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ABSTRACT

In this paper we apply perturbation techniques to the S-matrix dynamical equations specifying the ρ - meson state, thereby calculating the ρ electromagnetic mass splitting. Approximate solutions to the N/D equations (for the $J^{\pi} = 1^{-}$ partial wave) are employed in two models,

(a) $\pi\pi$ inelastic scattering with ρ exchange and photon exchange "driving" force,

(b) SU_3 octet-octet scattering of pseudoscalars.

The numerical results are

$$M_{\rho^o} - M_{\rho^{\pm}} \approx +13 \text{ MeV} \quad .$$

INTRODUCTION

There has recently been great success[1] in calculating various symmetry-breakings of baryon masses and baryon-meson couplings by applying perturbation techniques to assumed existent dynamical equations specifying the baryon states. In this study we employ similar techniques to calculate the electromagnetic mass splitting of the ρ vector meson. Experimentally, this meson has a mass of $\approx$750 MeV, occurs in $+$, 0 , $-$ charge states expected of an isovector state, and has a large width for 2π decay.

We will assume two dynamical "bootstrap" models for the ρ :

(a) We consider the ρ as occurring in a single π-π $I = 1$, $J = 1^-$ channel, with inelasticity present.

(b) We examine the model wherein the relevant channel consists of two SU_3 octets of pseudoscalar mesons (mass-degenerate). The ρ is then assumed to belong to an SU_3 octet of (mass-degenerate) vector mesons. In this model we incidentally obtain a crude estimate for the K^* mass splitting also.

Before proceeding to more detailed discussion, it seems appropriate to briefly and simply describe the general origin of the mass splitting in this approach, and the numerical results of this study. We first note that there are two sources of distinct character which cause electromagnetic mass splittings (henceforth EMS) of an isospin multiplet of strongly interacting particles. In the scattering channels in which the multiplet is dynamically generated, the EMS of participating external and exchanged particles affect kinematics and exchange forces.

(In Reference 1 it is demonstrated that many aspects of these effects are of a group theoretic character.) The second type of effect is due to forces explicitly involving photons. For example, if we consider the ρ simply as $\pi\pi$, then we expect that single photon exchange in the $\rho^0 = \pi^+\pi^-$ channel provides an extra attraction decreasing the ρ^0 mass with respect to the $\rho^\pm$ mass. In fact, however, the effects of photon forces can be substantially modified by the first type of EMS effect mentioned above. It should be emphasized that the present study of the π-π model differs from the simple group theoretic discussion of the ρ in Reference 1 by its detailed consideration of photon driving forces and of coupling shifts.

Finally, we will obtain an estimate in the $\pi\pi$ model of

$$M_{\rho^0} - M_{\rho^+} \approx 13 \text{ MeV} \quad .$$

The degenerate octet model implies a number possibly larger by 2 or 3 MeV.

We wish to draw the reader's attention to a recent calculation of the ρ EMS by S. Bose[1] (using the technique of Dashen and Frautschi, rather than the present approach). The result is numerically in agreement with our result. However, our calculation has the additional feature of considering forces which actually do give rise to a bootstrap; this necessitates including inelastic effects.

The large mass-splitting predicted (on the basis of an admittedly shaky dynamical model) should be measurable in the near future as ρ-production data accumulates. For this purpose one should compare di-pion mass spectra obtained in reactions at fixed momentum transfer. This avoids the complications arising from momentum-transfer dependence of the effective di-pion phase space.

The following comments, we hope, provide a useful guide for the reader.

Section 1 contains a discussion of the application of perturbations to S-matrix dynamical equations and illustrates with the example of the π-π channel with ρ exchange. Section 2 discusses briefly the dynamical equations and approximations to be employed. Section 3 presents our treatment of photon driving forces and contains final numerical results of the $\pi\pi$ model. Sections 4 and 5 contain additional discussion of the sign of the ρ mass difference, and of the effect of ρ-ω mixing (which is neglible). In Section 6 we examine the degenerate SU_3 model. The main plot is contained in Sections 1, 3, and 6.

1. DYNAMICAL EQUATIONS AND PERTURBATION THEREOF

The basic assumption of the present approach is that the particle studied results dynamically from the forces in a scattering channel, i.e., from a "bootstrap"[2] calculation. At this point it is necessary to comment on two different types of current dynamical calculation. The first type may start with single-particle exchange forces, but then attempts to fully satisfy the required crossing symmetry[3] of the scattering amplitude. This usually requires complicated machine calculations and makes intractable the calculation of effects due to the EMS of participating particles. The second calculational approach assumes only single-particle exchange forces;[4] this permits simple examination of participating particle EMS effects and is the approach adopted here.

We shall furthermore assume that the relevant partial-wave amplitude has been calculated with the N/D formalism.[5] We will eventually use an approximate bootstrap calculation with several parameters characterizing the forces. In this study we are not interested in all the details of how such a modified force may have arisen, but merely seek a working bootstrap calculation to perturb. Having disposed of these preliminary comments, we devote the remainder of this first section to perturbations of our N/D calculation.

For simplicity, consider elastic single-channel scattering, with the following defined quantities:

(a) $t = N/D$ is a partial-wave amplitude, in state of definite J^{π}, divided by appropriate threshold factor,

(b) γ symbolically represents photon forces,

(c) $G_i \equiv g_i^2/4\pi$ are the various coupling constants involved,

(d) A,B are the external particles, masses m_A, m_B,

(e) C is the main exchanged particle, providing the force responsible for generating output particle E,

(f) s is the square of the c.m. energy.

If more than one exchange force is important, the generalization will be obvious.

We then have the mass of output particle E specified by

$$D(s = m_E^2;\ m_A^2,\ m_B^2,\ m_c^2, \gamma\ ,\ G_i) = 0 \quad . \tag{1}$$

We also require that our amplitude at least satisfy the requirements of crossing symmetry at the pole. This requires the output residue at the pole to equal the appropriate input coupling constant, to within specified kinematic factors. Stated mathematically,

$$G_{ABE} = N/\frac{\partial D}{\partial s}\bigg|_{D=0} \times \frac{1}{K} , \tag{2}$$

where K is an appropriate kinematic factor. Our approach is now to perturb Eqs. (1) and (2) as follows. For self-charge-conjugate meson multiplets with isospin ≤ 1, there is only one mass difference within a multiplet, denoted by δm_i^2. If we consider the channel AB for two different total charge states (of E), then the various particles (A,B,C) will have masses differing in the two channels by $C_i \delta m_i^2$, where C_i will be 0 or ± 1. Now, we write Eq. (1) for the two total charge states, and relate the two D functions by a Taylor expansion in the mass differences, etc. In this manner we obtain

$$\sum_{i=A,B,C^2} C_i \delta m_i^2 \frac{\partial D}{\partial m_i^2} + \sum_i \frac{\partial D}{\partial G_i} \delta G_i + " \frac{\partial D}{\partial \delta} \delta\gamma "$$

$$+ \frac{\partial D}{\partial s} C_E \delta m_E^2 = 0 \Big]_{s=m_E^2} . \qquad (3)$$

If we let $h(s, \cdots) = N/(K\partial D/\partial s)$, then similarly

$$\delta G_{ABE} = \sum C_i \delta m_i^2 \frac{\partial h}{\partial m_i^2} + " \frac{\partial h}{\partial \gamma} \delta\gamma " + \frac{\partial h}{\partial s} C_E \delta m_E^2$$

$$+ \delta G \text{ terms} \Big]_{s=m_E^2} . \qquad (4)$$

Our basic equations are then (3) and (4), simplified by the requirement that the bootstrap equations be scale-invariant. This means that keeping all G's fixed and increasing all $(\text{mass})^2$ and $(\text{energy})^2$ by a common factor β, we again obtain a consistent solution with (m_E^2) new $= \beta(m_E^2)$ old . If we let $\beta = 1 + \epsilon$, ϵ small, and again perform a Taylor expansion of $D(\beta s, \beta m_i^2)$ about $D(s, m_i^2)$, we obtain

$$\sum m_i^2 \frac{\partial D}{\partial m_i^2} + s \frac{\partial D}{\partial s} \Big]_{s=m_E^2} = 0, \qquad (5)$$

and similarly for h . The multichannel generalization, where t , N, and D are matrices, will be obvious when we need it later. At this point we comment on the correspondence of this approach with that of Dashen and Frautschi and on the motivation for this approach. We note that (if $C_E \neq 0$) , Eq. (3) has the form

$$\delta m_E^2 = - \frac{1}{C_E \partial D/\partial s} \left[\sum C_i \frac{\partial D}{\partial m_i^2} \delta m_i^2 \text{ etc.} + \text{"} \frac{\partial D}{\partial \gamma} \delta \gamma \text{"} \right] . \qquad (6a)$$

