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Author
Bardakci, K.

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Korkut Bardakci and Luis M. Bernardo

Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
Berkeley, CA 94720, U.S.A.

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String Field Equations from Generalized Sigma Model

Korkut Bardakci[♯] and Luis M. Bernardo[♭]

Department of Physics[†]
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
Berkeley, CA 94720, U.S.A.

Abstract

We propose a new approach for deriving the string field equations from a general sigma model on the world sheet. This approach leads to an equation which combines some of the attractive features of both the renormalization group method and the covariant beta function treatment of the massless excitations. It has the advantage of being covariant under a very general set of both local and non-local transformations in the field space. We apply it to the tachyon, massless and first massive level, and show that the resulting field equations reproduce the correct spectrum of a left-right symmetric closed bosonic string.

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[♯] e-mail: kbardakci@lbl.gov

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1. Introduction

A satisfactory formulation of string field theory continues to be one of the important open problems of string theory. There have been two main lines of approach to this problem in the past which have enjoyed varying degrees of success. The first approach (see refs. [3–7]) starts with the BRST formalism, developed first in the context of free strings, and generalizes it to interacting strings. This approach was first successfully applied to the open string theory, and, making use of the extension of the BRST method due to Batalin and Vilkovisky [15], it was later generalized to include closed strings [6]. The great advantage of this approach is that, compared to the alternatives, it is the most developed one from the technical stand point and its correctness is beyond doubt. However, so far it has not led to any substantial advances in our understanding of string theory. This is no doubt due in part to the complexity of this method, but also, it is due to the fact that initially a fixed background has to be specified. Although there are proofs of background independence [6], to our knowledge, there is no manifestly background independent formulation. A great advantage of such a formulation would be its role in unmasking various possible hidden symmetries of string theory. From the very beginning of string theory, symmetries like invariance under the local transformations of the target space coordinates, connected with the existence of the graviton, were difficult to understand in the usual formulation that uses a flat background metric. A manifestly background invariant formulation of string theory should improve our understanding of these symmetries, and also possibly shed light on the recently discovered symmetries such as duality.

The second main line of approach (see refs. [8], [1,2] and [9–14]) to understanding string dynamics starts with a two dimensional sigma model defined on the world sheet. In the earlier versions, the field content was restricted to massless excitations of the closed bosonic string, namely the graviton, the antisymmetric tensor and the dilaton [8]. The massive modes were neglected in order to have a renormalizable and classically scale invariant theory. The field equations are then derived by imposing quantum scale invariance, which amounts to demanding that the beta function vanish. In practice, one usually works with the beta function computed in the one loop approximation, using the background field method which preserves manifest covariance under local field redefinitions. This approach has many advantages over the first one: The background independence and the symmetries are made manifest, and the connection between conformal invariance and renormalization is made clear. There are also serious drawbacks: The massive modes are neglected and an off-shell formulation that goes beyond the equations of motion is missing. There is also the question about what happens beyond one loop; in many important cases the higher loop contributions lead to field redefinitions without changing the content of the equations of motion, although it is not clear how general this result is [14].

An important variation of this approach, which seems to overcome many of the drawbacks mentioned above, is originally due to Banks and Martinec [2]. The basic idea is to apply the renormalization group equations of Wilson and Polchinski to the two dimensional sigma model. The starting point is the most general sigma model, which includes all the massive levels of the string and is non-renormalizable in the conventional sense. A cutoff is introduced to make the model well-defined, and the equations of motion satisfied by the string fields are obtained by requiring the resulting partition function to be scale invariant. Although the classical action is scale non-invariant, and the cutoff introduces further scale breaking, the cancellation between these two effects makes the final scale invariance possible. Hughes, Liu and Polchinski [1] refined and extended this method, and they showed that the closed bosonic string scattering amplitudes in the classical (tree) limit can be derived from these equations. This approach has many nice features: It treats the whole string all at once and not just the massless levels, and it is also apparently exact and not limited to the one loop approximation. Finally, the emergence of the string amplitudes as a solution provides a stringent check on the resulting equations. Nevertheless, this approach also has some unsatisfactory features. As already noticed in ref. [1], the equations do not seem powerful enough to eliminate all the unwanted states of the string spectrum; some additional gauge invariance needed to eliminate them is apparently missing. Another drawback is that the coordinate system in the field space is fixed right from the beginning, and as a result, covariance under field transformations, which was such an attractive feature of ref. [8], is lost. We suspect that these problems are connected, and we offer some evidence in support of it.

In this paper, we propose a new approach which combines some of the advantageous features of both the renormalization group method and the covariant beta function treatment of the massless excitations. We start with a general sigma model that is supposed to represent all the levels of the closed bosonic string, with flat metric on the worldsheet. In section 2, the functional integral is written in the presence of a general background in a form completely covariant under coordinate (field) transformations, subject only to the condition that the determinant of the transformation is unity. These include not just the local transformations associated with gravity, but also non-local transformations which mix up different levels of the string. This covariant formulation requires the introduction of an as yet unknown connection with the correct transformation properties. To regulate the functional integral, just as in ref. [1], we introduce a cutoff in the free propagator, although our cutoff differs from theirs in some details. The effective action is then required to be invariant under the conformal (Virasoro) group in order to obtain the field equations. At this point, one has to specify the variations of the fields and the cutoff under conformal transformations. We chose the cutoff to transform exactly as in the renormalization group method. On the other hand, the standard expression for the conformal generators, is not covariant under coordinate transformations, and so it had to be promoted to a Killing vector with correct transformation properties. Putting everything together, we finally arrive at our version of the renormalization group equations written in covariant form (see (2.37)). This equation is not very useful as it stands; for one thing, it is an equation in two variables, and also it has an unknown function in it. From this equation, however, one can derive an infinite set of equations in a single variable free of the unknown function. The first of these is a one loop result, which is given explicitly (by (2.38)), and which forms the basis of all the following work. The remaining equations correspond to higher loops and they will not be considered in this paper. We stress that, in our approach, one loop result is exact; higher loops can provide more information, but they do not modify the one loop result [14].

Eq. (2.38) is still too formal to be useful as it stands; one needs explicit results for the connection and the Killing vector. At the moment, we do not have an exact expression for either of these, and to make progress, we resort to an expansion which we call the quasi-local expansion. This is an expansion in the number of derivatives on the world sheet and it is explained in section 3. The local coordinate transformations associated with gravity appear at zeroth order, and each new power of the expansion parameter b brings in transformations with two more derivatives on the world sheet. The levels of the string can also be similarly organized; the nth level goes with the power b^{n-1}. In section 3, we study the zeroth order term in the expansion. This corresponds to considering only the tachyon and the massless levels and imposing only local coordinate invariance. We are, therefore, back in familiar territory of renormalizable sigma model [8], where the connection and the generators of conformal algebra are well-known. Our reason for reexploring it is twofold: Firstly, we would like to check our formalism against standard results. This check is non-trivial since the way we treat the dilaton is different from the standard treatment [8], where the dilaton field is introduced as an independent field in the action from the beginning.

In our approach, the determinant of the metric plays the role of the dilaton field, and everything works out alright. We note that this determinant cannot be gauged away, since the coordinate transformations under which the model is invariant are restricted to have unit determinant. The second question we would like to answer is what happens if we abandon covariance by, for example, setting the connection equal to zero. In this case, we recover the gravitational equations in a fixed gauge, but we loose the equation of motion for the dilaton. Therefore, covariance is important in obtaining a complete set of equations.

The next step is to go to first order in the expansion, which is the subject of section 4. The first massive level of the string enters at this order, and also the coordinate (field) transformations include non-local terms for the first time. The important question is whether in this case, a suitable metric and a Killing vector that generates the conformal algebra exist. We show how to construct both the metric and the Killing vector to this order, and we derive the resulting equations of motion for the first massive level. An important check on the method is to find out whether the level structure agrees with that of the first massive level of the string. Again, to see what difference covariance makes, we check this for the non-covariant version, when the connection is set equal to zero. Just as in the case of ref. [1], we find that there are too many states, and there is not enough gauge invariance to eliminate the spurious states. In section 5, we investigate the first massive level in the covariant case. Here, the situation is the opposite; for a general left-right non-symmetric model, there are too few states. Only when the model is left-right symmetric, there is an exact match. We have to conclude that our approach works only for left-right symmetric models, although at this time, we do not have a good understanding of this restriction.

In our opinion, the main contribution of this paper is that, at least in the context of a natural expansion, the field equations of motion that follow from the general sigma model can be made covariant under not only local, but also non-local transformations in the field space. Furthermore, this covariance is crucial in eliminating spurious states of the first massive level. Clearly, as far as this question is concerned, we have only scratched the surface in this paper. It would be very desirable to go beyond the expansion we have used and to establish exact covariance under non-local transformations. A subsidiary result of this paper is the one loop basic equation, which, to some extent, bridges the gap between the standard treatment of the renormalizable sigma model, and the Wilson renormalization group approach. This equation may be of use in other applications.

2. Covariant Renormalization Group Equations

In this section, we derive a set of renormalization group equations for a general sigma model in a classical background. These equations are derived by imposing conformal invariance on the sigma model in the presence of a background field; they are covariant generalizations of the string equations of motion derived in ref. [1]. Throughout, we also work with flat worldsheet. We found the renormalization group approach of [1] advantageous for the following reason: When the generalized sigma model action S contains all the levels of the string and not just the massless ones, one is dealing with a conventionally non-renormalizable theory. In their approach, the conventionally non-renormalizable interactions coming from massive states, as well as the superrenormalizable interaction resulting from the tachyon, are treated on equal footing with the renormalizable interactions of the massless states. However, there are some problems with this approach. One of them is lack of covariance under the transformation of the target space coordinates. For example, the equations derived in [1] had a flat background; as a result, they were not explicitly covariant even under the usual coordinate transformations (local coordinate invariance) associated with gravity. Also, as pointed out by them, the equations do not seem strong enough to eliminate the states that are absent from the string spectrum. We will overcome both of these problems, at least for the first massive level, by combining the renormalization group approach with the traditional background field approach (see, for example [16]). Our approach will ensure covariance under not only local but also arbitrary non-local coordinate transformations, and by both considering a non-renormalizable action and also non-local coordinate transformations which mix up levels with different masses, the traditional treatment [8] will be extended to include massive levels of the string.

Our starting point is the partition function

$$Z[X_o,\Lambda] = \int [DX]\, e^{S'[X,\Lambda]}. \tag{2.1}$$

We will specify X_o and S' in terms of X and the action S shortly. The action S, which is a functional of the string coordinate[1] $X^{\mu\sigma}$ and a function of the cutoff parameter Λ, can be written as

$$\begin{aligned} S[X,\Lambda] &= \int d^2\sigma\, \mathcal{L}(X,\Lambda) \\ &= X^{\mu\sigma}\Delta_{\mu\sigma,\nu\sigma'}(\Lambda)X^{\nu\sigma'} + S_{\rm int}[X]. \end{aligned} \tag{2.2}$$

[1] $X^{\mu\sigma}$ is the same as $X^{\mu}(\sigma)$. In this paper σ always stands for worldsheet coordinates. All other greek indices refer to spacetime coordinates.

The cutoff appears only in the quadratic part of the action through the regularized free inverse propagator Δ; $S_{\rm int}$ is independent of the cutoff. When not essential, we will suppress the dependence on the cutoff; later, the cutoff dependence will be specified more precisely.

The primary goal of this paper is to formulate the string field equations in a form covariant under arbitrary functional transformations of the background field X_o:

$$X_o^{\mu\sigma} \rightarrow F^{\mu\sigma}(X_o). \tag{2.3}$$

We shall adopt the usual language of differential geometry: Tensors will be labeled by a composite index like $\mu\sigma$, and upper and lower indices will undergo the standard transformations of contravariant and covariant tensor indices. Also, when no confusion can arise, we follow the convention of summation over repeated discrete indices $\mu\nu$, and integration over repeated continuous indices σ, σ'. Here, σ stands for the worldsheet coordinates σ_0 and σ_1; the worldsheet metric is Euclidean. In the standard background field method, it is convenient to define a new coordinate variable $X^{\mu\sigma}(s)$ as a function of an internal parameter s through the geodesic equation

$$\frac{d}{ds}X^{\mu\sigma}(s) + \Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''}(X)\frac{d}{ds}X^{\alpha\sigma'}(s)\frac{d}{ds}X^{\beta\sigma''}(s) = 0, \tag{2.4}$$

with the boundary condition that, at $s=1$, $X^{\mu\sigma}(s=1) \equiv X^{\mu\sigma}$, where X is the original variable that appears in (2.1). As in this case, when the parameter s is omitted, this will mean X at $s=1$. The classical background field X_o is given by $X^{\mu\sigma}(s=0) \equiv X_o^{\mu\sigma}$, and it is also useful to define the tangent at $s=0$ by $(dX^{\mu\sigma}(s)/ds)_{s=0} \equiv \xi^{\mu\sigma}$. The connection Γ is yet unspecified; it is introduced in order to have covariance under (2.3). We shall see later on that quantum corrections break this group down to transformations with unit functional determinant:

$$\det\left(\frac{\delta F^{\mu\sigma}}{\delta X^{\nu\sigma'}}\right) = 1. \tag{2.5}$$

