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On Robustness Analysis of a Dynamic Average Consensus Algorithm to Communication Delay

Hossein Moradian and Solmaz S. Kia

***Abstract*—This paper studies the robustness of a dynamic average consensus algorithm to communication delay over strongly connected and weight-balanced (SCWB) digraphs. Under delay-free communication, the algorithm of interest achieves a practical asymptotic tracking of the dynamic average of the time-varying agents' reference signals. For this algorithm, in both its continuous-time and discrete-time implementations, we characterize the admissible communication delay range and study the effect of the delay on the rate of convergence and the tracking error bound. Our study also includes establishing a relationship between the admissible delay bound and the maximum degree of the SCWB digraphs. We also show that for delays in the admissible bound, for static signals the algorithms achieve perfect tracking. Moreover, when the interaction topology is an connected undirected graph, we show that the discrete-time implementation is guaranteed to tolerate at least one step delay. Simulations demonstrate our results.**

***Keywords*—Communication Delay, Dynamic Average Consensus, Dynamic Input Signals, Directed Graphs, Convergence Rate**

I. Introduction

In a network of agents each endowed with a dynamic reference input signal, the dynamic average consensus problem consists of designing a distributed algorithm that allows each agent to track the dynamic average of the reference inputs (a global task) across the network. The solution to this problem is of interest in numerous applications such as multi-robot coordination [1], sensor fusion [2]–[4], distributed optimal resource allocation [5], [6], distributed estimation [7], and distributed tracking [8]. Motivated by the fact that delays are inevitable in real systems, our aim here is to study the robustness of a dynamic average consensus algorithm to fixed communication delays. The methods we develop can be applied to other dynamic consensus algorithms, as well.

Distributed solutions to the dynamic and the static average consensus problems have attracted increasing attention in the last decade. Static average consensus, in which the reference signal at each agent is a constant static value, has been studied extensively in the literature (see e.g., [9]–[11]). Many aspects of the static average consensus problem including analyzing the convergence of the proposed algorithms in the presence of communication delays are examined in the literature (see e.g., [9], [12]–[14]). Also, the static average consensus problem for the multi-agent systems with second-order dynamics in the presence of communication delay has been studied in [15]–[17]. The dynamic average consensus problem has been studied in the literature, as well. The solutions for this problem normally guarantee convergence to some neighborhood of the network's dynamic average of the reference signals (see. e.g., [2], [18]–[21] for the continuous-time algorithms and [19], [22], [23] for the discrete-time algorithms). Zero error tracking has been achieved under restrictive assumptions on the type of the reference signals [24] or via non-smooth algorithms, which assume an upper bound on the derivative of the agents' reference signals is known [25]. Some of these references address important practical considerations such as the dynamic average consensus over changing topologies and over networks with event-triggered communication strategy, however, the dynamic average consensus in the presence of communication time delay has not been addressed. This paper intends to fill this gap as delays are inevitable in the real systems and are known to cause disruptive behavior such as network instability or the network desynchronization [26]–[28].

In this paper, we study the effect of fixed communication delay on a variation of the delay-free continuous-time dynamic average consensus algorithm of [19] as well as its discrete-time implementation. Unlike the (Laplacian) static average consensus algorithm, in dynamic average consensus algorithms instead of initial conditions, the reference inputs enter the algorithm as an external input. Therefore, the frequency domain Nyquist approach and the Lyapunov method that have been used to study the robustness to delay of the static average consensus algorithm do not readily generalize to the dynamic average consensus algorithms. Here, we use a set of time domain approaches to perform our studies. Specifically, in continuous-time, we model our dynamic average consensus algorithm with communication delay as a delay differential equation (DDE) and use the characterization of solutions of DDEs and their convergence analysis (c.f. e.g., [29], [30]) to preform our studies. In discrete-time, we model our algorithm as a delay difference equation whose solution characterization can be found, for example, in [31], [32]. For both of the continuous- and the discrete-time algorithms that we study, we carefully characterize the admissible delay bound over strongly connected and weight-balanced (SCWB) interaction topologies. We also characterize the convergence rate and show that in both of the continuous- and the discrete-time cases, the rate is a function of the time delay and network topology. Moreover, we show that the admissible delay bound is an inverse function of the maximum degree of the SCWB digraph, a result that previously only was established for undirected graphs in the case of the static average consensus algorithm. Our convergence analysis also includes establishing a practical bound on the tracking error, and showing that for static signals the algorithms achieve perfect tracking for delays in

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the admissible bound. We also show for connected undirected graphs, the discrete-time algorithm is guaranteed to tolerate, at least, one step delay. Simulations demonstrate our results and also show that for delays beyond the admissible bound the algorithms under study become unstable. A preliminary version of a variation of the continuous-time case of this paper appeared in [33].

Notations: We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}_{>0}$ $\mathbb{Z}_{\geq 0}$ and $\mathbb{C}$ denote the set of real, positive real, nonnegative real, integer, positive integer, nonnegative integer, and complex numbers respectively. $\mathbb{Z}_i^j$ is the set $\{i, i+1, \cdots, j\}$, where $i, j \in \mathbb{Z}$ and $i < j$. For a $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ represent its real and imaginary parts, respectively. Moreover, $|s|$ and $\arg(s)$ represent its magnitude and argument, respectively, i.e., $|s| = \sqrt{\mathrm{Re}(s)^2 + \mathrm{Im}(s)^2}$ and $\arg(s) = \mathrm{atan2}(\mathrm{Im}(s), \mathrm{Re}(s))$. When $s \in \mathbb{R}$, $|s|$ is its absolute value. For $\mathbf{s} \in \mathbb{R}^d$, $\|\mathbf{s}\| = \sqrt{\mathbf{s}^\top \mathbf{s}}$ denotes the standard Euclidean norm, We let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) denote the vector of n ones (resp. n zeros), and denote by $\mathbf{I}_n$ the $n \times n$ identity matrix. We let $\mathbf{\Pi}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. When clear from the context, we do not specify the matrix dimensions. For a dynamic signal $\mathbf{u}$, we denote by $\|\mathbf{u}\|_{\text{ess}}$, the supremum norm , i.e., $\|\mathbf{u}\|_{\text{ess}} = \sup\{\|\mathbf{u}(t)\|, \ t \geq 0\} < \infty$. For $x \in \mathbb{R}$, ceiling of x demonstrated by $\lceil x \rceil$ is the smallest integer greater than or equal to x. For a given $d \in \mathbb{Z}_{\geq 0}$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$, we define delayed exponential of the matrix $\mathbf{A}$ as

$$e_{d}^{\mathbf{A}k} = \sum_{l=0}^{m_k} \binom{k-(l-1)d}{l} \mathbf{A}^l, \ m_k = \lceil \frac{k}{d+1} \rceil, \ k \in \mathbb{Z}_{\geq 0}. \quad (1)$$

In a network of N agents, the aggregate vector of local variables p^i, $i \in \{1, \dots, N\}$, is denoted by $\mathbf{p} = (p^1, \dots, p^N)^\top \in \mathbb{R}^N$. We define $\mathbf{R} \in \mathbb{R}^{N \times (N-1)}$ such that

$$\begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^\top \\ \mathbf{R}^\top \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & \mathbf{R} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^\top \\ \mathbf{R}^\top \end{bmatrix} = \mathbf{I}_N. \quad (2)$$

Since $\mathbf{R}\mathbf{R}^\top = \mathbf{\Pi}_N$ and $\mathbf{\Pi}_N\mathbf{\Pi}_N = \mathbf{\Pi}_N$, we have

$$\|\mathbf{R}^\top \mathbf{y}\| = \|\mathbf{\Pi}\,\mathbf{y}\|, \quad \forall \mathbf{y} \in \mathbb{R}^N. \quad (3)$$

For a given $z \in \mathbb{C}$, Lambert W function is defined as the solution of the equation $s\,e^s = z$, i.e., $s = W(z)$ (c.f. [34], [35]). Except for $z = 0$ which gives $W(0) = 0$, W is a multivalued function with the infinite number of solutions denoted by $W_k(z)$ with $k \in \mathbb{Z}$, where W_k is called the k^{th} branch of W function. Matlab and Mathematica have functions to evaluate $W_k(z)$. For any $z \in \{z \in \mathbb{C} \,|\, |z| \leq \frac{1}{e}\}$, we have