The correspondence with Reference 1 is elucidated by examination of their expression for δm_E^2 (transcribing some notation),

$$\delta m_E^2 = \frac{K}{ND'} \frac{1}{\pi} \left[\int_L \frac{D^2(s')}{s' - m_E^2} \text{Im} \left[\sum_i C_i \delta m_i^2 \frac{\partial}{\partial m_i^2} (\text{Force Amplitude}) \right] \right.$$

$$\left. + \int \frac{\text{Im } [D^2 \delta A]}{s' - m_E^2} \right] . \qquad (6b)$$

Here $\int_{L(R)}$ means integration along the left- (right)-hand cuts of the amplitude. Im $[D^2 \delta A]$ contains the kinematic effects of EMS of external particles. Thus, for exchanged particles, the first term of (6b) is equivalent to the first term of (6a). This first term in (6b) can also include the photon exchange amplitude, and obviously must correspond to the "$\delta\gamma$" term in (6a).

In the static model calculations of Reference 1, D was required only on a relatively short region of the real axis, and therefore a linear approximation to D was useful. This need not be true in the present more fully relativistic calculations. Because simple approximations to forces can give explicit, manually manageable expressions for D, this author decided to avoid Eq. (6b) and its probably attendant use of computing machines in the present problem. (It is easy to perturb these expressions for D, but an ugly situation to have to integrate them.) The price paid for this decision will be seen to be clumsiness in dealing with photon exchange forces.

We conclude this section by illustration, using our first model, the inelastic π-π calculation. The inelasticity parameter we denote by $R = \frac{\sigma_{TOTAL}}{\sigma_{ELASTIC}}$. Exchange of the ρ is assumed to produce the dominant force. First we require some definitions. Let the partial-wave amplitude be $\bar{t}$, corresponding to

$$\frac{(s)^{1/2}}{2q} \frac{1}{4q^2} e^{i\delta} \sin \delta .$$

(q is the 3-momentum). We define

$$\delta\mu^2 = \mu^2_{\pi^+} - \mu^2_{\pi^o} , \quad \delta M^2 = M^2_{\rho^+} - M^2_{\rho^o} , \quad \delta G = G_{\rho^o\pi^+\pi^-} - G_{\rho^+\pi^+\pi^o} .$$

The ρ^o occurs in the $\pi^+\pi^-$ channel with $\rho^\pm$ exchange, while the ρ^+ occurs in the $\pi^+\pi^o$ channel with ρ^+ exchange. M_{ex}^2 represents the (mass)^2 of the exchanged ρ meson. As usual, let "γ" symbolically represent photon forces; $\frac{\partial D}{\partial \mu^2}$ represents the dependence of D on the mass of <u>one</u> of the four external particles. Equations (3), (4), (5) now specialize to

$$\frac{\partial D}{\partial \gamma} \text{"}\delta\gamma\text{"} - 2\delta\mu^2 \frac{\partial D}{\partial \mu^2} + \delta M^2 \left[\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2} \right] - \frac{\partial G}{G} + \text{"}\frac{\partial D}{\partial R}\ \delta R\text{"} = 0 \quad , \qquad (7)$$

$$4\mu^2 \frac{\partial D}{\partial \mu^2} + M^2 \left[\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2} \right] + \text{"}\frac{\partial D}{\partial R} \sum m_\alpha{}^2 \frac{\partial R}{\partial m\alpha^2}\text{"} = 0 \quad , \qquad (8)$$

$$\delta G = -2\delta\mu^2 \frac{\partial h}{\partial \mu^2} + \delta M^2 \left[\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2} \right] + \text{"}\frac{\partial h}{\partial \gamma}\ \delta\gamma\text{"} + \text{"}\frac{\partial h}{\partial R}\ \delta R\text{"} \quad , \qquad (9)$$

$$4\mu^2 \frac{\partial h}{\partial \mu^2} + M^2 \left[\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2} \right] + \text{"}\frac{\partial h}{\partial R} \sum m_\alpha{}^2 \frac{\partial R}{\partial m_\alpha{}^2}\text{"} = 0 \ . \qquad (10)$$

Equations (8) and (10) are statements of scale invariance. However, since R is itself dimensionless, it is scale-invariant. Consequently the R terms in (8) and (10) separately vanish in a simple model of R with R = constant above a fixed energy. We now extract from Eqs. (7) through (10) the final result:

$$\frac{\delta G}{G} - \text{"} \frac{\partial D}{\partial \gamma} \delta\gamma \text{"} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2\right)\left(\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2}\right) + \text{"} \frac{\partial D}{\partial R} \delta R \text{"}, \qquad (11a)$$

$$\delta G - \text{"} \frac{\partial h}{\partial \gamma} \delta\gamma \text{"} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2\right)\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2}\right) + \text{"} \frac{\partial h}{\partial R} \delta R \text{"} . \qquad (11b)$$

We immediately recover the group theoretic result of Dashen and Frautschi[1] if we assume $R = 0$ and neglect photon forces. Except in the event of accidental equality of the "D" and "h" brackets on the right sides of Eqs. (11a,b) (such equality is not required by the bootstrap equations), we obtain the solution

$$\delta G = 0,$$

$$\frac{\delta M^2}{M^2} = -\frac{1}{2} \frac{\delta\mu^2}{\mu^2} . \qquad (12)$$

2. N/D EQUATIONS AND APPROXIMATIONS

In this section, we shall first present the equations satisfied by N and D, and then develop an approximate representation of an inelastic ρ bootstrap. We finally indicate how we obtain the various mass derivatives occurring in Eq. (11).

We have already defined the partial-wave amplitude,

$$t = N/D = \frac{(s)^{1/2}}{2q} e^{i\delta} \sin\delta \times \frac{1}{4q^2} .$$

Physical threshold for $\pi\pi$ scattering is denoted by $s_1 \approx 4\mu^2$; let $\rho = 2q(s)^{-1/2}$. Let $B(s)$ be a function possessing the left-hand discontinuities of t. We then have[4]

$$N = B + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{ds'}{s' - s} 4q^2(s')\rho(s') \left[B(s') - \frac{s - s_t}{s' - s_t} B(s) \right] ds' , \tag{13}$$

$$D = 1 - \frac{s - s_t}{\pi} \int_{s_1}^{\infty} \frac{ds' 4q^2(s')\rho(s')}{(s' - s)(s' - s_t)} N(s')R(s') . \tag{14}$$

The quotient ND^{-1} is independent of the subtraction point s_t.

Taking account of appropriate boson symmetrization of π-π states, we find that the $I = 1$, $\ell = 1$ Born ρ-exchange amplitude, which we use as an approximation to B, is given by

$$t_\rho = \frac{G_{\rho\pi\pi}}{32q^2} \left(4 + \frac{s + M^2}{q^2} \right) Q_1 \left(1 + \frac{M^2}{2q^2} \right) . \tag{15a}$$

Here Q_1 is the Legendre function of the 2nd kind and

$$G_{\rho\pi\pi} \equiv g^2_{\rho\pi\pi}/4\pi = 4(\gamma^2_{\rho\pi\pi}/4\pi \text{ of reference 3}).$$

The ρ pole diagram contributes and amplitude

$$t_{pole} = -\frac{G_{\rho\pi\pi}}{12(s - M^2)} . \tag{15b}$$

In order to avoid machine calculation, we now make the following approximations:

(a) $\rho \approx 1$ in integrals. This errs most as $s \to s_1$, but in this part of the integration range $q^2 \to 0$, so that the approximation is good.