The idea of the background field method is to change variables in (2.1) from $X = X(1)$ to ξ at fixed X_o in order to exhibit the dependence on the classical field explicitly. This is conveniently done by expanding X and also the action in powers of the parameter s and setting $s=1$ at the end. For later use, here we write down the first three terms of the expansion of X:

$$X^{\mu\sigma} = X_o^{\mu\sigma} + \xi^{\mu\sigma} - \frac{1}{2}\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''}(X_o)\xi^{\alpha\sigma'}\xi^{\beta\sigma''} + \cdots \tag{2.6}$$

In the same way, the action can be expanded:

$$
\begin{aligned}
S[X] &= S[X_o] + \frac{d}{ds} S[X(s)]\Big|_{s=0} + \frac{1}{2}\frac{d^2}{ds^2} S[X(s)]\Big|_{s=0} + \cdots \\
&= S[X_o] + \frac{\delta S[X_o]}{\delta X_o^{\alpha\sigma}} \xi^{\alpha\sigma} + \frac{1}{2} G_{\alpha\sigma,\beta\sigma'}(X_o) \xi^{\alpha\sigma}\xi^{\beta\sigma'} + S_R, \\
&\equiv S[X_o, \xi].
\end{aligned}
\tag{2.7}
$$

Here, S_R denotes the cubic and higher order terms in ξ in the expansion of S. The propagator G in the presence of the background field is given by

$$
G_{\alpha\sigma,\beta\sigma'}(X_o) \equiv \frac{\delta^2 S[X_o]}{\delta X_o^{\alpha\sigma}\delta X_o^{\beta\sigma'}} - \frac{\delta S[X_o]}{\delta X_o^{\gamma\sigma''}} \Gamma^{\gamma\sigma''}_{\alpha\sigma,\beta\sigma'}(X_o). \tag{2.8}
$$

We are now ready to define S': it is gotten from S by subtracting the term linear in ξ:

$$
S'[X_o, \xi] = S[X(1)] - \frac{\delta S[X_o]}{\delta X_o^{\mu\sigma}} \xi^{\mu\sigma}. \tag{2.9}
$$

It is well-known that the transition from S to S' in the background field approach is equivalent to the introduction of a background field dependent source. Changing variables of integration from X to ξ in (2.1), the partition function can be written as

$$
\begin{aligned}
Z[X_o] &= \int [DX]\, e^{S'} \\
&= \int [D\xi]\, e^{(S'+\mathcal{M})},
\end{aligned}
\tag{2.10}
$$

where we have defined

$$
\det\left(\frac{\partial X}{\partial \xi}\right) \equiv \exp(\mathcal{M}). \tag{2.11}
$$

From now on, we will drop the subscript on X_o; X will stand for the classical background field, and in order to avoid confusion, the original field X will be denoted by $X(1)$. $\mathcal{M}$, the log of the jacobian, can be computed from (2.6); we write down the result to quadratic order in ξ:

$$
\mathcal{M} = -\left(\frac{1}{6} R_{\beta\sigma,\gamma\sigma'} + \frac{1}{2} D_{\beta\sigma}\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\gamma\sigma'}\right) \xi^{\beta\sigma}\xi^{\gamma\sigma'} + \mathcal{M}_R, \tag{2.12}
$$

where $\mathcal{M}_R$ is at least cubic in ξ. The combination of the Ricci tensor and the covariant derivative that appears on the right hand side of this equation is explicitly given by

$$
\begin{aligned}
&\frac{1}{6} R_{\beta\sigma,\gamma\sigma'} + \frac{1}{2} D_{\beta\sigma}\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\gamma\sigma'} = \\
&= \frac{1}{6}\left(\frac{\delta\Gamma^{\alpha\sigma''}_{\beta\sigma,\gamma\sigma'}}{\delta X^{\alpha\sigma''}} + \frac{\delta\Gamma^{\alpha\sigma''}_{\gamma\sigma',\alpha\sigma''}}{\delta X^{\beta\sigma}} + \frac{\delta\Gamma^{\alpha\sigma''}_{\beta\sigma,\alpha\sigma''}}{\delta X^{\gamma\sigma'}}\right) - \frac{1}{6}\left(\Gamma^{\alpha\sigma''}_{\nu\sigma''',\beta\sigma}\Gamma^{\nu\sigma'''}_{\alpha\sigma'',\gamma\sigma'} + 2\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\nu\sigma'''}\Gamma^{\nu\sigma'''}_{\beta\sigma,\gamma\sigma'}\right).
\end{aligned}
\tag{2.13}
$$

We note that

a) $\mathcal{M}$ is of order $\hbar$; it is a quantum correction to the classical action.

b) We dropped the term linear in ξ in (2.12); this can be taken care of by redefining S'.

c) Referring to (2.12), we see that the first term on the right, the Ricci tensor, is covariant; however, the second term, which is the covariant derivative of the contracted connection, is not. If the connection is derived from a metric, as will be the case here, we have

$$\Gamma^{\beta\sigma'}_{\alpha\sigma,\beta\sigma'} = \frac{1}{2g}\frac{\delta g}{\delta X^{\alpha\sigma}}. \tag{2.14}$$

Here, g is the determinant of the metric. From this, one sees that this term is covariant only under coordinate transformations with unit determinant. Therefore, although we have started with a fully covariant classical formulation, quantum corrections break the full diffeomorphism group down to transformations with unit determinant.

The next step in our program is to expand the partition function (see (2.10)) in a perturbation series. However, in contrast to the usual perturbation series, each term in our series is invariant under the restricted (unit determinant) transformations ((2.3), (2.5)). In deriving the perturbation expansion, we follow the standard functional approach discussed in the textbooks (see for example [17]). First, we define a free partition function Z_o in the absence of interaction (except with the external field), coupled to an external source J:

$$\begin{aligned} Z_o[X,J] &= \int [D\xi]\, \exp\left(S(X) + \frac{1}{2}G_{\alpha\sigma,\beta\sigma'}(X)\xi^{\alpha\sigma}\xi^{\beta\sigma'} + iJ_{\mu\sigma}\xi^{\mu\sigma}\right) \\ &= \exp\left(S(X) - \frac{1}{2}\mathrm{Tr}\log G + \frac{1}{2}J_{\mu\sigma}G^{\mu\sigma,\nu\sigma'}(X)J_{\nu\sigma'}\right). \end{aligned} \tag{2.15}$$

Here, G with the upper indices is the inverse of G with the lower indices. The full partition function can now be written as

$$Z(X) = \exp\left(S_I(X,P)\right) Z_o(X,J)|_{J=0}, \tag{2.16}$$

where,

$$S_I(X,\xi) = S_R(X,\xi) + \mathcal{M}(X,\xi), \tag{2.17}$$

and P, which replaces ξ as the argument of S_I in (2.16), is given by

$$P^{\mu\sigma} \equiv -i\frac{\delta}{\delta J_{\mu\sigma}}. \tag{2.18}$$

In (2.16), after the functional derivatives with respect to J act on $Z_o(X, J)$, J is set equal to zero. Eq. (2.16)can be used as the starting point of a perturbation expansion in powers of S_I. It is easy to see that the invariance under the coordinate transformations (2.3), subject to the constraint (2.5), are preserved in this expansion, if at the same time, J and P are transformed by

$$J_{\mu\sigma} \to \left(\frac{\delta F^{\nu\sigma'}}{\delta X^{\mu\sigma}}\right) J_{\nu\sigma'}, \qquad P^{\mu\sigma} \to \left(\frac{\delta F^{\mu\sigma}}{\delta X^{\nu\sigma'}}\right) P^{\nu\sigma'}. \tag{2.19}$$

Since we are dealing with a non-renormalizable interaction, the series is badly divergent. To have a well defined answer, we introduce a cutoff in the quadratic term in the action (see (2.2)). This cutoff in general violates the coordinate invariance described above. We shall later see how to deal with this problem; in fact, the solution will be at the heart of the derivation of the string equations.

Among the coordinate diffeomorphisms, conformal transformations on the world sheet will play a special role. They are given by

$$\sigma_+ \to f_+(\sigma_+, \sigma_-), \qquad \sigma_- \to f_-(\sigma_+, \sigma_-), \tag{2.20}$$

where

$$\sigma_+ \equiv \sigma_0 + i\sigma_1, \qquad \sigma_- \equiv \sigma_0 - i\sigma_1. \tag{2.21}$$

In what follows, to save writing, we will only exhibit the formulas corresponding to the f_+ transformations; the f_- expressions can be obtained from these by an interchange of $+$ with $-$. The string field equations follow from demanding that the partition function (2.1) be invariant under the conformal transformations. The first thing to check is the invariance of the quadratic part of the action in (2.2); in the absence of the cutoff, Δ is given by[2]

$$\Delta_{\mu\sigma,\nu\sigma'}(\Lambda = 0) = -\partial_{\sigma_+}\partial_{\sigma_-}\delta^2(\sigma - \sigma')\eta_{\mu\nu}, \tag{2.22}$$

and is conformally invariant. Here, $\eta_{\mu\nu}$ is the flat Minkowski metric. We introduce the cutoff by defining

$$\Delta_{\mu\sigma,\nu\sigma'}(\Lambda) = \eta_{\mu\nu}\Delta_{\sigma,\sigma'}(\Lambda). \tag{2.23}$$

[2] Whenever ∂_{σ_+} acts on a function of only σ, not σ and σ', it will be written as just ∂_+.

It will turn out to be useful to also define the following related functions:

$$\begin{aligned}&\Delta_{\sigma,\sigma'}(\Lambda) = -\partial_{\sigma_+}\partial_{\sigma_-}\delta^2_\Lambda(\sigma,\sigma'), \qquad && \partial_{\sigma_+}\partial_{\sigma_-}\Delta^{\sigma,\sigma'}(\Lambda) = -\tilde{\delta}^2_\Lambda(\sigma,\sigma'),\\ &\int d^2\sigma'\, \Delta_{\sigma,\sigma'}(\Lambda)\Delta^{\sigma',\sigma''}(\Lambda) = \delta^2(\sigma-\sigma''), \qquad && \int d^2\sigma'\, \tilde{\delta}^2_\Lambda(\sigma,\sigma')\delta^2_\Lambda(\sigma',\sigma'') = \delta^2(\sigma-\sigma'').\end{aligned} \tag{2.24}$$

The detailed structure of these functions is not important; all one needs to know is that $\delta^2_\Lambda(\sigma,\sigma')$ and $\tilde{\delta}^2_\Lambda(\sigma,\sigma')$ are smoothed out versions of the two dimensional Dirac delta function, and they are chosen so that $\Delta_{\sigma,\sigma'}(\Lambda)$, and as many derivatives of it as needed, are finite at $\sigma=\sigma'$, when the cutoff is finite. Notice that, unlike the propagator used in [1], our propagator need not vanish at $\sigma=\sigma'$. As a result, in contrast to [1], we shall encounter cutoff dependent terms in our equations. In some cases, these can be eliminated by renormalizing, for example, the slope parameter. In other cases, when such a renormalization is not possible, we will consider it as an anomaly and set its coefficient equal to zero. This will then provide additional useful information. For example, the field equation for the dilaton is derived in this fashion.

The cutoff violates conformal invariance; Δ with cutoff is no longer conformal invariant. To restore the conformal invariance, we have to supplement the transformations (2.20) by a suitable variation of the cutoff parameter(s). Specializing to infinitesimal variations, we define

$$\delta = \delta_\Lambda + \delta_v, \tag{2.25}$$

where δ_v is a "+" infinitesimal conformal transformation, which corresponds to taking the F in (2.3) and (2.19) to be[3]

$$F^{\mu\sigma}(X) \to F^{\mu\sigma}_v(X) = v(\sigma_+)\partial_+ X^{\mu\sigma}, \tag{2.26}$$

with a similar expression for the "$-$" transformations. Here v is an arbitrary function of σ_+, parametrizing conformal transformations. The variation δ_Λ is defined so that the quadratic part of the action in (2.2) is invariant under the total variation δ, resulting in the equation

$$\partial_{\sigma_+}\left(v(\sigma_+)\Delta_{\sigma,\sigma'}(\Lambda)\right) + \partial_{\sigma'_+}\left(v(\sigma'_+)\Delta_{\sigma,\sigma'}(\Lambda)\right) - \delta_\Lambda\left(\Delta_{\sigma,\sigma'}(\Lambda)\right) = 0. \tag{2.27}$$

---

[3] Then $\delta_v = \int d^2\sigma\, v(\sigma_+)\partial_+ X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}$, but in general $\delta_v = F^{\mu\sigma}_v(X)\frac{\delta}{\delta X^{\mu\sigma}}$. To be more precise, this F is not the same defined in (2.3) and (2.19), but is the $\tilde{F}$ defined by $F^{\mu\sigma}(X) = X^{\mu\sigma} + \tilde{F}^{\mu\sigma}(X)$, with the tilde dropped.