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n. \quad (4)$$

Next, we review some basic concepts from graph theory following [36]. A weighted *digraph*, is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*, and $\mathsf{A} \in \mathbb{R}^{N \times N}$ is a weighted *adjacency* matrix such that $\mathsf{a}_{ij} > 0$ if $(i,j) \in \mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. An edge (i,j) from i to j means that agent j can send information to agent i. Here, i is called an *in-neighbor* of j and j is called an *out-neighbor* of i. A *directed path* is a sequence of nodes connected by edges. A digraph is *strongly connected* if for every pair of nodes there is a directed path connecting them. A weighted digraph is *undirected* if $\mathsf{a}_{ij} = \mathsf{a}_{ji}$ for all $i, j \in \mathcal{V}$. A *connected* undirected graph is an undirected graph in which any two nodes are connected to each other by paths. The *weighted in-* and *out-degrees* of a node i are, respectively, $\mathsf{d}_{\text{in}}^i = \Sigma_{j=1}^N \mathsf{a}_{ji}$ and $\mathsf{d}_{\text{out}}^i = \Sigma_{j=1}^N \mathsf{a}_{ij}$. Maximum in- and out-degree of digraph are, respectively, $\mathsf{d}_{\text{out}}^{\max} = \max\{\mathsf{d}_{\text{out}}^i\}_{i=1}^N$, and $\mathsf{d}_{\text{in}}^{\max} = \max\{\mathsf{d}_{\text{in}}^i\}_{i=1}^N$. The *(out-) Laplacian* matrix is $\mathsf{L} = \mathsf{D}^{\text{out}} - \mathsf{A}$, where $\mathsf{D}^{\text{out}} = \mathrm{Diag}(\mathsf{d}_{\text{out}}^1, \cdots, \mathsf{d}_{\text{out}}^N) \in \mathbb{R}^{N \times N}$. Note that $\mathsf{L}\mathbf{1}_N = \mathbf{0}$. A digraph is *weight-balanced* iff at each node $i \in \mathcal{V}$, the weighted out-degree and weighted in-degree coincide, (iff $\mathbf{1}_N^T \mathsf{L} = \mathbf{0}$). In a weight-balanced digraph we have $\mathsf{d}_{\text{out}}^{\max} = \mathsf{d}_{\text{in}}^{\max} = \mathsf{d}^{\max}$. For a strongly connected and weight-balanced digraph, L has one eigenvalue $\lambda_1 = 0$, and the rest of the eigenvalues have positive real parts. Moreover,

$$0 < \hat{\lambda}_2 \mathbf{I} \leq \mathbf{R}^\top \mathrm{Sym}(\mathsf{L}) \mathbf{R} \leq \hat{\lambda}_N \mathbf{I}, \quad (5)$$

where $\hat{\lambda}_2$ and $\hat{\lambda}_N$, are, respectively, the smallest non-zero eigenvalue and maximum eigenvalue of $\mathrm{Sym}(\mathsf{L}) = (\mathsf{L} + \mathsf{L}^\top)/2$. For connected undirected graphs $\{\hat{\lambda}_i\}_{i=1}^N$, are equal to eigenvalues $\{\lambda_i\}_{i=1}^N$ of L, therefore, $0 < \lambda_2 \mathbf{I} \leq \mathbf{R}^\top \mathsf{L} \mathbf{R} \leq \lambda_N \mathbf{I}$.

II. Problem Statement

We consider a network of N first order integrator dynamics $\dot{x}^i = u^i$, $i \in \mathcal{V}$. The interaction topology $\mathcal{G}$ of the agents is a SCWB digraph in which the communication messages are subject to a fixed common transmission delay $\tau \in \mathbb{R}_{>0}$. In this network, each agent has access to a time-varying one-sided reference input signal

$$\mathsf{r}^i(t) = \begin{cases} r^i(t) \in \mathbb{R}, & t \in \mathbb{R}_{\geq 0} \\ 0, & t \in \mathbb{R}_{<0}. \end{cases} \quad (6)$$

The problem of interest is to devise a distributed solution for u^i such that the agreement state x^i of each agent $i \in \mathcal{V}$ tracks $\frac{1}{N}\sum_{j=1}^N \mathsf{r}^j(t)$, asymptotically. To solve this problem, we consider the continuous-time dynamic average consensus algorithm

$$\begin{aligned} &\dot{v}^i = \alpha\beta \sum_{j=1}^N \mathsf{a}_{ij}(x^i(t-\tau) - x^j(t-\tau)), \quad i \in \mathcal{V}, \quad (7) \\ &\dot{x}^i = -\alpha(x^i - \mathsf{r}^i) - \beta \sum_{j=1}^N \mathsf{a}_{ij}(x^i(t-\tau) - x^j(t-\tau)) - v^i, \\ &x^i(0), v^i(0) \in \mathbb{R}, \ \sum_{i=0}^N v^i(0) = 0, \ x^i(t) = 0 \text{ for } t \in [-\tau, 0). \end{aligned}$$

Algorithm (7) when $\tau = 0$ is proposed in [19], but with a summand $\dot{\mathsf{r}}^i$ in the right hand side of the $\dot{x}^i$ dynamics. Here, we have dropped this term to explore solutions that do not require access to $\dot{\mathsf{r}}^i$. Delay free form of algorithm (7) has also been used in [5], [6] to design distributed optimal resource allocation algorithms.

Theorem II.1 (Convergence of (7) over a SCWB digraph when $\tau = 0$)**.** *Let $\mathcal{G}$ be a SCWB digraph and assume that there is no communication delay, i.e., $\tau = 0$. Let $\|\dot{\mathbf{r}}\|_{ess} = \gamma < \infty$. Then, for* *any $\alpha,\beta\in\mathbb{R}_{>0}$, the trajectories of algorithm (7) are bounded and satisfy*

$$\lim_{t\to\infty}\big|x^i(t)-\frac{1}{N}\sum_{i=1}^{N}\mathsf{r}^j(t)\big|\leq\gamma\sqrt{(\frac{1}{\beta\hat{\lambda}_2})^2+(\frac{1}{\alpha})^2},\quad i\in\mathcal{V}. \tag{8}$$

The convergence rate to this error bound is no worse than $\min\{\alpha,\beta Re(\lambda_2)\}$.

The proof of this theorem can be easily deduced from the results in [19], and is omitted for brevity. An iterative form of algorithm (7) with step-size $\delta\in\mathbb{R}_{>0}$ and integer communication time-step delay $d\in\mathbb{Z}_{\geq0}$, is (in compact representation)

$$\begin{aligned}&\mathbf{v}(k+1)=\mathbf{v}(k)+\delta\alpha\beta\,\mathbf{L}\mathbf{x}(k-d), &(9)\\&\mathbf{x}(k+1)=\mathbf{x}(k)-\alpha\delta(\mathbf{x}(k)-\mathsf{r}(k))-\delta\beta\,\mathbf{L}\mathbf{x}(k-d)-\delta\mathbf{v}(k),\\&x^i(0),v^i(0)\in\mathbb{R},\ \sum\nolimits_{i=0}^{N}v^i(0)=0,\ x^i(k)=0\text{ for }k\in\mathbb{Z}_{-d}^{-1}.\end{aligned}$$

Theorem II.2 (Convergence of (9) over a SCWB digraph when $d=0$)**.** *Let $\mathcal{G}$ be a SCWB digraph and assume that there is no communication delay, i.e., $d=0$. Let $\|\Delta\mathsf{r}\|_{ess}=\gamma<\infty$. Then, for any $\alpha,\beta\in\mathbb{R}_{>0}$, the trajectories of algorithm (9) are bounded and satisfy*

$$\lim_{k\to\infty}\big|x^i(k)-\frac{1}{N}\sum_{i=1}^{N}\mathsf{r}^j(k)\big|\leq\gamma\sqrt{(\frac{\delta}{\beta\hat{\lambda}_2})^2+(\frac{1}{\alpha})^2}, \tag{10}$$

for $i\in\mathcal{V}$, provided $\delta\in(0,\min\{\alpha^{-1},\beta^{-1}(\mathsf{d}^{\max})^{-1}\})$.

The proof of this theorem also can be deduced from the results in [19], and is omitted for brevity. Note that the initialization condition $\sum_{i=1}^{N}v^i(0)=0$ in algorithm (7) and its discrete-time implementation can be easily satisfied if each agent $i\in\mathcal{V}$ uses $v^i(0)=0$. Initialization conditions appear in other dynamic average consensus algorithms, as well [2], [22], [24], [25].

Our objective is to characterize the admissible communication delay ranges for $\tau\in\mathbb{R}_{\geq0}$ and $d\in\mathbb{Z}_{\geq0}$ for the dynamic average consensus algorithms above. We also want to study the effect of a fixed communication delay on the tracking error bound and the rate of convergence of these algorithms. By admissible delay value we mean values of delay for which the algorithms stay internally exponentially stable. A short review of the definition of the exponential stability of linear delayed systems is provided in the appendix.

Remark II.1 (Alternative solutions)**.** Algorithm (7) is a practical solution for the applications where the agreement state x^i, $i\in\mathcal{V}$, is defined by an integrator dynamics that should be driven by command u^i. For such applications, if the interaction topology is a connected undirected graph, one can also use the proportional-integral dynamic average consensus algorithm of [18]. This algorithm has the advantage of being robust to initialization perturbations. On the other hand, for the applications where the agreement state x^i is a local computational state, one can use the simpler alternative dynamic average consensus algorithm

$$\begin{aligned}&x^i(t)=z^i(t)+\mathsf{r}^i,\qquad i\in\mathcal{V} &(11)\\&\dot{z}^i=-\beta\sum\nolimits_{j=1}^{N}\mathsf{a}_{ij}(x^i(t)-x^j(t)),\quad z^i(0)=0,\end{aligned}$$

$i\in\mathcal{V}$, of [18]. For SCWB digraphs, we can show that if $\|(\mathbf{I}_N-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\dot{\mathsf{r}}\|_{\text{ess}}=\gamma<\infty$, then for any $\beta\in\mathbb{R}_{>0}$, the trajectories of (II.1) are bounded and satisfy $\lim_{t\to\infty}\big|x^i(t)-\frac{1}{N}\sum_{i=1}^{N}\mathsf{r}^j(t)\big|\leq\frac{\gamma}{\beta\hat{\lambda}_2}$, $i\in\mathcal{V}$. Lastly, when $\dot{\mathsf{r}}^i$, $i\in\mathcal{V}$, is also available, one can use $\dot{\mathbf{x}}=-\beta\mathbf{L}\mathbf{x}+\dot{\mathsf{r}}$, the equivalent representation of algorithm (II.1), to obtain the driving command u^i of the first order integrator dynamics. The methods we develop can be used to analyze the robustness to delay of the aforementioned algorithms and their respective iterative representations, as well. In particular, the analysis for algorithm and its iterative implementation closely relates to our results. □

III. Convergence and Stability Analysis in the Presence of a Constant Communication Delay

In this section, we study the stability and convergence properties of algorithm (7) and its discrete-time implementation (9) in the presence of a constant communication delay.

