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**ON DAMAGE INDUCED ANISOTROPY
FOR FIBER-COMPOSITES**

BY

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AND
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On Damaged Induced Anisotropy for Fiber-Composites

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Abstract

A stress-strain relation on the basis of a homogenized continuum is developed from a potential energy function for an elastic material with anisotropic damage. It is assumed that the influence of the state of damage can be represented by a vector-valued function, resulting from an array of surface discontinuities with coinciding orientation such as disk-like, parallel cracks. Usually, only low order polynomial expansions in terms of the damage variable have been considered in the literature, limiting the results to non-interacting microcracks. In this paper a complete polynomial expansion of the potential function with respect to the damage variable is developed and the general form of the constitutive tensor for the damaged material is derived. It allows to account for nonlinear dependencies of the effective elastic properties on the damage variable in case of interacting microcracks.

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1 Introduction

Various authors, e.g. [2], [9], [12], [1], [22], [23], [25], [10], have developed constitutive laws for damaged materials from a potential function, expanded in terms of the damage variables and either stress or strain invariants. In this paper it is assumed that the state of defects in a solid consists of an array of parallel crack surfaces. In order to account for the influence of the defects on the elastic properties, a vector-valued function is introduced for the representation of the state of damage. Damage is understood as a phenomenological concept to model the degradation of the elastic properties due to surface-like defects. Despite the ongoing discussion on the description of damage by higher order tensors, the state of damage is represented here by means of vectors. The vectorial treatment of anisotropic damage is common to [2], [12], [22] or [1] among others, because of its simplicity and its geometrical significance for representing surface-like defects.

The invariance requirements on the strain energy function under superimposed rigid body motions have been enforced

- by means of constructing a second-order damage tensor and including it in the argument list of the strain energy, e.g. [9], [25]

or

- by adding the second invariant of the damage vector as an element to the integrity basis, which is formed by the strain tensor and in case of non-isotropic elasticity by other variables, accounting for the anisotropic properties of the material structure. The integrity basis also includes all the invariants, generated from products between the even powers of the damage vector and the strain tensor or the variables for the anisotropic material structure - see [22], [2].

In both cases the potential function can be expanded as a polynomial in terms of the invariants. In the latter case the polynomial representation of the strain energy contains only even powers of the damage parameter ω , i.e. it is a function of (ω^2) . Thus, linear damage models, as given for example in [8] or [18], are not recoverable without reinterpretation of the physical quantities. This inadequacy has been listed as a disadvantage of the vectorial representation of damage [10], [24]. The influence of the defect state on the effective elastic properties is nonproportional with regard to increasing crack sizes and crack densities, indicated by the scalar ω , unless the distance between neighboring cracks is large enough, i.e. dilute concentrations of defects, $\omega \ll 1$. In the dilute case the interaction of strain concentrations at adjacent crack tips is negligible under "homogeneous deformations", which means: deformations, generating states of homogeneous fields of strains in a homogeneous body.

2 Kinematics and Strain Energy

A comprehensive theory of the mechanical behavior of brittle solids, gradually degrading by the evolution of planar microdefects may be found in [2]. The orientation of an array of parallel cracks in an initially isotropic solid is given by the normal vector $\mathbf{n}$, also denoted as the damage normal.

A non-negative damage parameter ω is introduced, which measures the degradation of the elastic material properties (stiffness loss or flexibility increase). It may be identified as the void area density. The damage vector $\boldsymbol{\omega}$ is defined as:

$$\boldsymbol{\omega} = \omega \mathbf{n}$$

The current position $\mathbf{x}$ of the body is described by a function χ of its initial position $\mathbf{X}$ and the time t .

$$\mathbf{x} = \chi(\mathbf{X}, t)$$

The strain energy W is postulated as a function, depending formally on the deformation gradient $\mathbf{F}$ and the damage vector $\boldsymbol{\omega}$. The principle of invariance of W under superimposed rigid body motion [4] requires that the strain energy W depends on $\mathbf{F}$ through the right Cauchy-Green-deformation tensor $\mathbf{C}$.

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad , \quad \mathbf{C} = \mathbf{F}^T \mathbf{F} \quad , \quad W = \hat{W}(\mathbf{C}, \boldsymbol{\omega})$$

