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Combinatorics of polytopes, orthogonal polynomials, and Markov chains

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Combinatorics of polytopes, orthogonal polynomials, and Markov chains

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Abstract

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Doctor of Philosophy in Mathematics

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We study various combinatorial formulas arising in the asymmetric exclusion process, orthogonal polynomials, and Ehrhart theory. In particular, we give combinatorial formulas for polynomials with positive coefficients. Explaining the positivity of such polynomials is an interesting problem by itself, and giving combinatorial formulas is useful as they provide a fast and compact way to compute those polynomials.

The asymmetric exclusion process (ASEP) is an important model from statistical mechanics which describes a system of particles on a lattice hopping left and right. This process was introduced in the 1970s independently in the context of biology and mathematics. Since then, this model has many variants and was studied extensively in various fields. The ASEP on a line is a Markov chain on a one- dimensional lattice of length N with open boundaries. A particle can hop to the right with the rate 1 and can hop to the left with rate q, as long as the neighboring site is empty. And on each boundary, a particle can enter from the left or right with rate α or δ respectively and can exit to the left or right with rate γ or β respectively. Papers of Sasamoto [32] and subsequently, Uchiyama, Sasamoto, and Wadati [36] revealed a surprising connection between the ASEP on a line and orthogonal polynomials, in particular the Askey-Wilson polynomials which lie in the top hierarchy of (basic) hypergeometric orthogonal polynomials in the sense that all other polynomials in this hierarchy are limiting cases or specializations of the Askey-Wilson polynomials. In Chapter 2, we give combinatorial formulas for the Al-Salam-Chihara polynomials, which are related to the ASEP when $\gamma = \delta = 0$. The totally asymmetric exclusion process (TASEP) on a ring is a Markov chain on a periodic one-dimensional lattice of length N where each lattice site can be either occupied by a particle or empty. A particle can hop to its right (when it is empty) with the rate 1. In Chapter 3, we study the inhomogeneous version of the TASEP and show that many steady-state probabilities are proportional to the product of Schubert polynomials.

In the 1960s, Ehrhart introduced Ehrhart polynomials and Ehrhart series to study the num-

ber of lattice points inside polytopes. Since then, there has been a lot of study on Ehrhart polynomials and Ehrhart series of many well-known polytopes. The (k, n)*-th hypersimplex* $\Delta_{k,n}$ is a lattice polytope inside $\mathbb{R}^n$ whose vertices are (0,1)-vectors with exactly k 1's. The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of GL_n. In Chapter4, we prove the first combinatorial formula for the Ehrhart series of the hypersimplex, proving a conjecture of Early [17].

Dedicated to my family.

Contents

Chapter 1

Introduction

We summarize the main results.

1.1 Chapter 2: Combinatorial formulas for the coefficients of the Al-Salam-Chihara polynomials

We say that a family $(p_n(x))_{n\geq 0}$ of polynomials in one variable is *orthogonal* if the degree of $p_n(x)$ is n and they are orthogonal with respect to a certain measure ω, that is

$$\int p_n(x)p_m(x)d\omega = 0, \text{ for } m \neq n \ .$$

And the *N-th moments* μ_N of $(p_n(x))_{n\geq 0}$ are defined as $\mu_N = \int x^N d\omega$, for $N \geq 0$. The most widely used orthogonal polynomials are hypergeometric orthogonal polynomials, which include the Hermite polynomials, the Laguerre polynomials, and the Jacobi polynomials.

The Askey-Wilson polynomials are a family of orthogonal polynomials which include many of the other orthogonal polynomials as special or limiting cases. They lie at the top of the hierarchy in the Askey scheme. Surprisingly, in [36], the moments of this family are connected to the steady state probability of the ASEP on a line. Later in [12], Corteel and Williams gave a combinatorial formula for the steady state probabilities in particular showing that the moments of the Askey-Wilson polynomials are polynomials in $\alpha, \beta, \gamma, \delta$ and q with positive coefficients. Conjecturally, the coefficients of Askey-Wilson polynomials are polynomials in $\alpha, \beta, \gamma, \delta$ and q with positive coefficients.

In [24], we studied the coefficients of the Al-Salam-Chihara polynomials, which are obtained as the specialization of the Askey-Wilson polynomials at $\gamma = \delta = 0$. We gave a combinatorial formula for those coefficients, which explains positivity in the case $\gamma = \delta = 0$. To do this, we introduced a generalized q-binomial coefficient $M_n^{\mu}(b)$. Here n and b are non-negative integers and μ is a weakly increasing composition of length a. It is a polynomial in q and α that recovers ordinary q binomial coefficient $\binom{n+a+b}{b}_q$ when $\alpha = 0$. The construction of $M_n^{\mu}(b)$ was motivated by our bijective proof of the identity $\binom{n+a}{a}_q \binom{n+a+b}{b}_q = \binom{n+a+b}{a}_q \binom{n+b}{b}_q$.

We also showed that the well-known identity (1.1.1)

$$\binom{n+a+b+1}{b}_q = q^{n+a+1}\binom{n+a+b}{b-1}_q + \binom{n+a+b}{b}_q \tag{1.1.1}$$

lifts to (1.1.2)

$$M^{\mu}_{n+1}(b) = (q^{n+a+b} + [n+a+b]_q\alpha)M^{\mu}_{n+1}(b-1) + M^{\mu-1}_{n}(b). \tag{1.1.2}$$

So far, there seems to be no previous work that can be related to a generalized q-binomial coefficient. It would be interesting to study this and find more applications. Our main result in [24] expresses the coefficients of the (transformed) Al-Salam-Chihara polynomials in terms of the generalized q-binomial coefficients.

Theorem 1.1.1 ([24])**.** *The coefficient of x^n in the (transformed) Al-Salam-Chiara polynomial $\tilde{p}_{n+k}(x)$ is given by*

$$[x^n]\hat{p}_{n+k}(x) = \sum_{a+b=k}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\ 0\le\mu_1\le\cdots\le\mu_a\le n}} (\xi\alpha)^a\beta^b X_\mu M^{\mu}_{n}(b)\Big),$$

where $X_\mu = \prod_{i=1}^{a}(q^{\mu_i+i-1} + [\mu_i+i-1]_q\beta)$.

Theorem 1.1.1 makes clear that the coefficients of the (transformed) Al-Salam-Chihara polynomials are polynomials with positive coefficients.

It is an open problem to understand the minors of the coefficient matrix $G = (g_{n,i})_{n,i}$ where $g_{n,i}$ is the coefficient of x^i of the Askey-Wilson polynomial $p_n(x)$ if $i \le n$, otherwise zero.

$$G = \begin{bmatrix} 1 & g_{1,0} & g_{2,0} & g_{3,0}\cdots \\ 0 & 1 & g_{2,1} & g_{3,1}\cdots \\ 0 & 0 & 1 & g_{3,2}\cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The minors of G are also conjectured to be polynomials with positive coefficients. Unfortunately, techniques showing a positivity of minors in the literature (for example, Lindström-Gessel-Viennot lemma) do not apply to the coefficient matrix G. In [10], a combinatorial formula for some minors was given by relating it to the stationary distributions of 2-species ASEP. And in [24], we proved the positivity for some 2 by 2 minors when $\gamma = \delta = 0$ using generalized q-binomial coefficients.

Theorem 1.1.2 ([24])**.** *Let $g_{n+k,n}$ be the coefficient of x^n of the (transformed) Al-Salam-Chihara polynomial $\tilde{p}_{n+k}(x)$. Then*

$$(g_{n+a+b,n+a}g_{n+a,n} - g_{n+a+b,n})$$

is a polynomial with positive coefficients.

1.2 Chapter 3: Schubert polynomials and the inhomogeneous TASEP on a ring

The inhomogeneous multispecies TASEP on a ring is a Markov chain on a periodic lattice of length N where each lattice site is occupied by positive integers. This Markov chain is indexed by $m = (m_1, m_2, \cdots)$ such that $\sum m_i = N$ where m_i is the number of i's on a lattice. There are a total of $\binom{N}{m_1, m_2, \cdots}$ possible states in this case. The adjacent integers i and j (i is on the left of j) can swap their positions with a rate $r_{i,j}$ given as follows

$$r_{i,j} = \begin{cases} x_i & \text{if } i < j \\ 0 & \text{otherwise} \end{cases} .$$

When each $m_i = 1$, the possible states are permutations. In [28], Lam and Williams conjec-

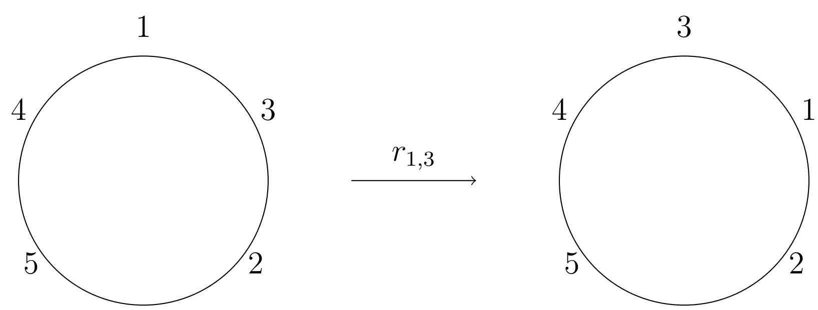

Figure 1.1: The figure shows a transitions rate of the inhomogeneous multispecies TASEP on a ring.

tured that each steady state probability is proportional to a positive linear combination of Schubert polynomials. Subsequently, a combinatorial formula for the steady state probability was given in terms of objects called multiline queues [19, 3, 1]. In particular, the steady state probability for the state w is given as a weighted sum over multiline queues of type w. However, the conjecture about Schubert polynomials remained open.

In [6], Cantini introduced a z-deformed steady state probability for the spectral parameters $z_1, z_2, \cdots$ which recovers the (usual) steady state probability when specialized to $z_i = \infty$. He gave an explicit formula for z-deformed steady state probability of a few states, explaining the appearance of Schubert polynomials in those cases.

In a joint with Williams [25], we introduced a special subset $\mathrm{St}(n,k)$ of S_n and gave an explicit formula for z-deformed steady state probabilities of $w \in \mathrm{St}(n,k)$ thereby generalizing Cantini's work. As a corollary we have the following statement.

Theorem 1.2.1 ([25]). *For $w \in \mathrm{St}(n,k)$, the steady state probability of the state w is proportional to a product of k Schubert polynomials.*

1.3 Chapter4: A combinatorial formula for the Ehrhart h^*-vector of the hypersimplex

For an n-dimensional lattice polytope $\mathcal{P} \subset \mathbb{R}^N$, it is well known from Ehrhart theory that the map $r \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n, which we call *Ehrhart polynomial*, and the corresponding *Ehrhart series* $\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r$ is a rational function of the form

$$\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r = \frac{h^*(t)}{(1-t)^{n+1}},$$

such that $h^*(t)$ is a polynomial of degree $\leq n$ (see [34]). Define h_d^* to be the coefficient of t^d in $h^*(t)$. The vector $(h_0^*, \cdots, h_n^*)$ is called the Ehrhart h^*-vector of $\mathcal{P}$. In [33], Stanley proved that the h^*-vector of a lattice polytope always consists of non-negative integers, so it became an interesting question to find a combinatorial interpretation of h^*-vectors for various polytopes.

In [17], Early conjectured a combinatorial interpretation of the h^*-vector of the hypersimplex $\Delta_{k,n}$ in terms of decorated ordered set partitions. A *decorated ordered set partition* $((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ of type (k,n) consists of an ordered partition $(L_1, \cdots, L_m)$ of $\{1, 2, \ldots, n\}$ and an m-tuple $(l_1, \cdots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \geq 1$. A decorated ordered set partition is called *hypersimplicial* if it satisfies $1 \leq l_i \leq |L_i| - 1$ for all i, and one can define a natural statistic called *winding number* for them. In [23], we proved the conjecture of Early.

Theorem 1.3.1 ([23])**.** *The h^*-vector $(h_0^*, h_1^*, \cdots)$ of the hypersimplex $\Delta_{k,n}$ has the property that h_d^* equals the number of hypersimplicial decorated ordered set partitions of type (k,n) with winding number d.*

Chapter 2

Combinatorial formulas for the coefficients of the Al-Salam-Chihara polynomials

The results of this chapter are based on [24].

2.1 Introduction

In the last few decades, there has been a lot of work on finding combinatorial formulas for moments of orthogonal polynomials (see [9], [11],[12], [13], [14], [36]), particularly when they are polynomials with positive coefficients. The Al-Salam-Chihara polynomials are an important class of orthogonal polynomials in one variable x which are connected to a model from statistical mechanics called the partially asymmetric simple exclusion process (PASEP). There have been some works on the combinatorics of the Al-Salam-Chihara polynomials (see [21]); this work has focused on the moments of the Al-Salam-Chihara polynomials, not the coefficients, as the coefficients fail to be positive polynomials. In this paper, we introduce the *transformed* Al-Salam-Chihara polynomials, which do have positive coefficients and give two manifestly positive combinatorial formulas for the coefficients.

Orthogonal polynomials in one variable $(p_n(x))_{n\geq 0}$ are a family of polynomials such that the degree of $p_n(x)$ is n and are orthogonal with respect to a certain measure ω, that is

$$\int p_n(x)p_m(x)d\omega = 0, \text{ for } m \neq n \ .$$

Monic orthogonal polynomials can be also defined by a three-term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x),$$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$ and where $(b_n)_{n\geq 0}$ and $(\lambda_n)_{n\geq 1}$ are constants (see [18]). We call $(b_n)_{n\geq 0}$ and $(\lambda_n)_{n\geq 1}$ the *structure constants* of $(p_n(x))_{n\geq 0}$. The *N-th moments* μ_N of $(p_n(x))_{n\geq 0}$ are defined as $\mu_N = \int x^N d\omega$, for $N \geq 0$.

The (monic) Al-Salam-Chihara polynomials are orthogonal polynomials with three free parameters (a, b, q). They are in the basic Askey scheme (see [2]) and can be obtained as the specialization of the Askey-Wilson polynomials at $c = d = 0$. The Al-Salam-Chihara polynomials may be defined by the following three-term recurrence relation (see [27])

$$\begin{aligned}
p_{n+1}(x) &= (x - b_n)p_n(x) - \lambda_n p_{n-1}(x)\\
b_n &= \frac{(a+b)q^n}{2}\\
\lambda_n &= \frac{(1-q^n)(1-abq^{n-1})}{4}.
\end{aligned}$$

Surprisingly, the moments of the Al-Salam Chihara polynomials are connected to a model from statistical mechanics called the partially asymmetric simple exclusion process (PASEP) (see [32], [36]). The PASEP is a model of interacting particles hopping left and right on a one-dimensional lattice of N sites. Each site can be either occupied by a particle or empty and transition rates between states are proportional to α, β and q (see Figure 2.1). Then the partition function Z_N of the PASEP can be written in terms of moments of the Al-Salam-Chihara polynomials (see [36], Section 6.1) as follows

$$Z_N = \sum_{k=0}^{N} \binom{N}{k} (\frac{2\alpha\beta}{1-q})^N \mu_{N-k}, \tag{2.1.1}$$

using the change of variables

$$a = \frac{1-q-\alpha}{\alpha}, \quad b = \frac{1-q-\beta}{\beta}. \tag{2.1.2}$$

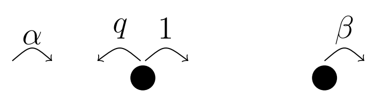

Figure 2.1: The figure shows transition rates of the PASEP

Motivated by the connection (2.1.1) with the PASEP, we consider a family $(p'_n(x))_{n\geq 0}$ of polynomials where

$$p'_n(x) = (\frac{2\alpha\beta}{q-1})^n p_n(\frac{q-1}{2\alpha\beta}x - 1),$$

using the change of variables (2.1.2). Then the N-th moment of $(p'_n(x))_{n\geq 0}$ becomes $(-1)^N Z_N$, and we have the following three-term recurrence relation

$$\begin{aligned}
p'_{n+1}(x) &= (x + b_n)p'_n(x) - \lambda_n p'_{n-1}(x) \qquad (2.1.3)\\
b_n &= (\alpha+\beta)q^n + 2\alpha\beta[n]_q\\
\lambda_n &= (\alpha\beta)^2[n]_q[n-1]_q + \alpha\beta(\alpha+\beta)q^{n-1}[n]_q + \alpha\beta(q^{2n-1} - q^{n-1}).
\end{aligned}$$

In [3], a combinatorial formula for Z_N was given in terms of permutation tableaux, showing in particular that it is a polynomial in α, β and q with positive coefficients.

Computing $p'_n(x)$ for small n we have

$$\begin{aligned}
p'_0(x) =&1\\
p'_1(x) =&x + (\alpha + \beta)\\
p'_2(x) =&x^2 + (\alpha + \beta + \alpha q + \beta q + 2\alpha\beta)x + (\alpha^2 q + \beta^2 q + \alpha\beta + \alpha\beta q + \alpha\beta^2 + \alpha^2\beta)\\
p'_3(x) =&x^3 + (\alpha + \beta + \alpha q + \beta q + \alpha q^2 + \beta q^2 + 4\alpha\beta + 2\alpha\beta q)x^2\\
&+ (\alpha\beta + 3\alpha^2\beta + 3\alpha\beta^2 + 3\alpha^2\beta^2 + \alpha^2 q + 2\alpha\beta q + 3\alpha^2\beta q + \beta^2 q + 3\alpha\beta^2 q + 3\alpha^2\beta^2 q\\
&+ \alpha^2 q^2 + 2\alpha\beta q^2 + 3\alpha^2\beta q^2 + \beta^2 q^2 + 3\alpha\beta^2 q^2 + \alpha^2 q^3 + \alpha\beta q^3 + \beta^2 q^3)x + (2\alpha^2\beta^2\\
&+ \alpha^3\beta^2 + \alpha^2\beta^3 + \alpha^2\beta q + \alpha^3\beta q + \alpha\beta^2 q + 2\alpha^2\beta^2 q + \alpha^3\beta^2 q + \alpha\beta^3 q + \alpha^2\beta^3 q + \alpha^2\beta q^2\\
&+ 2\alpha^3\beta q^2 + \alpha\beta^2 q^2 + 2\alpha^2\beta^2 q^2 + 2\alpha\beta^3 q^2 + \alpha^3 q^3 + \alpha^2\beta q^3 + \alpha\beta^2 q^3 + \beta^3 q^3).
\end{aligned}$$

Note that it is not obvious from the recurrence (2.1.3) that the coefficients are polynomials in α, β and q with positive coefficients.

It is worth noting that specializing to $q = 1$, the polynomial $p'_n(x)$ becomes

$$p'_n(x) = \sum_{k=0}^{n} (\binom{n}{k} \prod_{i=n-k}^{n-1} (\alpha + \beta + i\alpha\beta))x^{n-k},$$

which can be easily proved by induction. The coefficients of $p'_n(x)$ have a nice factorization formula in this case; however they do not factorize in general.

In this paper we will give two different combinatorial formulas for these coefficients making manifest that they are polynomials in α, β and q with positive coefficients. To do this we introduce the following more general orthogonal polynomials.

Definition 2.1.1. The *transformed Al-Salam-Chihara* polynomials $(\hat{p}_n(x))_{n\geq 0}$ are the family of orthogonal polynomials in one variable x depending on parameters α, β, ϵ_1, ϵ_2 and q defined by the following three-term recurrence relation

$$\begin{aligned}
\hat{p}_{n+1}(x) &= (x + b_n)\hat{p}_n(x) - \lambda_n \hat{p}_{n-1}(x) \qquad (2.1.4)\\
b_n &= (\alpha + \beta)q^n + (\epsilon_1 + \epsilon_2)[n]_q\\
\lambda_n &= \epsilon_1\epsilon_2[n]_q[n-1]_q + (\alpha\epsilon_2 + \beta\epsilon_1)q^{n-1}[n]_q + \alpha\beta(q^{2n-1} - q^{n-1}).
\end{aligned}$$

Remark 2.1.2. The connection with the PASEP provided some inspiration for Definition 2.1.1. In particular, there is a 1-parameter generalization of the partition function Z_N called the fugacity partition function $Z_N(\xi)$ where ξ is a variable keeping track of the number of particles for each state. This connection leads to the following family of orthogonal polynomials defined by the three-term recurrence relation

$$\begin{aligned}
p''_{n+1}(x) &= (x + b_n)p''_n(x) - \lambda_n p''_{n-1}(x) \qquad (2.1.5)\\
b_n &= (\xi\alpha + \beta)q^n + (1 + \xi)\alpha\beta[n]_q\\
\lambda_n &= \xi(\alpha\beta)^2[n]_q[n-1]_q + \xi\alpha\beta(\alpha + \beta)q^{n-1}[n]_q + \xi\alpha\beta(q^{2n-1} - q^{n-1}),
\end{aligned}$$

which is a ξ-analogue of (2.1.3) (see [15]). We can recover (2.1.5) from the more general setting of (2.1.4) by plugging $\alpha \to \xi\alpha$, $\epsilon_1 \to \xi\alpha\beta$ and $\epsilon_2 \to \alpha\beta$.

Remark 2.1.3. The coefficients $[x^n]p_{n+k}(x)$ of the Al-Salam-Chihara polynomials and the coefficients $[x^n]\hat{p}_{n+k}(x)$ of the transformed Al-Salam-Chihara polynomials are connected as follows

$$[x^n]p_{n+k}(x) = \sum_{i=0}^{k} \binom{n+i}{n} \left(\frac{q-1}{2\alpha\beta}\right)^{k-i} ([x^{n+i}]\hat{p}_{n+k}(x)),$$

where $\epsilon_1 = \epsilon_2 = \alpha\beta$, $a = \frac{1-q-\alpha}{\alpha}$ and $b = \frac{1-q-\beta}{\beta}$.

The rest of the paper studies the transformed Al-Salam-Chihara polynomials from Definition 2.1.1. We give two formulas for the coefficient $g_{n+k,n}$ of x^n in $\hat{p}_{n+k}(x)$. Our two formulas represent $g_{n+k,n}$ as polynomials in $X_i = \alpha q^i + \epsilon_1[i]_q$ and $Y_i = \beta q^i + \epsilon_2[i]_q$ (see (2.2.1)) where the coefficients lie in $\mathbb{Z}[q]$. For example, by our first result (Theorem 2.2.3) we have

$$g_{3,1} = \Big(\sum_{0\le i<j\le 2} X_iX_j\Big) + \Big(X_0Y_1 + Y_0X_2 + X_1Y_2 + \binom{3}{1}_q X_0Y_0\Big) + \Big(\sum_{0\le i<j\le 2} Y_iY_j\Big), \tag{2.1.6}$$

and by our second result (Theorem 2.2.8) we have

$$g_{3,1} = \Big(\sum_{0\le i<j\le 2} X_iX_j\Big) + \big(X_0Y_0 + X_0Y_1 + q^2X_0Y_0 + X_1Y_0 + X_1Y_1 + qX_1Y_1\big) + \Big(\sum_{0\le i<j\le 2} Y_iY_j\Big). \tag{2.1.7}$$

Note that (2.1.7) is invariant as a polynomial in X_i's and Y_i's under the exchange $X_i \leftrightarrow Y_i$. This is the case in general for our second formula and will be explained in Remark 2.4.6. The first formula, however, is not invariant as a polynomial in X_i's and Y_i's under the exchange $X_i \leftrightarrow Y_i$ as one can see from (2.1.6). So far, it is not clear how these two formulas are connected.

The structure of this paper is as follows. In Section 2, we will state the main results of this paper with examples. In Section 3, we will prove our first result (Theorem 2.2.3). In Section 4, we will prove our second result (Theorem 2.2.8). In Section 5, we will prove Theorem 2.2.14 which is a partial result of the conjecture regarding the minors of the matrix of coefficients $G = (g_{n,i})_{n,i}$.

Acknowledgments: The author is thankful to his advisor Lauren Williams for mentorship, valuable comments, and helping me revise the draft. The author would also like to thank Sylvie Corteel for her helpful comments and explanations.

2.2 Main Results

This section states the main results of this paper with examples. Throughout this section, we set

$$X_i = \alpha q^i + \epsilon_1[i]_q, \quad Y_i = \beta q^i + \epsilon_2[i]_q \tag{2.2.1}$$

for $i \ge 0$.

The first formula for the coefficients of $\hat{p}_n(x)$

Definition 2.2.1. Define a sequence Z_n for $n \geq 0$ by

$$Z_n = \begin{cases} X_{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is even} \\ Y_{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is odd} \end{cases}.$$

For a partition (weakly decreasing sequence of non-negative integers) $\mu = (\mu_1, \cdots, \mu_l)$, we define

$$s_m(\mu) = \begin{cases} min(|\{i|\mu_i = 0\}|, |\{i|\mu_i = 2m+1\}|) & \text{if } \mu_1 = 2m+1 \\ 0 & \text{otherwise} \end{cases}.$$

Denoting $k = s_m(\mu)$, we define a weight $u_m(\mu)$ to be

$$u_m(\mu) = (\prod_{i=1}^{l-k} Z_{\mu_{l+1-i}+2(i-1)})(\binom{m+l}{k}_q Y_0 \cdots Y_{k-1}).$$

Example 2.2.2. For $m = 1$, consider a partition $\mu = (3,3,1,0)$. Then $s_1((3,3,1,0)) = min(|\{i|\mu_i = 0\}| = 1, |\{i|\mu_i = 2m+1\}| = 2) = 1$, so we have

$$u_1((3,3,1,0)) = Z_0 Z_{1+2} Z_{3+4} \binom{5}{1}_q Y_0 = X_0 Y_1 Y_3 \binom{5}{1}_q Y_0.$$

For a partition $\mu = (3,3,0,0)$, we have $s_1((3,3,0,0)) = min(|\{i|\mu_i = 0\}| = 2, |\{i|\mu_i = 2m+1\}| = 2) = 2$, so this gives

$$u_1((3,3,0,0) = Z_0 Z_{0+2} \binom{5}{2}_q Y_0 Y_1 = X_0 X_1 \binom{5}{2}_q Y_0 Y_1.$$

Theorem 2.2.3. *The coefficient $g_{n+k,n}$ of x^n in $\hat{p}_{n+k}(x)$ is given by*

$$g_{n+k,n} = \sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu),$$

ie. it is the weighted sum over all Young diagrams contained in a $(k) \times (2n+1)$ rectangle, where the weight is given by Definition 2.2.1.

Example 2.2.4. By Theorem 2.2.3, we have

$$g_{k,0} = \sum_{\mu \subseteq (k) \times (1)} u_0(\mu) = \sum_{i=0}^{k} u_0((1^{k-i}, 0^i)).$$

If $k - i \leq i$, then

$$u_0((1^{k-i}, 0^i)) = X_0 \cdots X_{i-1} \binom{k}{k-i}_q Y_0 \cdots Y_{k-i-1} = X_0 \cdots X_{i-1} \binom{k}{i}_q Y_0 \cdots Y_{k-i-1}.$$

If $k-i>i$, then

$$u_0((1^{k-i},0^i)) = X_0\cdots X_{i-1}Y_i\cdots Y_{k-i-1}\binom{k}{i}_q Y_0\cdots Y_{i-1} = X_0\cdots X_{i-1}\binom{k}{i}_q Y_0\cdots Y_{k-i-1}.$$

In both cases, we have $u_0((1^{k-i},0^i)) = X_0\cdots X_{i-1}\binom{k}{i}_q Y_0\cdots Y_{k-i-1}$. Thus we have

$$g_{k,0} = \sum_{i=0}^{k}\binom{k}{i}_q X_0\cdots X_{i-1}Y_0\cdots Y_{k-i-1}. \tag{2.2.2}$$

Example 2.2.5. By Theorem 2.2.3, we have

$$\begin{aligned}
g_{3,1} = &\sum_{\mu\subseteq(2)\times(3)} u_1(\mu)\\
=&u_1((0,0))+u_1((1,0))+u_1((2,0))+u_1((3,0))+u_1((1,1))\\
&+u_1((2,1))+u_1((3,1))+u_1((2,2))+u_1((3,2))+u_1((3,3))\\
=&X_0X_1+X_0Y_1+X_0X_2+\binom{3}{1}_q X_0Y_0+Y_0Y_1\\
&+Y_0X_2+Y_0Y_2+X_1X_2+X_1Y_2+Y_1Y_2\\
=&(\sum_{0\le i<j\le 2}X_iX_j)+(X_0Y_1+Y_0X_2+X_1Y_2+\binom{3}{1}_q X_0Y_0)+(\sum_{0\le i<j\le 2}Y_iY_j).
\end{aligned}$$

The second formula for the coefficients of $\hat{p}_n(x)$

Definition 2.2.6. For a set S with integer elements, we define $S(k)$ to be the k-th smallest element of the set $(\{0\}\cup\mathbb{N})-S$. For example, when $S=\{1,4,7\}$, we have $S(1)=0$, $S(2)=2$, $S(3)=3$ and $S(4)=5$. We also define λ_S to be $(i_1,i_2-1,\cdots,i_s-s+1)$ where $S=\{i_1<\cdots<i_s\}$.

Definition 2.2.7. For a set $A=\{i_1<\cdots<i_a\}\subseteq\{0,\cdots,n+a-1\}$ and a set $B\subseteq\{0,\cdots,n+a+b-1\}$ with $|B|=b$, denoting $B\cap\{n+b,\cdots,n+b+a-1\}=\{n+b+a-j_k<\cdots<n+b+a-j_1\}$ and $\mu=\lambda_A=(i_1,i_2-1,\cdots,i_a-a+1)$, we define a weight $w_n(A,B)$ to be

$$w_n(A,B) = (\prod_{i\in A}X_i)(\prod_{i\in B\cap\{0,\cdots,n+b-1\}}Y_i)(\prod_{l=1}^{k}(q^{(n+b+a-j_l)-B(\mu_{j_l}+l)}Y_{B(\mu_{j_l}+l)})).$$

Definition 2.2.7 is motivated by the bijective proof of the simple identity

$$q^{\binom{a}{2}}\binom{n+a}{a}_q q^{\binom{b}{2}}\binom{n+a+b}{b}_q = q^{\binom{a}{2}}\binom{n+a+b}{a}_q q^{\binom{b}{2}}\binom{n+b}{b}_q \tag{2.2.3}$$

which will be given in Section 2.4.

Theorem 2.2.8. *The coefficient $g_{n+k,n}$ of x^n in $\hat{p}_{n+k}(x)$ is given by*

$$g_{n+k,n} = \sum_{a+b=k} \Big(\sum_{\substack{A\subseteq\{0,\cdots,(n+a-1)\} \\ |A|=a}} \Big(\sum_{\substack{B\subseteq\{0,\cdots,(n+a+b-1)\} \\ |B|=b}} w_n(A,B) \Big)\Big),$$

ie. it is the weighted sum over all pairs (A,B) such that $A \subseteq \{0,\cdots,(n+a-1)\}$ with $|A| = a$ and $B \subseteq \{0,\cdots,(n+a+b-1)\}$ with $|B| = b$, where the weight is given by Definition 2.2.7.

Example 2.2.9. We compute $w_0(A,B)$ as follows. Since $n = 0$ there is only one possible choice for A which is $\{0,\cdots,a-1\}$, so $\mu = \lambda_A = (0,\cdots,0)$. Suppose there are k elements in a set $B \cap \{b,\cdots,b+a-1\}$ then these elements will change to elements in $(\{0,\cdots,b-1\} - B)$. So we have $w_0(A,B) = X_0 \cdots X_{a-1}(q^{\sum B - \binom{b}{2}})Y_0\cdots Y_{b-1}$. Thus we have

$$\begin{aligned} g_{k,0} &= \sum_{a+b=k} \Big(\sum_{\substack{B\subseteq\{0,\cdots,a+b-1\} \\ |B|=b}} X_0\cdots X_{a-1}(q^{\sum B - \binom{b}{2}})Y_0\cdots Y_{b-1}\Big) \\ &= \sum_{a+b=k} \binom{a+b}{b}_q X_0\cdots X_{a-1}Y_0\cdots Y_{b-1}. \end{aligned}$$

In this case the formula is identical to (2.2.2).

Example 2.2.10. By Theorem 2.2.8, we have

$$\begin{aligned} g_{3,1} &= \sum_{\substack{B\subseteq\{0,1,2\} \\ |B|=2}} w_1(\phi,B) + \sum_{\substack{A\subseteq\{0,1\} \\ |A|=1}} \Big(\sum_{\substack{B\subseteq\{0,1,2\} \\ |B|=1}} w_1(A,B)\Big) + \sum_{\substack{A\subseteq\{0,1,2\} \\ |A|=2}} w_1(A,\phi) \\ &= \Big(\sum_{0\le i<j\le 2} X_iX_j\Big) + (X_0Y_0 + X_0Y_1 + q^2X_0Y_0 + X_1Y_0 + X_1Y_1 + qX_1Y_1) + \Big(\sum_{0\le i<j\le 2} Y_iY_j\Big). \end{aligned}$$

On the way to prove Theorem 2.2.8, we introduce the following, which extends q-binomial coefficient.

Definition 2.2.11. For a weakly increasing composition $\mu = (\mu_1,\cdots,\mu_a)$ such that $-1 \le \mu_1,\cdots,\mu_a \le n$ and a set $B \subseteq \{0,\cdots,n+a+b-1\}$ with $|B| = b$, denoting $B \cap \{n+b,\cdots,n+b+a-1\} = \{n+b+a-j_k < \cdots < n+b+a-j_1\}$, we define a weight $m_n^\mu(B)$ to be

$$m_n^\mu(B) = \Big(\prod_{i\in B\cap\{0,\cdots,n+b-1\}} Y_i\Big)\Big(\prod_{l=1}^{k} (q^{(n+b+a-j_l)-B(\mu_{j_l}+l)} Y_{B(\mu_{j_l}+l)}\Big),$$

where we define $B(0) = -1$ and $Y_{-1} = q^{-1}(\beta - \epsilon_2)$. We also define a *generalized q-binomial coefficient* $M_n^\mu(b)$ to be

$$M_n^\mu(b) = \sum_{\substack{B\subseteq\{0,\cdots,n+a+b-1\} \\ |B|=b}} m_n^\mu(B).$$

We named $M_n^\mu(b)$ a generalized q-binomial coefficient because when $\epsilon_2 = 0$, we have $M_n^\mu(b) = q^{\binom{b}{2}}\binom{n+a+b}{b}_q(\beta)^b$, where a is a number of components in $\mu = (\mu_1, \cdots, \mu_a)$. Note that q-binomial coefficients have the following well known identities

$$\binom{n+a+b+1}{b}_q = q^{n+a+1}\binom{n+a+b}{b-1}_q + \binom{n+a+b}{b}_q \tag{2.2.4}$$

$$[n+a+b+1]_q\binom{n+a+b}{b}_q = [n+a+1]_q\binom{n+a+b+1}{b}_q. \tag{2.2.5}$$

We will give a generalization of (2.2.4) in Lemma 2.4.10 and a generalization of (2.2.5) in Lemma 2.4.13. These two lemmas will be key ingredients for the proof of Theorem 2.2.8.

Positivity of minors of the matrix of coefficients

Motivated by [12] (Conjecture 4.4), we make the following conjecture.

Conjecture 2.2.12. Let $G = (g_{n,i})_{n,i}$ be the infinite array of coefficients $g_{n,i} = [x^i]\hat{p}_n(x)$ where $n, i \in \mathbb{Z}_{\geq 0}$ and $g_{n,i} = 0$ if $i > n$. Then the (non-vanishing) minors of G are polynomials with positive coefficients.

Specializing $\alpha \to \xi\alpha$, $\epsilon_1 \to \xi\alpha\beta$ and $\epsilon_2 \to \alpha\beta$, Conjecture 2.2.12 recovers the positivity conjecture for Koornwinder moments when $\gamma = \delta = 0$ (see [12], Conjecture 4.4).

Proposition 2.2.13. *Conjecture 2.2.12 is true for the following cases.*

(1) $\alpha = 0$ *(or* $\beta = 0$*)*

(2) $\alpha = \epsilon_1$ *(or* $\beta = \epsilon_2$*)*

(3) $\epsilon_1 = 0$ *(or* $\epsilon_2 = 0$*)*

Proof. By Remark 2.3.3 and Remark 2.4.7 together with the Lindström-Gessel-Viennot lemma (see [20]) proves the proposition. □

For polynomials f_1 and f_2 we will write $f_1 \geq f_2$ if $(f_1 - f_2)$ is a polynomial with positive coefficients. The following theorem shows that 2 by 2 minors of $G = (g_{n,i})_{n,i}$ having $g_{n,n} = 1$ as a lower left entry are polynomials with positive coefficients. The proof will use Theorem 2.2.8.

Theorem 2.2.14. *For non-negative integers* n, a *and* b*, we have*

$$g_{n+a+b,n+a}\,g_{n+a,n} \geq g_{n+a+b,n}.$$

Example 2.2.15. We have

$$\begin{aligned} g_{4,2}g_{2,1} - g_{4,1} = &\alpha^2\beta + \alpha\beta^2 + 2\alpha^2\epsilon_1 + 2\alpha\beta\epsilon_1 + \beta^2\epsilon_1 + \beta\epsilon_1^2 + \alpha^2\epsilon_2 + 2\alpha\beta\epsilon_2 + 2\beta^2\epsilon_2 + \alpha\epsilon_1\epsilon_2 \\ &+ \beta\epsilon_1\epsilon_2 + \alpha\epsilon_2^2 + \alpha^3 q + 2\alpha^2\beta q + 2\alpha\beta^2 q + \beta^3 q + \alpha^2\epsilon_1 q + 2\alpha\beta\epsilon_1 q + \beta^2\epsilon_1 q \\ &+ \beta\epsilon_1^2 q + \alpha^2\epsilon_2 q + 2\alpha\beta\epsilon_2 q + \beta^2\epsilon_2 q + \alpha\epsilon_1\epsilon_2 q + \beta\epsilon_1\epsilon_2 q + \alpha\epsilon_2^2 q + \alpha^3 q^2 + 2\alpha^2\beta q^2 \\ &+ 2\alpha\beta^2 q^2 + \beta^3 q^2 + 2\alpha\beta\epsilon_1 q^2 + \beta^2\epsilon_1 q^2 + \alpha^2\epsilon_2 q^2 + 2\alpha\beta\epsilon_2 q^2 + \alpha^2\beta q^3 + \alpha\beta^2 q^3. \end{aligned}$$

We conclude $g_{4,2}g_{2,1} \geq g_{4,1}$.

2.3 Proof of Theorem 2.2.3

Our goal in this section is to prove Theorem 2.2.3, which gives a combinatorial formula for the coefficients of the transformed Al-Salam-Chihara polynomials as a weighted sum over Young diagrams contained in a rectangle. One step along the way is to prove Proposition 2.3.2, which gives an analogous result in a simpler setting.

A motivating result

Consider the family of orthogonal polynomials $(\tilde{p}_n(x))_{n\geq 0}$ in one variable x, defined by the following three-term recurrence relation

$$\begin{aligned}\tilde{p}_{n+1}(x) &= (x+\tilde{b}_n)\tilde{p}_n(x) - \tilde{\lambda}_n\tilde{p}_{n-1}(x)\\ \tilde{b}_n &= X_n + Y_n\\ \tilde{\lambda}_n &= Y_{n-1}X_n,\end{aligned}$$

where X_i's and Y_i's are indeterminates.

Definition 2.3.1. For a partition $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$, we define a weight $w(\mu)$ to be

$$w(\mu) = \prod_{i=1}^{l} Z_{\mu_{l+1-i}+2(i-1)} = Z_{\mu_l} Z_{\mu_{l-1}+2} Z_{\mu_{l-2}+4} \cdots Z_{\mu_1+2(l-1)},$$

where Z_n is given by Definition 2.2.1.

Proposition 2.3.2. *The coefficient $\tilde{g}_{n+k,n}$ of x^n in $\tilde{p}_{n+k}(x)$ is given by*

$$\tilde{g}_{n+k,n} = \sum_{\mu\subseteq(k)\times(2n+1)} w(\mu),$$

ie. it is the weighted sum over all Young diagrams contained in a $(k)\times(2n+1)$ rectangle, where the weight is given by Definition 2.3.1.