A. Continuous-time case

For convenience in analysis, we start our study by applying the change of variables (recall $\mathbf{R}$ from (2))

$$\begin{bmatrix}\begin{bmatrix}q_1\\\mathbf{q}_{2:N}\end{bmatrix}\\\begin{bmatrix}p_1\\\mathbf{p}_{2:N}\end{bmatrix}\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{N}}\mathbf{1}_N^\top\\\mathbf{R}^\top\end{bmatrix}&\begin{bmatrix}\mathbf{0}\\\alpha\,\mathbf{R}^\top\end{bmatrix}\\\mathbf{0}&\begin{bmatrix}\frac{1}{\sqrt{N}}\mathbf{1}_N^\top\\\mathbf{R}^\top\end{bmatrix}\end{bmatrix}\begin{bmatrix}\mathbf{v}-\alpha\,\mathbf{\Pi}\,\mathsf{r}\\\mathbf{x}-\frac{1}{N}\sum\limits_{j=1}^{N}\mathsf{r}^j\mathbf{1}_N\end{bmatrix} \tag{12}$$

to represent algorithm (7) in the equivalent compact form

$$\dot{q}_1=0, \tag{13a}$$
$$\dot{p}_1=-\alpha p_1-q_1-\frac{1}{\sqrt{N}}\sum\nolimits_{j=1}^{N}\dot{\mathsf{r}}^j, \tag{13b}$$
$$\dot{\mathbf{q}}_{2:N}=-\alpha\mathbf{q}_{2:N}-\alpha\mathbf{R}^\top\dot{\mathsf{r}}, \tag{13c}$$
$$\dot{\mathbf{p}}_{2:N}=\mathbf{H}\,\mathbf{p}_{2:N}(t-\tau)-\mathbf{q}_{2:N}, \tag{13d}$$

where

$$\mathbf{H}=-\beta\,\mathbf{R}^\top\mathbf{L}\mathbf{R}. \tag{14}$$

Because of (2), $\|\mathbf{x}(t)-\frac{1}{N}\sum_{j=1}^{N}\mathsf{r}^j(t)\mathbf{1}_N\|^2=\|\mathbf{p}(t)\|^2=\|\mathbf{p}_{2:N}(t)\|^2+|p_1(t)|^2$. Therefore, we can write

$$\begin{aligned}\lim_{t\to\infty}\big|x^i(t)-\frac{1}{N}\sum_{j=1}^{N}\mathsf{r}^j(t)\big|^2&\leq\lim_{t\to\infty}\big\|\mathbf{x}(t)-\frac{1}{N}\sum_{j=1}^{N}\mathsf{r}^j(t)\mathbf{1}_N\big\|^2\\&=\lim_{t\to\infty}\Big(\|\mathbf{p}_{2:N}(t)\|^2+|p_1(t)|^2\Big),\quad i\in\mathcal{V}.\end{aligned} \tag{15}$$

For algorithm (7), using $\sum_{i=0}^{N} v^i(0)=0$, we obtain

$$q_1(0)=\frac{1}{\sqrt{N}}\mathbf{1}_N^\top(\mathbf{v}(0)-\alpha\,\mathbf{\Pi}\,\mathbf{r}(0))=\frac{1}{\sqrt{N}}\sum\nolimits_{i=0}^{N} v^i(0)=0, \quad (16)$$

Therefore, (13a) gives $q_1(t)=0$, $t\in\mathbb{R}_{\geq 0}$. Using standard results for linear systems, from (13b) and (13c) for $t\in\mathbb{R}_{\geq 0}$ we can also write

$$|p_1(t)|\leq \mathrm{e}^{-\alpha t}|p_1(0)|+\frac{\left|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\dot{\mathsf{r}}(t)\right|_{\text{ess}}}{\alpha}, \quad (17a)$$

$$\|\mathbf{q}_{2:N}(t)\|\leq \mathrm{e}^{-\alpha t}\|\mathbf{q}_{2:N}(0)\|+\|\mathbf{R}\dot{\mathbf{r}}\|_{\text{ess}}. \quad (17b)$$

To establish an upper bound on the tracking error of each agent, we need to obtain an upper bound on $\|\mathbf{p}_{2:N}(t)\|$, as well. Since (13d) is a DDE system with the system matrix H and a delay free input $-\mathbf{q}_{2:N}(t)$, the admissible delay bound for (13d) and subsequently, for the dynamic average consensus algorithm (7), is determined by the delay bound for the zero input dynamics of (13d), i.e.,

$$\dot{\mathbf{p}}_{2:N}(t)=\mathsf{H}\,\mathbf{p}_{2:N}(t-\tau). \quad (18)$$

Note that (18) is the Laplacian average consensus algorithm with the zero eigenvalue separated. For connected undirected graphs, [9] used the Nyquist criterion to characterize the admissible delay bound. Here, to obtain the admissible delay range of (18) over SCWB digraphs, we use a result based on the characteristic equation analysis for linear delay systems.

Lemma III.1 (Admissible range of τ for (18) over SCWB digraphs)**.** *Let $\mathcal{G}$ be a SCWB digraph. Then, for any $\tau\in[0,\bar{\tau})$ the zero-input dynamics (18) is exponentially stable if and only if*

$$\bar{\tau}=\min\left\{\tau\in\mathbb{R}_{>0}\,\middle|\,\tau=\frac{|atan(\frac{\mathrm{Re}(\lambda_i)}{\mathrm{Im}(\lambda_i)})|}{\beta\,|\lambda_i|},\ i\in\mathbb{Z}_2^N\right\}, \quad (19)$$

where $\{\lambda_i\}_{i=2}^N$ are the non-zero eigenvalues of L. Also, for connected undirected graphs, we have $\bar{\tau}=\frac{\pi}{2\,\beta\lambda_N}$.

Proof: Recall that for the SCWB digraphs, $\mathsf{H}\in\mathbb{R}^{(N-1)\times(N-1)}$ is a Hurwitz matrix whose eigenvalues are $\{-\beta\,\lambda_i\}_{i=2}^N$. Then, our proof is the straightforward application of [37, Theorem 1], which states that a linear delayed system $\dot{\boldsymbol{\chi}}=\mathbf{A}\boldsymbol{\chi}(t-\tau)$, is exponentially stable if and only if $\mathbf{A}\in\mathbb{R}^{n\times n}$ is Hurwitz and $0\leq\tau<\frac{|\text{atan}(\frac{\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)})|}{|\mu_i|}$, $i\in\{1,\cdots,n\}$, where μ_i is the i^{th} eigenvalue of $\mathbf{A}$. For connected undirected graphs, we have $\{\beta\,\lambda_i\}_{i=2}^N\subset\mathbb{R}_{>0}$, hence $|\text{atan}(\frac{\mathrm{Re}(\lambda_i)}{\mathrm{Im}(\lambda_i)})|=\frac{\pi}{2}$, and $\max\{\beta\,|\lambda_i|\}_{i=2}^N=\beta\lambda_N$. Therefore $\bar{\tau}$ in (19) is equal to $\frac{\pi}{2\,\beta\lambda_N}$. ■

For delays in the admissible bound characterized in Lemma III.1, the zero-input dynamics (18) is exponentially stable. Therefore, there are $\kappa_\tau\in\mathbb{R}_{>0}$ and $\rho_\tau\in\mathbb{R}_{>0}$ such that (see Definition A.1)

$$\|\mathbf{P}_{2:N}(t)\|\leq\kappa_\tau\mathrm{e}^{-\rho_\tau t}\sup_{\eta\in[-\tau,0]}\|\mathbf{p}_{2:N}(\eta)\|,\quad t\in\mathbb{R}_{\geq 0}. \quad (20)$$

Next, we use the results in [38] to obtain κ_τ and ρ_τ.

Lemma III.2 (Exponentially decaying upper bound for $\|\mathbf{p}_{2:N}(t)\|$ in (18))**.** *Let $\tau\in[0,\bar{\tau})$, where $\bar{\tau}$ is given in (19). Then, the trajectories of zero-input dynamics (18) over SCWB digraphs satisfy the exponential bound in (A.38) with*

$$\rho_\tau=\frac{1}{\tau}\max\{Re(W_0(-\beta\lambda_i\tau)\}_{i=2}^N, \quad (21a)$$

$$\kappa_\tau=\max(K_1,K_2)+\max(K_3,K_4), \quad (21b)$$

where $\{\lambda_i\}_{i=2}^N$ are the none zero eigenvalues of L and K_1,K_2,K_3,K_4 are gains that can be computed based on systems matrices (the closed form expressions are omitted for brevity, please see [38, Theorem 1]).

Proof: The proof is the direct application of [38, Theorem 1] which is omitted to avoid repetition. In applying [38, Theorem 1], one should recall that system matrix H in (18) is Hurwitz, its nullity is zero and, its eigenvalues are $\{-\beta\lambda_i\}_{i=2}^N$. ■

Using our preceding results, our main theorem below establishes admissible delay bound, an ultimate tracking bound, and the rate of convergence to this error bound for the distributed average consensus algorithm (7) over SCWB digraphs.

Theorem III.1 (Convergence of (7) over SCWB digraphs in the presence of communication delay)**.** *Let $\mathcal{G}$ be a SCWB digraph with communication delay in $\tau\in[0,\bar{\tau})$ where $\bar{\tau}$ is given in (19). Let $\|\dot{\mathsf{r}}\|_{ess}=\gamma<\infty$. Then, for any $\alpha,\beta\in\mathbb{R}_{>0}$, the trajectories of algorithm (7) are bounded and satisfy*

$$\lim_{t\to\infty}\left|x^i(t)-\frac{1}{N}\sum\nolimits_{j=1}^{N}\mathsf{r}^j(t)\right|\leq\gamma\sqrt{(\frac{\kappa_\tau}{\rho_\tau})^2+(\frac{1}{\alpha})^2}, \quad (22)$$

for $i\in\mathcal{V}$. Here, $\rho_\tau,\kappa_\tau\in\mathbb{R}_{>0}$ are given by (21). The rate of convergence to this error neighborhood is no worse than $\min\{\alpha,\rho_\tau\}$.