With regard to the symmetry properties of the elastic response for the initially isotropic material it is assumed that the anisotropic behavior is only due to the damage of the material, where the damage normal $\mathbf{n}$ specifies the direction of the privileged axis. The deformation and thus the strain energy does not change under rigid body rotations about the preferred (principal) direction $\mathbf{n}$ for a fixed state of damage. Therefore, a second reference configuration is considered, which is obtained by a rigid rotation $\mathbf{Q}$ of the undeformed body with its array of planar microdefects. Let $\bar{\mathbf{X}}$ be the position vector of the particles of the body in the new reference configuration. Then the current position vector $\bar{\mathbf{X}}$ is related to its initial position vector $\mathbf{X}$ by:

$$\bar{\mathbf{X}} = \mathbf{Q} \mathbf{X}$$

Likewise is the relation for the vector-valued damage variable between $\bar{\boldsymbol{\omega}}$ in the rotated reference configuration and $\boldsymbol{\omega}$ of the initial reference configuration:

$$\bar{\boldsymbol{\omega}} = \mathbf{Q} \boldsymbol{\omega}$$

Straight forward derivations lead to the deformation gradient $\bar{\mathbf{F}}$ and the right Cauchy-Green deformation tensor $\bar{\mathbf{C}}$ of the deformed body with respect to its new reference configuration:

$$\begin{aligned} \bar{\mathbf{F}} &= \frac{\partial \mathbf{x}}{\partial \bar{\mathbf{X}}} = \mathbf{F} \mathbf{Q}^T \\ \bar{\mathbf{C}} &= \bar{\mathbf{F}}^T \bar{\mathbf{F}} = \mathbf{Q} \mathbf{C} \mathbf{Q}^T \end{aligned}$$

As indicated, the associated strain energy function $\bar{W}$, depending on $\bar{\mathbf{C}}$ and $\bar{\omega}$, is required to be invariant under the group of all proper orthogonal transformations $\mathbf{Q}$, representing superimposed rigid rotations of the body.

$$\bar{W}(\bar{\mathbf{C}}, \bar{\omega}) = \bar{W}(\mathbf{Q} \mathbf{C} \mathbf{Q}^T, \mathbf{Q} \omega) \equiv \hat{W}(\mathbf{C}, \omega)$$

In addition, the strain energy W has to remain invariant with respect to the orientation (sign) of the damage normal $\mathbf{n}$. Therefore, W has to be an even function of the damage normal $\mathbf{n}$. As a sufficient condition for this last invariance requirement the dependency of the strain energy on ω is replaced by W as a function of the dyadic product $\omega \otimes \mathbf{n} = \omega \mathbf{n} \otimes \mathbf{n}$, formed by the damage vector ω with the damage normal $\mathbf{n}$.

$$W = \hat{W}(\mathbf{C}, \omega \mathbf{n} \otimes \mathbf{n})$$

The integrity basis needs to be constructed for the Taylor series expansion of W as an invariant function under the transformations $\mathbf{Q}$. The following properties of the symmetric second order tensor $\omega \otimes \mathbf{n} = \omega \mathbf{n} \otimes \mathbf{n}$ are employed next.

$$\begin{aligned} (\omega \mathbf{n} \otimes \mathbf{n})^m &= \omega^m (\mathbf{n} \otimes \mathbf{n}) \\ \text{tr}(\omega \mathbf{n} \otimes \mathbf{n}) &= \omega \\ \text{tr}[(\omega \mathbf{n} \otimes \mathbf{n}) \mathbf{C}] &= \omega \mathbf{n} \mathbf{C} \mathbf{n} \end{aligned} \tag{1}$$

The integrity basis - see for example [19] - for the two symmetric second order tensors $\mathbf{C}$ and $\omega \otimes \mathbf{n}$ consists of the six independent invariants, which are given in symbolic as well as in index notation.

$$\begin{aligned} I_1 &= \text{tr} \mathbf{C} &= C_{ii} \\ I_2 &= \frac{1}{2} \text{tr} \mathbf{C}^2 &= \frac{1}{2} C_{ij} C_{ji} \\ I_3 &= \frac{1}{3} \text{tr} \mathbf{C}^3 &= \frac{1}{3} C_{ij} C_{jk} C_{ki} \\ I_4 &= \text{tr}(\omega \mathbf{n} \otimes \mathbf{n}) &= \omega \\ I_5 &= \text{tr}[(\omega \mathbf{n} \otimes \mathbf{n}) \mathbf{C}] &= \omega \mathbf{n} \mathbf{C} \mathbf{n} = \omega n_i C_{ij} n_j \\ I_6 &= \text{tr}[(\omega \mathbf{n} \otimes \mathbf{n}) \mathbf{C}^2] &= \omega \mathbf{n} \mathbf{C}^2 \mathbf{n} = \omega n_i C_{ij} C_{jk} n_k \end{aligned} \tag{2}$$

It has to be noted that the invariants $\text{tr}(\omega \mathbf{n} \otimes \mathbf{n})^2$ and $\text{tr}[(\omega \mathbf{n} \otimes \mathbf{n})^2 \mathbf{C}]$ can be expressed as polynomials in terms of I_4 and I_5 . Therefore, they do not appear in the list of independent invariants, which form the irreducible integrity basis.

3 Polynomial Expansion of Strain Energy For Initially Isotropic Materials

A suitable linearized elasticity model, accounting for small strains (i.e. the stretch tensor is close to the second order identity tensor) but finite rotations is appropriate to demonstrate the methodology. The strain energy function W is approximated by a quadratic polynomial in terms of the Green-Lagrange-strains $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, which replace their related deformation measure $\mathbf{C}$. For the linear approximation of an elastic-brittle material law, only the second-order polynomial terms of the strains are taken into consideration. Eigenstrains can be easily incorporated by including all combinations of the elements from the irreducible integrity basis which produce linear expressions in the strains.

