.c., $\|$for some $\alpha, p_\alpha \Vdash \kappa_0 \in \bigcap\{\hat{j}(x_\beta) \colon \beta < \kappa_0\} \| = 1$. Hence $\|$ for some $\alpha, p_\alpha \Vdash \kappa_0 \in \hat{j}(\Delta\{x_\beta\}) \| = 1$. Thus $J^\smile$ is closed under diagonal intersections, so J is normal.

This shows that the map $x \mapsto \| x \in U \|$ induces a boolean algebra monomorphism from $P(\kappa_0)/J$ to $B(\mathrm{Col}(\omega, \omega_1))$. Hence, J is $\aleph_2$-saturated. Since J is normal, this map is a regular embedding and thus $P(\kappa_0)/J$ is isomorphic to a regular subalgebra of $B(\mathrm{Col}(\omega, \omega_1))$. Thus J is an $\aleph_1$-dense ideal. ∎

We now apply Claim 1.2 to a particular partial ordering. Let j_0, j_1 be the almost huge embeddings posited in the hypothesis of Theorem 1.1. Working inside V_1, construct the following "universal" partial ordering due to Kunen.

(1) $\mathbb{Q}$ will be a κ_1-stage iteration
(2) Let $\mathbb{Q}_0 = \mathrm{Add}(\omega_1)$. Then $V^{\mathbb{Q}_0} \models A.C. + \kappa_1$ is almost huge.
(3) $\mathbb{Q}$ is a countable support iteration over $V^{\mathbb{Q}_0}$.
(4) If α is inaccessible and $\mathbb{Q}_\alpha \cap V_\alpha$ is a regular subalgebra of $\mathbb{Q}_\alpha$, then we let $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * S^{\mathbb{Q}_\alpha \cap V_\alpha}(\alpha, \kappa_1)$.
(5) Otherwise, we let $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \{1\}$.

Then standard arguments due to Kunen and Laver show that $\mathbb{Q}$ is κ_1-c.c.

and countably closed. Hence we can apply Claim 1.2 to this partial ordering with $j = j_0$ to see that if $G \subset \mathbb{Q}$ is V_1-generic then $V_1[G] \models Z.F.C.+$ there is an $\aleph_1$-dense ideal on $P(\aleph_1) + \Diamond_{\omega_1}$. Further, for any inaccessible α such that $\mathbb{Q}_\alpha \cap V_\alpha$ is a regular subalgebra of $\mathbb{Q}_\alpha$ we have a canonical regular embedding of $(\mathbb{Q}_\alpha \cap V_\alpha) * S^{(\mathbb{Q}_\alpha \cap V_\alpha)}(\alpha, \kappa_1)$ into $\mathbb{Q}_{\alpha+1}$.

Fix a generic $G \subset \mathbb{Q}$ and let $V_2 = V_1[G]$. Let $H \subset S^{V_2}(\kappa_1, \kappa_2)$ be generic over V_2, and let $V_3 = V_2[H]$.

Since κ_1 is inaccessible and $\mathbb{Q}$ is κ_1-c.c., $\mathbb{Q}$ is a regular subalgebra of $j_1(\mathbb{Q})_{\kappa_1}$. Since $j_1(\mathbb{Q})$ is defined the same way as $\mathbb{Q}$ is we see that $j_1(\mathbb{Q})_{\kappa_1+1} = j_1(\mathbb{Q})_{\kappa_1} * S^{\mathbb{Q}}(\kappa_1, \kappa_2)$ and thus it makes sense to form $j_1(\mathbb{Q})/G * H$.

We now analyse the structure of $j_1(\mathbb{Q})/G * H$. This analysis is very similar to one in [Hu1] where a detailed discussion is given (similar arguments are also given in [F-L]).

To motivate the analysis, we consider the example of the Silver collapse $S(\omega_1, \lambda)$ for a Mahlo λ. Let $B = B(S(\omega_1, \lambda))$, and for inaccessible α, $B_\alpha = B(S(\omega_1, \alpha))$. For other β, let $B_\beta = \bigcup\langle B_\alpha : \alpha < \beta\rangle$. Then $\langle B_\alpha : \alpha < \lambda\rangle$ is a filtration of B. Further, for inaccessible α, the map $\pi_\alpha : S(\omega_1, \lambda) \to S(\omega_1, \alpha)$ given by $\pi_\alpha(q) = q \cap V_\alpha$ is a projection map. Thus we get a collection of witnesses for layering of $\mathcal{B}(S(\omega_1, \lambda))$ that preserve a countably closed dense set. The system of π_α's remains a collection of witnesses to layering in any forcing extension that preserves the stationarity of the old inaccessible cardinals. In particular, if we collapse λ to be $\aleph_2$ with countably closed conditions and then force a closed unbounded set through $\mathrm{cof}(\omega) \cap \{V - \textit{inaccessibles}\}$ using $\aleph_1$-sized conditions, $\mathcal{B}(S(\omega_1, \lambda))^V$ will have cardinality $\aleph_2$ and be strongly layered.

For notational simplicity, let $\hat{Q} = j_1(\mathbb{Q})/G * H$. Then, $\hat{Q}$ is an iteration of sorts with countable support and two kinds of coordinates. The first kind of coordinates are those arising from $\alpha < \kappa_1$. These coordinates yield the trivial forcing, unless α is inaccessible and falls under case (4) of the definition of $\mathbb{Q}$. In this case G induces a generic object G_α on $S^{\mathbb{Q}_\alpha \cap V_\alpha}(\alpha, \kappa_1)$. For these α, $\hat{Q}_\alpha$ is the forcing $S^{\mathbb{Q}_\alpha \cap V_\alpha}(\alpha, j_1(\kappa_1))/G_\alpha$. This partial ordering is countably closed. The coordinate $\alpha = \kappa_1$ is entirely swallowed by H. In coordinates $\alpha > \kappa_1$, we force with terms for elements of Silver collapses over models extending $V[G]$ that have the same ω-sequences as does $V[G]$. A decreasing ω-sequence of conditions in $\hat{Q}$ induces a decreasing ω-sequence of conditions in each such coordinate. Since each coordinate is ω-closed, by taking the coordinatewise meets we get a condition below our ω-sequence in $\hat{Q}$.

Let $B = B(\hat{Q})$. Then $\hat{Q}$ is a countably closed dense subset of B. We describe a filtration $B = \langle B_\alpha\colon \alpha < \kappa_2\rangle$ and a commuting family of projection maps from $\hat{Q}$ to $D_\alpha = \hat{Q} \cap B_\alpha$. For each coordinate γ in the iteration $\hat{Q}$ and each α between κ_1 and κ_2 that is inaccessible in V, let $Q(\gamma,\alpha) = \{q \in \hat{Q}\colon$ it is forced by the trivial condition that $\mathrm{dom}(q(\gamma)) \subset \gamma \times \alpha\}$. Let $D_\alpha = \{q \in \hat{Q}\colon$ for all $\gamma < \alpha q(\gamma) \in Q(\gamma,\alpha)\}$. Clearly, $\hat{Q} = \bigcup_{\alpha<\kappa_2} D_\alpha$.

For α inaccessible in V, let B_α be the boolean subalgebra generated in $\mathcal{B}(\hat{Q})$ by Q_α. For other α, let $B_\alpha = \bigcup\{B_\beta\colon \beta < \alpha\}$. This describes the filtration. For α inaccessible let $\pi_\alpha\colon \hat{Q} \to D_\alpha$ be the map described by setting $\pi_\alpha(q)(\gamma) = q(\gamma) \restriction \gamma \times \alpha$ for $\gamma < \alpha$ and $\pi_\alpha(q)(\gamma) = 1$ otherwise. It is easy to check that this is a commuting family of projection maps.

We have now shown:

Claim 1.3: *Working in V_3, let $B = B(\hat{Q})$. Then there is a dense ω-closed subset D of B and a filtration $B = \bigcup\langle B_\alpha\colon \alpha < \kappa_2\rangle$ and a commuting family of projection maps $\{\pi_\alpha\colon \alpha$ is inaccessible in $V\}$ with $\pi_\alpha\colon D \to D \cap B_\alpha$.*

Claim 1.4: *In V_3 there is an ideal I on ω_2 such that*

$$P(\omega_2)/I \cong B(j_1(\mathbb{Q})/G * H).$$

Proof: In this proof we will refer to claims proved in [F-M-S] on pages 529–531. Let $\hat{G} \subset j_1(\mathbb{Q})$ be $V_1[G * H]$-generic for $B(j_1(\mathbb{Q})/G * H)$. By Claims 6 and 7 of [F-M-S] and the discussion on pages 530–531 there is an ultrafilter $\mathfrak{F}$ in $V[\hat{G}]$ on $P(\kappa_1)^{V[G*H]}$ that is closed under diagonal intersections and intersections of $< -\kappa_1$-sequences that lie in $V[G * H]$. Further, for each $q \in j(\mathbb{Q})/G * H$, there is an $x \in P(\kappa_1)^{V[G*H]}$ such that $\| x \in \mathfrak{F} \| = q$.

Hence the map $i\colon P(\kappa_1) \to B(j(\mathbb{Q})/G * H)$ given by $i(x) = \| x \in \mathfrak{F} \|$ gives an order preserving map of $P(\kappa_1)$ onto a dense subset of $B(j(\mathbb{Q})/G * H)$. Letting $I = \{x\colon i(x) = 0\}$, we find that in $V[G * H]$:

$$i\colon P(\kappa_1)/I \to B(\mathbb{Q}/G * H)$$

is an order and incompatibility preserving map to a dense subset. Since $\mathfrak{F}$ is closed under diagonal intersections and $< \kappa_1$- intersections from $V[G * H]$, I is normal and κ_1-complete. Since $j(\mathbb{Q})/G*H$ is $j_1(\kappa_1)$- c.c., I is saturated, $P(\kappa_1)/I$ is a complete Boolean algebra, and hence i is an isomorphism. ∎

Claims 1.3 and 1.4 together show that in V_3, I is a layered ideal with a countably closed dense subset and witnesses for layering. Working in V_3, let S be the set of ordinals between ω_2 and ω_3 that were inaccessible in V (i.e. the set on which we have witnesses for layering). Then $S \subset \mathrm{cof}(\omega_2)$. Standard arguments

[A-S] show that there is a cardinal preserving partial ordering $\mathbb{P}$ that doesn't add new subsets of ω_2 but does add a closed unbounded subset of $S \cup \mathrm{cof}(\leq \omega_1)$. Let C' be generic for $\mathbb{P}$. Then in $V_3[C']$ the witnesses for layering still are projection maps and hence I is a very strongly layered ideal.

Finally, we must add $\square_{\omega_2}$ to get the desired model. To do this we use the standard partial ordering for adding $\square_{\omega_2}$ using "initial segments". Namely conditions in the partial ordering are partial $\square$ sequences $\langle C_\alpha \colon \alpha \leq \gamma \rangle$ for some $\gamma < \omega_3$, ordered by inclusion. Standard arguments show that this partial ordering is ω_2-strategically closed. Hence, the partial ordering doesn't add new subsets of ω_2 and therefore preserves the very-strongly-layered ideal.

We now verify the diamond conditions. To see $\Diamond_{\omega_1}$ we note that the partial ordering $\mathbb{Q}'$ adds a generic filter for $\mathrm{Add}(\omega_1)$ over the model $V(\mathbb{R}_0)^{\mathrm{Add}(\omega_1)}$ in such a way that the quotient forcing $\mathbb{Q}/\mathrm{Add}(\omega_1) * \mathrm{Add}(\omega_1)$ is countably closed. Since forcing with $\mathrm{Add}(\omega_1)$ adds diamond and countably closed forcing cannot kill diamond, we know that $\Diamond_{\omega_1}$ holds in $V(\mathbb{R}_0)^{\mathbb{Q}}$. Since the rest of the forcing adds no new subsets to ω_1, $\Diamond_{\omega_1}$ holds in the final model.

Similarly, over $V(\mathbb{R}_0)^{\mathbb{Q}}(= V_2)$, forcing with the Silver collapse making κ_2 the successor of κ_1 adds a $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$ sequence. Thus V_3 satisfies $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$. The rest of the forcing to get the final model adds no new subsets of ω_2 so $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$ holds in the final model. ∎

2. The transfer theorem

The main result of this section is the following theorem:

THEOREM 2.1: *Suppose there is a very strongly layered ideal I on $\omega_2, \square_{\omega_2}$, C.H. and $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$. Then there is a σ- complete uniform ideal $K \supset I$ on ω_2 such that*

$$P(\omega_2)/K \cong P(\omega_1)/\{\text{countable sets}\}.$$

Remark: This theorem is true if we replace ω_1 by a regular κ, the ideal of countable sets by the ideal of sets of cardinality $< \kappa$ and ω_2 by κ^+. The ideal on κ^+ will be $< \kappa$-complete.

COROLLARY 2.2: *Suppose there is a very strongly layered ideal I on $\omega_2, \square_{\omega_2}$, C.H. and $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$. Then for all uniform ideals J on ω_1, there is a uniform ideal K on ω_2 such that:*

$$P(\omega_2)/K \cong P(\omega_1)/J.$$

Further, the degree of completeness of K equals the degree of completeness of J, and if J is $\aleph_2$-saturated then K is weakly normal.

Remark: Even in the presence of a very strongly layered ideal on ω_2, the collection of Boolean Algebras arising from quotients of the form $P(\omega_1)/J$ can be exceedingly rich! For example as stated in §1 it can include all complete, $\aleph_2$-saturated Boolean Algebras of cardinality $\aleph_2$ that collapse ω_1. In particular, from this corollary we can get the consistency of a countably complete, uniform $\aleph_1$-dense ideal on $\aleph_2$.

Proof of 2.2: By Theorem 2.1 there is an ideal $K_0 \supset I$ on ω_2 such that $P(\omega_2)/K_0 \cong P(\omega_1)/\{\text{countable sets}\}$. Let J be any ideal on ω_1 containing all the countable sets. Then we have a surjective homomorphism

$$\phi\colon P(\omega_1)/\{\text{countable sets}\} \to P(\omega_1)/J.$$

This induces a surjective homomorphism $\phi'\colon P(\omega_2)/K_0 \to P(\omega_1)/J$. If we let K be the ideal generated over K_0 by $\ker(\phi')$ then $P(\omega_2)/K \cong P(\omega_1)/J$. If J is countably complete, then the kernel of ϕ' is countably complete in $P(\omega_2)/K_0$. Hence the ideal K is countably complete.

