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On the Existence of Stationary States in General Road Networks

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Abstract

Our daily driving experience and empirical observations suggest that traffic patterns in a road network are relatively stationary during peak periods. In numerous transportation network studies, there has been an implicit conjecture that stationary states exist in a network when origin demands, route choice proportions, and destination supplies are constant. In this study, we first rigorously formulate the conjecture within the framework of a network kinematic wave theory with an invariant junction model. After defining stationary states, we derive a system of algebraic equations in 3-tuples of stationary link flow-rates, demands, and supplies. We then introduce a new definition of junction critical demand levels based on effective demands and supplies. With a map in critical demand levels, we show that its fixed points and, therefore, stationary states exist with the help of Brouwer's fixed point theorem. For two simple road networks, we show that the map is well-defined and can be used to solve stationary states with a brute-force method. Finally we summarize the study and present some future extensions and applications.

Keywords: Network kinematic wave model; stationary states; demand and supply; critical demand levels; fixed point.

1. Introduction

During peak periods in an urban freeway network, a daily commuter would have almost the same schedule and route everyday and also experience congestion at similar locations and times. Therefore, from the viewpoint of the traffic system, “the traffic demand and origin-destination desires are relatively constant over the time period”, and the network reaches a stationary state, in which the locations and sizes of queues are nearly time-independent (Wattleworth, 1967). Such stationary traffic patterns can be observed from the snapshots of speed profiles in the Los Angeles freeway network during the morning peak hours on June 18, 2013, as shown in Figure 1: in the network, congested links, queue lengths, and bottleneck locations remain the same during the peak period from 7:30 to 9:00.

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(a) 7:30 AM



(b) 8:00 AM



(c) 8:30 AM



Figure 1: Stationary traffic patterns in the Los Angeles freeway network during the morning peak period (7:30-8:30) on June 18, 2013 (Data source: <http://pems.dot.ca.gov/>)

In many studies on analysis, control, management, planning, and design of road networks during peak periods, traffic patterns have been assumed to be stationary (Merchant and Nemhauser, 1978b; Yang and Yagar, 1995; Yang and Lam, 1996): in (Beckmann et al., 1956), the static traffic assignment problem was formulated to determine the aggregate route choice behaviors of vehicles; in (Godfrey, 1969), it was postulated that a network-wide macroscopic fundamental diagram (MFD) exists in such stationary, or steady, states, and this has been verified by observations (Geroliminis and Daganzo, 2008); in (Wattleworth, 1967), the local and global control problem of a freeway system was solved with linear programming methods; in (Potts and Oliver, 1972), network flow conservation problems are solved; and in (Payne and Thompson, 1974), the integrated traffic assignment and ramp metering problem was solved for stationary traffic patterns. Thus, there has been an implicit conjecture that constant demand and route choice patterns lead to stationary patterns in general networks. Even though an understanding of characteristics of stationary states is instrumental for studying various network problems, there has been no theoretical proof or disproof of it.

Furthermore, link performance functions have been widely used to determine travel times from flow-rates on stationary links during peak periods (Beckmann et al., 1956). For example, the BPR link performance functions have been critical for the advancement of transportation network analysis, planning, and design, since they enabled well-defined mathematical programming formulations and numerical solution methods of the static traffic assignment problem in large-scale road networks (Sheffi, 1984; Boyce et al., 2005). However, more and more evidences have shown that link performance functions fail to capture realistic traffic characteristics on links or through junctions in oversaturated networks, as (i) they contradict the fundamental diagram of traffic flow, which suggests that the travel time cannot be uniquely determined by the flow-rate (Greenshields, 1935); (ii) they cannot capture the interactions or competitions among different traffic streams at merging and diverging bottlenecks (Daganzo, 1995a). Such limitations of link performance functions have motivated many studies on dynamic traffic assignment problems (Merchant and Nemhauser, 1978a; Peeta and Ziliaskopoulos, 2001), in which more realistic traffic flow models are used. But such dynamic problems are much more challenging both analytically and computationally. In addition, even though link performance functions are physically limited for congested links, the existence of stationary traffic patterns during peak periods is a reasonable assumption and has been verified by our daily experience and observations. Therefore, a more reasonable next step for traffic assignment is to develop a physically meaningful link performance function for stationary links. Such an undertaking again requires an understanding of characteristics of stationary states in general networks during peak periods.

In this study, we first rigorously formulate the conjecture and then prove it affirmatively. This study is facilitated by the network kinematic wave model, in which traffic dynamics are described by the LWR model on links and macroscopic merging and diverging models at general junctions (Jin, 2012b,a). Then we define stationary states on a link and at a junction in terms of densities as well as 3-tuples of link flow-rates, demands, and supplies. We then formulate the traffic statics problem as solving a system of algebraic equations in 3-tuples for

all links. We further define a map in critical demand levels for all junctions and show that the fixed points of the map correspond to the stationary states in a network. Finally we will be able to prove the existence of such stationary states with constant demand patterns by proving the existence of fixed points of the map. We can see that stationary states defined in this study are physically realistic, since they satisfy both fundamental diagrams and junction models.

This study is an extension to (Jin, 2012c), where the traffic statics problem was defined as to find stationary states in a road network subject to constant origin demands, destination supplies, and route choice proportions within the framework of network kinematic wave theories and formulated as a system of algebraic equations in 3-tuples. However, in (Jin, 2012c), only a diverge-merge network is studied with separate diverging and merging models, and the problem was solved by a brute-force method, which cannot be extended for general road networks. In contrast, in this study we employ a unified junction model, which leads to a map and enables the definition and resolution of the conjecture for general networks.

In addition, the traffic statics problem is related to special network loading problems with constant demands and route choice proportions, which have been extensively studied either by assuming link performance functions (Xu et al., 1998; Prashker and Bekhor, 1998; Wu et al., 1998; Xu et al., 1999; Chabini, 2001; Astarita et al., 2001) or with microscopic or mesoscopic models (Barcelo and Casas, 2005; Bliemer, 2007). But the new model of stationary traffic flow is both mathematically tractable and physically meaningful, since it is analytically based on network kinematic wave models, which can lead to network models .

The rest of the paper is organized as follows. In Section 2, we review a network kinematic wave model, in particular, a general closed-form junction model, and formulate the conjecture within the framework of the network kinematic theory. In Section 3, we define stationary states on links and at junctions and derive a system of algebraic equations of 3-tuples in stationary states. In Section 4, we present a map in critical demand levels and prove the conjecture. In Section 5, we present two examples. Finally in Section 6, we conclude with some follow-up research directions.