Later on, we will also need the cutoff variation of the propagator, so the variation of the function $\Delta^{\sigma,\sigma'}(\Lambda)$, which is the inverse of $\Delta_{\sigma,\sigma'}(\Lambda)$, is needed. At first, it may seem that the inverse also satisfies the same equation; and this would be true if the inverse were unique. However, there is a well-known ambiguity in going from Δ to its inverse; for example, in the absence of cutoff

$$\Delta^{\sigma,\sigma'}(\Lambda=0)=\frac{1}{4\pi}\log\left((\sigma-\sigma')^2\right)+k_+(\sigma_+)+k_-(\sigma_-). \tag{2.28}$$

The functions k_+ and k_- are arbitrary, resulting in a non-unique inverse. The variation of the propagator under the change of the cutoff also suffers from the same ambiguity. This ambiguity can be resolved by demanding that the ultraviolet cutoff does not change the long distance behavior of the propagator[4]. Imposing this boundary condition, we have the following equation:

$$v(\sigma_+)\partial_{\sigma_+}\Delta^{\sigma,\sigma'}(\Lambda)+v(\sigma'_+)\partial_{\sigma'_+}\Delta^{\sigma,\sigma'}(\Lambda)-\delta_\Lambda\Delta^{\sigma,\sigma'}(\Lambda)=\frac{1}{4\pi}\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}. \tag{2.29}$$

Comparing (2.27) to (2.29), we note that, because of the boundary conditions at large distances, an extra term appeared on the right hand side of (2.29). This term is the source of the conformal anomaly; in the string equations of [1], this anomaly is cancelled by the explicitly conformal non-invariant terms in the action. The above equation will play an important role in the calculations that follow: In applying the fundamental equation (2.38) to special cases, one needs an explicit expression for the variation of the propagator under the change of the cutoff; namely the term $\delta_\Lambda\Delta^{\sigma,\sigma'}$ in the above equation. This equation therefore provides the needed explicit expression. Another point that needs to be clarified is the dependence of $\Delta^{\sigma,\sigma'}$ on the variables σ and σ'. We would like to impose two dimensional rotation and translation invariance on the world sheet even in the presence of the cutoff. There is no problem in imposing both of these invariances for a fixed cutoff, however, when the cutoff is changed infinitesimally from this fixed value, its variation is given by (2.29) and it is clearly no longer translation and rotation invariant. This is the consequence of the translation and rotation non-invariant long distance boundary condition imposed in determining the cutoff variation.

[4] For a detailed treatment of this question, see [1].

The generators of conformal transformations, in the form they are expressed in (2.26), are not covariant under general coordinate transformations (2.3). They can be cast into a covariant form by writing

$$F_v^{\mu\sigma}(X) = v(\sigma_+)\partial_+ X^{\mu\sigma} + \int d^2\sigma'\, v(\sigma'_+) f_{\sigma'}^{\mu\sigma}(X). \tag{2.30}$$

The function f is introduced to make F transform as a vector in the indices $\mu\sigma$; note that the subscript σ' in $f_{\sigma'}^{\mu\sigma}$ is not a tensor index. A further constraint on f comes from demanding that δ_v satisfy the Virasoro algebra. We will specify this function later on when we discuss concrete examples.

We are now ready to write down the fundamental string field equation; it is obtained by demanding the invariance of the partition function under the conformal variation (2.25):

$$\delta Z(X, \Lambda) = 0. \tag{2.31}$$

We note that this is not merely a requirement of covariance but one of invariance. In this respect, $F_v^{\mu\sigma}$ acts like a Killing vector that generates the conformal symmetry. Carrying out the operations represented by δ on the right hand side of (2.16) gives

$$\exp\left(S_I(X,P)\right)\mathcal{H}(X,P)\exp\left(\frac{1}{2}J_{\mu\sigma}G^{\mu\sigma,\nu\sigma'}J_{\nu\sigma'}\right)\Big|_{J=0} = 0, \tag{2.32}$$

where $\mathcal{H}(X,P)$ will be defined shortly. From this equation, it is tempting to conclude that

$$\mathcal{H}(X,P) = 0. \tag{2.33}$$

However, this conclusion is not correct; eq. (2.33) is too strong as it stands. This is because of the existence of an identity of the form

$$\exp\left(S_I(X,P)\right)K^{\mu\sigma}(X,P)\left(\frac{\delta S_I}{\delta P^{\mu\sigma}} + G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right)\exp\left(\frac{1}{2}J_{\alpha\sigma''}G^{\alpha\sigma'',\beta\sigma'''}J_{\beta\sigma'''}\right)\Big|_{J=0} = 0, \tag{2.34}$$

where K satisfies

$$\frac{\delta K^{\mu\sigma}}{\delta P^{\mu\sigma}} = 0, \tag{2.35}$$

but is otherwise an arbitrary function of X and P. This identity, easy to verify directly, can be understood as follows. Reversing the steps leading from (2.10) to (2.16), one can get rid of the operator P and write the above identity as an integral over the variable ξ. The identity is then satisfied by virtue of the integrand being a total derivative. A total derivative corresponds to an infinitesimal change of variable in the integral in (2.10); therefore (2.34) is equivalent to the invariance of (2.10) under such a change of variable. Eq. (2.35) expresses the restriction that the Jacobian of this transformation is unity so as to leave the action unchanged, and (2.33) amounts to deducing the vanishing of the integrand from the vanishing of an integral and it is therefore too strong; the correct equation should be

$$\mathcal{H} = K^{\mu\sigma}\left(\frac{\delta S_I}{\delta P^{\mu\sigma}} + G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right), \tag{2.36}$$

and (2.32) is satisfied by virtue of (2.34).

The function $\mathcal{H}$ can be determined by carrying out the variations indicated in (2.31); the result is

$$\begin{aligned}
&\left(F_v^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}} + P^{\nu\sigma'}\frac{\delta F_v^{\mu\sigma}}{\delta X^{\nu\sigma'}}\frac{\delta}{\delta P^{\mu\sigma}} + \delta_\Lambda\right)\left(S(X) + S_I(X,P) - \frac{1}{2}\mathrm{Tr}\log G(X)\right)\\
&\quad - \frac{1}{2}\left(F_v^{\alpha\sigma''}\frac{\delta G^{\mu\sigma,\nu\sigma'}}{\delta X^{\alpha\sigma''}} - \frac{\delta F_v^{\mu\sigma}}{\delta X^{\alpha\sigma''}}G^{\alpha\sigma'',\nu\sigma'} - \frac{\delta F_v^{\nu\sigma'}}{\delta X^{\alpha\sigma''}}G^{\mu\sigma,\alpha\sigma''} + \delta_\Lambda G^{\mu\sigma,\nu\sigma'}\right)\\
&\quad \times\left(\frac{\delta^2 S_I(X,P)}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}} + \frac{\delta S_I}{\delta P^{\mu\sigma}}\frac{\delta S_I}{\delta P^{\nu\sigma'}}\right) = K^{\mu\sigma}\left(\frac{\delta S_I}{\delta P^{\mu\sigma}} + G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right).
\end{aligned} \tag{2.37}$$

Eq. (2.37) is our version of renormalization group equations for the string action S. As it stands, it has two unusual features:

a) It is an equation in two variables X and P, whereas the standard renormalization group equations are in a single variable, the background field X.
b) It contains a function $K^{\mu\sigma}(X,P)$, arbitrary except for the constraint given by (2.35).

We will now show that these two seeming defects cancel each other; it is possible to convert (2.37) into an equation in a single variable X by taking advantage of the arbitrariness of the function K. To see this, imagine expanding S_I, $\mathcal{M}$ and K in a power series in the variable P. By equating different powers of P on both sides of the equation, we obtain an infinite set of equations, each in the single variable X. Let us now focus on the equation zeroth order in P. Since the right hand side of (2.37) starts with a linear term in P, this equation receives no contribution from K. This follows from the fact that S_R, $\mathcal{M}$ and therefore S_I all start at least quadratically in the expansion in powers of P. We therefore have an equation in the single variable X and free of the ambiguity coming from K:

$$E_G + E_{\mathcal{M}} = 0, \tag{2.38}$$

where,

$$E_G = \left(F_v^{\mu\sigma} \frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda \right) \left(S - \frac{1}{2} \mathrm{Tr} \log G \right), \tag{2.39}$$

and

$$\begin{aligned} E_{\mathcal{M}} =& \frac{1}{2} \left(-F_v^{\alpha\sigma''} \frac{\delta G^{\mu\sigma,\nu\sigma'}}{\delta X^{\alpha\sigma''}} + \frac{\delta F_v^{\mu\sigma}}{\delta X^{\alpha\sigma''}} G^{\alpha\sigma'',\nu\sigma'} + \frac{\delta F_v^{\nu\sigma}}{\delta X^{\alpha\sigma''}} G^{\mu\sigma,\alpha\sigma''} - \delta_\Lambda G^{\mu\sigma,\nu\sigma'} \right) \\ & \times \left[\frac{\delta^2 \mathcal{M}}{\delta P^{\mu\sigma} \delta P^{\nu\sigma'}} + \frac{\delta \mathcal{M}}{\delta P^{\mu\sigma}} \frac{\delta \mathcal{M}}{\delta P^{\nu\sigma'}} \right]_{P=0} = 0. \end{aligned} \tag{2.40}$$

This equation is the fundamental result of the section. In the next two sections, it will provide our starting point for the derivation of the string field equations.

We end this section with a couple of comments:

a) Eq. (2.38) is a a one loop result, which one can verify either by counting powers of $\hbar$ or more simply from the appearance of the "Tr log" type terms. It should therefore agree with the standard treatment [8] for a renormalizable action S. We will make this comparison in the next section.

b) This observation leads to an apparent paradox: No approximation has been made in deriving (2.38), yet it is clearly a one loop result. We have to conclude that the one loop result leads to exact string field equations.

c) There are of course an additional infinite number of equations coming from higher powers of P. These equations can then be used to determine the unknown function $K^{\mu\sigma}(X, P)$. One can then extract relations not involving K by using the constraint (2.35). These appear to come from two or more loops. We do not know whether these equations are redundant, or whether they contain additional information, which would then supplement the one loop result but not change it.

d) Eq. (2.38) contains two unknown functions: The Killing vector $F_v^{\mu\sigma}$ and implicitly, the connection $\Gamma^{\gamma\sigma''}_{\alpha\sigma,\beta\sigma'}$ (See (2.8)). They have to be expressed in terms of the fields that appear in S. This will be done in the following sections.

3. The Tachyon and the Massless Level

In this section, we will apply the formalism developed in the last section to the two lowest levels of a closed, unoriented bosonic string, the tachyon and the massless level. Ideally, one would like to start with S as an arbitrary functional of $X^{\mu\sigma}$ and try to solve (2.38) in all its generality. However, this direct approach seems hopelessly complicated and not particularly useful. Instead, the problem is made tractable by expanding S in powers of the σ derivatives of $X^{\mu\sigma}$. We will call this the quasi-local expansion. This expansion is quite natural from the point of two dimensional field theory on the world sheet and it has been the basis of most of the work done on this subject. From the string point of view, it is an expansion in the level number, two derivatives in σ corresponding to an increase of one unit in level number. We note that as a consequence of two-dimensional rotation invariance on the world sheet, which we shall always assume, there is always an equal number of derivatives with respect to σ_+ and σ_-. It is then convenient to introduce a parameter "b" to keep track of the expansion: The field representing the nth level of the string will be multiplied by $b^{(n-1)}$. For example, the tachyon has coefficient b^{-1} and the massless level is independent of b. The first two terms in the quasi-local expansion of S are then given by

$$S = b^{-1}S^{(-1)} + S^{(0)} + \cdots = \int d^2\sigma \, \left(b^{-1}\Phi(X(\sigma)) + \tilde{g}_{\mu\nu}(X(\sigma))\partial_+ X^{\mu\sigma}\partial_- X^{\nu\sigma}\right). \quad (3.1)$$

Here and in the sequel, we have adopted the following notation: The superscripts (-1), (0), etc., refer to terms in S proportional to the corresponding powers of b. Expressions like $\Phi(X(\sigma))$ denote local functions of the coordinate $X(\sigma)$, whereas expressions such as $F^{\mu\sigma}(X)$ denote functionals in the same coordinate. Also, we should make clear that the parameter "b" is merely a bookkeeping device and can be set equal to one at the end of the calculation.

In the same spirit, the coordinate transformations (2.3) have a quasi-local expansion:

$$F^{\mu\sigma}(X) = f^{\mu}(X(\sigma)) + bf^{\mu}{}_{\nu\lambda}(X(\sigma))\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + bf^{\mu}{}_{\nu}\partial_+\partial_- X^{\nu\sigma}(X(\sigma)) + \cdots. \quad (3.2)$$

The first term is the local coordinate transformation associated with gravity; terms with increasing powers of b contain higher derivatives of σ and become increasingly non-local. In this section, we will only be concerned with invariance under local transformations represented by the first term in (3.2). However, the general strategy, pursued in the next section, is to determine $F_v^{\mu\sigma}$, the generator of the conformal transformation (see (2.30)), and the connection Γ as a power series in b so as to achieve covariance under both local and non-local transformations. To zeroth order in b, F, the generator of the conformal transformations, is given by the first term in (2.30); the function $f_v^{\mu\sigma}$ is at least of first order in b.

To simplify the exposition, we have so far neglected the cutoff dependence in S (see (2.2)). With the cutoff restored, the second term in (3.1) should read

$$S^{(0)} = \int d^2\!\sigma d^2\!\sigma' \, \tilde{g}_{\mu\sigma,\nu\sigma'} \partial_{\sigma_+} X^{\mu\sigma} \partial_{\sigma'_-} X^{\nu\sigma'}, \tag{3.3}$$

where $\tilde{g}$ is given by

$$\tilde{g}_{\mu\sigma,\nu\sigma'} = \eta_{\mu\nu} \delta^2_\Lambda(\sigma,\sigma') + \tilde{h}_{\mu\nu}(X(\sigma))\delta^2(\sigma-\sigma'). \tag{3.4}$$

$\delta^2_\Lambda(\sigma,\sigma')$ is defined in (2.24) and $\tilde{h}$ is a cutoff independent local function of $X(\sigma)$.