Suppose now that J is $\aleph_2$-saturated. We must show that K is weakly normal. This is equivalent to the following statement:

If $f\colon \omega_2 \to \omega_2$ is a regressive function then there is a set $A \in K^{\smile}$ and a $\gamma \in \omega_2$ such that the range of f on A is bounded by γ.

Clearly ϕ' induces a homomorphism $\psi\colon P(\omega_2)/I \to P(\omega_1)/J$. Given a regressive function f, let $a_i = \{\delta\colon f(\delta) = i\}$. Then $\langle a_i\colon i \in \omega_2\rangle$ contains a maximal antichain in $P(\omega_2)/I$. Let $b_\alpha = \vee\{a_i\colon i \in \alpha\}$. Then $\langle b_\alpha\colon \alpha \in \omega_2\rangle$ is an increasing $\aleph_2$-sequence in $P(\omega_2)/I$. Since J is $\aleph_2$-saturated there is a $\gamma \in \omega_2$ such that for all $\gamma' > \gamma, \psi(b_{\gamma'}) = \psi(b_\gamma)$. Since the b_α's sup to 1, this constant value for ψ must be 1. But then $\psi(\{\delta\colon f(\delta) < \gamma\}) = 1$, hence $\{\delta\colon f(\delta) < \gamma\} \in K^{\smile}$. ∎

Remark: It is possible to eliminate the □ hypothesis in the previous theorems. The referee requested this be made explicit so at various places in the argument we will briefly outline the modifications necessary.

Remark: In the case where we have a very strongly layered ideal on κ^+ (κ regular), then we get a similar statement: for all ideals J on κ containing the ideal of sets of size $< \kappa$ there is a uniform ideal K on κ^+ with $P(\kappa^+)/K \cong P(\kappa)/J$. In this case the degree of completeness of K is exactly that of J. From this, one can use very strongly layered ideals on consecutive cardinals to "bootstrap" ideals. For example, if for all n there is a very strongly layered ideal on $\aleph_n$, then for all $n < m$ and all uniform ideals J on $\aleph_n$ there is a uniform ideal K on $\aleph_m$ such that

$$P(\aleph_n)/J \cong P(\aleph_m)/K$$

and the degrees of completeness of K and J are equal. To follow this route to the consistency of "for all $n \in \omega$ there is an uniform countably complete $\aleph_1$-dense ideal on $\aleph_n$, one must produce a model with an $\aleph_1$-dense ideal on $\aleph_1$ and very strongly layered ideals on all of the other $\aleph_n$'s. At the time of this writing it is not known how to get very strongly layered ideals on three consecutive successor cardinals.

Proof of Theorem 2.1: Fix a strongly layered ideal I, and witnesses $B_\alpha, D, \pi_\alpha, \ldots$ to the strong layering. To build a σ-complete ideal K we construct a surjective homomorphism

$$h\colon P(\omega_2)/I \to P(\omega_1)/\{\text{countable sets}\}$$

so that the kernel of h is countably complete. Letting K be the kernel of h, we see that

$$P(\omega_2)/K \cong P(\omega_1)/\{\text{countable sets}\}.$$

Given a subset of ω_2 we need to "measure" it by a subset of ω_1. Any function $f\colon \omega_1 \to D$ measures each set $x \subset \omega_2$ by the yielding the set $A_x = \{i\colon f(i) \subset_I x\}$. Unfortunately this measurement may be ambiguous in that $\omega_2 \backslash A_x \neq A_{\omega_2 \backslash x}$ (modulo countable sets), i.e. if it is not the case that for almost all i, either $f(i) \subset_I x$ or $f(i) \subset_I \omega_2 \backslash x$. But it is hopeless to unambiguously measure every subset of ω_2 in this way with one function f. Hence we need a family of functions. Further the measurements these functions make must agree with each other. This is the motivation for the first three clauses of the following definition. The last clause is a coding device to make the homomorphism surjective. In what follows we will use $\leq$ to mean $\subset_I$.

We will construct a matrix of functions:

$$\mathcal{F} = \{f^\delta_\gamma\colon \gamma < \omega_2, \delta \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})\}$$

such that for each δ, γ,

$$f^\delta_\gamma\colon \omega_1 \to D_\delta \quad (= \text{the } \delta\text{th layer of } D).$$

The family of functions $\mathcal{F}$ will satisfy the following four properties:

(1) Horizontal Coherence: For each δ and $\gamma < \gamma'$, for all but countably many i, $f^\delta_\gamma(i) \geq f^\delta_{\gamma'}(i)$.
(2) Vertical Coherence: For $\delta < \delta'$ there is an unbounded set of $\gamma < \omega_2$ such that for all but countably many i, $f^\delta_\gamma(i) = \pi_\delta(f^{\delta'}_\gamma(i))$.

(3) Genericity: For each $x \subset \omega_2$, with $x \in B_\delta$ there is a $\gamma < \omega_2$ such that for all but countably many i, either $f^\delta_\gamma(i) \leq x$ or $f^\delta_\gamma(i) \wedge x =_I 0$.
Let $D'_0 \subset D_0$ be dense with $D'_0 \cong \mathrm{Col}(\omega_1, \omega_2)$. If f^0_γ takes values in D'_0 then, by using this isomorphism, we can assume that for all i, $f^0_\gamma(i) \in \mathrm{Col}(\omega_1, \omega_2)$.

(4) Coding: Fix an enumeration $P(\omega_1) = \{y_\eta : \eta < \omega_2\}$. For each $\eta < \omega_2$, there is a γ_η such that for all i, $f^0_{\gamma_\eta}(i) \in D'_0$, and $\gamma_\eta \in$ range $f^0_{\gamma_\eta}(i)$. Further, letting $\beta_\eta(i)$ be the least β such that $f^0_{\gamma_\eta}(i)(\beta) = \gamma_\eta$, then $y_\eta = \{i : \beta_\eta(i)$ is a limit ordinal$\}$.

Remark: The purpose of the vertical coherence condition is to show that the homomorphism h defined in Claim 2.3 below is well defined. It can be weakened to the following statement:

WEAK VERTICAL COHERENCE. For all $\delta < \delta'$ and all γ' there is a $\gamma > \gamma'$ such that for all but countably many i, $f^\delta_\gamma(i) \leq \pi_\delta(f^{\delta'}_{\gamma'}(i))$.

This is useful in the variant of this argument that doesn't use square.

CLAIM 2.3: *If there is a set of functions $\mathcal{F}$ satisfying the conditions* (1)–(4) *then there is a surjective homomorphism h: $P(\omega_2)/I \to P(\omega_1)/\{$countable sets$\}$ with a countably complete kernel.*

Remark: In §4 we will refer to the kernel of h as "THE ideal determined by $\mathcal{F}$."

Proof: Given the set of functions $\mathcal{F}$, and an $x \subset \omega_2$, we look at the least δ such that $x \in B_\delta$. By genericity there is a $\gamma < \omega_2$, for all but countably many $i, f^\delta_\gamma(i) \subset_I x$ or $f^\delta_\gamma(i) \cap x =_I 0$. Let $A_x = \{i : f^\delta_\gamma(i) \subset_I x\}$.

We claim that for all $\delta' > \delta$ and all large enough γ' (depending on δ'), $A_x = \{i : f^{\delta'}_{\gamma'}(i) \subset_I x\}$ modulo countable sets. Namely, fix a δ' and choose a $\gamma' > \gamma$ where for all but countably many i, $f^\delta_{\gamma'}(i) = \pi_\delta(f^{\delta'}_{\gamma'}(i))$. Since $x \in B_\delta$ and π_δ is a projection map, $f^{\delta'}_{\gamma'}(i) \subset_I x$ iff $\pi_\delta(f^{\delta'}_{\gamma'}(i)) \subset_I x$. Since $\gamma' > \gamma$ for all but countable many i, $f^\delta_{\gamma'}(i) \subset_I x$ iff $f^\delta_\gamma(i) \subset_I x$ and similarly for $\omega_2 \backslash x$. Hence for all but countable many i, $f^\delta_{\gamma'}(i) \subset_I x$ or $f^\delta_{\gamma'}(i) \subset_I \omega_2 \backslash x$, and $A_x = \{i : f^\delta_{\gamma'}(i) \subset_I x\} = \{i : f^{\delta'}_{\gamma'}(i) \subset_I x\}$.

Define a function h: $P(\omega_2) \to P(\omega_1)/\{$countable sets$\}$ by setting $h(x) = [A_x]$. Then h is well defined by the remarks in the previous paragraph. To see that h is a homomorphism, it suffices to show that h preserves complements and intersections. Clearly, for all δ, γ and all x, y,

$$\{i : f^\delta_\gamma(i) \subset_I x \cap y\} = \{i : f^\delta_\gamma(i) \subset_I x\} \cap \{i : f^\delta_\gamma(i) \subset_I y\}.$$

Hence $h(x \cap y) = h(x) \cap h(y)$. Let $x \subset \omega_2$. Choose a large enough δ, γ such that $h(x) = \{i: f^\delta_\gamma(i) \subset_I x\}$ and for all but countably many i, $f^\delta_\gamma(i) \subset_I x$ or $f^\delta_\gamma(i) \cap x =_I 0$. Then

$$h(\omega_2 \backslash x) = \{i: f^\delta_\gamma(i) \subset_I \omega_2 \backslash x\} = \{i: f^\delta_\gamma(i) \cap x =_I 0\} = \omega_1 \backslash h(x).$$

To see that h is surjective, fix some $[y_\eta] \in P(\omega_1)$. By condition (4) on $\mathcal{F}$, there is a γ_η such that for all $i, f^0_{\gamma_\eta}(i) \in D'_0$, $\gamma_\eta \in$ range $f^0_{\gamma_\eta}(i)$ and $y_\eta = \{i$: the least β with $f^0_{\gamma_\eta}(i)(\beta) = \gamma_\eta$ is a limit ordinal$\}$. Since a generic ultrafilter for $P(\omega_2)/I$ canonically induces a generic ultrafilter on D'_0, we have a canonical term $\dot{G}_0$ for a generic object for $\mathrm{Col}(\omega_1, \omega_2)$.

Let $x = \|$the least β with $\dot{G}_0(\beta) = \gamma_\eta$ is a limit$\|$ where the Boolean value is taken in the forcing $P(\omega_2)/I$. Then $x \in B_\delta$, for some δ.

Hence $h(x) = [\{i: f^\delta_\gamma(i) \subset_I x\}]$ for all large enough γ.

By the Coding condition, for all i, $f^0_{\gamma_\eta}(i) \in \mathrm{Col}(\omega_1, \omega_2)$ and $\gamma_\eta \in$ range $f^0_{\gamma_\eta}(i)$. Hence $f^0_{\gamma_\eta}(i) \subset_I x$ iff the least β with $f^0_{\gamma_\eta}(i)(\beta) = \gamma_\eta$ is a limit. Otherwise, $f^0_{\gamma_\eta}(i) \subset_I \omega_2 \backslash x$.

Hence, by the coding condition, for $\gamma \geq \gamma_\eta$, and all but countably many i, $f^0_\gamma(i) \subset_I x$ or $f^0_\gamma(i) \subset_I \omega_2 \backslash x$. Hence we see that for all large enough $\gamma \geq \gamma_\eta, f^\delta_\gamma(i) \subset_I x$ iff $f^0_\gamma(i) \subset_I x$, and $h(x) = [\{i: f^0_{\gamma_\eta}(i) \subset_I x\}]$.

But $f^0_{\gamma_\eta}(i) \subset_I x$ iff $i \in y_\eta$, by the coding condition. Hence, $h(x) = [y_\eta]$, and we have shown that h is surjective.

Let K be the kernel of h. To see that K is countably complete, let $\{X_n: n \in \omega\} \subset K$. Then for all δ, γ and all n, $\{i: f^\delta_\gamma(i) \subset_I X_n\} =_{\{\text{countable sets}\}} 0$. Let δ, γ be so large that for all n, and all but countably many i, $f^\delta_\gamma(i) \subset_I X_n$ or $f^\delta_\gamma(i) \cap X_n =_I 0$ and that $h(\bigcup X_n) = \{i: f^\delta_\gamma(i) \subset_I \bigcup X_n\}$. Then $h(\bigcup X_n) = \bigcup\{i: f^\delta_\gamma(i) \subset_I X_n\}$ is countable. Hence, $\bigcup X_n \in K$. ∎

To construct the matrix of functions we use the powerful $\diamondsuit$ techniques forged by Shelah in his papers *Models with Second Order Properties* I–V [Sh].

In fact we will build the matrix of functions for a cofinal set of $\delta \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})$. This suffices since, if we have the f^δ_γ defined for a cofinal set T (with $0 \in T$) and this collection satisfies the properties (1)–(4), we can define them on all $\delta \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})$ to satisfy (1)–(4). To do this: let $\delta \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})$ be arbitrary. Let δ' be the least element of T greater than or equal to δ. Then we can define $f^\delta_\gamma(i) = \pi_\delta(f^{\delta'}_\gamma(i))$ for all i. It is easy to check that this new matrix of functions still satisfies the properties (1)–(4).

From now on we will use phrases such as "almost all", "almost every", "a.e." to mean "for all but a countable set". Similarly, $\underset{a.e.}{\leq}$ will mean $\leq$ on a co-countable set, etc. We will also adopt the convention that if $f\colon \omega_1 \to D$ and $\delta \in \mathrm{cof}(\omega_2)$, then $\pi_\delta(f)$ is the function π_δ composed with f, so $\pi_\delta(f)(i) = \pi_\delta(f(i))$.

$\diamondsuit$ AND DIAGONALIZATION. Our remaining task is to construct the matrix of functions $\mathcal{F}$.

The idea behind constructing the matrix $\mathcal{F}$ is to imitate the construction of a generic object for the reduced product

$$\langle P(\omega_2)/I\rangle^{\omega_1}/\{\text{countable sets}\}$$

over the model $V^{\omega_1}/\{\text{countable sets}\}$. To do this we must build a filter of compatible conditions that meet certain dense sets. The "coherence conditions" are restatements of the requirement that the conditions in the generic object be compatible and that the resulting filter be generated in an organized fashion. The "genericity" condition is a statement of the particular dense sets we are interested in meeting with our filter. (With a little more work, we could meet *all* the dense sets, but this seems irrelevant.) The coding condition is a trick (due to Woodin) to guarantee surjectivity of the homomorphism. It is easily satisfied and introduces no difficulties in the construction.