In the case of interacting microcracks the strain energy W is no longer only linear in the damage variable ω . Therefore, it is mandatory to expand W as a polynomial of arbitrary power in the strain independent invariant I_4 , which is equal to the damage parameter ω .

$$\begin{aligned}
 W = & I_1^2 \left(\frac{1}{2} \lambda_o + \lambda_1 I_4 + \lambda_2 I_4^2 + \lambda_3 I_4^3 + \dots \right) \\
 & + I_2 \left(2\mu_o + \mu_1 I_4 + \mu_2 I_4^2 + \mu_3 I_4^3 + \dots \right) \\
 & + I_1 I_5 \left(\alpha_o + \alpha_1 I_4 + \alpha_2 I_4^2 + \alpha_3 I_4^3 + \dots \right) \\
 & + I_5^2 \left(\beta_o + \beta_1 I_4 + \beta_2 I_4^2 + \beta_3 I_4^3 + \dots \right) \\
 & + I_6 \left(\gamma_o + \gamma_1 I_4 + \gamma_2 I_4^2 + \gamma_3 I_4^3 + \dots \right)
 \end{aligned} \tag{3a}$$

The contributions of I_1^2 and I_2 to the polynomial expansion of W are related to the isotropic properties of the undamaged solid. The associated power series in I_4 describe isotropic damage. Of primary concern in this paper, however, is the anisotropic part of the elastic behavior induced by damage. Therefore, the isotropic part due to damage is ignored for simplicity and it suffices to deal only with constant Lamé coefficients $\lambda(I_4) = \lambda_o$ and $\mu(I_4) = \mu_o$ with regard to the damage parameter ω for further considerations.

Only convergent power series in terms of the invariant I_4 are of practical interest. By denoting the limits of the convergent series as $\bar{\alpha}(\omega)$, $\bar{\beta}(\omega)$ and $\bar{\gamma}(\omega)$, it is possible to simplify eq. 3a to:

$$W = \frac{1}{2} \lambda_o I_1^2 + 2\mu_o I_2 + \bar{\alpha}(\omega) I_1 I_5 + \bar{\beta}(\omega) I_5^2 + \bar{\gamma}(\omega) I_6 \tag{3b}$$

The second Piola-Kirchhoff stresses follow from the gradient of W :

$$\begin{aligned}
 \mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = & \lambda_o (tr \mathbf{E}) \mathbf{I} + 2\mu_o \mathbf{E} + \alpha(\omega) [(tr \mathbf{E}) \mathbf{n} \otimes \mathbf{n} + (\mathbf{n} \mathbf{E} \mathbf{n}) \mathbf{I}] \\
 & + \beta(\omega) (\mathbf{n} \mathbf{E} \mathbf{n}) \mathbf{n} \otimes \mathbf{n} + \gamma(\omega) [\mathbf{E} \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{E} \mathbf{n}]
 \end{aligned} \tag{4a}$$

after the damage dependent coefficients $\omega \bar{\alpha}(\omega)$, $2\omega^2 \bar{\beta}(\omega)$ and $\omega \bar{\gamma}(\omega)$ are redefined in terms of $\alpha(\omega)$, $\beta(\omega)$ and $\gamma(\omega)$. This result resembles transverse isotropy, were we to interpret

the functions $\alpha(\omega)$, $\beta(\omega)$ and $\gamma(\omega)$ as material constants. In index notation eq. 4a may be written as:

$$\begin{aligned} S_{IJ} &= \lambda_o \delta_{IJ} E_{KK} + 2\mu_o E_{IJ} \\ &+ \alpha(\omega) [n_I n_J E_{KK} + \delta_{IJ} n_K n_L E_{KL}] \\ &+ \beta(\omega) n_I n_J n_K n_L E_{KL} + \gamma(\omega) [n_K n_J E_{IK} + n_I n_K E_{KJ}] \end{aligned} \quad (4b)$$

The constitutive tensor D_{IJKL} follows immediately from eq. 4b, if the state of damage does not change, i.e. the damage parameter ω remains constant.

$$\begin{aligned} D_{IJKL} &= \lambda_o \delta_{IJ} \delta_{KL} + 2\mu_o \delta_{IK} \delta_{JL} \\ &+ \alpha(\omega) [n_I n_J \delta_{KL} + \delta_{IJ} n_K n_L] \\ &+ \beta(\omega) n_I n_J n_K n_L + \gamma(\omega) [\delta_{IL} n_J n_K + n_I n_K \delta_{JL}] \end{aligned} \quad (5a)$$