Proof. The proof goes with the induction. We first prove $\tilde{g}_{k,0}$ is given by the above formula. The base cases $\tilde{g}_{0,0} = 1$ and $\tilde{g}_{1,0} = X_0 + Y_0$ are trivially satisfied. It suffices to prove that the formula satisfies the recurrence $\tilde{g}_{k+1,0} = \tilde{b}_k\tilde{g}_{k,0} - \tilde{\lambda}_k\tilde{g}_{k-1,0}$. Since $\sum_{\mu\subseteq k\times 1} w(\mu) = \sum_{i=0}^{k} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1}$, we have

$$\begin{aligned}
\sum_{\mu\subseteq(k+1)\times 1} w(\mu) &= \sum_{i=0}^{k+1} X_0\cdots X_{i-1}Y_i\cdots Y_k = X_0\cdots X_k + (\sum_{i=0}^{k} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1})Y_k \\
&= \left(\sum_{i=0}^{k} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1} - \sum_{i=0}^{k-1} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1}\right)X_k \\
&\quad + (\sum_{i=0}^{k} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1})Y_k \\
&= (\sum_{i=0}^{k} X_0\cdots X_{i-1}Y_i\cdots Y_{k-1})(X_k+Y_k) - (\sum_{i=0}^{k-1} X_0\cdots X_{i-1}Y_i\cdots Y_{k-2})Y_{k-1}X_k \\
&= \tilde{b}_k(\sum_{\mu\subseteq(k)\times 1} w(\mu)) - \tilde{\lambda}_k(\sum_{\mu\subseteq(k-1)\times 1} w(\mu)).
\end{aligned}$$

Now we will show that for $n>0$, the formula satisfies the recurrence

$$\tilde{g}_{n+k+1,n} = \tilde{g}_{n+k,n-1} + \tilde{b}_{n+k}(\tilde{g}_{n+k,n}) - \tilde{\lambda}_{n+k}(\tilde{g}_{n+k-1,n})$$

which is checked as follows

$$\begin{aligned}
&\sum_{\mu\subseteq(k+1)\times(2n+1)} w(\mu) \\
&= \sum_{\mu\subseteq(k+1)\times(2n-1)} w(\mu) + \sum_{\substack{\mu\subseteq(k+1)\times(2n+1)\\ \mu_1=2n}} w(\mu) + \sum_{\substack{\mu\subseteq(k+1)\times(2n+1)\\ \mu_1=2n+1}} w(\mu) \\
&= \sum_{\mu\subseteq(k+1)\times(2n-1)} w(\mu) + \sum_{\mu'\subseteq(k)\times 2n} w(\mu')X_{n+k} + \sum_{\mu'\subseteq(k)\times 2n+1} w(\mu')Y_{n+k} \\
&= \sum_{\mu\subseteq(k+1)\times(2n-1)} w(\mu) + \left(\sum_{\mu'\subseteq(k)\times(2n+1)} w(\mu') - \sum_{\substack{\mu'\subseteq(k)\times(2n+1)\\ \mu'_1=2n+1}} w(\mu')\right)X_{n+k} \\
&\quad + \sum_{\mu'\subseteq(k)\times(2n+1)} w(\mu')Y_{n+k} \\
&= \sum_{\mu\subseteq(k+1)\times(2n-1)} w(\mu) + (\sum_{\mu'\subseteq(k)\times(2n+1)} w(\mu'))(X_{n+k}+Y_{n+k}) - \sum_{\substack{\mu'\subseteq(k)\times(2n+1)\\ \mu'_1=2n+1}} w(\mu')X_{n+k} \\
&= \sum_{\mu\subseteq(k+1)\times(2n-1)} w(\mu) + \tilde{b}_{n+k}\left(\sum_{\mu'\subseteq(k)\times(2n+1)} w(\mu')\right) - \tilde{\lambda}_{n+k}\left(\sum_{\mu''\subseteq(k-1)\times(2n+1)} w(\mu'')\right).
\end{aligned}$$

□

Remark 2.3.3. Consider a vertex set $V = \{(i,j)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0} \mid -2j-1\leq i\leq 0\}$ and directed edges connecting every horizontally or vertically adjacent pair of vertices in an increasing direction (see Figure 2.2). We will assign weights to directed edges as follows. For a directed edge of the form $(i,j) \to (i+1,j)$, we give a weight 1 and for a directed edge of the form $(i,j) \to (i,j+1)$, we give a weight Z_{i+2j+1} (defined in Definition 2.2.1). The weight $W(P)$ of a path P is defined to be the product of the weights of its edges. Set $u_i = (-2i-1,i)$ and $v_i = (0,i)$ for $i \geq 0$. Then the formula for $\tilde{g}_{n+k,n}$ given in Proposition 2.3.2 is equivalent to $\tilde{g}_{n+k,n} = \sum_{P:u_n \to v_{n+k}} W(P)$, where the sum is over all paths P from u_n to v_{n+k}.

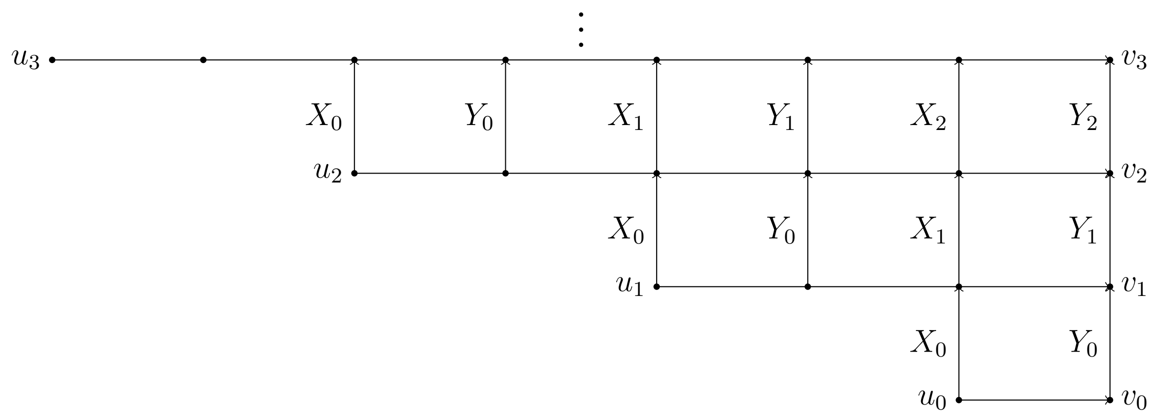

Figure 2.2: The figure shows the weighted directed graph constructed in Remark 2.3.3.

From now on, we specify X_i's and Y_i's as given in (2.2.1)

$$X_i = \alpha q^i + \epsilon_1 [i]_q, \;\; Y_i = \beta q^i + \epsilon_2 [i]_q.$$

Then the structure constants for the transformed Al-Salam-Chihara polynomials can be represented as follows

$$b_n = X_n + Y_n$$

$$\lambda_n = Y_{n-1} X_n - \alpha(\beta - \epsilon_2) q^{n-1}.$$

When $\alpha = 0$ or $\beta = \epsilon_2$, Proposition 2.3.2 gives the formula for $g_{n+k,n} = [x^n]\hat{p}_{n+k}(x)$. The weight $u_m(\mu)$ given by Definition 2.2.1 can be considered as a modification of the weight $w(\mu)$ given by Definition 2.3.1. They manifestly coincide when $\alpha = 0$ or $\beta = \epsilon_2$ (Remark 2.3.4).

Remark 2.3.4. Let $k = s_m(\mu)$ and assume $\alpha = 0$. If $k = 0$, then trivially $u_m(\mu) = w(\mu)$. If $k > 0$, then both $u_m(\mu)$ and $w(\mu)$ have a factor $Z_0 = X_0 = \alpha$ so they are both zero. Now assume $\beta = \epsilon_2$ then Y_i equals $[i+1]_q \beta$. This gives

$$\frac{w(\mu)}{u_m(\mu)} = \frac{(\prod_{i=1}^{l} Z_{\mu_{l+1-i}+2(i-1)})}{(\prod_{i=1}^{l-k} Z_{\mu_{l+1-i}+2(i-1)})(\binom{m+l}{k}_q Y_0 \cdots Y_{k-1})} = \frac{(\prod_{i=l-k+1}^{l} Z_{\mu_{l+1-i}+2(i-1)})}{(\binom{m+l}{k}_q Y_0 \cdots Y_{k-1})}$$

$$= \frac{Y_{m+l-k} \cdots Y_{m+l-1}}{(\binom{m+l}{k}_q Y_0 \cdots Y_{k-1})} = \frac{\beta^k [m+l-k+1]_q \cdots [m+l]_q}{\beta^k \binom{m+l}{k}_q [1]_q \cdots [k]_q} = 1.$$

We conclude that $w(\mu) = u_m(\mu)$ when $\alpha = 0$ or $\beta = \epsilon_2$.

Proof of Theorem 2.2.3

We will first prove Theorem 2.2.3 for $g_{k,0}$. It suffices to prove that (2.2.2) satisfies the recurrence $g_{k+1,0} = b_k g_{k,0} - \lambda_k g_{k-1,0}$ (Proposition 2.3.6). The base cases $g_{0,0} = 1$ and $g_{1,0} = X_0 + Y_0 = \alpha + \beta$ are trivially satisfied.

Lemma 2.3.5. *The following equality holds ($k \geq 1$)*

$$\binom{k+1}{i}_q X_{i-1} Y_{k-i} = \binom{k}{i-1}_q X_k Y_{k-i} + \binom{k}{i}_q X_{i-1} Y_k - \lambda_k \binom{k-1}{i-1}_q.$$

Proof. Moving the middle term on the right hand side to the left, the left hand side becomes

$$\binom{k+1}{i}_q X_{i-1} Y_{k-i} - \binom{k}{i}_q X_{i-1} Y_k = X_{i-1}(\binom{k+1}{i}_q Y_{k-i} - \binom{k}{i}_q Y_k)$$

$$= X_{i-1}(q^{k-i}\binom{k}{i-1}_q (\beta - \epsilon_2)).$$

The remaining right hand side becomes

$$\binom{k}{i-1}_q X_k Y_{k-i} - (X_k Y_{k-1} - \alpha(\beta - \epsilon_2) q^{k-1}) \binom{k-1}{i-1}_q$$

$$= X_k (\binom{k}{i-1}_q Y_{k-i} - \binom{k-1}{i-1}_q Y_{k-1}) + \alpha(\beta - \epsilon_2) q^{k-1} \binom{k-1}{i-1}_q$$

$$= X_k (q^{k-i} \binom{k-1}{i-2}_q (\beta - \epsilon_2)) + \alpha(\beta - \epsilon_2) q^{k-1} \binom{k-1}{i-1}_q$$

$$= q^{k-i}(\beta - \epsilon_2)(\binom{k-1}{i-2}_q X_k + \alpha q^{i-1} \binom{k-1}{i-1}_q)$$

$$= q^{k-i}(\beta - \epsilon_2)(\alpha(q^k \binom{k-1}{i-2}_q + q^{i-1}\binom{k-1}{i-1}_q)) + \epsilon_1 [k]_q \binom{k-1}{i-2}_q)$$

$$= q^{k-i}(\beta - \epsilon_2)(\alpha(q^{i-1}\binom{k}{i-1}_q) + \epsilon_1 [i-1]_q \binom{k}{i-1}_q)$$

$$= q^{k-i}(\beta - \epsilon_2)(\binom{k}{i-1}_q X_{i-1})).$$

So the equality holds. □

Proposition 2.3.6. *The following equality holds* ($k \geq 1$)

$$\sum_{i=0}^{k+1} \binom{k+1}{i}_q X_0 \cdots X_{i-1} Y_0 \cdots Y_{k-i} = b_k \Big(\sum_{i=0}^{k} \binom{k}{i}_q X_0 \cdots X_{i-1} Y_0 \cdots Y_{k-i-1} \Big) - \lambda_k \Big(\sum_{i=0}^{k-1} \binom{k-1}{i}_q X_0 \cdots X_{i-1} Y_0 \cdots Y_{k-i-2} \Big).$$

Proof. Multiplying the equality in Lemma 2.3.5 with $(X_0 \cdots X_{i-2} Y_0 \cdots Y_{k-i-1})$ gives

$$\binom{k+1}{i}_q X_0 \cdots X_{i-1} Y_0 \cdots Y_{k-i} = \Big(\binom{k}{i-1}_q X_0 \cdots X_{i-2} Y_0 \cdots Y_{k-i-1} \Big)(X_k) + \Big(\binom{k}{i}_q X_0 \cdots X_{i-1} Y_0 \cdots Y_{k-i-2} \Big)(Y_k) - \Big(\binom{k-1}{i-1}_q X_0 \cdots X_{i-2} Y_0 \cdots Y_{k-i-1} \Big)(\lambda_k). \tag{2.3.1}$$

Summing (2.3.1) for i from 0 to $(k+1)$ gives a desired equality. □

To prove Theorem 2.2.3, it remains to show that for $n > 0$, the formula satisfies the recurrence $g_{n+k+1,n} = g_{n+k,n-1} + b_{n+k} g_{n+k,n} - \lambda_{n+k} g_{n+k-1,n}$. In other words, we will show the identity

$$\sum_{\mu \subseteq (k+1) \times (2n+1)} u_n(\mu) = \sum_{\mu \subseteq (k+1) \times (2n-1)} u_{n-1}(\mu) + b_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) - \lambda_{n+k} \Big(\sum_{\mu \subseteq (k-1) \times (2n+1)} u_n(\mu) \Big). \tag{2.3.2}$$

The middle term on the right hand side of (2.3.2) becomes

$$\begin{aligned} b_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) &= X_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) + Y_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) \\ &= X_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n)} u_n(\mu) + \sum_{\substack{\mu \subseteq (k) \times (2n+1) \\ \mu_1 = 2n+1}} u_n(\mu) \Big) + Y_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) \\ &= X_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n)} u_n(\mu) \Big) + Y_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) \\ &\quad + X_{n+k} \Big(\sum_{\mu' \subseteq (k-1) \times (2n+1)} u_n((2n+1, \mu')) \Big). \end{aligned}$$

Plugging this to (2.3.2) gives

$$\sum_{\mu \subseteq (k+1) \times (2n+1)} u_n(\mu) = \sum_{\mu \subseteq (k+1) \times (2n-1)} u_{n-1}(\mu) + X_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n)} u_n(\mu) \Big) + Y_{n+k} \Big(\sum_{\mu \subseteq (k) \times (2n+1)} u_n(\mu) \Big) + \Big(\sum_{\mu \subseteq (k-1) \times (2n+1)} (X_{n+k} u_n((2n+1, \mu)) - \lambda_{n+k} u_n(\mu)) \Big). \tag{2.3.3}$$

For $\mu \subseteq (k+1)\times(2n+1)$ we define

$$\bar{u}_n(\mu) = \begin{cases} u_{n-1}(\mu) & \text{if } \mu \subseteq (k+1)\times(2n-1) \\ X_{n+k}u_n(\mu') & \text{if } \mu = (2n,\mu') \text{ for some } \mu' \subseteq (k)\times(2n) \\ Y_{n+k}u_n(\mu') & \text{if } \mu = (2n+1,\mu') \text{ for some } \mu' \subseteq (k)\times(2n+1), \end{cases}$$

and for $\mu \subseteq (k-1)\times(2n+1)$ we define

$$v_i(\mu) = X_{n+k}u_n((2n+1,\mu)) - \lambda_{n+k}u_n(\mu).$$

Then (2.3.3) is represented as follows

$$\sum_{\mu\subseteq(k+1)\times(2n+1)} u_n(\mu) = \sum_{\mu\subseteq(k+1)\times(2n+1)} \bar{u}_n(\mu) + \sum_{\mu\subseteq(k-1)\times(2n+1)} v_i(\mu). \tag{2.3.4}$$

To prove (2.3.4) we first partition the sets $\{\mu \subseteq (k+1)\times(2n+1)\}$ and $\{\mu \subseteq (k-1)\times(2n+1)\}$.

Definition 2.3.7. Define $\hat{B}_{n+k,n}$ to be a subset of $\{\mu \subseteq (k+1)\times(2n+1)\}$ consisting of μ such that $s_n(\mu) = 0$ and $s_{n-1}(\mu) = 0$. For a partition $\mu \subseteq (k+1)\times(2n-1)$ such that $s_{n-1}(\mu) = l > 0$, denoting $\mu = ((2n-1)^l, \mu')$, we define $B^{\mu}_{n+k,n}$ to be

$$B^{\mu}_{n+k,n} = \{((2n-1)^l,\mu'), ((2n+1),(2n-1)^{l-1},\mu'), \cdots, ((2n+1)^l,\mu')\}.$$

Definition 2.3.8. For a partition $\mu \subseteq (k-1)\times(2n+1)$ such that $\mu_{k-1} \geq 2$, define $\bar{s}_n(\mu)$ to be a mininum of two numbers $|\{i|\mu_i = 2\}|$ and $|\{i|\mu_i = 2n+1\}|$. If $\bar{s}_n(\mu) = l$, denoting $\mu = (\mu', 2^l)$, we define a set $C^{\mu}_{n+k,n}$ to be

$$C^{\mu}_{n+k,n} = \{(\mu',2^l), (\mu',2^{l-1},0), \cdots, (\mu',0^l)\}.$$

If $l = 0$, then $C^{\mu}_{n+k,n}$ consists of a single element μ.

Example 2.3.9. Consider a partition $\mu = (3,3,1,0,0,0)$. Since $s_1(\mu) = 2$, we have

$$B^{(3,3,1,0,0,0)}_{7,2} = \{(3,3,1,0,0,0), (5,3,1,0,0,0), (5,5,1,0,0,0)\}.$$

For a partition $\mu = (5,5,2,2)$, we have

$$C^{(5,5,2,2)}_{7,2} = \{(5,5,2,2), (5,5,2,0), (5,5,0,0)\}.$$

Definition 2.3.10. We define $B^X_{n+k,n}$, $B^Y_{n+k,n}$, $C^X_{n+k,n}$ and $C^Y_{n+k,n}$ as follows

$$\begin{aligned} B^X_{n+k,n} &= \{\mu \subseteq (k+1)\times(2n+1) |\ s_n(\mu) = l > 0 \text{ and } \mu = ((2n+1)^l, 2n, \mu') \text{ for some } \mu'\} \\ B^Y_{n+k,n} &= \{\mu \subseteq (k+1)\times(2n+1) |\ s_n(\mu) = l > 0 \text{ and } \mu = ((2n+1)^{l+1}, \mu') \text{ for some } \mu'\} \\ C^X_{n+k,n} &= \{\mu \subseteq (k-1)\times(2n+1) |\ s_n(\mu) = l \text{ and } \mu = (\mu', 0^{l+1}) \text{ for some } \mu'\} \\ C^Y_{n+k,n} &= \{\mu \subseteq (k-1)\times(2n+1) |\ s_n(\mu) = l \text{ and } \mu = (\mu', 1, 0^l) \text{ for some } \mu'\}. \end{aligned}$$

Proposition 2.3.11. *The set* $\{\mu \subseteq (k+1)\times(2n+1)\}$ *is a disjoint union of* B*'s and the set* $\{\mu \subseteq (k-1)\times(2n+1)\}$ *is a disjoint union of* C*'s. That is*

$$\{\mu \subseteq (k+1)\times(2n+1)\} = \hat{B}_{n+k,n} \uplus \Big(\biguplus_{\substack{\nu\subseteq(k+1)\times(2n-1)\\ s_{n-1}(\nu)>0}} B^{\nu}_{n+k,n}\Big) \uplus B^{X}_{n+k,n} \uplus B^{Y}_{n+k,n}$$

$$\{\mu \subseteq (k-1)\times(2n+1)\} = \Big(\biguplus_{\substack{\nu\subseteq(k-1)\times(2n+1)\\ \nu_{k-1}\geq 2}} C^{\nu}_{n+k,n}\Big) \uplus C^{X}_{n+k,n} \uplus C^{Y}_{n+k,n}.$$

Proof. For $\mu \subseteq (k+1)\times(2n+1)$, if $s_n(\mu)=0$ and $s_{n-1}(\mu)=0$ then $\mu\in\hat{B}_{n+k,n}$. If $s_{n-1}(\mu)>0$, then $\mu\in B^{\mu}_{n+k,n}$. Now assume $s_n(\mu)=l>0$. By the definition of $B^X_{n+k,n}$ and $B^Y_{n+k,n}$, we have $\mu\in B^X_{n+k,n}$ if $\mu_{l+1}=2n$ and $\mu\in B^Y_{n+k,n}$ if $\mu_{l+1}=2n+1$. For the remaining case $\mu_{l+1}\leq 2n-1$, let $\nu=((2n-1)^l,\mu_{l+1},\cdots,\mu_{k+1})$. Then ν is a partition inside $(k+1)\times(2n-1)$ with $s_{n-1}(\nu)\geq l>0$. Thus we have $\mu\in B^{\nu}_{n+k,n}$. This proves the first statement.

For the second statement, take $\mu\subseteq(k-1)\times(2n+1)$ such that $s_n(\mu)=l$. If $\mu_{k-1-l}=0$ then $\mu\in C^X_{n+k,n}$ and if $\mu_{k-1-l}=1$ then $\mu\in C^Y_{n+k,n}$. For the remaining case $\mu_{k-1-l}\geq 2$, let $\nu=(\mu_1,\cdots,\mu_{k-1-l},2^l)$, then we have $\mu\in C^{\nu}_{n+k,n}$. □

We have decompositions of the sets $\{\mu\subseteq(k+1)\times(2n+1)\}$ and $\{\mu\subseteq(k-1)\times(2n+1)\}$. The next proposition relates these two decompositions.

Proposition 2.3.12. *The followings hold.*

(a) There exists a bijection between $\{\mu\subseteq(k+1)\times(2n-1)|s_{n-1}(\mu)>0\}$ *and* $\{\mu\subseteq(k-1)\times(2n+1)|\mu_{k-1}\geq 2\}$.

(b) There exists a bijection between $B^X_{n+k,n}$ *and* $C^X_{n+k,n}$.

(c) There exists a bijection between $B^Y_{n+k,n}$ *and* $C^Y_{n+k,n}$.

Proof. (a) Given an element μ in the first set, the partition $(\mu_2+2,\cdots,\mu_k+2)$ is in the second set. Conversely given an element ν in the second set, the partition $(2n-1,\nu_1-2,\cdots,\nu_{k-1}-2,0)$ is in the first set. This gives a bijection.

(b) Given an element μ in the first set, let $s_n(\mu)=l>0$ and denote $\mu=((2n+1)^l,2n,\mu')$. Note that the tail of μ' contains at least l 0's. So we write $\mu=((2n+1)^l,2n,\mu'',0^l)$. Then the partition $\nu=((2n+1)^{l-1},\mu'',0^l)$ belongs to the second set. Conversely, given an element ν in the second set, let $s_n(\nu)=l'$ (possibly zero). Likewise, we can write $\nu=((2n+1)^{l'},\nu'',0^{l'+1})$. We send this to the partition $\mu=((2n+1)^{l'+1},2n,\nu'',0^{l'+1})$ then $s_n(\mu)=l'+1>0$ which implies that μ belongs to the first set. These processes are inverse to each other thus give a bijection.

(c) The argument goes similarly with that of (b). The partition $\mu=((2n+1)^{l+1},\mu'',0^l)$ goes to the partition $\nu=((2n+1)^{l-1},\mu'',1,0^{l-1})$ and conversely, the partition $\nu=((2n+1)^{l'},\nu'',1,0^{l'})$ goes to the partition $\mu=((2n+1)^{l'+2},\nu'',0^{l'+1})$. □

Proposition 2.3.13. *The following identities hold.*

(a) For a partition $\mu\in\hat{B}_{n+k,n}$, *we have* $u_n(\mu)=\bar{u}_n(\mu)$.

(b) For partitions $\mu \subseteq (k+1)\times(2n-1)$ such that $s_{n-1}(\mu) > 0$ and $\nu \subseteq (k-1)\times(2n+1)$ such that $\nu_{k-1} \geq 2$ which are corresponding pair under the bijection in Proposition 2.3.12 (a), we have

$$\sum_{\mu' \in B^{\mu}_{n+k,n}} u_n(\mu') = \sum_{\mu' \in B^{\mu}_{n+k,n}} \bar{u}_n(\mu') + \sum_{\nu' \in C^{\nu}_{n+k,n}} v_n(\nu').$$

(c) For partitions $\mu \in B^X_{n+k,n}$ and $\nu \in C^X_{n+k,n}$ which are corresponding pair under the bijection in Proposition 2.3.12 (b), we have

$$u_n(\mu) = \bar{u}_n(\mu) + v_n(\nu).$$

(d) For partitions $\mu \in B^Y_{n+k,n}$ and $\nu \in C^Y_{n+k,n}$ which are corresponding pair under the bijection in Proposition 2.3.12 (c), we have

$$u_n(\mu) = \bar{u}_n(\mu) + v_n(\nu).$$

Note that (2.3.4) follows from Proposition 2.3.11 and Proposition 2.3.13. So it suffices to prove Proposition 2.3.13. To do that, we first compute $v_n(\nu)$ explicitly.

Lemma 2.3.14. *The following holds.*
(a) For a partition $\nu \in C^X_{n+k,n}$ with $s_n(\nu) = l$, we have

$$v_n(\nu) = X_0 \cdots X_l X_{n+k-l-1} (\prod_{i=l+1}^{k-2-l} Z_{\nu_i + 2(k-1-i)}) Y_0 \cdots Y_{l-1} q^l \binom{n+k}{l+1}_q (\beta - \epsilon_2).$$

(b) For a partition $\nu \subseteq (k-1)\times(2n+1)$ such that $\nu \notin C^X_{n+k,n}$ with $s_n(\nu) = l$, we have

$$v_n(\nu) = X_0 \cdots X_l (\prod_{i=l+1}^{k-1-l} Z_{\nu_i + 2(k-1-i)}) Y_0 \cdots Y_{l-1} q^{n+k-l-1} \binom{n+k}{l}_q (\beta - \epsilon_2).$$

Proof. (a) Denoting $\nu = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-2-l}, 0^{l+1})$, we have

$$u_n(\nu) = X_0 \cdots X_l (\prod_{i=l+1}^{k-2-l} Z_{\nu_i + 2(k-1-i)}) \binom{n+k-1}{l}_q Y_0 \cdots Y_{l-1}.$$

And since $s_n(((2n+1), \nu)) = l+1$, we have

$$u_n(((2n+1), \nu)) = X_0 \cdots X_l (\prod_{i=l+1}^{k-2-l} Z_{\nu_i + 2(k-1-i)}) \binom{n+k}{l+1}_q Y_0 \cdots Y_l.$$

Setting the common factor $M = X_0 \cdots X_l (\prod_{i=l+1}^{k-2-l} Z_{\nu_i+2(k-1-i)}) Y_0 \cdots Y_{l-1}$, we have

$$\begin{aligned}
& v_i(\nu) \\
=& X_{n+k} u_n(((2n+1),\nu)) - \lambda_{n+k} u_n(\nu) = X_{n+k}(M \binom{n+k}{l+1}_q Y_l) - \lambda_{n+k}(M \binom{n+k-1}{l}_q) \\
=& M(\binom{n+k}{l+1}_q X_{n+k} Y_l - (Y_{n+k-1} X_{n+k} - \alpha(\beta-\epsilon_2) q^{n+k-1}) \binom{n+k-1}{l}_q) \\
=& M(X_{n+k}(\binom{n+k}{l+1}_q Y_l - \binom{n+k-1}{l}_q Y_{n+k-1}) + \alpha(\beta-\epsilon_2) q^{n+k-1} \binom{n+k-1}{l}_q) \\
=& M(X_{n+k}(q^l \binom{n+k-1}{l+1}_q (\beta-\epsilon_2)) + \alpha(\beta-\epsilon_2) q^{n+k-1} \binom{n+k-1}{l}_q) \\
=& M(\beta-\epsilon_2) q^l (X_{n+k} \binom{n+k-1}{l+1}_q + \alpha q^{n+k-l-1} \binom{n+k-1}{l}_q)) \\
=& M(\beta-\epsilon_2) q^l (\alpha(q^{n+k} \binom{n+k-1}{l+1} + q^{n+k-l-1} \binom{n+k-1}{l}_q) + \epsilon_1 \binom{n+k-1}{l+1} [n]_q) \\
=& M(\beta-\epsilon_2) q^l (\binom{n+k}{l+1}_q X_{n+k-l-1}) \\
=& X_0 \cdots X_l X_{n+k-l-1} (\prod_{i=l+1}^{k-2-l} Z_{\nu_i+2(k-1-i)}) Y_0 \cdots Y_{l-1} q^l \binom{n+k}{l+1}_q (\beta-\epsilon_2).
\end{aligned}$$

(b) Denoting $\nu = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-1-l}, 0^l)$, we have

$$u_n(\nu) = X_0 \cdots X_{l-1} (\prod_{i=l+1}^{k-1-l} Z_{\nu_i+2(k-1-i)}) \binom{n+k-1}{l}_q Y_0 \cdots Y_{l-1}.$$

And since $s_n(((2n+1),\nu)) = l$, we have

$$\begin{aligned}
u_n(((2n+1),\nu)) =& X_0 \cdots X_{l-1} (\prod_{i=l}^{k-1-l} Z_{\nu_i+2(k-1-i)}) \binom{n+k}{l}_q Y_0 \cdots Y_{l-1} \\
=& X_0 \cdots X_{l-1} (\prod_{i=l+1}^{k-1-l} Z_{\nu_i+2(k-1-i)}) \binom{n+k}{l}_q Y_0 \cdots Y_{l-1} Z_{\nu_l+2(k-1-l)} \\
=& X_0 \cdots X_{l-1} (\prod_{i=l+1}^{k-1-l} Z_{\nu_i+2(k-1-i)}) \binom{n+k}{l}_q Y_0 \cdots Y_{l-1} Y_{n+k-l-1}.
\end{aligned}$$

Setting the common factor $M = X_0 \cdots X_{l-1}(\prod_{i=l+1}^{k-1-l} Z_{\nu_i+2(k-1-i)})Y_0 \cdots Y_{l-1}$, we have

$$\begin{aligned}
v_n(\nu) =& M(\binom{n+k}{l}_q X_{n+k}Y_{n+k-l-1} - (Y_{n+k-1}X_{n+k} - \alpha(\beta-\epsilon_2)q^{n+k-1})\binom{n+k-1}{l}_q) \\
=& M(\beta-\epsilon_2)q^{n+k-l-1}(\binom{n+k}{l}_q X_l) \\
=& X_0 \cdots X_l (\prod_{i=l+1}^{k-1-l} Z_{\nu_i+2(k-1-i)})Y_0 \cdots Y_{l-1} q^{n+k-l-1}\binom{n+k}{l}_q (\beta-\epsilon_2).
\end{aligned}$$

□

The following lemma will be used for the proof of Proposition 2.3.13 (b).

Lemma 2.3.15. *For l and n such that $l \geq 0$ and $n-l-1 \geq 0$, we have*

$$\begin{aligned}
& Y_{n-l-1} \cdots Y_{n-1} + \sum_{i=1}^{l+1} Y_{n-l-1} \cdots Y_{n-i-1}\binom{n+1}{i}_q Y_0 \cdots Y_{i-1} \\
=& \binom{n}{l+1}_q Y_0 \cdots Y_l + Y_n(\sum_{i=1}^{l+1} Y_{n-l-1} \cdots Y_{n-i-1}\binom{n}{i-1}_q Y_0 \cdots Y_{i-2}) \\
& + (\beta-\epsilon_2)(\sum_{i=0}^{l} q^{n-i-1} Y_{n-l-1} \cdots Y_{n-i-2}\binom{n}{i}_q Y_0 \cdots Y_{i-1}).
\end{aligned}$$

Proof. For $l=0$, the identity becomes $Y_{n-1} + \binom{n+1}{1}_q Y_0 = \binom{n}{1}_q Y_0 + Y_n + (\beta-\epsilon_2)q^{n-1}$, which can be checked by a direct computation. Now assume that the identity holds for $(l-1)$ and for all valid n. From the identity corresponding to $(l-1)$ and n, we multiply both sides with Y_{n-l-1} which gives

$$\begin{aligned}
& Y_{n-l-1} \cdots Y_{n-1} + \sum_{i=1}^{l} Y_{n-l-1} \cdots Y_{n-i-1}\binom{n+1}{i}_q Y_0 \cdots Y_{i-1} \\
=& \binom{n}{l}_q Y_0 \cdots Y_{l-1}Y_{n-l-1} + Y_n(\sum_{i=1}^{l} Y_{n-l-1} \cdots Y_{n-i-1}\binom{n}{i-1}_q Y_0 \cdots Y_{i-2}) \\
& + (\beta-\epsilon_2)(\sum_{i=0}^{l-1} q^{n-i-1} Y_{n-l-1} \cdots Y_{n-i-2}\binom{n}{i}_q Y_0 \cdots Y_{i-1}).
\end{aligned} \tag{2.3.5}$$

We also have the identity

$$\binom{n+1}{l+1}_q Y_l = \binom{n}{l+1}_q Y_l - \binom{n}{l}_q Y_{n-l-1} + \binom{n}{l}_q Y_n + q^{n-l-1}\binom{n}{l}_q (\beta-\epsilon_2),$$

that can be checked by a direct computation. Multiplying both sides with $Y_0 \cdots Y_{l-1}$ gives

$$\begin{aligned}
\binom{n+1}{l+1}_q Y_0 \cdots Y_l =& \binom{n}{l+1}_q Y_0 \cdots Y_l - \binom{n}{l}_q Y_0 \cdots Y_{l-1}Y_{n-l-1} \\
& + \binom{n}{l}_q Y_0 \cdots Y_{l-1}Y_n + q^{n-l-1}\binom{n}{l}_q Y_0 \cdots Y_{l-1}(\beta-\epsilon_2).
\end{aligned} \tag{2.3.6}$$

Adding (2.3.5) and (2.3.6) gives the identity corresponding to l and n. The proof follows from the induction. □

Proof of Proposition 2.3.13.

(a) It is trivial to verify.

(b) Denote $\bar{s}_n(\nu) = l$ and $\nu = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-1-l}, 2^l)$. Then the corresponding partition μ is $((2n-1)^{l+1}, (\nu_{l+1}-2), \cdots, (\nu_{k-1-l}-2), 0^{l+1})$. The elements of $C^{\nu}_{n+k,n}$ are denoted as $\nu^i = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-1-l}, 2^{l-i}, 0^i)$ where i ranges from 0 to l. As $s_n(\nu^i) = i$, by Lemma 2.3.14 we have

$$\begin{aligned}
v_n(\nu^i) &= X_0 \cdots X_i (\prod_{j=i+1}^{k-1-i} Z_{(\nu^i)_j + 2(k-1-j)}) Y_0 \cdots Y_{i-1} q^{n+k-i-1} \binom{n+k}{i}_q (\beta - \epsilon_2) \\
&= X_0 \cdots X_i (\prod_{j=i+1}^{k-1-l} Z_{(\nu^i)_j + 2(k-1-j)})(\prod_{j=k-l}^{k-1-i} Z_{(\nu^i)_j + 2(k-1-j)}) Y_0 \cdots Y_{i-1} q^{n+k-i-1} \binom{n+k}{i}_q (\beta - \epsilon_2) \\
&= X_0 \cdots X_l (\prod_{j=i+1}^{k-1-l} Z_{(\nu^i)_j + 2(k-1-j)}) Y_0 \cdots Y_{i-1} q^{n+k-i-1} \binom{n+k}{i}_q (\beta - \epsilon_2) \\
&= X_0 \cdots X_l (\prod_{j=i+1}^{l} Z_{(\nu^i)_j + 2(k-1-j)})(\prod_{j=l+1}^{k-1-l} Z_{(\nu^i)_j + 2(k-1-j)}) Y_0 \cdots Y_{i-1} q^{n+k-i-1} \binom{n+k}{i}_q (\beta - \epsilon_2) \\
&= X_0 \cdots X_l (\prod_{j=l+1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}})(Y_{n+k-l-1} \cdots Y_{n+k-i-2}) Y_0 \cdots Y_{i-1} q^{n+k-i-1} \binom{n+k}{i}_q (\beta - \epsilon_2).
\end{aligned}$$

And the elements of $B^{\mu}_{n+k,n}$ are denoted as $\mu^i = ((2n+1)^i, (2n-1)^{l+1-i}, (\nu_{l+1}-2), \cdots, (\nu_{k-1-l}-2), 0^{l+1})$ where i ranges from 0 to $(l+1)$. We have

$$\begin{aligned}
u_n(\mu^0) &= X_0 \cdots X_l (\prod_{j=l+1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}})(Y_{n+k-l-1} \cdots Y_{n+k-1}) \\
\bar{u}_n(\mu^0) &= X_0 \cdots X_l (\prod_{j=l=1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}})(\binom{n+k}{l+1}_q Y_0 \cdots Y_l),
\end{aligned}$$

and for i from 1 to $(l+1)$ we have

$$\begin{aligned}
u_n(\mu^i) &= X_0 \cdots X_l (\prod_{j=l+1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}})(Y_{n+k-l-1} \cdots Y_{n+k-i-1} \binom{n+k+1}{i}_q Y_0 \cdots Y_{i-1}) \\
\bar{u}_n(\mu^i) &= X_0 \cdots X_l (\prod_{j=l+1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}})(Y_{n+k-l-1} \cdots Y_{n+k-i-1} \binom{n+k}{i-1}_q Y_0 \cdots Y_{i-2}) Y_n.
\end{aligned}$$

Taking out the common factor $(X_0 \cdots X_l (\prod_{j=l+1}^{k-1-l} Z_{\nu_{j+2(k-1-j)}}))$, the desired identity follows from Lemma 2.3.15.

(c) Denote $s_n(\nu) = l$ and $\nu = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-2-l}, 0^{l+1})$. Then the corresponding partition μ is $((2n+1)^{l+1}, 2n, \nu_{l+1}, \cdots, \nu_{k-2-l}, 0^{l+1})$. We have

$$\begin{aligned} u_n(\mu) =& X_0 \cdots X_l (\prod_{j=l+2}^{k-l} Z_{\mu_j + 2(k+1-j)}) \binom{n+k+1}{l+1}_q Y_0 \cdots Y_l \\ =& X_0 \cdots X_l X_{n+k-l-1} (\prod_{j=l+1}^{k-2-l} Z_{\nu_j + 2(k-1-j)}) \binom{n+k+1}{l+1}_q Y_0 \cdots Y_l, \end{aligned}$$

and

$$\bar{u}_n(\mu) = X_0 \cdots X_l X_{n+k-l-1} (\prod_{j=l+1}^{k-2-l} Z_{\nu_j + 2(k-1-j)}) \binom{n+k}{l}_q Y_0 \cdots Y_{l-1} Y_{n+k}.$$

So we have

$$\begin{aligned} & u_n(\mu) - \bar{u}_n(\mu) \\ =& (X_0 \cdots X_l X_{n+k-l-1} (\prod_{j=l+1}^{k-2-l} Z_{\nu_j + 2(k-1-j)}) Y_0 \cdots Y_{l-1}) (\binom{n+k+1}{l+1}_q Y_l - \binom{n+k}{l}_q Y_{n+k}) \\ =& (X_0 \cdots X_l X_{n+k-l-1} (\prod_{j=l+1}^{k-2-l} Z_{\nu_j + 2(k-1-j)}) Y_0 \cdots Y_{l-1}) q^l \binom{n+k}{l+1}_q (\beta - \epsilon_2). \end{aligned}$$

The proof follows from Lemma 2.3.14.

(d) Denote $s_n(\nu) = l$ and $\nu = ((2n+1)^l, \nu_{l+1}, \cdots, \nu_{k-2-l}, 1, 0^l)$. Then the corresponding partition μ is $((2n+1)^{l+2}, 2n, \nu_{l+1}, \cdots, \nu_{k-2-l}, 0^{l+1})$. The proof follows similarly with (c).

2.4 Proof of Theorem 2.2.8

Our goal in this section is to prove Theorem 2.2.8, which gives a combinatorial formula for the coefficients of the transformed Al-Salam-Chihara polynomials as a weighted sum over pairs of sets. In Section 2.4, we start by providing a bijective proof of a combinatorial identity (2.2.3), which motivates the formula in Theorem 2.2.8. In Section 2.4, we study generalized q-binomial coefficients $M_n^\mu(b)$ (Definition 2.2.7) to establish key lemmas (Lemma 2.4.10, Lemma 2.4.13) for the proof of Theorem 2.2.8. In Section 2.4, we finish the proof.

Bijective proof of the identity (2.2.3)

Definition 2.4.1. For non-negative integers n, a and b, we define $T(n,a,b)$ to be a set of pairs (S_1, S_2) such that $S_1 \subseteq \{0, \cdots, (n+a-1)\}$ with $|S_1| = a$ and $S_2 \subseteq \{0, \cdots, (n+a+b-1)\}$ with $|S_2| = b$. For a set S with integer elements, we define $\sum S = \sum_{i \in S} i$.

The identity (2.2.3) can be rephrased as follows

$$\sum_{(A_1,B_1)\in T(n,a,b)} q^{\sum A_1+\sum B_1} = \sum_{(B_2,A_2)\in T(n,b,a)} q^{\sum A_2+\sum B_2} \tag{2.4.1}$$

The identity (2.4.1) shows that there exists a bijection between $T(n,a,b)$ and $T(n,b,a)$ that preserves the sum. Now we construct a such bijection. Recall from Definition 2.2.6 that $S(k)$ is the k-th smallest element of the set $(\{0\}\cup\mathbb{N})-S$ and λ_S is $(i_1, i_2-1, \cdots, i_s-s+1)$ where $S=\{i_1<\cdots<i_s\}$.

Definition 2.4.2. For $(A_1,B_1)\in T(n,a,b)$, we denote $A_1=\{i_1<\cdots<i_a\}$, $B_1\cap\{n+b,\cdots,n+b+a-1\}=\{n+b+a-j_k<\cdots<n+b+a-j_1\}$ and $\mu=\lambda_{A_1}$. We define a map $\psi_{n,a,b}$ from $T(n,a,b)$ as follows

$$\begin{aligned}
&(B_2,A_2)=\psi_{n,a,b}((A_1,B_1))\\
&A_2=\{i_m|m\neq j_l\}\cup\{n+a+b-(B_1(\mu_{j_l}+l)-(i_{j_l}-j_l))|l=1,\cdots,k\}\\
&B_2=(B_1\cap\{0,\cdots,n+b-1\})\cup\{B_1(\mu_{j_l}+l)|l=1,\cdots,k\}.
\end{aligned}$$

Note that we have $\sum A_1+\sum B_1=\sum A_2+\sum B_2$ from the construction.