(b) $R(s) = R\theta(s - c)$, i.e., a step-function inelasticity is employed.

(c) We observe that $t_\rho(s)$, for $M_\rho^2 = 30$, actually increases slowly out to very high energies (e.g., $s = 100\ m_\pi^2$) before commencing a $(\ell n\ s/s)$ decrease. It might not be unreasonable to represent t_ρ, on the right-hand cut, by a pole form, with this "pseudopole" quite far out on the negative real axis. We now do this, setting

$$t_\rho(s) \approx \frac{aG \equiv A}{s - a} \quad \text{for} \quad s > s_1 , \tag{16}$$

and choosing the parameters to give the same strength force as t_ρ near threshold. This, in a sense, applies a moderate cutoff to t_ρ at energies around 100 (in units of m_π^2) ; we need not worry about the strength in the very-high-energy $1/s$ region because of the convergence of the subtracted integrals for N and D.

Taking advantage of the fact that Eqs. (14) give N/D independent of s_t, we choose $s_t = a$.

The consequence of these approximations is the following set of expressions for N, Re D, and D':

$$N = \frac{A}{s - a} \quad , \quad s > s_1 \quad , \tag{17a}$$

$$\mathrm{Re}\, D = 1 - \frac{A}{\pi}\left[1 + \frac{s - s_1}{s - a}\,\ell n\,\frac{s_1 - a}{s - s_1} - (R - 1)\left\{\frac{s - s_1}{s - a}\,\ell n\,\frac{c - s}{c - a} - \frac{s_1 - a}{c - a}\right\}\right] , \tag{17b}$$

$$\mathrm{Re}\, D' = \frac{A}{\pi(s - a)}\left[1 - \frac{s_1 - a}{s - a}\,\ell n\left(\frac{s_1 - a}{s - s_1}\right) - (R - 1)\left\{\frac{s - s_1}{c - s} - \frac{s_1 - a}{s - a}\,\ell n\left(\frac{c - s}{c - a}\right)\right\}\right] . \tag{17c}$$

From these equations we notice that there is effectively a strong attractive force below the inelastic threshold $(s = c)$, so that there is a tendency to have resonance[6] near (and below) $s = c$. Of course, P-wave inelasticity commences less abruptly than a step function, but these general features remain valid.

With the above pole-model approximation we can quite simply reproduce the results of most sophisticated bootstraps described in the literature,[5] when we set $R = 0$. In this elastic situation, we then find that, characteristically, the resultant ρ mass is around 350 MeV, with the width several times as large as experiment. Balasz,[7] however, considered inelasticity and found a ρ in the 600 to 700 MeV region with large inelasticity well above threshold $(R \geqslant 4)$. Indeed, from a survey of the literature on π-π calculations, one acquires the impression that sufficient inelasticity generally permits the bootstrapping of a reasonably physical ρ . Unfortunately, most self-consistent approaches characterize R in a manner not easily susceptible of our type of perturbation treatment. For simplicity, then, we adopt R(s) as above. The initial calculation will assume R independent of charge state; a subsequent model for R will enable us to examine possible modification of our results.

Those who have performed multichannel calculations[3] generally find the π-ω channel to be next in importance to the $\pi\pi$ channel in determining the ρ . We therefore arbitrarily select the inelastic threshold C to be the $\pi\omega$ threshold, namely $s = 45$ (in units of m_π^2) . The following parameters generate a "self-consistent" ρ at $s = 30$:

$$\begin{aligned} C &= 45 \quad , \\ R &= 4 \quad , \\ a &= -100 \quad , \quad \alpha = 0.18 \quad , \\ G &= 3.7 \quad . \end{aligned} \tag{18}$$

The final aspect of our approximation is in reference to the mass derivatives. We assume that the parameters α and a , used in simulating t_ρ , are dependent on the force parameters, including M_{ex}^2. Consequently we match the mass derivative of the pole term to that of the Born amplitude (15a) at two energies in the low-energy region, to obtain

$$\begin{pmatrix} \alpha_M \\ \\ a_M \end{pmatrix} \equiv \frac{\partial}{\partial M_{ex}^2} \begin{pmatrix} \alpha \\ \\ a \end{pmatrix} . \tag{19}$$

Admittedly, these mass derivatives should be energy-dependent. However, if we take some "average" numbers obtained by matching in the low-energy region (where one tends most to believe the Born estimate), then errors at high energies are damped by the subtractions in the relevant integrals. Taking the numbers obtained by matching at (a) $s = 8$ and 16 and (b) at $s = 16$ and 24, one obtains

$$a_M \approx +11.0 \quad , \quad \alpha_M \approx -0.025 \quad . \tag{20}$$

Using Eqs. (17), we now obtain

$$\frac{\partial D}{\partial M_{ex}^2} = -\frac{\alpha_M}{\alpha} - \frac{A}{\pi} a_M f_2(s),$$

$$f_2(s) = \frac{s - s_1}{s - a}\left[\frac{\ln\left|(s_1 - a)/(s - s_1)\right|}{s - a} - \frac{1}{s - a} - (R - 1)\right.$$
$$\left.\left\{\frac{\ln\left|\frac{c - s}{c - a}\right|}{s - a} + \frac{1}{c - a}\right\}\right] - (R - 1)\left(\frac{1}{c - a} - \frac{s_1 - a}{(c - a)^2}\right) \quad . \tag{21}$$

Also, from $G = -12\ N/D'\big|_{s = M_\rho^2}$ (note that the factor $\mathcal{a}$ cancels out of this expression), and using Eqs. (11),

$$\frac{\delta G}{G} = \left(\delta M^2 + \frac{M^2}{2\mu^2}\ \delta\mu^2\right) f_4(s) \Bigg/ \left(\frac{\pi(s - a)}{A}\ D'\right) + h_D\ , \qquad (22a)$$

where

$$f_4(s) = \frac{a_M}{s - a}\left(-1 - \frac{s - s_1}{s - a}\ \ell n\ \frac{s_1 - a}{s - s_1} + (R - 1)\left[\frac{s - s_1}{s - a}\ \ell n\ \frac{c - s}{c - a} - \frac{s_1 - a}{c - a}\right]\right)$$

$$- \frac{s_1 - a}{(s - a)^2}\ \ell n\ \left(\frac{s_1 - a}{s - s_1}\right) - \frac{s_1 - a}{(s - a)(s - s_1)}$$

$$+ (R - 1)\left[\frac{1}{c - s}\left(1 + \frac{s_1 - a}{s - a}\right) + \frac{s - s_1}{(c - s)^2} + \frac{s_1 - a}{(s - a)^2}\ \ell n\ \left(\frac{c - s}{c - a}\right)\right] \qquad (22b)$$

$$h_D = \left(\frac{\delta N}{N} - \frac{\delta D'}{D'}\right) \quad \text{photon force}$$

$$\equiv - \left(\frac{\delta N}{N} - \frac{\delta D'}{D'}\right) \quad \rho^0 \ \text{photon force} \ . \qquad (22c)$$

Inserting numbers, we find

$$\frac{\partial D}{\partial M_{ex}^2} \approx + 0.13$$

$$\frac{\partial D}{\partial s} \approx -0.016 \ . \qquad (22)$$

The sign of $\frac{\partial D}{\partial M^2_{ex}}$ is easily understood: a heavier exchange mass means a shorter-range force, hence more centrifugal repulsion, and therefore a weaker force. The sign is also verified with t_{Born} in the Dashen-Frautschi formalism.

The important aspect of (22) is that the crude dynamics (which is numerically representative of most efforts nowadays) implies $|\partial D/\partial M^2_{ex}| >> |\partial D/\partial s|$. (In the language of Dashen and Frautschi, $A_{MM} >> 1$). This very strong dependence on the exchanged mass is a characteristic of the present meson dynamics which did not occur in static model meson-baryon calculations.[1] We also wish to emphasize that the relative magnitude of these derivatives is a feature which is present if we calculate them in a model omitting inelasticity.