We build the generic filter by induction on the layering. At a particular level of the layering we build a generic object for the reduced product of the Boolean algebra at that layer by building a descending sequence of conditions of length $\aleph_2$. At horizontal limit stages of the construction we use the countably closed dense set to see that we can diagonalize directed sets of size $\aleph_1$ in the reduced product to get a condition below all of the elements of the filter we have constructed to that stage. At successor stages in the construction the $\diamondsuit$-sequence will present us with dense sets to meet and we choose a condition in the dense set below the filter we have built to that stage.

The main difficulty with this plan arises at ordinals $\alpha \in \omega_3$ of cofinality ω or ω_1 that are limits of ordinals of cofinality ω_2. To describe this difficulty let's let α be an ordinal of countable cofinality and $\{\lambda_n\}$ be an increasing sequence of ordinals of cofinality ω_2 with supremum α. Denote by $\mathcal{F}_{\lambda_n}$ the matrix of functions in $B^{\omega_1}_{\lambda_n}$ constructed at stage λ_n. Then on the complete subalgebra of B generated by $\bigcup B_{\lambda_n}$, we are committed to the filter generated by $\bigcup \mathcal{F}_{\lambda_n}$. But this may not be generic! (The analogous circumstance in a more common situation is that if we are given a coherent collection of generic filters $G_n \subset \mathrm{Col}(\omega_1, \lambda_n)$ there is no guarantee that $\bigcup G_n$ is generic for $\mathrm{Col}(\omega_1, \sup\{\lambda_n\})$.)

To avoid this difficulty we must make an elaborate $\diamondsuit$ construction so that when we are building our collection of functions at stage λ we are anticipating *ALL* possible future limit stages. We define the notion of a "risky" ordinal γ where the $\diamondsuit$ sequence makes a prediction about a potential future stage. At a particular ordinal δ in the filtration we then require that for a closed unbounded set of $\gamma \in \omega_2$, if γ is risky then the function f^δ_γ obeys the dictates of the prediction made by $\diamondsuit$ at that stage. (This means that f^δ_γ is the projection to δ of the dictate of the $\diamondsuit$-sequence.)

Then at the troublesome ordinal α described above, we intersect the countable collection of closed unbounded sets of $\gamma \in \omega_2$ corresponding to the λ_n to get a closed unbounded set of γ where all of the λ_n simultaneously obeyed the $\diamondsuit$-sequence's communiques about α. Hence for each dense set in $B_{\alpha+1}$ there is at least one stage γ where there is a fixed element f of the dense set such that each projection $\pi_{\lambda_n} \circ f = f^{\lambda_n}_\gamma$ is in $\mathcal{F}_{\lambda_n}$. Since for all $\gamma' > \gamma, f^{\lambda_n}_{\gamma'}(i) \leq f^{\lambda_n}_\gamma(i)$ for all but countably many i, and π_{λ_n} is a projection map, we see that for all γ', and all but countably many i, $f(i)$ is compatible with $f^{\lambda_n}_{\gamma'}(i)$. This allows us to set $f^{\alpha+1}_\gamma = f$ compatibly with $\bigcup \mathcal{F}_{\lambda_n}$, meeting the genericity requirement imposed by D.

The rest of this section of the paper is devoted to fleshing out these ideas.

Let λ be a large regular cardinal and I the very strongly layered ideal on ω_2 with witnesses $\langle \pi_\alpha\colon \alpha \in \omega_3\rangle, \langle D_\alpha\colon \alpha \in \omega_3\rangle$ etc. Let $\mathfrak{A} = \langle H(\lambda), \in, \lessdot, I, \langle B_\alpha\colon \alpha < \aleph_3\rangle, D, \langle \pi_\alpha\colon \alpha \in \omega_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})\rangle, \ldots\rangle$, where $\lessdot$ is a wellordering of $H(\lambda)$.

Let $\langle A_\gamma\colon \gamma < \aleph_2\rangle$ be a $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$ sequence. We will view each A_γ as "guessing":

(1) transitive structures $M_\gamma = \langle M, \in, \leq^M, I^M, \langle B^M_\alpha\colon \alpha \in \omega^M_3\rangle, D^M, \langle \pi^M_\alpha\colon \alpha \in \omega^M_3 \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})^M\rangle, \ldots\rangle$, where $M_\gamma \equiv \mathfrak{A}$, and $M^\omega \subset M$,
(2) a set x_γ, with $M_\gamma \models x_\gamma \subset \omega_2$,
(3) a matrix of functions $\langle g^\nu_{\gamma'}\colon \nu \in S, \gamma' < \gamma\rangle$, where $S \subset \omega^M_3 + 1$, $g^\nu_{\gamma'}\colon \omega_1 \to D^M$, and for each ν, $\langle g^\nu_{\gamma'}\colon \gamma' < \gamma\rangle$ is $\subset_{I^M}$ decreasing mod countable sets (i.e. for each ν, $\gamma' < \gamma* < \gamma$, and all but countably many $i, g^\nu_{\gamma*}(i) \subset_{I^M} g^\nu_{\gamma'}(i)$).

The $\diamondsuit$ property we want is that for all transitive $N \equiv \mathfrak{A}$ of cardinality $\aleph_2$, with $N^\omega \subset N$, and $\omega^N_2 = \omega_2$ and all filtrations $\langle N_{\gamma'}\colon \gamma' < \omega_2\rangle$ of N, all subsets x of ω_2 with $x \in N$ and all matrices of functions $\langle h^\nu_{\gamma'}\colon \gamma' < \omega_2, \nu \in R \subset \omega^N_3 + 1\rangle$ into D^N that are decreasing mod countable sets there is a $\gamma < \omega_2$ such that:

If $^-$ is the transitive collapsing map of N_γ then $^-\colon N_\gamma \to M_\gamma$ and is an isomorphism sending x to x_γ. Further

$$S = {}^-``(R \cap N_\gamma) \text{ and } \langle \overline{h}^\nu_{\gamma'}\colon \gamma' < \gamma, \nu \in (R \cap N\gamma)\rangle = \langle g^\nu_{\gamma'}\colon \gamma' < \gamma, \nu \in S\rangle,$$

where we define $\overline{h}^{\nu}_{\gamma}$ to be that function with domain ω_1 such that for all $i, \overline{h}^{\nu}_{\gamma}(i) = \overline{h^{\nu}_{\gamma}(i)}$.

Since the C.H. holds there is a closed unbounded set of $\gamma \in \mathrm{cof}(\omega_1)$ such that $N^{\omega}_{\gamma} \subset N_{\gamma}$. Hence the existence of such a $\diamondsuit$ sequence follows from $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$.

Fix a $\gamma \in \omega_2 \cap \mathrm{cof}(\omega_1)$. If there is a function $g\colon \omega_1 \to D^{M_\gamma}$ such that for all $\nu \in S$ and all $\gamma' < \gamma$ there is a co-countable set of $i \in \omega_1$ such that $g(i) \subset_{I^{M_\gamma}} g^{\nu}_{\gamma'}(i)$, choose such a function and define $\Delta(\gamma)\colon \omega_1 \to D^{M_\gamma}$ to be a function such that for all $i, \Delta(\gamma)(i) \subset g(i)$ and $\Delta(\gamma)(i) \subset_{I'} x_\gamma$ or $\Delta(\gamma)(i) \cap x_\gamma =_{I'} 0$. If such a g exists then this is possible since $x_\gamma \in M_\gamma$. If no such g exists then $\Delta(\gamma)$ will not be defined. (The choice of Δ here is quite free. We exploit this in other applications; see §4.)

We will need to see that $\Delta(\gamma)$ is frequently defined. Towards this end we prove two diagonalization lemmas:

LEMMA 2.4 (First diagonalization lemma): *Let M be a transitive structure with $M \equiv \mathfrak{A}$. Suppose that $M^{\omega} \subset M$ and $\langle f_\alpha : \alpha < \omega_1 \rangle \subset (D^M)^{\omega_1}$ is an a.e. decreasing sequence of functions (not necessarily in M). Then there is a $g\colon \omega_1 \to D^M$ such that for all $\alpha, g \underset{a.e.}{\leq} f_\alpha$.*

Proof: For each α, let $i_\alpha \in \omega_1$ be such that for all $\beta < \alpha$ and all $i > i_\alpha, f_\alpha(i) \leq f_\beta(i)$. Define $g(i) = \bigwedge\{f_\alpha(i): i > \max\{\alpha, i_\alpha\}\}$. Then $g(i)$ is a meet of a countable decreasing sequence of elements in D^M, and hence is in D^M. Further for all α, and $i > \max\{\alpha, i_\alpha\}$ we have $g(i) \leq f_\alpha(i)$. ∎

LEMMA 2.5 (Second diagonalization lemma): *Suppose we have $\gamma \in \omega_2 \cap \mathrm{cof}(\omega_1)$ and an A_γ from the $\diamondsuit$ sequence. (So A_γ "guesses" M_γ etc.) Suppose that $\delta = \sup S \in S$, $B \subset S$ is unbounded in $\sup(S \cap \delta)$ and for all $\nu < \nu' \in S \cap \delta$, $\{\gamma' : \pi^{M_\gamma}_{\nu}(g^{\nu'}_{\gamma'}(i)) \underset{a.e.}{=} g^{\nu}_{\gamma'}(i)\}$ is unbounded in γ, and for $\nu \in B$ and all $\gamma^* < \gamma$ there is a γ' between γ^* and γ such that $g^{\nu}_{\gamma'} \underset{a.e.}{\leq} \pi^{M_\gamma}_{\nu}(g^{\delta}_{\gamma'})$. Then $\Delta(\gamma)$ is defined.*

Remark: In the version of the proof of 2.1 without square, we have a less stringent vertical coherence property, and hence this lemma requires strengthening. In that case the condition that holds here for δ is required to hold for all elements of S, replacing the stronger property of strict vertical coherence for members of $S \cap \delta$.

Proof: We must show that there is a function $g\colon \omega_1 \to D$, such that for all $\nu \in S, \gamma' < \gamma$ and for almost all $i, g(i) \subset_{I^M} g^{\nu}_{\gamma'}(i)$.

CASE 1: $\mathrm{cof}(S \cap \delta) = \omega_1$. In this case we may assume that the order type of B is ω_1.

If $A \subset B$ is countable, $\nu \in B \setminus \sup(A), \gamma' < \gamma$ then for all large enough $\gamma^{\maltese} < \gamma$ for all $\nu' \in A$, there is a γ^* between γ' and $\gamma^{\maltese}$ such that $g^{\nu'}_{\gamma^*} \underset{a.e.}{=} \pi^{M_\gamma}_{\nu'}(g^{\nu}_{\gamma^*})$. Since $|B| = \omega_1$, this fact allows us to choose increasing cofinal sequences $\langle \nu_\alpha : \alpha \in \omega_1 \rangle \subset B \cap \delta$ and $\langle \gamma_\alpha : \alpha \in \omega_1 \rangle \subset \gamma$ such that we have:

(a) For all $\nu \in S \cap \delta, \gamma' < \gamma$ there is an $\alpha \in \omega_1$ and γ^* between γ' and γ_α such that $\pi^{M_\gamma}_{\nu}(g^{\nu_\alpha}_{\gamma^*}) \underset{a.e.}{=} g^{\nu}_{\gamma^*}$

(b) For all $\beta < \alpha$, there is a γ^* between γ_β and γ_α such that $\pi^{M_\gamma}_{\nu_\beta}(g^{\nu_\alpha}_{\gamma^*}) \underset{a.e.}{=} g^{\nu_\beta}_{\gamma^*}$.

For each $\alpha \in \omega_1$, there is a γ^* such that $g^{\nu_\alpha}_{\gamma^*} \underset{a.e.}{\le} \pi^{M_\gamma}_{\nu_\alpha}(g^{\delta}_{\gamma_\alpha})$. Hence for a.a. i, $g^{\nu_\alpha}_{\gamma_\alpha}(i) \wedge g^{\delta}_{\gamma_\alpha}(i) \neq 0$. Let $g_\alpha(i) = g^{\nu_\alpha}_{\gamma_\alpha}(i) \wedge g^{\delta}_{\gamma_\alpha}(i)$. Then each g_α is a function from ω_1 to D. By the first diagonalization lemma, it suffices to prove the following claim:

Claim:

(1) $\langle g_\alpha : \alpha \in \omega_1 \rangle$ is a $\underset{a.e.}{\le}$ decreasing sequence.

(2) If $g: \omega_1 \to D$ is such that for all $\alpha, g \underset{a.e.}{\le} g_\alpha$ then for all $\nu \in S, \gamma' < \gamma$ we have $g \underset{a.e.}{\le} g^{\nu}_{\gamma'}$.

Proof: (1) Let $\beta < \alpha < \omega_1$. Then $g^{\delta}_{\gamma_\alpha} \underset{a.e.}{\le} g^{\delta}_{\gamma_\beta}$. So we must show that $g^{\nu_\alpha}_{\gamma_\alpha} \underset{a.e.}{\le} g^{\nu_\beta}_{\gamma_\beta}$. Choosing γ^* as in clause (b), $\pi^{M_\gamma}_{\nu_\beta}(g^{\nu_\alpha}_{\gamma^*}) \underset{a.e.}{\le} g^{\nu_\beta}_{\gamma_\beta}$, and $g^{\nu_\alpha}_{\gamma_\alpha} \underset{a.e.}{\le} g^{\nu_\alpha}_{\gamma^*} \underset{a.e.}{\le} \pi^{M_\gamma}_{\nu_\beta}(g^{\nu_\alpha}_{\gamma^*})$.