Proposition 2.4.3. *With the notation in Definition 2.4.2, we have $(B_2,A_2)\in T(n,b,a)$. And we have $(A_1,B_1)=\psi_{n,b,a}((B_2,A_2))$.*

Proof. We have $B_1(\mu_{j_1}+1)<\cdots<B_1(\mu_{j_k}+k)$ and since $\mu_{j_k}\le n$, we have $B_1(\mu_{j_k}+k)\le B_1(n+k)$. As $|B_1\cap\{0,\cdots,(n+b-1)\}|=b-k$, we have $B_1(n+k)\le(n+b-1)$, which implies $B_2\subseteq\{0,\cdots,(n+b-1)\}$ with $|B_2|=b$. Now we set $r_l=B_1(\mu_{j_l}+l)-(i_{j_l}-j_l)$. Since $B_1(\mu_{j_l}+l)$ is the $(\mu_{j_l}+l)$-th smallest element in a set $\{0,\cdots,n+b-1\}-B_1$, there are $B_1(\mu_{j_l}+l)-(\mu_{j_l}+l+1)$ elements in B_1 smaller than $B_1(\mu_{j_l}+l)$. Also $B_1(\mu_{j_1}+1),\cdots,B_1(\mu_{j_{l-1}}+l-1)$ are smaller than $B_1(\mu_{j_l}+l)$ so there are total $(B_1(\mu_{j_l}+l)-(\mu_{j_l}+l+1)+l-1)$ elements smaller than $B_1(\mu_{j_l}+l)$ in B_2. So $B_1(\mu_{j_l}+l)$ is the $B_1(\mu_{j_l}+l)-(\mu_{j_l}+1)=(B_1(\mu_{j_l}+l)-(i_{j_l}-j_l))=r_l$-th smallest element in B_2. This implies that $1\le r_1<\cdots<r_k\le b$, so the sets $\{i_m|m\neq j_l\}$ and $\{n+a+b-r_l|l=1,\cdots,k\}$ are disjoint. Thus we have $A_2\subseteq\{0,\cdots,(n+a+b-1)\}$ with $|A_2|=a$.

Denoting $\mu'=\lambda_{B_2}$, we have $\mu'_{r_l}+l=B_1(\mu_{j_l}+l)-r_l+1+l=i_{j_l}-j_l+l+1$ and $A_2\cap\{n+a,\cdots,n+a+b-1\}=\{n+a+b-r_1<\cdots<n+a+b-r_k\}$. Since there are (j_l-l) elements smaller than i_{j_l} in A_2, we have $i_{j_l}=A_2(i_{j_l}-j_l+l+1)=A_2(\mu'_{r_l}+l)$. We see $(A_1,B_1)=\psi_{n,b,a}((B_2,A_2))$. □

Example 2.4.4. For $n=1$, $a=3$ and $b=4$, consider $A_1=\{0,2,3\}$ and $B_1=\{2,4,5,7\}$ denoting $\mu=\lambda_{A_1}=(0,1,1)$. The above process changes the element $7=8-1\in B_1$ to $B_1(\mu_1+1)=0$ and correspondingly change the element $0\in A_1$ to 7. Likewise, we change the element $5=8-3\in B_1$ to the element $B_1(\mu_3+2)=3$ and change the element $3\in A_1$ to 5. So we have $\psi_{1,3,4}((A_1,B_1))=(B_2,A_2)=(\{0,2,3,4\},\{2,5,7\})$. Since we have $\mu'=\lambda_{B_2}=(0,1,1,1)$, the element $7=8-1\in A_2$ goes to $A_2(\mu'_1+1)=0$ and $0\in B_2$ goes to 7. Likewise the element $5=8-3\in A_2$ goes to $A_2(\mu'_3+2)=3$ and $3\in B_2$ goes to 5. We see $\psi_{1,4,3}((B_2,A_2))=((A_1,B_1))$.

Remark 2.4.5. Consider $B \subseteq \{0, \cdots, (n+a+b-1)\}$ with $|B| = b$ and $|B \cap \{0, \cdots, (n+b-1)\}| = b-k$. For each positive integer l, there are $(B(l) - l + 1)$ elements in B smaller than $B(l)$. And $B(l)$ is the unique integer with that property. Since we have $B(1), \cdots, B(n+k) \le (n+b-1)$, for $C = B \cap \{0, \cdots, (n+b-2)\}$, we have $B(l) = C(l)$ for l from 1 to $(n+k)$.

Remark 2.4.6. With the notation in Definition 2.4.2, the weight $w_n(A_1, B_1)$ given in Definition 2.2.7 becomes

$$w_n(A_1, B_1) = q^{\sum B_1 - \sum B_2}(\prod_{i \in A_1} X_i)(\prod_{i \in B_2} Y_i).$$

Now let $\bar{w}_n(A_1, B_1)$ be the one obtained by exchanging $X_i \leftrightarrow Y_i$ from $w_n(A_1, B_1)$. Then we have

$$\bar{w}_n(A_1, B_1) = q^{\sum B_1 - \sum B_2}(\prod_{i \in A_1} Y_i)(\prod_{i \in B_2} X_i) = q^{\sum A_2 - \sum A_1}(\prod_{i \in B_2} X_i)(\prod_{i \in A_1} Y_i) = w_n(B_2, A_2).$$

This gives

$$\begin{aligned}
&\sum_{a+b=k}\Big(\sum_{\substack{A_1 \subseteq \{0, \cdots, n+a-1\} \\ |A_1| = a}} \Big(\sum_{\substack{B_1 \subseteq \{0, \cdots, n+a+b-1\} \\ |B_1| = b}} \bar{w}_n(A_1, B_1)\Big)\Big) \\
&= \sum_{a+b=k}\Big(\sum_{\substack{B_2 \subseteq \{0, \cdots, n+b-1\} \\ |B_2| = b}} \Big(\sum_{\substack{A_2 \subseteq \{0, \cdots, n+a+b-1\} \\ |A_2| = a}} w_n(B_2, A_2)\Big)\Big) \\
&= \sum_{a+b=k}\Big(\sum_{\substack{A_1 \subseteq \{0, \cdots, n+a-1\} \\ |A_1| = a}} \Big(\sum_{\substack{B_1 \subseteq \{0, \cdots, n+a+b-1\} \\ |B_1| = b}} w_n(A_1, B_1)\Big)\Big).
\end{aligned}$$

Thus the formula in Theorem 2.2.8 is invariant as a polynomial in X_i's and Y_i's under the exchange $X_i \leftrightarrow Y_i$. This was not true for the formula in Theorem 2.2.3.

Remark 2.4.7. When $\epsilon_2 = 0$, we have $Y_i = q^i \beta$. So the weight $w_n(A, B)$ simply becomes $(\prod_{i \in A} X_i)(\prod_{i \in B} Y_i)$. Then considering a directed graph in Figure 2.3, the formula for $g_{n+k,n}$ in Theorem 2.2.8 specializes to sum over weights of all paths from u_n to v_{n+k}. When $\epsilon_1 = 0$, we have an analogous result by Remark 2.4.6.

Generalized q-binomial coefficients

In this section, we prove Lemma 2.4.10 and Lemma 2.4.13.

It follows from definitions (Definition 2.2.7, Definitions 2.2.11) that

$$\sum_{\substack{B \subseteq \{0, \cdots, (n+a+b-1)\} \\ |B| = b}} w_n(A, B) = (\prod_{i \in A} X_i) M_n^{\lambda_A}(b)$$

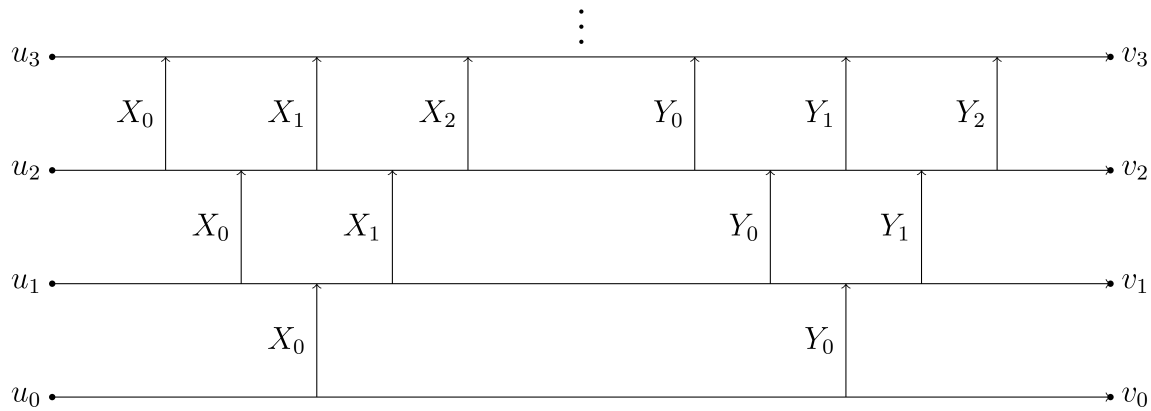

Figure 2.3: The figure shows the weighted directed graph that gives rise to $g_{n+k,n}$ when $\epsilon_2 = 0$.

where $A \subseteq \{0, \cdots, (n+a-1)\}$ with $|A| = a$. Thus the formula in Theorem 2.2.8 can be rephrased as follows

$$g_{n+k,k} = \sum_{a+b=k} \left(\sum_{\substack{A\subseteq\{0,\cdots,(n+a-1)\} \\ |A|=a}} (\prod_{i\in A} X_i) M_n^{\lambda_A}(b) \right). \tag{2.4.2}$$

Note that we defined $M_n^{\mu}(b)$ for a weakly increasing composition μ possibly starting with (-1). To do that we introduced a dummy variable $Y_{-1} = q^{-1}(\beta - \epsilon_1)$ which was defined accordingly to satisfy the recurrence $Y_{n+1} = qY_n + \epsilon_2$. As λ_A in (2.4.2) consists of non-negative integers, we do not see $M_n^{\mu}(b)$'s such that μ starts with (-1) in (2.4.2). However, we will need them for the proof.

When μ consists of non-negative integers, we see that $M_n^{\mu}(b)$ is a polynomial in $Y_0, \cdots, Y_{n+b-1}$ with $\mathbb{Z}[q]$ coefficients. The next proposition computes the coefficient of each monomial.

Proposition 2.4.8. *Let $E \subseteq \{0, \cdots, n+b-1\}$ with $|E| = b$ and $\mu = (\nu_1^{e_1}, \cdots, \nu_p^{e_p})$ such that $0 \le \nu_1 < \cdots < \nu_p \le n$ with $e_i > 0$. Denote the multiplicity of ν_i in λ_E (can be possibly zero) by f_i and write the corresponding elements of E with $c_i, \cdots, (c_i + f_i - 1)$. Then the coefficient of $\prod_{i\in E} Y_i$ in $M_n^{\mu}(b)$ is given as follows*

$$[\prod_{i\in E} Y_i] M_n^{\mu}(b) = \sum_{k_1,\cdots,k_p} \left(\prod_{i=1}^{p} (q^{k_i(d_i+k_i-1)} \binom{e_i}{k_i}_q \binom{f_i}{k_i}_q) \right),$$

where $d_i = (n + b + \sum_{j=i+1}^{p} e_j) - (c_i + f_i - 1)$ and k_i ranges from 0 to $min(e_i, f_i)$.

Proof. To get a monomial $\prod_{i\in E} Y_i$ in $M_n^{\mu}(b)$ we first pick $0 \le k_i \le min(e_i, f_i)$ and then k_i integers $(c_i + f_i - t_{i,1}) < \cdots < (c_i + f_i - t_{i,k_i})$ in a set $\{c_i, \cdots, c_i + f_i - 1\}$ and $(n + b + \sum_{j=i}^{p} e_j - u_{i,k_i}) <$

$\cdots < (n+b+\sum_{j=i}^{p} e_j - u_{i,1})$ in a set $\{(n+b+\sum_{j=i+1}^{p} e_j), \cdots, (n+b+\sum_{j=i}^{p} e_j - 1)\}$. Now for a set

$$B = (E - \bigcup_{i=1}^{p}\{c_i + f_i - t_{i,1} < \cdots < c_i + f_i - t_{i,k_i}\}) \cup (\bigcup_{i=1}^{p}\{n+b+\sum_{j=i}^{p} e_j - u_{i,k_i}, \cdots, n+b+\sum_{j=i}^{p} e_j - u_{i,1}\}),$$

we have

$$\begin{aligned} B(\nu_1 + j) &= (c_1 + f_1 - t_{1,j}) \text{ for } j \text{ from } 1 \text{ to } k_1 \\ B(\nu_2 + k_1 + j) &= (c_2 + f_2 - t_{2,j}) \text{ for } j \text{ from } 1 \text{ to } k_2 \\ &\vdots \\ B(\nu_p + \sum_{i=1}^{p-1} k_i + j) &= (c_p + f_p - t_{p,j}) \text{ for } j \text{ from } 1 \text{ to } k_p, \end{aligned}$$

which gives $m_n^{\mu}(B) = q^{\Sigma B - \Sigma E} \prod_{i\in E} Y_i$. Summing over all possible such B with fixed k_i's, we get

$$\prod_{i=1}^{p}(q^{k_i(d_i+k_i-1)}\binom{e_i}{k_i}_q \binom{f_i}{k_i}_q) \prod_{i\in E} Y_i. \tag{2.4.3}$$

Summing (2.4.3) over all possible k_i's gives the formula for the coefficient. □

Example 2.4.9. Let $E = \{0,1\}$ ($\lambda_E = (0,0)$) and $\mu = (0,0)$. Then the coefficient of Y_0Y_1 in $M_2^{\mu}(2)$ comes from the following terms

$m_2^{\mu}(\{0,1\}) = Y_0Y_1$
$m_2^{\mu}(\{0,4\}) = q^3Y_0Y_1, m_2^{\mu}(\{0,5\}) = q^4Y_0Y_1, m_2^{\mu}(\{1,4\}) = q^4Y_0Y_1, m_2^{\mu}(\{1,5\}) = q^5Y_0Y_1$
$m_2^{\mu}(\{4,5\}) = q^8Y_0Y_1.$

So we have

$$[Y_0Y_1]M_2^{\mu}(2) = 1 + (q^3 + 2q^4 + q^5) + q^8 = \binom{2}{0}_q\binom{2}{0}_q + q^{1\cdot 3}\binom{2}{1}_q\binom{2}{1}_q + q^{2\cdot 4}\binom{2}{2}_q\binom{2}{2}_q.$$

Now we give a generalization of (2.2.4).

Lemma 2.4.10. *For a weakly increasing composition $\mu = (\mu_1, \cdots, \mu_a)$ with $0 \le \mu_1, \cdots \mu_a \le n+1$, $n \ge 0$ and $b \ge 1$, the following identity holds*

$$M_{n+1}^{\mu}(b) = Y_{n+a+b}M_{n+1}^{\mu}(b-1) + M_n^{\mu-1}(b),$$

where $\mu - 1 = (\mu_1 - 1, \cdots, \mu_a - 1)$.

Proof. We have

$$\begin{aligned} & M^{\mu}_{n+1}(b) - (Y_{n+a+b}M^{\mu}_{n+1}(b-1) + M_n^{(\mu_1-1,\cdots,\mu_a-1)}(b)) \\ = & \sum_{\substack{B\subseteq\{0,\cdots,n+a+b\}\\ |B|=b,(n+a+b)\in B}} (m^{\mu}_{n+1}(B) - Y_{n+a+b}m^{\mu}_{n+1}(B-\{n+a+b\})) \\ & + \sum_{\substack{B\subseteq\{0,\cdots,n+a+b-1\}\\ |B|=b}} (m^{\mu}_{n+1}(B) - m_n^{(\mu_1-1,\cdots,\mu_a-1)}(B)). \end{aligned} \tag{2.4.4}$$

We may assume $Y_n = \beta_1 q^n + \epsilon$ after the change of variables, $\beta_1 = \beta - \frac{\epsilon_2}{1-q}$ and $\epsilon = \frac{\epsilon_2}{1-q}$. For a set $C \subseteq \{0,\cdots,n+b-1\}$ such that $|C| = b-k$ and an increasing integer sequence $J = (j_1,\cdots,j_k)$ such that $1 \le j_1,\cdots,j_k \le a+1$, we define

$$f^J_C(l) = \begin{cases} q^{(n+b+a+1-j_l)-C(\mu_{j_l}+l)}Y_{C(\mu_{j_l}+l)} & \text{if } j_l \ne a+1 \\ Y_{n+b} & \text{if } j_l = a+1 \end{cases}$$

$$d^J_C(k) = \begin{cases} (n+b+a+1-j_l) - C(\mu_{j_l}+l) & \text{if } j_l \ne a+1 \\ 0 & \text{if } j_l = a+1 \end{cases}.$$

We also define

$$\bar{f}^J_C(l) = \begin{cases} = q^{(n+b+a+1-j_l)-C(\mu_{j_l-1}+l-1)}Y_{C(\mu_{j_l-1}+l-1)} & \text{if } j_l \ne 1 \\ Y_{n+a+b} & \text{if } j_l = 1 \end{cases}$$

$$\bar{d}^J_C(k) = \begin{cases} (n+b+a+1-j_l) - C(\mu_{j_l-1}+l-1) & \text{if } j_l \ne 1 \\ 0 & \text{if } j_l = 1 \end{cases}.$$

Then by Definition 2.2.11 and Remark 2.4.5, for $B = C\cup\{n+b+a+1-j_k < \cdots < n+b+a+1-j_1\}$, we have

$$\begin{aligned} & m^{\mu}_{n+1}(B) = (\prod_{i\in C} Y_i)(f^J_C(k)\cdots f^J_C(1)) \\ & Y_{n+b+a}m^{\mu}_{n+1}(B-\{n+b+a\}) = (\prod_{i\in C} Y_i)(\bar{f}^J_C(k)\cdots \bar{f}^J_C(1)) \quad \text{if } (n+b+a)\in B \\ & m_n^{(\mu_1-1,\cdots,\mu_a-1)}(B) = (\prod_{i\in C} Y_i)(\bar{f}^J_C(k)\cdots \bar{f}^J_C(1)) \quad \text{if } (n+b+a)\notin B. \end{aligned}$$

So (2.4.4) can be written as follows

$$\sum_{k\ge 0}(\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\ |C|=b-k}} (\prod_{i\in C} Y_i)(\sum_{\substack{J=(j_1,\cdots,j_k)\\ 1\le j_1<\cdots<j_k\le a+1}} (f^J_C(k)\cdots f^J_C(1) - \bar{f}^J_C(k)\cdots \bar{f}^J_C(1)))). \tag{2.4.5}$$

We first rewrite the quantity

$$\sum_{\substack{J=(j_1,\cdots,j_k)\\ 1\le j_1<\cdots<j_k\le a+1}} (f^J_C(k)\cdots f^J_C(1) - \bar{f}^J_C(k)\cdots \bar{f}^J_C(1)) \tag{2.4.6}$$

as follows

$$\begin{aligned}
&\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} (f_C^J(k)\cdots f_C^J(1) - \bar{f}_C^J(k)\cdots\bar{f}_C^J(1))\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} (\sum_{l=1}^{k} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)\left(f_C^J(k-l+1) - \bar{f}_C^J(k-l+1)\right) f_C^J(k-l)\cdots f_C^J(1))\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} (\sum_{l=1}^{k} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)\left(q^{d_C^J(k-l+1)}\epsilon - q^{\bar{d}_C^J(k-l+1)}\epsilon\right) f_C^J(k-l)\cdots f_C^J(1))\\
&=\epsilon\sum_{l=1}^{k}(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)(q^{d_C^J(k-l+1)})f_C^J(k-l)\cdots f_C^J(1)) \qquad (2.4.7)\\
&\quad -\epsilon\sum_{l=1}^{k}(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)(q^{\bar{d}_C^J(k-l+1)})f_C^J(k-l)\cdots f_C^J(1)).
\end{aligned}$$

For $J = (j_1, \cdots, j_k)$ such that $j_{k-l+1} = j_{k-l} + 1$, we have

$$\bar{f}_C^J(k-l+1)(q^{d_C^J(k-l)}) = (q^{\bar{d}_C^J(k-l+1)})f_C^J(k-l)$$

since $C(\mu_{(j_{k-l+1}-1)} + k - l) = C(\mu_{j_{k-l}} + k - l)$. So we have

$$\begin{aligned}
&\bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+1)(q^{d_C^J(k-l)})f_C^J(k-l-1)\cdots f_C^J(1)\\
&= \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)(q^{\bar{d}_C^J(k-l+1)})f_C^J(k-l)\cdots f_C^J(1).
\end{aligned}$$

Applying this cancellation to (2.4.7), it becomes

$$\begin{aligned}
&\epsilon(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} q^{d_C^J}(k)f_C^J(k-1)\cdots f_C^J(1)) \qquad (2.4.8)\\
&+\epsilon\sum_{l=2}^{k}(\sum_{\substack{J=(j_1,\cdots,j_k)^{l-1}\\1\le j_1<\cdots<j_k\le a}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)(q^{d_C^J(k-l+1)})f_C^J(k-l)\cdots f_C^J(1))\\
&-\epsilon\sum_{l=1}^{k-1}(\sum_{\substack{J=(j_1,\cdots,j_k)^{l}\\1\le j_1<\cdots<j_k\le a}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(k-l+2)(q^{\bar{d}_C^J(k-l+1)})f_C^J(k-l)\cdots f_C^J(1))\\
&-\epsilon(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(2)q^{\bar{d}_C^J}(1)),
\end{aligned}$$

where $(j_1, \cdots, j_k)^l := (j_1, \cdots, j_{k-l}, (j_{k-l+1} + 1), \cdots, j_k + 1)$. We regard $(j_1, \cdots, j_k)^0 = (j_1, \cdots, j_k)$ by convention. Now define

$$W_C^J(l) = \bar{f}_C^{J'}(k)\cdots\bar{f}_C^{J'}(k-l+2)(q^{d_C^{J'}(k-l+1)})f_C^{J'}(k-l)\cdots f_C^{J'}(1),$$

where $J' = J^{l-1}$ and

$$\bar{W}_C^J(l) = \bar{f}_C^{J'}(k)\cdots\bar{f}_C^{J'}(k-l+2)(q^{\bar{d}_C^{J'}(k-l+1)})f_C^{J'}(k-l)\cdots f_C^{J'}(1)),$$

where $J' = J^l$. We also define

$$V_C^J = f_C^J(k)\cdots f_C^J(1),$$
$$\bar{V}_C^J = \bar{f}_C^{J'}(k+1)\cdots f_C^{J'}(2),$$

where $J' = (1, J^k)$ and k is a length of J. Since we have

$$\begin{aligned}
&\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} q^{d_C^J}(k)f_C^J(k-1)\cdots f_C^J(1)\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}} q^{d_C^J}(k)f_C^J(k-1)\cdots f_C^J(1) + \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k=a+1}} f_C^J(k-1)\cdots f_C^J(1)\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}} W_C^J(1) + \sum_{\substack{J'=(j_1,\cdots,j_{k-1})\\1\le j_1<\cdots<j_{k-1}\le a}} V_C^{J'},\\
&\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a+1}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(2)q^{\bar{d}_C^J}(1)\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)^k\\1\le j_1<\cdots<j_k\le a}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(2)q^{\bar{d}_C^J}(1) + \sum_{\substack{J=(j_1,\cdots,j_k)\\1=j_1<\cdots<j_k\le a+1}} \bar{f}_C^J(k)\cdots\bar{f}_C^J(2)\\
&= \sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}} \bar{W}_C^J(k) + \sum_{\substack{J'=(j_1,\cdots,j_{k-1})\\1\le j_1<\cdots<j_{k-1}\le a}} \bar{V}_C^{J'},
\end{aligned}$$

the quantity (2.4.8) can be written as

$$\epsilon(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}}\sum_{l=1}^{k}(W_C^J(l)-\bar{W}_C^J(l)))+\epsilon(\sum_{\substack{J'=(j_1,\cdots,j_{k-1})\\1\le j_1<\cdots<j_{k-1}=a}}(V_C^J-\bar{V}_C^J).$$

Plugging this to (2.4.5) and dividing with ϵ yields

$$\begin{aligned}
&\sum_{k\ge0}(\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}}(\prod_{i\in C}Y_i)(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}}\sum_{l=1}^{k}(W_C^J(l)-\bar{W}_C^J(l))))\\
&+\sum_{k\ge0}(\sum_{\substack{D\subseteq\{0,\cdots,n+b-1\}\\|D|=b-k-1}}(\prod_{i\in D}Y_i)(\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}}(V_D^J-\bar{V}_D^J))\\
&=\sum_{\substack{J=(j_1,\cdots,j_k)\\1\le j_1<\cdots<j_k\le a}}(\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}}(\prod_{i\in C}Y_i)\sum_{l=1}^{k}(W_C^J(l)-\bar{W}_C^J(l))\\
&+\sum_{\substack{D\subseteq\{0,\cdots,n+b-1\}\\|D|=b-k-1}}(\prod_{i\in D}Y_i)((V_D^J-\bar{V}_D^J))).
\end{aligned}\tag{2.4.9}$$

We will show that the quantity

$$\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}}(\prod_{i\in C}Y_i)\sum_{l=1}^{k}(W_C^J(l)-\bar{W}_C^J(l))+\sum_{\substack{D\subseteq\{0,\cdots,n+b-1\}\\|D|=b-k-1}}(\prod_{i\in D}Y_i)((V_D^J-\bar{V}_D^J)) \tag{2.4.10}$$

vanishes as a polynomial in Y_i's for every $J=(j_1,\cdots,j_k)$ with $1\le j_1<\cdots<j_k\le a$. Then it shows that (2.4.9) vanishes. We will denote $(\mu_{j_1},\cdots,\mu_{j_k})$ as $(\nu_1^{e_1},\cdots,\nu_p^{e_p})$ such that $\nu_1<\cdots<\nu_p$ and $e_i>0$. Then we have $W_l=\bar{W}_{l-1}$ if $l=(\sum_{i=r+1}^{p}e_i+2),\cdots,(\sum_{i=r}^{p}e_i)$ for $r=1,\cdots,p$. So (2.4.10) becomes

$$\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}}(\prod_{i\in C}Y_i)\sum_{r=1}^{p}(W_C^J(\sum_{i=r+1}^{p}e_i+1)-\bar{W}_C^J(\sum_{i=r}^{p}e_i)) \tag{2.4.11}$$
$$+\sum_{\substack{D\subseteq\{0,\cdots,n+b-1\}\\|D|=b-k-1}}(\prod_{i\in D}Y_i)((V_D^J-\bar{V}_D^J)).$$

Now we will take the coefficient of the monomial $\prod_{i\in E}Y_i$ in the quantity. Denote the multiplicity of ν_i in λ_E with f_i (can be possibly zero) and corresponding elements of E with $c_i,\cdots,(c_i+f_i-1)$. Let $d_{i,h}=(n+b+a-j_s)-(c_i+f_i)$ where $s=\sum_{l=1}^{i-1}e_l+h$ for $1\le h\le e_i$. To get a monomial $\prod_{i\in E}Y_i$ in $(\prod_{i\in C}Y_i)(W_C^J(\sum_{i=r+1}^{p}e_i+1)-\bar{W}_C^J(\sum_{i=r}^{p}e_i))$ we should take C as

$$C=E-(\bigcup_{\substack{i=1\\i\ne r}}^{p}\{c_i+f_i-t_{i,e_i}<\cdots<c_i+f_i-t_{i,1}\})-\{c_r+f_r-t_{r,e_r-1}<\cdots<c_r+f_r-t_{r,1}\}$$

such that $1\le t_{i,h}\le f_i$. Then for such C we have $(J'=J^{\sum_{i=r+1}^{p}e_i})$

$$f_C^{J'}(\sum_{l=1}^{i-1}e_l+h)=q^{d_{i,h}+t_{i,h}+1}Y_{c_i+f_i-t_{i,h}} \text{ for } 1\le i\le r-1 \text{ and } 1\le h\le e_i$$
$$f_C^{J'}(\sum_{l=1}^{r-1}e_l+h)=q^{d_{r,h}+t_{r,h}+1}Y_{c_r+f_r-t_{r,h}} \text{ for } 1\le h\le e_r-1$$
$$f_C^{J'}(\sum_{l=1}^{r}e_l)=q^{d_{r,e_r}+1}Y_{c_r+f_r}\to d_C^{J'}(\sum_{l=1}^{r}e_l)=d_{r,e_r}+1$$
$$\bar{f}_C^{J'}(\sum_{l=1}^{i-1}e_l+h)=q^{d_{i,h}+t_{i,h}}Y_{c_i+f_i-t_{i,h}} \text{ for } r+1\le i\le p \text{ and } 1\le h\le e_i,$$

which gives

$$(\prod_{i\in C}Y_i)W_C^J(\sum_{i=r+1}^{p}e_i+1)=q^{\sum_{i,h}d_{i,h}}q^{\sum_{l=1}^{r}e_l}q^{\sum_{i,h}t_{i,h}}\prod_{i\in E}Y_i.$$

And we have ($J' = J^{\sum_{i=r}^{p} e_i}$)

$$\bar{f}_C^{J'}(\sum_{l=1}^{r-1} e_l + 1) = q^{d_{r,1}+f_r+1} Y_{c_r - 1} \to d_C^{J'}(\sum_{l=1}^{r} e_l) = d_{r,1} + f_r + 1$$
$$\bar{f}_C^{J'}(\sum_{l=1}^{r-1} e_l + h + 1) = q^{d_{r,h}+t_{r,h}} Y_{c_r + f_r - t_{r,h}} \text{ for } 1 \le h \le e_r - 1,$$

which gives

$$(\prod_{i\in C} Y_i)\bar{W}_C^J(\sum_{i=r}^{p} e_i) = q^{\sum_{i,h} d_{i,h}} q^{\sum_{l=1}^{r-1} e_l} q^{(\sum_{i,h} t_{i,h})+f_r+1} \prod_{i\in E} Y_i.$$

So we have

$$\begin{aligned}&(\prod_{i\in C} Y_i)(W_C^J(\sum_{i=r+1}^{p} e_i + 1) - \bar{W}_C^J(\sum_{i=r}^{p} e_i))\\ =&q^{\sum_{i,h} d_{i,h}} q^{\sum_{l=1}^{r-1} e_l} q^{\sum_{i,h} t_{i,h}} (q^{e_r} - q^{f_r+1}) \prod_{i\in E} Y_i.\end{aligned}$$

Summing over all possible $t_{i,h}$'s we have

$$\begin{aligned}&[\prod_{i\in E} Y_i]\left(\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}} (\prod_{i\in C} Y_i)(W_C^J(\sum_{i=r+1}^{p} e_i + 1) - \bar{W}_C^J(\sum_{i=r}^{p} e_i)) \right)\\ =&q^{\sum_{i,h} d_{i,h}} q^{\sum_{l=1}^{r-1} e_l} q^{\binom{e_r}{2}} \binom{f_r}{e_r - 1}_q (\prod_{\substack{l=1\\l\neq r}}^{p} (q^{\binom{e_l+1}{2}} \binom{f_l}{e_l}_q))(q^{e_r} - q^{f_r+1})\\ =&q^{\sum_{i,h} d_{i,h}} q^{\sum_{l=1}^{r-1} e_l} (\prod_{l=1}^{p} (q^{\binom{e_l+1}{2}} \binom{f_l}{e_l}_q))(1 - q^{e_r}) = q^{\sum_{i,h} d_{i,h}} (\prod_{l=1}^{p} (q^{\binom{e_l+1}{2}} \binom{f_l}{e_l}_q))(q^{\sum_{l=1}^{r-1} e_l} - q^{\sum_{l=1}^{r} e_l}).\end{aligned}$$

Now summing over all $r = 1, \cdots, p$ we have

$$\begin{aligned}&[\prod_{i\in E} Y_i]\left(\sum_{\substack{C\subseteq\{0,\cdots,n+b-1\}\\|C|=b-k}} (\prod_{i\in C} Y_i) \sum_{r=1}^{p} (W_C^J(\sum_{i=r+1}^{p} e_i + 1) - \bar{W}_C^J(\sum_{i=r}^{p} e_i)) \right) \qquad (2.4.12)\\ =&q^{\sum_{i,h} d_{i,h}} (\prod_{l=1}^{p} (q^{\binom{e_l+1}{2}} \binom{f_l}{e_l}_q))(1 - q^{\sum_{l=1}^{p} e_l}).\end{aligned}$$

To get a monomial $\prod_{i\in E} Y_i$ in $(\prod_{i\in D} Y_i)((V_D^J - \bar{V}_D^J)$, we should take D as

$$D = E - \bigcup_{i=1}^{p} \{c_i + f_i - t_{i,e_i} < \cdots < c_i + m_i - t_{i,1}\}$$

such that $1 \le t_{i,h} \le f_i$. Then for such D we have

$$f_D^J(\sum_{l=1}^{i-1} e_l + h) = q^{d_{i,h}+t_{i,h}+1} Y_{c_i+f_i-t_{i,h}} \text{ for } 1 \le i \le p \text{ and } 1 \le h \le e_i$$
$$\bar{f}_D^{J'}(\sum_{l=1}^{i-1} e_l + h + 1) = q^{d_{i,h}+t_{i,h}} Y_{c_i+f_i-t_{i,h}} \text{ for } 1 \le i \le p \text{ and } 1 \le h \le e_i,$$

where $J' = (0, J^k)$. So we have

$$(\prod_{i\in D} Y_i) V_C^J = q^{\sum_{i,h} d_{i,h}} q^{\sum_{l=1}^{p} e_l} q^{\sum_{i,h} t_{i,h}} \prod_{i \in E} Y_i$$
$$(\prod_{i\in D} Y_i) \bar{V}_C^J = q^{\sum_{i,h} d_{i,h}} q^{\sum_{i,h} t_{i,h}} \prod_{i \in E} Y_i,$$

which gives

$$(\prod_{i\in D} Y_i)(V_C^J - \bar{V}_C^J) = q^{\sum_{i,h} d_{i,h}} q^{\sum_{i,h} t_{i,h}} (q^{\sum_{l=1}^{p} e_l} - 1) \prod_{i \in E} Y_i.$$

Summing over all possible $t_{i,h}$'s we have

$$\begin{aligned}
&[\prod_{i\in E} Y_i]\left(\sum_{\substack{D\subseteq\{0,\cdots,n+b-1\}\\ |D|=b-k-1}} (\prod_{i\in D} Y_i)((V_D^J - \bar{V}_D^J) \right) \qquad (2.4.13)\\
=&q^{\sum_{i,h} d_{i,h}} (\prod_{l=1}^{p} (q^{\binom{e_l+1}{2}} \binom{f_l}{e_l}_q))(q^{\sum_{l=1}^{p} e_l} - 1).
\end{aligned}$$

Adding (2.4.12) and (2.4.13), we see that the coefficient vanishes. □

Next, we generalize (2.2.5) (Lemma 2.4.13). Before stating and proving the generalization, we prepare with a definition and a lemma.

Definition 2.4.11. For $\nu = (\tau_1^{e_1}, \cdots, \tau_p^{e_p})$ such that $-1 \le \tau_1 < \cdots < \tau_p$ and $e_i > 0$, we define

$$\nu(i) = \begin{cases} (i, \tau_1^{e_1}, \cdots, \tau_p^{e_p}) & \text{if } -1 \le i \le \tau_1 - 1 \\ (\tau_1^{e_1}, \cdots, \tau_l^{e_l}, i, \tau_{l+1}^{e_{l+1}}, \cdots, \tau_p^{e_p}) & \text{if } \tau_l \le i \le \tau_{l+1} - 1 \\ (\tau_1^{e_1}, \cdots, \tau_p^{e_p}, i) & \text{if } \tau_p \le i \end{cases}.$$

For example, if $\nu = (-1, -1, 1)$, we have $\nu(-1) = (-1, -1, -1, 1)$, $\nu(0) = (-1, -1, 0, 1)$, $\nu(1) = (-1, -1, 1, 1)$ and $\nu(2) = (-1, -1, 1, 2)$.