Finally, omitting the charge-state dependence of R , and leaving photon forces for the next section, we obtain from Eqs. (11) our penultimate expressions in this procedure:

$$\frac{\delta G}{G} + \delta D_{\rho^{o}"\gamma"} = 3.4 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right), \tag{23a}$$

$$\frac{\delta G}{G} + \left(\frac{\delta N}{N} - \frac{\delta D'}{D'} \right)_{\rho^{o},"\gamma"} = -0.2 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right) \tag{23b}$$

or

$$\left(\frac{\delta D'}{D'} - \frac{\delta N}{N} + \delta D \right)_{\rho^{o}"\gamma"} = 3.6 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right), \tag{24a}$$

$$\frac{\delta G}{G} \approx \left(\frac{\delta D'}{D'} - \frac{\delta N}{N} \right)_{\rho^{o}"\gamma"} . \tag{24b}$$

3. PHOTON FORCES

In this section we discuss the nature of the explicit photon amplitudes, briefly review the treatment of the infrared difficulties (as in Reference 1), and eventually present a few formulae and numerical results. Detailed formulae are relegated to an appendix .

The photon amplitudes may be enumerated as follows:

(a) γ exchange in u and t channels. This is obtained from t_ρ exchange simply by letting $M_\rho^2 \to 0$ (only the isovector photon is exchanged, so all numerical factors are identical to ρ exchange).

(b) s-channel 1-photon pole. This can be found (numerically) to give only a small effect, and is subsequently ignored.

(c) γ-vertex corrections. Either these can be considered as coming from three-particle ($\pi\pi\gamma$) intermediate states--in which case calculation currently comes to a stop--or these corrections can be absorbed into what we call δG (neglecting the t dependence of the vertex correction); we adopt that latter point of view.

(d) $\gamma - \rho$ "straight ladder" diagrams. These may be considered, roughly, as a consequence of applying N/D calculations to pure ρ and pure γ exchanges as input forces.

(e) $\gamma - \rho$ "twisted ladder" diagrams. These correspond again to three-body intermediate states, and arise from the u-t double spectral function in the Mandelstam representation. We ignore these too, hopefully not omitting any large effects (in view of the fact that one-γ-exchange effects will later be seen to be rather small).

We can consider infrared divergence questions by initially assigning to the photon a fictitious mass $\lambda^{1/2}$ which we eventually set equal to zero.[8] In a P-wave amplitude, we expect that the S-matrix element on the right-hand cut acquires a divergent "Coulomb" phase factor, which is therefore exhibited by the D function. In practice D will possess terms finite as $\lambda \to 0$, and terms divergent as $\lambda \to 0$. These latter we interpret as arising from our perturbation expansion of the infinite phase factor of the S matrix, and therefore discard. The relevant strong-interaction S matrix is defined as the S matrix from which the divergent Coulomb phase factor, really arising from improper consideration of brehmstrahlung, is removed.

We arrive at an estimate for the divergent Coulomb phase in a P-wave amplitude as follows: The one-photon-exchange amplitude is proportional to $(t - \lambda)^{-1}$, which corresponds to a potential $V(r) \approx \exp(-\lambda^{-1/2} r)/r$. Nonrelativistically,[9]

$$\delta_1 \propto \int J_\ell^2(qr)V(r)r^2 dr$$

$$\propto \int_0^\infty dr \exp(-r\lambda^{-1/2})\, J_{3/2}^2(qr)$$

which, in turn, is proportional to[10]

$$Q_1(1 + \lambda/2q^2) \quad . \tag{25}$$

Therefore, whenever a $\ell n\, \lambda$ term occurs, we remove a term proportional to (25). Hereafter we use a subscript f to denote the "finite part" of a Coulomb divergent quantity.

The lowest-order approximation to the photon contributions is obtained when N = Born γ exchange term:

i.e., $N_\gamma \equiv B_\gamma$, and therefore ,

$$\delta D_{"\gamma"} = -\frac{s-a}{\pi} \oint_{s_1}^{\infty} \frac{(s'-s_1)ds' B_\gamma \rho(s')}{(s'-s)(s'-a)} \Big]_f \quad . \qquad (26a)$$

Note that $N_\lambda]_f = 0$, so that $\delta_{"\gamma"}N]_f = 0$. (26b)

If we solve the integral equations for N and D , then in our pole model for the strong forces, using $\delta B)_f = 0$ and Eq. (14), we obtain (using subscript I to denote integral equations)

$$\delta N_{"\gamma"I})_f = -\frac{A}{s-a} \times [\delta D_{"\gamma"} \text{ of Eq. (26a)}]_f \quad . \qquad (26c)$$

Unfortunately, and this is where our approach is clumsy compared with that of Dashen and Frautschi, $D_{"\gamma"I})_f$ is the finite part of an integral over $N_{"\gamma"I}$ and this can be exceptionally messy. We therefore use Eq. (26a). In calculating the photon exchange diagram, we employ[8] a reasonable approximation for the $\pi\pi\gamma$ form factor at $t < 0$, namely $F_\pi(t) = -\frac{M_\rho^2}{t - m_\rho^2}$. The calculation of (26a) is now displaced to Appendix A so that the reader may not suffer from a lengthy digression from the plot. We find

$$\delta D)_f \approx -0.6e^2/4\pi + \text{inelastic effect} \approx -2(e^2/4\pi) \quad , \text{ say} \quad . \qquad (27)$$

This effect, in the absence of other particle EMS effects, would give a mass shift

$$\delta M_\rho^{\ 2} \approx \frac{\delta D}{\partial D/\partial s} \quad \text{or} \quad \delta M_\rho \approx +\ 11.5 \text{ MeV} \quad . \tag{28}$$

We can easily argue about what to expect as follows: The size of the ρ will be somewhat less than the range of the forces, perhaps $(\frac{3}{2} M_\rho)^{-1}$. We therefore expect a Coulomb energy $(e^2/4\pi)\ (\frac{3}{2} M_\rho) \approx 8$ MeV . Furthermore, it can be argued that in this case, the determinantal approximation over-estimates the photon force. We shall retain Eq. (27), with the conviction that, if anything, we overestimate the photon driving force.

In the determinantal approximation $\delta N)_f = 0$ and we find that $\frac{\delta D'}{D'}$ is very small. Consequently, we obtain from Eqs. (24a) and (27)

$$\frac{\delta M^2}{M^2} = -\frac{\delta \mu^2}{2\mu^2} - 0.004 , \tag{28}$$

which corresponds to

$$M_{\rho^0} - M_{\rho^+} \approx +\ 12.5 \text{ MeV} \quad . \tag{29}$$

As the purely group theoretic result (Eq. 28 without 0.004) is about 11 MeV, we see that the driving forces are not nearly so important as the exchange-mass-sensitive strong force.

From determinantal calculation here, we also obtain δG very small (<< 1%) . If we use (26c) to estimate the nondeterminantal $\delta N)_f$, we obtain

$$\frac{\delta G}{G} \approx -1\% \quad . \tag{30}$$

4. FURTHER CONSIDERATIONS ON THE SIGN OF δM^2

We investigate the charge-state dependence of R in the following model: Well above the ρ energy, around $\pi\omega$ threshold,

$$R \approx \left| \frac{A\pi\pi \to \pi\omega}{A\pi\pi \to \pi\pi} \right|^2_{\text{Born exchange amplitudes}} \tag{31a}$$

$$A\pi\pi \to \pi\omega \equiv \frac{1}{2} (A\rho^0 \text{ exchange} + A\rho^+ \text{ exchange}) \text{ in } + \text{ channel}$$

$$\equiv (A\rho^+ \text{ exchange}) \text{ in the } 0 \text{ channel} \tag{31b}$$

(an initial π^0 can couple either to a final ω or a final π^+ in the + channel). Thus

$$\delta |A\pi\pi \to \pi\omega|^2 \approx (-\delta M^2) \times \left(\frac{\partial}{\partial M_\rho^2} |A|^2 \right) + \delta\mu^2 \text{ term} \quad .$$

The second factor is negative: a heavier exchanged mass gives a weaker force, being more affected by the centrifugal barrier). Also

$$\delta |A\pi\pi \to \pi\pi|^2 \approx (+\delta M^2) \times \left(\frac{\partial}{\partial M\rho^2} |A|^2 \right) .$$

Thus $\delta R \approx \delta M^2 \times (>0)$. Detailed examination of these Born diagrams confirms this and also gives us $\delta R \approx \delta\mu^2 \times (<0)$. Since $\frac{\partial D}{\partial R} < 0$ here, we can see from Eqs. (11) that we now have an expression of the form

$$[\delta M^2 (\times < 1) + \frac{M^2}{2\mu^2} \delta\mu^2 \times (>1)] \quad .$$

In other words, the mass shift with the $\pi\omega$ channel effects will be larger than estimated in Eq. (29), (and still of the same sign).

