RANDOM MATRICES, FREE PROBABILITY, PLANAR ALGEBRAS AND SUBFACTORS.

A. GUIONNET*, V. F. R. JONES†, D. SHLYAKHTENKO‡.

Abstract. Using a family of graded algebra structures on a planar algebra and a family of traces coming from random matrix theory, we obtain a tower of non-commutative probability spaces, naturally associated to a given planar algebra. The associated von Neumann algebras are II_1 factors whose inclusions realize the given planar algebra as a system of higher relative commutants. We thus give an alternative proof to a result of Popa that every planar algebra can be realized by a subfactor.

1. Introduction

It has been apparent for quite some time that there exists a strong connection between subfactors, large random matrices and free probability theory. Perhaps the most clear instance of this connection is that all three theories have an underlying *planar* structure. For example, the standard invariant of a subfactor (i.e., the system of higher relative commutants) is in a natural way a planar algebra [Jon99]. Traces of polynomials in random matrices naturally count certain planar objects ([tH74, BIPZ78, GMS06, MS06, CMŚS07, MŚS07, Zvo97]). Finally, the combinatorics of free probability theory is intimately tied with that of non-crossing (i.e., planar) partitions [Spe94]. Furthermore, techniques from some of these subjects proved useful for applications to others. For example, there are many connections between work in free probability theory and certain computations in the paper [BJ97]. Random matrices and free probability theory were used to construct subfactors [Ră d94, SU02, Pop95, PS03b]. More recently, Mingo and Speicher and Guionnet and Maurel-Segala [Gui06, GMS06, MS06, MŚS07, CMŚS07] have found combinatorial expressions, involving planar diagrams, for the that the large-N asymptotics of moments of polynomials in certain random matrices.

In this paper we exploit for the first time the *graded algebra* coming from a planar algebra P to obtain a subfactor $N \subset M$ whose standard invariant is P. The essential ingredient is a *trace* on the graded algebra coming from free probability/random matrices, whose use in this context was inspired by [Gui06, GMS06, GS08], which promises to be a source of further developments in this direction.

We take the point of view that all of the three subjects mentioned above are intimately related to the notion of a planar algebra. Specifically, the underlying idea is that a planar algebra, endowed with its graded multiplication $\wedge_0$ and trace Tr_0 is a natural replacement for the ring of polynomials occurring in both free probability theory [VDN92] and the theory of random matrices with a potential [Gui06, GMS06, GS08].

To be more precise, a *subfactor planar algebra* (SPA) P will be a graded vector space $P = (P_n, n > 0, P_0^{\pm})$ which is an algebra over the planar operad of [Jon01, Jon99, Jon00] and satisfies certain dimension and positivity conditions outlined in §2. Every extremal finite index subfactor has an SPA as its *standard invariant.*

Given an SPA P, we define the sequence Gr_kP, $k = 0, 1, 2, \ldots$ of complex $*$-algebras with $Gr_kP = \oplus_{n\geq k} P_n$ ($P_{0,+} \oplus \bigoplus_{n\geq 1} P_n$ if $k = 0$) and multiplications $\wedge_k : P_n \times P_m \to P_{n+m-k}$ given by tangles as in §2. On each Gr_kP we define a trace $Tr_k : Gr_kP \to \mathbb{C}$ using the sum of all Temperley-Lieb tangles. The trace Tr_0 comes directly from Wick's Theorem applied to large N limit of a certain Gaussian matrix model using Wishart matrices, defined in §3. But once calculated, this trace can be defined entirely in terms of planar algebras.

A rather important special case is when the matrix models may be taken as p independent Hermitian matrices. Then the algebra Gr_0P is the even degree subalgebra of $\mathbb{C}\langle\{X\}\rangle$, the non-commutative polynomials

*UMPA, ENS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, FRANCE, aguionne@umpa.ens-lyon.fr. A part of this project was supported by the Miller Institute for Basic Research in Science, University of California Berkeley.
† Department of Mathematics, UC Berkeley, Berkeley, CA 94720, vfr@math.berkeley.edu. Research supported by NSF grant DMS-0401734
‡ Department of Mathematics, UCLA, Los Angeles, CA 94720, shlyakht@math.ucla.edu. Research supported by NSF grant DMS-0555680.

in p self-adjoint variables $\{X\}$ (with $\#\{X\} = p$). The trace Tr_0 is then the one discovered by Voiculescu in the context of his free probability theory [VDN92, Voi85, Voi91]. It can be realized as the vacuum expectation value on the full Fock space on a real p-dimensional vector space with basis $\{X\}$, by the representation of $\mathbb{C}\langle\{X\}\rangle$ which sends X to $\ell_X + \ell_X^*$, ℓ_X being the left creation operator of X (see [Voi85, VDN92]). The higher multiplications $\wedge_k$ are then given (on monomials of even degree $\geq 2k$) by

$$(X_1X_2\cdots X_r)\wedge_k(Y_1Y_2\cdots Y_s) = (\prod_{i=1}^{k}\delta_{X_{r-i+1},Y_i})X_1\cdots X_{r-k}Y_{k+1}\cdots Y_s.$$

Our main result is, with notation as above and an SPA P, of index parameter δ,

Theorem. *(i) For each k, tr_k is a faithful tracial state on Gr_kP and the GNS completion of Gr_kP is a II_1 factor M_k as long as $\delta > 1$;*

(ii) There are unital inclusions $Gr_kP \subset Gr_{k+1}P$ which extend to $M_k \subset M_{k+1}$ and projections $\mathbf{e}_k \in Gr_{k+1}P$, such that $(M_{k+1}, \mathbf{e}_k)$ is the tower of basic constructions for the subfactor $M_0 \subset M_1$;

(iii) The relative commutants $M_0' \cap M_k$ are canonically identified with the vector spaces P_k and this identification is a homomorphism of planar $*$-algebras.

This theorem gives a new proof of the breakthrough result of Popa [Pop95], showing that any subfactor planar algebra P can indeed be realized by the system of higher relative commutants of a II_1 subfactor.

The key ingredient in the proofs will be representations of the algebras Gr_kP on Fock spaces. In order to define these we will suppose that P is given as a planar subalgebra of the full planar algebra P^Γ of some bipartite graph $\Gamma = \Gamma_+ \amalg \Gamma_-$ as in [Jon00]. This is always possible — one may for instance take Γ to be the principal graph of P. A basis of P^Γ is formed by loops on Γ starting and ending in Γ_+ but we will define a slightly different planar algebra structure from that of [Jon00], better adapted to graded multiplication.

The Fock space will then be spanned (orthogonally) by paths of varying lengths on Γ, ending in Γ_+. It is naturally $\mathbb{Z}/2\mathbb{Z}$-graded. Note that Γ may be infinite so we need to make a choice of Perron-Frobenius eigenvector and eigenvalue for the adjacency matrix of Γ. There will not necessarily be a Markov trace on P^Γ so we work instead with the center-valued trace. This will restrict to a Markov trace on P.

As in the theory of graph C^*-algebras [Rae05], each edge e of Γ defines an operator ℓ_e (of grading 1) on the Fock space, creating an edge on a path. A loop of edges $e_1\cdots e_{2p}$ in P^Γ is then represented by the product $c(e_1)c(e_2)\cdots c(e_{2p})$ where $c(e_i)$ is a version of $a_i\ell(e_i) + a_i^{-1}\ell(e_i^*)$ according to the parity of i, where the factors a_i are determined by the Perron-Frobenius eigenvector. We also make use of the fact that the Fock space of loops and the resulting II_1 factor can be embedded into a type III factor canonically associated to the graph and its Perron-Frobenius eigenvector using a free version of the second quantization procedure.

The material is organized as follows:

Contents

1.1. **Notations.** To aid the reader, we list here some notation used in the paper.

- Bi-partite graph (§2.4): Γ; vertices: Γ; even/odd vertices: $\Gamma_\pm$; Edges: E; positively/negatively oriented edges: $E_\pm$; edges starting at v: $\Gamma_+(v)$; edges ending at v: $\Gamma_-(v)$. All loops of length k starting at even/odd vertex: $L_k^\pm$; all loops starting at a positive/negative vertex: $L^\pm$.
- Planar algebra (Def. 4): P; k-box space: $P_{k,\pm}$; positive/negative part: $P_\pm$; $P_k = P_{k,+}$. Planar algebra of a graph (§2.4): P^Γ, $P^\Gamma_{k,\pm}$ etc. Subfactor planar algebra: Def. 5.
- Graded multiplications: $\wedge_k$ (Def. 7), the algebra $Gr_k P$. Trace on $Gr_k P$: Tr_k (Def. 8).

2. On planar algebras.

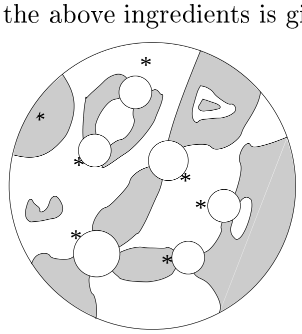
2.1. **Definition.** We begin with a definition of planar algebra which will be recognizably equivalent to other definitions [Jon99] and suited to the purposes of this paper.

Definition 1. (Planar k-tangles.) A planar k-tangle will consist of a smoothly embedded disc D $(= D_0)$ in $\mathbb{R}^2$ minus the interiors of a finite (possibly empty) set of disjoint smoothly embedded discs $D_1, D_2, \ldots, D_n$ in the interior of D. Each disc D_i, $i \geq 0$, will have an even number $2k_i \geq 0$ of marked points on its boundary (with $k = k_0$). Inside D and outside $D_1, D_2, \ldots, D_n$ there is also a finite set of disjoint smoothly embedded curves called strings which are either closed curves or whose boundaries are marked points of the D_i's. Each marked point is a boundary point of some string, and the strings meet the boundaries of the discs transversally, only in the marked points. The connected components of the complement of the strings in $\overset{\circ}{D} \setminus \bigcup_{i=1}^{n} D_i$ are called regions. Those parts of the boundaries of the discs between adjacent marked points (and the whole boundary if there are no marked points) will be called intervals. The regions of the tangle will be shaded black and white so that two regions whose boundaries intersect are shaded differently. (Such a shading is always possible, since there is an even number of marked points.) The shading will be considered to extend to the intervals which are part of the boundary of a region. Finally, to each disc in a tangle there is a distinguished interval on its boundary (which may be shaded black or white).

Definition 2. The set of internal discs of a tangle T will be denoted $\mathcal{D}_T$.

Remark 1. Observe that diffeomorphisms of $\mathbb{R}^2$ act on planar tangles in the obvious way. In particular if Φ is a diffeomorphism it induces a map $\Phi : \mathcal{D}_T \to \mathcal{D}_{\Phi(T)}$

We will often have to draw pictures of tangles. To indicate the distinguished interval on the boundary of a disc we will place a *, near to that disc, in the region whose boundary contains the distinguished interval. An example of a 4-tangle illustrating all the above ingredients is given below;



We will often use pictures with a given number of strings to illustrate a situation where the number of strings is arbitrary. We hope this will not lead to misinterpretation. Similarly if the shading is implicit or both possible shadings are intended we will suppress the shading.

Given planar k and k'-tangles T and S respectively, we say they are composable if

(1) The outside boundary of S is equal to the boundary of one of the inside discs of T where equality means that the marked points are the same, the shadings of the intervals are the same and the distinguished intervals are the same. And,
(2) The union of the strings of S and those of T are smooth curves.

Definition 3. If T and S are composable we define the composition $T \circ S$ to be the union $T \cup S$. The strings of $T \circ S$ are the unions of the strings of T and S.

Since the shadings of T and S agree on their common boundary curve, it is easy to see that $T \cup S$ is a planar k-tangle. This composition operation is often called "gluing" as one may think of S as being glued inside T.

We will now define a notion of planar algebra. Axioms can be subtracted to obtain more general objects but for convenience in this paper the term "planar algebra" will imply all the properties.

Before giving the formal definition we recall the notion of the Cartesian product of vector spaces over an index set $\mathcal{I}$, $\times_{i\in\mathcal{I}} V_i$. This is the set of functions f from $\mathcal{I}$ to the union of the V_i with $f(i) \in V_i$. Vector space operations are point-wise. Multilinearity is defined in the obvious way, and one converts multinearity into linearity in the usual way to obtain $\otimes_{i\in\mathcal{I}} V_i$, the tensor product indexed by $\mathcal{I}$. A Cartesian product over the empty set will mean the scalars.

Definition 4. A (unital) planar algebra P will be a family of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces indexed by the set $\{\mathbb{N} \cup \{0\}\}$, where $P_{k,\pm}$ will denote the $\pm$ graded space indexed by k. To each planar tangle T there will be a multilinear map

$$Z_T : \times_{D\in\mathcal{D}_T} P_D \to P_{D_0}$$

where P_D is the vector space indexed by half the number of marked boundary points of D and graded by $+$ if the distinguished interval of D is shaded white and $-$ if it is shaded black.

The maps Z_T are subject to the following two requirements:

(1) (Isotopy invariance) If φ is an <u>orientation preserving</u> diffeomorphism of $\mathbb{R}^2$ then

$$Z_T = Z_{\varphi(T)}$$

where the sets of internal discs of T and $\varphi(T)$ are identified using φ.

(2) (Naturality)

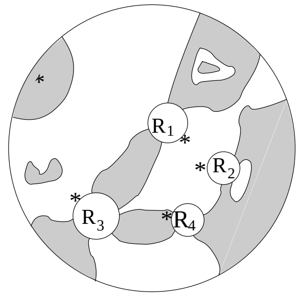
$$Z_{T\circ S} = Z_T \circ Z_S$$

Where the right hand side of the equation is defined as follows: first observe that $\mathcal{D}_{T\circ S}$ is naturally identified with $(\mathcal{D}_T \setminus \{D'\}) \cup \mathcal{D}_S$, where D' is the disc of T containing S. Thus given a function f on $\mathcal{D}_{T\circ S}$ to the appropriate vector spaces, we may define a function $\tilde{f}$ on $\mathcal{D}_T$ by

$$\tilde{f}(D) = \begin{cases} f(D) & \text{if } D \neq D' \\ Z_S(f|_{\mathcal{D}_S}) & \text{if } D = D' \end{cases}$$

Finally the formula $Z_T \circ Z_S(f) = Z_T(\tilde{f})$ defines the right hand side.

The natural notation for $Z_T(f)$ is to write in the $\{f(D),\ D \in \mathcal{D}_T\}$ into $\mathbb{D}$. This is just like the notation "$y(x_1, \ldots, x_n)$" for a function of several variables, where the x_i are the $f(D)$, and the internal discs correspond to the spaces between the commas. (We also call the internal disks "input discs"). Thus if R_1 and R_2 are in $P_{2,+}$, R_3 is in $P_{2,-}$ and R_4 is in $P_{3,+}$ then the following picture is an element of $P_{4,-}$:

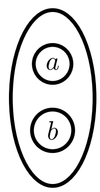


The vector spaces $P_{n,\pm}$ will possess a conjugate linear involution $*$, $x \to x^*$ with the compatibility requirement:

$$Z_T(f^*) = Z_{\Phi(T)}(f \circ \Phi)^*$$

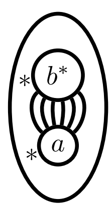
whenever Φ is an orientation reversing diffeomorphism.

Observe that $P_{0,\pm}$ become *unital commutative $*$-algebras* under the multiplication operation (with either shading):

$ab =$ 

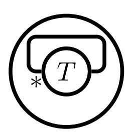
Definition 5. A subfactor planar algebra P will be a planar algebra satisfying the following four conditions:
(a) $\dim(P_{n,\pm}) < \infty$ for all $(n, \pm)$
(b) $\dim(P_{0,\pm}) = 1$
Condition (ii) allows us to canonically identify $P_{0,\pm}$ with $\mathbb{C}$ as *-algebras, 1 being Z (a 0-tangle with nothing in it).

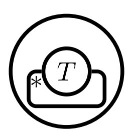
This further allows us to define a sesquilinear form on each $P_{n,\pm}$ by:

$\langle a, b\rangle =$ 

where the outside region is shaded according to $\pm$.

(3) The form $\langle\ ,\ \rangle$ is positive definite.
(4) $Z_{T_1} = Z_{T_2}$ where T_1 and T_2 are the following two 0-tangles:

$T_1 =$ 

$T_2 =$ 

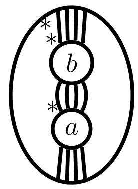
The last condition is topologically natural and corresponds to extremality of the subfactor ([PP86],[Pop94, 1.2.5]). This condition means that the partition function of a fully-labeled zero-tangle (when considered without its boundary disc) is actually well-defined for that zero-tangle on the sphere S^2 obtained by adding a point at ∞ to $\mathbb{R}^2$. It is natural then to suppress the outer disc of a 0-tangle in pictures.

Remark 2. Once $P_{0,\pm}$ have been identified with the scalars there is a canonical scalar δ associated with a subfactor planar algebra with the property that the multilinear map associated to any tangle containing a closed string is equal to δ times the multilinear map of the same tangle with the closed string removed. By positivity $\delta > 0$ and it is well known that in fact the possible values of δ form the set $\{4\cos^2 \pi/n : n = 3, 4, 5, \ldots\} \cup [4, \infty)$ [Jon83].

Remark 3. Since $\delta \neq 0$ it is clear that all the spaces $P_{n,-}$ are redundant and subfactor planar algebra could be axiomatized in terms of $P_{n,+}$. For this reason we will use in what follows P_n to denote $P_{n,+}$ (even in the non-subfactor case).