Proof: Recall the equivalent representation (13) of (7) and also (16). We have already shown that $q_1(t)=0$, and p_1 and $\mathbf{q}_{2:N}$ satisfy (17). for $t\in\mathbb{R}_{\geq 0}$. Since under the given initial condition we have $\mathbf{p}_{2:N}(t)=\mathbf{0}_{N-1}$ for $t\in[-\tau,0)$, the trajectories $t\mapsto\mathbf{p}_{2:N}$ of (13d) are

$$\mathbf{p}_{2:N}(t)=\mathbf{\Phi}(t)\,\mathbf{p}_{2:N}(0)-\int_0^t\mathbf{\Phi}(t-\zeta)\,\mathbf{q}_{2:N}(\zeta)\mathrm{d}\zeta,$$

where $\Phi(t-\zeta)=\sum_{k=-\infty}^{k=\infty}\mathrm{e}^{\mathbf{S}_k(t-\zeta)}\mathbf{C}_k$ with $\mathbf{S}_k=\frac{1}{\tau}W_k(\mathsf{H}\tau)$, and each coefficient $\mathbf{C}_k$ depending on τ and H (c.f. [30] for details). Then, we can write

$$\|\mathbf{p}_{2:N}(t)\|\leq\|\mathbf{\Phi}(t)\,\mathbf{p}_{2:N}(0)\|+\int_0^t\|\mathbf{\Phi}(t-\zeta)\|\left(\mathrm{e}^{-\alpha\zeta}\|\mathbf{q}_{2:N}(0)\|+\gamma\right)\mathrm{d}\zeta. \quad (23)$$

Here, we used (3) to write $\|\mathbf{R}^\top\dot{\mathbf{r}}\|_{\text{ess}}=\|\mathbf{\Pi}\,\dot{\mathbf{r}}\|_{\text{ess}}\leq\|\dot{\mathbf{r}}\|_{\text{ess}}=\gamma$. Because for the given τ the zero-input dynamics of (13d) is exponentially stable, by invoking the results of Lemma III.2 we can deduce that the trajectories of the zero-input dynamics of (13d) for $t\in\mathbb{R}_{\geq 0}$ satisfy $\|\mathbf{\Phi}(t)\mathbf{p}_{2:N}(0)\|\leq\kappa_\tau\mathrm{e}^{-\rho_\tau t}\|\mathbf{p}_{2:N}(0)\|$, where ρ_τ and κ_τ are described in the statement. Here, we used $\sup_{t\in[-\tau,0]} \|\mathbf{p}_{2:N}(t)\| = \|\mathbf{p}_{2:N}(0)\|$, which holds because of the given initial conditions. Because this bound has to hold for any initial condition including those satisfying $\|\mathbf{p}_{2:N}(0)\| = 1$, we can conclude that $\|\mathbf{\Phi}(t)\| \leq \kappa_\tau \mathrm{e}^{-\rho_\tau t}$, for $t \in \mathbb{R}_{\geq 0}$ (recall the definition of the matrix norm $\|\mathbf{\Phi}(t)\| = \sup\{\|\mathbf{\Phi}(t)\,\mathbf{p}_{2:N}(0)\| \,|\, \|\mathbf{p}_{2:N}(0)\| = 1\}$).

Therefore, from (23) we obtain $\|\mathbf{p}_{2:N}(t)\| \leq \kappa_\tau \mathrm{e}^{-\rho_\tau t}\|\mathbf{p}_{2:N}(0)\| + \int_0^t \kappa_\tau \mathrm{e}^{-\rho_\tau (t-\zeta)}(\|\mathbf{q}_{2:N}(0)\|\mathrm{e}^{-\alpha\zeta} + \gamma)\mathrm{d}\zeta$. Then, using $\int_0^t \mathrm{e}^{-\rho_\tau(t-\zeta)}d\zeta = \frac{1}{\rho_\tau}(1-\mathrm{e}^{-\rho_\tau t})$, and

$$\int_0^t \mathrm{e}^{-\rho_\tau(t-\zeta)}\mathrm{e}^{-\alpha\zeta}d\zeta = \begin{cases} t\,\mathrm{e}^{-\rho_\tau t} & \rho_\tau = \alpha, \\ \frac{1}{\rho_\tau - \alpha}(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\rho_\tau t}) & \rho_\tau \neq \alpha, \end{cases}$$

we can write

$$\|\mathbf{p}_{2:N}(t)\| \leq \kappa_\tau \mathrm{e}^{-\rho_\tau t}\|\mathbf{p}_{2:N}(0)\| + \frac{\kappa_\tau}{\rho_\tau}(1-\mathrm{e}^{-\rho_\tau t})\gamma + \quad (24)$$
$$\|\mathbf{q}_{2:N}(0)\| \begin{cases} t\mathrm{e}^{-\rho_\tau t} & \rho_\tau = \alpha, \\ \frac{1}{\rho_\tau-\alpha}(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\rho_\tau t}) & \rho_\tau \neq \alpha. \end{cases}$$

Given that $\|\frac{1}{\sqrt{N}}\mathbf{1}_N^\top \dot{\mathbf{r}}\|_{\text{ess}} \leq \|\dot{\mathbf{r}}\|_{\text{ess}} = \gamma$ and $q_1(t) = 0$ for $t \in \mathbb{R}_{\geq 0}$, the boundedness of the trajectories of (24) and the correctness of (23) follow from (15), (18) and (24). Moreover, the rate of convergence is also $\min\{\alpha, \rho_\tau\}$. ∎

Observe that

$$\lim_{\tau\to 0}\rho_\tau = \lim_{\tau\to 0} -\frac{1}{\tau}\max\{\mathrm{Re}(W_0(-\beta\lambda_i\,\tau))\}_{i=2}^N$$
$$= -\max\{\mathrm{Re}(-\beta\lambda_i)\}_{i=2}^N = \beta\,\mathrm{Re}(\lambda_2).$$

In other words, as $\tau \to 0$, the rate of convergence of algorithm (7) converges to its respective value of the delay free implementation given in Theorem II.2. Moreover, note that (22) indicates that if the reference inputs are static, i.e., $\|\dot{\mathbf{r}}\|_{\text{ess}} = \gamma = 0$, for any admissible delay algorithm (7) converges to the exact average of the reference inputs.

Next, we establish a relationship between the upper bound of the admissible delay bound and the maximum degree of the communication graph. For connected undirected graphs as $\lambda_N \leq 2\,\mathsf{d}^{\max}$ (see [9]), the admissible delay range that is identified in Lemma (III.1) satisfies $[0, \frac{\pi}{4\,\beta \mathsf{d}^{\max}}) \subseteq [0, \bar{\tau})$. We extend the results to SCWB digraphs.

Lemma III.3 (Admissible range of τ for (18) in terms of maximum degree of the digraph)**.** *Let $\mathcal{G}$ be a SCWB digraph. Then, $\bar{\tau}$ in (19) satisfies*

$$\bar{\tau} \geq 1/(2\,\beta\,\mathsf{d}^{\max}). \quad (25)$$

Proof: Recall that $\{\lambda_i\}_{i=2}^N$ are the non-zero eigenvalues of $\mathbf{L}$. By invoking the Gershgorin circle theorem [39], we can write $(\mathrm{Re}(\lambda_i) - \mathsf{d}^i_{\text{out}})^2 + \mathrm{Im}(\lambda_i)^2 \leq \mathsf{d}^i_{\text{out}}$, $i \in \{2, \cdots, N\}$. Given that $0 < \mathsf{d}^i_{\text{out}} \leq \mathsf{d}^{\max}$, then for any λ_i we have $(\mathrm{Re}(\lambda_i) - \mathsf{d}^{\max})^2 + \mathrm{Im}(\lambda_i)^2 \leq (\mathsf{d}^{\max})^2$, $i \in \{2, \cdots, N\}$, which is equivalent to $\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2 \leq 2\,\mathrm{Re}(\lambda_i)\,\mathsf{d}^{\max}$. Hence, we can write

$$\frac{1}{2\mathsf{d}^{\max}} \leq \frac{\mathrm{Re}(\lambda_i)}{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2} = \frac{\mathrm{Re}(\lambda_i)}{\sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}}\frac{1}{|\lambda_i|}.$$

Let $\phi = \mathrm{asin}(\frac{\mathrm{Re}(\lambda_i)}{\sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}})$. Since $\tan(\mathrm{asin}(x)) = \frac{x}{\sqrt{1-x^2}}$ holds for any $x \in \mathbb{R}$, one can yield that $\phi = \pm\mathrm{atan}(\frac{\mathrm{Re}(\lambda_i)}{\mathrm{Im}(\lambda_i)})$. Also, for any $\phi \in \mathbb{R}$ we have $\sin(\phi) \leq |\phi|$. Thus, we have

$$\frac{1}{2\beta\mathsf{d}^{\max}} \leq \frac{\mathrm{Re}(\lambda_i)}{\beta(\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2)} \leq \frac{|\mathrm{atan}(\frac{\mathrm{Re}(\lambda_i)}{\mathrm{Im}(\lambda_i)})|}{\beta|\lambda_i|},$$

proving that (25) is a sufficient condition for (19). ∎

The inverse relation between the maximum admissible delay and the maximum degree of the communication topology is aligned with the intuition. One can expect that the more links to arrive at some agents of the network, the more susceptible the convergence of the algorithm will be to the larger delays.