It is amusing also to consider the π as a bound $\pi\rho$ state. If we ignore "driving" photon forces and δG effects, it can be inferred from examination as in Section 1 that

$$(-\delta M^2) \approx < \frac{2M^2}{\mu^2} \cdot \delta\mu^2 \quad ;$$

at least this gives confirmation of the sign of our mass shift.

5. $\rho - \omega$ MIXING

In this section we consider possible mass shifts of the ρ^o due to its electromagnetic coupling with the ω_o (mass 787 MeV). To do this in the context of two-particle scattering let us consider the ω as occurring in a two-particle channel such as the π-ρ channel. We now have a two-channel system determining the ρ_o, whose position is specified by the zero of the determinant of the D matrix for these two channels. That is,

$$\text{Det } D(s = \rho_o) = 0 \quad .$$

When we expand the determinant, the charged - ρ - channel term $D_{\pi\pi}$ is now replaced by

$$D_{\pi\pi} - \frac{D_x^{\,2}}{D_{\pi\rho}} = 0 \quad , \tag{32}$$

where D_x arises from the ρ channel $\to \omega$ channel transition amplitude. The first point which we should observe is that D_x is of order e^2, so that we anticipate that $\omega - \rho$ mixing introduces a mass shift of order e^4. We now make this estimate more precise. Analogous to our earlier treatment, we have the equation

$$0 = \delta D_{\pi\pi} = +2\delta\mu^2 \frac{\partial D_{\pi\pi}}{\partial\mu^2} - \left(\frac{\partial D_{\pi\pi}}{\partial s} + \frac{\partial D_{\pi\pi}}{\partial M_{ex}^{\,2}}\right)\delta M^2 - \frac{D_x^{\,2}}{D_{\pi\rho}} + \text{photon terms} \quad , \tag{33a}$$

which, combined with scale invariance of $D_{\pi\pi}$, is reduced to

$$\delta M^2 = -\frac{M^2}{2\mu^2}\,\delta\mu^2 + \left(\frac{\delta G}{G} - \delta D_{\rho^o "\gamma"} - \frac{D_x^2}{D_{\pi\rho}}\right)\chi^{-1},$$

$$\chi = \frac{\partial D}{\partial M_{ex}^2} + \frac{\partial D}{\partial s} \quad . \tag{34}$$

Now, we have already seen χ to be positive, and if $M_\rho < M_\omega$, then $D_{\pi\rho} > 0$ below M_ω^2. The effect is therefore to further increase the ρ^o mass. Notice that in ordinary perturbation theory, the mixing of two nearby states pushes them apart, but here the bootstrapping effects (namely $\chi > 0$) reverse the usual tendencey.

A crude estimate of the extra mass shift can be obtained via the following assumptions:

(a) $D_{\pi\pi} \approx 1 - \frac{g^2}{e^2} D_x$, implying $D_x \approx \frac{e^2}{g^2}$ at the ρ mass .

(b) $D_{\pi\rho} \approx (M_{\rho^2} - M_{\omega^2}) \frac{\partial D}{\partial s}$.

Combined with previous numerical estimates of the mass derivatives, these assumptions imply an additional mass shift

$$\delta' M \approx \frac{-1}{2M} \frac{(e^4/g^4)\mu^4 \text{ MeV}}{(0.114)(0.016)(M_\omega^2 - M_\rho^2)} \approx \frac{-0.33}{M_\omega - M_\rho} \approx 0.01 \text{ MeV} \quad . \tag{35}$$

6. SU_3 OCTET MODEL

Another model which is relatively cleanly examined is that of SU_3 octets of degenerate pseudoscalar mesons generating a degenerate vector octet. It turns out that the crossing-matrix element relating the effective strength of exchange forces to the direct-channel pole strength is the same as for the $\pi\pi\rho$ problem. This implies that our dynamical statements about mass derivatives in the $\pi\pi$ problem should remain valid in this model. As the degenerate octet model disagrees badly with reality (with regards to masses), we do not intend to include the complicated effects of photon and δG terms in this discussion, but merely wish to examine the consequences of the particle EMS. This, we hope, will furnish additional insight into the role played by inelasticity (in this case arising from the $\pi\pi$ - $K\bar{K}$ process).

We shall consider both the ρ and the K^* in this examination of degenerate octet scattering. To begin, we now present the forces in these channels, in the absence of particle EMS. In the following, we denote the 1^- (Born) amplitude with C exchange,

$$t_{Born})_{C\ exchange} \equiv N)_c\, f(s), \tag{36}$$

where $f(s)$ is a standard amplitude, and N is a numerical factor arising from SU_3 Clebsch-Gordan coefficients. The resultant t matrices[12] are given below, assuming vector exchange as the dominant force:

$$t_\rho \equiv \begin{array}{c c} & \begin{array}{cc} \pi\pi & \quad K\bar{K} \end{array} \\ \begin{array}{c} \pi\pi \\ K\bar{K} \end{array} & \begin{pmatrix} 4)_\rho & 8^{1/2})_{K^*} \\ 8^{1/2})_{K^*} & 3)_\omega - 1)_\rho \end{pmatrix} \end{array} f(s) \quad , \tag{37a}$$

$$t_{K^*} = \begin{array}{c c} & \begin{array}{cc} K\pi & \quad K\eta \end{array} \\ \begin{array}{c} K\pi \\ K\eta \end{array} & \begin{pmatrix} 4)_\rho - 1)_{K^*} & 3)_{K^*} \\ 3)_{K^*} & 3)_{K^*} \end{pmatrix} \end{array} f(s) \quad . \tag{37b}$$

We shall next examine the effects of particle EMS on these forces. (Note that the present approach is merely a pedestrian manner of accomplishing what are really group theoretic calculations.) In order to do so, the following assorted facts are needed:

$$|K\pi\rangle^{+}_{I=1/2} = \frac{K^+\pi^0 - \sqrt{2}\,K^0\,\pi^+}{\sqrt{3}} \quad ,$$

$$|K\pi\rangle^{0}_{I=1/2} = \frac{K^0\pi^0 + \sqrt{2}\,K^+\pi^-}{\sqrt{3}} \quad ,$$

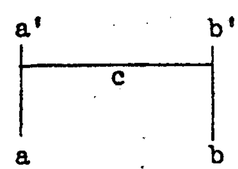
$$K\bar{K}\rangle^{0}_{I=1/2} = \frac{K^+K^- - K^0\bar{K}^0}{\sqrt{2}} \quad , \tag{38}$$

$$2|g_{\rho^0 K^+ K^+}| = |g_{\rho^0 \pi^+ \pi^-}| = |g_{\rho^+ \pi^+ \pi^0}| = |g_{K^{*0} K^0 \pi^0}| = 1,$$