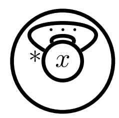
Remark 4. In the development of planar algebras the following structures played a major role:

(1) Multiplication: Each $P_{n,\pm}$ is a *-algebra with the involution defined above and the multiplications:

$ab =$ 

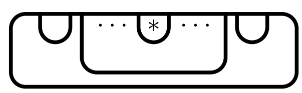
There are two choices of shadings which give in general non-isomorphic algebra structures. (We shall refer to this multiplication sometimes as the "usual" multiplication on $P_{n,\pm}$).

(2) Trace: Each $P_{n,\pm}$ is equipped with a linear map $Tr : P_{n,\pm} \to P_{0,\pm}$ which is given by

$Tr(x) =$ 

(3) The Temperley-Lieb tangles: each tangle consisting of an outside box all of whose $2k$ boundary points are connected by (non-crossing) strings inside of the box determines an element of $P_{k,\pm}$, pairing depending on the shading of the region containing $*$. The set of such tangles is denoted by $TL(k)$.

(4) The Jones projections: $\mathbf{e}_k \in P_{k,+}$ is given by the Temperley-Lieb tangle (having $2k$ boundary points):

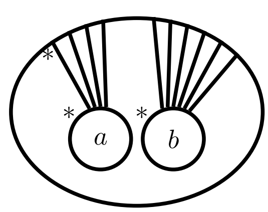
$\mathbf{e}_k =$  .

Remark 5. (Rectangles). It is sometimes very convenient to use rectangles rather than circles for the input and output discs. Strictly speaking this is not allowed since the boundaries are supposed to be smooth. But nothing will happen at the corners of the rectangles so one may simply interpret a picture of a rectangle as one with smoothed corners. Use of horizontal rectangles also makes it possible to avoid specifying the first interval which we will always suppose to be the one containing the left hand vertical part of the rectangle.

Remark 6. (Outer disks and shading). We occasionally omit the outer disk when describing a planar algebra element, especially in the case that there are no boundary points on the outer disk. Also, unless the shading is explicitly indicated in a picture, we follow the convention that the region adjacent to the boundary region marked with a $*$ is unshaded (white).

2.2. **Graded algebra structures.**

Definition 6. If P is a planar algebra we define a graded algebra GrP as follows. As a graded vector space $GrP = \bigoplus_{n=0}^{\infty} P_{n,+}$ and the graded product $\wedge : P_n \times P_m \to P_{m+n}$ is given by the tangle below which puts the element of P_n entirely to the left of the element of P_n:

$a \wedge b =$ 

(The shading in the picture above is determined by saying that the region adjacent to the marked interval on the outer box is unshaded (white); as before, $*$'s denote the marked intervals on the disks). Note that one could also define a dual structure changing $+$ to $-$ and changing the shading in the above figure.

As a graded algebra a subfactor planar algebra is just the free graded algebra on a certain graded vector space as we shall see.

If $\mathcal{P}$ is a subfactor planar algebra let $\mathfrak{M}$ be the 2-sided ideal of $Gr\mathcal{P}$ spanned by all elements of degree 1 or more. $\mathfrak{M}$. Each graded piece of $\mathfrak{M}$ has an innner product as defined above. For each $n \geq 1$ let $\mathfrak{N}_n$ be the orthogonal complement of $(\mathfrak{M}^2)_n$ in $\mathfrak{M}_n$.

Theorem 1. *With notation as above, $Gr\mathcal{P}$ is the free graded algebra generated freely by $\cup_{n=1}^{\infty}\mathfrak{N}_n$.*

Proof. Let $\pi = (\pi_1, \pi_2, ..., \pi_k)$ be an ordered k-tuple of integers with $\pi_i \geq 1$ and $\sum_{j=1}^{k} \pi_k = n$. Then multiplication defines linear maps

$$mult_\pi : \mathfrak{N}_{\pi_1} \otimes \mathfrak{N}_{\pi_2} \otimes ....\mathfrak{N}_{\pi_k} \to \mathcal{P}_n.$$

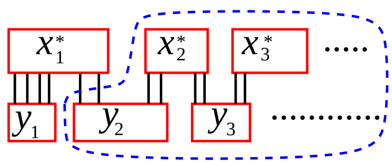
By induction the images of $mult_\pi$ span $\mathfrak{M}_n^2$ as π varies. So, together with $\mathfrak{N}_n$ they span $\mathcal{P}_n$. Thus the theorem follows from the two assertions:
i) Each $mult_\pi$ is injective.
ii) The images of the $mult_\pi$ are orthogonal for different π.

To see i), note that each $mult_\pi$ is an isometry if we give

$$\mathfrak{N}_{\pi_1} \otimes \mathfrak{N}_{\pi_2} \otimes ....\mathfrak{N}_{\pi_k}$$

the Hilbert space tensor product structure.

To see ii), let π and ρ be two distinct partitions of n as above. Suppose $\pi_1 > \rho_1$. Consider the following picture:



This is the inner product of an element $y_1 \otimes y_2 \otimes ...$ in $\mathfrak{N}_{\rho_1} \otimes \mathfrak{N}_{\rho_2} \otimes ....\mathfrak{N}_{\rho_k}$ with an element $x_1 \otimes x_2 \otimes ...$ in $\mathfrak{N}_{\pi_1} \otimes \mathfrak{N}_{\pi_2} \otimes ....\mathfrak{N}_{\pi_k}$. (Here $\pi_1 = 3, \pi_2 = 2$ and $\rho_1 = 2 = \rho_2 = \rho_3$. One may evaluate the tangle inside the dashed curve to obtain an element of $\mathfrak{M}_{\pi_1 - \rho_1}$. Thus the figure is actually the inner product of x_1 with an element of $\mathfrak{M}^2$, thus it is zero. So the images of $mult_\pi$ and $mult_\rho$ are orthogonal unless $\pi_1 = \rho_1$. Continuing in this way we see that the images of $mult_\pi$ and $mult_\rho$ are orthogonal unless $\pi = \rho$. □

Remark 7. Writing elements of $Gr\mathcal{P}$ as sums of products of elements orthogonal to $\mathfrak{M}^2$ times arbitrary elements gives, by an easy argument with generating functions,

$$\Psi_{\mathcal{P}}(z) = 1 - \frac{1}{\Phi_{\mathcal{P}}(z)}$$

Where $\Psi_{\mathcal{P}}(z)$ is the generating function for $\dim(\mathfrak{M}/\mathfrak{M}^2)_n$, and $\Phi_{\mathcal{P}}(z)$ is the generating function for $\dim P_n$. In general if Φ_n is the generating function for the dimensions of the graded vector space $\mathfrak{M}^n/\mathfrak{M}^{n+1}$ we have $\Phi_n = \Phi(\Phi_n - \Phi_{n+1})$ so that $\Phi_n = (1 - 1/\Phi)^n$.

Although the graded algebra structure is not commutative even up to a sign, the presence of the cyclic group action gives a kind of "cyclic commutativity" as follows where ρ denotes the action of the counter-clockwise rotation tangle on P_n:

Proposition 1. *If $\mathcal{P}$ is a planar algebra then*

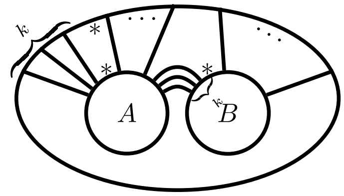
$$\rho^{\deg a}(a \wedge b) = b \wedge a$$

Proof. Just draw the picture. □

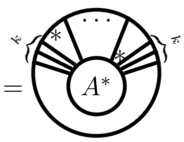
Remark 8. The multiplication in an exterior algebra can be made to satisfy exactly the same commutativity formula by making the cyclic group act by the appropriate sign in each degree.

Besides $Gr\mathcal{P}$ we will need other "shifted" graded *-algebra structures on $\mathcal{P}$ in order to define a subfactor and analyze its tower.

Definition 7. Given a planar algebra $P = (P_n)$ and an integer $k \geq 0$ we make $\oplus_{n=k}^{\infty} P_n$ into an associative (unital) *-algebra with multiplication $\wedge_k : P_m \times P_n \to P_{m+n-k}$ given by the following formula:

$A \wedge_k B =$ 

The involution (denoted by † to distinguish it from the usual involution $*$ on P_k) is given by

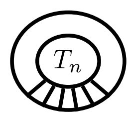
$A^\dagger =$ 

The shading in both figures above is determined by the condition that the marked boundary region $*$ is adjacent to an unshaded (white) region. Here A^* means $\phi(A)$ where ϕ is an orientation-reversing diffeomorphism (cf. Def. 4(2)).

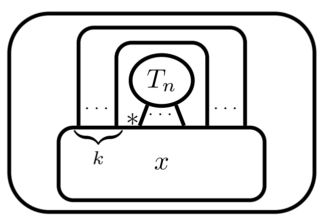
We denote this *-algebra by $Gr_k P$.

2.3. **The traces Tr_k.** Any planar algebra contains in a canonical way the Temperley-Lieb planar algebra TL. Indeed, TL is spanned by TL diagrams: a TL diagram is a diagram that has no inner disks, and all of whose strings connect points on the outer disk. Any such diagram is naturally an element of P.

Definition 8. Let T_n be the sum of all TL diagrams having $2n$ points on the outer disk represented pictorially below (for $n = 3$):



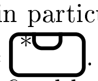
(The position of the $*$ is irrelevant, by since the set of TL diagrams is invariant under a rotation by $2\pi/n$). The trace $Tr_k(x)$ is defined for $x \in P_m$, $m \geq k$, and is valued in the zero box space of P:



where $n = m - k$ (in other words, there are k strings surrounding T_n).

Lemma 1. *Tr_k is a trace on $Gr_k P$ if endowed with the multiplication $\wedge_k$.*

Proof. This follows from the fact that the set of all TL diagrams on $2m$ points is invariant under rotations by $2\pi/m$. □

Before proceeding further, we consider an example. Let us assume that P is a subfactor planar algebra, so that in particular $P_{0,\pm}$ are one-dimensional and Tr_k is scalar-valued. Let $\cup$ be the following element of TL: $\cup =$ . Let us denote by Φ the moment generating function of $\cup$. Thus we let $\Phi(z)$ be the unique scalar defined by

$$\Phi(z) = \sum_{n=0}^{\infty} Tr_0(\underbrace{\cup \wedge_0 \cdots \wedge_0 \cup}_{n \text{ times}}) z^n.$$

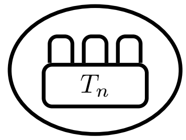
We shall presently compute $\Phi(z)$ by using planar algebra methods.

Definition 9. Let T_n be the element of the planar algebra defined as the sum of all the Temperley Lieb diagrams connecting the $2n$ boundary points,

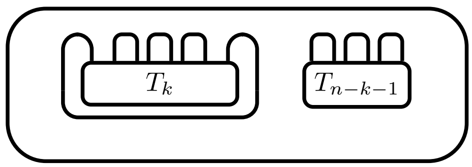
Lemma 2.

$$\Phi(z) = \frac{1-(\delta-1)z}{2z}\left(1-\sqrt{1-\frac{4z}{(1-(\delta-1)z)^2}}\right).$$

Proof. The trace of $\cup^n$ is given by the picture (corresponding to $n=3$):



Group the TL diagrams in T_n according to where the first boundary point of $\cup^n$ is connected. Adding all those diagrams where it is connected to its nearest neighbor we get $\delta Tr_0(\cup^{n-1})$. Proceeding similarly we get, for $k = 1, 2, \ldots, n-1$, contributions of the form:



If the first term in the picture is rotated by one we may use the rotational invariance of T_k to see that it is just $Tr_0(\cup^k)$. Thus we have, for each $n > 0$,

$$Tr_0(\cup^n) = (\delta-1)Tr_0(\cup^{n-1}) + \sum_{k=0}^{n-1} Tr_0(\cup^k)Tr_0(\cup^{n-k-1})$$

Multiplying both sides by z^n and summing from $n=1$ to ∞ we see that

$$\Phi - 1 = z(\delta-1)\Phi + z\Phi^2$$

Solving the quadratic equation and checking the first term to get the right solution we obtain our answer. □

The function Φ in Lemma 2 is that of a free Poisson random variable having R-transform $\delta(1-z)^{-1}$ (see [Voi00, p. 311]). We shall give an alternative computation using free convolution later in the paper (see Lemma 5).

The following lemma can be easily proved by drawing the appropriate pictures:

Lemma 3. *The obvious linear embedding of P_k into $Gr_k P = P_k \oplus P_{k+1} \oplus \cdots$ is an algebra $*$-homomorphism from P_k endowed with its usual $*$-algebra structure of Remark 4, to $Gr_k P$ taken with multiplication $\wedge_k$ and conjugation $\dagger$ as in Definition 7. Moreover, this embedding carries the trace Tr_k to the usual trace Tr on P_k.*

2.4. **The planar algebra of a bipartite graph.** Let $\Gamma = \Gamma_+ \cup \Gamma_-$ be a (locally finite) bipartite graph with adjacency matrix A_Γ possessing an eigenvector $\mu = \mu_v$ (v being a vertex of Γ) with $\mu_v > 0$ for all v and $A_\Gamma \mu = \delta\mu$. Note that although μ may be unbounded as a function of Γ, the ratios $\mu(v)/\mu(v')$ where v and v' are adjacent, are bounded by the eigenvector condition.

We shall denote by E the set of oriented edges of Γ, taken with all possible orientations. Thus $E = E_+ \cup E_-$ where E_+ consists of all edges of Γ oriented so as to start at a vertex in Γ_+ and end at a vertex in Γ_-, and E_- will consists of all edges of Γ oriented so as to start in Γ_- and end in Γ_+. For $e \in E$ we'll denote by e^o the edge with the opposite orientation.

In [Jon00] a planar algebra was associated with the above data with the property that closed strings may be removed multiplicatively as in remark 2. We quickly redo this planar algebra with a slightly different (but isomorphic) structure, emphasizing those elements that arise when Γ is infinite.

With Γ, μ as above we will define the planar algebra $P^\Gamma = P^\Gamma_{n,\pm}$ where $P^\Gamma_{n,\pm}$ is the vector space of bounded functions on loops on Γ of length $2n$ starting and ending in Γ_+ for the plus sign and Γ_- for the minus sign.

Definition 10. (Spin State) Given a planar tangle T, and a bipartite graph Γ as above a *spin state* σ will be a function from the regions of T to the vertices of Γ, shaded regions being mapped to Γ_+ and unshaded ones to Γ_-, together with a function from the strings of T to the edges of Γ such that if a string S is part of the boundary of the regions R_1 and R_2 then $\sigma(S)$ is an edge connecting $\sigma(R_1)$ and $\sigma(R_2)$.

Note that a state σ determines a function $\ell_\sigma : \mathcal{D}_T \cup \{\text{boundary disc}\} \to \{\text{loops on } \Gamma\}$ in the obvious way-if we follow a disc of T around clockwise, the intervals, beginning at the distinguished one, touch regions of T to which σ has assigned vertices of Γ and the strings connected to the marked boundary points of a disc D have been assigned edges of Γ connecting the vertices on either side. We will call $\ell_\sigma(D)$ the loop *induced* on D by σ.

Definition 11. (The curvature factor of a spin state.) Given a tangle and a spin state σ as above, define the curvature factor $c(\sigma)$ as follows. First isotope the tangle so that all discs are horizontal rectangles (with the first boundary interval on the left as in remark 5) and all marked points are on the top edges of the rectangles. Arrange also for all singularities of the y coordinate on the strings to be generic (maxima or minima). Near such a maximum (resp. minimum) we see regions above and below, one of which is convex, labeled by adjacent (on Γ) vertices v_{convex} and v_{concave} according to σ. Assign the number $\sqrt{\frac{\mu(v_{\text{convex}})}{\mu(v_{\text{concave}})}}$ to this singularity. Then the curvature factor is

$$c(\sigma) = \text{product over all maxima and minima of } \sqrt{\frac{\mu(v_{\text{convex}})}{\mu(v_{\text{concave}})}}.$$

Definition 12. (The planar algebra of a bipartite graph.) We now define the action of a planar tangle T on $\mathcal{P}^\Gamma$. We are given a function $R : \mathcal{D}_T \to$ functions on $\{\text{loops on } \Gamma\}$ and we have to define a function on loops appropriate to the boundary of T, in a multilinear way.

So given a loop γ appropriate to the boundary, define

$$Z_T(R)(\gamma) = \sum_\sigma \left\{ \prod_{D \in \mathcal{D}_T} R(D)(\ell_\sigma(D)) \right\} c(\sigma)$$

where the sum runs over all σ which induce γ on the boundary of T.

The main thing to note in this definition is that the sum is finite since there are only a finite number of states inducing γ on the boundary, and it defines a bounded function since all the R are bounded and so is the factor $c(\sigma)$.

We leave it as an exercise to show that this definition of Z_T is compatible with the gluing of tangles and the *-structure where the * of a loop is that loop read backwards. Also that the eigenvector property of μ guarantees that contractible closed strings in tangles can be removed with a multiplicative factor of δ. Also that this planar algebra structure is isomorphic to that of [Jon99], the only change being in how tangles are isotoped in order to define the factor $c(\sigma)$. The reason for the change is that we are mostly dealing with the *graded* algebra for which the isotopy we use is the most natural.