2. A network kinematic wave model

For a general road network, e.g., the Braess network shown in Figure 2, we have the following notations:

1. R : the set of origin links (dash-dotted red lines); W : the set of destination links (dash-dotted green lines); A : the set of regular links (solid black lines); $A' = R \cup W \cup A$: the set of all links. Here the origin and destination links are dummy links with zero lengths.
2. Ω : the set of commodities (blue dashed lines), where vehicles using the same path belong to a commodity; Ω_a : the set of commodities using link $a \in A'$.
3. J : the set of junctions (cyan dots); I_j : the set of upstream (incoming) links of junction $j \in J$; O_j : the set of downstream (outgoing) links of junction j .

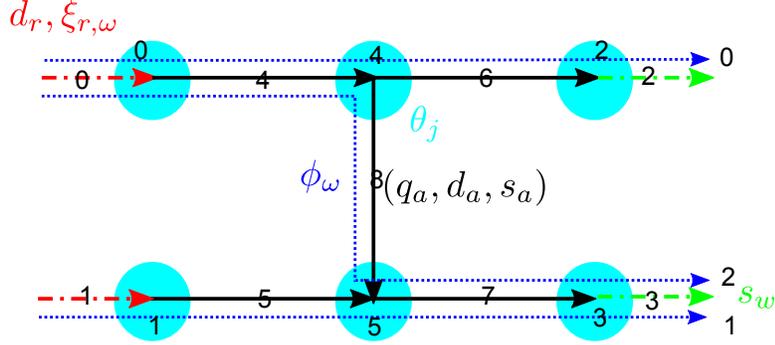


Figure 2: The Braess network

4. (a, x_a) : point x_a on link $a \in A$, where the positive direction of x_a is the same as traffic direction, and $x_a \in [X_a^-, X_a^+]$ with $-\infty \leq X_a^- \leq X_a^+ \leq \infty$.

For example, the Braess network in Figure 2 has two origin links, two destination links, five regular links, three commodities, and six junctions, and we can number them accordingly.

2.1. Link model

At a point (a, x_a) and time t , we denote the total density, speed, and flow-rate by $k_a(x_a, t)$, $v_a(x_a, t)$, and $q_a(x_a, t)$, respectively. We denote density, speed, and flow-rate of commodity $\omega \in \Omega_a$ by $k_{a,\omega}(x_a, t)$, $v_{a,\omega}(x_a, t)$, and $\phi_\omega(x_a, t)$, respectively. Hereafter we omit (x_a, t) from the variables unless necessary. Then the evolution of traffic density k_a is described by the following LWR model (Lighthill and Whitham, 1955; Richards, 1956):

$$\frac{\partial k_a}{\partial t} + \frac{\partial Q_a(k_a)}{\partial x_a} = 0, \quad (1)$$

which can be derived from the flow conservation equation, $\frac{\partial k_a}{\partial t} + \frac{\partial q_a}{\partial x_a} = 0$, and a fundamental diagram (Greenshields, 1935):¹, $q_a = Q_a(k_a)$, and $v_a = V_a(k_a)$. For commodity ω on link a , we have the following multi-commodity LWR model

$$\frac{\partial k_{a,\omega}}{\partial t} + \frac{\partial k_{a,\omega} V_a(k_a)}{\partial x_a} = 0, \quad (2)$$

from which the commodity proportion,

$$\xi_{a,\omega} = \frac{k_{a,\omega}}{k_a} = \frac{\phi_\omega}{q_a}, \quad (3)$$

¹Without loss of generality, we assume that all links are homogeneous; i.e., the fundamental diagram of a link is location-independent. But we allow different fundamental diagrams for different links, since they can have different numbers of lanes, free-flow speeds, and other characteristics.

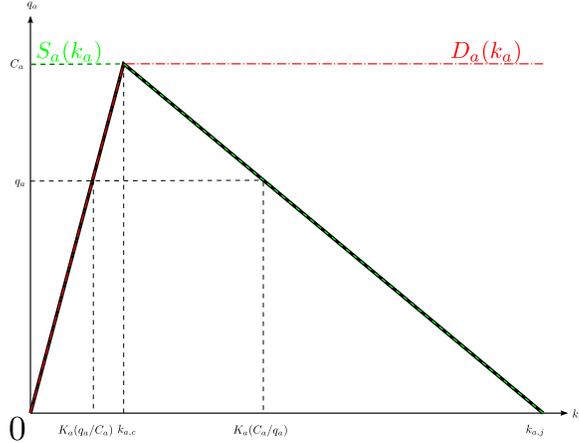


Figure 3: The fundamental diagram, demand (red dashed) and supply (green dash-dotted) functions, and flow-density relation

satisfies the following equation

$$\frac{\partial \xi_{a,\omega}}{\partial t} + V_a(k_a) \frac{\partial \xi_{a,\omega}}{\partial x_a} = 0.$$

We can see that $\xi_{a,\omega}$ always travels forward and the LWR model satisfies the First-In-First-Out principle (Lebacque, 1996).

Generally, $Q_a(k_a)$ is a unimodal function in k_a and reaches its capacity, C_a , when traffic density equals the critical density $k_{a,c}$. If traffic density k_a is strictly smaller than, equal to, or strictly greater than the critical density $k_{a,c}$, then we call the traffic state as strictly under-critical (SUC), critical (C), or strictly over-critical (SOC), respectively. An under-critical state (UC) can be SUC or C, and an over-critical state (OC) can be SOC or C.

Therefore, the link model, (1) and (2), is a system of network hyperbolic conservation laws. Due to the existence of shock waves, the solutions of (2) are defined in the weak sense. However, some entropy conditions have to be specified to pick out unique, physical weak solutions in such systems (Lax, 1972).

2.2. Junction model

In (Jin et al., 2009; Jin, 2010, 2014, 2012b), it was shown that macroscopic junction models, which were first presented in the discrete Cell Transmission Model (CTM) (Daganzo, 1995b; Lebacque, 1996), can be used as entropy conditions. That is, they can be used to complement the link model to form a network kinematic wave theory.