$$|g_{\rho^- K^0 K^+}| = |g_{K^{*+} K^0 \pi^-}| = \sqrt{2},$$

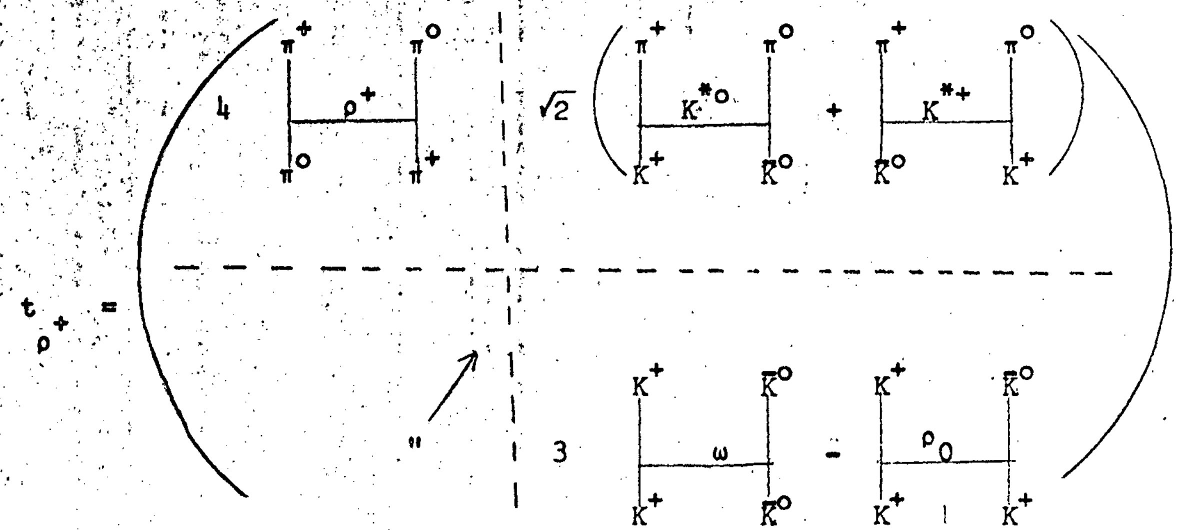
$$|g_{K^* K \eta}| = |g_{\omega K \bar{K}}| = \sqrt{3} \quad .$$

We now introduce a convenient pictorial representation of the exchange amplitudes. Denote the amplitude f , including the perturbing effects of the EMS appropriate to the labelled particles, by



. (39)

With this notation, Eqs. (37a,b), with particle EMS accounted for, are given in Eqs. (40a), (40b), (41a), and (41b).


t
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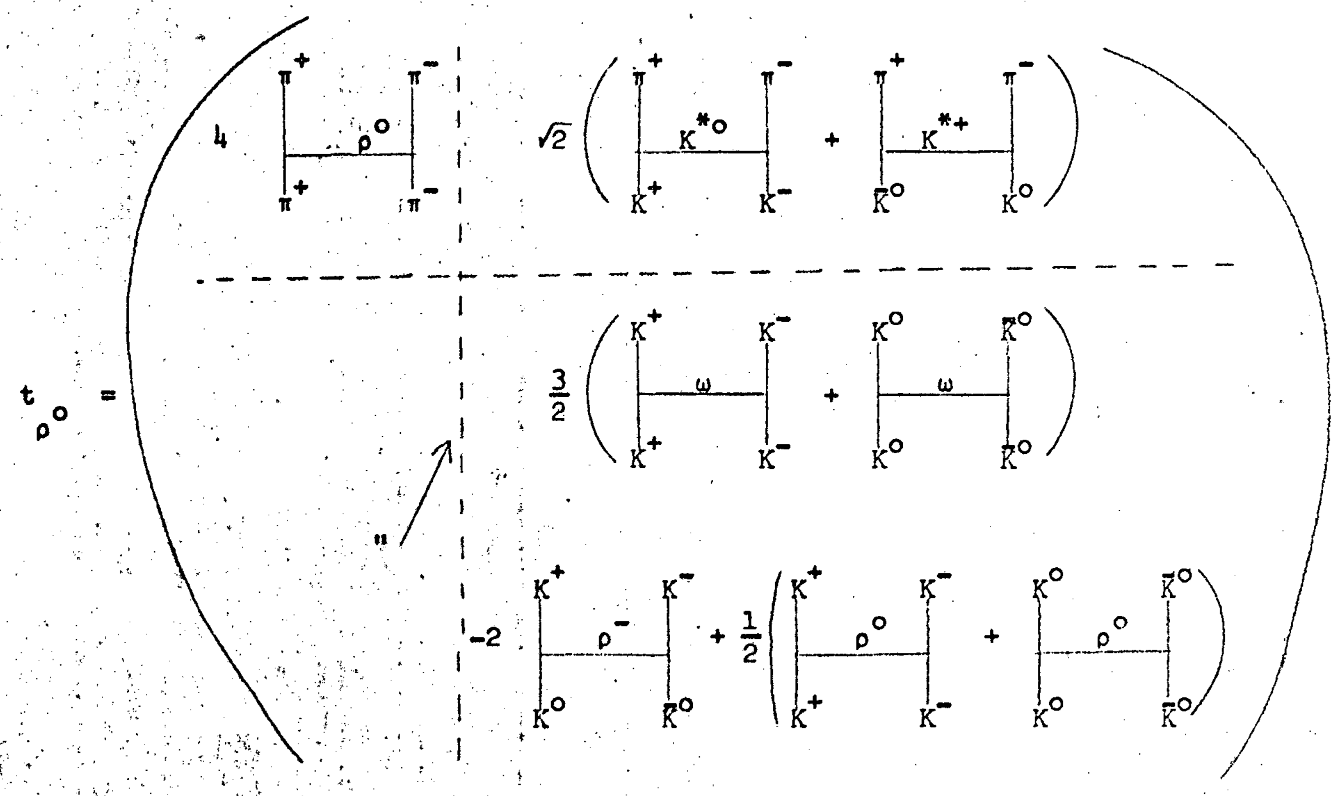
K+

(40a)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K0
K̄0
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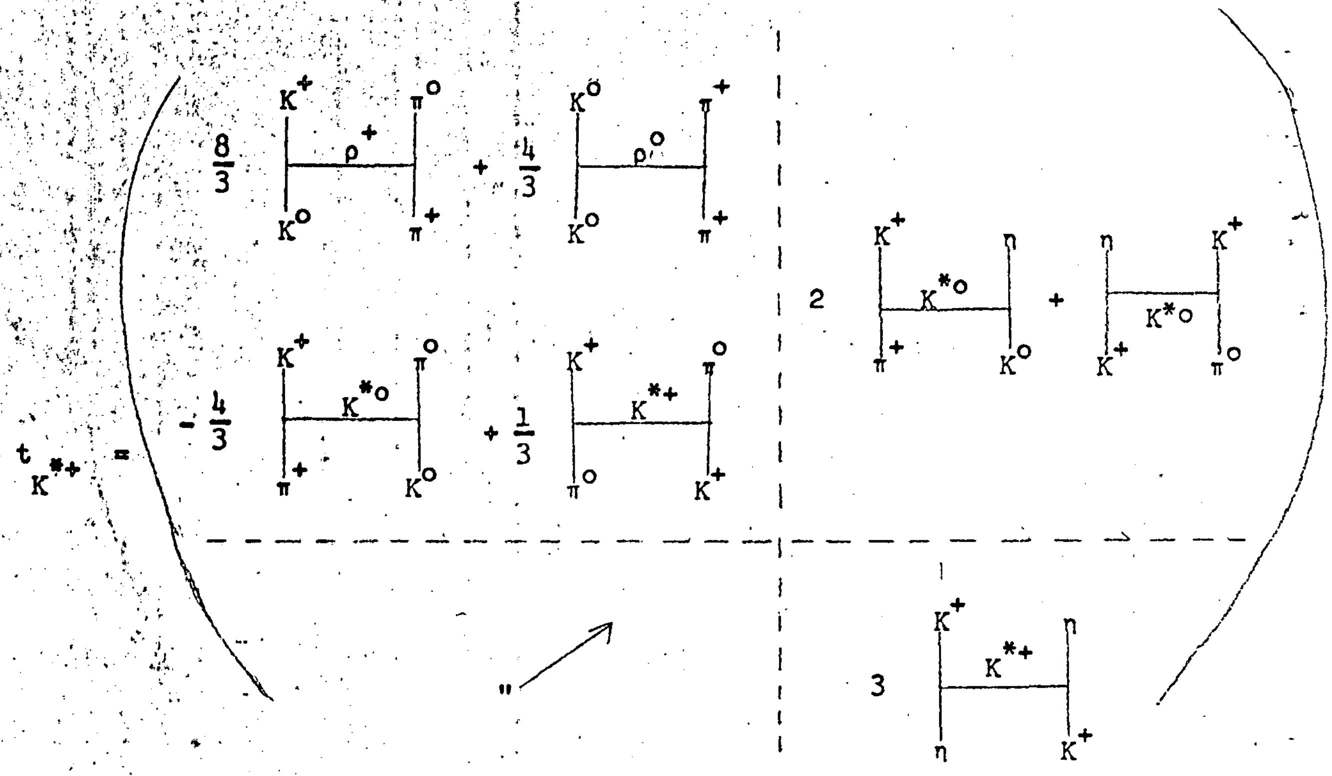
K0
K̄0