Each of the vector spaces $P^\Gamma_{n,\pm}$ is infinite dimensional if Γ is infinite. Moreover $P^\Gamma_{0,\pm}$ are the abelian von Neumann algebras $\ell^\infty(\Gamma_\pm)$ which act on the $P^\Gamma_{n,\pm}$. (Note that the graded product and the usual product are the same on these subalgebras). The trace tangle when applied to any element of the planar algebra $P^\Gamma_{0,\pm}$ produces an element of $\ell^\infty(\Gamma_\pm)$. We thus get a bilinear conditional expectation $\mathcal{E}$ from $P^\Gamma_{0,\pm}$ (taken with its usual product) onto $\ell^\infty(\Gamma_\pm)$.

The inner product tangles of definition 5 thus become $\ell^\infty(\Gamma_\pm)$-valued inner products, satisfying $\langle a, b\rangle = \mathcal{E}(a^*b)$. It will follow from a representation of the graph planar algebra on a Hilbert space that the conditional expectation $\mathcal{E}$ (and thus the inner product) is non-negative definite.

2.4.1. *Representing the planar algebra of a bipartite graph as loops.* In the next few sections, we shall work out several examples, which make explicit the operations of graded multiplication on P^Γ, and which will be useful in the rest of the paper. All of the facts mentioned below are straightforward consequences of the definition of the graph planar algebra.

We will sometimes use the word "loop" to also mean the planar algebra element given by the delta function on the set of all loops supported on the given loop.

As a matter of convenience, when inserting a loop into an internal disc of a tangle we will line up the edges of the loop with the boundary points of the disc, starting with the one first in clockwise order after $*$. This convention is useful, since given a string meeting the disc in question at a certain boundary point, any state σ which has a nonzero contribution to the sum Z_T of will have to assign the edge of this boundary point to that string.

For an edge e we'll write $s(e)$ for its starting vertex and $t(e)$ for its ending vertex. For a vertex v we'll write $\Gamma_+(v)$ for the set of all edges starting at v (i.e., $\Gamma_+(v) = \{e : s(e) = v\}$), and we'll denote by $\Gamma_-(v)$ the set of all edges that end at v. We'll also use the notation

$$\sigma(e) = \left[\frac{\mu(t(e))}{\mu(s(e))}\right]^{1/2}.$$

Let L_k^+ be the set of all loops of length $2k$ starting at an even vertex, and L_k^- be the set of all loops of length $2k$ starting at an odd vertex.

From now on, fix an integer t and consider the algebra $Gr_t P^\Gamma$ with its graded multiplication $\wedge_t$.

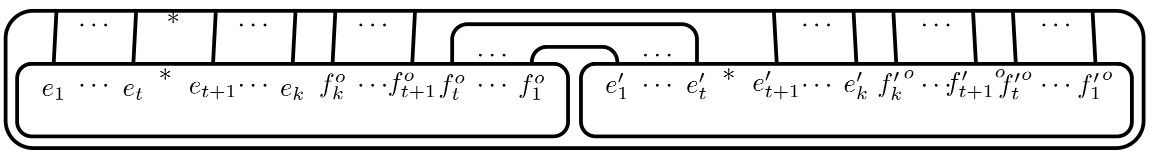
Let $a \in L_k^+$ be a loop,

$$a = e_{t+1} \cdots e_k f_k^o \cdots f_1^o e_1 \cdots e_t$$

where e_j and f_j are edges of Γ. Let

$$b = e'_{t+1} \cdots e'_k {f'_k}^o \cdots {f'_1}^o e'_1 \cdots e'_t$$

Then the graded product $a \wedge_t b$ is given by:



which translates into the following formula:

$$\begin{aligned} a \wedge_t b \quad = \quad & \delta_{s(f_1)=s(e'_1)} \prod_{j=1}^{t} \delta_{f_j = e'_j} \left[\frac{\mu(s(e'_j))}{\mu(t(e'_j))}\right]^{1/2} \cdot \\ & \cdot e_{t+1} \cdots e_k f_k^o \cdots f_{t+1}^o \; e'_{t+1} \cdots e'_{k'} {f'}^o_{k'} \cdots {f'}^o_1 \cdots e_1 \cdots e_t. \end{aligned}$$

Apart from the Perron-Frobenious factors, $a \wedge_t b$ corresponds to a kind of amalgamated concatenation of paths, although the edges of the path are should be cyclically permuted. If for a path $a \in L_k^\pm$ (parity according to t) we denote by $D_t(a)$ the path that starts at the $t+1$-th edge of a, then we have:

$$D_t(a) \wedge_t D_t(b) = \text{const} D_t(c)$$

where c is zero if the last t edges of a do not form the inverse of the path formed by the first t segments of b, and is the concatenation of a (with last t segments removed) and b (with first t segments removed) otherwise.

In particular, if $t = 0$, given two paths a, b in L_k^+ the graded multiplication $\wedge_0$ is just concatenation of paths (note that in this case D_t is the identity map).

The (usual) trace Tr is given by

$$Tr(e_1 \cdots e_k f_k^o \cdots f_1^o) = \prod_j \delta_{e_j = f_j} \sigma(e_j) s(e)$$

(where again the infinite sums are locally finite).

2.4.2. $TL \subset Gr_0 P^\Gamma$. Let us now set $t = 0$ and identify in terms of paths the element of $TL(k) \subset P_k^\Gamma \subset Gr_0 P^\Gamma$ corresponding to any TL picture. Suppose that we are given a box B with $2k$ boundary points (arranged so that all boundary points are at the top and $*$ is at position 0 from the top-left). Assume also that there are k non-crossing curves inside B which connect pairs of boundary points together. Let π be the associated non-crossing pairing. The associated element of the planar algebra is the function w_B on loops, defined on a loop a by:

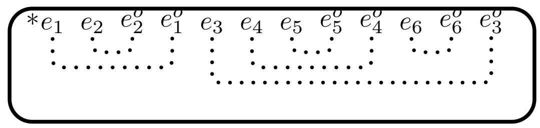
$$w_B(a) = \begin{cases} \sigma(e_1) \cdots \sigma(e_n) & \text{if } e_i = e_j^o \text{ whenever } i \overset{\pi}{\sim} j,\ i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi = \{i_1, j_1\} \cup \cdots \cup \{i_k, j_k\}$ where $i_1 < i_2 < \cdots$ and $i_p < j_p$, then one can think of w_B as the following locally finite sum of delta functions:

$$w_B = \sum_{e_1 \cdots e_{2k} \in L_k^+} \left\{ \prod \delta_{e_{i_p} = e_{j_p}^o} \sigma(e_{i_p}) \right\} (e_1 \cdots e_{2k}).$$

An example of $w_B \in Gr_0 P^\Gamma$ associated to the pairing $\{1,4\}, \{2,3\}, \{5,12\}, \{6,9\}, \{7,8\}, \{9,10\}$ (thus $k = 6$ and $t = 0$) is presented below:

$$w_B = \sum_{\substack{e_1, \ldots, e_6 : \\ e_1 e_2 e_2^o e_1^o e_3 e_4 e_5 e_5^o e_6 e_6^o e_3^o \in L_5^+}} \sigma(e_1) \cdots \sigma(e_6)$$



(The dotted lines are for illustration purposes only and are not part of the planar diagram). In this way, given a $TL(k)$ element B we get an associated element $w_B \in P_{k,+}^\Gamma \in Gr_k P^\Gamma$. This embedding is the canonical inclusion of the Temperley-Lieb planar algebra into P^Γ.

2.4.3. *The center-valued trace Tr_0 on $Gr_0 P^\Gamma$.* As before, we denote by T_k the element

$$T_k = \sum_{B \in TL(k)} w_B$$

obtained by summing over all $TL(k)$ diagrams.

Let $P_{0,\pm}$ be the zero-box space, i.e., as a linear space it is $\ell^\infty(\Gamma_\pm)$. The algebras $P_{0,\pm}^\Gamma$, when considered with the graded multiplication $\wedge_0$, are abelian, and are in the center of $Gr_0 P^\Gamma$. Recall that $\mathcal{E} : P_n^\Gamma \to P_0^\Gamma$ is a P_0^Γ-bilinear map determined by $\mathcal{E}(ab^*) = \langle a, b \rangle$; one can check that $\mathcal{E}(v) = \mu(v) v$, where v denotes the delta function at $\Gamma_\pm$.

The center-valued trace $Tr_0 : Gr_0 P^\Gamma \to P_0^\Gamma$ is given by the equation

$$Tr_0(x) = \langle x, T_k \rangle = \mathcal{E}(x \cdot T_k), \qquad v \in V^+, x \in P_k^\Gamma.$$

Here as before T_k is the sum of all TL diagrams.

Lemma 4. *Let $v \in \Gamma$ and let $\phi_v : Gr_0 P^\Gamma \to \mathbb{C}$ be defined by $\phi_v(x) v = Tr_0(x) \wedge_0 v$ (i.e., the value of $Tr_0(x)$, viewed as a function on Γ). Let $x = e_1 \cdots e_{2k} \in L_k^+$ be a loop. Then if x starts at v,*

$$\phi_v(x) = \sum_{\pi \in NCP(2k)} \prod_{\{i,j\} \subset \pi} \sigma(e_i) \delta_{e_i = e_j^o},$$

where the sum is over all non-crossing pairings of $2k$ integers and the product is taken over all tuples $\{i, j\}$, $i < j$ which are paired by π. If x does not start at v, $\phi_v(x) = 0$.

Furthermore, ϕ_v is uniquely determined by the recursive formula

$$\phi_v(x) = \sum_{x = e x_1 e^o x_2} \sigma(e) \phi_{t(e)}(x_1) \phi_v(x_2)$$

and the formula $\phi_v(e f^o) = \delta_{e=f}\ \delta_{s(e)=v} \sigma(e)$.

We note that although the support of an element $a \in Gr_0 P^\Gamma$, viewed as a function on paths, may not be finite, the support of $a \wedge_0 v$ is always finite, since this element is supported on paths of a fixed length starting and ending at v. Thus the value of ϕ_v is well-defined. Moreover, to know the value of ϕ_v, it is sufficient to know its value on elements of $Gr_0 P^\Gamma$ that have finite support.

Proof. Clearly, the recursive formula gives rise to a uniquely defined linear functional on all finitely-supported elements of $Gr_0 P^\Gamma$ (these elements are, of course, viewed as functions on paths). By the comments above, we shall therefore prove the lemma if we prove that both the functional ϕ_v and the functional

$$\phi_v'(x) = \delta_{s(e_1) = v} \sum_{\pi \in NCP(2k)} \prod_{\{i,j\} \subset \pi} \sigma(e_i) \delta_{e_i = e_j^o}$$

satisfy this recursive relation.

Let $\pi \in NCP(2k)$. Then 1 is paired with some integer q. Thus $NCP(2k) = \sqcup_{q>1} NC\{2,\dots,q-1\} \times NC\{q+1,\dots,2k\}$. Thus

$$\begin{aligned} \phi'_v(x) &= \delta_{s(e_1)=v} \sum_{\pi \in NCP(2k)} \prod_{\{i,j\}\subset\pi} \sigma(e_i)\delta_{e_i=e_j^o} \\ &= \sum_{q>1} \sum_{\substack{\pi_1 \in NCP\{1,\dots,q-1\} \\ \pi_2 \in NCP\{q+1,\dots,2l\}}} \delta_{e_1=e_q^o}\sigma(e_1) \prod_{\{i,j\}\subset\pi_1} \sigma(e_i)\delta_{e_i=e_j^o} \prod_{\{i,j\}\subset\pi_2} \sigma(e_i)\delta_{e_i=e_j^o} \\ &= \sum_{x=ex_1e^ox_2} \sigma(e)\phi'_{t(e)}(x_1)\phi'_v(x_2). \end{aligned}$$

Furthermore, $\phi'_v(ef^o)$ is given by the claimed formula. Thus ϕ'_v satisfies the recursive relation.

We now turn to showing that ϕ_v satisfies the same recursive relation. Note that $\phi_v(x) = 0$ unless x starts at v.

Note that if $x = e_1 \cdots e_{2k}$ and $y = f_1 \cdots f_{2k}$ then $\langle x, y\rangle = 0$ unless $x = y^o$ (an opposite of a path is a path with the order of edges and also all edges reversed). Furthermore, if $x = y^o$, then

$$\langle x, x^o\rangle = s(e_1)\prod_{i=1}^{2k} \sigma(e_i)$$

The set TL of all Temperley-Lieb diagrams can be written as a union

$$TL(2k) = \sqcup_q TL\{2,\dots,q-1\} \times TL\{q+1,\dots,2k\}$$

in a manner similar to decomposing the partitions (q denotes the other endpoint of the string ending at 1). Let us assume that x starts at v. Let us denote by $\widehat{1B_1q}B_2$ the diagram in which 1 is connected to q and $B_1 \in TL\{2,\dots,q-1\}$, $B_2 \in TL\{q+1,\dots,2k\}$. Then

$$\begin{aligned} Tr_0(x) &= \langle x, T_k\rangle = \sum_{B\in TL(2k)} \langle x, w_B\rangle \\ &= \sum_q \sum_{\substack{B_1 \in TL\{1,\dots,q-1\} \\ B_2 \in TL\{q+1,\dots,2k\}}} \langle x, w_{\widehat{1B_1q}B_2}\rangle. \end{aligned}$$

Now, recall that

$$w_B = \sum_{f_1\dots f_{2k}\in L_k^+} \left\{\prod \delta_{e_{i_p}=e_{j_p}^o}\sigma(f_{i_p})\right\} f_1\cdots f_{2k},$$

so that

$$\begin{aligned} w_{\widehat{1B_1q}B_2} &= \sum_{f_1\dots f_{q-1}e\dots f_{2k-1}e^o\in L_k^+} \sigma(e) \prod_{\substack{\{i_p,j_p\}\subset B_1 \\ \text{or } \{i_p,j_p\}\subset B_2}} \delta_{f_{i_p}=f_{j_p}^o}\sigma(f_{i_p})ef_1\cdots f_{q-1}e^of_{q+1}\cdots f_{2k} \\ &= \sum_e \sigma(e)ew_{B_1}e^ow_{B_2}. \end{aligned}$$

Moreover,

$$\langle x, w_{\widehat{1B_1q}B_2}\rangle = 0$$

unless x has the form $x = ex_1e^ox_2$ with x_1 a loop having length $q-2$ and e an edge. In this case,

$$\langle x, w_{\widehat{1B_1q}B_2}\rangle = \langle x_1, w_{B_1}\rangle\langle x_2, w_{B_2}\rangle\sigma(e)\sigma(e^o)\sigma(e) = \langle x_1, w_{B_1}\rangle\langle x_2, w_{B_2}\rangle\sigma(e).$$

Lastly, if $v = s(e)$ then

$$Tr_0(ee^o) = \langle e, e\rangle = \sigma(e)v.$$

It follows that ϕ_v satisfies the same recursive formula as ϕ'_v and, in particular, $\phi_v = \phi'_v$. □

2.4.4. *Examples.* Let us denote by $\cup$ the element $\cup = \sum_{ee^o \in L^+} \sigma(e) ee^o$. Then

$$\begin{aligned} Tr_0(\cup) &= \sum_{e\in E_+} \sum_f \mathcal{E}(ee^o \cdot ff^o)\sigma(e)\sigma(f) \\ &= \sum_e \mathcal{E}(ee^o \cdot ee^o)\left[\frac{\mu(t(e))}{\mu(s(e))}\right] = \sum_e \left[\frac{\mu(t(e))}{\mu(s(e))}\right] s(e) \\ &= \sum_{v\in\Gamma_+} v\frac{1}{\mu(v)} \sum_{s(e)=v} \mu(t(e)) = \sum_{v\in\Gamma_+} \delta v, \end{aligned}$$

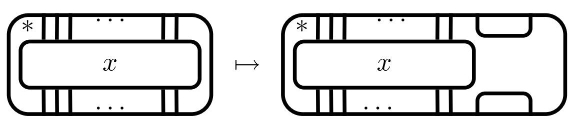
since $\sum_{s(e)=v} \mu(t(e)) = \sum_w \Gamma_{vv}\mu(w) = \delta\mu(v)$.

2.5. **Planar subalgebras of P^Γ.** It is not the planar algebras P^Γ that are of real interest, but some of their planar subalgebras. In particular those with finite dimensional P_n and 1-dimensional $P_{0,\pm}$ for which the inner product is thus scalar valued and inherits positive definiteness from P^Γ.

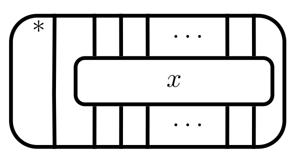
The following theorem, which follows from Popa's work on the theory of λ-lattices (see e.g. Theorem 2.9 (4) in [PS03b]), shows that any subfactor planar algebra is a sub-planar algebra of a planar algebra of a discrete bipartite graph.

Theorem 2. *Let P be an (extremal) subfactor planar algebra, realized as the λ-lattice A_{ij} with principal graph Γ and associated Perron-Frobenius eigenvector μ. Let $\mathcal{A}_i^j$ be as in Theorem 2.9(4) in* [PS03b]. *Then:*
(a) The graph planar algebra P^Γ is the planar algebra of the inclusion $\mathcal{A}_{-1}^{-1} \subset \mathcal{A}_0^{-1}$; in other words, $(\mathcal{A}_{-1}^{-1})' \cap \mathcal{A}_k^{-1} = \mathcal{P}_k^\Gamma$;
(b) The isomorphism $P_{j,+} = A_{-1j} \cong (\mathcal{A}_{-1}^0)' \cap \mathcal{A}_j^{-1} \subset (\mathcal{A}_{-1}^{-1})' \cap \mathcal{A}_j^{-1}$ gives rise to a planar algebra inclusion of P into P^Γ.