In the continuous CTM formulation of the network kinematic wave theory, traffic demand and supply functions, also known as sending and receiving flows, denoted by $d_a(x_a, t)$ and $s_a(x_a, t)$ respectively, are defined from local densities:

$$d_a = D_a(k_a) \equiv Q_a(\min\{k_a, k_{a,c}\}), \quad (4)$$

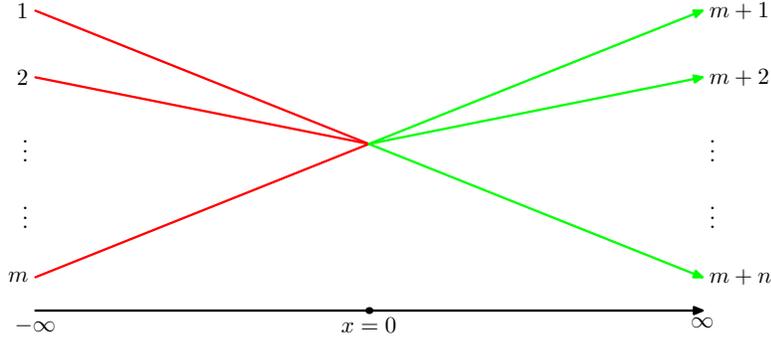


Figure 4: A general junction with m upstream links and n downstream links

$$s_a = S_a(k_a) \equiv Q_a(\max\{k_a, k_{a,c}\}). \quad (5)$$

Here the traffic demand function, $D_a(k_a)$, increases in k_a , and the traffic supply function, $S_a(k_a)$, decreases in k_a . Furthermore, $q_a = \min\{d_a, s_a\}$, $C_a = \max\{d_a, s_a\}$, and k_a can be uniquely determined by d_a/s_a , since $d_a/s_a = D_a(k_a)/S_a(k_a)$ is a strictly increasing function of k_a . We denote its inverse function by

$$k_a = K_a(d_a/s_a). \quad (6)$$

Therefore, instead of k_a , traffic demand and supply, $U_a = (d_a, s_a)$, can also be used as state variables: a traffic state at a point is SUC iff $d_a < s_a = C_a$, SOC iff $s_a < d_a = C_a$, and C iff $d_a = s_a = C_a$. Thus in UC states, $U_a = (q_a, C_a)$; and in OC states, $U_a = (C_a, q_a)$. Thus a flow-rate q_a corresponds to two densities: $K_a(q_a/C_a)$, and $K_a(C_a/q_a)$, where $Q_a(K_a(q_a/C_a)) = Q_a(K_a(C_a/q_a)) = q_a$, and $K_a(q_a/C_a) \leq k_{a,c} \leq K_a(C_a/q_a)$. A fundamental diagram, demand and supply functions, and flow-density relation are illustrated in Figure 3.

The core of a junction model is a flux function, which determines in- and out-fluxes from upstream demands, downstream supplies, and turning proportions. Here we use the following invariant² junction model based on the fair merging and FIFO diverging rules derived in (Jin, 2012b). At a junction j , as shown in Figure 4, we assume that $X_a^+ = 0$ for $a \in I_j \equiv \{1, 2, \dots, m\}$ and $X_b^- = 0$ for $b \in O_j \equiv \{m+1, m+2, \dots, m+n\}$ ($m \geq 1$ and $n \geq 1$). Here the upstream variables are defined at $(0^-, t)$, downstream variables at $(0^+, t)$, and boundary fluxes at $(0, t)$. The junction model consists of the following four steps: first, we calculate from commodity densities on a link upstream demands, downstream supplies, and turning proportions; second, we calculate the critical demand level and out-fluxes of upstream links; third, we calculate commodity fluxes; and finally, we calculate in-fluxes of downstream links. The details are in the following:

²The continuous CTM formulation of the network kinematic wave theory is still well defined with non-invariant junction models. But invariant junction models lead to much simpler analyses since we can ignore the existence of interior states (Jin, 2012b).

1. At any time t , from total and commodity densities on all links next to the junction, k_a ($a \in I_j$), $k_{a,\omega}$ ($a \in I_j, \omega \in \Omega_a$), k_b ($b \in O_j$), and $k_{b,\omega}$ ($b \in O_j, \omega \in \Omega_b$), we can calculate all upstream demands, d_a ($a \in I_j$), downstream supplies s_b ($b \in O_j$), and the turning proportions $\xi_{a \rightarrow b}$

$$\xi_{a \rightarrow b} = \sum_{\omega \in \Omega_a \cap \Omega_b} \xi_{a,\omega}. \quad (7a)$$

2. The out-flux of upstream link $a \in I_j$ is

$$g_a = \min\{d_a, \theta_j(t)C_a\}, \quad (7b)$$

where the critical demand level of junction j at time t , $\theta_j(t)$, uniquely solves the following min-max problem

$$\theta_j(t) = \min_{b \in O_j} \{1, \Gamma_b(s_b, \vec{d}, \vec{C}, \vec{\xi}_b)\}, \quad (7c)$$

where $\vec{d} = (d_a)_{a \in I_j}$, $\vec{C} = (C_a)_{a \in I_j}$, the remaining supply of downstream link b is $\pi_b = s_b - \sum_{a \in I_j} d_a \xi_{a \rightarrow b}$, $\vec{\xi}_b = (\xi_{a \rightarrow b})_{a \in I_j}$, and the critical demand level of downstream link b

$$\Gamma_b(s_b, \vec{d}, \vec{C}, \vec{\xi}_b) = \max_{B \subseteq I_j, B \neq \emptyset} \frac{\pi_b + \sum_{i \in B} d_i \xi_{i \rightarrow b}}{\sum_{i \in B} C_i \xi_{i \rightarrow b}}. \quad (7d)$$

Note that this formula is equivalent to but simpler than those in (Jin, 2012b) and (Jin, 2012a): in (Jin, 2012b), B was allowed to be empty; in (Jin, 2012a), the critical demand level $\theta_j(t)$ was also bounded by $\max_{a \in I_j} \frac{d_a}{C_a}$.

3. The commodity flux is ($a \in I_j, b \in O_j, \omega \in \Omega_a \cap \Omega_b$)

$$\phi_\omega = q_{b,\omega} = q_{a,\omega} = g_a \xi_{a,\omega}. \quad (7e)$$

4. The in-flux of downstream link $b \in O_j$ is

$$f_b = \sum_{a \in I_j} g_a \xi_{a \rightarrow b}. \quad (7f)$$

Since linear ($m = n = 1$), merging ($m > n = 1$), and diverging ($n > m = 1$) junctions are special cases of a general junction, (7) is a unified junction model with an arbitrary number of upstream links and an arbitrary number of downstream links. As shown in (Jin, 2012a), the junction model is consistent with the fair merging rule, since the out-fluxes of uncongested (UC) upstream links equal their demand ($g_a = d_a$), and the out-fluxes of congested upstream links are proportional to their capacities ($g_a = \theta_j(t)C_a$); it is consistent with the FIFO diverging rule, since all upstream links are in ‘‘one-pipe regime’’ and the commodity fluxes

are proportional to their proportions ($\phi_\omega = g_a \xi_{a,\omega}$). In an extreme case, when $s_b = 0$ for link b and all turning proportions to this link are positive, then $\Gamma_b(s_b, \vec{d}, \vec{C}, \vec{\xi}_b) = 0$, $\theta_j(t) = 0$, and all upstream links become jammed. That is, one jammed downstream link would block all upstream links, and this verifies the FIFO diverging principle.