(40b)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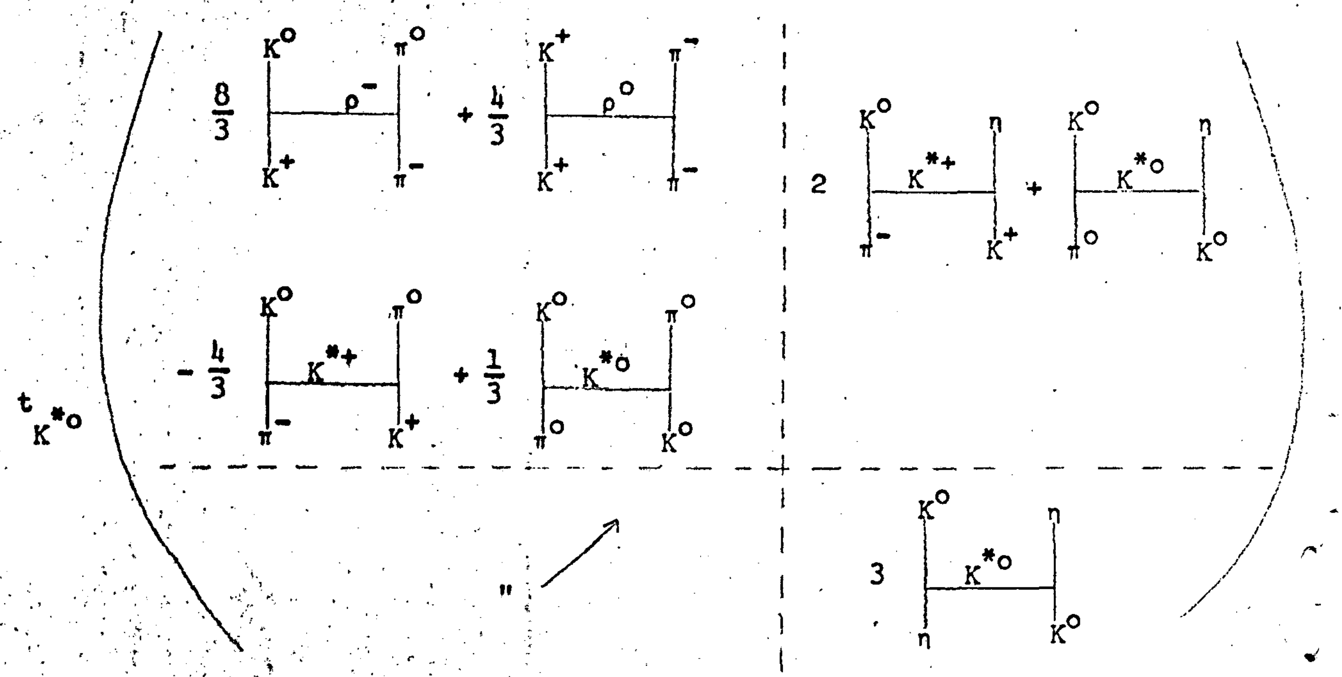
(41a)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1/3
2
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ρ-
ρo
K*+
K*o
K+
K0
π0
π-
η
"

(41b)

The final assumption in this model is that the matrix D is given by

$$D = 1 - \text{dispersion integral over } t . \qquad (42a)$$

This is obviously in the determinantal approximation, or in a one-pole approximation to the force. More generally, for degenerate forces, D - 1 is proportional to the matrix (37) even when the integral equations are solved for N and D. In addition, one can verify that our subsequent perturbation treatment provides results identical with those obtained via the Dashen-Frautschi technique (subject to the correspondence indicated in Section 1). Defining

$$d = \frac{s - s_t}{\pi} \int \frac{ds' \rho(s') n(s')}{(s'-s)(s'-s_t)}, \qquad (42b)$$

we obtain for the mass-degenerate case (Eqs. 37)

$$\det(D) \equiv |D| = 1 - 6d, \qquad (43)$$

which, of course, applies to both ρ and K^* channels because of the assumed SU_3 symmetry. At the vector octet mass M, $|D| = 0$; therefore $d(M^2) = 1/6$. We shall define the following quantities for the sake of typographical simplicity of the ensuing discussion:

$$\delta^{\pi} \equiv \mu^2_{\pi^+} - \mu^2_{\pi^o} \qquad \left.\frac{\partial d}{\partial(\mu^2 \text{ external})}\right]_{d=1/6} \equiv d_1 \ , \tag{44}$$

$$\delta^{K} \equiv \mu^2_{K^+} - \mu^2_{K^o}$$

$$\delta^{\rho} \equiv M^2_{\rho^+} - M^2_{\rho^o} \qquad \left.\frac{\partial d}{\partial(M^2 \text{ exchanged})}\right]_{d=1/6} \equiv d_2 \ ,$$

$$\delta^{K^*} \equiv M^2_{K^{*+}} - M^2_{K^{*o}}$$

$$\frac{\partial d}{\partial s} \equiv d_s \quad .$$

From Eqs. (40) and (41) we next obtain

$$\delta D_{\rho} = +\begin{pmatrix} 8d_1\delta^{\pi} - 4(d_2 + d_s)\delta^{\rho} & 8^{1/2}(d_1\delta^{\pi} - d_s\delta^{\rho}) \\ 8^{1/2}(d_1\delta^{\pi} - d_s\delta^{\rho}) & -2\delta^{\rho}(d_2 + d_s) \end{pmatrix}, \tag{45a}$$

$$\delta D_{K^*} = \begin{pmatrix} -2d_1\delta^K + \frac{5}{3}d_2\delta^{K^*} + 3d_s\delta^{K^*} & 2d_1\delta^K + (3d_s - d_2)\delta^{K^*} \\ (2d_1\delta^K + (3d_s - d_2)\delta^{K^*} & 6d_1\delta^K + 3(d_2 + d_s)\delta^{K^*} \end{pmatrix} . \tag{45b}$$

The expressions can be simplified by utilizing the scale invariance of $|D|$ to obtain the relation

$$\left. 4\mu^2 d_1 + M^2(d_2 + d_s) \right]_{d=1/6} = 0 \quad . \tag{46}$$

Finally, using the relation

$$\delta|D| = \text{Trace (co-factor matrix of } D \times \delta D), \qquad (47)$$

together with Eqs. (45) and (46), we obtain

$$\delta^{\rho} = -\delta^{\pi} \frac{M^2}{\mu^2} \left(\frac{d_2 + d_s}{\frac{5}{3} d_2 + 3d_s} \right) , \qquad (48a)$$

$$\delta^{K^*} = +\delta^{K} \frac{M^2}{2\mu^2} \left(\frac{d_2 + d_s}{\frac{2}{3} d_2 + 3d_s} \right) , \qquad (48b)$$

where we ignore the effects of δG's and of photon forces. With the dyanmical situation $d_2 > d_s$ (which might <u>not</u> suffice to provide for the dominance of d_2 in 48b) we obtain

$$\delta^{\rho} \approx - \frac{3}{5} (M^2/\mu^2)\delta^{\pi} , \qquad (49a)$$

$$\delta^{K^*} \approx + \frac{3}{4} (M^2/\mu^2)\delta^{K} . \qquad (49b)$$

Incidentally, if we consider the SU_3 model wherein the pseudoscalars are bound pseudoscalar-vector states, with pseudoscalar exchange dominant,[13] then we obtain the same signs as in Eqs. (49). Inserting the physical ρ, π masses in (49a), we find

$$\delta M^{\rho} = -13 \text{ MeV}$$

in the $\pi\pi$ model (without δG, $\delta\gamma$) . With the substitution of physical K, K^* masses,

$$\delta M^{K^*} = -5.5 \text{ MeV} .$$

DISCUSSION

The numerical results of this study indicate $(M_{\rho^0} - M_{\rho^+})$ from 12 to 15 MeV. Present experimental data are uncertain, although such a mass difference should be detectable, despite the large width of the ρ. There is currently a controversy as to the possibility that the $\pi^+\pi^-$ experimental mass spectra may exhibit a two-humped structure.[14] Some proponents of this interpretation believe the lower energy peak to be due to an $I = 0$ scalar meson,[15] leaving the higher peak to the ρ^0, which then would be over 20 MeV heavier than the $\rho^\pm$. However, this situation is freely acknowledged to be tentative, and the author does not yet infer any correspondence with the prediction of this study.