(2) If $\nu = \delta$ this is immediate. Otherwise, let $\gamma' < \gamma$ and $\nu \in S$. There is an $\alpha \in \omega_1$ and a γ^* between γ' and γ_α such that $\pi^{M_\gamma}_{\nu}(g^{\nu_\alpha}_{\gamma^*}) = g^{\nu}_{\gamma^*}$. Then:

$$g_\alpha \underset{a.e.}{\le} g^{\nu_\alpha}_{\gamma_\alpha} \underset{a.e.}{\le} g^{\nu_\alpha}_{\gamma^*} \underset{a.e.}{\le} \pi^{M_\gamma}_{\nu}(g^{\alpha}_{\gamma^*}) \underset{a.e.}{\le} g^{\nu}_{\gamma'}.$$

CASE 2: *$\mathrm{cof}(S \cap \delta)$ is countable.* In this case we can assume that B is an increasing sequence $\langle \nu_n : n \in \omega \rangle$. For each $\gamma' < \gamma$, we define a sequence of ordinals $\langle \gamma_n : n \in \omega \rangle \subset \gamma$ by induction. Let $\gamma_0 \ge \gamma'$ be the smallest ordinal such that $g^{\nu_0}_{\gamma_0} \underset{a.e.}{\le} \pi^{M_\gamma}_{\nu_0}(g^{\delta}_{\gamma'})$. Suppose we have defined $\gamma_0, \ldots, \gamma_n$. Let γ_{n+1} be the least ordinal above γ_n such that $\pi^{M_\gamma}_{\nu_n}(g^{\nu_{n+1}}_{\gamma_{n+1}}) = g^{\nu_n}_{\gamma_{n+1}}$, and for some γ^* between γ_n and $\gamma_{n+1}, g^{\nu_{n+1}}_{\gamma^*} \underset{a.e.}{\le} \pi^{M_\gamma}_{\nu_{n+1}}(g^{\delta}_{\gamma'})$. Let $h_n = g^{\nu_n}_{\gamma_n} \wedge g^{\delta}_{\gamma'}$. Then $\{h_n\}$ is an a.e. decreasing sequence of functions from ω_1 to D. Hence for almost all $i, \langle h_n(i) : n \in \omega \rangle$ is a decreasing sequence of elements of D. Define $h_{\gamma'}(i) = \bigwedge h_n(i)$. Since D is closed under descending ω-sequences, $h_{\gamma'} : \omega_1 \to D$ and $h_{\gamma'} \underset{a.e.}{\le} h_n$ for all n.

Clearly $\langle h_{\gamma'} : \gamma' < \gamma \rangle$ is an a.e.-descending sequence and for all $\nu \in S, \gamma' < \gamma$, there is a γ^* between γ' and γ with $h_{\gamma^*} \underset{a.e.}{\le} g^{\nu}_{\gamma'}$. By the first diagonalization lemma, there is a function $g: \omega_1 \to D$ such that for all $\gamma' < \gamma, g \underset{a.e.}{\le} h_{\gamma'}$. ∎

THE FILTRATIONS. Let $T \subset (\{\delta < \omega_3 : \mathrm{sk}^{\mathfrak{A}}(\delta) \cap \omega_3 = \delta\} \cup \{0\})$ be a closed unbounded set in ω_3 with $0 \in T$ and successor points of T having cofinality ω_2. By the remarks after Claim 2.3, it suffices to define the functions f^δ_γ for $\delta \in T$ having cofinality ω_2. For successor $\delta \in T$ we define δ^- be the immediate predecessor of δ in T. For each $\delta \in T\backslash\{0\}$, let $\mathfrak{B}^\delta = \mathrm{sk}^{\mathfrak{A}}(\delta)$. For $\delta = 0$, let d_0 be the least $\delta \in \mathrm{cof}(\omega_2) \cap \omega_3$ such that $\mathrm{sk}^{\mathfrak{A}}(\delta) \cap \omega_3 = \delta$. Let $\mathfrak{B}^0 = \mathrm{sk}^{\mathfrak{A}}(d_0)$. We build distinguished filtrations of each $\mathfrak{B}^\delta$.

Remark: In the version of the proof of 2.1 without square, arbitrary filtrations are taken. This causes some additional work in the construction of the matrix at ordinals $\delta \in T$ that are successors of limits of T of cofinality ω_2.

Let $\langle C_\alpha : \alpha \in \omega_3\rangle$ be a $\square_{\omega_2}$ sequence. For $\alpha \in T$, we replace C_α by $C_\alpha \cap T$ to get a new "$\square$"-sequence $\langle C_\alpha : \alpha \in T, \alpha = \sup(\alpha \cap T)\rangle$ such that:

(1) $C_\alpha \subset T$, o.t. $C_\alpha \leq \omega_2$, and if $T \cap \alpha$ has uncountable cofinality, C_α is closed and unbounded in α.
(2) If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

By choosing cofinal ω-sequences of elements of T through those $\alpha \in T$, that are limits of T where $C_\alpha \cap T$ is bounded we can get a genuine $\square$ sequence on T. Also, we can replace successor elements of each new C_α by their successors in T, to get a another new $\square$ sequence where we may assume that every successor point of each C_α is a member of T of cofinality ω_2. From now on we assume that our $\square$-sequence has these properties.

By induction on $\delta \in (T \cap \mathrm{cof}(\omega_2)) \cup \{0\}$, we define filtrations $\langle \mathfrak{B}^\delta_\gamma : \gamma \in \omega_2\rangle$.

For $\delta = 0$ or a successor point of T with δ^- having cofinality ω_2, define $\langle \mathfrak{B}^\delta_\gamma : \gamma \in \omega_2\rangle$ to be an arbitrary filtration of consisting of elementary substructures of $\mathfrak{B}^\delta$ and having $\omega_1 \subset \mathfrak{B}^\delta_0$.

If δ is a successor point of T, but δ^- has cofinality less then ω_2, we let $\gamma =$ o.t. C_{δ^-}. For $\gamma' < \gamma$ we let $\mathfrak{B}^\delta_{\gamma'} = \emptyset$. Let $\mathfrak{B}^\delta_\gamma = \bigcup\{\mathfrak{B}^\nu_{\gamma'} : \nu \in C_{\delta^-}, \gamma' < \gamma\}$. We let the rest of the iteration be an arbitrary filtration of elementary substructures of $\mathfrak{B}^\delta$.

If δ is a limit point of T and $\delta \in \mathrm{cof}(\omega_2)$, we let η be the γ^{th} element of C_δ and set $\mathfrak{B}^\delta_\gamma = \bigcup\{\mathfrak{B}^\nu_{\gamma'} : \nu \in C_\eta, \gamma' < \gamma\}$. This is a filtration of $\mathfrak{B}^\delta$.

The cogent property of this labored definition is that if δ' is a limit point of T of cofinality ω_2, and $\delta \in T$ is such that δ^- is the γ^{th} element of $C_{\delta'}$, then $\mathfrak{B}^{\delta'}_\gamma = \mathfrak{B}^\delta_\gamma$.

LEMMA 2.6: *Let $\delta \in T \cap \mathrm{cof}(\omega_2)$ be a successor point of T. If $\delta^- \in \mathrm{cof}(\omega_2) \cup \{0\}$, let $U = \{\delta^-\}$, otherwise let U be the successor points of C_{δ^-}.*

Then there is a closed unbounded set $C \subset \omega_2$ such that for all $\gamma \in C$ and all $\nu \in U, \mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^\nu = \mathfrak{B}^\nu_\gamma$. Further, for $\gamma \in C, \mathfrak{B}^\nu_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap (\nu \cup d_0))$.

LEMMA 2.7: *Let $\delta \in \mathrm{cof}(\omega_2) \cap T$ be a limit point of T, and $\langle \nu_j : j \in \omega_2 \rangle$ be an increasing enumeration of C_δ. Then there is a closed unbounded set of $\gamma \in \omega_2$ such that for all successor $j < \gamma$:*

(1) $\langle \nu_{j'} : j' < j \rangle \in \mathfrak{B}^\delta_\gamma$,
(2) $\mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\nu_j} = \mathfrak{B}^{\nu_j}_\gamma$,
(3) $\mathfrak{B}^{\nu_j}_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap \nu_j)$ *and*
(4) $\mathfrak{B}^\delta_\gamma = \mathfrak{B}^\nu_\gamma$ *where ν is the least element of T above ν_γ.*

Proof: At limit ordinals we have that $\mathfrak{B}^\delta_\gamma = \bigcup\{\mathfrak{B}^\mu_i : \mu \in C_{\nu_\gamma}, i < \gamma\} = \mathfrak{B}^\nu_\gamma$. For each successor $j, \{\mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\nu_j} : \gamma \in \omega_2\}$ is a filtration of $\mathfrak{B}^{\nu_j}$, hence there is a closed unbounded set $E_j \subset \omega_2$ such that for all $\gamma \in E_j, \mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\nu_j} = \mathfrak{B}^{\nu_j}_\gamma$ and $\langle \nu_{j'} : j' < j \rangle \in \mathfrak{B}^{\nu_j}_\gamma$. The conclusion of the lemma holds for all $\gamma \in \Delta E_j$. ∎

Let $\overline{\mathfrak{B}^\delta_\gamma}$ be the transitive collapse of $\mathfrak{B}^\delta_\gamma$. For $w \in \mathfrak{B}^\delta_\gamma$, let $\overline{w}$ be the image of w under the canonical transitive collapse map. We note that there is a difference between $\bar{\delta}$ and δ^-, and we hope that context will help reduce notational confusion.

THE DEFINITION OF OBEDIENT. *Definition:* Let $\delta \in T$ have cofinality ω_2. A sequence of functions $\langle f^\nu_\gamma : \gamma \in \omega_2, \nu \in T \cap (\delta + 1) \cap (\mathrm{cof}(\omega_2) \cup 0) \rangle$ from ω_1 to D_δ is **obedient** provided that for a closed unbounded set of $\gamma \in \omega_2 \cap \mathrm{cof}(\omega_1)$ if:

(1) $f^\nu_{\gamma'} : \omega_1 \to D_\delta \cap \mathfrak{B}^\delta_\gamma$ for each $\nu \in T \cap (\delta + 1)$ and $\gamma' < \gamma$,
(2) A_γ guesses M_γ and the sequence $\langle g^\nu_{\gamma'} : \gamma' < \gamma, \nu \in S \rangle$ and $\Delta(\gamma)$ is defined,
(3) if we denote $\omega_3^{\overline{\mathfrak{B}^\delta_\gamma}}$ by $\bar{\delta}$ then $\bar{\delta} \le \omega_3^M, \mathrm{sk}^{M_\gamma}(\bar{\delta}) \cap \omega_3^{M_\gamma} = \bar{\delta}$ and $M_\gamma \models \mathrm{cof}(\bar{\delta}) > \omega_1$ (since $\delta \notin \mathfrak{B}^\delta_\gamma$ this abuse of notation causes no inconsistency except perhaps at 0),
(4) $\overline{\mathfrak{B}}^\delta_\gamma \cong \mathrm{sk}^{M_\gamma}(\bar{\delta})$,
(5) for each $\gamma' < \gamma, \nu \in (T \cap \mathfrak{B}^\delta_\gamma) \cup \{\delta\}$, we have $\overline{f^\nu_{\gamma'}(i)} = g^{\overline{\nu}}_{\gamma'}(i)$ for all but countably many i,

then for all $i, \nu \in (T \cap \mathfrak{B}^\delta_\gamma) \cup \{\delta\}$, we have $\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\overline{\nu}}(\Delta(\gamma)(i))$. (If $\overline{\nu} = \overline{\delta} = \omega_3^{M_\gamma}$ then we take $\pi^{M_\gamma}_{\overline{\nu}}$ to be the identity.)

Note that properties (1)–(5) in the definition of obedience only involve the functions $f^\nu_{\gamma'}$ for $\nu \in (T \cap \mathfrak{B}^\delta_\gamma) \cup \{\delta\}$ and $\gamma' < \gamma$. Thus given a sequence $\langle f^\nu_{\gamma'} : \gamma' < \gamma, \nu \in (T \cap \mathfrak{B}^\delta_\gamma) \cup \{\delta\} \rangle$ we define an ordinal γ to be **risky** if it satisfies conditions (1)–(5).

To make this definition somewhat less obscure it is perhaps worth observing that if $\delta' \in T \cap \mathfrak{B}^\delta_\gamma$, $\mathfrak{B}^{\delta'}_\gamma = \mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\delta'}$ and $\mathfrak{B}^{\delta'}_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap \delta')$ (i.e. the conclusion of Lemma 2.6 for $\nu = \delta'$ and δ) and M_γ guesses $\overline{\mathfrak{B}^\delta_\gamma}$ then clauses (3) and (4) of the definition of risky are satisfied. (A very similar fact is verified in some detail in the proof of Lemma 2.8.) This is the mechanism that allows the diamond sequence to guess requirements from $\mathfrak{B}^\delta$ when constructing the sequence at δ'.

Remark: It is critical to the method we will use to observe that $D^{M_\gamma}_{\overline{\delta}} \subset \mathrm{sk}^{M_\gamma}(\overline{\delta})$ and hence $D^{M_\gamma}_{\overline{\delta}} \subset \overline{\mathfrak{B}}^\delta_\gamma$ in a natural way. Thus, if γ is risky, then for all $i, \pi_{\overline{\delta}}(\Delta(\gamma)(i))$ is in the transitive collapse $\overline{\mathfrak{B}}^\delta_\gamma$. This will allow us to define a function f taking values in $\mathfrak{B}^\delta_\gamma$ such that for all $i, \overline{f(i)} = \pi_{\overline{\delta}}(\Delta(\gamma)(i))$.

Definition: Suppose $U \subset T \cap (\mathrm{cof}(\omega_2) \cup \{0\})$. A sequence of functions $\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in U\rangle$ is **obedient** provided that for all $\delta \in U$ the sequence $\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in T \cap (\delta+1) \cap (\mathrm{cof}(\omega_2) \cup 0)\rangle$ is obedient. Suppose $|U| < \omega_2$. Then there is a closed unbounded set $C \subset \omega_2$ such that for all $\gamma \in C, \delta \in U$ if γ is risky for δ then for all $i, \nu \in (T \cap \mathfrak{B}^\delta_\gamma) \cup \{\delta\}$, we have $\overline{f^\nu_\gamma(i)} = \pi_{\overline{\nu}}(\Delta(\gamma)(i))$. (Intersect $|U|$ many club sets.) We will say that C is a witness for the obedience of the $\langle f^\delta_\gamma\rangle$ for $\delta \in U$.