We have now to determine the connection Γ and the generator of conformal transformations $F^{\mu\sigma}_v$ to zeroth order in b. We have already observed above that F is given by (2.26) to zeroth order, since f in (2.30) is already first order in b. As for the connection, it will be derived from a metric that transforms correctly under local transformations. The standard choice for the metric made in sigma model calculations, which we shall adopt, is the symmetric part of $\tilde{g}$ in (3.3):

$$g_{\mu\sigma,\nu\sigma'} = \frac{1}{2}\left(\tilde{g}_{\mu\sigma,\nu\sigma'} + \tilde{g}_{\nu\sigma',\mu\sigma}\right) = \eta_{\mu\nu}\delta^2_\Lambda(\sigma,\sigma') + h_{\mu\nu}(X(\sigma))\delta^2(\sigma-\sigma'), \tag{3.5}$$

where,

$$h_{\mu\nu} = \frac{1}{2}\left(\tilde{h}_{\mu\nu} + \tilde{h}_{\nu\mu}\right). \tag{3.6}$$

The connection, to zeroth order in b, is given in terms of the metric by the standard formula:

$$\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''} = \frac{1}{2} g^{\mu\sigma,\lambda\sigma'''}\left(\frac{\delta g_{\lambda\sigma''',\beta\sigma''}}{\delta X^{\alpha\sigma'}} + \frac{\delta g_{\alpha\sigma',\lambda\sigma'''}}{\delta X^{\beta\sigma''}} - \frac{\delta g_{\alpha\sigma',\beta\sigma''}}{\delta X^{\lambda\sigma'''}}\right). \tag{3.7}$$

With these preliminaries out of the way, we are ready to write down the field equation for the tachyon field. This we do by extracting terms lowest order in b, proportional to b^{-1}, from (2.38). Notice that only E_G contributes. The equation reduces to the following simple form

$$\left(\int d^2\!\sigma \, v(\sigma_+)\partial_+ X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda\right)\left(S^{(-1)}_{\text{int}}[X] - \frac{1}{2}\text{Tr}\log G^{(-1)}\right) = 0. \tag{3.8}$$

The first term on the left is easy to calculate:

$$\int d^2\!\sigma \, v(\sigma_+)\partial_+ X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}\left(S^{(-1)}_{\text{int}}[X]\right) = -\int d^2\!\sigma \, v'(\sigma_+)\Phi(X(\sigma)). \tag{3.9}$$

In calculating the contribution of the second term, we take advantage of the following simplifications: We are going to drop all the non-local terms that can arise in the expansion of this term. Such non-local terms are in general present since the action S that satisfies (2.37) is not necessarily one particle irreducible, and we wish to extract the one particle irreducible part that is local. Another simplification follows from the fact that the result clearly is going to be a covariant Klein-Gordon equation for the tachyon field Φ in the background metric g. We can then first linearize this equation by expanding to first order in $h_{\mu\nu}$ of (3.5) around the flat background, and then covariantize the result to arrive at the full answer in an arbitrary background. This will be our strategy in the rest of the paper; only the linear part of the field equations will be computed in the presence of a flat background, and the result will be generalized to a non-trivial background, making use of the powerful restrictions resulting from covariance.

Eq. (3.8) is linearized by setting

$$G_{\mu\sigma,\nu\sigma'}(X) = 2\Delta_{\mu\sigma,\nu\sigma'} + H_{\mu\sigma,\nu\sigma'}(X), \tag{3.10}$$

and by expanding the "Tr log" to first order in H:

$$\mathrm{Tr}\log G \cong \frac{1}{2}\Delta^{\mu\sigma,\nu\sigma'} H_{\mu\sigma,\nu\sigma'}. \tag{3.11}$$

The linear part of H, to order b^{-1}, is given by

$$H^{(-1)}_{\mu\sigma,\nu\sigma'} \cong \frac{\delta^2 S_{\rm int}[X]}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \cong \delta^2(\sigma-\sigma')\partial_\mu\partial_\nu\Phi(X(\sigma)). \tag{3.12}$$

In calculating the left hand side of (3.8), the following identity proves useful:

$$\begin{aligned}\Delta^{\mu\sigma,\nu\sigma'}\int d^2\sigma''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}}\left(\frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\right) = \Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\\ \times\left(\int d^2\sigma''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''}\frac{\delta S}{\delta X^{\lambda\sigma''}}\right) - \frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\left(v(\sigma_+)\partial_{\sigma_+} + v(\sigma'_+)\partial_{\sigma'_+}\right)\Delta^{\mu\sigma,\nu\sigma'}.\end{aligned} \tag{3.13}$$

The second term on the right hand side can be calculated making use of (2.29), leading to the equation

$$\int d^2\sigma\, v'(\sigma_+)\left[-\Phi(X(\sigma)) + \frac{1}{16\pi}\partial^\mu\partial_\mu\Phi(X(\sigma)) + \frac{1}{4}\Delta^{(0)}(\Lambda)\partial^\mu\partial_\mu\Phi(X(\sigma))\right] = 0, \tag{3.14}$$

where $\Delta^{(0)}(\Lambda)$ is the propagator $\Delta^{\sigma,\sigma'}(\Lambda)$, evaluated at $\sigma = \sigma'$. By translation invariance, it is independent of σ. Since Φ is only a function of $X^{\mu\sigma}$ and not of its derivatives with respect to σ, it follows that

$$\left(\frac{1}{16\pi} + \frac{1}{4}\Delta^{(0)}(\Lambda)\right) \partial^\mu \partial_\mu \Phi(X(\sigma)) - \Phi(X(\sigma)) = 0. \tag{3.15}$$

The term $\Delta^{(0)}(\Lambda)$ is cutoff dependent and it blows up as $\Lambda \to \infty$. This cutoff dependent term can be eliminated by explicitly introducing the slope parameter which we have suppressed and by renormalizing it. The same cutoff dependent term is encountered in the equations for the higher levels and it is again eliminated by the same slope renormalization. Finally, (3.15) can easily be generalized to an arbitrary background by using the metric given by (3.5) and casting it into a covariant form.

The next step is to derive the field equation for $\tilde{g}$, which includes both the metric ((3.5)), and the antisymmetric tensor

$$B_{\mu\nu} = \frac{1}{2}(\tilde{h}_{\mu\nu} - \tilde{h}_{\nu\mu}). \tag{3.16}$$

To do this, we have to extract zeroth order terms in b from (2.38). It is useful to distinguish between the two terms E_G and $E_{\mathcal{M}}$, the former coming from the variation of the $\mathrm{Tr}\log G$, and the latter coming from the variation of $\mathcal{M}$. The reason for this distinction is that the E_G is cutoff independent, whereas $E_{\mathcal{M}}$ is proportional to a cutoff dependent factor. We will argue later that these two terms must vanish separately, yielding two separate equations. The first of these will be the equation for the metric g and the antisymmetric tensor B; the second will provide the equation for the dilaton. Our strategy is again to expand around the flat background to first order in h and B, and use covariance to arrive at the full answer. We make use of (3.11) to calculate $\mathrm{Tr}\log G$, extracting the linear piece in $\tilde{h}$, and using (2.29), we find

$$\begin{aligned} E_G^{(0)} &= \left(\int d^2\!\tilde{\sigma}\, v(\sigma_+)\partial_+ X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda\right)\left(H^{(0)}_{\mu\sigma',\nu\sigma'}\Delta^{\mu\sigma',\nu\sigma'}(\Lambda)\right)\Big|_{\mathrm{lin}} = \\ &= -\frac{1}{4\pi}\int d^2\!\tilde{\sigma} d^2\!\tilde{\sigma}'\, \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\left(\frac{\delta^2 S^{(0)}_{\mathrm{int}}[X]}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}}\Big|_{\mathrm{lin}} + 2\delta^2(\sigma - \sigma')\Gamma^\lambda_{\mu\mu}(X(\sigma))\partial_+\partial_- X^{\lambda\sigma}\right). \end{aligned} \tag{3.17}$$

Here, the subscript "lin" refers to terms linear in $\tilde{h}$. The Γ that appears on the right hand side is the linearized form of the connection (3.7):

$$\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''} \cong \tilde{\delta}^2_\Lambda(\sigma,\sigma')\delta^2(\sigma' - \sigma'')\Gamma^\mu_{\alpha\beta}(X(\sigma')), \tag{3.18}$$

where,

$$\Gamma^{\mu}_{\alpha\beta}(X(\sigma)) \cong \frac{1}{2}\eta^{\mu\nu}\Big(\partial_\alpha h_{\nu\beta}(X(\sigma)) + \partial_\beta h_{\alpha\nu}(X(\sigma)) - \partial_\nu h_{\alpha\beta}(X(\sigma))\Big). \tag{3.19}$$

The functional derivative of $S^{(0)}_{\text{int}}$ can be calculated from (3.1), and repeating the steps that led to (3.15) gives the following field equation:

$$\Box \tilde{h}_{\nu\lambda} - \partial_\nu \partial_\mu \tilde{h}_{\mu\lambda} + \partial_\mu \partial_\lambda \tilde{h}_{\mu\nu} - 2\partial_\lambda \Gamma^{\nu}_{\mu\mu} = 0, \tag{3.20}$$

where $\Box = \partial_\mu \partial^\mu$. Here, since we use a flat metric to raise and lower indices, there is no real distinction between upper and lower indices. This will be understood whenever we have repeated upper or lower indices.

The equation above came from the conformal transformations in the variable σ_+. The other set of conformal transformations in the variable σ_- result in an additional equation:

$$\Box \tilde{h}_{\nu\lambda} - \partial_\lambda \partial_\mu \tilde{h}_{\nu\mu} + \partial_\nu \partial_\mu \tilde{h}_{\lambda\mu} - 2\partial_\nu \Gamma^{\lambda}_{\mu\mu} = 0. \tag{3.21}$$

It is now convenient to combine these two equations and rewrite them in terms of h and the antisymmetric tensor B. Interestingly, we find that, without any reference to (3.19), these equations fix the contracted connection Γ up to an arbitrary scalar field ϕ, which we shall identify with the dilaton field:

$$\Gamma^{\lambda}_{\mu\mu} = \partial_\mu h_{\mu\lambda} + \partial_\lambda \phi, \tag{3.22}$$

and we arrive at the following equations for h and B:

$$\Box h_{\nu\lambda} - \partial_\nu \partial_\mu h_{\mu\lambda} - \partial_\lambda \partial_\mu h_{\mu\nu} - 2\partial_\nu \partial_\lambda \phi = 0, \tag{3.23}$$

$$\Box B_{\nu\lambda} - \partial_\nu \partial_\mu B_{\mu\lambda} + \partial_\lambda \partial_\mu B_{\mu\nu} = 0. \tag{3.24}$$

Comparing with (3.19) determines ϕ:

$$\phi = -\frac{1}{2} h_{\mu\mu}, \tag{3.25}$$

and substituting this result back into (3.23) gives the standard equations of gravity without source in linearized form. The equation for the antisymmetric tensor $B_{\mu\nu}$ is the linearized version of the standard result of [8].

Up to this point, we have not taken into account $E_{\mathcal{M}}$. Let us first calculate the $\mathcal{M}$ dependent factor in (2.40). From equation (2.12), we find that

$$\begin{aligned} \left.\frac{\delta^2\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right|_{P=0} &= -\frac{1}{3}\left(\frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\mu\sigma}}{\delta X^{\nu\sigma'}} + \frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\nu\sigma'}}{\delta X^{\mu\sigma}} + \frac{\delta\Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'}}{\delta X^{\lambda\sigma''}}\right) \\ &= -\frac{1}{3}\Delta^{(0)}(\Lambda)\delta^2(\sigma-\sigma')\Big(\partial_\nu\Gamma^\lambda_{\lambda\mu}(X(\sigma)) + \partial_\mu\Gamma^\lambda_{\lambda\nu}(X(\sigma)) + \partial_\lambda\Gamma^\lambda_{\mu\nu}(X(\sigma))\Big). \end{aligned} \tag{3.26}$$

The term which is quadratic in $\mathcal{M}$ will not contribute since it is non-linear (and also non-local). To the order we are considering, the first factor on the right in (2.40) can be evaluated by setting $G^{\alpha\sigma,\beta\sigma'}$ equal to $\Delta^{\alpha\sigma,\beta\sigma'}$, with the result,

$$\begin{aligned} E^{(0)}_{\mathcal{M}} &= \frac{1}{2}\left(v(\sigma_+)\partial_{\sigma_+}\Delta^{\mu\sigma,\nu\sigma'} + v(\sigma'_+)\partial_{\sigma'_+}\Delta^{\mu\sigma,\nu\sigma'} - \delta_\Lambda\Delta^{\mu\sigma,\nu\sigma'}\right)\left.\frac{\delta^2\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right|_{P=0} \\ &= -\frac{1}{24\pi}\Delta^{(0)}(\Lambda)\eta^{\mu\nu}\int d^2\sigma\, v'(\sigma_+)\left(2\partial_\nu\Gamma^\lambda_{\lambda\mu}(X(\sigma)) + \partial_\lambda\Gamma^\lambda_{\mu\nu}(X(\sigma))\right). \end{aligned} \tag{3.27}$$