Lemma 2.4.12. *For $\nu = (\nu_1, \cdots, \nu_{a-1})$ such that $0 \le \nu_1 \le \cdots \le \nu_{a-1} \le n + 1$, we have*

$$M_n^{(-1,\nu-1)}(b) = q^{n+a+b} Y_{-1} M_{n+1}^{\nu}(b-1) + M_n^{\nu-1}(b).$$

Proof. We have $m_n^{(-1,\nu-1)}(B\cup\{n+a+b-1\}) = q^{n+a+b}Y_{-1}m_{n+1}^{\nu}(B)$ for $B\subseteq\{0,\cdots,n+a+b-2\}$ with $|B| = b-1$ and $m_n^{(-1,\nu-1)}(B) = m_n^{\nu-1}(B)$ for $B\subseteq\{0,\cdots,n+a+b-2\}$ with $|B| = b$. □

Lemma 2.4.13. *For $\nu = (\tau_1^{e_1},\cdots,\tau_p^{e_p})$ such that $0\le\tau_1<\cdots<\tau_p\le n$ and $e_i>0$, we have*

$$[n+b+1+\sum_{i=1}^{p}e_i]_q M_n^{\nu}(b) = (\sum_{i=1}^{p} q^{(\tau_i+\sum_{l=1}^{i-1}e_l)}[e_i+1]_q M_n^{\nu(\tau_i)}(b)) + \sum_{l=0}^{\tau_1-1} q^l M_n^{\nu(l)}(b) \tag{2.4.14}$$
$$+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i}e_j)+l} M_n^{\nu(l)}(b)) + \sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p}e_j)+l} M_n^{\nu(l)}(b).$$

For $\nu = ((-1)^{e_1},\tau_2^{e_2},\cdots,\tau_p^{e_p})$ such that $0\le\tau_2<\cdots<\tau_p\le n$ and $e_i>0$, we have

$$[n+b+1+\sum_{i=1}^{p}e_i]_q M_n^{\nu}(b) = (\sum_{i=2}^{p} q^{(\tau_i+\sum_{l=1}^{i-1}e_l)}[e_i+1]_q M_n^{\nu(\tau_i)}(b)) + [e_1]_q M_n^{\nu(-1)}(b) \tag{2.4.15}$$
$$+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i}e_j)+l} M_n^{\nu(l)}(b)) + \sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p}e_j)+l} M_n^{\nu(l)}(b).$$

Proof. We first prove (2.4.14). We will show that the equality holds as a polynomial in Y_i's. Consider a length b integer vector $\mu = (0^{f_0},\cdots,n^{f_n})$ such that $f_i\ge 0$, then we will compare the coefficients of $Y_\mu(:=\prod_{i=1}^{b} Y_{\mu_i+i-1})$ in both sides of (2.4.14). First define the following

$$g^{(k_1,\cdots,k_p)}(i) = \begin{cases} q^i + q^{n+(\sum_{j=i+1}^{n}f_j)+(\sum_{j=1}^{p}e_j)+1}[f_i]_q & \text{if } 0\le i\le\tau_1-1 \\ q^{i+(\sum_{j=1}^{l}k_j+(\sum_{j=1}^{l}e_j)} + q^{n+(\sum_{j=i+1}^{n}f_j)+(\sum_{j=1}^{p}e_j)+(\sum_{j=1}^{l}k_j)+1}[f_i]_q & \text{if } \tau_l+1\le i\le\tau_{l+1}-1 \\ q^{i+(\sum_{j=1}^{p}k_j+(\sum_{j=1}^{p}e_j)} + q^{n+(\sum_{j=i+1}^{n}f_j)+(\sum_{j=1}^{p}e_j)+(\sum_{j=1}^{p}k_j)+1}[f_i]_q & \text{if } \tau_p+1\le i\le n \\ q^{\tau_l+(\sum_{j=1}^{l-1}e_j)+(\sum_{j=1}^{l-1}k_j)}[e_l+k_l+1]_q & \\ +q^{n+(\sum_{j=\tau_l+1}^{n}f_j)+(\sum_{j=1}^{p}e_j)+(\sum_{j=1}^{l}k_j)+1}[f_{\tau_l}-k_l]_q & \text{if } i=\tau_l \end{cases}$$

$$d(i) = \begin{cases} n+b+(\sum_{j=1}^{p}e_j)-(i+(\sum_{j=0}^{i}f_j)-1) & \text{if } 0\le i\le\tau_1-1 \\ n+b+(\sum_{j=l+1}^{p}e_j)-(i+(\sum_{j=0}^{i}f_j)-1) & \text{if } \tau_l\le i\le\tau_{l+1}-1 \\ n+b-(i+(\sum_{j=0}^{i}f_j)-1) & \text{if } \tau_p\le i\le n \end{cases},$$

where k_i is an integer from 0 to $min(e_i, f_{\tau_i})$. Then we have

$$\sum_{i=0}^{n} g^{(k_1,\cdots,k_p)}(i) = [n+b+1+\sum_{i=1}^{p} e_i]_q. \tag{2.4.16}$$

Next we will define the following

$$S^{\nu}(k_1,\cdots,k_p) = q^{\sum\limits_{i=1}^{p} k_i(d(\tau_i)+k_i-1)} \prod_{i=1}^{p}(\binom{e_i}{k_i}_q \binom{f_{\tau_i}}{k_i}_q)$$

$$S^{\nu(i)}(k_1,\cdots,k_p)$$
$$= \begin{cases} q^{\sum\limits_{i=1}^{p} k_i(d(\tau_i)+k_i-1)} (\prod\limits_{i=1}^{p}(\binom{e_i}{k_i}_q \binom{f_{\tau_i}}{k_i}_q))(1+q^{d(i)}[f_i]_q) & \text{if } 0 \le i \le \tau_1 - 1 \\ q^{\sum\limits_{i=1}^{l} k_i} q^{\sum\limits_{i=1}^{p} k_i(d(\tau_i)+k_i-1)} (\prod\limits_{i=1}^{p}(\binom{e_i}{k_i}_q \binom{f_{\tau_i}}{k_i}_q))(1+q^{d(i)}[f_i]_q) & \text{if } \tau_l + 1 \le i \le \tau_{l+1} - 1 \\ q^{\sum\limits_{i=1}^{p} k_i} q^{\sum\limits_{i=1}^{p} k_i(d(\tau_i)+k_i-1)} (\prod\limits_{i=1}^{p}(\binom{e_i}{k_i}_q \binom{f_{\tau_i}}{k_i}_q))(1+q^{d(i)}[f_i]_q) & \text{if } \tau_p + 1 \le i \le n \\ q^{\sum\limits_{i=1}^{l-1} k_i} q^{\sum\limits_{i=1}^{p} k_i(d(\tau_i)+k_i-1)} (\prod\limits_{\substack{i=1 \\ i \ne l}}^{p}(\binom{e_i}{k_i}_q \binom{f_{\tau_i}}{k_i}_q))\binom{e_l+1}{k_l}_q \binom{f_{\tau_l}}{k_l}_q & \text{if } i = \tau_l \end{cases}.$$

By Proposition 2.4.3, we have

$$[Y_\mu](M_n^{\nu}(b)) = \sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p)$$
$$[Y_\mu](M_n^{\nu(i)}(b)) = \sum_{k_1,\cdots,k_p} S^{\nu(i)}(k_1,\cdots,k_p).$$

Taking the coefficient of Y_μ in the left hand side of (2.4.14) and using (2.4.16) gives

$$(\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p))[n+b+1+\sum_{i=1}^{p} e_i]_q = \sum_{i=0}^{n}(\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p))g^{(k_1,\cdots,k_p)}(i)).$$

It is straightforward to check the following

$$S^{\nu}(k_1,\cdots,k_p)g^{(k_1,\cdots,k_p)}(i) = \begin{cases} q^i S^{\nu(i)}(k_1,\cdots,k_p) & \text{if } 0 \le i \le \tau_1 - 1 \\ q^{(\sum\limits_{j=1}^{l} e_j)+i} S^{\nu(i)}(k_1,\cdots,k_p) & \text{if } \tau_l + 1 \le i \le \tau_{l+1} - 1 \\ q^{(\sum\limits_{j=1}^{p} e_j)+i} S^{\nu(i)}(k_1,\cdots,k_p) & \text{if } \tau_p + 1 \le i \le n \end{cases},$$

so it suffices to prove

$$\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p))g^{(k_1,\cdots,k_p)}(\tau_l) = [Y_\mu](q^{\tau_l+(\sum\limits_{i=1}^{l-1} e_i)}[e_l+1]_q M_n^{\nu(\tau_l)}(b)). \tag{2.4.17}$$

We first claim the following identity

$$q^{\tau_l+(\sum_{j=1}^{l-1} e_j)}[e_l+1]_q S^{\nu(\tau_l)}(k_1,\cdots,k_p) = S^{\nu}(k_1,\cdots,k_p)(q^{\tau_l+(\sum_{j=1}^{l-1} e_j)+(\sum_{j=1}^{l-1} k_j)}[e_l+k_l+1])_q \tag{2.4.18}$$
$$+S^{\nu}(k_1,\cdots,k_l-1,\cdots,k_p)(q^{n+(\sum_{j=\tau_l+1}^{n} f_j)+(\sum_{j=1}^{p} e_j)+(\sum_{j=1}^{l} k_j)}[f_{\tau_l}-k_l+1]_q),$$

which is equivalent to the following (after cancelling a common factor)

$$[e_l+1]_q\binom{e_l+1}{k_l}_q\binom{f_{\tau_l}}{k_l}_q = [e_l+k_l+1]_q\binom{e_l}{k_l}_q\binom{f_{\tau_l}}{k_l}_q$$
$$+q^{e_l-k_l+1}[f_{\tau_l}-k_l+1]_q\binom{e_l}{k_l-1}_q\binom{f_{\tau_l}}{k_l-1}_q.$$

And this can be checked by a direct computation. Now the left hand side of (2.4.17) becomes

$$\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p)g^{(k_1,\cdots,k_p)}(\tau_l)$$
$$=\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p))(q^{\tau_l+(\sum_{j=1}^{l-1} e_j)+(\sum_{j=1}^{l-1} k_j)}[e_l+k_l+1]_q + q^{n+(\sum_{j=\tau_l+1}^{n} f_j)+(\sum_{j=1}^{p} e_j)+(\sum_{j=1}^{l} k_j)+1}[f_{\tau_l}-k_l]_q)$$
$$=\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_p)(q^{\tau_l+(\sum_{j=1}^{l-1} e_j)+(\sum_{j=1}^{l-1} k_j)}[e_l+k_l+1])_q$$
$$+\sum_{k_1,\cdots,k_p} S^{\nu}(k_1,\cdots,k_l-1,\cdots,k_p)(q^{n+(\sum_{j=\tau_l+1}^{n} f_j)+(\sum_{j=1}^{p} e_j)+(\sum_{j=1}^{l} k_j)}[f_{\tau_l}-k_l+1]_q).$$
$$=\sum_{k_1,\cdots,k_p}(q^{\tau_l+(\sum_{j=1}^{l-1} e_j)}[e_l+1]_q S^{\nu(\tau_l)}(k_1,\cdots,k_p)),$$

where the last equality uses (2.4.18). So this proves (2.4.17).

We will show (2.4.15) using an induction on e_1. The base case $e_1=0$ is same as (2.4.14). Assume (2.4.15) holds for $\nu=((-1)^{e_1},\tau_2^{e_2},\cdots,\tau_p^{e_p})$. And let $\nu'=(-1,\nu)$ and $\nu^+=\nu+1$, then we have

$$\nu'(i)=(-1,\nu(i)),\ \ \nu^+(i)=\nu(i-1)+1. \tag{2.4.19}$$

Writing (2.4.15) for ν and (2.4.14) for ν^+, we have

$$[n+b+1+\sum_{i=1}^{p} e_i]_q M_n^{\nu}(b) =(\sum_{i=2}^{p} q^{(\tau_i+\sum_{l=1}^{i-1} e_l)}[e_i+1]_q M_n^{\nu(\tau_i)}(b)) + [e_1]_q M_n^{\nu(-1)}(b) \tag{2.4.20}$$
$$+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i} e_j)+l} M_n^{\nu(l)}(b)) + \sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p} e_j)+l} M_n^{\nu(l)}(b)$$

$$\begin{aligned}
&[n+b+1+\sum_{i=1}^{p} e_i]_q M_{n+1}^{\nu^+}(b-1) \qquad (2.4.21)\\
=&\sum_{i=2}^{p} q^{(\tau_i+\sum_{l=1}^{i-1} e_l+1)}[e_i+1]_q M_{n+1}^{\nu^+(\tau_i+1)}(b-1)) + [e_1+1]_q M_{n+1}^{\nu^+(0)}(b-1)\\
&+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i} e_j)+l+1} M_{n+1}^{\nu^+(l+1)}(b-1)) + \sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p} e_j)+l+1} M_{n+1}^{\nu^+(l+1)}(b-1).
\end{aligned}$$

Multiplying q to (2.4.20), multiplying $(q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1})$ to (2.4.21) and adding these two we have

$$\begin{aligned}
&q[n+b+1+\sum_{i=1}^{p} e_i]_q (M_n^{\nu}(b) + q^{n+b+1+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+}(b-1))\\
=&\sum_{i=2}^{p} q^{(\tau_i+\sum_{l=1}^{i-1} e_l+1)}[e_i+1]_q (M_n^{\nu(\tau_i)}(b) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(\tau_i+1)}(b-1))\\
&+q[e_1]_q (M_n^{\nu(-1)}(b) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(0)}(b-1)) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(0)}(b-1)\\
&+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i} e_j)+l+1} (M_n^{\nu(l)}(b) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(l+1)}(b-1))))\\
&+\sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p} e_j)+l+1} (M_n^{\nu(l)}(b) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(l+1)}(b-1))).
\end{aligned}$$

By Lemma 2.4.12 and (2.4.19), it becomes

$$\begin{aligned}
&q[n+b+1+\sum_{i=1}^{p} e_i]_q M_n^{\nu'}(b)\\
=&(\sum_{i=2}^{p} q^{(\tau_i+\sum_{l=1}^{i-1} e_l+1)}[e_i+1]_q M_n^{\nu'(\tau_i)}(b)) + q[e_1]_q M_n^{\nu'(-1)}(b) + q^{n+b+2+(\sum_{i=1}^{p} e_i)} Y_{-1} M_{n+1}^{\nu^+(0)}(b-1)\\
&+\sum_{i=1}^{p-1}(\sum_{l=\tau_i+1}^{\tau_{i+1}-1} q^{(\sum_{j=1}^{i} e_j)+l+1} M_n^{\nu'(l)}(b)) + \sum_{l=\tau_p+1}^{n} q^{(\sum_{j=1}^{p} e_j)+l+1} M_n^{\nu'(l)}(b).
\end{aligned}$$

Adding $M_n^{\nu'}(b)(= M_n^{\nu(-1)}(b))$ to both sides gives (2.4.15) for ν'. □

Proof of Theorem 2.2.8

For a weakly increasing composition $\mu = (\mu_1, \cdots, \mu_a)$, we define $X_\mu = \prod_{i=1}^{a} X_{\mu_i+i-1}$. Then (2.4.2) can be rephrased as follows

$$g_{n+k,n} = \sum_{a+b=k} (\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\ 0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu M_n^{\mu}(b)). \qquad (2.4.22)$$

Since we already know that Theorem 2.2.8 is true for $g_{k,0}$ (Example 2.2.9, Proposition 2.3.6), it suffices to prove that (2.4.22) satisfies the recurrence relation

$$g_{n+1+k,n+1} = g_{n+k,n} + (b_{n+k})g_{n+k,n+1} - (\lambda_{n+k})g_{n+k-1,n+1}. \tag{2.4.23}$$

We will show the identity

$$\begin{aligned}\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n+1}} X_\mu M^\mu_{n+1}(b) = \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu M^\mu_n(b) + X_{n+a+b}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu M^\mu_{n+1}(b)\Big)\\ +Y_{n+a+b}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n+1}} X_\mu M^\mu_{n+1}(b-1)\Big) - (\lambda_{n+a+b})\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu M^\mu_{n+1}(b-1)\Big),\end{aligned} \tag{2.4.24}$$

which gives (2.4.23) when summed over all possible a and b such that $a+b=k$. Using Lemma 2.4.10 and Lemma 2.4.12, the identity (2.4.24) becomes

$$\begin{aligned}\leftrightarrow \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n+1}} X_\mu(M^\mu_{n+1}(b) - Y_{n+a+b}(M^\mu_{n+1}(b-1)) = \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu M^\mu_n(b)\\ +X_{n+a+b}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu(M^\mu_{n+1}(b) - Y_{n+a+b-1}(M^\mu_{n+1}(b))\\ +(\alpha q^{n+a+b}Y_{-1})\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu M^\mu_{n+1}(b-1)\Big)\end{aligned}$$

$$\begin{aligned}\leftrightarrow \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n+1}} X_\mu(M^{\mu-1}_n(b)) = \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu M^\mu_n(b)\\ +X_{n+a+b}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu(M^{\mu-1}_n(b)) + (\alpha q^{n+a+b}Y_{-1})\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu M^\mu_{n+1}(b-1)\Big)\end{aligned}$$

$$\begin{aligned}\leftrightarrow \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu(M^{\mu-1}_n(b)) + \sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1},n+1)\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu(M^{\mu-1}_n(b)) = \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu M^\mu_n(b)\\ +X_{n+a+b}\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu(M^{\mu-1}_n(b)) + (\alpha q^{n+a+b}Y_{-1})\Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_{a-1})\\0\le\mu_1\le\cdots\le\mu_{a-1}\le n+1}} X_\mu M^\mu_{n+1}(b-1)\Big)\end{aligned}$$

$$\begin{aligned}\leftrightarrow\\ \sum_{\substack{\nu=(\nu_1,\cdots,\nu_{a-1})\\0\le\nu_1\le\cdots\le\nu_{a-1}\le n+1}} (X_{(\nu,n+1)}M^{(\nu,n+1)-1}_n(b) - X_{n+a+b}X_\nu M^{\nu-1}_n(b) - \alpha q^{n+a+b}Y_{-1}X_\nu M^\nu_{n+1}(b-1))\\ = \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu(M^\mu_n(b) - M^{\mu-1}_n(b)).\end{aligned}$$

$$\overleftrightarrow{\sum_{\substack{\nu=(\nu_1,\cdots,\nu_{a-1})\\0\le\nu_1\le\cdots\le\nu_{a-1}\le n+1}} (X_{(\nu,n+1)}M_n^{(\nu,n+1)-1}(b) - (X_{n+a+b}-X_0)X_\nu M_n^{\nu-1}(b) - X_0X_\nu M_n^{(-1,\nu-1)}(b))}$$

$$= \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu(M_n^\mu(b) - M_n^{\mu-1}(b)). \tag{2.4.25}$$

We will show (2.4.25) by showing its refinement (Proposition 2.4.15).

Definition 2.4.14. For $\nu = (\tau_1^{e_1},\cdots,\tau_p^{e_p})$ such that $0\le\tau_1<\cdots<\tau_p$ and $e_i>0$, we define

$$\bar{\nu}(i) = \begin{cases} (i,(\tau_1-1)^{e_1},\cdots,(\tau_p-1)^{e_p}) & \text{if } -1\le i\le\tau_1-1\\ (\tau_1^{e_1},\cdots,\tau_l^{e_l},i,(\tau_{l+1}-1)^{e_{l+1}},\cdots,(\tau_p-1)^{e_p}) & \text{if } \tau_l\le i\le\tau_{l+1}-1\\ (\tau_1^{e_1},\cdots,\tau_p^{e_p},i) & \text{if } \tau_p\le i\end{cases}$$

Proposition 2.4.15. *For $\nu=(\nu_1,\cdots,\nu_{a-1})$ such that $0\le\nu_1\le\cdots\le\nu_{a-1}\le n+1$, denoting $\nu^-=\nu-1$, we have*

$$X_{(\nu,n+1)}M_n^{(\nu,n+1)-1}(b) - (X_{n+a+b}-X_0)X_\nu M_n^{\nu-1}(b) - X_0X_\nu M_n^{(-1,\nu-1)}(b) = \sum_{i=0}^{n} X_{\bar{\nu}(i)}(M_n^{\nu^-(i)}(b) - M_n^{\nu^-(i-1)}(b)). \tag{2.4.26}$$

Proof. Note that we have

$$\nu^-(n) = (\nu,n+1)-1,\ \ \nu^-(-1) = (-1,\nu-1),\ \ \bar{\nu}(n+1) = (\nu,n+1).$$

So the identity (2.4.26) becomes

$$(X_{n+a+b}-X_0)X_\nu M_n^{\nu^-}(b) = M_n^{\nu^-(-1)}(b)(X_{\bar{\nu}(0)}-X_\nu X_0) + \sum_{i=0}^{n} M_n^{\nu^-(i)}(b)(X_{\bar{\nu}(i+1)}-X_{\bar{\nu}(i)}). \tag{2.4.27}$$

Writing $\nu^- = (\tau_1^{e_1},\cdots,\tau_p^{e_p})$ such that $-1\le\tau_1<\cdots<\tau_p$ and $e_i>0$, we have

$$X_{\bar{v}(0)} - X_vX_0 = \begin{cases} 0 & \text{if } \tau_1>-1\\ X_\nu(X_{e_1}-X_0) & \text{if } \tau_1=-1\end{cases}$$

$$X_{\bar{v}(i+1)} - X_{\bar{v}(i)} = \begin{cases} X_\nu(X_{i+1}-X_i) & \text{if } 0\le i\le\tau_1-1\\ X_\nu(X_{k+1}-X_k) & \text{if } \tau_l+1\le i\le\tau_{l+1}-1 \text{ and } k=(\sum_{j=1}^{l}e_j)+i\\ X_\nu(X_{k+1}-X_k) & \text{if } \tau_p+1\le i\le n \text{ and } k=(\sum_{j=1}^{p}e_j)+i\\ X_\nu(X_{k+e_l+1}-X_k) & \text{if } i=\tau_l \text{ and } k=(\sum_{j=1}^{l-1}e_j)+\tau_l\end{cases}.$$

Since $X_{k+i}-X_k = q^k[i]_q(X_1-X_0)$, dividing (2.4.27) with $X_\nu(X_1-X_0)$, we have (2.4.14) or (2.4.15) for ν^- which was proved in Lemma 2.4.13. □

Proof of the identity (2.4.25). By Proposition 2.4.15, the left hand side of (2.4.25) becomes

$$\sum_{\substack{\nu=(\nu_1,\cdots,\nu_{a-1})\\ 0\le\nu_1\le\cdots\le\nu_{a-1}\le n+1}} \left(\sum_{i=0}^{n} X_{\bar{v}(i)}(M_n^{\nu^-(i)}(b) - M_n^{\nu^-(i-1)}(b))\right), \tag{2.4.28}$$

where $\nu^- = \nu - 1$. For $\mu = (\mu_1, \cdots, \mu_a)$ such that $0 \le \mu_1 \le \cdots \le \mu_a \le n$, the term X_μ appears on (2.4.28) when $\nu = (\mu_1, \cdots, \mu_{l-1}, \mu_{l+1}+1, \cdots, \mu_a+1)$ with the coefficient $(M_n^{(\mu_1-1,\cdots,\mu_{l-1}-1,\mu_l,\cdots,\mu_a)}(b) - M_n^{(\mu_1-1,\cdots,\mu_l-1,\mu_{l+1},\cdots,\mu_a)}(b))$ for $l = 1, \cdots, a$. So (2.4.28) becomes

$$\begin{aligned}
&\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\ 0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu(\sum_{l=1}^{a}(M_n^{(\mu_1-1,\cdots,\mu_{l-1}-1,\mu_l,\cdots,\mu_a)}(b) - M_n^{(\mu_1-1,\cdots,\mu_l-1,\mu_{l+1},\cdots,\mu_a)}(b)))\\
&= \sum_{\substack{\mu=(\mu_1,\cdots,\mu_a)\\ 0\le\mu_1\le\cdots\le\mu_a\le n}} X_\mu(M_n^{\mu}(b) - M_n^{\mu-1}(b)).
\end{aligned}$$

□

2.5 Proof of Theorem 2.2.14

We prepare with lemmas to prove Theorem 2.2.14. Recall that we write $f_1 \ge f_2$ if $(f_1 - f_2)$ is a polynomial with positive coefficients.

Lemma 2.5.1. *Let $f_1, \cdots, f_i$ and $h_1, \cdots, h_i$ be polynomials such that $f_j \ge h_j$ for all j and $h_1, \cdots, h_{i-1} \ge 0$. Then $f_1 \cdots f_i \ge h_1 \cdots h_i$.*

Proof. We have $f_1\cdots f_i - h_1\cdots h_i = (f_1\cdots f_i - h_1\cdots h_{i-1} f_i) + h_1\cdots h_{i-1}(f_i - h_i)$ and it is trivial to see $(f_1\cdots f_i - h_1\cdots h_{i-1} f_i) \ge 0$ and $h_1\cdots h_{i-1}(f_i - h_i) \ge 0$. □

Lemma 2.5.2. *For weakly increasing compositions μ and ν such that $l_1 = length(\mu) \ge l_2 = length(\nu)$, $\mu_1 \ge -1$, $\nu_1 \ge 0$ and $\mu_i \le \nu_i$ for all possible i, we have $M^{\nu}_{n-l_2}(b) \ge M^{\mu}_{n-l_1}(b)$ for all valid n and b.*

Proof. It is enough to show $m^{\nu}_{n-l_2}(B) \le m^{\mu}_{n-l_1}(B)$ for all $B \subset \{0, \cdots, n+b-1\}$ with $|B| = b$. Let $B \cap \{n+b-l_2, \cdots, n+b-1\} = \{n+b-j_k \cdots < n+b-j_1\}$ and $B \cap \{n+b-l_1, \cdots, n+b-l_2-1\} = \{n+b-j_{k+k'} \cdots < n+b-j_{k+1}\}$. Then we have

$$\begin{aligned}
m^{\mu}_{n-l_1}(B) &= (\prod_{i\in B\cap\{0,\cdots,n+b-l_1-1\}} Y_i)(\prod_{i=1}^{k+k'} q^{(n+b-j_i)-B(\mu_{j_i}+i)} Y_{B(\mu_{j_i}+i)})\\
m^{\nu}_{n-l_2}(B) &= (\prod_{i\in B\cap\{0,\cdots,n+b-l_1-1\}} Y_i)(\prod_{i=k+1}^{k+k'} Y_{n+b-j_i})(\prod_{i=1}^{k'} q^{(n+b-j_i)-B(\nu_{j_i}+i)} Y_{B(\nu_{j_i}+i)}).
\end{aligned}$$

Since $\mu_{j_i} \le \nu_{j_i}$ we have $B(\mu_{j_i}+i) \le B(\nu_{j_i}+i)$, which implies

$$q^{(n+b-j_i)-B(\mu_{j_i}+i)} Y_{B(\mu_{j_i}+i)} \le q^{(n+b-j_i)-B(\nu_{j_i}+i)} Y_{B(\nu_{j_i}+i)}$$

for $1 \le i \le k$. We also have

$$q^{(n+b-j_i)-B(\mu_{j_i}+i)} Y_{B(\mu_{j_i}+i)} \le Y_{n+b-j_i}$$

for $k+1 \le i \le k+k'$. And every term is a polynomial with positive coefficients except for $q^{(n+b-1)-B(0)}Y_{B(0)} = q^{n+b}Y_{-1}$ when $(n+b-1) \in B$ and $\mu_1 = -1$. So we have $m^{\nu}_{n-l_2}(B) \le m^{\mu}_{n-l_1}(B)$ for all $B \subset \{0, \cdots, n+b-1\}$ by Lemma 2.5.1. □

Example 2.5.3. For $\mu = (-1,1,1)$, $\nu = (0,1)$ and $B = \{0,5,6,7\}$, we have

$$m^{\mu}_1(B) = m^{(-1,1,1)}_1(\{0,5,6,7\}) = Y_0(qY_4)(q^3Y_3)(q^8Y_{-1})$$
$$m^{\nu}_2(B) = m^{(0,1)}_2(\{0,5,6,7\}) = Y_0(Y_5)(q^3Y_3)(q^6Y_1).$$

Since $qY_4 \le Y_5$, $q^8Y_{-1} \le q^6Y_1$ and every term except (q^8Y_{-1}) is a polynomial with positive coefficients, we have $m^{\mu}_1(B) \le m^{\nu}_2(B)$.

Lemma 2.5.4. *For $\mu = (\mu_1, \cdots, \mu_l)$ such that $0 \le \mu_1 \le \cdots \le \mu_l \le n$, we have*

$$M^{\mu}_n(b) \le \sum_{j=1}^{l} \Big(\sum_{\substack{0 \le \nu_1 \le \cdots \le \nu_k \le n+l \\ \nu_1 = n+l+1-j}} \big(M^{(\mu_j,\cdots,\mu_l)}_n (b-k) \prod_{i=1}^{k} Y_{\nu_i+b-k+i-1} \big) \Big) \tag{2.5.1}$$
$$+ \sum_{\substack{0 \le \nu_1 \le \cdots \le \nu_k \le n+l \\ \nu_1 \le n}} \big(M_{\nu_1}(b-k) \prod_{i=1}^{k} Y_{\nu_i+b-k+i-1} \big).$$

Proof. By Lemma 2.4.10, we have $M^{\mu}_n(b) = Y_{n+l+b-1}M^{\mu}_n(b-1) + M^{\mu-1}_{n-1}(b)$ and by Lemma 2.5.4, we have $M^{\mu-1}_{n-1}(b) \le M^{(\mu_2,\cdots,\mu_l)}_n(b)$ which gives

$$M^{\mu}_n(b) \le Y_{n+l+b-1}M^{\mu}_n(b-1) + M^{(\mu_2,\cdots,\mu_l)}_n(b). \tag{2.5.2}$$

Applying (2.5.2) to $M^{(\mu_2,\cdots,\mu_l)}_n(b)$ on the right hand side, we have

$$M^{\mu}_n(b) \le Y_{n+l+b-1}M^{\mu}_n(b-1) + Y_{n+l+b-2}M^{(\mu_2,\cdots,\mu_l)}_n(b-1) + M^{(\mu_3,\cdots,\mu_l)}_n(b-1). \tag{2.5.3}$$

Keeping this process, we have

$$M^{\mu}_n(b) \le Y_{n+l+b-1}M^{\mu}_n(b-1) + Y_{n+l+b-2}M^{(\mu_2,\cdots,\mu_l)}_n(b-1) + \cdots \tag{2.5.4}$$
$$+ Y_{n+b}M^{(\mu_l)}_n(b-1) + M_n(b).$$

Since $M_n(b)$ is an elementary symmetric polynomial of degree b with variables from Y_0 to Y_{n+b-1}, we have

$$M_n(b) = Y_{n+b-1}M_n(b-1) + Y_{n+b-2}M_{n-1}(b-1) + \cdots + Y_{b-1}M_0(b-1). \tag{2.5.5}$$

Applying (2.5.5) to (2.5.4) gives

$$M_n^\mu(b) \leq Y_{n+l+b-1}M_n^\mu(b-1) + Y_{n+l+b-2}M_n^{(\mu_2,\cdots,\mu_l)}(b-1) + \cdots \tag{2.5.6}$$
$$+Y_{n+b}M_n^{(\mu_l)}(b-1) + Y_{n+b-1}M_n(b-1) + Y_{n+b-2}M_{n-1}(b-1) + \cdots + Y_{b-1}M_0(b-1).$$

The inequality (2.5.6) corresponds to the case $k = 1$ of (2.5.1). Applying (2.5.5) and (2.5.6) to $M_n^\mu(b-1), \cdots, M_n^{(\mu_l)}(b-1), M_n(b-1), \cdots, M_0(b-1)$ on the right hand side of (2.5.6) gives $k = 2$ of (2.5.1). Subsequently applying (2.5.5) and (2.5.6) gives (2.5.1) for any k. □

Note that we have $M_n^{(\mu_j,\cdots,\mu_l)}(b-k) \leq M_n^\mu(b-k)$ for $1 \leq j \leq l$ and $M_j(b-k) \leq M_n^\mu(b-k)$ for $0 \leq j \leq n$. Applying this to inequality (2.5.1) gives

$$M_n^\mu(b) \leq M_n^\mu(b-k)(\sum_{0\leq\nu_1\leq\cdots\leq\nu_k\leq n+l}(\prod_{i=1}^{k} Y_{\nu_i+b-k+i-1})),$$

and since $\sum_{0\leq\nu_1\leq\cdots\leq\nu_k\leq n+l}(\prod_{i=1}^{k} Y_{\nu_i+b-k+i-1}) \leq M_{n+l+b-k}(k)$, we have

$$M_n^\mu(b) \leq M_n^\mu(b-k)M_{n+l+b-k}(k). \tag{2.5.7}$$

Proof of Theorem 2.2.14. By Theorem 2.2.8, we have

$$g_{n+a+b,n} = \sum_{c=0}^{a+b}(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_c)\\ 0\leq\mu_1\leq\cdots\leq\mu_c\leq n}} X_\mu M_n^\mu(a+b-c)).$$

When $c \geq a$, let $\mu' = (\mu_1, \cdots, \mu_a)$ and $\nu = (\mu_{a+1}+a, \cdots, \mu_c+a)$ then $X_\mu = X_{\mu'}X_\nu$ and $M_n^\mu(a+b-c) \leq M_{n+a}^\nu(a+b-c)$ by Lemma 2.5.2. So we have

$$X_\mu M_n^\mu(a+b-c)) \leq X_{\mu'}(X_\nu M_{n+a}^\nu(a+b-c)). \tag{2.5.8}$$

When $c < a$, by (2.5.7), we have

$$M_n^\mu(a+b-c) \leq M_n^\mu(a-c)M_{n+a}^\mu(b),$$

which gives

$$X_\mu M_n^\mu(a+b-c)) \leq (X_\mu M_n^\mu(a-c))M_{n+a}(b). \tag{2.5.9}$$

Terms on the right hand sides of (2.5.8) and (2.5.9) appear in $g_{n+a+b,n+a}g_{n+a,n}$ and they do not overlap. So summing up (2.5.8) and (2.5.9) for all possible c and μ gives

$$g_{n+a+b,n} \preceq g_{n+a+b,n+a}g_{n+a,n}.$$

□

Chapter 3

Schubert polynomials and the inhomogeneous TASEP on a ring

The results of this chapter were announced in [26], and the proofs will appear in [25].

3.1 Introduction

In recent years, there has been a lot of work on interacting particle models such as the *asymmetric simple exclusion process* (ASEP), a model in which particles hop on a one-dimensional lattice subject to the condition that at most one particle may occupy a given site. The ASEP on a one-dimensional lattice with open boundaries has been linked to Askey-Wilson polynomials and Koornwinder polynomials [36, 13, 5, 12], while the ASEP on a ring has been linked to Macdonald polynomials [7, 10]. The *inhomogeneous totally asymmetric simple exclusion process* (TASEP) is a variant of the exclusion process on the ring in which the hopping rate depends on the weight of the particles. In this paper we build on works of Lam-Williams [28], Ayyer-Linusson [3], and especially Cantini [6] to give formulas for many steady state probabilities of the inhomogeneous TASEP on a ring in terms of Schubert polynomials.

Definition 3.1.1. Consider a lattice with n sites arranged in a ring. Let $\mathrm{St}(n)$ denote the $n!$ labelings of the lattice by distinct numbers $1, 2, \ldots, n$, where each number i is called a *particle of weight* i. The *inhomogeneous TASEP on a ring of size* n is a Markov chain with state space $\mathrm{St}(n)$ where at each time t a swap of two adjacent particles may occur: a particle of weight i on the left swaps its position with a particle of weight j on the right with transition rate $r_{i,j}$ given by:

$$r_{i,j} = \begin{cases} x_i - y_{n+1-j} & \text{if } i < j \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we will identify each state with a permutation in S_n. Following [28, 6], we multiply all steady state probabilities for $\mathrm{St}(n)$ by the same constant, obtaining "renormalized" steady state probabilities ψ_w, so that

$$\psi_{123\ldots n} = \prod_{i<j} (x_i - y_{n+1-j})^{j-i-1}. \tag{3.1.1}$$

See Figure 3.1 for the transition diagram and renormalized steady state probabilities when $n = 3$.

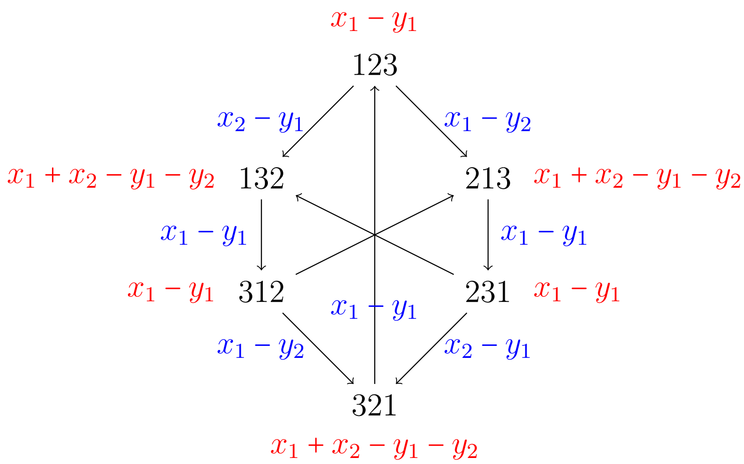

Figure 3.1: The transition diagram for the inhomogeneous TASEP on St(3), with transition rates shown in blue, and renormalized steady state probabilities ψ_w in red. Though not shown, the transition rate $312 \to 213$ is $x_2 - y_1$ and the transtition rate $231 \to 132$ is $x_1 - y_2$.

In the case that $y_i = 0$, Lam and Williams [28] studied this model[1] and conjectured that after a suitable normalization, each steady state probability ψ_w can be written as a monomial factor times a positive sum of Schubert polynomials, see Table 3.1 and Table 3.2. They also gave an explicit formula for the monomial factor, and conjectured that under certain conditions on w, ψ_w is a multiple of a particular Schubert polynomial. Subsequently Ayyer and Linusson [3] gave a conjectural combinatorial formula for the stationary distribution in terms of *multiline queues*, which was proved by Arita and Mallick [1]. In [6], Cantini introduced the version of the model given in Definition 3.1.1[2] with y_i general, and gave a series of *exchange equations* relating the components of the stationary distribution. This allowed him to give explicit formulas for the steady state probabilities for n of the $n!$ states as products of double Schubert polynomials.

In this paper we build on [6, 3, 1], and give many more explicit formulas for steady state probabilities in terms of Schubert polynomials: in particular, we give a formula for ψ_w as a

[1]However the convention of [28] was slightly different; it corresponds to labeling states by the inverse of the permutations we use here.

[2]We note that in [6], the rate $r_{i,j}$ was $x_i - y_j$ rather than $x_i - y_{n+1-j}$ as we use in Definition 3.1.1.

State w	Probability ψ_w
1234	$(x_1-y_1)^2(x_1-y_2)(x_2-y_1)$
1324	$(x_1-y_1)\mathfrak{S}_{1432}$
1342	$(x_1-y_1)(x_2-y_1)\mathfrak{S}_{1423}$
1423	$(x_1-y_1)(x_1-y_2)(x_2-y_1)\mathfrak{S}_{1243}$
1243	$(x_1-y_2)(x_1-y_1)\mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423}\mathfrak{S}_{1342}$

Table 3.1: The renormalized steady state probabilities for $n=4$.

product of (double) Schubert polynomials whenever w is *evil-avoiding*, that is, it avoids the patterns 2413, 4132, 4213 and 3214. We show that there are $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$ evil-avoiding permutations in S_n, so this gives a substantial generalization of Cantini's previous result [6] in this direction.

In order to state our main results, we need a few definitions. First, we say that two states w and w' are *equivalent*, and write $w\sim w'$, if one state is a cyclic shift of the other, e.g. $(w_1,\dots,w_n)\sim(w_2,\dots,w_n,w_1)$. Because of the cyclic symmetry inherent in the definition of the TASEP on a ring, it is clear that the probabilities of states w and w' are equal whenever $w\sim w'$. We will therefore often assume, without loss of generality, that $w_1=1$. Note that up to cyclic shift, $\mathrm{St}(n)$ contains $(n-1)!$ states.

We now introduce some definitions needed to characterize the Schubert polynomial factors that appear in the probabilities ψ_w.

Definition 3.1.2. Let $w=(w_1,\dots,w_n)\in\mathrm{St}(n)$. We say that w is a k-*Grassmannian permutation*, and we write $w\in\mathrm{St}(n,k)$ if: $w_1=1$; w is *evil-avoiding*, i.e. w avoids the patterns 2413, 3214, 4132, and 4213; and w has k *recoils*, that is, letters a in w such that $a+1$ appears to the left of a in w. (Equivalently, w^{-1} has exactly k *descents*.)

Definition 3.1.3. We associate to each $w\in\mathrm{St}(n,k)$ a sequence of partitions $\Psi(w)=(\lambda^1,\dots,\lambda^k)$ as follows. Write the Lehmer code (cf. Definition 3.2.3) of w^{-1} as $c(w^{-1})=c=(c_1,\dots,c_n)$; since w^{-1} has k descents, c has k *descents* in positions we denote by $a_1,\dots,a_k$. Set $a_0=0$. For $1\le i\le k$, define $\lambda^i=(n-a_i)^{a_i}-(\underbrace{0,\cdots,0}_{a_{i-1}},c_{a_{i-1}+1},c_{a_{i-1}+2},\dots,c_{a_i})$.

See Table 3.3 for examples of the map $\Psi(w)$.

Definition 3.1.4. Given a positive integer n and a partition λ properly contained in a $\mathrm{length}(\lambda)\times(n-\mathrm{length}(\lambda))$ rectangle (we will later use the notation $\lambda\in\mathrm{Val}(n)$), we define an integer vector $g_n(\lambda)=(v_1,\dots,v_n)$ of length n as follows. Write $\lambda=(\mu_1^{k_1},\cdots,\mu_l^{k_l})$ where $k_i>0$

State w	Probability ψ_w
12345	$\mathbf{x}^{(6,3,1)}$
12354	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$\mathbf{x}^{(2,1,0)}(\mathfrak{S}_{164325}+\mathfrak{S}_{25431})$
13542	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$\mathbf{x}^{(1,0,0)}(\mathfrak{S}_{1753246}+\mathfrak{S}_{265314}+\mathfrak{S}_{2743156}+\mathfrak{S}_{356214}+\mathfrak{S}_{364215}+\mathfrak{S}_{365124})$
14352	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$\mathbf{x}^{(3,1,0)}(\mathfrak{S}_{146325}+\mathfrak{S}_{24531})$
15324	$\mathbf{x}^{(2,1,1)}(\mathfrak{S}_{15432}+\mathfrak{S}_{164235})$
15342	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Table 3.2: The renormalized steady state probabilities for $n = 5$, when each $y_i = 0$. In the table, $\mathbf{x}^{(a,b,c)}$ denotes $x_1^a x_2^b x_3^c$.

and $\mu_1 > \cdots > \mu_l$. We assign values to the entries $(v_1, \ldots, v_n)$ by performing the following step for i from 1 to l.

- (Step i) Set $v_{n-\mu_i}$ equal to μ_i. Moving to the left, assign the value μ_i to the first $(k_i - 1)$ unassigned components.

After performing Step l, we assign the value 0 to any entry v_j which has not yet been given a value.

Remark 3.1.5. Note that in Step 1, we set $v_{n-\mu_1}, v_{n-\mu_1-1}, \cdots, v_{n-\mu_1-k_1+1}$ equal to μ_1.

Example 3.1.6.

$$g_5((2,1,1)) = (0,1,2,1,0)$$
$$g_6((3,2,2,1)) = (0,2,3,2,1,0)$$
$$g_6((3,1,1)) = (0,0,3,1,1,0).$$

The main result of this paper is Theorem 3.1.8. The definition of Schubert polynomial can be found in Section 3.2.

Definition 3.1.7. We write $a \to b \to c$ if the letters a, b, c appear in cyclic order in w. So for example, if $w = 1423$, we have that $1 \to 2 \to 3$ and $2 \to 3 \to 4$, but it is not the case that $3 \to 2 \to 1$ or $4 \to 3 \to 2$. We then define

$$\mathrm{xyFact}(w) = \prod_{i=1}^{n-2} \prod_{\substack{i+2 \le k \le n \\ i \to i+1 \to k}} (x_1 - y_{n+1-k}) \cdots (x_i - y_{n+1-k}). \tag{3.1.2}$$

Theorem 3.1.8. *Let $w \in \mathrm{St}(n,k)$ be a k-*Grassmannian *permutation, as in Definition 3.1.2, and write $\Psi(w) = (\lambda^1, \cdots, \lambda^k)$. Then the (renormalized) steady state probability is given by*

$$\psi_w = \mathrm{xyFact}(w) \prod_{i=1}^{k} \mathfrak{S}_{c^{-1}(g_n(\lambda^i))}, \tag{3.1.3}$$

where $\mathfrak{S}_{c^{-1}(g_n(\lambda^i))}$ is the double Schubert polynomial associated to the permutation with Lehmer code $g_n(\lambda^i)$, and g_n is given by Definition 3.1.4.

In the case that each $y_i = 0$, Theorem 3.1.8 becomes Theorem 3.1.9 below.

Theorem 3.1.9. *Let $w \in \mathrm{St}(n,k)$, and let $\Psi(w) = (\lambda^1, \dots, \lambda^k)$. Let μ be the vector $\mu := (\binom{n-1}{2}, \binom{n-2}{2}, \dots, \binom{2}{2}) - \sum_{i=1}^{k} \lambda^i$, where we view each partition λ^i as a vector in $\mathbb{Z}_{\ge 0}^{n-2}$, adding trailing 0's if necessary. Then when each $y_i = 0$, the renormalized steady state probability ψ_w is given by*

$$\psi_w = \mathbf{x}^{\mu} \prod_{i=1}^{k} \mathfrak{S}_{c^{-1}(g_n(\lambda^i))},$$

where $\mathfrak{S}_{c^{-1}(g_n(\lambda^i))}$ is the Schubert polynomial of the permutation with Lehmer code $g_n(\lambda^i)$.