In this model, the most important quantity is the junction critical demand level $\theta_j(t)$, which determines the criticality of the downstream state of an upstream link: when $d_a/C_a < \theta_j(t)$, or when the demand level is smaller than the critical demand level, an SUC state arises at the downstream boundary of link a ; when $d_a/C_a > \theta_j(t)$, or when the demand level of the link is higher than the critical demand level, an SOC state arises at the downstream boundary of link a ; when $d_a/C_a = \theta_j(t)$, or when the demand level equals the critical demand level, either an SOC, C, or SUC state can arise at the downstream boundary of link a . From (7d) we have $\Gamma_b(s_b, \vec{d}, \vec{C}, \vec{\xi}_b) \geq \frac{s_b}{\sum_{i \in I_j} C_i \xi_{i \rightarrow b}} \geq 0$. Thus the critical demand level θ_j is bounded:

$$0 \leq \theta_j(t) \leq 1. \quad (8)$$

Furthermore, from Theorem 4.3 of (Jin, 2012a), we can replace the upstream demands in (7c) by the corresponding out-fluxes:

$$\theta_j(t) = \min_{b \in O_j} \{1, \Gamma_b(s_b, \vec{g}, \vec{C}, \vec{\xi}_b)\}, \quad (9)$$

where $\vec{g} = (g_a)_{a \in I_j}$.

2.3. The conjecture

Traffic dynamics can be completely described by the network kinematic wave model, (1) and (2) with (7), whose solutions are determined by initial conditions in $k_a(x_a, t)$ and $k_{a,\omega}(x_a, t)$ ($a \in A$, $\omega \in \Omega_a$) and boundary conditions in the origin demands, $d_r(t)$ ($r \in R$), route choice proportions, $\xi_{r,\omega}(t)$ ($\omega \in \Omega_r$), and destination supplies, $s_w(t)$ ($w \in W$).

By a stationary state, we mean that, if a road network starts with a traffic state, the traffic patterns will remain the same along the time. Intuitively, if the boundary conditions vary with time, we do not expect to have such a stationary state. But if the boundary conditions are constant, we would expect to have a stationary state. Within the framework of the network kinematic wave theory, we formulate such a conjecture as follows.

Conjecture 2.1. *When the boundary conditions are time-independent, i.e., if the origin demands, $d_r(t) = d_r$ ($r \in R$), route choice proportions, $\xi_{r,\omega}(t) = \xi_{r,\omega}$ ($\omega \in \Omega_r$), and destination supplies, $s_w(t) = s_w$ ($w \in W$), there exist time-independent stationary solutions of $k_a(x_a, t) = k_a(x_a)$ and $k_{a,\omega}(x_a, t) = k_{a,\omega}(x_a)$ on each regular link $a \in A$ for the network kinematic wave model, (1) and (2) with (7).*

3. Stationary states in a road network

In this section we first define stationary states on regular links and at junctions and then derive a system of algebraic equations for stationary states in a road network.

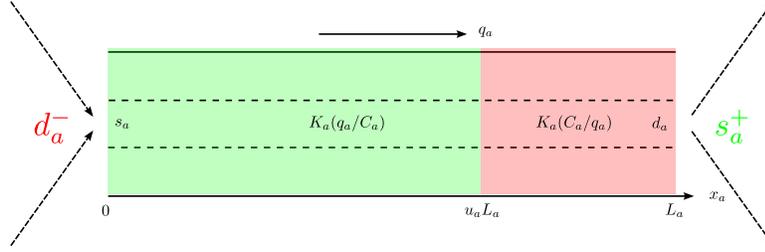


Figure 5: A stationary state on a link

3.1. Stationary states on regular links

Traffic is stationary on a regular link a , if and only if both commodity and total densities are time-independent: $\frac{\partial k_a(x_a, t)}{\partial t} = 0$ and $\frac{\partial k_{a, \omega}(x_a, t)}{\partial t} = 0$. Clearly, in stationary states both link flow-rates $q_a(x_a, t) = q_a$ and commodity flow-rates $\phi_\omega(x_a, t) = \phi_\omega$ are both time- and location-independent, and the commodity proportion $\xi_{a, \omega}(x_a, t) = \xi_{a, \omega}$ is also both time- and location-independent when link a is not jammed.³ In addition, when a regular link a becomes stationary, its density can be written as ($x_a \in [0, L_a]$)⁴

$$k_a(x_a, t) = H(u_a L_a - x_a) K_a(q_a/C_a) + (1 - H(u_a L_a - x_a)) K_a(C_a/q_a), \quad (10)$$

where $u_a \in [0, 1]$ is the uncongested fraction of the road, $H(\cdot)$ is the Heaviside function,

We refer to $s_a(0^+, t)$ as link supply and $d_a(L_a^-, t)$ as link demand. Thus in stationary states, they are also constant and denoted by $s_a(0^+, t) = s_a$, and $d_a(L_a^-, t) = d_a$. There can be four types of stationary states:

$$\text{Strictly under-critical (SUC): } u_a = 1, q_a = d_a < s_a = C_a \quad (11a)$$

$$\text{Strictly over-critical (SOC): } u_a = 0, q_a = s_a < d_a = C_a \quad (11b)$$

$$\text{Critical (C): } u_a \in [0, 1], q_a = d_a = s_a = C_a \quad (11c)$$

$$\text{Zero-speed shock wave (ZS): } u_a \in (0, 1), q_a < d_a = s_a = C_a \quad (11d)$$

A stationary state on link a is illustrated in Figure 5. Note that all types of stationary states can be considered as special cases of a zero-speed shock wave, depending on the fractions of UC (green) and OC (red) regions.

From (11) we can see that the flow-rate and type of stationary state of link a can be uniquely determined if we can find the 3-tuple, (q_a, d_a, s_a) , which satisfies (11). Thus (11) is the feasible condition for the 3-tuple. Note that for C stationary states traffic densities are the same for any u_a , but the interface of a zero-speed shock wave cannot be uniquely

³Note that, when link a is jammed, the commodity proportions may be location-dependent, but the commodity flow-rate is zero and therefore still location-independent.