B. Discrete-time case

For convenience in stability analysis, we use the change of variable (12), to represent (9) in the equivalent form

$$q_1(k+1) = q_1(k), \quad (26a)$$
$$p_1(k+1) = (1-\alpha\delta)p_1(k) - \delta\, q_1(k) - \frac{\delta}{\sqrt{N}}\sum_{l=1}^N \Delta\mathsf{r}^l(k), \quad (26b)$$
$$\mathbf{q}_{2:N}(k+1) = (1-\alpha\delta)\mathbf{q}_{2:N}(k) - \alpha\delta\mathbf{R}^\top\Delta\mathbf{r}(k), \quad (26c)$$
$$\mathbf{p}_{2:N}(k+1) = \mathbf{p}_{2:N}(k) + \delta\,\mathbf{H}\,\mathbf{p}_{2:N}(k-d) - \delta\mathbf{q}_{2:N}(k). \quad (26d)$$

where $\mathbf{H}$ is defined in (14) and $\Delta\mathsf{r}^i(k) = \mathsf{r}^i(k+1) - \mathsf{r}^i(k)$. In what follows, We conduct our analysis for $\delta \in (0, \min(\alpha^{-1}, \beta^{-1}(\mathsf{d}^{\max})^{-1})$, a stepsize for which the delay free algorithm is stable, see Theorem II.2.

Similar to the continuous-time case, the tracking error of algorithm (9) for each agent $i \in \mathcal{V}$ satisfies

$$\lim_{k\to\infty}\Big|x^i(k) - \frac{1}{N}\sum_{j=1}^N \mathsf{r}^j(k)\Big|^2 \leq \lim_{k\to\infty}\Big(\|\mathbf{p}_{2:N}(k)\|^2 + |p_1(k)|^2\Big). \quad (27)$$

Under the assumption that $\sum_{i=1}^N v^i(0) = 0$, which gives (16), (26a) results in $q_1(k) = q(0) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Subsequently, for any $k \in \mathbb{Z}_{\geq 0}$, (26b) and (26c) for $\delta \in (0, \alpha^{-1})$ give

$$|p_1(k)| \leq (1-\alpha\delta)^k|p_1(0)| + \delta\phi\Big|\sum_{l=1}^N \frac{1}{\sqrt{N}}\Delta\mathsf{r}^l(j)\Big|_{\text{ess}}, \quad (28a)$$
$$\|\mathbf{q}_{2:N}(k)\| \leq (1-\alpha\delta)^k\|\mathbf{q}_{2:N}(0)\| + \alpha\delta\phi\|\mathbf{R}^\top\Delta\mathbf{r}(j)\|_{\text{ess}}, \quad (28b)$$

where $\phi = \sum_{j=0}^{k-1}(1-\alpha\delta)^{k-j-1} = \frac{1-(1-\alpha\delta)^k}{\alpha\delta}$. To establish an upper bound on the tracking error of each agent, we need to obtain an upper bound on $\|\mathbf{p}_{2:N}(k)\|$. Following [31, Thoerems 3.1 and 3.5], the trajectories of (26d) are described by (recall (1))

$$\mathbf{p}_{2:N}(k) = \sum\nolimits_{j=-\mathscr{d}+1}^{0} \mathrm{e}_{\mathscr{d}}^{\delta\,\mathsf{H}\,(k-\mathscr{d}-j)}\Delta\mathbf{p}_{2:N}(j-1) - \delta\sum\nolimits_{j=0}^{k-1}\mathrm{e}_{\mathscr{d}}^{\delta\,\mathsf{H}(k-j-1-\mathscr{d})}\mathbf{q}_{2:N}(j),\ \ k\in\mathbb{Z}_{\geq 0}. \quad (29)$$

We start our analysis by characterizing the admissible ranges of time-step delay $\mathscr{d}$ for the zero-input dynamics of (26d), i.e.,

$$\mathbf{p}_{2:N}(k+1) = \mathbf{p}_{2:N}(k) + \delta\mathsf{H}\,\mathbf{p}_{2:N}(k-\mathscr{d}). \quad (30)$$

Lemma III.4 (Admissible range of $\mathscr{d}$ for (30) over SCWB digraphs)**.** *Let $\mathcal{G}$ be a SCWB digraph. Assume that $\delta\in(0,(\beta\,\mathsf{d}^{\max})^{-1})$. Then, for any $\mathscr{d}\in[0,\bar{\mathscr{d}})$ the zero-input dynamics (30) is exponentially stable if and only if*

$$\bar{\mathscr{d}} = \min\Big\{d\in\mathbb{Z}_{\geq 0}\,\big|\, d>\hat{d},\quad \hat{d}=\frac{1}{2}\big(\frac{\pi-2|\arg(\lambda_i)|}{2\arcsin(\frac{\beta|\lambda_i|\delta}{2})}-1\big),\quad i\in\{2,\cdots,N\}\Big\}, \quad (31)$$

where $\{\lambda_i\}_{i=2}^{N}$ are the non-zero eigenvalues of L. For connected undirected graphs, $\hat{d}$ in (31) is $\hat{d}=\frac{1}{2}\big(\frac{\pi}{2\arcsin(\frac{\beta\,\lambda_i\,\delta}{2})}-1\big)$.

Proof: When $\mathscr{d}=0$, by knowing that eigenvalues of $\mathbf{I}+\delta\,\mathsf{H}$ are $\{1-\beta\delta\lambda_i\}_{i=2}^{N}$ [19] shows that for (30) to be exponentially stable over SCWB digraphs, δ has to satisfy $\delta\in(0,(\beta\,\mathsf{d}^{\max})^{-1})$. To characterize an upper bound for admissible $\mathscr{d}\in\mathbb{Z}_{>0}$, we invoke [40, Theorem 1] which states that the discrete-time delayed system (30) with a fixed delay $\mathscr{d}\in\mathbb{Z}_{>0}$ is exponentially stable iff eigenvalues of $\delta\,\mathsf{H}$ lie inside the region of complex plane enclosed by the curve

$$\Gamma=\big\{z\in\mathbb{C}|z=2\,\mathrm{i}\,\sin(\frac{\phi}{2\mathscr{d}+1})\mathrm{e}^{\mathrm{i}\phi},-\frac{\pi}{2}\leq\phi\leq\frac{\pi}{2}\big\}. \quad (32)$$

First, we consider $0<\phi<\frac{\pi}{2}$. If we write z in (32) as $z=2\mathrm{i}\sin(\frac{\phi}{2\mathscr{d}+1})\mathrm{e}^{\mathrm{i}\phi}=2\sin(\frac{\phi}{2\mathscr{d}+1})\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\phi)}$, then eigenvalue $-\delta\beta\lambda_i=\delta\,\beta\,|\lambda_i|\mathrm{e}^{(\pi+\arg(\lambda_i))\mathrm{i}}$, $i\in\{2,\cdots,N\}$, of $\delta\,\mathsf{H}$ lies inside Γ if and only if $\pi+\arg(\lambda_i)=\phi+\frac{\pi}{2}$ and $\beta\delta|\,\lambda_i|<2\,\sin(\frac{\phi}{2\mathscr{d}+1})$, in which $-\frac{\pi}{2}<\arg(\lambda_i)<0$, thus, $\beta\delta|\lambda_i|<2\sin(\frac{\frac{\pi}{2}+\arg(\lambda_i)}{2\mathscr{d}+1})$. Similarly, for $-\frac{\pi}{2}<\phi<0$, (i.e. $0<\arg(\lambda_i)<\frac{\pi}{2}$,) we obtain $\beta\delta|\lambda_i|<2\sin(\frac{\frac{\pi}{2}-\arg(\lambda_i)}{2\mathscr{d}+1})$. Therefore, for (30) to be asymptotically stable, we have

$$\mathscr{d}<\frac{1}{2}\big(\frac{\pi-2|\arg(\lambda_i)|}{2\arcsin(\frac{\delta\,\beta\,|\lambda_i|}{2})}-1\big),\quad \forall i\in\{2,\cdots,N\}.$$

For connected undirected graphs, the eigenvalues of $\delta\,\mathsf{H}$, i.e., $\{-\beta\delta\lambda_i\}_{i=2}^{N}$ are real, hence $\arg(\lambda_i)=0$. As a result for (30) to be exponentially stable over connected undirected graphs, we obtain $\mathscr{d}<\frac{1}{2}\big(\frac{\pi}{2\arcsin(\frac{\delta\,\beta\,\lambda_i}{2})}-1\big)$. This completes our proof. ■

Using our preceding results and the auxiliary Lemma A.1 that we presented in Appendix, our main theorem below establishes admissible delay bound, an ultimate tracking bound, and the rate of convergence to the error bound for the discrete-time algorithm (9) over SCWB digraphs.