Remarks and Comparison to Other Results

- The second order damage tensors, used by [9], [10] and [1] for the representation of the local damage state, are constructed by the dyadic product between the "displacement discontinuity vector" $\mathbf{b}$ across the boundaries of the crack and the normal $\mathbf{n}$ to the crack surface. Frequently, instead of $\mathbf{b}$ only its component normal to the crack surface is considered as relevant in the outer product for the damage tensor - see also discussion in [13], p 117. As indicated, the strain energy function W in terms of the second order damage tensor is invariant with regard to the orientation of the normal $\mathbf{n}$.
- For the application of this model to plane stress the two dimensional case is considered, where the damage normal is taken as collinear with the X_1 -axis. The constitutive tensor can be written in Voigt-notation as:

$$\begin{aligned} [\mathbf{D}] &= \begin{bmatrix} \lambda_o + 2\mu_o & \lambda_o & 0 \\ \lambda_o + & \lambda_o + 2\mu_o & 0 \\ 0 & 0 & \mu_o \end{bmatrix} \\ &+ \alpha(\omega) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \beta(\omega) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma(\omega) \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \end{aligned} \quad (5b)$$

The matrix structure of the constitutive tensor for the damaged material clearly shows that the effective elasticity coefficients for the shear stiffness, both normal stiffnesses and Poisson's ratio are mutually independent unlike the case for the undamaged isotropic material. A comparison with [11] can be made, if only one set of cracks with the same orientation is considered. The last contribution in eq. 5b from $\gamma(\omega)$ does not find its counterpart in [11], eq. (31). A similar conclusion can be drawn concerning the results in [1].

- In [17] the "void orientation density function" $v(\mathbf{n})$ is introduced, which describes damage due to surface-like defects. The void orientation function is represented as a series of its spherical harmonics in terms of the normal $\mathbf{n}$ on the parallel cracks. The compliance or the stiffness tensor may be constructed from the second order identity tensor and the first three terms of the Fourier series for even functions of $v(\mathbf{n})$, which are invariant with respect to the orientation (sign) of the normal. Because of the minor and major symmetries of the constitutive tensor, $C_{ijkl} = C_{klij} = C_{lki j} = C_{lkji}$, it follows that the approach in [17] allows the same general form as in eq. 5a for an initially isotropic constitutive tensor with induced anisotropy due to surface-like defects.

4 Generalization to Transverse Isotropic Elasticity with Anisotropic Damage

The foregoing approach is applied next to continuum damage of transverse isotropic materials, which are important for modeling unidirectional fiber-reinforced composites. A unit vector $\mathbf{a}$ is introduced to indicate the direction of reinforcement. The previously used strain energy function W additionally depends on the vector $\mathbf{a}$.

$$W = \hat{W}(\mathbf{C}, \omega, \mathbf{a})$$

The same argument as in section 2 applies for invariance of W under superimposed rigid body motions in order to find the appropriate dependence of the strain energy function on its arguments and to investigate the material symmetry group under all orthogonal transformations $\mathbf{Q}$.

$$\hat{W}(\mathbf{C}, \omega, \mathbf{a}) = \bar{W}(\mathbf{Q} \mathbf{C} \mathbf{Q}^T, \mathbf{Q} \omega, \mathbf{Q} \mathbf{a})$$

Similar to the treatment of the damage normal $\mathbf{n}$ the strain energy $\hat{W}$ must also be invariant with respect to the orientation of the direction vector $\mathbf{a}$. A sufficient condition for orientation insensitivity of the strain energy implies that W is a function of the symmetric second order tensor $\mathbf{a} \otimes \mathbf{a}$ rather than of the vector $\mathbf{a}$, besides its dependence on $\mathbf{C}$ and $\omega \otimes \mathbf{n}$.

$$W = \bar{W}(\mathbf{C}, \omega \mathbf{n} \otimes \mathbf{n}, \mathbf{a} \otimes \mathbf{a})$$

The irreducible integrity basis, formed by the three symmetric second order tensor consists of ten invariants if use is made of the following identities:

$$(\mathbf{a} \otimes \mathbf{a})^m = \mathbf{a} \otimes \mathbf{a} \quad \text{and} \quad \text{tr}(\mathbf{a} \otimes \mathbf{a}) = a_I a_I = 1. \quad (6)$$

The first six elements I_1 to I_6 of the irreducible integrity basis for the strain energy W of transverse isotropic materials with damage are the same as in section 2, eq. 2. Further

invariants are:

$$\begin{aligned}
I_7 &= \text{tr}(\mathbf{a} \otimes \mathbf{a})\mathbf{C} &= \mathbf{a}\mathbf{C}\mathbf{a} &= a_I C_{IJ} a_J \\
I_8 &= \text{tr}(\mathbf{a} \otimes \mathbf{a})\mathbf{C}^2 &= \mathbf{a}\mathbf{C}^2\mathbf{a} &= a_I C_{IK} C_{KJ} a_J \\
I_9 &= \text{tr}[\mathbf{C}(\omega \mathbf{n} \otimes \mathbf{n})(\mathbf{a} \otimes \mathbf{a})] &= \omega(\mathbf{a} \cdot \mathbf{n}) \mathbf{a}\mathbf{C}\mathbf{n} &= \omega(a_K n_K) a_I C_{IJ} n_J \\
I_{10} &= \text{tr}[(\omega \mathbf{n} \otimes \mathbf{n})(\mathbf{a} \otimes \mathbf{a})] &= \omega(\mathbf{a} \cdot \mathbf{n})^2 &= \omega(a_I n_I)^2
\end{aligned} \tag{7}$$