We illustrate Theorem 3.1.9 in Table 3.3 in the case that $n = 5$. The quantity $s(w)$ is defined in Definition 3.5.8.

k	$w \in \mathrm{St}(5,k)$	$\Psi(w)$	probability ψ_w	$s(w)$
0	12345	$\varnothing$	$\mathbf{x}^{(6,3,1)}$	(0)
1	12354	$(1,1,1)$	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$	(0)
1	12435	$(2,2,1)$	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$	(0)
1	12453	$(2,2)$	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$	(0)
1	12534	$(1,1)$	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$	(0)
1	13245	$(3,2)$	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$	(0)
1	13425	$(3,1)$	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$	(0)
1	13452	(3)	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$	(0)
1	14235	$(2,1,1)$	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$	(0)
1	14253	$(2,1)$	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$	(0)
1	14523	(2)	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$	(0)
1	15234	(1)	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$	(0)
2	12543	$(2,2),(1,1,1)$	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1)$
2	13254	$(3,2),(1,1,1)$	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$	$(0,0)$
2	13542	$(3),(1,1,1)$	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$	$(0,-1)$
2	14352	$(3),(2,2,1)$	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$	$(0,-1)$
2	14532	$(3),(2,2)$	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$	$(0,-1)$
2	15342	$(3),(1,1)$	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$	$(0,-1)$
2	15423	$(2),(1,1,1)$	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$	$(0,-2)$
3	15432	$(3),(2,2),(1,1,1)$	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1,-2)$

Table 3.3: Special states $w \in \mathrm{St}(5,k)$ and the corresponding sequences of partitions $\Psi(w)$, together with steady state probabilities ψ_w and vectors $s(w)$.

Proposition 3.1.10. *The number of evil-avoiding permutation in S_n satisfies the recurrence $e(1)=1, e(2)=2, e(n)=4e(n-1)-2e(n-2)$ for $n \geq 3$, and is given explicitly as*

$$e(n) = \frac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}. \tag{3.1.4}$$

This sequence begins as $1,2,6,20,68,232$, and occurs in Sloane's encyclopedia as sequence A006012. The cardinalities $|\mathrm{St}(n,k)|$ also occur as sequence A331969.

Remark 3.1.11. Let $w(n,h) := (h, h-1, \ldots, 2, 1, h+1, h+2, \ldots, n) \in \mathrm{St}(n)$. In [6, Corollary 16], Cantini gives a formula for the steady state probability of state $w(n,h)$, as a trivial factor times a product of certain (double) Schubert polynomials. Note that our main result is a significant generalization of [6, Corollary 16]. For example, for $n=4$, Cantini's result gives a formula for the probabilities of three states – $(1,2,3,4), (1,3,4,2)$, and $(1,4,3,2)$. And for $n=5$, his result gives a formula for four states – $(1,2,3,4,5), (1,3,4,5,2), (1,4,5,3,2)$, and $(1,5,4,3,2)$. On the other hand, Theorem 3.1.9 gives a formula for all six states when $n=4$ (see Table 3.1) and 20 of the 24 states when $n=5$. Asymptotically, since the number of special states in S_n is given by (3.1.4), Theorem 3.1.9 gives a formula for roughly $\frac{(2+\sqrt{2})^{n-1}}{2}$ out of the $(n-1)!$ states of $\mathrm{St}(n)$.

Another point worth mentioning is that the Schubert polynomials that occur in the formulas of [6] are all of the form $\mathfrak{S}_{\sigma(a,n)}$, where $\sigma(a,n)$ denotes the permutation $(1, a+1, a+2, \dots, n, 2, 3, \dots, n)$. However, many of the Schubert polynomials arising as (factors) of steady probabilities are not of this form. Already we see for $n = 4$ the Schubert polynomials $\mathfrak{S}_{1432}$ and $\mathfrak{S}_{1243}$, which are not of this form.

3.2 Background on partitions, permutations and Schubert polynomials

We let S_n denote the symmetric group on n letters, which is a Coxeter group generated by the simple reflections $s_1, \dots, s_{n-1}$, where s_i is the simple transposition exchanging i and $i+1$. We let $w_0 = (n, n-1, \dots, 2, 1)$ denote the longest permutation.

For $1 \le i < n$, we have the *divided difference operator* ∂_i which acts on polynomials $P(x_1, \dots, x_n)$ as follows:

$$(\partial_i P)(x_1, \dots, x_n) = \frac{P(\dots, x_i, x_{i+1}, \dots) - P(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

If $s_{i_1} \dots s_{i_m}$ is a reduced expression for a permutation w, then $\partial_{i_1} \dots \partial_{i_m}$ depends only on w, so we denote this operator by ∂_w.

Definition 3.2.1. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two sets of variables, and let

$$\Delta(\mathbf{x}, \mathbf{y}) = \prod_{i+j \le n} (x_i - y_j).$$

To each permutation $w \in S_n$ we associate the *double Schubert polynomial*

$$\mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = \partial_{w^{-1} w_0} \Delta(\mathbf{x}, \mathbf{y}),$$

where the *divided difference operator* acts on the x-variables.

Definition 3.2.2. A *partition* $\lambda = (\lambda_1, \dots, \lambda_r)$ is a weakly decreasing sequence of positive integers. We say that r is the *length* of λ, and denote it $r = \mathrm{length}(\lambda)$. We also define λ_{last} to be the smallest part of λ.

Definition 3.2.3. The *Rothe diagram* of a permutation w is

$$D(w) = \{(i,j) \mid 1 \le i, j \le n, w(i) > j, w^{-1}(j) > i\}.$$

Graphically, we may construct the diagram by taking the complement of the hooks with vertices $(i, w(i))$, for $1 \le i \le n$.

The sequence of the numbers of the points of the diagram in successive rows is called the *Lehmer code* or *code* $c(w)$ of the permutation. If $(v_1, \ldots, v_n)$ is the code of a permutation, we also define $c^{-1}(v_1, \ldots, v_n)$ to be the permutation whose Lehmer code is $(v_1, \ldots, v_n)$. The partition obtained by sorting the components of the code is called the *shape* $\lambda(w)$ of w.

Example 3.2.4. The permutation $w = (1,3,5,4,2)$ has Lehmer code $(0,1,2,1,0)$ and shape $\lambda(w) = (2,1,1)$.

The following is well-known.

Lemma 3.2.5. *Given a vector* $(c_1, \cdots, c_n) \in \mathbb{Z}^n_{\geq 0}$*, there exists a permutation* $w \in S_n$ *such that* $c(w) = (c_1, \cdots, c_n)$ *if and only if* $c_i + i \leq n$ *for all* $1 \leq i \leq n$.

3.3 Combinatorics of evil-avoiding and k-Grassmannian permutations

In this section we study the special states of our Markov chain whose probabilities are proportional to products of Schubert polynomials. Recall our definition of k-Grassmannian permutations $\mathrm{St}(n,k)$ from Definition 3.1.2. As we will see, the set $\mathrm{St}(n,k)$ is in bijection with a certain set $\mathrm{ParSeq}(n,k)$ of sequences $(\lambda^1, \ldots, \lambda^k)$ of k partitions. Recall that our main result (see Theorem 3.1.8) states that the probability of each state in $\mathrm{St}(n,k)$ is proportional to a product of k Schubert polynomials, which are determined by the corresponding sequence of partitions.

Remark 3.3.1. Note that w contains a pattern p if and only if w^{-1} contains the pattern p^{-1}. So w is evil-avoiding if and only if w^{-1} avoids 3142, 2431, 3241, and 3214.

We say that a sequence $(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ has a *descent* in position j if $w_j > w_{j+1}$.

Proposition 3.3.2. *Let* $c = (c_1, c_2, \ldots, c_n)$ *be the code of* $w \in S_n$*. Then* w *avoids the patterns* 3142*,* 3214*,* 2431*, and* 3241 *(equivalently,* w^{-1} *is evil-avoiding) if and only if for each descent position* j*, if there is a* $b \leq j$ *such that:*

- $w_b < w_{b+1} < \cdots < w_j$ *and* $0 < c_b < n - j$*, and* b *is maximal with these properties,*

then we must have $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$.

Remark 3.3.3. Let $w = (w_1, \ldots, w_n) \in S_n$. Note that having the bth entry of the code $c = \mathrm{code}(w)$ equal to c_b means that there are precisely c_b letters less than w_b which do not occur in positions 1 through b. The condition $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$ means that these letters must occur in increasing order in positions $j+1, \ldots, j+c_b$.

Remark 3.3.4. If $k = 1$, then $w \in \mathrm{St}(n,1)$ implies that $\mathrm{code}(w^{-1})$ has a unique descent. By [30, Definition 2.2.3], this means that w^{-1} is a *Grassmannian permutation.* Equivalently, there is precisely one letter a in w such that $a+1$ appears to the left of a in w.

Proof of Proposition 3.3.2. We start by showing that if w fails to satisfy the condition of Proposition 3.3.2, then it must contain one of the patterns 3142, 3214, 2431, and 3241. If w fails to satisfy the condition of Proposition 3.3.2, then there is a descent position j and a $b \leq j$ with $w_b < w_{b+1} < \cdots < w_j$ and $0 < c_b < n-j$, and b is maximal with this property, but we do not have $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$.

1. We have $w_b < w_{b+1} < \cdots < w_j > w_{j+1}$.
2. By the definition of code, there must be c_b letters $\mathcal{C}$ smaller than w_b which appear to the right of w_b; they must therefore appear to the right of w_j.
3. Since $c_b < n-j$, there must be at least one letter $w_\ell > w_b$ which appears among positions $[j,n]$. Let ℓ be minimal with this property.
4. The fact that we do not have $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$ implies that the letter w_ℓ appears to the left of some letter $w_r \in \mathcal{C}$, i.e. $\ell < r$.

Let us first consider the case that $b = j$. If $c_{b+1} \neq 0$ then w_{b+1} is not the smallest letter to appear in positions $[b+1,n]$, so there exists some $m > b+1$ such that $w_{b+1} > w_m$. Therefore the letters $\{w_b, w_{b+1}, w_m, w_\ell\}$ give either an instance of the pattern 3214 or of the pattern 3241, depending on whether $m < \ell$ or $\ell < m$.

If $b = j$ and $c_{b+1} = 0$ then w_{b+1} is the smallest letter to appear in positions $[b+1,n]$. But then the letters $\{w_b, w_{b+1}, w_\ell, w_r\}$ form the pattern 3142.

If $b < j$, then the fact that b is maximal such that $j + c_b < n$ implies that $j + c_{b+1} \geq n$. Since $w_{b+1} < \cdots < w_j$, the c_{b+1} letters which are less than w_{b+1} and to the right of position $b+1$ must actually lie in positions $[j+1,n]$. However, now since $c_{b+1} \geq n-j$, *all* the letters in positions $[j+1,n]$ must be less than w_{b+1}. In particular the letter w_ℓ defined in (3) must be less than w_{b+1}. But now the letters $\{w_b, w_{b+1}, w_\ell, w_r\}$ form the pattern 2431.

Therefore, we have shown that if w avoids the patterns 3142, 3214, 2431, and 3241, then it satisfies the condition of Proposition 3.3.2.

In the other direction, suppose that w contains the pattern 3214. Let $i < k < \ell < m$ denote the positions of the letters of this pattern. Then $w_i > w_k$ implies that there exists some j with $i \leq j < k$ such that $w_i < w_{i+1} < \cdots < w_j > w_{j+1}$. We have $c_i \geq 2$ since w_k and w_ℓ are both less than w_i. And all of the c_i letters less than w_i which occur to the right of w_i must occur in positions $[j+1,n]$. Moreover, the 4 in the pattern, representing the letter $w_m > w_i$, occurs in a position in $[j+1,n]$. Therefore $c_i < n-j$.

Therefore there exists some b with $i \leq b \leq j$ with $w_i \leq w_b < w_{b+1} < \cdots < w_j$ and $0 < c_b < n-j$, and we choose b to be maximal with these properties. We have that w_b is greater than both w_k and w_ℓ. But then it is impossible for $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$: by Remark 3.3.3 this means that the c_b letters less than w_b that appear to the right of w_b must occur in increasing order in positions $j+1, j+2, \ldots, j+c_b$, but this is false since w_k and w_ℓ (the 2 and 1 of the pattern) occur in the wrong order. Exactly the same argument holds if w contains the pattern 3241.

Nearly the same argument holds if w contains the pattern 3142. Again, let $i < k < \ell < m$ denote the positions of the letters of this pattern. As before, since $w_i > w_k$, we can find $i \le j < k$ such that $w_i < w_{i+1} < \cdots < w_j > w_{j+1}$, and we have $2 \le c_i < n - j$. We then choose b maximal with $i \le b \le j$ such that $0 < c_b < n - j$. But again using Remark 3.3.3 we see it is impossible to have $c_{j+1} = c_{j+2} = \cdots = c_{j+c_b} = 0$ – the 142 in the pattern 3142 (i.e. the letters in positions k, ℓ, m) means that the letters less than w_b do not occur in increasing order in consecutive positions.

Finally, suppose that w contains the pattern 2431. Let $h < i < \ell < m$ denote the positions of the letters of this pattern. The fact that $w_i > w_\ell$ implies there must be a descent position j with $j < \ell$. Let j be minimal such that $w_h < w_{h+1} < \cdots < w_j > w_{j+1}$. Because $w_m < w_h$, we know that $c_h \ge 1$. Moreover, the letters less than w_h which are to the right of it must appear in positions $[j+1, n]$. Because $w_\ell > w_h$, we know that $c_h < n - j$. Therefore, there exists some $b \ge h$ with $w_b < w_{b+1} < \cdots < w_j$ and $0 < c_b < n - j$; we choose b to be maximal with this property. But then by Remark 3.3.3, the c_b letters less than w_b which appear to the right of w_b must appear in increasing order in positions $j+1, j+2, \ldots, j+c_b$. This is impossible, since w_ℓ and w_m lie weakly to the right of position $j+1$ but in the wrong order (since $w_\ell > w_m$). This completes the proof. □

Definition 3.3.5. We say that a partition λ is *valid* for n if λ is properly contained in a $\text{length}(\lambda) \times (n - \text{length}(\lambda))$ rectangle. Let $\text{Val}(n)$ denote the collection of all valid partitions for n.

Remark 3.3.6. One can show that $|\text{Val}(n)| = 2^{n-1} - (n-1) - 1$. They are in bijection with Grassmannian permutations in S_n that starts with 1.

Definition 3.3.7. For $1 \le k \le n-2$, we let $\text{ParSeq}(n,k)$ denote the set of all sequences of partitions $(\lambda^1, \ldots, \lambda^k)$ such that each λ^i is valid for n, and for all $1 \le i \le k-1$ we have:

- if ℓ is the smallest part of λ^i, then the first $(n-\ell)$ parts of λ^{i+1} are equal.

If $k = 0$ then $\text{ParSeq}(n,k)$ consists of one element, the empty sequence.

Example 3.3.8. If $n = 6$, then $((3),(2,2,2,1))$ and $((4,2),(1,1,1,1))$ lie in $\text{ParSeq}(6,2)$ but $((3),(2,2,1,1))$ and $((4,2),(1,1,1)$ do not lie in $\text{ParSeq}(6,2)$.

Remark 3.3.9. It follows from Definition 3.3.7 that if $(\lambda^1, \ldots, \lambda^k) \in \text{ParSeq}(n,k)$, then for all $1 \le i \le k-1$:

- the number of parts of λ^i is less than the number of parts of λ^{i+1}
- every part of λ^i is greater than every part of λ^{i+1}.

Remark 3.3.10. Let $v(n)$ denote the vector $v(n) := (\binom{n-1}{2}, \binom{n-2}{2}, \ldots, \binom{2}{2})$. Then it follows from Definition 3.3.7 that $\mu := v - \sum_{i=1}^k \lambda^i$ is a partition. (Here we think of each partition λ^i as a vector in $\mathbb{Z}_{\ge 0}^{n-2}$ by adding extra parts equal to 0 if necessary.) Note that the vector μ appears in the steady steady probability formula from Theorem 3.1.9.

Proposition 3.3.11. *The map* $\Psi : \mathrm{St}(n,k) \to \mathrm{ParSeq}(n,k)$ *(cf. Definition 3.1.3) is well-defined and bijective. The inverse map* $\Psi^{-1} : \mathrm{ParSeq}(n,k) \to \mathrm{St}(n,k)$ *can be described as follows. Let* $(\lambda^1,\dots,\lambda^k) \in \mathrm{ParSeq}(n,k)$*, and let* $(f_1,\dots,f_k)$ *be the sequence of first parts of* $\lambda^1,\dots,\lambda^k$*, i.e.* $f_i = \lambda^i_1$*. Then* $((f_1^{n-f_1} - \lambda^1) + (f_2^{n-f_2} - \lambda^2) + \cdots + (f_k^{n-f_k} - \lambda^k))$ *is the code of a permutation* w^{-1} *of* $w \in \mathrm{St}(n,k)$*. We define* $\Psi^{-1}(\lambda^1,\dots,\lambda^k) = w$*.*

Proof. We first show that when we apply the map Ψ, we obtain a vector that satisfies the properties of Definition 3.3.7. Write $\mathrm{code}(w^{-1}) = c = (c_1,\dots,c_n)$ and let $a_1,\dots,a_k$ denote the positions of the descents of c. We have $\lambda^j = (n-a_j)^{a_j} - (0,\cdots,0,c_{a_{j-1}+1},\cdots,c_{a_j})$. If we take the maximal b such that $a_{j-1} < b \le a_j$ and $a_j + c_b < n$, then by Proposition 3.3.2, we have $c_{a_j+1} = c_{a_j+2} = \cdots = c_{a_j+c_b} = 0$. Since $\lambda^{j+1} = (n-a_{j+1})^{a_{j+1}} - (0,\cdots,0,c_{a_j+1},\cdots,c_{a_{j+1}})$, t he first $a_j + c_b$ parts of λ^{j+1} are equal. Let l be the smallest part of λ_j, then $l = (n-a_j) - c_b$. So the first $(n-l) = a_j + c_b$ parts of λ^{j+1} are equal.

Now we show that when we apply the map Ψ^{-1} to $(\lambda_1,\dots,\lambda_k) \in \mathrm{ParSeq}(n,k)$, we get an element in $\mathrm{St}(n,k)$. Let $c = ((f_1^{n-f_1} - \lambda^1) + (f_2^{n-f_2} - \lambda^2) + \cdots + (f_k^{n-f_k} - \lambda^k))$. Since the smallest part of λ^i is no greater than f_i, the first $n-f_i$ parts of λ_{i+1} are equal. So the first $n-f_i$ components of the vector $(f_{i+1}^{n-f_{i+1}} - \lambda_{i+1})$ are zero. We write $c = (c_1,\dots,c_n)$ as follows

$$c_j = \begin{cases} f_1 - \lambda^1_j, & \text{if } 1 \le j \le n - f_1 \\ f_i - \lambda^i_j, & \text{if } n - f_{i-1} < j \le n - f_i, \text{ for } 2 \le i \le k \\ 0, & \text{if } n - f_k < j, \end{cases}$$

where we regard $\lambda^i_j = 0$ if j is bigger that the length of λ^i.

We claim that $c_{n-f_i} > c_{n-f_i+1}$. If $\lambda^i_{n-f_i} = 0$ then we have $c_{n-f_i} = f_i > f_{i+1} \ge f_{i+1} - \lambda^{i+1}_{n-f_i+1} = c_{n-f_i+1}$. If $\lambda^i_{n-f_i} > 0$ then $\lambda^i_{n-f_i} < f_i$ since $\lambda^i \in \mathrm{Val}(n)$. The first $n - \lambda^i_{n-f_i}$ parts of λ^{i+1} are equal so $\lambda^{i+1}_{n-f_i+1} = f_{i+1}$. Thus we have $c_{n-f_i} = f_i - \lambda^i_{n-f_i} > 0 = c_{n-f_i+1}$.

We see that the descents of c are at $n-f_1,\dots,n-f_k$. Now take the maximal b such that $n - f_{i-1} < b \le n - f_i$ and $n - f_i + c_b < n$. Then $c_b = f_i - l$ where l is the smallest part of λ^i. So the first $n-l$ parts of λ^{i+1} are equal which implies $c_{n-f_i+1} = \cdots = c_{n-f_i+c_b}$ as $n - f_i + c_b = n - l$. We conclude that c is the code of w^{-1} for some $w \in \mathrm{St}(n,k)$ by Proposition 3.3.2. □

Example 3.3.12. If $\mathrm{code}(w^{-1}) = (0,3,1,1,0)$ then $\Psi(w) = ((3),(1,1))$. If $\mathrm{code}(w^{-1}) = (0,2,2,1,0)$ then $\Psi(w) = ((2),(1,1,1))$.

Remark 3.3.13. While we have not found any previous works studying evil-avoiding permutations, we note that the sequence $\{e(n)\}$ of cardinalities of evil-avoiding permutations in S_n has several other combinatorial interpretations listed in Sloane:

- $e(n)$ counts permutations $\pi \in S_n$ for which the pairs $(i,\pi(i))$ with $i < \pi(i)$, considered as closed intervals $[i+1,\pi(i)]$, do not overlap; equivalently, for each $i \in [n]$ there is at most one $j \le i$ with $\pi(j) > i$.

- $e(n)$ is the number of permutations on $[n]$ with no subsequence abcd such that (i) bc are adjacent in position and (ii) $\max(a,c) < \min(b,d)$. For example, the 4 permutations of $[4]$ not counted by $a(4)$ are $1324, 1423, 2314, 2413$.

- $e(n)$ is the number of *rectangular permutations* on $[n]$, i.e. those permutations which avoid the four patterns 2413, 2431, 4213, 4231, see [8].

It would be interesting to find a bijection between the k-Grassmannian permutations in S_n (where we let k vary) and either of the above sets of permutations.

Remark 3.3.14. Given (the Young diagram of) a partition contained in an $r \times (n-r)$ rectangle, we identify it with the lattice path cutting out the Young diagram which takes unit steps south and east from the upper right corner $(r, n-r)$ to the lower left corner $(0,0)$ of the rectangle. Label the steps of that lattice path with the numbers 1 through n. If we first read off the labels of the vertical steps (in order) and then read off the labels of the horizontal steps, we obtain a permutation $w(\lambda)$ which is *Grassmannian*, i.e. it has a unique descent. For example, for $n = 5$ and $\lambda = (2,2)$, we have $w(\lambda) = (1,2,5,3,4)$.

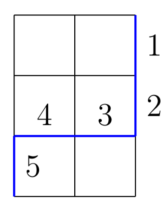

Figure 3.2: Reading off the (Grassmann) permutation from a partition

If $w \in \mathrm{St}(n,1)$, then $\Psi(w)$ consists of a single partition, the unique partition λ such that $w(\lambda) = w^{-1}$. For example, for $w = (1,2,4,3,5)$, we have $\Psi(w) = ((2,2,1))$, and for $w = (1,2,4,5,3)$, we have $\Psi(w) = ((2,2))$.

Proposition 3.3.15. *For $k \geq 1$, we have that*

$$|\operatorname{ParSeq}(n,k)| = 2|\operatorname{ParSeq}(n-1,k)| + \sum_{i=k+1}^{n-1} |\operatorname{ParSeq}(i,k-1)|.$$

Proof. To prove Proposition 3.3.15, we define two different injective maps $\Psi_i : \operatorname{ParSeq}(n-1,k) \to \operatorname{ParSeq}(n,k)$ for $i = 1,2$ as well as a family of injective maps $\Phi_{i,k,n} : \operatorname{ParSeq}(i,k-1) \to \operatorname{ParSeq}(n,k)$ for $k+1 \leq i \leq n-1$. The statement then follows from the claim that every element of $\operatorname{ParSeq}(n,k)$ lies in the image of precisely one of these maps.

We define $\Psi_1 : \operatorname{ParSeq}(n-1,k) \to \operatorname{ParSeq}(n,k)$ to be the map which takes $(\lambda^1, \dots, \lambda^k)$ to $(\mu^1, \dots, \mu^k)$, where μ^i is obtained from λ^i by duplicating its first part. That is, if $\lambda^i = (\lambda^i_1, \lambda^i_2, \dots, \lambda^i_r)$, then $\mu^i = (\lambda^i_1, \lambda^i_1, \lambda^i_2, \dots, \lambda^i_r)$. So for example,

$$\Psi_1((2),(1,1)) = ((2,2),(1,1,1)).$$

We define $\Psi_2 : \mathrm{ParSeq}(n-1,k) \to \mathrm{ParSeq}(n,k)$ to be the map which takes $(\lambda^1,\dots,\lambda^k)$ to $(\mu^1,\dots,\mu^k)$, where for $i \geq 2$, μ^i is obtained from λ^i by duplicating its first part. For $i = 1$, let $\lambda^i = (\lambda^i_1,\dots,\lambda^i_r)$. If all parts of λ^i are equal, then we define $\mu^i = (\lambda^i_1+1,\lambda^i_1,\dots,\lambda^i_r)$. Otherwise, we define $\mu^i = (\lambda^i_1+1,\lambda^i_2,\dots,\lambda^i_r)$. So if $\lambda^1 = (\lambda^1_1,\lambda^1_2,\dots,\lambda^1_r)$, then $\mu^1 = (\lambda^1_1+1,\lambda^1_1,\lambda^1_2,\dots,\lambda^1_r)$. So for example,

$$\Psi_2((2),(1,1)) = ((3,2),(1,1,1)).$$

For $k+1 \leq i \leq n-1$, we define $\Phi_{i,k,n} : \mathrm{ParSeq}(i,k-1) \to \mathrm{ParSeq}(n,k)$ to be the map which takes $(\lambda^1,\dots,\lambda^k)$ to $((i-1),\mu^1,\dots,\mu^{k-1})$, where μ^j is obtained from λ^j by duplicating the first part of λ^j $n-i$ times. That is, if $\lambda^j = (\lambda^j_1,\lambda^j_2,\dots,\lambda^j_r)$, then $\mu^j = (\underbrace{\lambda^j_1,\lambda^j_1,\dots,\lambda^j_1}_{n-i+1},\lambda^j_2,\dots,\lambda^j_r)$.

So for example, we have that

$$\begin{aligned}
\Phi_{4,2,5}((1)) &= ((3),(1,1)),\\
\Phi_{4,2,5}((2)) &= ((3),(2,2,)),\\
\Phi_{4,2,5}((1,1)) &= ((3),(1,1,1)),\\
\Phi_{4,2,5}((2,1)) &= ((3),(2,2,1)).
\end{aligned}$$

And

$$\Phi_{3,2,5}((1)) = ((2),(1,1,1)).$$

The above examples express the seven elements of $\mathrm{ParSeq}(5,2)$ as images of elements of $\mathrm{ParSeq}(4,2)$, $\mathrm{ParSeq}(4,1)$, and $\mathrm{ParSeq}(3,1)$. □

Corollary 3.3.16. *Define the number $T(n,k)$ by*

$$T(n,k) = \sum_{i=0}^{n-k-2} 2^i \binom{i+k-1}{k-1}\binom{n-2-i}{k}. \tag{3.3.1}$$

(These numbers appear in A331969.) Then we have

$$|\mathrm{St}(n,k)| = |\mathrm{ParSeq}(n,k)| = T(n,k).$$

Equivalently, the number of evil-avoiding permutations w in S_{n-1} such that w^{-1} has exactly k descents is $T(n,k)$.

Proof. The formula (3.3.1) is equivalent to the generating function given in A331969. We will prove that

$$T(n,k) = 2T(n-1,k) + \sum_{i=k+1}^{n-1} T(i,k-1).$$

Once we have done this, the result will follow from Proposition 3.3.15 and Proposition 3.3.11.

We have

$$\begin{aligned}T(n,k)-2T(n-1,k) &= \sum_{i=0}^{n-k-2} 2^i\binom{i+k-1}{k-1}\binom{n-2-i}{k} - 2\sum_{i=0}^{n-k-3} 2^i\binom{i+k-1}{k-1}\binom{n-3-i}{k}\\ &= \binom{n-2}{k} + 2\sum_{i=0}^{n-k-3} 2^i(\binom{i+k}{k-1}\binom{n-3-i}{k} - \binom{i+k-1}{k-1}\binom{n-3-i}{k})\\ &= \binom{n-2}{k} + 2\sum_{i=0}^{n-k-3} 2^{i+2}\binom{i+k-1}{k-2}\binom{n-3-i}{k}.\end{aligned}$$

Thus

$$\begin{aligned}&(T(n,k)-2T(n-1,k)) - (T(n-1,k)-2T(n-2,k))\\ &= (\binom{n-2}{k} + \sum_{i=0}^{n-k-3} 2^{i+1}\binom{i+k-1}{k-2}\binom{n-3-i}{k}) - (\binom{n-3}{k} + \sum_{i=0}^{n-k-4} 2^{i+1}\binom{i+k-1}{k-2}\binom{n-4-i}{k})\\ &= \binom{n-3}{k-1} + \sum_{i=0}^{n-k-4} 2^{i+1}\binom{i+k-1}{k-2}\binom{n-4-i}{k-1} + 2^{n-k-2}\binom{n-4}{k-2} = T(n-1,k-1).\end{aligned}$$

Now the proof follows from the induction on n. □

Proof of Proposition 3.1.10) Since $e(n) = \sum_{k=0}^{n-2}|St(n,k)|$, by Proposition 3.3.15 we have

$$\begin{aligned}e(n) = \sum_{k=0}^{n-2}|St(n,k)| &= 1 + \sum_{k=1}^{n-2}(2|St(n-1,k)| + \sum_{j=k+1}^{n-1}|St(j,k-1)|)\\ &= (3\sum_{k=0}^{n-3}|St(n-1,k)| - 1) + \sum_{k=1}^{n-3}\sum_{j=k+1}^{n-2}|St(j,k-1)|\\ &= 4e(n-1) - (\sum_{k=0}^{n-3}|St(n-1,k)| + 1 - \sum_{k=1}^{n-3}\sum_{j=k+1}^{n-2}|St(j,k-1)|)\\ &= 4e(n-1) - (1 + \sum_{k=1}^{n-3}(2|St(n-2,k)|) = 4e(n-1) - 2e(n-2).\end{aligned}$$

3.4 Cantini's z-deformation of steady state probabilities

In [6], Cantini associated to each $w \in \mathrm{St}(n)$ a *deformed steady state probability* $\psi_w(\mathbf{z}) = \psi_w(z_1,\cdots,z_n)$ which recovers the usual steady state probability ψ_w for w in the inhomogeneous TASEP when "$\mathbf{z} = \infty$," or in other words, when one reads off the coefficient $\mathrm{LC}_{\mathbf{z}}(\psi_w(\mathbf{z}))$ of the largest monomial in $\mathbf{z}$.

Note that the symmetric group S_n acts on states. Our convention is that for $1 \le i \le n-1$, the simple transposition s_i acts on a state $w = (w_1,\dots,w_n) \in \mathrm{St}(n)$ by

$$s_i(w_1,\dots,w_i,w_{i+1},\dots,w_n) = (w_1,\dots,w_{i+1},w_i,\dots,w_n).$$

In what follows, the indices of states are always considered modulo n, e.g. for $w = (w_1, \dots, w_n) \in \mathrm{St}(n)$, the notation w_{n+1} means w_1. Correspondingly we let s_n act by

$$s_n(w_1, w_2, \dots, w_{n-1}, w_n) = (w_n, w_2, \dots, w_{n-1}, w_1).$$

We use the notation $\mathrm{cyc}(w,k)$ to denote a cyclic shift of the state w by k positions, i.e. $\mathrm{cyc}(w,k) = (w_{1+k}, \cdots, w_{n+k})$. If f is a multivariate polynomial in $z_1, \dots, z_n$, we let s_i act on f by permuting variables $z_1, \dots, z_n$. And define $\mathrm{LC}_{\mathrm{z}}(f)$ to be the coefficient of the highest degree term with respect to the z variables. In this section we will also consider the indices for $\mathbf{z}$ modulo n.

Proposition 3.4.1. *[6, Equations (2), (24), (27), (28), (34)]*
We associate to each $w \in \mathrm{St}(n)$ the quantity $\psi_w(\mathbf{z}) = \psi_w(z_1, \dots, z_n)$, which is computed as follows:

$$\psi_{(1,2,\cdots,n)}(\mathbf{z}) = \prod_{1 \le i < j \le n} (x_i - y_{n+1-j})^{j-i-1} \prod_{i=1}^{n} \Big(\prod_{j=1}^{i-1} (z_i - x_j) \prod_{j=i+1}^{n} (z_i - y_{n+1-j}) \Big),$$
$$\psi_{s_l w}(\mathbf{z}) = \pi_l(w_l, w_{l+1}; n) \psi_w(\mathbf{z}) \quad \text{if } w_l > w_{l+1},$$

where $\pi_l(\beta, \alpha; n)$ is the isobaric divided difference operator *defined by*

$$\pi_l(\beta, \alpha; n) G(\boldsymbol{z}) = \frac{(z_l - y_{n+1-\beta})(z_{l+1} - x_\alpha)}{x_\alpha - y_{n+1-\beta}} \frac{G(\boldsymbol{z}) - s_l G(\boldsymbol{z})}{z_l - z_{l+1}}.$$

Then we have that for cyclically equivalent w and $w' = \mathrm{cyc}(w,k)$ in $St(n)$,

$$\psi_{w'}(z_1, \cdots, z_n) = \psi_w(z_{1+k}, \cdots, z_{n+k}), \tag{3.4.1}$$

where the indices on the z-variables are considered modulo n.
Moreover,

$$\mathrm{LC}_{\mathrm{z}}(\psi_w(\mathbf{z})) = \psi_w. \tag{3.4.2}$$

Because of (3.4.2)*, we refer to $\psi_w(\mathbf{z})$ as the* deformed steady state probability.

In what follows, we will often omit n in the operator $\pi_l(w_l, w_{l+1}; n)$ and just write $\pi_l(w_l, w_{l+1})$ when n is clear from the context.

Example 3.4.2. For $n = 3$, we have

$$\psi_{(1,2,3)}(\mathbf{z}) = (x_1 - y_1)(z_1 - y_2)(z_1 - y_1)(z_2 - x_1)(z_2 - y_1)(z_3 - x_1)(z_3 - x_2),$$

which gives

$$\begin{aligned}
&\psi_{(3,2,1)}(\mathbf{z}) = \pi_3(3,1)\psi_{(1,2,3)}(\mathbf{z}) = (z_1-x_1)(z_2-x_1)(z_2-y_1)(z_3-y_1)\times\\
&((x_1+x_2-y_1-y_2)z_3z_1+(x_1x_2-y_1y_2)(z_3+z_1)-x_1x_2y_1-x_1x_2y_2+x_1y_1y_2+x_2y_1y_2))\\
&\psi_{(2,3,1)}(\mathbf{z}) = \pi_1(3,2)\psi_{(3,2,1)}(\mathbf{z})\\
&=(x_1-y_1)(z_3-y_2)(z_3-y_1)(z_1-x_1)(z_1-y_1)(z_2-x_1)(z_2-x_2)\\
&\psi_{(3,1,2)}(\mathbf{z}) = \pi_2(2,1)\psi_{(3,2,1)}(\mathbf{z})\\
&=(x_1-y_1)(z_2-y_2)(z_2-y_1)(z_3-x_1)(z_3-y_1)(z_1-x_1)(z_1-x_2)\\
&\psi_{(1,3,2)}(\mathbf{z}) = \pi_1(3,1)\psi_{(3,1,2)}(\mathbf{z}) = (z_2-x_1)(z_3-x_1)(z_3-y_1)(z_1-y_1)\times\\
&((x_1+x_2-y_1-y_2)z_1z_2+(x_1x_2-y_1y_2)(z_1+z_2)-x_1x_2y_1-x_1x_2y_2+x_1y_1y_2+x_2y_1y_2))\\
&\psi_{(2,1,3)}(\mathbf{z}) = \pi_2(3,1)\psi_{(2,3,1)}(\mathbf{z}) = (z_3-x_1)(z_1-x_1)(z_1-y_1)(z_2-y_1)\times\\
&((x_1+x_2-y_1-y_2)z_2z_3+(x_1x_2-y_1y_2)(z_2+z_3)-x_1x_2y_1-x_1x_2y_2+x_1y_1y_2+x_2y_1y_2)).
\end{aligned}$$

Taking the leading coefficients with respect to z-variables recovers the probabilities from Figure 3.1, namely

$$\begin{aligned}
&\psi_{(1,2,3)} = \psi_{(2,3,1)} = \psi_{(3,1,2)} = x_1 - y_1\\
&\psi_{(1,3,2)} = \psi_{(2,1,3)} = \psi_{(3,2,1)} = x_1 + x_2 - y_1 - y_2.
\end{aligned}$$

3.5 The z-deformation of our main result and z-Schubert polynomials

In this section we present a z-deformation of our main result (see Theorem 3.5.9); it says that for $w \in \mathrm{St}(n,k)$, the z-deformed steady state probability $\psi_w(\mathbf{z})$ is equal to a "trivial factor" $\mathrm{TF}(w)$ times a product of *z-Schubert polynomials* – certain polynomials in $\mathbf{x},\mathbf{y},\mathbf{z}$ which reduce to double Schubert polynomials when $\mathbf{z}=\infty$.

We note that the z-Schubert polynomials $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}) \in \mathbb{R}[\mathbf{z};\mathbf{x};\mathbf{y}]$ (where $\mathbf{z}=\{z_1,\dots,z_n\}$, $\mathbf{x}=\{x_1,\dots,x_{n-1}\}$, $\mathbf{y}=\{y_1,\dots,y_{n-1}\}$) are not defined for any permutation but rather depend on a choice of positive integer n and a partition $\lambda \in \mathrm{Val}(n)$.

Given $\lambda=(\lambda_1,\dots,\lambda_k)$, the polynomial $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ has the property that: when $\mathbf{z}=\infty$, this polynomial reduces to the double Schubert polynomial of the permutation with Lehmer code $g_n(\lambda)$, see Lemma 3.5.7;

Definition 3.5.1. Let $\mathbf{x}$ be an ordered set of variables $\mathbf{x}=(x_1,x_2,x_3,\dots)$. Then we let $\sigma(\mathbf{x})$ denote $(x_2,x_3,\dots,)$ and $\sigma^m(\mathbf{x})$ denote $(x_{m+1},x_{m+2},\dots)$.

If I is a set of integers then the notation $\mathbf{x}_{\hat{I}}$ means $\mathbf{x}_{\hat{I}} = \mathbf{x}\setminus\{x_i \mid i\in I\}$, keeping the order inherited from $\mathbf{x}$. We abuse notation and use $\mathbf{x}_{\hat{i}}$ to denote $\mathbf{x}_{\{\hat{i}\}}$.

Example 3.5.2. For $f(\mathbf{z};\mathbf{x};\mathbf{y}) = x_2z_1z_2 + z_1 + z_2 + x_1 + x_2 + x_3$, we have

$$\begin{aligned} f(\sigma^2(\mathbf{z});\mathbf{x};\mathbf{y}) &= x_2z_3z_4 + z_3 + z_4 + x_1 + x_2 + x_3 \\ f(\mathbf{z};\mathbf{x}_{\hat{1}};\mathbf{y}) &= x_3z_1z_2 + z_1 + z_2 + x_2 + x_3 + x_4 \\ f(\mathbf{z};\mathbf{x}_{\hat{2}};\mathbf{y}) &= x_3z_1z_2 + z_1 + z_2 + x_1 + x_3 + x_4 \\ f(\mathbf{z};\mathbf{x}_{\widehat{\{2,3\}}};\mathbf{y}) &= x_4z_1z_2 + z_1 + z_2 + x_1 + x_4 + x_5. \end{aligned}$$

We now define z-Schubert polynomials $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$.

Definition 3.5.3. For a positive integer n and a partition $\lambda \in \mathrm{Val}(n)$, we define $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ recursively as follows

$$\partial_{n-\lambda_1-\mathrm{mul}(\lambda)}\cdots\partial_1\Big(\mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}\mathbf{z};\mathbf{x}_{\hat{1}};\mathbf{y})\prod_{l=1}^{n-\mathrm{mul}(\lambda)}(x_1-y_l)\prod_{i=1}^{(\lambda_1-\lambda_2+1)}\prod_{m=2}^{n-\lambda_1-\mathrm{mul}(\lambda)+1}(z_i-x_m)\Big). \quad (3.5.1)$$

where λ' is the partition obtained by deleting the first part of λ. If λ' is an empty partition then we set $\mathfrak{S}^{n-1}_{\lambda'}(\mathbf{z};\mathbf{x};\mathbf{y}) = 1$ by convention and regard $\lambda_2 = 0$. Note that the divided difference operators *act on the x-variables.*

Remark 3.5.4. It is easy to show that $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ depends on z-variables $z_1,\cdots,z_{\lambda_1+\mathrm{length}(\lambda)}$ using an induction on n.

Remark 3.5.5. For positive integers r and s, Cantini introduced a polynomial that he denoted $\mathfrak{S}^{r,s}$ in [6]. In our notation this is the same as $\mathfrak{S}^{r+s-1}_{((r-1)^{s-1})}(\mathbf{z};\mathbf{x};\mathbf{y})$.