Theorem III.2 (Convergence of (9) over SCWB digraphs in the presence of communication delay)**.** *Let $\mathcal{G}$ be a SCWB digraph with with communication delay in $d\in[0,\bar{\mathscr{d}})$ where $\bar{\mathscr{d}}$ is given in (31). Let $\|\Delta\mathbf{r}\|_{ess}=\gamma<\infty$. Then, for any $\alpha,\beta\in\mathbb{R}_{>0}$ and $\delta\in(0,\min(\alpha^{-1},\beta^{-1}(\mathsf{d}^{\max})^{-1})$, the trajectories of algorithm (9) are bounded and satisfy*

$$\lim_{k\to\infty}\Big|x^i(k)-\frac{1}{N}\sum_{j=1}^{N}\mathsf{r}^j(k)\Big|\leq\gamma\sqrt{(\frac{\delta\,\bar{\kappa}_{\mathscr{d}}}{1-\bar{\omega}_{\mathscr{d}}})^2+(\frac{1}{\alpha})^2}, \quad (33)$$

for $i\in\mathcal{V}$, where

$$(\bar{\omega}_{\mathscr{d}},\bar{\kappa}_{\mathscr{d}},\bar{\mathbf{Q}})=\underset{\omega_{\mathscr{d}},\kappa_{\mathscr{d}},\mathbf{Q}}{\operatorname{argmin}}\ \omega_{\mathscr{d}}^2,\quad \textit{subject to}\ \ (A.42), \quad (34)$$

with $\mathbf{A}_{aug}=\begin{bmatrix}\mathbf{0}_{\mathscr{d}(N-1)\times(N-1)} & \mathbf{I}_{\mathscr{d}(N-1)}\\ \delta\,\mathsf{H} & \begin{bmatrix}\mathbf{0}_{(N-1)\times(\mathscr{d}-1)(N-1)} & \mathbf{I}_{N-1}\end{bmatrix}\end{bmatrix}.$

Proof: Consider (26), the equivalent representation of algorithm (9). We have already established that $q_{1(k)=0}$, $p_1(k)$ and $\mathbf{q}_{2:N}(k)$ satisfy (28) for $k\in\mathbb{Z}_{\geq0}$. Next, we establish an upper bound on trajectories $k\mapsto\|\mathbf{p}_{2:N}\|$ for $k\in\mathbb{Z}_{\geq0}$. For a $\mathscr{d}$ in the admissible range defined in the statement, Lemma III.4 guarantees that the zero dynamic of (26d), i.e., (30), is exponentially stable. Therefore, invoking Lemma A.1, there always exist a positive definite $\mathbf{Q}\in\mathbb{R}^{(\mathscr{d}+1)(N-1)\times(\mathscr{d}+1)(N-1)}$ and scalars $0<\omega_{\mathscr{d}}<1$ and $\kappa_{\mathscr{d}}\geq1$ that satisfy (A.42) for $\mathbf{A}_{\text{aug}}$ as defined in the statement. The smallest $\omega_{\mathscr{d}}$ can be obtained from the matrix inequality optimization problem (34). Then, using the results of Lemma A.1, we have the guarantees that the solutions of the zero-state dynamics (30), for $k\in\mathbb{Z}_{\geq0}$ satisfy $\|\mathbf{p}_{2:N}(k)\|=\|\mathrm{e}_{\mathscr{d}}^{\delta\,\mathsf{H}(k-\mathscr{d})}\mathbf{p}_{2:N}(0)\|\leq\bar{\kappa}_{\mathscr{d}}\,\bar{\omega}_{\mathscr{d}}^{k}\,\|\mathbf{p}_{2:N}(0)\|$. Because this bound holds for any $\mathbf{p}_{2:N}(0)$ including those satisfying $\|\mathbf{p}_{2:N}(0)\|=1$, we can conclude that (recall the definition of a norm of a matrix) $\|\mathrm{e}_{\mathscr{d}}^{\delta\,\mathsf{H}(k-\mathscr{d})}\|\leq\bar{\kappa}_{\mathscr{d}}\,\bar{\omega}_{\mathscr{d}}^{k}$, for all $k\in\mathbb{Z}_{\geq0}$. Consequently, from (29), along with (28b) and $\|\Delta\mathbf{r}\|_{\text{ess}}=\gamma$, we can write

$$\|\mathbf{p}_{2:N}(k)\|\leq\bar{\kappa}_{\mathscr{d}}\,\bar{\omega}_{\mathscr{d}}^{k}\,\|\mathbf{p}_{2:N}(0)\|+\delta\sum\nolimits_{j=0}^{k-1}\bar{\kappa}_{\mathscr{d}}\,\bar{\omega}_{\mathscr{d}}^{(k-j-1)}\times\big(\|\mathbf{q}_{2:N}(0)\|(1-\alpha\delta)^j+\gamma(1-(1-\alpha\delta)^j)\big). \quad (35)$$

Note that $\sum_{j=0}^{k-1}\bar{\omega}_{\mathscr{d}}^{(k-j-1)}(1-\alpha\delta)^j=\bar{\omega}_{\mathscr{d}}^{k-1}\sum_{j=0}^{k-1}(\frac{1-\alpha\delta}{\bar{\omega}_{\mathscr{d}}})^j$ $=(1-\alpha\delta)^{k-1}\sum_{j=0}^{k-1}(\frac{\bar{\omega}_{\mathscr{d}}}{1-\alpha\delta})^j$. As a result, when $k\to\infty$ we obtain (recall that $0<\omega_{\mathscr{d}}<1$, $0<1-\alpha\delta<1$)

$$\lim_{k\to\infty}\sum\nolimits_{j=0}^{k-1}\bar{\omega}_{\mathscr{d}}^{(k-j-1)}(1-\alpha\delta)^j=\begin{cases}\lim_{k\to\infty}\omega_{\mathscr{d}}^{k-1}(1-(\frac{1-\alpha\delta}{\bar{\omega}_{\mathscr{d}}})^{-1})=0, & \bar{\omega}_{\mathscr{d}}>(1-\alpha\delta),\\ \lim_{k\to\infty}(1-\alpha\delta)^{k-1}(1-(\frac{\bar{\omega}_{\mathscr{d}}}{1-\alpha\delta})^{-1})=0, & \bar{\omega}_{\mathscr{d}}<(1-\alpha\delta),\\ \lim_{k\to\infty}k\,\bar{\omega}_{\mathscr{d}}^{(k-j-1)}=0, & \bar{\omega}_{\mathscr{d}}=(1-\alpha\delta).\end{cases}$$

Moreover, $\lim_{k\to\infty}\sum_{j=0}^{k-1}\bar{\omega}_{\mathscr{d}}^{(k-j-1)}=(1-\bar{\omega}_{\mathscr{d}})^{-1}$. Therefore, as $k\to\infty$, from (35), we can conclude that $\lim_{k\to\infty}\|\mathbf{p}_{2:N}(k)\|\leq\delta\bar{\kappa}_{\mathscr{d}}\gamma\lim_{k\to\infty}\sum_{j=0}^{k-1}\bar{\omega}_{\mathscr{d}}^{(k-j-1)}=$

$\frac{\bar{k}_d\,\delta\,\gamma}{(1-\bar{\omega}_d)}$. Then, given that from (17a), we can write $\lim_{k\to\infty}|p_1(k)| \le \frac{\gamma}{\alpha}$,we can obtain (33) from (27). ∎

Note that the optimization problem (34) is a convex linear matrix inequality (LMI) in variables $(\omega_d^2, \frac{1}{\bar{k}_d}, \mathbf{Q})$ and can be solved using efficient LMI solvers. Also notice that the tracking error bound (33) implies that if the local reference signals are static, i.e., $\|\Delta \mathsf{r}\|_{\text{ess}} = \gamma = 0$, for any admissible delay, algorithm (9) converges to the exact average of the reference inputs.

Because in connected undirected graphs all the non-zero eigenvalues of Laplacian matrix are real and satisfy $0 < \delta\beta\lambda_i < 1$, $i \in \{2,\cdots,N\}$, algorithm (9) is guaranteed to tolerate, at least, one step delay as $\frac{\pi}{2\arcsin(\frac{\delta\,\beta\,\lambda_i}{2})} > 3$. We close this section with establishing a relationship between admissible delay bound of (9) and the maximum degree of the communication graph. Here also similar to the continuous-time case, this relationship is inverse.

Lemma III.5 (Admissible range of d for (30) in terms of maximum degree of the digraph)**.** *Let $\mathcal{G}$ be a SCWB digraph. Then, $\bar{d}$ in* (31) *satisfies*

$$\bar{d} \ge \frac{1}{2}\left(\frac{1}{\beta\delta\mathsf{d}^{\max}} - 1\right). \tag{36}$$

Proof: Let Γ be the stability region introduced in (32) for a specific time delay, d. Invoking Gershgorin theorem, we know that all the eigenvalues of $\delta\mathbf{H}$ are located inside a circle which can be written in the polar form as $G = \{z = -re^{i\,\theta} \in \mathbb{C} | r = 2\beta\delta\mathsf{d}^{\max}\cos(\theta)| - \frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$. Due to symmetricity of G and Γ , we just consider $0 \le \theta \le \frac{\pi}{2}$ for simplification. G lies inside Γ if and only if $\phi = \frac{\pi}{2} - \theta$, and $|r| \le |z|$. Since $z = 2\sin(\frac{\phi}{2d+1})\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\phi)}$, it yields to $\beta\,\delta\,\mathsf{d}^{\max}\sin(\phi) \le \sin(\frac{\phi}{2d+1})$, or $\beta\,\delta\,\mathsf{d}^{\max} \le f(\phi) = \frac{\sin(\frac{\phi}{2d+1})}{\sin(\phi)}$. Moreover, $f(\phi)$ is a strictly increasing function over $\phi \in [0, \frac{\pi}{2}]$, since

$$\frac{\partial f(\phi)}{\partial\phi} = \frac{\cos(\phi)\cos(\frac{\phi}{2d+1})(\frac{1}{2d+1}\tan(\phi) - \tan(\frac{\phi}{2d+1}))}{\sin^2(\phi)} \ge 0.$$

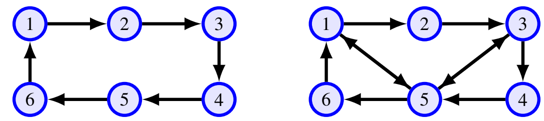
Thus, the least value of $f(\phi)$ is obtained as $\phi \to 0$ which is equal to $\frac{1}{2d+1}$. Thus, it implies that $\beta\delta\mathsf{d}^{\max} \le \frac{1}{2d+1}$, is a sufficient condition to guarantee the stability of the zero-input dynamics (30), which is equivalent to $\bar{d} \ge \frac{1}{2}(\frac{1}{\beta\delta\mathsf{d}^{\max}} - 1)$. ∎

IV. Numerical Simulations

We analyze the robustness of algorithm (7) and its discrete-time implementation (9) to communication delay for two academic examples taken from [19]. The network topology in these examples is given in Figure 1 (a). We also use the network given in Figure 1 (b) to study the effect of the maximum degree of a network on the admissible delay range.