Except of the invariant I_{11} ,

$$I_{11} = \text{tr}[\mathbf{C}^2(\omega \mathbf{n} \otimes \mathbf{n})(\mathbf{a} \otimes \mathbf{a})] = \omega(\mathbf{a} \cdot \mathbf{n}) \mathbf{a}\mathbf{C}^2\mathbf{n} = \omega(a_L n_L) a_I C_{IJ} C_{JK} n_K,$$

it is obvious that the additional invariants

$$I_{12} = \text{tr}[\mathbf{C}(\omega \mathbf{n} \otimes \mathbf{n})^2(\mathbf{a} \otimes \mathbf{a})] \quad \text{and} \quad I_{13} = \text{tr}[\mathbf{C}^2(\omega \mathbf{n} \otimes \mathbf{n})^2(\mathbf{a} \otimes \mathbf{a})]$$

are equal to polynomials in terms of I_4 , I_9 and I_{11} after use has been made of eqs. 1 and 6. Therefore, they do not qualify as elements of the irreducible integrity basis, listed in eqs. 2 and 7. The polynomial identity given in [21] allows I_{11} to be expressed in terms of the first ten invariants. For linear elasticity the most general form for the strain energy function W is written as a quadratic polynomial in terms the Green-Lagrange strains $\mathbf{E}$ and represented as a power series of the strain free invariants I_4 and I_{10} .

Thirteen different quadratic expressions in terms of the components of the strain tensor can be formed as product combinations of the invariants I_1 , I_2 and I_5 to I_9 . The complete polynomial expansion of the strain energy function with respect to the strain independent invariants I_4 and I_{10} results in the sum of at most thirteen different power series, associated with each quadratic term in the strains. Only the converging power series, formed by a double series with arguments I_4 and I_{10} are of further interest. The limit for each converging power series defines the i -th "coefficient function" ϕ_i . The thirteen scalars $\phi_i = \hat{\phi}_i(\omega, I_{10})$ are functions of the damage parameter ω , resulting from I_4 , and the cosine of the angle ψ , i.e. $\cos \psi = \cos \angle(\mathbf{a}, \mathbf{n}) = (a_I n_I)$, between the unit vectors $\mathbf{a}$ and $\mathbf{n}$, appearing in $I_{10} = \omega(a_I n_I)^2$. Invariance of the strain energy with respect to the orientation of the vectors $\mathbf{a}$ and $\mathbf{n}$ implies that $|\psi| = \pi/2$ and $\phi_i(\omega, I_{10}) = \bar{\phi}_i(\omega, \psi)$.

$$\begin{aligned}
\phi_i(\omega, I_{10}) &= \phi_{i00} + \phi_{i10} I_4 + \phi_{i20} I_4^2 + \dots \\
&+ \phi_{i01} I_{10} + \phi_{i11} I_4 I_{10} + \phi_{i21} I_4^2 I_{10} + \dots \\
&+ \phi_{i02} I_{10}^2 + \phi_{i12} I_4 I_{10}^2 + \phi_{i22} I_4^2 I_{10}^2 + \dots \\
&+ \dots + \phi_{imn} I_4^m I_{10}^n + \dots \\
&= \sum_n \sum_m \phi_{imn} I_4^m I_{10}^n
\end{aligned}$$

In addition to the five contributions occurring in eq. 3b another eight terms appear in the complete quadratic expansion of the potential function W with respect to the strains.

$$\begin{aligned}
W &= \phi_1 I_1^2 + \phi_2 I_2 + \phi_3 I_1 I_7 + \phi_4 I_7^2 + \phi_5 I_8 \\
&+ \phi_6 I_1 I_5 + \phi_7 I_5^2 + \phi_8 I_6 + \phi_9 I_5 I_7 + \phi_{10} I_1 I_9 + \phi_{11} I_5 I_9 + \phi_{12} I_7 I_9 + \phi_{13} I_9^2 \quad (8a)
\end{aligned}$$

Special Cases

Of interest for applications are the cases, when the vectors $\mathbf{a}$ and $\mathbf{n}$ are orthogonal or when they coincide, since cracks are either predominantly parallel or perpendicular to the fibers of reinforced resins such as glass-fiber vinylester composites. The second case easily follows from the well-known results of transverse isotropy with the elasticity constants depending on the damage parameter ω - see section 3.