⁴We denote $X_a^- = 0$ and $X_a^+ = L_a$, where L_a is the length of link a .

determined when $q_a < d_a = s_a = C_a$. However, this property does not impact the existence of stationary states.

To find (q_a, d_a, s_a) , we introduce two new variables for link a : the upstream demand, d_a^- , and downstream supply, s_a^+ . With the two new variables, we can decouple a network into $|A'|$ separate links: as shown in Figure 5, for link a , it has an artificial origin with a demand of d_a^- and an artificial destination with a supply s_a^+ . Note that for an origin link r , we just need to introduce s_r^+ ; and for a destination link w , we just need to introduce d_w^- .

3.2. Stationary states at junctions

In stationary states we have $f_a = g_a = q_a$ ⁵. In this subsection, we derive the conditions for all 3-tuples of flow-rates, demands, and supplies, (q_a, d_a, s_a) for $a \in A$, from the junction model (7) and (9).

First, in stationary states the critical demand level at junction j is also time-independent:

$$\theta_j = \min_{b \in O_j} \{1, \Gamma_b(s_b, \vec{q}, \vec{C}, \vec{\xi}_b)\}, \quad (12a)$$

where $\vec{q} = (q_a)_{a \in I_j}$, $\pi_b = s_b - \sum_{\alpha \in I_j} q_\alpha \xi_{\alpha \rightarrow b}$, and

$$\Gamma_b(s_b, \vec{q}, \vec{C}, \vec{\xi}_b) = \max_{B \subseteq I_j, B \neq \emptyset} \frac{\pi_b + \sum_{i \in B} q_i \xi_{i \rightarrow b}}{\sum_{i \in B} C_i \xi_{i \rightarrow b}}. \quad (12b)$$

In addition, we define the critical demand level excluding the potential bottleneck effect of link $\beta \in O_j$ by

$$\theta_{j \setminus \beta} = \min_{b \in O_j, b \neq \beta} \{1, \Gamma_b(s_b, \vec{q}, \vec{C}, \vec{\xi}_b)\}. \quad (12c)$$

Then we define the downstream supplies of entering links and upstream demands of exiting links by

$$s_a^+ = \theta_j C_a, \quad a \in I_j, \quad (12d)$$

$$d_\beta^- = \sum_{a \in I_j} \min\{d_a, \theta_{j \setminus \beta} C_a\} \xi_{a \rightarrow \beta}, \quad \beta \in O_j. \quad (12e)$$

Then from (7b) and the definition of the downstream supply in (12d) the out-flux of link a is given by

$$q_a = \min\{d_a, s_a^+\}; \quad (12f)$$

⁵In (Branston, 1976), it was observed that “the necessary condition for a steady state” “is simply that, during such a period, both entry and exit flows on a link will be equal and time independent.” But note that this is not a sufficient condition.

from the definition of the upstream demand in (12e) the in-flux of link a is

$$q_a = \min\{d_a^-, s_a\}, \quad (12g)$$

which was proved in Lemma 4.4 of (Jin, 2012a). From (12f) and (12g) we can see that s_a^+ and d_a^- are indeed downstream supply and upstream demand for link a .

In particular, at destination $w \in W$, the in-flux is given by

$$q_w = \min\{d_w^-, s_w\}, \quad (12h)$$

and at origin $r \in R$, the out-flux and commodity fluxes are given by

$$\begin{cases} q_r = \min\{d_r, s_r^+\} = s_r^+; \\ \phi_\omega = \sum_{r \in R} q_r \xi_{r,\omega}, \end{cases} \quad (12i)$$

where we set $C_r = d_r$. Then from the link-path relation the flow-rate on a regular link $a \in A$ is given by

$$q_a = \sum_{\omega \in \Omega_a} \phi_\omega, \quad (12j)$$

and the turning proportions at junction j are

$$\xi_{a \rightarrow b} = \sum_{\omega \in \Omega_a \cap \Omega_b} \phi_\omega / q_a, \quad (12k)$$

for $a \in I_j$ and $b \in O_j$. In addition, from flow conservation at junction j we have for $b \in O_j$

$$q_b = \sum_{a \in I_j} q_a \xi_{a \rightarrow b}. \quad (12l)$$

From (11) and (12), we obtain a system of algebraic equations in terms of the 3-tuples (q_a, d_a, s_a) ($a \in A$) for general road networks. In these equations, d_r , $\xi_{r,\omega}$, and s_w ($r \in R$, $\omega \in \Omega$, $w \in W$) are given, but there are other intermediate unknown variables: junction critical demand levels θ_j ($j \in J$), downstream supplies s_a^+ ($a \in R \cup A$), upstream demands d_a^- ($a \in A \cup W$), origin flow-rates q_r ($r \in R$), destination flow-rates q_w ($w \in W$), commodity flow-rates ϕ_ω , and turning proportions $\xi_{a \rightarrow b}$.

4. Proof of the conjecture

In this section, we present a proof of the existence of stationary states for Conjecture 2.1. That is, we prove that there exist solutions of (q_a, d_a, s_a) ($a \in A$) for the system of equations, (11) and (12), when d_r , $\xi_{r,\omega}$, and s_w are given.

4.1. A new definition of junction critical demand levels

We denote the effective link demand and supply by δ_a and σ_a , respectively. For origin $r \in R$, $\delta_r = d_r$; for destination $w \in W$, $\sigma_w = s_w$. For regular link $a \in A$ and $a \in I_j$,

$$\begin{cases} \delta_a = \min\{q_a, C_a\}, \\ \sigma_a = \theta_j C_a = s_a^+. \end{cases} \quad (13a)$$

Based on the effective demands and supplies, we have the following new definition of the critical demand levels at junction j :

$$\theta_j = \min_{b \in O_j} \{1, \Gamma_b(\sigma_b, \vec{\delta}, \vec{C}, \vec{\xi}_b)\}, \quad (13b)$$

$$\theta_{j \setminus \beta} = \min_{b \in O_j, b \neq \beta} \{1, \Gamma_b(\sigma_b, \vec{\delta}, \vec{C}, \vec{\xi}_b)\}, \quad (13c)$$

where $\vec{\delta} = (\delta_a)_{a \in I_j}$.

Lemma 4.1. *In stationary states, which satisfy (11) and (12), the critical demand levels in (13b) and (13c) are equivalent to those in (12a) and (12c), respectively.*

Proof. From (12d) and (12f), we can see that, in stationary states, $q_a \leq C_a$. Thus $\delta_a = \min\{q_a, C_a\} = q_a$; i.e., the effective demand is the same as the flow-rate, and we can replace \vec{q} by $\vec{\delta}$ in (12a) and (12c) to obtain (13b) and (13c), respectively.