Example 3.5.6. We have

$$\begin{aligned} \mathfrak{S}^3_{(1)}(\mathbf{z};\mathbf{x};\mathbf{y}) &= \partial_1\big((x_1-y_1)(x_1-y_2)(z_1-x_2)(z_2-x_2)\big) \\ &= \frac{(x_1-y_1)(x_1-y_2)(z_1-x_2)(z_2-x_2)-(x_2-y_1)(x_2-y_2)(z_1-x_1)(z_2-x_1)}{x_1-x_2} \\ &= z_1z_2(x_1+x_2-y_1-y_2)-(z_1+z_2)(x_1x_2+y_1y_2)+(x_1x_2(y_1+y_2)-y_1y_2(x_1+x_2)). \end{aligned}$$

Observe that $\mathrm{LC_z}(\mathfrak{S}^3_{(1)}(\mathbf{z};\mathbf{x};\mathbf{y})) = x_1+x_2-y_1-y_2$, the double Schubert polynomial $\mathfrak{S}_{(1,3,2)}(\mathbf{x},\mathbf{y})$. It also appears as a renormalized steady state probability in Figure 3.1.

The next proposition shows that in general the leading coefficient $\mathrm{LC_z}(\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}))$ is the double Schubert polynomial of the permutation whose Lehmer code is $g_n(\lambda)$, where $g_n(\lambda)$ is the vector from Figure 3.1.4.

Proposition 3.5.7. *Fix n and choose a partition $\lambda \in \mathrm{Val}(n)$. Then the leading coefficient $\mathrm{LC_z}(\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}))$ of $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ is the double Schubert polynomial $\mathfrak{S}_{c^{-1}(g_n(\lambda))}$ of the permutation $c^{-1}(g_n(\lambda)) \in S_n$ whose Lehmer code is $g_n(\lambda)$.*

We will prove Lemma 3.5.7 in the next section.

In Theorem 3.5.9 below we give a z-deformation of our main result (Theorem 3.1.8); it says that for $w \in \mathrm{St}(n,k)$, the z-deformed steady state probability $\psi_w(\mathbf{z})$ is equal to a "trivial factor" $\mathrm{TF}(w)$ times a product of z-Schubert polynomials.

Recall that we defined $\mathrm{xyFact}(w)$ in Definition 3.1.7. We also define

$$\mathrm{xzFact}(w) = \prod_{i=1}^{n} \prod_{\substack{j \neq w_i \\ \min\{w_i, w_{i-1}, w_{i-2}, \ldots, j\} = j}} (z_i - x_j) \tag{3.5.2}$$

and

$$\mathrm{yzFact}(w) = \prod_{i=1}^{n} \prod_{\substack{j \neq n+1-w_i \\ \max\{w_i, w_{i+1}, w_{i+2}, \ldots, n+1-j\} = n+1-j}} (z_i - y_j). \tag{3.5.3}$$

Finally we set

$$\mathrm{TF}(w) = \mathrm{xyFact}(w)\, \mathrm{xzFact}(w)\, \mathrm{yzFact}(w). \tag{3.5.4}$$

We associate to a partition sequence $(\lambda^1, \ldots, \lambda^k)$ a vector $(a_1, \ldots, a_k)$ as follows.

Definition 3.5.8. Let $(\lambda^1, \ldots, \lambda^k)$ be a sequence of partitions. We denote the parts of λ^j by λ^j_1, λ^j_2, etc. We define $s((\lambda^1, \ldots, \lambda^k); n) = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ by setting $a_1 = 0$, and for each $2 \le i \le k$,

$$a_i = a_{i-1} + \lambda^{i-1}_1 + \mathrm{length}(\lambda^{i-1}) - n.$$

If $w \in \mathrm{St}(n,k)$ such that $\Psi(w) = (\lambda^1, \ldots, \lambda^k)$, then abusing notation, we also refer to $s((\lambda^1, \ldots, \lambda^k); n)$ as $s(w)$.

See Table 3.3 for examples of $s(w)$.

Theorem 3.5.9. *Let $w \in \mathrm{St}(n,k)$, and write $\Psi(w) = (\lambda^1, \cdots, \lambda^k)$ and $s(w) = (a_1, \cdots, a_k)$. Then we have*

$$\psi_w(\mathbf{z}) = \mathrm{TF}(w) \prod_{i=1}^{k} \mathfrak{S}^n_{\lambda^i}(\sigma^{a_i}(\mathbf{z}); \mathbf{x}; \mathbf{y}) \tag{3.5.5}$$

where subscripts for z variables are considered modular n.

Theorem 3.5.9 will be proved in Section 3.7.

3.6 Properties of z-Schubert polynomials

Double Schubert polynomials

We review an algorithmic formula for computing double Schubert polynomials in terms of rc-graphs, based on [4].

$$\begin{array}{ccc} & j & j+1 \\ i-m & \cdot & \cdot \\ & + & + \\ & + & + \\ & + & + \\ i & + & \cdot \end{array} \quad \longrightarrow \quad \begin{array}{ccc} & j & j+1 \\ i-m & \cdot & + \\ & + & + \\ & + & + \\ & + & + \\ i & \cdot & \cdot \end{array}$$

Figure 3.3: The figure shows the ladder move $L_{i,j}$.

Definition 3.6.1. Given a finite subset $D \subsetneq \{1,2,\cdots\} \times \{1,2,\cdots\}$ we define the *weight* of D to be

$$\mathrm{wt}(D)(\mathbf{x},\mathbf{y}) = \prod_{(i,j)\in D} (x_i - y_j).$$

The *initial diagram* of the permutation w is

$$D_{in}(w) = \{(i,j) \mid 1 \le j \le c(w)_i\}.$$

Definition 3.6.2. For a finite subset $D \subsetneq \{1,2,\cdots\}\times\{1,2,\cdots\}$, assume the following conditions are satisfied for some i and j:

- $(i,j) \in D, (i,j+1) \notin D$,
- $(i-m,j),(i-m,j+1) \notin D$ for some $0 < m < i$,
- $(i-k,j),(i-k,j+1) \notin D$ for each $1 \le k < m$.

Then we define the *ladder move* $L_{i,j}$ to be $L_{i,j}(D) = D \cup \{(i-m,j+1)\} \setminus \{(i,j)\}$. We represent diagrams D as above as collections of +'s, see Figure 3.3. We also define $\mathcal{L}(D)$ to be the set of all D' that can be obtained by applying ladder moves to D.

Billey and Bergeron [4, Theorem 3.7] showed that $\mathcal{L}(D_{in}(w))$ gives the set of *rc-graphs* for w. Combining this with the well-known formula of double Schubert polynomials gives the following formula.

Theorem 3.6.3. *[4, Theorem 3.7 and Lemma 3.2] Let w be a permutation.*

(a) We have $\mathfrak{S}_w(\mathbf{x},\mathbf{y}) = \sum_{D'\in\mathcal{L}(D_{in}(w))} \mathrm{wt}(D')(\mathbf{x},\mathbf{y})$.

(b) The map sending D' to its transpose $(D')^t$ is a bijection between $\mathcal{L}(D_{in}(w))$ and $\mathcal{L}(D_{in}(w^{-1}))$.

Example 3.6.4. There are three diagrams in $\mathcal{L}(D_{in}(1,4,2,3))$, see Figure 3.4. So we have

$$\mathfrak{S}_{(1,4,2,3)}(\mathbf{x},\mathbf{y}) = (x_2-y_1)(x_2-y_2)+(x_2-y_1)(x_1-y_3)+(x_1-y_2)(x_1-y_3).$$

$$\begin{array}{cccc} & 1 & 2 & 3 \\ 1 & \cdot & \cdot & \cdot \\ 2 & + & + & \cdot \end{array} \longrightarrow \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & \cdot & \cdot & + \\ 2 & + & \cdot & \cdot \end{array} \longrightarrow \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & \cdot & + & + \\ 2 & \cdot & \cdot & \cdot \end{array}$$

$D_{in}((1,4,2,3))$

Figure 3.4: The figure above shows three diagrams in $\mathcal{L}(D_{in}(1,4,2,3))$

Note that rc-graphs are transparently in bijection with *reduced pipe dreams* (where each + becomes a crossing of two wires and each empty position becomes a pair of "elbows").

We will need the following result about linear factors of certain double Schubert polynomials. Note that in the case that each $y_i = 0$, Proposition 3.6.5 follows from [4, Corollary 3.11].

Proposition 3.6.5. *Let $w \in S_n$ be a permutation, and let $c(w) = (c_1,\dots,c_n)$ and $c(w^{-1}) = (\tilde{c}_1,\dots,\tilde{c}_n)$ denote the codes of w and w^{-1}. Suppose that $l \geq 0$ has the property that $\tilde{c}_m = 0$ for all $m > l$, and suppose that w' is the permutation with code $c(w') = (l, c_1,\dots,c_n)$ (whose existence follows from Lemma 3.2.5). Then*

$$\mathfrak{S}_{w'}(\mathbf{x},\mathbf{y}) = \mathfrak{S}_w(\mathbf{x}_{\hat{1}},\mathbf{y})\prod_{k=1}^{l}(x_1-y_k).$$

Remark 3.6.6. Note that the condition that $\tilde{c}_m = 0$ for all $m > l$ is equivalent to the condition that w has an increasing subsequence of the form $l+1, l+2, \dots, n$.

Proof. We first claim that no $D' \in \mathcal{L}(D_{in}(w))$ contains a + in a column greater than l, i.e. there is no $(i,j) \in D'$ with $j > l$. If there were such a D', then we'd have $(j,i) \in (D')^t$. But $D_{in}(w^{-1})$ does not have an element whose x-coordinate is bigger than l and ladder moves never increase the x-coordinates of the +'s involved. This proves the claim.

By the claim, $D_{in}(w)$ does not contain a + in a column greater than l, which implies the same is true for $D_{in}(w')$ and hence for $\mathcal{L}(D_{in}(w'))$.

Now we define a map $f : \mathcal{L}(D_{in}(w)) \to \mathcal{L}(D_{in}(w'))$ by

$$f(D') = \{(i+1,j) \mid (i,j) \in D'\} \cup \{(1,1),(1,2),\cdots,(1,l)\}.$$

This map is clearly injective, and is well defined since $f(D_{in}(w)) = D_{in}(w')$. We claim that f is surjective. Assume not. Then we can find $D_1, D_2 \in \mathcal{L}(D_{in}(w')$ such that $D_2 = L_{i,j}(D_1)$, and D_1 is in the image of f but D_2 is not.

Clearly the only way that there would be a viable ladder move L_{ij} on D_1 which does not have a counterpart for $f^{-1}(D_1)$ is if L_{ij} adds a + in row 1, necessarily in some column $j > l$ (since the first component of $c(w')$ is l). But we know that no diagram in $\mathcal{L}(D_{in}(w'))$ can have a + in a column greater than l. Therefore, the map f is surjective and hence bijective.

We conclude that

$$\begin{aligned}\mathfrak{S}_{w'}(\mathbf{x},\mathbf{y}) &= \sum_{D''\in\mathcal{L}(D_{in}(w'))} \mathrm{wt}(D'')(\mathbf{x},\mathbf{y}) = \sum_{D'\in\mathcal{L}(D_{in}(w))} \mathrm{wt}(f(D'))(\mathbf{x},\mathbf{y}) \\ &= \sum_{D'\in\mathcal{L}(D_{in}(w))} \prod_{k=1}^{l}(x_1-y_k)\ \mathrm{wt}(D')(\mathbf{x}_{\hat{1}},\mathbf{y}) \\ &= \prod_{k=1}^{l}(x_1-y_k) \sum_{D'\in\mathcal{L}(D_{in}(w))} \mathrm{wt}(D')(\mathbf{x}_{\hat{1}},\mathbf{y}) = \prod_{k=1}^{l}(x_1-y_k)\mathfrak{S}_w(\mathbf{x}_{\hat{1}},\mathbf{y}).\end{aligned}$$

□

The proof of Lemma 3.5.7.

The following lemma is easy to verify from the definitions.

Lemma 3.6.7. *Let $w \in S_n$ with code $c(w) = (v_1,\cdots,v_n)$. Then $w_i > w_{i+1}$ if and only if $v_i > v_{i+1}$. In this case we have*

$$\partial_i \mathfrak{S}_w = \mathfrak{S}_{ws_i}, \tag{3.6.1}$$

and $c(ws_i) = (v'_1,\cdots,v'_n)$, where $v'_i = v_{i+1}$, $v'_{i+1} = v_i - 1$, and $v'_j = v_j$ for $j \notin \{i,i+1\}$.

If we iterate (3.6.1), we find that if $v_1 - i \geq v_{i+1}$ for all $1 \leq i \leq r$, then

$$\partial_r\partial_{r-1}\ldots\partial_1\mathfrak{S}_{c^{-1}(v_1,\ldots,v_n)}(\mathbf{x},\mathbf{y}) = \mathfrak{S}_{c^{-1}(v_2,v_3,\ldots,v_{r+1},v_1-r,v_{r+2},\ldots,v_n)}(\mathbf{x},\mathbf{y}). \tag{3.6.2}$$

We are now ready to prove Lemma 3.5.7.

Proof of Lemma 3.5.7. We use induction on the number of parts of λ. If $\lambda = (\ell)$ has one part, then using the definition of z-Schubert polynomials together with (3.6.2), we obtain

$$\begin{aligned}\mathrm{LC}_z(\mathfrak{S}^n_{(\ell)}(\mathbf{z};\mathbf{x};\mathbf{y})) &= \partial_{n-\ell-1}\cdots\partial_1\Big(\prod_{l=1}^{n-1}(x_1-y_l)\Big) = \partial_{n-\ell-1}\cdots\partial_2\partial_1(\mathfrak{S}_{c^{-1}(n-1,0,\cdots,0)}(\mathbf{x},\mathbf{y})) \\ &= \partial_{n-\ell-1}\cdots\partial_2\big(\mathfrak{S}_{c^{-1}(0,n-2,0,\cdots,0)}(\mathbf{x},\mathbf{y})\big) = \mathfrak{S}_{c^{-1}(0,\cdots,0,\ell,0,\cdots,0)}(\mathbf{x},\mathbf{y}) \\ &= \mathfrak{S}_{c^{-1}(g_n((\ell)))}(\mathbf{x},\mathbf{y}).\end{aligned}$$

Now let $\lambda = (\lambda_1,\cdots,\lambda_k)$ with $k > 1$ and assume the statement holds for partitions with at most $k-1$ parts. Setting $\lambda' = (\lambda_2,\cdots,\lambda_k)$ and using the induction hypothesis, we have

$$\begin{aligned}\mathrm{LC}_z(\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})) &= \partial_{n-\lambda_1-\mathrm{mul}(\lambda)}\cdots\partial_1\Big(\mathrm{LC}_z(\mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}\mathbf{z};\mathbf{x}_{\hat{1}};\mathbf{y}))\prod_{l=1}^{n-\mathrm{mul}(\lambda))}(x_1-y_l)\Big) \\ &= \partial_{n-\lambda_1-\mathrm{mul}(\lambda)}\cdots\partial_1\Big(\mathfrak{S}_{c^{-1}(g_{n-1}(\lambda'))}(\mathbf{x}_{\hat{1}},\mathbf{y})\prod_{l=1}^{n-\mathrm{mul}(\lambda)}(x_1-y_l)\Big).\end{aligned}$$

Let $g_{n-1}(\lambda') = (v_1, \cdots, v_{n-1})$ and let $w = c^{-1}(v_1, \dots, v_{n-1}) \in S_{n-1}$. From Remark 3.1.5 we have $v_i = \lambda'_1$ for $n-1-\lambda'_1 \le i \le n - \mathrm{mul}(\lambda') - \lambda'_1$. This plus Definition 3.1.4 implies that

$$\begin{aligned} w_{n-\mathrm{mul}(\lambda')-\lambda'_1} &= n - \mathrm{mul}(\lambda') \\ &\vdots \\ w_{n-2-\lambda'_1} &= n-2 \\ w_{n-1-\lambda'_1} &= n-1. \end{aligned}$$

It follows from Remark 3.6.6 that if we write $c(w^{-1}) = (\tilde{c}_1, \dots, \tilde{c}_{n-1})$, then $\tilde{c}_m = 0$ for all $m > n - \mathrm{mul}(\lambda') - 1$. Since $n - \mathrm{mul}(\lambda) \ge n - \mathrm{mul}(\lambda') - 1$, we have $\tilde{c}_m = 0$ for all $m > n - \mathrm{mul}(\lambda)$, and Proposition 3.6.5 implies that

$$\mathfrak{S}_{c^{-1}(v_1,\dots,v_{n-1})}(\mathbf{x}_{\hat{1}}, \mathbf{y}) \prod_{l=1}^{n-\mathrm{mul}(\lambda)} (x_1 - y_l) = \mathfrak{S}_{c^{-1}(n-\mathrm{mul}(\lambda), v_1, \cdots, v_{n-1})}(\mathbf{x}, \mathbf{y}).$$

Therefore we have that

$$\mathrm{LC}_{\mathrm{z}}(\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})) = \partial_{n-\lambda_1-\mathrm{mul}(\lambda)} \cdots \partial_1 \big(\mathfrak{S}_{c^{-1}(n-\mathrm{mul}(\lambda), v_1, \cdots, v_{n-1})}(\mathbf{x}, \mathbf{y})\big).$$

We will apply (3.6.2), but need to first check that $n - \mathrm{mul}(\lambda_1) \ge v_i + i$ for $1 \le i \le n - \lambda_1 - \mathrm{mul}(\lambda)$. To see this, note that since $(v_1, \dots, v_{n-1}) = g_{n-1}(\lambda')$, each $v_i \le \lambda_1$. But then $v_i + i \le \lambda_1 + n - \lambda_1 - \mathrm{mul}(\lambda) = n - \mathrm{mul}(\lambda)$, as desired. Applying (3.6.2) now gives

$$\mathrm{LC}_{\mathrm{z}}(\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})) = \mathfrak{S}_{c^{-1}(v_1, \cdots, v_{n-\lambda_1-\mathrm{mul}(\lambda)}, \lambda_1, v_{n-\lambda_1-\mathrm{mul}(\lambda)+1}, \cdots, v_{n-1})}(\mathbf{x}, \mathbf{y}) = \mathfrak{S}_{c^{-1}(g_n(\lambda))}(\mathbf{x}, \mathbf{y}).$$

□

3.7 Proof of Theorem 3.5.9

In this section, we prove Theorem 3.5.9, which in turn implies Theorem 3.1.8 and Theorem 3.1.9. Our strategy will be to prove Theorem 3.5.9 first in the case of $w \in \mathrm{St}(n,1)$, and then use induction on k to prove it for $w \in \mathrm{St}(n,k)$.

We note that in this section, the divided difference operators *act on the z-variables.* For brevity we will often denote the z-Schubert polynomial $\mathfrak{S}^n_\lambda(\sigma^a(\mathbf{z});\mathbf{x};\mathbf{y})$ with shifted z-variables by $\mathfrak{S}^n_\lambda(\sigma^a(\mathbf{z}))$. As in the previous section, the subscripts for z-variables are considered modulo n.

Definition 3.7.1. For a partition $\lambda \in \mathrm{Val}(n)$, we identify it with the lattice path $L(\lambda;n)$ cutting out the Young diagram that takes unit steps south and east from the upper right corner $(\lambda_1, n-\lambda_1)$ to the lower left corner $(0,0)$ of the rectangle. Label the vertical steps from the top to bottom with numbers 1 through $n-\lambda_1$. Then label the horizontal steps from the right to the left with numbers $n-\lambda_1+1$ through n. We define $w(\lambda;n)$ to be the permutation of length n obtained by reading off the numbers through the lattice path.

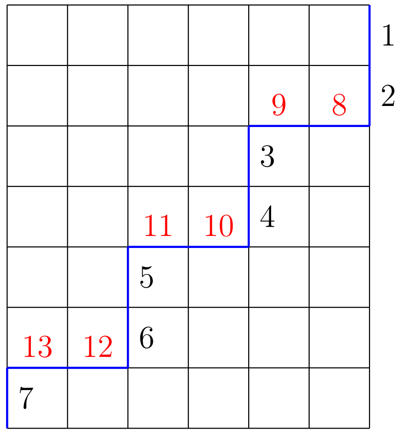

Figure 3.5: For $\lambda = (6,6,4,4,2,2) \in \mathrm{Val}(13)$, we can read off $w(\lambda;13) = (1,2,8,9,3,4,10,11,5,6,12,13,7)$ from the lattice path $L(\lambda;13)$ cutting out the Young diagram.

See Figure 3.5 for an example. Clearly $w(\lambda;n) \in \mathrm{St}(n,1)$. As we will see in Proposition 3.7.3 (1), $\Psi(w(\lambda;n)) = (\lambda)$.

Our first goal is to analyze the trivial factor $\mathrm{TF}(w)$ for $w = w(\lambda;n)$ (Corollary 3.7.5). This will help us prove Theorem 3.5.9 in the case of $w \in \mathrm{St}(n,1)$. We start by refining the quantities introduced in (3.1.2), (3.5.2), and (3.5.3).

Definition 3.7.2. Fix a positive integer n and choose $w \in \mathrm{St}(n)$. We define

$$\mathrm{xyFact}(w;i) = \prod_{\substack{i+1<k\leq n \\ i\to i+1\to k}} (x_1 - y_{n+1-k})\cdots(x_i - y_{n+1-k}) \qquad \text{for } 1 \leq i \leq n-2$$

$$\mathrm{xzFact}(w;i) = \prod_{\substack{j\neq w_i \\ \min\{w_i, w_{i-1}, w_{i-2}, \ldots, j\} = j}} (z_i - x_j) \quad \text{for } 1 \leq i \leq n$$

$$\mathrm{yzFact}(w;i) = \prod_{\substack{j\neq n+1-w_i \\ \max\{w_i, w_{i+1}, w_{i+2}, \ldots, n+1-j\} = n+1-j}} (z_i - y_j) \quad \text{for } 1 \leq i \leq n.$$

Clearly we have that

$$\mathrm{xyFact}(w) = \prod_{i=1}^{n-2} \mathrm{xyFact}(w;i), \mathrm{xzFact}(w) = \prod_{i=1}^{n} \mathrm{xzFact}(w;i), \mathrm{yzFact}(w) = \prod_{i=1}^{n} \mathrm{yzFact}(w;i).$$

Proposition 3.7.3. *Let $\lambda \in \mathrm{Val}(n)$ and $w = (w_1, \ldots, w_n) = w(\lambda;n)$. Recall that $w \in \mathrm{St}(n,1)$. The following statements hold:*

1. *We have that $\Psi(w(\lambda;n)) = (\lambda)$. Equivalently, $c(w^{-1}) = \lambda_1^{n-\lambda_1} - \lambda$, where we regard the vectors on the right-hand side as vectors of length n by adding trailing 0's.*

2. *Suppose that w_i lies on a vertical step of $L(\lambda; n)$. Let A be the set of numbers on the horizontal steps below w_i and B be the set of numbers on the vertical steps that are on the same vertical line as w_i and below w_i. We have*

$$\mathrm{xzFact}(w; i) = \prod_{k=1}^{w_i - 1} (z_i - x_k)$$
$$\mathrm{yzFact}(w; i) = \prod_{k \in A \cup B} (z_i - y_{n+1-k}).$$

3. *Suppose w_i lies on a horizontal step of $L(\lambda; n)$. Let C be the set of numbers on the vertical steps above w_i and D be the set of numbers on the horizontal steps that are on the same horizontal line of w_i and to the right of w_i. We have*

$$\mathrm{xzFact}(w; i) = \prod_{k \in C \cup D} (z_i - x_k)$$
$$\mathrm{yzFact}(w; i) = \prod_{k = w_i + 1}^{n} (z_i - y_{n+1-k}).$$

Proof. (1) The numbers $n - \lambda_1 + 1$ through n appear in increasing order in w so $c(w^{-1})$ vanishes after the $(n - \lambda_1)$st component. For $1 \le k \le n - \lambda_1$, let w_i be the letter on the kth vertical step of $L(\lambda; n)$. Then there are $\lambda_1 - \lambda_k$ numbers bigger than w_i in $w_1, \cdots, w_{i-1}$ (where we regard $\lambda_k = 0$ if $k > \mathrm{length}(\lambda)$. Thus the kth component of $c(w^{-1})$ is $\lambda_1 - \lambda_k$, and $c(w^{-1})$ has a unique descent in position $n - \lambda_1$. The fact that $\Psi(w(\lambda; n)) = \lambda$ now follows from the definition of Ψ.

(2) The numbers 1 through w_i appear in increasing order in w so we have $\mathrm{xzFact}(w; i) = \prod_{k=1}^{w_i - 1} (z_i - x_k)$. To compute $\mathrm{yzFact}(w; i)$, we need to find all letters ℓ which are maximum among $\{w_i, w_{i+1}, \ldots, \ell\}$ and for each one we pick up a factor of $(z_i - y_{n+1-\ell})$. Clearly these letters are precisely the ones in $A \cup B$.

(3) The proof is similar to part (2). □

Example 3.7.4. Let $\lambda = (6, 6, 4, 4, 2, 2) \in \mathrm{Val}(13)$ as in Figure 3.5. We have

$$w = w(\lambda; 13) = (1, 2, 8, 9, 3, 4, 10, 11, 5, 6, 12, 13, 7)$$
$$c(w^{-1}) = (0, 0, 2, 2, 4, 4, 6, 0, 0, 0, 0, 0, 0)$$
$$\lambda_1^{n-\lambda_1} - \lambda = (0, 0, 2, 2, 4, 4, 6).$$

For $w_5 = 3$, we have

$$\mathrm{xzFact}(w; 5) = (z_5 - x_1)(z_5 - x_2)$$
$$\mathrm{yzFact}(w; 5) = (z_5 - y_{10})(z_5 - y_1)(z_5 - y_2)(z_5 - y_3)(z_5 - y_4).$$

Proposition 3.7.5. *Let $\lambda \in \mathrm{Val}(n)$ with $\mathrm{mul}(\lambda) = b$. Write $\lambda = ((\lambda_1)^b, \tilde{\lambda})$ for some $\tilde{\lambda}$. For $w = w(((\lambda_1)^b, \tilde{\lambda}); n)$ and $w' = w(((\lambda_1)^{b-1}, \lambda_1 - 1, \tilde{\lambda}); n)$, the following statements are true.*

1. *If $b > 1$, we have $w' = s_b w$ and $w_b < w_{b+1}$.*

2. *If $b = 1$, we have $\mathrm{cyc}(w', 1) = s_1 w$ and $w_1 < w_2$.*

3. *If $b > 1$, we have*

$$\mathrm{TF}(s_b w) = M_1(x_b - y_{\lambda_1}) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1} (z_{b+1} - y_{n+1-b-i})$$

$$\mathrm{TF}(w) = M_1(z_b - y_{\lambda_1})(z_{b+1} - x_b) \prod_{i=1}^{b-1}(z_i - y_{n+1-b}) \prod_{i=b+2}^{b+\lambda_1-\tilde{\lambda}_1} (z_i - x_{n+1-\lambda_1}).$$

for some rational expression M_1 which is symmetric in variables z_b and z_{b+1}.

4. *If $b = 1$, we have*

$$\mathrm{TF}(s_1 w) = M_2(x_1 - y_{\lambda_1}) \prod_{i=1}^{n-\lambda_1} (z_1 - x_i) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1} (z_2 - y_{n-i})$$

$$\mathrm{TF}(w) = M_2(z_1 - y_{\lambda_1})(z_2 - x_1) \prod_{i=3}^{1+\lambda_1-\tilde{\lambda}_1} (z_i - x_{n+1-\lambda_1}).$$

for some rational expression M_2 which is symmetric in variables z_1 and z_2.

Proof. Part (1) and part (2) are straightforward from Definition 3.7.1.

(3) For $1 \le i \le b-1$, $w_b = b$ is on the same vertical line with w_i in $L(((\lambda_1)^b, \tilde{\lambda}); n)$ but $w'_{b+1} = b$ is not on the same vertical line with w_i in $L(((\lambda_1)^{b-1}, \lambda_1 - 1, \tilde{\lambda}); n)$. By Proposition 3.7.3 (2), whenever $1 \le i \le b-1$, we have

$$\mathrm{xzFact}(w; i)\, \mathrm{yzFact}(w; i) = \mathrm{xzFact}(w'; i)\, \mathrm{yzFact}(w'; i)(z_i - y_{n+1-b}). \tag{3.7.1}$$

For $b+2 \le i \le b+\lambda_1-\tilde{\lambda}_1$, $w_{b+1} = n+1-\lambda_1$ is on the same horizontal line with w_i in $L(((\lambda_1)^b, \tilde{\lambda}); n)$ but $w'_b = n+1-\lambda_1$ is not on the same horizontal line with w_i in $L(((\lambda_1)^{b-1}, \lambda_1 - 1, \tilde{\lambda}); n)$. By Proposition 3.7.3 (3), whenever $1 \le i \le b-1$, we have

$$\mathrm{xzFact}(w; i)\, \mathrm{yzFact}(w; i)(z_i - x_{n+1-\lambda_1}) = \mathrm{xzFact}(w'; i)\, \mathrm{yzFact}(w'; i). \tag{3.7.2}$$

Note that $w'_{b+1}, w'_{b+2}, \cdots, w'_{b+\mathrm{mul}(\tilde{\lambda})+1}$ are on the same vertical line if and only if $\lambda_1 - 1 = \tilde{\lambda}_1$. By Proposition 3.7.3 (2), we have

$$\mathrm{xzFact}(w; b)\, \mathrm{yzFact}(w; b) = \prod_{k=1}^{b-1}(z_b - x_k) \prod_{k=1}^{\lambda_1}(z_b - y_k) \tag{3.7.3}$$

$$\mathrm{xzFact}(w'; b+1)\, \mathrm{yzFact}(w'; b+1) \tag{3.7.4}$$

$$= \prod_{k=1}^{b-1}(z_{b+1} - x_k) \prod_{k=1}^{\lambda_1-1}(z_{b+1} - y_k) \prod_{k=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1} (z_{b+1} - y_{n+1-b-k}).$$

And by Proposition 3.7.3 (3), we have

$$\mathrm{xzFact}(w;b+1)\,\mathrm{yzFact}(w;b+1) = \prod_{k=1}^{b}(z_{b+1}-x_k)\prod_{k=1}^{\lambda_1-1}(z_{b+1}-y_k) \tag{3.7.5}$$

$$\mathrm{xzFact}(w';b)\,\mathrm{yzFact}(w';b) \tag{3.7.6}$$
$$= \prod_{k=1}^{b-1}(z_b-x_k)\prod_{k=1}^{\lambda_1-1}(z_b-y_k).$$

For w, we have $b-1\to b\to w_{b+1}=n+1-\lambda_1$ while for w' we have $b\to b+1\to w'_b=n+1-\lambda_1$. So we conclude

$$\frac{\mathrm{xyFact}(w)}{\mathrm{xyFact}(w')} = \frac{\prod_{k=1}^{b-1}(x_k-y_{\lambda_1})}{\prod_{k=1}^{b}(x_k-y_{\lambda_1})} = \frac{1}{x_b-y_{\lambda_1}}. \tag{3.7.7}$$

Combining (3.7.1), (3.7.2), (3.7.3),(3.7.5) and (3.7.7) proves the argument.

(4) The proof is similar to part (3). □

Proposition 3.7.6. *Theorem 3.5.9 is true for* $w\in\mathrm{St}(n,1)$.

Proof. We use induction on $|\lambda|$ for $\Psi(w)=(\lambda)$. The base case $|\lambda|=0$ corresponds to the identity permutation in $\mathrm{St}(n,0)$. Take any $w\in\mathrm{St}(n,1)$ such that $\Psi(w)=(\lambda)$. Let $b=\mathrm{mul}(\lambda)$ and write $\lambda=((\lambda_1)^b,\tilde\lambda)$ for some $\tilde\lambda$. Denoting $w'=w(((\lambda_1)^{b-1},\lambda_1-1,\tilde\lambda);n)$, by induction hypothesis we have

$$\psi_{w'}(\mathbf{z}) = \mathrm{TF}(w')\mathfrak{S}^n_{((\lambda_1)^{b-1},\lambda_1-1,\tilde\lambda)}(\mathbf{z}).$$

We first prove the induction step when $b>1$. By Corollary 3.7.5 (1) and Proposition 3.4.1, we have

$$\psi_w(\mathbf{z}) = \pi_b(w_{b+1},w_b)\psi_{w'}(\mathbf{z}) = \frac{(z_b-y_{\lambda_1})(z_{b+1}-x_b)}{x_b-y_{\lambda_1}}\partial_b\Big(\mathrm{TF}(w')\mathfrak{S}^n_{((\lambda_1)^{b-1},\lambda_1-1,\tilde\lambda)}(\mathbf{z})\Big). \tag{3.7.8}$$

By Corollary 3.7.5 (3) we can write

$$\mathrm{TF}(w') = M_1(x_b-y_{\lambda_1})\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde\lambda))-1}(z_{b+1}-y_{n+1-b-i})$$
$$\mathrm{TF}(w) = M_1(z_b-y_{\lambda_1})(z_{b+1}-x_b)\prod_{i=1}^{b-1}(z_i-y_{n+1-b})\prod_{i=b+2}^{b+\lambda_1-\tilde\lambda_1}(z_i-x_{n+1-\lambda_1})$$

for some M_1 that is symmetric in variables z_b and z_{b+1}. Plugging to (3.7.8) gives

$$\psi_w(\mathbf{z}) = M_1(z_b-y_{\lambda_1})(z_{b+1}-x_b)\partial_b\left(\mathfrak{S}^n_{((\lambda_1)^{b-1},\lambda_1-1,\tilde\lambda)}(\mathbf{z})\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde\lambda))-1}(z_{b+1}-y_{n+1-b-i})\right)$$

By Proposition 3.8.5 we have

$$\begin{aligned}\psi_w(\mathbf{z}) &= M_1(z_b - y_{\lambda_1})(z_{b+1} - x_b)\left(\mathfrak{S}^n_{((\lambda_1)^b,\tilde{\lambda})}(\mathbf{z})\prod_{i=1}^{b-1}(z_i - y_{n+1-b})\prod_{i=b+2}^{b+\lambda_1-\tilde{\lambda}_1}(z_i - x_{n+1-\lambda_1})\right)\\ &= \mathrm{TF}(w)\mathfrak{S}^n_\lambda(\mathbf{z}).\end{aligned}$$

Now consider the case $b = 1$. By Corollary 3.7.5 (2) and Proposition 3.4.1, we have

$$\psi_w(\mathbf{z}) = \pi_1(w_2, w_1)\psi_{\mathrm{cyc}(w',1)}(\mathbf{z}) = \frac{(z_1 - y_{\lambda_1})(z_2 - x_1)}{x_1 - y_{\lambda_1}}\partial_1\left(\mathrm{TF}(\mathrm{cyc}(w',1))\mathfrak{S}^n_{(\lambda_1-1,\tilde{\lambda})}(\sigma(\mathbf{z}))\right). \tag{3.7.9}$$

By Corollary 3.7.5 (4) we can write

$$\begin{aligned}\mathrm{TF}(\mathrm{cyc}(w',1)) &= M_2(x_1 - y_{\lambda_1})\prod_{i=1}^{n-\lambda_1}(z_1 - x_i)\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1}(z_2 - y_{n-i})\\ \mathrm{TF}(w) &= M_2(z_1 - y_{\lambda_1})(z_2 - x_1)\prod_{i=3}^{1+\lambda_1-\tilde{\lambda}_1}(z_i - x_{n+1-\lambda_1}).\end{aligned}$$

for some M_2 that is symmetric in variables z_1 and z_2. Plugging to (3.7.9) gives

$$\psi_w(\mathbf{z}) = M_2(z_1 - y_{\lambda_1})(z_2 - x_1)\partial_1\left(\mathfrak{S}^n_{(\lambda_1-1,\tilde{\lambda})}(\sigma(\mathbf{z}))\prod_{i=1}^{n-\lambda_1}(z_1 - x_i)\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1}(z_2 - y_{n-i})\right)$$

By Proposition 3.8.4 we have

$$\begin{aligned}\psi_w(\mathbf{z}) &= M_2(z_1 - y_{\lambda_1})(z_2 - x_1)\left(\mathfrak{S}^n_{(\lambda_1,\tilde{\lambda})}(\mathbf{z})\prod_{i=3}^{1+\lambda_1-\tilde{\lambda}_1}(z_i - x_{n+1-\lambda_1})\right)\\ &= \mathrm{TF}(w)\mathfrak{S}^n_\lambda(\mathbf{z}).\end{aligned}$$

□

Definition 3.7.7. If $\pi \in S_m$ and $\sigma \in S_p$, the *direct sum* $\pi \oplus \sigma \in S_{m+p}$ is the permutation defined by $(\pi \oplus \sigma)(i) = \begin{cases}\pi(i) & \text{if } 1 \le i \le m\\ \sigma(i-m) + m & \text{if } m+1 \le i \le m+n.\end{cases}$

For example, $(3,2,1) \oplus (3,1,2,5,4) = (3,2,1,6,4,5,8,7)$.

The following lemma is easy to verify.

Lemma 3.7.8. *Given* $\lambda \in \mathrm{Val}(n)$, *let*

$$u = (u_1, \ldots, u_n) := \mathrm{cyc}(w(\lambda; n), \mathrm{length}(\lambda) + \lambda_1 - n).$$

Let $\bar{w}(\lambda; n) := (u_1, \ldots, u_{n-\lambda_{\mathrm{last}}})$. *Then* $\bar{w}(\lambda; n) \in S_{n-\lambda_{\mathrm{last}}}$ *and* $u = \bar{w}(\lambda; n) \oplus \mathrm{id}_{\lambda_{\mathrm{last}}}$, *where* id_m *is the identity permutation on* m *letters.*

Example 3.7.9. Let $n = 5$ and $\lambda = (2,2) \in \mathrm{Val}(5)$, so that $\lambda_{\mathrm{last}} = 2$. Then $w(\lambda;5) = (1,2,4,5,3)$ and $\mathrm{length}(\lambda) + \lambda_1 - n = -1$. We have $u = \mathrm{cyc}((1,2,4,5,3),-1) = (3,1,2,4,5)$, so $\bar{w}(\lambda;5) = (3,1,2)$. We have $u = (3,1,2) \oplus \mathrm{id}_2$.

Proposition 3.7.10. *Let $w \in \mathrm{St}(n,k)$ for $k \geq 2$ with $\Psi(w) = (\lambda^1, \lambda^2, \cdots, \lambda^k)$ and $s(w) = (a_1, \ldots, a_k)$. Then we can write $\mathrm{cyc}(w, a_2) = \bar{w}(\lambda^1; n) \oplus w'$ for some w'. If we let $w^{\downarrow} := \mathrm{id}_{n-\lambda^1_{\mathrm{last}}} \oplus w'$, then $w^{\downarrow} \in \mathrm{St}(n, k-1)$ and $\Psi(w^{\downarrow}) = (\lambda^2, \cdots, \lambda^k)$.*

Example 3.7.11. Let $w = (1,2,5,4,3) \in \mathrm{St}(5,2)$. Then $\Psi(w) = ((2,2),(1,1,1))$ and $s(w) = (a_1, a_2) = (0,-1)$. From the previous example, $\bar{w}(\lambda;5) = (3,1,2)$. We have $\mathrm{cyc}(w, a_2) = (3,1,2,5,4) = (3,1,2) \oplus w'$ where $w' = (2,1)$. And we have $w^{\downarrow} = \mathrm{id}_{n-\lambda^1_{\mathrm{last}}} \oplus w' = (1,2,3) \oplus (2,1) = (1,2,3,5,4) \in \mathrm{St}(5,1)$ with $\Psi(w^{\downarrow}) = (\lambda^2) = ((1,1,1))$.

Proof. Note that $a_2 = \mathrm{length}(\lambda^1) + \lambda^1_1 - n$. We have $c(w^{-1}) = ((\lambda^1_1)^{n-\lambda^1_1} - \lambda^1) + ((\lambda^2_1)^{n-\lambda^2_1} - \lambda^2) + \cdots + ((\lambda^k_1)^{n-\lambda^k_1} - \lambda^k)$ and by the definition of $\mathrm{ParSeq}(n,k)$ we know that the first $(n - \lambda^1_{\mathrm{last}})$ parts of $((\lambda^i_1)^{n-\lambda^i_1} - \lambda^i)$ are zero for $2 \leq i \leq k$. So the first $(n - \lambda^1_{\mathrm{last}})$ parts of $c(w^{-1})$ equals $((\lambda^1_1)^{n-\lambda^1_1} - \lambda^1)$ which is same as $c((w(\lambda^1;n))^{-1})$ by Proposition 3.7.3 (1). So the positions of the numbers 1 through $(n - \lambda^1_{\mathrm{last}})$ are same in w and $w(\lambda^1;n)$. Since taking the first $(n - \lambda^1_{\mathrm{last}})$ parts of $\mathrm{cyc}(w(\lambda^1;n), \mathrm{length}(\lambda^1) + \lambda^1_1 - n)$ gives $\bar{w}(\lambda;n)$, we conclude taking the first $(n - \lambda^1_{\mathrm{last}})$ parts of $\mathrm{cyc}(w, \mathrm{length}(\lambda^1) + \lambda^1_1 - n)$ also gives $\bar{w}(\lambda;n)$.