Obedience Lemmas. Lemma 2.8 (Risky ordinals lemma): *Let $\delta' < \delta$ be elements of T having cofinality ω_2. Let $\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in T \cap (\delta+1)\rangle$ be a sequence of functions from ω_1 to D_δ such that if $\nu \le \delta'$ then f^ν_γ maps into $D_{\delta'}$. Suppose $\mathfrak{B}^{\delta'}_\gamma = \mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\delta'}$, $\delta' \in \mathfrak{B}^\delta_\gamma$ and $\mathfrak{B}^{\delta'}_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap \delta')$. Then if γ is risky for δ then γ is risky for δ'. Similarly for $\delta' = 0$ with the hypothesis that $\mathfrak{B}^{\delta'}_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap d_0)$.*

Proof: The first clause in the definition of "risky" is clear, since if $b \in D_{\delta'} \cap \mathfrak{B}^\delta_\gamma$, then $b \in D_{\delta'} \cap \mathfrak{B}^{\delta'}_\gamma$. The second clause doesn't mention either δ or δ'.

We verify that clauses (3) and (4) hold for δ' provided that they hold for δ.

Let $\overline{\delta'}$ be the image of ω_3 in the sense of $\mathfrak{B}^{\delta'}_\gamma$ under the transitive collapse of $\mathfrak{B}^{\delta'}_\gamma$. Since $\mathfrak{B}^{\delta'}_\gamma = \mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^{\delta'}$, we see that $\overline{\delta'}$ is also the image of δ' under the collapse map of $\mathfrak{B}^\delta_\gamma$ (hence there is no notational ambiguity). Clearly $\overline{\delta'} < \overline{\delta}$, and thus $\overline{\delta'} < \omega_3^{M_\gamma}$.

Since $\mathfrak{B}^{\delta'}_\gamma = \mathrm{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap \delta')$, when we take transitive collapses we find that $\mathfrak{B}^{\delta'}_\gamma$ is isomorphic to the skolem hull of $\overline{\delta'}$ in $\overline{\mathfrak{B}}^\delta_\gamma$. Since $\overline{\mathfrak{B}}^\delta_\gamma \cong \mathrm{sk}^{M_\gamma}(\bar{\delta}), \overline{\delta'} < \bar{\delta}$ we see that $\mathfrak{B}^{\delta'}_\gamma$ is isomorphic to the skolem hull of $\overline{\delta'}$ in M_γ, as required in clause (4).

Since $\mathfrak{B}^{\delta'}_\gamma \cong \mathrm{sk}^{M_\gamma}(\overline{\delta'})$, and $\overline{\delta'} = \omega_3^{\overline{\mathfrak{B}}^{\delta'}_\gamma}$ we see that $\mathrm{sk}^{M_\gamma}(\overline{\delta'}) \cap \omega_3^{M_\gamma} = \overline{\delta'}$. Finally, since $\delta' \in \mathfrak{B}^\delta_\gamma$ which is an elementary substructure of $\mathfrak{A}$, we have that $\mathfrak{B}^\delta_\gamma \models \mathrm{cof}(\delta') = \omega_2$. Since $\mathfrak{B}^\delta_\gamma \cong \mathrm{sk}^{M_\gamma}(\overline{\delta})$, we see that $M_\gamma \models \mathrm{cof}(\overline{\delta'}) = \omega_2$. We have thus verified clause (3).

To see clause (5), we first remark that for $\beta \in \delta' \cap \mathfrak{B}^\delta_\gamma$ we have $D_\beta \cap \mathfrak{B}^\delta_\gamma = D_\beta \cap \mathfrak{B}^{\delta'}_\gamma$. Further, elements of $D_\beta \cap \mathfrak{B}^\delta_\gamma$ are carried the same place by the transitive collapses of both $\mathfrak{B}^\delta_\gamma$ and $\mathfrak{B}^{\delta'}_\gamma$.

For $\nu \in (T \cap \mathfrak{B}^{\delta'}_\gamma) \cup \{\delta'\}$ and $\gamma' < \gamma$ we know that $f^\nu_{\gamma'}(i)$ is in D_β for some $\beta < \delta', \beta \in \mathfrak{B}^{\delta'}_\gamma$. Hence $f^\nu_{\gamma'}(i)$ is taken to the same set by both transitive collapses. Thus if (5) holds for δ, (5) holds for δ'.

The proof with $\delta' = 0$ is similar with $\overline{d_0}$ playing the role of $\overline{\delta'}$ in clauses 3 and 4. ∎

LEMMA 2.9 (Existence of Risky Ordinals): *Let $\delta \in T$ and $x \in B_\delta$. Suppose that $\langle f^\nu_\gamma \colon \nu \in T \cap (\delta + 1), \gamma \in \omega_2\rangle$ is a sequence of functions that satisfy horizontal coherence, satisfy vertical coherence for $\nu \in T \cap \delta$ and there is a cofinal set $B \subset T \cap \operatorname{cof}(\omega_2) \cap \delta$ such that for all $\nu \in B$ and each $\gamma^* \in \omega_2$ there is a $\gamma' \in \omega_2$ such that for all but countably many $i, f^\nu_{\gamma'}(i) \leq \pi_\nu(f^\delta_{\gamma^*}(i))$. Then there is a stationary set of $\gamma \in \omega_2$ that are risky for the matrix $\langle f^\nu_\gamma \colon \nu \in T \cap (\delta+1), \gamma \in \omega_2\rangle$ and where $x_\gamma = \overline{x}$.*

Proof: To show that there are stationarily many risky γ we will show that there is a closed unbounded set of γ such that whenever the $\Diamond$ sequence guesses $\overline{\mathfrak{B}}^\delta_\gamma$ and the matrix of functions up to that point, then γ is risky. This suffices because the $\Diamond$ property guarantees that there are stationarily many γ where this occurs and where $x_\gamma = \overline{x}$. If the $\Diamond$ sequence guesses $\overline{\mathfrak{B}}^\delta_\gamma$ and the matrix of functions up to γ, the only reason that that γ might not be risky is that $\Delta(\gamma)$ might not be defined.

Consider the expansion of the structure $\mathfrak{B}^\delta$, $\mathfrak{C} = \langle \mathfrak{B}^\delta, \langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu < \delta\rangle, \langle g_\gamma \colon \gamma < \omega_2\rangle\rangle$ where $g_\gamma = f^\delta_\gamma$. Let γ be arbitrary such that $\langle \mathfrak{B}^\delta_\gamma, \langle f^\nu_{\gamma'} \colon \nu \in \delta \cap \mathfrak{B}^\delta_\gamma, \gamma' < \gamma\rangle, \langle g_{\gamma'} \colon \gamma' < \gamma\rangle\rangle$ is an elementary substructure of $\mathfrak{C}$. Then for all $\nu < \nu' \in T \cap \mathfrak{B}^\delta_\gamma$ the collection of γ' witnessing vertical coherence is cofinal in γ and for all $\gamma^* < \gamma$ there is a γ' between γ^* and γ such that for almost all $i, f^\nu_{\gamma'}(i) \leq \pi_\nu(f^\delta_{\gamma^*}(i))$.

If A_γ guesses $\overline{\mathfrak{B}}^\delta_\gamma$ (i.e. $M_\gamma = \overline{\mathfrak{B}}^\delta_\gamma$) and the matrix of the $\overline{f^\nu_{\gamma'}}$ for $\nu \in T \cap \mathfrak{B}^\delta_\gamma \cup \{\delta\}$ then A_γ satisfies the hypothesis of the Second diagonalization lemma (Lemma 2.5). Hence $\Delta(\gamma)$ is defined and γ is risky. ∎

LEMMA 2.10: *Suppose that $\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in T \cap (\delta + 1) \cap (\operatorname{cof}(\omega_2) \cup \{0\})\rangle$ is an obedient matrix of functions from ω_1 to D_δ satisfying vertical and horizontal coherence. Then for all $\nu \in T \cap (\delta+1)$ the matrix satisfies the genericity condition for $x \in B_\nu$.*

Proof: Let $x \in B_\nu$, for $\nu \in T \cap (\delta+1)$. Then by the proof of the previous lemma there is a stationary set of risky $\gamma \in \mathrm{cof}(\omega_1)$ such that A_γ guesses $\mathfrak{B}^\nu_\gamma$, the matrix $\langle f^\nu_{\gamma'}\colon \gamma' \in \gamma, \nu \in T \cap (\delta+1) \cap \mathfrak{B}^\nu_\gamma\rangle$ and where $\overline{x} = x_\gamma$. For such a γ, we have that $\Delta(\gamma)(i) \subset_{I'} x_\gamma$ or $\Delta(\gamma)(i) \cap x_\gamma =_{I'} 0$ for all i. Since $\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\overline{\nu}}(\Delta(\gamma)(i))$ and $\overline{x} = x_\gamma$ we see that for all i, either $f^\nu_\gamma(i) \subset_I x$ or $f^\nu_\gamma(i) \cap x =_I 0$. Hence the genericity condition holds. ∎

THE ACTUAL CONSTRUCTION. From what we know now, it suffices to build a coherent, obedient matrix of functions satisfying the coding condition, defined for $\delta \in T$ that have cofinality ω_2. This almost completely determines our construction, namely at almost every risky ordinal γ we must take $f^\delta_\gamma(i)$ so that $\overline{f^\delta_\gamma(i)}$ is the projection $\pi^{M_\gamma}_{\overline{\delta}}(\Delta(\gamma)(i))$ for all but countably many i. We now verify that this works.

We define the functions $\langle f^\delta_\gamma\colon \gamma \in \omega_2\rangle$ by induction on $\delta \in T \cap (\mathrm{cof}(\omega_2) \cup \{0\})$ and for each such δ by induction on $\gamma \in \omega_2$.

INDUCTION HYPOTHESIS. We will maintain the induction hypothesis for $\delta \in T$, the matrix $\langle f^\nu_\gamma\colon \gamma \in \omega_2, \nu \in T \cap \delta\rangle$ is obedient, satisfies vertical and horizontal coherence, and the coding condition holds.

By Lemma 2.10, this suffices to prove that $\mathcal{F}$ has the desired properties.

We will treat successor γ and $\gamma \in \mathrm{cof}(\omega)$ uniformly for all δ. Suppose the induction hypothesis and that we have horizontal coherence for $\gamma' < \gamma$. We define $f^\delta_0 \equiv 1$ and $f^\delta_{\gamma+1} = f^\delta_\gamma$. For all γ of countable cofinality we choose a sequence $\{\gamma_n\colon n \in \omega\}$ cofinal in γ and for each i define $f^\delta_\gamma(i) = \bigwedge f^\delta_{\gamma_n}(i)$. Since the $f^\delta_{\gamma_n}$ are decreasing mod countable and take values in the ω-closed set D_δ, f^δ_γ takes values in D_δ for almost all i, and is clearly below each $f^\delta_{\gamma'}$ on a co-countable set.

CASE 1: $\delta = 0$. In this case, in addition to obedience, we must ensure that the coding condition holds. To do this we will use the stationary set of places where the $\diamondsuit$-sequence guesses "nonsense" to do our coding. Notice that whenever we have a $\gamma \in \mathrm{cof}(\omega_1)$ which is not risky, we are free to define f^0_γ arbitrarily subject to the requirement of horizontal coherence.

For each $\eta \in \omega_2$ we will say that y_η is coded by γ_η, if the coding condition (condition (4) on $\mathcal{F}$) is satisfied.

Suppose we have defined $\langle f^0_{\gamma'}\colon \gamma' < \gamma\rangle$. We now define f^0_γ for $\gamma \in \mathrm{cof}(\omega_1)$.

CASE 1a: *γ is risky.* Define $f^0_\gamma(i)$ so that $\overline{f^0_\gamma(i)} = \pi^{M_\gamma}_0(\Delta(\gamma)(i))$.

CASE 1b: *Otherwise.* Recall that D'_0 was the dense subset of D_0 isomorphic to $\mathrm{Col}(\omega_1, \omega_2)$. Let η be least such that y_η has not been coded.

Enumerate γ as $\langle \gamma_j \colon j \in \omega_1 \rangle$. We diagonalize against this enumeration to find the "maximal" ω_1-sequence of elements of D'_0 determined by the sequence $\langle f^0_{\gamma'} \colon \gamma' < \gamma \rangle$. We choose an increasing sequence $\langle \alpha_i \colon i \in \omega_1 \rangle \subset \omega_1$ such that for all $j < i$ and all $i^* > \alpha_i$, $f^0_{\gamma_i}(i^*)$ is comparible with $f^0_{\gamma_j}(i^*)$, and $\gamma_i \leq \gamma_j$ implies $f^0_{\gamma_j}(i^*) \leq f^0_{\gamma_i}(i^*)$.

Define $h(i) = \bigcup\{f^0_{\gamma_j}(i) \colon j < i, \alpha_j < i \text{ and } f^0_{\gamma_j}(i) \in D'_0\}$. Since D'_0 is closed under descending ω-sequences, for all $i, h(i) \in D'_0$. It may or may not happen that for all $\gamma' < \gamma$ there is a co- countable set of i with $f^0_{\gamma'}(i) \geq h(i)$. If this fails, or if for a cofinal set of $i \in \omega_1, \gamma \in \text{range } h(i)$, let f^0_γ be arbitrary satisfying horizontal coherence and taking all of its values in D'_0

If it happens that h is below each $f^0_{\gamma'}$ on a co-countable set *and* that $\gamma \notin \text{range}\, h(i)$ on a co-countable set then by modifying h on a countable set, we can arrange that h still lies below each $f^0_{\gamma'}$ and that for no i is $\gamma \in \text{range } h(i)$. Then define $f^0_\gamma(i) \in D'_0$ below $h(i)$ so that $\gamma \in \text{range } f^0_\gamma(i)$ and for all $i, i \in y_\eta$ iff the least β such that $f^0_\gamma(i)(\beta) = \gamma$ is a limit.

CLAIM: *This sequence is obedient and satisfies conditions (1)–(4).*

Proof: We first verify horizontal coherence.

If γ is a risky ordinal, then $\overline{f^0_\gamma(i)} = \pi_0^{M_\gamma}(\Delta(\gamma)(i))$. Since the function g used to define $\Delta(\gamma)$ is below each $g^0_{\gamma'}$ and $\overline{f^0_{\gamma'}(i)} = g^0_{\gamma'}$ for almost all i, we have that $\pi_0^{M_\gamma}(\Delta(\gamma)(i)) \leq \overline{f^0_{\gamma'}(i)}$ for all $\gamma' < \gamma$ and almost all i. Hence $f^0_\gamma(i) \leq f^0_{\gamma'}(i)$ for all but countably many i.

At non-risky γ we explicitly chose f_γ below all of the previous $f_{\gamma'}$. Hence horizontal coherence holds.