We see that unlike the cutoff independent E_G, $E_{\mathcal{M}}$ is proportional to the cutoff dependent factor $\Delta^{(0)}(\Lambda)$. It easy to show that this term cannot be eliminated by the addition of any local counterterm to S, and therefore, it must be set equal to zero by itself. This gives us the additional equation

$$2\partial_\mu\Gamma^\lambda_{\lambda\mu} + \partial_\lambda\Gamma^\lambda_{\mu\mu} = \frac{1}{2}\Box h_{\mu\mu} + \partial_\mu\partial_\nu h_{\mu\nu} = 0, \tag{3.28}$$

and combining this with (3.23) and (3.25), we find that

$$\Box h_{\mu\mu} = 0, \qquad \partial_\mu\partial_\nu h_{\mu\nu} = 0, \tag{3.29}$$

and

$$\Box h_{\mu\nu} - \partial_\mu\partial_\lambda h_{\lambda\nu} - \partial_\nu\partial_\lambda h_{\lambda\mu} - \partial_\mu\partial_\nu h_{\lambda\lambda} = 0. \tag{3.30}$$

Equations (3.29) and (3.30) describe the coupled graviton-dilaton system in the linear approximation. To see this, we note that these equations are invariant only under coordinate transformations of unit determinant, which, linearized, results in invariance under gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\kappa_\nu + \partial_\nu\kappa_\mu, \tag{3.31}$$

with the important restriction that $\partial_\mu\kappa^\mu = 0$. This restriction to unit determinant was explained in section 2 in the paragraph following (2.14). As a consequence, the trace of h, $h_{\mu\mu}$, which can be gauged away if there is invariance under unrestricted coordinate transformations, can no longer be eliminated and becomes a dynamical degree of freedom. Up to normalization, we identify it with the dilaton field ϕ. The natural candidate for the graviton field is the traceless component of h:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{D}\eta_{\mu\nu}h_{\lambda\lambda}, \tag{3.32}$$

where D is the dimension of space. We identify $\bar{h}$ with the graviton field in the gauge where the metric has unit determinant and as a consequence, the graviton field is traceless. $\bar{h}$ also satisfies (3.30), which is the correct equation for the graviton coupled to the dilaton in this gauge. We would like to point out the difference between our treatment of the dilaton and the standard approach. In the standard treatment, in addition to $X^{\mu\sigma}$, the dilaton field ϕ is introduced in the action from the beginning, and the theory is regularized by going from 2 to $2+\epsilon$ dimensions on the world sheet. We stay with a two dimensional world sheet, regularize only the free propagator (see (2.2)), and the dilaton field is identified with the log of the determinant of the metric. This identification is only possible because the full coordinate invariance is broken down to transformations of unit determinant.

We end this section with a few observations:

a) So far, we have worked out the coupled system of the graviton, dilaton and the antisymmetric tensor only in the linear approximation. As stressed earlier, the full dependence on the graviton field follows from covariance. However, we have not calculated the higher order contributions in the dilaton field ϕ and the antisymmetric tensor field $B_{\mu\nu}$. It would be interesting to compare these to the results of [8], although such a comparison is plagued with ambiguities due to possible field redefinitions involving the dilaton field. It is also not clear that we should even consider the antisymmetric tensor: Our approach works only for the left-right symmetric string models and the antisymmetric tensor decouples in that case.

b) In the presence of the cutoff, the coordinate transformation, given by (3.2), has to be modified to preserve the invariance of the action. In the linear approximation, the modification is

$$F^{\mu\sigma}(X) \cong \int d^2\sigma'\, \tilde{\delta}^2_\Lambda(\sigma,\sigma') f^\mu(X(\sigma')) + \cdots. \tag{3.33}$$

Although they are not needed in this paper, the non-linear corrections to (3.2) can, in principle, be worked out.

c) There is an ambiguity in the expression for the connection given by (3.19), which is the standard result of differential geometry derived from the metric. However, since we insist on invariance under transformations with unit determinant, we are free to modify the metric by, for example

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}(\det g)^k, \tag{3.34}$$

where k is an arbitrary constant. The modified metric leads to a modified connection, and to a new set of equations. These equations are not, however, physically different from (3.29) and (3.30); they correspond to field redefinitions involving the dilaton field mentioned above. This becomes clear by noticing that the dilaton field can be taken to be the log of the determinant of g; then (3.34) is a dressing of the metric by the dilaton field.

d) It is of some interest to find out what would have happened, if we had carried out a non-covariant calculation. This means setting the connection Γ equal to zero throughout, and referring to the equations (3.20) and (3.21), it amounts to choosing the gauge

$$\Gamma^\lambda_{\mu\mu} \cong \partial_\mu h_{\mu\lambda} - \frac{1}{2}\partial_\lambda h_{\mu\mu} = 0. \tag{3.35}$$

Therefore, the equation for the graviton comes out gauge fixed, but otherwise correct. What is missing is (3.29), the equation for the dilaton field. This is because the equation of motion for the dilaton comes entirely from $\mathcal{M}$, and with connection equal to zero, $\mathcal{M}$ is also zero.

4. The First Massive Level - Non-Covariant Approach

In this section, we shall investigate the first massive state, using the tools developed in section 2. The particular question we would like to address is whether the spectrum of states that follows from the linear (free) part of the equations of motion we are going to derive is consistent with the known spectrum of the first massive level of the string. This is clearly a necessary test any successful candidate for string field equations must pass. Of course, in addition, the non-linear part of the equations should reproduce the interactions of the string theory. We will not address the question of interactions here, apart from observing that the stringent requirements of covariance we are going to impose probably fix the interaction uniquely.

The field equations will again follow from (2.38), given F ((2.30)) and the connection Γ to first order in b. For the sake of comparison with the non-covariant renormalization group approach of [1], we will first carry out a calculation with vanishing connection and F given by (2.26). Comparing the resulting physical states to those of the string, we will find that there are too many of them. In the next section the calculation is done covariantly: We start with Γ and F derived from a metric, suitably defined so as to satisfy invariance under coordinate transformations (2.3) and (2.5) to first order in b. The resulting set of states appear to be consistent with those of the left-right symmetric string model. We conclude that only the covariant approach yields equations powerful enough to produce the spectrum of at least the left-right symmetric string theory; the equations resulting from the non-covariant approach turn out to be too weak.

The starting point is the first massive level, written out in full generality:

$$\begin{aligned}S^{(1)} = \int d^2\sigma \Big(& e^{(1)}_{\mu_1\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} + e^{(1)}_{\mu_1\mu_2\mu_3}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\mu_3} \\ & + e^{(2)}_{\mu_1\mu_2\mu_3}\partial_+^2X^{\mu_1}\partial_-X^{\mu_2}\partial_-X^{\mu_3} + e^{(3)}_{\mu_1\mu_2\mu_3}\partial_-^2X^{\mu_1}\partial_+X^{\mu_2}\partial_+X^{\mu_3} \\ & + e^{(1)}_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2} + e^{(2)}_{\mu_1\mu_2}\partial_+^2\partial_-X^{\mu_1}\partial_-X^{\mu_2} \\ & + e^{(3)}_{\mu_1\mu_2}\partial_-^2\partial_+X^{\mu_1}\partial_+X^{\mu_2} + e^{(4)}_{\mu_1\mu_2}\partial_+^2X^{\mu_1}\partial_-^2X^{\mu_2}\Big),\end{aligned} \tag{4.1}$$

where the e's in this expression are local functions of the field $X^{\mu\sigma}$. Here and in many of the equations that follow, we have also simplified writing by replacing, for example, $X^{\mu_1\sigma}$ by X^{μ_1}.

Eq. (4.1) is highly redundant because of the existence of linear gauges. These result from the possibility of adding zero to (4.1) by adding a total derivative in σ_+ or in σ_- to the integrand. Such a possibility already exists for the zero mass level; adding

$$0 = \int d^2\sigma \left(\partial_+\left(\partial_-X^{\mu\sigma}\Lambda_\mu(X(\sigma))\right) - \partial_-\left(\partial_+X^{\mu\sigma}\Lambda_\mu(X(\sigma))\right)\right) \tag{4.2}$$

to (3.1) amounts to the well-known gauge transformation of the antisymmetric tensor B:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu. \tag{4.3}$$

For the first massive level, the situation is more complicated; there are six distinct linear gauge transformations. These are discussed in Appendix A, where it is also shown that, making use of these gauges, all but three of the fields appearing in (4.1) can be eliminated. The resulting linear gauge fixed form of $S^{(1)}$ reads

$$\begin{aligned}S^{(1)} = \int d^2\sigma \big(& e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} \\ & + e_{\mu_1\mu_2\mu_3}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\mu_3} + e_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}\big).\end{aligned} \tag{4.4}$$

It is also shown in Appendix A that this form of $S^{(1)}$ is in fact completely gauge fixed; in contrast to the massless level, there are no linear gauge transformations left of the form (4.3) that map it into itself. It is now easy to carry out the non-covariant calculation by substituting S given by (4.4) in (2.38), and setting $\Gamma = 0$ and F to the value given by (2.26). The resulting equation is

$$\left(\int d\tilde{\sigma}''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''} \frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\left(S^{(1)} - \frac{1}{4}\Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\right) = 0. \tag{4.5}$$

The second term of this equation can be evaluated with help of the identities (3.13) and (2.29):

$$\left(\int d\tilde{\sigma}''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''} \frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} =$$
$$= \Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \int d\tilde{\sigma}'' v(\sigma''_+)\partial_{\sigma''_+} X^{\lambda\sigma''} \frac{\delta S^{(1)}}{\delta X^{\lambda\sigma''}} - \frac{1}{4\pi}\int d\tilde{\sigma} d\tilde{\sigma}' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}}. \tag{4.6}$$

Setting $S^{(1)} = \int d\tilde{\sigma}\, U(X(\sigma))$ and using the above results gives

$$\begin{aligned}\int d\tilde{\sigma}\, v'(\sigma_+)U(X(\sigma)) - \frac{1}{4}\Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \int d\tilde{\sigma}''\, v'(\sigma''_+)U(X(\sigma''))& \\ + \frac{1}{16\pi}\int d\tilde{\sigma} d\tilde{\sigma}' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} &= 0.\end{aligned} \tag{4.7}$$

The last term in this equation can be evaluated after a tedious but straightforward calculation, with the result

$$\begin{aligned}\int d\tilde{\sigma} d\tilde{\sigma}' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}} = \int d\tilde{\sigma}\, v'(\sigma_+) \\ \times \Big((\Box e_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1}\partial_\mu e_{\mu\mu_2,\nu_1\nu_2} + \frac{1}{3}\partial_{\mu_1}\partial_{\mu_2} e_{\mu\mu,\nu_1\nu_2})\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2} \\ (\Box e_{\mu_1\mu_2\mu_3} - 4\partial_\mu e_{\mu\mu_2,\mu_1\mu_3} - \partial_{\mu_2}\partial_\mu e_{\mu_1\mu\mu_3} + \frac{4}{3}\partial_{\mu_2} e_{\mu\mu,\mu_1\mu_3})\partial_+\partial_- X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\mu_3} \\ (\Box e_{\mu_1\mu_2} - \partial_\mu e_{\mu_1\mu\mu_2} + \frac{2}{3} e_{\mu\mu,\mu_1\mu_2})\partial_+\partial_- X^{\mu_1}\partial_+\partial_- X^{\mu_2} \\ (-2\partial_\mu e_{\mu\mu_1,\mu_2\mu_3} + \frac{1}{3}\partial_{\mu_1} e_{\mu\mu,\mu_2\mu_3})\partial_+^2 X^{\mu_1}\partial_- X^{\mu_2}\partial_- X^{\mu_3} \\ (-\partial_\mu e_{\mu_1\mu\mu_2} + \frac{2}{3} e_{\mu\mu,\mu_1\mu_2})\partial_+^2\partial_- X^{\mu_1}\partial_- X^{\mu_2}\Big).\end{aligned} \tag{4.8}$$

One has to take into account possible gauge invariance of the integral on the right hand side of this equation. Because of the presence of the factor $v'(\sigma_+)$, the gauges are generated by adding a total derivative with respect to σ_- only, and as a result, there are only three of them, as opposed to six in the case of (4.1). In writing down (4.8), we have already eliminated all redundant terms and fixed the linear gauges completely.

Let us now evaluate the second term in (4.7). As opposed to (4.8), which is cutoff independent, here we encounter only cutoff dependent terms. These terms are proportional to $\Delta^{\sigma,\sigma'}(\Lambda)$ and its derivatives, evaluated at $\sigma = \sigma'$. By rotation invariance on the world sheet, the number of derivatives with respect to σ_+ must match those with respect to σ_-. Defining

$$\begin{aligned} \partial_{\sigma_+}\partial_{\sigma_-}\Delta^{\sigma,\sigma'}(\Lambda)|_{\sigma=\sigma'} &\equiv \Delta_2^{(0)}(\Lambda), \\ \partial^2_{\sigma_+}\partial^2_{\sigma_-}\Delta^{\sigma,\sigma'}(\Lambda)|_{\sigma=\sigma'} &\equiv \Delta_4^{(0)}(\Lambda), \end{aligned} \tag{4.9}$$

we have,

$$\begin{aligned} &\Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \int d^2\sigma''\, v'(\sigma''_+)U(X(\sigma''))\Big|_{\text{sing}} = \\ &= \int d^2\sigma\, v'(\sigma_+)\Big(\Delta^{(0)}(\Lambda)\Box e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} \\ &\quad + \Box e_{\mu_1\mu_2\mu_3}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\mu_3} + \Box e_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}) \\ &\quad + \Delta_2^{(0)}(\Lambda)\,(-8e_{\mu\mu_1,\mu\mu_2} + 2\partial_\mu e_{\mu\mu_1\mu_2} + 2\partial_{\mu_2}e_{\mu_1\mu\mu} - 4\partial_\mu\partial_{\mu_2}e_{\mu\mu_1})\,\partial_+X^{\mu_1}\partial_-X^{\mu_2} \\ &\quad + 2\Delta_4^{(0)}(\Lambda)e_{\mu\mu}\Big). \end{aligned} \tag{4.10}$$

These singular terms can be eliminated by renormalization as follows: The same slope renormalization that got rid of the cutoff dependent term in the equation for the tachyon (see (3.15) and the discussion that follows) also eliminates the term proportional to $\Delta^{(0)}(\Lambda)$ here. The other cutoff dependent terms are due to the contraction of two X's in the same vertex, and they can be taken care of by vertex renormalization. This amounts to eliminating them by introducing local counterterms in S of the form, for example,

$$\Delta S = \text{const} \times \Delta_2^{(0)}(\Lambda) \int d^2\sigma\, e_{\mu\mu_1,\mu\mu_2}\partial_+X^{\mu_1}\partial_-X^{\mu_2}. \tag{4.11}$$

In the operator formulation of the string theory, these divergent terms are eliminated by the operator normal ordering of the vertex.