The algebras $\mathcal{A}_{-1}^{-1}$ and $\mathcal{A}_0^{-1}$ were constructed in [PS03b] as certain non-unital inductive limits of the algebra A_{ij}. Pictorially, this construction corresponds to e.g. taking $\mathcal{A}_0^{-1}$ to be the inductive limit of the algebras $\{P_k : k \text{ even}\}$ using the non-unital inclusion given by the following picture (the region containing $*$ is unshaded):

$P_k \ni$  $\in P_{k+2}$

The algebra $\mathcal{A}_{-1}^{-1}$ then consists of all diagrams having a vertical through-string on the left (again, region containing $*$ is unshaded):

$\mathcal{A}_{-1}^{-1} = \Big\{$  $\Big\}$

3. A random matrix model for Tr_0

3.1. **Random block matrices associated to the graph.** Given Γ as above, let $\Gamma_\pm$. Let $A = A^+ \oplus A^-$ where $A^\pm = \ell^\infty(\Gamma_\pm) = P_{0,\pm}^\Gamma$. We shall denote by $E_\pm$ the set of edges of Γ which are positively or negatively oriented (according to the sign $\pm$). We shall make the convention that together with any edge $e \in E_\pm$ there is also its opposite edge $e^o \in E_\mp$.

We endow A with a (semi-finite) trace tr given on the minimal projections of A by the formula

$$tr(\delta_v) = \mu(v), \qquad v \in \Gamma.$$

Let N, M be integers. For each choice of M choose integers $\{M_v : v \in \Gamma f\}$ with the property that for each fixed vertex v, $M_v/M \to \mu(v)$ as $M \to \infty$.

In the foregoing, we will consider (infinite if the graph Γ is infinite) matrices whose entries are indexed by the set $\sqcup_{v\in\Gamma}\{1,\ldots,N\} \times \{1,\ldots,M_v\}$. Such an entry will be denoted $A_{ij\ mn\ vw}$, where $i, j \in \{1,\ldots,N\}$,

$m \in \{1, \ldots, M_v\}$, $n \in \{1, \ldots, M_w\}$ and $v, w \in \Gamma$. Given such a matrix $A = (A_{ij\ mn\ vw})$, we compute its trace as follows:

$$tr(A) = \sum_v \frac{1}{N} \sum_{1 \le i \le N} \frac{1}{M} \sum_{1 \le n \le M_v} A_{ii\ nn\ vv}.$$

Our matrices will be such that $A_{ij\ mn\ vw} = 0$ unless v, w belong to a finite set, so that the sum above is finite.

For $v \in \Gamma$ consider the diagonal matrix d_v given by

$$(d_v)_{ij\ mn\ uw} = \delta_{i=j} \delta_{m=n} \delta_{u=w=v}.$$

Note that the joint law of $(d_v : v \in \Gamma)$ converges as $M \to \infty$ to the joint law of $(\delta_v : v \in \Gamma)$.

Consider then for a positively oriented edge $e \in E_+$ from v to w the $NM_v \times NM_w$ matrix X_e defined as follows. The entry $X^e_{ij\ mn\ tu}$ is zero unless $t = v$ and $u = w$. Otherwise, $X^e_{ij\ mn\ vw}$ is (up to scaling) a random Gaussian matrix; in other words, the entries form a family of independent complex Gaussian random variables, each of variance $(\mu(s(e))\mu(t(e)))^{-1/2}(NM)^{-1}$. We shall moreover choose the matrices X_e in such a way that the entries of matrices corresponding to different positively oriented edges are independent. Thus the variables $\{X^e_{ij\ mn\ vw} : e \in E_+, v = s(e), w = t(e), 1 \le i, j \le N, 1 \le m \le M_v, 1 \le n \le M_w\}$ are assumed to be independent.

For a negatively oriented edge f, set $X_f = X^*_{e^o}$. For a loop $w \in L^{\pm}_k, w = e_1 \cdots e_{2k}$, set $X_w = X_{e_1} \cdots X_{e_{2k}}$. Note that $w \mapsto X_w$ is a homomorphism from the algebra $(P^\Gamma, \wedge_0)$ to the algebra of random matrices.

3.2. Tr_0 via random matrices.

Proposition 2. *Let E denote the expected value of a random variable. Then the matrices X_e satisfy: (a) $d_v X_e d_w = \delta_{v=s(e)} \delta_{w=t(e)} X_e$; (b) $E(tr(X^*_e X_e)) = E(tr(X_e X^*_e))$ is independent of N and converges to $(\mu(s(e))\mu(t(e)))^{1/2}$ as $M \to \infty$; (c) For any $v \in V$, $w \in L^{\pm}_k$, $\lim_{M\to\infty} \lim_{N\to\infty} E(tr(d_v X_w)) = tr(\delta_v) Tr_0(w)(v)$ (here $Tr_0(w)(v)$ means the value of the function $Tr_0(w) \in \ell^\infty(\Gamma)$ at $v \in \Gamma$).*

Proof. (a) and (b) are both straightforward; note that

$$E(tr(X_e X^*_e)) = \frac{1}{(\mu(s(e))\mu(t(e)))^{1/2}} \frac{M_v M_w}{M^2} \to (\mu(s(e))\mu(t(e)))^{1/2}.$$

To see (c), we first note that if $w = ee^o$ then $E(tr(X_w)) \to (\mu(s(e))\mu(t(e))^{1/2}$ as $N \to \infty$ and then $M \to \infty$. On the other hand, $tr(Tr_0(w)) = \sigma(e) tr(w) = \sigma(e)\mu(s(e)) = (\mu(s(e))\mu(t(e)))^{1/2}$.

Denote by $\mathcal{E}$ the map the conditional expectation onto the algebra A. Then we have that if $v = s(e)$, $\mathcal{E}(X_e X^*_e)$ is a multiple of d_v. Since $\mathcal{E}$ is tr-preserving, we have

$$E(\mathcal{E}(X_e X^*_e)) = tr(v)^{-1} tr(\mathcal{E}(X_e X^*_e)) \delta_v = \sigma(e) d_v.$$

In particular, we see that

$$E(\mathcal{E}(X_e X^*_f)) = \delta_{e=f}\ \delta_v\ \sigma(e).$$

It is known (see e.g. [BG05, Shl96]) that the variables $\{X_e : e \in \Gamma\}$ converge in distribution (jointly also with elements of A) to a family of A-valued semicircular variables with variance

$$\theta_e : \delta_w \mapsto \delta_{w=v} \delta_v\ \sigma(e).$$

Hence if $w = e_1 \cdots e_{2k}$, then for any $a \in A$,

$$\lim_{N\to\infty} \lim_{M\to\infty} tr(d_v(E(\mathcal{E}(X_w))) = tr\left(\delta_v \sum_{\pi \in NC(2k)} \prod_{\{i,j\} \subset \pi} \delta_{e_i = f_i} \sigma(e)\right).$$

By Lemma 4, we see that

$$tr(E(\mathcal{E}(X_w)) d_v) \to tr(\delta_v) Tr_0(w)(v), \qquad \forall v \in V,$$

as claimed. □

Since the trace tr is positive and faithful on A we conclude that the center-valued trace Tr_0 is non-negative:

Corollary 1. $Tr_0(x^* \wedge_0 x) \ge 0$ *if* $x \in P^\Gamma$.

3.3. **Another construction of random block matrices.** Recall that a bi-partite graph can be used as a Bratteli diagram to describe an inclusion of two algebras.

Let $B \subset C$ be an inclusion of multi-matrix algebras corresponding to the graph Γ. This means $B = \bigoplus_{v\in V_+} M_{k(v)\times k(v)}$ and $C = \bigoplus_{w\in V_-} M_{l(w)\times l(w)}$. In particular, each $v \in \Gamma_+$ corresponds to a central projection p_v in B (the unit of the v-th direct summand), and each $w \in \Gamma_-$ corresponds to a central projection $q_w \in C$. The inclusion $B \subset M$ is such that $p_v q_w = 0$ if there is no edge between v and w. If there are r edges between v and w, then $M_{k(v)\times k(v)} = p_v B p_v$ is included into $q_w C q_w = M_{l(w)\times l(w)}$ with index r. In particular, this means that $l(w) = rk(v)$ and also that we can choose r orthogonal projections $\{P^e\}_{s(e)=v,t(e)=w}$ in $q_w C q_w$ with the property that $P^e q_w C q_w P^e \overset{\phi_e}{\cong} M_{k(v)\times k(v)}$ and the inclusion of $p_v B p_v$ into $q_w C q_w$ is given by $x \mapsto \sum_{s(e)=v,t(e)=w} P^e \phi_j(x) P^e$. Choose also isometries $V_{e,f}$ so that $P^e V_{e,f} = V_{e,f} P^f$.

Let Tr be the semi-finite trace on $B \oplus C$ determined by the requirement that $Tr(p_v) = \mu(v)$, $Tr(q_w) = \mu(w)$.

Let Y be a semicircular element, free from $B\oplus C$ (this only makes sense in the case that $Tr(1) < \infty$; more precisely, we shall consider a large projection Q in the center of $B \oplus C$ and consider an element Y free from $Q(B \oplus C)Q$ with respect to $Tr(Q)^{-1}Tr(\cdot)$; our computations will not depend on Q once it is large enough).

To a positive edge e, we associate: (i) a central projection $p_{s(e)} \in B$; (ii) a projection $P^e \in q_{t(e)} C q_{t(e)} \subset C$.

Let $Y_e = (\mu(t(e))\mu(s(e)))^{-1/4} \sum_{s(f)=s(e),t(f)=t(e)} (p_{s(e)} Y P_e) V_{e,f}$ if $e \in E_+$ and $Y_e = Y^*_{e^o}$ if $e \in E_-$.

Note that $Y_e Y_f = 0$ unless $t(e) = s(f)$. We can think of Y_e as a limit of a "$\mu(s(e)) \times \mu(t(e))$" random block matrix, since its left and right support projections, p_v and q_w, have traces $\mu(s(e))$ and $\mu(t(e))$. In fact, one can model Y by a suitable GUE random matrix in the limit when its size goes to infinity, in which case the variables Y_e are indeed approximated in law by random blocks as their sizes go to infinity.

Furthermore, if $e \in E_+$,

$$\begin{aligned} Tr(p_v Y_e q_w Y_f^*) &= (\mu(t(e))\mu(s(e)))^{-1/2} Tr(p_v p_{s(e)}) Tr(\sum_{e',f'} V_{e,e'} q_w V_{f',f}) \\ &= (\mu(t(e))\mu(s(e)))^{-1/2} \delta_{v=s(e)} Tr(p_v) \delta_{w=t(e)} \delta_{v=f} Tr(q_w). \end{aligned}$$

Thus also $Tr(q_w Y_{f^o} p_v Y^*_{e^o}) = (\mu(t(e))\mu(s(e)))^{-1/2} \delta_{v=s(e)} Tr(p_v) \delta_{w=t(e)} \delta_{v=f} Tr(q_w)$. It follows that if we denote by $\mathcal{E}$ the conditional expectation onto the center of $B \oplus C$, then, keeping in mind that $Tr(\delta_v) = \mu(v)$,

$$\mathcal{E}(Y_e \delta_v Y_f) = (\mu(t(e))\mu(s(e)))^{-1/2} \delta_{e=f^o} \delta_{v=t(e)} \delta_{s(e)} \mu(t(v)) \ = \delta_{e=f^o} \delta_{v=t(e)} \delta_{s(e)} \sigma(e).$$

Thus the variables $\{Y_e : e \in E\}$ have the same joint law as the variables $\{X_e : e \in E\}$ that we constructed in the previous section.

4. The Fock Space Model.

4.1. **A Hilbert bimodule associated to a bi-partite graph.** Let Γ be a bi-partite graph, as before. Consider the real vector space H with basis given by the (oriented) edges E of the graph; we denote, as before, by E_+ the set of positively oriented edges. Then H is equipped with a natural conjugation which takes an edge to its opposite, $e \mapsto e^o$ and can thus be endowed with a complex structure: $i(e + e^o) = (e - e^o)$ for any positively-oriented e. The inner product on H is determined by requiring that $\langle e, f\rangle = 0$ unless $e = f$ and

$$\|e\|^2 = \left[\frac{\mu(s(e))}{\mu(t(e))}\right]^{1/2}.$$

(Note that $e \mapsto e^o$ is *not* isometric). As before, we shall use the notation

$$\sigma(e) = \left[\frac{\mu(t(e))}{\mu(s(e))}\right]^{1/2} = \|e\|^{-2}.$$

Let A denote the abelian algebra $A = \ell^\infty(\Gamma)$, where as before Γ denotes the set of vertices of Γ. Then H is naturally an A, A-bimodule: given e an edge in E, define

$$v \cdot e \cdot v' = \delta_{v=s(e)} \delta_{v'=t(e)} e.$$

Moreover, H has a natural A-valued inner product:

$$\langle e, f^o\rangle_A = \langle e, f^o\rangle s(e) = \langle e, f^o\rangle t(f).$$

4.2. **The operators $c(e)$, the weight ϕ, and the A-valued conditional expectation E.** We now consider the Fock space [Pim97]

$$\mathcal{F} = A \oplus \bigoplus_{k\geq 0} H^{\otimes_A k}$$

(here $\otimes_A$ denotes the relative bimodule tensor product). For $e \in E$ we consider the operator

$$\ell(e) : \mathcal{F} \to \mathcal{F}, \qquad \ell(e)\xi = e \otimes \xi.$$

Its adjoint is given by

$$\ell(e)^*(e_1 \otimes \cdots \otimes e_n) = \langle e, e_1\rangle_A e_2 \otimes \cdots \otimes e_n.$$

Note that the norm of this operator is given by

$$\|\ell(e)\| = \|\ell^*(e)\ell(e)\|^{1/2} = \|e\|^{1/2}.$$

Let also

$$c(e) = \ell(e) + \ell(e^o)^*.$$

Note that $c(e)^* = c(e^o)$.

Let $B(\mathcal{F})$ be the algebra of bounded adjointable operators on $\mathcal{F}$ and let $E : B(\mathcal{F}) \to A$ be the natural conditional expectation given by

$$E(X) = \langle 1_A, X1_A\rangle_A. \tag{1}$$

Each vertex $v \in \Gamma$ determines a state on $B(\mathcal{F})$ given by

$$\phi_v(X) = \delta_v \circ E(X),$$

where $\delta_v : A \to \mathbb{C}$ is the point evaluation at v. Then

$$\phi = \sum_v \phi_v$$

is a weight on $B(\mathcal{F})$. Note that ϕ is finite on all finite words in $c(e) : e \in E$, and therefore defines a semifinite weight on the von Neumann algebra $W^*(c(e) : e \in E)$ (in the GNS representation associated to ϕ).

Lemma 5. *(i) The weight ϕ and the conditional expectation E are faithful on this algebra.*

(ii) The modular group of ϕ is determined by $\sigma_t^\phi(c(e)) = \left[\frac{\mu(t(e))}{\mu(s(e))}\right]^{it} c(e) = \sigma(e)^{2it}c(e)$.

(iii) Consider $\cup = \bigoplus_{v\ even} \sum_{e\in\Gamma_+(v)} \sigma(e)c(e)c(e^o)$, where $\Gamma_+(v)$ denotes the set of all edges that start at v. Then for each v, the law of $\cup$ with respect to ϕ_v has no atoms and is the free Poisson law with R-transform $\delta(1-z)^{-1}$. In particular, $v \cup v$ is bounded for all v and thus the (possibly infinite) direct sum defining $\cup$ yields a bounded operator.

Proof. The GNS vector space $\mathcal{F}_v$ associated to ϕ_v can be identified with the subspace of the Fock space $F(H) = \mathbb{C}v \oplus \bigoplus_{k\geq 1} H^{\otimes k}$ spanned by tensors of the form $e_1 \otimes \cdots \otimes e_n$, $e_j \in E$ for which $e_1 \cdots e_n$ form a path (i.e., are "composable": $s(e_j) = t(e_{j+1})$). If we denote by $\hat{\ell}(e) : F(H) \to F(H)$ the operator $\hat{\ell}(e)\xi = e \otimes \xi$ and by $\hat{c}(e)$ the operator $\hat{c}(e) = \hat{\ell}(e) + \hat{\ell}(e^o)^*$, then we have

$$P\hat{c}(e)P = c(e), \qquad P : F(H) \to \mathcal{F}_v \text{ orthogonal projection.}$$

Let P be the set of all paths in Γ and $P(v)$ be the set of all paths starting at v. For a path $w = e_1 \cdots e_n \in P(v)$, let $c(w) = c(e_1)\cdots c(e_n)$ and similarly for $\hat{c}$. We then see that the joint laws associated to the vacuum expectation state of the variables

$$\{c(w) : w \in P(v)\} \text{ and } \{\hat{c}(w) : w \in P(v)\}$$

have the same law. Indeed, $\hat{c}(w)v = c(w)v$ if $w \in P_v$.