Continuous-time case: the reference signals at each agent are

$r^1(t) = 0.55\,\sin(0.8\,t)$, $r^2(t) = 0.5\,\sin(0.7\,t) + 0.5\,\sin(0.6\,t)$,
$r^3(t) = 0.1\,t$, $r^4(t) = 2\,\text{atan}(0.5\,t)$,
$r^5(t) = 0.1\,\cos(2\,t)$, $r^6(t) = 0.5\,\sin(0.5\,t)$.



(a) (b)

Fig. 1: SCWB digraphs with edge weights 0 and 1.

We set the parameters of the algorithm (7) at $\alpha = 1$ and $\beta = 1$. The maximum admissible delay bounds obtained using the result of Lemma III.1 for the topologies depicted in Figure 1 are (a) $\bar{\tau} = 0.52$ and (b) $\bar{\tau} = 0.41$ seconds. Note that as expected from (25), for case (b) with $\mathsf{d}^{\max} = 3$, the maximum admissible delay is less than case (a) with $\mathsf{d}^{\max} = 1$. Figure 2 shows the time history of tracking error of algorithm (7) over the network topology of Figure 1 (a) for four different values of time delay (a) $\tau = 0$, (b) $\tau = 0.2s$, (c) $\tau = 0.4s$ and (d) $\tau = 0.6s$. As this figure shows, by increasing time delay, the rate of convergence of the algorithm decreases. Also, as can be predicted from (22), the tracking error increases. For case (d) the delay is beyond the admissible range, and as expected results in instability. The convergence rate for each of the other aforementioned time delay is given by, respectively (a) $\min\{\alpha = 1, \rho_\tau = 0.5\} = 0.5$, (b) $\min\{\alpha = 1, \rho_\tau = 0.28\} = 0.28$ and (c) $\min\{\alpha = 1, \rho_\tau = 0.11\} = 0.11$.

Discrete-time case: For discrete-time implementation, we use the same network depicted in Figure 1 (a) but with the reference input signal

$$\mathsf{r}^i(t(m)) = 2 + \sin\big(\omega(m)t(m) + \phi(m)\big) + b^i \qquad m \in \mathbb{Z}_{\ge 0}.$$

Here, $\mathbf{b}^\top = [-0.55, 1, 0.6, -0.9, -0.6, 0.4]$, and ω and ϕ are random signals with Gaussian distributions, $N(0, 0.25)$ and $N(0, (\frac{\pi}{2})^2)$, respectively. [19] states that these reference signals correspond to a group of sensors with bias b^i which sample a process at sampling times $t(m)$, $m \in \mathbb{Z}_{\ge 0}$. Here, we assume that the data is sampled at $\Delta\,t(m) = 5s$, for all $m \in \mathbb{Z}_{\ge 0}$. Each sensor needs to obtain the average of the measurements across the network before the next sampling time. To obtain the average we implement (9) with sampling stepsize δ and input $\mathsf{r}^i(k) = \mathsf{r}^i(k\delta)$, $i \in \mathcal{V}$, which between sampling times $t(m)$ and $t(m+1)$ is fixed at $\mathsf{r}^i(t(m))$. Figure 3 demonstrates the result of simulation for $\alpha = 5$ and $\beta = 1$ and $\delta = 0.19\,s$. The admissible delay for this case is $\bar{d} = 2$. Figure 3 shows the time history of tracking error for different amounts of delay (a) $d = 0$, (b) $d = 1$, (c) $d = 2$, and (d) $d = 3$. As shown, the steady error goes up as the delay increases. Moreover, the rate of convergence for each case is obtained by solving optimization problem (34). The optimal upper bound specifications for each case correspond to (a) $\bar{k}_d = 7.2$, $\bar{\omega}_d = 0.16$ (b) $\bar{k}_d = 68$, $\bar{\omega}_d = 0.96$ and (c) $\bar{k}_d = 70$, $\bar{\omega}_d = 0.99$, respectively. In case (d), the delay is outside admissible range and as expected the algorithm is unstable and diverges.

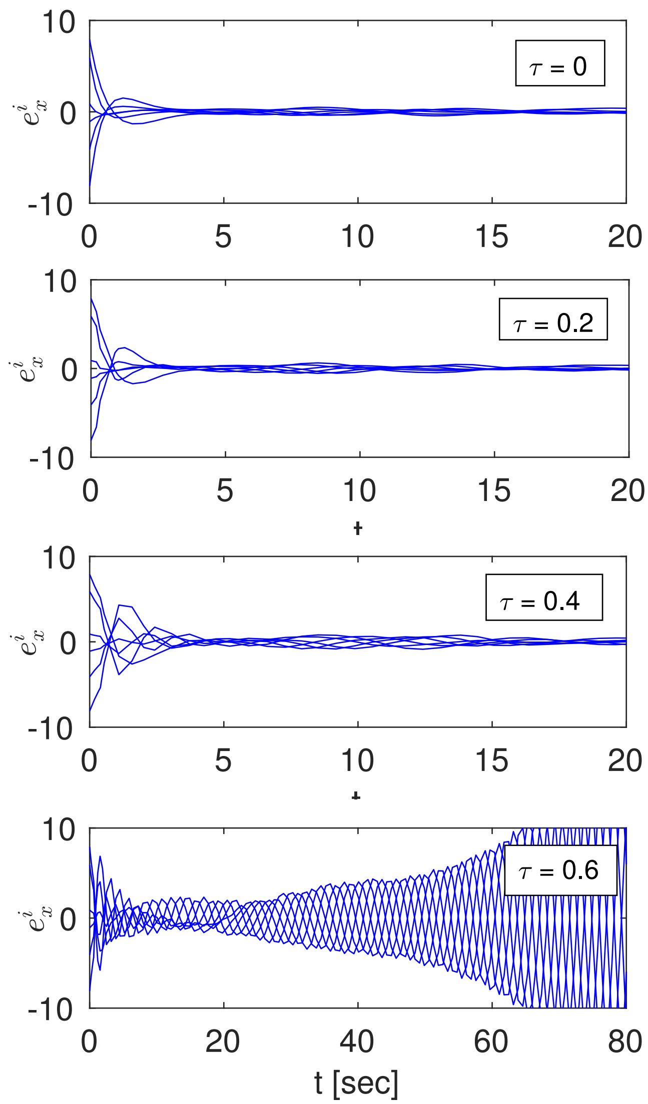


Fig. 2: Time history of the tracking error $e_x^i = x^i - \frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j$ of algorithm (7) over the network of Figure 1 (a) in the absence and presence of communication delay.

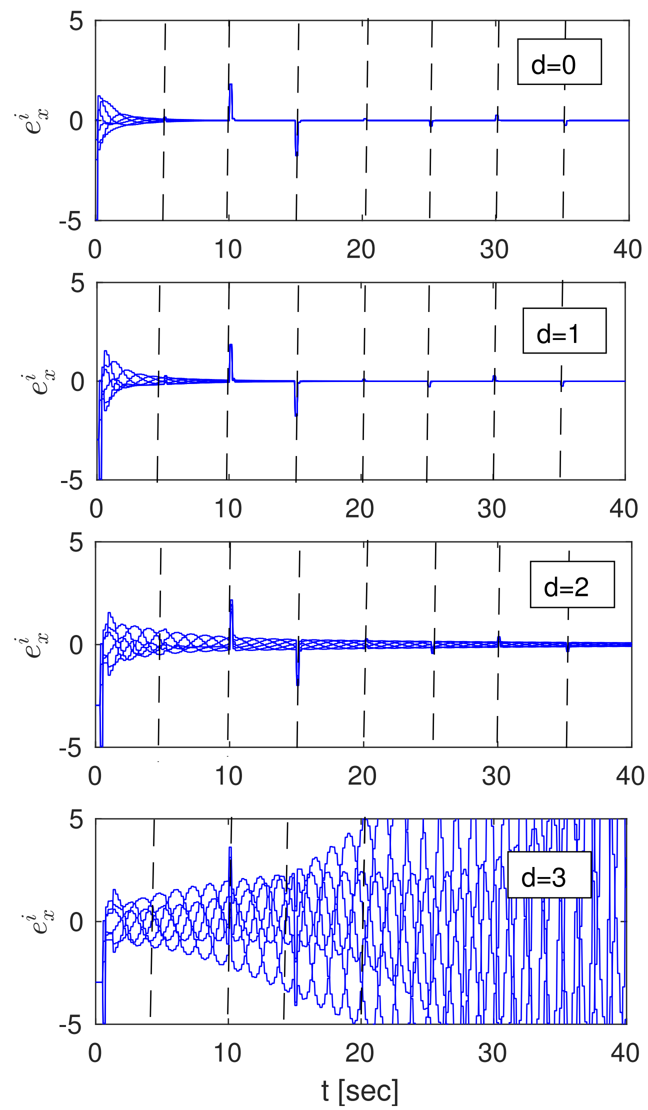


Fig. 3: Time history of the tracking error $e_x^i = x^i - \frac{1}{N}\sum_{j=1}^{N} \mathsf{r}^j$ of algorithm (9) over the network of Figure 1 (a) in the absence and presence of communication delay. The vertical dashed lines show the data sampling times (every 5 seconds).