Vertical coherence is irrelevant in this case.

We need to see that the coding condition holds. Suppose that $y = y_\eta \in P(\omega_1)$ is least such that the coding condition fails. Since $\langle A_\gamma \colon \gamma \in \omega_2 \rangle$ is a $\diamondsuit$ sequence there is a stationary set of cofinality ω_1 ordinals which fall under case 1b (e.g. where $A_\gamma = \emptyset$).

At any ordinal γ that falls under case 1b, f^0_γ takes all of its values in D'_0. If γ is a case 1b ordinal that is a limit of case 1b ordinals, then for all $\gamma' < \gamma$, $h \underset{a.e.}{\leq} f^0_{\gamma'}$. Further there is a closed unbounded set of ordinals γ such that if γ is a case 1b ordinal, then $\gamma \notin \bigcup\{\text{range}\, h(i) \colon i \in \omega_1\}$. If γ is such an ordinal and for all $\gamma' < \gamma, h \underset{a.e.}{\leq} f^0_{\gamma'}$. and each $y_{\eta'}$ with $\eta' < \eta$ has been coded by an ordinal below γ, then γ codes y.

Finally the sequence is obedient, since at every risky ordinal we defined $f^0_\gamma(i)$ so that $\overline{f^0_\gamma(i)} = \pi_0^{M_\gamma}(\Delta(\gamma)(i))$. ∎

CASE 2: *δ is a successor point of T.* Let δ^- be the immediate predecessor of δ in T. If $\delta^- \in \text{cof}(\omega_2)$, let $U = \{\delta^-\}$. Otherwise, let U be the successor points of C_{δ^-}.

Our sequence $\langle f^\delta_\gamma : \gamma < \omega_2 \rangle$ will have the property that at every stage $\gamma \in \omega_2$, and all $\nu \in U$ and each $\gamma^* \in \gamma$ there is a γ', between γ^* and γ such that for all but countably many $i, f^\nu_{\gamma'}(i) \leq \pi_\nu(f^\delta_{\gamma^*}(i))$. We will call this the "compatibility condition" for U. (Note the similarity to the hypothesis of Lemmas 2.5 and 2.9 with $B = U$.) It is easy to check that the compatibility condition is preserved at limit stages of countable cofinality.

By Lemma 2.6 there is a closed unbounded set C of $\gamma \in \omega_2$ where $\mathfrak{B}^\delta_\gamma \cap \mathfrak{B}^\nu = \mathfrak{B}^\nu_\gamma = \text{sk}^{\mathfrak{B}^\delta_\gamma}(\mathfrak{B}^\delta_\gamma \cap \nu)$ for all $\nu \in U$. Since $|U| < \omega_2$ we can assume that C is a witness to the obedience of $\langle f^\nu_\gamma \rangle$ for $\nu \in U$. By passing to a tail segment of C we may assume that for all $\gamma \in C$, and all $\nu \in U$, we have that $U \cap \nu \in \mathfrak{B}^\nu_\gamma$. We define a sequence $\langle f^\delta_\gamma : \gamma \in \omega_2 \rangle$ by induction on γ. This sequence will have the property that if $\gamma \in C, \nu \in U$ and γ is risky for δ, then for all i, $f^\nu_\gamma(i) = \pi_\nu(f^\delta_\gamma(i))$.

We now define f^δ_γ for $\gamma \in \text{cof}(\omega_1)$. To start with for all $\gamma < o.t.C_{\delta^-}$ we let $f^\delta_\gamma \equiv 1$.

CASE 2a: $\text{cof}(\delta^-) < \omega_2$ *and* $\gamma \leq \text{o.t.}C_{\delta^-}$.

If $\gamma <$o.t.C_{δ^-} let $f^\delta_\gamma \equiv 1$.

For $\gamma = \text{o.t.}C_{\delta^-}$ we know $\mathfrak{B}^\delta_\gamma = \bigcup\{\mathfrak{B}^{\nu'}_j : \nu' \in C_{\delta^-}, j < \gamma\}$. If γ is risky for all successor $\nu' \in C_{\delta^-}$, and $\text{cof}(\bar{\delta}) > \omega_1$ in M_γ, then (using the remarks following the definition of obedient) we can define $f^\delta_\gamma : \omega_1 \to D_\delta$ so that for all $i, \overline{f^\delta_\gamma(i)} = \pi^{M_\gamma}_{\bar{\delta}}(\Delta(\gamma)(i))$.

Then for successor elements $\nu \in C_{\delta^-}$, we have that

$$\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\bar{\nu}}(\Delta(\gamma)(i)) = \pi^{M_\gamma}_{\bar{\nu}}(\pi^{M_\gamma}_{\bar{\delta}}(\Delta(\gamma)(i))) = \pi^{M_\gamma}_{\bar{\nu}}(\overline{f^\delta_\gamma(i)}) = \overline{\pi_\nu(f^\delta_\gamma(i))}$$

for all i. Hence, $f^\nu_\gamma(i) = \pi_\nu(f^\delta_\gamma(i))$ for all i and thus the compatibility condition holds at $\gamma + 1$.

Since $f^\delta_{\gamma'} \equiv 1$ for $\gamma' < \gamma$, we trivially have that $f^\delta_\gamma \underset{a.e.}{\leq} f^\delta_{\gamma'}$ and horizontal coherence holds.

If $\gamma = \text{o.t.}C_{\delta^-}$ and the conditions above aren't met, we let $f^\delta_\gamma \equiv 1$.

CASE 2b: *$\gamma \in C$ and γ is risky.* Then $\overline{\mathfrak{B}^\delta_\gamma} \cong \text{sk}^{M_\gamma}(\bar{\delta})$, and for all $\gamma' < \gamma$, $\pi^{M_\gamma}_{\bar{\delta}}(\Delta(\gamma)) \underset{a.e.}{\leq} \bar{f}^\delta_{\gamma'}$. By Lemma 2.8, γ is risky for each $\nu \in U$. Hence for all $\nu \in U, i \in \omega_1$ we have that $\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\bar{\nu}}(\Delta(\gamma)(i))$. Define $f^\delta_\gamma : \omega_1 \to D_\gamma$ so as to ensure that for all $i, \overline{f^\delta_\gamma(i)} = \pi^{M_\gamma}_{\bar{\delta}}(\Delta(\gamma)(i))$. Then, doing a computation similar to the one in case 2a, for all $i, \nu \in U$, we have $\pi_\nu(f^\delta_\gamma(i)) = f^\nu_\gamma(i)$. Further, for all $\gamma' < \gamma$ and a.e. i, $f^\delta_\gamma(i) \leq f^\delta_{\gamma'}(i)$. Again, the compatibility condition is easy to verify.

CASE 2c: *Not case 2a or 2b, but the risky ordinals in C are cofinal in γ.* We define f^δ_γ so that for all $\nu \in U, f^\nu_\gamma \underset{a.e.}{\leq} \pi_\nu(f^\delta_\gamma)$. This implies the compatibility condition for δ and $\gamma + 1$.

Let $\langle \gamma_\alpha : \alpha \in \omega_1 \rangle$ be a sequence of risky ordinals in C cofinal in γ. Then for all $\alpha, i \in \omega_1, \nu \in U$, $\pi_\nu(f^\delta_{\gamma_\alpha}(i)) = f^\nu_{\gamma_\alpha}(i)$.

Enumerate $U = \langle \nu_\alpha : \alpha < \omega_1 \rangle$ (possibly with repetitions). Define an increasing sequence $\langle i_\alpha : \alpha \in \omega_1 \rangle$ by taking i_α to be so large that:

(1) for all $j > i_\alpha$ and $\beta < \alpha$, we have $f^\delta_{\gamma_\beta}(j) \geq f^\delta_{\gamma_\alpha}(j)$,

(2) for all $\beta, \beta' \leq \alpha$ and all $j > i_\alpha$ the inequality $f^{\nu_{\beta'}}_{\gamma_\beta}(j) \geq f^{\nu_{\beta'}}_\gamma(j)$ holds.

Define:

$$f^\delta_\gamma(j) = \bigwedge\{f^\delta_{\gamma_\alpha}(j) : j > i_\alpha\}.$$

Note that $\langle f^\delta_{\gamma_\alpha}(j) : j > i_\alpha \rangle$ is a countable decreasing sequence of elements of D_δ. Clearly for all $\alpha, f^\delta_\gamma \underset{a.e.}{\leq} f^\delta_{\gamma_\alpha}$, so horizontal coherence holds.

To see the compatibility condition for γ, note that for $\nu \in U$, if $\nu = \nu_\beta$ and $j > i_\beta$:

$$\begin{aligned}
\pi_\nu(f^\delta_\gamma(j)) &= \pi_\nu(\bigwedge\{f^\delta_{\gamma_\alpha}(j) : j > i_\alpha\}) \\
&= \bigwedge\{\pi_\nu(f^\delta_{\gamma_\alpha}(j)) : j > i_\alpha\} \\
&= \bigwedge\{f^\nu_{\gamma_\alpha}(j) : j > i_\alpha\} \geq f^\nu_\gamma(j).
\end{aligned}$$

This establishes the compatibility condition in case 2c. Note that we chose $f^\delta_\gamma \underset{a.e.}{\leq} f^\delta_{\gamma'}$, so the horizontal coherence hypothesis is also established.

CASE 2d: *Otherwise.* If $\langle f^\delta_{\gamma'} : \gamma' < \gamma \rangle$ is eventually constant, take f^δ_γ be this constant value. Otherwise, let f^δ_γ be arbitrary below each $f^\delta_{\gamma'}$ a.e.

In this case, let $\gamma*$ be an upper bound on the risky ordinals in $C \cap \gamma$. Then all of the ordinals of cofinality ω_1 between $\gamma*$ and γ fall into case 2d. Hence an easy induction shows that the sequence $f^\delta_{\gamma'}$ is constant between $\gamma*$ and γ. Hence f^δ_γ is this constant value. Since the sequence is constant the compatibility condition is easy to verify.

CLAIM: *$\langle f^\nu_\gamma : \gamma \in \omega_2, \nu \in T \cap (\delta + 1) \rangle$ is obedient and satisfies vertical and horizontal coherence, and the compatibility condition is satisfied.*

Proof: The compatibility condition and horizontal coherence were verified during the construction in case 2. Obedience is clear, since at each risky ordinal in C we defined f^δ_γ in the prescribed manner.

Since the matrix $\langle f^\nu_\gamma : \gamma \in \omega_2, \nu \in T \cap (\delta+1)\rangle$ satisfies the hypothesis of Lemma 2.9, there is a stationary set of risky ordinals. Let $\nu \in T \cap \delta$ be arbitrary. Let $\gamma \in C$ be a risky ordinal for δ in the closed unbounded set where the hypothesis of 2.8 holds for ν. Then γ is risky for ν. Hence, again, for all i,

$$\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\bar\nu}(\Delta(\gamma)(i)) = \pi^{M_\gamma}_{\bar\nu}(\pi^{M_\gamma}_{\bar\delta}(\Delta(\gamma)(i))) = \pi^{M_\gamma}_{\bar\nu}(\overline{f^\delta_\gamma(i)}) = \overline{\pi_\nu(f^\delta_\gamma(i))}$$

and thus $f^\nu_\gamma(i) = \pi_\nu(f^\delta_\gamma(i))$, and we have verified vertical coherence. ∎

CASE 3: $\delta \in \mathrm{cof}(\omega_2)$ *and* $T \cap \delta$ *is cofinal in* δ. In this case we will arrange that for all $\gamma \in \mathrm{cof}(\omega_1)$, there are ν, γ' such that $f^\delta_\gamma = f^\nu_{\gamma'}$. (In fact we are simply finding a descending ω_2-sequence from our matrix up to δ that generates the "generic filter" so far.) Towards this goal it is helpful to make the following remark:

Remark: Suppose that $\delta \in \mathrm{cof}(\omega_2)$ with $T \cap \delta$ cofinal in δ and $\langle f^\nu_\gamma : \nu \in (T \cap \delta \cap \mathrm{cof}(\omega_2)), \gamma < \omega_2\rangle$ is an obedient matrix of functions satisfying vertical and horizontal coherence. Then for any collection of functions $\{g_j : j \in \omega_1\}$ from this matrix, there are $\nu \in T \cap \delta, \gamma < \omega_2$ for all $j, f^\nu_\gamma \underset{a.e.}{\le} g_j$.

To see this we can choose a ν so large that for all j there are $\nu' < \nu$ and $\gamma < \omega_2$ such that $g_j = f^{\nu'}_\gamma$. Now choose a γ so large that for all j, if $g_j = f^{\nu'}_{\gamma'}$ then there is a $\gamma*, \gamma' < \gamma* < \gamma$ such that $\pi_{\nu'}(f^\nu_{\gamma*}(i)) \underset{a.e.}{=} f^{\nu'}_{\gamma*}$. Then for all j, if $g_j = f^{\nu'}_{\gamma'}$ then $f^\nu_\gamma \underset{a.e.}{\le} \pi_{\nu'}(f^\nu_\gamma) \underset{a.e.}{\le} \pi_{\nu'}(f^\nu_{\gamma*}) \underset{a.e.}{=} f^{\nu'}_{\gamma*} \underset{a.e.}{\le} g_j$. Hence this f^ν_γ satisfies the conclusion of the remark. ∎

In cases 3a–b we define f^δ_γ for $\gamma \in \mathrm{cof}(\omega_1)$.

Let $C_\delta = \langle \nu_j : j \in \omega_2\rangle$. For each successor $\nu \in C_\delta$, let E_ν be the closed unbounded set witnessing obedience. Let E be the closed unbounded set guaranteed to exist by the second filtration lemma (Lemma 2.7). Let $C = E \cap \Delta E_\nu$. Recall that for each γ, $\mathfrak{B}^\delta_\gamma = \bigcup\{\mathfrak{B}^{\nu_j}_{\gamma'} : j, \gamma' < \gamma\}$.

CASE 3a: $\gamma \in C$ *and* γ *is risky for* δ. Define f^δ_γ so that for all $i, \overline{f^\delta_\gamma(i)} = \pi^{M_\gamma}_{\bar\delta}(\Delta(\gamma)(i))$.