After renormalization, one is left with the finite equations given by (4.8). They fall into two classes: Propagating equations of motion are (with the unconventionally normalized mass squared given by 16 π)

$$\begin{aligned} \Box e_{\mu_1\mu_2,\nu_1\nu_2} + 16\pi e_{\mu_1\mu_2,\nu_1\nu_2} &= 0, \\ \Box e_{\mu_1\mu_2\mu_3} + 16\pi e_{\mu_1\mu_2\mu_3} &= 0, \\ \Box e_{\mu_1\mu_2} + 16\pi e_{\mu_1\mu_2} &= 0, \end{aligned} \tag{4.12}$$

plus constraints

$$\begin{aligned} \partial_\mu e_{\mu\mu_1,\nu_1\nu_2} - \frac{1}{6}\partial_{\mu_1} e_{\mu\mu,\nu_1\nu_2} &= 0, \\ \partial_\mu e_{\mu_1\mu\mu_2} - \frac{2}{3} e_{\mu\mu,\mu_1\mu_2} &= 0, \end{aligned} \tag{4.13}$$

and also the constraints that come from $v'(\sigma_-)$

$$\begin{aligned} \partial_\nu e_{\mu_1\mu_2,\nu\nu_2} - \frac{1}{6}\partial_{\nu_2} e_{\mu_1\mu_2,\nu\nu} &= 0, \\ \partial_\mu e_{\mu_1\mu_2\mu} - \frac{2}{3} e_{\mu_1\mu_2,\nu\nu} &= 0. \end{aligned} \tag{4.14}$$

Comparing with the structure of the first massive level of the string (see Appendix B), it is clear that the above constraints are too weak. For example, in string theory, everything is expressible in terms of the analogue of $e_{\mu_1\mu_2,\nu_1\nu_2}$, whereas here $e_{\mu_1\mu_2}$ and most of $e_{\mu_1\mu_2\mu_3}$ cannot be so expressed. Clearly, the latter fields are spurious should somehow be eliminated. In the next section, we will see that the covariant approach overcomes this problem.

5. The First Massive Level - Covariant Approach

In this section, the field equations for the first massive level will be rederived, this time imposing covariance under coordinate transformations given by (3.2). When treating the massless levels, covariance under only the local transformations (first term in (3.2)) was imposed; we now require, in addition, covariance under transformations first order in b. We will initially simplify the problem by starting with flat Minkowski metric, with $\tilde{h} = 0$, in (3.4), and with the action

$$S = X^{\mu\sigma}\Delta_{\mu\sigma,\nu\sigma'}(\Lambda)X^{\nu\sigma'} + bS^{(1)} = S^{(0)} + bS^{(1)}, \tag{5.1}$$

where $S^{(1)}$ is given by (4.4). Because the metric is flat, we have to set the first term in (3.2) equal to zero, and also take into account the introduction of the cutoff in (5.1) by modifying the transformations. The modification needed is similar to (3.33):

$$X^{\mu\sigma} \to X'^{\mu\sigma} = X^{\mu\sigma} + b \int d^2\tilde{\sigma}' \, \tilde{\delta}^2_\Lambda(\sigma,\sigma') \Big(f_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'} \partial_- X^{\lambda\sigma'} + f_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'} \Big). \tag{5.2}$$

It is easy to check that, to first order in b, (5.1) is invariant under (5.2), if at the same time, the fields transform by

$$\begin{aligned} e_{\mu\nu\lambda} &\to e_{\mu\nu\lambda} + 2 f_{\mu\nu\lambda}, \\ e_{\mu\nu} &\to e_{\mu\nu} + f_{\mu\nu} + f_{\nu\mu}. \end{aligned} \tag{5.3}$$

Since only the symmetric part of $f_{\mu\nu}$ appears, from now on we will impose the condition

$$f_{\mu\nu} = f_{\nu\mu}. \tag{5.4}$$

As we have mentioned earlier, we initially work with flat metric in order to simplify the exposition. After having derived the field equations with the flat metric as background, we will then show that everything can easily be generalized to accommodate an arbitrary metric.

The above transformations are subject to the condition of unit determinant (see (2.5)). This translates into

$$\begin{aligned} 0 &= \operatorname{Tr}\log\left(\frac{\delta X'}{\delta X}\right) \\ &\cong \int d^2\tilde{\sigma} d^2\tilde{\sigma}' \, \tilde{\delta}^2_\Lambda(\sigma,\sigma') \frac{\delta}{\delta X^{\mu\sigma}} \Big(f_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'} \partial_- X^{\lambda\sigma'} + f_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'} \Big). \end{aligned} \tag{5.5}$$

Using two dimensional rotational invariance, several cutoff dependent terms vanish, giving us

$$\tilde{\delta}^2_\Lambda(0) \int d^2\tilde{\sigma} \, \big(\partial_\mu f_{\mu\nu\lambda} \partial_+ X^{\nu\sigma} \partial_- X^{\lambda\sigma} + \partial_\mu f_{\mu\nu} \partial_+\partial_- X^{\nu\sigma} \big) = 0, \tag{5.6}$$

and

$$f_{\mu\mu} = 0, \tag{5.7}$$

where we have used the fact that, by translation invariance, $\tilde{\delta}^2_\Lambda(\sigma,\sigma)$, which will be shortened to $\tilde{\delta}^2_\Lambda(0)$, does not depend on σ.

The first condition is satisfied by setting

$$\partial_\mu f_{\mu\nu\lambda} - \partial_\lambda \partial_\mu f_{\mu\nu} + \partial_\lambda \Lambda_\nu - \partial_\nu \Lambda_\lambda = 0, \tag{5.8}$$

where Λ_μ is arbitrary. It is interesting to identify field combinations that are invariant under the transformations (5.3), subject to the constraints (5.7) and (5.8). $e_{\mu\mu}$ is clearly one such invariant; another combination which is almost invariant is given by

$$k_{\nu\lambda} = \partial_\mu e_{\mu\nu\lambda} - \partial_\lambda \partial_\mu e_{\mu\nu}. \tag{5.9}$$

Under (5.3), $k_{\nu\lambda}$ undergoes the following gauge transformation:

$$k_{\nu\lambda} \to k_{\nu\lambda} + 2(\partial_\nu \Lambda_\lambda - \partial_\lambda \Lambda_\nu), \tag{5.10}$$

and so it is the appropriate gauge invariant field strength constructed out of $k_{\nu\lambda}$ that is invariant. Later, we will see that this gauge invariance is broken for reasons that will become clear.

We can now extend the metric given by (3.5) to include the first order correction in b. The key observation is that if there were no restrictions on the f's, we could gauge away the fields $e_{\mu\nu\lambda}$ and $e_{\mu\nu}$ by a transformation of the form (5.2) by setting

$$\begin{aligned} f_{\mu\nu\lambda} &= \frac{1}{2} e_{\mu\nu\lambda}, \\ f_{\mu\nu} &= \frac{1}{2} e_{\mu\nu}. \end{aligned} \tag{5.11}$$

The metric extended to first order in b is then constructed starting with flat metric to zeroth order in b and carrying out the transformation (5.2), with the f's given by (5.11):

$$X^{\mu\sigma} \to X^{\mu\sigma} - \frac{b}{2} \int d\tilde{\sigma}' \, \tilde{\delta}^2_\Lambda(\sigma, \sigma') \left(e_{\mu\nu\lambda}(X(\sigma')) \partial_+ X^{\nu\sigma'} \partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma')) \partial_+ \partial_- X^{\nu\sigma'} \right). \tag{5.12}$$

The result is

$$\begin{aligned} g_{\mu\sigma,\nu\sigma'} &= \eta_{\mu\nu} \delta^2_\Lambda(\sigma, \sigma') + b h^{(1)}_{\mu\sigma,\nu\sigma'}, \\ h^{(1)}_{\mu\sigma,\nu\sigma'} &= -\frac{1}{2} \frac{\delta}{\delta X^{\nu\sigma'}} \left(e_{\mu\alpha\lambda}(X(\sigma)) \partial_+ X^{\alpha\sigma} \partial_- X^{\lambda\sigma} + e_{\mu\alpha}(X(\sigma)) \partial_+ \partial_- X^{\alpha\sigma} \right) \\ &\quad + (\mu\sigma \leftrightarrow \nu\sigma'). \end{aligned} \tag{5.13}$$

If the constraints (5.7) and (5.8) did not exist, this would be a trivial metric, equivalent to a flat metric. In that case, there would be no need to go to the trouble of constructing it;

it would have been simpler to fix gauge by eliminating the fields $e_{\mu\nu\lambda}$ and $e_{\mu\nu}$. However, the constraints on the f's make (5.13) a non-trivial metric: Because of these constraints, $h_{\mu\sigma,\nu\sigma'}$ can no longer be transformed away, and neither can the e's be completely eliminated. It is easy to check directly, using (5.3), that, even in the presence of the constraints, (5.13) transforms correctly to first order in b under (5.2).

From the metric given above, one can find the first order correction in b to the connection and the generators of the conformal transformations (see (2.30) and the related discussion). The standard formula of differential geometry expressing the connection in terms of metric gives

$$\begin{aligned}\Gamma^{(1)\lambda\sigma'''}_{\mu\sigma,\nu\sigma'} = & -\frac{1}{2}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\int d\tilde{\sigma}''\,\tilde{\delta}^2_\Lambda(\sigma''',\sigma'')\\ & \times\left(e_{\lambda\alpha\beta}(X(\sigma''))\partial_+ X^{\alpha\sigma''}\partial_- X^{\beta\sigma''} + e_{\lambda\alpha}(X(\sigma''))\partial_+\partial_- X^{\alpha\sigma''}\right).\end{aligned} \tag{5.14}$$

Now let us compute the corrected conformal generators. Write (2.30) as

$$\begin{aligned}F^{\mu\sigma}_v(X) =& \int d\tilde{\sigma}'\,v(\sigma'_+)F^{\mu\sigma}_{\sigma'}(X),\\ F^{\mu\sigma}_{\sigma'}(X) =& \,\delta^2(\sigma'-\sigma)\partial_+ X^{\mu\sigma} + f^{\mu\sigma}_{\sigma'}(X),\end{aligned}$$

where $f^{\mu\sigma}_{\sigma'}$ starts at first order in b. Taking advantage of the fact that F transforms like a vector in the indices $\mu\sigma$, the first order correction is computed exactly as in the case of the metric. Start with

$$F^{\mu\sigma}_{\sigma'}(X) = \delta^2(\sigma'-\sigma)\partial_+ X^{\mu\sigma}$$

at zeroth order, and apply the vector transformation law to it under coordinate transformation (5.12), with the result

$$\begin{aligned}\int d\tilde{\sigma}'\,v(\sigma'_+)f^{\mu\sigma}_{\sigma'} = & -\frac{b}{2}\int d\tilde{\sigma}'\,\partial_{\sigma_+}\tilde{\delta}^2_\Lambda(\sigma,\sigma')(v(\sigma_+)-v(\sigma'_+))\\ & \times\left(e_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\right).\end{aligned} \tag{5.15}$$

This result can be simplified in the limit of large cutoff. As Λ becomes large, $\tilde{\delta}^2_\Lambda(\sigma,\sigma')\to\delta^2(\sigma-\sigma')$, and $\partial_{\sigma_+}\tilde{\delta}^2_\Lambda(\sigma,\sigma')(v(\sigma_+)-v(\sigma'_+))\to -\tilde{\delta}^2_\Lambda(\sigma,\sigma')v'(\sigma'_+)$, so we can write

$$\begin{aligned}\int d\tilde{\sigma}'\,v(\sigma'_+)f^{\mu\sigma}_{\sigma'} \cong & \frac{b}{2}\int d\tilde{\sigma}'\,v'(\sigma'_+)\tilde{\delta}^2_\Lambda(\sigma,\sigma')\\ & \times\left(e_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\right).\end{aligned} \tag{5.16}$$