From (11) and (12f), we can have the following cases:

1. When link b is stationary at SOC, $s_b = q_b < d_b = C_b$. Thus $s_b = q_b = s_b^+ = \sigma_b$.
2. When link b is stationary at C, $s_b = q_b = d_b = C_b$. Thus $s_b^+ = C_b$. In this case, $s_b = \sigma_b$.
3. When link b is stationary at SUC, $s_b = C_b > q_b = d_b$. Thus $\sigma_b = s_b^+ \geq q_b$.
4. When link b is stationary at ZS, $s_b = C_b = d_b > q_b$. Thus $\sigma_b = s_b^+ = q_b < s_b$.

In the latter two cases, we show that σ_b can also be replaced by s_b in (13b) and (13c). At junction j ($b \in O_j$), we denote the set of its SOC upstream links by A_* . Then from (12l) we have $q_b = \sum_{\alpha \in I_j \setminus A_*} q_\alpha \xi_{\alpha \rightarrow b} + \theta_j \sum_{a \in A_*} C_a \xi_{a \rightarrow b}$.

1. If $A_* = \emptyset$, from (12a) we can see that $\theta_j \geq \max_{a \in I_j} \frac{q_a}{C_a}$. In this case if we replace s_b by q_b , then $\pi_b = 0$ and $\Gamma_b(q_b, \vec{q}, \vec{C}, \vec{\xi}_b) = \max_{i \in I_j} \frac{q_i}{C_i}$. Thus $\theta_j \geq \max_{a \in I_j} \frac{q_a}{C_a}$. Similarly if we replace s_b by $\sigma_b \geq q_b$, we also have $\theta_j \geq \max_{a \in I_j} \frac{q_a}{C_a}$.
2. If $A_* \neq \emptyset$, there must exist another bottleneck link β such that $\Gamma_\beta(s_\beta, \vec{q}, \vec{C}, \vec{\xi}_\beta) = \theta_j$. Otherwise, $\theta_j = \Gamma_b(s_b, \vec{q}, \vec{C}, \vec{\xi}_b) = \frac{\pi_b + \sum_{i \in A_*} q_i \xi_{i \rightarrow b}}{\sum_{i \in A_*} C_i \xi_{i \rightarrow b}}$, which leads to $s_b = \sum_{\alpha \in I_j \setminus A_*} q_\alpha \xi_{\alpha \rightarrow b} + \theta_j \sum_{a \in A_*} C_a \xi_{a \rightarrow b}$. This is not possible since $s_b = C_b > q_b$ in the latter two cases. Therefore, in this case, if we replace s_b by $\sigma_b \geq q_b$, θ_j remains the same.

Therefore, in all cases, we can replace s_b by σ_b in (12a) to obtain (13b). This is also true for (12c) and (13c).

Therefore, (13b) and (13c) are equivalent to (12a) and (12c), respectively. \blacksquare

Note that effective demands and supplies of all links in (13a) are determined by link flow-rates and junction critical demand levels. Also note that the new junction critical demand levels are determined by effective demands and supplies as well as turning proportions, which are calculated from both link and commodity flow-rates as in turning-proportion. This suggests that we can solve flow-rates and critical demand levels separately from (11) and (12).

In the following Lemma, we can see that the original link demand and supply, d_a and s_a , can be computed from the effective upstream demand and downstream supply.

Lemma 4.2. *For link a , given q_a , d_a^- , and s_a^+ , we can determine its link demand and supply as follows:*

1. When $q_a = d_a^- < \min\{C_a, s_a^+\}$, the link is stationary at SUC with $d_a = q_a < C_a = s_a$;
2. When $q_a = s_a^+ = C_a \leq d_a^-$, the link is stationary at C with $d_a = s_a = q_a = C_a$;
3. When $q_a = s_a^+ < \min\{d_a^-, C_a\}$, the link is stationary at SOC with $d_a = C_a > q_a = s_a$;
4. When $q_a = d_a^- = s_a^+ < C_a$, the link is stationary at either SUC, SOC, or ZS.

Proof. From (11), (12f), and (12g), we have $q_a = \min\{d_a^-, s_a, d_a, s_a^+\}$, which leads to $q_a = \min\{d_a^-, C_a, s_a^+\}$. Therefore we can determine the stationary state as follows:

1. Link a is stationary at C if and only if $C_a \leq \min\{d_a^-, s_a^+\}$;
2. When $d_a^- < \min\{s_a^+, C_a\}$, link a is stationary at SUC. When link a is stationary at SUC, $d_a^- \leq s_a^+$ and $d_a^- < C_a$.
3. When $s_a^+ < \min\{d_a^-, C_a\}$, link a is stationary at SOC. When link a is stationary at SOC, $s_a^+ \leq d_a^-$ and $s_a^+ < C_a$.
4. When $d_a^- = s_a^+ < C_a$, link a can be stationary at SUC, SOC, or ZS. When link a is stationary at ZS, $d_a^- = s_a^+ < C_a$.

From (11) we can determine d_a and s_a accordingly. \blacksquare

Since the upstream demand and downstream supply in (12e), and (12d) are calculated from flow-rates and critical demand levels, stationary states can be determined by flow-rates and critical demand levels. Furthermore, from (12i), (12j), and (12k), link flow-rates can be uniquely determined by the critical demand levels at junctions downstream to origins. That is, stationary states can be solved in four steps: first, we find all critical demand levels; second, we calculate link flow-rates from (12i) and (12j); third, we calculate upstream demands and downstream supplies; finally, we determine stationary states from Lemma 4.2. Therefore, Conjecture 2.1 is proved if we can prove that there exist critical demand levels satisfying (12) and (13).

4.2. A map in critical demand levels and proof of the conjecture

We denote the vector of critical demand levels by $\vec{\theta} = (\theta_j)_{j \in J}$. From (12d), (12i), (12j), (12k), and (13a), we can see that link flow-rates, effective demands and supplies, and all turning proportions are functions of $\vec{\theta}$. Therefore (13b) can be written as

$$\vec{\theta} = \mathbb{F}(\vec{\theta}). \quad (14)$$

That is, $\vec{\theta}$ is the fixed point for the following map from $\vec{\theta}$ to $\vec{\theta}'$:

$$\vec{\theta}' = \mathbb{F}(\vec{\theta}). \quad (15)$$

The updating sequence of the map in (15) is as follows:

1. Given $\vec{\theta}$, we have the critical demand level for each junction.
2. From (12d), we calculate all downstream supplies.
3. For origin link $r \in R$, from (12i) we calculate q_r and ϕ_ω .
4. From (12j), we calculate link flow-rates on all regular links.
5. From (12k), we calculate the turning proportions at all junctions.
6. From (13a) we calculate effective demands and supplies.
7. From (13b) we calculate new critical demand levels $\vec{\theta}'$ at all junctions.