Let ν be the least element of T above ν_γ. We claim that $f^\delta_\gamma = f^\nu_\gamma$. Since $C_{\nu^-} = C_{\nu_\gamma}, \nu$ falls under case 2a, and thus we have that $\mathfrak{B}^\nu_\gamma = \mathfrak{B}^\delta_\gamma$. Further for all $j < \gamma$ we have that $\nu_j \in \mathfrak{B}^\delta_\gamma$, by the conclusions of Lemma 2.7. By Lemmas 2.7 and 2.8 we see that γ is risky for each successor $\nu_j \in C_{\nu^-}$. Finally $\omega_3^{\mathfrak{B}^\nu_\gamma} = \omega_3^{\mathfrak{B}^\delta_\gamma}$, so if γ is risky for δ, we know that $\mathrm{cof}(\overline{\nu}) > \omega_1$ in M_γ. Thus Case 2a defines f^ν_γ so that for all i, $\overline{f^\nu_\gamma(i)} = \pi^{M_\gamma}_{\overline{\nu}}(\Delta(\gamma)(i)) = \pi^{M_\gamma}_{\overline{\delta}}(\Delta(\gamma)(i)) = \overline{f^\delta_\gamma(i)}$. Since $\mathfrak{B}^\delta_\gamma = \mathfrak{B}^\nu_\gamma$, this implies that for all i, $f^\nu_\gamma(i) = f^\delta_\gamma(i)$.

CASE 3b: *γ is not risky.* If γ is a limit of ordinals of cofinality ω_1 then by induction we can assume that for all $\gamma' < \gamma$ of cofinality ω_1 there are $\nu' < \delta$ and η' such that $f^\delta_{\gamma'} = f^{\nu'}_{\eta'}$. By the remark we can choose ν, η so that for all $\gamma' < \gamma$, $f^\nu_\eta \underset{a.e.}{\leq} f^\delta_{\gamma'}$. Let $f^\delta_\gamma = f^\nu_\eta$ for such ν, η.

If γ is not a limit of ordinals of uncountable cofinality, then for some $\alpha < \gamma$, $\gamma = \alpha + \omega_1$. It is easy to check that for all $\beta < \omega_1$, $f^\delta_{\alpha+\beta} \underset{a.e.}{=} f^\delta_\alpha$. Let $f^\delta_\gamma = f^\delta_\alpha$.

In either subcase of Case 3b, we have preserved horizontal coherence.

CLAIM: *$\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in T \cap (\delta + 1)\rangle$ is obedient and satisfies vertical and horizontal coherence, and the compatibility condition is satisfied.*

Proof: Horizontal coherence is clear since we chose the f^δ_γ's decreasing a.e. Obedience is clear, since at each risky ordinal in C we defined f^δ_γ in the prescribed manner.

We verify the hypothesis of Lemma 2.9 for the matrix $\langle f^\nu_\gamma \colon \gamma \in \omega_2, \nu \in T \cap (\delta + 1)\rangle$. The only hypothesis that is not evident is that there is a cofinal set $B \subset T \cap \mathrm{cof}(\omega_2) \cap \delta$ such that for all $\nu \in B$ and each $\gamma^* \in \omega_2$ there is a $\gamma' \in \omega_2$ such that for all but countably many i, $f^\nu_{\gamma'}(i) \leq \pi_\nu(f^\delta_{\gamma^*}(i))$. Letting B be the successor points of C_δ this follows easily from the fact that for all $\gamma* < \omega_2$ there are $\alpha < \delta, \eta < \omega_2$ with $f^\delta_{\gamma*} = f^\alpha_\eta$. Let ν be a successor point of C_δ, then we can choose a $\gamma' > \max\{\eta, \gamma*\}$ so that γ' is a witness for vertical coherence between α and ν. Then $f^\nu_{\gamma'} \underset{a.e.}{\leq} \pi_\nu(f^\alpha_{\gamma'}) \underset{a.e.}{\leq} \pi_\nu(f^\alpha_\eta) \underset{a.e.}{\leq} \pi_\nu(f^\delta_{\gamma*})$. Hence there is a stationary set of risky ordinals. Arguing exactly as in case 2, this implies vertical coherence. ∎

This completes the construction and the proof of Theorem 2.1.

We now discuss the modifications necessary to execute the proof without using □. The main difference comes in case 3 of the construction. In this case even with square we are simply rearranging the generic object up to that stage. The complication in the proof given above is caused by requiring that the timing of the enumeration of a decreasing sequence coincide exactly (on a club set) with the requirements of the diamond sequence. This is arranged by choosing our filtration carefully so that the dictates of the diamond sequence correspond exactly to what we did in case 2a, and hence obeying them causes no incompatibility problems.

To eliminate square we choose the f^δ_γ in case 3 to be an arbitrary enumeration of a decreasing dense subset of the matrix up to that stage. (This allows us to take arbitrary filtrations.) However we have to weaken the obedience property to only hold at successor points of T. This causes additional difficulties at T-successors of ordinals in case 3. The saving remark is that if δ is an ordinal in case 3 and $x \in D$ then $\pi_\delta(x) \in D_\beta$ for some $\beta < \delta$. Hence, at the T-successor of δ, δ^+ an obedience condition can be satisfied, as its projection to δ is really a projection to a $\beta < \delta$, and this β can be shown to be a member of the appropriate stage in the filtration of $\mathfrak{B}^{\delta^+}$.

3. Applications

A fundamental problem in the theory of ultraproducts is to calculate the cardinality of an ultraproduct, given the cardinalities of the structures involved, the cardinality of the index set and properties of the ultrafilter.

In the 1960's a basic distinction was made between *regular* ultrafilters, whose ultrapowers have predictable cardinalities, and the *non-regular* ultrafilters, about which little can be said. As an example let's consider the particular case where we are taking an ultrapower of ω. It is easy to show that for a countably incomplete ultrafilter D on an infinite cardinal κ:

$$2^\omega \leq |\omega^\kappa/D| \leq 2^\kappa.$$

For regular ultrafilters, the maximal cardinality is always attained.

It remained an open problem for many years whether non-regular ultrafilters could even exist on accessible cardinals. Following early work of Ketonen, work of Donder, Jensen and Koppelberg [D-J-K] showed that every ultrafilter in L is regular.

Work of Magidor [M], using huge cardinals, showed that it was consistent for ω^{ω_2}/D to have cardinality $\aleph_2$. This showed that it was consistent for there to be a non-regular ultrafilter on ω_2.

Laver, using Woodin's model for an $\aleph_1$-dense ideal on ω_1 constructed an ultrafilter D on ω_1 such that $|\omega^{\omega_1}/D| = \omega_1$. Later work in [F-M-S] showed that it was consistent to have non-regular ultrafilters on arbitrary successors of regular cardinals. These ultrafilters, however, did not have the minimal cardinality of ultrapower.

In this section we show that it is consistent to have an ultrafilter D on ω_2 such that

$$|\omega^{\omega_2}/D| = \aleph_1.$$

From this we are able to calculate the chromatic number of the Erdős–Hajnal graph.

By Theorem 1.1 and Corollary 2.2, we get the following theorem:

THEOREM 3.1: *Suppose that there is a huge cardinal. Then there is a partial ordering $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there is an $\aleph_1$-dense, countably complete, weakly normal, uniform ideal on $\aleph_2$. (Further in the model $V^{\mathbb{P}}$, $\diamondsuit_{\omega_1}, \diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1)), \square_{\omega_2}$ and the G.C.H. hold.)*

COROLLARY 3.2: *If there is a huge cardinal then it is consistent that there is an ultrafilter D on ω_2 such that*

$$|\omega^{\omega_2}/D| = \aleph_1.$$

Proof: Let I be an $\aleph_1$-dense ideal on $\aleph_2$. Let $\mathcal{D} \subset P(\omega_2)/I$ be the dense set of cardinality $\aleph_1$. Then there is an ultrafilter D on $\aleph_2$ extending $I^{\smile}$ such that if $A \subset \mathcal{D}$ has $\bigvee A = 1$ then for some countable subset $B \subset A, \bigvee B \in D$. In particular there are sets $\{a_\alpha \colon \alpha \in \omega_1\}$ such that D is generated by $I^{\smile} \bigcup \{a_\alpha \colon \alpha \in \omega_1\}$. (See [La, BS, Hu2] for proofs of this fact under the assumptions of $\diamondsuit$, CH and ZFC, respectively. The ZFC result was also established independently by Woodin.) Let D be any such ultrafilter.

Let $f \colon \omega_2 \to \omega$ be an arbitrary function. For each n, let A_n be the collection of elements of $\mathcal{D}$ below $f^{-1}(n)$. Then there is a countable set $B \subset \bigcup\{A_n \colon n \in \omega\}$ with $\bigvee B \in D$. Choose disjoint representatives of the elements of $B, \{b_m \colon m \in \omega\}$. Define $g \colon \bigcup B \to \omega$, by setting $g(\alpha) = n$ iff $\alpha \in b_m$ and $b_m \subset A_n$. Then $g \equiv f$ mod D. Since there are only $\aleph_1$ many such g's (mod D) we have that ω^{ω_2}/D has cardinality $\aleph_1$. ∎

We note that if D is an ultrafilter on ω_2 with ω^{ω_2} having cardinality $\aleph_1$, then $\omega_1^{\omega_2}/D$ has cardinality $\aleph_2$, the minimum possible. To see this we note that the structure $\mathfrak{A} = \langle H(\omega_3), \in, \Delta \rangle \models$ "ω_1 is the successor of ω". Hence $\mathfrak{B} = \mathfrak{A}^{\omega_2}/D \models$ "ω_1 is the successor of ω". Hence, $|\langle \omega_1, \in \rangle^{\omega_2}/D| \leq |\langle \omega, \in \rangle^{\omega_2}/D|^+$.

THE ERDŐS–HAJNAL GRAPH. Recall the definition of the Erdős–Hajnal graph:

$$\mathfrak{G}(\kappa, \lambda) = \langle \{f | f \colon \kappa \to \lambda\}, \perp \rangle,$$

where $f \perp g$ iff $|\{\alpha \colon f(\alpha) = g(\alpha)\}| < \kappa$. We now remind the reader of the universal properties of the Erdős–Hajnal graph. See [E-H] or [Ko] for a detailed analysis.

Definition: A graph G has **type** $[\kappa, \lambda]$ iff its domain has cardinality κ and the graph has chromatic number λ. We will write $[\kappa, \lambda] \twoheadrightarrow [\kappa', \lambda']$ iff every graph of type $[\kappa, \lambda]$ has a subgraph of type $[\kappa', \lambda']$

Hajnal, in unpublished work, has shown that $[\aleph_2, \aleph_1] \twoheadrightarrow [\aleph_1, \aleph_0]$. Erdős and Hajnal showed, under the assumption of the C.H., that there is a graph on $\aleph_2$ with uncountable chromatic number, all of whose $\aleph_1$ subgraphs are of type $[\aleph_1, \aleph_0]$. Erdős and Hajnal proposed studying the property $[\aleph_2, \aleph_2] \twoheadrightarrow [\aleph_1, \aleph_1]$, by looking at $\mathfrak{G}(\omega_2, \omega)$. They proved:

THEOREM (Erdős–Hajnal): *If G is a graph on ω_2 with no subgraph of type $[\aleph_1, \aleph_1]$, then G can be embedded in $\mathfrak{G}(\omega_2, \omega)$.*

Proof: Let G be such a graph and for each $\alpha \in \omega_2$, let $c_\alpha\colon \alpha \to \omega$ be a coloring of the graph induced by G on α. For each $\beta \in \omega_2$, define a function $f_\beta\colon \omega_2 \to \omega$, by setting

$$f_\beta(\alpha) = \begin{cases} c_\alpha(\beta), & \alpha > \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then, if β and β' are connected by an edge in G, for all $\alpha > \max\{\beta, \beta'\}, c_\alpha(\beta) \neq c_\alpha(\beta')$. Hence $f_\beta(\alpha) \neq f_{\beta'}(\alpha)$. This shows that the mapping $\beta \to f_\beta$ preserves adjacency. ∎

An immediate corollary of this is result is that assuming the C.H., $\mathfrak{G}(\omega_2, \omega)$ has uncountable chromatic number.

COROLLARY: *If $\mathfrak{G}(\omega_2, \omega)$ has chromatic number $\aleph_1$ then $[\aleph_2, \aleph_2] \twoheadrightarrow [\aleph_1, \aleph_1]$.*

Komjath has shown the consistency of $\mathfrak{G}(\omega_2, \omega)$ having chromatic number $\aleph_3$. Komjath remarked that in Magidor's model ([M]) $\mathfrak{G}(\omega_2, \omega)$ has chromatic number $\aleph_2$. See Komjath's paper ([Ko]) for details.

The conclusion $[\aleph_2, \aleph_2] \twoheadrightarrow [\aleph_1, \aleph_1]$ was shown to be consistent from a huge cardinal in [F-L].

THEOREM 3.3: *If there is a huge cardinal then it is consistent that $\mathfrak{G}(\omega_2, \omega)$ has chromatic number $\aleph_1$.*

Proof: We have seen that the hypotheses of the theorem imply that there is a model of set theory with a uniform ultrafilter D on ω_2 such that $|\omega^{\omega_2}/D| = \omega_1$. Since elements of $\mathfrak{G}(\omega_2, \omega)$ are functions from $\omega_2 \to \omega$, we get a natural map from $\mathfrak{G}(\omega_2, \omega)$ to $|\omega^{\omega_2}/D|$. Since the ultrafilter is uniform, if $f \perp g$ then $[f]_D \neq [g]_D$. Hence the induced map to the ultraproduct is a coloring of $\mathfrak{G}(\omega_2, \omega_1)$ into ω_1 colors. ∎

We end this section with a remark that is joint with A. Dow:

PROPOSITION 3.4: *Suppose there is a uniform, countably complete, $\aleph_1$-dense ideal on $\aleph_2$. Let (X, τ) be a topological space with the property that every point in X has a countable neighborhood base. Suppose further that whenever $A \subset X$ is a discrete set of size $\aleph_1$, then $A = \bigcup A_n$, where A_n is separated. Then for all discrete sets $B \subset X$ of size $\aleph_2$ there are separated sets $\{B_\alpha : \alpha \in \omega_1\}$ such that $B = \bigcup B_\alpha$.*

Proof: Recall a set A is separated if there is a collection of disjoint open sets $\{O_a : a \in A\}$ such that for all $a \in A, O_a \cap A = \{a\}$. If we fix in advance a neighborhood basis for each point in A, then we may assume that each O_a is in the neighborhood basis for a.