It follows that the von Neumann algebra generated by $(A, c(e) : e \in E)$ in the GNS representation π_v associated to ϕ_v can be embedded into the von Neumann algebra $W^*(\hat{c}(e) : e \in E)$ in such a way that the restriction of the state $\hat{\phi}_v = \langle v, \cdot v\rangle$ to the former algebra is exactly ϕ_v. But it is known [Shl97] that $\hat{\phi}_v$ is faithful, and so ϕ_v is faithful (on the image in the GNS construction π_v). Furthermore, the modular group of $\hat{\phi}_v$ is given by

$$\sigma_t^{\hat{\phi}_v}(\hat{c}(e)) = \left[\frac{\mu(t(e))}{\mu(s(e))}\right]^{it} \hat{c}(e).$$

It follows that

$$\sigma_t^{\phi_v}(\pi_v(c(e))) = \left[\frac{\mu(t(e))}{\mu(s(e))}\right]^{it} \pi_v(c(e)).$$

It is clear that the GNS vector space for ϕ is just the direct sum of the GNS vector spaces for ϕ_v taken over all vertices v. Thus ϕ is faithful and so E is faithful (on the possibly larger algebra $W^*(A, c(e) : e \in E)$. Thus (i) holds.

Let now

$$Y = \sum_{e\in\Gamma_+(v)} \sigma(e)\hat{c}(e)\hat{c}(e^o).$$

Then Y has the same law for $\hat{\phi}_v$ as does $\cup$ for ϕ_v. Note that $Y = \sum_{e\in\Gamma_+(v)} b(e)$, with $b(e) = \sigma(e)\hat{c}(e)\hat{c}(e^o)$. Thus $b(e)$ are free and so the law of Y satisfies

$$\mu_Y = \boxplus_{e\in\Gamma_+(v)}\mu_{b(e)}.$$

Now, for each e, $b(e)^{1/2}$ has free Poisson distribution with R-transform $\mu(t(e))/\mu(s(e))\cdot(1-z)^{-1}$ (see [Shl97, Remark 4.4 on p. 347]). Thus the law of $b(e)$ has only one atom of mass $\alpha(e) = 1 - \mu(t(e))/\mu(s(e))$ at zero (this expression is to be interpreted as zero if it is negative). It follows from additivity of R-transform [VDN92] that the law of Y is free Poisson with R-transform

$$(1-z)^{-1} \sum_{e\in\Gamma_+(v)} \frac{\mu(t(e))}{\mu(v)} = \frac{\delta}{1-z},$$

which will have an atom iff $\delta < 1$. Since $\delta \geq 1$, the law of Y has no atoms. Thus (iii) holds.

Finally, it is also clear that (ii) holds since a similar formula holds in the GNS representation of each ϕ_v and $\phi = \sum \phi_v$. □

Lemma 6. *Let L be the set of all loops in Γ. Then the algebra $W^*(c(w) : w \in L)$ belongs to the fixed point of the modular group acting on the algebra $W^*(c(w) : w \in P)$.*

Proof. Since w is a loop, the factors $\mu(t(e))/\mu(s(e))$ associated to each factor in $c(w)$ cancel. □

4.3. **The conditional expectation E realizes Tr_0.** Let $Y_w = c(w)$, $w \in L_k^+$ (the set of all loops starting at a positive vertex and of length $2k$).

Lemma 7. *Let $w \in L$ be a loop given by $w = e_1 \cdots e_n$. Then $\phi(Y_w) = \sum_{\pi\in NC(2k)} \prod_{\{i,j\}\subset\pi} \delta_{e_i=e_j^o}\sigma(e_i)$. In particular, $E(Y_w) = Tr_0(w)$.*

Proof. $E(Y_w) = \sum_{\pi\in NC(2k)} \prod_{\{i,j\}\subset\pi} \delta_{e_i=e_j^o}\cdot E(Y_{e_i}Y_{e_j})s(e_i)$. Moreover, $E(c(e_i)c(e_i)^*) = s(e_i)\sigma(e_i)$. The rest follows from Lemma 4. □

Lemma 8. *Let L^+ be the set of loops starting at an even vertex. Consider $\mathcal{M}_0 = W^*(c(w) : w \in L)$ with its semi-finite weight ϕ. Then each even $v \in \Gamma$, defines a central projection in $\mathcal{M}_0$ and*

$$(\mathcal{M}_0, \phi) = \bigoplus_{v\ even} (v\mathcal{M}_0 v, \phi_v).$$

For each v, the algebra $v\mathcal{M}_0 v$ can be canonically embedded into a free group factor.

Proof. If $w \in L$ is a loop starting at v, then $v'c(w) = c(w)v' = \delta_{v=v'}c(w)$.

We have seen before that $\pi_v(W^*(c(w) : w \in P))$ with its state ϕ_v can be embedded into a free Araki-Woods factor associated to H and taken with its free quasi-free state, in a state-preserving way. The image of $\mathcal{M}_0$ under π_v is precisely $v\mathcal{M}_0 v$, and this image clearly lies in the centralizer of the free quasi-free state. The free quasi-free state is periodic (the modular group, restricted to $c(H)$ has as its eigenvectors the edges of Γ) and therefore the centralizer is a free group factor. □

Note that $\phi = \bigoplus \phi_v$ is faithful.

Theorem 3. *Let $w \in L^+$ be a loop on Γ, starting at an even vertex. Then the map $w \mapsto c(w)$ extends to a trace-preserving embedding with dense range of $(Gr_0 P^\Gamma, \wedge_0, Tr_0)$ into $(\mathcal{M}_0, E)$. Thus Tr_0 is a faithful center-valued trace.*

Proof. Clearly the theorem is true on elements of P_+^Γ that have finite support, i.e., are finite linear combinations of loops.

We have to check that this embedding makes sense for elements of P_+^Γ which, as functions on loops, have infinite support.

Let $w \in P_k^\Gamma$. Then for any $v \in \Gamma_+$, $\delta_v \wedge_0 w = w \wedge_0 \delta_v = \delta_v \wedge_0 w \wedge_0 \delta_v$ has finite support. Moreover, by assumption $\langle w, w^* \rangle \in P_0^\Gamma = \ell^\infty(\Gamma_+)$ has finite ℓ^∞ norm. But the value of $\langle w, w^* \rangle$ at v is exactly $\|c(\delta_v \wedge_0 w \wedge_0 \delta_v)\|^2_{L^2(\phi_v)}$ and is therefore uniformly bounded as a function of v. Moreover, note that each $c(\delta_v \wedge_0 w \wedge_0 \delta_v)$ belongs to the span of words of length $2k$ in operators $c(e) : e \in E_+$.

The eigenvector condition implies that the ratios $\mu(v)/\mu(w)$ for v, w adjacent are bounded, and also that the valence of the graph is bounded.

It follows that the linear dimension of the space of all loops of length k starting at a vertex v is uniformly bounded, by a constant independent of v. Moreover, the norms of the orthogonal basis for this space (consisting of the various loops) are bounded both above and below uniformly in v. Thus the restrictions of the operator norm and the $L^2(\phi)$-norm to the finite-dimensional linear span of loops of length k starting at v are equivalent, and the constants in the equivalence can be chosen to be uniform in v.

It follows that $vc(w)v$ is uniformly bounded in norm (independent of v).

Since the projections $v : v \in \Gamma_+$ are orthogonal, it follows that $c(w)$, defined as the ultraweakly-convergent sum $\sum vc(w)v$, is a bounded operator in $\mathcal{M}_0$.

Since the map $w \mapsto c(w)$ is bilinear over P_0^Γ, and is an algebra homomorphism when restricted to finite linear combinations of loops, it is easy to see that it is an algebra homomorphism on all of $Gr_0 P^\Gamma, \wedge_0$. $\square$

4.4. **The operator $\cup$.** Let $\Gamma_+(v)$ denote the set of all edges starting at an even vertex v and let E_+ denote the set of all positively oriented edges (i.e., ones that start at an even vertex). Recall that

$$\cup = \sum_{e \in E_+} \sigma(e) c(e) c(e)^*.$$

If we let δ be the Perron-Frobenius eigenvalue, then

$$\begin{aligned}(\cup) &= \sum_{e\in E_+} \sigma(e)\left(\ell(e)\ell(e^o) + (\ell(e)\ell(e^o))^* + \ell(e)\ell(e)^* + \sigma(e)\right)\\ &= 2\sum_{e\in E_+}\sigma(e)\Re(\ell(e)\ell(e^o)) + \sum_v \sum_{e\in\Gamma_+(v)}\left[\frac{\mu(t(e))}{\mu(s(e))}\right] v + \sum_{e\in E_+}\sigma(e)\ell(e)\ell(e)^*\\ &= 2\sum_{e\in\Gamma_+(v)}\sigma(e)\Re(\ell(e)\ell(e^o)) + \delta + \sum_{e\in E_+}\sigma(e)\ell(e)\ell(e)^*.\end{aligned}$$

Here we used

$$\sum_{e\in\Gamma_+(v)} \mu(t(e)) = \sum_j \Gamma_{vj}\mu_j = \delta\mu(v)$$

so that

$$\sum_{e\in\Gamma_+(v)} \frac{\mu(t(e))}{\mu(s(e))} = \frac{1}{\mu(v)} \sum_{e\in\Gamma_+(v)} \mu(t(e)) = \frac{1}{\mu(v)}\delta\mu(v) = \delta.$$

Let $\mathcal{F}_+$ be the set of all vectors in $\mathcal{F}$ starting and ending in a positive vertex and let $A_+ = A \cap \mathcal{F}_+$. Since if $\zeta \in \mathcal{F} \ominus A_+$

$$\sum_{e\in\Gamma_+(v)} \sigma(e)\ell(e)\ell(e)^*\zeta = \zeta,$$

(because of the normalizations of the lengths of e, e^o we have that the sum $\sum_{e\in\Gamma_+(v)}\sigma(e)\ell(e)\ell(e)^*$ is the same as the sum $\sum_f \ell(f)\ell(f^*)$ where the summation is over an orthonormal basis). Thus

$$\cup|_{\mathcal{F}_+} = 2\sum_{e\in\Gamma_+(v)}\sigma(e)\Re(\ell(e)\ell(e^o)) + \delta + (1-P), \tag{2}$$

where $P : \mathcal{F}_+ \to \mathcal{F}_+$ is the projection onto $A_+ \subset \mathcal{F}_+$ and δ is the Perron-Frobenius eigenvalue.

As consequence, we note that we can now identify the position of $\mathcal{A}_v = \pi_v(W^*(Y)) = vW^*(Y)v$ inside of $v\mathcal{F}v = L^2(W^*(c(w) : w \in L), \phi_v)$ identified with a subspace of $\mathbb{C}v \oplus \bigoplus_{k\geq 1} H^{\otimes k}$:

Lemma 9. *Let v be an even vertex. Then $L^2(\mathcal{A}_v)$ is the closed linear span of the orthogonal system of vectors*

$$\xi^{\otimes k} = \left(\sum_{e \in \Gamma_+(v)} \sigma(e) e \otimes e^o \right)^{\otimes k}, \qquad k = 0, 1, \ldots.$$

Moreover,

$$\|\xi^{\otimes k}\|_2^2 = \delta^k,$$

where δ is the Perron-Frobenius eigenvalue.

Proof. We note that $Yv = \xi$. Moreover, the linear span of $\xi^{\otimes k}$ is clearly stable under the action of Y. Thus it is sufficient to prove that if $\xi^{\otimes r}$ for $r < k$ are in $L^2(\mathcal{A})$ then also $\xi^{\otimes k} \in L^2(\mathcal{A})$. But this follows from noting that $Y^k v = \xi^{\otimes k} + \zeta$, where ζ is a tensor of smaller degree in $L^2(\mathcal{A})$.

Furthermore,

$$\langle \xi, \xi \rangle = \sum_{e \in \Gamma_+(v)} \left[\frac{\mu(t(e))}{\mu(s(e))} \right] \|e \otimes e^o\|_2^2 = \sum_{e \in \Gamma_+(v)} \frac{\mu(t(e))}{\mu(v)} = \frac{1}{\mu(v)} \sum_j \Gamma_{vj} \mu_j = \delta.$$

□

4.5. **Relative commutant of $\cup$.** Recall that $L^2(\mathcal{M}_0, \phi) = \bigoplus_v L^2(v\mathcal{M}_0 v, \phi_v) = \bigoplus_v v\mathcal{F}v$. Let as before $Y = \cup$, $\mathcal{A} = W^*(Y)$ and $\mathcal{A}_v = v\mathcal{A}v$.

Lemma 10. *(i) $\mathcal{A}_v$ is a singular MASA in $v\mathcal{M}_0 v$.*
(ii) $W^*(\cup)' \cap \mathcal{M}_0 = \bigoplus_{v \text{ even}} v\mathcal{A}v$.
(iii) Consider the algebra $\mathcal{N}_+ = W^*(c(w) : w$ path in Γ starting and ending at an even vertex$)$. Then $\mathcal{A}' \cap \mathcal{N}_+ = \bigoplus_{v \in \Gamma_+} v\mathcal{A}v$.

Proof. We first note that any $v \in V$ commutes with $Y = \cup$. In particular, $v \in \mathcal{A}' \cap \mathcal{N}_+$. Hence $[Y, x] = 0$ implies that $v[Y, x]w = [Y, vxw] = 0$ for all $v, w \in V$. Hence $\mathcal{A}' \cap \mathcal{N}_+$ is the closure of $\sum_{v,w} (\mathcal{A}' \cap v\mathcal{N}_+ w)$.

Consider the full Fock space $F(H)$ as in the proof of Lemma 5, where H is as before a Hilbert space having as basis edges of Γ. Thus $F(H)$ is spanned by all tensors of the form $e_{i_1} \otimes \cdots \otimes e_{i_m}$, where $e_{i_k} \in E$. Let $\tilde{H} = H \otimes H$, and consider $\tilde{\mathcal{F}} \subset F(H)$ given by $\tilde{\mathcal{F}} = \bigoplus_{k \geq 0} \tilde{H}^{\otimes k}$. Let $T = W^*(\hat{\ell}(e) + \hat{\ell}(e^o)^* : e \in E)$ acting on $F(H)$, and consider the subalgebra $Q \subset T$ given by $Q = W^*([\hat{\ell}(e) + \hat{\ell}(e^o)^*][\hat{\ell}(f) + \hat{\ell}(f^o)^*] : e, f \in E)$. Then clearly $L^2(Q) \subset L^2(P)$ can be identified with $\tilde{F} \subset F(H)$. Furthermore, Q is invariant under the modular group associated to ϕ_v (the vector state associated to the vacuum vector in $\tilde{\mathcal{F}} \subset F(H)$). Thus the modular group of Q is the restriction to Q of the modular group of P.

Fix $v, w \in V$. Denote by λ_v the element $\sum_{e \in \Gamma_+(v)} \sigma(e)(\hat{\ell}(e) + \hat{\ell}(e^o)^*)(\hat{\ell}(e^o) + \hat{\ell}(e)^*) \in Q$. Denote by ρ_w the element $\sum_{e \in \Gamma_+(w)} \sigma(e)(\hat{r}(e) + \hat{r}(e^o)^*)(\hat{r}(e^o) + \hat{r}(e)^*) \in JQJ$ (here $\hat{r}$ denotes the right creation operator and J is the modular conjugation). Note that $\rho_v = J\lambda_v J$.

We now make the identification $U : L^2(v\mathcal{N}_+ w) \hookrightarrow L^2(Q)$ obtained by sending a tensor $e_1 \otimes_A \cdots \otimes_A e_{2n}$ associated to a *path* $e_1 \cdots e_{2n}$ starting at v and ending at w to the tensor $e_1 \otimes \cdots \otimes e_{2n}$. It is not hard to see that

$$\lambda_v U = UY, \qquad \rho_w U = UJYJ.$$

It follows that the laws of ρ_w and λ_v (with respect to the vacuum state on $\tilde{\mathcal{F}}(H)$) are the same as that of Y and have no atoms; thus $W^*(\lambda_v)$ and $W^*(\rho_w)$ are diffuse. In particular, if $\Xi \in L^2(v\mathcal{N}_+ w)$ satisfies $Y\Xi = JYJ\Xi$, then $U\Xi$ satisfies $\lambda_v U\Xi = \rho_w U\Xi$.

Consequently, we would prove (iii) if we could show:
(a) if $v \neq w$, $\lambda_v \zeta = \rho_w \zeta$ for $\zeta \in UL^2(v\mathcal{N}_+ w)$ only occurs if $\zeta = 0$ and
(b) if $v = w$ and $\lambda_v \zeta = \rho_v \zeta$ for some $\zeta \in UL^2(v\mathcal{N}_+ v)$, then $\zeta \in UL^2(v\mathcal{A}v)$.

Let $\xi_v = \sum_{e \in \Gamma_+(v)} \sigma(e) e \otimes e^o$.

Assume first that $u = v$. Let $K_v = \tilde{H} \ominus \mathbb{C}\xi_v$. Put $\mathcal{H}_v = \mathbb{C}\Omega \oplus \mathbb{C}\xi_v \oplus \mathbb{C}\xi_v^{\otimes 2} \oplus \cdots$. Then $\mathcal{H}_v = UL^2(v\mathcal{A}v)$ in such a way that the left and right multiplication by Y on $L^2(v\mathcal{A}v)$ correspond to the actions of λ_v and ρ_v. In particular, $\mathcal{H}_v$ is invariant under both λ_v and ρ_v.

The image of $L^2(v\mathcal{N}_+ v)$ under U lies in the closure of the direct sum

$$\mathcal{H}_v \oplus (\mathcal{H}_v \otimes K_v \otimes \mathcal{H}_v) \oplus (\mathcal{H}_v \otimes K_v \otimes \mathcal{H}_v \otimes K_v \otimes \mathcal{H}_v) \oplus \cdots.$$

(This direct sum is identified with a subspace $\tilde{\mathcal{F}}$ by identifying $\Omega \otimes \zeta$ and $\zeta \otimes \Omega$ with ζ if $\zeta \in F(H)$). Each direct summand in this sum is invariant under both ρ_v and λ_v since their actions respect the tensor product decompositions $\mathcal{H}_v \otimes K_v \otimes \cdots \otimes K_v \otimes \mathcal{H}_v$: ρ_v acts as $\mathrm{id} \otimes \rho_v|_{\mathcal{H}_v}$ and λ_v acts as $\lambda_v|_{\mathcal{H}_v} \otimes \mathrm{id}$.