Note that i-th component of $c(w^{-1})$ counts number of $w_j > i$ for $1 \leq j \leq w^{-1}(i)$. Thus $c(w^{-1})$ and $c((\mathrm{cyc}(w, \mathrm{length}(\lambda^1) + \lambda^1_1 - n))^{-1})$ coincide after $(n - \lambda^1_{\mathrm{last}})$-th component as taking the numbers smaller than $(n - \lambda^1_{\mathrm{last}})$ to the front does not affect the code of its inverse after $(n - \lambda^1_{\mathrm{last}})$-th component. Thus $c((\mathrm{id}_{n-\lambda^1_{\mathrm{last}}} \oplus w')^{-1})$ coincides with $c(w^{-1})$ after $(n - \lambda^1_{\mathrm{last}})$-th component and the first $(n - \lambda^1_{\mathrm{last}})$ parts are zero as the permutation starts with $\mathrm{id}_{n-\lambda^1_{\mathrm{last}}}$. We conclude $c((\mathrm{id}_{n-\lambda^1_{\mathrm{last}}} \oplus w')^{-1}) = ((\lambda^2_1)^{n-\lambda^2_1} - \lambda^2) + \cdots + ((\lambda^k_1)^{n-\lambda^k_1} - \lambda^k)$ and we are done. □

Lemma 3.7.12. *Let u and u' be permutations in S_n and w and w' be permutations in S_m. We have*

$$\frac{\mathrm{TF}(u' \oplus w)}{\mathrm{TF}(u \oplus w)} = \frac{\mathrm{TF}(u' \oplus w')}{\mathrm{TF}(u \oplus w')}.$$

Proof. It is enough to show the following three equations

$$\frac{\mathrm{xyFact}(u' \oplus w)}{\mathrm{xyFact}(u \oplus w)} = \frac{\mathrm{xyFact}(u' \oplus w')}{\mathrm{xyFact}(u \oplus w')} \tag{3.7.10}$$

$$\frac{\mathrm{xzFact}(u' \oplus w)}{\mathrm{xzFact}(u \oplus w)} = \frac{\mathrm{xzFact}(u' \oplus w')}{\mathrm{xzFact}(u \oplus w')} \tag{3.7.11}$$

$$\frac{\mathrm{yzFact}(u' \oplus w)}{\mathrm{yzFact}(u \oplus w)} = \frac{\mathrm{yzFact}(u' \oplus w')}{\mathrm{yzFact}(u \oplus w')}. \tag{3.7.12}$$

For $1 \leq i \leq n-1$ we have $\mathrm{xyFact}(u \oplus w; i) = \mathrm{xyFact}(u \oplus w'; i)$ and $\mathrm{xyFact}(u' \oplus w; i) = \mathrm{xyFact}(u' \oplus w'; i)$. For $n+1 \leq i \leq n+m-1$ we have $\mathrm{xyFact}(u \oplus w; i) = \mathrm{xyFact}(u' \oplus w; i)$ and

$\text{xyFact}(u \oplus w'; i) = \text{xyFact}(u' \oplus w'; i)$. And for $i = n$ we have $\text{xyFact}(u \oplus w; i) = \text{xyFact}(u' \oplus w; i)$ and $\text{xyFact}(u \oplus w'; i) = \text{xyFact}(u' \oplus w'; i)$. So the first equation (3.7.10) follows.

For $1 \le i \le n$ we have $\text{xzFact}(u \oplus w; i) = \text{xzFact}(u \oplus w'; i)$ and $\text{xzFact}(u' \oplus w; i) = \text{xzFact}(u' \oplus w'; i)$. Now consider the case $n+1 \le i \le n+m$. For $1 \le j \le n$, $\text{xzFact}(u \oplus w; i)$ has a factor $(z_i - x_j)$ for $\text{xzFact}(u \oplus w'; i)$ has a factor $(z_i - x_j)$. And the same is true for $\text{xzFact}(u' \oplus w; i)$ and $\text{xzFact}(u' \oplus w'; i)$. For $n+1 \le j \le n+m$ $\text{xzFact}(u \oplus w; i)$ has a factor $(z_i - x_j)$ for $\text{xzFact}(u' \oplus w; i)$ has a factor $(z_i - x_j)$. And the same is true for $\text{xzFact}(u \oplus w'; i)$ and $\text{xzFact}(u' \oplus w'; i)$. So the second equation (3.7.11) follows.

The proof for (3.7.12) is similar to the proof for (3.7.11). □

Proposition 3.7.13. *[6, Theorem 20] Let $u \in S_n$ and $w \in S_m$. Then we can write*

$$\psi_{u \oplus w}(\mathbf{z}) = \psi^1_u(\mathbf{z})\psi^2_w(\mathbf{z})$$

where $\psi^1_u(\mathbf{z})$ (respectively $\psi^2_w(\mathbf{z})$) depends only on u (respectively w).[3]

Corollary 3.7.14. *Let u and u' be permutations in S_n and w and w' be permutations in S_m. We have*

$$\frac{\psi_{u' \oplus w}(\mathbf{z})}{\psi_{u \oplus w}(\mathbf{z})} = \frac{\psi_{u' \oplus w'}(\mathbf{z})}{\psi_{u \oplus w'}(\mathbf{z})}.$$

Proof of Theorem 3.5.9. We prove Theorem 3.5.9 for $w \in \text{St}(n,k)$ using induction on k. Theorem 3.5.9 holds for $k = 1$ by Proposition 3.7.6. Suppose the theorem holds for all elements in $\text{St}(n, k-1)$.

We now consider $w \in \text{St}(n,k)$. Let $\Psi(w) = (\lambda^1, \lambda^2, \cdots, \lambda^k)$ and $s(w) = (a_1, a_2, \cdots, a_k)$. By Proposition 3.7.10, we have $\text{cyc}(w, a_2) = \bar{w}(\lambda^1; n) \oplus w'$ for some w'. Moreover, if we set $w^{\downarrow} := \text{id}_{n-\lambda^1_{\text{last}}} \oplus w'$, we have that $w^{\downarrow} \in \text{St}(n, k-1)$ and $\Psi(w^{\downarrow}) = (\lambda^2, \cdots, \lambda^k)$. Using Definition 3.5.8 it is easy to see that $s(w^{\downarrow}) = (0, a_3 - a_2, \cdots, a_k - a_2)$. By the induction hypothesis we have

$$\psi_{\text{id}_{n-\lambda^1_{\text{last}}} \oplus w'}(\mathbf{z}) = \text{TF}(\text{id}_{n-\lambda^1_{\text{last}}} \oplus w') \prod_{i=2}^{k} \mathfrak{S}^n_{\lambda^i}(\sigma^{a_i - a_2}(\mathbf{z})). \tag{3.7.13}$$

By Lemma 3.7.8, and using the fact that $a_2 = \text{length}(\lambda^1) + \lambda^1_1 - n$, we have that $\bar{w}(\lambda^1; n) \oplus \text{id}_{\lambda^1_{\text{last}}} = \text{cyc}(w(\lambda^1; n), a_2)$, where $w(\lambda^1; n) \in \text{St}(n, 1)$. Therefore the induction hypothesis and (3.4.1) implies that

$$\psi_{\bar{w}(\lambda^1;n) \oplus \text{id}_{\lambda^1_{\text{last}}}}(\mathbf{z}) = \text{TF}(\bar{w}(\lambda^1; n) \oplus \text{id}_{\lambda^1_{\text{last}}})\ \mathfrak{S}^n_{\lambda^1}(\sigma^{a_1 - a_2}(\mathbf{z})). \tag{3.7.14}$$

By Corollary 3.7.14 we have

$$\psi_{\bar{w}(\lambda^1;n) \oplus w'}(\mathbf{z}) = \frac{\psi_{\bar{w}(\lambda^1;n) \oplus \text{id}_{\lambda^1_{\text{last}}}}(\mathbf{z})\ \psi_{\text{id}_{\lambda^1_{n-\text{last}}} \oplus w'}(\mathbf{z})}{\psi_{\text{id}_{n-\lambda^1_{\text{last}}} \oplus \text{id}_{\lambda^1_{\text{last}}}}(\mathbf{z})}$$

[3]The result stated in [6, Theorem 20] concerns the skew sum of the permutations u and w, not the direct sum; however, the direct sum of u and w is a cyclic rotation of the skew sum of w and u, so the result we've stated follows.

Plugging in (3.7.13) and (3.7.14) together with Lemma 3.7.12 gives

$$\begin{aligned}\psi_{\bar{w}(\lambda^1;n)\oplus w'}(\mathbf{z}) &= \frac{\psi_{\bar{w}(\lambda^1;n)\oplus \mathrm{id}_{\lambda^1_{\mathrm{last}}}}(\mathbf{z})\ \psi_{\mathrm{id}_{\lambda^1_{n-\mathrm{last}}}\oplus w'}(\mathbf{z})}{\psi_{\mathrm{id}_{n-\lambda^1_{\mathrm{last}}}\oplus \mathrm{id}_{\lambda^1_{\mathrm{last}}}}(\mathbf{z})} \\ &= \frac{\mathrm{TF}(\bar{w}(\lambda^1;n)\oplus \mathrm{id}_{\lambda^1_{\mathrm{last}}})\,\mathrm{TF}(\mathrm{id}_{\lambda^1_{n-\mathrm{last}}}\oplus w')}{\mathrm{TF}(\mathrm{id}_{n-\lambda^1_{\mathrm{last}}}\oplus \mathrm{id}_{\lambda^1_{\mathrm{last}}})}\prod_{i=1}^{k}\mathfrak{S}^n_{\lambda^i}(\sigma^{a_i-a_2}(\mathbf{z})) \\ &= \mathrm{TF}(\bar{w}(\lambda^1;n)\oplus w')\prod_{i=1}^{k}\mathfrak{S}^n_{\lambda^i}(\sigma^{a_i-a_2}(\mathbf{z})).\end{aligned}$$

Now using (3.4.1) and cyclically shifting z-variables completes the proof.

3.8 Appendix: technical results for the proof of Theorem 3.5.9

In this section, we collect some technical results (Proposition 3.8.4 and Proposition 3.8.5) which are needed for the proof of Theorem 3.5.9.

Lemma 3.8.1. *[6, Lemma 22] Let $K(x_1;x_2,\cdots,x_{m+1})$ be a rational expression in $x_1,\ldots,x_{m+1}$ which is symmetric in the variables $x_2,\cdots,x_{m+1}$. We have*

$$\partial_m\cdots\partial_1 K = \sum_{i=1}^{m+1}\frac{K(x_i;x_1,\cdots,\hat{x}_i,\cdots,x_{m+1})}{\prod_{\substack{j=1\\ j\neq i}}^{m+1}(x_i-x_j)}.$$

Now we rewrite (3.5.1).

Lemma 3.8.2. *For $\lambda\in\mathrm{Val}(n)$, let $\mathrm{mul}(\lambda)\geq k$ for some k. Denote $\tilde{\lambda}$ to be the partition obtained by deleting the first k parts of λ. Then we have the equation*

$$\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}) = \sum_{\substack{I\subseteq[n-\lambda_1-\mathrm{mul}(\lambda)+k]\\ |I|=k}}\frac{\mathfrak{S}^{n-k}_{\tilde{\lambda}}(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat{I}};\mathbf{y})\prod_{\substack{1\leq l\leq n-\mathrm{mul}(\lambda)\\ i\in I}}(x_i-y_l)\prod_{\substack{1\leq i\leq\lambda_1-\lambda_{k+1}+k\\ j\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I}}(z_i-x_j)}{\prod_{\substack{i\in I\\ j\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I}}(x_i-x_j)}. \tag{3.8.1}$$

Proof. Note that (3.8.1) implies that $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ is symmetric in variables $x_1,\cdots,x_{n-\lambda_1-\mathrm{mul}(\lambda)+k}$. Since we can take $k=\mathrm{mul}(\lambda)$, (3.8.1) implies that $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ is symmetric in variables $x_1,\cdots,x_{n-\lambda_1}$. Now we proceed with the induction on n. Assume the equations hold for $(n-1)$.

Let λ' to be the partition obtained by deleting the first part of λ. By induction hypothesis, $\mathfrak{S}^{n-1}_{\lambda'}(\mathbf{z};\mathbf{x};\mathbf{y})$ is symmetric in variables $x_1,\cdots,x_{(n-1)-\lambda'_1}$. Note that we have $(n-1-\lambda'_1) \geq n-\lambda_1-\text{mul}(\lambda)$. Thus we rewrite (3.5.1) using Lemma 3.8.1 as follow

$$\mathfrak{S}^{n}_{\lambda}(\mathbf{z};\mathbf{x};\mathbf{y}) \tag{3.8.2}$$
$$= \sum_{i=1}^{n-\lambda_1-\text{mul}(\lambda)+1} \frac{\mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}(\mathbf{z});\mathbf{x}_{\hat{i}};\mathbf{y}) \prod_{1\leq l\leq n-\text{mul}(\lambda)} (x_i-y_l) \prod_{l=1}^{\lambda_1-\lambda_2+1} \prod_{\substack{j=1\\ j\neq i}}^{n-\lambda_1-\text{mul}(\lambda)+1} (z_l-x_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n-\lambda_1-\text{mul}(\lambda)+1} (x_i-x_j)},$$

which implies $k=1$ case of (3.8.1).

Assume $k>1$. In this case, we have $\text{mul}(\lambda)>1$, which implies $\text{mul}(\lambda')=\text{mul}(\lambda)-1$ and $\lambda'_1=\lambda_2=\lambda_1$. Applying (3.8.1) to $\mathfrak{S}^{n-1}_{\lambda'}(\mathbf{z};\mathbf{x};\mathbf{y})$ and $(k-1)$ gives

$$\mathfrak{S}^{n-1}_{\lambda'}(\mathbf{z};\mathbf{x};\mathbf{y}) = \sum_{\substack{I\subseteq[n-1-\lambda_1-\text{mul}(\lambda)+k]\\ |I|=k-1}} \tag{3.8.3}$$
$$\frac{\mathfrak{S}^{n-k}_{\tilde{\lambda}}(\sigma^{\lambda_1-\lambda_{k+1}+k-1}(\mathbf{z});\mathbf{x}_{\hat{I}};\mathbf{y}) \prod_{\substack{1\leq l\leq n-\text{mul}(\lambda)\\ i\in I}} (x_i-y_l) \prod_{\substack{1\leq i\leq \lambda_1-\lambda_{k+1}+k-1\\ j\in[n-1-\lambda_1-\text{mul}(\lambda)+k]-I}} (z_i-x_j)}{\prod_{\substack{i\in I\\ j\in[n-1-\lambda_1-\text{mul}(\lambda)+k]-I}} (x_i-x_j)}.$$

and (3.8.2) becomes

$$\mathfrak{S}^{n}_{\lambda}(\mathbf{z};\mathbf{x};\mathbf{y}) = \sum_{j=1}^{n-\lambda_1-\text{mul}(\lambda)+1} \frac{\mathfrak{S}^{n-1}_{\lambda'}(\sigma(\mathbf{z});\mathbf{x}_{\hat{j}};\mathbf{y}) \prod_{l=1}^{n-\text{mul}(\lambda)} (x_j-y_l) \prod_{\substack{m=1\\ m\neq j}}^{n-\lambda_1-\text{mul}(\lambda)+1} (z_1-x_m)}{\prod_{\substack{m=1\\ m\neq j}}^{n-\lambda_1-\text{mul}(\lambda)+1} (x_j-x_m)}. \tag{3.8.4}$$

Plugging in (3.8.3) to (3.8.4) gives

$$\mathfrak{S}^{n}_{\lambda}(\mathbf{z};\mathbf{x};\mathbf{y}) \tag{3.8.5}$$
$$= \sum_{\substack{I\subseteq[n-1-\lambda_1-\text{mul}(\lambda)+k]\\ |I|=k-1}} \sum_{j=1}^{n-\lambda_1-\text{mul}(\lambda)+1} f_I(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat{j}};\mathbf{y}) M_I(\mathbf{z};\mathbf{x}_{\hat{j}};\mathbf{y}) \prod_{l=1}^{n-\text{mul}(\lambda)} (x_j-y_l) \tag{3.8.6}$$

where

$$f_I(\mathbf{z};\mathbf{x};\mathbf{y}) = \mathfrak{S}^{n-k}_{\tilde{\lambda}}(\mathbf{z};\mathbf{x}_{\hat{I}};\mathbf{y})$$

$$M_I(\mathbf{z};\mathbf{x};\mathbf{y}) = \frac{\prod\limits_{\substack{1\le l\le n-\mathrm{mul}(\lambda)\\ i\in I}}(x_i-y_l)\prod\limits_{\substack{2\le i\le \lambda_1-\lambda_{k+1}+k\\ m\in[n-1-\lambda_1-\mathrm{mul}(\lambda)+k]-I}}(z_i-x_m)\prod\limits_{m=1}^{n-\lambda_1-\mathrm{mul}(\lambda)}(z_1-x_m)}{\prod\limits_{\substack{i\in I\\ m\in[n-1-\lambda_1-\mathrm{mul}(\lambda)+k]-I}}(x_i-x_m)\prod\limits_{m=1}^{n-\lambda_1-\mathrm{mul}(\lambda)}(x_j-x_m)}.$$

For a fixed $I_0=\{i_1<\cdots<i_k\}\subseteq[n-\lambda_1-\mathrm{mul}(\lambda)+k]$, to have

$$f_I(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat{j}};\mathbf{y}) = \mathfrak{S}^{n-k}_{\tilde{\lambda}}(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat{I_0}};\mathbf{y})$$

we need to take $I=I_h=\{i_1<\cdots<i_{h-1}<i_{h+1}-1<\cdots<i_k-1\}$ and $j=i_h\le n-\lambda_1-\mathrm{mul}(\lambda)+1$. For such $I=I_h$ and $j=i_h$ we have

$$M_{I_h}(\mathbf{z};\mathbf{x}_{\hat{i_h}};\mathbf{y}) = \frac{\prod\limits_{\substack{1\le l\le n-\mathrm{mul}(\lambda)\\ i\in I_0}}(x_i-y_l)\prod\limits_{\substack{2\le i\le \lambda_1-\lambda_{k+1}+k\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(z_i-x_m)\prod\limits_{\substack{m=1\\ m\neq i_h}}^{n-\lambda_1-\mathrm{mul}(\lambda)+1}(z_1-x_m)}{\prod\limits_{\substack{i\in I_0, i\neq i_h\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(x_i-x_m)\prod\limits_{\substack{m=1\\ m\neq i_h}}^{n-\lambda_1-\mathrm{mul}(\lambda)+1}(x_{i_h}-x_m)},$$

so taking the coefficient of $\mathfrak{S}^{n-k}_{\tilde{\lambda}}(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat{I_0}};\mathbf{y})$ in (3.8.5) gives

$$\sum_{\substack{h\ge 1\\ i_h\le n-\lambda_1-\mathrm{mul}(\lambda)+1}} M_{I_h}(\mathbf{z};\mathbf{x}_{\hat{i_h}};\mathbf{y}).$$

We claim

$$\frac{\prod\limits_{\substack{1\le l\le n-\mathrm{mul}(\lambda)\\ i\in I_0}}(x_i-y_l)\prod\limits_{\substack{1\le i\le \lambda_1-\lambda_{k+1}+k\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(z_i-x_m)}{\prod\limits_{\substack{i\in I_0\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(x_i-x_m)} = \sum_{\substack{h\ge 1\\ i_h\le n-\lambda_1-\mathrm{mul}(\lambda)+1}} M_{I_h}(\mathbf{z};\mathbf{x}_{\hat{i_h}};\mathbf{y}). \tag{3.8.7}$$

We view both sides as polynomials in z_1 in a degree at most $(n-\lambda_1-\mathrm{mul}(\lambda))$. So it is enough to show that they coincide when we plug in $z_1=x_1,\cdots,x_{n-\lambda_1-\mathrm{mul}(\lambda)+1}$. If we plug in $z=x_m$ for $m\notin I_0$ then both sides are zero. If we plug in $z_1=x_{i_t}$ for some $i_t\in I_0$ then only M_{I_t,i_t} does not vanish and we have

$$M_{I_t}(\mathbf{z};\mathbf{x}_{\hat{i_t}};\mathbf{y})|_{z_1=x_{i_t}} = \frac{\prod\limits_{\substack{1\le l\le n-\mathrm{mul}(\lambda)\\ i\in I_0}}(x_i-y_l)\prod\limits_{\substack{2\le i\le \lambda_1-\lambda_{k+1}+k\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(z_i-x_m)}{\prod\limits_{\substack{i\in I_0, i\neq i_t\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(x_i-x_m)}.$$

This is the same as the left hand side of (3.8.7) evaluated at $z_1 = x_{i_t}$.

In conclusion (3.8.4) becomes

$$\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}) = \sum_{\substack{I_0\subseteq[n-\lambda_1-\mathrm{mul}(\lambda)+k]\\ |I_0|=k}} \frac{\mathfrak{S}^{n-k}_{\tilde\lambda}(\sigma^{\lambda_1-\lambda_{k+1}+k}(\mathbf{z});\mathbf{x}_{\hat I_0};\mathbf{y}) \prod\limits_{\substack{1\le l\le n-\mathrm{mul}(\lambda)\\ i\in I_0}}(x_i-y_l) \prod\limits_{\substack{1\le i\le \lambda_1-\lambda_{k+1}+k\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(z_i-x_m)}{\prod\limits_{\substack{i\in I_0\\ m\in[n-\lambda_1-\mathrm{mul}(\lambda)+k]-I_0}}(x_i-x_m)},$$

which completes the proof. □

Proposition 3.8.3. *For $\lambda\in\mathrm{Val}(n)$, the z-Schubert polynomial $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ satisfies the following property. When we set $z_1 = x_a$ for $1\le a\le n-\lambda_1$ we have*

$$\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y}))|_{z_1=x_a} = \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}(\mathbf{z});\mathbf{x}_{\hat a};\mathbf{y}))\prod_{l=1}^{n-\mathrm{mul}(\lambda)}(x_a-y_l)\prod_{i=2}^{(\lambda_1-\lambda_2+1)}\prod_{\substack{m=1\\ m\ne a}}^{n-\lambda_1-\mathrm{mul}(\lambda)+1}(z_i-x_m), \tag{3.8.8}$$

where λ' is the partition obtained by deleting the first part of λ. If $\mathrm{length}(\lambda)=1$ *then we regard $\lambda_2=0$.*

Proof. Taking $k=\mathrm{mul}(\lambda)$ in (3.8.1), we deduce that $\mathfrak{S}^n_\lambda(\mathbf{z};\mathbf{x};\mathbf{y})$ is symmetric in variables $x_1,\cdots,x_{n-\lambda_1}$. So it is enough to show (3.8.8) for $a=1$. And the case $a=1$ is covered by (3.8.2). □

In Proposition 3.8.4 and Proposition 3.8.5 the divided difference operator ∂_i *acts on z-variables.*

Proposition 3.8.4. *For $\lambda\in\mathrm{Val}(n)$ such that $\lambda_1>\lambda_2$ we denote $\lambda=(\lambda_1,\lambda')$ for some λ'. We have*

$$\mathfrak{S}^n_{(\lambda_1,\lambda')}(\mathbf{z};\mathbf{x};\mathbf{y})\prod_{i=3}^{\lambda_1-\lambda_2+1}(z_i-x_{n+1-\lambda_1}) = \partial_1\Big(\mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})\prod_{i=1}^{n-\lambda_1}(z_1-x_i)\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\lambda'))-1}(z_2-y_{n-i})\Big). \tag{3.8.9}$$

Proof. We will view each side of the equation as a polynomial in z_1 and analyze its degree to use interpolation. Note that $\mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})$ does not depend on z_1 so exchanging variables z_1 and z_2 is same as plugging in z_1 in the place of z_2. So the right hand side of (3.8.9) becomes

$$RHS = \Big[\mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y}) \prod_{i=1}^{n-\lambda_1}(z_1-x_i) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\lambda'))-1}(z_2-y_{n-i}) - \mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})|_{z_2=z_1} \prod_{i=1}^{n-\lambda_1}(z_2-x_i) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\lambda'))-1}(z_1-y_{n-i})\Big]/(z_1-z_2).$$

By (3.5.1), $\mathfrak{S}^n_{(\lambda_1-1,\lambda)}(\mathbf{z};\mathbf{x};\mathbf{y})$ is of degree at most $(n-(\lambda_1-1)-\mathrm{mul}((\lambda-1,\lambda')))$ in z_1. So the numerator of RHS is of degree at most $(n-\lambda_1)$ in z_1, which implies RHS is of degree at most $(n-\lambda_1-1)$ in z_1. And the left-hand side of (3.8.9) is of degree at most $(n-\lambda_1-\mathrm{mul}(\lambda))=(n-\lambda_1-1)$ in z_1 by (3.5.1). So it is enough to show that they are the same when we plug in $z_1=x_h$ for any $1\le h\le n-\lambda_1$. For the right-hand side we have

$$\begin{aligned} RHS|_{z_1=x_h} &= \frac{-\mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})|_{z_2=x_h} \prod_{i=1}^{n-\lambda_1}(z_2-x_i) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\lambda'))-1}(x_h-y_{n-i})}{x_h-z_2} \\ &= \mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})|_{z_2=x_h} \prod_{\substack{i=1\\ i\neq h}}^{n-\lambda_1}(z_2-x_i) \prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\lambda'))-1}(x_h-y_{n-i}). \end{aligned}$$

And by Proposition 3.8.3 we have

$$\begin{aligned} &\mathfrak{S}^n_{(\lambda_1-1,\lambda')}(\mathbf{z};\mathbf{x};\mathbf{y}))|_{z_1=x_h} \\ &= \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2}(\mathbf{z});\mathbf{x}_{\hat{h}};\mathbf{y})) \prod_{l=1}^{n-\mathrm{mul}((\lambda_1-1,\lambda'))}(x_h-y_l) \prod_{i=2}^{(\lambda_1-\lambda_2)} \prod_{\substack{m=1\\ m\neq h}}^{n-\lambda_1-\mathrm{mul}((\lambda_1-1,\lambda'))+2}(z_i-x_m) \\ &= \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2}(\mathbf{z});\mathbf{x}_{\hat{h}};\mathbf{y})) \prod_{l=1}^{n-\mathrm{mul}((\lambda_1-1,\lambda'))}(x_h-y_l) \prod_{i=2}^{(\lambda_1-\lambda_2)} \prod_{\substack{m=1\\ m\neq h}}^{n-\lambda_1+1}(z_i-x_m), \end{aligned}$$

where the last equality uses the fact that when $\mathrm{mul}((\lambda_1-1,\lambda'))>1$ we have $\lambda_1-\lambda_2=1$ so the product over $i=2$ to $\lambda_1-\lambda_2$ is vacuous. Shifting variables by $\mathbf{z}\to\sigma(\mathbf{z})$ we deduce

$$\begin{aligned} &\mathfrak{S}^n_{(\lambda_1-1,\lambda)}(\sigma(\mathbf{z});\mathbf{x};\mathbf{y})|_{z_2=x_h} \\ &= \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}(\mathbf{z});\mathbf{x}_{\hat{h}};\mathbf{y}) \prod_{l=1}^{n-\mathrm{mul}((\lambda_1-1,\lambda'))}(x_h-y_l) \prod_{i=3}^{(\lambda_1-\lambda_2+1)} \prod_{\substack{m=1\\ m\neq h}}^{n-\lambda_1-1}(z_i-x_m). \end{aligned}$$

Plugging it to $RHS|_{z_1=x_h}$ gives

$$\begin{aligned}
RHS|_{z_1=x_h} &= \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}(\mathbf{z});\mathbf{x}_{\hat{h}};\mathbf{y})\prod_{i=1}^{n-1}(x_h-y_i)\prod_{\substack{i=1\\ i\neq h}}^{n-\lambda_1}(z_2-x_i)\prod_{i=3}^{(\lambda_1-\lambda_2+1)}\prod_{\substack{m=1\\ m\neq h}}^{n-\lambda_1+1}(z_i-x_m)\\
&= \mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1}(\mathbf{z});\mathbf{x}_{\hat{h}};\mathbf{y})\prod_{i=1}^{n-1}(x_h-y_i)\prod_{i=2}^{(\lambda_1-\lambda_2+1)}\prod_{\substack{m=1\\ m\neq h}}^{n-\lambda_1}(z_i-x_m)\prod_{i=3}^{\lambda_1-\lambda_2+1}(z_i-x_{n+1-\lambda_1})\\
&= \mathfrak{S}^{n}_{(\lambda_1,\lambda')}(\mathbf{z};\mathbf{x};\mathbf{y})|_{z_1=x_h}\prod_{i=3}^{\lambda_1-\lambda_2+1}(z_i-x_{n+1-\lambda_1}) = LHS|_{z_1=x_h}.
\end{aligned}$$

□

Proposition 3.8.5. *For* $\lambda\in\mathrm{Val}(n)$ *with* $\mathrm{mul}(\lambda)=b>1$. *Write* $\lambda=((\lambda_1)^b,\tilde{\lambda})$ *for some* $\tilde{\lambda}$. *We have*

$$\begin{aligned}
&\mathfrak{S}^{n}_{((\lambda_1)^b,\tilde{\lambda})}(\mathbf{z};\mathbf{x};\mathbf{y})\prod_{i=1}^{b-1}(z_i-y_{n+1-b})\prod_{i=b+2}^{b+\lambda_1-\tilde{\lambda}_1}(z_i-x_{n+1-\lambda_1}) \qquad (3.8.10)\\
&= \partial_b\left(\mathfrak{S}^{n}_{((\lambda_1)^{b-1},\lambda_1-1,\tilde{\lambda})}(\mathbf{z};\mathbf{x};\mathbf{y})\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1}(z_{b+1}-y_{n+1-b-i})\right)
\end{aligned}$$

Proof. We view both sides as polynomials in z_1 and analyze their degrees to use interpolation. By (3.5.1), $\mathfrak{S}^{n}_{((\lambda_1)^b,\tilde{\lambda})}(\mathbf{z};\mathbf{x};\mathbf{y})$ is of degree at most $(n-\lambda_1-b)$ in z_1 so the right hand side is of degree at most $(n-\lambda_1-b+1)$ in z_1. Likewise, $\mathfrak{S}^{n}_{((\lambda_1)^{b-1},\lambda_1-1,\tilde{\lambda})}(\mathbf{z};\mathbf{x};\mathbf{y})$ is of degree at most $(n-\lambda_1-b+1)$ in z_1 so the right hand side is of degree at most $(n-\lambda_1-b+1)$ in z_1. Since $b>1$, it is enough to show that they coincide when we plug in $z_1=x_a$ for $1\le a\le n-\lambda_1$.

We proceed with the induction on b. First assume $b=2$. Plugging in $z_1=x_a$ to both sides, Proposition 3.8.3 gives

$$\begin{aligned}
LHS|_{z_1=x_a} &= \mathfrak{S}^{n-1}_{(\lambda_1,\tilde{\lambda})}(\mathbf{z};\mathbf{x}_{\hat{a}};\mathbf{y})\prod_{l=1}^{n-1}(x_a-y_l)\prod_{i=4}^{2+\lambda_1-\tilde{\lambda}_1}(z_i-x_{n+1-\lambda_1})\\
RHS|_{z_1=x_a} &= \partial_2\left(\mathfrak{S}^{n-1}_{(\lambda_1-1,\tilde{\lambda})}(\sigma(\mathbf{z});\mathbf{x}_{\hat{a}};\mathbf{y})\prod_{l=1}^{n-1}(x_a-y_l)\prod_{\substack{m=1\\ m\neq a}}^{n-\lambda_1}(z_2-x_m)\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1}(z_3-y_{n-1-i})\right).
\end{aligned}$$

So the equality $LHS|_{z_1=x_a}=RHS|_{z_1=x_a}$ comes from Proposition 3.8.4 with variables shifted by $\mathbf{x}\to\mathbf{x}_{\hat{a}}$ and $\mathbf{z}\to\sigma(\mathbf{z})$.

For $b > 2$, assume we have the equation for $(b-1)$. Plugging in $z_1 = x_a$ to both sides, Proposition 3.8.3 gives

$$LHS|_{z_1=x_a} = \mathfrak{S}^{n-1}_{((\lambda_1)^{b-1},\tilde{\lambda})}(\mathbf{z};\mathbf{x}_{\hat{a}};\mathbf{y}) \prod_{l=1}^{n-b+1}(x_a - y_l)\prod_{i=2}^{b-1}(z_i - y_{n+1-b})\prod_{i=b+2}^{b+\lambda_1-\tilde{\lambda}_1}(z_i - x_{n+1-\lambda_1})$$

$$RHS|_{z_1=x_a} = \partial_b\left(\mathfrak{S}^{n-1}_{((\lambda_1)^{b-2},\lambda_1-1,\tilde{\lambda})}(\mathbf{z};\mathbf{x}_{\hat{a}};\mathbf{y}) \prod_{l=1}^{n-b+1}(x_a - y_l)\prod_{i=1}^{\mathrm{mul}((\lambda_1-1,\tilde{\lambda}))-1}(z_{b+1} - y_{n+1-b-i})\right).$$

The equality $LHS|_{z_1=x_a} = RHS|_{z_1=x_a}$ comes from induction hypothesis with variables shifted by $\mathbf{x} \to \mathbf{x}_{\hat{a}}$ and $\mathbf{z} \to \sigma(\mathbf{z})$. □

Chapter 4

A combinatorial formula for the Ehrhart h^*-vector of the hypersimplex

The results of this chapter are based on [23].

4.1 Introduction

For two integers $0 < k < n$, the (k,n)*-th hypersimplex* is defined to be

$$\Delta_{k,n} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1, x_1 + \cdots + x_n = k\}.$$

It is an $(n-1)$-dimensional polytope inside $\mathbb{R}^n$ whose vertices are (0,1)-vectors with exactly k 1's. In particular it is an integral polytope. The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of GL_n.

For an n-dimensional integral polytope $\mathcal{P} \subset \mathbb{R}^N$, it is well known from Ehrhart theory that the map $r \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n, which we call *Ehrhart polynomial*, and corresponding *Ehrhart series* $\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r$ is a rational function of the form

$$\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r = \frac{h^*(t)}{(1-t)^{n+1}},$$

such that $h^*(t)$ is a polynomial of degree $\le n$ (see [34]). Define h^*_d to be the coefficient of t^d in $h^*(t)$. The vector $(h^*_0, \cdots, h^*_n)$ is called the Ehrhart h^*-vector of $\mathcal{P}$ and $h^*(t)$ is called the h^*-polynomial of $\mathcal{P}$. A standard result from Ehrhart theory is that $\sum_{i=0}^{n} h^*_i$ equals the normalized volume of $\mathcal{P}$.

For a permutation $w \in S_n$, we say $i \in [n-1]$ is a *descent* of w if $w(i) > w(i+1)$ and define $des(w)$ to be the number of descents of w. The *Eulerian* number $A_{k,n-1}$ is the number of

$w \in S_{n-1}$ with $des(w) = k-1$. A well-known fact about the hypersimplex $\Delta_{k,n}$ is that its normalized volume is $A_{k,n-1}$ (see [35]). So we have

$$\sum_{d=0}^{n-1} h_d^*(\Delta_{k,n}) = A_{k,n-1}.$$

In general, the entries of the h^*-vector of an integral polyotope are nonnegative integers (see [33]). It has been an open problem for some time to give a combinatorial interpretation of $h_d^*(\Delta_{k,n})$. In [29], N. Li gave a combinatorial interpretation of $h_d^*(\Delta'_{k,n})$, where $\Delta'_{k,n}$ is the hypersimplex with the lowest facet removed, using permutations $w \in S_{n-1}$ and their descents, excedances, and covers. In [17], N. Early conjectured a combinatorial interpretation for $h_d^*(\Delta_{k,n})$ using hypersimplicial decorated ordered set partitions of type (k,n).

In [22], Katzman computed the Hilbert series of algebras of Veronese type, which gives a formula for the Ehrhart series of the hypersimplex $\Delta_{k,n}$ as a special case. The formula is

$$\frac{\sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\left(\sum_{j\geq 0}\binom{i}{j}(t-1)^j\left(\sum_{l\geq 0}\binom{n-j}{l(k-i)}_{k-i}t^l\right)\right)\right)}{(1-t)^n} \tag{4.1.1}$$

where the notation $\binom{n}{b}_a$ means the coefficient of t^b in $(1+t+\cdots+t^{a-1})^n$. For example, when $a=2$, it becomes an ordinary binomial coefficient. The numerator of (1.1.1) is the h^*-polynomial of the hypersimplex $\Delta_{k,n}$, thus giving an explicit formula for its h^*-vector. However, it doesn't give a combinatorial or manifestly positive formula for the h^*-vector.

In this paper, we prove N. Early's conjecture by relating it to ((4.1.1)). We now explain the conjecture. A *decorated ordered set partition* $((L_1)_{l_1},\cdots,(L_m)_{l_m})$ of type (k,n) consists of an ordered partition $(L_1,\cdots,L_m)$ of $\{1,2,...,n\}$ and an m-tuple $(l_1,\cdots,l_m)\in\mathbb{Z}^m$ such that $l_1+\cdots+l_m = k$ and $l_i \geq 1$. We call each L_i a *block* and we place them on a circle in a clockwise fashion then think of l_i as the clockwise distance between adjacent blocks L_i and L_{i+1} (indices are considered modulo m). So the circumference of the circle is $l_1+\cdots+l_m = k$. We regard decorated ordered set partitions up to cyclic rotation of blocks (together with corresponding l_i). For example, decorated ordered set partition $(\{1,2,7\}_2,\{3,5\}_3,\{4,6\}_1)$ is same as $(\{3,5\}_3,\{4,6\}_1,\{1,2,7\}_2)$. A decorated ordered set partition is called *hypersimplicial* if it satisfies $1\leq l_i\leq |L_i|-1$ for all i. For the motivation and more background on decorated ordered set partitions, see [16].

Example 4.1.1. Consider a decorated ordered set partition $(\{1,2,7\}_2,\{3,5\}_3,\{4,6\}_1)$ of type (6,7) (see Figure 4.1). This is not hypersimplicial as $3>|\{3,5\}|-1$.

By inserting empty spots, we can encode the distance information. For example, the (clockwise) distance between $\{1,2,7\}$ and $\{3,5\}$ is 2 so we insert one empty spot on the circle between those blocks. The distance between $\{3,5\}$ and $\{4,6\}$ is 3 so we insert two empty spots. We obtain the figure on the right as a result. Including empty spots, there will be $k=6$ spots total.

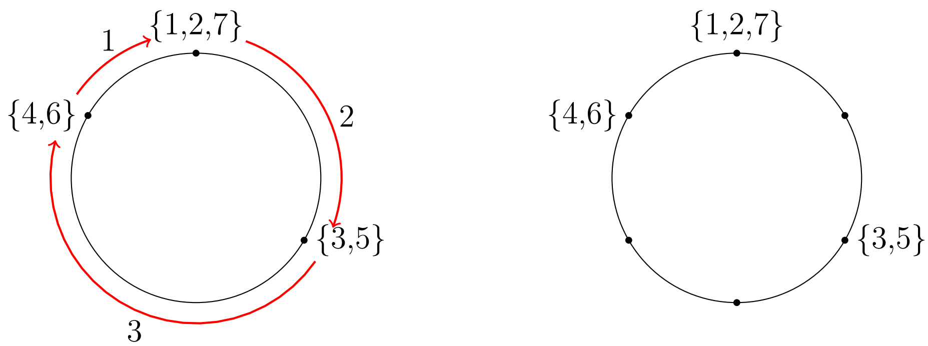

Figure 4.1: The figure on the left is the picture associated to the decorated ordered set partition $(\{1,2,7\}_2,\{3,5\}_3,\{4,6\}_1)$. The figure on the right is the picture obtained after inserting empty spots.

Given a decorated ordered set partition, we define the *winding vector* and the *winding number*. To define the winding vector, let w_i be the distance of the path starting from the block containing i to the block containing $(i+1)$ moving clockwise (where i and $(i+1)$ are considered modulo n). If i and $(i+1)$ are in the same block then $w_i = 0$. In Figure 4.1, the winding vector is $w = (0,2,3,3,3,1,0)$.

The total length of the path is $(w_1 + \cdots + w_n)$, which should be a multiple of k as we started from 1 and came back to 1 moving clockwise. If $(w_1 + \cdots + w_n) = kd$, then we define the winding number to be d. In Figure 4.1, the winding number is 2.

Remark 4.1.2. It is known that hypersimplicial decorated ordered set partitions of type (k,n) are in bijection with $w \in S_{n-1}$ such that $des(w) = k-1$ (see [31]).

Now we will state the conjectures of N. Early.

Conjecture 4.1.3 ([17], Conjecture 1)**.** The number of hypersimplicial decorated ordered set partitions of type (k,n) with winding number d is $h^*_d(\Delta_{k,n})$.

Next we will state a more general version of Conjecture 4.1.3 for a generic cross section of a hypercube.

Definition 4.1.4. For positive integers r,k, and n, the *generic cross section of a hypercube* is

$$I^n_{r,k} = \{(x_1,\cdots,x_n) \in [0,r]^n \mid \sum_{i=1}^n x_i = k\}.$$

When $r=1$, it is the hypersimplex $\Delta_{k,n}$.

Definition 4.1.5. A decorated ordered set partition $P = ((L_1)_{l_1},\cdots,(L_m)_{l_m})$ is r-*hypersimplicial* if $1 \le l_i \le r|L_i| - 1$ for all i.

Note that the notions of hypersimplicial and 1-hypersimplicial are equivalent. The decorated ordered set partition $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$ in Example 4.1.1 is not hypersimplicial, but it is r-hypersimplicial for $r \geq 2$.