In the case with $\mathbf{a}$ orthogonal to $\mathbf{n}$ the vanishing scalar product $\mathbf{a} \cdot \mathbf{n} = 0$ reduces the strain energy function in eq. 8a to the polynomial in eq. 8b. For notational simplicity the familiar coefficients λ_i and μ_i are used for the elasticity constants ϕ_1 to ϕ_5 of transverse isotropy. Their dependence on the damage parameter ω and the angle ψ between the fiber direction $\mathbf{a}$ and the damage normal $\mathbf{n}$ is not pursued further, since focus is only on the damaged induced anisotropy involving the invariants I_5 and I_6 here. The coefficient functions $\bar{\alpha}(\omega)$ to $\bar{\delta}(\omega)$ replace in eq. 8a the quantities ϕ_6 to ϕ_9 , which are fixed at the angle $\psi = \pi/2$.

$$\begin{aligned} W = & \frac{1}{2} \lambda_0 I_1^2 + 2\mu_0 I_2 + \lambda_1 I_1 I_7 + \mu_1 I_7^2 + \mu_2 I_8 \\ & + \bar{\alpha}(\omega) I_1 I_5 + \bar{\beta}(\omega) I_5^2 + \bar{\gamma}(\omega) I_6 + \bar{\delta}(\omega) I_5 I_7 \end{aligned} \quad (8b)$$

The stresses $\mathbf{S}$ follow as before:

$$\begin{aligned} \mathbf{S} = & \lambda_0 (\text{tr} \mathbf{E}) \mathbf{I} + 2\mu_0 \mathbf{E} \\ & + \lambda_1 [(\text{tr} \mathbf{E}) \mathbf{a} \otimes \mathbf{a} + (\mathbf{a} \mathbf{E} \mathbf{a}) \mathbf{I}] + 2\mu_1 (\mathbf{a} \mathbf{E} \mathbf{a}) \mathbf{a} \otimes \mathbf{a} + \mu_2 [\mathbf{E} \mathbf{a} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{E} \mathbf{a}] \\ & + \alpha(\omega) [(\text{tr} \mathbf{E}) \mathbf{n} \otimes \mathbf{n} + (\mathbf{n} \mathbf{E} \mathbf{n}) \mathbf{I}] + \beta(\omega) (\mathbf{n} \mathbf{E} \mathbf{n}) \mathbf{n} \otimes \mathbf{n} \\ & + \gamma(\omega) [\mathbf{E} \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{E} \mathbf{n}] + \delta(\omega) [(\mathbf{n} \mathbf{E} \mathbf{n}) \mathbf{a} \otimes \mathbf{a} + (\mathbf{a} \mathbf{E} \mathbf{a}) \mathbf{n} \otimes \mathbf{n}] , \end{aligned} \quad (9a)$$

where $\delta(\omega)$ is substituted for $\omega \bar{\delta}(\omega)$ and use has been made of the redefined coefficients employed in eq. 4a. The constitutive law in index notation becomes:

$$\begin{aligned} S_{IJ} = & \lambda_0 \delta_{IJ} E_{KK} + 2\mu_0 E_{IJ} \\ & + \lambda_1 [a_I a_J E_{KK} + \delta_{IJ} a_K a_L E_{KL}] + 2\mu_1 a_I a_J a_K a_L E_{KL} \\ & + \mu_2 [a_K a_J E_{IK} + a_I a_K E_{KJ}] \\ & + \alpha(\omega) [n_I n_J E_{KK} + \delta_{IJ} n_K n_L E_{KL}] + \beta(\omega) n_I n_J n_K n_L E_{KL} \\ & + \gamma(\omega) [n_K n_J E_{IK} + n_I n_K E_{KJ}] + \delta(\omega) [a_I a_J n_K n_L E_{KL} + n_I n_J a_K a_L E_{KL}] \end{aligned} \quad (9b)$$

The constitutive tensor D_{IJKL} for the initially transverse isotropic material with crack surfaces parallel to the fibers takes the form:

$$\begin{aligned} D_{IJKL} = & \lambda_0 \delta_{IJ} \delta_{KL} + 2\mu_0 \delta_{IK} \delta_{JL} \\ & + \lambda_1 [a_I a_J \delta_{KL} + \delta_{IJ} a_K a_L] + 2\mu_1 a_I a_J a_K a_L + \mu_2 [\delta_{IL} a_J a_K + a_I a_K \delta_{JL}] \\ & + \alpha(\omega) [n_I n_J \delta_{KL} + \delta_{IJ} n_K n_L] + \beta(\omega) n_I n_J n_K n_L \\ & + \gamma(\omega) [\delta_{IL} n_J n_K + n_I n_K \delta_{JL}] + \delta(\omega) [a_I a_J n_K n_L + n_I n_J a_K a_L] \end{aligned} \quad (10)$$