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Appendix A--Photon Contribution to D

The following technique is convenient for evaluating finite parts of integrals. Let $e^2/4\pi = \frac{1}{137}$. For now, ignore inelasticity. We start with

$$B_\gamma(s,t) = \left\{\frac{e^2}{16\pi}\left[\frac{-M^4}{(t-M^2)^2}\left(1+\frac{2s-4\mu^2}{t-M^2}\right)+\frac{2s-4\mu^2}{t-M^2}-\frac{2s-4\mu^2}{t-\lambda}\right]\times\frac{1}{2}\right.$$

$$\left. + \text{ U channel term giving identical partial wave}\right\} . \tag{A1}$$

The 3rd term contributes to D_ℓ:

$$-\frac{1}{2}\int_{-1}^{1} dxP_\ell(x)\,\frac{e^2}{16\pi}\,\frac{s-a}{\pi}\int_{s_1}^{\infty}\frac{ds'}{(s'-s)(s'-a)}\;\frac{2s'-s_1}{s'-\alpha}\,, \tag{A2}$$

where

$$\alpha = s_1 - \frac{2\lambda}{1-x}\,,$$

$$= -\frac{e^2}{16\pi^2}\int_{-1}^{1}\frac{P_\ell(x)}{1-x}\left[\ln(s_1-\alpha)\left\{\frac{2s-s_1}{s-\alpha}-\frac{2a-s_1}{a-\alpha}\right\} - \frac{2s-s_1}{s-\alpha}\ln(s_1-s)+\frac{2a-s_1}{a-\alpha}\ln(s_1-a)\right] . \tag{A3}$$

Now observe that

$$\frac{1}{2}\int_{-1}^{1}\frac{P_\ell(x)}{1-x}\,\frac{1}{s-\alpha} = \frac{1}{s-s_1}\times Q_\ell\left(1+\frac{\lambda}{2q^2}\right),$$

which we discard. Also, we have

$$\frac{1}{2}\int_{-1}^{1} \frac{P_\ell(x)\ \ell n\ (s_1 - \alpha)}{(1 - x)(s - \alpha)} = \frac{\ell n\ 2\lambda}{s - s_1}\ Q_\ell\left(1 + \frac{\lambda}{2q^2}\right)$$

$$- \frac{1}{2}\int_{-1}^{1} \frac{P_\ell(x)\ \ell n\ (1 - x)}{(1 + \frac{\lambda}{2q^2} - x)(s - s_1)} .$$

Again, we discard the first term, and find

$$= + \frac{1}{4q^2}\ (\ell n\ 2 - 2) - \frac{1 + \gamma_q}{\gamma q^2}\int_0^2 \frac{du\ \ell n\ u}{u + \gamma_q} , \qquad (A4)$$

where

$$\gamma_q = \frac{\lambda}{2q^2} .$$

Defining $4k^2 = a - s_1$ (analogously to $4q^2 = s - s_1$), we find that the $1/t - \lambda$ term contributes

$$\delta D \times \left(\frac{-e^2}{8\pi^2}\right)^{-1}$$

$$= \frac{(2a - s_1)\ \ell n\ (s_1 - a)}{8k^2}\ \ell n\left|\frac{k^2}{q^2}\right| + \frac{(2s - s_1)\pi^2}{48q^2}$$

$$- \frac{2a - s_1}{4k^2}\left[\frac{-5\pi^2}{24} + \ell n\left|\frac{k^2}{q^2}\right|\left(-\frac{\ell n^2}{2} + \frac{\ell n|k^2/q^2|}{4} + \frac{1}{2}\left(\ell n\ 8q^2\right)\right)\right] . \qquad (A5)$$

To obtain this we evaluated the integrals

$$(a)\quad I_q = \int_0^2 du \frac{\ln u}{u + \gamma_q} = \frac{(\ln 2)^2}{2} - \frac{(\ln \gamma_q)^2}{2} - \frac{\pi^2}{6} \cdots \text{ as } \lambda \to 0 \tag{A6}$$

$$(b)\quad I_k = \oint_0^2 du \frac{\ln u}{u - |\gamma_k|} \equiv \frac{5}{12}\pi^2 - \frac{(\ln |\gamma_k|)^2}{2} + \frac{(\ln 2)^2}{2} .$$

The expansions used for these integrals may be found in Reference 11. No new problem arises with the $(t - M^2)^{-1}$ term, which contributes

$$\begin{aligned}\delta D\left(\frac{+e^2}{8\pi^2}\right)^{-1} = & \frac{2s - s_1}{4q^2} \ln\left(\frac{2M^2}{s - s_1}\right) Q_1(a_q) \\ & - \frac{2a - s_1}{4k^2} \ln\left(\frac{2M^2}{s_1 - a}\right) Q_1(a_k) \\ & - \frac{2s - s_1}{4q^2} \left(1 - \ln 2 + \frac{a_q}{2} I_q\right) \\ & + \frac{2a - s_1}{4k^2} \left(1 - \ln 2 + \frac{a_k}{2} I_k\right) .\end{aligned} \tag{A7}$$

Here

$$a_q = 1 + \frac{M^2}{2q^2} .$$

The $(t - M^2)^{-2}$ terms contribute

$$\delta D \times \left(\frac{-e^2}{8\pi^2}\right)^{-1} = \frac{2s + M^2 - s_1}{4q^2}\, Q_1\left(1 + \frac{M^2}{2q^2}\right) - \frac{2a + M^2 - s_1}{4k^2}\, Q_1\left(1 + \frac{M^2}{2k^2}\right)$$

$$+2M^2\left\{\left(\frac{a_q}{a_q^{\,2} - 1} - Q_o\left(1 + \frac{M^2}{2q^2}\right)\right)\left(\frac{2s + M^2 - s_1}{16q^4}\right)\ln\left(\frac{s - s_1}{2M^2}\right)\right.$$

$$\left.-\left(\frac{a_k}{q_k^{\,2} - 1} - Q_o\left(\frac{1 + M^2}{2k^2}\right)\right)\left(\frac{2a + M^2 - s_1}{16k^4}\right)\ln\left(\frac{s_1 - a}{2M^2}\right)\right\}$$

$$+ M^2\left[\frac{2a + M^2 - s_1}{16k^4}\left(I_k + a_k \frac{d}{d\gamma_k} I_k\right) - \frac{2s + M^2 - s_1}{16q^4}\left(I_q + a_q \frac{d}{d\gamma_q} I_q\right)\right] \qquad \text{(A8)}$$

Here we used

$$\int \frac{\ln u\, du}{(u + \gamma)^2} = -\frac{d}{d\gamma}\int \frac{\ln u\, du}{u + \gamma} .$$

Finally, we obtain D from A5, A7, and A8.

The reader will appreciate by now that one would be loathe to execute the same procedure for the integration range $[c,\infty]$ arising from the assumed constant inelasticity in D . We therefore assume that the above estimate is multiplied by less than R (less because the lowest energy region is omitted in $[c,\infty]$). Therefore let us take as a crude estimate $(R - 1) \times$ above estimate.

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