Conjecture 4.1.6 ([17], Conjecture 6)**.** The number of r-hypersimplicial decorated ordered set partitions of type (k,n) with winding number d is $h^*_d(I^n_{r,k})$.

Our goal is to prove Conjecture 4.1.6 and derive Conjecture 4.1.3 as specializing to $r = 1$.

4.2 Proof of Conjecture 4.1.6

A simplification of Katzman's formula

Again using the formula for Hilbert series of algebras of Veronese type (see [22]), the Ehrhart series of $I^n_{r,k}$ is

$$\frac{\sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\sum_{j\geq 0}\binom{i}{j}(t-1)^j\left(\sum_{l\geq 0}\binom{n-j}{l(k-ri)}_{k-ri}t^l\right)\right)}{(1-t)^n}. \tag{4.2.1}$$

Now we simplify ((4.2.1)) to get a simple description for the h^*-vector of $I^n_{r,k}$.

Lemma 4.2.1. *For positive integers $n, m,$and a, we have*

$$\binom{n}{m}_a - \binom{n}{m-1}_a = \binom{n-1}{m}_a - \binom{n-1}{m-a}_a.$$

Proof. By a combinatorial argument, we have $\binom{n}{m}_a = \sum_{k=0}^{a-1}\binom{n-1}{m-k}_a$ and $\binom{n}{m-1}_a = \sum_{k=0}^{a-1}\binom{n-1}{m-1-k}_a$. Subtracting these two gives the lemma. □

Proposition 4.2.2. *For positive integers s and a, we have*

$$\sum_{j\geq 0}\binom{s}{j}(t-1)^j\left(\sum_{l\geq 0}\binom{n-j}{la}_a t^l\right) = \sum_{l\geq 0}\binom{n}{la-s}_a t^l.$$

Proof. We proceed by induction on s. For $s = 0$, this is a trivial identity. Let's assume that the proposition holds for $s = u-1$ and for all n, which means

$$\sum_{j\geq 0}\binom{u-1}{j}(t-1)^j\left(\sum_{l\geq 0}\binom{n-j}{la}_a t^l\right) = \sum_{l\geq 0}\binom{n}{la-u+1}_a t^l. \tag{4.2.2}$$

Now replacing n with $(n-1)$ and multiplying by $(t-1)$ we have

$$\sum_{j\geq 0}\binom{u-1}{j}(t-1)^{j+1}\left(\sum_{l\geq 0}\binom{n-1-j}{la}_a t^l\right) = \sum_{l\geq 0}\binom{n-1}{la-u+1}_a t^l(t-1).$$

Replacing j with $(j-1)$ and rearranging the right-hand side gives

$$\sum_{j\geq 0}\binom{u-1}{j-1}(t-1)^j(\sum_{l\geq 0}\binom{n-j}{la}_a t^l) = \sum_{l\geq 0}(\binom{n-1}{(l-1)a-u+1}_a - \binom{n-1}{la-u+1}_a)t^l. \tag{4.2.3}$$

Summing ((4.2.2)) and ((4.2.3)), and using Lemma 4.2.1 gives

$$\sum_{j\geq 0}\binom{u}{j}(t-1)^j(\sum_{l\geq 0}\binom{n-j}{la}_a t^l) = \sum_{l\geq 0}\binom{n}{la-u}_a t^l.$$

□

Using Proposition 2.3.6, the Ehrhart series of $I^n_{r,k}$ ((4.2.1)) becomes

$$\frac{\sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\sum_{l\geq 0}\binom{n}{l(k-ri)-i}_{k-ri}t^l\right)}{(1-t)^n}.$$

Thus we have

$$h^*_d(I^n_{r,k}) = \sum_{i\geq 0}(-1)^i\binom{n}{i}\binom{n}{(k-ri)d-i}_{k-ri}. \tag{4.2.4}$$

In Section 2.2, we will prove Conjecture 4.1.6 which contains Conjecture 4.1.3 as a special case when $r=1$. Since we have an explicit formula for $h^*_d(I^n_{r,k})$, our strategy is to count the number of r-hypersimplicial decorated ordered set partitions of type (k,n) with winding number d and compare the formulas.

Enumeration of r-hypersimplicial decorated ordered set partitions with a fixed winding number

We start with an elementary lemma, skipping the proof.

Lemma 4.2.3. *The $\mathbb{Z}/n\mathbb{Z}$ action on $\{1,2,\cdots,n\}$ by cyclic shift does not change the winding number of decorated ordered set partitions.*

For example, decorated ordered set partitions $(\{1,2,7\}_2,\{3,5\}_3,\{4,6\}_1)$ and $(\{2,3,1\}_2,\{4,6\}_3,\{5,7\}_1)$ have the same winding number.

Next we will show that a winding vector determines a decorated ordered set partition. We observed that when the winding number is d, then $w_1+\cdots+w_n = kd$. And $0\leq w_i\leq k-1$ since the circumference of the circle is k (if $w_i = k$, then i and $(i+1)$ are in a same block which means $w_i = 0$). It turns out that these are the only restrictions for winding vectors.

Proposition 4.2.4. *Decorated ordered set partitions of type (k,n) with winding number d are in bijection with elements of $\{(w_1,\cdots,w_n)\in\mathbb{Z}^n \mid 0\leq w_i\leq k-1,\ w_1+\cdots+w_n = kd\}$.*

Proof. It is enough to construct a decorated ordered set partition of type (k,n) with winding number d from a winding vector satisfying the above conditions. First, draw k spots on the circle in clockwise order and put 1 in one spot. Having put i in some spot, move clockwise w_i spots and put $i+1$ in that spot. After placing all elements, nonempty spots become blocks and the clockwise distance from L_i and L_{i+1} is l_i. □

Example 4.2.5. For type $(k,n) = (6,7)$, we will construct a decorated ordered set partition from the vector $(0,2,3,3,3,1,0)$. See Figure 4.2. First, draw $k = 6$ spots and put 1 in one spot (upper-left figure). Then put elements according to the given vector (upper-right figure). $\{1,2,7\}$, $\{3,5\}$, and $\{4,6\}$ will be blocks. There is one empty spot between $\{1,2,7\}$ and $\{3,5\}$ so the distance is 2. The distance between $\{3,5\}$ and $\{4,6\}$ is 3 as there are two empty spots. Resulting decorated ordered set partition is $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$ (lower figure). We recovered Example 4.1.1.

From Proposition 4.2.4, we know that the number of decorated ordered set partitions of type (k,n) with winding number d is $|\{(w_1,\cdots,w_n) \in \mathbb{Z}^n \mid 0 \le w_i \le k-1,\ w_1+\cdots+w_n = kd\}|$. A simple combinatorial argument shows this number is the same as the coefficient of t^{kd} in $(1+\cdots+t^{k-1})^n$, which is $\binom{n}{kd}_k$. So the number of decorated ordered set partitions of type (k,n) with winding number d is $\binom{n}{kd}_k$.

Recall that we are interested in the number of r-hypersimplicial decorated ordered set partitions of type (k,n) with winding number d. Throughout the rest of this section, when we say decorated ordered set partition, **we always assume it is of type (k,n) with winding number d.**

Definition 4.2.6. For a decorated ordered set partition $P = ((L_1)_{l_1}, (L_2)_{l_2}, \ldots, (L_m)_{l_m})$, a block L_i is *r-bad* if $l_i \ge r|L_i|$. Let $I_r(P) = \{L_i \mid L_i \text{ is } r\text{-bad}\}$.

For example, the set $I_1((\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1))$ is $\{\{3,5\}\}$. Recall that r-hypersimpilcial decorated ordered set partitions satisfy $1 \le l_i \le r|L_i| - 1$ for all blocks. So a decorated ordered set partition is r-hypersimplicial if and only if $I_r(P)$ is empty.

Definition 4.2.7. For a set T, define $UP(T)$ to be a set of all (unordered) partitions of T. For example, the partition $\{\{1,2,4\},\{3\},\{5\}\}$ is in $UP(\{1,2,3,4,5\})$.

Definition 4.2.8. For $T \subseteq \{1,2,\cdots,n\}$ and $S \in UP(T)$, define
$K_r(S) = \{P\text{: decorated ordered set partition such that } S \subseteq I_r(P)\}$.

In other words, the set $K_r(S)$ consists of all decorated ordered set partitions (of type (k,n) with winding number d) having elements of S as r-bad blocks. For example, when $S = \phi$, the set $K_r(\phi)$ consists of all decorated ordered set partitions (of type (k,n) with winding number d).

Definition 4.2.9. For $T \subseteq \{1,2,\cdots,n\}$, let $H_r(T) = \sum_{S \in UP(T)} (-1)^{|S|} |K_r(S)|$.

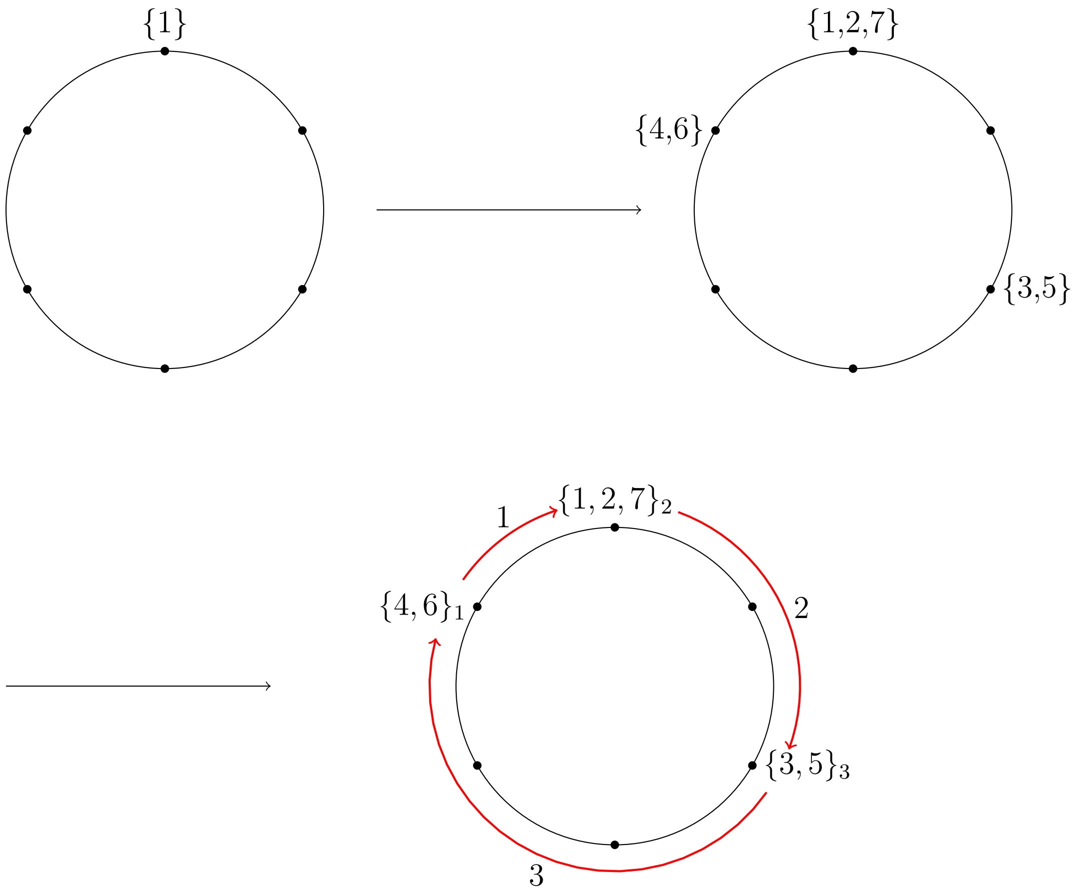

Figure 4.2: Constructing the decorated ordered set partition associated to the winding vector $(0,2,3,3,3,1,0)$.

Set	Elements
$K_1(\{\{1,2,3\}\})$	$(\{1,2,3\}_3, \{4,5\}_1)$
$K_1(\{\{1,2\},\{3\}\})$	$(\{1,2\}_2, \{3\}_1, \{4,5\}_1)$
$K_1(\{\{2,3\},\{1\}\})$	$(\{1\}_1, \{2,3\}_2, \{4,5\}_1)$
$K_1(\{\{1,3\},\{2\}\})$	
$K_1(\{\{1\},\{2\},\{3\}\})$	$(\{1\}_1, \{2\}_1, \{3\}_1, \{4,5\}_1)$

Table 4.1: Listing $K_1(S)$ for $S \in UP(\{1,2,3\})$, among decorated ordered set partitions of type $(4,5)$ with winding number 1.

Example 4.2.10. Table 4.1 shows the lists of $K_1(S)$ for $S \in UP(\{1,2,3\})$, among decorated ordered set partitions of type $(4,5)$ with winding number 1.

Note that $K_1(\{\{1,3\},\{2\}\})$ is an empty set, as it is impossible to have a winding number 1 with 1-bad blocks $\{1,3\}$ and $\{2\}$. In this case we have,

$$\begin{aligned} H_1(\{1,2,3\}) &= -|K_1(\{\{1,2,3\}\})| + |K_1(\{\{1,2\},\{3\}\})| + |K_1(\{\{2,3\},\{1\}\})| \\ &\quad + |K_1(\{\{1,3\},\{2\}\})| - |K_1(\{\{1\},\{2\},\{3\}\})| \\ &= -1 + 1 + 1 - 1 = 0. \end{aligned}$$

Now we relate $H_r(T)$ with the number of r-hypersimplicial decorated ordered set partitions (of type (k,n) with winding number d).

Proposition 4.2.11. *The number of r-hypersimplicial decorated ordered set partitions (of type (k,n) with winding number d) is*

$$\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T).$$

Proof. It is enough to compute $\sum_{T \subseteq \{1,2,\cdots,n\}} \left(\sum_{S \in UP(T)} (-1)^{|S|} |K_r(S)| \right)$, by the definition of $H_r(T)$. A decorated ordered set partition P belongs to $K_r(S)$ if and only if S is a subset of $I_r(P)$. So if $I_r(P)$ is empty then P will be counted once when $S = \phi$. If $I_r(P)$ is non empty, say $|I_r(P)| = m$, then P will be counted $\binom{m}{i}$ times with the sign $(-1)^i$ as S ranges over all i-element subsets of $I_r(P)$. Thus the contribution of P to $\left(\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T) \right)$ is $\sum_{i=0}^{m} (-1)^i \binom{m}{i} = 0$. So the above sum counts P such that $I_r(P)$ is empty, which means r-hypersimplicial. □

Now it remains to give a formula for $H_r(T)$. When $S \in UP(\{1,2,\cdots,n\})$, elements of $K_r(S)$ are decorated ordered set partitions $P = ((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ whose blocks are all r-bad, which means $l_i \geq r|L_i|$ for all i. Summing inequalities for all i gives $\sum l_i \geq r \sum |L_i|$ that implies $k \geq rn$ which is impossible as $k < n$. Thus $K_r(S)$ is an empty set, so $H_r(\{1,2,\cdots,n\}) = 0$. So we will only consider when T is a proper subset of $\{1,2,\cdots,n\}$. By Lemma 4.2.3, we may assume that $n \notin T$ since $H_r(T)$ is invariant under cyclic shifts of $\{1,2,\cdots,n\}$.

Definition 4.2.12. For a fixed $T \subsetneq \{1,2,\cdots,n\}$ such that $n \notin T$, a *T-singlet block* is a block with only one element t and $t \in T$. A sequence of consecutive blocks $(L_i, \cdots, L_{i+j})$ consisiting of T-singlet blocks in a decorated ordered set partition P (indices are considered modulo number of blocks in P) is *r-packed* if $l_i = \cdots = l_{i+j-1} = r$ and $l_{i+j} \geq r$. An r-packed sequence is *increasing r-packed* if elements in each block $(L_i, \cdots, L_{i+j})$ are in increasing order. Such a sequence is *maximal* if it is not a subsequence of another increasing r-packed sequence.

The increasing r-packed condition highly depends on T since it only applies to consecutive T-singlet blocks. Note that T-singlet blocks in r-packed sequence are all r-bad. It is the most concentrated arrangement that makes these blocks all r-bad. We allow an increasing r-packed sequence of length 1 by convention.

Example 4.2.13. For $T = \{1, 2, 4, 6\}$ and $r = 2$, Figure 4.3 is the picture for the decorated ordered set partition $(\{1\}_2, \{2\}_2, \{4\}_2, \{5, 8, 9, 10\}_1, \{6\}_2, \{7\}_2, \{11, 12, 13\}_1)$. The maximal increasing r-packed sequences here are $(\{1\}, \{2\}, \{4\})$ and $(\{6\})$. Note that the sequence $(\{6\}, \{7\})$ is not r-packed since $\{7\}$ is not a T-singlet block.

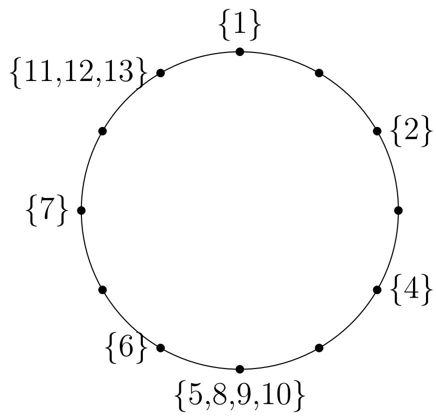

Figure 4.3: Reading off r-packed sequences for $r = 2$.

Lemma 4.2.14. *Let $S = \{M_1, M_2, \cdots, M_j\} \in UP(T)$, where $T = \{t_1 < t_2 < \cdots < t_m\}$ and $n \notin T$. Enumerate the elements of M_i in increasing order, so $M_i = \{t_{i_1} < t_{i_2} < \cdots < t_{i_w}\}$. Then elements of $K_r(S)$ are in bijection with elements of $K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})$ having an increasing r-packed sequence $(\{t_{i_1}\}, \{t_2\}, \cdots, \{t_{i_w}\})$ for all i.*

Proof. Given a decorated ordered set partition $P \in K_r(S)$, we pick a block $(M_i)_l$ which is r-bad. So $l \geq r|M_i| = rw$. Change $(M_i)_l$ to $\{t_{i_1}\}_r, \{t_{i_2}\}_r, \ldots, \{t_{i_w}\}_{l-r(w-1)}$. Since $l - r(w-1) \geq r$, the sequence $(\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\})$ will be increasing r-packed. This process does not change the winding number and new T-singlet blocks are all r-bad (see Example 4.2.15). Repeating this process for all i we get the desired correspondence. □

Example 4.2.15. See Figure 4.4. The figure on the left is a decorated ordered partition $(\{1, 2, 4\}_6, \{5, 8, 9, 10, 13\}_1, \{6, 7\}_4, \{11, 12\}_1)$. When $T = \{1, 2, 4, 6, 7\}$ and $r = 2$, the figure on the left has r-bad blocks $\{1, 2, 4\}$ and $\{6, 7\}$, so belongs to $K_r(\{\{1, 2, 4\}, \{6, 7, \}\})$. Under the correspondence stated in Lemma 4.2.14, this goes to

$(\{1\}_2, \{2\}_2, \{4\}_2, \{5, 8, 9, 10, 13\}_1, \{6, \}_2, \{7\}_2\{11, 12\}_1)$, a decorated ordered set partition for the figure on the right. The winding number does not change.

Remark 4.2.16. The condition $n \notin T$ is essential for Lemma 4.2.14. Without this condition, the correspondence might change the winding number as shown in Figure 4.5. The winding number on the left figure is 1 but the winding number on the right is 2. We spread elements

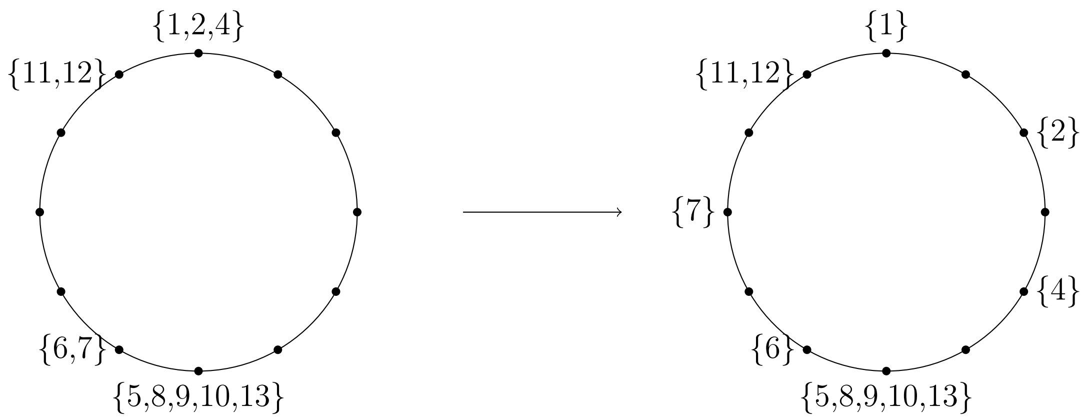

Figure 4.4: Correspondence in Lemma 4.2.14 for $T = \{1,2,4,6,7\}$ and $r = 2$.

in blocks in increasing order but since there is a cyclic symmetry, "increasing" might not be meaningful if $n \in T$.

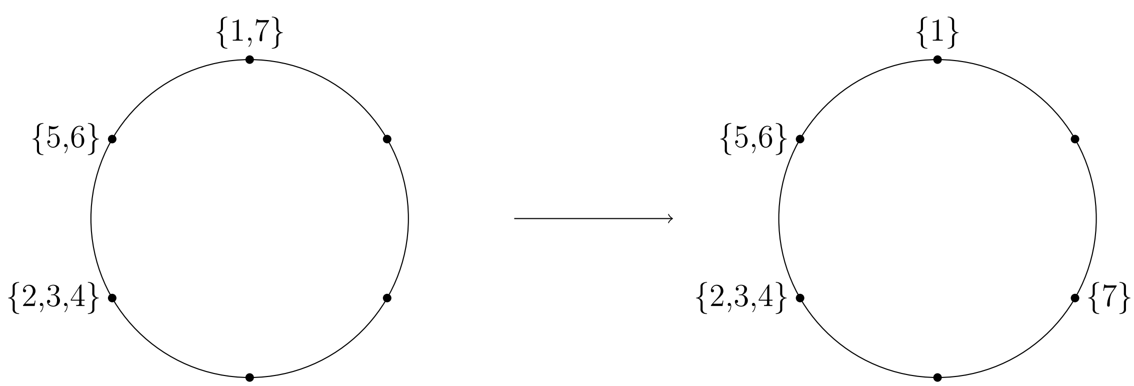

Figure 4.5: Correspondence in Lemma 4.2.14 for $T = \{1,7\}$ and $r = 2$.

Now fix $T = \{t_1 < t_2 < \cdots < t_m\} \subsetneq \{1,2,\cdots,n\}$ such that $n \notin T$. For $S \in UP(T)$, the correspondence in Lemma 4.2.14 gives an embedding

$$i_S : K_r(S) \to K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\}).$$

Let $\chi_S : K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\}) \to \{0,1\}$ to be the characteristic function of $i_S(K_r(S))$

which means $\chi_S(P) = 0$ if $P \notin i_S(K_r(S))$ and $\chi_S(P) = 1$ if $P \in i_S(K_r(S))$. Then we have

$$\begin{aligned} H_r(T) &= \sum_{S \in UP(T)} (-1)^{|S|} |K_r(S)| = \sum_{S \in UP(T)} (-1)^{|S|} |i_S(K_r(S))| \\ &= \sum_{S \in UP(T)} (-1)^{|S|} \Big(\sum_{P \in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})} \chi_S(P) \Big) \\ &= \sum_{P \in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})} \Big(\sum_{S \in UP(T)} (-1)^{|S|} \chi_S(P) \Big). \end{aligned} \tag{4.2.5}$$

Proposition 4.2.17. *For a fixed $T = \{t_1 < t_2 < \cdots < t_m\} \subsetneq \{1, 2, \cdots, n\}$ such that $n \notin T$, if a decorated ordered set partition $P \in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})$ does not have an increasing r-packed sequence of length greater than 1, then $\sum_{S \in UP(T)} (-1)^{|S|} \chi_S(P)$ equals $(-1)^{|T|}$. Otherwise it is zero.*

Proof. For $P \in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})$, define $\hat{S}(P)$ to be an unordered partition of T by putting t_i and t_j in same part if they belong to same increasing r-packed sequence (this will partition T by maximal increasing r-packed sequences of P). An unordered partition S is a finer partition than $\hat{S}(P)$ if and only if $\chi_S(P) = 1$. When P has no increasing r-packed sequence of length greater than 1, we have $\hat{S}(P) = \{\{t_1\},\{t_2\},\cdots,\{t_m\}\}$, the finest unordered partition of T. So $\chi_S(P) = 1$ only when $S = \hat{S}(P)$ thus $\sum_{S \in UP(T)} (-1)^{|S|} \chi_S(P) = (-1)^{|T|}$. Now assume there is $M = \{m_1 < \cdots < m_a\} \in \hat{S}(P)$ such that $|M| = a \geq 2$. To split M into b parts so that resulting finer partition S still satisfies $\chi_S(P) = 1$, we choose $(b-1)$ elements $i_1 < \cdots < i_{b-1}$ in a set $\{1, \cdots, a-1\}$ and split M into $\{m_1, \cdots, m_{i_1}\}, \{m_{i_1}, \cdots, m_{i_2}\}, \cdots, \{m_{i_{b-1}}, \cdots, m_a\}$. There are $\binom{a-1}{b-1}$ ways to do that and this process can be done independently on each $M \in \hat{S}(P)$ such that $|M| \geq 2$. So we have

$$\sum_{S \in UP(T)} (-1)^{|S|} \chi_S(P) = \prod_{M \in \hat{S}(P), |M| \geq 2} \Big(\sum_{b=1}^{|M|} (-1)^b \binom{|M|-1}{b-1} \Big) \prod_{M \in \hat{S}(P), |M|=1} (-1).$$

Since $\sum_{b=1}^{|M|} (-1)^b \binom{|M|-1}{b-1} = 0$, we have $\sum_{S \in UP(T)} (-1)^{|S|} \chi_S(P) = 0$ whenever P has an increasing r-packed sequence of length greater than 1, that is, the set $\hat{S}(P)$ has a part with more than one element.

□

Example 4.2.18. For $T = \{1, 2, 3, 4\}$, assume $P \in K_r(\{\{1\},\{2\},\{3\},\{4\}\})$ has (maximal) increasing r-packed sequence $(\{1\},\{2\},\{3\},\{4\})$. We will list $S \in UP(T)$ such that $\chi_S(P) = 1$ by number of elements.

$|S| = 1 \to \{\{1, 2, 3, 4\}\}$
$|S| = 2 \to \{\{1\},\{2, 3, 4\}\}, \{\{1, 2\},\{3, 4\}\}, \{\{1, 2, 3\},\{4\}\}$
$|S| = 3 \to \{\{1\},\{2\},\{3, 4\}\}, \{\{1\},\{2, 3\},\{4\}\}, \{\{1, 2\},\{3\},\{4\}\}$

$|S| = 4 \rightarrow \{\{1\},\{2\},\{3\},\{4\}\}$
So we have $\sum_{S \in UP(T)} -(1)^{|S|}\chi_S(P) = -1+3-3+1 = -\binom{3}{0}+\binom{3}{1}-\binom{3}{2}+\binom{3}{3} = 0.$

Definition 4.2.19. For a fixed $T = \{t_1 < t_2 < \cdots < t_m\} \subsetneq \{1,2,\cdots,n\}$ such that $n \notin T$, define $\hat{K}_r(T)$ to be the subset of $K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})$ consisting of decorated ordered set partitions without increasing r-packed sequence of length greater than 1.

By Proposition 4.2.17 and (4.2.5), we have

$$H_r(T) = (-1)^{|T|}|\hat{K}_r(T)|. \tag{4.2.6}$$

We will count the number of elements in $\hat{K}_r(T)$ by defining the second winding vector for each element. The second winding vector is a modified version of the winding vector that we previously defined.

Assume we are given $P \in \hat{K}_r(T)$. There are k spots total on the circle including empty spots that are recording distances and T-singlet blocks $\{t_1\},\{t_2\},\cdots,\{t_m\}$ are r-bad blocks so for each $\{t_i\}$, there will be at least $(r-1)$ empty spots after $\{t_i\}$ as the distance to the next block is at least r. Color these r spots, that is, the spot occupied by $\{t_i\}$ with $(r-1)$ empty spots after that **red**. Doing this for all i, total $r|T| = rm$ spots will be colored red. And color the remaining $(k-rm)$ spots **blue**.

Definition 4.2.20. For $P \in \hat{K}_r(T)$, *second winding vector* $v = (v_1, v_2, \cdots, v_n)$ is defined by setting v_i to be the number of **blue spots** passed while moving from i to $(i+1)$ in clockwise fashion. Do not include the starting point but include the arriving point (if it's blue) and when the starting point and the arriving point are in same block (spot), set $v_i = 0$.

Since the winding number is d, the whole path winds around the circle d times. So we have $v_1 + \cdots + v_n = (k-rm)d$.

If $i \notin T$, we are starting from the blue spot so v_i can range from 0 to $(k-rm-1)$. However when $i \in T$, we claim v_i cannot be zero. If $v_i = 0$, then the path from i to $i+1$ should not include any blue spots. So the path will be of the form $\{i\},\phi,\cdots,\phi,\{a_1\},\phi,\cdots,\phi,\cdots,\{a_q\},\phi,\cdots,\phi,\{i+1\}$ where ϕ means an empty spot. Thus the sequence $(\{i\},\{a_1\},\cdots,\{a_q\},\{i+1\})$ is *r-packed.* Since P does not have increasing r-packed sequence of length greater than 1, the sequence $(i, a_1, \cdots, a_2, i+1)$ should be a decreasing sequence which is impossible. It is possible to have $v_i = k - rm$ as the path can encounter every blue spot (see Example 4.2.21). We conclude $1 \le v_i \le k - rm$.

Example 4.2.21. Figure 4.6 explains the way to read off second winding vector. Let $P = (\{2\}_2,\{1\}_2,\{5,6\}_1,\{7,8\}_1,\{9\}_3,\{11,12,13\}_1,\{10,14\}_1,\{3,4\}_1)$, and fix $T = \{1,2,9\}$ and $r = 2$. The upper left figure is a picture for P. Note that the sequence $(\{2\},\{1\})$ is r-packed but not increasing r-packed. So P has no increasing r-packed sequence of length greater than 1. After coloring spots with the rule above we get the upper right figure. There will be $r|T|$ (=6) red spots and $(k-r|T|)$ (=6) blue spots. To get v_1, wind from 1 to 2 clockwise as shown in the lower figure, and count the number of blue spots passed. Here $v_1 = 6$. Continuing this process we have the second winding vector $v = (6,6,0,1,0,1,0,0,3,5,0,0,1,1)$.

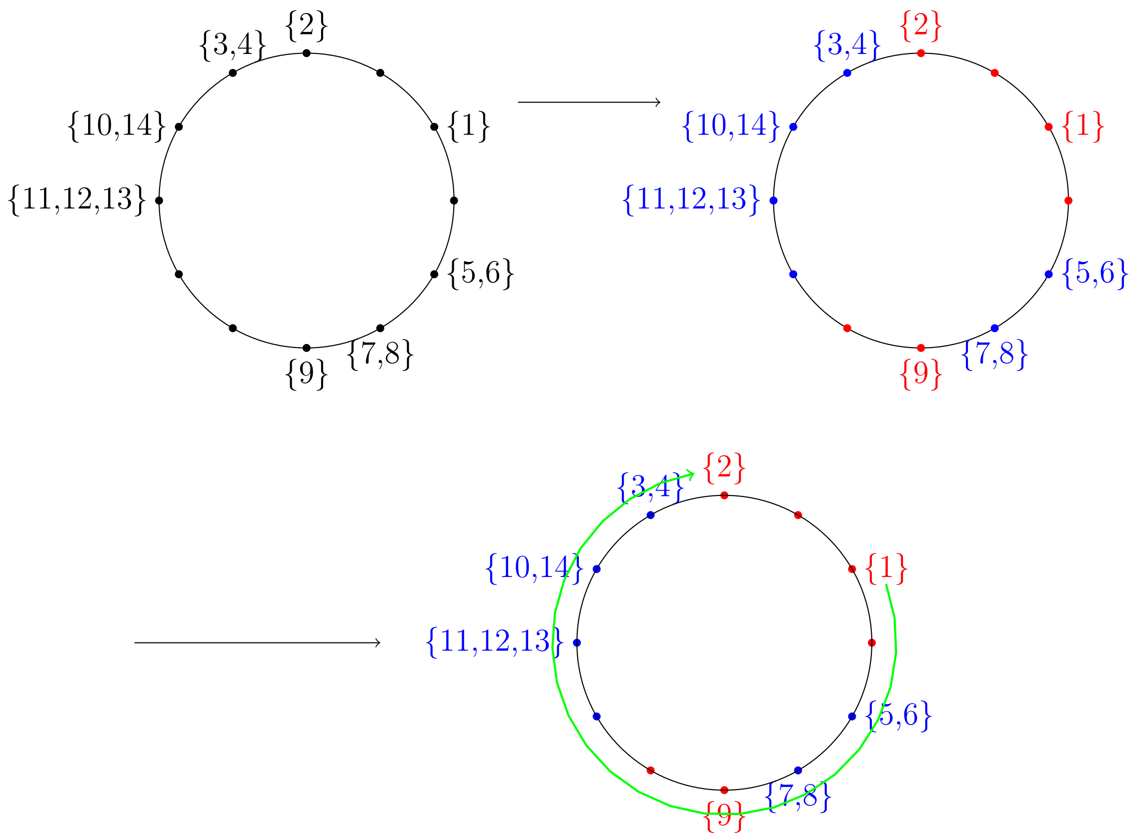

Figure 4.6: Reading off the second winding vector.

We saw that a second winding vector $v = (v_1, v_2, \cdots, v_n)$ satisfies $v_1 + \cdots + v_n = (k - rm)d$. And it also satisfies $0 \le v_i \le k - rm - 1$ if $i \notin T$, and $1 \le v_i \le k - rm$ if $i \in T$.

It turns out these are the only restrictions for the second winding vectors of the elements of $\hat{K}_r(T)$.

Proposition 4.2.22. *Elements of $\hat{K}_r(T)$, where $|T| = m$, are in bijection with elements of* $\{(v_1, v_2, \cdots, v_n) \in \mathbb{Z}^n \mid 0 \le v_i \le k - rm - 1 \text{ if } i \notin T, 1 \le v_i \le k - rm \text{ if } i \in T,\ v_1 + \cdots + v_n = (k - rm)d\}$.

Proof. The forward direction is done by the second winding vector. For the reverse direction, we should recover the decorated ordered set partition (in $\hat{K}_r(T)$) whose second winding vector is the specified vector $(v_1, v_2, \cdots, v_n)$. First draw $(k - rm)$ spots on the circle (recall $|T| = m$) and put 1 in one spot. Having put i in some spot, move clockwise w_i spots and put $i + 1$ in that spot. After placing every element, let's denote the resulting decorated ordered set partition with P. We construct $\tilde{P} \in \hat{K}_r(T)$ as follows. For each block B of P with $B \cap T \neq \phi$, let $B \cap T = \{i_1 < \cdots < i_s\}$. We replace B with $B \backslash T$ and then add (rs) spots immediately after $B \backslash T$ as follows: first a T-singlet block $\{i_s\}$ then $(r - 1)$ empty spots then T-singlet block $\{i_{s-1}\}$ then $(r - 1)$ empty spots $\cdots$ T-singlet block $\{i_1\}$ then $(r - 1)$ empty spots. Resulting decorated ordered set partition $\tilde{P}$ belongs to $\hat{K}_r(T)$ as all element in T are in T-singlet blocks and there is no increasing r-packed sequence of length greater than 1 since we placed $i_1 < \cdots < i_s$ in a decreasing order.

It remains to prove the second winding vector $(\tilde{v_1}, \tilde{v_2}, \cdots, \tilde{v_n})$ of $\tilde{P}$ is the given vector $(v_1, v_2, \cdots, v_n)$. If i and $(i+1)$ were in different blocks in P, then $v_i = \tilde{v_i}$ as we ignore red spots on the way. If i and $(i+1)$ were in a same block in P and $i \notin T$, then $v_i = 0$. From the construction of $\tilde{P}$, there is no blue spot on the way from i to $(i+1)$ except the starting spot so $\tilde{v_i} = 0$. If i and $(i+1)$ were in a same block in P and $i \in T$, then $v_i = k - rm$. Since $(i+1)$ is located behind i, to get from i to $(i+1)$ in $\tilde{P}$ the path winds the circle and encounters every blue spot. Thus $\tilde{v_i} = k - rm$. We conclude that the second winding vector of $\tilde{P}$ is the given vector. □

Example 4.2.23. Figure 4.7 shows how to recover a decorated ordered set partition from a second winding vector as stated in Proposition 4.2.22. We are given $T = \{1, 2, 9\}$, the number $r = 2$, and the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 0, 3, 5, 0, 0, 1, 1)$. In the upper left figure, there are $6 = k - r|T|$ spots ($k = 12$) on the circle and 1 is in one spot. Then put elements according to the second winding vector. The upper right figure shows this. The elements in T are denoted with a tilde. Consider the first block $\{\tilde{1}, \tilde{2}, 3, 4\}$. The numbers 3 and 4 will form a block and 1 and 2 will spread to the right into the space between blocks $\{\tilde{1}, \tilde{2}, 3, 4\}$ and $\{5, 6\}$, making four new red spots. The same thing happens for the block $\{7, 8, \tilde{9}\}$, making two new red spots. The lower figure is the picture for the resulting decorated ordered set partition in $\hat{K}_r(T)$. We recovered Example 4.2.21.

For a second winding vector $v = (v_1, \cdots, v_n)$, let $v' = (v'_1, \cdots, v'_n)$ be a vector such that $v'_i = v_i$ if $i \notin T$, and $v'_i = v_i - 1$ if $i \in T$. By the property of a second winding vector, we have $0 \le v'_i \le (k - rm - 1)$ and $v'_1 + \cdots + v'_n = (k - rm)d - |T| = (k - rm)d - m$. So the number of such v' is $\binom{n}{(k-rm)d-m}_{k-rm}$ which gives

$$H_r(T) = (-1)^{|T|} |\hat{K}_r(T)| = (-1)^m \binom{n}{(k - rm)d - m}_{k-rm}. \tag{4.2.7}$$

Proof of Conjecture 4.1.6) By Proposition 4.2.11, and the equation (4.2.7), the number of r-hypersimplicial decorated ordered set partitions (of type (k, n) with winding number d) is

$$\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T) = \sum_{m \ge 0} \left(\sum_{|T|=m} H_r(T) \right) = \sum_{m \ge 0} (-1)^m \binom{n}{m} \binom{n}{(k - rm)d - m}_{k-rm}.$$

Now comparing with the formula (4.2.4), we obtain *Conjecture* 4.1.6. By specializing to $r = 1$ we obtain *Conjecture* 4.1.3. □

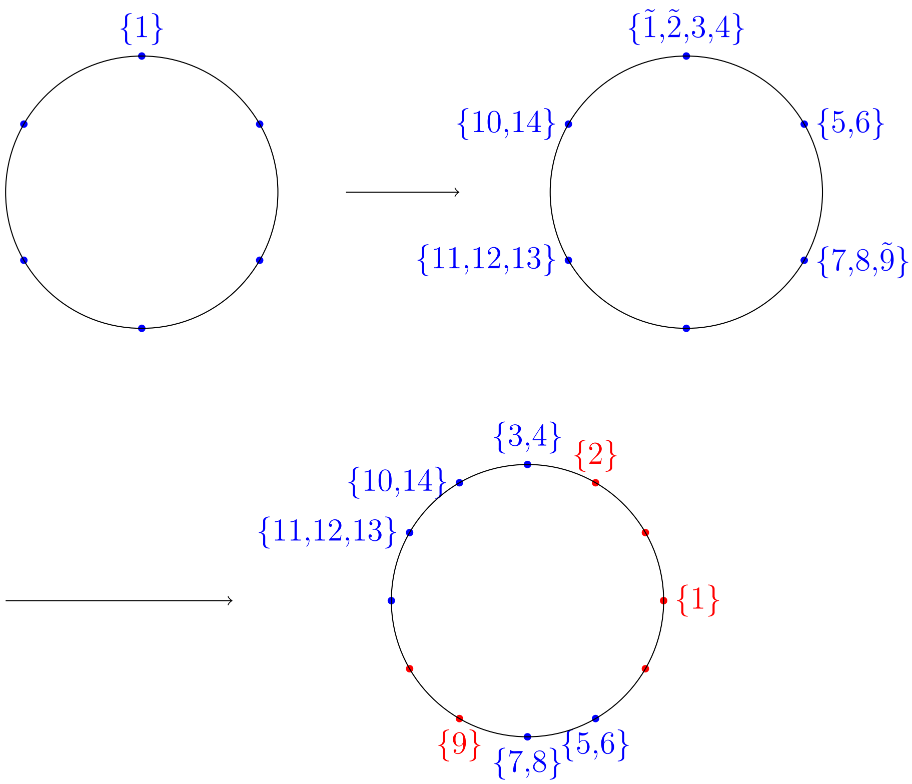

Figure 4.7: Constructing the decorated ordered set partition associated to the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 0, 3, 5, 0, 0, 1, 1)$.

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