Lemma 4.3. *There exists a fixed point for (15). That is, there exist critical demand levels in a general road network with constant demand patterns.*

Proof. First, since all functions in (12d), (12i), (12j), (12k), (13a), and (13b) are continuous, $\mathbb{F}(\vec{\theta})$ is a continuous function in $\vec{\theta}$.

Second, $\max_{A_1 \subseteq I_j} \frac{\sigma_b - \sum_{\alpha \in I_j \setminus A_1} \delta_\alpha \xi_{\alpha \rightarrow b}}{\sum_{a \in A_1} C_a \xi_{a \rightarrow b}} \geq \max_{A_1 \subseteq I_j} \frac{\sigma_b}{\sum_{a \in I_j} C_a \xi_{a \rightarrow b}} \geq 0$, since $\theta_j \geq 0$. In addition, since $0 \leq \delta_a \leq C_a$, we have that $0 \leq \theta'_j \leq 1$.

Third, the dimension of $\vec{\theta}$ equals the number of junctions in a road network and is finite.

Then from Brouwer's fixed point theorem (Zeidler, 1986, Section 2.3), there exists a fixed point for the map, $\vec{\theta}^*$, such that $\vec{\theta}^* = \mathbb{F}(\vec{\theta}^*)$. ■

Theorem 4.4. *[Existence of stationary states] Stationary states always exist in a road network with constant origin demands, route choice proportions, and destination supplies.*

Proof. From Lemma 4.3, there exist critical demand levels $\vec{\theta}$ that satisfy (15) as well as (12) and (13). From $\vec{\theta}$ we can calculate link flow-rates, q_a , from (12i) and (12j). Further from (12e) and (12d) we can calculate upstream demands and downstream supplies, and from Lemma 4.2 we can determine stationary states and the 3-tuples.⁶ Therefore there exist stationary states in a road network when the origin demands, route choice proportions, and destination supplies are constant. This proves Conjecture 2.1. ■

⁶Note that we can calculate a link's upstream demand from (12e), when all of its upstream links' demands are known. The computation sequence should start with links downstream to origins. It can be seen that, when there exists no ring roads in a network, all links' stationary states can be calculated.

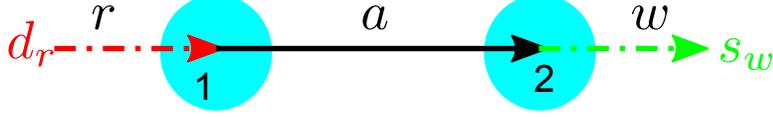


Figure 6: A network of a link

5. Examples for simple networks

For simple road networks, we can enumerate all combinations of stationary states and directly solve the algebraic equations to obtain the stationary states.

5.1. A single link

We consider the stationary state in a simple network, shown in Figure 6, with one origin r , one destination w , one regular link a , and two junctions 1 and 2. We assume that d_r and s_w are given.

From (12d), (12i), (12j), (12k), (13a), and (13b), we obtain the following map in critical demand levels, (15):

$$\begin{aligned}\theta'_1 &= \min\left\{1, \frac{\theta_2 C_a}{d_r}\right\}, \\ \theta'_2 &= \min\left\{1, \frac{s_w}{C_a}\right\},\end{aligned}$$

for which the fixed point clearly exists and can be calculated as $\theta_1^* = \min\left\{1, \frac{C_a}{d_r}, \frac{s_w}{d_r}\right\}$ and $\theta_2^* = \min\left\{1, \frac{s_w}{C_a}\right\}$. Then from (12i) and (12j), we find the link flow-rate $q_a = q_r = \theta_1 d_r = \min\{d_r, C_a, s_w\}$. Further from (12e) and (12d) we have the upstream demand and downstream supply $d_a^- = d_r$ and $s_a^+ = \theta_2 C_a = \min\{C_a, s_w\}$. Further from Lemma 4.2 we can determine stationary states: link a is stationary at SUC when $d_r < \min\{C_a, s_w\}$; C when $\min\{d_r, s_w\} \geq C_a$; SOC when $s_w < \min\{d_r, C_a\}$; either SUC, SOC, or ZS when $d_r = s_w < C_a$. Unsurprisingly for this example, the upstream demand and downstream supply are equivalent to the origin demand and destination supply: $d_a^- = d_r$ and $s_a^+ = s_w$.

Note that there exist multiple stationary states when $d_r = s_w < C_a$. From this example, we expect that there exist multiple stationary states in more general networks when a link can be stationary at ZS.

5.2. A diverge-merge network

We consider a diverge-merge network, shown in Figure 7, with one origin r , one destination w , two regular links 1 and 2, two junctions 1 and 2, and two paths 1 and 2.

From (12d), (12i), (12j), (12k), (13a), and (13b), we obtain the following map in critical demand levels, (15):

$$\theta'_1 = \min\left\{1, \frac{\theta_2 C_1}{\xi d_r}, \frac{\theta_2 C_2}{(1 - \xi) d_r}\right\}, \quad (16a)$$

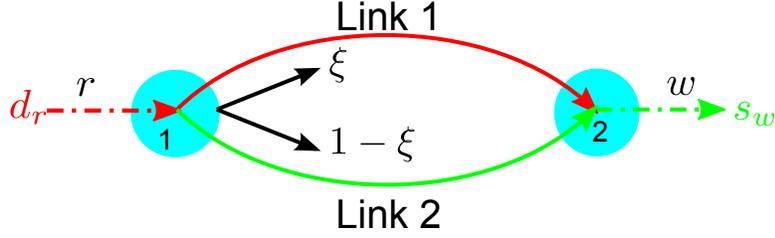


Figure 7: A diverge-merge network with unstable stationary states

$$\theta'_2 = \min\{1, \max\{\frac{s_w}{C_1 + C_2}, \frac{s_w - \min\{C_1, \theta_1 \xi d_r\}}{C_2}, \frac{s_w - \min\{C_2, \theta_1 (1 - \xi) d_r\}}{C_1}\}\}. \quad (16b)$$

From Theorem 4.4 we can see that, given d_r , s_w , and ξ , there exists a fixed point for (16). However, it is not straightforward to calculate such fixed points.