Now, for any choice of orthonormal basis for $K_v \otimes \mathcal{H}_v \otimes \cdots \otimes K_v$, ζ_α, we have for all $h, g \in \mathcal{H}_v$:

$$\langle h \otimes \zeta_\alpha \otimes g, h' \otimes \zeta_{\alpha'} \otimes g' \rangle = \delta_{\alpha=\alpha'} \langle h, h' \rangle \langle g, g' \rangle$$

and consequently $\mathcal{H}_v \otimes K_v \otimes \cdots \otimes \mathcal{H}_v$ is isomorphic to an (infinite) multiple of $\mathcal{H}_v \otimes \mathcal{H}_v$ as a bimodule over $W^*(\lambda_v)$ acting on the left copy of $\mathcal{H}_v$ and $W^*(\rho_v) = JW^*(\lambda_v)J$ acting on the right copy of $\mathcal{H}_v$. Since the spectral measure of λ_v is non-atomic, it follows that there can be no vector Ξ contained in $\mathcal{H}_v \otimes K_v \otimes \cdots \otimes \mathcal{H}_v$ satisfying $\lambda_v \Xi = \rho_v \Xi$, since such a vector would give rise (via an isomorphism of $\mathcal{H}_v \otimes \mathcal{H}_v$ with Hilbert-Schmidt operators on this space) to a Hilbert-Schmidt operator on $\mathcal{H}_v$, commuting with λ_v.

Thus the only possible Ξ satisfying $\lambda_v \Xi = \rho_v \Xi$ and lying in the image of $UL^2(v\mathcal{N}_+ v)$ must be contained in $\mathcal{H}_v = UL^2(v\mathcal{A}v)$. Thus we have proved (b).

To prove (a), we note that if $v \neq w$, and we let $K_{v,w} = \tilde{H} \ominus (\mathbb{C}\xi_v \oplus \mathbb{C}\xi_w)$, the image of $L^2(v\mathcal{N}_+ w)$ lies in

$$\mathcal{H}_v \otimes \left(\bigoplus_{k \geq 0} \bigoplus_{u_1, \ldots, u_k \in \{v,w\}} K_{v,w} \otimes \mathcal{H}_{u_1} \otimes K_{v,w} \otimes \mathcal{H}_{u_2} \cdots \otimes K_{v_w} \right) \otimes \mathcal{H}_w$$

(once again identified with a subspace of $\tilde{\mathcal{F}}$ as before), which is isomorphic to an infinite multiple of $\mathcal{H}_v \otimes \mathcal{H}_w$ as a bimodule over $W^*(\lambda_v)$ acting on the left copy of $\mathcal{H}_v$ and $W^*(\lambda_w)$ acting on the right copy of $\mathcal{H}_w$. Once again, we see that there can be no vector Ξ satisfying $\lambda_v \Xi = \rho_w \Xi$ in this space. Thus (a) is also proved. Thus we have proved (iii).

Note that we have actually proved that $L^2(v\mathcal{M}_0 v, \phi_v)$ when viewed as a bimodule over $\mathcal{A}_v = W^*(vYv)$ is the direct sum of $L^2(\mathcal{A}_v)$ and an (infinite) multiple of the coarse $\mathcal{A}_v, \mathcal{A}_v$-bimodule $L^2(\mathcal{A}_v) \bar{\otimes} L^2(\mathcal{A}_v)$. Recall (see e.g. [PS03a] or [FM77]) that the normalizer of $\mathcal{A}_v$ is contained in its quasi-normalizer $\mathcal{NQ}(\mathcal{A}_v)$, which consists of those elements ζ in $v\mathcal{M}_0 v$ for which the associated bimodule $\overline{\mathcal{A}_v \zeta \mathcal{A}_v}^{L^2}$ is "discrete". This bimodule cannot be discrete if it contains a sub-bimodule isomorphic to a compression of the coarse $\mathcal{A}_v, \mathcal{A}_v$-bimodule. Thus the only $\zeta \in \mathcal{QN}(\mathcal{A}_v)$ must lie in $L^2(\mathcal{A}_v)$ and thus in $\mathcal{A}_v$. It follows that the normalizer of $\mathcal{A}_v$ in $v\mathcal{M}_0 v$ is contained in $\mathcal{A}_v$. Thus $\mathcal{A}_v$ is a singular MASA and so (i) follows. Now (ii) easily follows from (i). □

4.6. **The operator $\Cup$, relative commutant of $W^*(\cup, \Cup)$ and factoriality.** We now consider the following sum

$$\Cup = \sum_{eff^o e^o \in L^+} \left[\frac{\mu(t(f))}{\mu(s(e))} \right]^{1/2} c(e)c(f)c(f)^* c(e)^*$$

taken over all loops that start at an even vertex. The pictorial representation of this planar algebra element is:

$\Cup =$ .

Lemma 11. *Let v be a fixed even vertex. Assume that there is a path of length 2 from v to v not of the form $ee^o ff^o$. Then algebra $vW^*(\cup, \Cup)v$ has a trivial relative commutant inside of the algebra $v\mathcal{N}_+ v$, where $\mathcal{N}_+ = W^*(c(w) : w$ path in Γ starting and ending at an even vertex$)$.*

Proof. Because of Lemma 10, we know that the relative commutant of $vW^*(\cup, \Cup)v$ inside of $v\mathcal{N}_+ v$ is contained in $vW^*(\cup)v = \mathcal{A}_v$.

Let $\eta = e_1 \otimes f_1 \otimes f_1^o \otimes e_1^o$ where $f_1 \neq e_1^o$ and $e_1 f_1 f_1^o e_1^o$ is a path from v to v.

Set

$$Z = v \Cup v = \sum_{e, f^o} \sigma(e)\sigma(f)c(e)c(f)c(f^o)c(e^o),$$

where the sum is over all paths eff^oe^o from v to v. Then if $k, l > 0$, and $\xi \in L^2(vW^*(\cup)v) = L^2(\mathcal{A}_v)$ is as in Lemma 9, we have:

$$\begin{aligned}\langle \eta\otimes\xi^{\otimes k}, [Z,\xi^{\otimes l}]\rangle &= \langle \eta\otimes\xi^{\otimes k}, \sum_{e,f}\sigma(e)\sigma(f)c(e)c(f)c(f^o)c(e^o)\xi^{\otimes l} - \sigma(e)\sigma(f)\xi^{\otimes l}c(e)c(f)c(f^o)c(e^o)\rangle \\ &= \langle \eta\otimes\xi^{\otimes k}, \sum_{e,f}\sigma(e)\sigma(f)e\otimes f\otimes f^o\otimes e^o\otimes\xi^{\otimes l} \\ &\quad + \sum_{e,f}\sigma(e)\sigma(f)\frac{\mu(t(e))^{1/2}}{\mu(v)^{1/2}}e\otimes f\otimes f^o\otimes\xi^{\otimes(l-1)}\rangle.\end{aligned}$$

Thus

$$\langle \eta\otimes\xi^{\otimes k}, [Z,\xi^{\otimes l}]\rangle = \begin{cases}\frac{\mu(t(f_1))^{1/2}}{\mu(v)^{1/2}}, & l=k\\ \frac{\mu(t(f_1))^{1/2}}{\mu(v)^{1/2}}\cdot\frac{\mu(t(e_1))^{1/2}}{\mu(v)^{1/2}}, & l=k+1\\ 0, & \text{otherwise.}\end{cases}$$

Thus if we consider

$$a = \sum \alpha_k \xi^{\otimes k} \in L^2(\mathcal{A}_v)$$

and assume that $[a, Z] = 0$ and $a \perp \mathbb{C}v$ (so that $\alpha_0 = 0$), we obtain:

$$\begin{aligned}0 &= \langle \eta\otimes\xi^{\otimes k}, [Z,a]\rangle\\ &= \frac{\mu(t(f_1))^{1/2}}{\mu(v)^{1/2}}\left(\alpha_k + \frac{\mu(t(e_1))^{1/2}}{\mu(v)^{1/2}}\alpha_{k+1}\right), \qquad k\geq 1.\end{aligned}$$

Since the choice of e_1 was arbitrary, we find that

$$\alpha_k = -\frac{\mu(t(e))^{1/2}}{\mu(v)^{1/2}}\alpha_{k+1}, \qquad \forall e\in\Gamma_+(v).$$

If $a \neq 0$, not all α_k are zero; from this recursive relation we deduce that $\mu(t(e))$ are all equal to the same value, μ', independent of $e \in \Gamma_+(v)$ and that (after rescaling a by a non-zero constant) we may assume that $\alpha_{k+1} = (-1)^k\lambda^{-(k+1)}$ where $\lambda = (\mu(t(e))/\mu(v))^{1/2} = (\mu'/\mu(v))^{1/2}$.

On the other hand,

$$\sum \Gamma_{vj}\mu' = \delta\mu(v)$$

so that

$$(\sum\Gamma_{vj})\mu'/\mu(v) = (\sum\Gamma_{vj})\lambda^2 = \delta.$$

Thus if $N \geq 1$ is the valence of Γ at v, we find that $N\lambda^2 = \delta$, so that $\lambda^2 = \delta/N$.

Using the fact that $\|\xi\|_2^2 = \delta$, we compute:

$$\|a\|_2^2 = \sum_k |\alpha_k|^2\|\xi^{\otimes k}\|_2^2 = \sum_k \lambda^{-2k}\delta^k = \sum_k (N/\delta)^k\delta^k = \sum_k N^k = \infty,$$

which is impossible. Thus $[Z, a] = 0$ forces $a \in \mathbb{C}v$. □

Lemma 12. *Let Γ be a connected bi-partite graph with $N+1$ vertices, $v \in \Gamma$ even and assume that the hypothesis of Lemma 11 is not satisfied. Then the remaining vertices $e_1, \ldots, e_N$ of Γ are all connected to v by a single edge, and Γ has no other edges.*

We can now prove that the relative commutant of $W^*(\cup, \Cup)$ can be controlled, if the graph Γ is not too small. The cases we exclude are A_1 (a graph with a single vertex and no edges) and A_2 (a graph with exactly two vertices connected by a single edge). In these cases, $\cup$ and $\Cup$ commute (in fact, they are equal). In either of these cases, the Perron-Frobenius eigenvalue is 1, which is of little interest to us.

Theorem 4. *Assume that $\Gamma \neq A_2$ and $\Gamma \neq A_1$ and let $v \in \Gamma$ even. Then (i) the relative commutant $(vW^*(\cup,\Cup)v)' \cap v\mathcal{M}_0v$ is trivial. In particular, the center of $\mathcal{M}_0$ is the algebra $A_+ = \ell^\infty$(even vertices). (ii) $W^*(\cup,\Cup)' \cap \mathcal{N}_+ = P_0^\Gamma$ (where $\mathcal{N}_+ = W^*(c(w)$: w path that starts and ends at an even vertex), and $P_0^\Gamma = \bigoplus_{v\ even} v\mathbb{C}$).*

Proof. Because of Lemma 11 and Lemma 12, it remains to consider the case in which Γ is a graph with $N+1>3$ vertices $v, e_1, \ldots, e_N$ with a single edge between v and each e_j and no other edges. Since $\Gamma = [1 \cdots 1]$, we find that $\|\Gamma\| = N$ and therefore $\delta = N$. Moreover, one can normalize the Perron-Frobenius eigenvector to be $\mu(e) = 1$ for all $e \in \{v, e_1, \ldots, e_n\}$.

Thus

$$\xi = \sum_j e_j \otimes e_j^o, \qquad Z = v \Cup v = \sum_i c(e_i)c(e_i^o)c(e_i)c(e_i^o).$$

Let $k > 1$. Then

$$\begin{aligned}
[Z, \xi^{\otimes k}] &= \sum_i c(e_i)c(e_i^o)c(e_i)c(e_i^o)\xi^{\otimes k} - \xi^{\otimes k} c(e_i)c(e_i^o)c(e_i)c(e_i^o) \\
&= \xi^{\otimes k-2} + 4\xi^{\otimes k-1} + 6\xi^{\otimes k} + 3\xi^{\otimes k+1} + \sum_i e_i \otimes e_i^o \otimes e_i \otimes e_i^o \otimes \xi^{\otimes k-1} + \sum_i e_i \otimes e_i^o \otimes e_i \otimes e_i^o \otimes \xi^{\otimes k} \\
&\quad - \xi^{\otimes k-2} - 4\xi^{\otimes k-1} - 6\xi^{\otimes k} - 3\xi^{\otimes k+1} - \sum_i \xi^{\otimes k-1} \otimes e_i \otimes e_i^o \otimes e_i \otimes e_i^o + \sum_i \xi^{\otimes k} e_i \otimes e_i^o \otimes e_i \otimes e_i^o \\
&= \sum_i e_i \otimes e_i^o \otimes e_i \otimes e_i^o \otimes (\xi^{\otimes k} + \xi^{\otimes k-1}) - (\xi^{\otimes k} + \xi^{\otimes k-1}) \otimes \sum_i e_i \otimes e_i^o \otimes e_i \otimes e_i^o \\
&= \zeta \otimes (\xi^{\otimes k} + \xi^{\otimes k-1}) - (\xi^{\otimes k} + \xi^{\otimes k-1}) \otimes \zeta,
\end{aligned}$$

where we have set $\zeta = \sum_i e_i \otimes e_i^o \otimes e_i \otimes e_i^o$.

Let $\eta = e_1 \otimes e_1^o \otimes e_2 \otimes e_2^o$. Then $\eta \otimes \xi^{\otimes l} \perp \zeta \otimes \xi^{\otimes k}$ for all l, k. On the other hand, $\langle \eta \otimes \xi^{\otimes l}, \xi^{\otimes k} \otimes \zeta \rangle = \delta_{k=l} \|\xi^{\otimes k-1}\|$.

It follows that for any $k > 1$,

$$\begin{aligned}
\langle \eta \otimes \xi^{\otimes l}, [Z, \xi^{\otimes k}] \rangle &= \langle \zeta \otimes (\xi^{\otimes k} + \xi^{\otimes k-1}) - (\xi^{\otimes k} + \xi^{\otimes k-1}) \otimes \zeta, \eta \otimes \xi^{\otimes l} \rangle \\
&= - \langle \xi^{\otimes k} \otimes \zeta, \eta \otimes \xi^{\otimes l} \rangle - \langle \xi^{\otimes k-1} \otimes \zeta, \eta \otimes \xi^{\otimes l} \rangle \\
&= - \delta_{l=k} \|\xi^{\otimes (l-1)}\| - \delta_{l=k-1} \|\xi^{\otimes (l-1)}\|.
\end{aligned}$$

It follows that if $a = \sum \alpha_k \xi^{\otimes k} \in L^2(\mathcal{A}_v)$, and we assume that $[Z, a] = 0$, then we get for all $l \geq 2$:

$$\begin{aligned}
0 = \langle [Z, a], \eta \otimes \xi^{\otimes l} \rangle &= \sum_k \alpha_k \langle [Z, \xi^{\otimes k}], \eta \otimes \xi^{\otimes l} \rangle \\
&= -\alpha_l \|\xi^{\otimes (l-1)}\| - \alpha_{l+1} \|\xi^{\otimes (l-1)}\|.
\end{aligned}$$

It follows that α_k, $k \geq 2$, is a constant sequence. But the sequence $\{\alpha_k\}$ is in L^2 and thus must be zero.

It follows that $a \in L^2(\mathcal{A}_v)$ commutes with Z, then $a = \alpha_0 1 + \alpha_1 \cup$. But in this case, $[a, Z]\Omega = \alpha_1 [\cup, Z]\Omega$. The only tensors of degree 6 in $\cup Z \Omega$ are $\xi \otimes \zeta$, and the only terms of this degree in $Z \cup \Omega$ are $\zeta \otimes \xi$, which are not equal. Thus $[a, Z] = 0$ implies that also $\alpha_1 = 0$ and so a must be a scalar.

To see (ii), note first that any $v \in \Gamma_+$ is a projection in the relative commutant of $W^*(\cup, \Cup)' \cap \mathcal{N}_+$. Since the projections corresponding to different vertices are orthogonal, it follows that any element x in the relative commutant is weakly-convergent infinite sum $\sum_{v \in \Gamma_+} vxv$, where $vxv \in vW^*(\cup, \Cup)'v \cap v\mathcal{N}_+ v = \mathbb{C}v$. □

4.7. **Factoriality of M_0.** Let P be an (extremal) subfactor planar algebra embedded into the planar algebra of some graph Γ. Thus $(Gr_0 P, Tr_0)$ can be viewed as a subalgebra of $(Gr_0 P^\Gamma, Tr_0) \subset \mathcal{M}_0$. Moreover, $TL(1), TL(2) \subset Gr_0 P$, and so $\cup$ and $\Cup$ both belong to $Gr_0 P$. Therefore, the center of $W^*(Gr_0 P, Tr_0)$ is contained in the relative commutant of $W^*(\cup, \Cup)$ inside of $\mathcal{M}_0$. By Theorem 4, this relative commutant is the intersection of the algebra A_0 identified with the zero box space in P^Γ.