V. Conclusions

We studied the robustness of a dynamic average consensus algorithm to fixed communication delays. Our study included both the continuous-time and the discrete-time implementations of this algorithm. For both implementations over SCWB digraphs, we (a) characterized the admissible delay range, (b) studied the effect of communication delay on the rate of convergence and the tracking error, (c) obtained upper bounds for them based on the value of the communication delay and the network's and the algorithm's parameters and (d) showed that the size of the admissible delay range has an inverse relation with the maximum degree of the interaction topology. Future work will be devoted to investigating the effect of uncommon and time-varying communication delay on the algorithms stability and convergence and expansion our results to switching graphs.

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Appendix: Stability of time-delay systems of retarded type

The exponential stability of a continuous-time linear DDE

$$\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t-\tau), \quad t \in \mathbb{R}_{\geq 0}, \\ &\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n, \quad \mathbf{x}(t) = \mathbf{g}(t) \text{ for } t \in [-\tau, 0),\end{aligned} \tag{A.37}$$

where $\mathbf{g}(t)$ is a continuously differential pre-shape function and $\tau \in \mathbb{R}_{>0}$ is the time-delay, is defined as follows.

Definition A.1 (Internal stability of (A.37) [29])**.** *For a given* $\tau \in \mathbb{R}_{>0}$ *the trivial solution* $\mathbf{x} \equiv \mathbf{0}$ *of the zero input system* *of* (A.37) *is said to be exponentially stable iff there exists a* $\mathscr{k}_\tau \in \mathbb{R}_{>0}$ *and an* $\rho_\tau \in \mathbb{R}_{>0}$ *such that for the given initial conditions the solution satisfies the inequality below*

$$\|\mathbf{x}(t)\| \leq \mathscr{k}_\tau e^{-\rho_\tau t} \sup_{\eta\in[-\tau,0]} \|\mathbf{x}(\eta)\|, \quad t \in \mathbb{R}_{\geq 0}. \tag{A.38}$$

The exponentially stability for discrete-time time-delay system

$$\dot{\mathbf{x}}(k+1) = \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k-\mathscr{d}), \quad k \in \mathbb{Z}_{k\geq 0}, \tag{A.39}$$
$$\mathbf{x}(0) = \mathbf{x}_0\mathbb{R}^n, \quad \mathbf{x}(k) = \mathbf{g}(k), \quad k \in \mathbb{Z}_{-\mathscr{d}}^{-1},$$

where $\mathscr{d} \in \mathbb{Z}_{k\geq 0}$ is the fixed time-step delay and $\mathbf{g}(k)$ is a pre-shape function, is defined as follows

Definition A.2 (Internal stability of (A.39))**.** *For a given* $\mathscr{d} \in \mathbb{Z}_{k\geq 0}$*, the trivial solution* $\mathbf{x} \equiv \mathbf{0}$ *of the zero input system of* (A.39) *is said to be exponentially stable iff there exists a* $\mathscr{k}_\mathscr{d} \in \mathbb{R}_{>0}$ *and an* $0 < \omega_\mathscr{d} < 1$ *such that for the given initial conditions the solution satisfies the inequality below*

$$\|\mathbf{x}(k)\| \leq \mathscr{k}_\mathscr{d}\, \omega_\mathscr{d}^k \sup_{k\in\mathbb{Z}_{-\mathscr{d}}^0} \|\mathbf{x}(k)\|, \quad k \in \mathbb{Z}_{\geq 0}. \tag{A.40}$$

Next, we develop an auxiliary result which we use in proof of our main result given in Theorem III.2.

Lemma A.1 (Exponential upper bound on a discrete-time delay dynamics)**.** *Consider the discrete-time time-delay system*

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k-\mathscr{d}), \quad k \in \mathbb{Z}_{\geq 0}, \tag{A.41}$$

where $\mathscr{d} \in \mathbb{Z}_{\geq 0}$*,* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and the initial conditions are* $\mathbf{x}(k) \in \mathbb{R}^n$ *for* $k \in \{-\mathscr{d}, \cdots, 0\}$*. Assume that the admissible delay range of* (A.41) *is non-empty, i.e., there exists a* $\bar{\mathscr{d}} \in \mathbb{Z}_{>0}$ *such that for* $\mathscr{d} \in [0, \bar{\mathscr{d}}]$ *the system* (A.41) *is exponentially stable. Then, for every* $\mathscr{d} \in [0, \bar{\mathscr{d}}]$ *there always exists a positive definite* $\mathbf{Q} \in \mathbb{R}^{(\mathscr{d}+1)n\times(\mathscr{d}+1)n}$ *and* $\mathscr{k}_\mathscr{d}, \omega_\mathscr{d} \in \mathbb{R}_{>0}$ *that satisfy*

$$\frac{1}{\mathscr{k}_\mathscr{d}}\mathbf{I} \leq \mathbf{Q} \leq \mathbf{I}, \quad 0 < \omega_\mathscr{d}^2 < 1, \quad \mathscr{k}_\mathscr{d} > 1, \tag{A.42a}$$
$$\mathbf{A}_{aug}^\top \mathbf{Q}\mathbf{A}_{aug} - \mathbf{Q} \leq -(1-\omega_\mathscr{d}^2)\,\mathbf{I}. \tag{A.42b}$$

Here, $\mathbf{A}_{aug} = \begin{bmatrix} \mathbf{0}_{\mathscr{d}n\times n} & \mathbf{I}_{\mathscr{d}n} \\ \mathbf{A} & \begin{bmatrix}\mathbf{0}_{n\times(\mathscr{d}-1)n} & \mathbf{I}_n\end{bmatrix}\end{bmatrix}$*. Moreover,*

(a) $\|\mathbf{x}(k)\| \leq \sqrt{\mathscr{d}+1}\,\mathscr{k}_\mathscr{d}\,\omega_\mathscr{d}^k \sup_{k\in\mathbb{Z}_{-\mathscr{d}}^0} \|\mathbf{x}(k)\|, \quad k \in \mathbb{Z}_{\geq 0}.$

(b) when $\mathbf{x}(k) = \mathbf{0}$ *for* $k \in \mathbb{Z}_{-\mathscr{d}}^{-1}$*, we have*

$$\|\mathbf{x}(k)\| \leq \mathscr{k}_\mathscr{d}\,\omega_\mathscr{d}^k\,\|\mathbf{x}(0)\|, \quad k \in \mathbb{Z}_{\geq 0}. \tag{A.43}$$

Proof: For a $\mathscr{d}$ in the admissible delay range of (A.41), let $\mathbf{x}_{\text{aug}}(k) = \begin{bmatrix}\mathbf{x}(k-\mathscr{d})^\top & \cdots & \mathbf{x}(k-1)^\top & \mathbf{x}(k)^\top\end{bmatrix}^\top$. Then,

$$\mathbf{x}_{\text{aug}}(k+1) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(k), \; k \in \mathbb{Z}_{\geq 0}, \; \mathbf{x}_{\text{aug}} \in \mathbb{R}^{(\mathscr{d}+1)n}, \tag{A.44}$$

with $\mathbf{A}_{\text{aug}}$ as defined in the statement. Because (A.41) is exponentially stable, the augmented state equation (A.44) is also exponentially stable. Then, by virtue of results on exponential stability of LTI discrete-time systems [41, Theorem 23.3], we have the guarantees that there exists a symmetric positive definite $\mathbf{P} \in \mathbb{R}^{n\times n}$ and scalars $\eta, \rho, \epsilon \in \mathbb{R}_{>0}$ that satisfy

$$\eta\,\mathbf{I} \leq \mathbf{P} \leq \rho\,\mathbf{I}, \quad \mathbf{A}_{\text{aug}}^\top \mathbf{P}\mathbf{A}_{\text{aug}} - \mathbf{P} \leq -\epsilon\,\mathbf{I}. \tag{A.45}$$

Moreover, for $\mathscr{k}_\mathscr{d} = \rho/\eta$ and $0 < \omega_\mathscr{d}^2 = 1 - \epsilon/\rho < 1$, we have

$$\|\mathbf{x}_{\text{aug}}(k)\| \leq \mathscr{k}_\mathscr{d}\,\omega_\mathscr{d}^k \|\mathbf{x}_{\text{aug}}(0)\|, \quad k \in \mathbb{Z}_{\geq 0}. \tag{A.46}$$

By applying the change of variables $\mathbf{Q} = \frac{1}{\rho}\mathbf{P}$, $\mathscr{k}_\mathscr{d} = \rho/\eta > 0$ and $0 < \omega_\mathscr{d}^2 = 1 - \epsilon/\rho < 1$, we can represent (A.45) in its equivalent form in (A.42). Moreover, because $\|\mathbf{x}(k)\| \leq \|\mathbf{x}_{\text{aug}}(k)\|$ and $\|\mathbf{x}_{\text{aug}}(0)\| = \sqrt{\|\sum_{j=0}^{-d}\mathbf{x}(j)\|^2} \leq \sqrt{\mathscr{d}+1}\sup_{k\in\mathbb{Z}_{-\mathscr{d}}^0}\|\mathbf{x}(k)\|$ and $\|\mathbf{x}(k)\| \leq \|\mathbf{x}_{\text{aug}}(k)\|$, we can use (A.46) to confirm claim (a) in the statement. On the other hand, because $\mathbf{x}(k) = \mathbf{0}$ for $k \in \mathbb{Z}_{-\mathscr{d}}^{-1}$, (A.46) also guarantees claim (b) in the statement. ■