Remarks

- The constitutive tensor for the damaged material changes from the class of transverse isotropy to the one of orthotropic symmetry.
- The last term in eq. 10 contains the coefficient function $\delta(\omega)$, which enters the constitutive tensor of the damaged material in addition to the three contributions from $\alpha(\omega)$, $\beta(\omega)$ and $\gamma(\omega)$. These three damage dependent scalar quantities are the same as for the initially isotropic material in eqs. 5a and 5b. Their contributions are independent of the fiber orientation in contrast to the $\delta(\omega)$ -terms.
- The matrix entries of the damage dependent coefficients α to δ are shown in eq. 11 for the constitutive tensor in principal coordinates. For convenience the first two coordinate axes are chosen to coincide with the vectors $\mathbf{a}$ and $\mathbf{n}$ for fiber and damage directions.

$$\mathbf{a} = [1 \ 0 \ 0]^T \quad \text{and} \quad \mathbf{n} = [0 \ 1 \ 0]^T$$

The essence of the constitutive tensor, splitted into $\mathbf{D} = \mathbf{D}_o + \mathbf{D}_\omega$, becomes already apparent in the plane strain case with the third coordinate axis collinear to the degenerated direction. $\mathbf{D}_o$ comprises the well-known transverse isotropic part of the defect-free material, whereas $\mathbf{D}_\omega$ includes all the contributions of the damage dependent coefficients.

$$\mathbf{D}_\omega = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 2\alpha + \beta & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta & 0 & 0 \\ \delta & 2\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma}{2} \end{bmatrix} \quad (11)$$

The contribution of $\delta(\omega)$ to $\mathbf{D}_\omega$ allows for stiffness degradation due to damage in the "off-diagonal" components D_{1122} and D_{2211} . These are stiffness coefficients, which describe the influence of the Poisson effect in the 1-direction, caused by stretching in the 2-direction and vice versa. Similarly, α monitors the Poisson effect in fiber-direction and in the X_3 -direction due to strains parallel to the damage normal $\mathbf{n}$. Thus, the change of each Poisson parameter for the 1- and 3-direction due to the strain component E_{22} can be independently specified by appropriate choices of α and δ . The physical meaning of the coefficients β and γ is obvious.

5 Application to Unidirectional Laminae

Fiber-reinforced composite materials with cracks parallel to the fibers serve as an example to find specific functions for the damage dependent coefficients. Normal stresses acting transverse to the fibers and shear stresses are transmitted through both the matrix and the fiber materials in non-ductile uniaxially reinforced composite laminae such as glass-fiber reinforced vinylester-resins. The growth of defects mainly takes place in the matrix or in the fiber-matrix interface, causing debond. Usually the bond strength of the interface zone

between fibers and matrix is the lowest in comparison to the data for the strength of the single constituents. Advancing cracks in the matrix soon pass into the fiber-matrix interface and propagate along the fibers without crossing into the fiber material. The defects in the composite material are treated in the mathematical model as having the equivalent effect on the elastic properties as disk-like cracks would exert, if they are tangential to the fiber direction.

The selection of the functions $\alpha(\omega)$, $\beta(\omega)$, $\gamma(\omega)$ and $\delta(\omega)$ is motivated by classical continuum damage mechanics - see [8], [18] - of the one-dimensional model, where the area of the defects ($A_{void} = \omega A$) is related to the cross section area A through the damage scalar ω . The well-known relationship of linear elasticity in terms of effective stresses, $\hat{\sigma} = \sigma/(1 - \omega) = E\epsilon$, is carried over to the transverse elastic stiffness, $E_{22} = \hat{E}_{22}(\omega)$, in the two-dimensional case: $E_{22} = (1 - \omega)E_{\perp}$, with $E_{\perp}$ as the Young's modulus transverse to the fiber direction $\mathbf{a}$ of the undamaged material. In [14] a strong influence on the spacing of parallel cracks is found for the effective Young's modulus E_{22} in polymer and metal matrix composites.

Damage of the matrix material has a secondary influence on the effective elastic stiffness for tension in fiber direction, since the low stiffness of even undamaged resins contribute little to the stiffness of the composite $E_{\parallel}$ for practical reinforcement ratios. This leads to the assumption that the effective elastic stiffness parameter $E_{11} = \hat{E}_{11}(\omega)$ in fiber direction is independent of the damage, i.e. $E_{11} = E_{\parallel}$. The same conclusions are drawn in [14].

Similar to the one dimensional case the shear stress can only be transmitted through the integral portion of the cross section area ($A - A_{void}$) in the representative volume element. However, rough crack surfaces are assumed, since friction is present and the disk-like cracks may also be penetrated by fibers. Therefore, the calculated loss areas for the normal and shear stresses differ and it is reasonable to introduce the damage parameter $\omega_s = \hat{\omega}_s(\omega)$ for the degradation of the shear stiffness as an dependent variable of ω . By denoting the shear stiffness of the damaged and undamaged material as $G = \hat{G}(\omega)$ and G^o , it follows: $G = (1 - \omega_s)G^o$.