We have now at hand all the information needed to evaluate (2.38) to first order in b; the action is given by (5.1), the connection by (5.14), and the conformal generator by (5.15). Since the calculation is straightforward but somewhat tedious, we skip the details and instead, indicate the main steps. Part of the calculation was already carried out for the non-covariant case in section 4; all we have to do is to add the extra terms that arise from the connection and from f in (5.15). We first calculate the terms that contribute to E_G in (2.38); a simple calculation gives

$$\int d\tilde{\sigma}' \, v(\sigma'_+) f^{\mu\sigma}_{\sigma'} \frac{\delta S^{(0)}}{\delta X^{\mu\sigma}} = -b \int d\tilde{\sigma} \, v'(\sigma_+) \partial_+ \partial_- X^{\mu\sigma} \left(e_{\mu\nu\lambda} \partial_+ X^{\nu\sigma} \partial_- X^{\lambda\sigma} + e_{\mu\nu} \partial_+ \partial_- X^{\nu\sigma} \right), \tag{5.17}$$

and therefore, to first order in b,

$$\left(\int d\tilde{\sigma}' \, v(\sigma'_+) F^{\mu\sigma}_{\sigma'} \frac{\delta S}{\delta X^{\mu\sigma}} \right)^{(1)} \cong \int d\tilde{\sigma} \, v'(\sigma_+) \left(e_{\mu_1\mu_2,\nu_1\nu_2} \partial_+ X^{\mu_1} \partial_+ X^{\mu_2} \partial_- X^{\nu_1} \partial_- X^{\nu_2} \right), \tag{5.18}$$

since the other two terms, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$, cancel. Next, expanding the Tr log as in (3.11), we compute $H_{\mu\sigma,\nu\sigma'}$ to first order in b:

$$\begin{aligned} H^{(1)}_{\mu\sigma,\nu\sigma'} =& \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma} \delta X^{\nu\sigma'}} - \Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'} \frac{\delta S^{(0)}}{\delta X^{\lambda\sigma''}} \\ =& \frac{\delta^2 S'}{\delta X^{\mu\sigma} \delta X^{\nu\sigma'}} + \partial_{\sigma_+} \partial_{\sigma_-} \left(\frac{\delta}{\delta X^{\nu\sigma'}} \left(e_{\mu\alpha\beta} \partial_+ X^{\alpha\sigma} \partial_- X^{\beta\sigma} + e_{\mu\alpha} \partial_+ \partial_- X^{\alpha\sigma} \right) \right) \\ &+ (\mu\sigma \leftrightarrow \nu\sigma'), \end{aligned} \tag{5.19}$$

where,

$$S' = \int d\tilde{\sigma} \, \left(e_{\mu_1\mu_2,\nu_1\nu_2} \partial_+ X^{\mu_1} \partial_+ X^{\mu_2} \partial_- X^{\nu_1} \partial_- X^{\nu_2} \right). \tag{5.20}$$

Next, we apply δ (see (2.25)) to $H_{\mu\sigma,\nu\sigma'}$. The contribution coming from the first term on the right in (5.19),

$$\left(\int d\tilde{\sigma}'' \, v(\sigma'') \partial_+ X^{\lambda\sigma''} \frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda \right) \left(\eta^{\mu\nu} \Delta^{\sigma,\sigma'}(\Lambda) \frac{\delta^2 S'}{\delta X^{\mu\sigma} \delta X^{\nu\sigma'}} \right),$$

has already been calculated in the last section; it is given by setting $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ in (4.8) equal to zero. The contribution of the second term in (5.19), after a somewhat lengthy computation, is given by

$$\begin{aligned} &\frac{1}{2} b \left(\int d\tilde{\sigma}'' \, v(\sigma''_+) \partial_+ X^{\lambda\sigma''} \frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda \right) \eta^{\mu\nu} \Delta^{\sigma,\sigma'}(\Lambda) \\ &\quad \times \left(\partial_{\sigma_+} \partial_{\sigma_-} \frac{\delta}{\delta X^{\nu\sigma'}} \left(e_{\mu\alpha\beta} \partial_+ X^{\alpha\sigma} \partial_- X^{\beta\sigma} + e_{\mu\alpha} \partial_+ \partial_- X^{\alpha\sigma} \right) + (\mu\sigma \leftrightarrow \nu\sigma') \right) = \\ &= -b \tilde{\delta}^2_\Lambda(0) \int d\tilde{\sigma} \, v'(\sigma_+) \partial_+ X^\alpha \partial_- X^\beta \left(\partial_\mu e_{\mu\alpha\beta} - \partial_\beta \partial_\mu e_{\mu\alpha} \right) - b \partial_+ \partial_- \tilde{\delta}^2_\Lambda(0) \int d\tilde{\sigma} \, v'(\sigma_+) e_{\mu\mu}, \end{aligned} \tag{5.21}$$

with

$$\partial_+\partial_-\tilde{\delta}^2_\Lambda(0) \equiv \left(\partial_{\sigma_+}\partial_{\sigma_-}\tilde{\delta}^2_\Lambda(\sigma,\sigma')\right)_{\sigma=\sigma'}.$$

The main steps in the computation are the following: The critical term to be evaluated turns out to be

$$\int d^2\!\sigma d^2\!\sigma' \left(\delta_\Lambda \Delta^{\sigma,\sigma'}(\Lambda)\right)\left(\partial_{\sigma_+}\partial_{\sigma_-}\frac{\delta}{\delta X^{\mu\sigma'}}\left(e_{\mu\alpha\beta}\partial_+X^{\alpha\sigma}\partial_-X^{\beta\sigma}+e_{\mu\alpha}\partial_+\partial_-X^{\alpha\sigma}\right)\right) =$$
$$= -\int d^2\!\sigma \Big(\big(\delta_\Lambda\tilde{\delta}^2_\Lambda(\sigma,\sigma')\big)_{\sigma=\sigma'}\partial_\mu e_{\mu\alpha\beta}\partial_+X^{\alpha\sigma}\partial_-X^{\beta\sigma}+\big(\partial_{\sigma_+}\partial_{\sigma_-}\delta_\Lambda\tilde{\delta}^2_\Lambda(\sigma,\sigma')\big)_{\sigma=\sigma'}e_{\mu\mu}\partial_+\partial_-X^{\alpha\sigma}\Big), \tag{5.22}$$

and the cutoff dependent factors can be simplified using (2.29):

$$\left(\delta_\Lambda\tilde{\delta}^2_\Lambda(\sigma,\sigma')\right)_{\sigma=\sigma'} = v'(\sigma_+)\tilde{\delta}^2_\Lambda(0). \tag{5.23}$$

To obtain E_G to first order, the above correction term should be added to (4.8), with $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ set equal to zero.

We now consider the term $E_{\mathcal{M}}$ in (2.38); a straightforward calculation gives the result

$$\begin{aligned}
E^{(1)}_{\mathcal{M}} &= \frac{1}{2}\int d^2\!\sigma''\, v(\sigma''_+)\left(\frac{\delta F^{\mu\sigma}_{\sigma''}}{\delta X^{\alpha\sigma'''}}G^{\alpha\sigma''',\nu\sigma'}+\frac{\delta F^{\nu\sigma'}_{\sigma''}}{\delta X^{\alpha\sigma'''}}G^{\mu\sigma,\alpha\sigma'''}-\delta_\Lambda G^{\mu\sigma,\nu\sigma'}\right)\left(\frac{\delta^2 \mathrm{Tr}\log\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right)_{P=0}\\
&= \frac{1}{2}\left(\left(v(\sigma_+)\partial_{\sigma_+}+v(\sigma'_+)\partial_{\sigma'_+}\right)\Delta^{\mu\sigma,\nu\sigma'}-\delta_\Lambda\Delta^{\mu\sigma,\nu\sigma'}\right)\left(\frac{\delta^2 \mathrm{Tr}\log\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right)_{P=0}\\
&= \frac{1}{24\pi}\int d^2\!\sigma d^2\!\sigma'\,\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}\eta^{\mu\nu}\left(\frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\mu\sigma}}{\delta X^{\nu\sigma'}}+\frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\nu\sigma'}}{\delta X^{\mu\sigma}}+\frac{\delta\Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'}}{\delta X^{\lambda\sigma''}}\right)\\
&= -\frac{b}{16\pi}\int d^2\!\sigma\, v'(\sigma_+)\Big(\partial_+\partial_-\tilde{\delta}^2_\Lambda(0)\Box e_{\lambda\lambda}\\
&\qquad +\tilde{\delta}^2_\Lambda(0)\left(\Box k_{\nu\lambda}-\partial_\nu\partial_\mu k_{\mu\lambda}+\partial_\lambda\partial_\mu k_{\mu\nu}\right)\partial_+X^\nu\partial_-X^\lambda\Big),
\end{aligned} \tag{5.24}$$

where Γ is given by (5.14), and k is defined by (5.9).

Putting together (4.8), (5.21) and (5.24) in (2.38), we finally get the equations for the first massive level. These equations contain cutoff independent terms, which come only from (4.8), and cutoff dependent terms, which all come from (5.21) and (5.24). We note that all the cutoff dependent contributions come from terms proportional to the connection Γ, and therefore they are absent from a non-covariant calculation. We first write down the cutoff independent equations:

$$\begin{gathered}
\Box e_{\mu_1\mu_2,\nu_1\nu_2}+16\pi e_{\mu_1\mu_2,\nu_1\nu_2}=0,\\
e_{\mu\mu,\nu_1\nu_2}=0, \qquad e_{\mu_1\mu_2,\nu\nu}=0,\\
\partial_\mu e_{\mu\mu_1,\nu_1\nu_2}=0, \quad \partial_\nu e_{\mu_1\mu_2,\nu\nu_1}=0.
\end{gathered} \tag{5.25}$$

We have one equation of motion and four constraints. In addition, we have three cutoff dependent equations. Two of them follow from the conformal transformations in σ_+:

$$\begin{aligned} \frac{1}{16\pi}\left(\Box k_{\nu\lambda} - \partial_\nu \partial_\mu k_{\mu\lambda} + \partial_\lambda \partial_\mu k_{\mu\nu}\right) + k_{\nu\lambda} &= 0, \\ \frac{1}{16\pi}\Box e_{\lambda\lambda} + e_{\lambda\lambda} &= 0, \end{aligned} \tag{5.26}$$

where k is defined by (5.9). The remaining equation (there is a fourth, repeated equation, for $e_{\lambda\lambda}$), results from conformal transformations in σ_- and it is conveniently written in terms of a field $\bar{k}$, defined by

$$\bar{k}_{\nu\lambda} = \partial_\mu e_{\mu\nu\lambda} - \partial_\nu \partial_\mu e_{\mu\lambda}, \tag{5.27}$$

and it reads

$$\frac{1}{16\pi}\left(\Box \bar{k}_{\nu\lambda} - \partial_\lambda \partial_\mu \bar{k}_{\nu\mu} + \partial_\nu \partial_\mu \bar{k}_{\lambda\mu}\right) + \bar{k}_{\nu\lambda} = 0. \tag{5.28}$$

The equation satisfied by k is not invariant under the gauge transformations given by (5.10). The reason for this is the following: In the computation of the determinant, the cutoff dependent factor $\tilde{\delta}^2_\Lambda(\sigma, \sigma')$ at $\sigma = \sigma'$ is σ independent and therefore it can be put in front of the integral in (5.6). The integral itself is then invariant under the gauge transformation (5.10). On the other hand, in the main step leading to (5.21), the cutoff variation of the same factor at $\sigma = \sigma'$ is σ dependent (see (5.22), (5.23), and also the discussion following (2.29)). As a consequence, an additional factor $v'(\sigma_+)$, as compared to (5.6), appears in the integral on the right hand side of (5.21), and this spoils gauge invariance under (5.10). It is, therefore, necessary to modify the condition (5.8); it should be replaced by

$$\begin{aligned} \partial_\mu f_{\mu\nu\lambda} &= 0, \\ \partial_\mu f_{\mu\nu} &= 0. \end{aligned} \tag{5.29}$$

Both k and $\bar{k}$ are invariant under the transformations satisfying these more stringent conditions.

Going back to the equations (5.25), we see that two of the constraints are too stringent,

$$e_{\mu\mu,\nu_1\nu_2} = 0, \qquad e_{\mu_1\mu_2,\nu\nu} = 0, \tag{5.30}$$

eliminating degrees of freedom from the field $e_{\mu_1\mu_2,\nu_1\nu_2}$ which are present in the string spectrum (see Appendix B). The hope is that k and $\bar{k}$ could supply the missing degrees of freedom. We shall see below that this happens in the left-right symmetric case, with parity invariance on the world sheet, which interchanges σ_+ and σ_-. In this case, e is invariant under the interchange of the μ's with ν's, and the components eliminated by (5.30) are the same as those of a symmetric second rank tensor. We have analyzed equations (5.26) and (5.28) in the left-right symmetric case, when

$$e_{\mu\nu\lambda} = e_{\mu\lambda\nu}.$$

Defining

$$l_{\nu\lambda} \equiv \partial_\mu e_{\mu\nu\lambda}, \qquad l_\nu \equiv \partial_\mu e_{\mu\nu},$$

and

$$A_{\mu\nu} \equiv 2l_{\mu\nu} - \partial_\mu l_\nu - \partial_\nu l_\mu, \qquad L_{\mu\nu} \equiv \partial_\mu l_\nu - \partial_\nu l_\mu,$$

one can easily show that equations (5.26) and (5.28) are equivalent to the equations

$$\begin{aligned} &\frac{1}{16\pi}\Box A_{\mu\nu} + A_{\mu\nu} = 0, \\ &\frac{1}{16\pi}\Box L_{\mu\nu} + L_{\mu\nu} = 0, \end{aligned} \tag{5.31}$$

plus the constraint

$$\partial_\mu \partial_\nu A_{\mu\lambda} - \partial_\mu \partial_\lambda A_{\mu\nu} = \Box L_{\lambda\nu}. \tag{5.32}$$

The number of independent degrees of freedom of the above system is the same as that of a symmetric second order tensor minus a scalar. The missing scalar is provided by $e_{\mu\mu}$, so in the final count, the fields k and $\bar{k}$ provide the missing degrees of freedom needed to establish agreement with the string theory spectrum. Unfortunately, in the general case with no left-right symmetry, there are still missing degrees of freedom, and at the present time, we have no solution to this problem. Our suspicion is that our method in its present form is applicable only in the symmetric case, and some new ideas are needed to extend it to the general case.