Lemma 13. *Assume that the zero-box space of P is one-dimensional. Then $W^*(Gr_0 P, Tr_0) \cap A_+ = \mathbb{C}1_{A_+}$.*

Proof. Note that tr_0 is the restriction to $W^*(Gr_0 P, \wedge_0, Tr_0)$ of the conditional expectation E from $\mathcal{M}_0$ onto A_+ (which is the center of $\mathcal{M}_0$). Since this conditional expectation is normal, if $z \in W^*(Gr_0 P, Tr_0) \cap A_+$, then $z = E(z)$. On the other hand, z is the limit (in the weak-operator topology) of some sequence $z_i \in Gr_0 P$. For each i, $E(z_i) = Tr_0(z_i)$ belongs to the zero-box space of P_+. Since $\cup \in P$ and the zero-box space is one-dimensional $E(z_i)$ must be a multiple of $\mathcal{E}(\cup) = \delta \mathbb{C} 1_{A_+}$ and hence $z = E(z) \in \mathbb{C}1_{A_+}$. □

Thus if the zero box space of P is one-dimensional, and since $W^*(\cup)$ is diffuse, we automatically get:

Theorem 5. *Let P be a planar algebra with one-dimensional zero box space and of index $\delta > 1$. Then $M_0 = W^*(Gr_0 P, Tr_0)$ is a type II_1 factor.*

Since $M_0 \subset \mathcal{M}_0$, we see by Lemma 8 that M_0 can be actually embedded into a direct sum of free group factors. In particular, M_0 has the Haagerup property and is R^ω-embeddable.

5. Higher relative commutants.

5.1. **The algebra $\mathfrak{M}_1$ and the trace ϕ_1.** We now proceed to define the algebra $M_1 = W^*(Gr_1 P, Tr_1)$, which will contain $M_0 = W^*(Gr_0 P, Tr_0)$ as a subfactor.

Let us denote by $\mathfrak{M}_0$ the image of the algebra $Gr_0 P$ inside $M_0 \subset \mathcal{M}_0$ acting on the Fock space $\mathcal{F}$ as in the previous section.

We first recall from Section 2 that if we identify elements of P^Γ with paths, then the multiplication $\wedge_1$ on GrP_1^Γ can be expressed as follows. Let $w = e_1 \cdots e_n$ and $w' = e'_1 \cdots e'_m$ be two paths. Denote by $D_1(w)$ the path obtained from w by following the path w, but starting at the first point of w (rather than its starting point). Then

$$D_1^{-1}(D_1(w) \wedge_1 D_1(w')) = \sigma(e_n)^{-1} \delta_{e_n = e'_1} e_1 \cdots e_{n-1} e'_2 \cdots e'_m.$$

(note that the factor $\sigma(e_n)^{-1}$ is exactly the norm $\|e_n\|^2$).

To a path $w = e_1 \cdots e_n = e_1 w_0 e_n$, where $w_0 = e_2 \cdots e_{n-1}$ we associate the variable

$$c_1(w) = \ell(e_1) c(w_0) \ell(e_n^o)^* \in B(\mathcal{F}).$$

Lemma 14. $Y^{(1)}_{D_1^{-1}(w)} Y^{(1)}_{D_1^{-1}(w')} = Y^{(1)}_{D_1^{-1}(w \wedge_1 w')}$.

Proof. This follows from the relation $\ell(e)^* \ell(g) = \delta_{e=g} \|e\|^2$. □

Let us introduce the notation

$$\mathfrak{M}_1 = \operatorname{span}\{c_1(w) : w \in L_-\}$$

where L_- is the set of all loops starting at an odd vertex.

The vector space $\mathfrak{M}_1$ is an algebra with multiplication $\wedge_1$. Thus $w \mapsto c_1(D_1^{-1}(w))$ is a $*$-homomorphism from $Gr_1 P^\Gamma$ onto $\mathfrak{M}_1$. The unit of $\mathfrak{M}_1$ is the element

$$\sum_{e \in E_-} \sigma(e) \ell(e) \ell(e)^*$$

(here E_- is the set of all odd edges, i.e., ones *ending* at an even vertex).

Lemma 15. *Let E_- be the set of all odd edges. Then the map*

$$i : Y \mapsto \sum_{e \in E_-} \sigma(e) \ell(e) Y \ell(e)^*$$

defines a unital $*$-homomorphism from the algebra $\mathfrak{M}_0$ to the algebra $\mathfrak{M}_1$.

Proof. We note that

$$\begin{aligned} i(Y_w) \cdot i(Y_{w'}) &= \sum_{e \in E_-} \sigma(e)^2 \|e\|^2 \ell(e) Y_w Y_{w'} \ell(e)^* \\ &= \sum_{e \in E_-} \sigma(e) \ell(e) Y_w Y_{w'} \ell(e)^*. \end{aligned}$$

Thus i is a homomorphism. Moreover, i is clearly $*$-preserving. □

We now define a tracial weight ϕ_1 on $\mathfrak{M}_1$:

$$\phi_1(\ell(e) Y_w \ell(f)^*) = \delta^{-1} \delta_{e=f} \sigma(e)^{-1} \phi(Y_w)$$

(the first δ is the Perron-Frobenius eigenvalue; note that $e = f$ forces $Y_w \in \mathfrak{M}_0$). In other words,

$$\phi_1(X) = \delta^{-1} \sum_{f \in E_-} \sigma(f)^{-1} \phi(\langle f, Xf \rangle_A).$$

The last observation shows that $\phi_1(X)$ is a non-negative functional.

Moreover, for any $v \in \mathfrak{M}_0$,

$$\begin{aligned}\phi_1(i(v)) &= \delta^{-1} \sum_{f\in\Gamma_-(v)} \sigma(f)^{-1}\|f\|_2^2 \\ &= \delta^{-1} \sum_{f\in\Gamma_-(v)} \left(\frac{\mu(s(f))}{\mu(t(f))}\right) = \delta^{-1}\sum_j \Gamma_{jv}\frac{\mu(j)}{\mu(v)} = \delta^{-1}\delta = 1.\end{aligned}$$

(here $\Gamma_-(v)$ denotes the set of edges ending at v).

Finally, ϕ_1 is a trace, since if w, w' are two loops of the form $e\hat{w}f^o$ and $e'\hat{w}'f'^o$ with $s(e) = t(f^o)$, $s(e') = t(f'^o)$ then

$$\begin{aligned}\phi_1(ww') &= \delta_{e=f'}\delta_{f=e'}\delta^{-1}\|e\|^4\|f\|^2\phi(\hat{w}\hat{w}') \\ &= \delta_{e'=f}\delta_{f'=e}\delta^{-1}\|e'\|^6\phi(\hat{w}'\hat{w}) = \phi_1(w'w)\end{aligned}$$

since $e = f'$ and $\|f'\| = \|e'\|$.

We finally note that

$$\begin{aligned}\phi_1(i(Y_w)) &= \sum_e \delta^{-1}\left(\frac{\mu(s(e))}{\mu(t(e))}\right)\phi(w) \\ &= \phi(w)\end{aligned}$$

because μ is an eigenvector for the graph matrix. We summarize these observations as the following

Lemma 16. *The weight ϕ_1 is a semifinite faithful trace, and the inclusion $i : (\mathfrak{M}_0, \phi) \to (\mathfrak{M}_1, \phi_1)$ is trace-preserving.*

Proof. Let us consider $Y = \sum_{g,h\in E_-} \ell(g)x_{g,h}\ell(h)^*$, $x_{g,h} \in W^*(c(e) : e \in \Gamma)$ with Y^*Y in the domain of ϕ_1. Then $Y^*Y = \sum_{g,h,g'} \ell(g)x_{g,h}x^*_{g',h}\ell(g')^*\|h\|^2$. Moreover,

$$\phi_1(YY^*) = \delta^{-1}\sum_{g,h}\|h\|^2\|g\|^4\phi(x_{g,h}x^*_{g,h})$$

and $x_{g,h}x^*_{g,h} \in \mathcal{M}_0$. Thus if $\phi_1(Y^*Y) = 0$, each of the positive terms in the sum above must be zero and so $\phi(x_{g,h}x^*_{g,h}) = 0$ for all g, h. It follows that $Y = 0$. □

Define now the map $E_1 : \mathfrak{M}_1 \to \mathfrak{M}_0$ by

$$E_1(\ell(e)c(w)\ell(f)^*) = \delta_{e=f}\delta^{-1}\left(\frac{\mu(s(e))}{\mu(t(e))}\right)c(w).$$

Note that

$$E_1(i(Y)) = Y$$

and moreover

$$\begin{aligned}E_1(i(c(w))\ell(e)c(w)\ell(f)^*) &= Y_w E_1(\ell(e)c(w')\ell(f)^*) \\ E_1(\ell(e)c(w)\ell(f)^* i(c(w)')) &= E_1(\ell(e)c(w)\ell(f)^*)c(w')\end{aligned}$$

so that $i \circ E_1 : \mathfrak{M}_1 \to i(\mathfrak{M}_0) \subset \mathfrak{M}_1$ is an $\mathcal{A}_0$-linear projection. Moreover, we see that $\phi_1 = \phi\circ(i\circ E_1)$ so that $\mathcal{E}_1 = i\circ E_1$ is the trace-preserving conditional expectation from $\mathfrak{M}_1 \to \mathfrak{M}_0$. It follows that $\mathcal{E}_1$ extends also to the von Neumann algebra generated by $\mathfrak{M}_1$.

5.2. **The algebras $\mathfrak{M}_n$ and traces ϕ_n.** The algebras $\mathfrak{M}_n$ with semi-finite traces ϕ_k are defined in a similar way. The algebra $\mathfrak{M}_n$ is the linear span

$$\mathfrak{M}_n = \operatorname{span}\{\ell(e_1)\cdots\ell(e_n)c(w)\ell(f_n)^*\cdots\ell(f_1)^* : e_1\cdots e_n w f_n^0\cdots f_1^o \in L^{\pm}\}$$

where the parity of the loops is chosen to match the parity of n. For a loop $e_1\cdots e_n w f_n^o\cdots f_1^o \in L_\pm$, we set

$$c_n(e_1\cdots e_n w f_n^o\cdots f_1^o) = \ell(e_1)\cdots\ell(e_n)c(w)\ell(f_n)^*\cdots\ell(f_1)^* \in \mathfrak{M}_n.$$

Then map $L^+ \ni w \mapsto c(D_k^{-1}(w))$ defines a $*$-homomorphism from $Gr_k P^\Gamma$ to $\mathfrak{M}_k$. (Recall that $D_k(w))$ denotes the loop obtained by replacing the starting point in w by the k-th point on the path w).

Let

$$\phi_n \quad = \quad \delta^{-n} \sum_{w=f_1\cdots f_n} \left(\frac{\mu(s(f_n))}{\mu(t(f_1))}\right)^{1/2} \phi \circ \langle \cdot f_n \otimes \cdots \otimes f_1, f_n \otimes \cdots \otimes f_1 \rangle_A.$$

The inclusion $i = i_n^{n-1} : \mathfrak{M}_{n-1} \to \mathfrak{M}_n$ is given by

$$c_{n-1}(w) \mapsto \sum_{ewe^o \in L} \sigma(e)^{-1} \ell(e) c_{n-1}(w) \ell(e)^*.$$

One can check that i is again a trace-preserving inclusion. The conditional expectation $E_n : \mathcal{M}_n \to \mathcal{M}_{n-1}$ is given by

$$E_n(\ell(e)c_{n-1}(w)\ell(f)^*) = \delta_{e=f}\delta^{-1} \left(\frac{\mu(s(e))}{\mu(t(e))}\right) c_{n-1}(w).$$

As before, set

$$\mathcal{E}_n = i \circ E_n : \mathfrak{M}_n \to i(\mathfrak{M}_{n-1}) \subset \mathfrak{M}_n.$$

It is not hard to check that this is the unique trace-preserving $\mathfrak{M}_{n-1}$ linear conditional expectation from $\mathfrak{M}_n$ to $i(\mathfrak{M}_{n-1})$ and that the trace ϕ_n is faithful (the argument is exactly the same as in the case $n = 1$). Moreover, one can easily check that $\mathcal{E}_n = Tr_n$ if we identify $\mathfrak{M}_n$ with $Gr_n P^\Gamma$.

Let us set

$$i_n^j = i_n^{n-1} \circ \cdots \circ i_{j+1}^j : \mathfrak{M}_j \to \mathfrak{M}_n, \qquad i_n = i_n^0.$$

Comparing these with the definitions of section 2 we get:

Theorem 6. *The map $w \mapsto c_n(w)$ is a $*$-isomorphism from $Gr_k P^\Gamma$ onto $\mathfrak{M}_k$. The semifinite weight ϕ_k satisfies $\phi_k(v \wedge_0 Tr_k(x)) = \phi_k(v \wedge_0 x)$. In particular, the trace Tr_k is positive and faithful.*

5.3. **Higher relative commutants.** We now let

$$\mathcal{M}_k = W^*(Gr_k P^\Gamma, \phi_k) = W^*(\mathfrak{M}_k).$$

Given a planar subalgebra $P \subset P^\Gamma$, we'll denote by M the subalgebra of $\mathcal{M}_k$ generated by elements from P. In other words, $M_k = W^*(Gr_k P, Tr_k)$.

We'll denote by $\cup_n$ and $\Cup_n$ the images in $\mathfrak{M}_n$ of $\cup, \Cup \in \mathfrak{M}_0$. Note that $\cup_n, \Cup_n \in M_n$.

Lemma 17. *Let $e_1 \cdots e_n f_1^o \cdots f_n^o$ be a loop in $L_\pm$ (parity according to n). Then the element $Z = \ell(e_1)\cdots\ell(e_n)\ell(f_1)^*\cdots\ell(f_n)^* \in W^*(\cup_n, \Cup_n)' \cap \mathfrak{M}_n \subset W^*(\cup_n, \Cup_n)' \cap \mathcal{M}_n$.*

The proof is a straightforward computation and is omitted.

Lemma 18. *Let $P \subset P^\Gamma$ be a subfactor planar algebra with index, and let $M_n = W^*(Gr_n P, Tr_n)$ as above. Then $W^*(\cup_n, \Cup_n)' \cap M_n = P_{n,+}$.*

Proof. Let Q_n be the set of all paths in Γ of length n ending at an even vertex (and starting at an even or odd vertex, according to the parity of n). For $w = e_1 \cdots e_n \in Q_n$, let $F_w = \ell(e_1)\cdots\ell(e_n)$. Then for any $Y \in \mathfrak{M}_n$,

$$\hat{Y}_{w,w'} = F_w^* Y F_{w'} \in \mathcal{N}_+. \tag{3}$$

Moreover,

$$Y = \sum_{w,w' \in Q_n} c_{w,w'} F_w \hat{Y}_{w,w'} F_{w'}^* \tag{4}$$

where $c_{w,w'}$ are some constants. Since the sum above is finite, it follows that equations (3) and (4) continue to hold whenever $Y \in \mathcal{M}_n$, i.e. after passing to weak limits.

Thus if $Y \in \mathcal{M}_n$, and we set $Z = F_w F_w^*$, $Z' = F_{w'} F_{w'}^*$, then $ZYZ' = F_w^* \hat{Y} F_w$, where $\hat{Y} \in \mathcal{N}_+$. Moreover, Y is equal to a fixed finite linear combination of terms $\{ZYZ' : w, w' \in Q_n\}$.

Let us assume now that $Y \in W^*(\cup_n, \Cup_n)' \cap \mathcal{M}_n$. Then by choosing Z, Z' as above, we see from Lemma 17 that $ZYZ' \in W^*(\cup_n, \Cup_n)' \cap \mathcal{M}_n$. Using equations (3) and (4), we conclude that Y is a finite linear combination of terms of the form

$$\ell(e_1)\cdots\ell(e_n) X \ell(f_n)^* \cdots \ell(f_1)^*, \qquad X \in \mathcal{N}_+,$$

and that each such term must belong to the relative commutant $W^*(\cup_n, \Cup_n)' \cap \mathcal{M}_k$.

We can thus assume that $Y = \ell(e_1)\cdots\ell(e_n)X\ell(f_n)^*\cdots\ell(f_1)^*$ with $X \in \mathcal{N}_+$. Then

$$[Y, i_n(\cup)] = \left(\frac{\mu(t(e_n))}{\mu(s(e_1))}\right)^{1/2} \ell(e_1)\cdots\ell(e_n)[X,\cup]\ell(f_n)^*\cdots\ell(f_1)^*$$

and similarly

$$[Y, i_n(\Cup)] = \left(\frac{\mu(t(e_n))}{\mu(s(e_1))}\right)^{1/2} \ell(e_1)\cdots\ell(e_n)[X,\Cup]\ell(f_n)^*\cdots\ell(f_1)^*.$$

Thus if Y is in the relative commutant of $i_n(\cup,\Cup)\cap\mathcal{M}_n$, then X must be in the relative commutant of $\{\cup,\Cup\}$ in $\mathcal{N}_+$, which we know to be A_+ (Theorem 4). It follows that

$$\begin{aligned} \{\cup,\Cup\}'\cap\mathcal{M}_n &\subset \operatorname{span}\{\ell(e_1)\cdots\ell(e_n)\ell(f_n)^*\cdots\ell(f_1)^* : e_1\cdots e_n f_n^o\cdots f_1^o \in L_\pm\} \\ &= \{c_n(w) : w \in L_\pm \text{ loop of length } 2n \text{ starting at even/odd vertex}\}. \end{aligned}$$

Since the reverse inclusion holds by Lemma 17, equality holds. In particular, $\{\cup_n,\Cup_n\}'\cap\mathcal{M}_n = \{\cup_n,\Cup_n\}'\cap\mathfrak{M}_n$. We now see from the definitions in section 2 that the latter algebra is exactly the planar algebra $P^\Gamma_{n,+}$ taken with its usual multiplication.