For the purpose of building the dependence on damage into multiaxial constitutive laws the compliance relationship, $\mathbf{H}(\omega)$, is more easily accessible in order to relate the material parameters to the mechanical response in the privileged axes. The compliance tensor has the same general algebraic form as the stiffness tensor in eq. 10. Use can be made of the previous relationships for the physical more meaningful material coefficients E_{11} , E_{22} , G and the yet to specify functions $h_{12}(\omega)$ and $h_{21}(\omega)$ for the major as well as the minor Poisson's ratios: $\nu_{12} = \hat{\nu}_{12}(\omega) = h_{12}(\omega) \nu_{12}^o$ and $\nu_{21} = \hat{\nu}_{21}(\omega) = h_{21}(\omega) \nu_{21}^o$ with ν_{12}^o and ν_{21}^o as the Poisson numbers of the undamaged composite and $\nu_{12} \nu_{21} < 1$ to assure positive definiteness of the compliance tensor. The plane stress condition is supposed to hold for the lamina model.

$$\mathbf{H}(\omega) = \begin{bmatrix} \frac{1}{E_{\parallel}} & \frac{-h_{21}(\omega) \nu_{21}^o}{E_{\parallel}} & 0 \\ \frac{-h_{12}(\omega) \nu_{12}^o}{(1-\omega)E_{\perp}} & \frac{1}{(1-\omega)E_{\perp}} & 0 \\ 0 & 0 & \frac{1}{(1-\omega_s)G^o} \end{bmatrix} \quad (12)$$

Quantitative assessments for the degradation of Poisson's ratio were made in [3], [7], [14], [23], [15] and [5] among others. The minor Poisson effect does not significantly change with varying crack spacings - see [14]. Thus, $\hat{\nu}_{21}(\omega)$ is assumed to be constant with respect to ω and $h_{21}(\omega) = 1$.

A thermodynamic potential is assumed to exist, if the deformation is elastic without changing the damage state under some loading conditions. Consequently, the equality $\frac{\nu_{12}^o}{E_{\perp}} = \frac{\nu_{21}^o}{E_{\parallel}}$ must remain valid not only for the virgin material - and in fact it does for most applications, but also for the elasticity moduli of the damaged material: $\frac{\nu_{12}}{E_{22}} = \frac{\nu_{21}}{E_{11}}$. The necessary major symmetry of the constitutive tensor is regained, if $h_{12}(\omega) = 1 - \omega$. As ω approaches one, the transverse stress tends to zero and the Poisson effect resulting from the (1,2)-matrix entry in eq. 12 has to vanish. Then the mechanical response of the lamina resembles the behavior of a "loose bundle of fibers" as the parameter ω indicates complete damage due to parallel cracks in the resin. Such strong dependence on the spacing of parallel cracks is reported in [14] for the major Poisson's ratio. The linear degradation of the effective major Poisson's ratio, $\hat{\nu}_{12} = (1 - \omega)\nu_{12}^o$, with ω is at least in qualitative agreement with the result of the cited authors, since no explicit relation between the geometry of the crack arrays and the phenomenological damage variable ω has been established. Under the aforementioned presuppositions the compliance tensor finally takes the form:

$$\mathbf{H}(\omega) = \begin{bmatrix} \frac{1}{E_{\parallel}} & -\frac{\nu_{21}^o}{E_{\parallel}} & 0 \\ -\frac{\nu_{12}^o}{E_{\perp}} & \frac{1}{(1-\omega)E_{\perp}} & 0 \\ 0 & 0 & \frac{1}{(1-\omega_s)G^o} \end{bmatrix}$$

as also proposed in [16]. The stiffness tensor follows from the inverse, which always exists as long as the damage variables are less than one.

$$\mathbf{C}(\omega) = \frac{1}{D} \begin{bmatrix} E_{\parallel} & (1 - \omega)\nu_{21}^o E_{\perp} & 0 \\ (1 - \omega)\nu_{12}^o E_{\parallel} & (1 - \omega)E_{\perp} & 0 \\ 0 & 0 & D(1 - \omega_s)G^o \end{bmatrix} \quad (13)$$

with $D = 1 - (1 - \omega)\nu_{12}^o \nu_{21}^o > 0$.

6 Conclusions and Summary

In combination with a proper invariance argument and a complete polynomial expansion of the strain energy function it is shown that the vector representation of the local damage state allows a linear as well as a nonlinear dependence of the constitutive tensor on the damage variable ω . By introducing an additional scalar variable ω_s it is possible to account for the effect of rough crack surfaces under shear loading on the effective elastic properties. This implies that the symmetric second order damage tensors of the form discussed do not offer more generality for continuum damage models.

It is demonstrated that a suitable choice of the damage dependent coefficient functions is possible for the derived constitutive law in order to yield a physically reasonable stiffness

matrix for the approximation of damaged laminae by means of a homogenized continuum. It shows that the uniaxial continuum damage model introduced by Kachanov and Rabotnov is a special case of the more general development outlined above.

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