We close this section by a brief description of the promised extension of the results of this section to the case of a general gravitational background. This means replacing the flat background given by $\eta_{\mu\nu}\delta^2_\Lambda(\sigma, \sigma')$ in (5.13) by the metric $g_{\mu\sigma,\nu\sigma'}$ of (3.5). We have to show that the equations of this section can be covariantized with respect to this metric. Most of the time, the task is trivial; one has to keep the upper and lower indices of tensors match correctly and use the metric to raise and lower indices as needed. For example, in

(4.4), the first term on the right is correctly written, since $\partial_+ X^\mu$ and $\partial_- X^\mu$ transform as contravariant vectors. On the other hand, $\partial_+\partial_- X^\mu$ is not a vector; it should be replaced by

$$\partial_+\partial_- X^\mu \rightarrow \partial_+\partial_- X^\mu + \partial_+ X^\alpha \partial_- X^\beta \Gamma^\mu_{\alpha\beta}(X(\sigma)),$$

where the connection Γ is given by (3.18). Similarly, the partial derivative with respect to X in (5.13) should be replaced by the covariant derivative using the same connection: For example,

$$\frac{\delta}{\delta X^{\mu\sigma}}(V_{\nu\sigma'}) \rightarrow \frac{\delta}{\delta X^{\mu\sigma}}(V_{\mu\sigma'}) - \Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'} V_{\lambda\sigma''},$$

for a vector $V_{\nu\sigma'}$. One can easily show that everything in this section goes through with these modifications. Notice, however, that in all this we have worked only with the metric, which is symmetric, and we have dropped the antisymmetric tensor altogether. This is clearly permissible only in a left-right symmetric model. It is clear that, in order to generalize our treatment to the left-right non-symmetric string, we have to figure out how to incorporate the antisymmetric tensor in the discussion above.

6. Conclusions

In this paper we have proposed a new approach for deriving the string field equations from a general sigma model on the world sheet. Those equations can be made covariant under not only local, but also non-local transformations in the field space. In this approach the world sheet one loop result is exact, although it may only give incomplete information, to be supplemented by higher loop results. We applied this method to derive the equations for the tachyon, massless and first massive level. The spectrum of states that follows from the linear part of these equations of motion was shown to agree with the known spectrum of strings. This is in contrast with a non-covariant approach, where the equations are too weak to produce the right spectrum.

In this paper we only analyzed the linear part of the equations. We did not address the question of string interactions, neither did we attempt to extend our results to higher string loops. It would be desirable to go beyond the expansion we have used, and establish exact covariance under non-local transformations. Also, natural generalizations such as a better treatment of left-right non-symmetric closed string, strings with boundaries (open strings) and fermionic strings are worthy of investigation.

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Appendix A.

In this section we fill up the steps that lead from (4.1) to (4.4). As it was said in the comments that follow (4.1), of the eight fields present there, all but three can be eliminated by linear gauge transformations. The six distinct linear gauge transformations that we can add to (4.1) are:

$$
\begin{aligned}
1)\quad & \partial_+(\varepsilon_{\mu,\nu_1\nu_2}\partial_+X^\mu\partial_-X^{\nu_1}\partial_-X^{\nu_2}) = \partial_{\mu_1}\varepsilon_{\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2}\\
&\quad + \varepsilon_{\mu,\nu_1\nu_2}(\partial_+^2X^\mu\partial_-X^{\nu_1}\partial_-X^{\nu_2} + 2\partial_+X^\mu\partial_+\partial_-X^{\nu_1}\partial_-X^{\nu_2})\\
2)\quad & \partial_-(\bar\varepsilon_{\mu_1\mu_2,\nu}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^\nu) = \partial_{\nu_1}\bar\varepsilon_{\mu_1\mu_2,\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2}\\
&\quad + \bar\varepsilon_{\mu_1\mu_2,\nu}(2\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^\nu + \partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-^2X^\nu)\\
3)\quad & \partial_+(\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_-X^{\mu_2}) = \partial_\nu\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+X^\nu\partial_+\partial_-X^{\mu_1}\partial_-X^{\mu_2}\\
&\quad + \varepsilon^{(1)}_{\mu_1\mu_2}(\partial_+^2\partial_-X^{\mu_1}\partial_-X^{\mu_2} + \partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2})\\
4)\quad & \partial_-(\bar\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}) = \partial_\nu\bar\varepsilon^{(1)}_{\mu_1\mu_2}\partial_-X^\nu\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\\
&\quad + \bar\varepsilon^{(1)}_{\mu_1\mu_2}(\partial_-^2\partial_+X^{\mu_1}\partial_+X^{\mu_2} + \partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2})\\
5)\quad & \partial_+(\varepsilon^{(2)}_{\mu_1\mu_2}\partial_+X^{\mu_1}\partial_-^2X^{\mu_2}) = \partial_\nu\varepsilon^{(2)}_{\mu_1\mu_2}\partial_+X^\nu\partial_+X^{\mu_1}\partial_-^2X^{\mu_2}\\
&\quad + \varepsilon^{(2)}_{\mu_1\mu_2}(\partial_+^2X^{\mu_1}\partial_-^2X^{\mu_2} + \partial_+X^{\mu_1}\partial_+\partial_-^2X^{\mu_2})\\
6)\quad & \partial_-(\bar\varepsilon^{(2)}_{\mu_1\mu_2}\partial_-X^{\mu_1}\partial_+^2X^{\mu_2}) = \partial_\nu\bar\varepsilon^{(2)}_{\mu_1\mu_2}\partial_-X^\nu\partial_-X^{\mu_1}\partial_+^2X^{\mu_2}\\
&\quad + \bar\varepsilon^{(2)}_{\mu_1\mu_2}(\partial_-^2X^{\mu_1}\partial_+^2X^{\mu_2} + \partial_-X^{\mu_1}\partial_+^2\partial_-X^{\mu_2}).
\end{aligned}
\tag{A.1}
$$

To eliminate $e^{(4)}_{\mu\nu}$ choose

$$e^{(4)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_1\mu_2} + \bar\varepsilon^{(2)}_{\mu_2\mu_1} = 0. \tag{A.2}$$

To eliminate $e^{(2)}_{\mu\nu}$ choose

$$e^{(2)}_{\mu_1\mu_2} + \varepsilon^{(1)}_{\mu_1\mu_2} + \bar\varepsilon^{(2)}_{\mu_2\mu_1} = 0. \tag{A.3}$$

To eliminate $e^{(3)}_{\mu\nu}$ choose

$$e^{(3)}_{\mu_1\mu_2} + \bar\varepsilon^{(1)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_2\mu_1} = 0. \tag{A.4}$$

To eliminate $e^{(2)}_{\mu\nu\lambda}$ choose

$$e^{(2)}_{\mu_1\mu_2\mu_3} + \varepsilon_{\mu_1,\mu_2\mu_3} + \frac{1}{2}(\partial_{\mu_2}\bar\varepsilon^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\bar\varepsilon^{(2)}_{\mu_2\mu_1}) = 0. \tag{A.5}$$

To eliminate $e^{(3)}_{\mu\nu\lambda}$ choose

$$e^{(3)}_{\mu_1\mu_2\mu_3} + \bar{\varepsilon}_{\mu_2\mu_3,\mu_1} + \frac{1}{2}(\partial_{\mu_2}\varepsilon^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\varepsilon^{(2)}_{\mu_2\mu_1}) = 0. \tag{A.6}$$

By choosing $\varepsilon^{(2)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1}$, $\varepsilon^{(1)}_{\mu_1\mu_2}$, $\bar{\varepsilon}^{(1)}_{\mu_1\mu_2}$, $\varepsilon_{\mu_1,\mu_2\mu_3}$, and $\bar{\varepsilon}_{\mu_2\mu_3,\mu_1}$ properly, we can eliminate everything except $e_{\mu_1\mu_2,\nu_1\nu_2}$, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$; we dropped the superscript (1) after gauge fixing. The transformations that preserve this gauge are

$$\begin{gathered}
\varepsilon^{(2)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \qquad \varepsilon^{(1)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \qquad \bar{\varepsilon}^{(1)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_2\mu_1} = 0, \\
2\varepsilon_{\mu_1,\mu_2\mu_3} + \partial_{\mu_2}\bar{\varepsilon}^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \\
2\bar{\varepsilon}_{\mu_2\mu_3,\mu_1} + \partial_{\mu_2}\varepsilon^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\varepsilon^{(2)}_{\mu_2\mu_1} = 0,
\end{gathered} \tag{A.7}$$

plus, we could also add

$$\partial_+(\varepsilon_{\mu_1}\partial_+\partial_-^2 X^{\mu_1}) - \partial_-(\varepsilon_{\mu_1}\partial_-\partial_+^2 X^{\mu_1}) = \partial_{\mu_1}\varepsilon_{\mu_2}(\partial_+ X^{\mu_1}\partial_+\partial_-^2 X^{\mu_2} - \partial_- X^{\mu_1}\partial_-\partial_+^2 X^{\mu_2}). \tag{A.8}$$

All of these linear transformations act trivially on (4.4); they leave $e_{\mu_1\mu_2,\nu_1\nu_2}$, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ invariant. This means that linear gauges are completely fixed in the form given by (4.4).

Appendix B.

In this section we show that the constraints (equations (4.13) and (4.14)) obtained in section 4 for the first massive level fields are too weak. This will be done by comparing those constraints with the analogue constraints of the first massive level of the string.

To obtain these consider the most general level 2 state $|s\rangle$ given by

$$\begin{aligned}
|s\rangle = \Big(&E_{\mu_1\mu_2,\nu_1\nu_2} a_1^{\dagger\mu_1} a_1^{\dagger\mu_2} b_1^{\dagger\nu_1} b_1^{\dagger\nu_2} + E_{\mu,\nu_1\nu_2} a_2^{\dagger\mu} b_1^{\dagger\nu_1} b_1^{\dagger\nu_2} \\
&+ E_{\mu_1\mu_2,\nu} a_1^{\dagger\mu_1} a_1^{\dagger\mu_2} b_2^{\dagger\nu} + E_{\mu\nu} a_2^{\dagger\mu} b_2^{\dagger\nu}\Big)|0\rangle,
\end{aligned} \tag{B.1}$$

where a_n^μ, $a_n^{\dagger\mu}$ and b_n^μ, $b_n^{\dagger\mu}$ are the closed string operators. This state satisfies the relations

$$(L_0 - 1)|s\rangle = (\bar{L}_0 - 1)|s\rangle = 0$$

and

$$L_1|s\rangle = \bar{L}_1|s\rangle = 0, \qquad L_2|s\rangle = \bar{L}_2|s\rangle = 0,$$

from which we get some conditions between the E's. The gauge freedom present in these conditions can be taken care of by adding zero norm states to $|s\rangle$. After that is done we get the constraints

$$
\begin{aligned}
p^{\mu} E_{\mu\mu_1,\nu_1\nu_2} + \sqrt{2} p_{\mu_1} E_{,\nu_1\nu_2} &= 0, \\
p^{\nu} E_{\mu_1\mu_2,\nu_1\nu} + \sqrt{2} p_{\nu_1} E_{\mu_1\mu_2,} &= 0, \\
p^{\mu} E_{\mu\mu_1,} + \sqrt{2} p_{\mu_1} E &= 0, \\
p^{\nu} E_{,\nu_1\nu} + \sqrt{2} p_{\nu_1} E &= 0, \\
4\sqrt{2} E_{,\nu_1\nu_2} - E_{\mu\mu,\nu_1\nu_2} &= 0, \\
4\sqrt{2} E_{\mu_1\mu_2,} - E_{\mu_1\mu_2,\nu\nu} &= 0,
\end{aligned}
\tag{B.2}
$$

where the new E's are related to the old ones by

$$
E_{\mu,\nu_1\nu_2} = p_{\mu} E_{,\nu_1\nu_2}, \qquad E_{\mu_1\mu_2,\nu} = p_{\nu} E_{\mu_1\mu_2,}, \qquad E_{\mu,\nu} = p_{\mu} p_{\nu} E.
$$

Combining, the only constraints on $E_{\mu_1\mu_2,\nu_1\nu_2}$ are

$$
\begin{aligned}
p^{\mu} E_{\mu\mu_1,\nu_1\nu_2} + \frac{1}{4} p_{\mu_1} E_{\mu\mu,\nu_1\nu_2} &= 0, \\
p^{\nu} E_{\mu_1\mu_2,\nu_1\nu} + \frac{1}{4} p_{\nu_1} E_{\mu_1\mu_2,\nu\nu} &= 0,
\end{aligned}
\tag{B.3}
$$

and all other E's are given in terms of $E_{\mu_1\mu_2,\nu_1\nu_2}$. If we consider that the E's play the analogue roll of the e's in section 4, then the constraints (4.13) and (4.14) are not powerfull enough, because for example, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ are not completely determined in terms of $e_{\mu_1\mu_2,\nu_1\nu_2}$.

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