In the following we present a systematic brute-force method, in which we enumerate all possible stationary states so as to solve all fixed points for (16):

1. From (12d) and (12e) we have $s_1^+ = \theta_2 C_1$, $s_2^+ = \theta_2 C_2$, $d_1^- = \min\{\xi d_r, \theta_2 C_2 \frac{\xi}{1 - \xi}\}$, and $d_2^- = \min\{(1 - \xi) d_r, \theta_2 C_1 \frac{1 - \xi}{\xi}\}$.
2. From Lemma 4.2 we can enumerate all 16 types of stationary states on the two links. Here we just consider one case: links 1 and 2 are stationary at SOC and SUC, respectively.⁷ In this case, $s_1^+ < \min\{d_1^-, C_1\}$ and $d_2^- < \min\{s_2^+, C_2\}$; i.e., $\theta_2 < 1$, $\theta_2 C_1 < \xi d_r$, and $\xi > \frac{C_1}{C_1 + C_2}$. Then (16) can be simplified as (here θ_1 and θ_2 are fixed points)

$$\theta_1 = \frac{\theta_2 C_1}{\xi d_r},$$

$$\theta_2 = \min\{1, \max\{\frac{s_w}{C_1 + C_2}, \frac{s_w - \theta_2 C_1 \frac{1 - \xi}{\xi}}{C_1}\}\}.$$

Since $\theta_2 < 1$, there are two possible solutions. First, when $\theta_2 = \frac{s_w}{C_1 + C_2}$, we have $\frac{s_w}{C_1 + C_2} \geq \frac{s_w - \theta_2 C_1 \frac{1 - \xi}{\xi}}{C_1}$, which leads to $\frac{1 - \xi}{\xi} \geq \frac{C_2}{C_1}$. This contradicts $\xi > \frac{C_1}{C_1 + C_2}$. Therefore, $\frac{s_w}{C_1 + C_2} \leq \theta_2 = \frac{s_w - \theta_2 C_1 \frac{1 - \xi}{\xi}}{C_1}$, which leads to $\theta_2 = \xi \frac{s_w}{C_1}$ and $\theta_1 = \frac{s_w}{d_r}$. Note that such stationary states occur when $s_w < d_r$ and $\frac{C_1}{C_1 + C_2} < \xi < \frac{C_1}{s_w}$. These conditions are consistent with those in Table 4 of (Jin, 2012c), in which $C_0 = d_r$, $C_3 = s_w$, and SOC and SUC states along with ZS states were also considered.

3. Then from (12i) and (12j) we can calculate the link flow-rates. In the special case above, we have $q_r = s_w$, $q_1 = \xi s_w$, and $q_2 = (1 - \xi) s_w$. Since we already know the types

⁷Here we exclude the cases when the links can also be in ZS states.

of stationary states on both links, their corresponding demands and supplies can be computed. Furthermore, if we know the fundamental diagrams on both links, we can obtain the densities from (10).

Note that this method is more streamlined than the brute-force method in (Jin, 2012c).

6. Conclusion

In this study we formulated and proved the existence of stationary solutions for a network kinematic wave model with an invariant junction model, when the origin demands, route choice proportions, and destination supplies are constant. After introducing the network kinematic wave model and defining stationary states on links and at junctions, we established a system of algebraic equations in 3-tuples of link flow-rates, demands, and supplies for stationary states. Then we derived a map in critical demand levels and proved the existence of stationary states by following Brouwer’s fixed point theorem. We further showed that fixed-points of the map and stationary states are well-defined and can be solved with a brute-force method for simple road networks.

To achieve the goal, a number of new concepts were introduced. First, algebraic equations, (11) and (12), were derived in terms of the 3-tuples. Thus we can prove the conjecture by showing the existence of solutions to such equations. Second, the upstream demand and downstream supply, d_a^- and s_a^+ , were introduced for link a . They help to decouple a general road network into links with artificial origins and destinations, as shown in Figure 5, and determine the stationary state on each link separately according to Lemma 4.2. Third, the effective demand and supply, δ_a and σ_a , were introduced for link a , and a new definition of the critical demand level at a junction was introduced in terms of effective demands and supplies in (13b). They help to isolate link flow-rates and junction critical demand levels from other variables and split the solution of stationary states into a number of steps as shown in the proof of Theorem 4.4. Finally, the most critical contribution of this study is the derivation of the map in critical demand levels, as it is instrumental for establishing the existence of stationary states and also for calculating them, as shown in the examples.

However, this study is only a starting point for a systematic theory of stationary states in a road network. Various extensions are possible along the line:

1. In (Jin, 2013), it was shown that stationary states can be unstable or converge to gridlocks when there exists a circular information propagation path in a general road network. In the future, we will be interested in analyzing such stability property of stationary states with the help of the map in critical demand levels. As shown in (Jin, 2012c), there can be multiple stationary states in a road network. We will examine the uniqueness of stationary critical demand levels for the map.
2. In this study, a brute-force method was proposed to solve the critical demand levels and stationary states by enumerating all possible combinations of stationary states. Even though effective for analyzing all possible stationary states in such small networks

as those studied in Section 5, for large networks with $|A|$ regular links, the number of possible combinations of stationary states is $4^{|A|}$, and the brute-force method is no longer applicable. In the future, we will be interested in developing efficient methods to numerically compute critical demand levels and stationary states with given origin demands, route choice proportions, and destination supplies in a specific network.

3. In the future we will study the same problem when turning proportions at all junctions, not route choice proportions, are given, or when other junction models are used. In this study, junctions are unsignalized. In the (Jin et al., 2013), stationary states are defined as periodic solutions in a signalized double-ring network. In the future, we will be interested in establishing the existence of stationary states in signalized networks.
4. In this study, networks are open with origins and destinations. Studies on closed networks, which have periodic boundary conditions, can also reveal important characteristics of a network. For example, in (Jin et al., 2013), the stationary states in a closed double-ring network were solved and used to study the macroscopic fundamental diagram as well as impacts of signals and route choice behaviors. In the future we will be interested in establishing the existence of stationary states in more general closed networks.
5. Finally, we will also be interested in empirically examining the existence of stationary states in a real-world network.

This study lays a theoretical foundation for a mathematically tractable and physically meaningful model of stationary traffic flow within the framework of the kinematic wave theory. In the future we will be interested in studying transportation networks during peak periods within this framework. For example, we can analyze bottleneck locations in a network, developing transportation network management, control, planning, and design strategies, including ramp metering algorithms, evacuation schemes, dynamic traffic assignment, advanced traveler information systems, congestion pricing, and network design.

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