Thus it follows that

$$\{\cup,\Cup\}'\cap M_n = P^\Gamma_{n,+}\cap M_n.$$

We claim that the latter intersection is exactly $P_{n,+}$. To see this, write any $Y\in\mathcal{M}_n$ as

$$Y = \sum_{w,w'\in Q_n} c_{w,w'}F_w\hat{Y}_{w,w;}F^*_{w'}$$

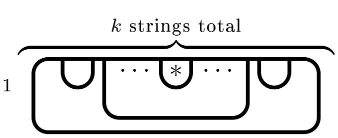
with $\hat{Y}_{w,w'} = F^*_wYF_{w'}\in\mathcal{N}_+$, as before. Let $E_n(Y) = \sum_{w,w'\in Q_n} c_{w,w'}F_wE(\hat{Y}_{w,w'})F^*_w$, where $E:\mathcal{N}_+\to A_+$ is the (normal) conditional expectation given by (1). Then E_n is a weakly-continuous map, and moreover $E_n(Y) = Y$ if $Y\in P^\Gamma_{n,+}$. Thus if $Y\in P^\Gamma_{n,+}\cap M_n$, then Y is the limit (in the weak operator topology) of a sequence $Y^{(j)}\in(P_{k_j,+})\subset M_n$. But then $Y = E(Y) = \lim_k E(Y^{(k)})$. Since the zero-box space of P is one-dimensional, it follows that $E(Y^{(k)})\in P_{n,+}$ (since then $E(\hat{Y}^{(k)}_{w,w'})\in\mathbb{C}1_{A_+}$) and so $Y\in P_{n,+}$. Thus $P^\Gamma_{n,+}\cap M_n = P_{n,+}$ and the theorem is proved. □

Theorem 7. *Let $P\subset P^\Gamma$ be a subfactor planar subalgebra of index $\delta\neq 1$. Let $M_k = W^*(Gr_kP, tr_k)$. Then $M'_0\cap M_k = P_{k,+}$ as algebras (here $P_{k,+}$ is taken with ordinary multiplication) in a way that preserves Jones projections.*

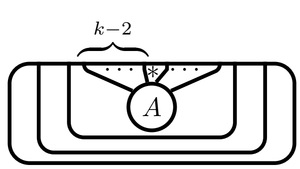
Proof. Since $\cup_k,\Cup_k\in M_k$, Lemma 18 shows that $P_{k,+}\supset M'_0\cap M_k$. Thus it is enough to prove that $P_{k,+}\subset M'_0\cap M_k$. But this is immediate, since P_k commutes with $i_k(\mathfrak{M}_0)$ and thus with M_0. The correspondence takes Jones projections to Jones projections (as is immediate from the pictures). □

Lemma 19. *Let $\mathbf{e}_k\in P_{k,+}$, $k\geq 2$ be the Jones projection. Then $\mathbf{e}_k$ is the Jones projection for the inclusion $M_{k-2}\subset M_{k-1}$.*

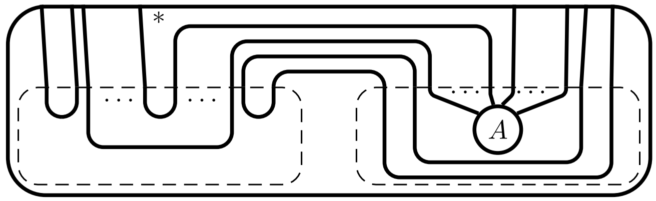
Proof. We first check that $\mathbf{e}_k\in M'_{k-2}\cap M_k$. Indeed,

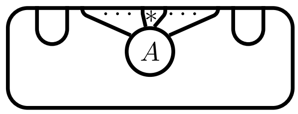
$\mathbf{e}_k = \delta^{-1}$ 

and since $x\in M_{k-2}$, it has the form

$x =$  .

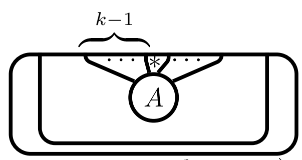
We now compute:

$\delta \mathbf{e}_k \wedge_k x \quad =$ 

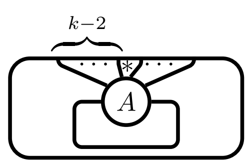
$=$ 

$= \quad \delta x \wedge_k \mathbf{e}_k \qquad$ (by symmetry).

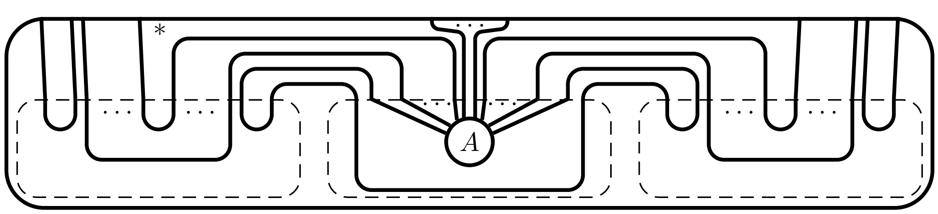
(here dashed lines indicate removed boxes).

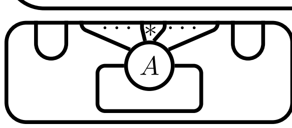
Next, we check that $\mathbf{e}_k \wedge_k x \wedge_k \mathbf{e}_k = \mathcal{E}_{k-2}(x) \wedge_k \mathbf{e}_k$ for $x =$ 

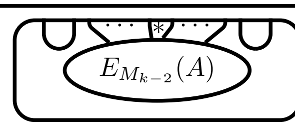
$\in M_{k-1}$. Note that it follows from the formula for $\mathcal{E}_{k-2}$ (or from an explicit computation using the trace) that

$E_{k-2}(A) = \delta^{-1}$ 

Now,

$\mathbf{e}_k \wedge_k x \wedge_k \mathbf{e}_k \quad = \quad \delta^{-2}$ 

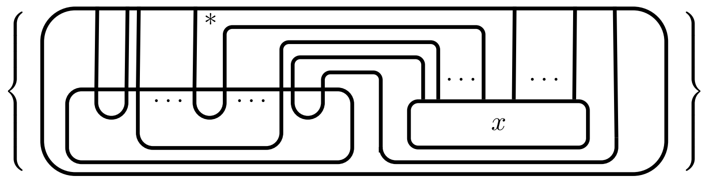
$= \quad \delta^{-2}$ 

$\quad = \quad \delta^{-1}$ 

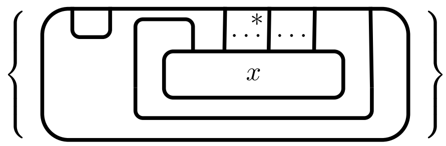
$= \quad \mathcal{E}_{k-2}(x) \wedge_k \mathbf{e}_k.$

Finally, we check that the trace Tr_k is λ-Markov. Let $x \in M_{k-1}$. Then:

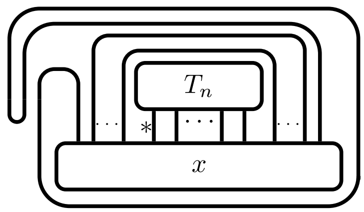
$\delta Tr_k(x \wedge_k \mathbf{e}_k) = Tr_k$



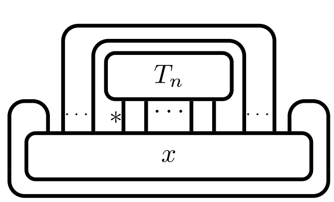
$= Tr_k$



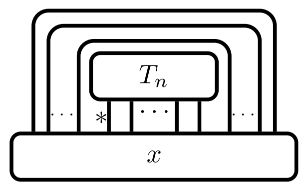
$=$



$=$



$=$



$= Tr_{k-1}(x).$

□

Lemma 20. *The algebras $M_0 \subset M_1 \subset M_2 \subset \cdots$ are exactly the tower obtained by iterating the basic construction for $M_0 \subset M_1$.*

Proof. We first note that because of the Markov property and the Jones relations between the projections $\mathbf{e}_n$, the algebras $\hat{M}_n = \langle M, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1} \rangle$, $n \geq 2$ are exactly the algebras appearing in the basic construction for $M_0 \subset M_1$. Hence clearly $\hat{M}_n \subset M_n$. Now suppose that for some n this inclusion were strict; choose smallest such n (necessarily > 1 since $M_0 = \hat{M}_0$ and $M_1 = \hat{M}_1$). Then the projection $\mathbf{e}_{n+1}$ is the Jones projection for $M_{n-1} \subset M_n$ and also for $M_{n-1} = \hat{M}_{n-1} \subset \hat{M}_n$. Thus the index of $M_{n-1} \subset M_n$ is the same as that of $M_{n-1} \subset \hat{M}_n$. But since $\hat{M}_n \subset M_n$, multiplicativity of index entails $[M_n : \hat{M}_n] = 1$ and thus $M = \hat{M}_n$, a contradiction. □

5.4. **The planar algebra structure on the higher relative commutants.** At this stage we have constructed a (II$_1$) subfactor $M_0 \subset M_1$ and its tower M_k as the completions of $Gr_k P$. We have also shown that $M_0' \cap M_k$ is precisely subspace $P_k \subset Gr_k(P)$.

Theorem 8. *The linear identification of P_k and $M_0' \cap M_j$ constructed in Theorem 7 is an isomorphism between P and the planar algebra of the subfactor $P(M_0 \subset M_1)$.*

In particular, any subfactor planar algebra can be naturally realized as the planar algebra of the II$_1$ subfactor $P(W^*(Gr_0 P, Tr_0) \subset W^*(Gr_1 P, Tr_1))$.

The second part of the theorem gives an alternative proof of a result of Popa [Pop93, Pop95, PS03b].

Proof. We have seen in Theorem 7 that the multiplication induced by $Gr_k(P)$ (hence M_k) on P_k is precisely that of the multiplication tangle. By [Jon99], to conclude that the planar algebra structure defined on P by this identification with the higher relative commutants for $M_0 \subset M_1$ we have to check the following.

1) That $M_0 \subset M_1$ is extremal (which means there is only one trace on the $M_0' \cap M_k$, that of M_k).

2) The Jones projections $\mathbf{e}_i$ of the tower are ($\frac{1}{\delta}$ times) the diagrammatic $\mathbf{e}_i$'s.

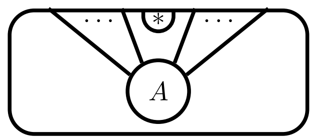
3) The inclusion of $M_0' \cap M_k$ in $M_0' \cap M_{k+1}$ is given by the appropriate tangle.

4) The trace on $M_0' \cap M_k$ given by restricting the trace on M_k is given by the appropriate tangle.

5) The projection from $M_0' \cap M_k$ onto $M_1' \cap M_k$ is given by the appropriate tangle.

For these, 1) follows from the definition of extremality in [PP86, Pop94] and a simple diagrammatic manipulation involving spherical invariance of the partition function. 2) was proved as part of Theorem 7. Properties 3) and 4) are just obvious pictures. The only one that requires any thought is 5) which we now prove.

Claim 1. Any element in $M_1' \cap M_k$ is in the image of the map from $M_0' \cap M_{k-1}$ to $M_0' \cap M_k$ defined by the following annular tangle:

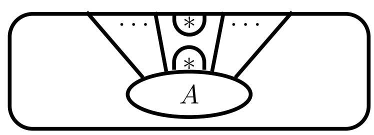
$A \mapsto$ 

.

(The shading is determined by the stars being in unshaded regions, the position of $*$ on the inside box being irrelevant.)

Proof of claim. It is a simple diagrammatic calculation to show that the image of this tangle does indeed commute with M_1. On the other hand the tangle defines an injective map (the inverse tangle is obvious) and from general subfactor theory the dimensions of $M_0' \cap M_{k-1}$ and $M_1' \cap M_k$ are the same. □

Claim 2. If A is in $M_0' \cap M_k$, identified with P_k, then

$E_{M_1'}(A) = \delta^{-1}$ 

.

Proof of claim. By extremality $E_{M_1'} = E_{M_1' \cap M_k}$ for elements of $M_0' \cap M_k$. Drawing the picture for $tr(AB)$ for $A \in M_0' \cap M_k$ and $B \in M_1' \cap M_k$, the result is visible. □

This conclude the proof of the Theorem. □

References

[BG05] F. Benaych-Georges. Rectangular random matrices, related free entropy and free Fisher's information. preprint arXiv.org:math/0512081, 2005.

[BIPZ78] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[BJ97] D. Bisch and V. F. R. Jones. Algebras associated to intermediate subfactors. *Invent. Math.*, 128(1):89–157, 1997.

[CMŚS07] B. Collins, J. Mingo, P. Śniady, and R. Speicher. Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants. *Doc. Math.*, 12:1–70 (electronic), 2007.

[FM77] J. Feldman and C. C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras I, II. *Trans. AMS*, 234:289–359, 1977.

[GMS06] A. Guionnet and E. Maurel-Segala. Combinatorial aspects of matrix models. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:241–279 (electronic), 2006.

[GS08] A. Guionnet and D. Shlyakhtenko. Free diffusion and matrix models with strictly convex potentials. *Geom. Func. Anal.*, 2008.

[Gui06] A. Guionnet. Random matrices and enumeration of maps. In *International Congress of Mathematicians. Vol. III*, pages 623–636. Eur. Math. Soc., Zürich, 2006.

[Jon83] V.F.R. Jones. Index for subfactors. *Invent. Math*, 72:1–25, 1983.

[Jon99] V.F.R. Jones. Planar algebras. Preprint, Berkeley, 1999.

[Jon00] V. F. R. Jones. The planar algebra of a bipartite graph. In *Knots in Hellas '98 (Delphi)*, volume 24 of *Ser. Knots Everything*, pages 94–117. World Sci. Publ., River Edge, NJ, 2000.

[Jon01] V.F.R. Jones. Annular structure of subfactors. In E. Ghys, P. de la Harpe, V.F.R. Jones, V. Sergiescu, and T. Tsuboi, editors, *Essays on Geometry and Related topics*, volume 32 of *L'Enseignment Mathematique*, 2001.

[MS06] J. Mingo and R. Speicher. Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces. *J. Funct. Anal.*, 235(1):226–270, 2006.

[MŚS07] J. Mingo, P. Śniady, and R. Speicher. Second order freeness and fluctuations of random matrices. II. Unitary random matrices. *Adv. Math.*, 209(1):212–240, 2007.

[Pim97] M. Pimsner. A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$. In D.-V. Voiculescu, editor, *Free Probability*, volume 12 of *Fields Institute Communications*, pages 189–212. American Mathematical Society, 1997.

[Pop93] S. Popa. Markov traces on universal Jones algebras and subfactors of finite index. *Invent. Math.*, 111:375–405, 1993.

[Pop94] S. Popa. Classification of amenable subfactors of type II. *Acta Math.*, 172(2):163–255, 1994.

[Pop95] S. Popa. An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.*, 120(3):427–445, 1995.

[PP86] M. Pimsner and S. Popa. Entropy and index for subfactors. *Ann. Sci. École Norm. Sup. (4)*, 19(1):57–106, 1986.

[PS03a] S. Popa and D. Shlyakhtenko. Cartan subalgebras and bimodule decompositions of II_1 factors. *Math. Scand.*, 92(1):93–102, 2003.

[PS03b] S. Popa and D. Shlyakhtenko. Universal properties of $L(\mathbf{F}_\infty)$ in subfactor theory. *Acta Math.*, 191(2):225–257, 2003.

[Răd94] F. Rădulescu. Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. *Invent. math.*, 115:347–389, 1994.

[Rae05] Iain Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005.

[Shl96] D. Shlyakhtenko. Random Gaussian band matrices and freeness with amalgamation. *Internat. Math. Res. Notices*, 20:1013–1026, 1996.

[Shl97] D. Shlyakhtenko. Free quasi-free states. *Pacific J. Math*, 177:329–368, 1997.

[Spe94] R. Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Math. Annalen*, 298:193–206, 1994.

[SU02] D. Shlyakhtenko and Y. Ueda. Irreducible subfactors of $L(F_\infty)$ of index $\lambda > 4$. *J. reine angew. Math*, 548:149–166, 2002.

[tH74] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys. B*, 72:461–473, 1974.

[Voi85] D.-V. Voicuelscu. Symmetries of some reduced free product C^*-algebras, in "Operator Algebras and their Connections with Topology and Ergodic Theory", Lecture Notes in Math. 1132, Springer Verlag, 1985, 556–588.

[Voi91] D.-V. Voiculescu. Limit Laws for Random Matrices and Free Products. Inv. Math., 104:201–220, 1991.

[VDN92] D.-V. Voiculescu, K. Dykema, and A. Nica. *Free Random Variables*, volume 1 of *CRM monograph series*. American Mathematical Society, 1992.

[Voi00] Dan Voiculescu. Lectures on free probability theory. In *Lectures on probability theory and statistics (Saint-Flour, 1998)*, volume 1738 of *Lecture Notes in Math.*, pages 279–349. Springer, Berlin, 2000.

[Zvo97] A. Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling*, 26(8-10):281–304, 1997. Combinatorics and physics (Marseilles, 1995).