We may assume that $B = \omega_2$. For each $\delta \in \omega_2$ choose a neighborhood basis $\{B_n^\delta : n \in \omega\}$. For each $\gamma \in \omega_2$, choose A_n^γ a partition of γ into separated sets and fix a seperating collection of basis elements. Define $f^\gamma : \gamma \to \omega \times \omega$, by setting $f^\gamma(\delta) = (m, n)$ iff $\delta \in A_m^\gamma$ and B_n^δ is the open neighborhood of δ in the seperation of A_m^δ. For $\delta \in \omega_2$, let $A^\delta_{(m,n)} = \{\gamma : f^\gamma(\delta) = (m, n)\}$.

Let K be the $\aleph_1$ dense ideal and $\{x_\eta : \eta \in \omega_1\}$ be a dense collection in $P(\omega_2)/K$. Let $\delta \in B_{\eta,m,n}$ iff $A^\delta_{m,n} \supset_K x_\eta$. Then $B = \bigcup\{B_{\eta,m,n} : \eta \in \omega_1, m, n \in \omega\}$. Further, for all $\delta_1, \delta_2 \in B_{\eta,m,n}$, there is a $\gamma \in x_\eta$ above δ_1 and δ_2. Since $f^\gamma(\delta_1) = f^\gamma(\delta_2), B_n^{\delta_1} \cap B_n^{\delta_2} = \emptyset$. Hence, $B_{\eta,m,n}$ is a separated set. ∎

4. Some Rudin–Keisler minimal ultrafilters

The results in this section are joint with A. Kanamori and M. Magidor.

THEOREM 4.1: *Suppose $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1)), \Box_{\omega_2}$ and that there is a very strongly layered ideal I on $\aleph_2$. Then for all functions $f : \omega_2 \to \omega_1$ which are not bounded in ω_1 on a set in $I^\smile$ there is a uniform, countably complete, weakly normal ideal K on ω_2 such that:*

1. *$P(\omega_2)/K \cong P(\omega_1)/\{$countable sets$\}$,*
2. *for all $g : \omega_2 \to \omega_1$ there is a $h : \omega_1 \to \omega_1$ with $g \equiv_K h \circ f$.*

Kanamori [Ka] calls the function f in the theorem a *finest partition* relative to K. We note the following corollary:

COROLLARY 4.2: *Under the hypothesis of Theorem 4.1 there is a uniform, countably complete ideal K on $\aleph_2$ such that, if D is any ultrafilter extending $K^\smile$, then*

$$|\omega_1^{\omega_2}/D| = \aleph_2.$$

To see the corollary from the theorem, we take K to be the ideal asserted to exist by the theorem. Then modulo K, there are only 2^{ω_1} many functions from

ω_2 to ω_1. We also note that the results in §1 imply that the hypothesis of this corollary are consistent.

Proof of Theorem 4.1: Let I be a very strongly layered ideal on ω_2. We will show that for every function f that is unbounded modulo I there is an ideal K satisfying the theorem. So fix such a function f. Then f induces a partition of $P(\omega_2)/I$, $\langle x_\alpha: \alpha \in \omega_1\rangle$. We will construct a matrix of functions $\mathcal{F}$ that satisfy the vertical and horizontal coherence and coding hypothesis and the following additional hypothesis which is a strengthening of the original genericity condition:

> **The selection hypothesis:** Let $\langle y_\beta: \beta \in \eta\rangle$ be a partition of $P(\omega_2)/I$, where $\eta \leq \omega_1$. Then there is some pair (γ, δ) and a function $h: \omega_1 \to \omega_1$ such that for all $j \in \omega_1$, $f^\delta_\gamma(j) \subset_I x_j \wedge y_{h(j)}$.

To see that this strengthens the original genericity condition we take an arbitrary set $x \subset \omega_2$ and apply the selection hypothesis to the partition $\{x, \omega_2 \backslash x\}$.

Having the new matrix of functions $\mathcal{F}$, we define the ideal K the same way as we did in §2 from $\mathcal{F}$ and I.

CLAIM: *Suppose that $\mathcal{F}$ satisfies the hypotheses of coding, coherence, and selection. Let K be the ideal defined in §2 from $\mathcal{F}$. For all $g: \omega_2 \to \omega_1$, let $y_\beta = g^{-1}(\beta)$ and let h be as in the selection hypothesis. Then we have that*

$$g =_K h \circ f.$$

Proof of Claim: We consider $X = \{\gamma: g(\gamma) = h(f(\gamma))\}$. Then $X \in K^\smile$ iff for all large enough γ, δ, $\{i: f^\delta_\gamma(i) \subset X\} \in I^\smile$. The latter condition is true iff there is some γ, δ such that for all i, $f^\delta_\gamma(i) \subset X$. Let δ, γ be as in the selection hypothesis. Then for *every* i, $f^\delta_\gamma(i) \subset X$. Hence $X \in K$. ∎

Thus to prove the theorem, we must show that we can build an $\mathcal{F}$ satisfying the original conditions and also satisfies the selection hypothesis.

Fix a partition of $P(\omega_2)/I, \langle x_i: i \in \omega_1\rangle$. Without loss of generality we may assume that each $x_i \in B_0$. Hence we can start our construction by setting $f^0_0(i) = x_i$.

We now must refine our notion of obedient.

We will follow the notation developed in the proof of Theorem 2.1. We first expand $\mathfrak{A}$ to include a relation denoting $\langle x_i: i \in \omega_1\rangle$.

We will view our $\Diamond$-sequence $\langle A_\gamma: \gamma \in \omega_2\rangle$ as guessing:

(1) transitive structures $M_\gamma = \langle M, \in, \leq^M, I^M, \langle B_\alpha: \alpha \in \omega_3^M\rangle, D^M, \langle \pi_\alpha: \alpha \in \omega_3^M \cap (\mathrm{cof}(\omega_2) \cup \mathrm{succ})^M\rangle, \langle x_i: i \in \omega_1\rangle^M \ldots\rangle$, where $M_\gamma \equiv \mathfrak{A}$, and $M^\omega \subset M$;

(2) a sequence $Y\gamma = \langle y_j \colon j \in \omega_1 \rangle \in M_\gamma$, with $M_\gamma \models Y_\gamma$ is a partition of $P(\omega_2)/I$;
(3) a matrix of functions $\langle g^\nu_{\gamma'} \colon \nu \in S, \gamma' < \gamma \rangle$, where $S \subset \omega_3^M$, $g^\nu_{\gamma'} \colon \omega_1 \to D^M$, and for each ν, $\langle g^\nu_{\gamma'} \colon \gamma' < \gamma \rangle$ is $\subset_{I'}$ decreasing mod countable sets (i.e. for each ν, $\gamma' < \gamma* < \gamma$, and all but countably many $i, g^\nu_{\gamma*}(i) \subset_{I^M} g^\nu_{\gamma'}(i)$);
(4) for all $i \in \omega_1, g^0_0(i) \subset x^M_i$.

We must also redefine our functions $\Delta(\gamma)$. Fix a $\gamma \in \omega_2 \cap \mathrm{cof}(\omega_1)$. If there is a function $g \colon \omega_1 \to D^{M_\gamma}$ such that for all $\nu \in S$ and all $\gamma' < \gamma$ there is a co-countable set of $i \in \omega_1$ such that $g(i) \subset g^\nu_{\gamma'}(i)$, choose such a function and define $\Delta(\gamma) \colon \omega_1 \to D^{M_\gamma}$ to be a function such that for all i, $\Delta(\gamma)(i) \subset g(i)$ and $\Delta(\gamma)(i) \subset_{I'} x_i \wedge y_j$ for some j. If such a g exists then this is possible since $Y\gamma \in M_\gamma$ and for almost all $i, g(i) \subset x_i$. Since Y_γ is a maximal antichain there is a j with $y_j \wedge x_i \wedge g(i) \neq 0$. If no such g exists then $\Delta(\gamma)$ will not be defined.

We now define obedience in exactly the same way with respect to the modified $\Delta(\gamma)$'s and $\Diamond$ sequence.

The rest of the proof goes as before: i.e. there are many risky ordinals, obedient sequences can be manufactured and obedient sequences satisfy the hypothesis on $\mathcal{F}$. $\blacksquare_{\text{Theorem 4.1}}$

In [Ka], Kanamori proved the following theorem, answering a question of A. Taylor [K-T].

THEOREM (Kanamori): *Assume that there is an $\aleph_1$-dense ideal on ω_1 and $\Diamond_{\omega_1}$. Suppose that D is any non-principal ultrafilter on ω, and $f \colon \omega_1 \to \omega$ is a map such that $f^{-1}(n) \in I^+$ for all $n \in \omega$. Then there is an $\aleph_1$-generated ultrafilter U over ω_1 extending I^* such that $f_*(U) = D$.*

The main result of this section is the following:

COROLLARY 4.3: *Assume there is an ideal on ω_2 that satisfies the conclusion of Theorem 4.1 for the function f. Assume that for all $\alpha \in \omega_1, f^{-1}(\alpha) \in K^+$. Let E be any non-principal ultrafilter on ω_1. Then there is an ultrafilter F over ω_2 extending I^* such that $f_*(F) = E$.*

Let $\leq_{RK}$ be the Rudin–Keisler ordering and $=_{RK}$ be the corresponding equivalence relation.

COROLLARY 4.4: *Assume:*
(1) $\Diamond_{\omega_1}, \Diamond_{\omega_2}(\mathrm{cof}(\omega_1)), \Box_{\omega_2}$,
(2) *there is a normal $\aleph_1$-dense ideal I on ω_1, and*
(3) *there is a very strongly layered ideal on ω_2.*

Then there are non-principal uniform ultrafilters D,E and F on ω, ω_1 and ω_2, respectively, such that for all non-principal ultrafilters U on some infinite set, if $U \leq_{RK} F$ then either $U =_{RK} D$ or $U =_{RK} E$ or $U =_{RK} F$.

Proof of 4.3 and 4.4: Fix a function f as in the proof of Theorem 4.1, and let K be the ideal constructed there. Let E be an arbitrary uniform ultrafilter on ω_1 extending the dual of some countably complete ideal I. Since the ideal I extends the countable sets and we know that $P(\omega_2)/K \cong P(\omega_1)/\{\textit{countable sets}\}$, we can extend K to an ideal L, where $P(\omega_2)/L \cong P(\omega_1)/I$. (Explicitly, we have that if $\mathcal{F}$ is the matrix of functions defined in the proof of 4.1, $X \in L$ iff for all large enough $\delta, \gamma, \{i\colon f^\delta_\gamma(i) \subset X\} \in I$.)

We construct an ultrafilter F on ω_2 extending $L^\smile$ such that $E = f_*(F)$.

Let F_0 be the filter on ω_2 induced by the basis $\{X\colon$ for some $Y \in E, X = \bigcup f^{-1}[Y]\}$. We claim that $F_0 \cup L^\smile$ has the finite intersection property. Let $Z \in L^\smile$ and $X \in F_0$. Then for all large enough $\delta, \gamma, \{i\colon f^\delta_\gamma(i) \subset Z\} \in I^\smile$ and for some $Y \in E, X = \bigcup\{x_i\colon i \in Y\}$ (where $x_i = f^{-1}(i)$). Since $f^0_0 = f$, for all $\delta, \gamma, f^\delta_\gamma(i) \subset X$ iff $i \in Y$. Since $E \supset I^\smile$, for all δ, γ there is an i with $f^\delta_\gamma(i) \subset X$ and $f^\delta_\gamma(i) \subset Z$. Hence $X \cap Z \supset f^\delta_\gamma(i)$ and we have shown that $F_0 \cup L^\smile$ has the finite intersection property. Extend $F_0 \cup L^\smile$ to a uniform ultrafilter F. Since $F \supset F_0$, we see easily that $E = f_*(F)$. This proves Corollary 4.3.

In [Ka] from the assumptions of Corollary 4.4, Kanamori shows the existence of ultrafilters D and E on ω and ω_1 respectively, that have the property that for all non-principal ultrafilters U, if $U \leq_{RK} E$ then either $U =_{RK} D$ or $U =_{RK} E$. Further, the ultrafilter E constructed by Kanamori extends the dual of the normal $\aleph_1$-dense ideal.

Apply the previous argument to E with I being the $\aleph_1$-dense ideal to get an ultrafilter F on ω_2. Since F extends $L^\smile$, Corollary 2.2 implies that F is weakly normal. Suppose now that $U \leq_{RK} F$ with ψ being the witness function. Then we may assume that $\psi\colon \omega_2 \to \omega_2$. By weak normalitiy if ψ is not $1-1$ on a set in F (i.e. not a Rudin–Keisler isomorphism), then ψ is bounded on a set in F. If ψ is bounded on a set in F, the there is a function $h\colon \omega_1 \to \omega_1$ such that $\psi = h \circ f$ modulo U. This function witnesses that $U \leq_{RK} E$. By the properties of the ultrafilters that Kanamori constructed, $U =_{RK} D$ or $U =_{RK} E$. ∎

5. Some open problems

In this section we list some open problems. The most obvious open problem is to get the consistency of countably complete $\aleph_1$-dense ideals on cardinals between $\aleph_2$ and $\aleph_\omega$. This looks like only a "technical problem", but perhaps not. The main obstacle is getting the correct κ^+-saturated ideals on the appropriate κ. Since these ideals don't feel exotic at all, it is possible to hope that there is not a major new idea required to do this.

A problem the author feels is related is the well known problem of getting the consistency of the statement "$\aleph_\omega$ is Jonsson". While it would be desirable to get this consistent in a model where the Chang property is manifest by the appropriate precipitous embedding, there is growing evidence that the Jonsson property should be proved using strong combinatorial properties of the $\aleph_n$'s.

An interesting class of problems remain open around the Erdős–Hajnal graph. For several of these there seems no obvious line of attack using saturated ideals. For example, it is not known how to calculate the chromatic number of the Erdős–Hajnal graphs for $\mathfrak{G}(\kappa, \omega)$ for $\aleph_\omega \leq \kappa \leq \aleph_{\omega_1}$. Can it be $\aleph_1$? The reason ideals of the type produced in this paper seem irrelevant is that the Kunen theorem prohibits highly saturated ideals on cardinals in the interval in question.

Finally, a basic problem seems to be to find some "representation theory" for ideals in general, perhaps along the lines of the matrix defined in this paper. The author will leave the details of this project to the interested reader.

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